HYPOELLIPTIC FUNCTIONAL INEQUALITIES

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Dedicated to the 75th birthday of Fulvio Ricci

Abstract. In this paper we derive a variety of functional inequalities for general homogeneous invariant hypoelliptic differential operators on nilpotent Lie groups. The obtained inequalities include Hardy, Sobolev, Rellich, Hardy-Littllewood-Sobolev, Gagliardo-Nirenberg, Caffarelli-Kohn-Nirenberg and Heisenberg-Pauli-Weyl type uncertainty inequalities. Some of these estimates have been known in the case of the sub-Laplacians, however, for more general hypoelliptic operators almost all of them appear to be new as no approaches for obtaining such estimates have been available. The approach developed in this paper relies on establishing integral versions of Hardy inequalities on homogeneous Lie groups, for which we also find necessary and sufficient conditions for the weights for such inequalities to be true. Consequently, we link such integral Hardy inequalities to different hypoelliptic inequalities by using the Riesz and Bessel kernels associated to the described hypoelliptic operators.

Contents

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1. Introduction

In this paper we are interested in developing approaches that allow one to derive a variety of functional inequalities for general homogeneous invariant hypoelliptic differential operators on nilpotent Lie groups. Inequalities of such type are important by themselves but also play an important role in wider analysis, in particular in view of the seminal results of Rothschild and Stein [36] linking the analysis of hypoelliptic differential operators on nilpotent (Lie) groups to differential operators on manifolds.

To give an idea of the obtained results and to put them in perspective we start by describing a collection of some of the obtained inequalities in the setting of sub-Laplacians on stratified (Lie) groups (homogeneous Carnot groups).

1.1. **Hardy-Sobolev-Rellich inequalities on stratified Lie groups.**

Hardy inequalities on stratified groups are extremely well investigated topic, with different versions of such inequalities known, also with best constants. While we can not possibly give a comprehensive bibliography for it here, we can refer to the recent book [37] for the literature reviews of the subject for the horizontal norm and for norms given in terms of the fundamental solutions of the sub-Laplacian, respectively.

However, the starting point for the investigation of this paper is the following version of the Hardy inequality recently obtained by Ciatti, Cowling and Ricci [6]. Let $\mathbb{G}$ be a stratified group of homogeneous dimension Q and let $\mathcal{L}$ be a sub-Laplacian on $\mathbb{G}$. Let $|\cdot|$ be homogeneous norm on $\mathbb{G}$. We refer to Section 2 for more details of this classical setting.

Let $1<p<\infty$ and let $T_\gamma f := |\cdot|^{-\gamma}\mathcal{L}^{-\gamma/2}f$ with $0<\gamma<Q/p$. Then, as it was shown in [6, Theorem A], the Hardy inequality for the fractional order operator $\mathcal{L}^{\gamma/2}$ can take the following form: the operator T_γ extends uniquely to a bounded operator on $L^p(\mathbb{G})$, and we have

$$\|T_\gamma\|_{L^p(\mathbb{G})\to L^p(\mathbb{G})} \lesssim 1 + C\gamma + O(\gamma^2). \tag{1.1}$$

We also refer to [6] for the history of (1.1).

Among other things, in this paper we extend the boundedness in (1.1) to the setting of general homogeneous invariant hypoelliptic differential operators taking place of the operator $\mathcal{L}$. Moreover, we extend such estimates to the range of $L^p - L^q$ estimates as well as give their critical versions in the case of $\gamma = Q/p$.

Let us list some of such results still in the simplified setting of the sub-Laplacians. First we observe that by combining (1.1) with Sobolev inequalities for the sub-Laplacian, we have the following extended version of (1.1):

- (**Hardy-Sobolev-Rellich inequalities on stratified groups**) Let $1<p\leq q<\infty$ and $0<a<Q/p$. Let $0\leq b<Q$ and $\frac{a}{Q}=\frac{1}{p}-\frac{1}{q}+\frac{b}{qQ}$. Then there exists a positive constant C such that

$$\left\|\frac{f}{|x|^{\frac{b}{q}}}\right\|_{L^q(\mathbb{G})} \leq C\|(-\mathcal{L})^{\frac{a}{2}}f\|_{L^p(\mathbb{G})} \tag{1.2}$$

  holds for all $f\in \dot{L}^p_a(\mathbb{G})$.

Here the space $\dot{L}^p_a(\mathbb{G})$ is the homogeneous Sobolev space over L^p of order a, based on the sub-Laplacian $\mathcal{L}$. The theory of such spaces has been extensively developed by Folland [19]. Consequently, more general results of this paper yield the following new version of the critical case of (1.2) for $a = Q/p$:

- (**Critical Hardy inequality for $a = Q/p$ on stratified groups**) Let $1 < p < r < \infty$ and $p \le q < (r-1)p'$, where $1/p + 1/p' = 1$. Then there exists a positive constant $C = C(p, q, r, Q)$ such that

$$\left\| \frac{f}{\left(\log\left(\mathrm{e} + \frac{1}{|x|}\right)\right)^{\frac{r}{q}} |x|^{\frac{Q}{q}}} \right\|_{L^q(\mathbb{G})} \le C(\|f\|_{L^p(\mathbb{G})} + \|(-\mathcal{L})^{\frac{Q}{2p}} f\|_{L^p(\mathbb{G})}) \tag{1.3}$$

  holds for all $f \in L^p_{Q/p}(\mathbb{G})$.

Thus, (1.3) gives the critical case of the Hardy type inequalities in [6, Theorem A].

Actually, in Section 3 we obtain all of the above inequalities for more general hypoelliptic operators on more general nilpotent groups, namely, on graded (Lie) groups. As far as we are aware there are no other Hardy type inequalities known on graded groups in the literature.

Note that the explicit best constants and explicit extremal functions to these inequalities are still an open problem, although some constants in their non-explicit form for these and other inequalities in this paper can be described in terms of the ground states of certain nonlinear PDEs and extremals of variational problems, see [42] and [44].

1.2. **Hardy-Sobolev-Rellich inequalities on graded Lie groups.** The setting of graded groups as developed by Folland and Stein [24] allows one to work efficiently with higher order hypoelliptic operators, contrary to only sub-Laplacians appearing on stratified groups.

We assume now that $\mathbb{G}$ is a nilpotent Lie group with a compatible dilation structure, i.e. a homogeneous (Lie) group. We refer to Section 2 for a precise (well-known) definition. Let Q be the homogeneous dimension of $\mathbb{G}$ and let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. Let $\mathcal{R}$ be a positive left-invariant homogeneous hypoelliptic invariant differential operator on $\mathbb{G}$ of homogeneous degree ν. Such operators are called *Rockland operators*. For instance, for the Heisenberg group, the sub-Laplacian and its powers are Rockland operators. If $\mathbb{G}$ is a stratified group with a basis $X_1, \ldots, X_n$ of the first stratum g_1, then the operators

$$\mathcal{R} = (-1)^m \sum_{j=1}^{n} a_j X_j^{2m}, \quad a_j > 0,$$

are positive Rockland operators for any $m \in \mathbb{N}$, yielding the sub-Laplacian for $m = 1$. More generally, for any graded group $\mathbb{G} \sim \mathbb{R}^n$ with dilation weights $\nu_1, \ldots, \nu_n$ and a basis $X_1, \ldots, X_n$ of the corresponding Lie algebra $\mathfrak{g}$ satisfying

$$D_r X_j = r^{\nu_j} X_j, \quad j = 1, \ldots, n, \ r > 0,$$

the operator

$$\mathcal{R} = \sum_{j=1}^{n} (-1)^{\frac{\nu_0}{\nu_j}} a_j X_j^{2\frac{\nu_0}{\nu_j}}, \quad a_j > 0, \tag{1.4}$$

is a Rockland operator of homogeneous degree $2\nu_0$, where ν_0 is any common multiple of $\nu_1, \ldots, \nu_n$. There are other examples of Rockland operators that can be adapted to special selections of vector fields generating the Lie algebra in a suitable way, such as for example the vector fields from the first stratum on the stratified groups. We can refer to [20, Section 4.1.2] for many other examples and a detailed discussion of Rockland operators.

In particular, the existence of such an operator is equivalent to the condition that the group is graded, and such operators can be characterised in terms of the representation theory of the group by the celebrated result of Helffer and Nourrigat [28]. We note that examples of graded groups include $\mathbb{R}^n$, the Heisenberg group, and general stratified groups. Again, for brevity, we refer to Section 2 for precise definitions.

Therefore, results for Rockland operators on graded groups can be viewed as the most general *differential* results in the setting on nilpotent Lie groups. As far as we know, none of the inequalities we now describe are known in such settings.

From now on we let $\mathcal{R}$ be a positive Rockland operator, that is, a positive left-invariant homogeneous hypoelliptic invariant differential operator on $\mathbb{G}$ of homogeneous degree ν. Its powers $\mathcal{R}^a$ are understood through the functional calculus on the whole of $\mathbb{G}$, extensively analysed in [20, 21].

We start with the following analogue of (1.2), which we also call the Hardy-Sobolev-Rellich inequalities since it contains the classical Hardy, Rellich and Sobolev inequalities:

- (**Hardy-Sobolev-Rellich inequalities on graded groups**) Let $1 < p \leq q < \infty$ and $0 < a < Q/p$. Let $0 \leq b < Q$ and $\frac{a}{Q} = \frac{1}{p} - \frac{1}{q} + \frac{b}{qQ}$. Then there exists a positive constant C such that
$$\left\| \frac{f}{|x|^{\frac{b}{q}}} \right\|_{L^q(\mathbb{G})} \leq C \|\mathcal{R}^{\frac{a}{\nu}} f\|_{L^p(\mathbb{G})} \tag{1.5}$$
holds for all $f \in \dot{L}^p_a(\mathbb{G})$.

In particular, for $q = p$ we obtain the general hypoelliptic family of the Hardy inequalities:
$$\left\| \frac{f}{|x|^{a}} \right\|_{L^p(\mathbb{G})} \leq C \|\mathcal{R}^{\frac{a}{\nu}} f\|_{L^p(\mathbb{G})}, \quad 1 < p < \infty, \ 0 < a < Q/p. \tag{1.6}$$
In particular, for $a = 1$ and $a = 2$ we obtain the hypoelliptic versions of **Hardy and Rellich inequalities**, respectively, which in this form are new already on the stratified groups since the operator $\mathcal{R}$ does not have to be a sub-Laplacian and can be of any order. At the same time, for $b = 0$, (1.5) gives an alternative proof of the **Sobolev inequality** obtained in [21]:
$$\|f\|_{L^q(\mathbb{G})} \leq C \|\mathcal{R}^{\frac{a}{\nu}} f\|_{L^p(\mathbb{G})}, \quad 1 < p < q < \infty, \ a = Q\left(\frac{1}{p} - \frac{1}{q}\right). \tag{1.7}$$

The homogeneous and inhomogeneous Sobolev spaces $\dot{L}^p_a(\mathbb{G})$ and $L^p_a(\mathbb{G})$ based on the positive left-invariant hypoelliptic differential Rockland operator $\mathcal{R}$ have been extensively investigated in [21] and [20, Section 4.4] to which we refer for the details of their properties. In these works, the authors generalised to graded groups the

Sobolev spaces based on the sub-Laplacian on stratified groups analysed by Folland in [19].

As a consequence of (1.5), we also get the following Heisenberg-Pauli-Weyl type uncertainty principle for general homogeneous invariant hypoelliptic differential operators:

- (**Uncertainty type principle on graded groups**). Let $1 < p \leq q < \infty$ and $0 < a < Q/p$. Let $0 \leq b < Q$ and $\frac{a}{Q} = \frac{1}{p} - \frac{1}{q} + \frac{b}{qQ}$. Then there exists a positive constant C such that
$$\|\mathcal{R}^{\frac{a}{\nu}} f\|_{L^p(\mathbb{G})} \||x|^{\frac{b}{q}} f\|_{L^{q'}(\mathbb{G})} \geq C \int_{\mathbb{G}} |f(x)|^2 dx \tag{1.8}$$
holds for all $f \in \dot{L}^p_a(\mathbb{G})$, where $1/q + 1/q' = 1$.

As in the stratified case, we have the following critical case of Hardy-Sobolev-Rellich inequalities:

- (**Critical Hardy inequality for $a = Q/p$ on graded groups**). Let $1 < p < r < \infty$ and $p \leq q < (r-1)p'$, where $1/p + 1/p' = 1$. Then there exists a positive constant $C = C(p, q, r, Q)$ such that
$$\left\| \frac{f}{\left(\log\left(\mathrm{e} + \frac{1}{|x|}\right)\right)^{\frac{r}{q}} |x|^{\frac{Q}{q}}} \right\|_{L^q(\mathbb{G})} \leq C \|f\|_{L^p_{Q/p}(\mathbb{G})} \tag{1.9}$$
holds for all $f \in L^p_{Q/p}(\mathbb{G})$.

Similarly to (1.9) was investigated in the Euclidean setting in [32].

1.3. Caffarelli-Kohn-Nirenberg and Gagliardo-Nirenberg inequalities on graded Lie groups.

First, let us recall the classical Caffarelli-Kohn-Nirenberg inequality [9]:

Theorem 1.1. *Let $n \in \mathbb{N}$ and let p, q, r, a, b, d, $\delta \in \mathbb{R}$ such that $p, q \geq 1$, $r > 0$, $0 \leq \delta \leq 1$, and*
$$\frac{1}{p} + \frac{a}{n}, \ \frac{1}{q} + \frac{b}{n}, \ \frac{1}{r} + \frac{c}{n} > 0 \tag{1.10}$$
where $c = \delta d + (1-\delta)b$. Then there exists a positive constant C such that
$$\||x|^c f\|_{L^r(\mathbb{R}^n)} \leq C \||x|^a |\nabla f|\|^{\delta}_{L^p(\mathbb{R}^n)} \||x|^b f\|^{1-\delta}_{L^q(\mathbb{R}^n)} \tag{1.11}$$
holds for all $f \in C_0^\infty(\mathbb{R}^n)$, if and only if the following conditions hold:
$$\frac{1}{r} + \frac{c}{n} = \delta\left(\frac{1}{p} + \frac{a-1}{n}\right) + (1-\delta)\left(\frac{1}{q} + \frac{b}{n}\right), \tag{1.12}$$
$$a - d \geq 0 \quad \text{if} \quad \delta > 0, \tag{1.13}$$
$$a - d \leq 1 \quad \text{if} \quad \delta > 0 \quad \text{and} \quad \frac{1}{r} + \frac{c}{n} = \frac{1}{p} + \frac{a-1}{n}. \tag{1.14}$$

The techniques developed in this paper also allow us to derive general hypoelliptic versions of Caffarelli-Kohn-Nirenberg and weighted Gagliardo-Nirenberg inequalities.

- **(Caffarelli-Kohn-Nirenberg inequalities on graded groups)**. Let $1 < p, q < \infty$, $\delta \in (0, 1]$ and $0 < r < \infty$ with $r \leq \frac{q}{1-\delta}$ for $\delta \neq 1$. Let $0 < a < Q/p$ and $\beta, \gamma \in \mathbb{R}$ with $\delta r(Q - ap - \beta p) \leq p(Q + r\gamma - r\beta)$ and $\beta(1-\delta) - \delta a \leq \gamma \leq \beta(1-\delta)$. Assume that $\frac{r(\delta Q + p(\beta(1-\delta)-\gamma-a\delta))}{pQ} + \frac{(1-\delta)r}{q} = 1$. Then there exists a positive constant C such that
$$\||x|^{\gamma} f\|_{L^r(\mathbb{G})} \leq C \left\|\mathcal{R}^{\frac{a}{\nu}} f\right\|^{\delta}_{L^p(\mathbb{G})} \left\||x|^{\beta} f\right\|^{1-\delta}_{L^q(\mathbb{G})} \tag{1.15}$$
holds for all $f \in \dot{L}^p_a(\mathbb{G})$.

In the Euclidean case $\mathbb{G} = (\mathbb{R}^n, +)$ with $Q = n$, if the conditions (1.10) are not satisfied, then the inequality (1.15) is not covered by Theorem 1.1. So, this actually also gives an extension of Theorem 1.1 with respect to the range of parameters. Let us give an example:

Example 1.2. *If $1 < p = q = r < n$, $a = 1$, $\mathcal{R} = -\Delta$ and $\gamma = \beta(1-\delta) - \delta$, then (1.15) takes the form*
$$\||x|^{\gamma} f\|_{L^p(\mathbb{R}^n)} \leq C \left\|(-\Delta)^{\frac{1}{2}} f\right\|^{\delta}_{L^p(\mathbb{R}^n)} \left\||x|^{\beta} f\right\|^{1-\delta}_{L^p(\mathbb{R}^n)}. \tag{1.16}$$
Here, we can take e.g. $\beta \leq -n/p$ or $\gamma \leq -n/p$ so that the conditions (1.10) are not satisfied, then the inequality (1.16) is not covered by Theorem 1.1.

We refer to [38] for the related analysis on stratified groups, [40] and [39] on homogeneous groups, namely, for Caffarelli-Kohn-Nirenberg type inequalities in terms of parameters but with radial derivative operator or horizontal gradient instead of Rockland operators.

We note that for $\beta = \gamma = 0$, inequality (1.15) also recovers the Garliardo-Nirenberg inequality (5.9) (see Remark 5.8), that is
$$\|f\|_{L^r(\mathbb{G})} \leq C \left\|\mathcal{R}^{\frac{a}{\nu}} f\right\|^{\delta}_{L^p(\mathbb{G})} \|f\|^{1-\delta}_{L^q(\mathbb{G})} \tag{1.17}$$
for all $f \in \dot{L}^p_a(\mathbb{G}) \cap L^q(\mathbb{G})$, previously established in [41] with an application to the global-in-time well-posedness of nonlinear damped wave equations related to Rockland operators on graded groups (see also [45] for nonlinear heat equations), where $a > 0$, $1 < p < Q/a$, $1 < q \leq r \leq pQ/(Q-ap)$ and $\delta = (1/q-1/r)(a/Q+1/q-1/p)^{-1}$.

We also refer to [4] for another type of Garliardo-Nirenberg inequality involving Besov norms on graded groups.

In [42] and [44] the best constant in the Gagliardo-Nirenberg inequality (1.17) with $q = p$ and its critical version ($a = Q/p$) and the Sobolev inequality with inhomogeneous norm are expressed in the variational form as well as in terms of the ground state solutions of the nonlinear Schrödinger equation.

1.4. **Integral Hardy inequalities on homogeneous Lie groups.** The described hypoelliptic Hardy-Sobolev-Rellich inequalities and their critical versions on graded groups follow from the following integral versions of Hardy inequalities that we can establish in the setting of general homogeneous groups. For example, we can obtain the Hardy-Sobolev-Rellich inequalities (1.18) by taking $T^{(1)}_a$ in the following result to be the Riesz kernel of a positive Rockland operator. Similarly, we obtain its critical versions by taking $T^{(2)}_a$ in (1.20) to be a combination of Riesz and Bessel kernels.

