

NEW SPINORIAL MASS-QUASILOCAL ANGULAR MOMENTUM INEQUALITY FOR INITIAL DATA WITH marginally FUTURE TRAPPED SURFACE

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ABSTRACT. We prove a new geometric inequality that relates the Arnowitt-Deser-Misner (ADM) mass of initial data to a quasilocal angular momentum of a marginally future trapped surface (MFTS) inner boundary. The inequality is expressed in terms of a 1-spinor, which satisfies an intrinsic first-order Dirac-type equation. Furthermore, we show that if the initial data is axisymmetric, then the divergence-free vector used to define the quasilocal angular momentum cannot be a Killing field of the generic boundary.

1. INTRODUCTION

Geometric inequalities arise naturally in General Relativity (GR) as relations involving quantities characterizing black holes, like mass, angular momentum, and horizon area. One of the most significant example of a geometric inequality is the positive (ADM) mass theorem. It was first proven by Schoen and Yau for the time-symmetric case in dimension three in [23] and [24] and later extended to dimension less than 8 in [22]. On the other hand, Witten [30] proved the spinorial version of the theorem in dimension 3, and Bartnik realised that the proof could be easily extended to higher dimensions provided the manifold was spin [6]. Witten's spinorial version of the theorem was extended to the case of initial data with trapped surfaces by Gibbons, Hawking, Horowitz, and Perry in [11]. The spinorial approach was further adapted by Ludvigsen and Vickers in the context of the Bondi mass in [18]. A refined version of the positivity of the ADM mass has been formulated by Penrose in the form of a lower bound on this quantity in terms of the horizon area of a black hole. If true, it would provide further evidence in favor of the weak cosmic censorship conjecture [20]. There exists a stronger version of the Penrose inequality involving the angular momentum of the initial data, namely

$$m \geq \left(\frac{|S|}{16\pi} + \frac{4\pi J^2}{|S|} \right)^{\frac{1}{2}}, \quad (1)$$

where m , J and $|S|$ are the ADM mass, angular momentum and the outermost apparent horizon area respectively—see e.g. [9, 10, 19] for more details. This inequality is expected to hold only in axial symmetry. It admits a rigidity case, where equality exclusively occurs for the Kerr black hole. A quasilocal version of this relation states that

$$m \geq \left(\frac{|S|}{16\pi} + \frac{4\pi J_{BH}^2}{|S|} \right)^{\frac{1}{2}},$$

where J_{BH} is the quasilocal angular momentum of the horizon.

Geometric inequalities for black holes remain a very active area of research with new interesting results being obtained. Among them is a bound on the ADM energy in terms of horizon area, angular

momentum, and charge obtained by Jaracz and Khuri in [12]. In [2, 3] a different approach was considered by Anglada, who used the monotonic properties of the Geroch and Hawking energy along the inverse mean curvature flow in order to prove a Penrose-like inequality with angular momentum. The first author and Tafel considered perturbations of Schwarzschild data and showed that (1) holds in this setting [15, 16]. Another refinement to the Penrose inequality with angular momentum has been proven by Alae and Kunduri for 4-dimensional biaxially symmetric maximal initial data [1]. Additionally, recent numerical results such as the ones obtained in [17] give support to the validity of (1) in the context of axial symmetry. The examples presented above are far from exhaustive, providing a glimpse into contemporary research in geometric inequalities in GR. We refer the reader to [10, 19] for further references.

In the present work a spinorial approach is used to obtain a geometric inequality involving the ADM mass of the initial data for the vacuum Einstein field equations and a quasilocal angular momentum (à la Szabados [27, 28]) of the MFTS inner boundary. It generalises the result presented in [14] to the case of non-vanishing connection 1-form on the normal bundle of the boundary. The solvability of the boundary value problem for the so-called *approximate twistor equation* is still an essential ingredient for deriving the main result. The existence of solution is used to obtain a basic mass inequality

$$4\pi m \geq \sqrt{2} \oint_{\partial\mathcal{S}} \widehat{\phi}^A \gamma_A{}^B \mathcal{D}_{BC} \phi^C dS, \quad (2)$$

where \mathcal{D}_{AB} and γ_{AB} are the 2-dimensional Sen connection and the complex metric on the boundary respectively (see below for details), while ϕ_A is a valence 1 spinor on $\partial\mathcal{S}$. The right-hand side of (2) can be rewritten in terms of the inner null expansion θ^- of the boundary and the aforementioned angular momentum, provided that ϕ_A satisfies a certain first-order Dirac-type equation. Ultimately,

$$4\pi m \geq \sqrt{2} \oint_{\mathbb{S}^2} \rho' |\widetilde{\phi}_0|^2 \Omega d\mathbb{S}^2 + \frac{\kappa}{\sqrt{2}} O[\widetilde{\phi}, U], \quad (3)$$

where $\rho' = -\frac{\theta^-}{2}$, $\widetilde{\phi}_A$ is a Dirac eigenspinor on \mathbb{S}^2 , Ω is a conformal factor relating the metrics of $\partial\mathcal{S}$ and \mathbb{S}^2 and $O[\widetilde{\phi}, U]$ a quasilocal angular momentum depending on $\widetilde{\phi}_A$ and a rotation potential U defined below. It should be noted that the integrals in (3) are now taken with respect to the 2-sphere volume element.

A natural symmetry associated with the angular momentum is the existence of axial Killing vector. Therefore, with such assumption we analyze a scenario where the quasilocal angular momentum is generated by such vector (on top of arising from a spinor ϕ_A) and show that it is in fact impossible for a generic MFTS inner boundary $\partial\mathcal{S}$.

The article is structured as follows: Section 2 provides a discussion of our main mathematical tools, in particular a new formalism for the 1 + 1 + 2 decomposition of spinors. Section 3 is an adaptation of the result of [14] to the case of non-vanishing connection 1-form on the normal bundle of $\partial\mathcal{S}$. In Section 4 we present the main result of this work, a new mass-quasilocal angular momentum inequality for the initial data with a MFTS. In the last section we particularise our analysis to the axisymmetric setting and show that the divergence-free vector generating the quasilocal angular momentum cannot arise simultaneously from a first-order Dirac-type equation and be a Killing vector of the boundary.

