Source Enumeration Utilizing Adaptive Diagonal Loading and Linear Shrinkage Coefficients

Ye Tian, *Member, IEEE,* Zhicheng Zhang, Wei Liu, *Senior Member, IEEE*,
Hua Chen, *Senior Member, IEEE*, and Gang Wang, *Senior Member, IEEE*

***Abstract*—Source enumeration is typically studied under the assumption of white noise, which may not be suitable for real-world applications. In this article, a source enumeration algorithm robust against both white and colored noises is presented, where the adaptive diagonal loading (ADL) technique combined with linear shrinkage (LS) coefficients are employed. First, a proper loading level is adaptively determined under the presumed number of sources v, which is then applied to mitigate correlated and heterogeneous noise, and simultaneously yields an encouraging result that the loaded eigenvalues match the asymptotic ratio behavior of the eigenvalues obtained under white noise. Then, by analyzing the characteristics of the loaded LS coefficients in various situations, source enumeration is successfully achieved by performing the second-order difference (SOD) operation on the loaded LS coefficients. The proposed algorithm does not require iteration or assume any *prior* information on noise, and therefore achieves high efficiency and robust enumeration results.**

***Index Terms*—Source enumeration, adaptive diagonal loading, linear shrinkage coefficient, second-order difference operation, random matrix theory.**

I. Introduction

Source enumeration is a critical problem in many array signal processing applications, such as multiple-input multiple-output (MIMO) wireless communications [1], [2], electroencephalography (EEG) [3], functional magnetic resonance imaging (fMRI) [4], sonar and radar systems [5]–[7], and normally regarded as an important input parameter for the following direction finding and localization processes (such as spatio-temporal dipole fitting and multiple signal classification, etc).

Many source enumeration algorithms have been developed based on different criterions, such as the Akaike information criterion (AIC) [8], the Bayesian information criterion (BIC) [9], the minimum description length (MDL) criterion [10], the multichannel time series based AIC and/or MDL criteria [11], the reduced-rank MDL estimation (MDLE) criterion [12], the minimum mean square error (MMSE)-based MDL (mMDL) criterion [13], the linear shrinkage based MDL (LS-MDL) criterion [14], the heuristic shrinkage coefficient ($\mathrm{SCD}_{\mathrm{heur}}$) based criterion [15], and the recently proposed two-step difference (TSD) criterion [16], etc. It should be noticed that the LS coefficient based enumerators in [15] and [16] have a robust performance in some difficult situations, such as a large number of sources and a small number of samples. Therefore, they hold great potential in more practical applications. Nevertheless, most of them are based on the assumption of additive white noise, and exploit the resulting multiplicity of the smallest eigenvalues to determine the number of signals; as a result, they are sensitive to any deviation from the white noise model, and thus may not perform consistently with data corrupted by unknown noise.

In practice, due to the application of various filters as well as different hardware configurations in receiving channels, correlated and heterogeneous noise models are more appropriate [17]. To handle such noise models, several algorithms have been proposed in recent years. In [18], an alternative detection criterion by deriving a new likelihood function was proposed for a nonuniform noise environment, where a proper transformation of the covariance matrix was applied to cope with the spatial non-uniformity of the noise. Through building a novel MDL objective function in the presence of unknown nonuniform noise, a global MDL minimization-based source enumeration algorithm was introduced in [19], which can yield a good estimate in both uniform and nonuniform noise cases. In [20], the Gerschgorin disk estimator (GDE) developed from the projection concept was presented, where the unitary transformation of the covariance matrix was employed, with robustness against unknown noise models. In [21], a modified GDE, called GDEWE, was proposed, which achieved source number estimation without eigendecompositon, and therefore computationally efficient. Different from [20] and [21], Eguizabal and Lameiro *et al*. modeled source enumeration as a regression problem, and then applied the information-theoretic criteria (ITC) to determine the model order of the regression [22]. Such an approach is suitable for colored noise, provided that the noise is sufficiently weaker than the signal. According to a non-asymptotic goodness-of-fit metric, a signal subspace matching (SSM) based algorithm was presented in [23], which is applicable to both white and colored noise, and to a very small number of samples. Based on the SSM, a modified version termed as invariant SSM (ISSM) was proposed in [24], which is aimed at matching a pair of translation invariant subspace and subsequently achieves an improved detection performance. Instead of applying SSM and ISSM, signal sub-

Manuscript received xx xx, 2023. This work was supported in part by the Zhejiang Provincial Natural Science Foundation of China under Grant LY23F010004, in part by the National Natural Science Foundation of China under Grants 62001256, 62222109, and in part by the UK Engineering and Physical Sciences Research Council (EPSRC) under grant EP/V009419/2.

Y. Tian, Z. Zhang, H. Chen and G. Wang are with the Faculty of Electrical Engineering and Computer Science, Ningbo University, Ningbo 315211, China (e-mail: tianfield@126.com; 1753240089@qq.com; dkchenhua0714@hotmail.com; wanggang@nbu.edu.cn).

W. Liu is with the School of Electronic Engineering and Computer Science, Queen Mary University of London, U.K. (e-mail: w.liu@qmul.ac.uk).

space projection (SSP) and eigen-subspace projection (ESP) were introduced in [25], where a good source enumeration performance for both white and colored noise scenarios was achieved. In addition, as verified by numerical experiments in [16], although TSD is proposed based on white noise, it has a reasonable performance under the case of banded colored noise, which is attributed to its full utilization of the distance between the signal and noise LS coefficients, as well as the distinct second-order difference feature of LS coefficients. Unfortunately, it fails to keep detection performance consistent with probability one.

In this article, a new source enumeration algorithm to cope with both white and non-white noises is proposed, where the asymptotic behavior of eigenvalues of the sample covariance matrix (SCM) in random matrix theory is exploited. Such an algorithm can be regarded as a modified version of TSD. By applying the linear shrinkage coefficient and developing the adaptive diagonal loading (ADL) technique properly, the detection stability and consistence of the proposed solution are guaranteed under various array configuration conditions and different types of noise. The main contributions of this work are summarized as follows:

- An ADL technique is developed to mitigate the problem of eigenvalue dispersion under unknown noise, which gives a clear guideline on how to choose the diagonal loading (DL) level reliably in comparison with the conventional DL [26]–[28], and subsequently exhibits improved robustness against different noise scenarios. To the best of our knowledge, it is the first time to adopt such a technique to solve the source enumeration problem.
- It is proved that the loaded eigenvalues obtained from ADL can match well with the asymptotic ratio behavior of the eigenvalues in white noise. With this specific characteristic, an improved source enumeration performance is achieved with enhanced flexibility.
- The asymptotic behavior of the loaded LS coefficients is analyzed in detail. With the aid of this asymptotic behavior and the ADL technique, the second-order difference (SOD) operation on loaded LS coefficients is exploited to achieve satisfactory source enumeration results.
- Extensive simulations are provided to show the effectiveness and superiority of the proposed algorithm in terms of the probability of correct detection (PCD) in different array configurations, different incident source powers and various noise scenarios.

The rest of the article is organized as follows. The problem formulation is presented in Section II. The developed ADL technique, the asymptotic characteristic analysis of the loaded LS coefficients, and the algorithm design are introduced in Section III. Simulation results are provided in Section IV, and conclusions are drawn in Section V.

II. Problem Formulation

Consider d narrowband source signals impinging on an array of m sensors from distinct directions $\{\theta_i\}_{i=1}^{d}$. The $m \times 1$ vector of the complex envelopes of the signals received by the array at time observation t can be written as

$$\mathbf{x}(t) = \mathbf{A}(\boldsymbol{\theta})\mathbf{s}(t) + \mathbf{w}(t), \quad t = 1, \ldots, n \tag{1}$$

where $\mathbf{A}(\theta) = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_d)]$ is the $m \times d$ array manifold matrix with its i-th column representing the steering vector of source i; $\mathbf{s}(t) = [s_1(t), \ldots, s_d(t)]^T$ and $\mathbf{w}(t) = [w_1(t), \ldots, w_m(t)]^T$ stand for the $d \times 1$ signal vector and $m \times 1$ noise vector, respectively, and $(\cdot)^T$ is the transpose operation. Here, it is assumed that the signals are uncorrelated with noise $\mathbf{w}(t)$ and independent and identically distributed (IID) complex Gaussian, i.e., $\mathbf{s}(t) \sim \mathcal{CN}(\mathbf{0}_d, \boldsymbol{\Sigma}_s)$, where $\mathbf{0}_d$, $\boldsymbol{\Sigma}_s$ and $\mathcal{CN}(\mathbf{0}_d, \boldsymbol{\Sigma}_s)$ represent the $d \times 1$ zero vector, $d \times d$ full-rank diagonal matrix and complex Gaussian distribution with mean $\mathbf{0}_d$ and covariance $\boldsymbol{\Sigma}_s = \mathbb{E}\{\mathbf{s}(t)\mathbf{s}(t)^H\}$, respectively, $\mathbb{E}\{\cdot\}$ denotes the statistical expectation operation, and $(\cdot)^H$ the conjugate transpose operation.

Based on (1) and above assumptions, the array covariance matrix is given by

$$\mathbf{R}_x = \mathbb{E}\{\mathbf{x}(t)\mathbf{x}(t)^H\} = \mathbf{A}\boldsymbol{\Sigma}_s\mathbf{A}^H + \mathbf{W} \tag{2}$$

where $\mathbf{W} = \mathbb{E}\{\mathbf{w}(t)\mathbf{w}(t)^H\}$ represents the unknown and arbitrary zero-mean noise covariance matrix. It can be spatially white noise with a diagonal covariance matrix having equal elements, nonuniform noise with a diagonal covariance matrix having non-equal elements [29], [30] or any other colored noise with a banded covariance matrix [24], [31], [32].

In practice, we can only obtain the sample covariance matrix (SCM) of $\mathbf{x}(t)$, which is used to replace $\mathbf{R}_x$ and calculated with n samples by

$$\hat{\mathbf{R}}_x = (1/n)\sum\nolimits_{t=1}^{n} \mathbf{x}(t)\mathbf{x}(t)^H. \tag{3}$$

According to [15], if $\mathbf{w}(t)$ is spatially white, $\mathbf{W}$ will reduce to $\sigma\mathbf{I}_m$, and the sample eigenvalues $l_1, \ldots, l_m$ of $\hat{\mathbf{R}}_x$ under the framework of general asymptotic theory (where $m, n \to \infty$ and $m/n = c \in (0, \infty)$) will follow

$$l_1 \geq \cdots \geq l_d \geq l_{d+1} \geq \cdots \geq l_m \tag{4}$$

$$(m-d)\left(\begin{bmatrix} \frac{1}{m-d}\sum\limits_{i=d+1}^{m} l_i \\ \frac{1}{m-d}\sum\limits_{i=d+1}^{m} l_i^2 \end{bmatrix} - \begin{bmatrix} \sigma \\ \sigma^2(1+c) \end{bmatrix}\right) \xrightarrow{\mathcal{D}} \mathcal{N}(\mathbf{0}_2, \mathbf{D}) \tag{5}$$

with

$$\mathbf{D} = \begin{bmatrix} \sigma^2 c & 2\sigma^3 c(c+1) \\ 2\sigma^3 c(c+1) & 2\sigma^4 c(2c^2+5c+2) \end{bmatrix} \tag{6}$$

$\xrightarrow{\mathcal{D}}$ denoting convergence in distribution, σ the noise variance, and $\mathbf{I}_m$ the $m \times m$ identity matrix. Such asymptotic behaviors indeed provide fundamental conditions for many source enumerators, such as LS-MDL, $\mathrm{SCD}_{\mathrm{heur}}$ and TSD. Unfortunately, when the noise is non-white, these asymptotic behaviors will no longer hold, and may result in failed detection for those enumerators. In what follows, we will first demonstrate how to exploit our designed technique to match a special asymptotic ratio behavior to the maximum extent possible for non-white noise, and subsequently propose a novel source enumerator with significantly improved performance.

