

The Challenge of Non-Markovian Energy Balance Models in Climate

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We first review the way in which Hasselmann’s paradigm, introduced in 1976 and recently honoured with the Nobel Prize, can, like many key innovations in complexity science, be understood on several different levels. It can be seen as a way to add variability into the pioneering energy balance models (EBMs) of Budyko and Sellers. On a more abstract level, however, it used the original stochastic mathematical model of Brownian motion to provide a conceptual superstructure to link slow climate variability to fast weather fluctuations, in a context broader than EBMs, and led Hasselmann to posit a need for negative feedback in climate modelling. Hasselmann’s paradigm has much still to offer us, but naturally, since the 1970s a number of newer developments have built on his pioneering ideas. One important one has been the development of a rigorous mathematical hierarchy that embeds Hasselmann-type models in the more comprehensive Mori-Zwanzig Generalised Langevin Equation (GLE) framework. Another has been the interest in stochastic EBMs with a memory that has slower decay and thus longer range than the exponential form seen in his EBMs. In this paper we argue that the Mori-Kubo overdamped GLE, as widely used in statistical mechanics, suggests the form of a relatively simple stochastic EBM with memory for the global temperature anomaly. We also explore how this EBM relates to Lovejoy et al’s Fractional Energy Balance Equation (FEBE).

Hasselmann’s stochastic energy balance model embodies a long-standing paradigm in climate science. Recent evidence that climate may show long ranged memory has thus motivated newer stochastic models such as Lovejoy et al’s FEBE. Physical arguments for extending the Hasselmann formalism are more general than this, however, and are motivated by many other well-known features of the system, including periodicity. We thus need a model that allows a more general range of dependency structures than either just the shortest possible or longest possible ranged behaviours, while allowing both as limiting cases. In this article we propose the use of such a formalism to extend Hasselmann’s EBM, the stochastic Mori-Kubo GLE.

I. INTRODUCTION

The value of climate science would be substantial even without anthropogenic interference in the climate system. The

presence of such interference, however, raises a number of categorically new challenges. The context of human-induced climate change has made it important to quantify the observed changes, understand their origin, and, as far as possible^{1,2}, predict their future scope, in the presence of the nontrivial intrinsic variability that is already present (see e.g.^{3,4}). The development of this need has coincided with the rise of the science of complexity, which has itself both pioneered and promoted new approaches to similar problems across a very wide range of topical application areas⁵.

Several key achievements in the study of fluctuations, at the interface^{6–8} of climate modelling and the fast-developing sciences of complexity, have recently been recognised by the award of the 2021 Nobel Prize for Physics to Hasselmann, Manabe and Parisi. The Nobel Committee have given a very useful and accessible short introduction⁹ to these branches of science, but two more recent articles have between them demonstrated an emerging dichotomy in responses to the Prize. Our paper seeks to reconcile them.

In Ref. 10 Franzke et al have shown how “the Hasselmann programme”, first enunciated in Ref. 11, of explaining how slow climate arises from fast weather using stochastic EBMs, has given rise to decades of productive and in-

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sightful research. The original stochastic EBMs were Markovian, depending only on the current values of their (stochastic) variables, and by construction assumed strongly separated fast and slow time scales. A reader might easily infer from this excellent review, however, that the broader ‘‘Hasselmann programme’’ built around such stochastic EBMs, their developments, and his subsequent spatiotemporal fingerprinting methods, has fully solved the problems of realistic climate modelling. This impression is reinforced by the fact that the paper does not delve into more recent, and maximally non-Markovian, EBMs which exhibit ‘‘long-range memory’’ (LRM; though see a complementary review of this by Franzke et al¹², and recent work by him with colleagues¹³).

In contrast, Lovejoy¹⁴ has stressed the differences between Hasselmann’s Markovian paradigm and more recent developments in complexity science, especially those related to LRM and fractional response. He has articulated^{15–17} the view that only models incorporating LRM can fully describe already known properties of climate data. Arguments for the relevance of LRM have at times been received with scepticism by some climate scientists¹⁸, while others (e.g. Ref. 19), who do acknowledge its importance, have nevertheless emphasised the deterministic behaviour present in the climate system because of the predictability that it affords. Recently, however the Nobel committee has remarked⁹ on an increasing awareness that ‘‘on decadal timescales of relevance to humans, evidence is consistent with the memory becoming effectively infinite ...’’.

An AGU Editor Highlight (Ana Baros, ‘‘Eminently Complex – Climate Science and the 2021 Nobel Prize’’, 1st January 2023) described Ref 14 as making ‘‘the case that for the last 100 years there has been little communication between atmospheric sciences and nonlinear geophysics, and this must change to improve useful predictability (i.e., lead time) of climate models.’’ In this paper we address the need for communication between these two fields in a number of ways.

i) Firstly, by reviewing the topic we will show that the situation is in some ways already better than it may appear, because in fact there has been work, notably that of Ref. 20, reviewed in Ref. 8, which while proceeding directly from the Hasselmann paradigm, has allowed the construction of models with non-Markovian structure, via the use of memory kernels. It seems not to have always been very accessible to a wider audience, perhaps because it exploits the Mori-Zwanzig formalism²¹, well known in applied mathematics, data assimilation and statistical mechanics, to decompose the effect of fast degrees of freedom into a memory term and a noise-like term. The fundamental (and nonlinear) equation which expresses the Mori-Zwanzig decomposition is an identity, and deterministic, but, confusingly to those already familiar with the linear stochastic equation first studied by Langevin in 1908, it is sometimes^{8,10,20–23} referred to as the Generalised Langevin Equation (GLE). This Mori-Zwanzig GLE is intractable without further approximations²¹ or other modifications⁸.

ii) In view of this intractability, our paper takes an alternative, more directly stochastic approach to an EBM with memory, based on an already existing extension of Langevin’s orig-

inal equation²⁴ of 1908. We advocate for the use of Mori’s and Kubo’s stochastic GLE, widely known and applied since the 1960s in condensed matter theory, quantum optics and physical chemistry^{25–29}. We show how the form of the Mori-Kubo GLE, occasionally known as the non-Markovian Langevin equation³⁰, suggests an analogous generalised Hasselmann stochastic EBM for the global temperature anomaly. Our paper expands on and revises our contribution to the 2020 International Conference on Complex Systems³¹, which was itself expanded as Ref. 32.

iii) We go on to show that Hasselmann’s model is one limiting case of our proposed EBM, that of short-ranged memory, while Lovejoy’s fractional model, FEBE^{16,33}, is closely related to the opposite special case, long-ranged memory. FEBE differs only by using a Riemann-Liouville fractional derivative instead of a Caputo. We thus bridge these two seemingly opposed perspectives. We further note that an EBM based on the Mori-Kubo GLE can handle not just the two extremes of delta-correlated (shortest ranged) or power law decaying (longest ranged) memory kernels, but also those with intermediate-ranged memory such as that which arises from a few widely separated timescales. In addition, modifications of such an EBM could in principle accommodate many of the types of periodic (or quasiperiodic) determinism which are present, such as ENSO. As early as 1976 Mitchell³⁴ asserted that ‘‘there are no known deterministic mechanisms of climatic variability that are internal to the atmosphere ... or any other part of the climatic system’’. From this viewpoint the emergence of any deterministic mode(s), even if nonlinear and only quasi-periodic, is thus crucial to predictability³⁵ and its inclusion essential in any compact description.

iv) We conclude by noting that an EBM based on the Mori-Kubo GLE would have the option of embodying a fluctuation-dissipation relation (FDR) **by construction**, because this was a main original purpose of their stochastic GLE.

We hope our contribution will open a new channel of communication between two disparate worlds and will facilitate construction of a new class of EBMs, which will in turn improve the study of climate predictability.

In the next section we will first briefly recap the basic notions of EBMs, and then discuss some ways in which nonlinearity and stochasticity have been incorporated in them.

II. ENERGY BALANCE MODELS

A. Nonlinearity in EBMs

The first energy balance models^{36–38} of Budyko and Sellers were deterministic. As the name implies they captured how the earth’s surface temperature T will, after a perturbation, evolve in time t to restore the balance between the incoming $R \downarrow$ and outgoing $R \uparrow$ heat fluxes at the earth’s surface expressed by the equation

$$C_P \frac{dT}{dt} = R \downarrow - R \uparrow. \quad (1)$$

Here C_P is effective heat capacity per unit area.

