Primitive factor rings of p-adic completions of enveloping algebras as arithmetic differential operators

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Abstract

We study the π-adic completion $\widehat{\mathcal{D}^{[1]}}$ of Berthelot's differential operators of level one on the projective line over a complete discrete valuation ring of mixed characteristic $(0,p)$. The global sections are shown to be isomorphic to a Morita context whose objects are certain fractional ideals of primitive factor rings of the π-adic completion of the universal enveloping algebra of $\mathfrak{sl}_2(R)$. We produce a bijection between the coadmissibly primitive ideals of the Arens Michael envelope of a nilpotent finite dimensional Lie algebra and the classical universal enveloping algebra. We make limited progress towards characterizing the primitive ideals of certain affinoid enveloping algebras of nilpotent Lie algebras under restrictive conditions on the Lie algebra. We produce an isomorphism between the primitive factor rings of these affinoid enveloping algebras and matrix rings over certain deformations of Berthelot's arithmetic differential operators over the affine line.

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Chapter 1

Introduction

1.1 Primitive Ideals

In non-commutative algebra, the notion of the prime spectrum and its associated geometric implications become significantly less useful. However, various people have tried to find an alternative geometric perspective on certain simple classes of mildly noncommutative rings. An example is the so-called Dixmier program, wherein Dixmier proposes to study simple modules over non-commutative rings by classifying their annihilators and their corresponding factor rings. This program has had mixed success. A great example of its strength is the complete classification of the primitive ideals of the enveloping algebra of a nilpotent lie algebra $\mathfrak{g}$ over a field of characteristic zero, and an isomorphism theorem for their factor rings. More precisely, he proves that the primitive ideals are parametrized by the coadjoint orbits on $\mathfrak{g}^*$ and that the factor ring of the enveloping algebra by any primitive ideal is isomorphic to a Weyl algebra over a finite field extension of the ground field. In this thesis, some mild progress will be made towards finding an analogue of this result in a rigid analytic setting.

1.2 Rigid Analytic Geometry

When working over $\mathbb{R}$ or $\mathbb{C}$ there is a notion of analytic manifolds and analytic functions on these manifolds. If one is to naively try and define an analytic manifold and K-analytic functions on that manifold over a p-adic field K in the same manner as one would over $\mathbb{R}$ or $\mathbb{C}$, one might find the results disappointing. For instance, due to the total disconnectedness of the topology on p-adic fields, manifolds may have a ring of analytic functions which is a domain, but find themselves being disconnected, so that there is a weaker correspondence between geometric and algebraic properties compared to the archimedean case. Tate solved this problem while studying elliptic curves by defining the category of rigid analytic spaces which carry a special Grothendieck topology. Since Tate's work there have been a plethora of results in algebraic number theory that have depended rigid analytic geometry.

One might wonder whether in the same manner that Dixmier creates a non-commutative geometry using the set of coadjoint orbits on $\mathfrak{g}^*$, there might be some similar rigid analytic non-commutative geometry. This paper does not answer that question. It does take some steps in the direction of describing the primitive spectrum of the π-adic completion of nilpotent enveloping algebras in a restrictive setting.

1.3 Affinoid Enveloping and Weyl Algebras

According to one point of view, one might view the universal enveloping algebra of a K-Lie algebra $\mathfrak{g}$ as an alternative multiplicative structure on $\mathrm{Sym}(\mathfrak{g})$. For each separated R-submodule $\mathscr{L}$ of $\mathfrak{g}$ such that $\mathscr{L} \otimes_R K = \mathfrak{g}$, where R is the ring of integers of K, there is an associated Tate algebra

$$\varprojlim_{i\in\mathbb{N}} \mathrm{Sym}(\mathscr{L})/\pi^i\mathrm{Sym}(\mathscr{L}) \otimes_R K.$$

It might seem interesting to ask whether one might define a similar alternative multiplicative structure on these Tate algebras.

Definition 1.3.0 *Let $\mathfrak{g}$ be a K-Lie algebra, and let $\mathscr{L}$ be a finitely generated R-submodule of $\mathfrak{g}$ such that $\mathscr{L} \otimes_R K = \mathfrak{g}$ and $[\mathscr{L}, \mathscr{L}] \subset \mathscr{L}$. Then we define the affinoid enveloping algebra of $\mathscr{L}$ to be the ring*

$$\widehat{U(\mathscr{L})}_K = \varprojlim_{i\in\mathbb{N}} U(\mathscr{L})/\pi^i U(\mathscr{L}) \otimes K.$$

There have been numerous advances in the study of affinoid enveloping algebras, mostly around semisimple Lie algebras, see for instance [2]. A version of Quillen's lemma holds. In this thesis, some modest results are given concerning the primitive spectrum of affinoid enveloping algebras over finite dimensional nilpotent Lie algebras under some strong restrictive conditions.

In studying affinoid enveloping algebras over a finite dimensional nilpotent Lie algebra $\mathfrak{g}$ over K, we will soon find that certain completions of the Weyl algebra play a central role. The Weyl algebra $W_s(K)$ over K for $s \in \mathbb{N} \cup \{0\}$ can be defined in various ways - we can think of it as being isomorphic as a K-vector space to the polynomial algebra in the variables t_i and ∂_i for $1 \leq i \leq s$, with multiplication defined by the relation $[\partial_i, t_j] = \delta_{ij}$. We define the affinoid Weyl algebras to be the rings

$$\widehat{W_{s,i,K}} = \varprojlim_{j\in\mathbb{N}} W_{s,j}/\pi^j W_{s,j} \otimes_R K$$

where $W_{s,i}$ is the R-subalgebra of $W_s(K)$ generated by $\pi^i\partial_j$ and $\pi^i t_j$ for $1 \leq j \leq s$ and $i \in \mathbb{N} \cup \{0\}$.

1.4 Fréchet-Stein Algebras and Coadmissible Modules

As an example of a rigid analytic space we could, for instance, wonder whether there is an analytification of the affine line over a p-adic field K. The answer is yes, but the construction is non-obvious. Affinoid spaces, which are the rigid analogue of affine spaces, have an inherent boundedness which the affine line lacks. If $\overline{K}$ is an algebraic closure of K, then we know that the affine line $\mathbb{A}^1_K$ over K can be viewed as the Galois orbits of $\overline{K}$. The norm $\|\|_K$ on K extends uniquely to a norm on $\overline{K}$. For any $r \in \|\overline{K}^\times\|$, the set $B_r(0) = \{\lambda \in \overline{K} : \|\lambda\|_K \leq r\}$ is an affinoid domain, and we can view the analytification of $\mathbb{A}^1_K$ as the direct limit of a sequence of these $B_{r_i}(0)$ for an increasing sequence $(r_i)_{i\in\mathbb{N}}$ with $r_i \in \|\overline{K}^\times\|$, $r_i \to \infty$ as $i \to \infty$.

The ring of analytic functions on each $B_{r_i}(0)$ can be viewed as the set of power series in a coordinate t for $\mathbb{A}^1_K$ which converge everywhere on $B_{r_i}(0)$, so that the global sections on the analytification of $\mathbb{A}^1_K$ are the set of power series in t converging for all values in $\overline{K}$. This ring is non-Noetherian and it is asking too much to understand its entire structure. However, we can restrict our attention to those modules over it which can be viewed as inverse limits of finitely generated modules over the coordinate rings of the $B_{r_i}(0)$ for $i \in \mathbb{N}$, with certain compatibility conditions. We call these modules coadmissible.

This construction can be generalized, as in [1], to a setting wherein we are given an almost commutative algebra B, and we define a Fréchet-Stein completion of B to be the projective limit of the completions B_q of B with respect to all of the seminorms q on B. A similar notion of coadmissibility can be defined. There have been a number of results concerning the coadmissible modules of such rings in papers such as [17] and [16]. These completions give examples of what Schneider and Teitelbaum call Fréchet-Stein algebras in [18]. See section 2.6 for the definition used in this thesis.

Given a K-Lie algebra $\mathfrak{g}$ its enveloping algebra $U(\mathfrak{g})$ is almost commutative and we call the associated Fréchet-Stein algebra the Arens-Michael envelope of $\mathfrak{g}$. Here is the precise definition:

Definition 1.4.0 *Let $\mathfrak{g}$ be a K-Lie algebra, and let J be the set of finitely generated R-submodules $\mathscr{L}$ of $\mathfrak{g}$ such that $\mathscr{L} \otimes_R K = \mathfrak{g}$ and $[\mathscr{L}, \mathscr{L}] \subset \mathscr{L}$. Then we define the Arens-Michael envelope $\widetilde{U(\mathfrak{g})}$ to be the ring*

$$\widetilde{U(\mathfrak{g})} = \varprojlim_{\mathscr{L} \in J} \widehat{U(\mathscr{L})}_K$$

Another example that will play a prominent role is the Fréchet-Stein completion $\widetilde{W_s}$ of the s-th Weyl algebra. It can be defined using the affinoid Weyl algebras as follows: $\widetilde{W_s} = \varprojlim_{i \in \mathbb{N}} \widehat{W_{s,i,K}}$.

1.5 Sheaves of Arithmetic Differential Operators

When studying the primitive factor rings of certain affinoid enveloping algebras, we shall see that they embed into the global sections of sheaves of arithmetic differential operators over the affine line. These sheaves were introduced by Berthelot.

Describing the sheaf of differential operators $\mathcal{D}_X$ of a smooth scheme X over the spectrum of a ring other than a field of characteristic zero presents new challenges. In the case of a field of characteristic zero the sheaf of differential operators is a locally Noetherian sheaf of rings generated by the structure sheaf $\mathcal{O}_X$ and its tangent sheaf $\mathcal{T}_X$. On the other hand, if we work over R, then the sheaf of differential operators is significantly more complicated - for instance if X is a copy of the affine line over R and ∂ is a generator for $\mathcal{T}(X)$ then $\mathcal{D}(X)$ is generated over $\mathcal{O}(X)$ by the operators $\partial^{[p^n]} = \frac{\partial^{p^n}}{p^n!}$ for all $n \in \mathbb{N}$, the 'divided powers' of ∂. This ring isn't even Noetherian. Reduction modulo p provides a similar example over a field of characteristic p. In [3], Berthelot introduced the sheaf $\mathcal{D}_X^{[m]}$ of divided powers of differential operators of level $m \in \mathbb{N}$ on a smooth R-scheme X. The sheaves $\mathcal{D}_X^{[m]}$ carry data about the classical sheaf of differential operators, but retain nice properties like Noetherianity by restricting attention to divided powers of a 'level' bounded by m. For example, over the affine line of R, the sheaf of partial differential operators of level m is generated as an algebra over the structure sheaf by the elements $\partial^{[p^i]}$ for $0 \leq i \leq m$.

The π-adic completion of $\mathcal{D}_X^{[m]}$ is an object of interest for various applications, see for instance [6]. In [11] a version of Beilinson-Bernstein localization is proved for these completions over flag varieties of semisimple algebraic groups.

1.6 The Main Results

1.6.1 Coadmissible Primitive Spectrum of Arens Michael Envelope of a Finite Dimensional Nilpotent K-Lie algebra

A bijection is given between the primitive spectrum $\mathrm{Prim}(U(\mathfrak{g}))$ of the enveloping algebra $U(\mathfrak{g})$ of a finite dimensional nilpotent Lie algebra $\mathfrak{g}$ and the set of annihilators $\mathrm{c.Prim}(\widetilde{U(\mathfrak{g})})$ of coadmissible simple modules of the Arens-Michael envelope $\widetilde{U(\mathfrak{g})}$ of $\mathfrak{g}$. The result is summed up in theorem 3.2.1:

Theorem 1.6.1 *Let $\mathfrak{g}$ be a finite dimensional nilpotent Lie algebra. Then the map $J \mapsto J \cap U(\mathfrak{g})$ induces a bijection* $\mathrm{c.Prim}(\widetilde{U(\mathfrak{g})}) \to \mathrm{Prim}(U(\mathfrak{g}))$.

In proving this theorem, we also get an isomorphism theorem regarding the factor rings of $\widetilde{U(\mathfrak{g})}$:

Proposition 1.6.1 *Let I be a closed ideal of $\widetilde{U(\mathfrak{g})}$ such that $Z(\widetilde{U(\mathfrak{g})}/I)$ is isomorphic to K. Then there is an surjection*

$$\widetilde{U(\mathfrak{g})} \to \widetilde{W_s}$$

with kernel I for some $s \in \mathbb{N}$.

1.6.2 The primitive spectrum of nilpotent affinoid enveloping algebras of powerful non-Abelian lattices with an Abelian ideal of codimension one

We say that an R-Lie algebra $\mathfrak{g}$ is powerful if $[\mathfrak{g}, \mathfrak{g}] \subset \pi\mathfrak{g}$. Given the strong result concerning the Arens-Michael envelope, we shouldn't be blamed for imagining that there exists some analogous result for the various affinoid enveloping algebras $\widehat{U(\mathscr{L})}_K$ over a finite dimensional nilpotent Lie algebra $\mathfrak{g}$.

Let $W_1(K)$ be the first Weyl algebra over K, and for $i \in \mathbb{N}$, let V_i be the R-subalgebra of $W_1(K)$ generated by $\pi^i t$ and ∂, and define $\widehat{V_{i,K}} = \varprojlim_{j\in\mathbb{N}} V_i/\pi^j V_i \otimes_R K$.

With a significant amount of work, it is possible to extract the following theorem:

Theorem 1.6.2 *Let $\mathfrak{g}$ be a non-Abelian finite dimensional nilpotent Lie algebra with an Abelian ideal of codimension one. Let $\mathscr{L}$ be a finitely generated R-submodule of $\mathfrak{g}$ such that $\mathscr{L} \otimes_R K = \mathfrak{g}$ and $[\mathscr{L}, \mathscr{L}] \subset \pi_K \mathscr{L}$. Let P be a primitive ideal of $\widehat{U(\mathscr{L})}_K$ such that $P \cap \mathfrak{g} = 0$. For some $m \in \mathbb{N}$, $i \in \mathbb{N}$ and finite Galois extension L of K, we have an isomorphism of K-algebras*

$$\widehat{U(\mathscr{L})}_K/P \to M_{p^m}(\widehat{V_{i,L}})^{\mathrm{Gal}(L/K)}$$

This theorem suggests that unlike in the classical case, primitive factors of nilpotent affinoid enveloping algebras need not be domains. One can produce an example of a

Lie algebra of dimension $p^m + 2$ which has a lattice whose associated affinoid enveloping algebra admits a primitive factor ring of uniform dimension p^m for any $m \in \mathbb{N}$.

From this, assuming the notation of the theorem, we can extract the following result:

Corollary 1.6.2 *Let $\mathfrak{g}$ and $\mathscr{L}$ be defined as in the above theorem.*

1. *If I is a primitive ideal of $\widehat{U(\mathscr{L})}_K$ then $I \cap U(\mathfrak{g})$ is a primitive ideal of $U(\mathfrak{g})$.*

2. *$J \mapsto J \cap U(\mathfrak{g})$ defines a map $\mathrm{Prim}(\widehat{U(\mathscr{L})}_K) \to \mathrm{Prim}(U(\mathfrak{g}))$ with finite fibres.*

3. *For an ideal $I \subset \widehat{U(\mathscr{L})}_K$ the following are equivalent:*

   (a) $Z(\widehat{U(\mathscr{L})}_K/I)$ is algebraic over K.

   (b) I is primitive.

   (c) I is maximal.

When there is no Abelian ideal in $\mathfrak{g}$ of codimension one, the methods used to prove the theorem fail, and there are many examples where no obvious analogue holds. There are numerous ways in which the conditions on $\mathfrak{g}$ might be relaxed but they will not be discussed in this thesis.

1.6.3 Artihmetic Differential Operators

Proving theorem 1.6.2 utilizes the Dixmier map, in which divided powers of a coordinate appear. Given that, we should not be surprised that Berthelot's notion of arithmetic differential operators plays a role in the proof.

An explicit description of the ring structure of the global sections of $\widehat{\mathcal{D}_X^{[m]}}$ for $m \in \mathbb{N}$ is given when $X = \mathbb{A}^1_R$. The following main result is proved in section 4.3.1:

Theorem 1.6.3 *Let X and Y be two copies of $\mathbb{A}^1_R$. Let t be a coordinate for X, τ a coordinate for Y, and let F be the morphism $X \to Y$; $\tau \mapsto t^p$. There is an isomorphism of $\mathcal{O}_Y$-rings*

$$M_{p^m}(\widehat{\mathcal{D}_Y^{[0]}}) \to F_*\widehat{\mathcal{D}_X^{[m]}}$$

such that, on global sections, $\mathrm{Id}\partial_\tau \mapsto \gamma\partial_t^{[p^m]}$ for some $\gamma \in 1 + \pi\widehat{\mathcal{D}_X^{[m]}}(X)$.

This result plays an essential role in our proof of theorem 1.6.2.

1.6.4 Over the Projective Line

Using theorem 1.6.3, an explicit description of the ring structure of the global sections of $\widehat{\mathcal{D}_X^{[1]}}$ is given when $X = \mathbb{P}^1_R$. The following main result is proven in section 6.1.8:

Theorem 1.6.4 *Let X and Y be two copies of $\mathbb{P}^1_R$. Let t be a coordinate for X, τ a coordinate for Y, and let F be the morphism $X \to Y$; $\tau \mapsto t^p$. Let $\mathcal{L}$ be the Serre twisting sheaf $\mathcal{O}_Y(1)$ of Y. Then there is an isomorphism of $\mathcal{O}_Y$-rings $F_*\widehat{\mathcal{D}_X^{[1]}} \to \mathcal{M}$, where $\mathcal{M}$ is the following Morita context of sheaves*

$$\mathcal{M} = \begin{bmatrix} M_{p-1}\left(\mathcal{L}^{\otimes -1} \otimes \widehat{\mathcal{D}_Y^{[0]}} \otimes \mathcal{L}\right) & \left(\mathcal{L}^{\otimes -1} \otimes \widehat{\mathcal{D}_Y^{[0]}}\right)^{p-1} \\ \left(\widehat{\mathcal{D}_Y^{[0]}} \otimes \mathcal{L}\right)^{p-1} & \widehat{\mathcal{D}_Y^{[0]}} \end{bmatrix}.$$

Using this information, in a manner similar to the classical case described in [20] we can describe the global sections of $\widehat{\mathcal{D}_X^{[1]}}$: if we let $\mathscr{L} = \mathfrak{sl}_2(R) = eR \oplus hR \oplus fR$, Ω be the Casimir invariant of $U(\mathscr{L})$ and let $\widehat{U_n} = \widehat{U(\mathscr{L})}/(\Omega - n^2 - 2n)$, where $\widehat{U(\mathscr{L})} = \varprojlim_{i\in\mathbb{N}} U(\mathscr{L})/\pi^i U(\mathscr{L})$ then we obtain the following corollary

Corollary 1.6.4 *Assume that* $\mathrm{char}(\kappa) \neq 2$. *The global sections of $\widehat{\mathcal{D}_X^{[1]}}$ are isomorphic to the following Morita context*

$$\widehat{\mathcal{D}^{[1]}}(X) \cong \begin{bmatrix} M_{p-1}\left(\widehat{U_{-1}}\right) & \widehat{P_0}^{p-1} \\ \widehat{I_0}^{p-1} & \widehat{U_0} \end{bmatrix}$$

where $\widehat{P_0}$ *is the right ideal of* U_0 *generated by* e *and* h *and* $\widehat{I_0}$ *is the left fractional ideal of* $\widehat{U_0}$ *generated by* 1 *and* he^{-1}*.*

There are various corollaries to this result, for instance it follows that $\widehat{\mathcal{D}^{[1]}}(X)$ is a prime ring of uniform dimension p, and that $\widehat{\mathcal{D}^{[1]}}(X)$ is Morita equivalent to $\widehat{\mathcal{D}^{[0]}}(X)$ (theorem 6.2.4). This was already known - see [5, Théorème 2.3.6].

Chapter 2

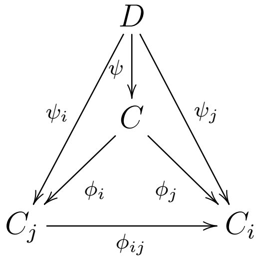
Preliminaries

2.1 Filtrations

2.1.1 Inverse Limits

Let $\mathcal{C}$ be a category, and let I be a directed partially ordered set (that is, for each $i, j \in I$, there is some $k \in I$ such that $i \leq k$ and $j \leq k$.) Then a directed system (C_i, ϕ_{ij}) in $\mathcal{C}$ (over I) is a collection of objects $C_i \in \mathcal{C}$ for each $i \in I$, along with morphisms $\phi_{ij} : C_j \to C_i$ for each $i, j \in I$ such that $i \leq j$, subject to the condition that ϕ_{ii} is the identity morphism on C_i for all $i \in I$ and $\phi_{ij}\phi_{jk} = \phi_{ik}$ for all $i, j, k \in I$ such that $i \leq j$ and $j \leq k$.

Then an inverse limit $C = \varprojlim_{i \in I} C_i$ for the directed system (C_i, ϕ_{ij}), if it exists, is an object in $C \in \mathcal{C}$ along with a set of morphisms $\phi_i : C \to C_i$ for $i \in I$ with the universal property that, for any $D \in \mathcal{C}$ and collection of morphisms $\psi_i : D \to C_i$ for $i \in I$ such that $\phi_{ij}\psi_i = \psi_j$ for all $i, j \in I$, there is a unique morphism $\psi : D \to C$ such that the following diagram commutes



. In this thesis, we will only have cause to deal with the inverse limit of directed systems in subcategories of the category of groups. In this case, we have a fundamental structure theorem:

Proposition 2.1.1 *Let I be a directed partially ordered set and let (G_i, ϕ_{ij}) be a directed system in the category of groups (over I). Then the group*

$$\left\{ (\alpha_i)_{i\in I} \in \prod_{i\in I} G_i \,:\, \phi_{ij}(\alpha_j) = \alpha_i \text{ for all } i, j \in I \text{ such that } i \leq j \right\}$$

is an inverse limit for (G_i, ϕ_{ij}).

2.2 Filtrations

2.2.1 Filtered Rings and Modules

Let A be a ring. Then a filtration on A is a set FA of additive subgroups F_iA of A for $i \in \mathbb{Z}$ such that

1. $1 \in F_0A$
2. $F_iA \subset F_{i+1}A$ for all $i \in \mathbb{Z}$.
3. $(F_iA)(F_jA) \subset F_{i+jA}$ for all $i, j \in \mathbb{Z}$.
4. $\bigcup_{i\in\mathbb{Z}} F_iA = A$.

A filtered ring (A, FA) is a ring A equipped with a filtration FA. We say that a ring homomorphism $\phi : A \to B$ between two filtered rings (A, FA) and (B, FB) is a filtered ring homomorphism of degree d if $\phi(F_iA) \subset F_{i+d}BS$ for all $i \in \mathbb{Z}$.

If (A, FA) is a filtered ring and M is an A-module, then a filtration on M is a set FM of additive subgroups F_iM of M for $i \in \mathbb{Z}$ such that

1. $F_iM \subset F_{i+1}M$ for all $i \in \mathbb{Z}$.

2. $(F_iR)(F_jM) \subset F_{i+j}M$ for all $i, j \in \mathbb{Z}$.

3. $\bigcup_{i\in\mathbb{Z}} F_iM = M$.

We define a filtered A-module (M, FM) to be an A-module M equipped with a filtration FM. The set of all filtered A-modules forms a category A-filt, with morphisms $M \to N$ defined to be A-linear maps $M \to N$ such that the image of F_iM is a subset of F_iN for all $i \in \mathbb{N}$.

If $\phi : M \to N$ is a filtered A-module homomorphism then we say that ϕ is strict if $\phi(F_iM) = \phi(M) \cap F_iN$ for all $i \in \mathbb{Z}$.

2.2.2 Filtration Topology

Let (A, FA) be a filtered ring and let (M, FM) be a filtered A-module. Then M carries a topology, which we will call the topology on M defined by FM, which is defined by taking the cosets $m + F_iM$ for $m \in M$ and $i \in \mathbb{Z}$ to be a base of open sets. A filtered ring carries a filtration topology if we consider it as a filtered module over itself. When references are made to topological properties of a filtered module without reference to the underlying topology it is always assumed that that topology is the filtration topology.

2.2.3 Completions

Let (A, FA) be a filtered ring and let (M, FM) be a filtered module in A-filt. Then we define the completion of M to be the object $\widehat{M} = \varprojlim_{i\in\mathbb{Z}} M/F_iM$, where the maps $M/F_{i+1}M \to M/F_iM$ are the natural projections. By proposition 2.1.1 we know that $\widehat{M}$ is isomorphic to

$$\left\{(\alpha_i)_{i\in\mathbb{Z}} \in \prod_{i\in\mathbb{Z}} M/F_{i+1}M \,:\, \alpha_{i+1} + F_iM = \alpha_i \text{ for all } i \in \mathbb{Z}\right\}.$$

There is a canonical homomorphism $M \to \widehat{M}$ obtained by sending $m \to (m + F_{i+1}M)_{i\in\mathbb{N}}$, called the diagonal homomorphism.

We say that the filtration FM is separated if $\bigcap_{i\in\mathbb{Z}} F_iM = 0$. The filtration FM is separated if and only if the diagonal homomorphism is an embedding. We say that M is complete if the diagonal homomorphism is an isomorphism.

2.2.4 Complete Discrete Valuation Rings

A discrete valuation ring R is a commutative PID with a unique maximal ideal $\mathfrak{m}$. A uniformizer π for R is a generator of $\mathfrak{m}$. The residue field κ of R is the factor ring $R/\mathfrak{m}$. R carries a separated filtration FR, called the π-adic filtration where $F_{-i}R = \pi^iR$ and $F_iR = R$ for $i \in \mathbb{N}$. If R is complete with respect to this filtration we say that R is a complete discrete valuation ring, or c.d.v.r. If K is the field of fractions of R, then the π-adic filtration on R can be extended to a separated filtration of K by setting $F_iK = \pi^{-i}R$ for $i \in \mathbb{Z}$. If R is a c.d.v.r, then K is complete with respect to this filtration.

2.2.5 Graded Rings and Modules

We say that a ring A is a graded ring if there exist some additive subgroups $A_i \subset A$ for $i \in \mathbb{Z}$ such that $A = \oplus_{i\in\mathbb{Z}} A_i$ and $A_iA_j \subset A_{i+j}$ for $i, j \in \mathbb{Z}$.

A graded module over a graded ring $A = \oplus_{i\in\mathbb{Z}} A_i$ is a left A-module M along with some additive subgroups M_i for $i \in \mathbb{Z}$ such that $M = \oplus_{i\in\mathbb{Z}} M_i$ and $A_i M_j \subset M_{i+j}$ for all $i, j \in \mathbb{Z}$.

We say that a ring homomorphism $A \to B$ between graded rings A and B is graded if $\phi(A_i) \subset B_i$ for all $i \in \mathbb{Z}$.

2.2.6 Associated Graded Rings and Modules

Let (A, FA) be a filtered ring, and define $A_i = F_i A/F_{i-1}A$ for $i \in \mathbb{Z}$. Then we can form a graded ring $\mathrm{Gr}(A) = \oplus_{i\in\mathbb{Z}} A_i$, where the multiplication is defined by setting $(a + F_{i-1}A)(b+F_{j-1}A) = ab+F_{i+j-1}A \in A_{i+j}$ for $a, b \in A$, and extending these rules bilinearly to all of $\mathrm{Gr}(A) \times \mathrm{Gr}(A)$. Given a filtered ring homomorphism $\phi : (A, FA) \to (B, FB)$ we can define a graded ring homomorphism

$$\mathrm{Gr}(\phi) : \mathrm{Gr}(A) \to \mathrm{Gr}(B) \; ; \; a + F_{i-1}A \mapsto \phi(a) + F_{i-1}B.$$

In this way Gr becomes a functor from filtered rings to graded rings.

Similarly, if (M, FM) is a filtered A-module, then we define $M_i = F_i M/F_{i-1}M$ for $i \in \mathbb{Z}$, and $\mathrm{Gr}(M) = \oplus_{i\in\mathbb{Z}} M_i$. $\mathrm{Gr}(M)$ becomes a graded A-module by setting $(a + F_{i-1}A)(m + F_{j-1}M) = am + F_{i+j-1}M$, and extending this definition linearly to all of $\mathrm{Gr}(A) \times \mathrm{Gr}(M)$. Given a homomorphism of filtered A-modules $\psi : (M, FM) \to (N, FN)$ we can define a homomorphism of graded A-modules

$$\mathrm{Gr}(\psi) : \mathrm{Gr}(M) \to \mathrm{Gr}(N) \; ; \; m + F_{i-1}M \mapsto \phi(m) + F_{i-1}N.$$

In this way Gr becomes a functor from filtered A-modules to graded $\mathrm{Gr}(A)$-modules.

Proposition 2.2.6 *Let M, M', and M'' be filtered A-modules, and let*

$$M \xrightarrow{\phi} M' \xrightarrow{\psi} M''$$

be a sequence of filtered A-module homomorphisms.

1. *If M and M' are complete then* $\mathrm{Gr}(\phi)$ *is a graded isomorphism if and only if ϕ is an isomorphism in A-filt.*

2. *If M is complete and M' is separated then the graded sequence*

$$\mathrm{Gr}(M) \xrightarrow{\mathrm{Gr}(\phi)} \mathrm{Gr}(M') \xrightarrow{\mathrm{Gr}(\phi)} \mathrm{Gr}(M'')$$

   is exact if and only if the original sequence is strict exact.

Proof: [9, Corollary 1.4.2.5(2)] for part 1, and [9, Theorem 1.4.2.4(5)] for part 2. □

2.2.7 Zariskian Filtrations

Let A be a ring with filtration F_*. Then we say that F_* is a Zariskian filtration on A if F_* is separated, $\mathrm{gr}(A)$ is a left Noetherian ring, $F_{-1}A$ is contained in the Jacobson radical of A, and A has the left Artin-Rees property, that is, for all finitely generated left ideals $I = \sum_{i=1}^{n} Ax_i$ of A, there exists some $c \in \mathbb{Z}$ such that for all $j \in \mathbb{Z}$,

$$F_jA \cap I = \sum_{i=1}^{n} F_{(j+c)}Ax_i.$$

The main result we need is the following:

Proposition 2.2.7 *1. Let A be a ring with left Zariskian filtration F_*. Then every left ideal of A is closed with respect to the F_*-topology.*

2. Suppose that A is complete with respect to the F_*-topology and $\mathrm{gr}(A)$ *is left Noetherian. Then F_* is a left Zariskian filtration.*

Proof:

1. [9, Theorem 2.1.2].