III. Proposed Algorithm

In this section, we first provide the definition of the LS coefficients, then present the ADL technique and a new expression for the signal LS coefficient after applying the designed ADL technique, and finally we combine them together to introduce our proposed algorithm.

A. LS Coefficient

The LS technique is an efficient tool for robust covariance matrix estimation [33]–[35] and reliable detection of the source number in the general asymptotic theory framework, which is quite suitable for high-dimensional covariance estimation and scenarios where the number of sensors is large and comparable with the number of samples [14]–[16].

The aim of LS is to shrink SCM or its corresponding subspace covariance matrix towards a target matrix ("unstructured") or a scaled identity matrix ("structured"), employing an appropriate LS coefficient that is asymptotically optimal for any distribution. For source enumeration under the IID Gaussian assumption of observations, the structured form is more appropriate [15]. Let $\mathbf{R}^{(v)}$, $\mathbf{\Sigma}_{\mathcal{N}}^{(v)} = \operatorname{diag}(\lambda_{v+1}, \ldots, \lambda_m)$, $\mathbf{S}_{\mathcal{N}}^{(v)} = \operatorname{diag}(l_{v+1}, \ldots, l_m)$, and $\mathbf{F} = [\operatorname{tr}\left(\mathbf{\Sigma}_{\mathcal{N}}^{(v)}\right)/(m-v)]\mathbf{I}_{m-v}$ denote the LS oracle estimate, unbiased noise subspace covariance matrix (NSCM), sample NSCM and shrinkage target under v presumed sources, respectively. Consequently, the oracle estimate $\mathbf{R}^{(v)}$ is the solution to the following constrained minimization of the mean square error (MSE) [33]:

$$\begin{aligned} &\min_{\alpha^{(v)}} g\left(\alpha^{(v)}\right) \triangleq \mathbb{E}\left[\left\|\mathbf{R}^{(v)} - \mathbf{\Sigma}_{\mathcal{N}}^{(v)}\right\|_F^2\right] \\ &\text{s.t. } \mathbf{R}^{(v)} = \alpha^{(v)}\mathbf{F} + \left(1-\alpha^{(k)}\right)\mathbf{S}_{\mathcal{N}}^{(v)} \end{aligned} \tag{7}$$

where $\alpha^{(v)} \in [0,1]$ is the shrinkage coefficients, $\mathbf{I}_{m-v}$ the $(m-v)\times(m-v)$ identity matrix; $\|\cdot\|_F$ and $\operatorname{tr}(\cdot)$ denote the Frobenius norm and the trace of a matrix, respectively.

Setting the derivative of $g\left(\alpha^{(v)}\right)$ to zero, the oracle estimate of $\alpha^{(v)}$ is calculated by

$$\begin{aligned} \alpha_o^{(v)} &= \frac{\mathbb{E}\left[\operatorname{tr}\left(\mathbf{F}-\mathbf{S}_{\mathcal{N}}^{(v)}\right)\left(\mathbf{\Sigma}_{\mathcal{N}}^{(v)}-\mathbf{S}_{\mathcal{N}}^{(v)}\right)^H\right]}{\mathbb{E}\left[\left\|\mathbf{S}_{\mathcal{N}}^{(v)}-\mathbf{F}\right\|_F^2\right]} \\ &= \frac{\mathbb{E}\left[\operatorname{tr}\left(\mathbf{S}_{\mathcal{N}}^{(v)}\left(\mathbf{S}_{\mathcal{N}}^{(v)}\right)^H\right)\right]-\operatorname{tr}\left(\mathbf{\Sigma}_{\mathcal{N}}^{(v)}\left(\mathbf{\Sigma}_{\mathcal{N}}^{(v)}\right)^H\right)}{\mathbb{E}\left[\operatorname{tr}\left(\mathbf{S}_{\mathcal{N}}^{(v)}\left(\mathbf{S}_{\mathcal{N}}^{(v)}\right)^H\right)\right]-\frac{1}{m-v}\left(\operatorname{tr}\left(\mathbf{\Sigma}_{\mathcal{N}}^{(v)}\right)\right)} \\ &= \frac{\mathbb{E}\left[\frac{1}{m-v}\sum_{i=v+1}^m l_i^2\right]+\frac{1}{m-v}\sum_{i=v+1}^m \lambda_i^2}{\mathbb{E}\left[\frac{1}{m-v}\sum_{i=v+1}^m l_i^2\right]-\left(\frac{\sum_{i=v+1}^m \lambda_i}{m-v}\right)^2}. \end{aligned} \tag{8}$$

For Gaussian observations, it can be derived according to the theoretical result in [36] that the consistent estimate of $\alpha_o^{(v)}$ can be replaced by

$$\hat{\alpha}_c^{(v)} = \frac{\frac{1}{m-v}\sum_{i=v+1}^m l_i^2 + (m-v)\left(\frac{1}{m-v}\sum_{i=v+1}^m l_i\right)^2}{(n+1)\left(\frac{1}{m-v}\sum_{i=v+1}^m l_i^2 - \left(\frac{\sum_{i=v+1}^m l_i}{m-v}\right)^2\right)}. \tag{9}$$

Given the fact that $\hat{\alpha}^{(v)}$ can be larger than one, it is normally replaced by $\hat{\alpha}^{(v)} = \min\left(\hat{\alpha}_c^{(v)}, 1\right)$ in practice. In what follows, we demonstrate in detail how to exploit the LS coefficient for robust source enumeration in various noise scenarios.

B. ADL Technique

The conventional DL technique is known as an effective approach to solve robust inversion problem of the low-rank or ill-conditioned SCM, and widely adopted in the field of beamforming [37] and covariance matrix reconstruction in hybrid analog-digital architectures [38]–[40]. Through adding a scaled identity matrix on the diagonal of the original SCM, one can not only improve the condition number of SCM and avoid ill-conditioned solutions, but also reduce the normalized variance of noise eigenvalues. Such properties provide a great potential for eigenvalue-based source enumeration in non-white noise. However, it should be emphasized that most of existing DL is designed by a user-defined constant loading value, which is only tailored for some certain scenarios. As a result, its flexibility and generalization ability are inadequate, and limit its application in non-white noises.

Instead of employing the conventional DL technique, the ADL technique established on the following proposition under the random matrix theory is investigated.

Proposition 1: Let $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_m$ represent the eigenvalues of the SCM $\hat{\mathbf{R}}_x$ in descending order under unknown noise, and $\frac{1}{m-v}\left(\sum_{i=v+1}^m \tilde{\lambda}_i\right) = \tau_v$. Further assume that $m-v \to m$ as $m, n \to \infty$, $m/n \to c$. Then, by selecting the positive loading level ϱ_v satisfying the following condition:

$$\varrho_v = -\tau_v + \sqrt{\tau_v^2 + \frac{\sum_{i=v+1}^m \tilde{\lambda}_i^2 - (m-v)\tau_v^2(1+c)}{(m-v)c}} \tag{10}$$

the ratio behavior of loaded eigenvalues $\bar{\lambda}_1, \ldots, \bar{\lambda}_m$ will match the ratio behavior in white noise, i.e.,

$$(m-v)\frac{\sum_{i=v+1}^m \bar{\lambda}_i^2}{\left(\sum_{i=v+1}^m \bar{\lambda}_i\right)^2} = (1+c). \tag{11}$$

Proof: According to (5), the ratio behavior of eigenvalues in white noise satisfies

$$(m-d)\frac{\sum_{i=d+1}^m l_i^2}{\left(\sum_{i=d+1}^m l_i\right)^2} \to (1+c). \tag{12}$$

By exploiting the ADL technique with loading level ϱ_v in non-white noise, the loaded eigenvalues $\bar{\lambda}_i$ can be expressed as $\bar{\lambda}_i = \tilde{\lambda}_i + \varrho_v$. If ϱ_v satisfies the condition in (10) under the presumed source number v, it can be further derived that

$$\sum_{i=v+1}^m \bar{\lambda}_i^2 = (m-v)(\tau_v+\varrho_v)^2(1+c) \tag{13}$$

which directly yields

$$(m-v)\frac{\sum_{i=v+1}^{m}\bar{\lambda}_i^2}{\left(\sum_{i=v+1}^{m}\bar{\lambda}_i\right)^2}=(1+c)\,. \tag{14}$$

Next, we prove $\varrho_v>0$. To this end, we only need to concentrate on the analysis of $\sum_{i=v+1}^{m}\tilde{\lambda}_i^2-(m-v)\tau_v^2(1+c)$. It can be observed that $(m-v)\tau_v^2(1+c)$ can be regarded as the quadratic sum of $\sum_{i=k+1}^{m}l_i^2$ in white noise under v sources. Without loss of generality, it is assumed that the eigenvalues under white noise and non-white noise have the same mean value or expectation, and we then have

$$\sum_{i=v+1}^{m}\tilde{\lambda}_i^2-\sum_{i=v+1}^{m}l_i^2=(m-v)\left[\mathbb{D}\left(\tilde{\lambda}_i\right)-\mathbb{D}\left(l_i\right)\right] \tag{15}$$

where $\mathbb{D}(\zeta)$ stands for the variance or degree of dispersion of eigenvalue ζ. As $\mathbb{D}\left(\tilde{\lambda}_i\right)>\mathbb{D}(l_i)$, we have $\sum_{i=v+1}^{m}\tilde{\lambda}_i^2-(m-v)\tau_v^2(1+c)>0$ and $\varrho_v>0$. ■

Remark 1: It can be seen that the loading level is adaptively determined once the SCM $\hat{\mathbf{R}}_x$, m,n and v are given, which provides a clear guideline and great flexibility on choosing ϱ_v properly. Moreover, it is worth emphasizing that the ratio behavior in (11) is of great importance for subsequent LS-based source enumerator, which provides an observable condition to distinguish signal LS coefficients from noise LS coefficients.

Remark 2: It is necessary to point out that we aim to make any presumed source number v meet the requirement of (11) for the proposed estimator, which is different from the case of white noise where only d is employed. In other words, we here form a set of hypothesized source number v to replace d; under a certain v, we further analyze the asymptotic behavior of the previous $k(\leq v)$ loaded LS coefficients, where k is defined as an auxiliary parameter. In what follows, we will first show that if the SOD of the loaded LS coefficients satisfies a special inequality condition (i.e., (41) in section III-D) for a given v and any $k\leq v$, it can be inferred that $v\leq d$; and then, according to this characteristic, we propose the new source enumerator.

