Mathematische Annalen
https://doi.org/10.1007/s00208-024-02909-6

Mathematische Annalen

Quantization of the energy for the inhomogeneous Allen–Cahn mean curvature

Huy The Nguyen[1] · **Shengwen Wang**[1]

Received: 27 June 2023 / Revised: 20 May 2024 / Accepted: 28 May 2024

Abstract

We consider the varifold associated to the Allen–Cahn phase transition problem in $\mathbb{R}^{n+1}$(or $n+1$-dimensional Riemannian manifolds with bounded curvature) with integral L^{q_0} bounds on the Allen–Cahn mean curvature (first variation of the Allen–Cahn energy) in this paper. It is shown here that there is an equidistribution of energy between the Dirichlet and Potential energy in the phase field limit and that the associated varifold to the total energy converges to an integer rectifiable varifold with mean curvature in L^{q_0}, $q_0 > n$. The latter is a diffused version of Allard's convergence theorem for integer rectifiable varifolds.

1 Introduction

Let $\Omega \subset (M^{n+1}, g)$ be an open subset in a Riemannian manifold with bounded curvature. Consider $u \in W^{2,p}(\Omega)$ satisfying the following equation

$$\varepsilon \Delta u_\varepsilon - \frac{W'(u_\varepsilon)}{\varepsilon} = f_\varepsilon, \tag{1.1}$$

where $W(t) = \frac{(1-t^2)^2}{2}$ is a double-well potential. The Eq. (1.1) can be viewed as a prescribed first variation problem to the Allen–Cahn energy

$$E_\varepsilon(u_\varepsilon) = \int_\Omega \left(\frac{\varepsilon |\nabla u_\varepsilon|^2}{2} + \frac{W(u_\varepsilon)}{\varepsilon} \right) dx.$$

✉ Shengwen Wang
shengwen.wang@qmul.ac.uk

Huy The Nguyen
h.nguyen@qmul.ac.uk

1 School of Mathematical Sciences, Queen Mary University of London, Mile End Road, London E1 4NS, UK

Published online: 14 June 2024

For any compactly supported test vector field $\eta \in C_c^\infty(\Omega, \mathbb{R}^{n+1})$, we have a variation $u_s(x) = u\,(x + s\eta(x))$ and the first variation formula at $u_0 = u_\varepsilon$ is given by

$$\begin{aligned}\frac{d}{ds}\bigg|_{s=0} E_\varepsilon\,(u_s) &= \int_\Omega \left(-\varepsilon\Delta u_\varepsilon + \frac{W'(u_\varepsilon)}{\varepsilon}\right)\langle\nabla u_\varepsilon, \eta\rangle dx \\ &= -\int_\Omega \left(\frac{f_\varepsilon}{\varepsilon|\nabla u_\varepsilon|}\right)\langle\nu, \eta\rangle\varepsilon|\nabla u_\varepsilon|^2 dx,\end{aligned} \tag{1.2}$$

where $\nu = \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}$ is a unit normal to the level sets at non-critical points of u.

By [5, 6, 9] using the framework of [3], the sequence of functionals E_ε Γ-converges to the n-dimensional area functional as $\varepsilon \to 0$. This shows that minimizing solutions to (1.1) with $f_\varepsilon = 0$ converge as $\varepsilon \to 0$ to area minimizing hypersurfaces. For general critical points ($f_\varepsilon = 0$) a deep theorem of Hutchinson–Tonegawa [4, Theorem 1] shows the diffuse varifold obtained by smearing out the level sets of u converges to limit which is a stationary varifold with $a.e.$ integer density. The main result of this paper is to prove Hutchinson–Tonegawa's Theorem [4, Theorem 1] in the context of natural integrability conditions on the first variation of E_ε. Under suitable controls on the first variation of the energy functional E_ε (the diffuse mean curvature) we can show comparable behaviour for the limit. In the case where $n = 2, 3$ Röger–Schätzle [8] have shown under the assumption

$$\liminf_{\varepsilon\to 0^+}\left(E_\varepsilon(u_\varepsilon) + \frac{1}{\varepsilon}\|f_\varepsilon\|^2_{L^2(\Omega)}\right) < \infty$$

that the limit is an integer rectifiable varifold with L^2 generalised mean curvature.

The main focus of this paper is to generalise this result to higher dimensions. Before we state our main theorem, we give a choice of the diffused analogue of "mean curvature" in the Allen–Cahn setting, which will be used to state our bounded L^{q_0} Allen–Cahn mean curvature condition in the theorem.

Recall that for an embedded hypersurface $\Sigma^n \subset \Omega \subset \mathbb{R}^{n+1}$ restricted to a bounded domain Ω and a compactly supported variation Σ_s with $\Sigma_0 = \Sigma$, we have the first variation area at $s = 0$ given by

$$\frac{d}{ds}\bigg|_{s=0} \text{Area}(\Sigma_s \cap \Omega) = -\int_{\mathbb{R}^{n+1}} \langle\mathbf{H}, \eta\rangle d\mu_\Sigma = \int_{\mathbb{R}^{n+1}} H\langle\nu, \eta\rangle d\mu_\Sigma, \tag{1.3}$$

where H is the mean curvature scalar, $\mathbf{H} = -H\nu$ is the mean curvature vector, ν is a unit normal vector field, η is the variation vector field, and $d\mu_\Sigma$ is the hypersurface measure. By comparing the first variation formula (1.2) for Allen–Cahn energy and the first variation formula (1.3) for area , we can see that $\left(\frac{f_\varepsilon}{\varepsilon|\nabla u|}\right)$ roughly plays the role of the mean curvature scalar in the Allen–Cahn setting. In [1], a result of Allard implies that if a sequence of integral varifolds has L^{q_0} integrable mean curvature scalar with $q_0 > n$, then after passing to a subsequence, there is a limit varifold which is also integer rectifiable.

Under similar conditions on L^{q_0} integrability of the term $\left(\frac{f_\varepsilon}{\varepsilon|\nabla u|}\right)$ with $q_0 > n$, we prove the integer rectifiability of the limit of sequences of Allen–Cahn varifolds :

Theorem 1.1 *Let $u_\varepsilon \in W^{1,2}(\Omega), \Omega \subset \mathbb{R}^{n+1}$ satisfy Eq. (1.1) with $\varepsilon \to 0$ and $f_\varepsilon \in L^1(\Omega)$. If any one of the following holds:*

(1) Bounds on the total energy

$$\int_\Omega \left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2} + \frac{W(u_\varepsilon)}{\varepsilon}\right) dx \leq E_0; \tag{1.4}$$

(2) Uniform L^∞ bounds

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \leq c_0; \tag{1.5}$$

(3) L^{q_0} bounds on the diffuse mean curvature

$$\int_\Omega \left(\frac{|f_\varepsilon|}{\varepsilon|\nabla u_\varepsilon|}\right)^{q_0} \varepsilon|\nabla u_\varepsilon|^2 dx \leq \Lambda_0 \tag{1.6}$$

for some $q_0 > n$;

then after passing to a subsequence, we have for the associated varifolds (see Sect. 2 for the definition) $V_{u_\varepsilon} \to V_\infty$ weakly and

(1) V_∞ is an integral n-rectifiable varifold;
(2) For any $B_r(x_0) \subset\subset \Omega$, the L^{q_0} norm of the generalized mean curvature of V_∞ is bounded by Λ_0;
(3) The discrepancy measure $\left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2} - \frac{W(u_\varepsilon)}{\varepsilon}\right) \to 0$ in L^1_{loc} as $\varepsilon \to 0$ (c.f. Proposition 4.4).

This theorem shows we can prove a result analagous to Hutchinson–Tonegawa [4], Tonegawa [10] and show as $\varepsilon \to 0$, the diffuse varifold associated to the Allen–Cahn functional converges to an integer rectifiable varifold. This has some similarities with Allard's compactness theorem for rectifiable varifolds and for integral varifolds but here the sequence consists of diffuse varifolds and hence we require stronger conditions on the proposed mean curvature. As we shall see in a later paper, these conditions are exactly what is required to prove a version of Allard's regularity theorem for Allen–Cahn varifolds.

In the proof of Theorem 1.1, we also obtained a variational approximation of a class of integral mean curvature functional via Γ - convergence by a sequence functionals from the phase-field model.

Corollary 1.2 *Let $u \in W^{1,2}(\Omega)$, $\Omega \subset \mathbb{R}^{n+1}$ satisfy Eq.* (1.1) *with $u_\varepsilon = u$ and $\mathcal{F}_\varepsilon : L^1(\Omega) \to \mathbb{R}$ be a sequence of functionals defined by*

$$\mathcal{F}_\varepsilon(u) = \int_\Omega \left(\frac{\varepsilon|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon} \right) dx + \int_\Omega \left(\frac{|\varepsilon \Delta u - \frac{W'(u)}{\varepsilon}|}{\varepsilon|\nabla u|} \right)^{q_0} \varepsilon|\nabla u|^2 dx,$$

for any $q_0 > n$. Then for any $\chi = 2\chi_E - 1$ with $E \subset \Omega$, $\partial E \cap \Omega \in C^2$ where χ_E is the characteristic function for E, there holds

$$\Gamma(L^1(\Omega)) - \lim_{\varepsilon \to 0} \mathcal{F}_\varepsilon(\chi) = \mathcal{F}(\chi) =: \alpha \mathcal{H}^n(\partial E \cap \Omega) + \alpha \int_{\partial E \cap \Omega} |\mathbf{H}_{\partial E}|^{q_0} d\mathcal{H}^n,$$

where $\alpha = \int_{-\infty}^{\infty} (\tanh' x)^2 dx$ is the total energy for the 1-d heteroclinic solution Allen-Cahn equation, $\mathcal{H}^n$ is the n-dimensional Hausdorff measure, and $\mathbf{H}_{\partial E}$ is the mean curvature of ∂E.

Our result can also imply some previous convergence results under various integrability conditions for the inhomogeneous term and its derivatives. (Notice that we do not require any integrability condition on the derivative of the inhomogeneous term f in Theorem 1.1).

Corollary 1.3 *If u_ε satisfies* (1.1) *and of one of the following conditions holds:*

(1)

$$\|f_\varepsilon\|_{L^s(\Omega)} \le C_1 \varepsilon^{\frac{1}{2}}, \quad \text{for some } 2 < s < n$$
$$\left\| \frac{f_\varepsilon}{\varepsilon|\nabla u_\varepsilon|} \right\|_{L^t(\Omega)} \le C_2, \quad \text{for some } t > \frac{n-2}{s-2} s > \max\{s, n-2\};$$

(2)

$$\left\| \frac{f_\varepsilon}{\varepsilon|\nabla u_\varepsilon|} \right\|_{W^{1,p}(\Omega)} \le C, \quad \text{for some } p > \frac{n+1}{2}, \quad \text{(c.f. [11])};$$

(3)

$$\|f_\varepsilon\|_{L^2(\Omega)} \le C_1 \varepsilon^{\frac{1}{2}}, \quad \text{if the ambient dimension } n+1 = 2, \quad \text{(c.f. [8])}$$
$$\left\| \frac{f_\varepsilon}{\varepsilon|\nabla u_\varepsilon|} \right\|_{L^\infty(\Omega)} \le C_2, \quad \text{if the ambient dimension } n+1 \ge 3;$$

then after passing to a subsequence as $\varepsilon \to 0$, the associated varifolds V_ε converge to an integral n-rectifiable varifold with generalized mean curvature in L^{q_0} for some $q_0 > n$.

Here we give an overview of our proof. In Sect. 2, we gather together some standard notation on varifolds and the first variation. In Sect. 3, we prove the main estimates required for the proof of the integrality and rectifiability. Specifically we will need a monotonicity formula. For the homogeneous Allen–Cahn equation and Allen–Cahn flow, a strict monotonicity formula can be proven due to Modica's estimate showing the discrepancy is negative. This estimate is not true without a homogeneous left hand side to Eq. (1.1). Instead we will use the integral bound (1.6) to derive a decay bound for L^1 norm of the discrepancy which we eventually show vanishes in the limit $\varepsilon \to 0$. This estimate constitutes one of the main advances of this paper. In Sect. 4 we show the limiting varifold we obtain as $\varepsilon \to 0$ is a rectifiable set and in Sect. 5 we show the limiting varifold is in addition integral. In Sect. 6, we prove Corollary 1.3 and Corollary 1.2.

2 Preliminaries and notations

Throughout the paper, we will denote a constant by C if it only depends on the constants n, E_0, c_0, Λ_0 which appear in the conditions of Theorem 1.1. At certain points we may increase this constant in some steps of the argument, but we will not relabel the constant unless there is a risk of confusion from the context. We associate to each solution of (1.1) a varifold in the following way : let $G(n+1, n)$ denote the Grassmannian (the space of unoriented n-dimensional subspaces in $\mathbb{R}^{n+1}$). We regard $S \in G(n+1, n)$ as the $(n+1) \times (n+1)$ matrix representing orthogonal projection of $\mathbb{R}^{n+1}$ onto S, that is

$$S^2 = S, \quad S^T S = I$$

and write $S_1 \cdot S_2 = \mathrm{tr}(S_1^T \cdot S_2)$. We say V is an n-varifold in $\Omega \subset \mathbb{R}^{n+1}$ if V is a Radon measure on $G_n(\Omega) = \Omega \times G(n+1, n)$. Varifold convergence means convergence of Radon measures or weak-$*$ convergence. We let $V \in \mathbb{V}_n(\Omega)$ and let $\|V\|$ denote the weight measure of V and we define the first variation of V by

$$\delta V(\eta) \equiv \int_{G_n(\Omega)} \nabla \eta(x) \cdot S dV(x, S) \quad \forall \eta \in C_c^1(\Omega; \mathbb{R}^{n+1}).$$

We let $\|\delta V\|$ be the total variation of δV. If $\|\delta V\|$ is absolutely continuous with respect to $\|\delta V\|$ then the Radon–Nikodym derivative $\frac{\delta V}{\|V\|}$ exists as vector valued measure. We denote by $H_V = -\frac{\delta V}{\|V\|}$, the generalised mean curvature.

Let $u = u_\varepsilon$ be a function in Theorem 1.1, we define the associated energy measure as a Radon measure given by

$$d\mu_\varepsilon \equiv \left(\frac{\varepsilon |\nabla u_\varepsilon|^2}{2} + \frac{W(u_\varepsilon)}{\varepsilon} \right) d\mathcal{L}^{n+1}$$

where $\mathcal{L}^{n+1}$ is the $(n+1)$ dimensional Lebesgue measure. We also denote the energy of the 1 dimensional solution by

$$\sigma = \int_{-1}^{1} \sqrt{2W(s)} ds.$$

There is an associated varifold $V \in \mathbb{V}_n(\Omega)$ to the functions u given by

$$
\begin{aligned}
V(\phi) &= \int_{\{|\nabla u| \neq 0\}} \phi\left(x, \left(\frac{\nabla u(x)}{|\nabla u(x)|}\right)^{\perp}\right) d\mu(x) \\
&= \int_{\{|\nabla u| \neq 0\}} \phi\left(x, I - \frac{\nabla u(x)}{|\nabla u(x)|} \otimes \frac{\nabla u(x)}{|\nabla u(x)|}\right) d\mu(x), \quad \phi \in C_c(G_n(\Omega)).
\end{aligned}
$$

where I is the $(n+1) \times (n+1)$ identity matrix and

$$
I - \frac{\nabla u(x)}{|\nabla u(x)|} \otimes \frac{\nabla u(x)}{|\nabla u(x)|}
$$

is orthogonal projection onto the space orthogonal to $\nabla u(x)$, that is $\{a \in \mathbb{R}^{n+1} \mid \langle a, \nabla u(x)\rangle = 0\}$. By definition $\|V\| = \mu \llcorner_{\{|\nabla u| \neq 0\}}$ and the first variation may be computed as

$$
\delta V(\eta) = \int_{\{|\nabla u| \neq 0\}} \nabla \eta \cdot \left(I - \frac{\nabla u(x)}{|\nabla u(x)|} \otimes \frac{\nabla u(x)}{|\nabla u(x)|}\right) d\mu(x), \forall \eta \in C_c^1(\Omega; \mathbb{R}^{n+1}). \tag{2.1}
$$

3 Discrepancy bounds and monotonicity formula

In this section, we deduce integral bounds on the discrepancy. There exists an almost monotonicity formula for the Allen–Cahn energy functional, we will give estimates on the terms appearing in the almost monotonicity formulas under the assumptions in Theorem 1.1 and obtain a monotonicity formula for the n-dimensional volume ratio. It will be used in the next section to deduce rectifiability and integrality of the limit varifold as $\varepsilon \to 0$. Conditions (1)–(3) in Theorem 1.1 are assumed to hold throughout this section.

The n-dimensional volume ratio of the energy measure satisfies the following almost monotonicity formula.

Proposition 3.1 (Almost Monotonicity Formula) *If u_ε satisfies (1.1) in $B_1 \subset \mathbb{R}^{n+1}$, then for $r < 1$, we have*

$$
\begin{aligned}
\frac{d}{dr}\left(\frac{\mu_\varepsilon(B_r)}{r^n}\right) &= -\frac{1}{r^{n+1}} \xi(B_r) + \frac{\varepsilon}{r^{n+2}} \int_{\partial B_r} \langle x, \nabla u_\varepsilon \rangle^2 \\
&\quad - \frac{1}{r^{n+1}} \int_{B_r} \langle x, \nabla u_\varepsilon \rangle f_\varepsilon.
\end{aligned} \tag{3.1}
$$

Here $\mu_\varepsilon(B_r) = \int_{B_r} d\mu_\varepsilon = \int_{B_r} \left(\frac{\varepsilon |\nabla u_\varepsilon|^2}{2} + \frac{W(u_\varepsilon)}{\varepsilon}\right)$ is the total energy and $\xi(B_r) = \int_{B_r} \left(\frac{\varepsilon |\nabla u_\varepsilon|^2}{2} - \frac{W(u_\varepsilon)}{\varepsilon}\right)$ is the discrepancy measure (difference between the Dirichlet and potential energy) in B_r.

Proof Multiplying Eq. (1.1) by $\langle x, \nabla u_\varepsilon \rangle$ and integrating by parts on B_r, we get

$$
\begin{aligned}
\int_{B_r} \langle x, \nabla u_\varepsilon \rangle f_\varepsilon &= \int_{B_r} \varepsilon \Delta u_\varepsilon \langle x, \nabla u_\varepsilon \rangle - \int_{B_r} \left\langle \frac{\nabla (W(u_\varepsilon))}{\varepsilon}, x \right\rangle \\
&= \int_{\partial B_r} \left(\varepsilon r \left| \frac{\partial u_\varepsilon}{\partial \nu} \right|^2 - r \frac{W(u_\varepsilon)}{\varepsilon} \right) \\
&\quad - \int_{B_r} \left(\varepsilon \delta_{ij} u_{x_i} u_{x_j} + \varepsilon \nabla^2 u (\nabla u_\varepsilon, x) - \frac{(n+1) W(u_\varepsilon)}{\varepsilon} \right) \\
&= \int_{\partial B_r} \left(\varepsilon r \left| \frac{\partial u_\varepsilon}{\partial \nu} \right|^2 - r \frac{W(u_\varepsilon)}{\varepsilon} \right) \\
&\quad - \int_{B_r} \left(\varepsilon |\nabla u_\varepsilon|^2 + \varepsilon \left\langle \nabla \frac{|\nabla u_\varepsilon|^2}{2}, x \right\rangle - \frac{(n+1) W(u_\varepsilon)}{\varepsilon} \right) \\
&= r \int_{\partial B_r} \left(\varepsilon \left| \frac{\partial u_\varepsilon}{\partial \nu} \right|^2 - \frac{W(u_\varepsilon)}{\varepsilon} - \varepsilon \frac{|\nabla u_\varepsilon|^2}{2} \right) \\
&\quad + \int_{B_r} \left(\varepsilon \frac{(n-1) |\nabla u_\varepsilon|^2}{2} + \frac{(n+1) W(u_\varepsilon)}{\varepsilon} \right) \\
&= n \int_{B_r} \left(\frac{\varepsilon |\nabla u_\varepsilon|^2}{2} + \frac{W(u_\varepsilon)}{\varepsilon} \right) - r \int_{\partial B_r} \left(\frac{\varepsilon |\nabla u_\varepsilon|^2}{2} + \frac{W(u_\varepsilon)}{\varepsilon} \right) \\
&\quad + \frac{\varepsilon}{r} \int_{\partial B_r} \langle x, \nabla u_\varepsilon \rangle^2 - \xi(B_r).
\end{aligned}
$$

The conclusion then follows by dividing both sides by r^{n+1} and noticing

$$
\frac{d}{dr} \left(\frac{\mu_\varepsilon(B_r)}{r^n} \right) = - \frac{n}{r^{n+1}} \int_{B_r} \left(\frac{\varepsilon |\nabla u_\varepsilon|^2}{2} + \frac{W(u_\varepsilon)}{\varepsilon} \right) + \frac{1}{r^n} \int_{\partial B_r} \left(\frac{\varepsilon |\nabla u_\varepsilon|^2}{2} + \frac{W(u_\varepsilon)}{\varepsilon} \right).
$$

□

Integrating the almost monotonicity formula (3.1) from ε to r_0 for $0 < \varepsilon < r_0 < 1$, we have

$$
\begin{aligned}
&\frac{\mu_\varepsilon(B_{r_0})}{r_0^n} - \frac{\mu_\varepsilon(B_\varepsilon)}{\varepsilon^n} \\
&= \int_\varepsilon^{r_0} \left(- \frac{1}{r^{n+1}} \xi(B_r) + \frac{\varepsilon}{r^{n+2}} \int_{\partial B_r} \langle x, \nabla u_\varepsilon \rangle^2 - \frac{1}{r^{n+1}} \int_{B_r} \langle x, \nabla u_\varepsilon \rangle f_\varepsilon \right) dr \\
&\geq -r_0 \sup_{B_{r_0}} \omega_{n+1} \left(\frac{\varepsilon |\nabla u_\varepsilon|^2}{2} - \frac{W(u_\varepsilon)}{\varepsilon} \right)_+ + \int_{B_{r_0} \setminus B_\varepsilon} \frac{\varepsilon \langle x, \nabla u_\varepsilon \rangle^2}{|x|^{n+2}} \\
&\quad - \int_\varepsilon^{r_0} \frac{1}{r^{n+1}} \int_{B_r} \langle x, \nabla u_\varepsilon \rangle f_\varepsilon dr, \qquad (3.2)
\end{aligned}
$$

where ω_{n+1} denotes the volume of unit ball in $\mathbb{R}^{n+1}$.

We need to estimate the first and third term on the right hand side to obtain a monotonicity formula. In order to estimate the third term, we derive an a priori gradient bound for u. Condition (3) of Theorem 1.1 states a combined integrability for the inhomogeneity f_ε and $|\nabla u|$. The following theorem allows us to obtain separate integrability and regularity for each quantity.