2. [9, Proposition 2.2.1].

□

2.3 Complete sliced K-algebras

2.3.1 Complete sliced K-vector spaces

Let R be a discrete valuation ring and let A be a right R-module. Then we define the π-adic filtration on A to be the filtration FA where $F_{-i}A = A\pi^i$ and $F_iA = A$ for $i \in \mathbb{N}$. This filtration can be extended to the π-adic filtration on $A \otimes_R K$ by setting $F_i(A \otimes_R K) = \pi^{-i}A$ for $i \in \mathbb{Z}$. If A is complete with respect to its π-adic filtration then so is $A \otimes_R K$.

If (V, FV) is a complete filtered K-vector space such that V has a presentation $V = L \otimes_R K$ where L is a flat right R-module such that FV is equal to the π-adic filtration on V induced by L, then we say that V is a complete sliced K-vector space. If L is itself complete with respect to its π-adic filtration, then we say that L is a lattice in V.

Given a π-adically filtered right R-module A, the K-vector space $\widehat{A} \otimes_R K$ is a complete sliced K-vector space with lattice $\widehat{A}$. We will often abbreviate $\widehat{A} \otimes_R K$ as $\widehat{A_K}$.

If V is a complete sliced K vector space with lattice L, $F_iV/F_{i-1}V = \pi^iL/\pi^{i-1}L$ for all $i \in \mathbb{Z}$. Since L is separated, there is an isomorphism of R-modules $\pi^iL/\pi^{i+1}L \to L/\pi L$; $\pi^i a + \pi^{i+1}L \mapsto a + \pi^{i+1}L$, and an isomorphism of graded $\mathrm{Gr}(R)$ modules

$$\mathrm{Gr}(L) \to L/\pi L\,[s] \; ; \pi^i a + \pi^{i+1}L \mapsto (a + \pi L)s^i,$$

and this isomorphism extends to an isomorphism of $\mathrm{Gr}(R)$ modules

$$\mathrm{Gr}(V) \to L/\pi L\left[s, s^{-1}\right].$$

We call $L/\pi L$ the slice of V.

A complete sliced K-algebra is a complete sliced K-vector space with the structure of a K-algebra.

Proposition 2.3.1 *1. Let A be a π-adically filtered Noetherian R-algebra, and assume that the filtration on A is separated. Then $\widehat{A}$ and $\widehat{A} \otimes_R K$ are flat A-modules.*

2. Let V, V', and V'' be complete sliced K-algebras with lattices L, L' and L''. Then a sequence

$$V \to V' \to V''$$

of filtered K-vector space homomorphisms is exact if the induced sequence

$$L/\pi L \to L'/\pi L' \to L''/\pi L''$$

on slices is exact.

Proof:

1. [4, 3.2.3(iv)] for the first statement, and the second follows by the transitivity of flatness and the fact that $\widehat{A} \otimes_R K$ is a flat $\widehat{A}$-module.

2. Since L is complete and L' is separated we can invoke [9, Theorem 1.4.2.4(5)], and it will be enough to show that if the sequence $L/\pi L \to L'/\pi L' \to L''/\pi L''$ is exact then the sequence $\mathrm{gr}(L) \to \mathrm{gr}(L') \to \mathrm{gr}(L'')$ is exact. Using the example above, since L, L', and L'' are π-torsion free, we have a commutative diagram

$$\begin{array}{ccccc} L/\pi L\,[s] & \longrightarrow & L'/\pi L'\,[s] & \longrightarrow & L''/\pi L''\,[s] \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{gr}(L) & \longrightarrow & \mathrm{gr}(L') & \longrightarrow & \mathrm{gr}(L'') \end{array}$$

where the vertical arrows are isomorphisms, and the top horizontal arrows correspond to the induced sequence $L/\pi L \to L'/\pi L' \to L''/\pi L''$ while sending $s \mapsto s$.

Therefore the if the induced sequence $L/\pi L \to L'/\pi L' \to L''/\pi L''$ is exact then the sequence $L \to L' \to L''$ is exact.

$\square$

2.3.2 Idempotents

Recall that if A is a ring then an idempotent $e \in A$ is an element with the property that $e^2 = e$.

Proposition 2.3.2 *Let K be a complete field of mixed characteristic $(0, p)$. Let A be a complete sliced K-algebra and let L be a multiplicatively closed lattice in A such that $1 \in L$. Let e be an idempotent in L.*

1. *If $\tilde{e} \in L$ is an element such that $e + \pi L$ is an idempotent of $L/\pi L$ then the sequence $(\tilde{e}^{p^i})_{i \in \mathbb{N}}$ converges to an idempotent of L (and A).*
2. *If e and f are commuting idempotents of A such that $e + \pi L = f + \pi L$ then $e = f$.*
3. *If f is an idempotent of A such that $ef = fe$ and $ef \in \pi L$ then $ef = 0$.*

Proof:

1. Suppose for induction that e_n is an idempotent of $L/\pi^n L$, so that $e_n^2 = e_n + \pi^n \lambda$ for some $\lambda \in L$. Then for $i \in \mathbb{N}$, working in $L/\pi^{n+1}L$, we have

   $e_n^i = e_n^{i-2}(e + \pi^n \lambda) = e_n^{i-3}(e + (1+e)\pi^n \lambda) = ... = e + (1 + (i-2)e)\pi^n \lambda \bmod \pi^{n+1}L$

   so that

   $$e_n^{2p} = e + (1 + (2p-2)e)\pi^n \lambda = e + (1 - 2e)\pi^n \lambda = e_n^p \bmod \pi^{n+1}L.$$

   Hence, the limit of the sequence $\tilde{e}^{p^n}$ as $n \to \infty$ is an idempotent of A.

2. Suppose that $e + \pi^n \lambda = f$ for some $\lambda \in L$, $n \in \mathbb{N}$. It will be enough to show that $e + \pi^{n+1}\lambda' = f$ some some $\lambda' \in L$. For this we simply observe that

$$f = f^p = \sum_{i=0}^{p} \binom{p}{i} e^i \pi^{n(p-i)} \lambda^{p-i} = e + \pi^{n+1}\lambda',$$

proving the claim.

3. Since e and f commute, ef is idempotent. Since $ef \in \pi L$, we have that $(ef)^n = ef \to 0$ as $n \to \infty$. Therefore $ef = 0$.

□

2.4 Lie Algebras

2.4.1 Lie Algebras

Let A be a commutative ring. Then an A-Lie algebra is a free A-module $\mathfrak{g}$ along with an operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the Lie bracket, such that

1. $[\cdot,\cdot]$ is A-bilinear.

2. $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$.

3. $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ for all $x, y, z \in \mathfrak{g}$.

Define $\mathfrak{g}_0 = \mathfrak{g}$, and for $i \in \mathbb{N}$ define $\mathfrak{g}_i = [\mathfrak{g}, \mathfrak{g}_i]$. Then we say that $\mathfrak{g}$ is nilpotent if for some $i \in \mathbb{N}$, $\mathfrak{g}_i = 0$. A homomorphism of Lie algebras $\phi : \mathfrak{g} \to \mathfrak{g}'$ is an A-linear map such that $\phi([x, y]) = [\phi(x), \phi(y)]$ for all $x, y \in \mathfrak{g}$.

If B is a free A-algebra then B can be given the structure of a A-Lie algebra by setting $[a, b] = ab - ba$ for all $a, b \in A$. Given a A-module M, a representation ρ of $\mathfrak{g}$ (in M) is an A-Lie algebra homomorphism $\mathfrak{g} \to \mathrm{End}_A(M)$.

If $x \in \mathfrak{g}$, then we define $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$ to be the K-linear map $y \mapsto [x, y]$.

A sub-Lie-algebra of a Lie algebra $\mathfrak{g}$ over A is an A-submodule $\mathfrak{h}$ closed under the Lie bracket. If $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$ then we say that $\mathfrak{h}$ is an ideal, and we may form the factor Lie algebra $\mathfrak{g}/\mathfrak{h}$, which is isomorphic as an A-module to $\mathfrak{g}/\mathfrak{h}$, with the Lie bracket define as $[x + \mathfrak{h}, y + \mathfrak{h}] = [x, y] + \mathfrak{h}$.

If R is a c.d.v.r, K is the field of fractions of R, and $\mathfrak{g}$ is a K-Lie algebra, then an R-lattice $\mathscr{L} \subset \mathfrak{g}$ is an R-Lie-subalgebra of $\mathfrak{g}$ such that $\mathscr{L} \otimes_R K = \mathfrak{g}$.

2.4.2 Universal Enveloping Algebras

Let A be a ring and let $\mathfrak{g}$ be an A-Lie algebra. Then the universal enveloping algebra $U(\mathfrak{g})$ is an A-algebra along with an A-Lie algebra homomorphism $\mathfrak{g} \to U(\mathfrak{g})$ such that for any A-Lie representation ρ of $\mathfrak{g}$ in M, there is a unique A-algebra homomorphism $U(\mathfrak{g}) \to \mathrm{End}_A(M)$ such that the following diagram commutes

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & U(\mathfrak{g}) \\ & \searrow & \downarrow \\ & & \mathrm{End}_A(M) \end{array}$$

Proposition 2.4.2 *1. Let K be a field and let L be a field extension of K. Let $\mathfrak{g}$ be a K-Lie algebra and let $\mathfrak{g}'$ be the L-Lie algebra $\mathfrak{g} \otimes_K L$. Then $U(\mathfrak{g}')$ is isomorphic to $U(\mathfrak{g}) \otimes_K L$.*

Proof:

1. [7, Section 2.2.20]

□

Theorem 2.4.2 *(Poincaré-Birkhoff-Witt (PBW) Theorem) Let A be a ring and let $\mathfrak{g}$ be an A-Lie algebra. Let $x_1, \ldots, x_n$ be a basis for $\mathfrak{g}$. Then $U(\mathfrak{g})$ is a free A-module, and the monomials $x_1^{i_1} \ldots x_n^{i_n}$ for $i_j \in \mathbb{N}$, $1 \leq j \leq n$, form a basis for $U(\mathfrak{g})$.*

Proof: [19] □

2.4.3 Quillens Lemma

Theorem 2.4.3 *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field K, and let M be a simple left $U(\mathfrak{g})$-module. Then $\mathrm{End}_{U(\mathfrak{g})}(M)$ is algebraic over K.*

Proof: [13] □

2.4.4 Primitive Ideals in Nilpotent Enveloping Algebras

Proposition 2.4.4 *Let $\mathfrak{g}$ be a finite dimensional nilpotent Lie algebra over a field K of characteristic zero. Then the following sets are equal:*

1. *The maximal ideals of $U(\mathfrak{g})$*
2. *The primitive ideals of $U(\mathfrak{g})$*
3. *The ideals I of $U(\mathfrak{g})$ such that the center of $U(\mathfrak{g})/I$ is algebraic over K.*

Proof: [7, Proposition 4.7.4] □

2.4.5 Weyl Algebras

The Weyl algebras are of central importance to understanding the primitive ideals of a nilpotent universal enveloping algebra over a field of characteristic zero.

Let A be a commutative ring and let $s \in \mathbb{N}$. Let $\mathfrak{h}_{2s}(A)$ (or just $\mathfrak{h}_{2s}$ when no confusion will arise) be the free left A-module

$$Az \oplus \left(\bigoplus_{i=1}^{s} At_i \oplus A\partial_i \right).$$

Then we can define an A-lie algebra structure on $\mathfrak{h}_{2s}$ by setting

1. $[z, x] = 0$ for all $x \in \mathfrak{h}_{2s}$.

2. $[t_i, t_j] = 0$ and $[\partial_i, \partial_j] = 0$ whenever $1 \leq i, j \leq s$.

3. $[t_i, \partial_j] = \delta_{ij} z$ whenever $1 \leq i, j \leq s$.

We define the s-th Weyl algebra over A to be the ring $W_s(A) = U(\mathfrak{h}_{2s})/(z-1)U(\mathfrak{h}_{2s})$.

When A is a field of characteristic zero, $W_s(A)$ is simple for all $s \in \mathbb{N}$ (see [7, Section 4.6.6]).

2.4.6 Affinoid Weyl Algebras

Let K be the field of fractions of a complete discrete valuation ring R of mixed characteristic $(0, p)$ with uniformizer π.

For $j \in \mathbb{N}$, let W_j be the R-subalgebra of $W_s(K)$ generated by $\pi^j \partial_i$ and $\pi^j t_i$ for $1 \leq i \leq s$. Then we define

$$\widehat{W_{s,j,K}} = \varprojlim_{i \in \mathbb{N}} W_j / \pi^i W_j \otimes_R K.$$

We also use an alternative presentation of these algebras in a special case: For $j \in \mathbb{N}$, let V_j be the R-subalgebra of $W_1(K)$ generated by $\pi^j t$ and ∂. Then we define

$$\widehat{V_{i,K}} = \varprojlim_{i \in \mathbb{N}} V_j / \pi^i V_j \otimes_R K.$$

2.4.7 Dixmier Map

Let K be a field and let A be a K-algebra. Then we say that an ideal $I \subset A$ is weakly rational if I is prime and the center of A/I is isomorphic to K. By 2.4.4, weakly rational ideals of the universal enveloping algebra of a finite dimensional nilpotent Lie algebra over a field K of characteristic zero are always primitive.

Let $\mathfrak{g}$ be a finite dimensional nilpotent Lie algebra over a field K of characteristic zero. We define a reducing quadruple for $\mathfrak{g}$ to be a quadruple $(x, y, z, \mathfrak{h})$ where

1. $[x, y] = z$.

2. $z \in \mathfrak{h}$ is central in $\mathfrak{g}$.

3. $\mathfrak{h}$ is a K-Lie-subalgebra of $\mathfrak{g}$ of codimension one such that $\mathfrak{h} \oplus Kx = \mathfrak{g}$.

4. y is central in $\mathfrak{h}$.

Proposition 2.4.7 *Let $(x, y, z, \mathfrak{h})$ be a reducing quadruple for $\mathfrak{g}$. Let $\lambda \in K^{\times}$. Let $J = yU(\mathfrak{h}) + (z - \lambda)U(\mathfrak{h})$ Then there is an isomorphism*

$$\Phi : U(\mathfrak{g})/(z - \lambda)U(\mathfrak{g}) \to U(\mathfrak{h})/J \otimes_K W_1(K)$$

sending

1. $\alpha + (z - \lambda)U(\mathfrak{g}) \mapsto \sum_{i \in \mathbb{N}} \left(\mathrm{ad}_x^i(\alpha) + J\right) \otimes \frac{t^i}{i!}$ *for all* $\alpha \in U(\mathfrak{h})$.

2. $x + (z - \lambda)U(\mathfrak{g}) \mapsto \partial$.

Φ has the property that $\Phi \mathrm{ad}_x \Phi^{-1} = \frac{d}{dt}$.

Proof: Use [7, Lemma 4.7.8(i)], reducing both sides by $(z - \lambda)$. □

If A is a ring, B is a subring, and I is an ideal of A, we say that I is controlled by B if $I = (I \cap B)A$.

The following corollary is a weakened version of a much stronger result that can be easily proved using similar methods.

Corollary 2.4.7 *Let I be a proper ideal of $U(\mathfrak{g})$ such that $z-\lambda \in I$ for some $\lambda \in K^{\times}$. Then I is controlled by $U(\mathfrak{h})$.*

Proof: Let $U = U(\mathfrak{g})/(z-\lambda)U(\mathfrak{g})$, let $H = U(\mathfrak{h})/(z-\lambda)U(\mathfrak{h})$, and let $H' = U(\mathfrak{h})/J$. Using proposition 2.4.7 we have an isomorphism

$$\Phi : U \to H' \otimes_K W_1(K)$$

Since $(z-\lambda)U(\mathfrak{g}) \subset I$ and $\Phi(x) = 1 \otimes \partial$, using [7, Lemma 4.5.1] we have $\Phi(\bar{I}) = I' \otimes W_1(K)$ for some ideal I' of H', where $\bar{I}$ is the image of I in U. Note that here $I' = \Phi(I) \cap H'$ and that from the theorem $\Phi(H) = H' \otimes K[t]$. Hence

$$\Phi(\bar{I}) \cap \Phi(H) = (I' \otimes W_1(K)) \cap (H' \otimes K[t]) = I' \otimes K[t]$$

and hence

$$(\Phi(\bar{I}) \cap \Phi(H))\Phi(U) = I' \otimes W_1(K) = \Phi(\bar{I}),$$

proving the claim. □

Theorem 2.4.7 *Let I be a weakly rational ideal of $U(\mathfrak{g})$. Then for some $s \in \mathbb{N}$ there is an isomorphism $U(\mathfrak{g})/I \to W_s(K)$.*

Proof: [7, Theorem 4.7.9]. □

2.5 Affinoid Enveloping Algebras

In this section we let R be a c.d.v.r. with uniformizer π, we let K be the field of fractions of R, and we let κ be the residue field of R.

2.5.1 Affinoid Enveloping Algebras

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over K. Then for each lattice $\mathscr{L} \subset \mathfrak{g}$ in $\mathfrak{g}$, the affinoid enveloping algebra defined by $\mathscr{L}$ (in $\mathfrak{g}$) is the ring $\widehat{U(\mathscr{L})}_K = \widehat{U(\mathscr{L})} \otimes_R K$, where we take $U(\mathscr{L})$ to be π-adically filtered, so that $\widehat{U(\mathscr{L})} = \varprojlim_{i\in\mathbb{N}} U(\mathscr{L})/\pi^i U(\mathscr{L})$.

Proposition 2.5.1 *1. Let $x_1, \ldots, x_n$ be an R-basis for $\mathscr{L}$. Then there is a natural isomorphism of filtered R-modules $R\langle x_1, \ldots, x_n\rangle \to \widehat{U(\mathscr{L})}$ and a natural isomorphism of K-vector spaces $K\langle x_1, \ldots, x_n\rangle \to \widehat{U(\mathscr{L})}_K$*

2. Let $\mathfrak{g}$ be a finite dimensional K-Lie algebra and let $\mathscr{L}$ be an R-Lie lattice in $\mathfrak{g}$. Then $\widehat{U(\mathscr{L})}_K$ is a flat $U(\mathfrak{g})$-module.

Proof:

1. The PBW theorem gives an filtered isomorphism of R-modules $R[x_1, \ldots, x_n] \to U(\mathscr{L})$ and a filtered isomorphism of K-vector spaces $R[x_1, \ldots, x_n] \to U(\mathscr{L})$. These lift to filtered isomorphisms $R\langle x_1, \ldots, x_n\rangle \to \widehat{U(\mathscr{L})}$ and $K\langle x_1, \ldots, x_n\rangle \to \widehat{U(\mathscr{L})}_K$.

2. This is a straightforward application of proposition 2.3.1(1)

□

Here is an affinoid version of Quillen's lemma

Theorem 2.5.1 *Let $\mathfrak{g}$ be a finite dimensional K-Lie algebra and let $\mathscr{L}$ be a lattice in $\mathfrak{g}$ such that $[\mathscr{L}, \mathscr{L}] \subset \pi\mathscr{L}$. Let M be a simple left $\widehat{U(\mathscr{L})}_K$-module. Then* $\mathrm{End}_{\widehat{U(\mathscr{L})}_K}(M)$ *is algebraic over K.*

Proof: [2, Corollary 8.6] □

2.6 Arens Michael Envelope

Let R be a c.d.v.r with uniformizer π, let K be its field of fractions, and let κ be its residue field

2.6.1 Fréchet-Stein Algebra

For the general definition see [18, Section 3]. Here, we will give a somewhat restricted definition.

Let A be a K-algebra. Then we say that A is a Fréchet algebra if there is a sequence $(L_i)_{i\in\mathbb{N}}$ of R-lattice subrings in A such that $L_{i+1} \subset L_i$ for $i \in \mathbb{N}$ $\bigcap_{i\in\mathbb{N}} L_i = 0$, and the diagonal homomorphism $A \to \varprojlim_{i\in\mathbb{N}} \widehat{L_i} \otimes_R K$ is an isomorphism. Let $\widehat{L_{i,K}} = \widehat{L_i} \otimes_R K$. We say that A is a Fréchet-Stein algebra if there exists such a sequence $(L_i)_{i\in\mathbb{N}}$ with the additional property that each $\widehat{L_i}$ is Noetherian and the canonical embedding $\widehat{L_{i+1,K}} \to \widehat{L_{i,K}}$ gives $\widehat{L_{i,K}}$ the structure of a flat $\widehat{L_{i+1,K}}$-module for all $i \in \mathbb{N}$. We will denote such a Fréchet-Stein algebra by (A, L_i) when we want to emphasize the defining lattices. The topologies induced by the lattice filtrations defined by the L_i on A define an inverse limit topology, which called the Fréchet topology on A.

A coadmissible left A-module is a left A-module M such that there exists some collection of finitely generated left $\widehat{L_{i,K}}$-modules M_i for $i \in \mathbb{N}$ along with $\widehat{L_{i+1,K}}$-linear maps $M_{i+1} \to M_i$ for $i \in \mathbb{N}$ such that M is isomorphic to $\varprojlim_{i\in\mathbb{N}} M_i$, and such that the canonical A-linear map

$$\widehat{L_{i,K}} \otimes_{\widehat{L_{i+1,K}}} M_{i+1} \to M_i$$

is an isomorphism for $i \in \mathbb{N}$. As a finitely generated $\widehat{L_{i,K}}$-module, M_i is naturally a complete filtered $\widehat{L_{i,K}}$-module, and the connecting maps $M_{i+1} \to M_i$ are filtered $\widehat{L_{i+1,K}}$-module homomorphisms. Then M carries an inverse limit topology as $\varprojlim_{i\in\mathbb{N}} M_i$, called the

Fréchet topology on M. In these circumstances, we call the sequence $(M_i)_{i\in\mathbb{N}}$ a coherent sheaf for A.

Proposition 2.6.1 *Let (A, L_i) be a K-Fréchet-Stein algebra.*

1. *$\widehat{L_{i,K}}$ is a flat A-module for all $i \in \mathbb{N}$.*

2. *Any finitely presented left A-module is coadmissible.*

3. *The kernel, image, cokernel, and coimage of an A-linear map between two coadmissible A-modules are all coadmissible.*

4. *If M is a coadmissible module and N is a submodule of M then the following are equivalent*

   (a) *N is coadmissible*

   (b) *M/N is coadmissible*

   (c) *N is closed w.r.t. the Fréchet topology on M*

5. *Let I be a closed ideal of a K-Fréchet-Stein algebra A. Then A/I is a K-Fréchet-Stein algebra, defined by the lattices $L_i/(I \cap L_i)$ for $i \in \mathbb{N}$.*

Proof: See section 3 of [18] □

Our principal interest in defining Fréchet-Stein algebras is the study of those defined by universal enveloping algebras of Lie algebras. Let $\mathfrak{g}$ be a finite-dimensional K-Lie algebra, and let $\mathscr{L}$ be a lattice in $\mathfrak{g}$. Then we define the Arens Michael envelope $\widetilde{U(\mathfrak{g})}$ of $\mathfrak{g}$ to be the ring $\varprojlim_{i\in\mathbb{N}} \widehat{U(\pi^i\mathscr{L})}_K$.

Proposition 2.6.1 *$\widetilde{U(\mathfrak{g})}$ is a K-Fréchet-Stein algebra.*

Proof: [15, 2.3] □

2.7 Morita Contexts

2.7.1 Morita Contexts

Let A and B be rings, M an A-B-bimodule, N a B-A-bimodule, $\phi : M \otimes_B N \to A$, $\psi : N \otimes_A M \to B$ homomorphisms of A-A and B-B bimodules respectively such that for all $m, m' \in M$ and $n, n' \in N$,

$$\phi(m \otimes n)m' = m\psi(n \otimes m') \text{ and } \psi(n \otimes m)n' = n\phi(m \otimes n').$$

Then we can construct a ring C from this data in the following manner: As an Abelian group, we define that $C = A \times M \times N \times B$. We write an element $(a, m, n, d) \in C$, where $a \in A$, $b \in B$, $m \in M$, and $n \in N$ in the form $\begin{bmatrix} a & m \\ n & b \end{bmatrix}$. Multiplication in C is defined by the equation

$$\begin{bmatrix} a & m \\ n & b \end{bmatrix} \begin{bmatrix} a' & m' \\ n' & b' \end{bmatrix} = \begin{bmatrix} aa' + \phi(m \otimes n') & am' + mb' \\ na' + bn' & \psi(n \otimes m') + bb' \end{bmatrix}.$$

We write

$$C = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$$

when the maps ϕ and ψ are understood. We call C the *Morita context* defined by A, M, N, B, ψ, and ϕ (see [10, Section 1.1.6].)

We say that A and B are *Morita equivalent* if there is a finitely generated projective right A-module M such that B is isomorphic as a ring to $\mathrm{End}_A(M)$ [10, Proposition 3.5.5]. If this is the case, then M has the structure of a B-A bimodule, and if $\mathcal{M}(A)$ and $\mathcal{M}(B)$ are the categories of right modules of A and B, then the functor $\mathcal{M}(A) \to \mathcal{M}(B)$; $N \mapsto N \otimes_A \mathrm{Hom}_A(M, A)$ is a natural isomorphism [10, Proposition 3.5.7(i)].

Proposition 2.7.1 *Two rings A and B are Morita equivalent if and only if there exist some A-B-bimodule M and some B-A-bimodule N such that there exists a surjective homomorphism $\phi : M \otimes_B N \to A$ and a surjective homomorphism $\psi : N \otimes_A M \to B$.*

Proof: [10, 3.5.4] $\square$

Chapter 3

Coadmissible primitive spectrum of the Arens-Michael Envelope of a nilpotent enveloping algebra

3.1 The Arens-Michael Envelope of a nilpotent enveloping algebra

3.1.1 Arens-Michael Envelope of $\mathfrak{g}$

Let $\mathfrak{g}$ be a K-Lie algebra. The Arens-Michael envelope of $\mathfrak{g}$ is the completion of $U(\mathfrak{g})$ with respect to all submultiplicative seminorms. For a Lie-lattice $\mathscr{L} \subset \mathfrak{g}$, we set $\widehat{U(\mathscr{L})} = \varprojlim_{i \in \mathbb{N}} U(\mathscr{L})/\pi^i U(\mathscr{L})$ and $\widehat{U(\mathscr{L})}_K = \widehat{U(\mathscr{L})} \otimes K$.

When $\mathscr{L} \subset \mathscr{T}$ there is an embedding $\widehat{U(\mathscr{L})} \to \widehat{U(\mathscr{T})}$. We set $\widetilde{U(\mathfrak{g})} = \varprojlim_{\mathscr{L}} \widehat{U(\mathscr{L})}_K$, where $\mathscr{L}$ runs over the set of all lattices in $\mathfrak{g}$. We can show that $\widetilde{U(\mathfrak{g})}$ is isomorphic to the Arens Michael envelope of $\mathfrak{g}$.

Theorem 3.1.1 *Let M be a simple, coadmissible left $\widetilde{U(\mathfrak{g})}$-module, and let $0 \neq \phi \in \mathrm{End}_{\widetilde{U(\mathfrak{g})}}(M)$. Then ϕ is algebraic over K.*

Proof: Since M is simple, every element of $\mathrm{End}_{\widetilde{U(\mathfrak{g})}}(M)$ is invertible, so the field $K(\phi) \subset \mathrm{End}_{\widetilde{U(\mathfrak{g})}}(M)$. Let $\mathscr{L}$ be a lattice in $\mathfrak{g}$ such that $[\mathscr{L}, \mathscr{L}] \subset \pi\mathscr{L}$. M is coadmissible, so by [18, Corollary 3.3], $M = \varprojlim_{i\in\mathbb{N}} \widehat{U(\pi^i\mathscr{L})}_K \otimes_{\widetilde{U(\mathfrak{g})}} M$. Hence, since M is non-zero, for some $i \in \mathbb{N}$, $\widehat{U(\pi^i\mathscr{L})}_K \otimes_{\widetilde{U(\mathfrak{g})}} M$ is non-zero. We assume without loss of generality that $N = \widehat{U(\mathscr{L})}_K \otimes_{\widetilde{U(\mathfrak{g})}} M \neq 0$. We can see that the map $\rho : \mathrm{End}_{\widetilde{U(\mathfrak{g})}}(M) \to \mathrm{End}_{\widehat{U(\mathscr{L})}_K}(N)$ that sends ψ to the linear extension of ψ to N is a ring homomorphism, so $\rho|_{K(\phi)}$ is a ring embedding with image $K(\phi')$, where ϕ' is the linear extension of ϕ to N. Hence every element of $K[\phi']$ is invertible. Of course N is finitely generated as a $\widehat{U(\mathscr{L})}_K$-module since M is finitely generated as a $\widetilde{U(\mathfrak{g})}$-module, $\widehat{U(\mathscr{L})}_K$ is an almost commutative affinoid algebra per the definition given in [2], and since $[\mathscr{L}, \mathscr{L}] \subset \pi\mathscr{L}$, $\widehat{U(\mathscr{L})}/\pi\widehat{U(\mathscr{L})}$ is isomorphic to $\mathrm{Sym}_{\kappa}(\mathscr{L})$, so is commutative and Gorenstein, so applying [2, Corollary 8.6] we have that ϕ' is algebraic over K, and hence ϕ is algebraic over K. $\square$

3.1.2 Affinoid Weyl Algebras

For a commutative ring S, the Weyl algebra $A_n(S)$ over S is generated over S by t_i and ∂_i for $1 \leq i \leq n$, subject only to the relations $[t_i, t_j] = 0$ and $[\partial_i, \partial_j] = 0$ for $1 \leq i, j \leq n$ and $[t_i, \partial_j] = \delta_{ij}$ for $1 \leq i, j \leq n$.