In the following, 4-dimensional metrics are considered to have the signature $(+---)$. As a result, Riemannian 3- and 2-dimensional metrics will be negative definite. Whenever appropriate, we will

expand spinorial expressions using either the Geroch-Held-Penrose (GHP) or Newman-Penrose (NP) formalism, following the conventions outlined in [21]. Throughout this paper, we employ abstract index notation, with lowercase letters representing tensorial indices and uppercase letters representing spinorial indices. Bold font will be used to denote components in a basis.

2. PRELIMINARIES

2.1. Basic setting. An initial data set $(\mathcal{S}, h_{ab}, K_{ab})$ for the vacuum Einstein field equations is said to be *asymptotically Schwarzschildian* if the metric h_{ab} and the second fundamental form K_{ab} satisfy the decay conditions

$$h_{ab} = - \left(1 + \frac{2m}{r} \right) \delta_{ab} + o_\infty(r^{-3/2}), \quad (4a)$$

$$K_{ab} = o_\infty(r^{-5/2}), \quad (4b)$$

with $r^2 \equiv (x^1)^2 + (x^2)^2 + (x^3)^2$, $(x^\alpha) = (x^1, x^2, x^3)$ being asymptotically Cartesian coordinates and m the ADM mass. In this work we assume that \mathcal{S} has an inner boundary $\partial\mathcal{S}$ which is a topological 2-sphere and is equipped with a metric σ_{ab} . We consider a $1+1+2$ spinor formalism, first proposed in [26] by Szabados, and based on the use of $SL(2, \mathbb{C})$ spinors. Maintaining the same philosophy as in [14], the so-called $SU(2, \mathbb{C})$ spinors (or space spinors) introduced in [25] will be essential to our purposes since they allow to work efficiently on spacelike hypersurfaces. For more information on the spinor formalism, we refer the reader to [4, 29].

Let $\tau^{AA'}$ be a spinorial counterpart of the orthogonal future vector τ^a to \mathcal{S} such that $\tau_{AA'}\tau^{AA'} = 2$. Likewise, we will denote a spinorial counterpart of the normal vector ρ^a to $\partial\mathcal{S}$ on \mathcal{S} as $\rho^{AA'}$ and assume that $\rho_{AA'}\rho^{AA'} = -2$. Let us choose $\rho_{AA'}$ so that it is pointing outwards, towards infinity. The spinors $\tau_{AA'}$ and $\rho_{AA'}$ are orthogonal —i.e. $\tau_{AA'}\rho^{AA'} = 0$. We consider dyads $\{o^A, \iota^A\}$ such that

$$\tau_{AA'} = o_A \bar{o}_{A'} + \iota_A \bar{\iota}_{A'},$$

$$\rho_{AA'} = o_A \bar{o}_{A'} - \iota_A \bar{\iota}_{A'}.$$

The spinor $\tau_{AA'}$ is used to construct a space-spinor version of a given spinor. In particular,

$$\gamma_{AB} \equiv \tau_{(B}{}^{A'} \rho_{A)A'}$$

is the space-spinor version of $\rho_{AA'}$, also called the *complex metric*. By construction, the complex metric can be understood as the space spinor version of the vector $\rho_{AA'}$, which is the spacelike normal to $\partial\mathcal{S}$ on \mathcal{S} (with the normalization $\rho_{AA'}\rho^{AA'} = -2$). It satisfies $\gamma_A{}^B \gamma_B{}^C = \delta_A{}^C$ and can be expressed as

$$\gamma_{AB} = o_A \iota_B + o_B \iota_A$$

with the use of spin dyad.

The spinorial counterpart of the projection operator $\Pi_a{}^b$ onto the 2-dimensional surface $\partial\mathcal{S}$ can now be defined as

$$\Pi_{AA'}{}^{BB'} \equiv \delta_A{}^B \delta_{A'}{}^{B'} - \frac{1}{2} \tau_{AA'} \tau^{BB'} + \frac{1}{2} \rho_{AA'} \rho^{BB'} = \frac{1}{2} \left(\delta_A{}^B \delta_{A'}{}^{B'} - \gamma_A{}^B \bar{\gamma}_{A'}{}^{B'} \right). \quad (5)$$

Similarly, the spinorial counterpart of the projector $T_{AA'BB'}$ onto \mathcal{S} reads

$$T_{AA'}{}^{BB'} \equiv \frac{1}{2} \left(\delta_A{}^B \delta_{A'}{}^{B'} - \frac{1}{2} \tau_{AA'} \tau^{BB'} \right).$$

Let $\nabla_{AA'}$ be the spinorial counterpart of the spacetime covariant derivative ∇_a . The $T_{AA'}{}^{BB'}$ projector allows to define the 3-dimensional Sen connection $\mathcal{D}_{AA'}$ associated to $\nabla_{AA'}$ as

$$\mathcal{D}_{AA'} \pi_C \equiv T_{AA'}{}^{BB'} \nabla_{BB'} \pi_C.$$

As mentioned above, the $SU(2, \mathbb{C})$ (i.e. space-spinor version) of $\mathcal{D}_{AA'}$ can be constructed by means of $\tau^{AA'}$ as

$$\mathcal{D}_{AB} = \tau_{(B}{}^{A'} \mathcal{D}_{A)A'}.$$

The space-spinor version ∇_{AB} of the 3-dimensional Levi-Civita connection on \mathcal{S} can be recovered from \mathcal{D}_{AB} via

$$\nabla_{AB}\pi_C = \mathcal{D}_{AB}\pi_C - \frac{1}{2}K_{ABC}{}^Q\pi_Q,$$

where $K_{ABCD} \equiv \tau_D{}^{C'} \mathcal{D}_{AB}\tau_{CC'}$ is the Weingarten spinor (note that symbol D_{AB} was chosen in [14] to denote the space version of the 3-dimensional Levi-Civita connection. Here we prefer to keep the symbol ∇ for Levi-Civita connections). The Weingarten spinor decomposes as

$$K_{ABCD} = \Omega_{ABCD} - \frac{1}{3}K\epsilon_{A(C}\epsilon_{D)B},$$

where $\Omega_{ABCD} \equiv K_{(ABCD)}$ is its fully symmetrized part, and $K \equiv K_{AB}{}^{AB}$ is the mean curvature of \mathcal{S} . The 3-dimensional Levi-Civita operator satisfies $\nabla_{AB}\epsilon_{CD} = 0$.