The Budyko-Sellers model in its simplest form then makes the outgoing and incoming radiation terms in (1) explicit. The former is modelled with the Stefan-Boltzman law and the latter includes an albedo feedback function $a(T)$:

$$C_P \frac{dT}{dt} = S_0(1 - a(T)) - \varepsilon \sigma_{SB} T^4 \quad (2)$$

where S_0 is the mean solar radiation incident per metre squared at the earth's surface³⁶, ε is the earth's emissivity, and σ_{SB} is the Stefan-Boltzmann constant.

Since the 1960s EBMs have been explored and extended in many directions. In particular much work has been done on nonlinear and stochastic climate dynamics in this framework, one topic of the recent edited book by Franzke and O'Kane⁶.

Considerable attention has been focused on the nonlinearity of the complicated potential created by the interplay of the T^4 term and the albedo feedback, for example. Taking a quadratic choice for the latter gives the model studied by Fraedrich³⁹

$$C_P \frac{dT}{dt} = \frac{1}{4} \mu I_0 (1 - a_2 + b_2 T^2) - \varepsilon \sigma_{SB} T^4 \quad (3)$$

where I_0 is the solar constant, μ is a fractional premultiplier³⁶ which gives the radiation incident at the earth's surface, and the quarter results in an average over the planet's surface.

B. Stochasticity in EBMs

Stochasticity can be introduced into nonlinear EBMs in a number of ways, such as the white noise $\xi(t)$ with a controllable variance σ^2 added as an extra term to Eq. (3) by Sutera⁴⁰. This changes it from an ODE to a stochastic differential equation (SDE), an important subtlety.

The Fraedrich-Sutera model is Markovian because its deterministic part depends only on the current value of its single variable, the temperature $T(t)$, while its stochastic part, Gaussian white noise, is delta-correlated and thus also has no memory of its previous state.

The marginal distributions of such a Markovian climate model need not be Gaussian, however⁴¹. Ref. 42 is a comprehensive study of α -stable Lévy noise of the more recent Ghil-Sellers extension to the original EBMs. The PDF of the noise here has an infinite variance and is heavy-tailed.

C. Memory in EBMs

The Markovian restriction on the earliest EBMs has since been relaxed in several different ways. A pioneering approach was that of Bhattacharya et al⁴³ who introduced memory into the deterministic part via a modification of the albedo term. Resulting work has been based on delay differential equations⁴⁴.

Multi-box (and thus multivariate) stochastic models have incorporated memory effects through the presence of a range of feedback time scales in their different "boxes"⁴⁵⁻⁴⁷.

Most of the remainder of this paper concerns memory in stochastic EBMs but in the next section we first examine the simplest linear Markovian stochastic models in more detail. In particular we consider the univariate autonomous and non-autonomous versions in which they have been used to model the earth's global temperature anomaly.

III. THE HASSELMANN PROGRAMME I: LINEARISED MARKOVIAN STOCHASTIC EBMS

Linear Markovian stochastic EBMs were explored in detail in the second of Hasselmann's classic mid 1970s papers, Ref. 48, complementing the treatment in the first paper¹¹ which was largely in PDE (Fokker-Planck) rather than SDE (Langevin) terms. They can be motivated in several ways.

The first is an ansatz, and was essentially the approach followed by Mitchell in the 1960s (section A below). It recognises that the discrete time autoregressive first order Markov process $AR(1)$, and the continuous time Langevin equation (9) whose solution is called the Ornstein-Uhlenbeck (O-U) process, are the simplest possible linear stochastic models of the response to a perturbation⁴⁹.

A more physically intuitive argument, summarised in section B, and given more fully by Lemke in Ref. 50, comes from linearising deterministic EBMs of the Budyko-Sellers type, and adding white noise. At this level of abstraction it is already apparent that the result is what we will refer to as Hasselmann's autonomous SDE. However Hasselmann's SDE is isomorphic to not one but two such equations arising in Brownian motion so section C describes the dilemma posed by the choice of possible mappings.

Hasselmann's mathematical analogy with Brownian motion was not limited simply to mapping an EBM to a Langevin equation. The most sophisticated and abstract level of his programme, proposed first in Ref 11, replaced Mitchell's ansatz, which has sometimes been seen as being purely statistical, with a postulate, inspired by statistical physics: that of the separation of fast and slow timescales, and the identification of the former with weather and the latter with climate.

It is very important to realise that Hasselmann's postulate is potentially quite generally applicable to climate variables, and not just to the global mean temperature anomaly-it was not simply a way to make a stochastic EBM, but kicked off a research programme.

A. Mitchell's stochastic models of 1966

Mitchell's pioneering chapter, Ref. 51 in a RAND Corporation conference proceedings volume from December 1966 proposed two stochastic climate models.

The first, as he put it, bore "*primarily on the origins of relatively rapid fluctuations, as for example those reflected in the interannual variability of temperature observed at a wide variety of locations during the past century*". He postulated that the air temperature anomaly in a marine environment was composed of two additive parts, one propor-

tional to the sea surface temperature (SST) anomaly and one independent of it. He then gave arguments suggesting that each component's data, comprising monthly values, would follow first order Markov, or "AR(1)", stochastic processes. An AR(1) process is defined at discrete times, in this case the set $t, t + dt, t + 2dt, \dots$, and takes the form:

$$x(t + dt) = \phi x(t) + \xi(t) \quad (4)$$

where $0 < \phi < 1$ and $\xi(t)$ is an iid normal random variable of zero mean and unit variance.

Mitchell derived the standard results for the autocorrelation function and power spectrum of the resulting compound first order linear Markov process, and pointed out that measurements of SST implied that its ϕ would be in the range 0.5 to 0.9, giving a red noise behaviour. He also fitted the compound model to the famous 269-year (1689 to 1957) Manley series of monthly temperatures for central England, illustrating how the two component model ACF gave a good description of the observed one. Hasselmann was made aware of Mitchell's work by a referee and took care to credit it, both in Ref. 11 and subsequently, e.g. the oral history of Ref. 52.

Mitchell's second model operated on longer time scales than his first and is less directly relevant to Hasselmann's work. He viewed it as bearing "*contrastingly on the origin of much longer fluctuations, and*" thought that it might "*have some unexpected relevance to the origins of the glacial-interglacial succession of the Pleistocene ice age*". In a sense it anticipated the previous section's Freidrich-Sutera model and is further evidence of how far Mitchell was ahead of his time.

B. Adding noise to a linearised Budyko-Sellers model

Following Ref. 9 we start with the Budyko-Sellers model (2) and expand T about the steady state average surface temperature T_S ,

$$T = T_S + \Delta T$$

so

$$C_P \frac{d(T_S + \Delta T)}{dt} = S_0(1 - a(T_S + \Delta T)) - \varepsilon \sigma_{SB}(T_S + \Delta T)^4$$

from which we can subtract the corresponding steady state values to get

$$C_P \frac{d\Delta T}{dt} = -S_0 \frac{\partial a}{\partial T} \Delta T - 4\varepsilon \sigma_{SB} T_S^3 \Delta T + O(\Delta T^3)$$

and thus, dropping terms in ΔT of third order and above, we obtain

$$C_P \frac{d\Delta T}{dt} = - \left[4\varepsilon \sigma_{SB} T_S^3 - S_0 \left| \frac{\partial a}{\partial T} \right| \right] \Delta T = -\Lambda \Delta T \quad (5)$$

where we have additionally used the fact that albedo sensitivity, $\partial a / \partial T$, is negative, and have defined a feedback parameter Λ .

As in the Fraedrich-Sutera model high frequency fluctuations are then introduced as additive white noise $\xi(t)$, a standard Gaussian random variable (with zero mean and unit variance) to give Hasselmann's stochastic EBM in its SDE form:

$$\boxed{C_P \frac{d\Delta T}{dt} = -\Lambda \Delta T + \sigma \xi(t)} \quad (6)$$

This SDE is Markovian in the sense that it depends only on the current value of its variable, which is now the global mean temperature (GMT) anomaly ΔT . This SDE is a Langevin equation governing the OU process $\Delta T(t)$. It expresses the competition between the Gaussian fluctuations and the mean-reverting $-\Lambda \Delta T$, which produces a dynamic steady state.

