Positivity and the electroweak hierarchy

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We point out that an unnatural hierarchy between certain higher-dimensional operator coefficients in a low-energy effective field theory (EFT) would automatically imply that the Higgs' vacuum expectation value is hierarchically smaller than the EFT cutoff, assuming the EFT emerged from a unitary, causal and local UV completion. Future colliders may have the sensitivity to infer such a pattern of coefficients for a little hierarchy with an EFT cutoff up to $\mathcal{O}(10)$ TeV.

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I. INTRODUCTION

The electroweak hierarchy problem is made all the more puzzling by the discovery of a Higgs boson with no signs of accompanying new physics beyond the Standard Model to solve its infamous naturalness problem; scalar masses unprotected by any symmetry would naturally lie at the effective field theory (EFT) cutoff scale unless fine-tuning occurs in the UV theory. Positivity bounds are powerful connections between certain EFT coefficients and basic properties of the underlying UV theory (see [1] and references therein), and here we explore a potential connection between positivity and the electroweak hierarchy.

The simplest positivity bounds carve the space of EFT coefficients into two regions: (1) values that satisfy the bounds and hence could have arisen from a unitary, causal and local UV completion; and (2) values that violate the bounds and have no such UV completion. Conventional efforts to solve the hierarchy problem live in Region 1. They typically aim to reconcile the electroweak hierarchy with EFT expectations by extending the symmetries of the Standard Model (SM). However, the absence of the necessary new physics at the weak scale has motivated looking for more "exotic" QFTs to address the hierarchy problem, e.g., Refs. [2–4]. They aim to break the rules of EFT by some as yet unknown UV/IR mixing, which is expected to violate decoupling and locality. Such exotic QFTs may live in either Regions 1 or 2. If measurements of EFT coefficients place us in Region 2, this would be a smoking gun for an exotic UV theory.

We point out that there is a subset of Region 1 in which positivity is only satisfied for a restricted range of the Higgs vacuum expectation value (vev), v, relative to the EFT cutoff scale, Λ, that we denote by Region $\bar{1}$. In particular, if there exist unitary, local, and causal UV theories that map to a specific pattern of dimension-8 and dimension-10 operator coefficients c_8 and c_{10}, respectively, with $|c_{10}| \gg |c_8|$ and appropriate signs, then by positivity they necessarily also have a hierarchy that satisfies

$$\frac{v^2}{\Lambda^2} < \frac{|c_8|}{|c_{10}|} \qquad \text{(Positively light)}. \tag{1}$$

We define a Higgs whose vev satisfies Eq. (1) living in the subset of Region $\bar{1}$ where $c_8 > 0$ and $c_{10} < 0$ as being "positively light." Another subset of Region $\bar{1}$ with the signs reversed requires instead a Higgs satisfying a lower bound on its vev, that we call "positively heavy,"

$$\frac{v^2}{\Lambda^2} > \frac{|c_8|}{|c_{10}|} \qquad \text{(Positively heavy)}. \tag{2}$$

The EFT coefficient space is illustrated in Fig. 1. Analogous positivity bounds involving dimension-6 operator coefficients can also be derived under additional UV assumptions.

Often, varying the parameters of a given UV theory can only produce a particular range of EFT coefficients at low energies. Our novel use of positivity shows that in any unitary, causal and local UV theory, once its parameters have been partially fixed so that $|c_8| \ll |c_{10}|$ (with $c_8 > 0$ and $c_{10} < 0$), then no matter how the remaining parameters are varied the Higgs vev may only take the unnaturally small

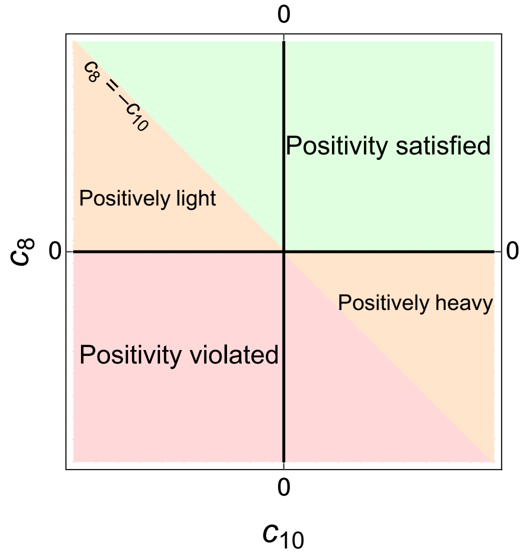

FIG. 1. Possible values for a particular dimension-8 (dimension-10) operator coefficient c_8 (c_{10}). The subset of Region 1 in which positivity bounds are satisfied for any vev is shown in green. The remainder of Region 1, in which positivity is only satisfied for a restricted range of vevs [Eqs. (1) or (2)] is colored orange—we refer to this as Region $\bar{1}$. Region 2, in which the positivity bounds are always violated, is shown in red.

values in Eq. (1). Since these UV theories produce a characteristic hierarchy between particular higher-dimension operators in the low-energy EFT, this scenario could be identified in future experiments.

A positively light Higgs would not, *per se*, be a solution to the hierarchy problem; as we will see, Region $\bar{1}$ from an EFT perspective is a fine-tuning in the ratio of higher-dimensional operator coefficients. Nevertheless, it is interesting that a correlated tuning of *a priori* unrelated quantities in the EFT—the dimension ≤ 4 scalar potential and higher-dimensional operator coefficients—could be a sign of some UV mechanism restricting the allowed spectrum of Higgs vevs, in the same spirit as Ref. [5] that proposes the weak gravity conjecture as an alternative UV consistency restriction on the Higgs vev. The argument advanced here is that self-consistency of the UV theory, rather than symmetry or cosmological dynamics, may provide a radically different perspective on the hierarchy problem. A potential measurement establishing us to be in Region $\bar{1}$ would strongly motivate looking for such a UV completion.