Thus, let now $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q, equipped with any fixed homogeneous quasi-norm $|\cdot|$. Then we have the following results:

- (**Integral Hardy inequality on homogeneous groups**) Let $1 < p \leq q < \infty$ and $0 < a < Q/p$. Let $0 \leq b < Q$ and $\frac{a}{Q} = \frac{1}{p} - \frac{1}{q} + \frac{b}{qQ}$. Assume that $|T_a^{(1)}(x)| \leq C_1 |x|^{a-Q}$ for some positive $C_1 = C_1(a, Q)$. Then there exists a positive constant $C = C(p, q, a, b)$ such that
$$\left\| \frac{f * T_a^{(1)}}{|x|^{\frac{b}{q}}} \right\|_{L^q(\mathbb{G})} \leq C \|f\|_{L^p(\mathbb{G})} \tag{1.18}$$
holds for all $f \in L^p(\mathbb{G})$.
- (**Critical integral Hardy inequality on homogeneous groups**) Let $1 < p < r < \infty$ and $p \leq q < (r-1)p'$, where $1/p + 1/p' = 1$. Assume that for $a = Q/p$ and for every $N > Q$ we have
$$|T_a^{(2)}(x)| \leq C_2 \begin{cases} |x|^{a-Q}, \text{ for } x \in \mathbb{G} \backslash \{0\}, \\ |x|^{-N}, \text{ for } x \in \mathbb{G} \text{ with } |x| \geq 1, \end{cases} \tag{1.19}$$
for some positive $C_2 = C_2(a, Q)$. Then there exists a positive constant $C = C(p, q, r, Q)$ such that
$$\left\| \frac{f * T_{Q/p}^{(2)}}{\left(\log\left(\mathrm{e} + \frac{1}{|x|}\right)\right)^{\frac{r}{q}} |x|^{\frac{Q}{q}}} \right\|_{L^q(\mathbb{G})} \leq C \|f\|_{L^p(\mathbb{G})} \tag{1.20}$$
holds for all $f \in L^p(\mathbb{G})$.

In the proof of (1.18) and (1.20) the following characterisation of weighted integral Hardy type inequalities plays an important role. In fact, the following results provide the characterisation of pairs of weights for the integral versions of Hardy inequalities to hold. For brevity, we only indicate the type of the obtained results referring to the corresponding theorems for precise characterising conditions.

- (**Integral Hardy inequality for $p \leq q$ on homogeneous groups**) Let $\{\phi_i\}_{i=1}^2$ and $\{\psi_i\}_{i=1}^2$ be positive functions on $\mathbb{G}$, and $1 < p \leq q < \infty$. Then we have
$$\left(\int_{\mathbb{G}} \left(\int_{B(0,|x|)} f(z) dz \right)^q \phi_1(x) dx \right)^{\frac{1}{q}} \leq C_3 \left(\int_{\mathbb{G}} (f(x))^p \psi_1(x) dx \right)^{\frac{1}{p}} \tag{1.21}$$
and
$$\left(\int_{\mathbb{G}} \left(\int_{\mathbb{G} \backslash B(0,|x|)} f(z) dz \right)^q \phi_2(x) dx \right)^{\frac{1}{q}} \leq C_4 \left(\int_{\mathbb{G}} (f(x))^p \psi_2(x) dx \right)^{\frac{1}{p}} \tag{1.22}$$
hold for all $f \geq 0$ a.e. on $\mathbb{G}$ if and only if $A_i(\phi_i, \psi_i) < \infty$, $i = 1, 2$, where $\{A_i\}_{i=1}^2$ are given in (3.3)-(3.4).
- (**Integral Hardy inequality for $p > q$ on homogeneous groups**) Let $\{\phi_i\}_{i=3}^4$ and $\{\psi_i\}_{i=3}^4$ be positive functions on $\mathbb{G}$, and $1 < q < p < \infty$ with

$1/\delta = 1/q - 1/p$. Then we have

$$\left(\int_{\mathbb{G}}\left(\int_{B(0,|x|)} f(z)dz\right)^q \phi_3(x)dx\right)^{\frac{1}{q}} \leq C_5\left(\int_{\mathbb{G}} (f(x))^p \psi_3(x)dx\right)^{\frac{1}{p}} \tag{1.23}$$

and

$$\left(\int_{\mathbb{G}}\left(\int_{\mathbb{G}\setminus B(0,|x|)} f(z)dz\right)^q \phi_4(x)dx\right)^{\frac{1}{q}} \leq C_6\left(\int_{\mathbb{G}} (f(x))^p \psi_4(x)dx\right)^{\frac{1}{p}} \tag{1.24}$$

hold for all $f \geq 0$ if and only if $A_i(\phi_i, \psi_i) < \infty$, $i = 3, 4$, where $\{A_i\}_{i=3}^4$ are given in (3.22)-(3.23).

- (**Weighted Hardy-Sobolev type inequality on homogeneous groups**) Let ϕ_5, ψ_5 be positive weight functions on $\mathbb{G}$ and let $1 < p \leq q < \infty$. Then there exists a positive constant C such that

$$\left(\int_{\mathbb{G}} \phi_5(x)|f(x)|^q dx\right)^{1/q} \leq C\left(\int_{\mathbb{G}} \psi_5(x)|\mathcal{R}_{|x|}f(x)|^p dx\right)^{1/p} \tag{1.25}$$

holds for radial functions f with $f(0) = 0$ if and only if $A_5(\phi_5, \psi_5) < \infty$, where A_5 is given in (3.69) and $\mathcal{R}_{|x|} := \frac{d}{d|x|}$ is the radial derivative.

We note that Hardy, Rellich and other related inequalities with respect to the radial derivative $\mathcal{R}_{|x|}$ have been investigated in [37] and [43].

1.5. **Weighted Hardy-Littlewood-Sobolev inequalities.** Let us give another illustration of the method of applying inequalities on homogeneous groups to obtain the corresponding hypoelliptic inequalities. First, in this paper we show that the integral Hardy inequalities (1.18) and (1.20) imply the following weighted versions of Hardy-Littlewood-Sobolev inequalities, still on general homogeneous groups:

- (**Weighted Hardy-Littlewood-Sobolev, or Stein-Weiss inequalities on homogeneous groups**) Let $0 < \lambda < Q$ and $1 < p, q < \infty$ be such that $1/p + 1/q + (\alpha + \lambda)/Q = 2$ with $0 \leq \alpha < Q/p'$ and $\alpha + \lambda \leq Q$, where $1/p + 1/p' = 1$. Then there exists a positive constant $C = C(Q, \lambda, p, \alpha)$ such that

$$\left|\int_{\mathbb{G}}\int_{\mathbb{G}} \frac{\overline{f(x)}g(y)}{|x|^\alpha |y^{-1}x|^\lambda} dxdy\right| \leq C\|f\|_{L^p(\mathbb{G})}\|g\|_{L^q(\mathbb{G})} \tag{1.26}$$

holds for all $f \in L^p(\mathbb{G})$ and $g \in L^q(\mathbb{G})$.

- (**Critical Hardy-Littlewood-Sobolev inequalities on homogeneous groups**) Let $1 < p < \infty$, $1 < q \leq p' < (r-1)q'$ and $q < r < \infty$, where $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. Let $T^{(2)}_{Q/p}(x)$ be as in (1.19). Then there exists a positive constant $C = C(p, q, r, Q)$ such that

$$\left|\int_{\mathbb{G}}\int_{\mathbb{G}} \frac{\overline{f(x)}g(y)T^{(2)}_{Q/q}(y^{-1}x)}{\left(\log\left(e + \frac{1}{|x|}\right)\right)^{\frac{r}{p'}} |x|^{\frac{Q}{p'}}} dxdy\right| \leq C\|f\|_{L^p(\mathbb{G})}\|g\|_{L^q(\mathbb{G})} \tag{1.27}$$

holds for all $f \in L^p(\mathbb{G})$ and $g \in L^q(\mathbb{G})$.

Consequently, similar to the outline of Section 1.4, by working with Riesz kernels of positive Rockland operators, we subsequently obtain the following hypoelliptic differential versions of Hardy-Littlewood-Sobolev inequalities:

- **(Weighted Hardy-Littlewood-Sobolev inequalities on graded groups)**. Let $1 < p, q < \infty$, $0 \leq a < Q/p$ and $0 \leq b < Q/q$. Let $0 < \lambda < Q$, $0 \leq \alpha < a + Q/p'$ and $0 \leq \beta \leq b$ be such that $(Q - ap)/(pQ) + (Q - q(b - \beta))/(qQ) + (\alpha + \lambda)/Q = 2$ and $\alpha + \lambda \leq Q$, where $1/p + 1/p' = 1$. Then there exists a positive constant $C = C(Q, \lambda, p, \alpha, \beta, a, b)$ such that
$$\left| \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{\overline{f(x)} g(y)}{|x|^{\alpha} |y^{-1}x|^{\lambda} |y|^{\beta}} dxdy \right| \leq C \|f\|_{\dot{L}^p_a(\mathbb{G})} \|g\|_{\dot{L}^q_b(\mathbb{G})} \tag{1.28}$$
holds for all $f \in \dot{L}^p_a(\mathbb{G})$ and $g \in \dot{L}^q_b(\mathbb{G})$.
- **(Critical Hardy-Littlewood-Sobolev inequalities on graded groups)**. Let $1 < p, q < \infty$, $0 \leq a < Q/p$, $0 \leq \beta \leq b < Q/q$. $Q(1/p + 1/q - 1) + \beta - a - b \geq 0$, $\max\{\frac{Qq}{Q - bq + \beta q}, \frac{pq(a + b - \beta + 2Q) - Q(p + q)}{pq(Q + a) - Qq}\} < r < \infty$. Then there exists a positive constant $C = C(p, q, a, b, \beta, r, Q)$ such that
$$\left| \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{\overline{f(x)} g(y) \mathcal{B}_{Q/q}(y^{-1}x)}{\left(\log\left(\mathrm{e} + \frac{1}{|x|}\right)\right)^{\frac{r(pQ - Q + ap)}{pQ}} |x|^{a + \frac{Q}{p'}} |y|^{\beta}} dxdy \right| \leq C \|f\|_{\dot{L}^p_a(\mathbb{G})} \|g\|_{\dot{L}^q_b(\mathbb{G})} \tag{1.29}$$
holds for all $f \in \dot{L}^p_a(\mathbb{G})$ and $g \in \dot{L}^q_b(\mathbb{G})$, where $\mathcal{B}_{Q/p}$ is the Bessel kernel from (2.7).

Certainly, the Hardy-Littlewood-Sobolev inequalities is a very classical subject going back to Hardy-Littlewood [26], [27] and Sobolev [50]. In the setting of homogeneous groups, it was established by Folland and Stein [23] on the Heisenberg group, and its sharp constants were also investigated in [29] and [18] in the Euclidean and in the Heisenberg group settings. As for the logarithmic Hardy-Littlewood-Sobolev inequalities we can refer to e.g. [10], [30], [5] and the recent paper [15] as well as the references therein. In the appendix in Section 6 we note a simple equality between best constants in certain Hardy-Littlewood-Sobolev and Sobolev inequalities.

The organisation of the paper is as follows. In Section 2 we briefly recall the necessary concepts of homogeneous Lie groups and fix the notation. In Section 3 we introduce the weighted integral Hardy inequalities and in Section 4 we apply them to obtain the Hardy-Littlewood-Sobolev inequalities on homogeneous groups. The Hardy-Sobolev-Rellich and Caffarelli-Kohn-Nirenberg inequalities on graded groups are established in Section 5. In the appendix in Section 6 we breifly discuss the best constants in certain Hardy-Littlewood-Sobolev and Sobolev inequalities.

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2. Preliminaries

Following Folland and Stein [24, Chapter 1] and the recent exposition in [20, Chapter 3] let us recall that a family of dilations of a Lie algebra $\mathfrak{g}$ is a family of linear mappings of the following form

$$D_\lambda = \mathrm{Exp}(A\ln\lambda) = \sum_{k=0}^{\infty}\frac{1}{k!}(\ln(\lambda)A)^k,$$

where A is a diagonalisable linear operator on $\mathfrak{g}$ with positive eigenvalues. We also recall that D_λ is a morphism of $\mathfrak{g}$ if it is a linear mapping from $\mathfrak{g}$ to itself satisfying the property

$$\forall X, Y \in \mathfrak{g},\ \lambda > 0,\ [D_\lambda X, D_\lambda Y] = D_\lambda[X, Y],$$

where $[X, Y] := XY - YX$ is the Lie bracket. Then, a *homogeneous group* $\mathbb{G}$ is a connected simply connected Lie group whose Lie algebra is equipped with a morphism family of dilations. It induces the dilation structure on $\mathbb{G}$ which we denote by $D_\lambda x$ or just by λx.

We call $\mathbb{G}$ a *graded Lie group* if its Lie algebra $\mathfrak{g}$ admits a gradation

$$\mathfrak{g} = \bigoplus_{i=1}^{\infty}\mathfrak{g}_i,$$

where the $\mathfrak{g}_1, \mathfrak{g}_2, ...,$ are vector subspaces of the Lie algebra $\mathfrak{g}$, all but finitely many equal to $\{0\}$, and satisfying

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \ \ \forall i, j \in \mathbb{N}.$$

Every graded Lie group is also a homogeneous group with the dilation structure induced by the commutator relations.

The triple $\mathbb{G} = (\mathbb{R}^n, \circ, D_\lambda)$ is called a *stratified group* if it satisfies the conditions:

- For some natural numbers $N = N_1, N_2, ..., N_r$ with $N + N_2 + \ldots + N_r = n$, the following decomposition $\mathbb{R}^n = \mathbb{R}^N \times \ldots \times \mathbb{R}^{N_r}$ is valid, and for each $\lambda > 0$ the dilation $D_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ defined by
$$D_\lambda(x) = D_\lambda(x', x^{(2)}, \ldots, x^{(r)}) := (\lambda x', \lambda^2 x^{(2)}, \ldots, \lambda^r x^{(r)})$$
is an automorphism of the stratified group $\mathbb{G}$. Here $x' \equiv x^{(1)} \in \mathbb{R}^N$ and $x^{(k)} \in \mathbb{R}^{N_k}$ for $k = 2, \ldots, r$.
- Let N be as in above and let $X_1, \ldots, X_N$ be the left invariant vector fields on the stratified group $\mathbb{G}$ such that $X_k(0) = \frac{\partial}{\partial x_k}|_0$ for $k = 1, \ldots, N$. Then
$$\mathrm{rank}(\mathrm{Lie}\{X_1, \ldots, X_N\}) = n,$$
for each $x \in \mathbb{R}^n$, that is, the iterated commutators of $X_1, \ldots, X_N$ span the Lie algebra of the stratified group $\mathbb{G}$.

Note that the left invariant vector fields $X_1, \ldots, X_N$ are called the (Jacobian) generators of the stratified group $\mathbb{G}$ and r is called a step of this stratified group $\mathbb{G}$. For the expressions for left invariant vector fields on $\mathbb{G}$ in terms of the usual (Euclidean) derivatives and further properties see e.g. [20, Section 3.1.5].

As usual we always assume that $\mathbb{G}$ is connected and simply connected. If we fix a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$ adapted to the gradation, then by the exponential mapping $\exp_{\mathbb{G}} : \mathfrak{g} \to \mathbb{G}$ we obtain points $x \in \mathbb{G}$:

$$x = \exp_{\mathbb{G}}(x_1X_1 + \ldots + x_nX_n).$$

Let A be a diagonalisable linear operator on the Lie algebra $\mathfrak{g}$ with positive eigenvalues. Then, a family of linear mappings of the form

$$D_r = \mathrm{Exp}(A \ln r) = \sum_{k=0}^{\infty} \frac{1}{k!}(\ln(r)A)^k$$

is a family of dilations of the Lie algebra $\mathfrak{g}$. Each D_r is a morphism of $\mathfrak{g}$, that is, D_r is a linear mapping from the Lie algebra $\mathfrak{g}$ to itself with the following property

$$\forall X, Y \in \mathfrak{g},\ r > 0,\ [D_r X, D_r Y] = D_r[X, Y],$$

where $[X, Y] := XY - YX$ is the Lie bracket. We can always extend these dilations through the exponential mapping to the group $\mathbb{G}$ by

$$D_r(x) = rx := (r^{\nu_1} x_1, \ldots, r^{\nu_n} x_n), \quad x = (x_1, \ldots, x_n) \in \mathbb{G}, \quad r > 0, \tag{2.1}$$

where $\nu_1, \ldots, \nu_n$ are weights of the dilations. The sum of these weights

$$Q := \mathrm{Tr}\, A = \nu_1 + \cdots + \nu_n$$

is called the homogeneous dimension of $\mathbb{G}$. Recall the fact that the standard Lebesgue measure dx on $\mathbb{R}^n$ is the Haar measure for $\mathbb{G}$ (see, e.g. [20, Proposition 1.6.6]). The continuous non-negative function

$$\mathbb{G} \ni x \mapsto |x| \in [0, \infty)$$

satisfying the following properties:

- $|x^{-1}| = |x|$ for any $x \in \mathbb{G}$,
- $|\lambda x| = \lambda |x|$ for any $x \in \mathbb{G}$ and $\lambda > 0$,
- $|x| = 0$ if and only if $x = 0$,

is called a homogeneous quasi-norm on $\mathbb{G}$.

In the sequel we will need the following well-known facts, see e.g. [20, Proposition 3.1.38 and Theorem 3.1.39]:

Proposition 2.1. *Let $\mathbb{G}$ be a homogeneous Lie group and let $|\cdot|$ be an arbitrary homogeneous quasi-norm on $\mathbb{G}$. Then there exists a constant C_0 such that*

$$|xy| \le C_0(|x| + |y|) \tag{2.2}$$

holds for all $x, y \in \mathbb{G}$. At the same time, there always exists a homogeneous quasi-norm $|\cdot|$ on $\mathbb{G}$ which satisfies the triangle inequality

$$|xy| \le |x| + |y| \tag{2.3}$$

for all $x, y \in \mathbb{G}$.

The quasi-ball centred at $x \in \mathbb{G}$ with radius $R > 0$ can be defined by

$$B(x, R) := \{y \in \mathbb{G} : |x^{-1} y| < R\}.$$

There exists a (unique) positive Borel measure σ on the sphere

$$\mathfrak{S} := \{x \in \mathbb{G} :\ |x| = 1\}, \tag{2.4}$$

such that for all $f \in L^1(\mathbb{G})$ there holds

$$\int_{\mathbb{G}} f(x)dx = \int_0^{\infty} \int_{\mathfrak{S}} f(ry) r^{Q-1} d\sigma(y) dr. \tag{2.5}$$

We denote by $\widehat{\mathbb{G}}$ the unitary dual of $\mathbb{G}$ and by $\mathcal{H}_\pi^\infty$ the space of smooth vectors for a representation $\pi \in \widehat{\mathbb{G}}$. If the left-invariant differential operator $\mathcal{R}$ on $\mathbb{G}$, which is homogeneous of positive degree, satisfies the following condition:

(**Rockland condition**) for every representation $\pi \in \widehat{\mathbb{G}}$, except for the trivial representation, the operator $\pi(\mathcal{R})$ is injective on $\mathcal{H}_\pi^\infty$, that is,

$$\forall v \in \mathcal{H}_\pi^\infty, \quad \pi(\mathcal{R})v = 0 \Rightarrow v = 0,$$

then the left-invariant differential operator $\mathcal{R}$ is called a Rockland operator. Here, $\pi(\mathcal{R}) := d\pi(\mathcal{R})$ is the infinitesimal representation of the Rockland operator $\mathcal{R}$ as of an element of the universal enveloping algebra of $\mathbb{G}$.