There is also a non-autonomous version of Hasselmann's model, in which $F(t)$ represents the deterministic part of the forcing, in particular that arising from anthropogenic carbon emissions:

$$C_P \frac{d\Delta T}{dt} = F(t) - \Lambda \Delta T + \sigma \xi(t). \quad (7)$$

Equations (6) and (7) are formal, and, like all SDEs, contain objects whose meanings must be made explicit and rigorous. For white noise-driven SDEs this process has taken decades of mathematical work in developing the now mature field of stochastic calculus^{21,53}. In that Markovian case there is a widely used convention to write SDEs as differentials rather than derivatives. We do not use this notation here because many of the equations we study have coloured or even fractional noise driving terms for which there is less of a notational consensus.

C. Relation of EBM to the equations of Brownian motion

Hasselmann was directly inspired by a conceptual and mathematical analogy with Brownian motion. As he observed in an AIP oral history interview⁵²: "*my stochastic model is an application of the concept of Brownian motion . . . One of the simplest stochastic processes. The idea that one could explain long term climate variability very simply by the short term fluctuations of the atmosphere in analogy with Brownian motion, is really rather obvious and I thought I would write it up somewhere in a little note*".

His paper explored an important distinction between the mathematical and physical usages of the term Brownian motion (BM). For mathematicians the Wiener process defines BM, while for physicists the underdamped LE defines BM. Hasselmann showed that the former was already insightful in addition to the then-current paradigms for fluctuations in climate, but that the latter was necessary to create a power spectrum that more closely resembled those already being observed.

Interestingly, however, there are not one but two possible SDEs in physical BM to choose from, the underdamped and overdamped Langevin equations. If Hasselmann had proceeded purely by mapping one of these onto an EBM he could

ostensibly have used either, as we now show. While this ambiguity was relatively unimportant to his work, we will then argue that it becomes significant when thinking about extensions to the Hasselmann EBM.

1. Underdamped Langevin equation

The fundamental SDE used in Brownian motion is the underdamped Langevin equation:

$$M \frac{dv}{dt} = -V'(y) - \gamma v + \sigma \xi(t) \quad (8)$$

It describes the acceleration dv/dt , where velocity $v = dy/dt$, of a particle of mass M which experiences a deterministic force due to a potential $V(y)$, a stochastic force $\sigma \xi(t)$, and a linear damping force $-\gamma v$.

If we i) map velocity into GMT anomaly, and mass into heat capacity C_P , ii) ignore the question of the mapping and meaning of the position variable y , and iii) replace the derivative of a potential by a deterministic forcing we obtain the non-autonomous version (7) of the Hasselmann EBM.

The autonomous version (6), with $F(t) = 0$ is also frequently studied, e.g. Ref. 54 where an attempt has been made to remove the prompt effects of the deterministic forcing from the time series.

2. Overdamped Langevin equation

We now consider forced Brownian motion in a potential. If the mass is very large and the particle motion is close to a minimum of the potential, we can neglect the acceleration term in (8) and set $V'(y) \approx -ky$. This gives the overdamped Langevin equation

$$\gamma \frac{dy}{dt} = -ky + \sigma \xi(t) \quad (9)$$

The solution of this for the position of the overdamped particle is the O-U process.

Equation (9) thus offers an alternative conceptual starting point for a stochastic EBM, where instead of velocity it is the position which is mapped to ΔT . The result is an autonomous equation:

$$C_P \frac{d\Delta T}{dt} = -\Lambda \Delta T + \sigma \xi(t). \quad (10)$$

Although mathematically identical to the unforced version of (7), rather than the mass M being mapped to the heat capacity C_P , it is now the friction constant γ which is so mapped. This is why although γ is usually absorbed into a \bar{k} and $\bar{\sigma}$ we did not do so in equation (9). Another conceptual difference is that the linear term in ΔT of equation (10) arises from mapping a linearised potential rather than a friction coefficient.

The overdamped Langevin equation can be modified to study problems with additional deterministic forcing of the

velocity, such as the ion trap modelled by Ref. 55, and so its analogous Hasselmann EBM could also have an additional deterministic forcing term.

3. Which of the underdamped or overdamped Langevin equations is closer to Hasselmann's EBM ?

The conceptual differences between the underdamped and overdamped Langevin equations start to become important when we try to derive EBMs from first principles. This is even more true if we want to extend them to include effects such as long range memory and fluctuation-dissipation relations. We illustrate this in the next section by considering Hasselmann's own motivation¹¹ of his stochastic model embodied in Equation (10), and the outline derivation of it offered by the Nobel committee⁹. We will find that the latter in particular has more in common with the overdamped than the underdamped Langevin equation.

IV. THE HASSELMANN PROGRAMME II: FLUCTUATIONS, FAST AND SLOW.

A. Assumptions of the Hasselmann programme

A modernized version of Hasselmann's argument is sketched by the Nobel committee⁹, who describe it as an "interpretative and notationally uncluttered outline". We follow it and its notation here, supplemented by reference to Hasselmann¹¹.

1. Fast and slow variables

Hasselmann assumes that the instantaneous state of the atmosphere-ocean-cryosphere-land system is described by a finite set of discrete variables $\mathbf{z} = (z_1, z_2, \dots)$. He then assumes its evolution is described by a series of prognostic equations like

$$\frac{dz_i}{dt} = w_i(\mathbf{z}), \quad (11)$$

and then divides the system into two subsystems $\mathbf{z} = (\mathbf{x}, \mathbf{y})$.

The set of fast "weather" variables \mathbf{x} have a response time scale τ_x of order a few days, and evolve under one set of ODEs

$$\dot{x}_i = f_i(x, y), \quad (12)$$

while the slow "climate" variables \mathbf{y} like sea surface temperature, ice coverage, land foliage etc have a longer response time scale τ_y , of order months, years or more:

$$\dot{y}_i = g_i(x, y). \quad (13)$$

Here f, g correspond to u, v in Hasselmann's original notation¹¹.

2. Conditional average over fast dynamics

In Ref 11 Hasselmann then discusses the key idea of averaging over the fast dynamics. The Nobel committee’s argument⁹ writes the dynamics of the fast variables in terms of the slow ones, and is simplified without loss of generality by considering scalar x and y , so:

$$x = \langle x|y \rangle + x^* \quad (14)$$

where, following Hasselmann, the first term is an ensemble average of a set of values of x for a given y , i.e. a conditional average, and the second term defines a fluctuation with respect to this average.

B. The Nobel committee’s derivation.

1. Drift force and multiplicative noise

At this stage the Nobel committee⁹ gives a simpler but more explicit argument for a Langevin-based EBM than Hasselmann’s own. His original paper, in contrast, dealt with both Langevin and Fokker-Planck approaches.

Their approach is to substitute for x using Eq. 14 in the equation for the slowly varying dynamics \dot{y} , and then Taylor expand in the relatively small but fast fluctuations x^* :

$$\begin{aligned} \dot{y} &= g(x, y) \\ &= g(\langle x|y \rangle + x^*, y) \approx g(\langle x|y \rangle, y) + \partial_x g(\langle x|y \rangle, y)x^* \end{aligned}$$

One can then model the slowly varying first term as a drift force arising from a potential $U(y)$, while the fast fluctuations are modelled as a white noise $\xi(t)$ whose strength $\bar{\sigma}$ is dependent on the value of y , i.e. this noise is not additive but multiplicative:

$$\dot{y} = -\frac{dU(y)}{dy} + \bar{\sigma}(y)\xi(t) \quad (15)$$

2. Linearising the drift force, and assuming additive noise.

To transform the more general model Eq. 15 into the specific Hasselmann form with linear damping and additive noise the Nobel committee then argues that noise intensity is small so its amplitude can be taken to be approximately constant:

$$\bar{\sigma}(y) \approx \bar{\sigma} \quad (16)$$

and that in the case of interest the system loiters near a dynamical fixed point y_E around which the potential and thus the drift force can be expanded:

$$U(y) = U(y_E) + (y - y_E) \frac{dU(y_E)}{dy} + \frac{1}{2!} (y - y_E)^2 \frac{d^2U(y_E)}{dy^2} + \dots$$

and linearised by dropping terms above second order in $y - y_E$.

The SDE for the resulting dynamics then has the well known O-U solution, but we can see from the above arguments that the SDE must be in the variable $y^* = y - y_E$ which tracks the relative distance from the fixed point, rather than y :

$$\frac{dy^*}{dt} = -\bar{\gamma}y^* + \bar{\sigma}\xi(t). \quad (17)$$

The result can be compared to the autonomous Hasselmann model (10) which is in ΔT , not T .