II. POSITIVITY WITHOUT VEVS

We begin by recalling some known positivity bounds for EFTs in which every field has already been expanded around a unique stable vev. The central object that bridges between the EFT and the UV is the scattering amplitude for the elastic process $AB \to AB$. This can be computed using the low-energy EFT as a series expansion in the center-of-mass energy s and momentum transfer t,

$$\mathcal{A}(s,t) = \sum_{i,j=0} a_{i,j} \frac{s^i t^j}{\Lambda^{2i+2j}}. \tag{3}$$

If the underlying UV completion is to be causal, unitary and local,[1] the $a_{2n,0}$ coefficients in the EFT expansion must obey the bounds

$$a_{2n,0} \geq 0, \tag{4}$$

for all $n \geq 1$ [6]. Analogous bounds can also be derived for the $a_{2n,1}$ coefficients (again with $n \geq 1$), and these take the form [7–12],

$$-\beta_n a_{2n,0} \leq a_{2n,1} \leq +\alpha_n a_{2n,0}. \tag{5}$$

Here, the positive parameters α_n and β_n depend on which scattering process is considered, but ultimately there is always a two-sided bound which forbids arbitrarily small tunings of $a_{2n,0}$ relative to $a_{2n,1}$. We provide further details in the Appendix.

In practice, the positivity bounds of Eqs. (4) and (5) are most useful when applied to the EFT coefficients appearing in a Lagrangian. For instance, consider,

$$\mathcal{L}_{\text{EFT}} = \sum_{d=0}^{\infty} \bar{c}_d \frac{\mathcal{O}_d}{\Lambda^{d-4}}, \tag{6}$$

where $\mathcal{O}_d$ denotes a generic dimension-d operator, $\bar{c}_d$ is its constant coefficient, and there is an implicit sum over all operators at each dimension. At tree-level, it is the dimension-8 operators which contribute to the s^2 part of the amplitude and hence $a_{2,0}$ is a linear combination of the $\bar{c}_8$ coefficients.[2] The bound Eq. (4) then implies that some $\bar{c}_8$ (or a linear combination of them) must be sign-definite if a unitary, causal and local UV completion is to exist. Similarly, $a_{2,1}$ is a linear combination of the $\bar{c}_{10}$ coefficients and Eq. (5) implies that these $\bar{c}_{10}$ cannot be tuned arbitrarily large relative to $\bar{c}_8$. As a concrete example, $\bar{c}_8\mathcal{O}_8 = \bar{c}_8(\partial\phi)^4$ for a real scalar field ϕ must have $\bar{c}_8 \geq 0$, and the higher-derivative correction $\bar{c}_{10}\mathcal{O}_{10} = \bar{c}_{10}(\partial\phi)^2(\partial\partial\phi)^2$ may not be hierarchically larger than $\bar{c}_8$.

Finally, it was recently noticed in Ref. [8] that for sufficiently large helicities the parameter β_n can vanish, and as a result $a_{2n,1} \geq 0$ obeys a simple positivity bound. In particular, for the scattering of two spin-1/2 fields with helicities $h_A = h_B = +1/2$, the amplitude must obey $a_{0,1} \geq 0$ in order to have a UV completion which is causal, unitary

[1]We assume Lorentz invariance throughout, and also the existence of an S-matrix at high-energies.

[2]Any contribution to $a_{2,0}$ from pairs of dimension-6 operators may be absorbed into the $\bar{c}_8$ coefficient by a suitable field redefinition.

and converges at large s slightly faster than the Froissart bound. This subset of local UV completions contains, for instance, all tree-level completions with no t-channel exchange [13–15]. Since $a_{0,1}$ is a linear combination of the $\bar{c}_6$ coefficients, such a UV completion can only exist if some dimension-6 operator coefficients are sign-definite.

III. POSITIVITY WITH VEVS

Now we turn to the main focus of this work, which is the effect on this story of a scalar field having a nontrivial vev. To illustrate this most simply, consider an EFT which contains a dimension-8 and dimension-10 operator of the form,

$$\mathcal{L}_{\rm EFT}[H] = c_8 \frac{\mathcal{O}_8}{\Lambda^4} + c_{10} \frac{|H|^2 \mathcal{O}_8}{\Lambda^6}, \tag{7}$$

where $\mathcal{O}_8$ is the only interaction that could contribute to the s^2 part of an $AB \to AB$ scattering amplitude, and H is an additional (possibly complex) scalar field. We will often identify H with the complex Higgs doublet of the SM,[3] although the discussion here is more general. We assume the potential for H has a stable vev at $|H| = v$. We also assume that a well-defined S-matrix element for $AB \to AB$ scattering exists on this background.

Focussing on low-energy perturbations about this vacuum, i.e., integrating out H, produces the simpler EFT,

$$\mathcal{L}_{\rm EFT}[v] = \bar{c}_8 \frac{\mathcal{O}_8}{\Lambda^4}, \tag{8}$$

where

$$\bar{c}_8 = c_8 + \frac{v^2}{\Lambda^2} c_{10}. \tag{9}$$

Provided we have normalized $\mathcal{O}_8$ so that $a_{2,0} = \bar{c}_8$, the positivity bound of Eq. (4) then requires,

$$c_8 + \frac{v^2}{\Lambda^2} c_{10} \geq 0. \tag{10}$$

The underlying UV theory may only be unitary, causal, local *and contain v as a stable vev* if this bound is satisfied.[4] The implication of positivity bounds around different vevs was recently studied in Ref. [17] in the context of cosmological EFTs, where demanding the existence of different vacua (e.g., flat versus expanding spacetime [18,19]) places different constraints on the low-energy coefficients. Here, we turn that logic around and point out that for certain values of the coefficients, the vev is effectively constrained by the requirements of unitarity, causality and locality in the UV, as in Eq. (1). In particular, when there is a hierarchy $|c_8| \ll |c_{10}|$, a negative value of c_{10} (with $c_8 > 0$) can only satisfy this bound if v is hierarchically smaller than Λ. Consequently, any unitary, causal, and local UV theory that produces a low-energy EFT with $|c_8|/|c_{10}| \ll 1$ with $c_8 c_{10} < 0$ could only ever produce a restricted range of Higgs vevs.