Given a spinor $\pi_{A_1\dots A_K}$, its *Hermitian conjugate* is defined as follows,

$$\widehat{\pi}_{A_1\dots A_K} \equiv \tau_{A_1}{}^{A'_1} \dots \tau_{A_K}{}^{A'_K} \overline{\pi}_{A'_1\dots A'_K}.$$

A spinor $\pi_{A_1\dots A_K}$ is said to be real if

$$\widehat{\pi}_{A_1B_1\dots A_kB_k}{}^{C_1D_1\dots C_mD_m} = (-1)^{(k+m)}\pi_{A_1B_1\dots A_kB_k}{}^{C_1D_1\dots C_mD_m}.$$

The space counterpart of the Levi-Civita connection ∇_{AB} is real in the sense that $\widehat{\nabla_{AB}\pi_C} = -\nabla_{AB}\widehat{\pi}_C$, while

$$\widehat{\mathcal{D}_{AB}\pi_C} = -\mathcal{D}_{AB}\widehat{\pi}_C + K_{ABC}{}^D\widehat{\pi}_D.$$

2.2. On the inner boundary. A 2-dimensional Sen connection $\mathcal{D}_{AA'}$ on $\partial\mathcal{S}$ arises as a Π -projection of $\nabla_{AA'}$, i.e.

$$\mathcal{D}_{AA'} \equiv \Pi_{AA'}{}^{BB'} \nabla_{BB'}, \quad (6)$$

and its associated $SU(2, \mathbb{C})$ version is given by $\mathcal{D}_{AB} \equiv \tau_{(B}{}^{A'} \mathcal{D}_{A)A'}$. It can be promoted to the 2-dimensional Levi-Civita connection $\mathcal{V}_{AA'}$ with the use of the transition spinor $Q_{AA'BC}$,

$$\mathcal{V}_{AA'}v_{BB'} = \mathcal{D}_{AA'}v_{BB'} - Q_{AA'B}{}^C v_{CB'} - \overline{Q}_{AA'B'}{}^{C'} v_{BC'}, \quad (7)$$

where

$$Q_{AA'BC} \equiv -\frac{1}{2}\gamma_C{}^D \mathcal{D}_{AA'}\gamma_{BD}. \quad (8)$$

The $\mathcal{V}_{AA'}$ connection is torsion-free by definition, i.e. $(\mathcal{V}_{AA'}\mathcal{V}_{BB'} - \mathcal{V}_{BB'}\mathcal{V}_{AA'})\phi = 0$, and its curvature spinor $\mathcal{V}'_{CC'DD'AA'BB'}$ can be defined with the use of the following relation

$$(\mathcal{V}_{AA'}\mathcal{V}_{BB'} - \mathcal{V}_{BB'}\mathcal{V}_{AA'})\pi^C = \mathcal{V}'^C{}_{QAA'BB'}\pi^Q = (m_a\overline{m}_b - \overline{m}_a m_b)(\rho\rho' - \sigma\sigma' + \Psi_2)\gamma^{CD}\pi_D, \quad (9)$$

where $m^{AA'} \equiv o^A\overline{\iota}^{A'}$, $\overline{m}^{AA'} \equiv \iota^A\overline{\sigma}^{A'}$, and $\Psi_2 \equiv \Psi_{ABCD}o^A o^B \iota^C \iota^D$ is a component of the Weyl spinor Ψ_{ABCD} .

Another 2-dimensional connection (\mathcal{D}_{AB}) can be obtained by considering a space-spinor counterpart of $\mathcal{V}_{AA'}$, i.e.

$$\mathcal{D}_{AB} \equiv \tau_{(B}{}^{A'} \mathcal{V}_{A)A'}. \quad (10)$$

It is particularly useful in some calculations and can be related to \mathcal{D}_{AB} via

$$\mathcal{D}_{AB}\pi_C = \mathcal{D}_{AB}\pi_C - Q_{AB}{}^Q\pi_Q, \quad (11)$$

where the transition spinor is now given by

$$Q_{AB}{}^C{}_D \equiv -\frac{1}{2}\gamma_D{}^Q \mathcal{D}_{AB}\gamma_{QC} = \sigma' o_A o_B o_C o_D + \sigma \iota_A \iota_B \iota_C \iota_D - \rho o_A o_B \iota_C \iota_D - \rho' \iota_A \iota_B o_C o_D. \quad (12)$$

A natural choice for the ingoing and outgoing null vectors k^a and l^a spanning the normal bundle to $\partial\mathcal{S}$ is given by

$$k^a = \frac{1}{2}(\tau^a - \rho^a), \quad l^a = \frac{1}{2}(\tau^a + \rho^a).$$

The nature of a trapped surfaces is determined by the causal character and orientation of its mean curvature vector, or equivalently, by the signs of the associated inner and outer null expansions,

$$\theta^- = \sigma^{ab}\nabla_a k_b, \quad \theta^+ = \sigma^{ab}\nabla_a l_b.$$

Making use of Proposition 4.14.2 in [21] we can express θ^- and θ^+ in terms of a GHP spin coefficients,

$$\theta^- = -2\rho', \quad \theta^+ = -2\rho.$$

We are now ready to define a marginally future trapped surface.