V. STOCHASTIC EBMS AND IAMs

Economic assessments of climate change typically rely on a suite of integrated assessment models (IAMs) descended from the work of another Nobel laureate, Nordhaus.^{56,57} These models have now been refined and extended in many directions—from increasing their spatial and sectoral resolutions^{58–60} to explicitly modelling technological choices⁶¹ to incorporating agent-based and non-equilibrium models of the economy.^{62–64} Yet, though Hasselmann himself contributed to the early development of IAMs,^{65,66} the climate modules of IAMs are generally based on deterministic EBMs.⁶⁷

Climate stochasticity, when it is acknowledged at all, most commonly enters in the form of uncertain temperature thresholds (effectively “tipping points”) at which the model’s parameter values suddenly change.⁶⁸

One strand of economic research has added noise to the temperature dynamics directly.^{69–71} The main objective has been to study how fast we can learn the model’s parameter values—the equilibrium climate sensitivity, in particular—when we observe a noisy temperature time series. Anticipated learning tends to reduce the effect of parameter uncertainty on the optimal mitigation policy, but there is no agreement on whether this anticipated learning-channel is quantitatively meaningful.⁷²

One possible reason that the research on IAMs has paid so little attention to temperature stochasticity is that this source of uncertainty does not affect the *marginal* warming resulting from an additional tonne of carbon emissions. As a consequence, temperature stochasticity itself has very little effect on the optimal mitigation policy.^{73,74} However, a recent paper by Calel et al.,⁷³ which perhaps offers the first example of a Hasselmann EBM coupled to an economic damage function, points out that temperature variability could still greatly affect optimal investment in adaptation. They conclude that a forward-looking social planner would be willing to make substantial investments to avoid or reduce the effects of temperature fluctuations. Evidently, Hasselmann’s “obvious” idea still has much to teach us.

VI. BUT SHOULD STOCHASTIC CLIMATE MODELS REALLY BE MARKOVIAN ?

A. Leith's "infrared climate problem"

As well as Mitchell and Hasselmann, a third great climate pioneer, Leith, argued for the use of Langevin models in the mid 1970s. In his case this was explicitly because of their connection with fluctuation-dissipation relations. As he put it at an NCAR conference on statistics and climate in 1994,⁷⁵: *"a stochastic climate model that incorporates the fluctuation dissipation relation can be devised and would provide a crude estimate of climate sensitivity. This model would be of the Langevin type"*.

However by then he had identified a problem of enduring relevance, that *"there is evidence that this kind of model, would not on its own be satisfactory to capture some of the low frequency phenomena observed in the atmosphere"*. He referred to this, in a phrase that has sadly not gained currency, as the *"infrared climate problem"*. This was not a literal reference to thermal radiation but rather to an "infrared catastrophe" in fluctuations in temperature, i.e the *"piling up of extra variance at low frequencies"*.

The Nobel committee⁹ illustrates and acknowledges the continuing relevance of Leith's worry. It points out that the Markovian nature of Hasselmann-type models results in an exponential decay of their autocorrelation functions:

$$R(\tau) = \langle y^*(t)y^*(t + \tau) \rangle. \quad (18)$$

Such a model predicts that the typical power spectrum of a climate variable would be Lorentzian:

$$P(\omega) = \bar{\sigma}^2 / (\bar{\gamma}^2 + \omega^2) \quad (19)$$

and that the ratio of the strength $\bar{\sigma}$ of the rescaled noise term and $\bar{\gamma}$, the rescaled damping term would control the spectrum's shape. The power spectral density would thus be flat at low frequencies, a property that was reported in early comparisons of the Hasselmann paradigm with data from weather ships.

B. Phenomenological evidence for LRM in climate

The simplest stochastic process with memory, AR(1) Eq. (4) is Markovian, being formulated only in instantaneous variables. It can additionally be given random initial conditions, although this is not always the relevant case for climate, where for example the initial temperature anomaly may be set to zero.

The emergent physics of the earth system on larger scales need not be similarly memoryless, though, and in fact the Nobel committee's summary states that in contrast to the atmosphere *"the ocean has a very long memory of events, at least in the hundreds ... 1000 year range"*. As we noted in our introduction, the committee then remarks on the increasing awareness of effectively infinite memory and that *"this suggests a potential self similarity or fractal character"*, citing Moon

et al's Ref. 76 in support of their statement. Moon et al in their turn refer to the by now quite extensive literature on long range dependence in climate, from several research teams. We note in particular the groups at Tromso (Refs. 47, 77, and 78) and Montreal,^{16,33,79} and the recent PAGES CVAS group¹².

In Ref. 14 Lovejoy addressed the issue of LRM directly. He compared Frankignoul and Hasselmann's original Markovian power spectral fit⁴⁸ to 16 years of sea surface temperature data from the Ocean Weather Ship India, with the data now available from 452 stations over about 100 years. He argued that a compound power law fit, $S(f) \sim f^{-0.6}$ at low frequencies and $\sim f^{-1.8}$ at higher frequencies is much more satisfactory.

C. Theoretical approaches to LRM in climate.

In other papers^{16,33} Lovejoy and colleagues have argued for a fractional energy balance equation (FEBE) as a method for incorporating long range memory into EBMs.

In the next section we will emphasise that there was *already* a well-studied and standard way, the Mori-Kubo GLE⁸⁰⁻⁸³, to incorporate memory in a Langevin equation. The Mori-Kubo GLE is suitable for use as a univariate stochastic equation, and thus to add memory to EBMs of the Hasselmann type. We show that FEBE, but with a Caputo fractional derivative rather than the Riemann-Liouville type used by Lovejoy et al, corresponds to one particular limit of this equation.

VII. WHAT IS A GENERALISED LANGEVIN EQUATION ?

As frequently happens when mathematics and physics intersect, the terminology used in this subject varies, which has sometimes caused confusion.

An example is the otherwise excellent Nobel committee article, Ref. 9, which describes the SDE corresponding to the O-U process as a "generalised Langevin equation". It is literally so, but only in the relatively limited sense that both the underdamped and overdamped Langevin equations have the same mathematical form, and that any other linearly damped model with additive white noise will also obey an SDE of this same form, whether physical in origin or not.

Two other equations which much more substantially merit the description "generalised" are

- the equation which has been widely studied in applied mathematics as a concise statement of the Mori-Zwanzig decomposition, and which is occasionally referred to as the GLE^{8,22} and
- the stochastic Mori-Kubo GLE which has found widespread application in statistical physics.

The Mori-Zwanzig decomposition has played an important conceptual role in formalising an approach to modelling the dynamics of a few preferred degrees of freedom in the presence of many other unobserved ones in many areas of science, notably quantum mechanics. The resulting equation are sometimes directly derived from Hamiltonian equations of motion

for the whole system, for example in a reservoir plus subsystem model²². The technique is not restricted to such cases, though, and it has also become arguably the most influential paradigm for thinking about how to make the Hasselmann programme mathematically rigorous, and has been employed in climate science to order empirical modes²⁰, see also Ref. 8.

In contrast, throughout the statistical mechanics community the more standard usage of the term GLE^{25–29} is to refer to the linear integrodifferential Langevin equation of Mori and Kubo with a memory kernel on its velocity term:

$$M \frac{d^2 y}{dt^2} = -V'(y) - \int_{t_m}^t dt' \gamma(t-t') \frac{dy(t')}{dt} + v(t) \quad (20)$$

where the Markovian friction term $\gamma dy/dt$ is replaced by a non-Markovian integral over past values of the velocity starting at a “memory time” t_m (c.f. Ref 55).

In the Mori-Kubo formalism the autocorrelation function of the noise $v(t)$ is proportional to $\gamma(t-t')$ with the prefactor depending on the temperature of the medium

$$\langle v(t)v(t') \rangle = k_B T \gamma(t-t'),$$

where T is the temperature of the medium and k_B is the Boltzmann constant. It is thus a non-white noise.

VIII. THE MORI-KUBO GENERALISED LANGEVIN EQUATION: A NEW APPROACH TO THE INFRARED CLIMATE PROBLEM ?

A. The overdamped Mori-Kubo GLE

Just as there is an overdamped Langevin equation whose solution is the O-U process (9) there is a corresponding overdamped generalised Langevin equation, shown here for motion near the minimum of the potential $V(y)$:

$$\int_{t_m}^t dt' \gamma(t-t') \frac{dy(t')}{dt} = -ky + v(t). \quad (21)$$

We note that the memory kernel is on the left hand side of this equation, on the term in dy/dt . We also remark that the same FDR as that for underdamped Mori-Kubo GLE also applies here. The power spectrum of the noise $v(t)$ is matched to that of the kernel $\gamma(t-t')$, and so as above $v(t)$ is no longer white noise.