The new insight here is that the low-energy coefficient $\bar{c}_8$ in $\mathcal{L}_{\rm EFT}[v]$ receives contributions from coefficients of higher-dimension operators once we partially UV complete the theory by introducing the radial modes of H. More generally than Eq. (7), this partial UV completion takes the form,

$$\mathcal{L}_{\rm EFT}[H] = f_8(\xi) \frac{\mathcal{O}_8}{\Lambda^4}, \qquad \xi \coloneqq \frac{|H|^2}{\Lambda^2}. \tag{11}$$

Positivity of low-energy $AB \to AB$ scattering around the $|H| = v$ vacuum then requires,

$$f_8(v^2/\Lambda^2) \geq 0. \tag{12}$$

This can impose a restriction on the vev v whenever the function $f_8(\xi)$ violates the natural EFT power counting in which $f_8(\xi) = \sum_n f_{8,n} \xi^n$ with Taylor coefficients $f_{8,n} \sim \mathcal{O}(1)$. With this power counting, $f_8(\xi)$ is expected to be very flat near the vacuum point $\xi \approx v^2/\Lambda^2$, since all derivatives would be suppressed by powers of $v^2/\Lambda^2 \ll 1$. Any violation of this power counting, in which one or more derivatives of f_8 become large and with appropriate signs, would produce a bound on v^2/Λ^2 in terms of f_8 evaluated at a different value of ξ (away from the vacuum point). For instance, when $f_8(0)$ is unnaturally small the positivity bound Eq. (12) can be written as,

$$\frac{v^2}{\Lambda^2} + \mathcal{O}\left(\frac{v^4}{\Lambda^4}\right) \leq \frac{|f_8(0)|}{|f_8'(v^2/\Lambda^2)|} \ll 1, \tag{13}$$

whenever $f_8(0) > 0$ and $f_8'(v^2/\Lambda^2) < 0$.

Furthermore, for UV theories with a super-Froissart convergence we would arrive at the same conclusion for dimension-6 operators of the form,

$$\mathcal{L}_{\rm EFT}[H] = f_6(\xi) \frac{\mathcal{O}_6}{\Lambda^2}, \tag{14}$$

where $\mathcal{O}_6$ contributes to $a_{0,1}$ for four-Fermi scattering with aligned helicities. While the restriction to super-Froissart

[3] In the context of the electroweak theory, our discussion applies only to UV theories that match onto the SMEFT at low energy. We do not consider more general UV theories that can be matched at low energies only onto a nonlinearly realized electroweak chiral Lagrangian with a singlet Higgs, known as the "Higgs EFT (HEFT)" (see, e.g., Ref. [16] and references therein).

[4] If there is more than one stable vacuum, then there will be one bound of the form Eq. (10) for each vev. Some regions of the EFT parameter space may only admit UV completions (satisfy the positivity bounds) around some of these vevs [17], which corresponds to the fact that distant minima in the EFT potential can be destabilized when "integrating in" heavy states.

growth makes such bounds less general, observational prospects are better for these dimension-6 operators (see, for example, Refs. [20,21] for current constraints from recent global fits and Ref. [22] for future sensitivity projections).

Finally, notice that since c_{10} (f'_8) does not contribute to $a_{2,1}$, it is not bounded in relation to c_8 (f_8) by other positivity bounds like Eq. (5). Unlike dimension-10 operators which contain more derivatives, a dimension-10 operator which contains more fields may be tuned much larger than c_8—at least, until its loop corrections grow to dominate $a_{2,0}$.[5] Of course, in practice, the "natural" expectation is that $|c_8| \sim |c_{10}|$ are comparable when there are no symmetries or selection rules to suggest otherwise, since this is invariably what happens in the simplest UV completions.

IV. EXAMPLE UV COMPLETION

The unnatural hierarchy in higher-dimensional operator coefficients that a UV theory must have to belong to Region $\bar{1}$ leads us to expect a UV-completion that goes beyond the simplest possibilities. In the absence of an explicit realization of such a UV theory, it is nonetheless useful to see a concrete example of a simple model in which the obstacles can be made explicit.

Consider a toy model with a heavy real scalar ϕ of mass M, a light complex scalar H and a Dirac fermion Ψ. Their UV interactions include

$$\mathcal{L}_{\mathrm{UV}} \supset -(y\phi\bar{\Psi}\Psi + \mathrm{H.c.}) - \mu g_3\phi^3 - \mu g_1\phi|H|^2, \tag{15}$$

where y, g_1, g_3 are dimensionless couplings and μ is a dimensionful scale. Matching to the EFT Lagrangian equation (7), with

$$\mathcal{O}_8 = -\bar{\Psi}\Psi\partial_\mu\bar{\Psi}\partial^\mu\Psi, \tag{16}$$

using MATCHETE [23] yields

$$c_8 = (y + \bar{y})^2, \tag{17}$$

$$c_{10} = \frac{4g_1(3g_3 - g_1)\mu^2}{M^2}(y + \bar{y})^2. \tag{18}$$

We see that c_8 has a definite positive sign, while c_{10} can have arbitrary sign depending on our choice for the couplings g_1, g_3. Substituting into Eq. (10), assuming c_{10} is negative, then gives an apparent bound on the vev,

$$\frac{v^2}{M^2} < \frac{|c_8|}{|c_{10}|} = \frac{1}{|g_1(3g_3 - g_1)|}\frac{M^2}{\mu^2}. \tag{19}$$

However, this cannot impose a hierarchy $v^2/M^2 \ll 1$ by more than a loop factor, since it would require taking $\mu \gg M$ which is associated with a breakdown of perturbation theory when $(g_i\mu)^2 \geq 4\pi M^2$. A more systematic exploration of potential UV completions is warranted.

V. EXPERIMENTAL PROSPECTS

We have so far discussed the positivity bound Eq. (1) as a theoretical constraint on the EFT parameter space. To potentially establish this bound experimentally depends on whether a low-energy observer has access to measurements at different energy scales within the EFT. From hereon we fix the scalar field H to be the complex Higgs doublet of the electroweak theory. Expanding around the vev, we also define the singlet radial mode h via $H = (0, v + h(x))^T/\sqrt{2}$.

In the $|H|^2 = v^2/2$ vacuum of the SM, and at very low energies E below the Higgs mass m_h, i.e., $E < m_h < \Lambda$, an experimentalist can only measure the physical combination $\bar{c}_8$ of Eq. (9), for example via $AB \to AB$ scattering. One cannot determine the vev contribution to $\bar{c}_8$ by doing measurements at these low energy scales. At higher energies, $m_H < E < \Lambda$, but still within the EFT, the Higgs' radial mode h can be produced on-shell and participate in scattering processes, so that c_{10} can be measured *e.g.* via the $AB \to ABh$ or $AB \to ABhh$ processes. See the Appendix for more detailed discussion.