For $i \in \mathbb{N}$ let $\widehat{W_{s,i,K}}$ be defined as in section 2.4.6 and define $\widetilde{W_s} = \varprojlim_{i\in\mathbb{N}} \widehat{W_{s,i,K}}$. If we set $\mathfrak{h}_{2s}$ to be the K-lie algebra generated by $z, x_1, y_1, ..., x_s, y_s$ with the x_i pairwise commuting for $1 \leq i \leq s$, the y_i commuting for $1 \leq i \leq s$, z central, and $[x_i, y_j] = \delta_{ij} z$ for $1 \leq i, j \leq s$, and for $i \in \mathbb{N}$ we set $\mathcal{H}_i$ to be the R-lie lattice in $\mathfrak{g}$ generated by

$\pi^i z, \pi^i x_1, \pi^i y_1, ..., \pi^i x_s, \pi^i y_s$, then

$$\widetilde{U(\mathfrak{h}_3)} = \varprojlim_{i\in\mathbb{N}} \widehat{U(\mathcal{H}_i)}_K,$$

while $\widehat{W}_{s,i,K} = \widehat{U(\mathcal{H}_i)}_K / z\widehat{U(\mathcal{H}_i)}_K$, so that $\widetilde{W}_s = \widetilde{U(\mathfrak{h}_3)}/z\widetilde{U(\mathfrak{h}_3)}$, a Fréchet-Stein algebra.

For each $i \in \mathbb{N}$, there is a natural left action of $\widehat{W_{s,i,K}}$ on $M_{s,i} = K\langle \pi^i t_1, \ldots, \pi^i t_s\rangle$, where t_j acts by multiplication and ∂_j acts by $\alpha \mapsto [\partial_j, \alpha]$ for $\alpha \in M_{s,i}$, $1 \leq j \leq s$. There is a natural embedding $\sigma_i : M_{s,i} \to \widehat{W_{s,i,K}}$. σ_i is characterized by the fact that $\sigma_i(\alpha) \cdot 1 = \alpha$ for all $\alpha \in M_{s,i}$.

Proposition 3.1.2 *Let $s \in \mathbb{N}$. Then the $M_{s,i}$ form a coherent sheaf for $(\widetilde{W}_s, q_i)$, where q_i is the norm induced on $\widetilde{W}_s$ by the norm on $\widehat{W_{s,i,K}}$.*

Proof: For each $i \in \mathbb{N}$ we need to produce an isomorphism $\widehat{W_{s,i,K}} \otimes_{\widehat{W_{s,i+1,K}}} M_{s,i+1} \to M_{s,i}$. Let $i \in \mathbb{N}$ and set $N = M_{s,i}$, $N' = M_{s,i+1}$, $V = \widehat{W_{s,i,K}}$ and $V' = \widehat{W_{s,i+1,K}}$. Let ϕ be the homomorphism of left V-modules

$$\phi : V \otimes_{V'} N' \to N \; ; \; \alpha \otimes n \mapsto \alpha \cdot \rho_i(n)$$

where ρ_i is the natural embedding of left V'-modules $N' \to N$. It is clear that ϕ is a well defined homomorphism of left V-modules. Consider the map

$$\phi' : N \to V \otimes_{V'} N' \; ; \; \alpha \mapsto \sigma_i(\alpha) \otimes 1.$$

Since $\sigma_i(\alpha) \cdot 1 = \alpha$, we deduce that $\phi(\sigma_i(\alpha) \otimes 1) = \alpha$. Then ϕ' is a right inverse to ϕ, so ϕ is surjective. It remains to show that ϕ is injective. For this we first observe that $\alpha \otimes n = \alpha\sigma_i(\rho_i(n)) \otimes 1$ for all $n \in N'$. Now we can write

$$\alpha\sigma_i(\rho_i(n)) = \sum_{\lambda\in\mathbb{N}^s} \sigma_i(\gamma_\lambda)\partial^\lambda$$

where $\gamma_\lambda \in N$, $\pi^{-i|\lambda|}\gamma_\lambda \to 0$ as $|\lambda| \to \infty$, and $\partial^\lambda = \partial_1^{\lambda_1} \dots \partial_n^{\lambda_n}$. Now, whenever $\lambda \neq \underline{0}$, we have that $\partial^\lambda \otimes 1 = 1 \otimes (\partial^\lambda \cdot 1) = 0$, so by producing an obvious convergent series we find that $\alpha\sigma_i(\rho_i(n)) \otimes 1 = \sigma_i(\gamma_{\underline{0}}) \otimes 1$. Now we find that

$$\phi'\phi(\alpha \otimes n) = \phi'\phi(\sigma_i(\gamma_{\underline{0}}) \otimes 1) = \phi'(\gamma_{\underline{0}}) = \sigma_i(\gamma_{\underline{0}}) \otimes 1.$$

Hence ϕ' is also a left inverse to ϕ, and ϕ is an isomorphism of left V_i-modules. □

By proposition 3.1.2 we can form a coadmissible module $\widetilde{M_s} = \varprojlim_{i\in\mathbb{N}} M_{s,i}$ with a natural left action from $\widetilde{W_s}$. Clearly the following diagram commutes

$$\begin{array}{ccc} \widetilde{W_s} \times \widetilde{M_s} & \longrightarrow & \widetilde{M_s} \\ \downarrow \iota & & \downarrow \iota \\ \widehat{W_{s,i,K}} \times M_{s,i} & \longrightarrow & M_{s,i} \end{array}$$

where the vertical arrows are homeomorphisms onto their images, and the bottom horizontal map is continuous, so it follows the action of $\widetilde{W_s}$ on $\widetilde{M_s}$ is continuous.

Now, we identify $M_{s-1,i}$ with its image in $M_{s,i}$ under the map sending $t_i \mapsto t_i$ for $1 \leq i \leq j$, and $\widetilde{M_{s-1}}$ with the induced image in $\widetilde{M_s}$.

Given a K-Banach space B, we define $B\langle t\rangle$ as the algebra

$$B\langle t\rangle = \left\{ \sum_{i=0}^{\infty} a_i t^i \in B\,[[t]] \,:\, a_i \to 0 \text{ as } i \to \infty \right\}.$$

as in [1, 4.1]. Note that B is a $K\langle t\rangle$ module.

Lemma 3.1.2 *1. Let B be a K-Banach space. For $j \in \mathbb{N}$, let $\omega_j = 1 - \frac{t\partial}{j}$ and let $\Omega_j = \omega_j \dots \omega_1$. Then if $\alpha = \alpha_0 + \alpha' \in B\langle t\rangle$ where $\alpha_0 \in B$ and $\alpha' \in tB\langle t\rangle$, then $\Omega_j(\alpha) \to \alpha_0$ as $j \to \infty$.*

2. Let $0 \neq \alpha \in \widetilde{M_s}$. Then $\widetilde{W_s}\alpha$ contains a sequence converging to a non-zero element of K.

3. $\widetilde{W_s}$ has no non-trivial closed ideals.

Proof:

1. Let $\alpha \in B\langle t\rangle$. Let $\partial = \frac{d}{dt} \in \mathrm{End}_K(B\langle t\rangle)$. Write $\alpha = \sum_{i\in\mathbb{N}} \alpha_i t^i$ for $\alpha_i \in B$, where $\alpha_i \to 0$ in B as $i \to \infty$. We can calculate that

$$\omega_j \alpha = \sum_{i\in\mathbb{N}} (1 - \frac{i}{j})\alpha_i t^i.$$

Noting that $\binom{j}{i} = 0$ when $j < i$, for all $l \in \mathbb{N}$ we have

$$\prod_{j=1}^{l} (1 - \frac{i}{j}) = (-1)^l \binom{i-1}{l}$$

for $i > 0$, and is equal to 1 for all $l \in \mathbb{N}$ when $i = 0$. Then we find

$$\Omega_l \alpha = \alpha_0 + \sum_{i\geq 1} (-1)^l \binom{i}{l} \alpha_i t^i.$$

Since $v_K\left(\binom{i}{l}\right) \geq 0$ for all $i, l \in \mathbb{N}$, the sequence $(\Omega_j \alpha)_{j\in\mathbb{N}}$ converges to α_0 in $B\langle t\rangle$, proving the claim.

2. We proceed by induction on s. The base case $s = 0$ is obvious since $\widetilde{M_0} = K$. Now assume that the theorem holds for $\widetilde{M_{s-1}}$. For each $i \in \mathbb{N}$, let N_i be the closed submodule of $M_{s,i}$ generated by the t_j for $1 \leq j \leq s-1$, so that $M_{s,i} = N_i\langle \pi^i t_s\rangle$.

For $j \in \mathbb{N}$, let ρ_j be the natural embedding $\widetilde{M_s} \to \widehat{M_{s,j,K}}$. We can write $\alpha = \sum_{i\in\mathbb{N}} \alpha_i t_s^i$ for some $\alpha_i \in \widetilde{M_{s-1}}$ such that $\pi^{-ni}\alpha_i \to 0$ as $i \to \infty$ for all $n \in \mathbb{N}$. Then by part 1 $\Omega_k \rho_j(\alpha) \to \rho_j(\alpha_0)$ as $k \to \infty$ for all $j \in \mathbb{N}$. It follows that $\Omega_k \alpha \to \alpha_0$ as $k \to \infty$.

In the case that $\alpha_0 = 0$, since $\alpha \neq 0$ for some $i \in \mathbb{N}$ we must have that $\partial^i(\alpha)_0 \neq 0$, so we simply replace α with $\partial^i(\alpha)$.

3. From [12, Proposition 1.4.6] $\widehat{W_{s,i,K}}$ is simple for $i \in \mathbb{N}$. Let I be a closed non-zero ideal of $\widetilde{W_s}$. Then the closure of I in $\widehat{W_{s,i,K}}$ is equal to $\widehat{W_{s,i,K}}$ for $i \in \mathbb{N}$. It follows that $I = \varprojlim_{i\in\mathbb{N}} \widehat{W_{s,i,K}} = \widetilde{W_s}$, so $\widetilde{W_s}$ has no non-trivial ideals.

□

Theorem 3.1.2 *$\widetilde{W_s}$ has a faithful coadmissible simple module.*

Proof: We will show that $\widetilde{M_s}$ is a coadmissible simple faithful left $\widetilde{W_s}$-module. We have already seen that the action of $\widetilde{W_s}$ on $\widetilde{M_s}$ is continuous. By [18, 3.4(ii)], the coimage of any linear map between coadmissible modules is coadmissible, hence for any $\alpha \in \widetilde{M_s}$ we have that $\widetilde{W_s}\alpha$ is coadmissible in M_s. Now, by lemma 3.1.2(2) $\widetilde{W_s}\alpha$ contains an element of K, and since $\widetilde{W_s}\alpha$ is closed it follows that it contains a non-zero element of K. The map $\widetilde{M_s} \to \widetilde{W_s} \times \{1\}$; $\alpha \mapsto (\alpha, 1)$ is a section to the restriction of the action of $\widetilde{W_s}$ on $\widetilde{M_s}$ to $\widetilde{W_s} \times \{1\}$, so it follows $\widetilde{W_s}\alpha = \widetilde{M_s}$, and hence $\widetilde{M_s}$ is simple.

To see that $\widetilde{M_s}$ is faithful, let $a, b \in \widetilde{W_s}$. Write $a = \sum_{i \in \mathbb{N}} a_i \partial^i$ and $b = \sum_{i \in \mathbb{N}} b_i \partial^i$ with a_i, b_i in the image of $\widetilde{M_s}$ in $\widetilde{W_s}$. Choose the smallest $i \in \mathbb{N}$ such that $a_i \neq b_i$. Then $at^i = a_0 t^i + i a_1 t^{i-1} + ... + i! a_i$ while $bt^i = a_0 t^i + ... + (i-1)! a_{i-1} t^{i-1} + i! b_i$. Therefore $at^i \neq bt^i$, proving the claim. □

3.2 Coadmissible Primitive Spectrum of the Arens Michael Envelope of a Primitive Lie Algebra

3.2.1 Correspondence Theorem

If A is a K algebra and I is an ideal of A such that $Z(A/I)$ is isomorphic to K, then we say that I is weakly rational.

In this section, we let $\mathfrak{g}$ be a finite dimensional nilpotent K-Lie algebra. By [7, Proposition 4.7.4, Theorem 4.7.8(ii)], the set of rational, primitive, and maximal ideals of $U(\mathfrak{g})$ are equal, and if I is a weakly rational ideal, then for some $s \in \mathbb{N}$ we have a surjection of K-algebras $U(\mathfrak{g})/I \to A_s(K)$ with kernel I, and call such a map a Dixmier map. We abuse notation and use ι to refer to the canonical embedding of any topological ring in its completion.

Lemma 3.2.1 *1. If J is a closed primitive ideal of $\widetilde{U(\mathfrak{g})}$ then $J \cap U(\mathfrak{g})$ is primitive, and if J is a closed weakly rational ideal then $J \cap U(\mathfrak{g})$ is weakly rational.*

2. Let J be a closed weakly rational ideal of $\widetilde{U(\mathfrak{g})}$, and let I be a closed weakly rational ideal of $U(\mathfrak{g})$. Then $I\widetilde{U(\mathfrak{g})}$ is a closed weakly rational ideal of $\widetilde{U(\mathfrak{g})}$, and $(J \cap U(\mathfrak{g}))\widetilde{U(\mathfrak{g})} = J$.

3. $\widetilde{U(\mathfrak{g})}$ is flat as a $U(\mathfrak{g})$-module.

Proof:

1. Let $J' = J \cap U(\mathfrak{g})$. By theorem 3.1.1 the center L of $\widetilde{U(\mathfrak{g})}/J$ is an algebraic field extension of K. $U(\mathfrak{g})$ is dense in $\widetilde{U(\mathfrak{g})}$, so $U(\mathfrak{g})/J'$ is dense in $\widetilde{U(\mathfrak{g})}/J$ and

$$K \subset Z(U(\mathfrak{g})/J') \subset L.$$

It follows that $Z(U(\mathfrak{g})/J')$ is a field, and by [7, Proposition 4.7.4] it follows that J' is primitive. Clearly when $L = K$ we have $Z(U(\mathfrak{g})/J') = K$, so that J' is weakly rational.

2. First of all, $I\widetilde{U(\mathfrak{g})}$ is a finitely generated right $\widetilde{U(\mathfrak{g})}$-module so it is closed, and since I is dense in $I\widetilde{U(\mathfrak{g})}$, I is an ideal since the closure of an ideal is an ideal.

Let Ψ be a Dixmier map $U(\mathfrak{g}) \to A_s(K)$ with kernel I. Choose a lattice $\mathscr{L}$ in $\mathfrak{g}$ and for $i \in \mathbb{N}$ let $\mathscr{L}_i = \pi^i \mathscr{L}$. Then $\Psi(\mathscr{L})$ is a finitely generated R-module in $A_s(K)$. Fix $j \in \mathbb{N}$. Then $W_{s,j} \otimes K = A_s(K)$, so it follows that for some $n \in \mathbb{N}$ we have $\Psi(\mathscr{L}_n) \subset W_{s,j}$. Hence $\Psi(U(\mathscr{L}_n)) \subset W_{s,j}$, so Ψ is bounded with respect to the norms induced on $U(\mathfrak{g})$ by $U(\mathscr{L}_n)$ and on $A_s(K)$ by $W_{s,j}$. Then there is a unique continuous extension

$$\Psi_j : \widehat{U(\mathscr{L}_n)}_K \to \widehat{W_{s,j,K}}$$

of Ψ. $\widehat{U(\mathscr{L}_n)}_K$ is a flat $U(\mathfrak{g})$-module by [4, 3.2.3(iv)], so the kernel of Ψ_j is $I\widehat{U(\mathscr{L}_n)}_K$. Let ρ_j be the canonical embedding of $\widetilde{U(\mathfrak{g})}$ into $\widehat{U(\mathscr{L}_n)}_K$, and let $\Psi'_j = \Psi_j \rho_j$ (Ψ'_j doesn't depend on the choice of n.) Then for all $j \in \mathbb{N}$ we have $\Psi_j|_{U(\mathfrak{g})} = \iota\Psi$, where ι is the embedding $W_s \to \widehat{W_{s,j,K}}$. Noting that $U(\mathfrak{g})$ is dense in $\widetilde{U(\mathfrak{g})}$, we deduce that the following diagram commutes

$$\begin{array}{ccc} & \widetilde{U(\mathfrak{g})} & \\ \swarrow^{\Psi'_j} & & \searrow^{\Psi'_i} \\ \widehat{W_{s,j,K}} & \longrightarrow & \widehat{W_{s,i,K}} \end{array}$$

whenever $j \geq i$. Hence we obtain a continuous map $\widetilde{U(\mathfrak{g})} \to \widetilde{W}_s$ whose restriction to $U(\mathfrak{g})$ is Ψ and whose kernel contains $I\widetilde{U(\mathfrak{g})}$. Let χ be the induced map $\widetilde{U(\mathfrak{g})}/I\widetilde{U(\mathfrak{g})} \to \widetilde{W}_s$. We will produce an inverse map to χ.

From the proof of [18, 3.7], we have that

$$\varprojlim_{i \in \mathbb{N}} \widehat{U(\mathscr{L}_i)}_K / I\widehat{U(\mathscr{L}_i)}_K = \widetilde{U(\mathfrak{g})}/I\widetilde{U(\mathfrak{g})}.$$

I is a maximal ideal of $U(\mathfrak{g})$ so clearly we have that $I = I\widetilde{U(\mathfrak{g})} \cap U(\mathfrak{g})$. Hence $U(\mathfrak{g})/I$ is dense in each $\widehat{U(\mathscr{L}_i)}_K / I\widehat{U(\mathscr{L}_i)}_K$. By similar reasoning to above, if we fix $i \in \mathbb{N}$ we can find some $n \in \mathbb{N}$ such that $\Psi^{-1}(W_n) \subset U(\mathscr{L}_i)/(I \cap U(\mathscr{L}_i))$ and construct a continuous morphism $\chi' : \widetilde{W}_s \to \widetilde{U(\mathfrak{g})}/I\widetilde{U(\mathfrak{g})}$ such that $\chi'|_{A_s(K)} = \Psi^{-1}$. Then $(\chi\chi')|_{A_s(K)} = \mathrm{id}_{A_s(K)}$ and $(\chi'\chi)|_{U(\mathfrak{g})/I} = \mathrm{id}_{U(\mathfrak{g})/I}$. Since $U(\mathfrak{g})/I$ is dense in $\widetilde{U(\mathfrak{g})}/I\widetilde{U(\mathfrak{g})}$ and $A_s(K)$ is dense in $\widetilde{W}_s$, it follows that χ is an isomorphism. Since the center of $\widetilde{W}_s$ is K, this proves the first statement.

By lemma 3.1.2(3) $\widetilde{W_s}$ has no non-trivial closed ideals so it follows that $I\widetilde{U(\mathfrak{g})}$ is maximal within the lattice of closed ideals of $\widetilde{U(\mathfrak{g})}$. Then if we take I to be the ideal $J \cap U(\widetilde{U(\mathfrak{g})})$, a weakly rational ideal by part 1, then $(J \cap U(\mathfrak{g}))\widetilde{U(\mathfrak{g})} \subset J$ implies $J = (J \cap U(\mathfrak{g}))\widetilde{U(\mathfrak{g})}$, proving the second statement.

3. By [18, Remark 3.2], for any lattice $\mathscr{L} \subset \mathfrak{g}$, $\widehat{U(\mathscr{L})}_K$ is a flat $\widetilde{U(\mathfrak{g})}$-module, and by [4, 3.2.3(iv)] $\widehat{U(\mathscr{L})}_K$ is a flat $U(\mathfrak{g})$ module. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of $U(\mathfrak{g})$ modules. Let $\mathscr{L}$ be a lattice in $\mathfrak{g}$ and for $n \in \mathbb{N}$ let $\mathscr{L}_n = \pi^n \mathscr{L}$. Then the map

$$\widehat{U(\mathscr{L}_n)}_K \otimes_{U(\mathfrak{g})} M_1 \to \widehat{U(\mathscr{L}_n)}_K \otimes_{U(\mathfrak{g})} M_2$$

is an embedding for all $n \in \mathbb{N}$ since $\widehat{U(\mathscr{L}_n)}_K$ is flat over $U(\mathfrak{g})$.

It follows that the map $\varprojlim_{n\in\mathbb{N}} \widehat{U(\mathscr{L}_n)}_K \otimes M_1 \to \varprojlim_{n\in\mathbb{N}} \widehat{U(\mathscr{L}_n)}_K \otimes M_2$ is an embedding. Let $\widetilde{M_i} = \widetilde{U(\mathfrak{g})} \otimes_{U(\mathfrak{g})} M_i$ for $i = 1, 2$. From [18, Corollary 3.3], for $i = 1, 2$ we have that $\widetilde{M_i} = \varprojlim_{n\in\mathbb{N}} \widehat{U(\mathscr{L}_n)}_K \otimes_{\widetilde{U(\mathfrak{g})}} \widetilde{M_i}$. Of course, for $n \in \mathbb{N}$ we have that

$$\varprojlim_{n\in\mathbb{N}} \widehat{U(\mathscr{L}_n)}_K \otimes_{\widetilde{U(\mathfrak{g})}} \widetilde{M_i} = \varprojlim_{n\in\mathbb{N}} \widehat{U(\mathscr{L}_n)}_K \otimes_{U(\mathfrak{g})} M_i,$$

so the map $\widetilde{M_1} \to \widetilde{M_2}$ is an embedding, and $\widetilde{U(\mathfrak{g})}$ is a flat $U(\mathfrak{g})$-module.

□

We define c.Prim$(\widetilde{U(\mathfrak{g})})$ to be the set of ideals which annihilate simple coadmissible $\widetilde{U(\mathfrak{g})}$-modules.

Theorem 3.2.1 *The map $J \mapsto J \cap U(\mathfrak{g})$ induces a bijection between* c.Prim$(\widetilde{U(\mathfrak{g})})$ *and* Prim$(U(\mathfrak{g}))$.

Proof: By lemma 3.2.1(i) the map Ψ : c.Prim$(\widetilde{U(\mathfrak{g})}) \to$ Prim$(U(\mathfrak{g}))$; $J \mapsto J \cap U(\mathfrak{g})$ is well defined. We aim to prove that the map $\Psi' : I \mapsto \widetilde{U(\mathfrak{g})}I$ is inverse to Ψ. Let I

be a primitive ideal of $U(\mathfrak{g})$. Let L be the center of $U(\mathfrak{g})/I$. Then L is an algebraic field extension of K by Quillen's lemma. Let M be a simple left $U(\mathfrak{g})$-module such that I is the annihilator of M. Then there is a natural extension of the action of $U(\mathfrak{g})$ to $U(\mathfrak{g})_L = U(\mathfrak{g}) \otimes L$. Let I' be the annihilator of M in $U(\mathfrak{g})_L$ with respect to this action. Then since $L \subset U(\mathfrak{g})/I$, we deduce that $U(\mathfrak{g})/I$ is isomorphic as a K-algebra to $U(\mathfrak{g})_L/I'$. Of course I' is a weakly rational ideal of $U(\mathfrak{g})_L$ so by lemma 3.2.1(ii), $\widetilde{U(\mathfrak{g})}_L I'$ is a closed weakly rational ideal of $\widetilde{U(\mathfrak{g})}_L = \widetilde{U(\mathfrak{g})} \otimes_K L$.

Now, the sequence

$$0 \to \widetilde{U(\mathfrak{g})}I \to \widetilde{U(\mathfrak{g})} \to \widetilde{U(\mathfrak{g})} \otimes_{U(\mathfrak{g})} U(\mathfrak{g})/I \to 0$$

is exact. But since $\widetilde{U(\mathfrak{g})} \otimes_{U(\mathfrak{g})} (U(\mathfrak{g})/I)$ is isomorphic to $\widetilde{U(\mathfrak{g})} \otimes_{U(\mathfrak{g})} (U(\mathfrak{g})_L/I')$, a primitive ring, it follows that $\widetilde{U(\mathfrak{g})}I$ is primitive. Moreover, since $\widetilde{U(\mathfrak{g})} \otimes_{U(\mathfrak{g})} (U(\mathfrak{g})/I')$ has no non-trivial closed ideals, we find that $\widetilde{U(\mathfrak{g})}I$ is maximal within the lattice of closed ideals of $\widetilde{U(\mathfrak{g})}$.

Since $\widetilde{U(\mathfrak{g})}I$ is finitely generated it is coadmissible, so the map Ψ' is well defined, and it is trivial that if $I \in \text{Prim}(U(\mathfrak{g}))$ then $I \subset \Psi\Psi'(I)$. But by [7, Proposition 4.7.4] I is maximal, so $I = \Psi\Psi'(I)$. On the other hand if $J \in \text{c.Prim}(\widetilde{U(\mathfrak{g})})$ then $\Psi(J)$ is primitive, and as shown above $\Psi'\Psi(J)$ is maximal within the lattice of closed ideals. But $\Psi'\Psi(J) \subset J$, so $\Psi'\Psi(J) = J$. Hence Ψ' is inverse to Ψ. □

Chapter 4

Arithmetic differential operators over the affine line

4.1 Some Notation From Algebraic Geometry

It is assumed that the reader is familiar with the basic notions of algebraic geometry as an account here would be awkward. For an introduction you could see, for instance, Hartshorne or EGA. I will try and explain the important mechanics at play in the paper, which are all quite simple and fundamental.

Let R be a c.d.v.r, π a uniformizer of R, κ its residue field, and K its field of fractions. Let $S = \mathrm{Spec}(R)$.

4.1.1 Affine and Projective Line

Let A be a ring, and let $V = \mathrm{Spec}(A)$. Then we define the affine line $\mathbb{A}^1_A$ over A to be the spectrum of the one dimensional polynomial algebra over A. By choosing a coordinate t for $\mathbb{A}^1_A$ we are simply choosing a presentation $\mathbb{A}^1_A = \mathrm{Spec}(A\,[t])$.

Now let $\mathcal{A}$ and $\mathcal{A}'$ be two copies of $\mathbb{A}^1_A$. Let t be a coordinate for $\mathcal{A}$ and let s be a coordinate for $\mathcal{A}'$. Then the natural A-algebra homomorphism

$$\mathcal{O}(A) \to \mathcal{O}(A \backslash 0)$$

and the A-algebra homomorphism

$$\mathcal{O}(B) \to \mathcal{O}(A \backslash 0) \ ; \ s \mapsto t^{-1}$$

induce a diagram of X-schemes

$$\begin{array}{ccc} \mathcal{A} \backslash \{0\} & \longrightarrow & \mathcal{A} \\ \downarrow & & \\ \mathcal{A}' & & \end{array}$$

. We define $\mathbb{P}^1_A$ to be the colimit of of this diagram. When we choose a coordinate t for $\mathbb{P}^1_A$, we are simply choosing a presentation of $\mathbb{P}^1_A$ as the colimit of a diagram as above (and abusing notation by identifying s with t^{-1}.)

4.1.2 Completion of $\mathcal{O}_X$-modules

Now let X be an S-scheme, and let M be an $\mathcal{O}_X$-module. Then we define $\widehat{M} = \varprojlim_{i \in \mathbb{N}} M/\pi^i M$, whose module of sections on an open $U \subset X$ can be shown to be the R-module $\varprojlim_{i \in \mathbb{N}} M(U)/\pi^i M(U)$ (see [8, Proposition 9.2].) We define $\widehat{M_K}$ to be the $\mathcal{O}_X$-module whose sections on an open subset U of X are the R-module $\widehat{M}(U) \otimes_R K$.

4.2 Berthelot's Arithmetic Differential Operators

4.2.1 The Sheaf of Divided Powers of level m over $\mathbb{A}^1$

The sheaf of divided powers $\mathcal{D}^{[m]}_X$ of a smooth S-scheme X of level m is defined in [4, Section 2.2.1] ($\mathcal{D}^{[m]}_X$ is written $\mathcal{D}^{(m)}_{X/S}$ in that paper.) $\mathcal{D}^{[0]}_X$ is naturally an $\mathcal{O}_X$-module by [4, Section 2.2.1, Equation 2.2.1.4]. We will investigate the structure of the π-adic completion of the sheaf $\mathcal{D}_X^{[m]}$ where X is in dimension one, so we will only recall the properties we will need for this specific case.

First, let $X = \mathbb{A}^1_R$. Let t be a coordinate for X, let dt be the basis element for $\Omega^1_{X/S}(X)$ corresponding to t, and let ∂_t be the dual operator to dt in $\mathcal{T}_X(X)$, where $\mathcal{T}_X$ is the tangent sheaf of X. Then $\mathcal{D}_X^{[0]}$ is a sheaf of Noetherian rings generated over $\mathcal{O}_X$ by ∂_t (for a proof see [4, Corollaire 2.2.5], and [4, Remarque 2.2.5(i)].) Now, let p be the residue characteristic of R, let $i \in \mathbb{N}$ and let $q_i \in \mathbb{N}$ be the unique integer such that $i = p^m q_i + r$ with $0 \leq r < p^m$.

Using [4, Proposition 2.2.4] we set $\mathcal{D}_X^{[m]} = \bigoplus_{i\in\mathbb{N}} \partial_t^{[i]} \mathcal{O}_X$ as an $\mathcal{O}_X$-module, with multiplication defined on an open $U \subset X$ by the following equations:

1. For all $i \in \mathbb{N}$, the action of $\partial_t^{[i]}$ on $\mathcal{O}_X(U)$ is given by
$$\partial_t^{[i]}(t^j) = q_i! \binom{j}{i} t^{j-i} \in \mathcal{O}_X(U) \text{ for all } j \in \mathbb{N}.$$
2. For all $i, j \in \mathbb{N}$,
$$\partial_t^{[i]} \partial_t^{[j]} = \binom{i+j}{i} \frac{q_i! q_j!}{q_{i+j}!} \partial_t^{[i+j]} \in \mathcal{D}_X^{[1]}.$$
3. For all $i \in \mathbb{N}$, $\alpha \in \mathcal{O}_X(U)$
$$\partial_t^{[i]} \alpha = \sum_{k+l=i} \frac{q_i!}{q_k! q_l!} \partial_t^{[k]}(\alpha) \partial_t^{[l]}.$$

($\partial_t^{[p]}$ is written $\partial_t^{\langle p \rangle (m)}$ or just $\partial_t^{\langle p \rangle}$ in [4].)