It is important to realise that while mathematically this is adding memory to the Langevin model, physically it is simply making explicit something which was already known, the presence of finite collision times in physical Brownian motion.

B. An EBM with memory.

The significance of the Mori-Kubo GLE for our purposes is that it allows us to suggest a Mori-Kubo GLE-based extension

of the Hasselmann EBM. We show our suggested EBM below in its autonomous version:

$$\boxed{\int_{t_m}^t dt' C_P(t-t') \frac{d\Delta T(t')}{dt} = -\Lambda \Delta T + v(t).} \quad (22)$$

We remark that an integral over a time dependent heat capacity $C_P(t-t')$ has replaced the previous constant C . We do not prescribe the form of the noise, so the notation $v(t)$ does not presume that a fluctuation-dissipation relation exists.

C. A Mori-Kubo GLE with power law kernel: the Fractional Langevin Equation.

An interesting special case of the overdamped Mori-Kubo GLE arises when we take a decaying power law form for its memory kernel

$$\gamma(t-t') = \frac{(t-t')^{-\alpha}}{\Gamma(1-\alpha)} \quad (23)$$

instead of an arbitrary memory function.

The resulting integrodifferential SDE is written (for non-integer α):

$$\boxed{\frac{1}{\Gamma(1-\alpha)} \int_{t_m}^t dt' (t-t')^{-\alpha} \frac{dy(t')}{dt} = -ky + v_\alpha(t)} \quad (24)$$

and is known in the physics literature as the overdamped fractional Langevin equation (FLE). Both the overdamped and underdamped cases of the FLE have been extensively studied^{84–87} following the pioneering investigation by Mainardi and Pironi⁸⁸ of the $\alpha = 3/2$ case.

The notation v_α here acknowledges the presence of a relationship, the FDR^{89,90}, between the noise and the damping kernel.

D. Fractional derivatives.

The integrodifferential form of the FLE is sometimes simplified by the use of a fractional derivative. Fractional calculus forms a large and growing field of research in itself, see e.g. Refs. 91–93, so to aid the reader we have included Appendix A to summarise those aspects we have used here. We again emphasise that the FLE is only one special case of the more general Mori-GLE, and that we remain “agnostic” about its relevance to the climate system, preferring to suggest a model with a more flexible memory kernel.

The (left-handed) Caputo derivative of order α for a suitable function $f(x)$ is defined^{94,95} for $0 < \alpha < 1$ as

$${}^C D_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x du f'(u) (x-u)^{-\alpha} \quad (25)$$

where $f'(u) = df/du$.

As well as its order α the Caputo derivative has a second parameter, the lower limit a of the integral in its definition.

The Caputo fractional derivative has physically more intuitive initial conditions than the most widely used alternative to it, the Riemann-Liouville derivative. This is because they are specified as integer rather fractional powers of physical quantities. However, as noted by⁹⁵, the problem can be circumvented by the Schneider-Weiss formulation of the Riemann-Liouville derivative.

The Caputo derivative also has the nice property that while of fractional order for $\alpha < 1$ it becomes the familiar integer derivative in the $\alpha = 1$ case, i.e. ${}^C D_t^\alpha f(x) = df/du$, as is shown in Section 2.4.1 of Ref. 92. A Caputo FLE can thus be defined

$${}^C D_t^\alpha y = -ky + v_\alpha(t) \quad (26)$$

that straightforwardly encompasses the familiar Langevin equation as its $\alpha = 1$ case. This equation and its close relatives are increasingly being studied in several disciplines⁹⁶⁻¹⁰⁰, typically under the name ‘‘Caputo fractional SDE’’, though typically specialised to the case of white noise driving.

E. The LRM EBM special case and its relation to FEBE.

In the same way we can postulate a decaying power law form for the kernel in the EBM above

$$C(t-t') = \frac{(t-t')^{-\alpha}}{\Gamma(1-\alpha)} \quad (27)$$

instead of an arbitrary memory function, and can thus write the integral over the kernel in our EBM as a Caputo fractional derivative, giving an autonomous version as:

$$\boxed{{}^C D_t^\alpha \Delta T = -\Lambda \Delta T + v_\alpha(t)} \quad (28)$$

In this picture the external climate forcing can then be incorporated as an additional term $F_2(t)$, so we obtain

$${}^C D_t^\alpha \Delta T = -\Lambda \Delta T + v_\alpha(t) + F_2(t). \quad (29)$$

which can be seen to be closely related to Lovejoy and coworkers’ FEBE by comparison with, for example, Equations (13) and (14) of Ref. 16. The key difference is that we have a Caputo derivative rather than their Riemann-Liouville one. We postpone to future work the exploration of the implications of this choice in the model’s design. Importantly the $\alpha = 1, H = 1/2$ limit of both derivatives is of course the same, and gives the Hasselmann model as required. The determination of the order of the Caputo derivative empirically is an area of current research, both generally and in the specific climate context. It is directly related to the form of the response function of the system under study, if the Caputo model is itself applicable, and indirectly related to the power spectral density of the main observable, here ΔT . The specific relationship if any between the response function and the noise driving depends on the presence or absence of a fluctuation-dissipation relation.

We note that although the integral in the Caputo derivative is frequently started at $t_m = 0$ (including for example in its current implementation in *Mathematica*), its definition allows it to start anywhere on the real line¹⁰¹. It thus naturally encompasses the $t_m = -\infty$, ‘‘Weyl’’ case to which Lovejoy³³ has drawn attention in the climate context, without additional effort.

IX. SOLVING THE MORI-KUBO GLE AND THE FLE ANALYTICALLY.

In this section and the following one we have used the words ‘‘analytic’’, ‘‘numerical’’ and ‘‘solution’’ as is commonly done in the physics literature. The mathematics literature has different conventions and the reader is advised to beware !

A. General solution of the overdamped Mori-Kubo GLE

There are by now numerous sources in the literature for analytic solutions of the Mori-Kubo GLE, starting from near-contemporary work such as that of Fox⁸³. They typically use Laplace transform methods e.g. those of Ref. 86.

In our present application to climate it is important to note that solutions of the GLE have not been limited to assuming a starting time t_m of zero. A notable recent example where t_m has been more general is Ref. 55 whose authors studied an overdamped Mori-Kubo GLE

$$\int_{t_m}^t dt' K(t-t') \dot{x}(t') + kx(t) = F_2(t) + \xi(t),$$

which described a particle of mass M moving in a harmonic confining potential $V(x) = (1/2)kx^2$ (in their case an optical trap) and experiencing an additional deterministic forcing (from dragging of the trap) $F_2(t)$ and a stochastic forcing which they assumed to be white noise $\xi(t)$.

The authors obtained the general solution of their Mori-Kubo GLE as

$$x(t) = x_{t_m}(1 - k\chi(t-t_m)) + \int_{t_m}^t dt' \chi_x(t-t')[F_2(t') + \xi(t')] \quad (30)$$

where the x_{t_m} denoted the initial condition for position, $x(t=t_m)$. The position susceptibility $\chi_x(t)$ here was defined through its Laplace transform

$$\hat{\chi}_x(s) = [Ms^2 + s\hat{K}(s) + k]^{-1}$$

and the integral of it appearing in their solution was defined as

$$\chi(t) = \int_0^t dt' \chi_x(t').$$

The cases treated in Ref. 55 were relatively tractable, so they should in principle also allow some special cases of our proposed stochastic EBM to be solved analytically—a problem which we expect to return to in future work.

B. Solving the FLE by Green's functions

1. The overdamped FLE

For now, though, we will specialise to the fractional special case of the overdamped Mori-Kubo GLE, i.e. the overdamped FLE. We recall that this will specialise further for $\alpha = 1$ to the familiar exponential solutions of the Hasselmann model.

The overdamped FLE, both with and without deterministic forcing, has been studied in the mathematics and statistical physics literatures, e.g. in 2017 by Li et al in Ref. 102. With an Riemann-Liouville derivative rather than the Caputo type it has been studied by Lovejoy and co-workers as a stochastic climate model since 2019, and dubbed "the fractional energy balance equation".