In the absence of a direct determination of $c_8 = f_8(0)$ it is not possible to experimentally establish a bound on the vev from measurements alone, but we can still obtain indirect evidence for being in Region $\bar{1}$ if we were to measure $\bar{c}_8/\Lambda^4$ and *e.g.* vc_{10}/Λ^6, and thence infer that $\bar{c}_8 \ll |c_{10}|$ assuming the cutoff scale Λ is at least a TeV, for instance—a reasonable assumption given current null results in direct searches.[6] This is expected to be the case in Region $\bar{1}$ since $\bar{c}_8$ is bounded from above by c_{10}. Therefore, even if we are not able to explicitly establish an upper bound on the Higgs vev through measurements, if we indeed live in Region $\bar{1}$ then a consequence is that we should find an unnatural suppression of a particular ratio of higher-dimensional operator coefficients and no new physics below a certain scale.

Dimension-8 operators have been constrained at the LHC in various processes [24–37], with promising prospects for much higher sensitivity at future colliders [36–39]. As an illustrative example, we consider a set of flavor-universal,

[5] $\mathcal{O}_{10}$ will generically contribute to $a_{2,0}$ via two- and three-loop processes, and we describe in the Appendix how to include these in the positivity bound when the EFT is strongly coupled.

[6] If evidence of nonzero dimension-6 EFT coefficients were also to be established in experiment, one could put a more meaningful "prior" on the scale Λ that would then imply an even smaller ratio of $\bar{c}_8/|c_{10}|$ the higher the scale Λ is above a TeV.

four-lepton, dimension-8 operators and their dimension-10 counterparts where an $|H|^2$ is attached to the corresponding dimension-8 operator. The dimension-8 operators are defined as

$$\begin{aligned}\mathcal{O}_8^{(1)} &= \partial^\nu(\bar{e}_i\gamma^\mu e_i)\partial_\nu(\bar{e}_i\gamma_\mu e_i),\\ \mathcal{O}_8^{(2)} &= \partial^\nu(\bar{e}_i\gamma^\mu e_i)\partial_\nu(\bar{L}_i\gamma_\mu L_i),\\ \mathcal{O}_8^{(3)} &= D^\nu(\bar{e}_i L_i)D_\nu(\bar{L}_i e_i),\\ \mathcal{O}_8^{(4)} &= \partial^\nu(\bar{L}_i\gamma^\mu L_i)\partial_\nu(\bar{L}_i\gamma_\mu L_i),\end{aligned} \tag{20}$$

where i labels the lepton flavor, and L and e denote the left-handed lepton doublets and right-handed (charged) lepton singlets respectively. The full set of positivity bounds that can be obtained from four-lepton scattering amplitudes was computed in Ref. [39].

We compute projected sensitivities at future lepton colliders to the corresponding $\bar{c}_8^{(i)}$ and $c_{10}^{(i)}$, taking FCC-ee [40], CLIC [41], and a hypothetical 10 TeV muon collider [42] (μC) as typical examples across a range of center-of-mass energies. The processes that are sensitive to $\bar{c}_8^{(i)}$ and $c_{10}^{(i)}$ are $e^+e^- \to e^+e^-$ and $e^+e^- \to e^+e^-h$, respectively (replacing e with μ for the μC). The FCC-ee and CLIC dimension-8 projections are taken from Ref. [39], and we computed the μC projections assuming 10 ab^{-1} of integrated luminosity.

For the $e^+e^- \to e^+e^-$ projections, the differential cross-section with respect to the cosine of the lepton polar angle, $\cos\theta$, was used as a discriminating variable, taking 25 equally spaced bins in the interval $[-1, 1]$. We performed an identical analysis to Ref. [39] for the μC projections, also assuming a 1% systematic uncertainty in each bin.

For $e^+e^- \to e^+e^-h$, we instead used the cosine of the angular separation between the lepton and antilepton, $\cos(\Delta\theta_{\ell^+\ell^-})$ and assumed only statistical uncertainties. For a given collider energy, $E_{\mathrm{c.m.}}$, the dominant $e^+e^- \to Zh$ background, with the Z boson decaying on-shell to a pair of leptons, has a shoulder at

$$\cos(\Delta\theta_{\ell^+\ell^-}) = 1 - 2\left(\frac{2m_Z E_{\mathrm{c.m.}}}{E_{\mathrm{c.m.}}^2 + m_Z^2 - m_h^2}\right)^2, \tag{21}$$

where m_Z and m_h denote the Z and Higgs boson masses, respectively. In the high-energy limit, the position of the shoulder tends to 1. We exploited this feature to define optimized bin boundaries for the intermediate energy experiments where $E_{\mathrm{c.m.}} > m_Z + m_h$, finding that the sensitivity did not depend on increasing the number of bins, as long as one boundary was defined suitably near the predicted $\cos(\Delta\theta_{\ell^+\ell^-})$ shoulder and another was placed about halfway between -1 and the shoulder. For the higher energy experiments, good sensitivity was obtained by having a few bins near $\cos(\Delta\theta_{\ell^+\ell^-}) = -1$, where the majority of the SM background resides. The specific binnings used for each collider and energy stage are given in Table I. The e^+e^- collider run conditions were chosen to match Ref. [39] and summarized in Table II with the exception of the polarized beam at CLIC.

TABLE I. Binnings used for the differential distributions in $\cos(\Delta\theta_{\ell^+\ell^-})$ to determine our sensitivity projections for dimension-10 operators in $\ell^+\ell^- \to \ell^+\ell^-h$ at future lepton collider experiments.

Collider	$E_{\mathrm{c.m.}}$ [GeV]	$\cos(\Delta\theta_{\ell^+\ell^-})$ bin edges
FCC-ee	161	10 equally spaced bins in $[-1, 1]$
	240	$[-1, -0.38, 1]$
	350	$[-1, -0.31, 0.49, 1]$
	365	$[-1, -0.28, 0.54, 1]$
CLIC	380	$[-1, -0.26, 0.59, 1]$
	1500	$[-1, -0.95, -0.9, 1]$
	3000	$[-1, -0.95, -0.9, -0.85, -0.8, 1]$
μC	10000	$[-1, -0.95, -0.9, 1]$

TABLE II. Run conditions assumed for the projected sensitivities shown in Fig. 2.