Definition 1. *The boundary $\partial\mathcal{S}$ is said to be a marginally future trapped surface (MFTS) if $\theta^+ = 0$ and $\theta^- \leq 0$ or if $\theta^- = 0$ and $\theta^+ \leq 0$, i.e. if $\rho = 0$ and $\rho' \geq 0$ or if $\rho' = 0$ and $\rho \geq 0$ on $\partial\mathcal{S}$.*

The 2-dimensional connection \mathcal{D}_{AB} annihilates ϵ_{AB} and γ_{AB} :

$$\mathcal{D}_{AB}\epsilon_{CD} = 0, \quad \mathcal{D}_{AB}\gamma_{CD} = 0.$$

However, \mathcal{D}_{AB} is not a Levi-Civita connection on $\partial\mathcal{S}$ as it has a non-vanishing torsion,

$$\mathcal{D}_{AB}\mathcal{D}_{CD}\phi - \mathcal{D}_{CD}\mathcal{D}_{AB}\phi = \frac{1}{2}(A_{AB}\gamma_C^X\delta_D^Y - A_{CD}\gamma_A^X\delta_B^Y)\mathcal{D}_{XY}\phi, \quad (13)$$

where A_{AB} is defined as

$$A_{AB} \equiv \tau^{CC'}\mathcal{D}_{AB}\rho_{CC'} = 2(\alpha + \bar{\beta})\iota_A\iota_B - 2(\bar{\alpha} + \beta)\iota_A\iota_B. \quad (14)$$

Notice that \mathcal{D}_{AB} was considered in [14] to be the space version of the 2-dimensional Levi-Civita connection, being defined in the same way as in this work. This was possible since the boundary was torsion-free in [14] ($\alpha + \bar{\beta} = 0$ on $\partial\mathcal{S}$). In the current work this restriction is dropped, and $A_{AB} \neq 0$. This spinor is real, $\widehat{A}_{AB} = -A_{AB}$, and satisfies $\gamma^{AB}A_{AB} = 0$. We can use it to recover a space-spinor version of the Levi-Civita connection ∇_{AB} ,

$$\nabla_{AB}\pi_C \equiv \mathcal{D}_{AB}\pi_C - \frac{1}{4}A_{AB}\gamma_C^D\pi_D. \quad (15)$$

Indeed,

$$\nabla_{AB}\epsilon_{CD} = \nabla_{AB}\gamma_{CD} = 0, \quad \gamma^{AB}\nabla_{AB}\pi_C = 0, \quad (16)$$

and ∇_{AB} has vanishing torsion. Moreover,

$$(\nabla_{AC}\nabla_B^C - \nabla_B^C\nabla_{AC})\pi^B = \frac{1}{2}(\Psi_2 + \rho\rho' - \sigma\sigma' + \text{c.c.})\pi_A,$$

where c.c. denotes the complex conjugation of the expression in the brackets above. Ultimately, the 2-dimensional Sen and Levi-Civita connections on the boundary are related in the following way,

$$\mathcal{D}_{AB}\phi_C = \nabla_{AB}\phi_C + \frac{1}{4}A_{AB}\gamma_C^D\phi_D + Q_{AB}{}^L{}_C\phi_L. \quad (17)$$

The connection ∇_{AB} is real, i.e.

$$\widehat{\nabla_{AB}\pi_C} = -\nabla_{AB}\widehat{\pi_C},$$

while,

$$\widehat{\mathcal{D}_{AB}\pi_C} = -\mathcal{D}_{AB}\widehat{\pi_C} + (Q_{ABC}{}^D + \widehat{Q}_{ABC}{}^D)\widehat{\pi_D} + \frac{1}{2}\gamma_C^D\widehat{\pi_D}A_{AB}.$$

In the following we will also require an expression for the Hermitian conjugate of the connection \mathcal{D}_{AB} . A direct computation yields

$$\widehat{\mathcal{D}_{AB}\pi_C} = -\mathcal{D}_{AB}\widehat{\pi_C} + \frac{1}{2}\gamma_C^D\widehat{\pi_D}A_{AB}. \quad (18)$$

In [27], Szabados proposed the following definition of quasilocal angular momentum associated with $\partial\mathcal{S}$,

$$O[N] \equiv -\frac{1}{2\kappa} \oint_{\partial\mathcal{S}} N^c A_c dS, \quad (19)$$

where N^a is a divergence-free vector on $\partial\mathcal{S}$, $A_c = \rho_a \Pi^f{}_c \nabla_f \tau^a$ is the connection 1-form on the normal bundle of $\partial\mathcal{S}$ and $\kappa = 8\pi G$ the gravitational coupling constant. After inspecting the definition (14) of a spinor A_{AB} we immediately see that it is in fact a space-spinor counterpart of A_b in the expression above.

In the sequel we will also make use of a Hodge decomposition on $\partial\mathcal{S}$. Specifically, given any 1-form V_a on $\partial\mathcal{S}$ there exist two functions f and f' such that

$$V_a = \epsilon_a{}^b \nabla_b f + \nabla_a f',$$

where ϵ_{ab} is the 2-dimensional Riemannian volume form of the boundary $\partial\mathcal{S}$.

2.3. Conformal rescaling of the 2-dimensional Dirac operator. An action of a 2-dimensional (Levi-Civita) *Dirac operator* on a spinor π_A is given by $\nabla_A{}^B \pi_B$. The purpose of this subsection is to explore its properties under conformal rescalings of the inner boundary metric σ_{ab} . Indeed, according to the uniformization theorem for compact Riemannian surfaces (*cf.* [13, Theorem 4.4.1]), σ_{ab} is conformal to the spherical metric, i.e.

$$\sigma_{ab} = \Omega^2 \tilde{\sigma}_{ab},$$

where $\Omega : \mathbb{S}^2 \rightarrow \mathbb{R}$ is a non-negative smooth function and $\tilde{\sigma}_{ab}$ a round 2-sphere metric. Given any holonomic basis $\{\partial_{AB}\}$ on $\partial\mathcal{S}$, the covariant derivative of π_A can be expressed as $\nabla_{AB} \pi_C = \partial_{AB}(\pi_C) - \Gamma_{AB}{}^Q{}_C \pi_Q$, where $\Gamma_{AB}{}^Q{}_C \equiv \frac{1}{2} \Gamma_{AB}{}^{QL}{}_{CL}$ are the spin coefficients associated to the Christoffel symbols of σ_{ab} . Since $\sigma_{ABCD} = \Omega^2 \tilde{\sigma}_{ABCD}$, one arrives at

$$\nabla_{AB} \pi_C = \tilde{\nabla}_{AB} \pi_C - \frac{1}{2} \left(\partial_{AB}(\log \Omega) \tilde{\sigma}{}^{EF}{}_{CF} + \partial_{CF}(\log \Omega) \tilde{\sigma}{}^{EF}{}_{AB} - \partial^{EF}(\log \Omega) \tilde{\sigma}_{ABCF} \right) \pi_E, \quad (20)$$