Different characterisations of the Rockland operators have been obtained by Rockland [34] and Beals [3]. We refer to [21] and [20, Chapter 4] for an extensive presentation about Rockland operators and for the theory of Sobolev spaces on graded groups, and refer to [12] for the Besov spaces on graded Lie groups.

By Helffer and Nourrigat [28], we know that one can also define *Rockland operators as left-invariant homogeneous hypoelliptic differential operators on* $\mathbb{G}$, since this is equivalent to the Rockland condition.

Since we will deal with the Riesz and Bessel potentials, let us recall them on graded groups, and prove some useful estimates. Let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν. Then, the operators $\mathcal{R}^{-a/\nu}$ for $\{a \in \mathbb{R}, 0 < a < Q\}$ and $(I + \mathcal{R})^{-a/\nu}$ for $a \in \mathbb{R}_+$ are called Riesz and Bessel potentials, respectively. If we denote their kernels by $\mathcal{I}_a$ and $\mathcal{B}_a$, then we have

$$\mathcal{I}_a(x) := \frac{1}{\Gamma\left(\frac{a}{\nu}\right)} \int_0^\infty t^{\frac{a}{\nu}-1} h_t(x) dt \tag{2.6}$$

for $0 < a < Q$ with $a \in \mathbb{R}$, and

$$\mathcal{B}_a(x) := \frac{1}{\Gamma\left(\frac{a}{\nu}\right)} \int_0^\infty t^{\frac{a}{\nu}-1} \mathrm{e}^{-t} h_t(x) dt \tag{2.7}$$

for $a > 0$, where Γ denotes the Gamma function, and h_t is the heat kernel associated to the positive Rockland operator $\mathcal{R}$. We refer for more details to [20, Section 4.3.4].

Before using $\mathcal{I}_a(x)$ and $\mathcal{B}_a(x)$, we recall the following results:

Theorem 2.2 ([20, Theorem 4.2.7])**.** *Let $\mathcal{R}$ be a positive Rockland operator on a graded Lie group $\mathbb{G}$. Let $|\cdot|$ be a fixed homogeneous quasi-norm. Let h_t be a heat kernel associated with the Rockland operator. Then each h_t is Schwartz and we have*

$$\forall s, t > 0 \quad h_t * h_s = h_{t+s}, \tag{2.8}$$

$$\forall x \in \mathbb{G}, r, t > 0 \quad h_{r^\nu t}(rx) = r^{-Q} h_t(x), \tag{2.9}$$

$$\forall x \in \mathbb{G} \quad h_t(x) = \overline{h_t(x^{-1})}, \tag{2.10}$$

$$\int_{\mathbb{G}} h_t(x) dx = 1. \tag{2.11}$$

Moreover, we have

$$\exists C = C_{\alpha,N,\ell} > 0 \;\; \forall t \in (0,1] \quad \sup_{|x|=1} |\partial_t^\ell X^\alpha h_t(x)| \leq C_{\alpha,N} t^N \tag{2.12}$$

for any $N \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^n$ and $\ell \in \mathbb{N}_0$.

Lemma 2.3 ([20, Lemma 4.3.8]). *Let $\mathcal{R}$ be a positive Rockland operator on graded Lie group $\mathbb{G}$ and let h_t be its heat kernel as in Theorem 2.2. Let $|\cdot|$ be a homogeneous quasi-norm and $\alpha \in \mathbb{N}_0^n$ be a multi-index. Then for any real number a with $0 < a < (Q + [\alpha])/\nu$ there exists a positive constant C such that*

$$\int_0^\infty t^{a-1}|X^\alpha h_t(x)|dt \le C|x|^{-Q-[\alpha]+\nu a}. \tag{2.13}$$

Replacing a by a/ν and putting $\alpha = 0$ in Lemma 2.3, and using the representation (2.6) for $\mathcal{I}_a(x)$, we obtain

Lemma 2.4. *Let $|\cdot|$ be a homogeneous quasi-norm. Let $0 < a < Q$ and $a \in \mathbb{R}$. Then there exists a positive constant $C = C(Q, a)$ such that*

$$|\mathcal{I}_a(x)| \le C|x|^{-(Q-a)}. \tag{2.14}$$

Now let us prove the following useful lemma for $\mathcal{B}_a$, which may be not optimal (for example, the exponential decay is known on $\mathbb{R}^n$, see [1]), but sufficient for our purposes.

Lemma 2.5. *Let $|\cdot|$ be a homogeneous quasi-norm. Let $0 < a < Q$ and $a \in \mathbb{R}$. Then there exists a positive constant $C = C(Q, a)$ such that*

$$|\mathcal{B}_a(x)| \le C \begin{cases} |x|^{-(Q-a)}, \text{ for } x \in \mathbb{G}\backslash\{0\}, \\ |x|^{-N}, \text{ for } x \in \mathbb{G} \ \text{ with } |x| \ge 1 \end{cases} \tag{2.15}$$

for every N.

Proof of Lemma 2.5. We split the integral in (2.15) as follows

$$\begin{aligned} \mathcal{B}_a(x) &= \frac{1}{\Gamma\left(\frac{a}{\nu}\right)} \int_0^\infty t^{\frac{a}{\nu}-1} \mathrm{e}^{-t} h_t(x) dt \\ &= \frac{1}{\Gamma\left(\frac{a}{\nu}\right)} \int_0^{|x|^\nu} t^{\frac{a}{\nu}-1} \mathrm{e}^{-t} h_t(x) dt + \frac{1}{\Gamma\left(\frac{a}{\nu}\right)} \int_{|x|^\nu}^\infty t^{\frac{a}{\nu}-1} \mathrm{e}^{-t} h_t(x) dt \\ &=: J_1(x) + J_2(x). \end{aligned} \tag{2.16}$$

To estimate J_1 using the property of homogeneity of h_t in (2.9), we calculate

$$\begin{aligned} |J_1(x)| &= \left| \frac{1}{\Gamma\left(\frac{a}{\nu}\right)} \int_0^{|x|^\nu} t^{\frac{a}{\nu}-1} \mathrm{e}^{-t} |x|^{-Q} h_{|x|^{-\nu} t}\left(\frac{x}{|x|}\right) dt \right| \\ &\le \frac{1}{\Gamma\left(\frac{a}{\nu}\right)} |x|^{-Q} \left(\sup_{|y|=1, 0\le t_1\le 1} |h_{t_1}(y)| \right) \int_0^{|x|^\nu} t^{\frac{a}{\nu}-1} dt \\ &= \frac{\nu}{a\Gamma\left(\frac{a}{\nu}\right)} |x|^{a-Q} \left(\sup_{|y|=1, 0\le t_1\le 1} |h_{t_1}(y)| \right) \\ &\le C|x|^{a-Q}, \end{aligned} \tag{2.17}$$

where we have used that $\sup\limits_{|y|=1,\, 0\le t_1\le 1} |h_{t_1}(y)|$ is finite by (2.12).

Now we estimate J_2. A direct calculation gives that

$$\begin{aligned} |J_2(x)| &= \left|\frac{1}{\Gamma\left(\frac{a}{\nu}\right)}\int_{|x|^\nu}^\infty t^{\frac{a}{\nu}-1}\mathrm{e}^{-t}h_t(x)dt\right| \\ &\le \frac{1}{\Gamma\left(\frac{a}{\nu}\right)}\int_{|x|^\nu}^\infty t^{\frac{a}{\nu}-1}t^{-\frac{Q}{\nu}}|h_1(t^{-\frac{1}{\nu}}x)|dt \\ &\le \frac{1}{\Gamma\left(\frac{a}{\nu}\right)}\|h_1\|_{L^\infty(\mathbb{G})}\int_{|x|^\nu}^\infty t^{\frac{a}{\nu}-1-\frac{Q}{\nu}}dt \\ &\le C|x|^{a-Q}, \end{aligned} \tag{2.18}$$

where we have used that $\|h_1\|_{L^\infty(\mathbb{G})}$ is finite since h_1 is Schwartz. Combining (2.16), (2.17) and (2.18), we obtain (2.15).

On the other hand, when $|x| \ge 1$, using that h_1 is Schwartz, one has for every N that

$$\begin{aligned} |J_1(x)| &\lesssim \int_0^{|x|^\nu} t^{\frac{a-Q}{\nu}-1}\mathrm{e}^{-t}h_1\left(t^{-\frac{1}{\nu}}x\right)dt \\ &\lesssim |x|^{-N}\int_0^{|x|^\nu} t^{\frac{a-Q+N}{\nu}-1}\mathrm{e}^{-t}dt \\ &\lesssim |x|^{-N}\int_0^\infty t^{\frac{a-Q+N}{\nu}-1}\mathrm{e}^{-t}dt \\ &\lesssim |x|^{-N}, \end{aligned} \tag{2.19}$$

and, again using the first line in (2.19), and that h_1 is Schwartz, we get

$$|J_2(x)| \lesssim \int_{|x|^\nu}^\infty t^{\frac{a-Q}{\nu}-1}\mathrm{e}^{-t}dt \lesssim \int_{|x|^\nu}^\infty t^{\frac{a-Q-N}{\nu}-1}dt, \tag{2.20}$$

showing that $\mathcal{B}_a(x)$ is rapidly decreasing at ∞. Combining (2.16), (2.19) and (2.20), we obtain (2.15) for $|x| \ge 1$. ☐

3. Weighted integral Hardy inequalities on homogeneous Lie groups

In this section we introduce various types of weighted $L^p - L^q$ inequalities for the Hardy operator on homogeneous groups for different ranges of indices $1 < p, q < \infty$. We obtain necessary and sufficient condition on weights for such inequalities to be true. Subsequently, we apply them (Theorem 3.1) to obtain an integral Hardy inequality on general homogeneous groups which will be crucial for the further investigation of this paper. For a version of this result on more general metric measure spaces with polar decomposition see also [46].

Theorem 3.1. *Let $\mathbb{G}$ be a homogeneous Lie group of homogeneous dimension Q. Let $\{\phi_i\}_{i=1}^2$ and $\{\psi_i\}_{i=1}^2$ be positive functions on $\mathbb{G}$, and let $1 < p \le q < \infty$. Then the inequalities*

$$\left(\int_{\mathbb{G}}\left(\int_{B(0,|x|)} f(z)dz\right)^q \phi_1(x)dx\right)^{\frac{1}{q}} \le C_3\left(\int_{\mathbb{G}}(f(x))^p\psi_1(x)dx\right)^{\frac{1}{p}} \tag{3.1}$$

and

$$\left(\int_{\mathbb{G}}\left(\int_{\mathbb{G}\setminus B(0,|x|)} f(z)dz\right)^{q}\phi_2(x)dx\right)^{\frac{1}{q}} \leq C_4\left(\int_{\mathbb{G}}(f(x))^p\psi_2(x)dx\right)^{\frac{1}{p}} \tag{3.2}$$

hold for all $f \geq 0$ a.e. on $\mathbb{G}$ if and only if, respectively, we have

$$A_1 := \sup_{R>0}\left(\int_{\{|x|\geq R\}}\phi_1(x)dx\right)^{\frac{1}{q}}\left(\int_{\{|x|\leq R\}}(\psi_1(x))^{-(p'-1)}dx\right)^{\frac{1}{p'}} < \infty \tag{3.3}$$

and

$$A_2 := \sup_{R>0}\left(\int_{\{|x|\leq R\}}\phi_2(x)dx\right)^{\frac{1}{q}}\left(\int_{\{|x|\geq R\}}(\psi_2(x))^{-(p'-1)}dx\right)^{\frac{1}{p'}} < \infty. \tag{3.4}$$

Moreover, if $\{C_i\}_{i=3}^4$ are the smallest constants for which (3.1) and (3.2) hold, then

$$A_i \leq C_i \leq (p')^{\frac{1}{p'}}p^{\frac{1}{q}}A_i, \quad i = 3,4. \tag{3.5}$$

Remark 3.2. In the abelian case $\mathbb{G} = (\mathbb{R}^n, +)$ and $Q = n$, if we take $q = p > 1$ and $\phi_1(x) = |B(0,|x|)|^{-p}$ and $\psi_1(x) = 1$ in (3.1), then we have $A_1 = (p-1)^{-1/p}$ and

$$\left(\int_{\mathbb{R}^n}\left|\frac{1}{|B(0,|x|)|}\int_{B(0,|x|)} f(z)dz\right|^p dx\right)^{\frac{1}{p}} \leq \frac{p}{p-1}\left(\int_{\mathbb{R}^n}|f(x)|^p dx\right)^{\frac{1}{p}}, \tag{3.6}$$

where $|B(0,|x|)|$ is the volume of the ball $B(0,|x|)$. The inequality (3.6) was obtained in [8].

Proof of Theorem 3.1. We prove (3.1)⇔(3.3), the case (3.2)⇔(3.4) can be proved similarly.

First, we show (3.3)⇒(3.1). Then, using polar coordinates on $\mathbb{G}$ and denoting $r = |x|$, we write

$$\begin{aligned}&\int_{\mathbb{G}}\phi_1(x)\left[\int_{B(0,r)} f(z)dz\right]^q dx\\ &= \int_0^\infty\int_{\mathfrak{S}} r^{Q-1}\phi_1(ry)\left[\int_0^r\int_{\mathfrak{S}} s^{Q-1}f(sy)d\sigma(y)ds\right]^q d\sigma(y)dr.\end{aligned} \tag{3.7}$$

Setting

$$g(r) = \left\{\int_{\mathfrak{S}}\int_0^r s^{Q-1}(\psi_1(sy))^{1-p'}dsd\sigma(y)\right\}^{1/(pp')}, \tag{3.8}$$

and using Hölder's inequality, we calculate

$$\begin{aligned}\int_0^r\int_{\mathfrak{S}} s^{Q-1}f(sy)d\sigma(y)ds &= \int_{\mathfrak{S}}\int_0^r s^{(Q-1)/p}f(sy)(\psi_1(sy))^{1/p}g(s)s^{(Q-1)/p'}\\ &\quad\times\left((\psi_1(sy))^{1/p}g(s)\right)^{-1}dsd\sigma(y)\\ &\leq \left(\int_{\mathfrak{S}}\int_0^r s^{Q-1}\left[f(sy)(\psi_1(sy))^{1/p}g(s)\right]^p dsd\sigma(y)\right)^{1/p}\\ &\quad\times\left(\int_{\mathfrak{S}}\int_0^r s^{Q-1}\left[(\psi_1(sy))^{1/p}g(s)\right]^{-p'}dsd\sigma(y)\right)^{1/p'}.\end{aligned} \tag{3.9}$$

If we define U, V and W_1 by

$$U(s)=\int_{\mathfrak{S}} s^{Q-1}\left(f(sy)(\psi_1(sy))^{1/p}g(s)\right)^{p}d\sigma(y), \tag{3.10}$$

$$V(r)=\int_0^r\int_{\mathfrak{S}} s^{Q-1}\left((\psi_1(sy))^{1/p}g(s)\right)^{-p'}d\sigma(y)ds, \tag{3.11}$$

$$W_1(r)=\int_{\mathfrak{S}} r^{Q-1}\phi_1(ry)d\sigma(y), \tag{3.12}$$

for $s, r>0$, respectively, then plugging (3.9) into (3.7) we obtain

$$\int_{\mathbb{G}}\phi_1(x)\left(\int_{B(0,r)}f(z)dz\right)^{q}dx\le\int_0^\infty W_1(r)\left(\int_0^r U(s)ds\right)^{q/p}(V(r))^{q/p'}dr. \tag{3.13}$$

Now we need to use the following continuous version of Minkowski's inequality (see e.g. [14, Formula 2.1]): Let $\theta\ge 1$. Then for all $f_1(x), f_2(x)\ge 0$ on $(0,\infty)$, we have

$$\int_0^\infty f_1(x)\left(\int_0^x f_2(z)dz\right)^{\theta}dx\le\left(\int_0^\infty f_2(z)\left(\int_z^\infty f_1(x)dx\right)^{1/\theta}dz\right)^{\theta}. \tag{3.14}$$

Using this with $\theta=q/p\ge 1$ on the right-hand side of (3.13), we get

$$\int_{\mathbb{G}}\phi_1(x)\left(\int_{B(0,r)}f(z)dz\right)^{q}dx$$
$$\le\left(\int_0^\infty U(s)\left(\int_s^\infty W_1(r)(V(r))^{q/p'}dr\right)^{p/q}ds\right)^{q/p}. \tag{3.15}$$

In order to simplify the right-hand side of above, denoting

$$T(s):=\int_{\mathfrak{S}} s^{Q-1}(\psi_1(sy))^{1-p'}d\sigma(y),$$

and using (3.8), (3.11), the integration by parts, (3.3) and (3.12) we compute

$$\begin{aligned}
V(r)&=\int_{\mathfrak{S}}\int_0^r s^{Q-1}(\psi_1(sy))^{1-p'}\left(\int_0^s\int_{\mathfrak{S}} t^{Q-1}(\psi_1(tw))^{1-p'}d\sigma(w)dt\right)^{-1/p}dsd\sigma(y)\\
&=\int_0^r T(s)\left(\int_0^s T(t)dt\right)^{-1/p}ds=p'\int_0^r\frac{d}{ds}\left(\int_0^s T(t)dt\right)^{1/p'}ds\\
&=p'\left(\int_0^r T(s)ds\right)^{1/p'}=p'\left(\int_0^r\int_{\mathfrak{S}} s^{Q-1}(\psi_1(sy))^{1-p'}d\sigma(y)ds\right)^{1/p'}\\
&\le p'A_1\left(\int_r^\infty s^{Q-1}\int_{\mathfrak{S}}\phi_1(sw)d\sigma(w)ds\right)^{-1/q}=p'A_1\left(\int_r^\infty W_1(s)ds\right)^{-1/q}.
\end{aligned}$$

Similarly, applying the integration by parts and (3.3), we have from above

$$\begin{aligned}
&\int_s^\infty W_1(r)(V(r))^{q/p'}dr \\
&\quad\le (p'A_1)^{q/p'}\int_s^\infty W_1(r)\left(\int_r^\infty W_1(s)ds\right)^{-1/p'}dr \\
&\quad= (p'A_1)^{q/p'}p\left(\int_s^\infty W_1(r)dr\right)^{1/p} \\
&\quad= (p'A_1)^{q/p'}p\left(\int_s^\infty\int_{\mathfrak{S}} r^{Q-1}\phi_1(ry)d\sigma(y)dr\right)^{1/p} \\
&\quad\le (p'A_1)^{q/p'}pA_1^{q/p}\left(\int_0^s r^{Q-1}\int_{\mathfrak{S}}(\psi_1(ry))^{1-p'}d\sigma(y)dr\right)^{-q/(p'p)} \\
&\quad= A_1^q(p')^{q/p'}p(g(s))^{-q},
\end{aligned} \tag{3.16}$$

where we have used (3.8) in the last line. Putting (3.16) in (3.15) and recalling (3.10), we obtain

$$\begin{aligned}
\int_{\mathbb{G}}\phi_1(x)\left(\int_{B(0,r)}f(z)dz\right)^q dx &\le \left(\int_0^\infty U(s)A_1^p(p')^{p-1}p^{p/q}(g(s))^{-p}ds\right)^{q/p} \\
&= A_1^q(p')^{q/p'}p\left(\int_0^\infty U(s)(g(s))^{-p}ds\right)^{q/p} \\
&= A_1^q(p')^{q/p'}p\left(\int_0^\infty\int_{\mathfrak{S}} s^{Q-1}(f(sy))^p\psi_1(sy)d\sigma(y)ds\right)^{q/p} \\
&= A_1^q(p')^{q/p'}p\left(\int_{\mathbb{G}}\psi_1(x)(f(x))^p dx\right)^{q/p},
\end{aligned} \tag{3.17}$$

yielding (3.1) with $C_3 = A_1(p')^{1/p'}p^{1/q}$.