Theorem 3.2 *There exists $C, \varepsilon_0 > 0$ depending on E_0, c_0, Λ_0 as defined in Theorem 1.1 such that if u_ε satisfies (1.1) in $B_1 \subset \mathbb{R}^{n+1}$ with $\varepsilon < \varepsilon_0$ and if $q_0 > n+1$, then*

$$\sup_{B_{1-\varepsilon}} \varepsilon |\nabla u_\varepsilon| \leq C, \tag{3.3}$$

and

$$\varepsilon^{2-\frac{n+1}{q_0}} \|u_\varepsilon\|_{C^{1,1-\frac{n+1}{q_0}}(B_{1-\varepsilon})} \leq C. \tag{3.4}$$

If $n < q_0 \leq n+1$, then

$$\varepsilon^{\frac{1}{2}} \|u_\varepsilon\|_{C^{0,\frac{1}{2}}(B_{1-\varepsilon})} \leq C. \tag{3.5}$$

Furthermore, there exists a $\delta_0 > 0$ so that f has the following improved integrability

$$\|f_\varepsilon\|_{L^{\frac{n+1}{2}+\delta_0}(B_{1-\varepsilon}(x_0))} \leq C\varepsilon^{-\frac{n}{q_0}}. \tag{3.6}$$

Proof We first consider the case $q_0 > n+1$: Define the rescaled solution $\tilde{u}(x) := u(\varepsilon x)$ and $\tilde{f}(x) = \varepsilon f_\varepsilon(\varepsilon x)$ which satisfies the equation

$$\Delta \tilde{u} - W'(\tilde{u}) = \tilde{f}, \quad \text{in } B_{\frac{1}{\varepsilon}} \subset \mathbb{R}^{n+1}. \tag{3.7}$$

By condition (3) in Theorem 1.1, we have by rescaling

$$\begin{aligned}\int_{B_{\frac{1}{\varepsilon}}} \tilde{f}^{q_0} \varepsilon^{n-q_0} |\nabla \tilde{u}|^{2-q_0} &= \int_{B_{\frac{1}{\varepsilon}}} \varepsilon^{-2q_0} \tilde{f}^{q_0} \varepsilon |\nabla \tilde{u}|^{2-q_0} \varepsilon^{q_0-2} \varepsilon^{n+1} \\ &= \int_{B_1} \varepsilon^{-q_0} f^{q_0} \varepsilon |\nabla u|^{2-q_0} \leq \Lambda_0. \end{aligned} \tag{3.8}$$

□

Claim For any $\bar{B}_1(x_0) \subset B_{\frac{1}{\varepsilon}-1}$, we have

$$\|\nabla \tilde{u}\|_{L^2(B_1(x_0))} \leq C(c_0, \Lambda_0, q_0, n).$$

Proof of Claim By the hypothesis $\bar{B}_1(x_0) \subset B_{\frac{1}{\varepsilon}-1}$ we have $B_2(x_0) \subset B_{\frac{1}{\varepsilon}}$. We choose a smooth cutoff function $\phi \in C_c^\infty(B_2(x_0)), [0, 1])$ with $\phi \equiv 1$ in $B_1(x_0)$ and $|\nabla\phi| \leq 4$. By integration by parts and Young's inequality, we obtain

$$
\begin{aligned}
\int_{B_2(x_0)} |\nabla\tilde{u}|^2\phi^2 &\leq \int_{B_2(x_0)} 2c_0|\nabla\tilde{u}||\phi||\nabla\phi| + \int_{B_2(x_0)} c_0\phi^2|\Delta\tilde{u}| \\
&\leq \int_{B_2(x_0)} 2c_0|\nabla\tilde{u}||\phi||\nabla\phi| + \int_{B_2(x_0)} c_0\phi^2|W'(\tilde{u})| \\
&\quad + \int_{B_2(x_0)} c_0\phi^2|\tilde{f}| \\
&\leq \frac{1}{2}\int_{B_2(x_0)} |\nabla\tilde{u}|^2\phi^2 + \int_{B_2(x_0)} 2c_0^2|\nabla\phi|^2 + \int_{B_2(x_0)} c_0\phi^2 C_{c_0} + \int_{B_2(x_0)} c_0\phi^2|\tilde{f}|.
\end{aligned} \tag{3.9}
$$

We write $c_0\phi^2|\tilde{f}| = c_0|\tilde{f}|\varepsilon^{\frac{n}{q_0}-1}|\nabla\tilde{u}|^{\frac{2}{q_0}-1} \times \phi^2\varepsilon^{1-\frac{n}{q_0}}|\nabla\tilde{u}|^{1-\frac{2}{q_0}}$ and use Young's inequality with exponent q_0 to get

$$
\begin{aligned}
\int_{B_2(x_0)} c_0\phi^2|\tilde{f}| &\leq \frac{1}{\delta q_0}\int_{B_2(x_0)} \left|c_0|\tilde{f}|\varepsilon^{\frac{n}{q_0}-1}|\nabla\tilde{u}|^{\frac{2}{q_0}-1}\right|^{q_0} \\
&\quad + \frac{\delta(q_0-1)}{q_0}\int_{B_2(x_0)} \left|\phi^2\varepsilon^{1-\frac{n}{q_0}}|\nabla\tilde{u}|^{1-\frac{2}{q_0}}\right|^{\frac{q_0}{q_0-1}} \\
&\leq \frac{c_0^{q_0}}{\delta q_0}\Lambda_0 + \frac{\delta(q_0-1)}{q_0}\int_{B_2(x_0)} \phi^{\frac{2q_0}{q_0-1}}|\nabla\tilde{u}|^{\frac{q_0-2}{q_0-1}} \\
&\leq \frac{c_0^{q_0}}{\delta q_0}\Lambda_0 + \frac{C_n\delta(q_0-1)}{q_0}\left(\int_{B_2(x_0)} \phi^{\frac{4q_0}{q_0-2}}|\nabla\tilde{u}|^2\right)^{\frac{q_0-2}{2(q_0-1)}} \\
&\leq \frac{4C_n(q_0-1)c_0^n}{q_0^2}\Lambda_0 + \frac{1}{4}\max\left\{\left[\int_{B_2(x_0)} \phi^2|\nabla\tilde{u}|^2\right]^{\frac{4q_0}{q_0-2}}, 1\right\}.
\end{aligned}
$$

Here we used (3.8) to bound $\int_{B_2(x_0)} \left|c_0\tilde{f}\varepsilon^{\frac{n}{q_0}-1}|\nabla\tilde{u}|^{\frac{2}{q_0}-1}\right|^{q_0}$ and the fact that $\varepsilon^{1-\frac{n}{q_0}} < 1$ in the second inequality, Hölder's inequality with exponent $\frac{2(q_0-1)}{q_0-2}$ in the third inequality. And the fourth inequality is obtained from the third by choosing δ to be $\frac{q_0}{4C_n(q_0-1)}$. We assume $\int_{B_2(x_0)} \phi^2|\nabla\tilde{u}|^2 \geq 1$, otherwise the desired bound holds trivially. Inserting the above inequality into (3.9), we get

$$
\begin{aligned}
\int_{B_2(x_0)} |\nabla\tilde{u}|^2\phi^2 &\leq \frac{1}{2}\int_{B_2(x_0)} |\nabla\tilde{u}|^2\phi^2 + \int_{B_2(x_0)} 2c_0^2|\nabla\phi|^2 + \int_{B_2(x_0)} c_0\phi^2 C_{c_0} \\
&\quad + \frac{4C_n(q_0-1)c_0^n}{q_0^2}\Lambda_0 + \frac{1}{4}\max\left\{\int_{B_2(x_0)} \phi^2|\nabla\tilde{u}|^2, 1\right\}.
\end{aligned}
$$

Then by moving the first term $\frac{1}{2}\int_{B_2(x_0)} |\nabla \tilde{u}|^2\phi^2$ and the fifth term $\int_{B_2(x_0)} \phi^2|\nabla \tilde{u}|^2$ on the right to the left, we prove the claim. □

Now suppose $\|\nabla \tilde{u}\|_{L^{p_0}(B_1(x_0))} \le C(c_0, \Lambda_0, q_0, n)$ (independent of ε) for some $p_0 > 1$ (p_0 can be chosen to be 2 by the claim above). For any $B_2(x_0) \in B_{\frac{1}{\varepsilon}}(0)$, we have by Hölder's inequality

$$
\begin{aligned}
\|\tilde{f}\|_{L^{\frac{p_0q_0}{p_0+q_0-2}}(B_1(x_0))} &= \left(\int_{B_1(x_0)} |\tilde{f}|^{\frac{p_0q_0}{p_0+q_0-2}}\right)^{\frac{p_0+q_0-2}{p_0q_0}} \\
&\le \Bigg[\left\|\left|\tilde{f}\varepsilon^{\frac{n-q_0}{q_0}}|\nabla\tilde{u}|^{\frac{2}{q_0}-1}\right|^{\frac{p_0q_0}{p_0+q_0-2}}\right\|_{L^{\frac{p_0+q_0-2}{p_0}}(B_1(x_0))} \\
&\quad\cdot \left\|\left(\varepsilon^{\frac{q_0-n}{q_0}}|\nabla\tilde{u}|^{1-\frac{2}{q_0}}\right)^{\frac{p_0q_0}{p_0+q_0-2}}\right\|_{L^{\frac{p_0+q_0-2}{q_0-2}}(B_1(x_0))}\Bigg]^{\frac{p_0+q_0-2}{p_0q_0}} \\
&\le \left[\Lambda_0^{\frac{p_0}{p_0+q_0-2}}\varepsilon^{\frac{(q_0-n)p_0}{p_0+q_0-2}} \cdot \left(\int_{B_1(x_0)} |\nabla\tilde{u}|^{p_0}\right)^{\frac{q_0-2}{p_0+q_0-2}}\right]^{\frac{p_0+q_0-2}{p_0q_0}} \\
&= \Lambda_0^{\frac{1}{q_0}} \cdot \varepsilon^{\frac{q_0-n}{q_0}} \cdot \left(\int_{B_1(x_0)} |\nabla\tilde{u}|^{p_0}\right)^{\frac{q_0-2}{p_0q_0}} \\
&\le C(c_0, \Lambda_0, q_0, n)\varepsilon^{\frac{q_0-n}{q_0}} \le C(c_0, \Lambda_0, q_0, n). \qquad (3.10)
\end{aligned}
$$

Remark 3.3 Here $q_0 > n$ will make the scaling subcritical and ensures a uniform bound of $\|\tilde{f}\|_{L^{\frac{p_0q_0}{p_0+q_0-2}}(B_1(x_0))}$ independent of ε.

Thus $\tilde{f}$ is uniformly bounded in $L^{\frac{p_0q_0}{p_0+q_0-2}}(B_1(x_0))$ independent of ε. By applying the Sobolev inequality to (3.7), standard Calderon–Zygmund estimates and finally using the L^∞ bound of u in condition (2) of Theorem 1.1, we have

$$
\begin{aligned}
\|\nabla\tilde{u}\|_{L^{\frac{p_0q_0}{p_0+q_0-2-p_0\frac{q_0}{n+1}}}(B_1(x_0))} &\le \|\tilde{u}\|_{W^{1,\frac{p_0q_0}{p_0+q_0-2-p_0\frac{q_0}{n+1}}}(B_1(x_0))} \\
&\le C\|\tilde{u}\|_{W^{2,\frac{p_0q_0}{p_0+q_0-2}}(B_1(x_0))} \\
&\le C\|\tilde{f}\|_{L^{\frac{p_0q_0}{p_0+q_0-2}}(B_1(x_0))} + C\|W'(\tilde{u})\|_{L^{\frac{p_0q_0}{p_0+q_0-2}}(B_1(x_0))} \\
&\le C\Lambda_0^{\frac{1}{q_0}} \cdot \varepsilon^{\frac{q_0-n}{q_0}} \cdot \left(\int_{B_1(x_0)} |\nabla\tilde{u}|^{p_0}\right)^{\frac{q_0-2}{p_0q_0}} \\
&\quad + C\|W'(\tilde{u})\|_{L^\infty(B_1(x_0))}
\end{aligned}
$$

$$\leq C(c_0, \Lambda_0, q_0, n)\left(\varepsilon^{\frac{q_0-n}{q_0}}+1\right) \leq \tilde{C}(c_0, \Lambda_0, q_0, n). \tag{3.11}$$

We remark that $q_0 > n$ ensures the coefficient $\varepsilon^{\frac{q_0-n}{q_0}}$ stays uniformly bounded as $\varepsilon \to 0$.

In the case $\frac{p_0 q_0}{p_0+q_0-2} > n+1$, by Calderon–Zygmund estimates we have

$$\|\tilde{u}\|_{W^{2,\frac{p_0 q_0}{p_0+q_0-2}}(B_1(x_0))} \leq C(c_0, \Lambda_0, q_0, n)\left(\varepsilon^{\frac{q_0-n}{q_0}}+1\right) \leq \tilde{C}(c_0, \Lambda_0, q_0, n).$$

The Sobolev inequality then gives $\|\nabla\tilde{u}\|_{L^\infty} \leq C$.

In the case $\frac{p_0 q_0}{p_0+q_0-2} \leq n+1$, using $q_0 > n+1$, we have $p_0 < p_0\frac{q_0}{n+1}$. Namely

$$\frac{q_0}{p_0+q_0-2-p_0\frac{q_0}{n+1}}p_0 = \frac{q_0}{(p_0-p_0\frac{q_0}{n+1})+(q_0-2)}p_0 \geq \frac{q_0}{q_0-2}p_0. \tag{3.12}$$

So we improved $\nabla\tilde{u}$ from L^{p_0} to $L^{\frac{q_0}{q_0-2}p_0}$. Define $p_i = \frac{q_0}{q_0-2}p_{i-1}$. Using $q_0 > n+1$, we iterate finitely many times until $p_i > \frac{(n+1)(q_0-2)}{q_0-(n+1)}$, i.e. $\frac{p_i q_0}{p_i+q_0-2} > n+1$. The Sobolev inequality gives $\nabla\tilde{u} \in L^\infty$. So if $q_0 > n+1$, we get $\nabla\tilde{u} \in L^\infty$. Rescaling back, we get (3.3). By (3.8) where ($q_0 > n+1 \geq 2$) and $\nabla\tilde{u} \in L^\infty$, we have $\tilde{f} \in L^{q_0}$. Standard Calderon–Zygmund estimates give

$$\begin{aligned}\|\nabla\tilde{u}\|_{C^{0,1-\frac{n+1}{q_0}}(B_1(x_0))} &\leq \|\tilde{u}\|_{W^{2,q_0}(B_1(x_0))} \leq \|\tilde{f}\|_{L^{q_0}(B_1(x_0))}\\ &\quad + \|W'(\tilde{u})\|_{L^{q_0}(B_1(x_0))} < \infty,\end{aligned}$$

which gives (3.4).

Consider now the case $n < q_0 \leq n+1$. For any

$$p_i \leq \frac{2(n+1)}{n+1-q_0} - \delta, \tag{3.13}$$

we have

$$\begin{aligned}p_i + q_0 - 2 - p_i\frac{q_0}{n+1} &= p_i\frac{n+1-q_0}{n+1} + q_0 - 2\\ &= \left(\frac{2(n+1)}{n+1-q_0} - \delta\right)\frac{n+1-q_0}{n+1} + q_0 - 2\\ &= q_0 - \frac{n+1-q_0}{n+1}\delta.\end{aligned}$$

And thus

$$\frac{q_0}{p_i+q_0-2-p_i\frac{q_0}{n+1}}p_i \geq \frac{q_0}{q_0-\frac{n+1-q_0}{n+1}\delta}p_i \geq p_i. \tag{3.14}$$

So (3.11) increases the integrability of $\nabla\tilde{u}$ from L^{p_i} to $L^{\frac{q_0}{q_0-\frac{n+1-q_0}{n+1}\delta}p_i}$. And we can iterate until (3.13) fails, namely

$$\|\nabla\tilde{u}\|_{L^{\frac{2(n+1)}{n+1-q_0}-\delta}(B_1(x_0))} \leq C(c_0, \Lambda_0, q_0, n)\varepsilon^{\frac{q_0-n}{q_0}} \leq C(c_0, \Lambda_0, q_0, n), \tag{3.15}$$

for any $x_0 \in B_{\frac{1}{\varepsilon}-2}$(so that the condition in the claim above is satisfied). By Sobolev inequalities, we then have for any $x_0 \in B_{\frac{1}{\varepsilon}-2}$

$$\begin{aligned}\|\tilde{u}\|_{C^{0,\frac{1}{2}}(B_1(x_0))} &\leq C\|\tilde{u}\|_{W^{1,2(n+1)}(B_1(x_0))}\\ &\leq C\|\tilde{u}\|_{W^{1,\frac{2(n+1)}{n+1-q_0}-\delta}(B_1(x_0))}\\ &\leq C(c_0, \Lambda, 0, q_0, n)\varepsilon^{\frac{q_0-n}{q_0}} \leq C(c_0, \Lambda, 0, q_0, n).\end{aligned}$$

Rescaling back gives

$$\varepsilon^{\frac{1}{2}}\|u\|_{C^{0,\frac{1}{2}}(B_{1-\varepsilon})} \leq \|\tilde{u}\|_{C^{0,\frac{1}{2}}\left(B_{\frac{1}{\varepsilon}-1}\right)} \leq C(c_0, \Lambda_0, q_0, n)\varepsilon^{\frac{q_0-n}{q_0}} \leq C(c_0, \Lambda_0, q_0, n),$$

which is (3.5). By (3.10) we improve the integrability of $\tilde{f}$ in (3.10) up to

$$\|\tilde{f}\|_{L^{\frac{p_i q_0}{p_i+q_0-2}}(B_1(x_0))} \leq C\varepsilon^{\frac{q_0-n}{q_0}},$$

for $p_i \leq \frac{2(n+1)}{n+1-q_0} - \delta$. So if $q_0 \in (n, n+1]$, by choosing $p_i = 2(n+1)$, we have

$$\frac{p_i q_0}{p_i + q_0 - 2} = \frac{p_i}{\frac{p_i-2}{q_0}+1} > \frac{p_i}{\frac{p_i-2}{n}+1} = \frac{2(n+1)}{\frac{2(n+1)-2}{n}+1} = \frac{2(n+1)}{3}, \tag{3.16}$$

rearranging gives $\frac{p_i q_0}{p_i+q_0-2} > \frac{2(n+1)}{3} \geq \frac{n+1}{2} + \delta_0$ for some $\delta_0 > 0$. On the other hand, if $q_0 > n+1$, using (3.8) and the uniform gradient bound of u in Theorem 3.2, we have $\|\tilde{f}\|_{L^{q_0}(B_1(x_0))} \leq C\varepsilon^{\frac{q_0-n}{q_0}}$, where $q_0 > n+1 > \frac{n+1}{2} + \delta_0$. Combining both cases, for any $q_0 > n$

$$\|\tilde{f}\|_{L^{\frac{n+1}{2}+\delta_0}(B_1(x_0))} \leq C\varepsilon^{\frac{q_0-n}{q_0}}. \tag{3.17}$$

and

$$\|\tilde{f}\|_{L^{\frac{n+1}{2}+\delta_0}\left(B_{\frac{1}{\varepsilon}-1}(x_0)\right)} \leq C\varepsilon^{\frac{q_0-n}{q_0}}\varepsilon^{-n-1}, \tag{3.18}$$

Rescaling back gives the bound on f,

$$\|f\|_{L^{\frac{n+1}{2}+\delta_0}(B_{1-\varepsilon}(x_0))} \leq C\varepsilon^{-\frac{n}{q_0}}.$$

□

Since in the case $q_0 \in (n, n+1]$, we lack gradient bounds of u as in the case $q_0 > n+1$. In order to get better estimates of the discrepancy terms in the almost monotonicity formula, we use some ideas from [8]. We will apply the following Lemma to (3.7) for ε sufficiently small such that $C\varepsilon^{\frac{q_0-n}{q_0}} \leq \omega$.

Lemma 3.4 (cf [8, Lemma 3.2]) *Let $n+1 \geq 3$, $0 < \delta \leq \delta_1$ and $R(\delta) = \frac{1}{\delta^{p_1}}$, $\omega(\delta) = \delta^{p_2}$, where $p_1 = 5$, $p_2 = 35$. If $\tilde{u} \in C^2(B_R)$, $\tilde{f} \in C^0(B_R)$, $B_R = B_R(0) \subset \mathbb{R}^{n+1}$ where*

$$\begin{aligned} -\Delta\tilde{u} + W'(\tilde{u}) &= \tilde{f} \quad \text{in } B_R, \\ |\tilde{u}| &\leq c_0 \quad \text{in } B_R, \\ \|\tilde{f}\|_{L^{\frac{n+1}{2}+\delta_0}(B_R)} &\leq \omega, \end{aligned}$$

c_0 is as assumed in condition (2) of Theorem 1.1 and δ_0 is as in Theorem 3.2. Then

$$\int_{B_1} \left(\frac{|\nabla\tilde{u}|^2}{2} - W(\tilde{u})\right)_+ \leq C\delta. \tag{3.19}$$

And for $\tau = \delta^{p_3}$, where $p_3 = \frac{2\delta_0}{(n+1)^2+(n+1)\delta_0+6\delta_0}$, we get

$$\int_{B_{\frac{1}{2}}} \left(\frac{|\nabla\tilde{u}|^2}{2} - W(\tilde{u})\right)_+ \leq c\tau \int_{B_{\frac{1}{2}}} \left(\frac{|\nabla\tilde{u}|^2}{2} + W(\tilde{u})\right) + \int_{B_{\frac{1}{2}}\cap\{|\tilde{u}|\geq 1-\tau\}} \frac{|\nabla\tilde{u}|^2}{2}. \tag{3.20}$$

Proof Let us consider the auxiliary function ψ which solves the Dirichlet problem

$$\begin{aligned} \Delta\psi &= -\tilde{f}, && \text{in } B_R \\ \psi &= 0, && \text{on } \partial B_R. \end{aligned} \tag{3.21}$$

The auxiliary function will allows us to control the inhomogeneous part of the equation.

□

Claim The function ψ defined in (3.21) satisfies the bounds

$$\|\psi\|_{L^\infty(B_R)} \leq C\delta^{25+5\frac{n+1}{\frac{n+1}{2}+\delta_0}} \ll 1, \tag{3.22}$$

$$\|\nabla\psi\|_{L^{\frac{(n+1)(n+1+2\delta_0)}{n+1-2\delta_0}}(B_R)} \leq C\omega = C\delta^{35}. \tag{3.23}$$

Proof Rescaling by $\frac{1}{R}$, we have

$$\begin{aligned} \Delta\psi_R &= \tilde{f}_R, && \text{in } B_1 \\ \psi_R &= 0, && \text{on } \partial B_1, \end{aligned} \tag{3.24}$$

where $\psi_R(x) = \psi(Rx)$, $\tilde{f}_R(x) = R^2\tilde{f}(Rx)$. Standard Calderon–Zygmund estimates give

$$\begin{aligned} \|\psi_R\|_{W^{2,\frac{n+1}{2}+\delta_0}(B_1)} \leq \|\tilde{f}_R\|_{L^{\frac{n+1}{2}+\delta_0}(B_1)} &= R^{2-\frac{n+1}{\frac{n+1}{2}+\delta_0}}\|\tilde{f}\|_{L^{\frac{n+1}{2}+\delta_0}(B_R)} \\ &\leq CR^{2-\frac{n+1}{\frac{n+1}{2}+\delta_0}}\omega, \end{aligned}$$

where $2 - \frac{n+1}{\frac{n+1}{2}+\delta_0} > 0$. Rescaling back yields

$$\begin{aligned} &\|\psi\|_{L^{\frac{n+1}{2}+\delta_0}(B_R)} + R\|\nabla\psi\|_{L^{\frac{n+1}{2}+\delta_0}(B_R)} + R^2\|\nabla^2\psi\|_{L^{\frac{n+1}{2}+\delta_0}(B_R)} \\ &\quad = R^{\frac{n+1}{\frac{n+1}{2}+\delta_0}}\|\psi_R\|_{L^{\frac{n+1}{2}+\delta_0}(B_1)} + R^{\frac{n+1}{\frac{n+1}{2}+\delta_0}}\|\nabla\psi_R\|_{L^{\frac{n+1}{2}+\delta_0}(B_1)} \\ &\qquad + R^{\frac{n+1}{\frac{n+1}{2}+\delta_0}}\|\nabla^2\psi_R\|_{L^{\frac{n+1}{2}+\delta_0}(B_1)} \\ &\quad = R^{\frac{n+1}{\frac{n+1}{2}+\delta_0}}\|\psi_R\|_{W^{2,\frac{n+1}{2}+\delta_0}(B_1)} \\ &\quad \leq CR^{\frac{n+1}{\frac{n+1}{2}+\delta_0}}R^{2-\frac{n+1}{\frac{n+1}{2}+\delta_0}}\omega \\ &\quad = CR^2\omega \\ &\quad = C\delta^{25}. \end{aligned}$$

Here we prove (3.22): by the Sobolev inequality since $\delta_0 > 0 \implies \frac{n+1}{2} + \delta_0 > \frac{n+1}{2}$, we have

$$\begin{aligned} \|\psi\|_{L^\infty(B_R)} = \|\psi_R\|_{L^\infty(B_1)} &\leq C\|\psi_R\|_{W^{2,\frac{n+1}{2}+\delta_0}(B_1)} \\ &\leq CR^{2-\frac{n+1}{\frac{n+1}{2}+\delta_0}}\omega \\ &= C\delta^{25+5\frac{n+1}{\frac{n+1}{2}+\delta_0}} \ll 1, \end{aligned}$$

due to the choice of ω, where we used $\frac{(n+1)(n+1+2\delta_0)}{n+1-2\delta_0} > n+1$. *Here we prove the gradient bound* (3.23):

$$\|\nabla\psi\|_{L^{\frac{(n+1)(n+1+2\delta_0)}{n+1-2\delta_0}}(B_R)} \leq R^{\frac{n+1-2\delta_0}{n+1+2\delta_0}-1}\|\nabla\psi_R\|_{L^{\frac{(n+1)(n+1+2\delta_0)}{n+1-2\delta_0}}(B_1)}$$

$$
\begin{aligned}
&\le C R^{\frac{n+1-2\delta_0}{n+1+2\delta_0}-1}\|\psi_R\|_{W^{2,\frac{n+1}{2}+\delta_0}(B_1)} \\
&\le C R^{\frac{n+1-2\delta_0}{n+1+2\delta_0}-1} R^{2-\frac{n+1}{\frac{n+1}{2}+\delta_0}}\omega \\
&= C R^0 \omega \\
&= C\omega = C\delta^{35}.
\end{aligned}
$$

□

We define $\tilde{u}_0 := \tilde{u} + \psi \in W^{2,\frac{n+1}{2}+\delta_0}(B_R)$. By (3.21), (3.22), $\tilde{u}_0$ satisfies

$$
\begin{aligned}
&|\tilde{u}_0| \le c_0 + 1, \\
&\Delta \tilde{u}_0 = W'(\tilde{u}).
\end{aligned} \tag{3.25}
$$

We compute for any $\beta > 0$,

$$
\begin{aligned}
\frac{|\nabla \tilde{u}|^2}{2} - W(\tilde{u}) &= \frac{|\nabla \tilde{u}_0 - \nabla \psi|^2}{2} - W(\tilde{u}_0 - \psi) \\
&\le \left(\frac{1}{2} + \beta\right)|\nabla \tilde{u}_0|^2 + \left(\frac{1}{2} + \frac{1}{\beta}\right)|\nabla \tilde{\psi}|^2 - W(\tilde{u}_0) + C|\psi|,
\end{aligned}
$$

for some $C > 0$. Thus by (3.22) and (3.23), we have

$$
\begin{aligned}
&\int_{B_1}\left(\frac{|\nabla \tilde{u}|^2}{2} - W(\tilde{u})\right)_+ \\
&\le \int_{B_1}\left(\frac{|\nabla \tilde{u}_0|^2}{2} - W(\tilde{u}_0)\right)_+ + \int_{B_1}\left(\beta|\nabla \tilde{u}_0|^2 + C|\psi| + \left(\frac{1}{2} + \frac{1}{\beta}\right)|\nabla \psi|^2\right) \\
&\le \int_{B_1}\left(\frac{|\nabla \tilde{u}_0|^2}{2} - W(\tilde{u}_0)\right)_+ + C\left(\beta + R^{2-\frac{n+1}{\frac{n+1}{2}+\delta_0}}\omega + \left(\frac{1}{2} + \frac{1}{\beta}\right)\omega^2\right).
\end{aligned}
$$

By choosing $\beta = \omega \le \delta^{p_2}$ and using our hypothesis on ω : $R^{2-\frac{n+1}{\frac{n+1}{2}+\delta_0}}\omega = \delta^{25+\frac{5(n+1)}{\frac{n+1}{2}+\delta_0}}$. By our choice of $p_1 = 2$, $p_2 = 15$, we ensure

$$
\begin{aligned}
\beta = \delta^{35} &\le C\delta, \\
R^{2-\frac{n+1}{\frac{n+1}{2}+\delta_0}}\omega = \delta^{25+5\frac{n+1}{\frac{n+1}{2}+\delta_0}} &\le C\delta, \\
\left(\frac{1}{2} + \frac{1}{\beta}\right)\omega^2 = \frac{1}{2}\delta^{70} + \delta^{35} &\le C\delta,
\end{aligned}
$$

for $n \ge 2$. Thus

$$
\int_{B_1}\left(\frac{|\nabla \tilde{u}|^2}{2} - W(\tilde{u})\right)_+ \le \int_{B_1}\left(\frac{|\nabla \tilde{u}_0|^2}{2} - W(\tilde{u}_0)\right)_+ + C\delta.
$$