where $\tilde{\nabla}_{AB}$ is a space-spinor counterpart of the Levi-Civita connection on \mathbb{S}^2 . Contracting the second and third indices in the above with $\tilde{\epsilon}{}^{AB} = \Omega^{-1} \epsilon^{AB}$ gives

$$\nabla_A{}^B \pi_B = \Omega^{-1} \left[\tilde{\nabla}_A{}^B \pi_B - \frac{1}{2} \left(\partial_A{}^B(\log \Omega) \tilde{\sigma}{}^{EF}{}_{BF} + \partial_{BF}(\log \Omega) \tilde{\sigma}{}^{EF}{}_{A}{}^B - \partial^{EF}(\log \Omega) \tilde{\sigma}_A{}^B{}_{BF} \right) \pi_E \right],$$

with ∂_{AB} defined as

$$\partial_{AB} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} -\partial_{x^1} - i\partial_{x^2} & \partial_{x^3} \\ \partial_{x^3} & \partial_{x^1} - i\partial_{x^2} \end{pmatrix}, \quad (21)$$

where the same the same convention for the Infeld-Van der Waerden symbols as in [5] has been assumed. However, because $\tilde{\sigma}_{ABCD} = \frac{1}{2} (\tilde{\epsilon}_{AC} \tilde{\epsilon}_{BD} + \tilde{\gamma}_{AC} \tilde{\gamma}_{BD})$ and $\tilde{\gamma}{}^{AB} e_{AB}(\log \Omega) = 0$ ($\tilde{\gamma}_{AB}$ is orthogonal to $\partial\mathcal{S}$) the equation above reduces to

$$\nabla_A{}^B \pi_B = \Omega^{-1} \tilde{\nabla}_A{}^B \pi_B. \quad (22)$$

This can be used to establish the following fact: if $\tilde{\pi}_A$ is a Dirac eigenspinor on \mathbb{S}^2 with an eigenvalue $\lambda \in \mathbb{R}$, i.e.

$$\tilde{\nabla}_A{}^B \tilde{\pi}_B = i\lambda \tilde{\pi}_A,$$

then

$$\nabla_A{}^B \pi_B = i\frac{\lambda}{\Omega} \pi_A, \quad (23)$$

where the spin basis transforms in a following way, $\tilde{o}_A = \frac{1}{\sqrt{\Omega}}o_A$, $\tilde{l}_A = \frac{1}{\sqrt{\Omega}}l_A$ and $\tilde{\pi}_0 = \sqrt{\Omega}\pi_0$, $\tilde{\pi}_1 = \sqrt{\Omega}\pi_1$.

3. APPROXIMATE TWISTOR EQUATION

3.1. Setup. Let \mathfrak{S}_1 , \mathfrak{S}_3 be the spaces of symmetric valence 1 and 3 spinors over the hypersurface \mathcal{S} . The (overdetermined) spatial twistor operator can be defined as follows,

$$\mathbf{T} : \mathfrak{S}_1 \rightarrow \mathfrak{S}_3, \quad \mathbf{T}(\kappa)_{ABC} \equiv \mathcal{D}_{(AB}\kappa_{C)},$$

and is a space-spinor counterpart of the twistor operator $\nabla_{A'(A}\kappa_{B)}$ (see [5] for more details). The formal adjoint of \mathbf{T} is given by

$$\mathbf{T}^* : \mathfrak{S}_3 \rightarrow \mathfrak{S}_1, \quad \mathbf{T}^*(\zeta)_A \equiv \mathcal{D}^{BC}\zeta_{ABC} - \Omega_A{}^{BCD}\zeta_{BCD},$$

and allows to define the *approximate twistor operator* $\mathbf{L} \equiv \mathbf{T}^* \circ \mathbf{T} : \mathfrak{S}_1 \rightarrow \mathfrak{S}_1$,

$$\mathbf{L}(\kappa)_A \equiv \mathcal{D}^{BC}\mathcal{D}_{(AB}\kappa_{C)} - \Omega_A{}^{BCD}\mathcal{D}_{BC}\kappa_D, \quad (24)$$

which is formally self-adjoint —i.e. $\mathbf{L}^* = \mathbf{L}$.

Let κ_A be a solution of the approximate twistor equation $\mathbf{L}(\kappa)_A = 0$. The spinors

$$\xi_A \equiv \frac{2}{3}\mathcal{D}_A{}^Q\kappa_Q, \quad \xi_{ABC} \equiv \mathcal{D}_{(AB}\kappa_{C)}$$

encode independent components of $\mathcal{D}_{AB}\kappa_C$. Moreover, one has that

$$\mathbf{L}(\widehat{\xi})_A = 0.$$

Given the set of asymptotically Cartesian coordinates (x^α) on \mathcal{S} , the position spinor can be defined as follows,

$$x_{\mathbf{AB}} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} x^1 + ix^2 & -x^3 \\ -x^3 & -x^1 + ix^2 \end{pmatrix}.$$

We will consider a solution of the approximate twistor equation on *asymptotically Schwarzschildian* initial data for the vacuum Einstein field equations with an asymptotic behaviour of the form

$$\kappa_{\mathbf{A}} = \left(1 + \frac{m}{r}\right)x_{\mathbf{AB}}o^{\mathbf{B}} + o_\infty(r^{-1/2}). \quad (25)$$

A direct computation shows that

$$\xi_{\mathbf{A}} = \left(1 - \frac{m}{r}\right)o_{\mathbf{A}} + o_\infty(r^{-3/2}), \quad (26a)$$

$$\xi_{\mathbf{ABC}} = -\frac{3m}{2r^3}x_{(\mathbf{AB}}o_{\mathbf{C})} + o_\infty(r^{-5/2}). \quad (26b)$$

As a consequence of the above asymptotic expansion of κ_A , one arrives at the following inequality relating the ADM mass of \mathcal{S} and an integral of concomitants of the spinor κ_A , provided that the inner boundary $\partial\mathcal{S}$ is a MFTS [14],

$$4\pi m \geq \oint_{\partial\mathcal{S}} n_{AB}\zeta_C \widehat{\mathcal{D}^{(AB}\zeta^C)} dS, \quad (27)$$

where n_{AB} is the outer directed (i.e. towards $r = \infty$) unit normal on $\partial\mathcal{S}$ as a surface of \mathcal{S} and $\zeta_A \equiv \widehat{\xi}_A$. Since the space version of $\rho_{AA'}$ is the complex metric $\gamma_{AB} \equiv \tau_{(B}{}^{A'}\rho_{A)A'}$, relation $n_{AB} = \gamma_{AB}/\sqrt{2}$ holds. In the sequel we will use a boundary condition for κ_A to refine the inequality (27).