Now it remains to show (3.1)⇒(3.3). For that, we take $f(x) = (\psi_1(x))^{1-p'}\chi_{(0,R)}(|x|)$ with $R > 0$ to get

$$\begin{aligned}
&\left(\int_{\mathbb{G}}\psi_1(x)(f(x))^p dx\right)^{1/p}\left(\int_{|x|\le R}(\psi_1(x))^{1-p'}dx\right)^{-1/p} \\
&\qquad= \left(\int_{|x|\le R}(\psi_1(x))^{1-p'}dx\right)^{1/p}\left(\int_{|x|\le R}(\psi_1(x))^{1-p'}dx\right)^{-1/p} = 1.
\end{aligned} \tag{3.18}$$

Consequently, by (3.1) we have

$$\begin{aligned} C=C\left(\int_{\mathbb{G}}\psi_1(x)(f(x))^p dx\right)^{1/p}\left(\int_{|x|\leq R}(\psi_1(x))^{1-p'}dx\right)^{-1/p} \\ \geq\left(\int_{\mathbb{G}}\phi_1(x)\left(\int_{|z|\leq|x|}f(z)dz\right)^q dx\right)^{1/q}\left(\int_{|x|\leq R}(\psi_1(x))^{1-p'}dx\right)^{-1/p} \\ \geq\left(\int_{|x|\geq R}\phi_1(x)\left(\int_{|z|\leq|x|}f(z)dz\right)^q dx\right)^{1/q}\left(\int_{|x|\leq R}(\psi_1(x))^{1-p'}dx\right)^{-1/p} \\ =\left(\int_{|x|\geq R}\phi_1(x)dx\right)^{1/q}\left(\int_{|z|\leq R}(\psi_1(z))^{1-p'}dz\right)^{1/p'}. \end{aligned} \tag{3.19}$$

Combining (3.18) and (3.19), we obtain (3.3) with $C\geq A_1$. □

Now we show the case $q<p$ of Theorem 3.1. For a (later) version of this result on metric measure spaces see also [47].

Theorem 3.3. *Let $\mathbb{G}$ be a homogeneous Lie group of homogeneous dimension Q. Let $\{\phi_i\}_{i=3}^4$ and $\{\psi_i\}_{i=3}^4$ be positive functions on $\mathbb{G}$, and let $1<q<p<\infty$ with $1/\delta=1/q-1/p$. Then the inequalities*

$$\left(\int_{\mathbb{G}}\left(\int_{B(0,|x|)}f(z)dz\right)^q\phi_3(x)dx\right)^{\frac{1}{q}}\leq C_5\left(\int_{\mathbb{G}}(f(x))^p\psi_3(x)dx\right)^{\frac{1}{p}} \tag{3.20}$$

and

$$\left(\int_{\mathbb{G}}\left(\int_{\mathbb{G}\backslash B(0,|x|)}f(z)dz\right)^q\phi_4(x)dx\right)^{\frac{1}{q}}\leq C_6\left(\int_{\mathbb{G}}(f(x))^p\psi_4(x)dx\right)^{\frac{1}{p}} \tag{3.21}$$

hold for all $f\geq 0$ if and only if, respectively, we have

$$A_3:=\int_{\mathbb{G}}\left(\int_{\mathbb{G}\backslash B(0,|x|)}\phi_3(z)dz\right)^{\delta/q}\left(\int_{B(0,|x|)}(\psi_3(z))^{1-p'}dz\right)^{\delta/q'}(\psi_3(x))^{1-p'}dx<\infty \tag{3.22}$$

and

$$A_4:=\int_{\mathbb{G}}\left(\int_{B(0,|x|)}\phi_4(z)dz\right)^{\delta/q}\left(\int_{\mathbb{G}\backslash B(0,|x|)}(\psi_4(z))^{1-p'}dz\right)^{\delta/q'}(\psi_4(x))^{1-p'}dx<\infty. \tag{3.23}$$

Proof of Theorem 3.3. We show (3.20)⇔(3.22), the case (3.21)⇔(3.23) can be proved similarly.

First, we prove (3.22)⇒(3.20). Denote

$$W_2(r):=\int_{\mathfrak{S}}r^{Q-1}\phi_3(ry)d\sigma(y) \tag{3.24}$$

and

$$G(s):=\int_{\mathfrak{S}}s^{Q-1}h(sy)(\psi_3(sy))^{1-p'}d\sigma(y) \tag{3.25}$$

for $h \geq 0$ on $\mathbb{G}$. Then using polar coordinates on $\mathbb{G}$, we calculate

$$\begin{aligned}
&\int_{\mathbb{G}} \phi_3(x) \left(\int_{B(0,|x|)} h(z)(\psi_3(z))^{1-p'} dz \right)^q dx \\
&= \int_0^\infty \int_{\mathfrak{S}} r^{Q-1} \phi_3(rw) d\sigma(w) \left(\int_0^r \int_{\mathfrak{S}} s^{Q-1} h(sy)(\psi_3(sy))^{1-p'} d\sigma(y) ds \right)^q dr \\
&= \int_0^\infty W_2(r) \left(\int_0^r G(s) ds \right)^q dr \\
&= q \int_0^\infty G(s) \left(\int_0^s G(r) dr \right)^{q-1} \left(\int_s^\infty W_2(r) dr \right) ds \\
&= q \int_{\mathfrak{S}} \int_0^\infty s^{Q-1} h(sy)(\psi_3(sy))^{1-p'} \left(\int_0^s \int_{\mathfrak{S}} r^{Q-1} h(rw)(\psi_3(rw))^{1-p'} d\sigma(w) dr \right)^{q-1} \\
&\times \left(\int_s^\infty W_2(r) dr \right) ds d\sigma(y) \\
&= q \int_{\mathfrak{S}} \int_0^\infty s^{Q-1} h(sy)(\psi_3(sy))^{(1-p')(\frac{1}{p} + \frac{q-1}{p} + \frac{p-q}{p})} \\
&\times \left(\frac{\int_{\mathfrak{S}} \int_0^s r^{Q-1} h(rw)(\psi_3(rw))^{1-p'} dr d\sigma(w)}{\int_{\mathfrak{S}} \int_0^s r^{Q-1} (\psi_3(rw))^{1-p'} dr d\sigma(w)} \right)^{q-1} \\
&\times \left(\left(\int_{\mathfrak{S}} \int_0^s r^{Q-1} (\psi_3(rw))^{1-p'} dr d\sigma(w) \right)^{q-1} \left(\int_s^\infty W_2(r) dr \right) \right) ds d\sigma(y).
\end{aligned}$$

Here, using Hölder's inequality (with three factors) for $\frac{1}{p} + \frac{q-1}{p} + \frac{p-q}{p} = 1$ we get

$$\int_{\mathbb{G}} \phi_3(x) \left(\int_{B(0,|x|)} h(z)(\psi_3(z))^{1-p'} dz \right)^q dx \leq q K_1 K_2 K_3, \tag{3.26}$$

where

$$K_1 = \left(\int_{\mathfrak{S}} \int_0^\infty s^{Q-1} (h(sy))^p (\psi_3(sy))^{1-p'} ds d\sigma(y) \right)^{1/p} = \left(\int_{\mathbb{G}} (h(x))^p (\psi_3(x))^{1-p'} dx \right)^{1/p}, \tag{3.27}$$

$$K_2 = \left(\int_{\mathfrak{S}} \int_0^\infty s^{Q-1} (\psi_3(sy))^{1-p'} \left(\frac{\int_{\mathfrak{S}} \int_0^s r^{Q-1} h(rw)(\psi_3(rw))^{1-p'} dr d\sigma(w)}{\int_{\mathfrak{S}} \int_0^s r^{Q-1} (\psi_3(rw))^{1-p'} dr d\sigma(w)} \right)^p ds d\sigma(y) \right)^{\frac{q-1}{p}} \tag{3.28}$$

and

$$\begin{aligned}
K_3 = \Bigg(\int_{\mathfrak{S}} \int_0^\infty s^{Q-1} (\psi_3(sy))^{1-p'} & \left(\int_{\mathfrak{S}} \int_0^s r^{Q-1} (\psi_3(rw))^{1-p'} dr d\sigma(w) \right)^{\frac{(q-1)p}{p-q}} \\
& \times \left(\int_s^\infty W_2(r) dr \right)^{\frac{p}{p-q}} ds d\sigma(y) \Bigg)^{\frac{p-q}{p}}.
\end{aligned} \tag{3.29}$$

We have for K_2 that

$$K_2=\left(\int_{\mathbb{G}}\frac{(\psi_3(x))^{1-p'}}{(\int_{B(0,|x|)}(\psi_3(z))^{1-p'}dz)^p}\left(\int_{B(0,|x|)}(\psi_3(z))^{1-p'}h(z)dz\right)^p dx\right)^{\frac{q-1}{p}}.$$

To apply (3.1) for K_2 with $q=p$, $f(x)=(\psi_3(x))^{1-p'}h(x)$ and

$$\phi_1(x)=\frac{(\psi_3(x))^{1-p'}}{(\int_{B(0,|x|)}(\psi_3(z))^{1-p'}dz)^p},\quad \psi_1(x)=(\psi_3(x))^{(1-p')(1-p)},$$

we need to check the condition that

$$A_1(R)=\left(\int_{|x|\geq R}(\psi_3(x))^{1-p'}\left(\int_{B(0,|x|)}(\psi_3(z))^{1-p'}dz\right)^{-p}dx\right)^{1/p}$$
$$\times\left(\int_{|x|\leq R}(\psi_3(x))^{1-p'}dx\right)^{1/p'}<\infty \quad (3.30)$$

holds uniformly for all $R>0$. Indeed, once (3.30) has been established, the inequality (3.1) implies that

$$K_2\leq C\left(\int_{\mathbb{G}}(\psi_3(x))^{(1-p')(1-p+p)}(h(x))^p dx\right)^{\frac{q-1}{p}}=C\left(\int_{\mathbb{G}}(h(x))^p(\psi_3(x))^{1-p'}dx\right)^{\frac{q-1}{p}}. \quad (3.31)$$

To check (3.30), denoting $S(s)=\int_{\mathfrak{S}}s^{Q-1}(\psi_3(sw))^{1-p'}d\sigma(w)$ and using the integration by parts we compute

$$\begin{aligned}A_1(R)&=\left(\int_{\mathfrak{S}}\int_R^\infty r^{Q-1}(\psi_3(rw))^{1-p'}\left(\int_0^r S(s)ds\right)^{-p}drd\sigma(w)\right)^{1/p}\left(\int_0^R S(s)ds\right)^{1/p'}\\
&=\left(\int_R^\infty\left(\int_0^r S(s)ds\right)^{-p}S(r)dr\right)^{1/p}\left(\int_0^R S(s)ds\right)^{1/p'}\\
&\leq\left(\frac{1}{p-1}\left(\int_0^R S(s)ds\right)^{1-p}\right)^{1/p}\left(\int_0^R S(s)ds\right)^{1/p'}=(p-1)^{-1/p}<\infty.\end{aligned}$$

Next, for K_3, taking into account $\frac{1}{\delta}=\frac{1}{q}-\frac{1}{p}=\frac{p-q}{pq}$ and using (3.22), we have

$$\begin{aligned}K_3&=\left(\int_0^\infty\int_{\mathfrak{S}}\left(\int_s^\infty W_2(r)dr\right)^{\delta/q}\left(\int_{\mathfrak{S}}\int_0^r r^{Q-1}(\psi_3(rw))^{1-p'}drd\sigma(w)\right)^{\delta/q'}\right.\\
&\quad\left.\times s^{Q-1}(\psi_3(sy))^{1-p'}d\sigma(y)ds\right)^{\frac{p-q}{p}}\\
&=\left(\int_{\mathbb{G}}\left(\int_{\mathbb{G}\setminus B(0,|x|)}\phi_3(z)dz\right)^{\delta/q}\left(\int_{B(0,|x|)}(\psi_3(z))^{1-p'}dz\right)^{\delta/q'}(\psi_3(x))^{1-p'}dx\right)^{\frac{p-q}{p}}\\
&=A_3^{\frac{p-q}{p}}<\infty.\end{aligned} \quad (3.32)$$

Now, plugging (3.27), (3.31) and (3.32) into (3.26), we obtain

$$\int_{\mathbb{G}}\phi_3(x)\left(\int_{B(0,|x|)}h(z)(\psi_3(z))^{1-p'}dz\right)^q dx \le CA_3^{\frac{p-q}{p}}\left(\int_{\mathbb{G}}(h(x))^p(\psi_3(x))^{1-p'}dx\right)^{\frac{1}{p}+\frac{q-1}{p}},$$

which implies (3.20) after the setting $h := f\psi_3^{p'-1}$.

To show (3.20)⇒(3.22), as in the Euclidean case [14, Theorem 2.2] we put the functions

$$f_k(x) = \left(\int_{|z|\ge|x|}\phi_3(z)dz\right)^{\delta/(pq)}\left(\int_{\alpha_k\le|z|\le|x|}(\psi_3(z))^{1-p'}dz\right)^{\delta/(pq')}$$
$$\times(\psi_3(x))^{1-p'}\chi_{(\alpha_k,\beta_k)}(|x|),\ k=1,2,\ldots,$$

instead of $f(x)$ in (3.20) to get (3.22), where $0<\alpha_k<\beta_k$ with $\alpha_k \searrow 0$ and $\beta_k \nearrow \infty$ for $k\to\infty$. □

Now we introduce another integral Hardy inequality.

Theorem 3.4. *Let $\mathbb{G}$ be a homogeneous Lie group of homogeneous dimension Q. Let $|\cdot|$ be an arbitrary homogeneous quasi-norm. Let $1<p\le q<\infty$ and $0<a<Q/p$. Let $0\le b<Q$ and $\frac{a}{Q}=\frac{1}{p}-\frac{1}{q}+\frac{b}{qQ}$. Assume that $|T_a^{(1)}(x)|\le C_1|x|^{a-Q}$ for some positive $C_1=C_1(a,Q)$. Then there exists a positive constant $C=C(p,q,a,b)$ such that*

$$\left\|\frac{f*T_a^{(1)}}{|x|^{\frac{b}{q}}}\right\|_{L^q(\mathbb{G})}\le C\|f\|_{L^p(\mathbb{G})} \tag{3.33}$$

holds for all $f\in L^p(\mathbb{G})$.

Proof of Theorem 3.4. We split the integral into three parts:

$$\int_{\mathbb{G}}|(f*T_a^{(1)})(x)|^q\frac{dx}{|x|^b}\le 3^q(M_1+M_2+M_3), \tag{3.34}$$

where

$$M_1 := \int_{\mathbb{G}}\left(\int_{\{2|y|<|x|\}}|T_a^{(1)}(y^{-1}x)f(y)|dy\right)^q\frac{dx}{|x|^b},$$
$$M_2 := \int_{\mathbb{G}}\left(\int_{\{|x|\le2|y|<4|x|\}}|T_a^{(1)}(y^{-1}x)f(y)|dy\right)^q\frac{dx}{|x|^b}$$

and

$$M_3 := \int_{\mathbb{G}}\left(\int_{\{|y|>2|x|\}}|T_a^{(1)}(y^{-1}x)f(y)|dy\right)^q\frac{dx}{|x|^b}.$$

First, let us estimate M_1. We can assume that $|\cdot|$ is a norm without loss of generality because of the existence of a homogeneous norm (Proposition 2.1) and since replacing the seminorm by an equivalent one only changes the appearing constants. Although we could give a proof without this hypothesis, it simplifies the arguments below. Then, by the reverse triangle inequality and $2|y|<|x|$ we have

$$|y^{-1}x|\ge|x|-|y|>|x|-\frac{|x|}{2}=\frac{|x|}{2}, \tag{3.35}$$

which is $|x| < 2|y^{-1}x|$. Taking into account this and that $T_a^{(1)}(x)$ is bounded by a radial function which is non-increasing with respect to $|x|$, we calculate

$$\begin{aligned} M_1 &\leq \int_{\mathbb{G}} \left(\int_{\{2|y|<|x|\}} |f(y)|dy \right)^q \left(\sup_{\{|x|<2|z|\}} |T_a^{(1)}(z)| \right)^q \frac{dx}{|x|^b} \\ &\leq C \int_{\mathbb{G}} \left(\int_{\{2|y|<|x|\}} |f(y)|dy \right)^q \left(\frac{|x|}{2} \right)^{(a-Q)q} \frac{dx}{|x|^b}. \end{aligned} \tag{3.36}$$

In order to apply (3.1) for M_1, let us check the condition (3.3), that is, that

$$\left(\int_{\{2R<|x|\}} \left(\frac{|x|}{2} \right)^{(a-Q)q} \frac{dx}{|x|^b} \right)^{\frac{1}{q}} \left(\int_{\{|x|<R\}} dx \right)^{\frac{1}{p'}} \leq A_1 \tag{3.37}$$

holds for all $R > 0$. Indeed, taking into account $\frac{a}{Q} = \frac{1}{p} - \frac{1}{q} + \frac{b}{qQ}$, hence $(a-Q)q-b+Q = -\frac{Qq}{p'} \neq 0$, we have

$$\begin{aligned} &\left(\int_{\{2R<|x|\}} \left(\frac{|x|}{2} \right)^{(a-Q)q} \frac{dx}{|x|^b} \right)^{\frac{1}{q}} \left(\int_{\{|x|<R\}} dx \right)^{\frac{1}{p'}} \\ &\qquad \leq CR^{\frac{Q}{p'}} \left(\int_{\{2R<|x|\}} \left(\frac{|x|}{2} \right)^{(a-Q)q} \frac{dx}{|x|^b} \right)^{\frac{1}{q}} \\ &\qquad \leq CR^{\frac{Q}{p'}} \left(\int_{\{2R<|x|\}} |x|^{(a-Q)q-b} dx \right)^{\frac{1}{q}} \\ &\qquad \leq CR^{\frac{Q}{p'}} R^{\frac{(a-Q)q-b+Q}{q}} \\ &\qquad \leq C, \end{aligned} \tag{3.38}$$

since $(a-Q)q - b + Q = -\frac{Qq}{p'} \neq 0$.