To prove (3.19), it suffices to show

$$\int_{B_1}\left(\frac{|\nabla\tilde{u}_0|^2}{2}-W(\tilde{u}_0)\right)_+\leq C\delta. \tag{3.26}$$

Here we estimate $\tilde{u}$. Define $\tilde{u}_R(x)=\tilde{u}(Rx)$ then by the Calderon–Zygmund estimates we have

$$\begin{aligned}\|\tilde{u}_R\|_{W^{2,\frac{n+1}{2}+\delta_0}(B_{\frac{1}{2}})} &\leq C\|\Delta\tilde{u}_R\|_{L^{\frac{n+1}{2}+\delta_0}(B_1)}+C\|\tilde{u}_R\|_{L^{\frac{n+1}{2}+\delta_0}(B_1)}\\ &\leq C\left(R^{2-\frac{n+1}{\frac{n+1}{2}+\delta_0}}\|\Delta\tilde{u}\|_{L^{\frac{n+1}{2}+\delta_0}(B_R)}+1\right)\\ &\leq C\left(R^{2-\frac{n+1}{\frac{n+1}{2}+\delta_0}}\left(\|W'(\tilde{u})\|_{L^{\frac{n+1}{2}+\delta_0}(B_R)}+\|\tilde{f}\|_{L^{\frac{n+1}{2}+\delta_0}(B_R)}\right)+1\right)\\ &\leq C\left(R^{2-\frac{n+1}{\frac{n+1}{2}+\delta_0}}(R^{\frac{n+1}{\frac{n+1}{2}+\delta_0}}+\omega)+1\right)\\ &\leq CR^2. \end{aligned}\tag{3.27}$$

By the Sobolev embedding

$$\begin{aligned}\|\nabla\tilde{u}\|_{L^{\frac{(n+1)(n+1+2\delta_0)}{n+1-2\delta_0}}(B_{\frac{R}{2}})} &\leq R^{\frac{n+1-2\delta}{n+1+2\delta}-1}\|\nabla\tilde{u}_R\|_{L^{\frac{(n+1)(n+1+2\delta_0)}{n+1-2\delta_0}}(B_{\frac{1}{2}})}\\ &\leq R^{\frac{n+1-2\delta_0}{n+1+2\delta_0}-1}\|\tilde{u}_R\|_{W^{2,\frac{n+1}{2}+\delta_0}(B_{\frac{1}{2}})}\\ &\leq R^{\frac{n+1-2\delta_0}{n+1+2\delta_0}-1}\cdot CR^2=CR^{\frac{n+1-2\delta}{n+1+2\delta}+1}. \end{aligned}\tag{3.28}$$

We define

$$\begin{aligned}\tilde{f}_0 &:= -\Delta\tilde{u}_0+W'(\tilde{u}_0)\\ &= -\Delta\psi-\Delta\tilde{u}+W'(\tilde{u})+W''(\tilde{u})\psi+\frac{1}{2}W^{(3)}(\tilde{u})\psi^2+\frac{1}{6}W^{(4)}(\tilde{u})\psi^3\\ &= W''(\tilde{u})\psi+\frac{1}{2}W^{(3)}(\tilde{u})\psi^2+\frac{1}{6}W^{(4)}(\tilde{u})\psi^3, \end{aligned}$$

since the derivatives of order 5 or higher of the potential $W(u)=\frac{(1-u^2)^2}{2}$ vanish. By (3.22), (3.23) and (3.28), we have

$$\|\tilde{f}_0\|_{L^\infty(B_R)}\leq C\|\tilde{\psi}\|^3_{L^\infty(B_R)}\leq C\left(R^{2-\frac{n+1}{\frac{n+1}{2}+\delta_0}}\omega\right)^3\leq C\delta^{75+15\frac{n+1}{\frac{n+1}{2}+\delta_0}}\ll 1, \tag{3.29}$$

and

$$
\begin{aligned}
&\|\nabla \tilde{f}_0\|_{L^{\frac{(n+1)(n+1+2\delta_0)}{n+1-2\delta_0}}(B_R)} \\
&\quad \le C\left(\|\nabla \tilde{u}\|_{L^{\frac{(n+1)(n+1+2\delta_0)}{n+1-2\delta_0}}(B_R)} \cdot \|\psi\|_{L^\infty(B_R)} + \|\nabla\psi\|_{L^{\frac{(n+1)(n+1+2\delta_0)}{n+1-2\delta_0}}(B_R)}\right) \\
&\quad \le C\left(R^{\frac{n+1-2\delta_0}{n+1+2\delta_0}+1} \cdot R^{2-\frac{n+1}{\frac{n+1}{2}+\delta_0}}\omega + \omega\right) \\
&\quad = C(R^2\omega + \omega) \\
&\quad \le CR^2\omega \\
&\quad = C\delta^{25} \ll 1.
\end{aligned} \tag{3.30}
$$

Since we have $|\tilde{u}_0| \le c_0$, we apply Calderon–Zygmund to (3.25), for any $B_1(x) \subset B_R$ and $1 < r < \infty$ and we get

$$
\|\tilde{u}_0\|_{W^{2,r}\left(B_{\frac{1}{2}}(x)\right)} \le C_r. \tag{3.31}
$$

Hence by the Morrey embedding

$$
\|\nabla \tilde{u}_0\|_{L^\infty(B_{R-1})} \le C.
$$

We define a modified discrepancy

$$
\xi_G := \frac{|\nabla \tilde{u}_0|^2}{2} - W(\tilde{u}_0) - G(\tilde{u}_0) - \varphi, \tag{3.32}
$$

for some function $G \in C^\infty(\mathbb{R})$ and $\varphi \in W^{2,2}(B_R)$ that we choose as in the following claims

Claim If we make the following choice of G,

$$
G_\delta(r) := \delta\left(1 + \int_{-c_0-1}^{r} \exp\left(-\int_{-c_0-1}^{t} \frac{|W'(s)| + \delta}{2(W(s)+\delta)} ds\right) dt\right) \tag{3.33}
$$

then we have the properties

$$
\begin{aligned}
&\delta \le G_\delta(\tilde{u}_0) \le C\delta, \\
&0 < G'_\delta(\tilde{u}_0) \le \delta, \\
&0 < -G''_\delta(\tilde{u}_0) = G'_\delta(\tilde{u}_0)\frac{|W'(\tilde{u}_0)| + \delta}{2(W(\tilde{u}_0) + \delta)} \le C.
\end{aligned} \tag{3.34}
$$

Furthermore we have

$$
G'_\delta W' - 2G''_\delta(W + G_\delta) \ge \delta G'_\delta \tag{3.35}
$$

and

$$G'_\delta(\tilde{u}_0) \geq C\delta^3. \tag{3.36}$$

Proof of Claim The first three equations of (3.34) follow from the direct computations. For (3.35), since $G_\delta \geq \delta$, we obtain

$$\begin{aligned} G'_\delta W' - 2G''_\delta(W + G_\delta) &= G'_\delta \left(W' + \frac{|W'| + \delta}{(W + \delta)}(W + G_\delta) \right) \\ &\geq G'_\delta \left(W' + \frac{|W'| + \delta}{(W + \delta)}(W + \delta) \right) \\ &= G'_\delta \left(W' + |W'| + \delta \right) \\ &\geq \delta G'_\delta. \end{aligned}$$

For (3.36), from the definition of G_δ (3.33) and the bound $|\tilde{u}_0| \leq c_0 + 1$, we compute

$$\begin{aligned} G'_\delta(\tilde{u}_0) &\geq \delta \exp\left(- \int_{-c_0-1}^{c_0+1} \frac{|W'(s)| + \delta}{2(W(s) + \delta)} ds \right) \\ &\geq \delta \exp\left(- \int_{-c_0-1}^{-1} \left| \frac{d}{ds} \log(W(s) + \delta) \right| ds \right. \\ &\quad \left. - \int_{-1}^{0} \left| \frac{d}{ds} \log(W(s) + \delta) \right| ds - (c_0 + 1) \right) \\ &\geq \delta \exp\left(- (\log(W(-c_0 - 1) + \delta) - \log \delta) - (\log(1 + \delta) - \log \delta) - (c_0 + 1) \right) \\ &\geq \delta \exp\left(\tilde{C} - \log(\delta^2) \right) \\ &\geq C\delta^3, \end{aligned}$$

where we used W is an even function, increasing in $[-1, 0]$ and decreasing in $[-c_0 - 1, -1]$. □

Claim If we choose φ to satisfy the Dirichlet problem

$$\begin{aligned} -\Delta\varphi &= |\langle \nabla\tilde{u}_0, \nabla\tilde{f}_0 \rangle - (W' + G'_\delta)\tilde{f}_0| > 0 \quad \text{in } B_{\frac{R}{2}}, \\ \varphi &= 0 \quad \text{on } \partial B_{\frac{R}{2}} \end{aligned} \tag{3.37}$$

then we have

$$\varphi \geq 0 \quad \text{in } B_{\frac{R}{2}} \tag{3.38}$$

and

$$\|\varphi\|_{W^{1,\infty}(B_{\frac{R}{2}})} \leq CR^{4 - \frac{n+1-2\delta_0}{n+1+2\delta_0}} \omega = C\delta^{15 + 5\frac{n+1-2\delta_0}{n+1+2\delta_0}}. \tag{3.39}$$

Proof Since we have $\varphi \geq 0$ in $\partial B_{\frac{R}{2}}$ by applying the maximum principle, we have $\varphi \geq 0$ in $B_{\frac{R}{2}}$ which gives us (3.38). The estimates (3.31), (3.29) and (3.30) bound the right hand side of (3.37), that is

$$\begin{aligned}\|\Delta\varphi\|_{L^{\frac{(n+1)(n+1+2\delta_0)}{n+1-2\delta_0}}(B_{\frac{R}{2}})} &= \left|\langle\nabla\tilde{u}_0, \nabla\tilde{f}_0\rangle - (W' + G'_\delta)\tilde{f}_0\right|_{L^{\frac{(n+1)(n+1+2\delta_0)}{n+1-2\delta_0}}\left(B_{\frac{R}{2}}\right)} \\ &\leq CR^2\omega = C\delta^{25}.\end{aligned}$$

Denote by $\varphi_R(x) = \varphi(\frac{Rx}{2})$, then the Calderon–Zygmund estimates give

$$\begin{aligned}\|\varphi\|_{W^{1,\infty}(B_{\frac{R}{2}})} = \|\varphi_R\|_{W^{1,\infty}(B_1)} &\leq C\|\varphi_R\|_{W^{2,\frac{(n+1)(n+1+2\delta_0)}{n+1-2\delta_0}}(B_1)} \\ &\leq C\|\Delta\varphi_R\|_{L^{\frac{(n+1)(n+1+2\delta_0)}{n+1-2\delta_0}}(B_1)} \\ &\leq CR^{2-\frac{n+1-2\delta_0}{n+1+2\delta_0}}\|\Delta\varphi\|_{L^{\frac{(n+1)(n+1+2\delta_0)}{n+1-2\delta_0}}\left(B_{\frac{R}{2}}\right)} \\ &\leq CR^{4-\frac{n+1-2\delta_0}{n+1+2\delta_0}}\omega \\ &= C\delta^{15+5\frac{n+1-2\delta_0}{n+1+2\delta_0}}\end{aligned}$$

and hence we obtain (3.39). □

We choose φ according to (3.37). Notice if $\xi_G > 0$, then we have $\nabla\tilde{u}_0 \neq 0$ and

$$W(\tilde{u}_0) \leq \frac{1}{2}|\nabla\tilde{u}_0|^2. \tag{3.40}$$

The case $\xi_G \leq 0$ immediately gives us our desired estimate since we are seeking an upper bound.

Claim For the choice of G as in (3.33) and φ as in (3.37) we have the differential inequality

$$\Delta\xi_G \geq -C\left(1 + \frac{\delta}{|\nabla\tilde{u}_0|}\right)\left(|\nabla\xi_G| + R^{4-\frac{n+1-2\delta_0}{n+1+2\delta_0}}\omega\right) + C(\delta^6 + \delta^4) \tag{3.41}$$

in $B_{\frac{R}{2}} \cap \{\xi_G > 0\} \cap \{\nabla\tilde{u}_0 \neq 0\}$.

Proof We compute the Laplacian of the modified discrepancy

$$\begin{aligned}\Delta\xi_G &= |\nabla^2\tilde{u}_0|^2 + \langle\nabla\tilde{u}_0, \nabla\Delta\tilde{u}_0\rangle - \Delta\varphi - (W' + G')\Delta\tilde{u}_0 - (W'' + G'')|\nabla\tilde{u}_0|^2 \\ &= |\nabla^2\tilde{u}_0|^2 + \langle\nabla\tilde{u}_0, W''\nabla\tilde{u}_0 - \nabla\tilde{f}_0\rangle - \Delta\varphi - (W' + G')(W'(\tilde{u}_0) - \tilde{f}_0) \\ &\quad - (W'' + G'')|\nabla\tilde{u}_0|^2 \\ &= |\nabla^2\tilde{u}_0|^2 - \langle\nabla\tilde{u}_0, \nabla\tilde{f}_0\rangle - \Delta\varphi - (W' + G')(W'(\tilde{u}_0) - \tilde{f}_0) - G''|\nabla\tilde{u}_0|^2.\end{aligned} \tag{3.42}$$

By differentiating (3.32), we have

$$\nabla \xi_G = \nabla^2 \tilde{u}_0 \nabla \tilde{u}_0 - (W' + G')\nabla \tilde{u}_0 - \nabla \varphi,$$

and thus

$$\begin{aligned}
|\nabla^2 \tilde{u}_0|^2 |\nabla \tilde{u}_0|^2 &\geq |\nabla^2 \tilde{u}_0 \nabla \tilde{u}_0|^2 \\
&\geq |\nabla \xi_G + (W' + G')\nabla \tilde{u}_0 + \nabla \varphi|^2 \\
&\geq 2(W' + G') \langle \nabla \tilde{u}_0, \nabla(\xi_G + \varphi)\rangle + (W' + G')^2 |\nabla \tilde{u}_0|^2.
\end{aligned}$$

Dividing by $|\nabla \tilde{u}_0|^2$, the first term in (3.42), $|\nabla^2 \tilde{u}_0|^2$, is bounded as follows

$$|\nabla^2 \tilde{u}_0|^2 \geq \frac{2(W' + G')}{|\nabla \tilde{u}_0|^2} \langle \nabla \tilde{u}_0, \nabla(\xi_G + \varphi)\rangle + (W' + G')^2.$$

The last term in (3.42) is

$$|\nabla \tilde{u}_0|^2 = 2(\xi_G + W + G + \varphi).$$

Substituting these into (3.42) and rearranging, we have in $B_R \subset \{\nabla \tilde{u}_0 = 0\}$

$$\begin{aligned}
&\Delta \xi_G - \frac{2(W' + G')}{|\nabla \tilde{u}_0|^2} \langle \nabla \tilde{u}_0, \nabla \xi_G\rangle + 2G'' \xi_G \\
&\quad \geq (W' + G')^2 - W'(W' + G') - 2G''(W + G) + \frac{2(W' + G')}{|\nabla \tilde{u}_0|^2} \langle \nabla \tilde{u}_0, \nabla \varphi\rangle \\
&\quad\quad - 2G''\varphi - \Delta\varphi - \langle \nabla \tilde{u}_0, \nabla \tilde{f}_0\rangle + (W' + G')\tilde{f}_0 \\
&\quad = (G')^2 + \big(G'W' - 2G''(W + G)\big) + \frac{2(W' + G')}{|\nabla \tilde{u}_0|^2} \langle \nabla \tilde{u}_0, \nabla \varphi\rangle - 2G''\varphi - \Delta\varphi \\
&\quad\quad - \langle \nabla \tilde{u}_0, \nabla \tilde{f}_0\rangle + (W' + G')\tilde{f}_0.
\end{aligned}$$

We choose G to be (3.33) which allows us to apply the estimates (3.34) and (3.35) so that ξ_G satisfies

$$\begin{aligned}
\Delta \xi_G \geq\ & \frac{2(W' + G')}{|\nabla \tilde{u}_0|^2} \langle \nabla \tilde{u}_0, (\nabla \xi_G + \nabla \varphi)\rangle - 2G''_\delta \xi_G \\
& + (G'_\delta)^2 + \delta G'_\delta - 2G''_\delta \varphi - \Delta\varphi - \langle \nabla \tilde{u}_0, \nabla \tilde{f}_0\rangle + (W' + G'_\delta)\tilde{f}_0,
\end{aligned} \tag{3.43}$$

in $B_R \cap \{\nabla \tilde{u}_0 \neq 0\}$. Furthermore we have by (3.40)

$$|W'(\tilde{u}_0)|^2 = |\tilde{u}_0|^2 (1 - |\tilde{u}_0|^2)^2 \leq CW(\tilde{u}_0) \leq C|\nabla \tilde{u}_0|^2.$$

From (3.34), the bounds on G_δ and its derivatives, we get

$$\frac{|(W' + G'_\delta)(\tilde{u}_0)\nabla\tilde{u}_0|}{|\nabla\tilde{u}_0|^2} \le \frac{\frac{1}{2}|\nabla\tilde{u}_0|^3 + \delta|\nabla\tilde{u}_0|}{|\nabla\tilde{u}_0|^2} \le C\left(1 + \frac{\delta}{|\nabla\tilde{u}_0|}\right). \tag{3.44}$$

Substituting in (3.37), (3.39), and (3.44) into (3.43) and using the fact that $G'' < 0$, we have

$$\begin{aligned}\Delta\xi_G &\ge -C\left(1 + \frac{\delta}{|\nabla\tilde{u}_0|}\right)(|\nabla\xi_G| + |\nabla\varphi|) + (G'_\delta)^2 + \delta G'_\delta - \Delta\varphi + \Delta\varphi \\ &\ge -C\left(1 + \frac{\delta}{|\nabla\tilde{u}_0|}\right)\left(|\nabla\xi_G| + R^{4-\frac{n+1-2\delta_0}{n+1+2\delta_0}}\omega\right) + (G'_\delta)^2 + \delta G'_\delta.\end{aligned}$$

Thus applying Eq. (3.36) in $B_{\frac{R}{2}} \cap \{\xi_G > 0\} \cap \{\nabla\tilde{u}_0 \ne 0\}$, we have (3.41)

$$\Delta\xi_G \ge -C\left(1 + \frac{\delta}{|\nabla\tilde{u}_0|}\right)(|\nabla\xi_G| + R^{4-\frac{n+1-2\delta_0}{n+1+2\delta_0}}\omega) + C(\delta^6 + \delta^4). \tag{3.45}$$

□

We define

$$\eta := \sup_{B_1}\xi_G \tag{3.46}$$

and consider two cases :
case i) $\eta := \sup_{B_1}\xi_G < \delta$. Since

$$\xi_G := \frac{|\nabla\tilde{u}_0|^2}{2} - W(\tilde{u}_0) - G(\tilde{u}_0) - \varphi < \delta,$$

by (3.34) and (3.39) this implies

$$\frac{|\nabla\tilde{u}_0|^2}{2} - W(\tilde{u}_0) \le \delta + G(\tilde{u}_0) + \varphi \le \delta + C\delta + CR^{4-\frac{n+1-2\delta_0}{n+1+2\delta_0}}\omega.$$

Our choices give $CR^{4-\frac{n+1-2\delta_0}{n+1+2\delta_0}}\omega = C\delta^{15+5\frac{n+1-2\delta_0}{n+1+2\delta_0}} \le C\delta$ so

$$\frac{|\nabla\tilde{u}_0|^2}{2} - W(\tilde{u}_0) \le C\delta$$

which, after integrating proves (3.26).
case ii) $\eta := \sup_{B_1}\xi_G \ge \delta > 0$. We choose a cutoff function $\lambda \in C_0^2(B_{\frac{R}{2}})$ satisfying $0 \le \lambda \le 1$, $\lambda \equiv 1$ on $B_{\frac{R}{4}}$ and $|\nabla^j\lambda| \le CR^{-j}$ for $j = 1, 2$. Then $\exists x_0 \in B_{\frac{R}{2}}$ such that

$$(\lambda\xi_G)(x_0) = \max\left\{(\lambda\xi_G)(x) : x \in \bar{B}_{\frac{R}{2}}\right\} \ge \eta > 0.$$

By (3.31) we have $\xi_G \le C$ for some $C(c_0, \Lambda_0, E_0, n) > 0$ in B_{R-1}, and thus

$$\lambda(x_0) \ge \frac{\eta}{C}.$$

Moreover,

$$|\nabla \tilde{u}_0(x_0)|^2 \ge 2\xi_G(x_0) \ge 2(\lambda\xi_G)(x_0) \ge 2\eta \ge 2\delta > 0.$$

Since x_0 is a critical point, $\nabla(\lambda\xi_G)(x_0) = 0$, and we get

$$|\nabla\xi_G(x_0)| = \lambda(x_0)^{-1}|\nabla\lambda(x_0)|\xi_G(x_0) \le C(R\eta)^{-1}.$$

At a maximum point x_0, the Laplacian of the function $\lambda\xi_G$ satisfies

$$\begin{aligned} 0 &\ge \Delta(\lambda\xi_G)(x_0) \\ &= \lambda(x_0)\Delta\xi_G(x_0) + 2\langle\nabla\lambda(x_0), \nabla\xi_G(x_0)\rangle + \Delta\lambda(x_0)\xi_G(x_0), \end{aligned}$$

and thus

$$\begin{aligned} \Delta\xi_G(x_0) &\le \lambda(x_0)^{-1}\left(C|\nabla\lambda(x_0)||\nabla\xi_G(x_0)| + |\Delta\lambda(x_0)||\xi_G(x_0)|\right) \\ &\le C\eta^{-1}\left(CR^{-1}(R\eta)^{-1} + CR^{-2}\right) \\ &\le CR^{-2}\eta^{-1}(1+\eta^{-1}) \\ &\le CR^{-2}\eta^{-1}(1+\delta^{-1}) \\ &\le CR^{-2}\eta^{-1}\delta^{-1}, \end{aligned} \tag{3.47}$$

since $\delta \ll 1$. Combining (3.41) and (3.47) we have

$$\begin{aligned} CR^{-2}\eta^{-1}\delta^{-1} &\ge -C\left(1 + \frac{\delta}{|\nabla\tilde{u}_0(x_0)|}\right)\left(|\nabla\xi_G| + R^{4-\frac{n+1-2\delta_0}{n+1+2\delta_0}}\omega\right) + C(\delta^6 + \delta^4) \\ &\ge C\left[\left(1 + \frac{\delta}{2\delta}\right)\left((R\eta)^{-1} + R^{4-\frac{n+1-2\delta_0}{n+1+2\delta_0}}\omega\right) + \delta^4\right]. \end{aligned}$$

Thus the last term above is bounded by

$$\delta^4 \le C\left(R^{-2}\eta^{-1}\delta^{-1} + (R\eta)^{-1}\right) + CR^{4-\frac{n+1-2\delta_0}{n+1+2\delta_0}}\omega.$$

By our choice of $p_1 = 2$, $p_2 = 15$, we have $R^{4-\frac{n+1-2\delta_0}{n+1+2\delta_0}}\omega = R^{15+5\frac{n+1-2\delta_0}{n+1+2\delta_0}} \ll \delta^4$. So

$$\delta^4 \le C\left(R^{-2}\eta^{-1}\delta^{-1} + (R\eta)^{-1}\right),$$

dividing both sides by $\delta^4\eta^{-1}$ gives

$$\begin{aligned}\eta &\le C\left(R^{-2}\delta^{-4}\delta^{-1} + R^{-1}\delta^{-4}\right)\\ &\le C\delta.\end{aligned}$$

Namely, assuming (3.46) or not, we have

$$\xi_G \le C\delta,$$

and thus by (3.39)

$$\begin{aligned}\frac{|\nabla\tilde{u}_0|^2}{2} - W(\tilde{u}_0) &= \xi_G + G_\delta(\tilde{u}_0) + \varphi\\ &\le C\delta + R^{4-\frac{n+1-2\delta_0}{n+1+2\delta_0}}\omega\\ &\le C\delta + \delta^{15+5\frac{n+1-2\delta_0}{n+1+2\delta_0}}\\ &\le C\delta.\end{aligned}$$

This proves (3.26) and as a consequence (3.19). If $|\tilde{u}| \ge 1-\tau$ in $B_{\frac{1}{2}}$, then (3.20) follows because the left hand side is less than the second term on the right. So we only need to consider the case there exists $x_0 \in B_{\frac{1}{2}}$ with $\tilde{u}(x_0) \le 1-\tau$. By the Sobolev inequality and Calderon–Zygmund estimates we bound $\tilde{u}$ in the Hölder norm as follows

$$\begin{aligned}\|\tilde{u}\|_{C^{0,\frac{2\delta_0}{(n+1)+\delta_0}}(B_1)} &\le \|\tilde{u}\|_{W^{2,\frac{n+1}{2}+\delta_0}(B_1)}\\ &\le \tilde{C}\left(\|W'(\tilde{u})\|_{L^{\frac{n+1}{2}+\delta_0}(B_1)} + \|\tilde{f}\|_{L^{\frac{n+1}{2}+\delta_0}(B_1)} + \|\tilde{u}\|_{L^{\frac{n+1}{2}+\delta_0}(B_1)}\right)\\ &\le C.\end{aligned}$$

Therefore $|\tilde{u}| \le 1-\frac{\tau}{2}$ and $W(\tilde{u}) \ge \frac{\tau^2}{4}$ in $B_{(\frac{\tau}{2C})^{\frac{(n+1)+\delta_0}{2\delta_0}}} \subset B_1$. So

$$\int_{B_{\frac{1}{2}}} W(\tilde{u}) \ge \frac{\tau^2}{4}\left(\frac{\tau}{2\tilde{C}_2}\right)^{\frac{(n+1)[(n+1)+\delta_0]}{2\delta_0}} = C\tau^{\frac{(n+1)^2+(n+1)\delta_0+4\delta_0}{2\delta_0}}.$$

By our choice $p_3 = \frac{2\delta_0}{(n+1)^2+(n+1)\delta_0+6\delta_0}$,

$$\begin{aligned}\int_{B_{\frac{1}{2}}}\left(\frac{|\nabla\tilde{u}|^2}{2} - W(\tilde{u})\right)_+ &\le C\delta\\ &\le C\tau^{\frac{(n+1)^2+(n+1)\delta_0+6\delta_0}{2\delta_0}}\end{aligned}$$

$$\le C\tau\tau^{\frac{(n+1)^2+(n+1)\delta_0+4\delta_0}{2\delta_0}}$$
$$\le C\tau\int_{B_{\frac{1}{2}}}\left(\frac{|\nabla\tilde{u}|^2}{2}+W(\tilde{u})\right),$$

which proves (3.20). ◻

Next we derive energy estimates away from transition regions.