$\mathcal{D}_X^{[m]}$ is a sheaf of Noetherian rings generated over $\mathcal{O}_X$ by $\partial_t^{[p^i]}$ for $0 \leq i \leq m$ by [4, Corollaire 2.2.5]. From the equations we can see that the morphism of sheaves $(X, \mathcal{D}_X^{[0]}) \to (X, \mathcal{D}_X^{[m]})$ which is the identity on X and sends $\partial_t^i \mapsto \frac{i!}{q_i!} \partial_t^{[i]}$ is injective. For this reason we write $\partial_t^{[1]} = \partial_t$ and think of $\mathcal{D}_X^{[0]}$ as a subsheaf of $\mathcal{D}_X^{[m]}$. Whenever $0 \leq i \leq p^m$, we have that $q_i! = 1$, so $\partial_t \partial_t^{[i]} = (i+1) \partial_t^{[i+1]}$. Hence $\partial_t^{p^m} = p^m! \partial_t^{[p^m]}$.

4.2.2 Completion of the Sheaf of Divided Powers

Let X be a copy of $\mathbb{P}^1_R$ or $\mathbb{A}^1_R$. Then we set $\widehat{\mathcal{D}_X^{[m]}} = \varprojlim_{i \in \mathbb{N}} \widehat{\mathcal{D}_X^{[m]}}/\pi^i \widehat{\mathcal{D}_X^{[m]}}$ for $m \in \mathbb{N}$ (The definition of an inverse limit of sheaves is given in [8, Chapter 2, Proposition 9.2].)

Proposition 4.2.2 *Let X be a copy of $\mathbb{A}^1_R$.*

1. *$\widehat{\mathcal{D}_X^{[0]}}(X) \otimes K$ is a simple domain.*

2. *Let $\mathbb{N}_p = \{0, \ldots, p-1\}$. Every element α of $\widehat{\mathcal{D}_X^{[m]}}(X)$ can be written uniquely in the form*

$$\alpha = \sum_{i,j \in \mathbb{N}, \lambda \in \mathbb{N}_p^m} \alpha_{ij\lambda} t^i \partial_t^{\lambda_1} (\partial_t^{[p]})^{\lambda_2} \ldots (\partial_t^{[p^{m-1}]})^{\lambda_m} (\partial_t^{[p^m]})^j$$

   with $\alpha_{ij\lambda} \in R$, $\alpha_{ij\lambda} \to 0$ as $i + j \to \infty$.

3. *$\widehat{\mathcal{D}_X^{[m]}}$ is flat over R.*

4. *$\frac{\partial_t^i}{i!} \in \mathcal{D}_X^{[m]}(X)$ for all $i < p^{m+1}$.*

Proof:

1. From the definitions we can see that $\widehat{\mathcal{D}_X^{[0]}}(X)$ is the π-adic completion of the R-algebra generated by t and ∂_t, subject only to the relation $[\partial_t, t] = 1$. Then [2, 7.3] provides a proof that $\widehat{\mathcal{D}_X^{[0]}}(X) \otimes K$ is simple.

2. Since X is affine, we know from [8, Proposition 9.2] that

$$\widehat{\mathcal{D}_X^{[m]}}(X) = \varprojlim_{i \in \mathbb{N}} \mathcal{D}_X^{[m]}(X)/\pi^i \mathcal{D}_X^{[m]}(X) = \widehat{\mathcal{D}_X^{[m]}(X)}.$$

   Let

$$A = R\left[a, b_0, \ldots, b_m\right] / (b_{i-1}^p - \frac{p^i!}{(p^{i-1}!)^p} b_i)$$

Then, if we give the R module A the structure of a topological R-module with the πA-topology, and give $\mathcal{D}_X^{[m]}(X)$ its $\pi\mathcal{D}_X^{[m]}(X)$-adic topology, then there is a topological isomorphism

$$A \to \mathcal{D}_X^{[m]}(X) \; ; \; a \mapsto t \; ; \; b_i \mapsto \partial_t^{[p^i]},$$

which gives us an isomorphism

$$\varprojlim_{i\in\mathbb{N}} A/\pi^i A \to \widehat{\mathcal{D}_X^{[m]}}(X).$$

Now, from the definition of A, there is exactly one way of writing each element $\alpha \in A$ (and each element $\alpha \in \varprojlim_{i\in\mathbb{N}} A/\pi^i A$ in the form

$$\alpha = \sum_{i,j\in\mathbb{N},\lambda\in\mathbb{N}_p^m} \alpha_{ij\lambda} a^i b_0^{\lambda_1} b_1^{\lambda_2} \dots b_{m-1}^{\lambda_m} b_m^j$$

proving the proposition.

3. Since $\mathcal{D}_X^{[m]}(X)$ is a free R-module it is flat over R, and by [4, 3.2.3(4)], $\widehat{\mathcal{D}_X^{[m]}}(X)$ is flat over $\mathcal{D}_X^{[m]}(X)$, it follows $\widehat{\mathcal{D}_X^{[m]}}(X)$ is flat over R.

4. Since $i < p^{m+1}$ we can write $i = a_m p^m + a_{m-1}p^{m-1} + \ldots + a_0$ with $0 \leq a_j < p$ for $0 \leq j \leq m$. Then by [6, 1.2.3.3]

$$v_K(i!) = \sum_{j=1}^{m} a_j v_K(p^j!).$$

Then, for some $\varepsilon \in R^\times$

$$\frac{\partial^i}{i!} = \varepsilon \partial^{a_0} \left(\partial^{[p]}\right)^{a_1} \dots (\partial^{[p^m]})^{a_m},$$

proving the claim.

□

4.3 Sections over the Affine Line

4.3.1 The Main Theorem

Let X and Y be two copies of $\mathbb{A}^1_R$, and let t be a coordinate for X and τ a coordinate for Y. Let $F : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be the morphism of formal schemes induced by the ring homomorphism $\mathcal{O}(Y) \to \mathcal{O}(X)$; $\tau \mapsto t^{p^m}$. Let $\partial_\tau \in \mathcal{T}(Y)$ be the operator dual to $d\tau$ and let $\partial_t \in \mathcal{T}(X)$ be the operator dual to dt.

If $\mathcal{A}$ is a sheaf of R-algebras over an S-scheme Z, we define $M_n(\mathcal{A})$ to be the sheaf such that $M_n(\mathcal{A})(U) = M_n(\mathcal{A}(U))$ for all open $U \subset Z$, where $M_n(\mathcal{A}(U))$ is the n-th matrix ring over $\mathcal{A}(U)$. We set $\{\tilde{e}_{ij}\}_{1\leq i,j\leq n}$ to be the set of standard matrix units for $M_n(\mathcal{A}(Z))$.

We will prove the following theorem:

Theorem 4.3.1 *There is an isomorphism of $\mathcal{O}_Y$-rings*

$$M_{p^m}(\widehat{\mathcal{D}_Y^{[0]}}) \to F_*\widehat{\mathcal{D}_X^{[m]}}$$

such that, on global sections, $\partial_\tau \mapsto \gamma\partial_t^{[p^m]}$ for some $\gamma \in 1 + \pi\widehat{\mathcal{D}_X^{[m]}}(X)$.

Let $A_m = F_*\widehat{\mathcal{D}_X^{[m]}}(Y)$ and $A_0 = \widehat{\mathcal{D}_Y^{[0]}}(Y)$.

Lemma 4.3.1 *Suppose that there is a map $\omega : M_{p^m}(A_0) \to A_m$ which is an isomorphism of $\mathcal{O}_Y(Y)$-rings such that $\partial_\tau \mapsto \gamma\partial_t^{[p^m]}$ for some $\gamma \in 1 + \pi A_m$. Then there exists an isomorphism of $\mathcal{O}_Y$-rings $\Omega : M_{p^m}(\widehat{\mathcal{D}_Y^{[0]}}) \to F_*\widehat{\mathcal{D}_X^{[m]}}$ such that $\Omega(Y) = \omega$.*

Proof: Set $\mathcal{A}_0 = \widehat{\mathcal{D}_Y^{[0]}}$ and $\mathcal{A}_m = \widehat{\mathcal{D}_Y^{[m]}}$. Let U be an open subset of Y. For $n \in \mathbb{N}$, the isomorphism $\omega : M_{p^m}(\mathcal{A}_0(X)) \to \mathcal{A}_m(Y)$ induces an isomorphism

$$\omega_n : M_p(\mathcal{A}_0(Y)/\pi^n\mathcal{A}_0(Y)) \to \mathcal{A}_m(Y)/\pi^n\mathcal{A}_m(Y).$$

For $i = 0, m$, $n \in \mathbb{N}$, we have $\mathcal{A}_i(Y)/\pi^n\mathcal{A}_i(Y) = \mathcal{D}_Y^{[i]}/\pi^n\mathcal{D}_Y^{[i]}$. As explained in section 4.2.1, $\mathcal{D}_Y^{[i]}$ is generated over $\mathcal{O}_Y$ by its global sections, so we can deduce that $\mathcal{D}_Y^{[i]}/\pi^n\mathcal{D}_Y^{[i]}(U) = \mathcal{O}_Y(U) \otimes_{\mathcal{O}(Y)} \mathcal{D}_Y^{[i]}/\pi^n\mathcal{D}_Y^{[i]}(Y)$. So we can construct a commutative square

$$\begin{array}{ccc} \mathcal{O}_Y(U) \otimes_{\mathcal{O}(Y)} M_{p^m}(\mathcal{D}_Y^{[0]}/\pi^n\mathcal{D}_Y^{[0]})(Y) & \longrightarrow & \mathcal{O}_Y(U) \otimes_{\mathcal{O}(Y)} \mathcal{D}_Y^{[m]}/\pi^n\mathcal{D}_Y^{[m]}(Y) \\ \downarrow & & \downarrow \\ M_{p^m}(\mathcal{D}_Y^{[0]}/\pi^n\mathcal{D}_Y^{[0]})(U) & \longrightarrow & \mathcal{D}_Y^{[m]}/\pi^n\mathcal{D}_Y^{[m]}(U) \end{array}$$

where the vertical arrows are equalities, and the top horizontal arrow is induced by ω_n. Now, we construct Ω by setting $\Omega(U) : M_p(\mathcal{A}_0)(U) \to \mathcal{A}_m(U)$ to be the inverse limit of the bottom horizontal arrows. $\square$

Now, using lemma 4.3.1, we can prove theorem 4.3.1 by constructing an isomorphism

$$\omega : \widehat{\mathcal{D}^{[m]}}(Y) \to M_{p^m}(\widehat{\mathcal{D}^{[0]}}(Y)).$$

We will proceed in the following manner:

1. Identify a commutative subalgebra $C \subset A_m$ containing a complete set of non-zero distinct orthogonal idempotents $\{e_{ii}\}_{0 \leq i \leq p^m - 1}$.

2. Find $\gamma \in 1 + \pi C$ such that $\left[\gamma\partial_t^{[p^m]}, t^{p^m}\right] = 1$, and prove that the closed R-algebra generated by $\mathcal{O}_Y(Y)$ and $\gamma\partial_t^{[p^m]}$ is isomorphic to A_0.

3. Define a set of elements $\{e_{ij}\}_{1 \leq i,j \leq p^m} \subset A_m$ which form a set of matrix units for A_m, and show that they commute with $\mathcal{O}_Y(Y)$ and $\gamma\partial_t^{[p^m]}$.

4. Use the set $\{e_{ij}\}_{1 \leq i,j \leq p^m}$ and the element $\gamma\partial_t^{[p^m]}$ to construct the required isomorphism of $\mathcal{O}_Y(Y)$-rings $\omega : M_p(A_0) \to A_m$.

Throughout this section, we will abbreviate ∂_t to ∂.

4.3.2 Facts about Binomials

For the rest of this section, we define $w = p^m$, and if $0 \le i \le w-1$, we set $i^* = w - 1 - i$.

Lemma 4.3.2 *1. Let x and y be some formal variables. Then*

$$\binom{x+y}{n} = \sum_{i=0}^{n} \binom{x}{j}\binom{y}{n-j}$$

2. Let $x \in \mathbb{N}$ and let $n \in \mathbb{N}$. Then

$$\sum_{i=0}^{n} \binom{i}{x} = \binom{n+1}{x+1}$$

3. For all $k \in \mathbb{Z}$

$$\binom{k}{w-1} \equiv \begin{cases} 1 \ mod\ p\mathbb{Z}_p & if\ k \equiv -1 \ mod\ w\mathbb{Z}_p \\ 0 \ mod\ p\mathbb{Z}_p & otherwise \end{cases}$$

4. (Newton Interpolation Formula:) Let $f(x) \in K[x]$, and for $j \in \mathbb{N}$ let $C_j(f) = \sum_{k \le j} (-1)^{j-k} \binom{j}{k} f(k)$. Then $f(x) = \sum_{j \in \mathbb{N}} C_j(f) \binom{x}{j}$.

5. For all $0 \le i \le w-1$, $\binom{w-1}{i} \equiv (-1)^i \ mod\ p\mathbb{Z}$.

Proof: Parts 1 and 2 are well known identities that can be easily found in a set of introductory lecture notes - the first is known as the Vandermonde identity and the second is the known as the "sum of binomial coefficients over upper index" identity. Part 4 can be viewed as a special case of Mahler's theorem. For part 5, use the identity

$$(1-X)^{w-1} \equiv \frac{(1+X^w)}{1-X} = (1+X^w)(\sum_{i \in \mathbb{N}} (-1)^i X^i) \bmod pK[[X]].$$

For part 3, write $k = a_{m'}p^{m'} + a_{m-1}p^{m-1} + \ldots + a_0$ with $m' \ge m$, $0 \le a_i < p$ for $0 \le i \le m'$. Then by Lucas' theorem

$$\binom{k}{w-1} \equiv \binom{a_{m'}}{0}\binom{a_{m'-1}}{0} \cdots \binom{a_m}{0}\binom{a_{m-1}}{p-1}\binom{a_{m-2}}{p-1} \cdots \binom{a_1}{p-1} \bmod p\mathbb{Z}.$$

from which part 3 follows immediately. □

4.3.3 The Diagonal Algebra

Let C be the closed R-subalgebra of A_m generated over R by $d_i = t^i\partial^{[i]}$ for $0 \leq i < p^{m+1}$. To ease notation, set $c_i = t^{p^i}\partial^{[p^i]}$ for $0 \leq i \leq m$.

Proposition 4.3.3 *1. For all $n \in \mathbb{N}$, $\prod_{i=0}^{n-1}(t\partial - i) = t^n\partial^n$*

2. $i!d_i = \prod_{j=0}^{i}(c_0 - j)$ for $1 \leq i \leq w$ (recall $w = p^m$), $a_i d_i = \prod_{j=0}^{i}(c_0 - j)$ for some $a_i \in R$ for all $i \in \mathbb{N}$, and C is a commutative R-algebra.

3. For all $0 \leq a \leq w$ and $b \in \mathbb{Z}$ the element $\binom{c_0+b}{a} \in C \otimes_R K$ belongs to the image of C in $C \otimes_R K$.

4. There is a continuous automorphism ϕ of C such that $\phi(c_0) = c_0 + 1$. ϕ can be extended to an automorphism of $\widehat{\mathcal{D}_X^{[m]}}(X\backslash\{0\})$ which sends $\alpha \mapsto t^{-1}\alpha t$ for all $\alpha \in \widehat{\mathcal{D}_X^{[m]}}(X\backslash\{0\})$.

5. For all $\alpha \in C$, $\partial\alpha = \phi(\alpha)\partial$ and $t\alpha = \phi^{-1}(\alpha)t$.

6. For all $i \in \mathbb{N}$, $\frac{v_K(i!)}{i} < \frac{v_K(p)}{p-1}$.

Proof:

1. We proceed by induction on n. For $n = 0$ the statement is tautological, so suppose the statement is true for all $m < n$. Then $\prod_{i=0}^{n}(t\partial - i) = (t^{n-1}\partial^{n-1})(t\partial - n)$. But $[\partial^{n-1}, t] = (n-1)\partial^{n-2}$, so

$$(t^{n-1}\partial^{n-1})(t\partial - (n-1)) = t^n\partial^n + (n-1)t^{n-1}\partial^{n-1} - (n-1)t^{n-1}\partial^{n-1} = t^n\partial^n,$$

completing the proof.

2. Since $d_i = t^i\partial^{[i]}$, from section 4.2.1 we have $i!d_i = t^i\partial^i$ when $1 \leq i \leq w$. From the definition of $\partial^{[i]}$ we have $a_i\partial^{[i]} = \partial^i$ for some $a_i \in \mathbb{Z}$, and we know $a_i = i!$ for

$1 \leq i \leq w$. Using part 1 we have

$$a_i d_i = t^i \partial^i = \prod_{j=0}^{i-1} (t\partial - j) = \prod_{j=0}^{i-1} (c_0 - j),$$

proving the first two parts of the statement. By proposition 4.2.2(3), C is contained in a flat R-algebra, so it follows that the d_i pairwise commute for $1 \leq i \leq w-1$, and C is a commutative R-algebra.

3. Let $0 \leq a \leq w-1$. Using lemma 4.3.2(1) we can see that

$$\binom{c_0 + b}{a} = \sum_{i=0}^{b} \binom{c_0}{j} \binom{b}{n-j} = \sum_{i=0}^{k} \binom{b}{n-j} d_j$$

proving the claim.

4. By proposition 4.2.2(2), C can be viewed as an R-subalgebra of $\widehat{\mathcal{D}_X^{[m]}}(X \backslash \{0\})$, we will prove that C is an invariant of the automorphism $a \mapsto t^{-1}at$ of $\widehat{\mathcal{D}_X^{[m]}}(X \backslash \{0\})$. This will be enough, as we can directly calculate that $t^{-1}c_0 t = \partial t = t\partial + 1 = c_0 + 1$. Clearly $c_0 + 1 \in C$, so we only need to show that $t^{-1}d_i t \in C$ for $1 \leq i \leq w$.

   Let $1 \leq i \leq w$. Using the equation $i!d_i = \prod_{j=0}^{i-1}(c_0 - j)$ from part 2 gives us the equation (working in $C \otimes_R K$)

$$t^{-1}c_i t = t^{-1} \binom{c_0}{i} t = \binom{c_0 + 1}{i}$$

   which belongs to C by part 3.

5. Again treating C as an R-subalgebra of $\widehat{\mathcal{D}_X^{[m]}}(X \backslash \{0\})$, we have that $\phi(\alpha) = t^{-1}\alpha t$ for all $\alpha \in C$. Then we can calculate that $t\alpha = t\alpha t^{-1} t = \phi^{-1}(\alpha) t$ for all $\alpha \in C$. We can see that $\partial c_0 = (c_0 + 1)\partial$, so $\partial i! d_i = \prod_{i=0}^{p-1}(d + 1 - i)\partial = \phi(i!d_i)\partial$ for $0 \leq i \leq w$. Since C is contained in a flat R-algebra by 4.2.2(3), it follows that $\partial\alpha = \phi(\alpha)\partial$ for all $\alpha \in C$.

6. If $i = a_n p^n + a_{n-1}p^{n-1} + \ldots + a_0$ for some $n \in \mathbb{N}$, $a_n \neq 0$, $0 \leq a_j < p$ for $0 \leq j \leq n$, then

$$v_K(i!) = v_K(p)\left(a_n \frac{p^n - 1}{p-1} + a_{n-1}\frac{p^{n-1}-1}{p-1} + \ldots + a_1 \frac{p-1}{p-1}\right) \leq \frac{v_K(p)}{p-1} i,$$

proving the claim.

□

Lemma 4.3.3 *let A be a π-adically complete R-algebra. If $\hat{e} \in A$ is an element such that $\hat{e} + \pi A$ is an idempotent of $A/\pi A$, then $\lim_{n\to\infty} \hat{e}^{p^n}$ is an idempotent of A.*

If e and f are idempotents of A such that $e \equiv f \mod \pi$ then $e = f$ Also, if $ef = fe$, then $ef \in \pi A$ implies $ef = 0$.

Proof: These facts are well known but a proof is given for the benefit of the reader.

Set $A_n = A/\pi^n A$. For the first statement, we will prove that given an element $e_n \in A$ such that $e_n + \pi^n A$ is an idempotent of A_n then $e_n^p + \pi^{n+1}A$ is an idempotent of A_{n+1}. This is enough since the condition of idempotence then guarantees that if $e_1 + \pi A$ is idempotent, then $e_1^{p^i} \equiv e_1^{p^j} \mod \pi^j A$ whenever $j \leq i$, so $(e_i^{p^i} + \pi^i A)_{i\in\mathbb{N}} \in \varprojlim_{n\in\mathbb{N}} A_n$ is an idempotent.

So let $e_n \in A$ such that $e_n + \pi^n A$ is an idempotent of A_n. Then $e_n^2 = e_n + \pi^n\alpha$ for some $\alpha \in A$. Of course $\pi^n\alpha = e_n^2 - e_n$, so $\pi^n\alpha$ commutes with e_n. Then we can apply the binomial theorem to see that

$$e_n^{2p} = \sum_{i=0}^{p} \binom{p}{i} e_n^i (\pi^n\alpha)^{p-i} \equiv e_n^p \mod \pi^{n+1}A.$$

So $e_n^p + \pi^{n+1}A$ is an idempotent of A_{n+1}, as required.

We now prove the second claim. Let e and f be idempotents of A such that $e \equiv f \mod \pi A$. We aim to show that $e \equiv f \mod \pi^n A$ for all $n \in \mathbb{N}$. We proceed by induction on n, so suppose that $f \equiv e \bmod \pi^n A$, i.e. $f = e + \pi^n \alpha$ for some $\alpha \in A$. Then we can calculate that

$$f = f^p = \sum_{i=0}^{p} \binom{p}{i} e^i \pi^{n(p-i)} \alpha^{p-i} \equiv e \bmod \pi^{n+1} A.$$

Now suppose that $ef \equiv 0 \bmod \pi A$ and assume that A is commutative. From commutativity we have that ef is idempotent, so $ef = (ef)^n$ for all $n \in \mathbb{N}$. Of course $ef \in \pi A$, so $((ef)^n)_{n\in\mathbb{N}} \to 0$ as $n \to \infty$. Then the sequence $(ef)_{n\in\mathbb{N}} \to 0$ as $n \to \infty$, so $ef = 0$. □

4.3.4 Idempotents in the Slice of C

For $0 \leq i \leq w-1$, set

$$\hat{e}_{ii} = \binom{c_0 + i}{w-1}.$$

The following proposition will be used to demonstrate that the $\hat{e}_{ii} + \pi C$ form a complete set of orthogonal idempotents in $C/\pi C$ (that is, a set of idempotents e_i such that $e_i e_j = \delta_{ij} e_i$ and $\sum e_i = 1$).

Proposition 4.3.4 *1.* $\sum_{i=0}^{w-1} \hat{e}_{ii} + \pi C = 1 + \pi C$.

2. $\phi(\hat{e}_{ii}) \equiv \hat{e}_{(i+1)(i+1)}$ *mod* πC *for* $0 \leq i \leq w-2$, *and* $\phi(\hat{e}_{(p-1)(p-1)}) \equiv \hat{e}_{00}$ *mod* πC.

3. $\hat{e}_{00}\hat{e}_{ii} \in pC$ *for* $1 \leq i \leq w-1$, *and* $\hat{e}_{00}\hat{e}_{00} + pC = \hat{e}_{00} + pC$.

Proof:

1. Using lemma 4.3.2(1) we have $\binom{x+y}{n} = \sum_{i=0}^{n} \binom{x}{j}\binom{y}{n-j}$, so we can write

$$\hat{e}_{ii} = \binom{c_0 + i}{w-1} = \sum_{j=0}^{w-1} \binom{c_0}{j} \binom{i}{w-1-j}.$$

Then we conclude that

$$\sum_{i=0}^{w-1} \hat{e}_{ii} = \sum_{j=0}^{w-1} \binom{c_0}{j} \left(\sum_{i=0}^{w-1} \binom{i}{w-1-j} \right).$$

Using lemma 4.3.2(2) we have

$$\sum_{i=0}^{w-1}\binom{i}{w-1-j}=\binom{w}{w-j},$$

hence

$$\sum_{i=0}^{w-1}\hat{e}_{ii}=\sum_{j=0}^{w-1}\binom{w}{j}d_j.$$

Since $\binom{w}{j}\in pR$ for $1\leq j\leq w-1$ we find that $\sum_{i=0}^{w-1}\hat{e}_{ii}\in 1+\pi C$.

2. Since $\phi(\hat{e}_{ii})=\phi(\binom{c_0+i}{w-1})=\binom{c_0+i+1}{w-1}$, the statement is obvious for $0\leq i\leq w-2$. Applying lemma 4.3.2(2)

$$\phi(\hat{e}_{(w-1)(w-1)})=\binom{c_0+w}{w-1}=\sum_{i=0}^{w-1}\binom{c_0}{j}\binom{w}{w-1-j}.$$

Since $\binom{w}{w-1-j}\in\pi R$ for $0\leq j\leq w-2$ we find that $\phi(\hat{e}_{w-1)(w-1)})+\pi C=\binom{c_0}{w-1}+\pi C=\hat{e}_{00}+\pi C$.

3. First assume that $1\leq i\leq w-1$. Then, treating the $\hat{e}_{jj}$ as elements of $K\left[c_0\right]$ we can write

$$\hat{e}_{00}\hat{e}_{ii}=\binom{c_0}{w-1}\binom{c_0+i}{w-1}.$$

Now, by lemma 4.3.2(4) we have $\hat{e}_{00}\hat{e}_{ii}=\sum_{j\in\mathbb{N}}C_j(\hat{e}_{00}\hat{e}_{ii})\binom{c_0}{j}$ where

$$C_j(\hat{e}_{00}\hat{e}_{ii})=\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}\binom{k}{w-1}\binom{k+i}{w-1}.$$

Now, suppose that $i\neq 0$. By lemma 4.3.2(3) we have that $\binom{k}{w-1}\in p\mathbb{Z}_p$ whenever $k\not\equiv -1 \bmod w$. Since $1\leq i\leq w-1$, it is true that for all $k\in\mathbb{N}$, either $k\not\equiv -1 \bmod w$ or $k+i\not\equiv -1 \bmod w$. Therefore $\binom{k}{w-1}\binom{k+i}{w-1}\in p\mathbb{Z}_p$ for all $k\in\mathbb{N}$, and consequently $C_j(\hat{e}_{00}e_{ii})\in p\mathbb{Z}_p$ for all $j\in\mathbb{N}$. On the other hand, if $i=0$, then

$$\binom{k}{w-1}\binom{k}{w-1}\in\begin{cases}p\mathbb{Z}_p & \text{if } k\not\equiv -1 \bmod w-1\\ 1+p\mathbb{Z}_p & \text{if } k\equiv -1 \bmod w-1\end{cases}$$

Consequently we find that

$$\hat{e}_{00}\hat{e}_{ii} \in \begin{cases} pC & \text{if } i \neq 0 \\ \hat{e}_{00} + pC & \text{if } i = 0 \end{cases}.$$

□

4.3.5 Idempotents of the Diagonal algebra

For $0 \leq i \leq w-1$, set

$$\alpha_i = \frac{1}{(w-1)!} \prod_{0 \leq j \leq w-1 \text{ and } w-1-i \neq j} (c_0 - j) \in C \otimes_R K.$$

The following lemma is an example of lemma 4.3.2(4)

Lemma 4.3.5 *Let* $0 \leq i \leq w-1$. *Then*

$$\alpha_i = \sum_{j=0}^{w-1} (-1)^j \binom{i}{j} \binom{w-1}{j}^{-1} d_{j^*}.$$

Proof: Let $0 \leq j \leq w-1$. Let B be the K-subalgebra of $C \otimes_R K$ generated by c_0. Then we can see that α is the unique solution in B to the equation $(c_0 - (w-1-i))X = wc_m$. Hence, it will be enough to show that

$$(c_0 - i^*) \left(\sum_{j=0}^{w-1} (-1)^j \binom{i}{j} \binom{w-1}{j}^{-1} d_{w-1-j} \right) = wc_m.$$

We already know that $c_0 d_j = (j+1)d_{j+1} + jd_j$, so the above product can be written as a telescoping sum. We can calculate that

$$(c_0 - i^*) \binom{i}{j} \binom{w-1}{j}^{-1} d_{j*} = \binom{i}{j} \binom{w-1}{j}^{-1} ((j^*+1)d_{j^*+1} + (i-j)d_{j^*}).$$

So the coefficient of d_{j^*} in the sum is

$$\binom{i}{j+1} \binom{w-1}{j+1}^{-1} (w-(j+1)) - \binom{i}{j} \binom{w-1}{j}^{-1} (i-j)$$

whenever $0 \le j \le w-1$. The boundary terms are

$$\binom{i}{0}\binom{w-1}{0}^{-1} wc_m - i^*(-1)^{w-1}\binom{i}{w-1}\binom{w-1}{w-1}^{-1} d_0 = wc_m,$$

proving the claim. □

Proposition 4.3.5 *1. For $0 \le i \le w-1$ we have $\alpha_i \in C$ and*

$$\hat{e}_{ii} \equiv \alpha_i \ mod\ pC.$$

2. *The $\hat{e}_{ii} + \pi C$ for $0 \le i \le w-1$ form a complete set of non-zero orthogonal idempotents in $C/\pi C$.*

3. *For $0 \le i \le w-1$, the sequences $(\hat{e}_{ii}^{p^n})_{n\in\mathbb{N}}$ are Cauchy and setting $e_{ii} := \lim_{n\to\infty} \hat{e}_{ii}^{p^n}$, the idempotents e_{ii} for $0 \le i \le w-1$ are a complete set of non-zero orthogonal idempotents in C.*

4. *$\phi(e_{ii}) = e_{(i+1)(i+1)}$ for $0 \le i \le w-2$, and $\phi(e_{(w-1)(w-1)}) = e_{00}$. Furthermore $\phi^w(e_{ii}) = e_{ii}$ for $0 \le i \le w-1$.*

Proof:

1. Let $0 \le i \le w-1$. We will prove that $\hat{e}_{ii} - \alpha_i \in \pi C$. Using lemma 4.3.5 we can write

$$\alpha_i = \sum_{j=0}^{w-1} (-1)^j \binom{i}{j}\binom{w-1}{j}^{-1} d_{w-1-j}$$

and using lemma 4.3.2(1) we can write

$$\hat{e}_{ii} = \binom{c_0+i}{w-1} = \sum_{j=0}^{w-1}\binom{i}{j} d_{w-1-i}$$

So that $\hat{e}_{ii} - \alpha = \sum_{j=0}^{w-1}(1-(-1)^j\binom{w-1}{j}^{-1})\binom{i}{j} d_{w-1-i}$. Then it will be enough to show that

$$1-(-1)^j\binom{w-1}{j}^{-1} \in pC$$

whenever $0 \leq j \leq w-1$. $\binom{w-1}{j} \equiv (-1)^j \bmod p\mathbb{Z}_p$ by 4.3.2(5), proving the claim.