C. Characteristic of Loaded LS Coefficients

In this section, we first introduce two special parameters α_v^k and γ_v^k, and then analyze their asymptotic characteristics. Different from the definitions of α^k and γ_k in [14]–[16], as well as $\alpha^{(v)}$ above, here α_v^k and γ_v^k are calculated by the loaded eigenvalues with the presumed source number v and auxiliary parameter k. The relationship between α_v^k and γ_v^k is expressed as

$$\alpha_v^k=\frac{m-k+\gamma_v^k}{(n+1)\left(\gamma_v^k-1\right)} \tag{16}$$

where

$$\gamma_v^k=(m-k)\frac{\sum_{i=k+1}^{m}\bar{\lambda}_i^2}{\left(\sum_{i=k+1}^{m}\bar{\lambda}_i\right)^2},\ k,v\leq m-1. \tag{17}$$

According to Proposition 1, it can be further derived for $k\leq v$ that

$$\begin{aligned}\gamma_v^k&=(m-k)\frac{\left[\sum_{i=k+1}^{v}\bar{\lambda}_i^2+\sum_{i=v+1}^{m}\left(\bar{\lambda}_i\right)^2\right]}{\left[\sum_{i=k+1}^{v}\bar{\lambda}_i+(m-v)\left(\tau_v+\varrho_v\right)\right]^2}\\&=(m-k)\frac{\sum_{i=k+1}^{v}\bar{\lambda}_i^2}{\left[\sum_{i=k+1}^{v}\bar{\lambda}_i+(m-v)\left(\tau_v+\varrho_v\right)\right]^2}\\&+(m-k)\frac{(m-v)\left(\tau_v+\varrho_v\right)^2(1+c)}{\left[\sum_{i=k+1}^{v}\bar{\lambda}_i+(m-v)\left(\tau_v+\varrho_v\right)\right]^2}.\end{aligned} \tag{18}$$

With (18), we give the following proposition 2.

Proposition 2: For large-scale antenna arrays and a sufficiently high signal-to-noise ratio (SNR), where signal eigenvalues $\tilde{\lambda}_k\simeq\mathcal{O}(m)\gg\tau_v$ [16], if the difference between the two sorted signal eigenvalues $\tilde{\lambda}_k$ and $\tilde{\lambda}_v$ is not significant, or precisely $\tilde{\lambda}_v\leq\tilde{\lambda}_k\leq\sqrt{\kappa}\tilde{\lambda}_v$, where $\kappa=2+2\frac{v-k}{d-v}$, and $\simeq$ implies that the values of its two segments are of the same order or comparable, then, the following inequality holds

$$2\left(\alpha_v^v-\alpha_v^{v-1}\right)>\alpha_v^k-\alpha_v^{k-1},k\leq v<d. \tag{19}$$

Proof: For scenarios of $v<d$ and signal eigenvalues $\tilde{\lambda}_k\simeq\mathcal{O}(m)\gg\tau_v$, it can be regarded that they dominate all eigenvalues. By ignoring the term τ_v in ϱ_v, we have

$$\varrho_v\approx\sqrt{\frac{\sum_{i=v+1}^{m}\tilde{\lambda}_i^2}{(m-v)c}}\approx\sqrt{\frac{\sum_{i=v+1}^{d}\tilde{\lambda}_i^2+\sum_{i=d+1}^{m}\tilde{\lambda}_i^2}{mc}}. \tag{20}$$

Define $\sum_{i=v+1}^{d}\tilde{\lambda}_i^2=(d-v)\breve{\lambda}$, and $\sum_{i=d+1}^{m}\tilde{\lambda}_i^2=\breve{c}\,(d-v)\,\breve{\lambda}$, where $\breve{c}$ is a very small constant under the scenarios of large-scale arrays and high SNRs. Then, (20) can be written as

$$\varrho_v\approx\sqrt{\left(1+\breve{c}\right)(d-v)}\sqrt{\breve{\lambda}/mc}=c_1\sqrt{\breve{\lambda}/mc} \tag{21}$$

where $c_1=\sqrt{\left(1+\breve{c}\right)(d-v)}\approx\sqrt{d-v}$.

Noting that $m-k,m-v\to m$ as $m\to\infty$. Subsequently, according to the derivation in Appendix, it can be obtained for $k<v$ that

$$\alpha_v^k\to\frac{c}{\left(c+\sum_{i=k+1}^{v}c_i^2\right)} \tag{22}$$

where $c_i=\frac{\sqrt{c}\tilde{\lambda}_i}{c_1\sqrt{\breve{\lambda}}}$.

For $k=v\geq 2$, it can be obtained from (17) that $\gamma_v^v=1+c$, which directly yields

$$\alpha_v^v=\frac{m-v+2+c}{c(n+1)}=\frac{m-v+2+c}{m+c}\prec 1,v\geq 2, \tag{23}$$

where $a\prec b$ represents that $a<b$ but strictly $a\approx b$.

Subsequently, it can be derived that

$$\begin{aligned}\tau&=2\left(\alpha_v^v-\alpha_v^{v-1}\right)-\alpha_v^k-\alpha_v^{k-1}\\&\to 2\left(1-\frac{c}{c+c_v^2}\right)-\left(\frac{c}{c+\sum_{i=k+1}^{v}c_i^2}-\frac{c}{c+\sum_{i=k}^{v}c_i^2}\right)\\&=\frac{2c_v^2}{c+c_v^2}-\frac{cc_k^2}{\left(c+\sum_{i=k+1}^{v}c_i^2\right)\left(c+\sum_{i=k}^{v}c_i^2\right)}.\end{aligned} \tag{24}$$

Define $\varsigma = \left(c + \sum_{i=k+1}^{v} c_i^2\right)\left(c + \sum_{i=k}^{v} c_i^2\right)$, and then (24) can be simplified as

$$\tau \to \frac{2c_v^2\varsigma - cc_k^2\left(c + c_v^2\right)}{\left(c + c_v^2\right)\varsigma}. \tag{25}$$

Since $\tilde{\lambda}_v \leq \tilde{\lambda}_k \leq \sqrt{\kappa}\tilde{\lambda}_v$, it can be derived according to the expression of c_i that

$$c_v^2 \leq c_k^2 \leq \kappa c_v^2 \tag{26}$$

$$c_v^2 \to \frac{c\tilde{\lambda}_v^2}{(d-v)\,\widehat{\lambda}} \geq \frac{c}{d-v}. \tag{27}$$

Based on (26) and (27), and the fact that

$$\begin{aligned}\varsigma &\geq \left[c + (v-k)\,c_v^2\right]\left[c + (v-k+1)\,c_v^2\right] \\ &> \left[c + (v-k)\,c_v^2\right]^2 \geq \left[1 + \frac{v-k}{d-v}\right]^2 c^2\end{aligned} \tag{28}$$

$$c + (v-k+1)\,c_v^2 \geq c + c_v^2 \tag{29}$$

we have

$$\begin{aligned}\tau &\geq \frac{c_v^2}{\left(c + c_v^2\right)\varsigma}\left[2\varsigma - \kappa c\left(c + c_v^2\right)\right] \\ &> \frac{c_v^2\left(c + c_v^2\right)c}{\left(c + c_v^2\right)\varsigma}\left(2\left[1 + \frac{v-k}{d-v}\right] - \kappa\right) = 0.\end{aligned} \tag{30}$$

That is, $2\left(\alpha_v^v - \alpha_v^{v-1}\right) > \alpha_v^k - \alpha_v^{k-1}$ for $k \leq v < d$. ■

Remark 3: It should be noticed that $\kappa = 2 + 2\frac{v-k}{d-v}$ is a tight condition for proposition 2. Since some constraints were relaxed during the above derivation process, the value of κ can be larger in practice. In addition, it is seen that κ increases with the decrease of k for given v and d, which conforms to the characteristics of the distribution of eigenvalues. On the other hand, it is worth emphasizing that the value of κ also reflects the requirement for the powers of the incident signals, if the signals are of equal power or the difference between signal powers is not significant, (19) holds almost surely as shown later by simulations, which provides a significant property for the subsequent difference based source enumerator.

Proposition 2 discusses the situation for $v < d$, the following proposition 3 further discusses the situation for $v \geq d$.

Proposition 3: For $d \leq k \leq v$, $\gamma_v^{k-1} - \gamma_v^k \ll c$ and $\alpha_v^k - \alpha_v^{k-1} \to 0$, provided that $m-v, m-k \to m$, and for $k \leq d-1$, $\alpha_v^k \to 0$.

Proof: For $m - v \to m$, we have

$$\begin{aligned}\varrho_v &\to -\tau_v + \sqrt{\tau_v^2 + \frac{(m-v)(\tau_v^2 + \mathbb{D}(\tilde{\lambda})) - m\tau_v^2(1+c)}{(m-v)c}} \\ &= -\tau_v + \sqrt{\mathbb{D}(\tilde{\lambda})/c} = -\tau_v + \sqrt{\xi/c}\end{aligned} \tag{31}$$

where $\xi = \mathbb{D}(\tilde{\lambda})$ is a constant represents the variance corresponding to $m - v$ small eigenvalues for given observed data.

Subsequently, it can be derived that

$$\begin{aligned}&\gamma_v^{k-1} - \gamma_v^k \\ &= (m-k+1)\frac{\sum_{i=k}^m \bar{\lambda}_i^2}{\left(\sum_{i=k}^m \bar{\lambda}_i\right)^2} - (m-k)\frac{\sum_{i=k+1}^m \bar{\lambda}_i^2}{\left(\sum_{i=k+1}^m \bar{\lambda}_i\right)^2} \\ &= \frac{\sum_{i=k}^m \bar{\lambda}_i^2}{(m-k+1)\left(\tau_v + \varrho_v\right)^2} - \frac{\sum_{i=k+1}^m \bar{\lambda}_i^2}{(m-k)\left(\tau_v + \varrho_v\right)^2}\end{aligned} \tag{32}$$

which directly yields

$$\begin{aligned}\gamma_v^{k-1} - \gamma_v^k \to \frac{(\tilde{\lambda}_k + \varrho_v)^2}{m(\tau_v + \varrho_v)^2} &= \frac{\left(\tilde{\lambda}_k - \tau_v + \sqrt{\xi/c}\right)^2}{m\left(\sqrt{\xi/c}\right)^2} \\ &= \frac{\left(1 + (\tilde{\lambda}_k - \tau_v)\sqrt{c/\xi}\right)^2}{m}.\end{aligned} \tag{33}$$

Given the following

$$\begin{aligned}&\frac{\left(\tilde{\lambda}_k - \tau_v\right)^2 + \sum_{i=k+1}^m \left(\tilde{\lambda}_i - \tau_v\right)^2}{m-k+1} = \xi, \\ &\left(\tilde{\lambda}_k - \tau_v\right)^2 \ll \sum\nolimits_{i=k+1}^m \left(\tilde{\lambda}_i - \tau_v\right)^2 \\ &\left(\tilde{\lambda}_k - \tau_v\right)^2 \to m\xi - \sum\nolimits_{i=k+1}^m \left(\tilde{\lambda}_i - \tau_v\right)^2 \ll m\xi\end{aligned}$$

we have

$$\frac{\tilde{\lambda}_k - \tau_v}{\sqrt{\xi}} \ll \sqrt{m} \tag{34}$$

and

$$\gamma_v^{k-1} - \gamma_v^k \ll \frac{\left(1 + \sqrt{mc}\right)^2}{m} \approx c. \tag{35}$$

Note from (17) that $\gamma_v^v = 1 + c$, and $\gamma_v^{v-1}, \cdots, \gamma_v^k, \gamma_v^{k-1} \to 1 + c$ for a small $v - d$, and we then have

$$\begin{aligned}\alpha_v^k - \alpha_v^{k-1} &= \frac{m-k+1}{(n+1)\left(\gamma_v^k - 1\right)} - \frac{m-k+2}{(n+1)\left(\gamma_v^{k-1} - 1\right)} \\ &\to \frac{m-k}{n+1}\frac{\gamma_v^{k-1} - \gamma_v^k}{\left(\gamma_v^k - 1\right)\left(\gamma_v^{k-1} - 1\right)} \to 0\end{aligned} \tag{36}$$

for $k = d, \cdots, v$.