Proposition 3.5 ([8, Proposition 3.4]) *For any $n \geq 2$, $0 \leq \delta \leq \delta_1$, $\varepsilon > 0$, $u_\varepsilon \in C^2(\Omega)$, $f_\varepsilon \in C^0(\Omega)$, if*

$$-\varepsilon\Delta u_\varepsilon + \frac{W'(u_\varepsilon)}{\varepsilon} = f_\varepsilon \quad \textit{in } \Omega$$

and

$$\Omega' \subset\subset \Omega, 0 < r \leq d(\Omega', \partial\Omega)$$

then

$$\int_{\{|u_\varepsilon|\geq 1-\delta\}\cap\Omega'}\left(\varepsilon|\nabla u_\varepsilon|^2+\frac{W(u_\varepsilon)}{\varepsilon}+\frac{W'(u_\varepsilon)^2}{\varepsilon}\right)$$
$$\leq C\delta\int_{\{|u_\varepsilon|\leq 1-\delta\}\cap\Omega}\varepsilon|\nabla u_\varepsilon|^2+C\varepsilon\int_\Omega|f_\varepsilon|^2+C\left(\frac{\delta}{r}+\frac{\delta^2}{r^2}\right)\varepsilon\mathcal{L}^{n+1}(\Omega)$$
$$+\frac{C\varepsilon}{r^2}\int_{\{|u_\varepsilon|\geq 1\}\cap\Omega}W'(u_\varepsilon)^2.$$

(Notice the power of f_ε in the above inequality will still be 2 instead of $\frac{n+1}{2}+\delta_0$.)

Proof Define a continuous function

$$g(t)=\begin{cases}W'(t), & \text{for } |t|\geq 1-\delta\\ 0, & \text{for } |t|\leq t_0\\ \text{linear}, & \text{for } t\in[-1+\delta,-t_0]\cup[t_0,1-\delta],\end{cases}$$

where $t_0=\frac{1}{\sqrt{3}}$ is chosen to be the number in $(0,1)$ such that $W''(t_0)=0$. Clearly $|g|\leq|W'|$. For $\eta\in C_0^1(\Omega)$ satisfying $0\leq\eta\leq 1$, $\eta\equiv 1$ on Ω' and $|\nabla\eta|\leq Cr^{-1}$, we get by integration by parts

$$\int_\Omega f_\varepsilon g(u_\varepsilon)\eta^2=\int_\Omega\left(-\varepsilon\Delta u_\varepsilon+\frac{W'(u_\varepsilon)}{\varepsilon}\right)g(u_\varepsilon)\eta^2$$
$$=\int_\Omega\varepsilon g'(u_\varepsilon)|\nabla u_\varepsilon|^2\eta^2+2\int_\Omega\varepsilon g(u_\varepsilon)\eta\langle\nabla u_\varepsilon,\nabla\eta\rangle$$
$$+\int_\Omega\frac{W'(u_\varepsilon)}{\varepsilon}g(u_\varepsilon)\eta^2.$$

$$(3.48)$$

The left hand side of (3.48) can be bounded by

$$\int_\Omega f_\varepsilon g(u_\varepsilon)\eta^2 \le \frac{\varepsilon}{2}\int_\Omega |f_\varepsilon|^2 + \frac{1}{2\varepsilon}\int_\Omega g(u_\varepsilon)^2\eta^2 \le \frac{\varepsilon}{2}\int_\Omega |f_\varepsilon|^2 + \frac{1}{2\varepsilon}\int_\Omega W'(u_\varepsilon)g(u_\varepsilon)\eta^2. \tag{3.49}$$

By the definition of g above, we have

$$|g(t)| \le |g(1-\delta)| = W'(1-\delta) \le C\delta,$$
$$|g'(t)| \le \frac{|g(1-\delta)|}{1-\delta} \le \frac{|g(1-\delta)|}{1-\delta_1} \le C\delta,$$

for $|t| \le 1-\delta$. Applying these estimates to the second term on the right hand side of (3.48) we get the bound

$$\begin{aligned}&\left|2\int_\Omega \varepsilon g(u_\varepsilon)\eta\langle\nabla u_\varepsilon, \nabla\eta\rangle\right| \\ &\quad\le 2\delta\int_{\{|u_\varepsilon|\le 1-\delta\}} \varepsilon\eta|\nabla u_\varepsilon||\nabla\eta| + \left|\int_{\{|u_\varepsilon|\ge 1-\delta\}} \varepsilon W'(u_\varepsilon)\langle\nabla u_\varepsilon, \nabla\eta\rangle\right| \\ &\quad\le C\delta\int_{\{|u_\varepsilon|\le 1-\delta\}} \varepsilon|\nabla u_\varepsilon|^2 + \varepsilon\delta r^{-1}\mathcal{L}^{n+1}(\Omega) \\ &\qquad + \tau\int_{\{|u_\varepsilon|\ge 1-\delta\}} \varepsilon|\nabla u_\varepsilon|^2\eta^2 + C\varepsilon\tau^{-1}r^{-2}\int_{\{|u_\varepsilon|\ge 1-\delta\}} W'(u_\varepsilon)^2,\end{aligned} \tag{3.50}$$

for $\tau > 0$. As $g'(t) = W''(t) \ge C_W > 0$ for $|t| \ge 1-\delta$, we obtain from (3.48), (3.49) and (3.50)

$$\begin{aligned}&C_W\int_{\{|u_\varepsilon|\ge 1-\delta\}} \varepsilon|\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon}\int_\Omega W'(u_\varepsilon)g(u_\varepsilon)\eta^2 \\ &\quad\le C_W\delta\int_{\{|u_\varepsilon|\le 1-\delta\}} \varepsilon|\nabla u_\varepsilon|^2 + \tau\int_{\{|u_\varepsilon|\ge 1-\delta\}} \varepsilon|\nabla u_\varepsilon|^2\eta^2 + \frac{\varepsilon}{2}\int_\Omega |f_\varepsilon|^2 \\ &\qquad + \left(\delta r^{-1} + C\delta^2\tau^{-1}r^{-2}\right)\mathcal{L}^{n+1}(\Omega) + C\varepsilon\tau^{-1}r^{-2}\int_{\{|u_\varepsilon|\ge 1\}} W'(u_\varepsilon)^2.\end{aligned}$$

Choosing $\tau = \frac{C_W}{2}$, and using $W(t) \le C_W W'(t)^2$ for $|t| \ge 1-\delta$ we get

$$\begin{aligned}&\int_{\{|u_\varepsilon|\ge 1-\delta\}\cap\Omega'} \left(\varepsilon|\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon} + \frac{W'(u_\varepsilon)^2}{\varepsilon}\right) \\ &\quad\le C\int_{\{|u_\varepsilon|\ge 1-\delta\}\cap\Omega'} \left(\varepsilon|\nabla u_\varepsilon|^2 + \frac{W'(u_\varepsilon)^2}{\varepsilon}\right)\end{aligned}$$

$$\leq C\delta \int_{\{|u_\varepsilon|\leq 1-\delta\}} \varepsilon|\nabla u_\varepsilon|^2 + C\varepsilon \int_\Omega |f_\varepsilon|^2 + C\varepsilon\left(\delta r^{-1} + \delta^2 r^{-2}\right)\mathcal{L}^{n+1}(\Omega)$$
$$+ C\varepsilon r^{-2} \int_{\{|u_\varepsilon|\geq 1\}} W'(u_\varepsilon)^2,$$

which completes the proof. □

The following proposition shows for all ε sufficiently small, if u_ε satisfies the inhomogeneous Allen–Cahn equation then we can control the last term $\int_{\{|u_\varepsilon|\geq 1\}\cap\Omega'} W'(u_\varepsilon)^2$ in Proposition 3.5 by applying the proposition inductively.

Proposition 3.6 ([8, Proposition 3.5]) *For $n \geq 2$, $\varepsilon > 0$, $u_\varepsilon \in C^2(\Omega)$, $f_\varepsilon \in C^0(\Omega)$, if*

$$-\varepsilon\Delta u_\varepsilon + \frac{W'(u_\varepsilon)}{\varepsilon} = f_\varepsilon \quad in\ \Omega$$

and $\Omega' \subset\subset \Omega$, $0 < r \leq d(\Omega', \partial\Omega)$ then

$$\int_{\{|u_\varepsilon|\geq 1\}\cap\Omega'} W'(u_\varepsilon)^2 \leq C_k(1 + r^{-2k}\varepsilon^{2k})\varepsilon^2 \int_{\Omega'_{i-1}} |f_\varepsilon|^2$$
$$+ C_k r^{-2k}\varepsilon^{2k} \int_{\{|u_\varepsilon|\geq 1\}\cap\Omega} W'(u_\varepsilon)^2$$

for all $k \in \mathbb{N}_0$.

Proof For any $k \in \mathbb{N}^+$ we choose a sequence of open sets

$$\Omega'_i := \begin{cases} \Omega & \text{for } i = 0 \\ \left\{x \in \Omega | d(x, \Omega') < \frac{(k-i)r}{k}\right\} & \text{for } i = 1, ..., k-1, \\ \Omega' & \text{for } i = k. \end{cases}$$

This sequence satisfies

$$\Omega' = \Omega'_k \subset\subset \Omega'_{k-1} \subset\subset ... \subset\subset \Omega'_0 = \Omega,$$

with $d(\Omega'_i, \Omega'_{i-1}) \geq \frac{r}{k}$ for $i = 1, ..., k$. Applying Proposition (3.5) with $\delta = 0$, we have

$$\int_{\{|u_\varepsilon|\geq 1\}\cap\Omega'_i} W'(u_\varepsilon)^2 \leq C\varepsilon^2 \int_{\Omega'_{i-1}} |f_\varepsilon|^2 + Ck^2 r^{-2}\varepsilon^2 \int_{\{|u_\varepsilon|\geq 1\}\cap\Omega'_{i-1}} W'(u_\varepsilon)^2,$$

for $i = 1, ..., k$. The conclusion is obtained by applying the above inequality inductively k times. □

We conclude with the following integral bound for positive part of discrepancy measure.

Lemma 3.7 ([8, Lemma 3.1] for all n) *Let $n \geq 2$, $0 < \delta \leq \delta_1$ (where δ_1 given as in Lemma 3.4), $0 < \varepsilon \leq \rho$, $\rho_0 := \max\{2, 1+\delta^{-M}\varepsilon\}\rho$ for some large universal constant M. If $u_\varepsilon \in C^2(B_{\rho_0})$, $f_\varepsilon \in C^0(B_{\rho_0})$ satisfies (1.1) in $B_{\rho_0}(0)$ then the positive part of the discrepancy measure satisfies*

$$\begin{aligned}
&\rho^{-n}\int_{B_\rho}\left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2}-\frac{W(u_\varepsilon)}{\varepsilon}\right)_+\\
&\quad\leq C\delta^{p_3}\rho^{-n}\int_{B_{2\rho}}\left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2}+\frac{W(u_\varepsilon)}{\varepsilon}\right)+C\delta^{-M}\varepsilon\rho^{-n}\int_{B_{\rho_0}}|f_\varepsilon|^2\\
&\qquad+C\delta^{-M}\rho^{-n}\int_{B_{\rho_0}\cap\{|u_\varepsilon|\geq 1\}}\frac{W'(u_\varepsilon)^2}{\varepsilon}+C\left(\frac{\varepsilon}{\rho}\right)\delta.
\end{aligned}$$

Proof We prove the case $0 < \varepsilon \leq \rho = 1$. The case for other $\rho > 0$ follows by rescaling to $\rho = 1$. For $0 < \delta \leq \delta_1$ we choose $R(\delta) = \frac{1}{\delta^{p_1}}$ and $\omega(\delta) = C_\omega\delta^{p_2}$ as in Lemma 3.4. Let $\{x_i\}_{i\in\mathbf{I}} \subset B_1$, $\mathbf{I} \subset \mathbb{N}$ be a maximal collection of points satisfying

$$\min_{i\neq j}|x_i - x_j| \geq \frac{\varepsilon}{2}.$$

Since $\varepsilon \leq 1$, we have

$$\begin{aligned}
&B_1(0) \subset \cup_{i\in\mathbf{I}}\bar{B}_{\frac{\varepsilon}{2}}(x_i) \subset B_{\frac{3}{2}}(0),\\
&\sum_{i\in\mathbf{I}}\chi_{B_\varepsilon(x_i)} \leq C_n\chi_{B_2(0)},\\
&\sum_{i\in\mathbf{I}}\chi_{B_{2R\varepsilon}(x_i)} \leq C_nR^{n+1}\chi_{B_{1+2R\varepsilon}(0)}.
\end{aligned}$$

For $i \in \mathbf{I}$ and $x \in B_{2R}$, we define the rescaled and translated functions as

$$\begin{aligned}
\tilde{u}_i(x) &:= u_\varepsilon(x_i + \varepsilon x),\\
\tilde{f}_i(x) &:= \varepsilon f_\varepsilon(x_i + \varepsilon x),
\end{aligned}$$

which satisfy the rescaled equation

$$-\Delta\tilde{u}_i + W'(\tilde{u}_i) = \tilde{f}_i, \quad \text{in } B_{2R}(0). \tag{3.51}$$

For $\tilde{u}_i$, $\tilde{f}_i$ to be well-defined, we choose $M \geq 5n+6$ and $\delta_1 \leq \frac{1}{2}$ so that

$$x_i + \varepsilon x \in B_{1+2R\varepsilon}(0) \subset B_{1+\delta^{-M}\varepsilon}(0) \subset B_{\rho_0}(0).$$

We decompose the index set $\mathbf{I}$ into

$$\mathbf{I}_1 := \left\{i \in \mathbf{I} : \|f_\varepsilon\|_{L^{\frac{n+1}{2}+\delta_0}(B_{2R\varepsilon}(x_i))} < \varepsilon^{\frac{n+1}{2}-1+\delta_0}\omega,\ \|(|u_\varepsilon|-1)_+\|_{L^1(B_{2R\varepsilon}(x_i))} < C_\omega\varepsilon^{n+1}\right\},$$

$\mathbf{I}_2 := \mathbf{I} \setminus \mathbf{I}_1.$

For $i \in \mathbf{I}_1$, we have

$$\|\tilde{f}_i\|_{L^{\frac{n+1}{2}+\delta_0}(B_{2R}(0))} = \varepsilon^{-\frac{n+1}{2}-\delta_0}\|\varepsilon f_\varepsilon\|_{L^{\frac{n+1}{2}+\delta_0}(B_{2R\varepsilon}(x_i))} < \omega \le C_\omega,$$
$$\|(|\tilde{u}_i|-1)_+\|_{L^1(B_{2R}(x_i))} = \varepsilon^{-n-1}\|(|u_\varepsilon|-1)_+\|_{L^1(B_{2R\varepsilon}(x_i))} < C_\omega.$$

By the condition $\|u\|_{L^\infty} \le c_0$ in the condition of Theorem 1.1, and choosing C_ω sufficiently small, we have

$$\|\tilde{u}_i\|_{L^\infty(B_R)} \le 1 + C \cdot C_\omega \le 2.$$

Applying Lemma 3.4 to $\tilde{u}_i$ gives (with p_3 from Lemma 3.4)

$$\int_{B_{\frac{1}{2}}} \left(\frac{|\nabla \tilde{u}_i|^2}{2} - W(\tilde{u}_i)\right)_+ \le C\delta^{p_3} \int_{B_{\frac{1}{2}}} \left(\frac{|\nabla \tilde{u}_i|^2}{2} + W(\tilde{u}_i)\right) + \int_{B_{\frac{1}{2}} \cap \{|\tilde{u}_i| \ge 1-\delta\}} \frac{|\nabla \tilde{u}_i|^2}{2}.$$

Rescaling back, we get

$$\int_{B_{\frac{\varepsilon}{2}}(x_i)} \left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2} - \frac{W(u_\varepsilon)}{\varepsilon}\right)_+ \le C\delta^{p_3} \int_{B_{\frac{\varepsilon}{2}}(x_i)} \left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2} + \frac{W(u_\varepsilon)}{\varepsilon}\right) + \int_{B_{\frac{\varepsilon}{2}}(x_i) \cap \{|u_\varepsilon| \ge 1-\delta\}} \frac{\varepsilon|\nabla u_\varepsilon|^2}{2}.$$

Summing over $i \in \mathbf{I}_1$ and noticing $B_{\frac{\varepsilon}{2}}(x_i)$ are disjoint, we get

$$\begin{aligned}\sum_{i\in\mathbf{I}_1} \int_{B_{\frac{\varepsilon}{2}}(x_i)} \left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2} - \frac{W(u_\varepsilon)}{\varepsilon}\right)_+ &\le C\delta^{p_3} \int_{B_{\frac{3}{2}}(0)} \left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2} + \frac{W(u_\varepsilon)}{\varepsilon}\right) \\ &\quad + C\int_{B_{\frac{3}{2}}(0)\cap\{|u_\varepsilon|\ge 1-\delta\}} \frac{\varepsilon|\nabla u_\varepsilon|^2}{2} \\ &\le C\delta^{p_3} \int_{B_2(0)} \left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2} + \frac{W(u_\varepsilon)}{\varepsilon}\right) \\ &\quad + C\varepsilon \int_{B_2(0)} |f_\varepsilon|^2 \\ &\quad + C\varepsilon\left(\delta + \int_{B_2(0)\cap\{|u_\varepsilon|\ge 1\}} W'(u_\varepsilon)^2\right), \quad (3.52)\end{aligned}$$

where we used Proposition 3.5 in the last line. Since for $n \geq 3$ (the $n = 2$ case requires $\delta_0 \geq \frac{1}{2}$, but has already been addressed in [8])

$$W'(t)^2 \geq 4t^2(1+t)^2(1-t)^2 \geq C_W t^2(|t|-1)^2 \geq C_W(|t|-1)_+^{\frac{n+1}{2}+\delta_0}.$$

Thus for $i \in \mathbf{I}_2$ (at least one of the bounds in $\mathbf{I}_1$ does not hold), we have

$$\begin{aligned} C_\omega &\leq \int_{B_{2R}(0)} (|\tilde{u}_i|-1)_+^{\frac{n+1}{2}+\delta_0} + \omega^{-2}\int_{B_{2R}} \tilde{f}_i^2 \\ &\leq C\int_{B_{2R}(0)\cap\{|\tilde{u}_i|\geq 1\}} W'(\tilde{u}_i)^2 + \omega^{-2}\int_{B_{2R}} \tilde{f}_i^2. \end{aligned}$$

By elliptic estimates applied to the rescaled Eq. (3.51), we get

$$\begin{aligned} \int_{B_{\frac{1}{2}}} |\nabla \tilde{u}_i|^2 &\leq \tilde{C}\int_{B_1}\left(W'(\tilde{u}_i)^2 + \tilde{u}_i^2 + \tilde{f}_i^2\right) \\ &\leq \tilde{C}\int_{B_{2R}}\left(W'(\tilde{u}_i)^2 + \tilde{f}_i^2\right) + \tilde{C}\omega_n c_0^2 C_\omega^{-1} C_\omega \\ &\leq C\int_{B_{2R}}\left(W'(\tilde{u}_i)^2 + \omega^{-2}\tilde{f}_i^2\right), \end{aligned}$$

where we used $\|\tilde{u}_i\|_{L^\infty} \leq c_0$. Rescaling back gives

$$\int_{B_{\frac{\varepsilon}{2}}(x_i)} \varepsilon|\nabla u_\varepsilon|^2 \leq C\int_{B_{2R\varepsilon}(x_i)}\left(\frac{W'(u_\varepsilon)^2}{\varepsilon} + \varepsilon\omega^{-2}|f_\varepsilon|^2\right).$$

Then summing over $i \in \mathbf{I}_2$ we get

$$\begin{aligned} \sum_{i\in\mathbf{I}_2}\int_{B_{\frac{\varepsilon}{2}}(x_i)} \varepsilon|\nabla u_\varepsilon|^2 &\leq \sum_{i\in\mathbf{I}_2} C\int_{B_{2R\varepsilon}(x_i)}\left(\frac{W'(u_\varepsilon)^2}{\varepsilon} + \varepsilon\omega^{-2}|f_\varepsilon|^2\right) \\ &\leq CR^{n+1}\int_{B_{1+2R\varepsilon}(0)}\left(\frac{W'(u_\varepsilon)^2}{\varepsilon} + \varepsilon\omega^{-2}|f_\varepsilon|^2\right) \\ &\leq C\delta^{-M}\int_{B_{1+\delta^{-M}\varepsilon}(0)}\left(\frac{W'(u_\varepsilon)^2}{\varepsilon} + \varepsilon|f_\varepsilon|^2\right), \end{aligned} \tag{3.53}$$

for large enough M since both $R = \delta^{-p_1}$ and $\omega = \delta^{p_2}$ are fixed powers of δ. Combining (3.52) and (3.53) we get

$$
\begin{aligned}
&\int_{B_1}\left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2}-\frac{W(u_\varepsilon)}{\varepsilon}\right)_+\\
&\quad\le \sum_{i\in\mathbf{I}_1}\int_{B_{\frac{\varepsilon}{2}}(x_i)}\left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2}-\frac{W(u_\varepsilon)}{\varepsilon}\right)_+ +\sum_{i\in\mathbf{I}_2}\int_{B_{\frac{\varepsilon}{2}}(x_i)}\frac{\varepsilon|\nabla u_\varepsilon|^2}{2}\\
&\quad\le C\delta^{p_3}\int_{B_2(0)}\left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2}+\frac{W(u_\varepsilon)}{\varepsilon}\right)\\
&\qquad + C\varepsilon\int_{B_2(0)}|f_\varepsilon|^2 + C\varepsilon\left(\delta+\int_{B_2(0)\cap\{|u_\varepsilon|\ge 1\}}W'(u_\varepsilon)^2\right)\\
&\qquad + C\delta^{-M}\int_{B_{1+\delta^{-M}\varepsilon}(0)}\left(\frac{W'(u_\varepsilon)^2}{\varepsilon}+\varepsilon|f_\varepsilon|^2\right)\\
&\quad\le C\delta^{p_3}\int_{B_2(0)}\left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2}+\frac{W(u_\varepsilon)}{\varepsilon}\right)+C\varepsilon\delta+C\varepsilon\delta^{-M}\int_{B_{\max\{2,1+\delta^{-M}\varepsilon\}}(0)}|f_\varepsilon|^2\\
&\qquad + C\delta^{-M}\int_{B_{\max\{2,1+\delta^{-M}\varepsilon\}}(0)}\frac{W'(u_\varepsilon)^2}{\varepsilon}.
\end{aligned}
$$

This completes the proof for $\rho = 1$ and rescaling gives the cases for other $\rho > 0$. □

As a result of these, we have the L^1 convergence of the positive part of the discrepancy measure as $\varepsilon \to 0$.

Lemma 3.8 *If we consider $\xi_\varepsilon = \xi_{\varepsilon,+} - \xi_{\varepsilon,-}$ the decomposition of ξ_ε into positive and negative variations then*

$$\xi_{\varepsilon,+} \to 0 \quad \textit{as } \varepsilon \to 0.$$

Furthermore this shows $\xi \le 0$.