Collider	Runs: Energy [GeV] (Luminosity [ab^{-1}])			
FCC-ee	161 (10)	240 (5)	350 (0.2)	365 (1.5)
CLIC	380 (0.5)	1500 (2)	3000 (4)	
μC	10000 (10)			

The projected sensitivities are show in Fig. 2, represented by a scale, Λ, with solid bars assuming $\bar{c}_8^i, c_{10}^i = 1$ and hatched bars corresponding to the maximally strongly coupled scenario of $\bar{c}_8^{(i)}, c_{10}^{(i)} = (4\pi)^2$. The projected reach in Λ increases (decreases) for coefficients induced by strongly (weakly) coupled new physics. All colliders probe Λ above their centre of mass energies for $c^{(i)} < (4\pi)^2$,[7] thus ensuring the validity of the EFT for at least some range of couplings. A 3 TeV CLIC or 10 TeV μC offer the best prospects for probing the 4-lepton operators we consider, while FCC-ee, even though it is primarily a precision machine, can still access the multi-TeV scale for large $\bar{c}_8^{(i)}$. The scale that can be probed for dimension-10 (dimension-8) operators is $\sim 10(200)$ TeV at the upper limit of strongly coupled scenarios. For such values of $c_{10}^{(i)}$, the dimension-8 coefficient corresponding to a little hierarchy in the vev with a 10 TeV EFT cutoff would be $|\bar{c}_8^{(i)}| \sim 0.1$, which Fig. 2 shows is within experimental reach.

We note that it is not possible to experimentally establish a bound on the vev without a measurement in a different

[7] An FCC-hh machine [40] could probe even higher energy scales for similar operators involving quarks, but we limit our comparison to lepton colliders here.

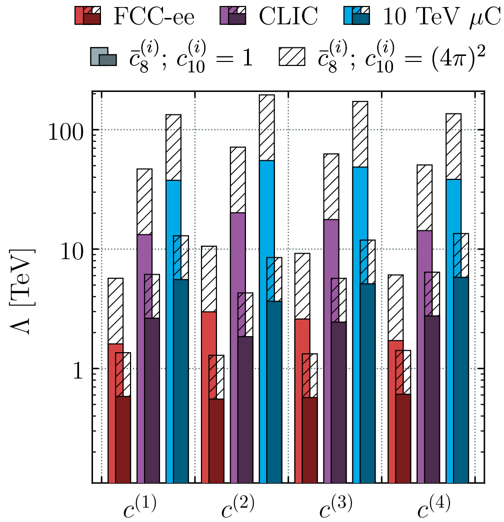

FIG. 2. 95% CL projected sensitivity of FCC-ee (red), CLIC 3 TeV (purple), and a 10 TeV muon collider (blue) to four-lepton dimension-8 operators (lighter shade, taken from Ref. [39] for FCC-ee and CLIC) and to their corresponding dimension-10 operators with a $|H|^2$ attached (darker shade). The filled bars assume a Wilson coefficient of 1, while the hatched bars assume a value of $(4\pi)^2$.

vacuum phase within the EFT or at least another independent determination of the EFT cutoff scale. More precisely, starting from the more general EFT described by Eq. (11), one can relate the function f_8 to these scattering processes by expanding the Lagrangian around the vacuum:

$$\mathcal{L} = \left[f_8\left(\frac{v^2}{2\Lambda^2}\right) + f_8'\left(\frac{v^2}{2\Lambda^2}\right)\frac{vh}{\Lambda^2} + \mathcal{O}(h^2)\right]\frac{\mathcal{O}_8}{\Lambda^4}. \quad (22)$$

So, by measuring $AB \to AB$ and $AB \to ABh$ scattering, we can extract the quantities $\frac{1}{\Lambda^4} f_8(v^2/2\Lambda^2) = \frac{\bar{c}_8}{\Lambda^4}$ and $\frac{v}{\Lambda^6} f_8'(v^2/2\Lambda^2)$ respectively. From the former alone, it is possible to ascertain whether the positivity bound Eq. (12) is satisfied. The latter measurement is of a dimension-9 HEFT operator $h\mathcal{O}_8$, that comes from a dimension-10 SMEFT operator whose coefficient in units of $1/\Lambda^6$ is c_{10}.

However, with these measurements alone one cannot use positivity [say, in the form of Eq. (13)] to infer whether the vev is bounded to be hierarchically small. The vev contribution to $\bar{c}_8$ is $f_8(v^2/2\Lambda^2) - f_8(0) \approx \frac{v^2}{2\Lambda^2} f_8'(v^2/2\Lambda^2)$ up to a dimension-12 contribution, and we are sensitive to the right-hand side (rhs) by doing measurements of $AB \to ABh$ at these intermediate energy scales. But without an independent extraction of the cutoff scale Λ we cannot turn this into a bound on v. Alternatively, as can be seen from Eq. (13), one could unambiguously establish the bound on v/Λ by accessing the value of f_8 (or its derivative) at a different value of $|H|^2/\Lambda^2$ (e.g. zero), say by measuring $AB \to AB(h)$ scattering in a different (meta-)stable vaccum. While this is in principle possible within the EFT, it is difficult to imagine doing any such measurement in practice.

Finally, we note that a vev contribution can also arise from dimension-8 operators contributing to dimension-6 positivity bounds [8,14,15], which require additional UV assumptions but are experimentally more accessible. We leave a detailed phenomenological study of experimental prospects to future work.

VI. CONCLUSION

We proposed a novel interpretation of positivity bounds when scalar vevs are taken into account. A positivity bound on a single higher-dimensional operator coefficient at low energies may subsume contributions from the vevs of scalars that only become apparent at higher energies in the next layer of EFT, where the scalar degrees of freedom can be produced on-shell. One consequence is the existence of a region of EFT parameter space where positivity is conditional upon a hierarchy in the scalar vev and the EFT cutoff. This is illustrated in Fig. 3.