For $k \leq d-1$, γ_v^k can be written as

$$\begin{aligned}\gamma_v^k &= (m-k)\frac{\sum\limits_{i=k+1}^{d}\left(\tilde{\lambda}_i + \varrho_v\right)^2 + \sum\limits_{i=d+1}^{m}(\tilde{\lambda}_i + \varrho_v)^2}{\left(\sum\limits_{i=k+1}^{d}(\tilde{\lambda}_k + \varrho_v) + \sum\limits_{i=d+1}^{m}(\tilde{\lambda}_i + \varrho_v)\right)^2} \\ &\to m\frac{\sum\limits_{i=k+1}^{d}(\rho_k\vartheta m + \varrho_v)^2 + (m-d)(\tau_v + \varrho_v)^2(1+c)}{\left(\sum\limits_{i=k+1}^{d}(\rho_k\vartheta m + \varrho_v) + (m-d)(\tau_v + \varrho_v)\right)^2}\end{aligned} \tag{37}$$

where ϑ stands for SNR, $\rho_k \gg 0$ is a constant, and $\rho_k\vartheta m \geq \mathcal{O}(m-d)$ represents the required condition for relatively high SNRs defined in [16]. On the other hand, $\varrho_v \ll m$ for $v \geq d$. Subsequently, we obtain that $\gamma_v^k \to m$ and $\alpha_v^k \to 0$. ■

An intuitive illustration for propositions 2 and 3 is shown in Fig. 1, where $m = 100$, $n = 500$, $d = 4$, and the auxiliary

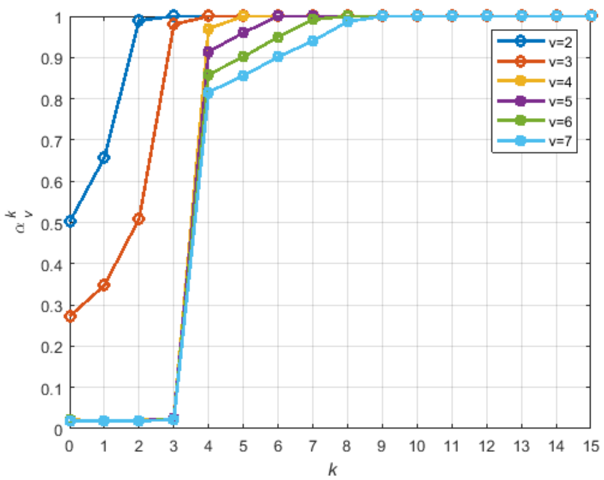

Fig. 1. α_v^k versus the auxiliary parameter k with various v under banded noise, $m = 100$, $n = 500$ and $d = 4$ with their corresponding DOAs $-24°, -12°, 12°, 24°$.

parameter k varies from 0 to 15, while v changes from 3 to 8. As can be seen clearly from Fig. 1,

$$2\left(\alpha_v^v - \alpha_v^{v-1}\right) > \alpha_v^k - \alpha_v^{k-1},\ k \le v < d \tag{38}$$

$$\alpha_v^k - \alpha_v^{k-1} < 0.05 \to 0,\ d \le k \le v \tag{39}$$

$$\alpha_v^k \to 0,\ k \le d-1, v \ge d \tag{40}$$

for each given v, which is consistent with the theoretical results of propositions 2 and 3.

D. Source Enumeration

Define $\Delta^{(k)} = \alpha_v^k - \alpha_v^{k-1}$, and then, according to the above analysis, we can indeed obtain

$$\Delta^{(v)} - \Delta^{(v+1)} > \Delta^{(k)} - \Delta^{(v)}, k \le v \le d \tag{41}$$

$$\Delta^{(v)} - \Delta^{(v+1)} < \Delta^{(d)} - \Delta^{(v)}, v > d \tag{42}$$

which indicates that if $\Delta^{(v)} - \Delta^{(v+1)} > \Delta^{(k)} - \Delta^{(v)}$ holds for a given v and any auxiliary parameter $k \le v$, we can obtain that $v \le d$. In other words, the condition of (41) remains valid as v increases until $v = d$. Such a conclusion enables us to achieve source enumeration via the following criterion

$$\hat{d} = \max v,\ s.\ t.\ \bar{\Delta}^{(v,v+1)} > \bar{\Delta}^{(k,v)}, k \le v \tag{43}$$

where $\bar{\Delta}^{(k,v)} = \Delta^{(k)} - \Delta^{(v)}$ denotes the second-order differencing (SOD) of LS coefficient α.

The proposed algorithm is summarized in Algorithm 1, where $\hat{\chi}$ denotes the estimate of χ.

Remark 4: In order to construct SOD effectively, implementation of the proposed algorithm should start with $v = 2$, and increases v step by step. When $d \ge 2$, we achieve source enumeration through (43). In particular, when $d = 1 < v$, it can be seen that (42) is established directly. Under such a circumstance, we can obtain that $\hat{d} = 1$.

Remark 5: It can be observed from Fig. 1 that the first-order differencing (FOD) operation can achieve source enumeration successfully, while the proposed approach exploits the SOD operation for source enumeration. The reason can be explained as follows: i) The SOD operation can provide a larger distance between $\bar{\Delta}^{(v,v+1)}$ and $\bar{\Delta}^{(k,v)}$ for $k \le v$, in comparison with the FOD operation, enabling the proposed algorithm to provide better estimation performance, especially for relatively low SNRs; ii) as analyzed in Remark 3, $2\left(\alpha_v^v - \alpha_v^{v-1}\right) > \alpha_v^k - \alpha_v^{k-1}, k \le v < d$, holds almost surely if the difference between signal powers is not significant, which implies that the SOD operation provides greater capacity for source enumeration under situations of unequal-power sources.

Algorithm 1: Proposed Source Enumerator

Input: $\mathbf{x}(t), m, n$
Output: $\hat{d}$

```
Collect n snapshots x(1), ..., x(n).
Calculate SCM via R̂_x = (1/n) Σ_{t=1}^{n} x(t)x(t)^H.
Perform eigenvalue decomposition on R̂_x to obtain m
  eigenvalues λ̃_1, ..., λ̃_m in descending order.
for v = 2 : m do
    Calculate ϱ_v according to (10) and α_v^(k) according to
      (16) for k ≤ v;
    Calculate Δ^(k) = α_v^k − α_v^(k−1) and its corresponding
      SOD values Δ̄^(v,v+1) = Δ^(v) − Δ^(v+1) and
      Δ̄^(k,v) = Δ^(k) − Δ^(v);
    if Δ̄^(v,v+1) > Δ̄^(k,v) then
        v=v + 1 ;
    else
        break;
    end
end
Return d̂ = v
```

Remark 6: The advantages of the proposed algorithm can be summarized as follows:

- According to (5), for IID Gaussian white noise, $\xi \to \tau_v c$, and $\varrho_v \to 0$, and therefore, the proposed source enumerator becomes a TSD-like enumerator; for non-white noise, $\varrho_v > 0$, with the established condition of (11), the proposed algorithm can still achieve satisfactory source enumeration via (43). In addition, by some preliminary simulation results, the proposed algorithm still works for the more general families of noise, such as mixed white, banded and nonuniform noise, as well as elliptical and/or generalized elliptical distribution noise. This is because that with the application of ADL, the behavior of LS coefficient and its corresponding SOD relationship are still valid under such circumstances; and it could be a topic of our future research for more rigorous interpretation and simulation verification. In a word, the proposed algorithm exhibits great robustness against various types of noise.
- The computational complexity of the proposed algorithm mainly lies in the SCM calculation and performing its EVD, which requires $\mathcal{O}\left(m^2 n + \frac{4}{3}m^2\right)$; as the SOD operation involved in (43) does not include complex multiplications, it is ignored. Note that the major complexity of the most representative LS-MDL, $\mathrm{SCD}_{\mathrm{heur}}$

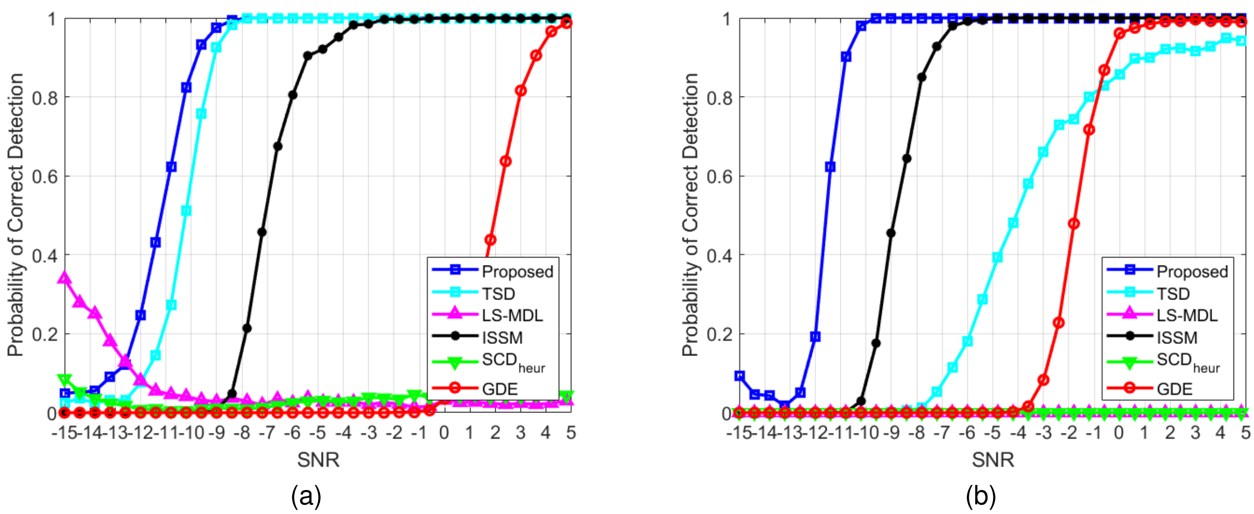

Fig. 2. PCD versus SNR under BCN, with four sources located at $\{-24°, -12°, 12°, 24°\}$: (a) $m = 80$, $n = 60$; (b) $m = 50$, $n = 300$.