2. By proposition 4.3.4(3) $\hat{e}_{00} + \pi C$ is an idempotent of $C/\pi C$, and since the automorphism ϕ acts transitively on the $\hat{e}_{ii}$ for $0 \leq i \leq w-1 \bmod \pi C$ by proposition 4.3.4(2), all of the $\hat{e}_{ii} + \pi C$ are idempotent. By proposition 4.3.4(1), the $\hat{e}_{ii}$ form a complete set of idempotents. Finally, if $0 \leq i, j \leq w-1$ and $i < j$ then $\hat{e}_{ii}\hat{e}_{jj} = \phi^i(\hat{e}_{00}\hat{e}_{(j-i)(j-i)}) \in \pi C$ by proposition 4.3.4(3), so the $\hat{e}_{ii} + \pi C$ are orthogonal.
3. Given part 2, this is a straightforward application of lemma 4.3.3.
4. Given part 3 and proposition 4.3.4(2), this is a straightforward application of lemma 4.3.3.

□

4.3.6 Existence of a differential for τ

In this section we will find an element $\gamma \in 1 + \pi C$ such that $\left[\gamma\partial^{[w]}, \tau\right] = 1$.

Lemma 4.3.6 *There exists $\gamma_i \in e_{ii}(1+\pi C)$ such that $\phi^{-w}(\gamma_i)\alpha_i = e_{ii}$. Moreover, for $0 \leq i \leq w-1$,*

$$wc_m\phi^{-w}(\gamma_i) = e_{ii}(c_0 - (w-1-i)).$$

Proof: By proposition 4.3.5(1) we have

$$\alpha_i \equiv \hat{e_{ii}} \bmod \pi C.$$

It follows that $e_{ii}\alpha_i \equiv e_{ii} \bmod \pi C$, so $e_{ii}\alpha_i = e_{ii}(1+\pi\lambda_i)$ for some $\lambda_i \in C$. Since C is π-adically complete, for all $x \in C$, $1 + \pi x$ is invertible, so set

$$\gamma_i = \phi^w\left(e_{ii}(1+\pi\lambda_i)^{-1}\right).$$

Then we have that $\phi^{-w}(\gamma_i)\alpha_i = e_{ii}(1+\pi\lambda)(1+\pi\lambda_i)^{-1} = e_{ii}$.

Using the equation $w!c_m = \prod_{i=0}^{w-1}(c_0 - i)$, we have that $(c_0 - (p-1-i))\alpha_i = wc_m$. Then

$$wc_m\phi^{-w}(\gamma_i) = (c_0 - (w-1-i))\alpha_i\phi^{-w}(\gamma_i) = e_{ii}(c_0 - (w-1-i)).$$

□

Theorem 4.3.6 *Let the elements $\gamma_i \in C$ for $0 \leq i \leq w-1$ be defined as in lemma 4.3.6. Set $\gamma = \sum_{i=0}^{w-1}\gamma_i$. Then $\left[\gamma\partial^{[w]}, t^w\right] = 1$.*

Proof:

Again using proposition 4.3.3, and (calculating inside the over-ring $\widehat{\mathcal{D}_X^{[m]}}(X\backslash\{0\})$) the fact that $\phi^w(c_m) = \phi^w(t^w\partial^{[w]}) = t^{-w}t^w\partial^{[w]}t^w = \partial^{[w]}t^w$, we can calculate that

$$\left[\gamma\partial^{[w]}, t^w\right] = \gamma\partial^{[w]}t^w - t^w\gamma\partial^{[w]} = \phi^w(c_m)\gamma - c_m\phi^{-w}(\gamma) = (\phi^w - \mathrm{id})(c_m\phi^{-w}(\gamma)).$$

So we need to show that $(\phi^w - \mathrm{id})(c_m\phi^{-w}(\gamma)) = 1$.

Fix $0 \leq i \leq w-1$. Since $\phi^w(c_0) = c_0 + w$, we can see that $(\phi^w - \mathrm{id})(x-(w-1-i)) = w$, so using the fact that $\phi^w(e_{ii}) = e_{ii}$, by lemma 4.3.6 we see that

$$(\phi^w - \mathrm{id})(wc_m\phi^{-w}(\gamma_i)) = e_{ii}\left(\phi^w - \mathrm{id}\right)\left(x - (w-1-i)\right) = we_{ii}.$$

Finally, by linearity we deduce that

$$(\phi^w - \mathrm{id})(c_m\phi^{-w}(\gamma)) = \sum_{i=0}^{w-1}(\phi^w - \mathrm{id})(c_m\phi^{-w}(\gamma_i)) = \sum_{i=0}^{w-1} e_{ii} = 1.$$

□

4.3.7 Matrix Units

Lemma 4.3.7 *For $0 \leq i, j \leq w-1$, set $e_{ij} = e_{ii}t^{j-i}e_{jj} \in \widehat{\mathcal{D}_X^{[m]}}(X\backslash\{0\})$. Then, if we consider A_m as a subset of $\widehat{\mathcal{D}_X^{[m]}}(X\backslash\{0\})$, each of the e_{ij} belong to A_m.*

Proof: A_m is complete, and $A_m/\pi A_m$ is finite over $A_0/\pi A_0$, so it is Noetherian. Therefore by proposition 2.2.7(2) A_m is a Zariskian ring, and its ideals are closed by proposition 2.2.7(1) By proposition 4.3.3 and proposition 4.3.3(1) we know that $(\hat{e}_{ii})^{p^n} = \phi^i(\hat{e}_{00})^{p^n} = (t^{-i}t^{p-1}\partial_t^{[p-1]}t^i)^{p^n} = \left(t^{p-1-i}\partial_t^{[p-1]}t^i\right)^{p^n} \in A_m t^i$ for all $n \in \mathbb{N}$, so it follows that $e_{ii} = \lim_{n\to\infty} \hat{e}_{ii}^{p^n} \in A_m t^i$, so that $e_{ii}t^{-i} \in A_m$. Then $e_{ij} = e_{ii}t^{-i}(t^j e_{jj}) \in A_m$. □

Proposition 4.3.7 *The $\{e_{ij}\}_{0\leq i,j\leq w-1}$ form a set of matrix units in A_m.*

Proof: We need to check that $e_{ij}e_{i'j'} = \delta_{ji'}e_{ij'}$. Note that since $e_{jj}e_{i'i'} = \delta_{ji'}e_{jj}$, so

$$e_{ij}e_{i'j'} = e_{ii}t^{j-i}e_{jj}e_{i'i'}t^{j'-i'}e_{j'j'} = \delta_{ji'}e_{ii}t^{j-i+j'-i'}e_{j'j'}$$

It is easy to see that this coincides with $e_{ij'}$ when $j = i'$. □

4.3.8 Proof of Theorem 4.3.1

For this section, we define that $\delta = \gamma\partial^{[w]}$, where γ is defined as in Theorem 4.3.6, so that $[\delta, \tau] = 1$.

Proposition 4.3.8 *1. For all $\alpha \in C$, $\alpha\tau = \phi^w(\alpha)\tau$, $\partial_t^{[w]}\alpha = \phi^w(\alpha)\partial_t^{[w]}$ and $\delta\alpha = \phi^w(\alpha)\delta$.*

2. Let $0 \leq j < w$. Then for $0 \leq i \leq w-1$

$$\phi^j(e_{ii}\tau\delta) = \begin{cases} e_{(i+j)(i+j)}\tau\delta & \text{if } i+j < w \\ e_{(i+j-p)(i+j-p)}(\tau\delta+1) & \text{if } i+j \geq w \end{cases}$$

3. $\tau e_{ij} = e_{ij}\tau$ and $\delta e_{ij} = e_{ij}\delta$ for $0 \leq i,j \leq w-1$.

4. $\phi^w(\tau\delta) = \tau\delta + 1$

5. For $0 \leq i \leq w-1$, $e_{ii}(c_0 - (w-1-i)) = e_{ii}w\tau\delta$.

6. *The map $A_0 \to A_m$; $\tau \mapsto \tau$; $\partial_\tau \mapsto \delta$ is a well defined injective homomorphism of $\mathcal{O}(Y)$-rings.*

Proof:

1. This follows from proposition 4.3.3 since $\tau = t^w$, and δ differs from $\partial^{[w]}$ by an element of C.

2. From lemma 4.3.6, using the fact that $wy\phi^{-w}(\gamma_i) = we_{ii}\tau\delta$ we can write

$$e_{ii}w\tau\delta = e_{ii}(c_0 - (w-1-i)).$$

Write $k = i+j$. By proposition 4.3.5(4) $\phi^j(e_{ii}) = e_{kk}$ when $k < w$ and $e_{(k-w)(k-w)}$ when $w \le k < 2w$. Now, $\phi(c_0) = c_0 + 1$, so $\phi^j(c_0 - (p-1-i)) = c_0 - (p-1-k)$. Putting these together, when $k < w$ it is clear that $\phi^j(e_{ii}\tau\delta) = e_{kk}\tau\delta$. When $w \le k < 2w$ observe that $c_0 - k^* = c_0 - (w-1-(k-w)) - w$ to get that

$$\begin{aligned} \phi^j\left(e_{ii}(c_0 - i^*)\right) &= e_{(k-w)(k-w)}(c_0 - (w-1-(k-w)) + w) \\ &= e_{(k-w)(k-w)}(w(\tau\delta+1)). \end{aligned}$$

3. Since τ commutes with t^{i-j} and the e_{ii}, τ commutes with the e_{ij}. Then to show that δ commutes with the e_{ij} it will be enough to show that $\tau\delta$ commutes with the e_{ij}: if so, then working in $\widehat{\mathcal{D}_X^{[m]}}(X\backslash\{0\})$, we have $e_{ij}\delta = \tau^{-1}e_{ij}\tau\delta = \delta e_{ij}$. Now, $e_{ij}\tau\delta = e_{ii}t^{j-i}\tau\delta e_{jj} = e_{ii}\phi^{i-j}(e_{jj}\tau\delta)t^{j-i}e_{jj}$. Using part 2, since $(i-j)+j = i < w$, and both e_{ii} and $\tau\delta$ belong to the commutative algebra C, $\phi^{i-j}(e_{jj}\tau\delta) = e_{ii}\tau\delta = \tau\delta e_{ii}$, proving the statement.

4. Since $\tau\delta = \phi^{-w}(\gamma)c_m$, theorem 4.3.6 tells us that $(\phi^w - \mathrm{id})(\tau\delta) = 1$.

5. Of course, $\tau\delta = \phi^{-w}(\gamma)c_m$, so lemma 4.3.6 tells us that $e_{ii}w\tau\delta = e_{ii}wc_m\phi^{-w}(\gamma) = e_{ii}(c_0 - (w-1-i))$.

6. The homomorphism of $\mathcal{O}(Y)$-rings $A_0 \to A_m$; $\partial_\tau \mapsto \delta$; $\tau \mapsto \tau$ is well defined since $[\delta, \tau] = 1$. It induces a K-algebra homomorphism $A_0 \otimes K \to A_m \otimes K$ which is injective since $A_0 \otimes K = \widehat{\mathcal{D}_Y^{[0]}}(Y) \otimes K$ is simple by proposition 4.2.2. Since A_0 embeds into $A_0 \otimes K$, the map $A_0 \to A_m$ is injective.

□

Lemma 4.3.8 *1. Let A be a ring and $\{e_{ij}\}_{1\le i,j\le n}$ be a set of matrix units for A. Let C be the centralizer of the set $\{e_{ij}\}_{1\le i,j\le n}$. Then there is an isomorphism $M_n(C) \to A$ sending $\tilde{e}_{ij} \mapsto e_{ij}$ and mapping C onto C.*

2. Let $\alpha = \sum_{i,j=0}^{w-1} \binom{i}{j} e_{ij} \in M_w(K)$. Then α is invertible and $\alpha^{-1} = \sum_{i,j=0}^{w-1} (-1)^{i-j} \binom{i}{j} e_{ij}$

3. $e_{ij} d_k \equiv (-1)^{k^*-j} \binom{k^*}{j} e_{ij}$ mod pC for all $0 \le i, j, k \le w-1$ (recall $d_i = t^i \partial^{[i]}$ for $0 \le i \le m$.)

Proof:

1. By [14, 1.10.34] if we set $T = \{\sum_{i=1}^n e_{i1} a e_{1i} : a \in A\}$, then there is an isomorphism of rings $M_n(T) \to A$ sending $\tilde{e}_{ij} \mapsto e_{ij}$ and sending $t \mapsto t$ for $t \in T$, so it will be enough to show that $T = C$. Since the map $M_n(T) \to A$ is a ring homomorphism and T centralizes the $\{\tilde{e}_{ij}\}$ by the definition of $M_n(T)$, we must have $T \subset C$. On the other hand, let $c \in C$. Then c commutes with the e_{ij}, so we have $\sum_{i=1}^n e_{i1} c e_{1i} = \sum_{i=1}^n e_{i1} e_{1i} c = c$. Hence $c \in T$.

2. The algebra homomorphism $\psi : K[t] \to K[t]$; $t \mapsto t+1$ is an isomorphism with inverse $t \mapsto t-1$. ψ preserves the K-submodule $V = \bigoplus_{i=0}^{w-1} K t^i$ of $K[t]$, and $\psi(t^i) = \sum_{j=0}^{w-1} \binom{i}{j} t^j$. Then the matrix of $\psi|_V$ is $\sum_{i,j=0}^{w-1} \binom{i}{j} e_{ij}$, and the matrix of $\psi^{-1}|_V$ is $\sum_{i,j=0}^{w-1} (-1)^{i-j} \binom{i}{j} e_{ij}$.

3. First we observe that using the lemma 4.3.2

$$\hat{e}_{ii} = \binom{x+i}{w-1} = \sum_{j=0}^{w-1} \binom{i}{j}\binom{x}{j^*} = \sum_{j=0}^{w-1} \binom{i}{j} d_{j^*}$$

So using part 2 we have

$$d_{i^*} = \sum_{j=0}^{w-1} (-1)^{i-j} \binom{i}{j} \hat{e}_{jj}.$$

So we can write

$$e_{ij} d_k = e_{ij} d_{(k^*)^*} = e_{ij} \sum_{a=0}^{w-1} (-1)^{k^*-a} \binom{k^*}{a} e_{aa} = (-1)^{k^*-j} \binom{k^*}{j} e_{ij},$$

working in C/pC.

□

We are now ready to prove the main theorem.

Proof:(Proof of Theorem 4.3.1) By 4.3.8(6) the R-algebra homomorphism $A_0 \to A_m$ which sends $\partial_\tau \mapsto \delta$ and $\tau \mapsto \tau$ is an embedding. By proposition 4.3.8(3) τ and δ commute with the e_{ij} so the image of A_0 is contained in the centralizer of the matrix units $\{e_{ij}\}_{0 \le i,j < w}$, therefore by lemma 4.3.8 the induced R-algebra homomorphism $M_w(A_0) \to A_m$ which sends $\tilde{e}_{ij} \mapsto e_{ij}$ and corresponds to the above homomorphism $A_0 \to A_m$ on A_0 is an embedding. We will prove the theorem by showing that it is an isomorphism. Using proposition 2.2.6(2) it will be enough to show that the induced map $f : M_w(A_0/\pi A_0) \to A_m/\pi A_m$ is an isomorphism. Let B be the sub-κ-algebra of $A_m/\pi A_m$ generated by $\partial^{[w]}$ and τ.

We have that $\delta \equiv \partial^{[w]} \bmod \pi A_m$, and so the map f maps $A_0/\pi A_0$ onto B. $A_m/\pi A_m$ is generated over κ by t, and the $\partial^{[p^i]}$ for $0 \le i \le m$. Then it is clear that $A_m/\pi A_m = \sum_{i,j=0}^{w-1} t^i \partial^{[j]} B$, while $M_w(A_0/\pi A_0) = \bigoplus_{i,j=0}^{w-1} \tilde{e}_{ij} A_0/\pi A_0$. From section 4.2.1 we know that $\partial^w = w! \partial^{[w]} \equiv 0 \bmod \pi A_m$ and $t^w = \tau$, so $A_m/\pi A_m$ is generated as a B-module by the set $\{t^i \partial^{[j]} : 0 \le i,j \le w-1\}$. Therefore, if we can show that t and the $\partial^{[p^i]}$ belongs to the image of f for $0 \leq i \leq w-1$, then f is surjective. First of all we can calculate that $e_{ii}t = e_{ii}(e_{ii}t) = e_{ii}te_{i+1,i+1} = e_{i,i+1}$ for $0 \leq i \leq w-1$, and we can calculate that $e_{w-1,w-1}t = t^w e_{w-1,0}$ so that $t = \left(\sum_{i=0}^{w-2} e_{i(i+1)}\right) + t^w e_{w-1,0}$. Using this, we claim that

$$e_{ii}\partial^{[j]} \equiv \begin{cases} (-1)^{i^*}\binom{j^*}{i-j} e_{i,i-j} & \text{if } j \leq i \leq w-1 \\ 0 & \text{if } j > i. \end{cases} \mod pA_m$$

To see this, let $0 \leq i, j \leq w-1$, and first assume that $j \leq i$. Then we can write $e_{ii}\partial^{[j]} = e_{i,i-j}t^j\partial^{[j]} = e_{i,i-j}d_j$. By lemma 4.3.8(3), noting that $j^* - (i-j) = i^*$, we find that

$$e_{i,i-j}d_j = (-1)^{j^*-(i-j)}\binom{j^*}{i-j}e_{i,i-j} = (-1)^{i^*}\binom{j^*}{i-j}e_{i,i-j}.$$

Now assume that $i < j$. Then

$$t^w e_{ii}\partial^{[j]} = e_{ii}t^{w-j}d_j = e_{i,i+w-j}d_j.$$

By lemma 4.3.8(3) we find that

$$e_{i,i+w-j}d_j = (-1)^{j^*-i-w+j}\binom{j^*}{i+w-j}e_{i,i+w-j}.$$

Now, $j^* = w-1-j < w+i-j$ since $i \geq 0$, so $t^w e_{ii}\partial^{[j]} = 0$. t is a regular element of $A_m/\pi A_m$, so $e_{ii}\partial^{[j]} = 0$.

□

Chapter 5

Description of the primitive spectrum of certain nilpotent affinoid enveloping algebra

5.1 Some results around the Newton Polygon Theorem

5.1.1 The Newton Polygon Theorem

Fix an algebraic closure $\overline{K}$ of K and let $\overline{R}$ be the integral closure of R in $\overline{K}$. We implicitly extend v_K to $\overline{K}$, so that the valuation $v_K(\lambda)$ of an element $\lambda \in \overline{K}$ is a well defined element of $\mathbb{Q}$.

Let $g(t) = a_0 + \ldots + a_n t^n \in K[t]$, and assue that a_0 and a_n are non-zero. Set $S = \{(i, v(a_i)) : 0 \leq i \leq n\}$. Then we define $N(g)$ to be the smallest subset of S such that $(0, v(a_0))$, $(n, v(a_n)) \in N(g)$, the slopes of the lines between the points of $N(g)$ are strictly increasing, and every point of S lies above the path traced by these lines ($N(g)$ can be viewed as the vertices of the lower convex hull or the lower convex envelope of the set S.)

Theorem 5.1.1 *Let $g(t) = a_0 + \ldots + a_n t^n \in K[t]$, and assume that a_0 and a_n are non-zero. Let $(j_1, v(a_{j_1})), \ldots, (j_s, v(a_{j_s}))$ be the vertices of $N(g)$. Then there are precisely $j_r - j_{r-1}$ roots of $g(t)$ of valuation $\frac{v(a_{j_{r-1}})-v(a_{j_r})}{j_r - j_{r-1}}$ for $1 \leq r \leq s$.*

Proof: (proposition 1.6.3) neukirch, but the proof is fairly instructive so it is given below.

First of all, changing the value of a_n only shifts the polygon up and down so we assume that $a_n = 1$. Let $\mu_1, \ldots, \mu_n$ be the roots of $g(t)$ in $\overline{K}$, organized so that $v_K(\mu_1) \leq v_K(\mu_2) \leq \ldots \leq v_K(\mu_n)$. We let $\{i_1, \ldots, i_s\}$ be the largest set of numbers between 1 and n where $v_K(\mu_{i_r}) < v_K(\mu_{i_r+1})$ for $1 \leq r \leq s$, so that setting $i_0 = 0$ the sets $\{\mu_{i_r+1}, \ldots, \mu_{i_{r+1}}\}$ partition the roots of $g(t)$ by value.

For $1 \leq i \leq n$, let I_i be the set of subsets of $\{1, \ldots, n\}$ of cardinality $n - i$. Then we have the equality

$$a_i = \pm \sum_{J \in I_i} \prod_{j \in J} \mu_j.$$

Applying the ultrametric ineqality to these sums, we find that if $i_r < i \leq i_{r+1}$ then setting $m_r = i_0 + \ldots + i_{r-1}$. Then

$$v_K(a_{n-i}) \geq i_1 v_K(\mu_{i_1}) + (i_2 - m_2) v_K(\mu_{i_2}) + \ldots + (i_r - m_r) v_K(\mu_{i_r}) + (i - m_{r+1}) v_K(\mu_{i_{r+1}})$$

with equality when $i = i_{r+1}$. From this we find

$$\{(0, v(a_0)), (i_1, v(a_{i_1})), \ldots, (i_s, v(a_{i_s})), (n, v(a_n))\} = N(g),$$

and setting $i_0 = 0$ and $i_{s+1} = n$, we can calculate that

$$\frac{v(a_{i_{r-1}}) - v(a_{i_r})}{i_r - i_{r-1}} = v_K(\mu_{i_r}),$$

proving the claim. □

5.1.2 Some seemingly arbitrary calculations

Let $g(t) = a_0 + \ldots + a_n t^n \in K[t]$ such that $a_0 \in R$ and $a_0, a_n \neq 0$. Then we define

$$\chi(g) := \max_{1 \leq i \leq n} \left\{ -\frac{v_K(a_i)}{i} \right\}.$$

Lemma 5.1.2 *Let M be the number of roots of $g(t)$ of valuation greater than or equal to $\chi(g)$. Let $\mu_1, \ldots, \mu_n$ be the roots of $g(t)$, ordered so that*

$$v_K(\mu_1) \leq v_k(\mu_2) \leq \ldots \leq v_K(\mu_n).$$

1. *$M > 0$.*
2. *$v_K(a_M) = v_K(a_n) + \sum_{i=1}^{n-M} v_K(\mu_i)$.*
3. *$\chi(g) = -\frac{v(a_M)}{M}$.*
4. *Let $\alpha \in \overline{K}$. If $v_K(\alpha) \geq \chi(g)$ then $v(g(\alpha)) \geq 0$.*
5. *For some $l \in \mathbb{Q} \cup \{-\infty\}$ such that $l < \chi(g)$ we have that if $l \leq v(\alpha) < \chi(g)$ then $v_K(g(\alpha)) < 0$.*

Proof: Let $\{(i_0, v(a_{i_1})), \ldots, (i_s, v(a_{i_s}))\} = N(g)$, organized so that $i_0 < \ldots < i_s$. Choose $1 \leq S \leq n$ such that $\chi(g) = \frac{-v_K(a_S)}{S}$. For $1 \leq i \leq n$, let I_i be the set of subsets of $\{1, \ldots, n\}$ of cardinality $n - i$. Recall that for $1 \leq i \leq n$, we have

$$a_i a_n^{-1} = \pm \sum_{J \in I_i} \prod_{j \in J} \mu_j.$$

We can deduce that

$$v_K(a_i) \geq v_K(a_n) + \sum_{j=0}^{n-i} v_K(\mu_j).$$

We will use this fact repeatedly throughout the proof without reference.

1. By theorem 5.1.1, for some root μ of $g(t)$, since $i_0 = 0$, we must have

$$v_K(\mu) = \frac{v_K(a_0) - v_K(a_{i_1})}{i_1}.$$

Since the slopes of the line segments between the points of $N(g)$ are strictly increasing and all of the points $(i, v(a_i))$ for $1 \leq i \leq n$ lie above these lines we have

$$v_K(\mu) = \max_{1 \leq i \leq n} \left\{ \frac{v_K(a_0) - v_K(a_i)}{i} \right\}.$$

Since $v_K(a_0) \geq 0$, we deduce that

$$v_K(\mu) \geq \max_{1 \leq i \leq n} \left\{ \frac{-v_K(a_i)}{i} \right\} = \chi(g).$$

2. From the ultrametric inequality we know that

$$v_K(a_M) \geq \min_{J \in I_M} \{ v_K(a_n) + \sum_{j \in J} v_K(\mu_j) \}$$

with equality when the minimum is attained uniquely. From the definition of M we know that for $n - M < i \leq n$ we have $v_K(\mu_i) \geq \chi(g) = \frac{-v_K(a_S)}{S}$, and for $1 \leq i \leq n - M$ we have $v_K(\mu_i) < \chi(g)$. Therefore, the minimum is attained uniquely at $v_K(a_n) + \sum_{j=1}^{n-M} v_K(mu_i)$.

3. It will be enough to show that $\frac{v_K(a_M)}{M} \leq \frac{v_K(a_S)}{S}$ (If $S = M$ we are done so we only need to prove the statement for $S < M$ and $S > M$). From the definition of M we know that for $n - M < i \leq n$ we have $v_K(\mu_i) \geq \chi(g) = \frac{-v_K(a_S)}{S}$. . First assume that $S < M$. Then using part 2,

$$\begin{aligned} v_K(a_S) \geq v_K(a_n) + \sum_{i=1}^{n-S} v_K(\mu_i) &= v_K(a_n) + \sum_{i=1}^{n-M} v_K(\mu_i) + \sum_{i=n-M+1}^{n-S} v_K(\mu_i) \\ &\geq v_K(a_M) - (n - S - (n - M)) \frac{-v_K(a_S)}{S} \\ &= v_K(a_M) + v_K(a_S) - \frac{M}{S} v_K(a_S), \end{aligned}$$

proving the claim. Now assume that $M < S$. Then, again using part 2 and the fact that $v_K(\mu_i) < \frac{-v_K(a_S)}{S}$ for $1 \leq i \leq n - M$, we can write

$$\begin{aligned} v_K(a_M) = v_K(a_n) + \sum_{i=1}^{n-M} v_K(\mu_i) &= v_K(a_n) + \sum_{i=1}^{n-S} v_K(\mu_i) + \sum_{i=n-S+1}^{n-M} v_K(\mu_i) \\ &\leq v_K(a_S) - (n - M - (n-S))\frac{v_K(a_S)}{S} = \frac{M}{S} v_K(a_S) \end{aligned}$$

proving the claim.

4. Let $r = \chi(g)$. If $v_K(\alpha) \geq \chi(g)$, then write $\alpha = \pi^r \varepsilon$ with $\varepsilon \in \overline{R}$. Then we can write

$$g(\alpha) = g(\pi^r \varepsilon) = a_0 + (\pi_K^r a_1)\varepsilon + \ldots + (\pi_K^{rn} a_n)\varepsilon^n.$$

By the definition of r, we of course have that $\pi_K^{ri} a_i \in \overline{R}$ for $1 \leq i \leq n$, so the statement follows immediately.

5. Let l be a rational number strictly between the value of a root of $g(t)$ that is strictly less than r and r if such a root exists, otherwise set $l = -\infty$. Suppose that $l \leq v_K(\alpha) < r$. Since $v_K(\mu_i) \geq \chi(g) = r$ if and only if $n - M < i \leq n$ we have that $v_K(\alpha - \mu_i) = v_K(\alpha)$ whenever $n - M < i \leq n$ and since $v_K(\alpha) \geq l$, we deduce that $v_K(\alpha - \mu_i) = v_K(\mu_i)$ whenever $1 \leq i \leq n - M$. Since $g(\alpha) = a_n \prod_{j=1}^{n} (\alpha - \mu_j)$ we have

$$v_K(g(\alpha)) = \sum_{i=1}^{n-M} v_K(\mu_i) + M v_K(\alpha) + v_K(a_n) = v_K(a_M) + M v_K(\alpha).$$

Of course, using part 3, since $v_K(\alpha) < r$ we have $v_K(\alpha) < -\frac{v_K(a_M)}{M}$, so it follows that $v_K(g(\alpha)) < 0$.

$\square$

Theorem 5.1.2 *Let $g(t) = a_0 + \ldots + a_n t^n \in K[t]$ such that $a_0 \in R$ and $a_n \neq 0$. Define X to be the set $\{\lambda \in \mathbb{A}_K^{1,an} : v_K(g(\lambda)) \geq 0\}$. Then X is an affinoid subdomain of $\mathbb{A}_K^{1,an}$ and the G-connected component of X about 0 is the closed disk $\{\lambda \in \mathbb{A}_K^{1,an} : v_K(\lambda) \geq \chi(g)\}$.*

Proof: Note that $a_0 \in R$ forces $0 \in X$, so that the G-connected component of X about 0 is non-empty. For $v_K(\lambda) < -N$ for N large we will have $v_K(g(\lambda)) < 0$, so X can be realized as the spectrum of a Weierstrass extension $K\langle \pi^N t\rangle\langle g(t)\rangle$, and hence X is an affinoid subdomain of $\mathbb{A}_K^{1,an}$.