3.2. A boundary value problem for the approximate twistor equation. Let

$$\mathcal{D}_A \mathcal{Q} \kappa_Q = -\frac{3}{2} \widehat{\phi}_A \quad \text{on } \partial\mathcal{S}, \quad (28)$$

where ϕ_A is a smooth spinorial field. The approximate twistor equation together with (28) satisfy the *Lopatinskij-Shapiro* compatibility conditions (see eg. [8, 31]). This implies that the associated boundary value problem is elliptic. Moreover, the decay conditions (4a) and (4b) for the first and second fundamental forms of the initial data make the approximate twistor operator \mathbf{L} asymptotically homogeneous. In the sequel we will make use of an operator \mathbf{B} , defined in the in the following way,

$$\mathbf{B} : \mathfrak{S}_1 \rightarrow \mathfrak{S}_1, \quad \mathbf{B}(\kappa)_A \equiv -\sqrt{2} \gamma_A^P \xi_P = -\frac{2\sqrt{2}}{3} \gamma_A^P \mathcal{D}^Q \rho \kappa_Q.$$

The equation (28) now becomes

$$\mathbf{B}(\kappa)_A|_{\partial\mathcal{S}} = \sqrt{2} \gamma_A^P \widehat{\phi}_P,$$

and the associated boundary value problem is

$$\mathbf{L}(\kappa)_A = 0, \quad \mathbf{B}(\kappa)_A|_{\partial\mathcal{S}} = \sqrt{2} \gamma_A^P \widehat{\phi}_P. \quad (29)$$

To discuss the solvability of (29) one has to look at the adjoint operators \mathbf{L}^* and \mathbf{B}^* . A similar computation as in [14] (in this case the extrinsic geometry of the boundary is non-trivial) leads to the following:

Proposition 1. *If $\partial\mathcal{S}$ is a MFTS on the asymptotically Schwarzschildian initial data set $(\mathcal{S}, h_{ab}, K_{ab})$ for the vacuum Einstein field equations, then the boundary value problem*

$$\mathbf{L}(\kappa)_A = 0, \quad \mathbf{B}(\kappa)_A|_{\partial\mathcal{S}} = \sqrt{2} \gamma_A^P \widehat{\phi}_P,$$

with a smooth spinorial field ϕ_A over $\partial\mathcal{S}$ admits a unique solution of the form

$$\kappa_A = \mathring{\kappa}_A + \theta_A, \quad \theta_A \in H_{-1/2}^2, \quad (30)$$

with $\mathring{\kappa}_A$ given by the leading term in (25) and where H_β^s with $s \in \mathbb{Z}^+$ and $\beta \in \mathbb{R}$ denotes the weighted L^2 Sobolev spaces.

3.3. Inequality with the connection 1-form on the normal bundle of the boundary. The boundary condition (28) allows to simplify the inequality (27) to the following form

$$4\pi m \geq \sqrt{2} \oint_{\partial\mathcal{S}} \widehat{\phi}^A \gamma_A^B \mathcal{D}_{BC} \phi^C dS, \quad (31)$$

where ϕ_A is a free data in the boundary value problem (29). All quantities in the integral are now intrinsic to the boundary. A relation (17) between the 2-dimensional Sen and Levi-Civita connections implies that

$$\widehat{\phi}^A \gamma_A^B \mathcal{D}_{BC} \phi^C = \widehat{\phi}^A \gamma_A^B (\nabla_{BC} \phi^C) + \frac{1}{4} \widehat{\phi}^A \gamma_A^B A_{BC} \gamma^{CL} \phi_L + \widehat{\phi}^A \gamma_A^B Q_{BC}{}^{LC} \phi_L.$$

Using (12), the GHP expression for Q_{ABCD} , it is easy to see that $Q_{AB}{}^{CB} = \rho' \iota_{AO}{}^C$ on a MFTS. This can be combined with the fact that $\gamma_A^B A_B{}^C \gamma_C{}^D = -A_A{}^D$ to yield

$$4\pi m \geq -\sqrt{2} \oint_{\partial\mathcal{S}} \widehat{\phi}^A \gamma_A^B (\nabla_B{}^C \phi_C) dS + \sqrt{2} \oint_{\partial\mathcal{S}} \rho' |\phi_0|^2 dS - \frac{1}{2\sqrt{2}} \oint_{\partial\mathcal{S}} A_{AB} \widehat{\phi}^A \phi^B dS. \quad (32)$$

It should be noted that since A_{AB} is intrinsic to $\partial\mathcal{S}$ the last term can be expressed as

$$-\frac{1}{2\sqrt{2}} \oint_{\partial\mathcal{S}} \widehat{\phi}^A \phi^B \sigma_{AB}{}^{CD} A_{CD} dS, \quad (33)$$

where $\sigma_{ABCD} = \frac{1}{2}(\epsilon_{AC}\epsilon_{BD} + \gamma_{AC}\gamma_{BD})$ is the spinorial counterpart of the 2-dimensional metric.

4. MASS-QUASILOCAL ANGULAR MOMENTUM INEQUALITY

In this section we present the main result of this article – the mass-quasilocal angular momentum inequality for the *asymptotically Schwarzschildian* initial data $(\mathcal{S}, h_{ab}, K_{ab})$ for the vacuum Einstein field equations. It is based on a simplification of (32) under suitable choice of the boundary spinor ϕ_A . A natural condition for ϕ_A arises after inspecting the first term on the right-hand side of (32) – its 2-dimensional Dirac derivative should be controlled. Indeed, we will proceed with a following choice,

$$\nabla_A{}^B\phi_B = i\frac{\lambda}{\Omega}\phi_A, \quad (34)$$

where Ω is a conformal factor relating a metric σ_{ab} on $\partial\mathcal{S}$ with that on a round sphere \mathbb{S}^2 . It can be showed that with a suitable choice of the conformal rescaling of the spin basis the equation (34) corresponds to a Dirac eigenproblem on \mathbb{S}^2 (see Subsection 2.3 for more details).