While EFTs in this region feature an unnatural ratio of higher-dimensional operator coefficients, it is intriguing that such ratios may be related to fine-tuning in the electroweak hierarchy assuming only unitarity, causality and locality in the UV. There are not many other cases where potential phenomena in the IR, together with reasonable assumptions on the UV, lead to a restricted spectrum of allowed Higgs vevs. Perhaps the closest example of a UV assumption relating the Higgs vev to a different IR observable is the connection between a feeble

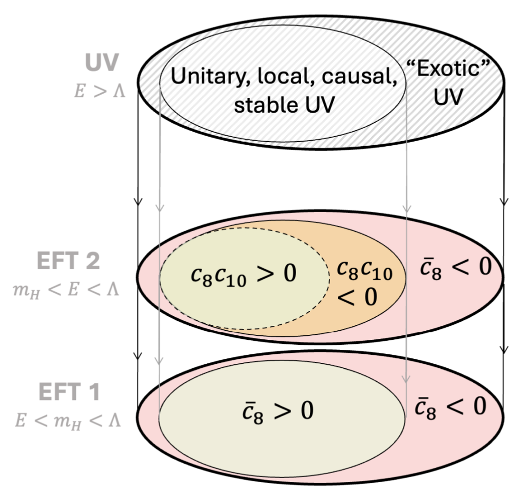

FIG. 3. Illustration of how the assumptions of a unitary, local, and causal UV with a stable vacuum v translate to constraints on the EFT parameter space in the IR. Region 1 where positivity is satisfied corresponds to the green subset in the lowest energy EFT 1 below a scalar mass m_H, which is split into further subsets in an EFT 2 at an intermediate scale above m_H but below the EFT cutoff Λ. Scalar vev contributions are relevant for interpreting the bounds in the orange Region $\bar{1}$.

fifth force and an upper bound on the Higgs vev assuming the weak gravity conjecture holds in the UV [3,5,43]. More generally, there can be other UV mechanisms that restrict the range of scalar field vevs, for instance the swampland distance conjecture [44]. It is worthwhile exploring such unconventional relations between vevs in the IR and properties of the underlying UV physics that may help us better understand the hierarchy problem.

The possibility of living in the special EFT region we have identified may not be so far-fetched. After all, the coefficients of operators at dimensions 0, 2, and 4 in the Higgs potential all indicate that our universe is highly nongeneric in many ways: not only is the Higgs quadratic term finely tuned to lie at the boundary of broken and unbroken phases, the Higgs quartic in the SM places us in a sliver of parameter space between vacuum stability and instability, while the cosmological constant value is precariously balanced between implosion and explosion. Dimension-6 operators could furthermore extend the connection between near-criticality and parameters of the SM [45–47]. It may well be that dimension-8 and −10 operators similarly place us on another boundary—at the edge of positive and nonpositive theory space.

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APPENDIX

Here we collect various technical details about the positivity bounds applied in the main text. In particular, we summarize the two-sided bounds that forbid unnatural hierarchies between successively higher-dimensional operator coefficients (that are not in conflict with the positivity bounds with vevs that we have studied), and recall the assumptions under which one may derive dimension-6 positivity bounds.

We assume that the scattering amplitude for the elastic process $AB \to AB$ can be computed using the low-energy EFT at center-of-mass energies $s < s_{\max}$, where $s_{\max}$ is the precise EFT cutoff (which may differ from the fiducial scale Λ by factors of the coupling constants, as well as 4π's and other combinatoric factors). In the complex s-plane, we therefore have an accurate determination of the amplitude inside a disc of radius $s_{\max}$. Subtracting any singularities which appear inside this disc[8] leads to an analytic amplitude which can be written as in Eq. (3). The EFT coefficients $a_{i,j}$ can be extracted from Eq. (3) by $\partial_s^i \partial_t^j \mathcal{A}(s,t)|_{s=t=0}$, or equivalently by integrating $\partial_t^j \mathcal{A}/s^{i+1}$ around a closed contour which encircles the origin in the complex s-plane. The assumption of causality (or, more precisely, analyticity) in the UV implies that this low-energy contour can be deformed into a high-energy contour that encircles the singularities of the UV theory [6,7],

$$a_{i,j} = \frac{\partial_t^j}{j!}\left[\int_{s_{\max}}^{\infty} \frac{ds}{2\pi i}\left(\frac{\mathrm{Disc}_s \mathcal{A}(s,t)}{s^{i+1}} - \frac{\mathrm{Disc}_u \mathcal{A}(u,t)}{u^{i+1}}\right) - \mathrm{Res}_{s=\infty}\left(\frac{\mathcal{A}(s,t)}{s^i}\right)\right], \tag{A1}$$

where $u = -s - t + 2m_A^2 + 2m_B^2$ is the third Mandelstam variable, Disc_z is the discontinuity across the real z-axis, and $\mathrm{Res}_{s=\infty}$ is the residue at infinity. This "sum rule" explicitly connects the EFT coefficients to the underlying UV physics.

The second assumption we make of the UV is that time evolution is unitary. In terms of the amplitude, this implies $\mathrm{Disc}_s \mathcal{A} \geq 0$ via the well-known optical theorem.[9] If the quantum numbers of A and B are chosen so that there is a trivial crossing relation between the s- and u-channel, then unitarity also implies that $\mathrm{Disc}_u \mathcal{A} \geq 0$.[10] Our third UV assumption, locality (in the form of the Froissart bound), guarantees that the residue at infinity vanishes for all $i \geq 2$. Consequently, the $a_{2n,0}$ coefficients in the EFT expansion must obey the bound in Eq. (4) for all $n \geq 1$ if its UV completion is to be causal, unitary and local. This bound can only be saturated for any particular $a_{2n,0}$ if every EFT coefficient vanishes and the theory is trivially free at all energies [53].

Positivity bounds for t derivatives of the amplitude follow similarly from Eq. (A1) and suitable generalizations of the optical theorem. These take the general form given in Eq. (5), where the strongest α_n and β_n depend on details of the scattering process considered. Concretely, $2\beta_n = 2n+1$ for scalar fields [7] and $2\beta_n = 2n+1-|h_A+h_B|-|h_A-h_B|$ for massless spinning fields with helicities h_A and

[8]It will be convenient to also subtract any t-channel poles appearing in the EFT amplitude. Note that we are working with a nongravitational EFT, i.e., in the decoupling limit $G_N \to 0$ where the t-channel pole from graviton exchange can be neglected. Otherwise the subtraction of this s^2/t term would lead to an object that violates the Froissart bound, and for which the residue at $s = \infty$ in Eq. (A1) only vanishes for $i \geq 3$ [6,48].