Thus, we have checked (3.37), then we can apply (3.1) for M_1 to obtain

$$M_1^{\frac{1}{q}} \leq (p')^{\frac{1}{p'}} p^{\frac{1}{q}} A_1 \|f\|_{L^p(\mathbb{G})}. \tag{3.39}$$

Now let us estimate M_3. Without loss of generality, we may assume again $|\cdot|$ is the norm. Then, similarly to (3.35) we note that $2|x| < |y|$ implies $|y| < 2|y^{-1}x|$. Taking into account this we obtain for M_3 that

$$M_3 \leq \int_{\mathbb{G}} \left(\int_{\{|y|>2|x|\}} \left(\frac{|y|}{2} \right)^{(a-Q)} |f(y)|dy \right)^q \frac{dx}{|x|^b}.$$

To apply (3.2) for M_3, we check the following condition:

$$\left(\int_{\{|x|<R\}} \frac{dx}{|x|^b} \right)^{\frac{1}{q}} \left(\int_{\{2R<|x|\}} \left(\frac{|x|}{2} \right)^{(a-Q)p'} dx \right)^{\frac{1}{p'}} \leq A_2. \tag{3.40}$$

Taking into account $Q \neq ap$, one gets

$$\left(\int_{\{2R<|x|\}}\left(\frac{|x|}{2}\right)^{(a-Q)p'}dx\right)^{\frac{1}{p'}} \leq C\left(\int_{\{2R<|x|\}}|x|^{(a-Q)p'}dx\right)^{\frac{1}{p'}} \leq CR^{a-\frac{Q}{p}}, \tag{3.41}$$

that is,

$$\left(\int_{\{|x|<R\}}\frac{dx}{|x|^b}\right)^{\frac{1}{q}}\left(\int_{\{2R<|x|\}}\left(\frac{|x|}{2}\right)^{(a-Q)p'}dx\right)^{\frac{1}{p'}} \leq CR^{a-\frac{Q}{p}+\frac{Q-b}{q}} \leq C,$$

since $b<Q$ and $a-\frac{Q}{p}+\frac{Q-b}{q}=0$ due to $\frac{a}{Q}=\frac{1}{p}-\frac{1}{q}+\frac{b}{qQ}$.

Thus, we have checked (3.40), then we can apply (3.2) for M_3 to get

$$M_3^{\frac{1}{q}} \leq (p')^{\frac{1}{p'}}p^{\frac{1}{q}}A_2\|f\|_{L^p(\mathbb{G})}. \tag{3.42}$$

Finally, we estimate M_2. We write

$$M_2=\sum_{k\in\mathbb{Z}}\int_{\{2^k\leqslant|x|<2^{k+1}\}}\left(\int_{\{|x|\leqslant 2|y|\leqslant 4|x|\}}|T_a^{(1)}(y^{-1}x)f(y)|dy\right)^q\frac{dx}{|x|^b}.$$

Since $|x|\leqslant 2|y|\leqslant 4|x|$ and $2^k\leqslant|x|<2^{k+1}$, we have $2^{k-1}\leqslant|y|<2^{k+2}$. As in (3.35), assuming $|\cdot|$ is the norm and using the triangle inequality, we have

$$3|x|=|x|+2|x|\geq|x|+|y|\geq|y^{-1}x|, \tag{3.43}$$

which implies $0\leq|y^{-1}x|\leq 3|x|<3\cdot 2^{k+1}$. If we denote $\widetilde{I}_a(x):=C_1|x|^{a-Q}$, then $|T_a^{(1)}(x)|\leq\widetilde{I}_a(x)$. Taking into account these, applying Young's inequality (well-known, see e.g. [20, Proposition 1.5.2]) for $1+\frac{1}{q}=\frac{1}{r}+\frac{1}{p}$ with $r\in[1,\infty]$ we estimate M_2 by

$$\begin{aligned}
M_2 &\leq \sum_{k\in\mathbb{Z}}2^{-kb}\int_{\mathbb{G}}(([f\cdot\chi_{\{2^{k-1}\leqslant|\cdot|<2^{k+2}\}}]*\widetilde{I}_a)(x))^q dx\\
&= \sum_{k\in\mathbb{Z}}2^{-kb}\|[f\cdot\chi_{\{2^{k-1}\leqslant|\cdot|<2^{k+2}\}}]*\widetilde{I}_a\|^q_{L^q(\mathbb{G})}\\
&\leq \sum_{k\in\mathbb{Z}}2^{-kb}\|\widetilde{I}_a\cdot\chi_{\{0\leqslant|\cdot|<3\cdot 2^{k+1}\}}\|^q_{L^r(\mathbb{G})}\|f\cdot\chi_{\{2^{k-1}\leqslant|\cdot|<2^{k+2}\}}\|^q_{L^p(\mathbb{G})}\\
&= C_1\sum_{k\in\mathbb{Z}}2^{-kb}\left(\int_{|x|<3\cdot 2^{k+1}}|x|^{(a-Q)r}dx\right)^{\frac{q}{r}}\|f\cdot\chi_{\{2^{k-1}\leqslant|x|<2^{k+2}\}}\|^q_{L^p(\mathbb{G})}\\
&\leq C\sum_{k\in\mathbb{Z}}2^{-kb}(3\cdot 2^{k+1})^{\left(\frac{(a-Q)pq}{pq+p-q}+Q\right)\frac{pq+p-q}{p}}\|f\cdot\chi_{\{2^{k-1}\leqslant|x|<2^{k+2}\}}\|^q_{L^p(\mathbb{G})}\\
&= C\sum_{k\in\mathbb{Z}}2^{-kb}(3\cdot 2^{k+1})^{b}\|f\cdot\chi_{\{2^{k-1}\leqslant|x|<2^{k+2}\}}\|^q_{L^p(\mathbb{G})}\\
&\leq C\sum_{k\in\mathbb{Z}}\|f\cdot\chi_{\{2^{k-1}\leqslant|x|<2^{k+2}\}}\|^q_{L^p(\mathbb{G})}\\
&\leq C\|f\|^q_{L^p(\mathbb{G})},
\end{aligned} \tag{3.44}$$

since $\frac{(a-Q)pq}{pq+p-q}+Q=\frac{bp}{pq+p-q}>0$ and $q\geq p$.

Thus, (3.39), (3.42) and (3.44) complete the proof of Theorem 3.4. □

Remark 3.5. Let us now very briefly discuss an alternative proof of Theorem 3.4 by using Schur's test [22].

In the case $q = p$, we have $b = ap$ from $\frac{a}{Q} = \frac{1}{p} - \frac{1}{q} + \frac{b}{qQ}$. Let $S_a f := |x|^{-b/p}(f * |x|^{a-Q})$, then $S_a^* g := (|x|^{-b/p} g) * |x|^{a-Q}$, where $(f, S_a^* g) = (S_a f, g)$. Since the integral kernel of S_a is positive, by Schur's test we see that instead of proving the estimate

$$\|S_a f\|_{L^p(\mathbb{G})} \leq A_{a,p}^{1/p'} B_{a,p}^{1/p} \|f\|_{L^p(\mathbb{G})}$$

for all $f \in L^p(\mathbb{G})$, it is enough to exhibit a positive function h and constants $A_{a,p}$ and $B_{a,p}$ such that

$$S_a(h^{p'})(x) \leq A_{a,p}(h(x))^{p'} \text{ and } S_a^*(h^p)(x) \leq B_{a,p}(h(x))^p$$

for almost all $x \in \mathbb{G}$.

Let us take $h_c(x) := |x|^{c-Q}$ with $c > 0$ and consider the convolution integrals

$$h_c^{p'} * |x|^{a-Q} \text{ and } (|x|^{-b/p} h_c^p) * |x|^{a-Q},$$

which arise in the computation of $S_a(h_c^{p'})$ and $S_a^*(h_c^p)$. We see that the homogeneity orders of $h_c^{p'}$ and $|x|^{-b/p} h_c^p$ are $(c - Q)p'$ and $(c - Q)p - b/p$, respectively. Then, the homogeneity of $h_c^{p'} * |x|^{a-Q}$ and $(|x|^{-b/p} h_c^p) * |x|^{a-Q}$ are $a - Q + (c - Q)p'$ and $a - Q + (c - Q)p - b/p$, respectively. Therefore, these convolution integrals converge absolutely in $\mathbb{G} \backslash \{0\}$ if and only if $0 < (c-Q)p' + Q < Q - a$ and $0 < (c-Q)p - b/p + Q < Q - a$, that is,

$$\max\left(\frac{Q}{p}, \frac{a}{p} + \frac{Q}{p'}\right) < c < Q - \frac{a}{p'}$$

since $b = ap$. This condition is true if $0 < a < Q/p$.

Thus, we have obtained

$$\||x|^{-b/p}(f * |x|^{a-Q})\|_{L^p(\mathbb{G})} \leq A_{a,p}^{1/p'} B_{a,p}^{1/p} \|f\|_{L^p(\mathbb{G})},$$

where $0 < a < Q/p$, $1 < p < \infty$, $f \in L^p(\mathbb{G})$ and $b = ap$.

Taking into account this and $|T_a^{(1)}(x)| \leq C|x|^{a-Q}$, we obtain

$$\left\| \frac{f * T_a^{(1)}}{|x|^{\frac{b}{p}}} \right\|_{L^p(\mathbb{G})} \leq C \left\| \frac{|f| * |T_a^{(1)}|}{|x|^{\frac{b}{p}}} \right\|_{L^p(\mathbb{G})}$$
$$\leq C \||x|^{-b/p}(|f| * |x|^{a-Q})\|_{L^p(\mathbb{G})} \leq C\|f\|_{L^p(\mathbb{G})}. \quad (3.45)$$

In the case $q > p$, the operator in (3.33) is dominated pointwise by

$$|x|^{-\frac{b}{q}} \left(\left(|f| * K_{a-\frac{b}{q}} \right) * K_{\frac{b}{q}} \right),$$

where $K_a(x) = |x|^{-Q+a}$, and the above is the composition of $g \mapsto g * K_{a-\frac{b}{q}}$, which maps L^p into L^q, followed by $h \mapsto |x|^{-\frac{b}{q}} \left(h * K_{\frac{b}{q}} \right)$ which falls in the $q = p$ case.

Now we also show the critical case $a = Q/p$ of Theorem 3.4.

Theorem 3.6. *Let $\mathbb{G}$ be a homogeneous Lie group of homogeneous dimension Q. Let $|\cdot|$ be an arbitrary homogeneous quasi-norm and let $1 < p < r < \infty$ and $p \le q < (r-1)p'$, where $1/p + 1/p' = 1$. Assume that for $a = Q/p$ we have*

$$|T_a^{(2)}(x)| \le C_2 \begin{cases} |x|^{a-Q}, \text{ for } x \in \mathbb{G}\backslash\{0\}, \\ |x|^{-N}, \text{ for } x \in \mathbb{G} \text{ with } |x| \ge 1, \end{cases} \tag{3.46}$$

for some positive $C_2 = C_2(a, Q)$ and for every $N > Q$. Then there exists a positive constant $C = C(p, q, r, Q)$ such that

$$\left\| \frac{f * T_{Q/p}^{(2)}}{\left(\log\left(\mathrm{e} + \frac{1}{|x|}\right)\right)^{\frac{r}{q}} |x|^{\frac{Q}{q}}} \right\|_{L^q(\mathbb{G})} \le C\|f\|_{L^p(\mathbb{G})} \tag{3.47}$$

holds for all $f \in L^p(\mathbb{G})$.

Proof of Theorem 3.6. Let us split the integral into three parts

$$\int_{\mathbb{G}} |(f * T_{Q/p}^{(2)})(x)|^q \frac{dx}{\left|\log\left(\mathrm{e} + \frac{1}{|x|}\right)\right|^r |x|^Q} \le 3^q(N_1 + N_2 + N_3), \tag{3.48}$$

where

$$N_1 := \int_{\mathbb{G}} \left(\int_{\{2|y|<|x|\}} |T_{Q/p}^{(2)}(y^{-1}x) f(y)| dy \right)^q \frac{dx}{\left|\log\left(\mathrm{e} + \frac{1}{|x|}\right)\right|^r |x|^Q},$$

$$N_2 := \int_{\mathbb{G}} \left(\int_{\{|x|\le 2|y|<4|x|\}} |T_{Q/p}^{(2)}(y^{-1}x) f(y)| dy \right)^q \frac{dx}{\left|\log\left(\mathrm{e} + \frac{1}{|x|}\right)\right|^r |x|^Q}$$

and

$$N_3 := \int_{\mathbb{G}} \left(\int_{\{|y|>2|x|\}} |T_{Q/p}^{(2)}(y^{-1}x) f(y)| dy \right)^q \frac{dx}{\left|\log\left(\mathrm{e} + \frac{1}{|x|}\right)\right|^r |x|^Q}.$$

First, let us estimate N_1. Similar to (3.35) from $2|y| < |x|$ we get

$$|y^{-1}x| \ge |x| - |y| > |x| - \frac{|x|}{2} = \frac{|x|}{2}, \tag{3.49}$$

which is $|x| < 2|y^{-1}x|$. Denote

$$|T_a^{(2)}(x)| \le \widetilde{B}_a(x) := C_2 \begin{cases} |x|^{a-Q}, \text{ for } x \in \mathbb{G}\backslash\{0\}, \\ |x|^{-N}, \text{ for } x \in \mathbb{G} \text{ with } |x| \ge 1, \end{cases} \tag{3.50}$$

for every $N > Q$. Since $T_{Q/p}^{(2)}(x)$ is bounded by $\widetilde{B}_{Q/p}(x)$ which is non-increasing with respect to $|x|$, then using (3.49) we get

$$\begin{aligned} N_1 &\le \int_{\mathbb{G}} \left(\int_{\{2|y|<|x|\}} |f(y)| dy \right)^q \left(\sup_{\{|x|<2|z|\}} |T_{Q/p}^{(2)}(z)| \right)^q \frac{dx}{\left|\log\left(\mathrm{e} + \frac{1}{|x|}\right)\right|^r |x|^Q} \\ &\le \int_{\mathbb{G}} \left(\int_{\{2|y|<|x|\}} |f(y)| dy \right)^q \left(\widetilde{B}_{Q/p}\left(\frac{x}{2}\right) \right)^q \frac{dx}{\left|\log\left(\mathrm{e} + \frac{1}{|x|}\right)\right|^r |x|^Q}. \end{aligned} \tag{3.51}$$

To apply (3.1) for N_1, we need to check the condition (3.3), that is, that

$$\left(\int_{\{2R<|x|\}}\left(\widetilde{B}_{Q/p}\left(\frac{x}{2}\right)\right)^q\frac{dx}{\left|\log\left(\mathrm{e}+\frac{1}{|x|}\right)\right|^r|x|^Q}\right)^{\frac{1}{q}}\left(\int_{\{|x|<R\}}dx\right)^{\frac{1}{p'}}\leq A_1 \tag{3.52}$$

holds for all $R>0$. In order to check this, let us consider two cases: $R\geq 1$ and $0<R<1$. Then, for $R\geq 1$ using the second equality in (3.50) and $N>Q$, one calculates

$$\begin{aligned}\left(\int_{\{2R<|x|\}}\left(\widetilde{B}_{Q/p}\left(\frac{x}{2}\right)\right)^q\frac{dx}{\left|\log\left(\mathrm{e}+\frac{1}{|x|}\right)\right|^r|x|^Q}\right)^{\frac{1}{q}}&\left(\int_{\{|x|<R\}}dx\right)^{\frac{1}{p'}}\\ &\leq CR^{\frac{Q}{p'}}\left(\int_{\{2R<|x|\}}\left(\widetilde{B}_{Q/p}\left(\frac{x}{2}\right)\right)^q\frac{dx}{|x|^Q}\right)^{\frac{1}{q}}\\ &=CR^{\frac{Q}{p'}}\left(\int_{\{2R<|x|\}}|x|^{-Nq-Q}dx\right)^{\frac{1}{q}}\\ &\leq CR^{-N}R^{\frac{Q}{p'}}\\ &\leq C.\end{aligned}\tag{3.53}$$

Now let us check (3.52) for $0<R<1$. We write

$$\begin{aligned}\int_{\{2R<|x|\}}&\left(\widetilde{B}_{Q/p}\left(\frac{x}{2}\right)\right)^q\frac{dx}{\left|\log\left(\mathrm{e}+\frac{1}{|x|}\right)\right|^r|x|^Q}\\ &=\int_{\{2R<|x|<2\}}\left(\widetilde{B}_{Q/p}\left(\frac{x}{2}\right)\right)^q\frac{dx}{\left|\log\left(\mathrm{e}+\frac{1}{|x|}\right)\right|^r|x|^Q}\\ &\quad+\int_{\{|x|\geqslant 2\}}\left(\widetilde{B}_{Q/p}\left(\frac{x}{2}\right)\right)^q\frac{dx}{\left|\log\left(\mathrm{e}+\frac{1}{|x|}\right)\right|^r|x|^Q}.\end{aligned}\tag{3.54}$$

We note that the second integral on the right-hand side of (3.54) is integrable by the second equality in (3.50). Then, using the first equality in (3.50) we get for the first integral that

$$\begin{aligned}\int_{\{2R<|x|<2\}}\left|\widetilde{B}_{Q/p}\left(\frac{x}{2}\right)\right|^q&\frac{dx}{\left|\log\left(\mathrm{e}+\frac{1}{|x|}\right)\right|^r|x|^Q}\\ &\leq\int_{\{2R<|x|<2\}}\left|\widetilde{B}_{Q/p}\left(\frac{x}{2}\right)\right|^q\frac{dx}{|x|^Q}\\ &\leq C\int_{\{2R<|x|<2\}}|x|^{-Qq/p'-Q}dx\\ &\leq CR^{-Qq/p'}.\end{aligned}$$

It implies with (3.54) that

$$\left(\int_{\{2R<|x|\}}\left|\widetilde{B}_{Q/p}\left(\frac{x}{2}\right)\right|^{q}\frac{dx}{\left|\log\left(\mathrm{e}+\frac{1}{|x|}\right)\right|^{r}|x|^{Q}}\right)^{\frac{1}{q}}\left(\int_{\{|x|<R\}}dx\right)^{\frac{1}{p'}}$$
$$\leq C(R^{-Q/p'}+1)R^{Q/p'}\leq C$$

for any $0<R<1$. Thus, we have checked (3.52), then applying (3.1) for N_1 one gets

$$N_1^{\frac{1}{q}}\leq (p')^{\frac{1}{p'}}p^{\frac{1}{q}}A_1\|f\|_{L^p(\mathbb{G})}. \tag{3.55}$$

Now we estimate N_3. Without loss of generality, we may assume again that $|\cdot|$ is the norm. Similarly to (3.49) we obtain $|y|<2|y^{-1}x|$ from $2|x|<|y|$. Then, we have for N_3 that

$$N_3\leq\int_{\mathbb{G}}\left(\int_{\{|y|>2|x|\}}\left(\widetilde{B}_{Q/p}\left(\frac{y}{2}\right)\right)|f(y)|dy\right)^{q}\frac{dx}{\left|\log\left(\mathrm{e}+\frac{1}{|x|}\right)\right|^{r}|x|^{Q}}.$$

In order to apply (3.2) for N_3, we need to check the following condition:

$$\left(\int_{\{|x|<R\}}\frac{dx}{\left|\log\left(\mathrm{e}+\frac{1}{|x|}\right)\right|^{r}|x|^{Q}}\right)^{\frac{1}{q}}\left(\int_{\{2R<|x|\}}\left(\widetilde{B}_{Q/p}\left(\frac{x}{2}\right)\right)^{p'}dx\right)^{\frac{1}{p'}}\leq A_2. \tag{3.56}$$