The overdamped forced FLE

$$[{}_{t_m}^C D_t^\alpha + k] x(t) = F_2(t) + v_\alpha(t) \quad (31)$$

can be seen to have an entirely deterministic fractional operator on its left hand side, while the right hand side incorporates both the deterministic forcing term $F_2(t)$ and a stochastic term v_α .

2. The delta function-forced Caputo FDE and the Green's function of the FLE.

One way to solve the overdamped forced FLE is to first obtain the solution of the deterministic relaxation and oscillation fractional differential equation, i.e. the Caputo FDE, when forced by a δ -function:

$$[{}_{t_m}^C D_t^\alpha + k] x(t) = \delta(x) \quad (32)$$

which can be done using Laplace transforms e.g. Ref. 103.

With an appropriate initial conditions, this gives the impulse response of the fractional operator, which can then be used to construct a Green's function to solve either the deterministic or stochastic versions of the inhomogenous equation.

The Laplace transform solution of (32) for $t_m = 0$ and $x(0) = 0$ is

$$x(t) = t^{\alpha-1} E_{\alpha,\alpha}(-kt^\alpha)$$

as can be checked in version 13.1 or later of Mathematica using its CaputoD and DiracDelta functions.

Fig. 1 plots $x(t)$ for values of α from 1 to 1/2, and $k = 1$. On the semilog axes chosen we can see that the $\alpha = 1$ case is exponential while smaller values of α correspond to more slowly decaying tails in the Mittag-Leffler function $E_{\alpha,\alpha}(t)$ and to an increasingly singular "spike" at zero from the power law prefactor. It may be compared with Figs 2 and 3 of Ref. 16 which used a Riemann-Liouville derivative.

3. Step function forcing of the Caputo FDE and FLE.

If we then include the deterministic forcing $F_2(t)$, and take $t_m = 0, x(t_m) = x_0$ the solution becomes

$$x(t) = x_0 E_\alpha(-kt^\alpha) + \int_0^t F_2(\tau) (t-\tau)^{\alpha-1} E_{\alpha,\alpha}[-k(t-\tau)^\alpha] d\tau.$$

In the well studied case most relevant to climate science, when the forcing becomes a Heaviside step function applied from $t = 0$ onwards, i.e. $F(t) = \theta(t)$, the integral can be evaluated and the solution becomes

$$x(t) = x_0 E_\alpha(-kt^\alpha) + t^\alpha E_{\alpha,1+\alpha}[-kt^\alpha].$$

which again can be confirmed by Mathematica.

This function is illustrated in Fig. 2 which plots $x(t)$, again for values of α from 1 to 0.5 but now for $k = 0.1$.

The effect of memory for the cases where $\alpha < 1$ is clear in lengthening the time it takes for the solution to take its asymptotic value. In a simplest forcing experiment, such as doubling CO_2 , for a stochastic EBM this would correspond to the approach to a new equilibrium value of the GMT anomaly.

4. The overdamped FLE with arbitrary deterministic and stochastic forcing

We come finally to the full overdamped forced FLE (31), again with $t_m = 0$, when the stochastic term has also been included. This is too general to have a useful analytic solution, but we can make some comments to help elucidate its meaning.

When written in integral form the SDE has three parts. The first and second are those seen in the previous sections, while the third must be interpreted as a stochastic integral. We will not deal with this term, corresponding to $v_\alpha(t)$ in (31), in full generality but will instead consider the case of fractional Gaussian driving:

$$x(t) = x_0 E_\alpha(-kt^\alpha) + \int_0^t (t-\tau)^{\alpha-1} F_2(\tau) E_{\alpha,\alpha}[-k(t-\tau)^\alpha] d\tau + C_H \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}[-k(t-\tau)^\alpha] dB_H(\tau) \quad (33)$$

The above stochastic integral is best understood in the $H = 1/2$ case of white noise when it can be described by Itô calculus, although there is by now also a substantial mathematical literature concerning integrals with respect to fractional Brownian motion^{94,104} (fBm).

Furthermore, when we then specialise to $F_2 = 0$ and $\alpha = 1$ the solution then corresponds to the very well studied O-U process:

$$x(t) = x_0 \exp(-kt) + \int_0^t e^{-k(t-\tau)} dW(\tau)$$

The stochastic integral in this case can be defined as the limit of a sum and dW as a suitably rescaled increment of

FIG. 1. Impulse response of the fractional relaxation-oscillation equation for $\alpha = 1$ to 0.5.

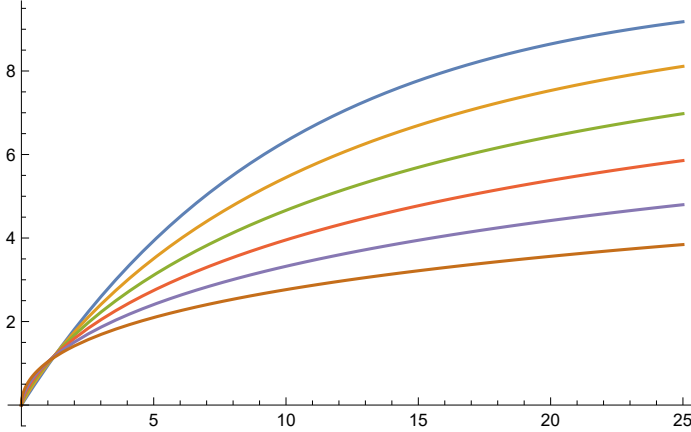


FIG. 2. Step response of the fractional relaxation-oscillation equation for $k = 0.1, \alpha = 1$ to 0.5.

Brownian motion,

$$\int_0^t e^{-k(t-\tau)} dW(\tau) = \lim_{\Delta \rightarrow 0} \sum_{j=0}^{n-1} e^{-k(j\Delta-\tau)} (W((j+1)\Delta) - W(j\Delta))$$

as described on pages 90-91 of Ref. 21.

X. NUMERICALLY EVALUATING TRAJECTORIES OF THE MORI-KUBO GLE AND FLE.

As is so often the case, the analytical intractability of the most physically interesting and relevant SDEs means that we must evaluate them numerically. We discuss this below, first for the Mori-Kubo GLE and then for its fractional special case.

A. Numerics for the Mori-Kubo GLE

There are by now several modern sources of numerical algorithm for the stochastic Mori-Kubo GLE e.g. Refs. 28, 29, 105, and 106.

The algorithm of Lim²⁸ requires that the system of SDEs to be solved must be ‘‘Bohl’’ i.e. the matrix elements of the memory kernel $K(t)$ are finite linear combinations of the functions of the form $t^k e^{\beta t} \cos(\omega t)$ and $t^k e^{\beta t} \sin(\omega t)$, where k is an integer and β and ω are real numbers. As Lim²⁸ notes, when these functions decay as a power law, for example, the resulting fractional Mori-Kubo GLE cannot be studied as a finite-dimensional SDE system. Although one can work formally in an infinite-dimensional setting, numerics will typically resort (e.g. Ref. 107) to approximating the power law, frequently by using a weighted sum of exponentials.

FIG. 3. Box plot of an ensemble of 10 sample paths of the overdamped fractional Langevin equation, forced by a Heaviside step function $\Theta(t)$ and a stochastic term of amplitude $\bar{\sigma} = 0.16$. The parameters were $t_m = 0, x(t_m) = 0, k = 0.1$, and $\alpha = 0.7$. Overplotted is the analytic solution, which can be seen to coincide well with the means of the ensemble.

B. Numerics for the overdamped FLE

The overdamped FLE can be solved in a number of ways. Lovejoy et al’s work^{15,16} on FEBE has so far used Fourier methods similar to those widely used to simulate fBm and its increment, fGn, and have focused on an initial condition specified at $t_m = -\infty$. In our paper we will instead consider finite t_m , which allows the infinitely distant case as a limit, and wish to use a non-Fourier based algorithm for its flexibility in potential future developments.

Our approach is illustrative, rather than rigorous. We use the explicit solution of the overdamped FLE and evaluate it at each required value of t . We do this by discretising t using the definitions of the stochastic and deterministic integrals as the limits of sums. However, we must stress that even in the O-U case it is known (e.g. Ref. 24) that this algorithm will **not** ensure that a solution will have the required autocorrelation properties on the solution. This is why our results must be considered illustrative rather than definitive for now.