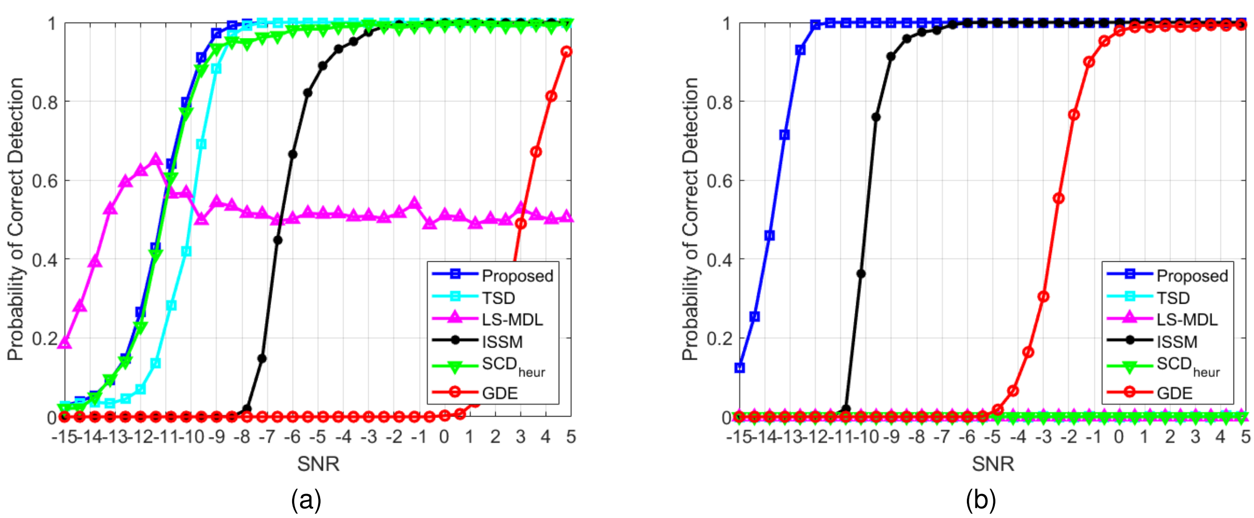

Fig. 3. PCD versus SNR under DCN, with four sources located at $\{-24°, -12°, 12°, 24°\}$: (a) $m = 80$, $n = 60$; (b) $m = 50$, $n = 300$.

and TSD algorithms also involve the computations of the SCM and EVD, and therefore, the proposed algorithm is almost the same as/slightly lower than these enumerators in terms of computational complexity, showing good practical applicability. Although there are some reduced-complexity enumerators, such as MDLE, mMDL and GDEWE, which avoid the calculation of SCM and its EVD, and roughly require $\mathcal{O}(mnq)$, with q denoting the dimension of the reduced-rank observation space, it is necessary to point out that the proposed algorithm can provide significant improvement in enumeration performance (see related simulation results for details)."

- Analyzing from the algorithm derivation process, it can be seen that the proposed algorithm does not rely on array geometries nor the prerequisite of $c > 1$ or $c < 1$, and thus it is applicable to a wide range of array geometries and array configurations, as shown later by simulations.
- Note that the proposed algorithm essentially relies on the SCM and LS coefficients. Due to similarity, it is possible to achieve further improved performance by combining the proposed solution with the shrinkage fixed-point covariance estimators such as the shrinkage Tyler's M-estimator [41], [42], which also exploits the LS coefficient and is more suitable for elliptical distribution models. Such a proposal could be another interesting topic for our future research.

IV. Simulation Results

In this section, simulations are performed to demonstrate the effectiveness of the proposed algorithm. The LS-MDL [14], MDLE [12], $\mathrm{SCD_{heur}}$ [15] and TSD [16] algorithms tailored to white noise, and the GDE [20], GDEWE [21] and ISSM [24] algorithms tailored to colored noise are selected for comparison. The white noise is IID zero-mean complex

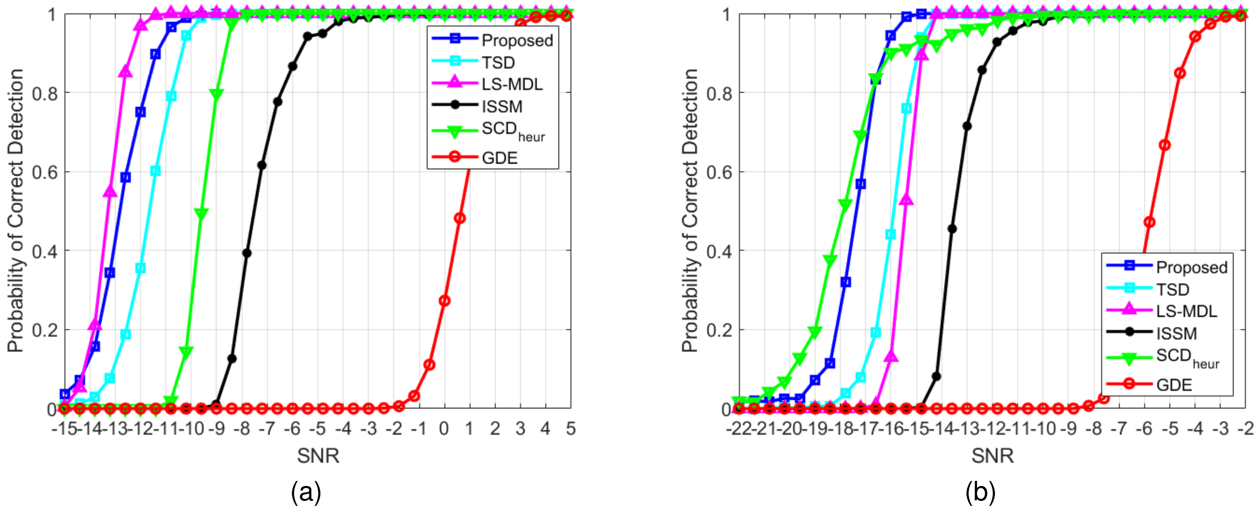

Fig. 4. PCD versus SNR under IID white noise, with four sources located at $\{-24°, -12°, 12°, 24°\}$: (a) $m = 80$, $n = 60$; (b) $m = 50$, $n = 300$.

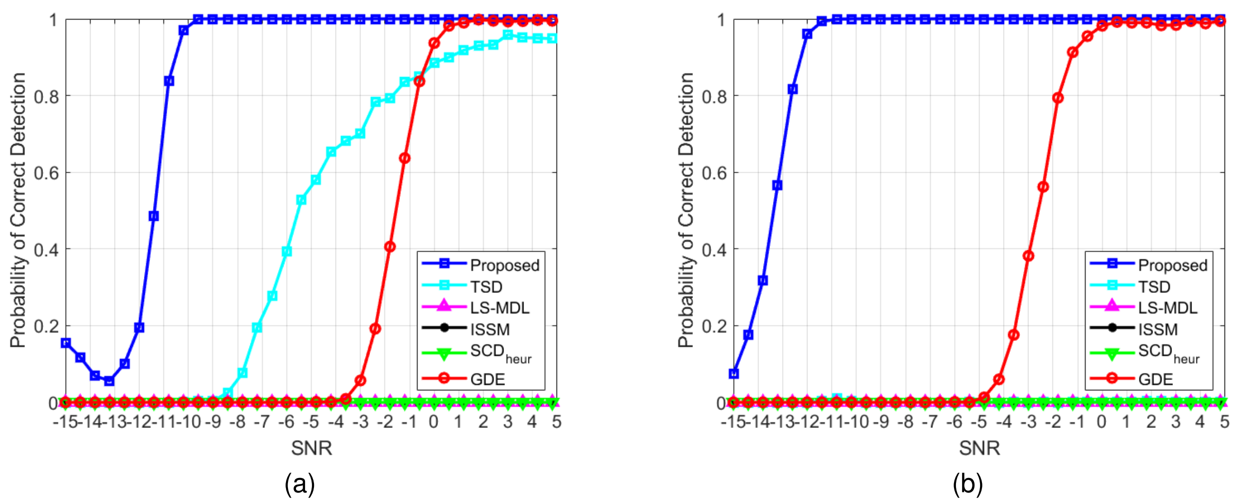

Fig. 5. PCD versus SNR for randomly spaced linear arrays under different types of noise, with four sources located at $\{-24°, -12°, 12°, 24°\}$, $m = 50$ and $n = 300$: (a) under BCN; (b) under DCN.

Gaussian with covariance matrix $\sigma \mathbf{I}_m$, while the colored noise is generated from the white noise by two different parametric models: the first generates noise with a diagonal covariance matrix (DCN) with non-equal elements, whose i-th element at time index t is given by

$$n_i(t) = \eta * w_i(t) \tag{44}$$

where $w_i(t)$ is the white noise corresponding to the i-th sensor, η is the nonuniform scaling factor, given by

$$\eta = \begin{cases} 1 & i \ mod \ 3 = 1 \\ 1 - \beta & i \ mod \ 3 = 2 \\ 1 + \beta & i \ mod \ 3 = 3 \end{cases} \tag{45}$$

and the second model generates a banded covariance noise (BCN), given by

$$\mathbf{n}(t) = \boldsymbol{\Upsilon}^{\frac{1}{2}} * \mathbf{W}(t) \tag{46}$$

with $\mathbf{W}(t)$ being an $m \times n$ matrix generated by Gaussian white process, and the (p, q)-th element of $\boldsymbol{\Upsilon}$ is expressed as

$$\boldsymbol{\Upsilon}(p, q) = \beta^{|p-q|} * e^{j\pi(p-q)/2} \tag{47}$$

with $\beta < 1$ representing a parameter controlling the level of non-whiteness. In particular, $\beta = 0$ indicates the case of white noise. It is necessary to emphasize here that some of the compared algorithms, such as LS-MDL and MDLE, are ITC-based enumerators, while the proposed one, $\mathrm{SCD}_{\mathrm{heur}}$ and TSD are essentially threshold-detection enumerators. Hence, it may be unfair to compare them directly under the considered scenarios. However, because the selected ITC-based solutions are some of the most representative algorithms in the field of source number estimation, they are also selected for comparison as good references.