The inequality (32) can now be simplified with the use of (34) to the following form,

$$4\pi m \geq \sqrt{2} \oint_{\partial\mathcal{S}} \rho' |\phi_0|^2 dS - \frac{1}{2\sqrt{2}} \oint_{\partial\mathcal{S}} \widehat{\phi}^A \phi^B \sigma_{AB}{}^{CD} A_{CD} dS, \quad (35)$$

where the reality of the ADM mass m has been used to eliminate a (purely imaginary) term with the eigenvalue λ , i.e.

$$\lambda \oint_{\partial\mathcal{S}} (|\phi_0|^2 - |\phi_1|^2) \Omega^{-1} dS = 0. \quad (36)$$

To make a connection between the second term on the right-hand side of (35) and the quasilocal angular momentum (19) we will introduce a spinor N^{AB} , defined as follows,

$$N^{AB} \equiv \sigma^{ABCD} \phi_C \widehat{\phi}_D = \phi_0 \overline{\phi}_1 \iota^A \iota^B - \phi_1 \overline{\phi}_0 \sigma^A \sigma^B. \quad (37)$$

One can verify that N^{AB} is real, i.e. $\widehat{N}^{AB} = -N^{AB}$, so it corresponds to a real 3-vector. Moreover, $\gamma^{AB} N_{AB} = 0$ and

$$\begin{aligned} \nabla_a N^a &= \nabla_{AB} (\sigma^{ABCD} \phi_C \widehat{\phi}_D) = \nabla_{AB} (\phi^A \widehat{\phi}^B) \\ &= -(\nabla_B{}^A \phi_A) \widehat{\phi}^B + \phi^A (\widehat{\nabla}_A{}^B \phi_B) = 0, \end{aligned} \quad (38)$$

where (34) has been used in the last equality. Hence, N^a is intrinsic to the boundary $\partial\mathcal{S}$ and ∇ -divergence-free, so we can identify it with N^a generating the quasilocal angular momentum (19). With this choice the inequality (35) yields

$$4\pi m \geq \sqrt{2} \oint_{\partial\mathcal{S}} \rho' |\phi_0|^2 dS + \frac{\kappa}{\sqrt{2}} O [\sigma^{ABCD} \phi_C \widehat{\phi}_D]. \quad (39)$$

In the remainder of this section we will simplify (39) and express it in terms of integrals over a round sphere \mathbb{S}^2 and the eigenspinor of the \mathbb{S}^2 -Dirac operator.

The Hodge decomposition can be applied to the connection 1-form on the normal bundle of the boundary A_b to yield $A_b = \epsilon_b{}^c \nabla_c U + \nabla_b U'$, where U is a rotation potential. This allows to simplify the quasilocal angular momentum term from (39), i.e.

$$\oint_{\partial\mathcal{S}} N^a A_a dS = \oint_{\partial\mathcal{S}} U \epsilon^{ab} \nabla_a N_b dS. \quad (40)$$

The spinorial counterpart of the volume element ϵ_{ab} of $\partial\mathcal{S}$ is $\epsilon_{ABCD} = \frac{i}{2} (\epsilon_{AC} \gamma_{BD} + \epsilon_{BD} \gamma_{AC})$, and

$$\epsilon^{ab} \nabla_a N_b = \epsilon^{ABCD} \nabla_{AB} N_{CD} = \frac{2\lambda}{\Omega} \gamma^{AB} \phi_A \widehat{\phi}_B = \frac{2\lambda}{\Omega} (|\phi_0|^2 - |\phi_1|^2), \quad (41)$$

where the definition (37) has been taken into account. Inserting this expression into (40) and using $dS = \Omega^2 d\mathbb{S}^2$ yields

$$\oint_{\partial\mathcal{S}} N^a A_a dS = 2\lambda \oint_{\mathbb{S}^2} U (|\tilde{\phi}_0|^2 - |\tilde{\phi}_1|^2) d\mathbb{S}^2, \quad (42)$$

where $\tilde{\phi}_A = \sqrt{\Omega} \phi_A$. The relation between the volume elements of $\partial\mathcal{S}$ and \mathbb{S}^2 can also be utilized to write the first term on the right-hand side of (39) in terms of an integral over \mathbb{S}^2 . Ultimately,

$$4\pi m \geq \sqrt{2} \oint_{\mathbb{S}^2} \rho' |\tilde{\phi}_0|^2 \Omega d\mathbb{S}^2 + \frac{\kappa}{\sqrt{2}} O[\tilde{\phi}, U], \quad (43)$$

where

$$O[\tilde{\phi}, U] \equiv -\frac{\lambda}{\kappa} \oint_{\mathbb{S}^2} U (|\tilde{\phi}_0|^2 - |\tilde{\phi}_1|^2) d\mathbb{S}^2,$$

i.e. the quasilocal angular momentum term can now be written only in terms of the geometry of \mathbb{S}^2 and the rotation potential U . Note that the conformal factor Ω appears in the first term in the right-hand side of the above inequality and it is non-unique since the Möbius group acting on \mathbb{S}^2 gives rise to different spherical metrics. The inequality obtained is therefore with respect to a given spherical metric, which can be thought of as a gauge choice here.

5. AXISYMMETRIC INNER BOUNDARY AND THE DIRAC-KILLING SYSTEM.

A natural assumption associated with the existence of a well-defined angular momentum is that the initial data is axisymmetric, i.e. there exists 1-form v_a such that

$$\nabla_{(a} v_b) = 0 \quad \text{on } \mathcal{S}.$$

If the inner boundary $\partial\mathcal{S}$ is invariant under the action of the 1-parameter group of isometries generated by v_a , then $v_a = \Pi_a{}^b v_b$ (v_a is intrinsic to $\partial\mathcal{S}$) and the projection of the Killing equation gives

$$\nabla_{(a} v_b) = 0 \implies \nabla_a v^a = 0.$$

This suggests that a natural choice for the vector N^a defining the quasilocal angular momentum (19) is that it arises as a solution to the boundary Killing equation, i.e. $\nabla_{(a} N_b) = 0$. However, N^a has already been constructed from a spinor ϕ_A satisfying a first-order Dirac-type equation (34) on $\partial\mathcal{S}$. Hence, a natural question arises —can such N_a be also a Killing vector of the boundary? We will show that this cannot be the case on a generic $\partial\mathcal{S}$.