Using lemma 5.1.2(4, 5) there is an some $l \in \mathbb{Q}$, $l > \chi(g)$ such that the intersection of X with the disk $B = \{\lambda \in \mathbb{A}_K^{1,an} : v_K(\lambda) \geq l\}$ is the disk $\{\lambda \in \mathbb{A}_K^{1,an} : v_K(\lambda) \geq \chi(g)\}$. Then $X \cap B$ and $X \cap \{\lambda \in \mathbb{A}_K^{1,an} : v_K(\lambda) \leq l\}$ is a disjoint admissible open covering of X, proving the second part of the statement. □

Corollary 5.1.2 *Adopting the notation of the theorem, X is a finite union of disjoint closed disks.*

Proof: For $f \in K[t]$ let $X(f) = \{\lambda \in X : v_K(f(\lambda)) \geq 0\}$. Suppose $\alpha \in X(f)$. Then it will suffice to prove that the G-connected component of X about α is a closed disk of finite radius. If α is a root of $f(t)$, then replace α with some other point in the G-connected component of α which is not a root. Let $f'(t) = f(\alpha - t)$. Then using theorem 5.1.2, since $0 \in X(f')$, the G-connected component about 0 in $X(f')$ is a closed disk of finite radius. But this implies the G-connected component of $X(f)$ about α is a closed disk of finite radius. It is a fact of p-adic geometry that two disks are either disjoint, or one is contained in the other, proving the claim. □

5.2 Working with ∂_t-stable disks

5.2.1 Defining Skew Tate Extensions

Let A be an affinoid algebra, and let δ be a derivation of A. Then we say that a sub-R-algebra B of $A^\circ = \{\alpha \in A : \|\alpha\| \leq 1\}$ is a δ-lattice if $\delta(\beta) \in B$ for all $\beta \in B$, and A° is a

B-module of finite type. We define

$$B\langle x\ ;\ \delta\rangle = \varprojlim_{i\in\mathbb{N}} B\left[x\ ;\ \delta\right]/\pi^i B\left[x\ ;\ \delta\right].$$

(where the ring $B\left[x\ ;\ \delta\right]$ is isomorphic as a left B-module to $B\left[x\right]$, with multiplication defined by the rule $xb - bx = \delta(b)$.

Proposition 5.2.1 *Let A be an affinoid algebra, let δ be a derivation of A, and suppose that B and B' are δ-lattices in A such that $B \subset B'$. Then the natural homomorphism*

$$B\langle x\ ;\ \delta\rangle \otimes_R K \to B'\langle x\ ;\ \delta\rangle \otimes_R K$$

is an isomorphism.

Proof: The decomposition $B\left[x\ ;\ \delta\right] = \bigoplus_{i\in\mathbb{N}} Bx^i$ induces an isomorphism of R-modules $B\langle Z\rangle \to B\langle x\ ;\ \delta\rangle$, and using these isomorphisms we get a commutative square

$$\begin{array}{ccc} B\langle Z\rangle \otimes_R K & \longrightarrow & B'\langle Z\rangle \otimes_R K \\ \downarrow & & \downarrow \\ B\langle x\ ;\ \delta\rangle \otimes_R K & \longrightarrow & B'\langle x\ ;\ \delta\rangle \otimes_R K \end{array}$$

Since B and B' are lattices in A, the top arrow is an isomorphism, and thus the bottom arrow is an isomorphism. $\square$

We say that A has a δ-stable lattice if there exists some δ-lattice in A, and if A has a δ-stable lattice B then we define

$$A\langle x\ ;\ \delta\rangle = B\langle x\ ;\ \delta\rangle \otimes_R K.$$

In light of proposition 5.2.1, this definition is independent of the choice of a δ-lattice.

5.2.2 Skew Tate Example

Similarly to the previous chapter, for an element $\alpha \in A$ where A is some complete sliced K-algebra, we define $\alpha^{[i]} = \frac{\alpha^i}{i!}$. For $i \in \mathbb{N}$, we set

$$N_i = v_K(p)\frac{p^i - 1}{p^i(p-1)}.$$

Note that N_i is a strictly increasing sequence of rational numbers converging to $\frac{v_K(p)}{p-1}$. Let $r = \frac{a}{b}$, a, b coprime, and $b \in \mathbb{N}$.

Let $A = K\langle \pi_K^{-r} y\rangle$ with $r \in \mathbb{Z}$ and suppose $N_{m-1} < r \leq N_m$, and let $s = p^m(N_m - r) \in \mathbb{Z}$. Let T be the set of power series α which can be written uniquely in the form

$$\alpha = \sum_{\lambda \in \mathbb{N}_p^m, i \in \mathbb{N}} \alpha_{\lambda i} y^{\lambda_1}(y^{[p]})^{\lambda_2} \dots (y^{[p^{m-1}]})^{\lambda_m}(\pi_K^s y^{[p^m]})^i$$

where $\mathbb{N}_i = \{0, 1, ...i-1\}$, with $\alpha_{\lambda i} \in R$, $\alpha_{\lambda i} \to 0$ as $i \to \infty$.

Proposition 5.2.2 *1. $T \subset A^\circ$.*

2. T is a ∂_y-lattice in A.

Proof:

1. To show $T \subset A^\circ = R\langle \pi_K^{-r} y\rangle$, we first observe that $y^{[p^i]} \in A^\circ$ for $0 \leq i \leq m-1$ since $-r < -N_i = \frac{-v_K(p^i!)}{p^i}$ for $0 \leq i \leq m-1$. Furthermore we have that

$$\pi_K^s y^{[p^m]} = \pi_K^{p^m(N_m - r)} \frac{y^{p^m}}{p^m!} = \varepsilon(\pi_K^{-r} y)^{p^m}$$

for $\varepsilon \in R^\times$. It follows that $\pi_K^s y^{[p^m]} \in A^\circ$.

2. T is a lattice in A because $\pi_K^{p^m r} R\langle \pi_K^{-r} y\rangle \subset T \subset R\langle \pi_K^{-r} y\rangle$ so T is a lattice in A since $R\langle \pi_K^{-r} y\rangle$ is a lattice in A. On the other hand we have $\partial_y(y^{[p^i]}) = y^{[p^i - 1]}$ for $0 \leq i \leq m-1$. But for some $\varepsilon \in R^\times$,

$$y^{[p^i - 1]} = \varepsilon y^{p-1}(y^{[p]})^{p-1} \dots (y^{[p^{i-1}]})^{p-1} \in T$$

Similarly

$$\pi^s y^{[p^m-1]} = \varepsilon \pi^s y^{p-1}(y^{[p]})^{p-1} \dots (y^{[p^{m-1}]})^{p-1} \in T$$

since $s \geq 0$. Since T is an R-subalgebra of A°, T is ∂_y-stable and thus a ∂_y-lattice in A.

□

5.2.3 Base Change

Let A be a K-affinoid algebra. Let $\alpha \in A$ and let $\frac{a}{b} \in \mathbb{Q}$, a and b coprime, $b \in \mathbb{N}$. Let K' be a finite Galois extension of K containing $K(\pi_K^{\frac{1}{b}})$. Then we define

$$A\langle \pi_K^r \alpha \rangle = (A \otimes_K K') \langle \pi_{K'}^a \alpha \rangle^{\mathrm{Gal}(K'/K)}.$$

Lemma 5.2.3 *1. Let A be a K-affinoid algebra and let B be a multiplicatively closed lattice in $A_{K'} = A \otimes_K K'$. Let $G = \mathrm{Gal}(K'/K)$. Then $A = B^G \otimes_R K$ and $B^G \otimes_R R'$ is a lattice in $A_{K'}$.*

2. If A is a simple K'-algebra and B is a K-algebra such that $A = B \otimes_K K'$ then B is simple.

Proof:

1. Every element of $B \otimes_{R'} K'$ can be written in the form $\beta \otimes \lambda$ with $\beta \in B$ and $\lambda \in K'$. B is a lattice in $A_{K'}$ and $A_{K'}^G = A$, so taking A as a subset of $A_{K'}$, and writing $\beta \otimes \lambda$ as $\beta\lambda$ for brevity, we have

$$A = \{\beta\lambda \in A_{K'}^G : \beta \in B, \lambda \in K'\}$$

Therefore, to show that $A = B^G \otimes_R K$ it is enough to observe $\beta\lambda \in A_{K'}^G$ with $\beta \in B$ and $\lambda \in (K')^\times$, then for some $\mu \in K^\times$ we have $\beta\lambda\mu^{-1} \in B$ or $\beta\lambda\mu \in B$, as then we can write either $\beta\lambda = (\beta\lambda\mu)\mu^{-1}$ or $\beta\lambda = (\beta\lambda\mu^{-1})\mu$. Set

$$\mu = \prod_{\sigma \in G} \sigma(\lambda).$$

Then $\mu \in K$, and $v_L(\mu) = \#(G)v_{K'}(\lambda)$. Since $\#(G) \geq 1$, we must have $v_{K'}(\mu\lambda) \geq 0$ or $v_{K'}(\mu^{-1}\lambda) \geq 0$. It follows that either $\beta\lambda\mu^{-1} \in B$ or $\beta\lambda\mu \in B$, proving that $A = B^G \otimes_R K$. Finally, since B is an R'-module we must have $B^G \otimes_R R' \subset B$, and using the fact that $A = B^G \otimes_R K$ and $A_{K'} = A \otimes_K K'$, we have $(B^G \otimes_R R') \otimes_{R'} K' = A_{K'}$.

2. Let I be an ideal of B. Then $I \otimes_K K'$ is an ideal of A, so $I \otimes_K K' = 0$ or $I \otimes_K K' = A$. But K' is a faithfully flat K-module so $I = 0$ or $I = A$.

□

Proposition 5.2.3 *Let A be a K-affinoid algebra and let δ be a derivation of A. Let K' be an algebraic field extension of K and let δ' be the linear extension of δ to $A_{K'} = A \otimes_K K'$. Let $G = \mathrm{Gal}(K'/K)$*

1. *A has a δ-lattice if and only if $A_{K'}$ has a δ'-lattice.*

2. *If B is a δ'-lattice in $A_{K'}$ then B^G is a δ-lattice in A and $B^G \otimes_R R'$ is a δ-lattice in $A_{K'}$.*

Proof: Suppose that $A_{K'}$ has a δ' lattice B. δ' fixes A°, so $(\delta')^i$ and consequently δ^i fixes B^G for all $i \in \mathbb{N}$. By lemma 5.2.3 $B^G \otimes_R K = A$ so B^G is a δ-lattice in A, proving the first statement of part 2.

Now suppose that A has a δ lattice B'. K'et $\alpha \in B'$ and let $\lambda \in R'$. Then $(\delta')^i(\alpha \otimes \lambda) = \delta^i(\alpha) \otimes \lambda$. Since $\alpha \in C$, $\delta^i(\alpha) \in B'$, and consequently $\delta^i(\alpha) \otimes \lambda \in B' \otimes_R R'$. It follows that $B' \otimes_R R'$ is $(\delta')^i$ invariant for all $i \in \mathbb{N}$. On the other hand $B' \otimes_R R'$ is a lattice in $A_{K'}$, so $B' \otimes_R R'$ is a δ'-lattice, proving part 1, and if B is a δ'-lattice in $A_{K'}$, then applying the same proof to B^G yields that $B^G \otimes_R R'$ is a δ-lattice in $A_{K'}$, completing the proof.

□

5.2.4 Computing the ∂_t lattice for disks

Let Z be a copy of $\mathbb{A}_K^{1,an}$. Let t be a coordinate for Z. Let $\partial_t \in \mathcal{T}_Z(Z)$ be the dual operator to $dt \in \Omega_Z^1(Z)$.

Let $r \in \mathbb{Q}$, $\mu \in \mathbb{A}_K^{1,an}$. Then we define $U_r(\mu) = \{\lambda \in \mathbb{A}_K^{1,an} : v_K(\lambda - \mu) \geq r\}$. Define $A_r(\mu) = \mathcal{O}(U_r(\mu)) = K\langle \pi_K^{-r}(t+\mu)\rangle$.

Then set $K' = K(\pi_K^{\frac{1}{b}})$. Let R' be the unit ball of K'. Let T be defined as in section 5.2.2 over K'.

Lemma 5.2.4 *Set $y = t - \mu$. Let $A = \mathcal{O}_Z(U_r(\mu))$. Let $G = \mathrm{Gal}(K'/K)$. Then T^G is a ∂_t-lattice in A.*

Proof: Using lemma 5.2.2(2), T is a ∂_t-lattice in $A \otimes_K K'$, so by lemma 5.2.3(2) T^G is a ∂_t-lattice in A. □

5.3 Skew-Tate extension of disks as Matrix Algebras over Affinoid Weyl Algebras

5.3.1 Building the isomorphism

Let $r \in \mathbb{Q}$, $\mu \in Z$, $0 < r < v_K(p)\frac{1}{p-1}$. Choose m such that $N_{m-1} < r \leq N_m$. Let $s = p^m(N_m - r)$. Let $r = \frac{a}{b}$, a, b coprime, and $b \in \mathbb{N}$. Let K' be a Galois extension of K such that $\pi_K^{\frac{1}{b}} \in K'$. Define $D_r(\mu) = \mathcal{O}(U_r(\mu))\langle x\ ;\partial\rangle$. Let T be defined as in 5.2.2 over

R'. Let $\mathscr{D} = T\langle\partial_t\rangle$.

Lemma 5.3.1 *Let* $X = \mathbb{A}^1_{R'}$. *Define*

$$\omega : \mathscr{D} \to \widehat{\mathcal{D}_X^{[m]}}(X) \; ; \; t^{[p^i]} \to \partial_t^{[p^i]} \; ; \; \partial_t \mapsto -t$$

Let C *be the diagonal algebra of* $\widehat{\mathcal{D}_X^{[m]}}(X)$ *defined in section 4.3.3.*

1. ω *is a* G*-equivariant* R'*-algebra embedding.*

2. $C' = \omega(\mathscr{D}) \cap C$ *is* ϕ*-invariant.*

3. *Let* $f(c_0) \in K[c_0]$ *and suppose that* $\deg(f) \leq 2w - 1$*. Then* $f(c_0) \in pC$ *implies* $f(c_0) \in \pi C'$.

4. *Let* e_{ij} *be defined as in section 4.3. Then* $e_{ij} \in \omega(\mathscr{D})$ *for all* $0 \leq i, j \leq w - 1$.

5. *Let* γ *be defined as in section 4.3. Then* $\gamma \in \omega(\mathscr{D})$.

Proof:

1. We will prove that every element $\alpha \in \widehat{\mathcal{D}_X^{[m]}}(X)$ can be written uniquely in the form

$$\alpha = \sum_{i,j \in \mathbb{N}, \lambda \in \mathbb{N}_p^m} \alpha_{ij\lambda} t^i (\partial_t)^{\lambda_1} \ldots (\partial_t^{[p^{m-1}]})^{\lambda_m} (\partial_t^{[p^m]})^j$$

where $\mathbb{N}_p = \{0, \ldots, p-1\}$, with $\alpha_{ij\lambda} \in R'$, $\alpha_{ij\lambda} \to 0$ as $i + j \to \infty$. Thus, there is an embedding of R'-modules

$$\mathscr{D} \to \widehat{\mathcal{D}_X^{[m]}}(X) \; ; \; \partial \mapsto -t \; ; \; t^{[p^i]} \mapsto \partial_t^{[p^i]}.$$

It is trivial to verify this is an R'-algebra homomorphism as an extension of the Fourier transform. G-equivariance is obvious from the definition of ω.

2. Let C_{m-1} be the closed sub-R'-algebra of C generated by the c_i for $0 \leq i \leq m - 1$. Then $C' = C_{m-1}\langle\pi^s c_m\rangle$. C_{m-1} is fixed by ϕ, and $\phi(c_m) \in c_m + C_{m-1}$, so ϕ fixes C'.

3. Recalling the definition $d_i = \binom{c_0}{i}$ for $0 \leq i \leq 2w$, we can write

$$f(c_0) = \sum_{i=0}^{2q-1} a_i d_i$$

for some $a_i \in K$. Then

$$f(c_0) \in pC \text{ if and only if } a_i \in pR' \text{ for } 0 \leq i \leq 2q-1.$$

Then it will suffice to prove that $pd_i \in \pi C'$ for $0 \leq i \leq 2q-1$. Since $C' = \omega(\mathscr{D}) \cap C$, it will be enough to show $pd_i \in \pi\omega(\mathscr{D})$. For $0 \leq i \leq q-1$ this is obvious so suppose that $i = q+j$, with $0 \leq j \leq q-1$. Then we can write

$$d_i = t^i \partial_t^{[i]} = \binom{i}{q}^{-1} t^i \partial_t^{[p^m]} \partial_t^{[j]}.$$

We have $i = q + i_{m-1}p^{m-1} + \ldots + i_0$ for some $0 \leq i_j < p$ for $0 \leq j \leq m-1$, so using Lucas' theorem

$$\binom{i}{q} \equiv \binom{1}{1} \prod_{j=0}^{m-1} \binom{i_j}{0} = 1 \bmod p\mathbb{Z}.$$

So that $\binom{i}{q}^{-1} \in (R')^\times$. Then for some $\varepsilon \in (R')^\times$ we have

$$pd_i = \pi_K^{v_K(p)-s} \varepsilon \left(t^i (\pi_K^s \partial_t^{[p^m]}) \partial_t^{[j]} \right)$$

proving the claim since $v_K(p) > s$ and $0 \leq j < p^m$.

4. We will first show that $e_{ii} \in C'$ for $0 \leq i \leq q-1$. ϕ fixes C' by part 2 so since $\phi(e_{ii}) = e_{(i+1)(i+1)}$ for $0 \leq i \leq q-1$ it will be enough to show that $e_{00} \in C'$. For this, it will be enough to show that $\hat{e}_{00}$ is idempotent in $C'/\pi C'$ as then $\hat{e}_{00}$ will converge to e_{00} in C'.

   If we consider $\hat{e}_{00} \in K[c_0]$, then $\deg(\hat{e}_{00}^2 - \hat{e}_{00}) < 2q-1$, so to show that $\hat{e}_{00}$ is idempotent in $C'/\pi C'$ it will be enough to show that $\hat{e}_{00}^2 - \hat{e}_{00} \in pC$ by part 3. But this follows by proposition 4.3.4(3).

   To see that the $e_{ij} \in \omega C$ one uses a proof similar to that in lemma 4.3.7.

5. By part 4 we have each of the $e_{ii} \in C'$. From the definition of γ, we can see that is will be sufficient to prove that $\alpha_i - e_{ii} \in \pi C'$ for all $0 \leq i \leq p^m - 1$. By proposition 4.3.5(1), $\alpha_i - \hat{e}_{ii} \in pC$, and if we consider $\alpha_i - \hat{e}_{ii}$ as an element of $K[c_0]$ then $\deg(\alpha_i - \hat{e}_{ii}) < 2p^m$, so it follows that $\alpha_i - \hat{e}_{ii} \in \pi C'$ and hence $\alpha_i - e_{ii} \in \pi C'$ by part 3, proving the claim.

□

Theorem 5.3.1 *There is a G-equivariant isomorphism $\mathscr{D} \to M_{p^m}(\widehat{V_{s,K'}})$.*

Proof: Since $e_{ij} \in \omega(\mathscr{D})$ for $0 \leq i, j \leq p^m - 1$ by part 4 of lemma 5.3.1,, using lemma 4.3.8, there is an isomorphism $M_{p^m}(Z) \to \omega(\mathscr{D})$, where Z is the centralizer of the e_{ij} in $\omega(\mathscr{D})$ for $0 \leq i, j \leq p^m - 1$. Now, if Z' is the centralizer of the e_{ij} in $\widehat{\mathcal{D}_X^{[m]}}(X)$ then it is clear that $Z = Z' \cap \omega\mathscr{D}$, and we know that Z' is the closed subring of $\widehat{\mathcal{D}_X^{[m]}}(X)$ generated by $\gamma\partial_t^{[p^m]}$ and t^{p^m}. $\gamma \in C'$ by part 5 of lemma 5.3.1 and we have a ring isomorphism $Z' \to \widehat{\mathcal{D}_X^{[0]}}(X)$ sending $\gamma\partial_t^{[p^m]}$ to ∂_t and $t^{p^m} \mapsto t$. Now, let V_s be the R'-subalgebra of $\widehat{\mathcal{D}_X^{[0]}}(X)$ generated by t and $\pi_K^s\partial_t$. We can see that $Z/\pi Z$ is isomorphic to the commutative κ-algebra generated by $t^{[p^m]}$ and $\pi_K^s\partial_t^{p^m}$, so the induced map $V_s/\pi V_s \to Z/\pi Z$ is an isomorphism. It follows that $V_{s,K'} \to Z$ is an isomorphism, proving the claim.

□

Corollary 5.3.1 *There is an isomorphism of K-algebra $D_r(\mu) \to M_q(\widehat{V_{s,K'}})^G$.*

Proof: Using theorem 5.3.1, we have a G-equivariant isomorphism of K'-algebras

$$D_r(\mu) \to M_q(\widehat{V_{s,K'}}).$$

The statement immediately follows. □

5.3.2 Skew-Tate Extensions of Affinoid Algebras Defined by Polynomials

Let Z be a copy of $\mathbb{A}_K^{1,an}$ and let $f(t) \in K[t]$. Then define $X(f) = \{\lambda \in Z : v_K(f(\lambda)) \geq 0\}$. Let $f_1(t), \ldots, f_n(t) \in K[t]$. Let $\partial = \frac{d}{dt}$. Then we define

$$X_\partial(f_1, \ldots, f_n) = \{\lambda \in Z : v_K(\partial^j(f_i(\lambda))) \geq 0 \text{ for } 1 \leq i \leq n \text{ and } j \in \mathbb{N}\}.$$

Theorem 5.3.2 *Let $f_1(t), \ldots, f_n(t) \in K[t]$. If $X_\partial(f_1(t), \ldots, f_n(t))$ is non-empty, it is a finite union of closed disks $U_{r_i}(\mu_i)$ for some $r_i \in \mathbb{Q}$, $\mu_i \in Z$ for $1 \leq i \leq s$. Each $r_i < \frac{-v_K(p)}{p-1}$.*

Proof: From the definition, we can see that

$$X_\partial(f_1, \ldots, f_n) = \bigcap_{1\leq i\leq n} \bigcap_{j\in\mathbb{N}} X(\partial^j f_i).$$

Since the intersection of two disks in Z is either empty or the disk of lesser radius, it will suffice to prove the statement for $X_\partial(f)$, where $f(t) \in K[t]$. Using corollary 5.1.2 each $X(\partial^i f)$ is a finite union of disks and hence $X_\partial(f)$ is a finite union of disks. Now, to prove the theorem it will be enough to show that each of the disks is equal to $U_{r_i}(\mu_i)$ for some $r_i \in \mathbb{Q}$ such that $r_i < \frac{-v_K(p)}{p-1}$. So let μ be an arbitrary point in $X_\partial f$. By translating Z we can assume w.l.o.g. that $\mu = 0$. Write $f(t) = a_0 + \ldots + a_m t^m$.Then the fact that $0 \in X(\partial^j(f))$ for all $j \in \mathbb{N}$ forces $i!a_i \in R$ for $1 \leq i \leq m$. The radius of the connected component of $X(\partial^j(f))$ about 0 is $\chi(\partial^j f)$ by theorem 5.1.2. Then the radius $r \in \mathbb{Q}$ of the connected component of $X_\partial(f)$ about 0 is

$$r = \max_{j\in\mathbb{N}}\{\chi(\partial^j f)\} = \max_{0\leq j\leq m-1} \max_{j+1\leq i\leq m} \left\{-\frac{1}{i-j} v_K\left(\frac{i!a_i}{(i-j)!}\right)\right\}$$

Then $i!a_i \in R$ for $1 \leq i \leq m$ forces

$$r \leq \max_{0\leq j\leq m \text{ and } j\leq i\leq m} \left\{\frac{v_K((i-j)!}{i-j}\right\}.$$

But for all $i \in \mathbb{N}$, $\frac{v_K(i!)}{i} < \frac{v_K(p)}{p-1}$ by lemma 4.3.3(6), proving the claim. $\square$

In light of this proof, given $f_1(t), \ldots, f_n(t) \in K[t]$, we define

$$K\langle f_1(t)^{(\partial)}, \ldots, f_n(t)^{(\partial)}\rangle = \mathcal{O}_Z(X_\partial(f_1, \ldots, f_n))).$$

Corollary 5.3.2 *Let $A = K\langle f_1(t)^{(\partial)}, \ldots, f_n(t)^{(\partial)}\rangle$. Then if A is non-zero, then for some $s \in \mathbb{N}$, $m_i \in \mathbb{N}$, $r_i \in \mathbb{N}$, and L_i extending K_i for $1 \le i \le s$, there is an isomorphism*

$$A\langle x\ ;\ \partial\rangle \to \prod_{i=1}^{s} M_{p^{m_i}}\left(\widehat{V_{r_i,L_i}}^{G_i}\right)$$

where $G_i = \mathrm{Gal}(L_i/K)$ for $1 \le i \le s$.

Proof: By theorem 5.3.2 there is an open immersion $\mathrm{Sp}(A) \to \mathbb{A}_K^{1,an}$ whose image is a finite union of closed disks $U_{r_i}(\mu_i)$ with $r_i \in \mathbb{Q}$, $\mu_i \in \mathbb{A}_K^{1,an}$, and $r_i < v_K(p)\frac{1}{p-1}$. By theorem 5.3.1 each $\mathcal{O}_{r_i}(\mu_i)\langle x\ ;\ \frac{d}{dt}\rangle$ is isomorphic to $M_{p^{m_i}}(\widehat{V_{r_i,L_i}})^{G_i}$ for some $m_i \in \mathbb{N}$ and $r_i \in \mathbb{N}$, and L_i/K. Now, there is an isomorphism $A \to \prod_{i=1}^{s} L_i\langle \pi_K^{r_i}(t+\mu_i)\rangle$. Let $e_1, ..., e_s$ be the primitive idempotents of this presentation. Then each $I_j = \sum_{i\neq j} e_i L_i\langle \pi_K^{r_i}(t+\mu_i)\rangle$ is a minimal prime over the ideal $\{0\}$ in A, so by [7, Lemma 3.3.3] I_j is invariant under ∂_t. It follows that each $L_i\langle \pi_K^{r_i}(t+\mu_i)\rangle$ is invariant under ∂_t, so we get a chain of isomorphisms

$$A\langle x\ ;\ \partial\rangle \to \prod_{i=1}^{s} L_i\langle \pi_K^{r_i}(t+\mu_i)\rangle\langle x\ ;\ \partial\rangle \to \prod_{i=1}^{s} M_{p^{m_i}}\left(\widehat{V_{r_i,L_i}}^{G_i}\right).$$

$\square$

5.4 Primitive Ideals in Weight One Powerful Nilpotent Enveloping Algebras

Let $\mathfrak{g}$ be a finite dimensional non-Abelian nilpotent K-Lie algebra, containing an Abelian sub-Lie algebra $\mathfrak{h}$ of codimension one. Let $x \in \mathfrak{g}\backslash\mathfrak{h}$. Let $D = \mathrm{ad}_x$. Then $\ker(D) \cap \mathfrak{h} = Z(\mathfrak{g})$

Let $0 \neq y$ be an element of $\mathfrak{h}$ such that $y \notin Z(\mathfrak{g})$ and $y + Z(\mathfrak{g})$ is central in $\mathfrak{g}/Z(\mathfrak{g})$. Let $z = [x, y]$. Then $0 \neq z \in Z(\mathfrak{g})$ since $y + Z(\mathfrak{g}) = y + (\ker(D) \cap \mathfrak{h}) \neq 0$. It follows that $(x, y, z, \mathfrak{h})$ forms a reducing quadruple for $\mathfrak{g}$.

5.4.1 The factor ring as a Skew-Tate-Extension

Let $\mathscr{L} \subset \mathfrak{g}$ be an R-Lie lattice in $\mathfrak{g}$ such that $[\mathscr{L}, \mathscr{L}] \subset \pi\mathscr{L}$. Let $\mathscr{H} = \mathscr{L} \cap \mathfrak{h}$. Let P be a weakly rational ideal of $U(\mathfrak{g})$ such that $P \cap \mathfrak{g} = 0$, and set $P' = \widehat{U(\mathscr{L})}_K P$. Let $P_{\mathfrak{h}} = P \cap U(\mathfrak{h})$ and let $P'_{\mathfrak{h}} = \widehat{U(\mathscr{H})}_K \cap P'$. For ease of notation we further define

1. $U = U(\mathfrak{g})$
2. $\hat{U} = \widehat{U(\mathscr{L})}_K$.
3. $\hat{H} = \widehat{U(\mathscr{H})}_K \subset \hat{U}$.

Let I be an ideal of $\hat{U}$ such that $Z(\hat{U}/I)$ is isomorphic to K. Let $I' = I \cap U$. Let $\mathscr{A}$ be the R-subalgebra of $\hat{U}$ generated by $\widehat{U(\mathscr{H})}$ and x.

Lemma 5.4.1 *1. I' is a weakly rational ideal of U.*

2. $P' = \hat{U} P_{\mathfrak{h}}$.

3. $\hat{H} P_{\mathfrak{h}} = P'_{\mathfrak{h}}$.

4. Let $A = \hat{H}/(P' \cap \hat{H})$. Then A is a lattice in $\hat{H}/P'_{\mathfrak{h}}$.

5. $\mathscr{A}$ is dense in $\hat{U}$. $P' \cap \mathscr{A}$ is controlled by $\widehat{U(\mathscr{H})}$. $\mathscr{A}$ is isomorphic to $\widehat{U(\mathscr{H})}\,[x\ ;\ D]$.

Proof:

1. $K \subset Z(U/I') \subset Z\left(\hat{U}/I\right) = K$.

2. $P' = \hat{U}P$ and $P = UP_{\mathfrak{h}}$ by corollary 2.4.7. Of course $\hat{U}U = \hat{U}$, so

$$\hat{U}P_{\mathfrak{h}} = \hat{U}UP_{\mathfrak{h}} = P'.$$

3. We need to show that $\hat{H}P_{\mathfrak{h}} \subset P'_{\mathfrak{h}}$ and $P'_{\mathfrak{h}} \subset \hat{H}P_{\mathfrak{h}}$. We have

$$P'_{\mathfrak{h}} = \hat{H} \cap \hat{U}P_{\mathfrak{h}},$$

so clearly $\hat{H}P_{\mathfrak{h}} \subset P'_{\mathfrak{h}}$. Now let $\alpha \in P'_{\mathfrak{h}}$. Then we can write $\alpha = u\beta$ for some $u \in \hat{U}$ and $\beta \in P_{\mathfrak{h}}$. Now, by proposition 2.5.1(1), u can be written uniquely in the form

$$u = \sum_{i \in \mathbb{N}} x^i u_i \text{ with } u_i \in \hat{H} \text{ and } u_i \to 0 \text{ as } i \to \infty.$$

Then since $\hat{U}$ is a domain using the uniqueness of the expression and the fact that $u_i\beta \in \hat{H}$ for all $i \in \mathbb{N}$,

$$u\beta = \sum_{i \in \mathbb{N}} x^i u_i \beta \in \hat{H} \text{ if and only if } u_i\beta = 0 \text{ for all } i > 0.$$

This only occurs when $u \in \hat{H}$, so that $\alpha = u\beta \in \hat{H}P_{\mathfrak{h}}$. It follows that $P'_{\mathfrak{h}} \subset \hat{H}P_{\mathfrak{h}}$, proving the claim.