In the simulations, unless otherwise specified, uncorrelated zero-mean complex Gaussian signals combined with uniform

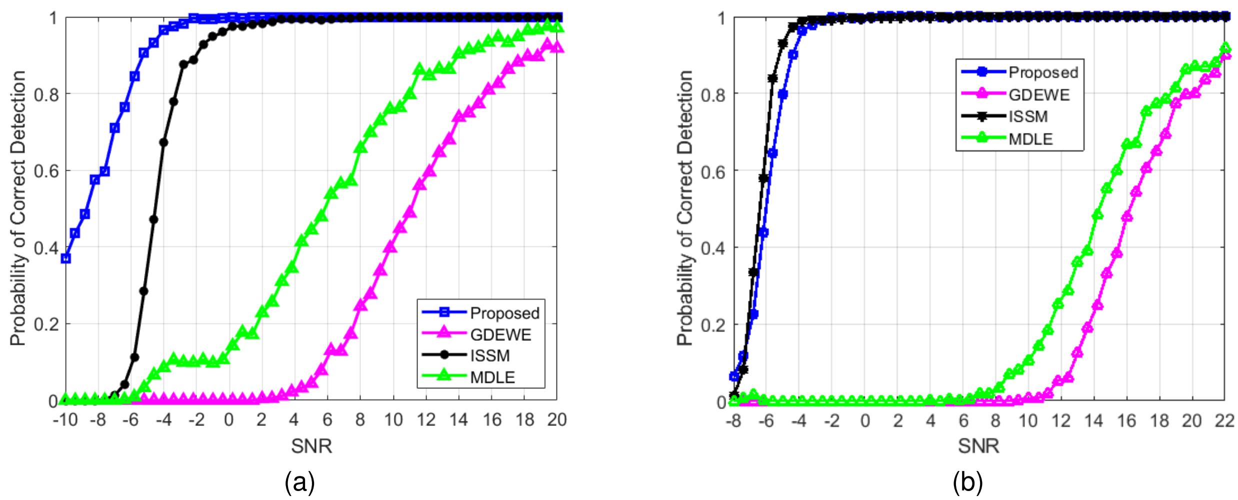

Fig. 6. Comparison of the PCD between the proposed algorithm and reduced-rank based enumerators for small-scale arrays with two sources located at $\{20°, 40°\}$: (a) $m = 12$, $n = 100$, DCN; (b) $m = 18$, $n = 256$, BCN.

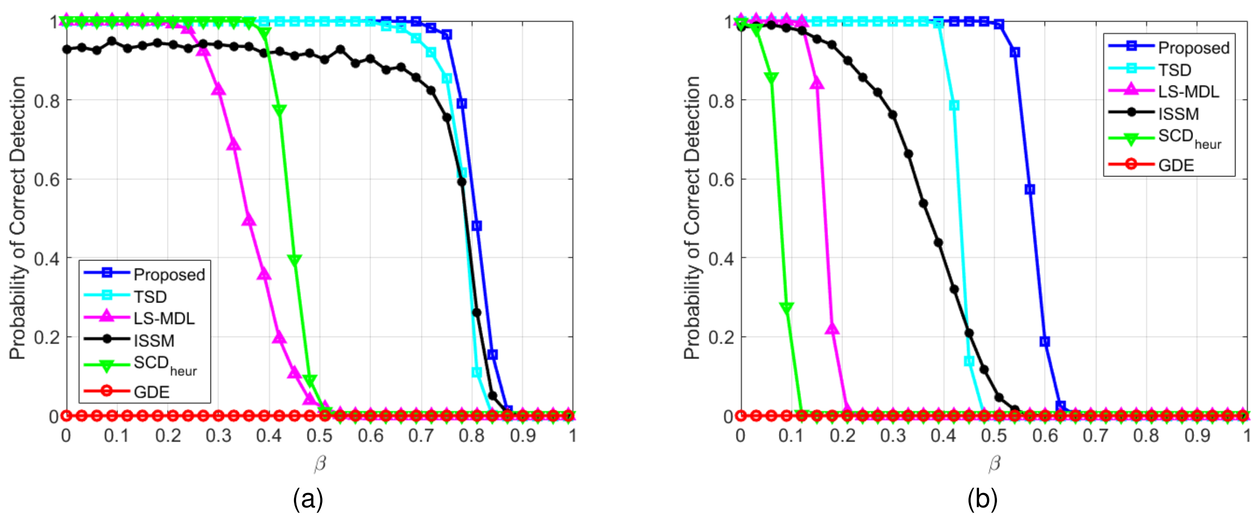

Fig. 7. PCD versus β under BCN, with four sources located at $\{-24°, -12°, 12°, 24°\}$: (a) SNR=-5, $m = 80$ and $n = 60$; (b) SNR=-10, $m = 50$ and $n = 300$.

linear arrays with half-wavelength spacing are considered. The SNR of the i-th signal is defined as $\mathrm{SNR} = \log \frac{\sigma_{si}}{\sigma_n}$, where σ_{si} denotes the power of the i-th signal. The probability of correct detection (PCD) is utilized to measure the source enumeration performance, which is calculated from 1000 independent Monte Carlo trials.

A. Simulation Results for Equal-Power Signals

1) PCD versus SNR: To examine the ability of the proposed algorithm against different noises, BCN and DCN with $\beta = 0.5$, and IID white noise are considered, respectively. For each type of noise, the array configurations for sample enough ($c < 1$) and sample starving ($c > 1$) are also considered. Figs. 2-4 show the PCD versus SNR for BCN, DCN and white noise, respectively, where four equal-power sources are located at $\{-24°, -12°, 12°, 24°\}$, and SNR varies from -15 dB to 5 dB. In Figs. 2(a), 3(a) and 4(a), $m = 80$ and $n = 60$ ($c > 1$), while in Figs. 2(b), 3(b) and 4(b), $m = 50$ and $n = 300$ ($c < 1$).

As can be seen in Fig. 2, the proposed algorithm performs better than the compared algorithms in the whole SNR region. Meanwhile, it is noted that only the proposed one and ISSM can provide accurate detection in low SNRs for both $c > 1$ and $c < 1$. In contrast, TSD can achieve source enumeration accurately in the sample-starving scenario, but lose consistence in sample-enough scenario, while GDE requires a high SNR for precise enumeration. Moreover, it can be further seen that the required SNR for the proposed algorithm to achieve 100% accurate detection are 10 dB and 7 dB lower than that for ISSM in sample-starving and sample-enough scenarios, respectively, showing great robustness of the proposed algorithm.

From Fig. 3, it can be observed that the proposed algorithm

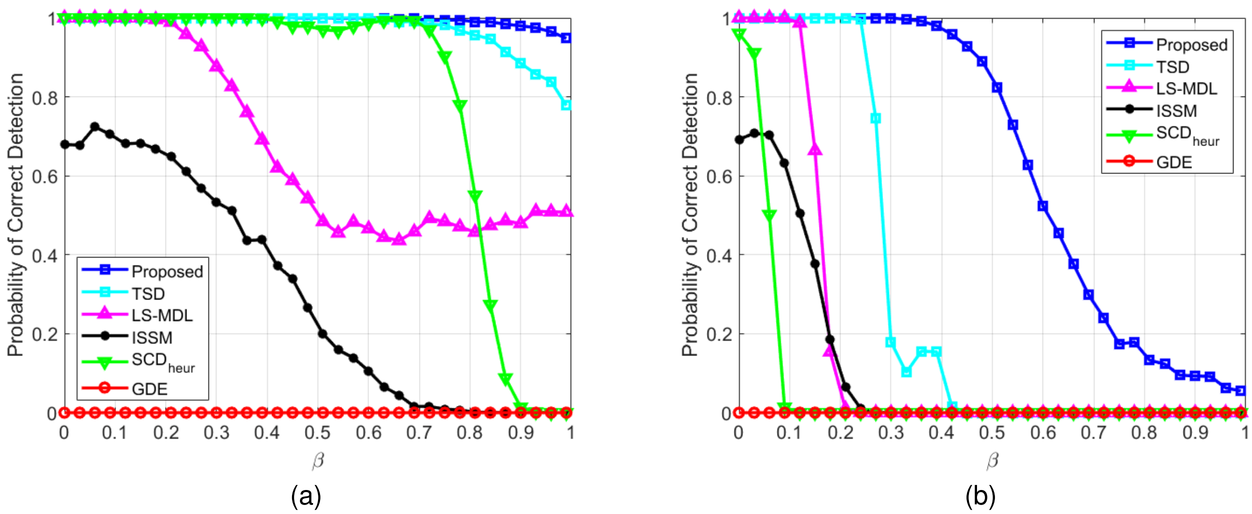

Fig. 8. PCD versus β under DCN, with four sources located at $\{-24°, -12°, 12°, 24°\}$: (a) SNR=-5, $m = 80$ and $n = 60$; (b) SNR=-10, $m = 50$ and $n = 300$.

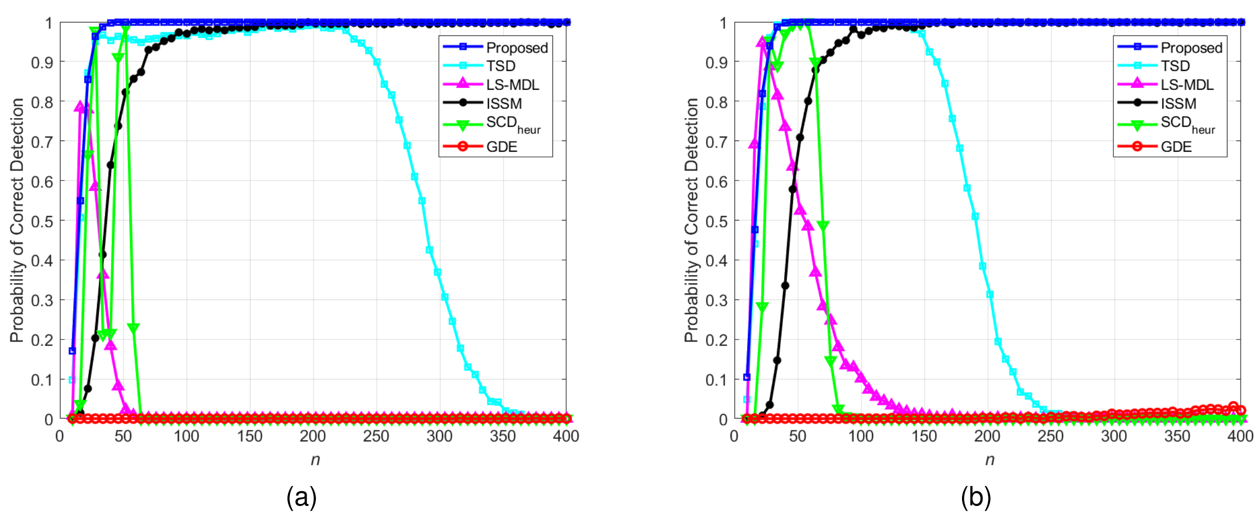

Fig. 9. PCD versus n in colored noise, with four sources located at $\{-24°, -12°, 12°, 24°\}$, SNR=-5 dB and $m = 50$: (a) BCN; (b) DCN.

can still provide accurate source enumeration for $c > 1$ under white noise, followed by the compared $\mathrm{SCD_{heur}}$, TSD, ISSM and GDE, while LS-MDL cannot achieve 100% accurate detection regardless of SNR, due to the model mismatch problem. Particularly, for $c < 1$, only three algorithms can detect the number of sources effectively, and the remaining two tailored to white noise fail completely.

Further observation from Fig. 4 shows that the proposed algorithm can still provide a satisfactory source enumeration result. In particular, the performance of the proposed one is slightly worse than LS-MDL, but better than the other state-of-the-art algorithms for the sample-starving scenario. For the sample-enough scenario, the proposed algorithm even obtains 100% detection accuracy at the lowest SNR. That is, the proposed solution cannot only handle white noise well, but also provide a competitive enumeration result. Through the above simulations, we can conclude that the proposed source enumerator is a general one, capable of dealing with both white and colored noises.