[9]While there are different conventions in use for the overall phase of the amplitude, this can always be chosen so that the optical theorem implies a positive discontinuity.

[10]Positivity bounds can also be derived more generally with a nontrivial crossing relation between s- and u-channel [8,49–52].

h_B [8].[11] For a general $\mathcal{A}_{AB\to AB}$ amplitude, $\alpha_1/(16\pi^2) = 5!/a_{2,1}$ [9,10] and the upper bound in Eq. (5) roughly corresponds to $a_{2,0}$ being one loop factor away from $a_{2,1}^2$. For a maximally crossing symmetric amplitude like $\mathcal{A}_{AA\to AA}$ for a real scalar A, the additional constraints from crossing can be used to derive further bounds [11], the strongest of which in four spacetime dimensions is $\alpha_1 \approx 5.3$ [12]. Similar two-sided bounds can also be derived by applying crossing transformations to amplitudes which are not manifestly crossing symmetric: see [54] for one recent example. A very general feature of this different bounds is that they forbid a large hierarchy between $a_{2,1}$ and $a_{2,0}$.

In this work, we have studied the application of positivity to the coefficients c_8 and c_{10} of two operators which differ by the insertion of $|H|^2$. At tree level, these operators contribute only to the amplitude coefficient $a_{2,0}$, as shown in Eq. (10). However, as discussed in [7,9,55] and more recently in [10,56], including loop contributions from the EFT operators always leads to stronger positivity bounds. For instance, when $|c_{10}| \gg |c_8|$, the dominant corrections to $a_{2,0}$ are the two-loop process $AB\to ABh\to AB$ and three-loop process $AB\to ABhh\to AB$, which contribute to the branch cut discontinuity of the low-energy amplitude,

$$\frac{\mathrm{Disc}_s\mathcal{A}(s,0)}{2\pi i} = \frac{|c_{10}|^2 s^6}{\Lambda^{12}}\left(\frac{N_3}{(16\pi^2)^3} + \frac{N_2}{(16\pi^2)^2}\frac{v^2}{s}\right), \tag{A2}$$

where N_2 and N_3 are positive numerical constants that depend on the details of $\mathcal{O}_8$, and we have neglected further corrections in $m_{A,B}^2/s$ and in $|c_8|/|c_{10}|$. Subtracting this branch cut up to $s_{\max}$ produces the stronger positivity bound,

$$c_8 + c_{10}\frac{v^2}{\Lambda^2} > \frac{|c_{10}|^2 s_{\max}^4}{\Lambda^8}\left(\frac{2N_3}{3(16\pi^2)^3} + \frac{N_2}{(16\pi^2)^2}\frac{v^2}{s_{\max}}\right).$$

Supposing that the EFT cutoff lies in the range $v^2 \ll s_{\max} \lesssim \Lambda$ and that the coefficients $c_8 > 0$, $c_{10} < 0$ as in the positively light quadrant of Fig. 1, this positivity bound can be written as[12]

$$\frac{v^2}{\Lambda^2} < \frac{|c_8|}{|c_{10}|} - \frac{|c_{10}|s_{\max}^4}{\Lambda^8}\frac{2N_3}{3(16\pi^2)^3}, \tag{A3}$$

which is indeed *strictly stronger* than (1). Equation (A3) shows how stronger positivity bounds can place further constraints on the vev when in the Region $\bar{1}$ of the EFT parameter space.

Interestingly, it also demonstrates that while the bound on $a_{2,1}$ may not forbid a hierarchy in $|c_{10}|/|c_8|$, this ratio *cannot be made arbitrarily large*. In particular, the EFT energy up to which we can reliably subtract the branch cut may not exceed $s_{\max}$, satisfying

$$s_{\max}^4 \approx \Lambda^8\frac{3(16\pi^2)^3}{2N_3}\frac{|c_8|}{|c_{10}|^2}, \tag{A4}$$

otherwise this positivity bound is violated for any positive value of v^2. While the constant N_3 will differ between different $\mathcal{O}_8$, in practice this correction in Eq. (A3) will always be suppressed by three loop factors, in addition to the hierarchy $s_{\max}^4/\Lambda^8$. So while an arbitrary tuning of $|c_{10}|/|c_8|$ cannot be used to place arbitrarily tight bounds on the vev (without making the EFT cutoff dangerously low), there remains a large region of EFT parameter space in which both (i) the vev is constrained by positivity arguments, and (ii) the cutoff remains large and loops are under control.

In addition to the finite branch cut discontinuity on the right-hand-side of the positivity bound, loops can further affect the bounds by introducing an RG running of the Wilson coefficients. For instance in the case of (7), the dimension-10 interaction leads to a one-loop running of $c_8(\mu)$. This does not affect our argument since the positivity bound can be applied to the Lagrangian at any fixed RG scale μ. One could always fix μ at the outset (to some experimentally relevant value) and interpret the bounds above as constraints on the c_8/c_{10} hierarchy at that particular μ.

[11] For massive spinning fields, $2\beta_n = 2n + 1 - |h_A + h_B| - |h_A - h_B|_{\min}$, where the minimum is over $h_{A,B}$ at fixed total $h_A + h_B$.

[12] We have neglected the N_2 term since it is subleading when the parametric separation between $s_{\max}$ and v^2 is sufficiently large. Regardless of this separation, including both N_2 and N_3 contributions results in a stronger bound.

[1] C. de Rham, S. Kundu, M. Reece, A. J. Tolley, and S. Y. Zhou, arXiv:2203.06805.
[2] N. Craig, Eur. Phys. J. C 83, 825 (2023).
[3] D. Lust and E. Palti, J. High Energy Phys. 02 (2018) 040.
[4] N. Craig and S. Koren, J. High Energy Phys. 03 (2020) 037.
[5] C. Cheung and G. N. Remmen, J. High Energy Phys. 12 (2014) 087.