Proof For $B_{2\rho} = B_{2\rho}(x) \subset \Omega' \subset\subset \Omega$, $0 < \delta < \delta_0$ and $0 < \varepsilon \le \delta^M$ then applying Lemma 3.7 we have

$$
\begin{aligned}
\int_{B_\rho}\left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2}-\frac{W(u_\varepsilon)}{\varepsilon}\right)_+ \le{}& C\delta^{p_3}\int_{B_{2\rho}}\left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2}+\frac{W(u_\varepsilon)}{\varepsilon}\right)\\
&+C\delta^{-M}\varepsilon\int_{B_\rho}|f_\varepsilon|^2\\
&+C\delta^{-M}\int_{B_\rho\cap\{|u_\varepsilon|\ge 1\}}\frac{W'(u_\varepsilon)^2}{\varepsilon}\\
&+C\left(\frac{\varepsilon}{\rho}\right)\delta\rho^n.
\end{aligned}
\tag{3.54}
$$

Proposition 3.6 gives us

$$\int_{\{|u_\varepsilon|\geq 1\}\cap B_\rho} W'(u_\varepsilon)^2 \leq C_k(1+\rho^{-2k}\varepsilon^{2k})\varepsilon^2 \int_{B_{2\rho}} |f_\varepsilon|^2 + C_k\rho^{-2k}\varepsilon^{2k} \int_{\{|u_\varepsilon|\geq 1\}\cap B_{2\rho}} W'(u_\varepsilon)^2$$

for all $k \in \mathbb{N}_0$. Choosing $k=2$ and applying the bound

$$\int_{\{|u_\varepsilon|\geq 1\}\cap B_{2\rho}} W'(u_\varepsilon)^2 \leq C(\Omega')$$

and inserting these estimates into (3.54), we get

$$\begin{aligned}\int_{B_\rho}\left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2}-\frac{W(u_\varepsilon)}{\varepsilon}\right)_+ &\leq C\delta^{p_3}\int_{B_{2\rho}}\left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2}+\frac{W(u_\varepsilon)}{\varepsilon}\right)\\ &\quad + C(\delta^{-M}\varepsilon+\varepsilon^2)\int_{B_\rho}|f_\varepsilon|^2\\ &\quad + C\delta^{-M}\varepsilon^3 + C\left(\frac{\varepsilon}{\rho}\right)\delta\rho^n.\end{aligned}$$

By the Hölder inequality with exponent $q_0/2$, we estimate

$$\begin{aligned}\varepsilon\int_{B_{r/2}}|f_\varepsilon|^2 &= \varepsilon^2\int_{B_{r/2}}\left(\frac{f_\varepsilon}{|\varepsilon|\nabla u_\varepsilon|}\right)^2\varepsilon|\nabla u_\varepsilon|^2\\ &\leq \varepsilon^2\left(\int_{B_{r/2}}\left|\frac{f_\varepsilon}{|\varepsilon|\nabla u_\varepsilon|}\right|^{q_0}\varepsilon|\nabla u_\varepsilon|^2\right)^{2/q_0}\left(\int_{B_{r/2}}\varepsilon|\nabla u_\varepsilon|^2\right)^{\frac{q_0}{q_0-2}}\\ &\leq \varepsilon^2 C(\Lambda_0, E_0),\end{aligned}\tag{3.55}$$

and obtain

$$\begin{aligned}\int_{B_\rho}\left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2}-\frac{W(u_\varepsilon)}{\varepsilon}\right)_+ &\leq \tilde{C}\delta^{p_3}+\tilde{C}\delta^{-M}\varepsilon^2+\tilde{C}\varepsilon^2+\tilde{C}\delta^{-M}\varepsilon^3+\tilde{C}\delta\varepsilon\\ &\leq \tilde{C}\delta.\end{aligned}$$

Letting $\varepsilon \to 0$ we get $\xi_{\varepsilon,+}(B_\rho) \to 0$. □

4 Rectifiability

We will proceed by proving upper and lower density bounds for the energy measure. Combining the estimates obtained in the previous section, we get an upper bound on the density ratio of the limit energy measure.

Theorem 4.1 *If we consider* $\Omega' \subset\subset \Omega$ *and* $r_0(\Omega') := \min\left\{1, \frac{d(\Omega', \partial\Omega)}{2}\right\}$ *then for all* $x_0 \in \Omega'$, $0 < r < r_0$ *there exists a function* $\phi(\varepsilon)$ *with* $\lim_{\varepsilon \to 0} \phi(\varepsilon) = 0$ *such that*

$$r^{-n}\mu_\varepsilon(B_r(x_0)) \leq C(\Lambda_0, \Omega') + \frac{\phi(\varepsilon)}{r^n}. \tag{4.1}$$

Letting $\varepsilon \to 0$ *we get*

$$r^{-n}\mu(B_r(x_0)) \leq C(\Lambda_0, \Omega'),$$

where $\mu = \lim_{\varepsilon \to 0} \mu_\varepsilon$ *is the weak-* limit of* $\mu_\varepsilon = \left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2} + \frac{W(u_\varepsilon)}{\varepsilon}\right) dx$ *in the sense of Radon measures.*

Proof For the sake of simplicity we set $x_0 = 0$ and set $B_\rho(0) = B_\rho$. By the almost monotonicity formula (3.1), Lemma 3.8 and Holder's inequality

$$\begin{aligned}\frac{d}{d\rho}\left(\frac{\mu_\varepsilon(B_\rho)}{\rho^n}\right) = &-\frac{1}{\rho^{n+1}}\xi_\varepsilon(B_\rho) + \frac{\varepsilon}{\rho^{n+2}}\int_{\partial B_\rho}\langle x, \nabla u\rangle^2 \\ &- \frac{1}{\rho^{n+1}}\int_{B_\rho}\langle x, \nabla u\rangle f_\varepsilon.\end{aligned} \tag{4.2}$$

We estimate the last term above as follows

$$\begin{aligned}\frac{1}{\rho^{n+1}}\left|\int_{B_\rho}\langle x, \nabla u\rangle f_\varepsilon\right| &\leq \frac{1}{\rho^{n+1}}\int_{B_\rho}|\langle x, \nabla u\rangle|\left|\frac{f_\varepsilon}{\varepsilon|\nabla u|}\right|\varepsilon|\nabla u| \\ &\leq \frac{1}{\rho^n}\int_{B_\rho}\left|\frac{f_\varepsilon}{\varepsilon|\nabla u|}\right|\varepsilon|\nabla u|^2 \\ &\leq \frac{1}{\rho^n}\left(\int_{B_\rho}\left|\frac{f_\varepsilon}{\varepsilon|\nabla u|}\right|^{q_0}\varepsilon|\nabla u|^2\right)^{\frac{1}{q_0}}\left(\int_{B_\rho}\varepsilon|\nabla u|^2\right)^{\frac{q_0-1}{q_0}} \\ &\leq \left(\frac{1}{\rho^n}\right)^{\frac{1}{q_0}}\left(\int_{B_\rho}\left|\frac{f_\varepsilon}{\varepsilon|\nabla u|}\right|^{q_0}\varepsilon|\nabla u|^2\right)^{\frac{1}{q_0}} \\ &\quad \left[\frac{1}{\rho^n}\right]^{\frac{q_0-1}{q_0}}\left(2\mu_\varepsilon(B_\rho)\right)^{\frac{q_0-1}{q_0}} \\ &\leq C(\Lambda_0)\rho^{-\frac{n}{q_0}}\left(\frac{\mu_\varepsilon(B_\rho)}{\rho^n}\right)^{\frac{q_0-1}{q_0}} \\ &\leq C(\Lambda_0)\rho^{-\frac{n}{q_0}}\left(1 + \frac{\mu_\varepsilon(B_\rho)}{\rho^n}\right)\end{aligned} \tag{4.3}$$

where we used the inequality $a^{1-\frac{1}{q_0}} \leq 1 + a$ which holds for all $a \geq 0$. Inserting this inequality into (4.2) and discarding the positive second term on the right had side, we get

$$\frac{d}{d\rho}\left(1+\frac{\mu_\varepsilon(B_\rho)}{\rho^n}\right)=\frac{d}{d\rho}\left(\frac{\mu_\varepsilon(B_\rho)}{\rho^n}\right)\geq-\frac{1}{\rho^{n+1}}\xi_\varepsilon(B_\rho)$$
$$-C(\Lambda_0)\rho^{-\frac{n}{q_0}}\left(1+\frac{\mu_\varepsilon(B_\rho)}{\rho^n}\right). \tag{4.4}$$

Multiplying both sides by $\exp\left(\int C(\Lambda_0)\rho^{-\frac{n}{q_0}}d\rho\right)=\exp\left(\frac{q_0}{q_0-n}C(\Lambda_0)\rho^{1-\frac{n}{q_0}}\right)$ we have

$$\frac{d}{d\rho}\left[\exp\left(\frac{q_0}{q_0-n}C(\Lambda_0)\rho^{1-\frac{n}{q_0}}\right)\left(1+\frac{\mu_\varepsilon(B_\rho)}{\rho^n}\right)\right]$$
$$\geq-\exp\left(\frac{q_0}{q_0-n}C(\Lambda_0)\rho^{1-\frac{n}{q_0}}\right)\frac{\xi_\varepsilon(B_\rho)}{\rho^{n+1}}.$$

Integrating from r to r_0 gives

$$\exp\left(\frac{q_0}{q_0-n}C(\Lambda_0)r_0^{1-\frac{n}{q_0}}\right)\left(1+\frac{\mu_\varepsilon(B_{r_0})}{r_0^n}\right)-\exp\left(\frac{q_0}{q_0-n}C(\Lambda_0)r^{1-\frac{n}{q_0}}\right)\left(1+\frac{\mu_\varepsilon(B_r)}{r^n}\right)$$
$$\geq-\int_r^{r_0}\exp\left(\frac{q_0}{q_0-n}C(\Lambda_0)\rho^{1-\frac{n}{q_0}}\right)\frac{\xi_{\varepsilon,+}(B_\rho)}{\rho^{n+1}}$$
$$\geq-\exp\left(\frac{q_0}{q_0-n}C(\Lambda_0)r_0^{1-\frac{n}{q_0}}\right)\int_r^{r_0}\frac{\xi_{\varepsilon,+}(B_\rho)}{\rho^{n+1}}.$$

Namely

$$\exp\left(\frac{q_0}{q_0-n}C(\Lambda_0)r_0^{1-\frac{n}{q_0}}\right)\left(1+\frac{\mu_\varepsilon(B_{r_0})}{r_0^n}\right)-\frac{\mu_\varepsilon(B_r)}{r^n}\geq-C(\Lambda_0,\Omega')\int_r^{r_0}\frac{\xi_{\varepsilon,+}(B_\rho)}{\rho^{n+1}}$$
$$\geq-C(\Lambda_0,\Omega')\int_r^{r_0}\frac{\xi_{\varepsilon,+}(B_{r_0})}{\rho^{n+1}}, \tag{4.5}$$

where we used $\exp\left(\frac{q_0}{q_0-n}C(\Lambda_0)r^{1-\frac{n}{q_0}}\right)>1$ for $r>0$. Passing to the limit as $\varepsilon\to0$ and using Lemma 3.8, we have

$$\frac{\mu(B_r)}{r^n}\leq C(\Lambda_0,\Omega',n,q_0).$$

□

Next, we obtain estimates of the discrepancy measure for each ε.

Proposition 4.2 *Let* $\delta=\rho^\gamma,\varepsilon\leq\rho\leq r$ *for* $0<\gamma<\frac{1}{M}\leq\frac{1}{2}$, *we have* $\delta^{-M}\varepsilon\leq\rho^{1-M\gamma}\leq1$. *For* $B_{3\rho^{1-\beta}}(x)\subset\subset\Omega$, *we have*

$$\rho^{-n-1}\xi_{\varepsilon,+}(B_\rho(x))\leq C\rho^{p_3\gamma-n-1}\mu_\varepsilon(B_{2\rho}(x))+\tilde{C}_k\varepsilon\rho^{-M\gamma-n-1}\int_{B_{3\rho^{1-\beta}}(x)}|f_\varepsilon|^2$$

$$+\tilde{C}_\beta \varepsilon \rho^{\gamma-2}\left(1+\int_{\{|u_\varepsilon|\geq 1\}\cap B_{3r^{1-\beta}}(x)} W'(u_\varepsilon)^2\right). \tag{4.6}$$

Proof For $0<\gamma<\frac{1}{M}\leq\frac{1}{2}$, by choosing $\delta^{-M}\varepsilon\leq\rho^{1-M\gamma}\leq 1$ we get $\max\{2, 1+\delta^{-M}\varepsilon\}=2$. Therefore substituting $\delta=\rho^\gamma$ into Lemma 3.7 we have

$$\begin{aligned}
\rho^{-n-1}\xi_{\varepsilon,+}(B_\rho) &= \rho^{-n-1}\int_{B_\rho(x)}\left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2}-\frac{W(u_\varepsilon)}{\varepsilon}\right)_+ \\
&\leq C\rho^{p_3\gamma-n-1}\int_{B_{2\rho}(x)}\left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2}+\frac{W(u_\varepsilon)}{\varepsilon}\right) \\
&\quad + C\varepsilon\rho^{-M\gamma-n-1}\int_{B_{2\rho}(x)}|f_\varepsilon|^2 \\
&\quad + C\varepsilon^{-1}\rho^{-M\gamma-n-1}\int_{B_{2\rho}(x)\cap\{|u_\varepsilon|\geq 1\}} W'(u_\varepsilon)^2 + C\varepsilon\rho^{\gamma-2}.
\end{aligned}$$

On the other hand we have by Proposition 3.6 with $r := d(B_{2\rho}(x), \partial B_{3\rho^{1-\beta}}(x)) = 3\rho^{1-\beta}-2\rho\geq\rho^{1-\beta}$

$$\begin{aligned}
\int_{\{|u_\varepsilon|\geq 1\}\cap B_{2\rho}} W'(u_\varepsilon)^2 &\leq C_k(1+\rho^{-2k(1-\beta)}\varepsilon^{2k})\varepsilon^2\int_{B_{3\rho^{1-\beta}}}|f_\varepsilon|^2 \\
&\quad + C_k\rho^{-2k(1-\beta)}\varepsilon^{2k}\int_{\{|u_\varepsilon|\geq 1\}\cap B_{3\rho^{1-\beta}}} W'(u_\varepsilon)^2.
\end{aligned}$$

Substituting this into our above estimate, we get

$$\begin{aligned}
\rho^{-n-1}\xi_{\varepsilon,+}(B_\rho) &\leq C\rho^{p_3\gamma-n-1}\int_{B_{2\rho}(x)}\left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2}+\frac{W(u_\varepsilon)}{\varepsilon}\right) \\
&\quad + \tilde{C}_k\varepsilon\rho^{-M\gamma-n-1}\int_{B_{3\rho^{1-\beta}}(x)}|f_\varepsilon|^2 \\
&\quad + C\varepsilon^{-1}\rho^{-M\gamma-n-1}\tilde{C}_k\rho^{2k\beta-2k}\varepsilon^{2k}\int_{\{|u_\varepsilon|\geq 1\}\cap B_{3\rho^{1-\beta}}(x)} W'(u_\varepsilon)^2 \\
&\quad + C\varepsilon\rho^{\gamma-2} \\
&\leq C\rho^{p_3\gamma-n-1}\mu_\varepsilon(B_{2\rho}(x)) + \tilde{C}_k\varepsilon\rho^{-M\gamma-n-1}\int_{B_{3\rho^{1-\beta}}(x)}|f_\varepsilon|^2 \\
&\quad + C\left(\varepsilon\rho^{\gamma-2}+\varepsilon^{-1}\rho^{-M\gamma-n-1}\varepsilon^{2k\beta}\int_{\{|u_\varepsilon|\geq 1\}\cap B_{3\rho^{1-\beta}}(x)} W'(u_\varepsilon)^2\right) \\
&\leq C\rho^{p_3\gamma-n-1}\mu_\varepsilon(B_{2\rho}(x)) + \tilde{C}_{k,\beta}\varepsilon\rho^{-M\gamma-n-1}\int_{B_{3\rho^{1-\beta}}(x)}|f_\varepsilon|^2
\end{aligned}$$

$$+\tilde{C}_\beta \varepsilon \rho^{\gamma-2}\left(1+\int_{\{|u_\varepsilon|\ge 1\}\cap B_{3\rho^{1-\beta}}(x)} W'(u_\varepsilon)^2\right),$$

where we have chosen $-M\gamma - n + 2k\beta + 1 \ge \gamma - 2$ or $k > \frac{\gamma-2+M\gamma+n+1}{2\beta}$ sufficiently large. □

In the following theorem we prove the density lower bound for the limit measure.

Theorem 4.3 *There exists $\bar{\theta} > 0$ such that for any $\Omega' \subset\subset \Omega$ and $r_1(\Omega') \le \frac{d(\Omega',\partial\Omega)}{2}$ sufficiently small, we have*

$$r^{-n}\mu(B_r(x)) \ge \bar{\theta} - Cr^\gamma,$$

for some $\gamma > 0$, and all $x \in \operatorname{spt}\mu \cap \Omega'$ and $0 < r \le r_1$. In particular,

$$\theta_*^n(\mu) \ge \frac{\bar{\theta}}{\omega_n}$$

for μ-a.e. in Ω.

Proof Without loss of generality, we assume $0 \in \operatorname{spt}\mu \cap \Omega'$ and want to prove a density lower bound at 0. We first integrate (4.4) from s to r.

$$\begin{aligned}\frac{\mu_\varepsilon(B_r(x))}{r^n} - \frac{\mu_\varepsilon(B_s(x))}{s^n} \ge &-\int_s^r \frac{1}{\rho^{n+1}}\xi_{\varepsilon,+}(B_\rho(x))d\rho \\ &-\int_s^r C(\Lambda_0)\rho^{-\frac{n}{q_0}}\left(\frac{\mu_\varepsilon(B_\rho(x))}{\rho^n}\right)^{\frac{q_0-1}{q_0}}.\end{aligned} \tag{4.7}$$

By (4.6) in Proposition 4.2, the discrepancy term

$$\begin{aligned}-\int_s^r \rho^{-n-1}\xi_{\varepsilon,+}(B_\rho(x)) \ge &-\int_s^r C\rho^{p_3\gamma-n-1}\mu_\varepsilon(B_{2\rho}(x)) \\ &-\int_s^r \tilde{C}_k\varepsilon\rho^{-M\gamma-n-1}\int_{B_{3\rho^{1-\beta}}(x)}|f_\varepsilon|^2 \\ &-\int_s^r \tilde{C}_\beta\varepsilon\rho^{\gamma-2}\left(1+\int_{\{|u_\varepsilon|\ge 1\}\cap\Omega}W'(u_\varepsilon)^2\right).\end{aligned} \tag{4.8}$$

By the ε-Upper Density Bound (4.1) we get

$$\begin{aligned}-\int_s^r \rho^{p_3\gamma-n-1}\mu_\varepsilon(B_{2\rho}(x)) &= -\int_s^r 2^n\rho^{p_3\gamma-1}\frac{\mu_\varepsilon(B_{2\rho}(x))}{(2\rho)^n} \\ &\ge -\int_s^r 2^n\rho^{p_3\gamma-1}\left(C(\Lambda_0,\Omega')+\frac{\phi(\varepsilon)}{\rho^n}\right)\end{aligned}$$

$$\geq -C(\Lambda_0, \Omega')\left(r^{p_3\gamma} - s^{p_3\gamma}\right) - \frac{\phi(\varepsilon)}{p_3\gamma - n + 1}\left(r^{p_3\gamma-n} - s^{p_3\gamma-n}\right).$$

The last term in (4.8) may be estimated as follows

$$-\int_s^r \tilde{C}_\beta \varepsilon \rho^{\gamma-2}\left(1 + \int_{\{|u_\varepsilon|\geq 1\}\cap\Omega} W'(u_\varepsilon)^2\right) \geq -\tilde{C}_\beta \int_s^t \rho^{\gamma-1} d\rho \leq -\tilde{C}_\beta(r^\gamma - s^\gamma).$$

Using the bound

$$\left(\frac{\mu_\varepsilon(B_\rho(x))}{\rho^n}\right)^{\frac{q_0-1}{q_0}} \leq \left(1 + \frac{\mu_\varepsilon(B_\rho(x))}{\rho^n}\right)$$

and the ε-Upper Density Bound (4.1), we get

$$\begin{aligned}
-\int_s^r C(\Lambda_0)\rho^{-\frac{n}{q_0}}\left(\frac{\mu_\varepsilon(B_\rho(x))}{\rho^n}\right)^{\frac{q_0-1}{q_0}} &\geq -\int_s^r C(\Lambda_0, \Omega')\rho^{-\frac{n}{q_0}}\left(1 + \frac{\mu_\varepsilon(B_\rho(x))}{\rho^n}\right) \\
&\geq -\int_s^r C(\Lambda_0, \Omega')\rho^{-\frac{n}{q_0}}\left(1 + C(\Lambda_0, \Omega') + \frac{\phi(\varepsilon)}{\rho^n}\right) \\
&\geq -C(\Lambda_0, \Omega')\left(r^{1-\frac{n}{q_0}} - s^{1-\frac{n}{q_0}}\right) \\
&\quad - C(\Lambda_0, \Omega')\phi(\varepsilon)\left(r^{1-n-\frac{n}{q_0}} - s^{1-n-\frac{n}{q_0}}\right).
\end{aligned}$$

Thus, plug all the above estimates of terms in (4.7), we get

$$\begin{aligned}
\frac{\mu_\varepsilon(B_r(x))}{r^n} - \frac{\mu_\varepsilon(B_s(x))}{s^n} &\geq -C(\Lambda_0, \Omega')\left(r^{p_3\gamma} - s^{p_3\gamma}\right) \\
&\quad - \frac{\phi(\varepsilon)}{p_3\gamma - n + 1}\left(r^{p_3\gamma-n} - s^{p_3\gamma-n}\right) \\
&\quad - \int_s^r \tilde{C}_\beta \varepsilon \rho^{-M\gamma-n-1}\left(\int_{B_{3\rho^{1-\beta}}(x)} |f_\varepsilon|^2\right) d\rho - \tilde{C}_\beta(r^\gamma - s^\gamma) \\
&\quad - C(\Lambda_0, \Omega')\left(r^{1-\frac{n}{q_0}} - s^{1-\frac{n}{q_0}}\right) \\
&\quad - C(\Lambda_0, \Omega')\phi(\varepsilon)\left(r^{1-n-\frac{n}{q_0}} - s^{1-n-\frac{n}{q_0}}\right).
\end{aligned} \tag{4.9}$$

Next, we estimate the term $\int_s^r \tilde{C}_\beta \varepsilon \rho^{-M\gamma-n-1}\left(\int_{B_{3\rho^{1-\beta}}(x)} |f_\varepsilon|^2\right) d\rho$ in the following claim. □

Claim There exists $x \in B_{\frac{r}{2}}$ such that

$$\varepsilon^{-n}\mu_\varepsilon(B_\varepsilon(x)) \geq 2\bar{\theta}_0 > \bar{\theta}_0 \geq \int_\varepsilon^{\frac{r}{4}} \tilde{C}_\beta \varepsilon \rho^{-M\gamma-n-1}\left(\int_{B_{3\rho^{1-\beta}}(x)} |f_\varepsilon|^2\right) d\rho, \tag{4.10}$$

for some universal constant $\bar{\theta}_0 > 0$.

Proof of Claim Consider a point $x \in B_{\frac{r}{2}}$ with $|u_\varepsilon(x)| \le 1-\tau$, for some $0 < \tau < 1$. We can assume $\varepsilon^{-n}\mu_\varepsilon(B_\varepsilon(x)) \le 1$(otherwise the conclusion automatically follows), and so

$$\varepsilon^{-n-1}\int_{B_\varepsilon(x)} u_\varepsilon^p \le \varepsilon^{-n-1}\int_{B_\varepsilon(x)} c_0^p \le c_0^p\omega_{n+1}, \forall p > 1.$$

From Theorem 3.2 we have

$$\varepsilon^{\frac{1}{2}}\|u\|_{C^{0,\frac{1}{2}}(B_{1-\varepsilon}(x))} \le C,$$

and thus

$$|u_\varepsilon| \le 1 - \frac{\tau}{2}, \quad \text{in } B_{\frac{\tau^2\varepsilon}{4C^2}}(x).$$

So since $W(t) = (1-t^2)^2 = (1+t)^2(1-t)^2$ we find in $B_{\frac{\tau^2\varepsilon}{4C^2}}(x)$

$$W(u_\varepsilon) = (1+|u_\varepsilon|)^2(1-|u_\varepsilon|)^2 \ge \frac{\tau^2}{4}$$

$$\begin{aligned}\varepsilon^{-n}\mu_\varepsilon(B_\varepsilon(x)) \ge \varepsilon^{-n}\int_{B_{\frac{\tau^2\varepsilon}{4C^2}}(x)} \frac{W(u_\varepsilon)}{\varepsilon} &\ge \varepsilon^{-n-1}\omega_{n+1}\left(\frac{\tau^2\varepsilon}{4C^2}\right)^{n+1}\frac{\tau^2}{4} \\ &\ge C_n\tau^{2n+4}. \end{aligned} \tag{4.11}$$

Denote

$$2\bar{\theta}_0 := \min\{1, C_n\tau^{2n+4}\},$$

then for $x \in B_{\frac{r}{2}} \cap \{|u_\varepsilon| \le 1-\tau\}$ the first inequality in the conclusion of the claim holds. Applying the error estimates Proposition 3.5 with the choice $\Omega' = B_{\frac{r}{4}}$ and $\Omega = B_{\frac{r}{2}}$, for sufficiently small τ

$$\begin{aligned}\mu_\varepsilon(B_{\frac{r}{4}}) &= \mu_\varepsilon\left(B_{\frac{r}{4}} \cap \{|u_\varepsilon| < 1-\tau\}\right) + \mu_\varepsilon\left(B_{\frac{r}{4}} \cap \{|u_\varepsilon| \ge 1-\tau\}\right) \\ &\le C\mu_\varepsilon\left(B_{\frac{r}{4}} \cap \{|u_\varepsilon| < 1-\tau\}\right) + C\varepsilon\int_{B_{\frac{r}{2}}} |f_\varepsilon|^2 + C\varepsilon(\tau r^n + \tau^2 r^{n-1}) \\ &\quad + Cr^{-2}\varepsilon.\end{aligned}$$

Notice by (3.55), the second term $\varepsilon \int_{B_{r/2}} |f_\varepsilon|^2 \le \varepsilon^2 C(\Lambda_0, E_0)$. So the last three terms are at most of order $O(\varepsilon)$. Hence, as $0 \in \operatorname{spt} \mu$, by passing to limit $\varepsilon \to 0$ we have

$$0 < \mu(B_{\frac{r}{4}}) \le \liminf_{\varepsilon\to 0} \mu_\varepsilon(B_{\frac{r}{4}}) \le \liminf_{\varepsilon\to 0} \mu_\varepsilon \left(B_{\frac{r}{4}} \cap \{|u_\varepsilon| < 1-\tau\}\right).$$

And in the set $\{|u_\varepsilon| \le 1-\tau\}$, we get by Lemma 3.8 that

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \varepsilon^{-1}\mathcal{L}^{n+1}(B_{\frac{r}{2}} \cap \{|u_\varepsilon| \le 1-\tau\})\\
&\quad\ge \liminf_{\varepsilon\to 0} \varepsilon^{-1} \int_{B_{\frac{r}{2}}\cap\{|u_\varepsilon|\le 1-\tau\}} \frac{W(u_\varepsilon)}{\tau^2}\\
&\quad= \liminf_{\varepsilon\to 0} \frac{1}{\tau^2}\left(\mu_\varepsilon - \xi_\varepsilon\right)\left(B_{\frac{r}{2}} \cap \{|u_\varepsilon| \le 1-\tau\}\right)\\
&\quad\ge \frac{1}{\tau^2}\liminf_{\varepsilon\to 0} \mu_\varepsilon\left(B_{\frac{r}{4}} \cap \{|u_\varepsilon| < 1-\tau\}\right) - \liminf_{\varepsilon\to 0} \frac{1}{\tau^2}\xi_{\varepsilon,+}(B_{\frac{r}{2}} \cap \{|u_\varepsilon| \le 1-\tau\})\\
&\quad\ge \frac{\mu(B_{\frac{r}{4}})}{\tau^2} > 0.
\end{aligned} \tag{4.12}$$