2) PCD versus SNR for randomly spaced linear arrays: The proposed algorithm is applicable for various array geometries. To show this, a linear array with randomly spaced elements is employed here. The PCD result under BCN and DCN is shown in Fig. 5, where $m = 50$, $n = 300$, and the remaining settings are the same as in the first simulation. From the result, we can observe that the PCD of the proposed solution is almost the same as that under the uniform linear array, demonstrating its robustness to the change of the array layout. Moreover, it can also be observed that ISSM is invalid under such a circumstance, since it is specifically designed for the uniform array, and needs to exploit the translation invariant subspace for source number estimation.

3) PCD versus SNR for small-scale arrays: This simulation shows the performance of the proposed algorithm under a

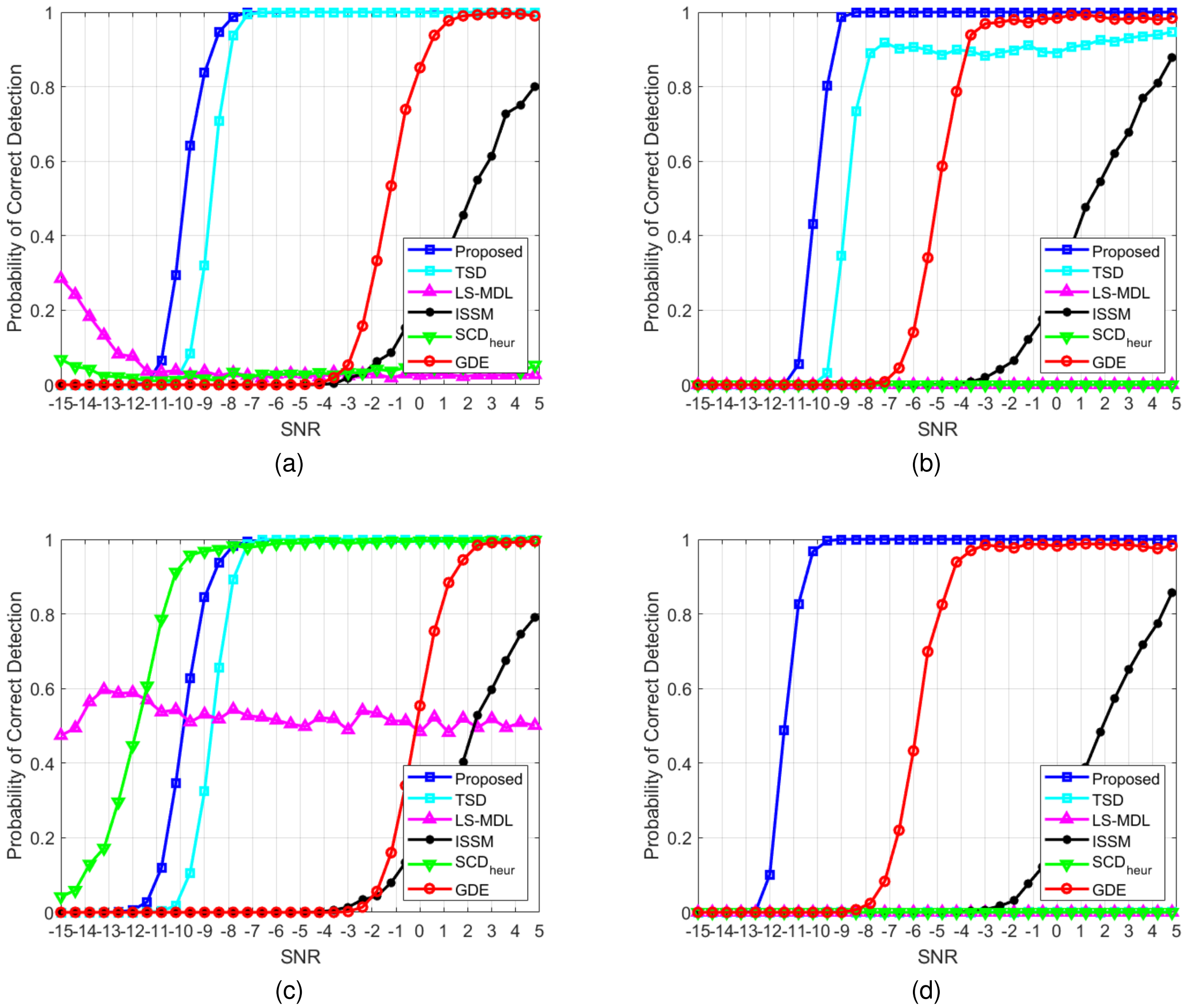

Fig. 10. PCD versus SNR for unequal-power sources (SNR+6dB, SNR+6dB, SNR, SNR), with four sources located at $\{-24°, -12°, 12°, 24°\}$: (a) $c > 1$ under BCN; (b) $c < 1$ under BCN; (c) $c > 1$ under DCN; (d) $c < 1$ under DCN.

small-scale uniform linear array, and compares with the performance of the computationally efficient MDLE and GDEWE. The simulation result for two sources located at $\{20°, 40°\}$ is shown in Fig. 6, where $m = 12$ and $n = 100$ for Fig. 6(a), $m = 18$ and $n = 256$ for Fig. 6(b), and both DCN and BCN are again considered. It can be seen that the proposed algorithm is still effective in small-scale arrays and its performance is significantly better than that of MDLE and GDEWE.

4) PCD versus parameter β: In this simulation, the impact of β is examined, where the impinging sources are the same as in the first simulation, both BCN and DCN are taken into consideration, and their corresponding results are illustrated in Figs. 7 and 8, respectively. In Figs. 7(a) and 8(a), SNR is fixed as -5 dB, $m = 80$ and $n = 60$, while in Figs. 7(b) and 8(b), SNR is set to -10 dB, $m = 50$ and $n = 300$. It can be seen that the proposed algorithm outperforms the compared ones for various β values. In detail, for BCN and $c > 1$, only ISSM, TSD and the proposed algorithm can work effectively when $\beta > 0.5$, but it is noted that ISSM cannot converge to one; for $c > 1$ and a lower SNR, the proposed algorithm can provide almost 100% PCD at $\beta = 0.5$, while the other ones basically fail under such heavily colored noise, demonstrating great robustness of proposed algorithm against colored noise.

5) PCD versus n in colored noise: This simulation shows the PCD results of various algorithms with different number of samples n, where $m = 50$, SNR=-5 dB, $\beta = 0.5$, and n varies from 10 to 400. As can be seen in Fig. 9, GDE cannot work in such a noisy condition since SNR< 0 dB, while LS-MDL, TSD and $\mathrm{SCD_{heur}}$ show obvious fluctuations, and only the proposed algorithm and ISSM can always provide stable estimate in both sample-starving and sample-enough scenarios. Nevertheless, it is worth emphasizing that the proposed algorithm outperforms ISSM significantly.

B. Simulation Results for Unequal-Power Signals

This simulation examines the performance of the proposed algorithm under unequal-power source signals, where both

BCN and DCN are considered again. The simulation configurations are the same as in Figs. 2 and 3, except that the SNRs of first two sources have increased by 6 dB on the basis of equal power, respectively. That is, there is a 6 dB SNR difference between sources. However, it can be seen that the proposed algorithm can not only provide satisfactory source enumeration results for both $c > 1$ and $c < 1$, but also yield 100% accurate enumeration when SNR≥-8 dB. As a comparison, it can be seen from Fig. 10 that ISSM is greatly affected under different SNR conditions, making it difficult for its PCD to reach 100% within the observed SNR region. On the other hand, it can also be observed that, due to the SNR increase for part of impinging sources, GDE has shown significant performance improvement compared to equal-power conditions, and $\mathrm{SCD}_{\mathrm{heur}}$ provides a competitive result for $c > 1$ and DCN scenario. However, they are not robust and consistent for all considered scenarios. This simulation again demonstrates robustness and superiorities of the proposed algorithm clearly.

V. Conclusion

A novel and computationally efficient source enumeration algorithm has been proposed, which is applicable to both spatially white and colored noise, and can provide an improved performance for scenarios of large-scale arrays with small number of samples. The solution is established on the ADL and LS techniques, by matching the asymptotic ratio behavior of the eigenvalues in white noise. We have formed a set of hypothesized source number v, and their corresponding LS coefficients α_v^k, and further analyzed the characteristics of loaded LS coefficients in detail. With the aid of such characteristics, a novel SOD based source enumerator was designed. Simulation results have shown that the proposed algorithm can not only exhibit improved robustness against white and colored noise, but also provide a higher PCD for source enumeration in various array configurations, than existing state-of-the-art algorithms.

Appendix

Substituting (21) into (18) yields

$$\gamma_v^k \to m \frac{\sum_{i=k+1}^{v}\left(\tilde{\lambda}_i + c_1\sqrt{\breve{\lambda}/mc}\right)^2}{\left[\left(\tau_v + c_1\sqrt{\breve{\lambda}/mc}\right)m + \sum_{i=k+1}^{v}\tilde{\lambda}_i\right]^2} + \frac{m^2\left[\left(\tau_v + c_1\sqrt{\breve{\lambda}/mc}\right)\right]^2(1+c)}{\left[\left(\tau_v + c_1\sqrt{\breve{\lambda}/mc}\right)m + \sum_{i=k+1}^{v}\tilde{\lambda}_i\right]^2}. \tag{48}$$

With the definition of $\breve{\lambda}$ and the condition of $\tilde{\lambda}_k \simeq \mathcal{O}(m) \gg \tau_v$, it can be derived for $k < v$ that $\sqrt{\breve{\lambda}} \simeq \mathcal{O}(m)$, yielding

$$c_1\sqrt{\breve{\lambda}/c} \simeq \sum_{i=k+1}^{v}\tilde{\lambda}_i \simeq \mathcal{O}(m) \tag{49}$$

$$\begin{aligned}\tau_v &= \frac{1}{m-v}\sum_{i=v+1}^{m}\tilde{\lambda}_i \approx \frac{1}{m-v}\sum_{i=v+1}^{d}\tilde{\lambda}_i \\ &\simeq \sqrt{\breve{\lambda}}/m \ll \mathcal{O}(m)\end{aligned} \tag{50}$$

which implies that

$$\begin{aligned}&\left(\tau_v + c_1\sqrt{\breve{\lambda}/mc}\right)m + \sum_{i=k+1}^{v}\tilde{\lambda}_i \\ &\qquad \to c_1\sqrt{m\breve{\lambda}/c} \simeq \mathcal{O}\left(m^{\frac{3}{2}}\right)\end{aligned} \tag{51}$$

Consequently, we have

$$\begin{aligned}\gamma_v^k - \gamma_v^{k-1} &\to -\frac{\left(\tilde{\lambda}_k + c_1\sqrt{\breve{\lambda}/mc}\right)^2}{\left[c_1\sqrt{\breve{\lambda}/c}\right]^2} \\ &= -\left(\frac{\sqrt{c}\tilde{\lambda}_k}{c_1\sqrt{\breve{\lambda}}} + \frac{1}{\sqrt{m}}\right)^2 \approx -c_k^2\end{aligned} \tag{52}$$

where $c_k = \frac{\sqrt{c}\tilde{\lambda}_k}{c_1\sqrt{\breve{\lambda}}}$. Due to $\tilde{\lambda}_k \leq \tilde{\lambda}_{k-1}$ leading to $c_k^2 \leq c_{k-1}^2$, and $\gamma_v^v = 1 + c$, $\gamma_v^k \approx 1 + c + \sum_{i=k+1}^{v} c_i^2$. It can be derived that

$$\begin{aligned}\alpha_v^k &= \frac{m-k+\gamma_v^k}{(n+1)\left(\gamma_v^k - 1\right)} = \frac{m-k+1+\left(\gamma_v^k-1\right)}{(n+1)\left(\gamma_v^k-1\right)} \\ &= \frac{m-k+1}{(n+1)\left(\gamma_v^k-1\right)} + \frac{1}{(n+1)} \\ &= \frac{m-k+1}{(n+1)\left(c+\sum_{i=k+1}^{v}c_i^2\right)} + \frac{1}{(n+1)} \\ &\to \frac{c}{\left(c+\sum_{i=k+1}^{v}c_i^2\right)}.\end{aligned} \tag{53}$$

References

[1] K. Yuan, L. Guo, C. Dong, and T. Kang, "Detection of active eavesdropper using source enumeration method in massive MIMO," in *Proc. IEEE Int. Conf. Commun. (ICC)*, Paris, France, 2017, pp. 1-5.