[6] A. Adams, N. Arkani-Hamed, S. Dubovsky, A. Nicolis, and R. Rattazzi, J. High Energy Phys. 10 (2006) 014.
[7] C. de Rham, S. Melville, A. J. Tolley, and S. Y. Zhou, Phys. Rev. D **96**, 081702 (2017).
[8] J. Davighi, S. Melville, and T. You, J. High Energy Phys. 02 (2022) 167.
[9] B. Bellazzini, F. Riva, J. Serra, and F. Sgarlata, Phys. Rev. Lett. **120**, 161101 (2018).
[10] B. Bellazzini, J. Elias Miró, R. Rattazzi, M. Riembau, and F. Riva, Phys. Rev. D **104**, 036006 (2021).
[11] A. J. Tolley, Z. Y. Wang, and S. Y. Zhou, J. High Energy Phys. 05 (2021) 255.
[12] S. Caron-Huot and V. Van Duong, J. High Energy Phys. 05 (2021) 280.
[13] A. Adams, A. Jenkins, and D. O'Connell, arXiv:0802.4081.
[14] G. N. Remmen and N. L. Rodd, Phys. Rev. D **105**, 036006 (2022).
[15] G. N. Remmen and N. L. Rodd, J. High Energy Phys. 09 (2022) 030.
[16] T. Cohen, N. Craig, X. Lu, and D. Sutherland, J. High Energy Phys. 03 (2021) 237.
[17] S. Melville and J. Noller, J. Cosmol. Astropart. Phys. 06 (2022) 031.
[18] S. Melville and J. Noller, Phys. Rev. D **101**, 021502 (2020); **102**, 049902(E) (2020).
[19] C. de Rham, S. Melville, and J. Noller, J. Cosmol. Astropart. Phys. 08 (2021) 018.
[20] J. Ellis, M. Madigan, K. Mimasu, V. Sanz, and T. You, J. High Energy Phys. 04 (2021) 279.
[21] J. J. Ethier, G. Magni, F. Maltoni, L. Mantani, E. R. Nocera, J. Rojo, E. Slade, E. Vryonidou, and C. Zhang (SMEFiT Collaboration), J. High Energy Phys. 11 (2021) 089.
[22] J. de Blas, Y. Du, C. Grojean, J. Gu, V. Miralles, M. E. Peskin, J. Tian, M. Vos, and E. Vryonidou, arXiv:2206.08326.
[23] J. Fuentes-Martín, M. König, J. Pagès, A. E. Thomsen, and F. Wilsch, Eur. Phys. J. C **83**, 662 (2023).
[24] B. Bellazzini, F. Riva, J. Serra, and F. Sgarlata, J. High Energy Phys. 11 (2017) 020.
[25] B. Bellazzini and F. Riva, Phys. Rev. D **98**, 095021 (2018).
[26] C. Zhang and S. Y. Zhou, Phys. Rev. D **100**, 095003 (2019).
[27] Q. Bi, C. Zhang, and S. Y. Zhou, J. High Energy Phys. 06 (2019) 137.
[28] G. N. Remmen and N. L. Rodd, J. High Energy Phys. 12 (2019) 032.
[29] C. Englert, G. F. Giudice, A. Greljo, and M. Mccullough, J. High Energy Phys. 09 (2019) 041.
[30] G. N. Remmen and N. L. Rodd, Phys. Rev. Lett. **125**, 081601 (2020); **127**, 149901(E) (2021).
[31] Q. Bonnefoy, E. Gendy, and C. Grojean, J. High Energy Phys. 04 (2021) 115.
[32] D. Ghosh, R. Sharma, and F. Ullah, J. High Energy Phys. 02 (2023) 199.
[33] J. Henriksson, B. McPeak, F. Russo, and A. Vichi, J. High Energy Phys. 06 (2022) 158.
[34] X. Li, K. Mimasu, K. Yamashita, C. Yang, C. Zhang, and S. Y. Zhou, J. High Energy Phys. 10 (2022) 107.
[35] J. Ellis, K. Mimasu, and F. Zampedri, J. High Energy Phys. 10 (2023) 051.
[36] J. Ellis, H. J. He, and R. Q. Xiao, Phys. Rev. D **107**, 035005 (2023).
[37] J. Ellis, S. F. Ge, and K. Ma, J. High Energy Phys. 04 (2022) 123.
[38] J. Gu, L. T. Wang, and C. Zhang, Phys. Rev. Lett. **129**, 011805 (2022).
[39] B. Fuks, Y. Liu, C. Zhang, and S. Y. Zhou, Chin. Phys. C **45**, 023108 (2021).
[40] A. Abada *et al.* (FCC Collaboration), Eur. Phys. J. C **79**, 474 (2019).
[41] J. de Blas *et al.* (CLIC Collaboration), The CLIC potential for new physics, 10.23731/CYRM-2018-003 (2018).
[42] C. Accettura, D. Adams, R. Agarwal, C. Ahdida, C. Aimè, N. Amapane, D. Amorim, P. Andreetto, F. Anulli, R. Appleby *et al.*, Eur. Phys. J. C **83**, 864 (2023).
[43] N. Craig, I. Garcia Garcia, and S. Koren, J. High Energy Phys. 09 (2019) 081.
[44] H. Ooguri and C. Vafa, Nucl. Phys. **B766**, 21 (2007).
[45] G. F. Giudice, M. McCullough, and T. You, J. High Energy Phys. 10 (2021) 093.
[46] J. Khoury and T. Steingasser, Phys. Rev. D **105**, 055031 (2022).
[47] T. Steingasser and D. I. Kaiser, Phys. Rev. D **108**, 095035 (2023).
[48] B. Bellazzini, C. Cheung, and G. N. Remmen, Phys. Rev. D **93**, 064076 (2016).
[49] B. Bellazzini, J. High Energy Phys. 02 (2017) 034.
[50] C. de Rham, S. Melville, A. J. Tolley, and S. Y. Zhou, J. High Energy Phys. 03 (2018) 011.
[51] C. de Rham, S. Melville, A. J. Tolley, and S. Y. Zhou, J. High Energy Phys. 03 (2019) 182.
[52] S. Melville, D. Roest, and D. Stefanyszyn, J. High Energy Phys. 02 (2020) 185.
[53] C. de Rham, S. Melville, A. J. Tolley, and S. Y. Zhou, J. High Energy Phys. 09 (2017) 072.
[54] B. Bellazzini, G. Isabella, S. Ricossa, and F. Riva, arXiv:2304.02550.
[55] C. de Rham, S. Melville, and A. J. Tolley, J. High Energy Phys. 04 (2018) 083.
[56] B. Bellazzini, M. Riembau, and F. Riva, Phys. Rev. D **106**, 105008 (2022).