To check this, let us consider the cases: $R\geq 1$ and $0<R<1$. Then, for $R\geq 1$ by the second equality in (3.50), we get

$$\left(\int_{\{2R<|x|\}}\left(\widetilde{B}_{Q/p}\left(\frac{x}{2}\right)\right)^{p'}dx\right)^{\frac{1}{p'}}\leq C\left(\int_{\{2R<|x|\}}|x|^{-Np'}dx\right)^{\frac{1}{p'}}\leq CR^{-\frac{N}{p}}. \tag{3.57}$$

Moreover, we have

$$\int_{\{|x|<R\}}\frac{dx}{\left|\log\left(\mathrm{e}+\frac{1}{|x|}\right)\right|^{r}|x|^{Q}}=\int_{\{|x|<\frac{1}{2}\}}\frac{dx}{\left|\log\left(\mathrm{e}+\frac{1}{|x|}\right)\right|^{r}|x|^{Q}}$$
$$+\int_{\{\frac{1}{2}\leq|x|<R\}}\frac{dx}{\left|\log\left(\mathrm{e}+\frac{1}{|x|}\right)\right|^{r}|x|^{Q}},$$

and we note that the first summand on the right-hand side of above is integrable since $r>1$. For the second term, we get

$$\int_{\{\frac{1}{2}\leq|x|<R\}}\frac{dx}{\left|\log\left(\mathrm{e}+\frac{1}{|x|}\right)\right|^{r}|x|^{Q}}\leq\int_{\{\frac{1}{2}\leq|x|<R\}}\frac{dx}{|x|^{Q}}\leq C(1+\log R). \tag{3.58}$$

Combining (3.57) and (3.58), we have for $R\geq 1$ that

$$\left(\int_{\{|x|<R\}}\frac{dx}{\left|\log\left(\mathrm{e}+\frac{1}{|x|}\right)\right|^{r}|x|^{Q}}\right)^{\frac{1}{q}}\left(\int_{\{2R<|x|\}}\left|\widetilde{B}_{Q/p}\left(\frac{x}{2}\right)\right|^{p'}dx\right)^{\frac{1}{p'}}$$
$$\leq CR^{-\frac{N}{p}}(1+\log R)^{\frac{1}{q}}\leq C.$$

Now let us check the condition (3.56) for $0 < R < 1$. We split the integral

$$\int_{\{2R<|x|\}} \left(\widetilde{B}_{Q/p}\left(\frac{x}{2}\right)\right)^{p'} dx = \int_{\{2R<|x|<2\}} \left(\widetilde{B}_{Q/p}\left(\frac{x}{2}\right)\right)^{p'} dx + \int_{\{|x|\geqslant 2\}} \left(\widetilde{B}_{Q/p}\left(\frac{x}{2}\right)\right)^{p'} dx. \tag{3.59}$$

We note that the second integral on the right-hand side of above is integrable by the second equality in (3.50). Then, using the first equality in (3.50) we get for the first integral that

$$\int_{\{2R<|x|<2\}} \left(\widetilde{B}_{Q/p}\left(\frac{x}{2}\right)\right)^{p'} dx \leq C \int_{\{2R<|x|<2\}} |x|^{-Q} dx \leq C \log\left(\frac{1}{R}\right),$$

which implies with (3.59) that

$$\int_{\{2R<|x|\}} \left(\widetilde{B}_{Q/p}\left(\frac{x}{2}\right)\right)^{p'} dx \leq C\left(1+\log\left(\frac{1}{R}\right)\right). \tag{3.60}$$

Since

$$\int_{\{|x|<R\}} \frac{dx}{\left|\log\left(\mathrm{e}+\frac{1}{|x|}\right)\right|^{r} |x|^{Q}} \leqslant C\left(\log\left(\mathrm{e}+\frac{1}{R}\right)\right)^{-(r-1)},$$

and (3.60), and taking into account $r > 1$ and $q < (r-1)p'$ we obtain that

$$\begin{aligned}\left(\int_{\{|x|<R\}} \frac{dx}{\left|\log\left(\mathrm{e}+\frac{1}{|x|}\right)\right|^{r} |x|^{Q}}\right)^{\frac{1}{q}} &\left(\int_{\{2R<|x|\}} \left(\widetilde{B}_{Q/p}\left(\frac{x}{2}\right)\right)^{p'} dx\right)^{\frac{1}{p'}} \\ &\leqslant C\left(\log\left(\mathrm{e}+\frac{1}{R}\right)\right)^{-\frac{r-1}{q}} \left(1+\left(\log\left(\frac{1}{R}\right)\right)^{\frac{1}{p'}}\right) \\ &\leqslant C.\end{aligned} \tag{3.61}$$

Thus, we have checked (3.56), then applying (3.2) for N_3 we obtain

$$N_3^{\frac{1}{q}} \leq (p')^{\frac{1}{p'}} p^{\frac{1}{q}} A_2 \|f\|_{L^p(\mathbb{G})}. \tag{3.62}$$

Now let us estimate N_2. We write

$$N_2 = \sum_{k\in\mathbb{Z}} \int_{\{2^k\leqslant|x|<2^{k+1}\}} \left(\int_{\{|x|\leqslant 2|y|\leqslant 4|x|\}} |T^{(2)}_{Q/p}(y^{-1}x)f(y)|dy\right)^{q} \frac{dx}{\left|\log\left(\mathrm{e}+\frac{1}{|x|}\right)\right|^{r} |x|^{Q}}.$$

Since the function $\left(\log\left(\frac{1}{|x|}\right)\right)^{r} |x|^{Q}$ is non-decreasing with respect to $|x|$ near the origin, there exists an integer $k_0 \in \mathbb{Z}$ with $k_0 \leqslant -3$ such that this function is non-decreasing in $|x| \in (0, 2^{k_0+1})$. We decompose N_2 with k_0 as follows

$$N_2 = N_{21} + N_{22}, \tag{3.63}$$

where

$$N_{21} := \sum_{k=-\infty}^{k_0} \int_{\{2^k\leqslant|x|<2^{k+1}\}} \left(\int_{\{|x|\leqslant 2|y|\leqslant 4|x|\}} |T^{(2)}_{Q/p}(y^{-1}x)f(y)|dy\right)^{q} \frac{dx}{\left|\log\left(\mathrm{e}+\frac{1}{|x|}\right)\right|^{r} |x|^{Q}}$$

and

$$N_{22} := \sum_{k=k_0+1}^{\infty} \int_{\{2^k \leqslant |x| < 2^{k+1}\}} \left(\int_{\{|x| \leqslant 2|y| \leqslant 4|x|\}} |T^{(2)}_{Q/p}(y^{-1}x) f(y)| dy \right)^q \frac{dx}{\left|\log\left(\mathrm{e} + \frac{1}{|x|}\right)\right|^r |x|^Q}.$$

Let us first estimate N_{22}. Since $|x| \leqslant 2|y| \leqslant 4|x|$ and $2^k \leqslant |x| < 2^{k+1}$, we have $2^{k-1} \leqslant |y| < 2^{k+2}$. Before starting to estimate N_{22}, using (3.46), $N > Q$ and $q \geq p$, let us show that

$$\begin{aligned}\int_{\mathbb{G}} |T^{(2)}_{Q/p}(x)|^{\tilde{r}} dx &= \int_{|x|<1} |T^{(2)}_{Q/p}(x)|^{\tilde{r}} dx + \int_{|x|\geq 1} |T^{(2)}_{Q/p}(x)|^{\tilde{r}} dx \\ &\leq C_2 \left(\int_{|x|<1} |x|^{-\frac{Qq(p-1)}{pq+p-q}} dx + \int_{|x|\geq 1} |x|^{-\frac{Npq}{pq+p-q}} dx \right) < \infty,\end{aligned} \tag{3.64}$$

where $\tilde{r} \in [1, \infty)$ is such that $1 + \frac{1}{q} = \frac{1}{\tilde{r}} + \frac{1}{p}$.

Then, (3.64) and Young's inequality (e.g. [20, Proposition 1.5.2]) for $1 + \frac{1}{q} = \frac{1}{\tilde{r}} + \frac{1}{p}$ with $\tilde{r} \in [1, \infty)$ imply that

$$\begin{aligned}N_{22} &\leqslant C \sum_{k=k_0+1}^{\infty} \int_{\{2^k \leqslant |x| < 2^{k+1}\}} \left(\int_{\{|x| \leqslant 2|y| \leqslant 4|x|\}} |T^{(2)}_{Q/p}(y^{-1}x) f(y)| dy \right)^q dx \\ &\leqslant C \sum_{k=k_0+1}^{\infty} \|[f \cdot \chi_{\{2^{k-1} \leqslant |\cdot| < 2^{k+2}\}}] * T^{(2)}_{Q/p}\|^q_{L^q(\mathbb{G})} \\ &\leqslant C \|T^{(2)}_{Q/p}\|^q_{L^{\tilde{r}}(\mathbb{G})} \sum_{k=k_0+1}^{\infty} \|f \cdot \chi_{\{2^{k-1} \leqslant |\cdot| < 2^{k+2}\}}\|^q_{L^p(\mathbb{G})} \\ &= C \sum_{k=k_0+1}^{\infty} \left(\int_{\{2^k \leqslant |x| < 2^{k+1}\}} |f(x)|^p dx \right)^{\frac{q}{p}} \\ &\leqslant C \left(\sum_{k\in\mathbb{Z}} \int_{\{2^k \leqslant |x| < 2^{k+1}\}} |f(x)|^p dx \right)^{\frac{q}{p}} \\ &= C\|f\|^q_{L^p(\mathbb{G})}.\end{aligned} \tag{3.65}$$

To complete the proof it is left to estimate N_{21}. As in (3.49), assuming $|\cdot|$ is the norm and using the triangle inequality, we have

$$3|x| = |x| + 2|x| \geq |x| + |y| \geq |y^{-1}x|, \tag{3.66}$$

where we have used $|y| \leqslant 2|x|$. Since $\left(\log\left(\frac{1}{|x|}\right)\right)^r |x|^Q$ is non-decreasing in $|x| \in (0, 2^{k_0+1})$ and $3|x| \geqslant |y^{-1}x|$, we have

$$\left(\log\left(\frac{1}{|x|}\right)\right)^r |x|^Q \geq \left(\log\left(\frac{1}{\left|\frac{y^{-1}x}{3}\right|}\right)\right)^r \left|\frac{y^{-1}x}{3}\right|^Q.$$

Then, these and (3.46) yield

$$N_{21} \leq C \sum_{k=-\infty}^{k_0} \int_{\{2^k \leq |x| < 2^{k+1}\}} \left(\int_{\{|x| \leq 2|y| \leq 4|x|\}} |y^{-1}x|^{-\frac{Q}{p'}} |f(y)| dy \right)^q \frac{dx}{\left(\log\left(\frac{1}{|x|}\right)\right)^r |x|^Q}$$

$$= C \sum_{k=-\infty}^{k_0} \int_{\{2^k \leq |x| < 2^{k+1}\}} \left(\int_{\{|x| \leq 2|y| \leq 4|x|\}} \frac{|y^{-1}x|^{-\frac{Q}{p'}} |f(y)|}{\left(\left(\log\left(\frac{1}{|x|}\right)\right)^r |x|^Q\right)^{\frac{1}{q}}} dy \right)^q dx$$

$$\leq C \sum_{k=-\infty}^{k_0} \int_{\{2^k \leq |x| < 2^{k+1}\}} \left(\int_{\{|x| \leq 2|y| \leq 4|x|\}} \frac{|y^{-1}x|^{-\frac{Q}{p'}} |f(y)|}{\left(\left(\log\left(\frac{1}{|(y^{-1}x)/3|}\right)\right)^r |(y^{-1}x)/3|^Q\right)^{\frac{1}{q}}} dy \right)^q dx.$$

Since $|x| \leq 2|y| \leq 4|x|$ and $2^k \leq |x| < 2^{k+1}$ with $k \leq k_0$, we get $2^{k-1} \leq |y| < 2^{k+2}$ and $|y^{-1}x| \leq 3|x| < 3 \cdot 2^{k_0+1} \leq 3/4$ by (3.66) and $k_0 \leq -3$. Taking into account these and setting

$$g(x) := \frac{\chi_{B_{\frac{3}{4}}(0)}(x)}{\left(\log\left(\frac{1}{|x|}\right)\right)^{\frac{r}{q}} |x|^{\frac{Q}{q}+\frac{Q}{p'}}},$$

we have for N_{21} that

$$N_{21} \leq C \sum_{k=-\infty}^{k_0} \int_{\{2^k \leq |x| < 2^{k+1}\}} \left(\int_{\{|x| \leq 2|y| \leq 4|x|\}} \frac{|f(y)|}{\left(\log\left(\frac{1}{|y^{-1}x|}\right)\right)^{\frac{r}{q}} |y^{-1}x|^{\frac{Q}{q}+\frac{Q}{p'}}} dy \right)^q dx$$

$$\leq C \sum_{k=-\infty}^{k_0} \|[f \cdot \chi_{\{2^{k-1} \leq |\cdot| < 2^{k+2}\}}] * g\|^q_{L^q(\mathbb{G})}.$$

Since $p \leq q < (r-1)p'$, we use Young's inequality for $1 + \frac{1}{q} = \frac{1}{\tilde{r}} + \frac{1}{p}$ with $\tilde{r} \in [1, \infty)$ to get

$$N_{21} \leq C\|g\|^q_{L^{\tilde{r}}(\mathbb{G})} \sum_{k=-\infty}^{k_0} \|f \cdot \chi_{\{2^{k-1} \leq |\cdot| < 2^{k+2}\}}\|^q_{L^p(\mathbb{G})} \leq C\|f\|^q_{L^p(\mathbb{G})}, \tag{3.67}$$

provided that $g \in L^{\tilde{r}}(\mathbb{G})$. Since $\left(\frac{Q}{q} + \frac{Q}{p'}\right)\tilde{r} = Q$, $\frac{r\tilde{r}}{q} = \frac{rp'}{p'+q}$ and $q < (r-1)p'$, then changing variables, we obtain

$$\|g\|^{\tilde{r}}_{L^{\tilde{r}}(\mathbb{G})} = \int_{B(0,3/4)} \frac{dx}{\left(\log\left(\frac{1}{x}\right)\right)^{\frac{rp'}{p'+q}} |x|^Q} = C \int_{\log\left(\frac{4}{3}\right)}^{\infty} \frac{dt}{t^{\frac{rp'}{p'+q}}} < \infty.$$

Thus, (3.55), (3.62), (3.63), (3.65), (3.67) and (3.48) complete the proof of Theorem 3.6. □

As an application of Theorem 3.1, we can also obtain the following weighted $L^p - L^q$ differential Hardy-Sobolev type inequality with the radial derivative:

Theorem 3.7. *Let $\mathbb{G}$ be a homogeneous Lie group of homogeneous dimension Q. Let ϕ_5, ψ_5 be positive weight functions on $\mathbb{G}$ and let $1 < p \leq q < \infty$. Then there exists a positive constant C such that*

$$\left(\int_{\mathbb{G}} \phi_5(x)|f(x)|^q dx\right)^{1/q} \leq C\left(\int_{\mathbb{G}} \psi_5(x)|\mathcal{R}_{|x|}f(x)|^p dx\right)^{1/p} \tag{3.68}$$

holds for all radial functions f with $f(0) = 0$ if and only if

$$A_5 := \sup_{R>0}\left(\int_{|x|\geq R} \phi_5(x)dx\right)^{1/q}\left(\int_0^R\left(\int_{\mathfrak{S}} r^{Q-1}\psi_5(ry)d\sigma(y)\right)^{1-p'} dr\right)^{1/p'} < \infty, \tag{3.69}$$

where $\mathcal{R}_{|x|} := \frac{d}{d|x|}$ is the radial derivative.

In the abelian case $\mathbb{G} = (\mathbb{R}^n, +)$ and $Q = n$, (3.68) was obtained in [14] and in [49].

Proof of Theorem 3.7. If we denote $\tilde{f}(r) = f(x)$ for $r = |x|$ and

$$\Phi(r) = \int_{\mathfrak{S}} r^{Q-1}\phi_5(ry)d\sigma(y), \qquad \Psi(r) = \int_{\mathfrak{S}} r^{Q-1}\psi_5(ry)d\sigma(y),$$

then using $\tilde{f}(0) = 0$ we have

$$\left(\int_{\mathbb{G}} \phi_5(x)|f(x)|^q dx\right)^{1/q} = \left(\int_{\mathfrak{S}}\int_0^\infty r^{Q-1}\phi_5(ry)|\tilde{f}(r)|^q dr d\sigma(y)\right)^{1/q}$$

$$= \left(\int_0^\infty \Phi(r)|\tilde{f}(r)|^q dr\right)^{1/q} = \left(\int_0^\infty \Phi(r)\left|\int_0^r \mathcal{R}_r\tilde{f}(r)dr\right|^q dr\right)^{1/q}$$

$$\leq C\left(\int_0^\infty \Psi(r)\left|\mathcal{R}_r\tilde{f}(r)\right|^p dr\right)^{1/p} = C\left(\int_{\mathbb{G}} \psi_5(x)|\mathcal{R}_{|x|}f(x)|^p dx\right)^{1/p}$$

if and only if the condition (3.69) holds by Theorem 3.1, namely by (3.1) and (3.3). □

4. Hardy-Littlewood-Sobolev inequalities on homogeneous Lie groups

In this section we apply the integral Hardy inequalities from the previous section to obtain the Hardy-Littlewood-Sobolev and logarithmic Hardy-Littlewood-Sobolev type inequalities on homogeneous Lie groups. We also discuss the reversed Hardy-Littlewood-Sobolev inequalities on general homogeneous Lie groups.

Now we start with the Hardy-Littlewood-Sobolev inequality (see [26], [27] and [50]). We also refer to [23] for the case of the Heisenberg group and to [29] and [18] for sharp constants of the Hardy-Littlewood-Sobolev inequality. Here, we investigate the weighted Hardy-Littlewood-Sobolev inequalities on general homogeneous groups.

Theorem 4.1. *Let $\mathbb{G}$ be a homogeneous Lie group of homogeneous dimension Q and let $|\cdot|$ be an arbitrary homogeneous quasi-norm. Let $0 < \lambda < Q$ and $1 < p, q < \infty$ be such that $1/p + 1/q + (\alpha + \lambda)/Q = 2$ with $0 \leq \alpha < Q/p'$ and $\alpha + \lambda \leq Q$, where $1/p + 1/p' = 1$. Then there exists a positive constant $C = C(Q, \lambda, p, \alpha)$ such that*

$$\left|\int_{\mathbb{G}}\int_{\mathbb{G}} \frac{\overline{f(x)}g(y)}{|x|^\alpha|y^{-1}x|^\lambda} dxdy\right| \leq C\|f\|_{L^p(\mathbb{G})}\|g\|_{L^q(\mathbb{G})} \tag{4.1}$$

holds for all $f \in L^p(\mathbb{G})$ and $g \in L^q(\mathbb{G})$.