[2] X. Ke, Y. Zhao, and L. Huang, "On accurate source enumeration: A new Bayesian information criterion," *IEEE Trans. Signal Process.*, vol. 69, pp. 1012-1027, Jan. 2021.

[3] X. Bai and B. He, "Estimation of number of independent brain electric sources from the scalp EEGs," *IEEE Trans. Biomed. Eng.*, vol. 53, no. 10, pp. 1883-1892, Oct. 2006.

[4] Y. O. Li, T. Adalı, and V. D. Calhoun, "Estimating the number of independent components for functional magnetic resonance imaging data," *Hum. Brain Mapping*, no. 28, pp. 1251-1266, 2007.

[5] T. Talebi and T. Pratt, "Model prder selection for complex sinusoids in the presence of unknown correlated Gaussian noise," *IEEE Trans. Signal Process.*, vol. 63, no. 7, pp. 1664-1674, Apr. 2015.

[6] Y. Yang, F. Gao, C. Qian, and G. Liao, "Model-aided deep neural network for source number detection," *IEEE Signal Process. Lett.,* vol. 27, pp. 91-95, Dec. 2020.

[7] P. V. Nagesha, G. V. Anand, N. Kalyanasundaram, and S. Gurugopinath, "Detection, enumeration and localization of underwater acoustic sources," in *Proc. 27th Eur. Signal Process. Conf. (EUSIPCO)*, A Coruna, Spain, 2019, pp. 1-5.

[8] H. Akaike, "A new book at the statistical model identification," *IEEE Trans. Autom. Control*, vol. 19, pp. 716-723, Dec. 1974.

[9] G. Schwartz, "Estimating the dimension of a model," *Ann. Stat.*, vol. 6, pp. 461-464, 1978.

[10] J. Rissanen, "Modeling by shortest data description," *Automatica*, vol. 14, pp. 465-471, 1978.

[11] M. Wax and T. Kailath, "Detection of signals by information theoretic criteria," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 33, no. 2, pp. 387-392, Apr. 1985.

[12] L. Huang, S. Wu, and X. Li, "Reduced-rank MDL method for source enumeration in high-resolution array processing," *IEEE Trans. Signal Process.*, vol. 55, no. 12, pp. 5658-5667, Dec. 2007.

[13] L. Huang, T. Long, E. Mao, and H.C. So, "MMSE-based MDL method for robust estimation of number of sources without eigendecomposition," *IEEE Trans. Signal Process.*, vol. 57, no. 10, pp. 4135-4142, Oct. 2009.

[14] L. Huang and H. C. So, "Source enumeration via MDL criterion based on linear shrinkage estimation of noise subspace covariance matrix," *IEEE Trans. Signal Process.*, vol. 61, no. 19, pp. 4806-4821, Oct. 2013.

[15] L. Huang, C. Qian, H. C. So, and J. Fang, "Source enumeration for large array using shrinkage-based detectors with small samples," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 51, no. 1, pp. 344-357, Jan. 2015.

[16] Z. Zhang, Y. Tian, W. Liu, and H. Chen, "Enumeration for a large number of sources based on a two-step difference operation of linear shrinkage coefficients," *IEEE Trans. Signal Process.*, vol. 71, pp. 2283-2295, Jun. 2023.

[17] A. Eguizabal, C. Lameiro, D. Ramírez, and P. J. Schreier, "Source enumeration in the presence of colored noise," *IEEE Signal Process. Lett.*, vol. 26, no. 3, pp. 475-479, Mar. 2019.

[18] S. Aouada, A. M. Zoubir, and C. M. Samson See, "Source detection in the presence of nonuniform noise," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process. (ICASSP)*, Montreal, QC, Canada, 2004, pp. 165-168.

[19] A. Liu, H. Guom and Y. Arnatovich, "Global MDL minimization-based method for detection of the number of sources in presence of unknown nonuniform noise," in *Proc. 30th Eur. Signal Process. Conf. (EUSIPCO)*, Belgrade, Serbia, 2022, pp. 1936-1940.

[20] H.-T. Wu, J.-F. Yang, and F.-K. Chen, "Source number estimators using transformed Gerschgorin radii," *IEEE Trans. Signal Process.*, vol. 43, no. 6, pp. 1325-1333, Jun. 1995.

[21] L. Huang, T. Long and S, Wu, "Source enumeration for high-resolution array processing using improved Gerschgorin radii without eigendecomposition," *IEEE Trans. Signal Process.*, vol. 56, no. 12, pp. 5916-5925, Dec. 2008.

[22] A. Eguizabal, C. Lameiro, D. Ramírez, and P. J. Schreier, "Source enumeration in the presence of colored noise," *IEEE Signal Process. Lett.*, vol. 26, no. 3, pp. 475-479, Mar. 2019.

[23] M. Wax and A. Adler, "Detection of the number of signals by signal subspace matching," *IEEE Trans. Signal Process.*, vol. 69, pp. 973-985, Jan. 2021.

[24] M. Wax and A. Adler, "Detection of the number of signals in uniform arrays by invariant-signal-subspace matching," *IEEE Trans. Signal Process.*, vol. 70, pp. 1270-1281, Feb. 2022.

[25] L. Wan, L. Sun, K. Liu, X. Wang, Q. Lin, and T. Zhu, "Autonomous vehicle source enumeration exploiting non-cooperative UAV in software defined internet of vehicles," *IEEE Trans. Intell. Transp. Syst.*, vol. 22, no. 6, pp. 3603-3615, Jun. 2021.

[26] B. D. Carlson, "Covariance matrix estimation errors and diagonal loading in adaptive arrays," *IEEE Trans. Aerosp. Electron. Syst*., vol. 24, no. 4, pp. 397-401, Jul. 1988. 24-39.

[27] X. Mestre and M. A. Lagunas, "Finite sample size effect on minimum variance beamformers: optimum diagonal loading factor for large arrays," *IEEE Trans. Signal Process.*, vol. 54, no. 1, pp. 69-82, Jan. 2006.

[28] M. Muhammad, M. Li, Q. H. Abbasi, C. Goh, and M. A. Imran, "Adaptive diagonal loading technique to improve direction of arrival estimation accuracy for linear antenna array sensors, *IEEE Sensors J.*, vol. 22, no. 11, pp. 10986-10994, Jun. 2022.

[29] P. Setlur, M. Sahmoudi, and F. Gagnon, "Source number estimation for unbalanced arrays using robust outlier detection in the eigenvalue domain," in *Proc. IEEE/SP 15th Workshop on Stat. Signal Process.*, Cardiff, UK, 2009, pp. 441-444.

[30] W. Zuo, J. Xin, N. Zheng, H. Ohmori, and A. Sano, "Subspace-based near-field source localization in unknown spatially nonuniform noise environment," *IEEE Trans. Signal Process.*, vol. 68, pp. 4713-4726, Aug. 2020.

[31] P. Stoica and M. Cedervall, "Detection tests for array processing in unknown correlated noise fields," *IEEE Trans. Signal Process.*, vol. 45, no. 9, pp. 2351-2362, Sep. 1997.

[32] W. Chen, J. P. Reilly, and K. M. Wong, "Detection of the number of signals in noise with banded covariance matrices," *IEEE Proc. - Radar, Sonar Navi.*, vol. 143, no. 5, pp. 289-294, 1996.

[33] Y. Chen, A. Wiesel, Y. C. Eldar, and A. O. Hero, "Shrinkage algorithms for MMSE covariance estimation," *IEEE Trans. Signal Process*., vol. 58, no. 10, pp. 5016-5029, Oct. 2010.

[34] H. Xu, Y. Tian, and S. Liu, "Linear-shrinkage-based DOA estimation for coherently distributed sources considering mutual coupling in massive MIMO systems," *Int. J. Electron. Commun.*, vol. 126, pp. 153398, Aug. 2020.

[35] B. Zhang, J. Zhou, and J. Li, "Improved shrinkage estimators of covariance matrices with Toeplitz-structured targets in small sample scenarios," *IEEE Access*, vol. 7, pp. 116785-116798, Aug. 2019.

[36] G. Letac and H. Massam, "All invariant moments of the Wishart distribution," *Scand. J. Statist.*, vol. 31, no. 2, pp. 285-318, 2004.

[37] C. Liu and J. Zhen, "Diagonal loading beamforming based on aquila optimizer," *IEEE Access*, vol. 11, pp. 69091-69100, Jul. 2023.

[38] S. Li, Y. Liu, L. You, W. Wang, H. Duan and X. Li, "Covariance Matrix Reconstruction for DOA Estimation in Hybrid Massive MIMO Systems," *IEEE Wireless Commun. Lett.*, vol. 9, no. 8, pp. 1196-1200, Aug. 2020.

[39] Y. Liu, Y. Yan, L. You, W. Wang, and H. Duan, "Spatial covariance matrix reconstruction for DOA estimation in hybrid massive MIMO systems with multiple radio frequency chains," *IEEE Trans. Veh. Technol.*, vol. 70, no. 11, pp. 12185-12190, Nov. 2021.

[40] Y. Zhou, B. Dang, Y. Li, and G. Liu, "An efficient spatial covariance matrix reconstruction algorithm in the hybrid analog-digital structure," *IEEE Trans. Veh. Technol.*, vol. 71, no. 7, pp. 7930-7935, Jul. 2022.

[41] E. Ollila, D. P. Palomar, and F. Pascal, "Affine equivariant Tyler's M-estimator applied to tail parameter learning of elliptical distributions," IEEE Signal Process. Lett., vol. 30, pp. 1017-1021, Aug. 2023.

[42] A. Breloy, E. Ollila, and F. Pascal, "Spectral shrinkage of Tyler's M-estimator of covariance matrix," *Proc. IEEE 8th Int. Workshop. Comput. Adv. Multi-Sensor Adaptive Process*., 2019, pp. 535-538.