In our future work, in order to achieve the required ACF, a time evolution equation rather than an integral will be needed. A prototype of such algorithms was derived by Gillespie^{24,108} for the O-U case. Candidates include the spectral method of Ref. 98, and the recently developed fast Euler-Maruyama algorithm given in Refs. 109 and 110.

Here we also restrict ourselves to the white noise-forced case, but with the additional deterministic forcing term, and intend to explore coloured noise forcing in future work.

We first approximate the stochastic integral using

$$\begin{aligned} & \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-k(t-\tau)^\alpha) dW(\tau) \\ & \approx \sum_{j=0}^{n-1} (t-j\Delta)^{\alpha-1} E_{\alpha,\alpha}(-k(t-j\Delta)^\alpha) (W[(j+1)\Delta] - W(j\Delta)) \\ & = \sum_{j=0}^{n-1} \Delta^{\alpha-1} (n-j)^{\alpha-1} E_{\alpha,\alpha}(-k\Delta^\alpha (n-j)^\alpha) \Delta^{1/2} N_{j\Delta}^{(j+1)\Delta} \end{aligned}$$

where, following Ref. 24 each finite difference of W has been replaced by the scaled Gaussian random number $\Delta^{1/2} N$.

The deterministic integral is approximated by

$$\begin{aligned} & \int_0^t F_2(\tau) (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-k(t-\tau)^\alpha) d\tau \\ & \approx \sum_{j=0}^{n-1} \Delta^{\alpha-1} (n-j)^{\alpha-1} E_{\alpha,\alpha}(-k\Delta^\alpha (n-j)^\alpha) \Delta \quad (34) \end{aligned}$$

where $F_2(\tau) = \theta(\tau)$ for the problem specified in equation. In Figure 3 we illustrate the behaviour of the FLE by plotting ten sample paths obtained this way for forcing by a Heaviside step function $\Theta(t)$ and a stochastic term of amplitude

FIG. 4. Box plot of 64 paths for ΔT from white noise fractional Caputo EBM driven by RCP 8.5 (shown in the inset). $\alpha = 0.7$, $\Lambda = 0.96$ and $\bar{\sigma} = 0.1$

$\bar{\sigma} = 0.16$. We used MATLAB's function `boxplot` where for each box, the central mark indicates the median, and the bottom and top edges of the box indicate the 25th and 75th percentiles, respectively. The whiskers extend to the most extreme data points not considered outliers, and the outliers are plotted individually using the '+' marker symbol. Parameters were $t_m = 0, x(t_m) = 0, k = 0.1, \alpha = 0.7$. The mean of the paths is overplotted and is seen to compare well with its known analytic result.

C. Numerics for our proposed Mori-Kubo GLE-type EBM driven by RCP 8.5.

We now briefly illustrate how our stochastic GLE-based EBM, in its fractional special case (29), can be used to study the global mean surface temperature. We do this in the white noise driven ($v_\alpha = \bar{\sigma}\xi$), fractional Caputo case by repeating the the previous subsection's approach, but this time taking the deterministic part of the driving from the RCP 8.5 Representative Concentration Pathway¹¹¹, a (worst case) model of anthropogenic forcing. Figure 4 is a box plot of 64 paths for ΔT , again with the choice $\alpha = 0.7$, and with the values $\Lambda = 0.96$ and $\bar{\sigma} = 0.1$ that were also chosen in Ref. 67. Further investigations will employ a better algorithm to solve the SDE, as discussed above.

XI. ARE THERE ALTERNATIVE STOCHASTIC GLE-BASED EBMS ?

Although this paper is focused on a stochastic EBM inspired by a mapping to the overdamped Mori-Kubo Langevin equation, other options exist, some of which have been explored so far in the literature.

A. Mori-Kubo GLE- based EBMs and other intrinsically stochastic approaches.

As well as Lovejoy et al's FEBE, discussed in some detail above, and the overdamped GLE-based EBM we have proposed here, there have been several other proposals that extend the Hasselmann formalism to include multiscale memory. We enumerate some below. As noted in the introduction, Ref.¹⁹ Ghil et al have argued that a key requirement of any such model is the ability to add quasi periodic determinism as well as a stochasticity.

1. Modified fractional Brownian motion

In several pioneering papers^{77,78} Rypdal and colleagues discussed various extensions of fractional Brownian motion as generalisations of the Hasselmann EBM. The main limitation of their idea was the absence of the damping term, as noted by Lovejoy^{16,33}.

2. Direct mapping to underdamped generalised Langevin equation rather than overdamped GLE.

In our earlier conference proceedings chapter³¹ and subsequent preprint³² we considered a different GLE-based EBM. Here the mapping was made from the underdamped Langevin equation rather than the overdamped Langevin equation.

This results in a mapping from velocity onto GMT anomaly, rather than from position. We erroneously thought that in its fractional special case our suggested stochastic EBM was isomorphic to FEBE after suitable conversions between their parameters, but in fact the comparisons made in previous sections of the present paper show that FEBE is much closer to the overdamped Langevin equation and thus to our newer proposal.

3. Two box models

The two box EBM is an interesting alternative which may be useful in some cases. It replaces the single decay scale of Hasselmann's model with two. This allows, for example, the atmosphere and ocean timescales to be explicitly represented.

As shown in Ref. 45 the two box model results in a second order SDE with constant coefficients, isomorphic to the underdamped Langevin equation for position.

Interestingly this second order equation is a also an alternative representation for the Mori-Kubo GLE when its memory kernel is of exponential form, as shown in Ref. 112, a special case of a more general result given in Ref. 106.

4. Multi box models

The above methods can be extended to the case of many relaxation time scales, an approach studied extensively for SDEs in the applied mathematics literature and also in climate science^{22,46}. Eigenvalue analysis can be used to predict some of the spectral properties of such multibox models, as done for example by Fredriksen and Rypdal⁴⁷

5. The fractional Ornstein-Uhlenbeck equation

The Langevin equation can be modified to be driven by fractional Gaussian noise rather than white noise, while still keeping the friction term constant. This has been studied in the mathematical finance literature extensively⁹⁴ and has been

suggested in the climate context¹¹³. These authors also develop a time series model^{113,114} based on the discrete time autoregressive fractionally integrated moving average ARFIMA process, which generalises AR(1).

6. Yuan et al's generalised stochastic time series model.

A fractional time series model, not simply isomorphic to ARFIMA or other existing models, has been proposed¹³, to systematically separate the “forcing-induced direct” and the “memory-induced indirect” trends in data.

B. Other stochastic Mori-Zwanzig GLE-based EBMs.

As noted in our introduction, the Mori-Kubo GLE is in principle a subset of the more complete Mori-Zwanzig formalism, which has at its heart a deterministic identity expressing the division of a system into observed and unobserved degrees of freedom²⁷. Space does not permit a full discussion of this here, but we refer the reader to a short introduction²³, and reviews of the method²¹ and its applications to climate⁸.

XII. FLUCTUATION-DISSIPATION RELATIONS

Fluctuation-dissipation relations (FDRs) have been studied extensively in climate science, see e.g. the reviews of Refs. 3 and 4 and recent examples like Ref. 115. We do not discuss them at length here, but simply note some aspects that relate to the stochastic EBMs we study.

A. Emergent parameters in the OU-process.

The meaning of the phrase “Fluctuation Dissipation Relation” can sometimes be as simple as its usage by the Nobel committee⁹, who noted that the variance $\langle y^2 \rangle$ of the OU process which solves the overdamped Langevin equation (9) is proportional to the ratio of the noise amplitude $\bar{\sigma}^2$ to the mean reversion term $\bar{\gamma}$. The Nobel document appears to us to be stressing that **any** system to which the O-U process has been applied must in consequence obey this limited kind of FDR. This is because there will then always be a ratio of $\sigma^2/2\bar{\gamma}$ and it will empirically determine the value of $\langle y^2 \rangle$. This is not a trivial observation in the climate context because either the value of $\langle y^2 \rangle$ or the noise amplitude $\bar{\sigma}$ are in this sense “emergent” parameters whose values are unknown *a priori*, c.f. the FDR-like relation found in Ref. 54.