4. We first observe that $(P \cap \widehat{U(\mathscr{H})}) \otimes_R K = P'_{\mathfrak{h}}$. Then since K is a flat R-module we have that following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P' \cap \widehat{U(\mathscr{H})} & \longrightarrow & \widehat{U(\mathscr{H})} & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P'_{\mathfrak{h}} & \longrightarrow & \hat{H} & \longrightarrow & A \otimes_R K & \longrightarrow & 0
\end{array}$$

with exact rows. It follows that A is a lattice in $\hat{H}/P'_{\mathfrak{h}}$.

5. The natural $\widehat{U(\mathscr{H})}$-module homomorphism

$$\bigoplus_{i \in \mathbb{N}} \widehat{U(\mathscr{H})} x^i \to \hat{U}$$

is an embedding by proposition 2.5.1(1). $\mathscr{A}$ is the image of this homomorphism, so it is clear that $\mathscr{A}$ is isomorphic as an R-algebra to $\widehat{U(\mathscr{H})}\,[x\ ;\ D]$.

Let $Q = P' \cap \mathscr{A}$, and let $\alpha = \sum_{i=0}^{n} a_i x^i \in Q$, $a_i \in \widehat{U(\mathscr{H})}$ for $0 \leq i \leq n$. Then to show that Q is controlled by $\widehat{U(\mathscr{H})}$, it is enough to show that $a_i \in Q$ for $0 \leq i \leq n$. By induction on the degree of α, it will be enough to simply show that $a_n \in Q$. For this we note that $\alpha \in Q$ implies $\mathrm{ad}_y^n(\alpha) = n!z^n a_n \in Q$. Now, $Q = P' \cap \mathscr{A}$, $a_n \in \widehat{U(\mathscr{H})}$. $Z(\hat{U}/P') = K$ and $P' \cap Z(\mathfrak{g}) = 0$ so $z - \mu \in P'$ for some $\mu \in K^{\times}$. It follows that $n!\mu^n a_n \in P'$. Since P' is a K-vector space, $n!\mu^n a_n \in P'$ implies $a_n \in P'$, so $a_n \in \widehat{U(\mathscr{H})} \cap Q$, proving the claim. Since $\mathscr{A}$ contains $U(\mathscr{L})$, $\mathscr{A}$ is dense in $\widehat{U(\mathscr{L})}$.

□

Now, let B be the R-subalgebra of $\hat{U}$ generated by H and x. Then $U(\mathscr{L}) \subset B \subset \widehat{U(\mathscr{L})}$, so B is a lattice in $\hat{U}$.

Proposition 5.4.1 *Let $A = \hat{H}/P'_{\mathfrak{h}}$ as defined in lemma 5.4.1(3).*

1. *Let δ be the restriction of the action of $D|_{\hat{H}}$ to A. Then there is an isomorphism of K-algebras*

$$\hat{U}/P' \to A\langle x\ ;\ \delta\rangle.$$

2. *For some collection of polynomials $f_i(t) \in K\,[t]$ for $1 \leq i \leq s$, there is an isomorphism $A \to K\langle f_1(t)^{(\partial)}, \ldots, f_s(t)^{(\partial)}\rangle$.*

Proof:

1. Let $J = P' \cap \mathscr{A}$. $\mathscr{A}$ is a lattice in $\hat{U}$, J is controlled by $\widehat{U(\mathscr{H})}$ by lemma 5.4.1(5). Let $J' = P' \cap \hat{H}$. $\mathscr{A}$ is isomorphic to $\widehat{U(\mathscr{H})}\,[x\ ;\ \delta]$ by lemma 5.4.1(5), so we have

an isomorphism

$$\mathscr{A}/J \to B\,[x\,;\,\delta]$$

where $B = \widehat{U(\mathscr{H})}/J'$. As $\mathscr{A}/J$ is a lattice in $\hat{U}/P'$, we get an isomorphism

$$\hat{U}/P' \to \varprojlim_{i\in\mathbb{N}} B\,[x\,;\,\delta]\,/\pi^i B\,[x\,;\,\delta] \otimes_R K.$$

Of course, since B is a δ-lattice in A, by definition $\varprojlim_{i\in\mathbb{N}} B\,[x\,;\,\delta]\,/\pi^i B\,[x\,;\,\delta] \otimes_R K = A\langle x\,;\,\delta\rangle$, completing the proof.

2. By proposition 2.4.7 we have a ring homomorphism

$$\Phi : U(\mathfrak{g}) \to W_1(K)\,;\, x \mapsto \partial \text{ and } \alpha \mapsto \sum_{i\in\mathbb{N}} \overline{D^i(\alpha)} t^{[i]}$$

with primitive kernel P generated by the ideal $(z-1)U(\mathfrak{g})$ and the preimage of $U(\mathfrak{h})/J$ as described in proposition 2.4.7. Let $\Phi_{\mathfrak{h}}$ be the restriction of Φ to $U(\mathfrak{h})$. Then $\Phi_{\mathfrak{h}}(U(\mathfrak{h})) = K\,[t]$. If we choose a basis $h_1, \ldots, h_n$ for $\mathscr{H}$, we can see that $\Phi_{\mathfrak{h}}(U(\mathscr{H}))$ is the sub-R-algebra generated by

$$f_i(t) = \sum_{j\in\mathbb{N}} \overline{D^j(h_i)} t^{[j]} \text{ for } 0 \le i \le d-1$$

Thus, we have an exact sequence

$$0 \to \widehat{P_{\mathfrak{h}}} \to \hat{H} \to K\langle f_i(t)\,;\, 0 \le i \le d-1\rangle \to 0.$$

By [4, 3.2.3(iii)] $\widehat{P_{\mathfrak{h}}} = \hat{H}P_{\mathfrak{h}}$. Then by lemma 5.4.1(2) $\widehat{P_{\mathfrak{h}}} = P'_{\mathfrak{h}}$. Now, it is enough to note that $\Phi\delta\Phi^{-1} = \partial_t$, so the fact that $\widehat{U(\mathscr{H})}$ is closed under the Lie bracket implies that $\partial_t^j(f_i(t)))$ is power bounded for all $j \in \mathbb{N}$ and $0 \le i \le d-1$. Then we can choose a presentation of $K\langle f_i(t) : 0 \le i \le d-1\rangle$ as in the statement.

□

Theorem 5.4.1 *For some $n \in \mathbb{N}$, some $m_i \in \mathbb{N}$, $s_i \in \mathbb{N}$ and some finite Galois extensions L_i of K with Galois groups G_i for $1 \le i \le n$, we have an isomorphism of K-algebras*

$$\hat{U}/P' \to \prod_{i=1}^{n} M_{p^{m_i}}(\widehat{V_{s_i,L_i}})^{G_i}$$

Proof: Using proposition 5.4.1(1) and (2), noting that the Dixmier map sends δ to ∂_t, there is an isomorphism

$$\hat{U}/P' \to K\langle f_1(t)^{(\partial)}, \dots, f_s(t)^{(\partial)}\rangle\langle x \, ; \, \partial\rangle.$$

Then the statement follows from corollary 5.3.2. □

Corollary 5.4.1 *1. If I is a primitive ideal of $\hat{U}$ then $I \cap U(\mathfrak{g})$ is a primitive ideal of $U(\mathfrak{g})$.*

2. There is a surjective map $\mathrm{Prim}(\hat{U}) \to \mathrm{Prim}(U(\mathfrak{g}))$ with finite fibres.

3. For an ideal $I \subset \hat{U}$ the following are equivalent:

(a) $Z(\hat{U}/I)$ is algebraic over K.

(b) I is primitive.

(c) I is maximal.

Proof: We first observe that each $\widehat{V_{s_i,L_i}}$ is simple: to see this we can use a similar method to that in lemma 3.1.2 part 1 to show each ideal in $\widehat{V_{s_i,L_i}}$ contains an element of K.

Let I be a primitive ideal of $\hat{U}$. Let $J = I \cap U(\mathfrak{g})$. Then by theorem 2.5 $L = Z(\hat{U}/I)$ is an algebraic field extension of K. It follows that $Z(U(\mathfrak{g})/J = L$, so J is primitive by proposition 2.4.4, proving part 1. On the other hand, let M be a simple $U(\mathfrak{g})$-module with annihilator J. Then there is a natural $U_L = U \otimes_K L$-module structure on M. Let J' be the annihilator of M in U_L. Then J' is weakly rational, so there are only finitely many primitives ideals in $\hat{U}_L/J'\hat{U}_L$ by theorem 5.4.1. $\hat{U}_L/J'\hat{U}_L$ is isomorphic to $\hat{U}/J\hat{U}$, so $J\hat{U}$ is semiprime as well, proving the second claim.

Now, for the the third claim, we first observe that we know (b) implies (a) by theorem 2.5. It is trivial that (c) implies (b) so it will be enough to prove that (a) implies (c). So suppose that $Z(\hat{U}/I)$ is algebraic over K. Then by theorem 5.4.1 we find that $\hat{U}/(I \cap U)\hat{U}$ is a product of galois invariants of matrix rings over deformed affinoid Weyl algebras, which are simple by lemma 5.2.3(2). Then it is clear that the condition on I implies that I is maximal. $\square$

Chapter 6

An analogue to Beilinson-Bernstein for the global sections of the arithmetic differential operators over the projective line

6.1 Definitions

6.1.1 $\mathcal{O}_X$-rings

We define the category of $\mathcal{O}_X$-rings over a scheme X in the following manner: Objects are pairs $(\mathcal{F}, \iota)$ where $\mathcal{F}$ is a sheaf of R-algebras over a scheme X and ι is a morphism of sheaves of rings $\mathcal{O}_X \to \mathcal{F}$. Morphisms $(\mathcal{F}, \iota) \to (\mathcal{G}, \iota)$ between $\mathcal{O}_X$-rings are defined to be morphisms of sheaves of R-algebras $\mathcal{F} \to \mathcal{G}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{F} & & \\ \uparrow{\scriptstyle \iota} & \searrow & \\ \mathcal{O}_X & \xrightarrow{\iota} & \mathcal{G} \end{array}$$

. We simply refer to $(\mathcal{F}, \iota)$ as $\mathcal{F}$ when no confusion will arise.

6.1.2 Notation and Preliminaries

Let X and Y be two copies of $\mathbb{P}^1_R$. Let t be a coordinate for X and τ a coordinate for Y. Let $F : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be the morphism of schemes defined by $\tau \mapsto t^p$. We explicitly construct $\mathcal{O}_Y$ in the following manner: Let X and X' be two copies of $\mathbb{A}^1_R$. Let x, x' be coordinates for X and X' respectively. Then we view $(Y, \mathcal{O}_Y)$ as the colimit of the following diagram

$$\begin{array}{ccc} \mathcal{O}_X & \longrightarrow & \mathcal{O}_X(X\backslash\{0\}) \\ & & \uparrow \\ & & \mathcal{O}_{X'} \end{array}$$

where the horizontal arrow is the sheaf map, and the vertical arrow maps $x' \mapsto x^{-1}$. We identify x with τ, x' with τ^{-1}, and define Y_0 to be the image of X in Y, Y_∞ the image of X' in Y, and $Y_{0\infty}$ the image of $X\backslash\{0\}$ in Y.

Let $\partial_t, \partial_{t^{-1}}$ be the dual operators to dt and dt^{-1} respectively.

Using theorem 4.3.1 we have isomorphisms of $\mathcal{O}_Y$-rings $M_p(\widehat{\mathcal{D}_Y^{[0]}})|_{Y_0} \to F_*\widehat{\mathcal{D}_X^{[1]}}|_{Y_0}$ and $F_*\widehat{\mathcal{D}_X^{[1]}}|_{Y_\infty} \to M_p(\widehat{\mathcal{D}_Y^{[0]}})|_{Y_\infty}$, so using the language of section 4.3 we define the following variables:

1. Let $D_0 = \widehat{\mathcal{D}_X^{[1]}}(X_0)$, $D_\infty = \widehat{\mathcal{D}_X^{[1]}}(X_\infty)$, and $D_{0\infty} = \widehat{\mathcal{D}_X^{[1]}}(X_{0\infty})$.

2. Let C_0 be the closed sub-R-algebra of D_0 generated by $x_0 = t\partial_t$ and $y_0 = t^p\partial_t^{[p]}$ (so that they satisfy the conditions of c_0 and c_1 as in 4.3.3), and let C_∞ be the closed

sub-R-algebra of D_∞ generated by $x_\infty = t^{-1}\partial_{t^{-1}}$ and $y_\infty = t^{-p}\partial_{t^{-1}}^{[p]}$. Let ϕ_0 be the automorphism $x_0 \mapsto x_0 + 1$ of C_0 and let ϕ_∞ be the automorphism $x_\infty \mapsto x_\infty + 1$ of C_∞.

3. For $0 \leq i \leq p-1$ let $\hat{e}_{ii} = \binom{t\partial_t + i}{p-1}$ and let $\hat{f}_{ii} = \binom{t^{-1}\partial_{t^{-1}} + i}{p-1}$. Let e_{ii} be the unique lifts of the idempotents $\hat{e}_{ii} + \pi C_0$ to C_0, and let f_{ii} be the unique lifts of the idempotents $\hat{f}_{ii} + \pi C_\infty$ to C_∞. Set $e_{ij} = e_{ii}t^{j-i}e_{jj} \in D_0$ and $f_{ij} = f_{ii}t^{i-j}f_{jj} \in D_\infty$, so that the $\{e_{ij}\}_{0 \leq i,j \leq p-1}$ form a set of matrix units for D_0 and $\{f_{ij}\}_{0 \leq i,j \leq p-1}$ form a set of matrix units for D_∞.

4. Let γ_0 be the element given by lemma 4.3.6 such that $\left[\gamma_0\partial_t^{[p]}, \tau\right] = 1$. Set $\delta_0 = \gamma_0\partial_t^{[p]}$. Similarly let γ_∞ be the element given by lemma 4.3.6 such that $\left[\gamma_\infty\partial_{t^{-1}}^{[p]}, \tau^{-1}\right] = 1$ and set $\delta_\infty = \gamma_\infty\partial_{t^{-1}}^{[p]}$.

We will see that in transferring between charts, the indices of the matrix units are shifted. To ease notation, for $0 \leq i \leq p-1$ we set

$$i^* = \begin{cases} p-2-i & \text{if } 0 \leq i \leq p-2 \\ p-1 & \text{if } i = p-1. \end{cases}$$

6.1.3 The Global Diagonal Algebra

Proposition 6.1.3 *The image of C_0 under the restriction map $D_0 \to D_{0\infty}$ and the image of $C_{0\infty}$ under the restriction map $D_\infty \to D_{0\infty}$ are identical. The automorphisms ϕ_0 and ϕ_∞ are inverse.*

Proof: In this proof, we identify C_0 and C_∞ with their images in $D_{0\infty}$. C_0 is the closed sub-R-algebra generated by x_0 and y_0 and C_∞ is the closed sub-R-algebra generated by x_∞ and y_∞, so to prove $C_0 = C_\infty$ it will be enough to check that x_0, $y_0 \in C_\infty$ and x_∞, $y_\infty \in C_0$.

Since $\partial_{t^{-1}} \mapsto -t^2\partial_t$, we can calculate that $x_0 = t\partial = -t^{-1}\partial_{t^{-1}} = -x_\infty$. Hence $x_0 \in C_\infty$ and $x_\infty \in C_0$. Using proposition 4.3.3(1) we can see that $p!y_\infty = t^{-p}\partial_{t^{-1}}^p = \prod_{i=0}^{p-1}(t^{-1}\partial_{t^{-1}} - i) = (-1)^p \prod_{i=0}^{p-1}(x_0+i)$. Of course, $(-1)^p \prod_{i=0}^{p-1}(x_0+i) = -\prod_{i=0}^{p-1}(x_0+p-1-i)$, so recalling that by definition $\phi_0^{p-1}(x_0) = x_0 + p - 1$, and again using lemma 4.3.3(1) we have

$$p!y_\infty = t^{-p}\partial_{t^{-1}}^p = -\phi_0^{p-1}\left(\prod_{i=0}^{p-1}(x_0 - i)\right) = -\phi_0^{p-1}(t^p\partial_t^p) = -p!\phi_0^{p-1}(y_0).$$

Since $D_{0\infty}$ is π-torsion free, it follows that $y_\infty = -\phi_0^{p-1}(y_0)$. Hence $y_\infty \in C_0$. By symmetry $y_0 = -\phi_\infty(y_\infty)^{p-1}$ so $y_0 \in C_\infty$, proving that $C_\infty = C_0$. Then ϕ_0 and ϕ_∞ are inverse, since ϕ_0 sends $x_0 \mapsto x_0 + 1$ and ϕ_∞ sends $x_0 = -x_\infty \mapsto -(x_\infty + 1) = x_0 - 1$. $\square$ Due to proposition 6.1.3 we now let C be the image of C_0 (and C_∞) in $D_{0\infty}$.

Corollary 6.1.3 *All of the idempotents e_{ii} and the elements $\hat{e}_{ii}$ are global.*

Proof: Each e_{ii} is global as they all belong to C, so they form a complete set of orthogonal idempotents in $\widehat{\mathcal{D}_X^{[1]}}(X)$. $\square$

6.1.4 Restriction of Matrix Units

Recall that we have defined, for $0 \leq i \leq p-1$, that

$$i^* = \begin{cases} p-2-i & \text{if } 0 \leq i \leq p-2 \\ p-1 & \text{if } i = p-1. \end{cases}$$

Let θ be the restriction map $\widehat{\mathcal{D}_X^{[1]}}(X_\infty) \to \widehat{\mathcal{D}_X^{[1]}}(X_{0\infty})$.

Proposition 6.1.4 *Let $\mu = e_{(p-1)(p-1)} + \tau^{-1}(1 - e_{(p-1)(p-1)})$. Then the restriction map $\theta : \widehat{\mathcal{D}_X^{[1]}}(X_\infty) \to \widehat{\mathcal{D}_X^{[1]}}(X_{0\infty})$ acts on the matrix units $\{f_{ij}\}_{0 \leq i,j \leq p-1}$ so that $\theta(f_{ij}) = \mu e_{i^*j^*}\mu^{-1}$.*

Proof:

From proposition 4.3.3(1) we can calculate that that $\hat{f}_{00} = t^{-(p-1)}\partial_{t^{-1}}^{[p-1]} = \binom{t^{-1}\partial_{t^{-1}}}{p-1}$. From proposition 4.3.3(1) we know that $\theta(t^{-1}\partial_{t^{-1}}) = -t\partial_t$, so we deduce that $\theta(\hat{f}_{00}) = \binom{-t\partial_t}{p-1} = \binom{t\partial_t+p-2}{p-1}$. Then for $0 \leq i \leq p-2$, using the fact that ϕ_0 and ϕ_∞ are inverse from proposition 6.1.3, and that $\phi_0(t\partial_t) = t\partial_t + 1$,

$$\theta(\hat{f}_{ii}) = \theta(\phi_\infty^i(\hat{f}_{00})) = \phi_0^{-i}\left(\binom{t\partial_t + p - 2}{p-1}\right) = \binom{t\partial_t + i^*}{p-1} = \hat{e}_{i^*i^*}.$$

so we can see that for $0 \leq i \leq p-2$,

$$\theta(f_{ii}) \equiv \theta(\phi_\infty^i(\hat{f}_{00})) = \hat{e}_{i^*i^*} \equiv e_{i^*i^*} \bmod \pi\widehat{\mathcal{D}_X^{[1]}}(X_{0\infty}).$$

Now, by lemma 4.3.3, since $\theta(f_{ii})$ is equivalent to $e_{i^*i^*}$ mod $\pi\widehat{\mathcal{D}_X^{[1]}}(X_{0\infty})$ we must have $\theta(f_{ii}) = e_{i^*i^*}$.

From the definitions we know that $\theta(t) = t$, and noting that $i - j = j^* - i^*$ for $0 \leq i, j \leq p-2$, we can observe that for $0 \leq i, j \leq p-2$,

$$\theta(f_{ij}) = \theta(f_{ii}t^{i-j}f_{jj}) = e_{i^*i^*}t^{j^*-i^*}e_{j^*j^*} = e_{i^*j^*}.$$

Since τ commutes with the $e_{i^*i^*}$, we find that $\theta(f_{ii}) = \mu e_{i^*i^*}\mu^{-1}$ for $0 \leq i, j \leq p-2$.

In a similar manner to above we can calculate that $\theta(\hat{f}_{(p-1)(p-1)}) = \theta(\phi_\infty^{p-1}(\hat{f}_{00})) = \phi_0^{-(p-1)}\binom{t\partial_t+p-2}{p-1} = \binom{t\partial_t-1}{p-1} = \phi_0^{-p}(\hat{e}_{(p-1)(p-1)})$. ϕ_0^{-p} is the identity mod $\pi\widehat{\mathcal{D}_X^{[1]}}(X_{0\infty})$, so $\phi_0^{-p}(\hat{e}_{(p-1)(p-1)}) \equiv \hat{e}_{(p-1)(p-1)}$ mod $\pi\widehat{\mathcal{D}_X^{[1]}}(X_{0\infty})$. Hence

$$\theta(f_{(p-1)(p-1)}) \equiv \theta(\hat{f}_{(p-1)(p-1)}) \equiv \hat{e}_{(p-1)(p-1)} \equiv e_{(p-1)(p-1)} \bmod \pi\widehat{\mathcal{D}_X^{[1]}}(X_{0\infty}).$$

As before, we conclude that $\theta(f_{(p-1)(p-1)}) = e_{(p-1)(p-1)}$. Since $(p-1)^* = p-1$, we have that $\theta(f_{(p-1)(p-1)}) = \mu e_{(p-1)^*(p-1)^*}\mu^{-1}$.

Finally, for $0 \leq i \leq p-2$, noting that $\tau = t^p$ commutes with the e_{ii}, we can calculate that

$$\theta(f_{(p-1)i}) = \theta(f_{(p-1)(p-1)}t^{(p-1)-i}f_{ii}) = e_{(p-1)(p-1)}t^{1+i^*}e_{i^*i^*}$$
$$= \tau e_{(p-1)(p-1)} = t^{i^*-(p-1)}e_{i^*i^*} = \tau e_{(p-1)i^*} = \mu e_{(p-1)i^*}\mu^{-1}$$

and similarly

$$\theta(f_{i(p-1)}) = e_{i^*i^*}t^{-(i^*+1)}e_{(p-1)(p-1)} = \tau^{-1}e_{i^*i^*}t^{(p-1)-i^*}e_{(p-1)(p-1)}$$
$$= \tau^{-1}e_{i^*(p-1)} = \mu e_{i^*(p-1)}\mu^{-1}.$$

□

6.1.5 Restriction of the τ-differential

Proposition 6.1.5 *Let $\mu = e_{(p-1)(p-1)} + \tau^{-1}(1 - e_{(p-1)(p-1)})$. The restriction $\widehat{\mathcal{D}_X^{[1]}}(X_\infty) \to \widehat{\mathcal{D}_X^{[1]}}(X_{0\infty})$ sends*

$$\delta_\infty \mapsto -\left(e_{(p-1)(p-1)}\tau^2\delta_0 + (1 - e_{(p-1)(p-1)})\tau\delta_0\tau\right) = \mu(-\tau^2\delta_0)\mu^{-1}.$$

Proof: Since the e_{ii} commute with τ, δ_0, and δ_∞ and $-\tau\delta_0\tau = \tau^{-1}(-\tau^2\delta_0)\tau$, we have

$$-\left(e_{(p-1)(p-1)}\tau^2\delta_0 + (1 - e_{(p-1)(p-1)})\tau\delta_0\tau\right) = \mu(-\tau^2\delta_0)\mu^{-1}.$$

Let θ be the restriction map $\widehat{\mathcal{D}_X^{[1]}}(X_\infty) \to \widehat{\mathcal{D}_X^{[1]}}(X_{0\infty})$.

Applying proposition 4.3.8(5) to $\widehat{\mathcal{D}_X^{[1]}}(X_\infty)$ we know that $f_{ii}p\tau^{-1}\delta_\infty = f_{ii}(t^{-1}\partial_{t^{-1}} - (p-1-i))$, so we can write

$$p\tau^{-1}\delta_\infty = \sum_{i=0}^{p-1} f_{ii}\left(t^{-1}\partial_{t^{-1}} - (p-1-i)\right)$$

From the definitions we know that $\theta(t^{-1}\partial_{t^{-1}}) = -t\partial$, and using proposition 6.1.4 we know that for $0 \leq i \leq p-1$, $\theta(f_{ii}) = e_{i^*i^*}$. From the definition of ϕ_0 we know $\phi_0^p(t\partial_t + i) = (t\partial_t + p + i)$, so for $0 \leq i \leq p-2$ we can calculate

$$\theta(f_{ii}p\tau^{-1}\delta_\infty) = -e_{i^*i^*}(t\partial_t + 1 + i^*) = -e_{i^*i^*}\phi_0^p(t\partial_t - (p-1-i^*)) = -\phi_0^p(e_{i^*i^*}p\tau\delta_0).$$

From proposition 4.3.8(4) we have $\phi_0^p(\tau\delta_0) = \tau\delta_0 + 1 = \delta_0\tau$, and we know that ϕ_0^p fixes $e_{i^*i^*}$ so we deduce that

$$\theta(f_{ii}\delta_\infty) = -e_{i^*i^*}\tau\phi_0^p(\tau\delta_0) = -e_{i^*i^*}\tau\delta_0\tau.$$

Summing over $0 \leq i \leq p-2$ we have that

$$\theta((1 - f_{(p-1)(p-1)})\delta_\infty) = -(1 - e_{(p-1)(p-1)})\tau\delta_0\tau.$$

Finally, using proposition 4.3.8(5) we can see that $f_{(p-1)(p-1)}p\tau^{-1}\delta_\infty = f_{(p-1)(p-1)}t^{-1}\partial_{t^{-1}}$ and $e_{(p-1)(p-1)}p\tau\delta_0 = e_{(p-1)(p-1)}t\partial_t$, so

$$\theta(f_{(p-1)(p-1)}p\tau^{-1}\delta_\infty) = -e_{(p-1)(p-1)}(t\partial_t) = -e_{(p-1)(p-1)}p\tau\delta_0.$$

So we get that $\theta(f_{(p-1)(p-1)}\delta_\infty) = -e_{(p-1)(p-1)}\tau^2\delta_0$, proving the claim. □

6.1.6 Twisted Sheaves of Algebras

Let X be a scheme, let $\mathcal{C}$ be some category, and let $\mathcal{F}$ be a $\mathcal{C}$-sheaf over X.

We define an $\mathcal{F}$-twist to be a $\mathcal{C}$-sheaf $\mathcal{G}$ such that there exists an open cover $\{U_i\}_{i\in I}$ of X and isomorphisms of $\mathcal{C}$-sheaves $\omega_i : \mathcal{G}|_{U_i} \to \mathcal{F}|_{U_i}$ for $i \in I$.

Lemma 6.1.6 *Suppose that $\mathcal{F}$ is a $\mathcal{C}$-sheaf on X, and that $\mathcal{G}$ and $\mathcal{H}$ are $\mathcal{F}$-twists. Choose an open cover $\{U_i\}_{i\in I}$ such that there exist isomorphisms $g_i : \mathcal{G}|_{U_i} \to \mathcal{F}|_{U_i}$ and $h_i : \mathcal{H}|_{U_i} \to \mathcal{F}|_{U_i}$ for $i \in I$. Write U_{ij} for $U_i \cap U_j$ for $i, j \in I$. Suppose that $g_i|_{U_{ij}} g_j|_{U_{ij}}^{-1} = h_i|_{U_{ij}} h_j|_{U_{ij}}^{-1}$. Then the morphism $\rho : \mathcal{G} \to \mathcal{H}$ defined locally by $\rho|_{U_i} = h_i^{-1} g_i$ for $i \in I$ is an isomorphism of $\mathcal{C}$-sheaves.*

Proof: Since g_i and h_i are isomorphisms of $\mathcal{C}$-sheaves for $i \in I$, $\rho|_{U_i}$ is an isomorphism of $\mathcal{C}$-sheaves for $i \in I$. Since $g_i|_{U_{ij}} g_j|_{U_{ij}}^{-1} = h_i|_{U_{ij}} h_j|_{U_{ij}}^{-1}$ we have

$$\rho|_{U_i}|_{U_{ij}} = h_i|_{U_{ij}}^{-1} g_i|_{U_{ij}} = h_j|_{U_{ij}}^{-1} g_j|_{U_{ij}} = \rho|_{U_j}|_{U_{ij}}$$

for all $i, j \in I$. Hence ρ is a well defined isomorphism. $\square$

Now let $\mathcal{L}$ and $\mathcal{K}$ be invertible $\mathcal{O}_X$-modules. If $\mathcal{F}$ is an $\mathcal{O}_X$-module then we define

$$^{\mathcal{L}}\mathcal{F}^{\mathcal{K}} = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}.$$

If X trivializes both $\mathcal{L}$ and $\mathcal{K}$, and $a \in \mathcal{L}(X)$, $b \in \mathcal{K}(X)$ are global generators for $\mathcal{L}$ and $\mathcal{K}$, then for all open $U \subset X$, every element of $^{\mathcal{L}}\mathcal{F}^{\mathcal{K}}(U)$ can be written in the form $a \otimes \alpha \otimes b$ for some $\alpha \in \mathcal{F}(U)$.

If U is an open subset of X which trivializes $\mathcal{L}$ and $a \in \mathcal{L}(U)$ then we write a^* for the element of $\mathcal{L}^{-1}(U)$ such that the canonical isomorphism $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^{-1}(U) \to \mathcal{O}_X(U)$ sends $a \otimes a^*$ to 1.