In the sequel we will use an adapted system of coordinates (ψ, φ) on the boundary, such that its metric σ_{ab} can be written in the following form,

$$\sigma = -R^2 \left(\frac{1}{F^2} d\psi \otimes d\psi + F^2 d\varphi \otimes d\varphi \right), \quad \psi \in [\psi_0, \psi_1], \quad \varphi \in [0, 2\pi],$$

where $F = F(\psi)$, R is a constant and the axisymmetric Killing vector is now proportional to ∂_φ . To avoid the conical singularities on the poles we will assume that $F(\psi_0) = F(\psi_1) = 0$. The NP operators δ and $\bar{\delta}$ reduce to

$$\delta = \frac{1}{\sqrt{2}R} (F \partial_\psi + \frac{i}{F} \partial_\varphi), \quad \bar{\delta} = \frac{1}{\sqrt{2}R} (F \partial_\psi - \frac{i}{F} \partial_\varphi),$$

in this setting. Moreover, $\alpha - \bar{\beta} = \frac{1}{\sqrt{2}R} \partial_\psi F$ (see [7] for details).

A straightforward computation yields

$$\begin{aligned} \not{\nabla}_{AB}N_{CD} + \not{\nabla}_{CD}N_{AB} &= 2\delta'(\bar{\phi}_0\phi_1) o_A o_B o_C o_D - (\delta(\bar{\phi}_0\phi_1) + \delta'(\phi_0\bar{\phi}_1)) o_A o_B \iota_C \iota_D \\ &\quad - (\delta(\bar{\phi}_0\phi_1) + \delta'(\phi_0\bar{\phi}_1)) \iota_A \iota_B o_C o_D + 2\delta(\phi_0\bar{\phi}_1) \iota_A \iota_B \iota_C \iota_D, \end{aligned}$$

where we have used a decomposition of vector N^a in terms of a spinor ϕ_A in accordance with (37). Ultimately, the condition $\not{\nabla}_{(a}N_{b)} = 0$ implies that

$$\phi_0\bar{\phi}_1 = icF, \quad c \in \mathbb{R}. \quad (44)$$

Additionally, the spinor ϕ_A satisfies a first-order Dirac-type equation (34), which can now be written as

$$\begin{aligned} F\partial_\psi\phi_1 + \frac{\phi_1}{2}\partial_\psi F - \frac{i\sqrt{2}\lambda R}{\Omega}\phi_0 &= 0, \\ F\partial_\psi\phi_0 + \frac{\phi_0}{2}\partial_\psi F - \frac{i\sqrt{2}\lambda R}{\Omega}\phi_1 &= 0. \end{aligned} \quad (45)$$

After multiplying the first equation by $\bar{\phi}_1$ and the second by $\bar{\phi}_0$ and performing some manipulations one arrives at

$$\begin{aligned} \partial_\psi(F|\phi_1|^2) + \frac{2\sqrt{2}\lambda RcF}{\Omega} &= 0, \\ \partial_\psi(F|\phi_0|^2) - \frac{2\sqrt{2}\lambda RcF}{\Omega} &= 0, \end{aligned}$$

where (44) has been used. Hence

$$|\phi_0|^2 = \frac{c_0}{F} + \frac{2\sqrt{2}\lambda Rc}{F} \int_{\psi_0}^{\psi} \frac{F}{\Omega} dz, \quad |\phi_1|^2 = \frac{c_1}{F} - \frac{2\sqrt{2}\lambda Rc}{F} \int_{\psi_0}^{\psi} \frac{F}{\Omega} dz, \quad (46)$$

for some $c_0, c_1 \in \mathbb{R}$. On the other hand, one can apply $F\partial_\psi$ to both sides of (44) and use (45) to get

$$\sqrt{2}\lambda R(|\phi_1|^2 - |\phi_0|^2) = 2c\Omega F\partial_\psi F.$$

After using (46) we obtain the following compatibility condition for F ,

$$\sqrt{2}\lambda R \left(\frac{c_1 - c_0}{F} - \frac{4\sqrt{2}\lambda Rc}{F} \int_{\psi_0}^{\psi} \frac{F}{\Omega} dz \right) = 2c\Omega F\partial_\psi F. \quad (47)$$

Multiplying the above relation by F and differentiating with respect to ψ we arrive at

$$c[\partial_\psi(\Omega F^2\partial_\psi F) + 4R^2\lambda^2\Omega^{-1}F] = 0, \quad (48)$$

where $F(\psi_0) = 0$ has been used. This equation implies that the metric of the inner boundary (via the function F) depends on the choice of the eigenvalue λ . This cannot be the case, as the former arises as part of the fixed geometric data associated with the initial hypersurface and the latter from the first-order Dirac-type equation, which is an auxiliary condition used to simplify the mass inequality. Hence, the only way to solve (48) is to assume that $c = 0$. In this case $\phi_A = 0$ and the right-hand side of the mass-quasilocal angular momentum inequality (39) vanishes.

Remark 1. *In case of axisymmetric initial data, the inequality (43) reduces to the positivity of the ADM mass because ϕ_A vanishes. This suggests that the Szabados's quasi-local angular momentum cannot give rise to an ADM angular momentum appearing in the Penrose inequality.*

6. CONCLUSIONS

We have obtained a new bound for the ADM mass of *asymptotically Schwarzschildian* initial data for the vacuum Einstein field equations in terms of the future inner null expansion of the inner MFTS boundary and its quasilocal angular momentum. Our approach bears similarities to the one presented in [14], but we extend it here to allow for boundaries with nontrivial extrinsic geometry. An expression for quasilocal angular momentum (in the sense of Szabados [27]) has been recovered in the bound for the ADM mass by assuming a specific type of boundary condition for the approximate twistor equation— a spinor ϕ_A solving a first-order Dirac-type equation. The strategy developed in this work could also be applied to obtain Penrose-type inequalities with different type of asymptotics (e.g. asymptotically hyperbolic) —as long as the concept of quasilocal angular momentum is well-defined.

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