Proof of Theorem 4.1. Let $T_a^{(3)}(x) := |x|^{a-Q}$ with $0 < a < Q/r$ for some $1 < r < \infty$. Then, using Hölder's inequality we calculate

$$\begin{aligned} \left| \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{\overline{f(x)} g(y)}{|x|^{\alpha} |y^{-1}x|^{\lambda}} dx dy \right| &= \left| \int_{\mathbb{G}} \overline{f(x)} \frac{(g * T_{Q-\lambda}^{(3)})(x)}{|x|^{\alpha}} dx \right| \\ &\leq \|f\|_{L^p(\mathbb{G})} \left\| \frac{g * T_{Q-\lambda}^{(3)}}{|x|^{\alpha}} \right\|_{L^{p'}(\mathbb{G})}. \end{aligned} \tag{4.2}$$

Note that the conditions $\alpha + \lambda \leq Q$ and $1/p + 1/q + (\alpha + \lambda)/Q = 2$ imply $q \leq p'$, while $0 < \lambda < Q$, $\alpha < Q/p'$ and $1/p + 1/q + (\alpha + \lambda)/Q = 2$ give

$$0 < Q - \lambda = Q - Q\left(2 - \frac{1}{p} - \frac{1}{q}\right) + \alpha < Q - Q\left(2 - \frac{1}{p} - \frac{1}{q}\right) + \frac{Q}{p'} = Q/q.$$

Since we have $1 < q \leq p' < \infty$, $0 \leq \alpha p' < Q$, $0 < Q - \lambda < Q/q$ and $(Q - \lambda)/Q = 1/q - 1/p' + \alpha/Q$, using Theorem 3.4 in (4.2) we obtain (4.1). ☐

Let us now introduce the critical case $\alpha = Q/p'$ of the Hardy-Littlewood-Sobolev inequality (4.1):

Theorem 4.2. *Let $\mathbb{G}$ be a homogeneous Lie group of homogeneous dimension Q and let $|\cdot|$ be an arbitrary homogeneous quasi-norm. Let $1 < p < \infty$, $1 < q \leq p' < (r-1)q'$ and $q < r < \infty$, where $1/p+1/p' = 1$ and $1/q+1/q' = 1$. Let $T_{Q/p}^{(2)}(x)$ be as in Theorem 3.6. Then there exists a positive constant $C = C(p, q, r, Q)$ such that*

$$\left| \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{\overline{f(x)} g(y) T_{Q/q}^{(2)}(y^{-1}x)}{\left(\log\left(e + \frac{1}{|x|}\right)\right)^{\frac{r}{p'}} |x|^{\frac{Q}{p'}}} dx dy \right| \leq C \|f\|_{L^p(\mathbb{G})} \|g\|_{L^q(\mathbb{G})} \tag{4.3}$$

holds for all $f \in L^p(\mathbb{G})$ and $g \in L^q(\mathbb{G})$.

Proof of Theorem 4.2. By Hölder's inequality we have

$$\begin{aligned} \left| \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{\overline{f(x)} g(y) T_{Q/q}^{(2)}(y^{-1}x)}{\left(\log\left(e + \frac{1}{|x|}\right)\right)^{\frac{r}{p'}} |x|^{\frac{Q}{p'}}} dx dy \right| &= \left| \int_{\mathbb{G}} \overline{f(x)} \frac{(g * T_{Q/q}^{(2)})(x)}{\left(\log\left(e + \frac{1}{|x|}\right)\right)^{\frac{r}{p'}} |x|^{\frac{Q}{p'}}} dx \right| \\ &\leq \|f\|_{L^p(\mathbb{G})} \left\| \frac{g * T_{Q/q}^{(2)}}{\left(\log\left(e + \frac{1}{|x|}\right)\right)^{\frac{r}{p'}} |x|^{\frac{Q}{p'}}} \right\|_{L^{p'}(\mathbb{G})}. \end{aligned} \tag{4.4}$$

Since we have $1 < q < r < \infty$ and $q \leq p' < (r - 1)q'$, then by applying Theorem 3.6 we derive (4.3) from (4.4). ☐

Remark 4.3. Let us make some remarks concerning the reversed Hardy-Littlewood-Sobolev inequality on homogeneous groups (see [16], [31] and [13] for the recent

Euclidean analysis of such inequalities). Namely, let us look at the validity of the inequality

$$\int_{\mathbb{G}}\int_{\mathbb{G}} f(x)|y^{-1}x|^{\lambda} f(y)dxdy \geq C_{Q,\lambda,p}\|f\|^{\theta}_{L^1(\mathbb{G})}\|f\|^{2-\theta}_{L^p(\mathbb{G})} \tag{4.5}$$

for any $0 \leq f \in L^1 \cap L^p(\mathbb{G})$ with $f \not\equiv 0$ and $0 < p < 1$, where $\lambda > 0$ and $\theta := (2Q - p(2Q+\lambda))/(Q(1-p))$. When $\mathbb{G} = (\mathbb{R}^n, +)$, hence $Q = n$, the case $p = 2n/(2n+\lambda)$ is investigated in [16] and [31], and the case $p > n/(n+\lambda)$ is studied in [13].

We show that in the case $0 < p \leq Q/(Q+\lambda)$ the inequality (4.5) is not valid, namely we show that (4.5) fails for any $C_{Q,\lambda,p} > 0$. This is showed in the Euclidean case in [7] when $p < n/(n+\lambda)$ and in [13] when $p \leq n/(n+\lambda)$.

We consider

$$f_\varepsilon(x) := f(x) + A\varepsilon^{-Q}h(x/\varepsilon),$$

for a non-negative function f with compact support and for a non-negative smooth fuction h with the property $\int_{\mathbb{G}} h(x)dx = 1$, and for some $A > 0$. Suppose (4.5) holds for some $C_{Q,\lambda,p} > 0$. Putting this f_ε in the inequality (4.5), we obtain

$$C_{Q,\lambda,p} \leq \frac{\int_{\mathbb{G}}\int_{\mathbb{G}} f_\varepsilon(x)|y^{-1}x|^{\lambda} f_\varepsilon(y)dxdy}{\|f_\varepsilon\|^{\theta}_{L^1(\mathbb{G})}\|f_\varepsilon\|^{2-\theta}_{L^p(\mathbb{G})}}$$
$$\to \frac{\int_{\mathbb{G}}\int_{\mathbb{G}} f(x)|y^{-1}x|^{\lambda} f(y)dxdy + 2A\int_{\mathbb{G}}|x|^{\lambda}f(x)dx}{(\int_{\mathbb{G}} f(x)dx + A)^{\theta}(\int_{\mathbb{G}}(f(x))^p dx)^{(2-\theta)/p}} \tag{4.6}$$

as $\varepsilon \to 0_+$, where we have used $\int_{\mathbb{G}} f_\varepsilon(x)dx = \int_{\mathbb{G}} f(x)dx + A$, and when $\varepsilon \to 0_+$ the following facts

$$\int_{\mathbb{G}} (f_\varepsilon(x))^p dx \to \int_{\mathbb{G}} (f(x))^p dx$$

and

$$\int_{\mathbb{G}}\int_{\mathbb{G}} f_\varepsilon(x)|y^{-1}x|^{\lambda} f_\varepsilon(y)dxdy = \int_{\mathbb{G}}\int_{\mathbb{G}} f(x)|y^{-1}x|^{\lambda} f(y)dxdy$$
$$+ 2A\int_{\mathbb{G}}\int_{\mathbb{G}} f(x)|(\varepsilon^{-1}y)^{-1}x|^{\lambda}h(y)dxdy + A^2\varepsilon^{-2Q}\int_{\mathbb{G}}\int_{\mathbb{G}} h\left(\frac{x}{\varepsilon}\right)h\left(\frac{y}{\varepsilon}\right)dxdy$$
$$\to \int_{\mathbb{G}}\int_{\mathbb{G}} f(x)|y^{-1}x|^{\lambda} f(y)dxdy + 2A\int_{\mathbb{G}}|x|^{\lambda}f(x)dx,$$

since $\int_{\mathbb{G}} h(x)dx = 1$. Note that we can take the limit as $A \to +\infty$ in (4.6), since it is valid for all $A > 0$. Then, when $\theta > 1$, i.e., $p < Q/(Q+\lambda)$, taking $A \to +\infty$ in (4.6) we see that $C_{Q,\lambda,p} = 0$. In the case $\theta = 1$, that is, $p = Q/(Q+\lambda)$, taking again the limit as $A \to +\infty$ in (4.6) we get

$$C_{Q,\lambda,p} \leq \frac{2\int_{\mathbb{G}}|x|^{\lambda}f(x)dx}{(\int_{\mathbb{G}}(f(x))^p dx)^{1/p}}. \tag{4.7}$$

Now we show that the right-hand side of (4.7) goes to zero when $R \to \infty$ if we put there the function

$$f_R(x) = \begin{cases} |x|^{-(Q+\lambda)}, \text{ for } 1 \leq |x| \leq R, \\ 0, \text{ otherwise}, \end{cases} \tag{4.8}$$

for any $R > 1$. Indeed, taking into account $p = Q/(Q+\lambda)$ we obtain from (4.7) that

$$C_{Q,\lambda,p} \leq \frac{2\int_{\mathbb{G}} |x|^{\lambda} f_R(x)dx}{(\int_{\mathbb{G}} (f_R(x))^p dx)^{1/p}} = 2(|\mathfrak{S}| \log R)^{-\lambda/Q} \to 0 \tag{4.9}$$

as $R \to \infty$, where $|\mathfrak{S}|$ is a $Q-1$ dimensional surface measure of the unit quasi-sphere in $\mathbb{G}$.

Thus, we have proved that the reversed Hardy-Littlewood-Sobolev inequality (4.5) is not valid with any positive constant $C_{Q,\lambda,p}$ for $0 < p \leq Q/(Q+\lambda)$.

5. Hypoelliptic Hardy, Sobolev, Rellich, Caffarelli-Kohn-Nirenberg and Hardy-Littlewood-Sobolev inequalities

In this section we obtain Hardy-Sobolev-Rellich inequality on graded groups, which implies Hardy, Sobolev and Rellich inequalities on graded groups. Moreover, we establish Caffarelli-Kohn-Nirenberg and Hardy-Littlewood-Sobolev inequalities, and uncertainty type principle on graded Lie groups.

Since we have (2.14) for the Riesz kernel $\mathcal{I}_\alpha$ from (2.6), taking $T_a^{(1)}(x) = \mathcal{I}_a(x)$ in Theorem 3.4 and noting that $\mathcal{R}^{-\frac{a}{\nu}} f = f * \mathcal{I}_a$ by [20, Corollary 4.3.11], we obtain the following Hardy-Sobolev-Rellich inequality:

Theorem 5.1. *Let $\mathbb{G}$ be a graded Lie group of homogeneous dimension Q and let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν. Let $|\cdot|$ be an arbitrary homogeneous quasi-norm. Let $1 < p \leq q < \infty$ and $0 < a < Q/p$. Let $0 \leq b < Q$ and $\frac{a}{Q} = \frac{1}{p} - \frac{1}{q} + \frac{b}{qQ}$. Then there exists a positive constant C such that*

$$\left\| \frac{f}{|x|^{\frac{b}{q}}} \right\|_{L^q(\mathbb{G})} \leq C \|\mathcal{R}^{\frac{a}{\nu}} f\|_{L^p(\mathbb{G})} \tag{5.1}$$

holds for all $f \in \dot{L}_a^p(\mathbb{G})$.

Remark 5.2. In the case $b = 0$, the inequality (5.1) implies the **Sobolev inequality** on graded groups [20, Proposition 4.4.13, (5)]: Let $1 < p < q < \infty$ and $0 < a < Q/p$ with $\frac{a}{Q} = \frac{1}{p} - \frac{1}{q}$. Then there exists a positive constant C such that

$$\|f\|_{L^q(\mathbb{G})} \leq C \|\mathcal{R}^{\frac{a}{\nu}} f\|_{L^p(\mathbb{G})} \tag{5.2}$$

holds for all $f \in \dot{L}_a^p(\mathbb{G})$.

Remark 5.3. In particular, for $q = p$ from (5.1) we derive the general hypoelliptic family of the Hardy inequalities:

$$\left\| \frac{f}{|x|^a} \right\|_{L^p(\mathbb{G})} \leq C \|\mathcal{R}^{\frac{a}{\nu}} f\|_{L^p(\mathbb{G})}, \quad 1 < p < \infty, \ 0 < a < Q/p, \tag{5.3}$$

for all $f \in \dot{L}_a^p(\mathbb{G})$.

Remark 5.4. In the case $q = p$, the inequality (5.1) gives on graded groups the **Hardy inequality**

$$\left\| \frac{f}{|x|} \right\|_{L^p(\mathbb{G})} \leq C \|\mathcal{R}^{\frac{1}{\nu}} f\|_{L^p(\mathbb{G})}, \quad 1 < p < Q, \quad f \in \dot{L}_1^p(\mathbb{G}), \tag{5.4}$$

when $a = 1$, and the **Rellich inequality**

$$\left\| \frac{f}{|x|^2} \right\|_{L^p(\mathbb{G})} \leq C \|\mathcal{R}^{\frac{2}{\nu}} f\|_{L^p(\mathbb{G})}, \quad 1 < p < \frac{Q}{2}, \quad f \in \dot{L}_2^p(\mathbb{G}), \tag{5.5}$$

when $a = 2$.

Similarly, putting $T_a^{(2)}(x) = \mathcal{B}_a(x)$ in Theorem 3.6 and using (2.15) with the Bessel kernel $\mathcal{B}_a$ from (2.7), by noting $(I + \mathcal{R})^{-\frac{a}{\nu}} f = f * \mathcal{B}_a$ by [20, Corollary 4.3.11], we obtain the critical case $a = Q/p$ of Theorem 5.1:

Theorem 5.5. *Let $\mathbb{G}$ be a graded Lie group of homogeneous dimension Q and let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν. Let $|\cdot|$ be an arbitrary homogeneous quasi-norm and let $1 < p < r < \infty$ and $p \leq q < (r-1)p'$, where $1/p + 1/p' = 1$. Then there exists a positive constant $C = C(p, q, r, Q)$ such that*

$$\left\| \frac{f}{\left(\log\left(\mathrm{e} + \frac{1}{|x|}\right)\right)^{\frac{r}{q}} |x|^{\frac{Q}{q}}} \right\|_{L^q(\mathbb{G})} \leq C \|f\|_{L^p_{Q/p}(\mathbb{G})} \tag{5.6}$$

holds for all $f \in L^p_{Q/p}(\mathbb{G})$.

The Hardy-Sobolev-Rellich inequality (5.1) implies the following Heisenberg-Pauli-Weyl type uncertainty principle for general homogeneous invariant hypoelliptic differential operators:

Corollary 5.6. *Let $\mathbb{G}$ be a graded Lie group of homogeneous dimension Q and let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν. Let $|\cdot|$ be an arbitrary homogeneous quasi-norm. Let $1 < p \leq q < \infty$ and $0 < a < Q/p$. Let $0 \leq b < Q$ and $\frac{a}{Q} = \frac{1}{p} - \frac{1}{q} + \frac{b}{qQ}$. Then there exists a positive constant C such that*

$$\|\mathcal{R}^{\frac{a}{\nu}} f\|_{L^p(\mathbb{G})} \| |x|^{\frac{b}{q}} f\|_{L^{q'}(\mathbb{G})} \geq C \int_{\mathbb{G}} |f(x)|^2 dx \tag{5.7}$$

holds for all $f \in \dot{L}^p_a(\mathbb{G})$, where $1/q + 1/q' = 1$.

Proof of Theorem 5.6. Using Hölder's inequality and (5.1), we have

$$\|\mathcal{R}^{\frac{a}{\nu}} f\|_{L^p(\mathbb{G})} \| |x|^{\frac{b}{q}} f\|_{L^{q'}(\mathbb{G})} \geq C \left\| \frac{f}{|x|^{b/q}} \right\|_{L^q(\mathbb{G})} \| |x|^{\frac{b}{q}} f\|_{L^{q'}(\mathbb{G})} \geq C \int_{\mathbb{G}} |f(x)|^2 dx,$$

which is (5.7). □

As another consequence of Theorem 5.1, we also obtain a family of extended Caffarelli-Kohn-Nirenberg inequalities on graded groups.

Theorem 5.7. *Let $\mathbb{G}$ be a graded Lie group of homogeneous dimension Q and let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν. Let $|\cdot|$ be an arbitrary homogeneous quasi-norm. Let $1 < p, q < \infty$, $\delta \in (0, 1]$ and $0 < r < \infty$ with $r \leq \frac{q}{1-\delta}$ for $\delta \neq 1$. Let $0 < a < Q/p$ and $\beta, \gamma \in \mathbb{R}$ with $\delta r(Q - ap - \beta p) \leq p(Q + r\gamma - r\beta)$ and $\beta(1-\delta) - \delta a \leq \gamma \leq \beta(1-\delta)$. Assume that $\frac{r(\delta Q + p(\beta(1-\delta) - \gamma - a\delta))}{pQ} + \frac{(1-\delta)r}{q} = 1$. Then there exists a positive constant C such that*

$$\| |x|^{\gamma} f\|_{L^r(\mathbb{G})} \leq C \left\| \mathcal{R}^{\frac{a}{\nu}} f \right\|^{\delta}_{L^p(\mathbb{G})} \left\| |x|^{\beta} f \right\|^{1-\delta}_{L^q(\mathbb{G})} \tag{5.8}$$

holds for all $f \in \dot{L}^p_a(\mathbb{G})$.

Remark 5.8. We note that the conditions $\beta = \gamma = 0$, $a > 0$, $1 < p < Q/a$, $1 < q \le r \le pQ/(Q-ap)$ and $\delta = (1/q - 1/r)(a/Q + 1/q - 1/p)^{-1}$ satisfy all the conditions of Theorem 5.7. Indeed, $\delta = (1/q - 1/r)(a/Q + 1/q - 1/p)^{-1}$, $r \ge q$ and $Q - ap > 0$ imply $r \le \frac{q}{1-\delta}$, while $r \le pQ/(Q-ap)$ gives $\delta r(Q - ap - \beta p) \le p(Q + r\gamma - r\beta)$ since $\beta = \gamma = 0$ and $\delta \le 1$. In this case, $\delta = (1/q - 1/r)(a/Q + 1/q - 1/p)^{-1}$ and $\beta(1-\delta) - \delta a \le \gamma \le \beta(1-\delta)$ are equivalent to $\frac{r(\delta Q + p(\beta(1-\delta)-\gamma-a\delta))}{pQ} + \frac{(1-\delta)r}{q} = 1$ and $a\delta \ge 0$, respectively. Thus, (5.8) recovers also the Gagliardo-Nirenberg inequality previously obtained in [41] and [42] on graded groups

$$\|f\|_{L^r(\mathbb{G})} \le C \left\|\mathcal{R}^{\frac{a}{\nu}} f\right\|^{\delta}_{L^p(\mathbb{G})} \|f\|^{1-\delta}_{L^q(\mathbb{G})} \tag{5.9}$$

for all $f \in \dot{L}^p_a(\mathbb{G}) \cap L^q(\mathbb{G})$.

We also note that when $\mathbb{G} = (\mathbb{R}^n, +)$, $Q = n$ and $\mathcal{R} = -\Delta$, in the special case $p = q = 2$ and $a = 1$, the inequality (5.9) essentially gives the classical Gagliardo-Nirenberg inequality [25] and [33].