B. The FDR in Brownian motion

In the physical context (the motion of a tracer particle in a fluid) from which the underdamped and overdamped Langevin equations originated^{4,24}, however, this ratio was not a free or empirically determined parameter. It was instead

constrained to be proportional to the temperature T of the surrounding heat bath. A proportionality to T is seen both in the Brownian motion special case of Kubo's “FDR2”^{89,90,116–118}, i.e. the ratio of the noise amplitude to the mean reversion (friction) term in the underdamped Langevin equation, and in a corresponding expression arising from the overdamped Langevin equation. Importantly these fluctuations are entirely “internal”, and the fast and slow terms in the Langevin equation arose from the same physics. This is not straightforwardly the case when an EBM is written down or derived for the climate problem, and extensive debates have occurred about the applicability or otherwise of the FDR in a climate context^{3,4}.

C. The FDR as seen in the Mori-Kubo GLE

As we noted above the Mori-Kubo GLE (here in its overdamped form)

$$\int_{t_m}^t dt' \gamma(t-t') \frac{dy(t')}{dt} = -ky + v(t) \quad (35)$$

was designed to embody an FDR which relates the covariant structure of the noise $v(t)$ to the auto-correlation structure of the kernel $\gamma(t-t')$, so that they are not free to vary independently. In the original context for the GLE, Brownian motion, this FDR is literally a fluctuation-dissipation relation because the $v(t)$ term is a fluctuating noise and the integral operator over $\gamma(t)$ is a friction term acting to reduce the magnitude of the velocity dy/dt .

If we naively take this functional form of the GLE over to the climate context, however, the integral operator is doing a physically different job. It is now an integral over a heat capacity $C_P(t-t')$ and so an “FDR” of the classic GLE type would be relating the covariance structure of the heat capacity to that of the noise $v(t)$.

This would still be doing essentially the same thing in the sense of balancing the noise and kernel, scale by scale, in order to allow the system to relax to a steady state, but further investigation would be necessary to establish whether it made physical sense.

XIII. CONCLUSIONS

Hasselmann's realisation that Brownian motion could be used as a mathematical superstructure to organise fast weather and slow climate fluctuations was a very powerful one, and his recent Nobel prize provided a timely context for our review. As we have shown, recent progress in going beyond his paradigm has generated what at first sight seem to be divergent positions, exemplified by the recent reviews of Franzke et al in Ref. 10 and Lovejoy Ref. 14.

In this paper we have sought to facilitate an improved dialogue between these views, and have found them to have more points of contact than was initially apparent. We first drew attention to the existing climate work using the Mori-Zwanzig approach, and proposed using the stochastic GLE of Mori and

Kubo as a framework to study memory in one dimensional stochastic EBMs. We then related the stochastic GLE-based EBM to Lovejoy's Fractional Energy Balance Equation, and concluded with some speculations on fluctuation dissipation relations.

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XV. DATA AVAILABILITY STATEMENT.

Data sharing is not applicable to this article as no new data were created or analyzed in this study

Appendix A: Summary of Fractional Calculus

Because of their relationship to long range dependence in time series and heavy tailed amplitude distributions, fractional differential equations (FDEs) and fractional stochastic differential equations (FSDEs) such as those in the present paper, have also been extensively studied in complexity science. In order to make the paper more accessible to readers who have

had little need to deal with fractional integration and differentiation, we give here a very brief summary of the results which we use. Fractional derivatives and the associated FDEs are now implemented directly in some numerical packages, like Mathematica whose demonstrations we encourage the reader to explore.

This Appendix follows the notational conventions of the physics literature. Readers seeking a more detailed but similarly physics-inspired treatment are referred to Ref. 119 and also the very useful general overview article of Ref. 120.

Readers who prefer a more mathematical development may consult Refs. 92, 93, 103, 121, and 122. In addition two books which study SDEs driven by fractional Brownian motion are Refs. 94 and 104.

1. Fractional integral

It is possible to begin by defining a fractional derivative but we follow most authors by starting with the inductive definition of integration, which the fractional integral generalises. We are used to the idea that the taking of a derivative of a function $f(t)$ with respect to time t is the inverse of integrating it with respect to t' up to the time t :

$$\int_a^t f(t') dt'. \quad (A1)$$

The n^{th} derivative is thus an inverse to the n -fold repeated integration:

$$\int_a^t dt'_1 \int_a^{t'_1} dt'_2 \dots \int_a^{t'_n} dt'_n f(t'_n) \quad (A2)$$

and requires us to use the integral identity:

$$\int_a^t \int_a^{t'_1} \dots \int_a^{t'_{n-1}} f(t'_n) dt'_n \dots dt'_1 = \frac{1}{(n-1)!} \int_a^t f(t')(t-t')^{n-1} dt' \quad (A3)$$

The form of the right hand side of this expression shows why we can then allow n to be replaced by a non-integer value α , thus defining an integral of fractional order. In defining a fractional integral we use the relationship between the factorial and the gamma function, $\Gamma(n) = (n-1)!$ the discovery of which was a consequence of the earliest work on fractional calculus¹²³.

The left-sided Riemann-Liouville fractional integral operator of order α on a finite interval (a, b) of the real line, acting on a function $f(t)$ is then

$$({}_a I_t^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t f(t')(t-t')^{\alpha-1} dt' \quad (A4)$$

Note that this definition requires that $\alpha > 0$. Typically $\alpha < 1$ is also assumed, but this is not obligatory¹⁰¹ Other notations exist for this integral e.g. $(I_{a+}^\alpha f)(x)$ used widely in the mathematics literature.

Taking $\alpha = 1$ recovers the usual integral of $f(t)$ from a to t ,

$$({}_a I_t^1 f)(t) = \int_a^t f(t') dt' \quad (A5)$$

whereas $\alpha = 1/2$ gives

$$({}_a I_t^{1/2} f)(t) = \frac{1}{\Gamma(1/2)} \int_a^t \frac{f(t')}{\sqrt{t-t'}} dt'. \quad (\text{A6})$$

which was one of the first fractional integrals to receive study in physics, chemistry, and engineering, notably in the heat transfer problem⁹¹.

Instead of being over a finite interval we can define the left-sided Riemann-Liouville fractional integral of $f(t)$ on the complete real line:

$$({}_{-\infty} I_t^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^t f(t') (t-t')^{\alpha-1} dt' \quad (\text{A7})$$

2. Riemann-Liouville Fractional derivative

We can now define fractional differentiation as the inverse of fractional integration. In practise this is done by taking a fractional integral of order 1 higher than the desired order of the derivative, and then using the usual integer derivative to lower its order by one. Because fractional integration and differentiation do not commute, two possible fractional derivatives exist, depending on whether the fractional integral of $f(t)$ is done first or the integer derivative. These are the Riemann-Liouville and Caputo cases respectively.

Taking the Riemann-Liouville case first, and considering the finite time interval (a, b) we define a derivative which is based on the inverse of the left-sided Riemann-Liouville integral. We use the fact that the Riemann-Liouville fractional derivative of the function $f(t)$ of order α , ${}_a D_t^\alpha f(t)$, is its Riemann-Liouville integral of order $-\alpha$. We calculate this by first writing the integral of order $1 - \alpha$, where $0 < \alpha < 1$:

$$({}_a I_t^{1-\alpha} f)(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t f(t') (t-t')^{-\alpha} dt' \quad (\text{A8})$$

and then differentiating:

$$({}_a I_t^{-\alpha} f)(t) = ({}_a D_t^\alpha f)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t f(t') (t-t')^{-\alpha} dt'. \quad (\text{A9})$$

The lower limit of this integral is usually taken to be 0, though this is not in fact essential¹⁰¹. As in the case of the fractional integral we can define the integral in the derivative over the whole real line, so a can be $-\infty$. This case is sometimes referred to as the Weyl or Riemann-Weyl derivative.

3. Caputo Fractional derivative

If instead we take the integer derivative first, before doing the fractional integral, we can define the Caputo fractional derivative:

$$({}_a^C D_t^\alpha f)(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t dt' (t-t')^{-\alpha} \frac{df}{dt}(t'). \quad (\text{A10})$$

The Caputo derivative can be defined via the Riemann-Liouville derivative:

$$({}_a^C D_t^\alpha f)(t) = ({}_a D_t^\alpha f)(t) - \frac{t^{-\alpha}}{\Gamma(-\alpha)} f(t), \quad (\text{A11})$$

so the Caputo and Riemann-Liouville fractional derivatives will coincide for negative orders α . One advantage of the Caputo definition over the Riemann-Liouville case is that it uses the values of $f(t)$ and its integer rather than fractional derivatives at 0 (or, in general, at any lower-limit point a). This is well suited to solving fractional-order initial-value problems using Laplace transforms, a popular approach for both FDEs and FSDEs.

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