(This guarantees we can always choose such a point $x \in B_{\frac{r}{2}}$ with $|u_\varepsilon(x)| \le 1-\tau$ if $0 \in \operatorname{spt}\mu$.) To complete the proof, we define for $0 < \rho < r_1$ the convolution

$$\omega_{\varepsilon,\rho}(x) := \rho^{-n-1}\left(\chi_{B_\rho} * \frac{1}{\varepsilon}|f_\varepsilon|^2\right)(x) = \rho^{-n-1}\int_{B_\rho(x)} \frac{1}{\varepsilon}|f_\varepsilon|^2,$$

with

$$\|\omega_{\varepsilon,\rho}(x)\|_{L^1(B_{\frac{r_1}{2}})} \le \int_{B_{\frac{r_1}{2}+r_1}} \frac{1}{\varepsilon}|f_\varepsilon|^2 \le C(\Lambda_0, E_0) < \infty,$$

by (3.55). Denote by $\omega_\varepsilon(x) := \int_0^{r_1} \omega_{\varepsilon,\rho}(x)d\rho$, we have

$$\|\omega_\varepsilon(x)\|_{L^1(B_{\frac{r_0}{2}})} \le r_1 C(\Lambda_0, E_0) < \infty.$$

Now we can estimate the term on the right hand side in the claim, by a change of variables $t = 3\rho^{1-\beta}$. Here $\beta := \beta(r_1)$ is chosen small enough such that $3\left(\frac{r_1}{4}\right)^{1-\beta} \le r_1$. We calculate, setting $t = 3\rho^{1-\beta}$

$$\begin{aligned}
&\int_\varepsilon^{\frac{r}{4}} \rho^{-M\gamma-n-1}\left(\int_{B_{3\rho^{1-\beta}}(x)} \frac{1}{\varepsilon}|f_\varepsilon|^2\right)d\rho\\
&\quad= \int_{3\varepsilon^{1-\beta}}^{3(\frac{r}{4})^{1-\beta}} \left(\frac{t}{3}\right)^{\frac{-M\gamma-n-1}{1-\beta}}\left(\int_{B_t(x)} \frac{1}{\varepsilon}|f_\varepsilon|^2\right) d\left(\frac{t}{3}\right)^{\frac{1}{1-\beta}}
\end{aligned}$$

$$\leq C_\beta \int_{3\varepsilon^{1-\beta}}^{3(\frac{r}{4})^{1-\beta}} t^{\frac{-M\gamma-n-1+\beta}{1-\beta}} \left(\int_{B_t} \frac{1}{\varepsilon} |f_\varepsilon|^2 \right) dt$$
$$\leq C_\beta \int_{3\varepsilon^{1-\beta}}^{3(\frac{r_1}{4})^{1-\beta}} t^{\frac{-M\gamma-n-1+\beta}{1-\beta}+(n+1)} \omega_{\varepsilon,t}(x) dt.$$

We find

$$\frac{-M\gamma - n - 1 + \beta}{1-\beta} + (n+1) = \frac{-M\gamma - n\beta}{1-\beta} < 0$$

so that $t^{\frac{-M\gamma-n\beta}{1-\beta}}$ is a decreasing function. Hence we get the bound

$$\begin{aligned} \int_\varepsilon^{\frac{r}{4}} \rho^{-M\gamma-n-1} \left(\int_{B_{3\rho^{1-\beta}}(x)} \frac{1}{\varepsilon} |f_\varepsilon|^2 \right) d\rho &\leq C_\beta \int_{3\varepsilon^{1-\beta}}^{3(\frac{r_1}{4})^{1-\beta}} \left(3\varepsilon^{1-\beta}\right)^{\frac{-M\gamma-n\beta}{1-\beta}} \omega_{\varepsilon,t}(x) dt \\ &\leq C_\beta \varepsilon^{-M\gamma-n\beta} \int_0^{r_1} \omega_{\varepsilon,t}(x) dt \\ &\leq C_\beta \varepsilon^{-M\gamma-n\beta} \omega_\varepsilon(x). \end{aligned} \tag{4.13}$$

Choosing $M\gamma < \frac{1}{2}$ and β sufficiently small so that $M\gamma + n\beta < \frac{1}{2}$, and applying the weak L^1 inequality for the distribution function and (4.13), we get for some $\tilde{C}_\beta$ depending on β

$$\begin{aligned} &\mathcal{L}^{n+1} \left(B_{\frac{r}{2}} \cap \left\{ \int_\varepsilon^{\frac{r}{4}} \tilde{C}_\beta \varepsilon \rho^{-M\gamma-n-1} \left(\int_{B_{3\rho^{1-\beta}}(x)} |f_\varepsilon|^2 \right) d\rho \geq \bar{\theta}_0 \right\} \right) \\ &\leq \mathcal{L}^{n+1} \left(B_{\frac{r}{2}} \cap \left\{ C_\beta \varepsilon^2 \varepsilon^{-M\gamma-n\beta} \omega_\varepsilon(x) \geq \bar{\theta}_0 \right\} \right) \\ &\leq C_\beta \varepsilon^{2-(M\gamma+n\beta)} \bar{\theta}_0^{-1} \|\omega_\varepsilon\|_{L^1(B_{\frac{r}{2}})} \\ &\leq C_\beta \varepsilon^{2-(M\gamma+n\beta)} \bar{\theta}_0^{-1} \|\omega_{\varepsilon,\rho}(x)\|_{L^1(B_{\frac{r_1}{2}})} \\ &\leq C_\beta \varepsilon^{2-(M\gamma+n\beta)} \bar{\theta}_0^{-1} C(\Lambda_0, E_0) \\ &\to 0, \end{aligned} \tag{4.14}$$

as $\varepsilon \to 0$. This guarantees we can always choose such a point $x' \in B_{\frac{r}{2}}$ with

$$\left\{ \int_\varepsilon^{\frac{r}{4}} \tilde{C}_\beta \varepsilon \rho^{-M\gamma-n-1} \left(\int_{B_{3\rho^{1-\beta}}(x')} |f_\varepsilon|^2 \right) d\rho \leq \bar{\theta}_0 \right\}.$$

We can thus combine (4.12) with (4.14) to find an $x \in B_{\frac{r}{2}}$ so that the upper bound and lower bound in the claim holds. □

With this claim, we proceed with the proof of the density lower bound. For the $\bar{\theta}_0$ obtained from the claim, we denote by $s := \sup\{0 \leq \rho \leq \frac{r}{4} : \frac{\mu_\varepsilon(B_\rho(x))}{\rho^n} \geq 2\bar{\theta}_0\}$. And it is obvious from (4.11)

$$s \geq \varepsilon.$$

By this choice of s, we have

$$\frac{\mu_\varepsilon(B_s(x))}{s^n} \geq 2\bar{\theta}_0,$$
$$\frac{\mu_\varepsilon(B_\rho(x))}{\rho^n} \leq 2\bar{\theta}_0, \forall \rho \in \left[s, \frac{r}{4}\right].$$

Substituting $\frac{r}{4}$ for r in the integral form of the almost monotonicity formula (4.9), we get from (4.10) the following density lower bound

$$\begin{aligned}
2^n \left[\frac{\mu_\varepsilon(B_{\frac{r}{2}}(x))}{\left(\frac{r}{2}\right)^n}\right] &\geq \frac{\mu_\varepsilon(B_{\frac{r}{4}}(x))}{\left(\frac{r}{4}\right)^n} \\
&\geq \frac{\mu_\varepsilon(B_s(x))}{s^n} - C(\Lambda_0, \Omega')\left(\left(\frac{r}{4}\right)^{p_3\gamma} - s^{p_3\gamma}\right) - \frac{\phi(\varepsilon)}{p_3\gamma - n + 1} \\
&\quad \times \left(\left(\frac{r}{4}\right)^{p_3\gamma - n} - s^{p_3\gamma - n}\right) - \int_s^{r/4} \tilde{C}_\beta \varepsilon \rho^{-M\gamma - n - 1}\left(\int_{B_{3\rho^{1-\beta}}(x)} |f_\varepsilon|^2\right) d\rho \\
&\quad - \tilde{C}_\beta(\left(\frac{r}{4}\right)^\gamma - s^\gamma) \\
&\quad - C(\Lambda_0, \Omega')\left(\left(\frac{r}{4}\right)^{1-\frac{n}{q_0}} - s^{1-\frac{n}{q_0}}\right) \\
&\quad - C(\Lambda_0, \Omega')\phi(\varepsilon)\left(\left(\frac{r}{4}\right)^{1-n-\frac{n}{q_0}} - s^{1-n-\frac{n}{q_0}}\right) \\
&\geq 2\overline{\theta}_0 - C(\Lambda_0, \Omega')r^{\gamma_n} - C(\Lambda_0, \Omega')\phi(\varepsilon)r^{-n-\frac{n}{q_0}} \\
&\quad - C(\Lambda_0, \Omega')\phi(\varepsilon)r^{p_3\gamma - n + 1} - \overline{\theta}_0 \\
&\geq \overline{\theta}_0 - C(\Lambda_0, \Omega')r^{\gamma_n} - C(\Lambda_0, \Omega')\phi(\varepsilon)r^{-n-\frac{n}{q_0}} - C(\Lambda_0, \Omega')\phi(\varepsilon)r^{p_3\gamma - n + 1},
\end{aligned}$$

where $\gamma_n := \min\{p_3\gamma, \gamma, 1 - \frac{n}{q_0}\} > 0$, and $\phi(\varepsilon) \to$ as $\varepsilon \to 0$ by Theorem 4.1. As $B_{\frac{r}{2}}(x) \subseteq B_r(0)$ we let $\varepsilon \to 0$ and get for some $\gamma_n > 0$

$$\frac{\mu(\overline{B_r})}{r^n} \geq \limsup_{\varepsilon \to 0} \frac{\mu_\varepsilon(B_r)}{r^n} \geq \limsup_{\varepsilon \to 0} \frac{\mu_\varepsilon(B_{\frac{r}{2}}(x))}{r^n} \geq C_n\bar{\theta}_0 - C_n r^{\gamma_n}.$$

Approximating $r' \nearrow r$ we get for $0 < r < r_1(\Omega')$

$$\frac{\mu(B_r(0))}{r^n} \geq c_0\overline{\theta}_0$$

and hence

$$\theta^n_*(\mu) \geq \frac{\overline{\theta}}{\omega_n} \quad \mu\text{-a.e. in } \Omega.$$

which completes the proof. □

Before proving the rectifiability of the limit measure, we need to show that the full discrepancy vanishes as the limit $\varepsilon \to 0$.

Proposition 4.4

$$|\xi_\varepsilon| \to 0 \quad \& \quad |\xi| = 0.$$

Proof We first prove the lower n-dimensional density of the discrepancy measure vanishes. Namely

$$\theta^n_*(|\xi|) = \liminf_{\rho\to 0} \frac{|\xi|(B_\rho)}{\rho^n} = 0.$$

If not, there exists $0 < \rho_0, \delta < 1$ and $B_{\rho_0} \subset \Omega$ such that

$$\frac{|\xi|(B_\rho(x))}{\rho^n} \geq \delta, \quad \forall 0 < \rho \leq \rho_0.$$

Multiplying both sides of (4.2) by an integrating factor and integrating from r to ρ_0 as in the proof of Theorem 4.1 we get

$$C(\Lambda_0, \Omega')\left(\frac{\mu_\varepsilon(B_{\rho_0})}{\rho_0^n}\right) - C(\Lambda_0, \Omega')\left(\frac{\mu_\varepsilon(B_r)}{r^n}\right) \geq -C(\Lambda_0, \Omega')\int_r^{\rho_0} \frac{\xi_\varepsilon(B_{r_0})}{\rho^{n+1}} d\rho.$$

Using Lemma 3.8, that is $\xi_+ = 0$ and Theorem 4.1, we have when passing to the limit $\varepsilon \to 0$

$$\begin{aligned}
\tilde{C}(\Lambda_0, \Omega') &\geq C(\Lambda_0, \Omega')\int_r^{\rho_0} \frac{\xi_-(B_\rho)}{\rho^{n+1}} d\rho = C(\Lambda_0, \Omega')\int_r^{\rho_0} \frac{\xi_-(B_\rho) + \xi_+(B_\rho)}{\rho^{n+1}} d\rho \\
&= C(\Lambda_0, \Omega')\int_r^{\rho_0} \frac{|\xi|(B_\rho)}{\rho^{n+1}} d\rho \\
&\geq C(\Lambda_0, \Omega')\int_r^{\rho_0} \frac{\delta}{\rho} d\rho \\
&= C(\Lambda_0, \Omega')\delta \ln\left(\frac{\rho_0}{r}\right).
\end{aligned}$$

This gives a contradiction by letting $r \to 0$. By the density lower bound Theorem 4.3 and differentiation theorem for measures, we have

$$D_\mu|\xi|(x) = \liminf_{\rho\to 0} \frac{|\xi|(B_\rho(x))}{\mu(B_\rho(x))} \leq \frac{\liminf_{\rho\to 0} \frac{|\xi|(B_\rho(x))}{\rho^n}}{\limsup_{\rho\to 0} \frac{\mu(B_\rho(x))}{\rho^n}}$$

$$\leq \frac{\theta_*^n(|\xi|, x)\omega_n}{\bar{\theta}} = 0$$

and this shows

$$|\xi| = D_\mu|\xi| \cdot \mu = 0.$$

□

Proposition 4.5 *We choose a Borel measurable function $\nu_\varepsilon : \Omega \to \partial B_1(0)$ extending $\frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}$ on $\nabla u_\varepsilon \neq 0$ and consider the varifold $V_\varepsilon = \mu_\varepsilon \otimes \nu_\varepsilon$ that is*

$$\int_{\{|\nabla u| \neq 0\}} \phi\left(x, I - \frac{\nabla u(x)}{|\nabla u(x)|} \otimes \frac{\nabla u(x)}{|\nabla u(x)|}\right) d\mu_i(x), \quad \phi \in C_c(G_n(\Omega)). \tag{4.15}$$

The first variation is given by

$$\delta V_\varepsilon(\eta) = -\int f_\varepsilon \langle \nabla u_\varepsilon, \eta \rangle dx + \int \nabla\eta\left(\frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}, \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}\right) d\xi_\varepsilon,$$
$$\forall \eta \in C_c^1(\Omega \times \mathbb{R}^{n+1}). \tag{4.16}$$

Proof By Eq. (2.1), we have

$$\begin{aligned}
\delta V_\varepsilon(\eta) &= \int_{\Omega \times G(n+1,n)} \operatorname{div}_S \eta(x) dV_\varepsilon(x, S) \\
&= \int_\Omega (\operatorname{div} \eta - \nabla\eta(\nu_\varepsilon, \nu_\varepsilon)) d\mu_\varepsilon \\
&= \int_\Omega (\operatorname{div} \eta - \nabla\eta(\nu_\varepsilon, \nu_\varepsilon)) \left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2} + \frac{W(u_\varepsilon)}{\varepsilon}\right) d\mathcal{L}^{n+1}.
\end{aligned}$$

The Stress-Energy tensor for the Allen–Cahn equation is given by

$$\begin{aligned}
T_{ij} &= \varepsilon \frac{|\nabla u_\varepsilon|^2}{2} \delta_{ij} - \varepsilon \nabla_i u_\varepsilon \nabla_j u_\varepsilon + W(u_\varepsilon) \delta_{ij}, \\
\nabla_i T_{ij} &= \varepsilon \nabla_i \nabla_k u_\varepsilon \nabla_k u_\varepsilon \delta_{ij} - \varepsilon \Delta u_\varepsilon \nabla_j u_\varepsilon - \varepsilon \nabla_i u_\varepsilon \nabla_i \nabla_k u_\varepsilon \\
&\quad + W'(u_\varepsilon) \nabla_i u_\varepsilon \delta_{ij} \\
&= \left(-\varepsilon \Delta u_\varepsilon + W'(u_\varepsilon)\right) \nabla_j u_\varepsilon.
\end{aligned}$$

Now

$$\begin{aligned}
T_{ij} \nabla_i \eta_j &= \left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2} + W(u_\varepsilon)\right) \operatorname{div} \eta - \varepsilon \nabla\eta(\nabla u_\varepsilon, \nabla u_\varepsilon) \\
&= \left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2} + W(u_\varepsilon)\right) \operatorname{div} \eta - \nabla\eta(\nu_\varepsilon, \nu_\varepsilon) \varepsilon |\nabla u_\varepsilon|^2.
\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}
&\int_\Omega \left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2} + W(u_\varepsilon)\right) \operatorname{div}\eta - \nabla\eta(\nu_\varepsilon, \nu_\varepsilon)\varepsilon|\nabla u_\varepsilon|^2 \\
&\quad = -\int_\Omega \nabla_i T_{ij}\eta_j \\
&\quad = \int_\Omega \left(-\varepsilon\Delta u_\varepsilon + W'(u_\varepsilon)\right)\langle\nabla u_\varepsilon, \eta\rangle.
\end{aligned}$$

Hence inserting this into our expression for the first variation we get

$$\delta V_\varepsilon(\eta) = -\int_\Omega \left(-\varepsilon\Delta u_\varepsilon + \frac{W'(u_\varepsilon)}{\varepsilon}\right)\langle\nabla u_\varepsilon, \eta\rangle d\mathcal{L}^{n+1} + \int_\Omega \nabla\eta(\nu_\varepsilon, \nu_\varepsilon) d\xi_\varepsilon.$$

□

Combining Theorem 4.1, Theorem 4.3 and Proposition 4.4, we obtain

Theorem 4.6 *After passing to a subsequence, the associated varifolds $V_\varepsilon \to V$ where V is a rectifiable n-varifold with the weak mean curvature in $L^{q_0}_{loc}(\mu_V)$.*

Proof We first compute the first variation of the associated varifolds V_ε to the energy measure μ_ε(c.f. [8, Proposition 4.10], [11, Equation 4.3]). For any $\eta \in C^1_0(\Omega; \mathbb{R}^{n+1})$, using Proposition 4.5 and Proposition 4.4

$$\begin{aligned}
|(\delta V)(\eta)| &= |\lim_{\varepsilon\to 0}(\delta V_\varepsilon)(\eta)| \\
&= \left|\lim_{\varepsilon\to 0}\left(-\int f_\varepsilon\langle\nabla u_\varepsilon, \eta\rangle dx + \int \nabla\eta\left(\frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}, \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}\right) d\xi_\varepsilon\right)\right| \\
&\le \lim_{\varepsilon\to 0}\int |f_\varepsilon||\nabla u_\varepsilon||\eta| dx + \lim_{\varepsilon\to 0}\int |\nabla\eta| d|\xi_\varepsilon| \\
&\le \lim_{\varepsilon\to 0}\int \left|\frac{f_\varepsilon}{\varepsilon|\nabla u|}\right| |\eta|\varepsilon|\nabla u_\varepsilon|^2 dx \\
&\le \lim_{\varepsilon\to 0}\left(\int \left|\frac{f_\varepsilon}{\varepsilon|\nabla u_\varepsilon|}\right|^{q_0} \varepsilon|\nabla u_\varepsilon|^2\right)^{\frac{1}{q_0}} \left(\int |\eta|^{\frac{q_0}{q_0-1}}\varepsilon|\nabla u_\varepsilon|^2\right)^{\frac{q_0-1}{q_0}} \\
&\le \Lambda_0^{\frac{1}{q_0}} \|\eta\|_{L^{\frac{q_0}{q_0-1}}(\mu_V)} \left(\le C(\Lambda_0, E_0)|\eta|\right).
\end{aligned} \tag{4.17}$$

So we see the limit varifold has locally bounded first variation, combining with the density lower bound Theorem 4.3 we conclude the limit varifold is rectifiable by Allard's rectifiability theorem. Moreover, the above calculation shows δV is a bounded linear functional on $L^{\frac{q_0}{q_0-1}}_{loc}(\mu_V)$ and thus itself is in $L^{q_0}_{loc}(\mu_V)$. □

5 Integrality

In this section, we prove the integrality of the limit varifold.

Theorem 5.1 *Let μ be defined by (4.15). Then $\frac{1}{\alpha}\mu$ is an integral n-varifold where $\alpha = \int_{-\infty}^{\infty}(\tanh' x)^2 dx$ is the total energy of the heteroclinic 1-d solution.*

From the previous section, we have already shown the limiting varifold V is rectifiable. And thus for a.e. $x_0 \in \operatorname{spt} \mu_V$, we have for any sequence $\rho_i \to 0$

$$\mathcal{D}_{\rho_i,\#} \circ \mathcal{T}_{x_0,\#}(\mu_V) \to \theta_{x_0} P_0, \quad \text{for some } P_0 \in G(n+1, n),$$

where $\mathcal{D}_{\rho_i}(x) = \rho_i^{-1} x$ and $\mathcal{T}_{x_0}(x) = x - x_0$ represent dilations and translations in $\mathbb{R}^{n+1}$ and θ_{x_0} is the density of μ_V at x_0. By choosing a sequence of rescaling factors ρ_i such that

$$\tilde{\varepsilon}_i := \frac{\varepsilon_i}{\rho_i} \to 0, \tag{5.1}$$

the new sequence $\tilde{u}_{\tilde{\varepsilon}_i}(x) := u_{\varepsilon_i}(\rho_i x + x_0)$, $\tilde{f}_{\tilde{\varepsilon}_i}(x) := \rho_i \tilde{f}_i(\rho_i x + x_0)$ satisfies

$$\tilde{\varepsilon}_i \Delta \tilde{u}_{\tilde{\varepsilon}_i} - \frac{W'(\tilde{u}_{\tilde{\varepsilon}_i})}{\tilde{\varepsilon}_i} = \tilde{f}_{\tilde{\varepsilon}_i}$$

and the associated varifold $\tilde{V}_i$ of this new sequence $\tilde{u}_{\tilde{\varepsilon}_i}$ converges to $\theta_{x_0} P_0$. By (3.55), we also have

$$\begin{aligned}
\frac{1}{\tilde{\varepsilon}_i}\int_{B_\rho} f_{\tilde{\varepsilon}_i}^2 &\leq C\left(\int_{B_\rho}\left(\frac{f_{\tilde{\varepsilon}_i}}{\tilde{\varepsilon}_i|\nabla u_{\tilde{\varepsilon}_i}|}\right)^{q_0} \tilde{\varepsilon}_i|\nabla u_{\tilde{\varepsilon}_i}|^2\right)^{\frac{2}{q_0}} \\
&= C\left(\rho_i^{q_0+1-(n+1)}\int_{B_{\rho_i\rho}}\left(\frac{f_{\varepsilon_i}}{\varepsilon_i|\nabla u_{\varepsilon_i}|}\right)^{q_0} \varepsilon_i|\nabla u_{\varepsilon_i}|^2\right)^{\frac{2}{q_0}} \\
&\leq C\rho_i^{\frac{2(q_0-n)}{q_0}} \to 0,
\end{aligned}$$

as $q_0 > n$. Furthermore, by choosing more carefully so that $\rho_i := \tilde{\varepsilon}_i^{\frac{(n-1)q_0}{2(q_0-n)}} = \varepsilon_i^{\frac{1}{1+\frac{2(q_0-n)}{(n-1)q_0}}}$, we have

$$\frac{1}{\tilde{\varepsilon}_i}\int_{B_\rho} f_{\tilde{\varepsilon}_i}^2 \leq \tilde{\varepsilon}_i^{n-1}, \quad \text{for } \rho > \tilde{\varepsilon}_i$$

and thus

$$\frac{1}{\tilde{\varepsilon}_i}\int_{B_\rho} f_{\tilde{\varepsilon}_i}^2 \leq \rho^{n-1}. \tag{5.2}$$

Therefore we have reduced Theorem 5.1 to the following proposition

Proposition 5.2 *If the limit varifold is $\theta_0\mathcal{H}^n\lfloor P_0$ for some $P_0 \in G(n+1,n)$ and $\theta_0 > 0$, then $\alpha^{-1}\theta_0$ is a nonnegative integer, where $\alpha = \int_{-\infty}^{\infty}(\tanh' x)^2 dx$ is the total energy of the heteroclinic 1-d solution.*

In order to prove Proposition 5.2, we need two lemmas. The first Lemma 5.5 is a multi-sheet monotonicity formula (c.f. [1, Theorem 6.2] for the version for integral varifolds, which is used to prove the integrality of the limits of sequences of integral varifolds). The second Lemma 5.7 says at small scales, the energy of each layers are almost integer multiple of the 1-d solution. We first gather some apriori bounds on energy ratio for μ_ε.