Note that another type of Garliardo-Nirenberg inequality involving Besov norms on graded groups was obtained in [4].

Proof of Theorem 5.7. **Case** $\delta = 1$. Notice that in this case, $\frac{r(\delta Q + p(\beta(1-\delta)-\gamma-a\delta))}{pQ} + \frac{(1-\delta)r}{q} = 1$ gives $\frac{a}{Q} = \frac{1}{p} - \frac{1}{r} - \frac{\gamma}{Q}$, which implies that the condition $\delta r(Q - ap - \beta p) \le p(Q + r\gamma - r\beta)$ is equivalent to the trivial estimate $pQ \le pQ$. The condition $\beta(1-\delta) - \delta a \le \gamma \le \beta(1-\delta)$ gives $-a \le \gamma \le 0$, which implies $r \ge p$ with $\frac{a}{Q} = \frac{1}{p} - \frac{1}{r} - \frac{\gamma}{Q}$. Taking into account these we see that (5.8) is equivalent to (5.1).

Case $\delta \in (0, 1)$. We write

$$\||x|^{\gamma} f\|_{L^r(\mathbb{G})} = \left(\int_{\mathbb{G}} |x|^{\gamma r} |f(x)|^r dx\right)^{\frac{1}{r}} = \left(\int_{\mathbb{G}} \frac{|f(x)|^{\delta r}}{|x|^{r(\beta(1-\delta)-\gamma)}} \cdot \frac{|f(x)|^{(1-\delta)r}}{|x|^{-\beta r(1-\delta)}} dx\right)^{\frac{1}{r}}.$$

Note that $\delta > 0$, $Q > ap$ and $\beta(1-\delta) - \gamma \ge 0$ imply $r(\delta Q + p(\beta(1-\delta) - \gamma - a\delta)) > 0$, while $\delta r(Q - ap - \beta p) \le p(Q + r\gamma - r\beta)$, $\delta < 1$ and $r \le \frac{q}{1-\delta}$ give $\frac{pQ}{r(\delta Q + p(\beta(1-\delta)-\gamma-a\delta))} \ge 1$ and $\frac{q}{(1-\delta)r} \ge 1$, respectively. Then by using Hölder's inequality for $\frac{r(\delta Q + p(\beta(1-\delta)-\gamma-a\delta))}{pQ} + \frac{(1-\delta)r}{q} = 1$, we obtain

$$\begin{aligned}\||x|^{\gamma} f\|_{L^r(\mathbb{G})} &\le \left(\int_{\mathbb{G}} \frac{|f(x)|^{\frac{\delta pQ}{\delta Q + p(\beta(1-\delta)-\gamma-a\delta)}}}{|x|^{\frac{pQ(\beta(1-\delta)-\gamma)}{\delta Q + p(\beta(1-\delta)-\gamma-a\delta)}}} dx\right)^{\frac{\delta Q + p(\beta(1-\delta)-\gamma-a\delta)}{pQ}} \left(\int_{\mathbb{G}} \frac{|f(x)|^q}{|x|^{-\beta q}} dx\right)^{\frac{1-\delta}{q}} \\ &= \left\|\frac{f}{|x|^{\frac{\beta(1-\delta)-\gamma}{\delta}}}\right\|^{\delta}_{L^{\frac{\delta pQ}{\delta Q + p(\beta(1-\delta)-\gamma-a\delta)}}(\mathbb{G})} \left\|\frac{f}{|x|^{-\beta}}\right\|^{1-\delta}_{L^q(\mathbb{G})}.\end{aligned} \tag{5.10}$$

We also note that the conditions $\frac{\delta pQ}{\delta Q + p(\beta(1-\delta)-\gamma-a\delta)} \ge \delta r > 0$ and $\beta(1-\delta) - \gamma \ge 0$ imply

$$\frac{\delta pQ}{\delta Q + p(\beta(1-\delta) - \gamma - a\delta)} \cdot \frac{\beta(1-\delta) - \gamma}{\delta} \ge 0, \tag{5.11}$$

while $Q > ap$ and $\delta > 0$ give

$$\frac{\delta pQ}{\delta Q + p(\beta(1-\delta) - \gamma - a\delta)} \cdot \frac{\beta(1-\delta)-\gamma}{\delta} < Q. \tag{5.12}$$

Then, (5.11), (5.12) and

$$\frac{a}{Q} = \frac{1}{p} - \frac{1}{\frac{\delta pQ}{\delta Q + p(\beta(1-\delta)-\gamma-a\delta)}} + \frac{\frac{\beta(1-\delta)-\gamma}{\delta}}{Q}$$

with $\gamma \geq \beta(1-\delta) - \delta a$ imply $\frac{\delta pQ}{\delta Q+p(\beta(1-\delta)-\gamma-a\delta)} \geq p$, so that we can use Theorem 5.1 in (5.10) to obtain (5.8). ☐

Now we show the weighted improved Hardy-Littlewood-Sobolev/Stein-Weiss inequality on graded groups. Note that in this version we can put derivatives on the right-hand side.

Theorem 5.9. *Let $\mathbb{G}$ be a graded Lie group of homogeneous dimension Q and let $|\cdot|$ be an arbitrary homogeneous quasi-norm. Let $1 < p, q < \infty$, $0 \leq a < Q/p$ and $0 \leq b < Q/q$. Let $0 < \lambda < Q$, $0 \leq \alpha < a + Q/p'$ and $0 \leq \beta \leq b$ be such that $(Q-ap)/(pQ) + (Q-q(b-\beta))/(qQ) + (\alpha+\lambda)/Q = 2$ and $\alpha + \lambda \leq Q$, where $1/p + 1/p' = 1$. Then there exists a positive constant $C = C(Q,\lambda,p,\alpha,\beta,a,b)$ such that*

$$\left| \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{\overline{f(x)} g(y)}{|x|^{\alpha} |y^{-1}x|^{\lambda} |y|^{\beta}} dx dy \right| \leq C \|f\|_{\dot{L}^p_a(\mathbb{G})} \|g\|_{\dot{L}^q_b(\mathbb{G})} \tag{5.13}$$

holds for all $f \in \dot{L}^p_a(\mathbb{G})$ and $g \in \dot{L}^q_b(\mathbb{G})$.

Proof of Theorem 5.9. We first prove it for $a \neq 0$ and $b \neq 0$. We want to use Theorem 4.1 on the left-hand side of (5.13) to get

$$\left| \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{\overline{f(x)} g(y)}{|x|^{\alpha} |y^{-1}x|^{\lambda} |y|^{\beta}} dx dy \right| \leq C \|f\|_{L^{p_1}(\mathbb{G})} \left\| \frac{g}{|y|^{\beta}} \right\|_{L^{q_1}(\mathbb{G})}, \tag{5.14}$$

where $p_1 := \frac{pQ}{Q-ap}$ and $q_1 := \frac{qQ}{Q-q(b-\beta)}$. For this, let us check conditions of Theorem 4.1. Note that $0 < a < Q/p$ together with $1 < p < \infty$ implies $1 < p_1 < \infty$, while $0 < b < Q/q$ and $0 \leq \beta \leq b$ give $1 < q_1 < \infty$. We also note that $0 \leq \alpha < a + Q/p' \Rightarrow 0 \leq \alpha < Q/p_1'$ with $p_1' = p_1/(p_1 - 1)$ and $2 = (Q-ap)/(pQ) + (Q-q(b-\beta))/(qQ) + (\alpha+\lambda)/Q = 1/p_1 + 1/q_1 + (\alpha+\lambda)/Q$. Thus, since we also have $0 < \lambda < Q$ and $\alpha + \lambda \leq Q$, we obtain (5.14).

We have $1 < p < p_1 < \infty$, $0 < a < Q/p$ and $\frac{a}{Q} = \frac{1}{p} - \frac{1}{p_1}$ since $p_1 = \frac{pQ}{Q-ap}$, then applying the Sobolev inequality (5.2) on graded groups (or [20, Proposition 4.4.13, (5)]) we get

$$\|f\|_{L^{p_1}(\mathbb{G})} \leq C \|f\|_{\dot{L}^p_a(\mathbb{G})}. \tag{5.15}$$

Since $Q - q(b-\beta) > 0$ and $Q - qb > 0$ we have $0 \leq \frac{\beta qQ}{Q-q(b-\beta)} < Q$, that is, $0 \leq \beta q_1 < Q$ since $q_1 = \frac{qQ}{Q-q(b-\beta)}$. We also have $b/Q = 1/q - 1/q_1 + \beta/Q$ since $q_1 = \frac{qQ}{Q-q(b-\beta)}$ and $1 < q \leq q_1 < \infty$. Then we can use (5.1), i.e.

$$\left\| \frac{g}{|y|^{\beta}} \right\|_{L^{q_1}(\mathbb{G})} \leq C \|g\|_{\dot{L}^q_b(\mathbb{G})}. \tag{5.16}$$

Finally, putting (5.15) and (5.16) in (5.14), we obtain (5.13).

In the case $a = 0$, the inequalities (5.16) and (5.14) give (5.13).

When $b = 0$, we have $\beta = 0$ since $0 \leq \beta \leq b$, then (5.14) with (5.15) implies (5.13). $\square$

Let us now discuss the critical case $\alpha = a + Q/p'$ of the Hardy-Littlewood-Sobolev inequality (5.13) on graded Lie groups.

Theorem 5.10. *Let $\mathbb{G}$ be a graded Lie group of homogeneous dimension Q and let $|\cdot|$ be an arbitrary homogeneous quasi-norm. Let $1 < p, q < \infty$, $0 \leq a < Q/p$, $0 \leq \beta \leq b < Q/q$. $Q(1/p+1/q-1)+\beta-a-b \geq 0$, $\max\{\frac{Qq}{Q-bq+\beta q}, \frac{pq(a+b-\beta+2Q)-Q(p+q)}{pq(Q+a)-Qq}\} < r < \infty$. Then there exists a positive constant $C = C(p, q, a, b, \beta, r, Q)$ such that*

$$\left| \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{\overline{f(x)} g(y) \mathcal{B}_{Q/q}(y^{-1}x)}{\left(\log\left(\mathrm{e} + \frac{1}{|x|}\right)\right)^{\frac{r(pQ-Q+ap)}{pQ}} |x|^{a+\frac{Q}{p'}} |y|^{\beta}} dxdy \right| \leq C \|f\|_{\dot{L}^p_a(\mathbb{G})} \|g\|_{\dot{L}^q_b(\mathbb{G})} \tag{5.17}$$

holds for all $f \in \dot{L}^p_a(\mathbb{G})$ and $g \in \dot{L}^q_b(\mathbb{G})$, where $\mathcal{B}_{Q/p}$ is the Bessel kernel from (2.7).

Proof of Theorem 5.10. As in the previous case, let us first show it for $a \neq 0$ and $b \neq 0$. If we use Theorem 4.2 on the left-hand side of (5.17), then we obtain

$$\left| \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{\overline{f(x)} g(y) \mathcal{B}_{Q/q}(y^{-1}x)}{\left(\log\left(\mathrm{e} + \frac{1}{|x|}\right)\right)^{\frac{r(pQ-Q+ap)}{pQ}} |x|^{a+\frac{Q}{p'}} |y|^{\beta}} dxdy \right| \leq C \|f\|_{L^{p_1}(\mathbb{G})} \left\| \frac{g}{|y|^{\beta}} \right\|_{L^{q_1}(\mathbb{G})}, \tag{5.18}$$

where $p_1 := \frac{pQ}{Q-ap}$ and $q_1 := \frac{qQ}{Q-q(b-\beta)}$. For this, we need to check conditions of Theorem 4.2. Observe that $0 < a < Q/p$ and $1 < p < \infty$ give $1 < p_1 < \infty$, while $0 < b < Q/q$ and $0 \leq \beta \leq b$ imply $1 < q_1 < \infty$. We also note that $Q(1/p + 1/q - 1) + \beta - a - b \geq 0$, $Q > bq$, $p, q > 1$, $a > 0$ and $\beta \geq 0$ yield $q_1 \leq p_1'$, while $\max\{\frac{Qq}{Q-bq+\beta q}, \frac{pq(a+b-\beta+2Q)-Q(p+q)}{pq(Q+a)-Qq}\} < r < \infty$ gives $p_1' < (r-1)q_1'$ and $q_1 < r < \infty$ since $Q - bq + \beta q > 0$ and $Qq(p-1) + apq > 0$. Thus, we obtain (5.18).

Since we have $1 < p < p_1 < \infty$, $0 < a < Q/p$ and $\frac{a}{Q} = \frac{1}{p} - \frac{1}{p_1}$, then we can use the Sobolev inequality (5.2) on graded groups (or [20, Proposition 4.4.13, (5)]) to get

$$\|f\|_{L^{p_1}(\mathbb{G})} \leq C \|f\|_{\dot{L}^p_a(\mathbb{G})}. \tag{5.19}$$

Note that $0 \leq \frac{\beta q Q}{Q-q(b-\beta)} < Q$ due to $Q - q(b-\beta) > 0$ and $Q - qb > 0$, that is, $0 \leq \beta q_1 < Q$ since $q_1 = \frac{qQ}{Q-q(b-\beta)}$. Moreover, we have $b/Q = 1/q - 1/q_1 + \beta/Q$ in virtue of $q_1 = \frac{qQ}{Q-q(b-\beta)}$ and $1 < q \leq q_1 < \infty$. Then, the Hardy-Sobolev-Rellich inequality (5.1) yields

$$\left\| \frac{g}{|y|^{\beta}} \right\|_{L^{q_1}(\mathbb{G})} \leq C \|g\|_{\dot{L}^q_b(\mathbb{G})}. \tag{5.20}$$

Finally, using (5.19) and (5.20) in (5.18) implies (5.17).

When $a = 0$ we obtain (5.17) from (5.20) and (5.18).

In the case $b = 0$, we have $\beta = 0$ since $0 \leq \beta \leq b$, then (5.18) with (5.19) concludes (5.17). $\square$

6. Appendix: On best constants in HLS and Sobolev inequalities

In this section we discuss the relation between the best constants of Sobolev and Hardy-Littlewood-Sobolev inequalities on graded groups for certain families of parameters.

Since the homogeneous order of $I_{Q-\lambda}(y^{-1}x)$ (where I_a is the Riesz potential for a positive Rockland operator $\mathcal{R}$, see (2.6)) and $|y^{-1}x|^{-\lambda}$ is $-\lambda$, then putting $p=q$, $f=g$ and $\lambda = Q-2a$ for $0<a<Q/2$ in (4.1) with $\alpha=0$ we obtain the following version of the Hardy-Littlewood-Sobolev inequality on graded groups

$$\left|\int_{\mathbb{G}}\int_{\mathbb{G}} I_{2a}(y^{-1}x)\overline{f(x)}f(y)dxdy\right| \leq C\|f\|^2_{L^{\frac{2Q}{Q+2a}}(\mathbb{G})} \tag{6.1}$$

for all $f\in L^{\frac{2Q}{Q+2a}}(\mathbb{G})$.

Let $C_{\rm HLS}$ be the best constant in (6.1). We show the relation between this constant and the best constant C_S in the Sobolev inequality (5.2) with $p=2Q/(Q+2a)$, $0<a<Q/2$, and $q=2$, that is, C_S is the best constant in the ineqality

$$\|f\|_{L^2(\mathbb{G})} \leq C\|f\|_{\dot{L}_a^{\frac{2Q}{Q+2a}}(\mathbb{G})} \tag{6.2}$$

for all $f\in \dot{L}_a^{2Q/(Q+2a)}(\mathbb{G})$.

We note here that the Riesz potential I_a as well as homogeneous Sobolev spaces norm in (6.2) above correspond to the particular fixed positive Rockland operator $\mathcal{R}$ of homogeneous degree ν, and is defined by $\|f\|_{\dot{L}^p_a(\mathbb{G})} = \|\mathcal{R}^{\frac{a}{\nu}}f\|_{L^p(\mathbb{G})}$. While it is known [20, 21] that these Sobolev spaces are independent of the choice of a positive Rockland operator $\mathcal{R}$, the best constants clearly depend on the precise expressions of the norms.

Theorem 6.1. *Let $\mathbb{G}$ be a graded Lie group of homogeneous dimension Q, and let $0<a<Q/2$. Then the Hardy-Littlewood-Sobolev (6.1) and Sobolev (6.2) inequalities are dual. Moreover, we have the equality between their best constants,*

$$C_S = C_{HLS}. \tag{6.3}$$

Remark 6.2. In the Euclidean case, we refer to [29], [5], [17] for the best constant in Hardy-Littlewood-Sobolev inequality, and refer to [35], [2], [51] and [29] for the best constant in Sobolev inequality. We also note that according to our knowledge the best constant in the Hardy-Littlewood-Sobolev inequality is not known yet on general stratified groups (beyond the Heisenberg group). Indeed, in [18], Frank and Lieb found the value of the best constant in the Hardy-Littlewood-Sobolev inequality on the Heisenberg group, however, using $|y^{-1}x|^{-\lambda}$ instead of the Riesz potential in (6.1). Although the homogeneous functions $|\cdot|^{-\lambda}$ and $I_{Q-\lambda}(\cdot)$ are equivalent, the best constant in the Hardy-Littlewood-Sobolev inequality does depend on the choice of this weight. In our case, it is the use of the Riesz potential that implies the validity of Theorem 6.1.

Remark 6.3. In [52], [11] and [42] the best constant in the Sobolev inequality with inhomogeneous norm for the parameters different than those in (6.2) is expressed in the variational form as well as in terms of the ground state solutions of the nonlinear

Schrödinger equation when $\mathbb{G}$ is $(\mathbb{R}^n, +)$, the Heisenberg group, and a general graded Lie group, respectively.

Proof of Theorem 6.1. Taking into account that $\mathcal{R}^{-\frac{a}{\nu}}f = f * \mathcal{I}_a$ (see [20, Corollary 4.3.11]), we rewrite the left-hand side of (6.1) as

$$\begin{aligned}\left|\int_{\mathbb{G}}\int_{\mathbb{G}} I_{2a}(y^{-1}x)\overline{f(x)}f(y)dxdy\right| &= \left|\int_{\mathbb{G}} \overline{f}\mathcal{R}^{-\frac{2a}{\nu}}f dx\right| = |(\mathcal{R}^{-\frac{2a}{\nu}}f, f)| \\ &= |(\mathcal{R}^{-\frac{a}{\nu}}f, \mathcal{R}^{-\frac{a}{\nu}}f)| = \|\mathcal{R}^{-\frac{a}{\nu}}f\|^2_{L^2(\mathbb{G})}.\end{aligned} \tag{6.4}$$

Putting (6.4) in (6.1), we arrive at

$$\|f\|_{L^2(\mathbb{G})} \leq C_{\mathrm{HLS}}\|f\|_{\dot{L}_a^{\frac{2Q}{Q+2a}}(\mathbb{G})}. \tag{6.5}$$

Since C_S is best constant in (6.2), that is, C_S is the best constant in (6.5), we have $C_S \leq C_{\mathrm{HLS}}$. On the other hand, similarly, one can obtain (6.1) with the constant C_S from (6.2) using (6.4), which means that also $C_{\mathrm{HLS}} \leq C_S$. ☐

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