Proposition 5.3 *Let $\delta = \rho^\gamma$, $\varepsilon \le \rho \le r$ for $0 < \gamma < \frac{1}{M} \le \frac{1}{2}$, we have $\delta^{-M}\varepsilon \le \rho^{1-M\gamma} \le 1$. Furthermore we choose $r := d(B_{2\rho}(x), \partial B_{3\rho^{1-\beta}}(x)) \ge \rho^{1-\beta}$. Then*

$$\begin{aligned} Cr^{-n}\mu_\varepsilon(B_r(x)) \ge s^{-n}\mu_\varepsilon(B_s(x)) - C\int_s^r \rho^{p_3\gamma-n-1}\mu_\varepsilon(B_{2\rho}(x))d\rho \\ - C_\beta\varepsilon\int_s^r \rho^{-M\gamma-n-1}\left(\int_{B_{3\rho^{1-\beta}}(x)} |f_\varepsilon|^2\right)d\rho \\ - \tilde{C}_\beta\left(1+\int_{\{|u_\varepsilon|\ge 1\}\cap B_{3\rho^{1-\beta}}(x)} W'(u_\varepsilon)^2\right)\int_s^r \rho^{\gamma-1}d\rho - C. \end{aligned} \tag{5.3}$$

Proof Substitute (4.6) into the Eq. (4.5) in the proof of Theorem 4.1, we have for $\varepsilon \le s \le \rho \le r \le 1$

$$\begin{aligned} C(\Lambda_0, q_0)\left(\frac{\mu_\varepsilon(B_r)}{r^n}\right) &\ge \left(\frac{\mu_\varepsilon(B_s)}{s^n}\right) - C(\Lambda_0,q_0) - C\int_s^r \frac{\xi_+(B_\rho)}{\rho^{n+1}} \\ &\ge \left(\frac{\mu_\varepsilon(B_s)}{s^n}\right) - C(\Lambda_0,q_0) - C\int_s^r \rho^{p_3\gamma-n-1}\mu_\varepsilon(B_{2\rho}(x))d\rho \\ &\quad - C_\beta\varepsilon\int_s^r \rho^{-M\gamma-n-1}\left(\int_{B_{3\rho^{1-\beta}}(x)}|f_\varepsilon|^2\right)d\rho \qquad (5.4) \\ &\quad - \int_s^r \tilde{C}_\beta\varepsilon\rho^{\gamma-2}\left(1+\int_{\{|u_\varepsilon|\ge 1\}\cap B_{3\rho^{1-\beta}}(x)} W'(u_\varepsilon)^2\right)d\rho. \end{aligned} \tag{5.5}$$

Noticing $\varepsilon \le \rho$ in the last term, we then conclude the desired energy ratio bound. □

As a corollary, we have

Corollary 5.4 *If in addition to the conditions in Proposition 5.3, we assume*

$$\frac{1}{\varepsilon}\int_{B_\rho} f_\varepsilon^2 \le \rho^{n-1}, \quad \text{for } \rho \ge \varepsilon, \tag{5.6}$$

and

$$\beta \in \left(0, \frac{1 - M\gamma}{2(n-1)}\right),$$

then the following upper bound for the energy ratio for μ_ε holds

$$\frac{\mu_\varepsilon(B_s(x))}{s^n} \leq C\frac{\mu_\varepsilon(B_r(x))}{r^n} + C(\Lambda_0, E_0, q_0, n), \tag{5.7}$$

for $\varepsilon \leq s \leq r$.

Proof We have

$$p_3\gamma - 1, -M\gamma + \beta(n-1), \gamma - 1 > -1.$$

Thus by Proposition 5.3 and $\varepsilon \leq \rho$, we have

$$\begin{aligned} C\left(\frac{\mu_\varepsilon(B_r(x))}{r^n}\right) \geq& \left(\frac{\mu_\varepsilon(B_s(x))}{s^n}\right) - C\int_s^r \rho^{p_3\gamma-1}\left(\frac{\mu_\varepsilon(B_{2\rho}(x))}{\rho^n}\right)d\rho \\ &- C_\beta\varepsilon^2\int_s^r \rho^{-M\gamma-2-\beta(n-1)}\left(\frac{\int_{B_{3\rho^{1-\beta}}(x)}|f_\varepsilon|^2}{\varepsilon\rho^{(1-\beta)(n-1)}}\right)d\rho \\ &- \tilde{C}_\beta\left(1 + \int_{\{|u_\varepsilon|\geq 1\}\cap\Omega} W'(u_\varepsilon)^2\right)\int_s^r \rho^{\gamma-1}d\rho - C \\ \geq& \left(\frac{\mu_\varepsilon(B_s(x))}{s^n}\right) - C\int_s^r \rho^{p_3\gamma-1}\left(\frac{\mu_\varepsilon(B_{2\rho}(x))}{\rho^n}\right)d\rho - C. \end{aligned}$$

The conclusion then follows by substituting in (5.6) and applying Gronwall's inequality to the above differential inequality. □

Lemma 5.5 *For any $N \in \mathbb{N}$, $\delta > 0$ small, $\Lambda > 0$ large and $\beta \in (0, \frac{1-M\gamma}{2(n-1)})$ where M, γ are from Proposition 4.2, there exists $\omega > 0$ such that the following holds: Suppose u_ε satisfies (1.1) and the conditions(1)-(3) in Theorem 1.1 are satisfied, then for any finite set $X \subset \{0^n\} \times \mathbb{R} \subset \mathbb{R}^{n+1}$, and the number of elements in X is no more than N. If moreover for some $0 < \varepsilon \leq d \leq R \leq \omega$, the followings are satisfied*

$$\operatorname{diam}(X) < \omega R, \tag{5.8}$$

$$|x - y| > 3d, \quad \textit{for } x, y \in X \textit{ and } x \neq y, \tag{5.9}$$

$$|\xi_\varepsilon|(B_\rho(x)) + \int_{B_\rho(x)} \varepsilon|\nabla u_\varepsilon|^2\sqrt{1 - \nu_{\varepsilon,n+1}^2} \leq \omega\rho^n, \quad \textit{for } x \in X \textit{ and } d \leq \rho \leq R, \tag{5.10}$$

$$\frac{1}{\varepsilon}\int_{B_\rho(x)} |f_\varepsilon|^2 \leq \Lambda\rho^{n-1}, \quad \textit{for } 3d^{1-\beta} \leq \rho \leq 3R^{1-\beta}. \tag{5.11}$$

Then we have

$$\sum_{x\in X} d^{-n}\mu_\varepsilon(B_d(x)) \le (1+\delta)R^{-n}\mu_\varepsilon(\cup_{x\in X}B_R(x)) + \delta. \tag{5.12}$$

The proof of the lemma is based on an inductive application of the sheets-separation proposition, along with appropriate choices of parameters γ and ω. To simplify notation in the remainder of this section, we introduce a shorthand for the sheets-separation term

$$\mathcal{S}_{y,x} =: (y_{n+1}-x_{n+1})\left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2}+\frac{W(u_\varepsilon)}{\varepsilon}\right) - \varepsilon\frac{\partial u_\varepsilon}{\partial x_{n+1}}\langle y-x, \nabla u_\varepsilon\rangle, \tag{5.13}$$

for any pair of points $x, y \in \mathbb{R}^{n+1}$.

Proposition 5.6 *Suppose the conditions in Theorem 1.1 are satisfied and let* $X \subset \{0^n\}\times[t_1+d, t_2-d] \subset \mathbb{R}^{n+1}$ *consist of no more than* $N \in \mathbb{N}$ *elements and* $\cup_{x\in X}B_{3R^{1-\beta}} \subset \Omega \subset \mathbb{R}^{n+1}$*. Furthermore suppose for* $-\infty \le t_1 < t_2 \le \infty$, $0 < \varepsilon \le d \le R \le \frac{1}{2}$, $\beta \in (0, \frac{1-M\gamma}{2(n-1)})$ *the following are satisfied:*

$$(\Gamma+1)\mathrm{diam}(X) < R, \quad \textit{for some } \Gamma \ge 1, \tag{5.14}$$

$$|x-y| > 3d, \quad \textit{for } x \ne y \in X, \tag{5.15}$$

$$\int_d^R \rho^{-n-1}\left|\int_{B_\rho(x)\cap\{y_{n+1}=t_j\}} \mathcal{S}_{y,x}\, d\mathcal{H}^n_y\right| d\rho \le \omega \tag{5.16}$$

for any $x \in X$, $j = 1, 2$ *and for some* $\omega > 0$,

$$|\xi_\varepsilon|(B_\rho(x)) + \int_{B_\rho(x)} \varepsilon|\nabla u_\varepsilon|^2\sqrt{1-\nu^2_{\varepsilon,n+1}} \le \omega\rho^n, \quad \textit{for } d \le \rho \le R \tag{5.17}$$

$$\frac{1}{\varepsilon}\int_{B_\rho(x)} |f_\varepsilon|^2 \le \Lambda\rho^{n-1}, \quad \textit{for } 3d^{1-\beta} \le \rho \le 3R^{1-\beta}, \tag{5.18}$$

$$\frac{\mu_\varepsilon(B_{2R}(x))}{R^n} \le \Lambda, \quad \forall x \in X \quad \textit{(this is implied by Corollary 5.4 as } R \ge \varepsilon\textit{)}. \tag{5.19}$$

Then by denoting $S_t^{t'} := \{t \le y_{n+1} \le t'\}$, *we have*

$$d^{-n}\mu_\varepsilon(B_d(x)) \le R^{-n}\mu_\varepsilon(B_R(x)\cap S_{t_1}^{t_2}) + CR^{\gamma_0} + 2\omega, \tag{5.20}$$

for some $\gamma_0 > 0$ *and for all* $x \in X$*. Furthermore, if* X *consists of more than one point, then there exists* $t_3 \in (t_1, t_2)$ *such that* $\forall x \in X$

$$|x_{n+1} - t_3| > d, \tag{5.21}$$

$$\int_d^{\tilde{R}} \rho^{-n-1}\int_{B_\rho(x)\cap\{y_{n+1}=t_3\}} \left|\mathcal{S}_{y,x}\right| d\mathcal{H}^n_y d\rho \le 3N\Gamma\omega, \tag{5.22}$$

where $\tilde{R} := \Gamma \operatorname{diam}(X)$ *and* $\mathcal{S}_{y,x}$ *as defined in* (5.13). *Moreover, both* $X \cap X_{t_1}^{t_3}$ *and* $X \cap X_{t_3}^{t_2}$ *are non-empty and*

$$\tilde{R}^{-n}\left(\mu_\varepsilon(\cup_{x\in X\cap X_{t_1}^{t_3}} B_{\tilde{R}}(x)\cap S_{t_1}^{t_3}) + \mu_\varepsilon(\cup_{x\in X\cap X_{t_3}^{t_2}} B_{\tilde{R}}(x)\cap S_{t_3}^{t_2})\right)$$
$$\leq \left(1+\frac{1}{\Gamma}\right)^n R^{-n}\mu_\varepsilon\left(\cup_{x\in X} B_R(x)\cap S_{t_1}^{t_2}\right) + CR^{\gamma_0} + 2\omega.$$

Proof First we choose ϕ to be a non-increasing function satisfying

$$\phi_{\delta,\rho} = \begin{cases} 1, & \text{on } [0,\rho] \\ 0, & \text{on } [\rho+\delta,\infty), \end{cases}$$

and χ_δ satisfying

$$\chi_\delta \equiv \begin{cases} 1, & \text{on } [t_1+\delta, t_2-\delta], \\ 0, & \text{on } (-\infty, t_1]\cup[t_2,\infty), \end{cases}$$

with $\chi_\delta' \geq 0$ on $[t_1, t_1+\delta]$ and $\chi_\delta' \leq 0$ on $[t_2-\delta, t_2]$. Then we multiply (1.1) on both sides by $\langle \nabla u, \eta\rangle$, where $\eta \in C_0^1(\Omega, \mathbb{R}^{n+1})$ is defined by $\eta(y) := (y-x)\phi_{\delta,\rho}(|y-x|)\chi_\delta(y_{n+1})$. Using integration by parts, we have

$$\begin{aligned}
&\int f_\varepsilon \langle y-x, \nabla u_\varepsilon\rangle \phi_{\delta,\rho}(|y-x|)\chi_\delta(y_{n+1}) \\
&= \int f_\varepsilon \langle \nabla u, \eta\rangle \\
&= \int \left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2} + \frac{W(u_\varepsilon)}{\varepsilon}\right)\operatorname{div}\eta - \varepsilon \nabla u\otimes\nabla u : \nabla\eta \\
&= \int \left(|y-x|\phi_{\delta,\rho}'\chi_\delta + (n+1)\phi_{\delta,\rho}\chi_\delta + (y_{n+1}-x_{n+1})\phi_{\delta,\rho}\chi_\delta'\right) d\mu_\varepsilon \\
&\quad - \int \varepsilon \frac{\phi_{\delta,\rho}'\chi_\delta}{|y-x|}\langle y-x, \nabla u_\varepsilon\rangle^2 - \int \varepsilon|\nabla u_\varepsilon|^2\phi_{\delta,\rho}\chi_\delta \\
&\quad - \int \varepsilon \frac{\partial u}{\partial x_{n+1}}\langle y-x, \nabla u_\varepsilon\rangle \phi_{\delta,\rho}\chi_\delta'.
\end{aligned}$$

Letting $\delta \to 0$, we have

$$\begin{aligned}
&\int_{B_\rho(x)\cap S_{t_1}^{t_2}} f_\varepsilon\langle y-x, \nabla u_\varepsilon\rangle \\
&= -\int_{\partial B_\rho\cap S_{t_1}^{t_2}} \rho\, d\mu_\varepsilon + (n+1)\int_{B_\rho\cap S_{t_1}^{t_2}} d\mu_\varepsilon \\
&\quad + \int_{B_\rho\cap\{y_{n+1}=t_2\}} (y_{n+1}-x_{n+1})d\mu_\varepsilon - \int_{B_\rho\cap\{y_{n+1}=t_1\}} (y_{n+1}-x_{n+1})d\mu_\varepsilon
\end{aligned}$$

$$+\int_{\partial B_\rho\cap S_{t_1}^{t_2}}\varepsilon\rho^{-1}\langle y-x,\nabla u_\varepsilon\rangle^2-\int_{B_\rho\cap S_{t_1}^{t_2}}\varepsilon|\nabla u_\varepsilon|^2$$
$$+\int_{B_\rho\cap\{y_{n+1}=t_2\}}\varepsilon\frac{\partial u}{\partial x_{n+1}}\langle y-x,\nabla u_\varepsilon\rangle-\int_{B_\rho\cap\{y_{n+1}=t_1\}}\varepsilon\frac{\partial u}{\partial x_{n+1}}\langle y-x,\nabla u_\varepsilon\rangle.$$

Dividing both sides by ρ^{n+1} and rearranging gives the following weighted monotonicity formula

$$\begin{aligned}
&\frac{d}{d\rho}\left(\rho^{-n}\mu_\varepsilon(B_\rho(x)\cap S_{t_1}^{t_2})\right)\\
&\quad=-n\rho^{-n-1}\mu_\varepsilon(B_\rho(x)\cap S_{t_1}^{t_2})+\rho^{-n}\mu_\varepsilon(\partial B_\rho\cap S_{t_1}^{t_2})\\
&\quad=-(n+1)\rho^{-n-1}\int_{B_\rho(x)\cap S_{t_1}^{t_2}}d\mu_\varepsilon+\rho^{-n}\int_{\partial B_\rho(x)\cap S_{t_1}^{t_2}}d\mu_\varepsilon\\
&\qquad+\rho^{-n-1}\int_{B_\rho(x)\cap S_{t_1}^{t_2}}\varepsilon|\nabla u_\varepsilon|^2-\rho^{-n-1}\int_{B_\rho(x)\cap S_{t_1}^{t_2}}d\xi_\varepsilon\\
&\quad=\rho^{-n-1}\int_{B_\rho\cap\{y_{n+1}=t_2\}}(y_{n+1}-x_{n+1})d\mu_\varepsilon\\
&\qquad-\rho^{-n-1}\int_{B_\rho\cap\{y_{n+1}=t_1\}}(y_{n+1}-x_{n+1})d\mu_\varepsilon\\
&\qquad+\rho^{-n-1}\int_{B_\rho\cap\{y_{n+1}=t_2\}}\varepsilon\frac{\partial u}{\partial x_{n+1}}\langle y-x,\nabla u_\varepsilon\rangle\\
&\qquad-\rho^{-n-1}\int_{B_\rho\cap\{y_{n+1}=t_1\}}\varepsilon\frac{\partial u}{\partial x_{n+1}}\langle y-x,\nabla u_\varepsilon\rangle\\
&\qquad-\rho^{-n-1}\int_{B_\rho(x)\cap S_{t_1}^{t_2}}d\xi_\varepsilon-\rho^{-n-1}\int_{B_\rho(x)\cap S_{t_1}^{t_2}}f_\varepsilon\langle y-x,\nabla u_\varepsilon\rangle\\
&\qquad+\rho^{-n-1}\int_{\partial B_\rho\cap S_{t_1}^{t_2}}\varepsilon\rho^{-1}\langle y-x,\nabla u_\varepsilon\rangle^2.
\end{aligned}\tag{5.23}$$

By the condition given by (5.16), the sum of norms of the first fours terms are bounded by 2ω. And by (4.6) and (5.18), the discrepancy term is bounded by

$$\begin{aligned}
&\rho^{-n-1}\int_{B_\rho(x)\cap S_{t_1}^{t_2}}d\xi_{\varepsilon,+}\\
&\quad\le C\rho^{p_3\gamma-n-1}\mu_\varepsilon(B_{2\rho}(x))+\tilde{C}_k\varepsilon\rho^{-M\gamma-n-1}\int_{B_{3\rho^{1-\beta}}(x)}|f_\varepsilon|^2\\
&\qquad+\tilde{C}_\beta\varepsilon\rho^{\gamma-2}\left(1+\int_{\{|u_\varepsilon|\ge1\}\cap\Omega}W'(u_\varepsilon)^2\right)\\
&\quad\le C\rho^{p_3\gamma-1}\left(\frac{\mu_\varepsilon(B_{2\rho}(x)\cap S_{t_1}^{t_2})}{\rho^n}\right)+C\varepsilon\rho^{-M\gamma-n-1}\Lambda\varepsilon\rho^{(1-\beta)(n-1)}+C\varepsilon\rho^{\gamma-2}
\end{aligned}$$

$$\leq C\rho^{p_3\gamma-1}\left(\frac{\mu_\varepsilon(B_{2\rho}(x)\cap S_{t_1}^{t_2})}{\rho^n}\right)+C\rho^{2-M\gamma-n-1+(n-1)-\beta(n-1)}+C\rho^{\gamma-1}$$
$$\leq C\rho^{p_3\gamma-1}+C\rho^{-1+\frac{1-M\gamma}{2}}+C\rho^{\gamma-1},$$

where we used (5.7) and $\varepsilon\leq\rho$ in the last line. By (4.3) in the proof of Theorem 4.1 and (5.7), we have

$$\rho^{-n-1}\left|\int_{B_\rho(x)\cap S_{t_1}^{t_2}} f_\varepsilon\langle y-x,\nabla u_\varepsilon\rangle\right|\leq C\rho^{-\frac{n}{q_0}}\left(1+\frac{\mu_\varepsilon(B_\rho\cap S_{t_1}^{t_2})}{\rho^n}\right)\leq\tilde{C}\rho^{-\frac{n}{q_0}}.$$

By integrating (5.23) from d to R and noting $B_d(x)\subset S_{t_1}^{t_2}$, we obtain the following upper bound of energy density for μ_ε,

$$d^{-n}\mu_\varepsilon(B_d(x))=d^{-n}\mu_\varepsilon(B_d(x)\cap S_{t_1}^{t_2})\leq R^{-n}\mu_\varepsilon(B_R(x)\cap S_{t_1}^{t_2})+CR^{\gamma_0}+2\omega,$$

where $\gamma_0=\min\{\frac{q_0-n}{q_0},\,p_3\gamma,\,\frac{1-M\gamma}{2},\,\gamma\}>0$. This proves (5.20).
Next, if X contains more than one point, then we can choose $x_\pm\in X$ such that $x_{+,n+1}-x_{-,n+1}>\frac{\operatorname{diam}X}{N}$ (where $x_{\pm,n+1}$ denotes the $(n+1)$-th coordinate of $x_\pm$) and there is no other element of X in $\{0\}\times(x_{-,n+1},x_{+,n+1})$. Let $\tilde{t}_1:=x_{-,n+1}+\frac{x_{+,n+1}-x_{-,n+1}}{3}$ and $\tilde{t}_2:=x_{+,n+1}-\frac{x_{+,n+1}-x_{-,n+1}}{3}$. For $x\in X$, $y\in B_\rho(x)$, $d\leq\rho\leq\tilde{R}$, we have

$$\begin{aligned}
|\mathcal{S}_{y,x}|&=\left|(y_{n+1}-x_{n+1})\left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2}+\frac{W(u_\varepsilon)}{\varepsilon}\right)-\varepsilon\frac{\partial u_\varepsilon}{\partial x_{n+1}}\langle y-x,\nabla u_\varepsilon\rangle\right|\\
&=\left|(y_{n+1}-x_{n+1})\left(\frac{W(u_\varepsilon)}{\varepsilon}-\frac{\varepsilon|\nabla u_\varepsilon|^2}{2}\right)+|(y_{n+1}-x_{n+1})\varepsilon|\nabla u_\varepsilon|^2\right.\\
&\quad\left.-\varepsilon\frac{\partial u_\varepsilon}{\partial x_{n+1}}\langle y-x,\nabla u_\varepsilon\rangle\right|\\
&\leq\rho\left|\frac{\varepsilon|\nabla u_\varepsilon|^2}{2}-\frac{W(u_\varepsilon)}{\varepsilon}\right|+\varepsilon|\nabla u_\varepsilon|^2|\langle y-x,e_{n+1}\rangle-\langle y-x,\nu_\varepsilon\rangle\langle e_{n+1},\nu_\varepsilon\rangle|\\
&\leq\rho\left|\frac{\varepsilon|\nabla u_\varepsilon|^2}{2}-\frac{W(u_\varepsilon)}{\varepsilon}\right|+\varepsilon|\nabla u_\varepsilon|^2|y-x|\sqrt{1-\nu_{\varepsilon,n+1}^2}\\
&\leq\rho\left|\frac{\varepsilon|\nabla u_\varepsilon|^2}{2}-\frac{W(u_\varepsilon)}{\varepsilon}\right|+\rho\varepsilon|\nabla u_\varepsilon|^2\sqrt{1-\nu_{\varepsilon,n+1}^2}.
\end{aligned}$$

And thus by condition (5.17), we have

$$\begin{aligned}
&\int_{\tilde{t}_1}^{\tilde{t}_2}\int_d^{\tilde{R}}\rho^{-n-1}\int_{B_\rho(x)\cap\{y_{n+1}=t\}}\left|\mathcal{S}_{y,x}\right|d\mathcal{H}^n_{\{y_{n+1}=t\}}d\rho dt\\
&\quad=\int_d^{\tilde{R}}\rho^{-n-1}\int_{B_\rho(x)\cap S_{\tilde{t}_1}^{\tilde{t}_2}}\left|(y_{n+1}-x_{n+1})\left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2}+\frac{W(u_\varepsilon)}{\varepsilon}\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&-\varepsilon \frac{\partial u_\varepsilon}{\partial x_{n+1}} \langle y-x, \nabla u_\varepsilon \rangle \Bigg| dy d\rho \\
&\leq \int_d^{\tilde{R}} \rho^{-n} \int_{B_\rho(x)\cap S_{\tilde{t}_1}^{\tilde{t}_2}} \left| \frac{\varepsilon |\nabla u_\varepsilon|^2}{2} - \frac{W(u_\varepsilon)}{\varepsilon} \right| + \varepsilon |\nabla u_\varepsilon|^2 \sqrt{1-\nu_{\varepsilon,n+1}^2} dy d\rho \\
&\leq \int_d^{\tilde{R}} \rho^{-n} \omega \rho^n d\rho \\
&\leq \omega \tilde{R}.
\end{aligned}
$$

So there must exist a $t_3 \in [\tilde{t}_1, \tilde{t}_2]$ such that

$$
\begin{aligned}
&\int_d^{\tilde{R}} \rho^{-n-1} \int_{B_\rho(x)\cap \{y_{n+1}=t_3\}} \left| \mathcal{S}_{y,x} \right| d\mathcal{H}^n_{\{y_{n+1}=t\}} d\rho \\
&\quad \leq \frac{\omega \tilde{R}}{(\tilde{t}_2 - \tilde{t}_1)} \leq \frac{3N\omega \tilde{R}}{\mathrm{diam}(X)} = 3N\Gamma\omega.
\end{aligned}
$$

By the choice of $t_3 \in [\tilde{t}_1, \tilde{t}_2]$, we automatically have $|x_{n+1} - t_3| > d$ for all $x \in X$. Finally, by denoting

$$
X_+ := \{x \in X, x_n \geq t_3\}, X_- := \{x \in X, x_n < t_3\},
$$

we have $X_\pm \neq \emptyset$ and

$$
\left(\cup_{x\in X_-} B_{\tilde{R}}(x) \cap S_{t_1}^{t_3}\right) \cup \left(\cup_{x\in X_+} B_{\tilde{R}}(x) \cap S_{t_3}^{t_2}\right) \subset B_{\tilde{R}+\mathrm{diam}(X)}(x_0) \cap S_{t_1}^{t_2},
$$

for any $x_0 \in X$. By (5.20)(with $\tilde{R} + \mathrm{diam}(X)$ in place of d), we then have

$$
\begin{aligned}
&\tilde{R}^{-n} \left(\mu_\varepsilon \left(\cup_{x\in X_-} B_{\tilde{R}}(x) \cap S_{t_1}^{t_3}\right) + \mu_\varepsilon \left(\cup_{x\in X_+} B_{\tilde{R}}(x) \cap S_{t_3}^{t_2}\right)\right) \\
&\quad \leq \tilde{R}^{-n} \mu_\varepsilon (B_{\tilde{R}+\mathrm{diam}(X)}(x_0) \cap S_{t_1}^{t_2}) \\
&\quad = \left(1 + \frac{1}{\Gamma}\right)^n (\tilde{R} + \mathrm{diam}(X))^{-n} \mu_\varepsilon (B_{\tilde{R}+\mathrm{diam}(X)}(x_0) \cap S_{t_1}^{t_2}) \\
&\quad \leq \left(1 + \frac{1}{\Gamma}\right)^n \left(R^{-n} \mu_\varepsilon (B_R(x_0) \cap S_{t_1}^{t_2}) + CR^{\gamma_0} + 2\omega\right) \\
&\quad \leq \left(1 + \frac{1}{\Gamma}\right)^n \left(R^{-n} \mu_\varepsilon (\cup_{x\in X} B_R(x) \cap S_{t_1}^{t_2})\right) + CR^{\gamma_0} + 2\omega.
\end{aligned}
$$

□

The next Lemma taken from [8] shows the energy ratio at small scales are very close to the 1-d solution.

Lemma 5.7 (Lemma 5.5 of [8]) *Suppose the conditions in Theorem 1.1 are satisfied. For any $\tau \in (0, 1)$, $\delta > 0$ small, $\Lambda > 0$ large, there exists $\omega > 0$ sufficiently small and*

$L > 1$ sufficiently large such that the following holds: Suppose u_ε satisfies condition of Theorem 1.1 in $B_{4L\varepsilon}(0) \subset \mathbb{R}^{n+1}$ and

$$|u_\varepsilon(0)| \leq 1 - \tau \tag{5.24}$$

$$|\xi_\varepsilon(B_{4L\varepsilon}(0))| + \int_{B_{4L\varepsilon}(0)} \varepsilon |\nabla u_\varepsilon|^2 \sqrt{1 - \nu_{\varepsilon,n+1}^2} \leq \omega(4L\varepsilon)^n \tag{5.25}$$

$$\frac{1}{\varepsilon} \int_{B_{4L\varepsilon}(0)} |f_\varepsilon|^2 \leq \Lambda(4L\varepsilon)^{n-2} \tag{5.26}$$

$$\mu_\varepsilon(B_{4L\varepsilon}(0)) \leq \Lambda(4L\varepsilon)^n. \tag{5.27}$$

Then by denoting $(0, t) \in \mathbb{R}^{n+1}$ to be the point with first n-th coordinate functions being 0 and the $(n+1)$-th coordinate functions being t, we have

$$|u(0,t)| \geq 1 - \frac{\tau}{2}, \quad \text{for all } L\varepsilon \leq |t| \leq 3L\varepsilon \tag{5.28}$$

$$\left| \frac{1}{\omega_n (L\varepsilon)^n} \mu_\varepsilon(B_{L\varepsilon}(0)) - \alpha \right| \leq \delta \tag{5.29}$$

$$\left| \int_{-L\varepsilon}^{L\varepsilon} W(u_\varepsilon(0,t))dt - \frac{\alpha}{2} \right| \leq \delta. \tag{5.30}$$

Proof First we consider the 1-dimensional solution

$$q_0'(t) = \sqrt{W(q_0(t))} \quad \forall t \in \mathbb{R},$$
$$q_0(0) = u(0).$$

We will use q_0 to choose L depending on $\tau, \delta > 0$. On $\mathbb{R}^{n+1}$ we write $q(x) = q_0(x_{n+1})$ and choose $L > 1$ large enough depending on τ, δ so that

$$\begin{aligned} &|q(0,t)| \geq 1 - \frac{\tau}{3}, \quad \text{for all } L \leq |t| \leq 3L, \\ &\left| \frac{1}{\omega_n L^{n-1}} \int_{B_L(0)} \left(\frac{|\nabla q|^2}{2} + W(q) \right) - \alpha \right| \leq \frac{\delta}{2} \\ &\left| \int_{-L}^{L} W(q(0,t))dt - \frac{\alpha}{2} \right| \leq \frac{\delta}{2} \end{aligned} \tag{5.31}$$

whenever $|q(0)| \leq 1 - \tau$. The function u satisfies the Allen–Cahn equation

$$-\Delta u + W'(u) = f,$$

and by our condition (2) in Theorem 1.1 we get $\|u_\varepsilon\|_{L^\infty(B_{1/2}(x))} \leq c_0$. Hence by Calderon–Zygmund estimates we get uniform $W^{2,\frac{n+1}{2}+\delta_0}$ estimates on $B_{3L}(0)$ of the form

$$\|u\|_{W^{2,\frac{n+1}{2}+\delta_0}(B_{3L}(0))} \leq C(\Lambda, L). \tag{5.32}$$

If there is no such $\omega > 0$ such that (5.28), (5.29) and (5.30) holds then this implies there exists $\omega_j \to 0$ and u_j, f_j satisfying the above estimates but that do not satisfy (5.28), (5.29) and (5.30). By (5.32), we get after passing to a suitable subsequence that $u_j \rightharpoonup u$ weakly in $W^{2,\frac{n+1}{2}+\delta_0}(B_{3L}(0))$ and $f_j \rightharpoonup f$ weakly in $L^{\frac{n+1}{2}+\delta_0}(B_{3L}(0))$. By the Sobolev embedding we have $W^{2,\frac{n+1}{2}+\delta_0}(B_{3L}(0)) \hookrightarrow C^0$ for $\delta_0 > 0$ and hence we get $u_j \to u$ uniformly in $C^0(B_{3L}(0))$. □

Claim The functions $u_j \to u = q$ strongly in $W^{1,2}(B_{3L}(0))$.

Proof Writing $\nabla = (\nabla', \partial_{n+1})$ we get (5.25)

$$\int_{B_{3L(0)}} \left| \frac{|\nabla u|^2}{2} - W(u) \right| \leq \liminf_{j\to\infty} \int_{B_{3L}(0)} \left| \frac{|\nabla u_j|^2}{2} - W(u_j) \right|$$
$$\leq \liminf_{j\to\infty} |\xi_j|(B_{3L}(0)) = 0$$

and

$$\int_{B_{3L}(0)} |\nabla' u| \leq \liminf_{j\to\infty} \int_{B_{3L}(0)} |\nabla' u_j| \leq C(L) \left(\int_{B_{3L}(0)} |\nabla u_j|^2 \sqrt{1 - \nu_{j,n}^2} \right)^{1/2} = 0,$$

where $\nu_j = \frac{\nabla u_j}{|\nabla u|}$ for $\nabla u_j \neq 0$. Therefore $|\nabla u|^2 = 2W(u)$ and $u(y,t) = u_0(t)$ for some $u_0 \in W^{2,\frac{n+1}{2}+\delta_0}((-L,L)) \hookrightarrow C^{1,\alpha}((-L,L))$ and $|u_0'| = 2\sqrt{2W(u_0)}$. As $|u_0(0)| \leq 1 - \tau$ by uniform convergence, we see $|u_0| < 1$ and $|u_0'| > 0$. After a reflection of the form $(y, x_n) \mapsto (y, -x_n)$ if necessary, we may assume $u_0' > 0$ and hence $u_0' = \sqrt{2W(u_0)}$. This gives us $u_0 = q_0$ and $u = q$. This shows $u_j \to u = q$ strongly in $W^{1,2}(B_{3L}(0))$. □

From this claim and (5.31) we conclude u_j satisfies (5.28), (5.29) and (5.30) for sufficiently large j which is a contradiction. □

Now we prove Proposition 5.2.

Proof of Proposition 5.2 Without loss of generality, we assume $P_0 = \{x \in \mathbb{R}^{n+1}, x_{n+1} = 0\}$ and let $\pi : \mathbb{R}^{n+1} \to P_0$ denote the associated orthogonal projection. Furthermore we know

$$V_\varepsilon = \mu_\varepsilon \otimes \nu_\varepsilon \to V$$

is rectifiable and

$$\mu_V = \mu$$
$$V = \theta_0 \mathcal{H}^n \llcorner P_0 \otimes \delta_{P_0}$$

and

$$\lim_{\varepsilon\to 0} \int_{B_4(0)} \varepsilon |\nabla u_\varepsilon|^2 \sqrt{1 - \nu_{\varepsilon,n+1}^2} = 0. \tag{5.33}$$

Let $N \in \mathbb{N}$ be the smallest integer with

$$N > \frac{\theta_0}{\alpha}$$

and let $0 < \delta \leq 1$ be small. By Proposition 3.5 and the L^∞ bound condition of u_ε in Theorem 1.1, we can fix $\tau > 0$ such that $\forall \varepsilon(\delta) > 0$ sufficiently small we have

$$\int_{\{|u_\varepsilon| \geq 1-\tau\} \cap B_4(0)} \frac{W'^2(u_\varepsilon)}{\varepsilon} + \frac{W(u_\varepsilon)}{\varepsilon} \leq \delta.$$

We have by Lemma 3.8,

$$\mu_\varepsilon(\{|u_\varepsilon| \geq 1 - \tau \cap B_4(0)\}) \leq |\xi_\varepsilon(B_4(0))| \\ + 2 \int_{\{|u_\varepsilon| \geq 1-\tau\} \cap B_4(0)} \frac{W(u_\varepsilon)}{\varepsilon} \leq 3\delta. \tag{5.34}$$

We want to apply Lemma 5.5 and Lemma 5.7. We choose $0 < \omega = \omega(N, \delta, \frac{1}{2}, \frac{1}{2}, C)$ and $\omega(\delta, \tau, C) \leq 1$ where $L = L(\delta, \tau)$ which are the parameters that appear in Lemma 5.5 and Proposition 5.6 and C is the constant so that

$$\mu_\varepsilon(\Omega) + \frac{1}{\varepsilon} \int_\Omega |f_\varepsilon|^2 \leq C \quad \Omega = B_4(0).$$

We define A_ε to be the set where the hypotheses for our Propositions hold, that is

$$A_\varepsilon = \left\{ x \in B_1(0) \,\middle|\, \begin{array}{l} |u_\varepsilon(x)| \leq 1 - \tau, \\ \forall \varepsilon \leq \rho \leq 3 : |\xi_\varepsilon(B_\rho(x))| + \int_{B_\rho(x)} \varepsilon |\nabla u_\varepsilon|^2 \sqrt{1 - \nu_{\varepsilon,n+1}^2} \leq \omega \rho^n, \\ \forall \varepsilon \leq \rho \leq 3 : \frac{1}{\varepsilon} \int_{B_\rho(x)} |f_\varepsilon|^2 \leq \omega \rho^{n-1}. \end{array} \right\}.$$

We show the complement of the set A_ε has small measure. By Besicovitch's covering theorem, we find a countable sub-covering $\cup_i B_{\rho_i}(x_i)$, $\rho_i \in [\varepsilon, 3]$ of $\{|u_\varepsilon| \leq 1-\tau\} \setminus A_\varepsilon$ such that every point $x \in \{|u_\varepsilon| \leq 1-\tau\} \setminus A_\varepsilon$ belongs to at most $\mathbf{B}_n$ balls in the covering, where $\mathbf{B}_n$ depends only on the dimension n. For each i, either

$$|\xi_\varepsilon(B_{\rho_i}(x_i))| + \int_{B_{\rho_i}(x_i)} \varepsilon |\nabla u_\varepsilon|^2 \sqrt{1 - \nu_{\varepsilon,n+1}^2} \geq \omega \rho_i^n,$$

or

$$\frac{1}{\varepsilon} \int_{B_{\rho_i}(x_i)} |f_\varepsilon|^2 \geq \omega \rho_i^{n-1} \geq C \omega \rho_i^n.$$

On the other hand, by (5.2), for sufficiently small ε, we have

$$\frac{1}{\varepsilon} \int_{B_\rho(x_i)} |f_\varepsilon|^2 \leq \omega \rho^{n-1}, \forall \rho \in [\varepsilon, 3].$$

By (5.7), for each i, we obtain

$$\mu_\varepsilon\left(\overline{B_{\rho_i}(x_i)}\right) \le C\rho_i^n.$$

Since the overlap in the Besicovitch covering is finite and (5.34), we get

$$\begin{aligned}\mu_\varepsilon\left(B_1(0)\setminus A_\varepsilon\right) &\le 3\delta + \sum_i C\rho_i^n \\ &\le 3\delta + C\omega^{-1}\left(|\xi_\varepsilon|(B_4(0)) + \int_{B_4(0)} \varepsilon|\nabla u_\varepsilon|^2\sqrt{1-\nu_{\varepsilon,n+1}^2}\right. \\ &\quad \left. + \frac{1}{\varepsilon}\int_{B_4(0)} |f_\varepsilon|^2\right) \\ &\le 4\delta,\end{aligned} \tag{5.35}$$

for ε sufficiently small. First by Lemma 5.5 and Lemma 5.7 we have $x \in A_\varepsilon$, $\forall L\varepsilon \le R \le \omega$,

$$\alpha\omega_n - \delta \le (1+\delta)R^{-n}\mu_\varepsilon(B_R(x)) + \delta.$$

By the reduction to the conditions in Proposition 5.2, we obtain

$$\mu_\varepsilon\left(\Omega \setminus \{|x_{n+1}| \le \zeta\}\right) \to 0, \quad \text{for any fixed } \zeta > 0.$$

Thus, for sufficiently small $\delta > 0$, we get

$$A_\varepsilon \subset \{|x_{n+1}| \le \zeta_\varepsilon\}, \quad \text{with } \zeta_\varepsilon \to 0 \text{ as } \varepsilon \to 0.$$

For any $\hat{y} \in B_1^n(0) \subset \mathbb{R}^n$, consider a maximal subset

$$X = \{y\} \times \{t_1 < \ldots < t_K\} \subset A_\varepsilon \cap \pi^{-1}(y)$$

with $|x - x'| > 3\,L\varepsilon$ if $x \ne x' \in X$, where π denotes the projection to $\{x_{n+1} = 0\}$. If $K \ge N$, we apply Lemma 5.5 with $d = 3L\varepsilon$, $R = \omega$ and Lemma 5.7 to get

$$N\alpha\omega_n - N\delta \le (1+\delta)R^{-n}\mu_\varepsilon\left(B_R(y)\right) + \delta \le (1+\delta)R^{-n}\mu_\varepsilon\left(B_{R+\zeta_\varepsilon}(y)\right) + \delta.$$

As

$$\limsup_{\varepsilon\to 0}(1+\delta)R^{-n}\mu_\varepsilon\left(B_{R+\zeta_\varepsilon}(y)\right) \le R^{-n}\mu(\overline{B_R(y)}) + C\delta = \theta\omega_n + C\delta,$$

and $\delta > 0$ is arbitrarily small, we have

$$N\alpha \le \theta,$$

which is a contradiction to our definition of N. So we obtain

$$K \leq N - 1.$$

Since X is maximal, we get

$$A_\varepsilon \cap \pi^{-1}(y) \subset \{y\} \times \cup_{k=1}^{K} (t_k - 3L\varepsilon, t_k + 3L\varepsilon).$$

By (5.28),

$$\begin{aligned} A_\varepsilon \cap \pi^{-1}(y) \cap & \left(\{y\} \times \cup_{k=1}^{K} (t_k - 3L\varepsilon, t_k + 3L\varepsilon) \right) \\ &= A_\varepsilon \cap \pi^{-1}(y) \cap \left(\{y\} \times \cup_{k=1}^{K} (t_k - L\varepsilon, t_k + L\varepsilon) \right). \end{aligned}$$

So

$$A_\varepsilon \cap \pi^{-1}(y) \subset \{y\} \times \cup_{k=1}^{K} (t_k - L\varepsilon, t_k + L\varepsilon)$$

and by (5.30),

$$\int_{t_k - L\varepsilon}^{t_k + L\varepsilon} \frac{W(u_\varepsilon(y,t))}{\varepsilon} dt \leq \frac{\alpha}{2} + \delta, \quad \forall k = 1, ..., K.$$

Hence summing over k gives

$$\int_{A_\varepsilon \cap \pi^{-1}(y)} \frac{W(u_\varepsilon)}{\varepsilon} d\mathcal{H}^1 \leq \frac{(N-1)\alpha}{2} + (N-1)\delta$$

and integrating over $B_1^n(0) \subset \mathbb{R}^n$ we obtain

$$\begin{aligned} \int_{B_1^{n+1}(0) \cap A_\varepsilon} \frac{1}{\varepsilon} W(u_\varepsilon) d\mathcal{H}^{n+1} &\leq \int_{B_1^n(0)} \int_{A_\varepsilon \cap \pi^{-1}(y)} \frac{W(u_\varepsilon)}{\varepsilon} d\mathcal{H}^1 dy \\ &\leq \frac{(N-1)\alpha\omega}{2} + C\delta. \end{aligned}$$

Recalling (5.35), we get

$$\begin{aligned} \mu_\varepsilon (B_1(0)) \leq & \int_{B_1^{n+1}(0) \cap A_\varepsilon} \frac{1}{\varepsilon} W(u_\varepsilon) d\mathcal{H}^{n+1} + |\xi_\varepsilon (B_1(0))| + \mu_\varepsilon (B_1(0) \setminus A_\varepsilon) \\ \leq & (N-1)\alpha\omega_n + C\delta. \end{aligned}$$

On the other hand, since $\lim_{\varepsilon \to 0} \mu_\varepsilon (B_1(0)) = \theta\omega_n$ and $\delta > 0$ is arbitrarily small, we obtain

$$\theta \leq (N-1)\alpha.$$

And since by definition N is the smallest integer such that $\theta < N\alpha$, we have

$$\theta = (N-1)\alpha.$$

□

6 Proof of corollaries and applications of Theorem 1.1

In this section, we provide the proof of Corollary 1.3 and Corollary 1.2, which are applications of Theorem 1.1.

We first prove the convergence result under various other Sobolev conditions on the inhomogeneous term.

Proof of Corollary 1.3 (1) To see the first condition implies the conditions in Theorem 1.1, we choose $q_0 = \frac{t(s-2)}{s} + 2$ ($q_0 > n$ is satisfied due to the choice of t and s above). Then we have

$$\begin{aligned}\int_\Omega \left|\frac{f_\varepsilon}{\varepsilon|\nabla u_\varepsilon|}\right|^{q_0} \varepsilon|\nabla u_\varepsilon|^2 dx &= \int_\Omega \left|\frac{f_\varepsilon}{\varepsilon|\nabla u_\varepsilon|}\right|^{q_0-2} \frac{|f_\varepsilon|^2}{\varepsilon} dx \\ &\leq \frac{1}{\varepsilon}\left(\int_\Omega \left|\frac{f_\varepsilon}{\varepsilon|\nabla u_\varepsilon|}\right|^{\frac{(q_0-2)s}{s-2}}\right)^{\frac{s-2}{s}} \left(\int_\Omega f_\varepsilon^{2\frac{s}{2}}\right)^{\frac{2}{s}} \\ &= \frac{1}{\varepsilon}\cdot \left\|\frac{f_\varepsilon}{\varepsilon|\nabla u_\varepsilon|}\right\|_{L^t(\Omega)}^{q_0-2} \cdot \|f_\varepsilon\|_{L^s(\Omega)}^2 \\ &\leq C_1^2 C_2^{q_0-2} \leq \Lambda_0\end{aligned}$$

where we used Hölder's inequality in the second line with exponent $\frac{s}{s-2}$.

(2) In the paper [11], assuming condition (2) above, the authors proved the same integer rectifiability and L^{q_0} mean curvature bound for the limit varifold. We show this conditions implies the integral bounds in the hypothesis of Theorem 1.1 for some $q_0 > n$. To see this, we compute

$$\nabla\left(\phi^{\frac{np}{n+1-p}}\right) = \frac{np}{n+1-p}\phi^{\frac{(n+1)(p-1)}{n+1-p}}\nabla\phi.$$

and applying [12, 5.12.4](c.f. [11, Theorem 3.7]) and [11, Theorem 3.8], and Hölder's inequality, with $\varphi = \phi^{\frac{np}{n+1-p}}$ and $d\mu = \varepsilon|\nabla u_\varepsilon|^2 d\mathcal{L}^{n+1}$.

$$\left|\int_{\mathbb{R}^n} \varphi d\mu\right| \leq c(n)K(\mu)\int_{\mathbb{R}^n} |\nabla\varphi| d\mathcal{L}^n \quad \forall\varphi \in C_c^1\left(\mathbb{R}^{n+1}\right)$$

which implies

$$\left|\int_{\mathbb{R}^{n+1}} |\phi|^{\frac{np}{n+1-p}} \varepsilon|\nabla u_\varepsilon|^2 d\mathcal{L}^{n+1}\right| \leq \left|\int_{\mathbb{R}^{n+1}} \varphi d\mu\right|$$

$$
\begin{aligned}
&\leq C(n)K(\mu)\left|\int_{\mathbb{R}^{n+1}} \frac{np}{n+1-p}|\nabla\phi||\phi|^{\frac{(n+1)(p-1)}{n+1-p}}\,d\mathcal{L}^{n+1}\right| \\
&\leq C(n,p)K(\mu)\left|\int_{\mathbb{R}^{n+1}} |\nabla\phi||\phi|^{\frac{(n+1)(p-1)}{n+1-p}}\,d\mathcal{L}^{n+1}\right| \\
&\leq C(n,p)\left(\int_{\mathbb{R}^{n+1}}|\nabla\phi|^p\right)^{1/p}\left(\int_{\mathbb{R}^{n+1}}|\phi|^{\frac{p(n+1)}{n+1-p}}\right)^{\frac{p-1}{p}} \\
&= C(n,p)\|\nabla\phi\|_{L^p(\mathbb{R}^{n+1})}\|\phi\|^{\frac{(p-1)(n+1)}{n+1-p}}_{L^{\frac{p(n+1)}{n+1-p}}(\mathbb{R}^{n+1})}.
\end{aligned}
$$

where $C(n,p)\to\infty$ as $p\to n+1$. We apply the above inequality with $\phi = \psi\frac{f_\varepsilon}{\varepsilon|\nabla u_\varepsilon|}$ and $d\mu = \varepsilon|\nabla u_\varepsilon|^2$ together the Sobolev inequality to get for $\psi\in C_0^1(\Omega)$

$$
\begin{aligned}
&\int_\Omega \left|\psi\frac{f_\varepsilon}{\varepsilon|\nabla u_\varepsilon|}\right|^{\frac{pn}{n+1-p}} \varepsilon|\nabla u_\varepsilon|^2 d\mathcal{L}^{n+1} \\
&\leq C\left\|\nabla\left(\psi\frac{f_\varepsilon}{\varepsilon|\nabla u_\varepsilon|}\right)\right\|_{L^p(\Omega)}\left\|\psi\frac{f_\varepsilon}{\varepsilon|\nabla u_\varepsilon|}\right\|^{\frac{(p-1)(n+1)}{n+1-p}}_{L^{\frac{p(n+1)}{n+1-p}}(\Omega)} \\
&\leq C\left\|\nabla\left(\psi\frac{f_\varepsilon}{\varepsilon|\nabla u_\varepsilon|}\right)\right\|_{L^p(\Omega)}\left\|\nabla\left(\psi\frac{f_\varepsilon}{\varepsilon|\nabla u_\varepsilon|}\right)\right\|^{\frac{(p-1)(n+1)}{n+1-p}}_{L^p(\Omega)} \\
&\leq C_\psi\left\|\frac{f_\varepsilon}{\varepsilon|\nabla u_\varepsilon|}\right\|_{W^{1,p}(\Omega}
\end{aligned}
$$

where we have $q_0 = \frac{pn}{n+1-p} > n$ since $p > \frac{n+1}{2}$.

(3) If $n+1 = 2$ then this is proven in [8]. For $n+1\geq 3$ it can be directly verified that the condition (3) implies the conditions in Theorem 1.1.

□

Secondly, we prove the Γ - convergence of the L^{q_0}, $q_0 > n$ "Allen-Cahn" mean curvature functional to the L^{q_0} mean curvature functional for hypersurfaces in $\mathbb{R}^{n+1}$.

Proof of 1.2 The Γ - convergence of the first term in the functional $\int_\Omega\left(\frac{\varepsilon|\nabla u|^2}{2}+\frac{W(u)}{\varepsilon}\right)dx$ to the perimeter functional $\alpha\mathcal{H}^n(\partial E\cap\Omega)$ was proved by Modica [5].

The limsup inequality for the Γ - convergence of $\int_\Omega\left(\frac{|\varepsilon\Delta u-\frac{W'(u)}{\varepsilon}|}{\varepsilon|\nabla u|}^{q_0}\right)\varepsilon|\nabla u|^2dx$ follows from a similar argument as in [2], using a smooth approximation of the boundary measure and a diagonal argument (see also [7]).

The liminf inequality for the Γ - convergence of $\int_\Omega\left(\frac{|\varepsilon\Delta u-\frac{W'(u)}{\varepsilon}|}{\varepsilon|\nabla u|}^{q_0}\right)\varepsilon|\nabla u|^2dx$ to the L^{q_0} functional follows from (4.17) in the proof of Theorem 4.6. □

Acknowledgements The first author was supported by EPSRC grant EP/S012907/1. The second author was supported by EPSRC grants EP/S012907/1 and EP/T019824/1. We would also like to thank the referee for the comments and suggestions that help improve our manuscript.

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare that there are no financial, competing or conflict of interest.

References

1. Allard, W.K.: On the first variation of a varifold. Ann. Math. **2**(95), 417–491 (1972)
2. Bellettini, G., Mugnai, L.: Approximation of Helfrich's functional via diffuse interfaces. SIAM J. Math. Anal. **42**(6), 2402–2433 (2010)
3. De Giorgi, E.: Convergence problems for functionals and operators. In Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), pp. 131–188. Pitagora, Bologna, (1979)
4. Hutchinson, J.E., Tonegawa, Y.: Convergence of phase interfaces in the van der Waals–Cahn–Hilliard theory. Calc. Var. Partial Differ. Equ. **10**(1), 49–84 (2000)
5. Modica, L.: The gradient theory of phase transitions and the minimal interface criterion. Arch. Rational Mech. Anal. **98**(2), 123–142 (1987)
6. Modica, L., Mortola, S.: Un esempio di Γ^--convergenza. Boll. Un. Mat. Ital. B (5) **14**(1), 285–299 (1977)
7. Mugnai, L.: Gamma-convergence results for phase-field approximations of the 2d-Euler elastica functional. ESAIM Control Optim. Calc. Var. **19**(3), 740–753 (2013)
8. Röger, M., Schätzle, R.: On a modified conjecture of De Giorgi. Math. Z. **254**(4), 675–714 (2006)
9. Sternberg, P.: The effect of a singular perturbation on nonconvex variational problems. Arch. Rational Mech. Anal. **101**(3), 209–260 (1988)
10. Tonegawa, Y.: Integrality of varifolds in the singular limit of reaction-diffusion equations. Hiroshima Math. J. **33**(3), 323–341 (2003)
11. Tonegawa, Y., Tsukamoto, Y.: A diffused interface with the advection term in a Sobolev space. Calc. Var. Partial Differ. Equ. **59**(6), 184 (2020)
12. Ziemer, W.P.: Weakly differentiable functions. In: Volume 120 of Graduate Texts in Mathematics. Springer-Verlag, New York (1989). (**Sobolev spaces and functions of bounded variation**)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.