If $\mathcal{A}$ is an $\mathcal{O}_X$-rings then $^{\mathcal{L}}\mathcal{A}^{\mathcal{L}^{-1}}$ is also an $\mathcal{O}_X$-ring with multiplication defined as the composite of the canonical isomorphism $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^{-1} \to \mathcal{O}_X$ and the multiplication on $\mathcal{A}$.

If $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{O}_X$-rings, and $\mathcal{M}$ is an $\mathcal{A}$-$\mathcal{B}$-bimodule, then $^{\mathcal{L}}\mathcal{M}^{\mathcal{K}}$ is an $^{\mathcal{L}}\mathcal{A}^{\mathcal{L}^{-1}}$-$^{\mathcal{K}^{-1}}\mathcal{B}^{\mathcal{K}}$-bimodule, with left action defined as the composite of the canonical isomorphism $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^{-1} \to \mathcal{O}_X$ and the left action of $\mathcal{A}$ on $\mathcal{M}$, and the right action defined similarly. This construction is functorial in the sense that if $\mathcal{M} \to \mathcal{N}$ is a homomorphism of $\mathcal{A}$-$\mathcal{B}$-bimodules then $^{\mathcal{L}}\mathcal{M}^{\mathcal{K}} \to {}^{\mathcal{L}}\mathcal{N}^{\mathcal{K}}$ is a homomorphism of $^{\mathcal{L}}\mathcal{A}^{\mathcal{L}^{-1}}$-$^{\mathcal{K}^{-1}}\mathcal{B}^{\mathcal{K}}$-bimodules.

Proposition 6.1.6 *Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{O}_X$-ring, let $\mathcal{M}$ be an $\mathcal{A}$-$\mathcal{B}$ bimodule and let $\mathcal{N}$ be a $\mathcal{B}$-$\mathcal{A}$-bimodule. Let*

$$\Phi : \mathcal{M} \otimes_{\mathcal{B}} \mathcal{N} \to \mathcal{A}$$

be a homomorphism of $\mathcal{A}$-bimodules. Then the morphism

$$\Phi(\mathcal{L}, \mathcal{K}) : {}^{\mathcal{L}}\mathcal{M}^{\mathcal{K}} \otimes_{^{\mathcal{K}^{-1}}\mathcal{B}^{\mathcal{K}}} {}^{\mathcal{K}^{-1}}\mathcal{N}^{\mathcal{L}^{-1}} \to {}^{\mathcal{L}}\mathcal{A}^{\mathcal{L}^{-1}}$$

defined as the composite of the canonical isomorphism $\mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{K}^{-1} \to \mathcal{O}_X$ and Φ is a well defined homomorphism of $^{\mathcal{L}}\mathcal{A}^{\mathcal{L}^{-1}}$-bimodules.

Proof: Let

$$\psi : {}^{\mathcal{L}}\mathcal{M}^{\mathcal{K}} \otimes_{\mathcal{O}_X} {}^{\mathcal{K}^{-1}}\mathcal{N}^{\mathcal{L}^{-1}} \to {}^{\mathcal{L}}\mathcal{A}^{\mathcal{L}^{-1}}$$

be the homomorphism of ${}^{\mathcal{L}}\mathcal{A}^{\mathcal{L}^{-1}}$-bimodules defined as the composite of the canonical isomorphism $\mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{K}^{-1} \to \mathcal{O}_X$ and Φ. Then to prove that $\Phi(\mathcal{L}, \mathcal{K})$ is well defined it will be enough to show that for all open $U \subset X$, for all $\mu \in {}^{\mathcal{L}}\mathcal{M}^{\mathcal{K}}(U)$, $\nu \in {}^{\mathcal{K}^{-1}}\mathcal{N}^{\mathcal{L}^{-1}}(U)$, and $\beta \in {}^{\mathcal{K}^{-1}}\mathcal{B}^{\mathcal{K}}(U)$ we have $\psi(\mu\beta \otimes \nu) = \psi(\mu \otimes \beta\nu)$. We can assume without loss of generality that U trivializes $\mathcal{L}$ and $\mathcal{K}$, so that we have global generators $a \in \mathcal{L}(U)$ of $\mathcal{L}|_U$ and $b \in \mathcal{K}(U)$ of $\mathcal{K}|_U$. Then it will be equivalent to show that for all $\mu \in \mathcal{M}(U)$, $\nu \in \mathcal{N}(U)$, and $\beta \in \mathcal{B}(U)$ we have $\psi((a \otimes \mu\beta \otimes b) \otimes (b^* \otimes \nu \otimes a^*)) = \psi((a \otimes \mu \otimes b) \otimes (b^* \otimes \beta\nu \otimes a^*))$. Now we can calculate that

$$\begin{aligned} &\psi((a \otimes \mu\beta \otimes b) \otimes (b^* \otimes \nu \otimes a^*)) \\ = a \otimes \Phi(\mu\beta \otimes \nu) \otimes a^* &= a \otimes \Phi(\mu \otimes \beta\nu) \otimes a^* \\ &= \psi((a \otimes \mu \otimes b) \otimes (b^* \otimes \beta\nu \otimes a^*)), \end{aligned}$$

proving the claim. □

6.1.7 Twisted Morita Contexts

Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{O}_X$-rings over an S-scheme X, $\mathcal{M}$ a sheaf of $\mathcal{A}$-$\mathcal{B}$-bimodules, $\mathcal{N}$ a sheaf $\mathcal{B}$-$\mathcal{A}$-bimodules, $\Phi : \mathcal{M} \otimes_{\mathcal{B}} \mathcal{N} \to \mathcal{A}$ a morphism of $\mathcal{A}$-$\mathcal{A}$-bimodules, and $\Psi : \mathcal{N} \otimes_{\mathcal{A}} \mathcal{M} \to \mathcal{B}$ a morphism of $\mathcal{B}$-$\mathcal{B}$-bimodules, with the condition that for all open $U \subset X$, and all $m, m' \in \mathcal{M}(U)$ and $n, n' \in \mathcal{N}(U)$ we have

$$\Phi(U)(m \otimes n)m' = m\Psi(U)(n \otimes m') \text{ and } n\Phi(U)(m \otimes n') = \Psi(U)(n \otimes m)n'.$$

Then we can construct an $\mathcal{O}_X$-ring $\mathcal{C}$ over X from this data in the following manner: As a sheaf of Abelian groups we define that $\mathcal{C} = \mathcal{A} \oplus \mathcal{M} \oplus \mathcal{N} \oplus \mathcal{B}$. If $U \subset X$ is an open set, we write an element $(a, m, n, d) \in \mathcal{C}(U)$, where $a \in \mathcal{A}(U)$, $b \in \mathcal{B}(U)$, $m \in \mathcal{M}(U)$, and $n \in \mathcal{N}(U)$ in the form $\begin{bmatrix} a & m \\ n & b \end{bmatrix}$. Multiplication in $\mathcal{C}(U)$ is defined by the equation

$$\begin{bmatrix} a & m \\ n & b \end{bmatrix} \begin{bmatrix} a' & m' \\ n' & b' \end{bmatrix} = \begin{bmatrix} aa' + \Phi(m \otimes n') & am' + mb' \\ na' + bn' & \Psi(n \otimes m') + bb' \end{bmatrix}.$$

We write

$$\mathcal{C} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{bmatrix}$$

when the maps Φ and Ψ are understood. We call $\mathcal{C}$ the *Morita context* over X defined by $\mathcal{A}$, $\mathcal{M}$, $\mathcal{N}$, $\mathcal{B}$, Ψ, and Φ.

Now suppose that $\mathcal{L}$ and $\mathcal{K}$ are invertible $\mathcal{O}_X$-modules. Then we define a Morita context

$$\mathcal{C}(\mathcal{L}, \mathcal{K}) = \begin{bmatrix} {}^{\mathcal{L}}\mathcal{A}^{\mathcal{L}^{-1}} & {}^{\mathcal{L}}\mathcal{M}^{\mathcal{K}^{-1}} \\ {}^{\mathcal{K}}\mathcal{N}^{\mathcal{L}^{-1}} & {}^{\mathcal{K}}\mathcal{B}^{\mathcal{K}^{-1}} \end{bmatrix}$$

with bimodule morphisms

$$\Phi(\mathcal{L}, \mathcal{K}) : {}^{\mathcal{L}}\mathcal{M}^{\mathcal{K}^{-1}} \otimes_{{}^{\mathcal{K}}\mathcal{B}^{\mathcal{K}^{-1}}} {}^{\mathcal{K}}\mathcal{N}^{\mathcal{L}^{-1}} \to {}^{\mathcal{L}}\mathcal{A}^{\mathcal{L}^{-1}}$$

and

$$\Psi(\mathcal{L}, \mathcal{K}) : {}^{\mathcal{K}}\mathcal{N}^{\mathcal{L}^{-1}} \otimes_{{}^{\mathcal{L}}\mathcal{A}^{\mathcal{L}^{-1}}} {}^{\mathcal{L}}\mathcal{M}^{\mathcal{K}^{-1}} \to {}^{\mathcal{K}}\mathcal{B}^{\mathcal{K}^{-1}}$$

defined as in proposition 6.1.6.

Proposition 6.1.7 *1. Assume that $a \in \mathcal{L}(X)$, $b \in \mathcal{K}(X)$ are global generators of $\mathcal{L}$ and $\mathcal{K}$ over $\mathcal{O}_X$. Then there exists an isomorphism of $\mathcal{O}_X$-rings*

$$\rho_X(a, b) : \mathcal{C}(\mathcal{L}, \mathcal{K}) \to \mathcal{C} \ ; \ \begin{bmatrix} a \otimes \alpha \otimes a^* & a \otimes \mu \otimes b^* \\ b \otimes \nu \otimes a^* & b \otimes \beta \otimes b^* \end{bmatrix} \mapsto \begin{bmatrix} \alpha & \mu \\ \nu & \beta \end{bmatrix}.$$

2. *Choose open U, $V \subset X$ and suppose that U and V trivialize $\mathcal{L}$ and $\mathcal{K}$. Let a_U and a_V be global generators for $\mathcal{L}|_U$ and $\mathcal{L}|_V$ respectively, and let b_U and b_V be global generators for $\mathcal{K}|_U$ and $\mathcal{K}|_V$ respectively. Let $\varepsilon, \theta \in \mathcal{O}_X(U \cap V)^\times$ such that $a_V|_{U\cap V} = \varepsilon a_U|_{U\cap V}$ and $b_V|_{U\cap V} = \theta b_U|_{U\cap V}$. Let $x = \begin{bmatrix} \varepsilon & 0 \\ 0 & \theta \end{bmatrix} \in \mathcal{C}(U \cap V)$. Let $\rho_U = \rho_U(a_U, b_U)$ and $\rho_V = \rho_V(a_V, b_V)$ as defined in part 1. Then $\rho_U|_{U\cap V}\rho_V|_{U\cap V}^{-1}$ sends c to xcx^{-1} for all $c \in \mathcal{C}(W)$ and all open $W \subset U \cap V$.*

Proof:

1. $\rho_X(a, b)$ is clearly an isomorphism of $\mathcal{O}_X$-modules, so we only need to check multiplicativity. We will drop tensor symbols for ease of notation, so that e.g. $a \otimes \alpha \otimes a^*$ is written $a\alpha a^*$. Let U be an open subset of $\mathcal{X}$, let $\alpha, \alpha' \in \mathcal{A}(U)$, $\beta, \beta' \in \mathcal{B}(U)$, $\mu, \mu' \in \mathcal{M}(U)$, and $\nu, \nu' \in \mathcal{N}(U)$. Then

$$\rho_X(a,b)\left(\begin{bmatrix} a\alpha a^* & a\mu b^* \\ b\nu a^* & b\beta b^* \end{bmatrix}\begin{bmatrix} a\alpha' a^* & a\mu' b^* \\ b\nu' a^* & b\beta' b^* \end{bmatrix}\right)$$

$$=\rho_X(a,b)\left(\begin{bmatrix} a\alpha\alpha' a^* + \Phi(\mathcal{L},\mathcal{K})(a\mu b^* \otimes b\nu' a^*) & a\alpha\mu' b^* + a\mu\beta' b^* \\ b\nu\beta' a^* + b\beta\nu' a^* & b\beta\beta' b^* + \Psi(\mathcal{L},\mathcal{K})(b\nu a^* \otimes a\mu' b^*) \end{bmatrix}\right)$$

$$=\rho_X(a,b)\left(\begin{bmatrix} a(\alpha\alpha' + \Phi(\mu\otimes\nu'))a^* & a(\alpha\mu' + \mu\beta')b^* \\ b(\nu\beta' + \beta\nu')a^* & b(\beta\beta' + \Psi(\nu\otimes\mu'))b^* \end{bmatrix}\right)$$

$$= \begin{bmatrix} \alpha\alpha' + \Phi(\mu\otimes\nu') & \alpha\mu' + \mu\beta' \\ \nu\beta' + \beta\nu' & \beta\beta' + \Psi(\nu\otimes\mu') \end{bmatrix} = \begin{bmatrix} \alpha & \mu \\ \nu & \beta \end{bmatrix}\begin{bmatrix} \alpha' & \mu' \\ \nu' & \beta' \end{bmatrix}.$$

Hence $\rho_X(a, b)$ is multiplicative, proving the claim.

2. Let W be an open subset of $U \cap V$ and let $c = \begin{bmatrix} \alpha & \mu \\ \nu & \beta \end{bmatrix} \in \mathcal{C}(W)$. Then

$$\rho_V(W)^{-1}(c) = \begin{bmatrix} a_V \alpha a_V^* & a_V \mu b_V^* \\ b_V \nu a_V^* & b_V \beta b_V^* \end{bmatrix} = \begin{bmatrix} a_U \varepsilon \alpha \varepsilon^{-1} a_U^* & a_U \varepsilon \mu \theta^{-1} b_U^* \\ b_U \theta \nu \varepsilon^{-1} a_U^* & b_U \theta \beta \theta^{-1} b_U^* \end{bmatrix}.$$

Hence

$$\rho_U(W)\rho_V(W)^{-1}(c) = \begin{bmatrix} \varepsilon \alpha \varepsilon^{-1} & \varepsilon \mu \theta^{-1} \\ \theta \nu \varepsilon^{-1} & \theta \beta \theta^{-1} \end{bmatrix} = xcx^{-1},$$

proving the claim.

□

6.1.8 The Main Theorem

Let $\mathcal{C} = M_p(\widehat{\mathcal{D}_Y^{[0]}})$, let $e = \tilde{e}_{(p-1)(p-1)} \in \mathcal{C}(Y)$ and let $f = (1 - \tilde{e}_{(p-1)(p-1)})$, so that we have

$$\mathcal{C} = \begin{bmatrix} f\mathcal{C}f & f\mathcal{C}e \\ e\mathcal{C}f & e\mathcal{C}e \end{bmatrix},$$

and throughout this section we will consider $\mathcal{C}$ as a Morita context with respect to this structure. Let $\mathcal{L} = \mathcal{O}_Y(1)$, the Serre twisting sheaf. Given an $\mathcal{O}_Y$-module $\mathcal{F}$, for $i, j \in \mathbb{Z}$ we will write $(i)\mathcal{F}(j)$ for ${}^{\mathcal{L}^{\otimes i}}\mathcal{F}^{\mathcal{L}^{\otimes j}}$, and ${}^{(i)}\mathcal{F}$ for $(i)\mathcal{F}(-i)$. We will prove the following theorem

Theorem 6.1.8 *Let*

$$\mathcal{M} = \mathcal{C}(\mathcal{L}^{-1}, \mathcal{O}_Y) = \begin{bmatrix} {}^{(-1)}f\mathcal{C}f & (-1)f\mathcal{C}e \\ e\mathcal{C}f(1) & e\mathcal{C}e \end{bmatrix}.$$

Then there exists an isomorphism of $\mathcal{O}_Y$-rings $F_*\widehat{\mathcal{D}_X^{[1]}} \to \mathcal{M}$.

In order to do so we will need the following proposition.

Proposition 6.1.8 *$F_*\widehat{\mathcal{D}_X^{[1]}}|_{Y_0}$ is an $M_p(\widehat{\mathcal{D}_Y^{[0]}})$-twist, with local isomorphisms of $\mathcal{O}_Y$-rings*

$$\theta_0 : F_*\widehat{\mathcal{D}_X^{[1]}}|_{Y_0} \to M_p(\widehat{\mathcal{D}_Y^{[0]}})|_{Y_0} \ ; \delta_0 \mapsto \partial_\tau \textit{ and } e_{ij} \mapsto \tilde{e}_{ij} \textit{ for } 0 \leq i,j \leq p-1$$

and

$$\theta_\infty : F_*\widehat{\mathcal{D}_X^{[1]}}|_{Y_\infty} \to M_p(\widehat{\mathcal{D}_Y^{[0]}})|_{Y_\infty} \ ; \delta_\infty \mapsto \partial_{\tau^{-1}} \textit{ and } f_{ij} \mapsto \tilde{e}_{i^*j^*} \textit{ for } 0 \leq i,j \leq p-1.$$

Let $\chi = e + \tau^{-1}f$. Then $\theta_0|_{Y_{0\infty}}\theta_\infty|_{Y_{0\infty}}^{-1} : M_p(\widehat{\mathcal{D}_Y^{[0]}})|_{Y_{0\infty}} \to M_p(\widehat{\mathcal{D}_Y^{[0]}})|_{Y_{0\infty}}$ sends $\alpha \mapsto \chi\alpha\chi^{-1}$.

Proof:

Applying theorem 4.3.1 to Y_0 and Y_∞, θ_0 and θ_∞ are isomorphisms of $\mathcal{O}_Y$-rings Hence from the definitions in section 6.1.6, $F_*\widehat{\mathcal{D}_X^{[1]}}$ is an $M_p(\widehat{\mathcal{D}_Y^{[0]}})$-twist.

So we only need to prove that $\theta_0|_{Y_{0\infty}}\theta_\infty|_{Y_{0\infty}}^{-1} : M_p(\widehat{\mathcal{D}_Y^{[0]}})|_{Y_{0\infty}} \to M_p(\widehat{\mathcal{D}_Y^{[0]}})|_{Y_{0\infty}}$ is defined by $\alpha \mapsto \chi\alpha\chi^{-1}$.

Let U be an open subset of $Y_{0\infty}$. We will abuse notation and write $\theta_\dagger$ for $\theta_\dagger(U)$ for $\dagger \in \{0,\infty\}$. Then it will suffice to prove that for all $\alpha \in M_p(\widehat{\mathcal{D}_Y^{[0]}})(U)$ we have $\theta_0\theta_\infty^{-1}(\alpha) = \chi\alpha\chi^{-1}$. Since $\theta_0\theta_\infty^{-1}$ is a morphism of $\mathcal{O}(U)$-rings, we have $\theta_0\theta_\infty^{-1}(\tau) = \tau = \chi\tau\chi^{-1}$, so it will suffice to prove that $\theta_0\theta_\infty^{-1}(\partial_{\tau^{-1}}) = \chi\partial_{\tau^{-1}}\chi^{-1}$ and $\theta_0\theta_\infty^{-1}(\tilde{e}_{ij}) = \chi\tilde{e}_{ij}\chi^{-1}$ for $0 \leq i,j \leq p-1$.

Let $\mu = e_{(p-1)(p-1)} + \tau^{-1}(1 - e_{(p-1)(p-1)}) = \theta_0^{-1}(\chi) \in F_*\widehat{\mathcal{D}_X^{[1]}}(U)$. By proposition 6.1.5 we have $\theta_\infty^{-1}(\partial_{\tau^{-1}}) = \delta_\infty|_U = \mu(-\tau^2\delta_0)|_U\mu^{-1}$, so

$$\theta_0\theta_\infty^{-1}(\partial_{\tau^{-1}}) = \chi(-\tau^2\partial_\tau)\chi^{-1} = \chi\partial_{\tau^{-1}}\chi^{-1}.$$

By proposition 6.1.4 we have $\theta_\infty^{-1}(\tilde{e}_{ij}) = f_{i^*j^*}|_U = \mu e_{ij}|_U\mu^{-1}$, so

$$\theta_0\theta_\infty^{-1}(\tilde{e}_{ij}) = \chi\tilde{e}_{ij}\chi^{-1},$$

proving the claim. □

Now we are ready to prove the main theorem. Proof:[Proof of the Main Theorem] For $\dagger \in \{0, \infty\}$ choose a generator $a_\dagger \in \mathcal{L}(X_\dagger)$ for $\mathcal{L}|_{X_\dagger}$ such that $a_\infty = \tau^{-1} a_0$, and let $\rho_\dagger = \rho_{X_\dagger}(a_\dagger, 1)$ be defined as in proposition 6.1.6(1).

Then by proposition 6.1.7(1) $\rho_0|_{X_{0\infty}} \rho_\infty|_{X_{0\infty}}^{-1}$ sends

$$\alpha \to \begin{bmatrix} \tau^{-1} & 0 \\ 0 & 1 \end{bmatrix} \alpha \begin{bmatrix} \tau & 0 \\ 0 & 1 \end{bmatrix}$$

for all $\alpha \in \mathcal{C}(U)$ and all open $U \subset Y_{0\infty}$. Of course, $\begin{bmatrix} \tau^{-1} & 0 \\ 0 & 1 \end{bmatrix} = f\tau^{-1} + e$, so by proposition 6.1.8 $\rho_0|_{X_{0\infty}} \rho_\infty|_{X_{0\infty}}^{-1} = \theta_0|_{X_{0\infty}} \theta_\infty|_{X_{0\infty}}^{-1}$. Hence, by lemma 6.1.6 there exists an isomorphism of $\mathcal{O}_Y$-rings $\Phi : F_* \widehat{\mathcal{D}_X^{[1]}} \to \mathcal{M}$ such that $\Phi|_{Y_\dagger} = \rho_\dagger^{-1} \theta_\dagger$ for $\dagger \in \{0, \infty\}$. □ Now we can prove the theorem 1.6 from the introduction.

Corollary 6.1.8 *There exists an isomorphism of $\mathcal{O}_Y$-rings*

$$F_* \widehat{\mathcal{D}_X^{[1]}} \to \begin{bmatrix} {}^{(-1)}M_{p-1}(\widehat{\mathcal{D}_Y^{[0]}}) & ((-1)\widehat{\mathcal{D}_Y^{[0]}})^{p-1} \\ (\widehat{\mathcal{D}_Y^{[0]}}(1))^{p-1} & \widehat{\mathcal{D}_Y^{[0]}} \end{bmatrix}.$$

Proof: This just follows from the fact that there are isomorphisms of $\mathcal{O}_Y$-rings $f\mathcal{C}f \to M_{p-1}(\widehat{\mathcal{D}_Y^{[0]}})$ and $e\mathcal{C}e \to \widehat{\mathcal{D}_Y^{[0]}}$, as well as isomorphisms of $f\mathcal{C}f$-$e\mathcal{C}e$-bimodules $f\mathcal{C}e \to (\widehat{\mathcal{D}_Y^{[0]}})^{p-1}$ and $e\mathcal{C}e$-$f\mathcal{C}f$-bimodules $e\mathcal{C}f \to (\widehat{\mathcal{D}_Y^{[0]}})^{p-1}$, and that $(n)\mathcal{F}^i(m) = ((n)\mathcal{F}(m))^i$ for all $\mathcal{O}_Y$-modules $\mathcal{F}$. □

6.2 Global Sections of Twists

6.2.1 Definitions

Set $\mathcal{D} = \widehat{\mathcal{D}_Y^{[0]}}$. $\mathcal{L} = \mathcal{O}_Y(1)$ and for $\dagger \in \{0, \infty\}$ let $\varepsilon_\dagger : \mathcal{L}|_{Y_\dagger} \to \mathcal{O}_Y|_{Y_\dagger}$ be defined so that $\varepsilon_0|_{Y_{0\infty}}\varepsilon_\infty|_{Y_{0\infty}}^{-1}$ sends $1 \mapsto \tau$. Let $\mathcal{C}$, e, and f be defined as in section 6.1.8, so that

$$\mathcal{C} = M_p(\mathcal{D}) = \begin{bmatrix} eM_p(\mathcal{D})e & eM_p(\mathcal{D})f \\ fM_p(\mathcal{D})e & fM_p(\mathcal{D})f \end{bmatrix}.$$

For $n \in \mathbb{Z}$ set $\mathcal{M}^n = \mathcal{C}(\mathcal{L}^{\otimes n-1}, \mathcal{L}^{\otimes n})$. For $\dagger \in \{0, \infty\}$, let $a_\dagger$ be a global generator for $\mathcal{L}^{\otimes n-1}$ such that $\varepsilon_\dagger^{\otimes n-1}(a_\dagger) = 1$ and let $b_\dagger$ be a global generator for $\mathcal{L}^{\otimes n}|_{Y_\dagger}$ such that $\varepsilon_\dagger^{\otimes n}(b_\dagger) = 1$. Let $\rho_\dagger = \rho_{Y_\dagger}(a, b)$ be defined as in proposition 6.1.7.

For $n \in \mathbb{Z}$ set

$$D^n = \{\alpha \in \mathcal{D}(Y_0) : \tau^{-n}\alpha|_{Y_{0\infty}}\tau^n \in \operatorname{res}_{Y_{0\infty}}^{Y_\infty}\mathcal{D}(Y_\infty)\}.$$

and

$$R^n = \{\alpha \in \mathcal{D}(Y_0) : \tau^{-(n-1)}\alpha|_{Y_{0\infty}}\tau^n \in \operatorname{res}_{Y_{0\infty}}^{Y_\infty}\mathcal{D}(Y_\infty)\}$$

$$L^n = \{\alpha \in \mathcal{D}(Y_0) : \tau^{-n}\alpha|_{Y_{0\infty}}\tau^{n-1} \in \operatorname{res}_{Y_{0\infty}}^{Y_\infty}\mathcal{D}(Y_\infty)\}.$$

Proposition 6.2.1 *The image of the R-algebra embedding $\rho_0(Y_0)\operatorname{res}_{Y_0}^Y : \mathcal{M}^n(Y) \to \mathcal{C}(Y_0)$ is the set*

$$A = \begin{bmatrix} M_{p-1}(D^{n-1}) & (R^n)^{p-1} \\ (L^n)^{p-1} & D^n \end{bmatrix}.$$

Proof: From the Čech complex we know that

$$\operatorname{res}_{Y_0}^Y(\mathcal{M}^n(Y)) = \{\alpha \in \mathcal{M}^n(Y_0) : \alpha|_{Y_{0\infty}} \in \operatorname{res}_{Y_{0\infty}}^{Y_\infty}\mathcal{M}^n(Y_\infty)\}.$$

Hence, noting that $\rho_0(Y_{0\infty})$ is an isomorphism, we have

$$\rho_0(Y_0)\operatorname{res}_{Y_0}^Y(\mathcal{M}^n(Y)) = \{\alpha \in \mathcal{C}(Y_0) : \alpha|_{Y_{0\infty}} \in \rho_0(Y_{0\infty})\operatorname{res}_{Y_{0\infty}}^{Y_\infty}\mathcal{M}^n(Y_\infty)\}.$$

Noting that $\rho_\infty(Y_{0\infty})(\mathrm{res}^{Y_\infty}_{Y_{0\infty}}\mathcal{M}^n(Y_\infty)) = \mathrm{res}^{Y_\infty}_{Y_{0\infty}}\mathcal{C}(Y_\infty)$ we conclude that

$$\rho_0(Y_0)\mathrm{res}^{Y}_{Y_0}(\mathcal{M}^n(Y)) = \{\alpha \in \mathcal{C}(Y_0) : \rho_\infty(Y_{0\infty})\rho_0(Y_{0\infty})^{-1}(\alpha|_{Y_{0\infty}}) \in \mathrm{res}^{Y_\infty}_{Y_{0\infty}}\mathcal{C}(Y_\infty)\}.$$

Now, by proposition 6.1.7(2), $\phi = \rho_\infty(Y_{0\infty})\rho_0(Y_{0\infty})^{-1}$ sends $\alpha \mapsto \begin{bmatrix} \tau^{-(n-1)} & 0 \\ 0 & \tau^{-n} \end{bmatrix} \alpha \begin{bmatrix} \tau^{n-1} & 0 \\ 0 & \tau^{n} \end{bmatrix}$ for all $\alpha \in \mathcal{C}(Y_{0\infty})$. Now, let $\{\tilde{e}_i\}_{0\le i\le p-1}$ be an orthogonal set of idempotents for $\mathcal{C}(Y_0)$ such that $\sum_{i=0}^{p-2}\tilde{e}_i = f$ and $\tilde{e}_{p-1} = e$. Let $\alpha \in \tilde{e}_i\mathcal{C}(Y_0)\tilde{e}_j$. We identify α with an element of $\mathcal{D}(Y_0)$ and we find

$$\phi(\alpha|_{Y_{0\infty}}) = \begin{cases} \tau^{-(n-1)}\alpha|_{Y_{0\infty}}\tau^{n-1} & \text{if } 0 \le i,j \le p-2 \\ \tau^{-(n-1)}\alpha|_{Y_{0\infty}}\tau^{n} & \text{if } 0 \le i \le p-2 \text{ and } j = p-1 \\ \tau^{-n}\alpha|_{Y_{0\infty}}\tau^{n-1} & \text{if } 0 \le j \le p-2 \text{ and } i = p-1 \\ \tau^{-n}\alpha|_{Y_{0\infty}}\tau^{n} & \text{if } i,j = p-1 \end{cases}$$

Proving the claim. □

6.2.2 Beilinson Bernstein for $\mathfrak{sl}_2$

In this section we assume the $\mathrm{char}(\kappa) \neq 2$. Let $\mathfrak{g} = \mathfrak{sl}(2,R)$, the R-Lie algebra which is free as an R-module on the basis $\{E,F,H\}$ where $[E,F] = H$, $[H,E] = 2E$, and $[H,F] = -2F$. We set $U = U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$. Set $\Omega = 4FE + H^2 + 2H$ be the Casimir element of U. Ω is a central element of U. For $\lambda \in R$, set $U_\lambda = U/U(\Omega - \lambda^2 - 2\lambda)$. Set $U_\kappa = U/\pi U$, and $U_{\lambda,\kappa} = U_\lambda/\pi U_\lambda$. U_κ carries the PBW filtration, and $U_{\lambda,\kappa}$ carries the quotient of the PBW filtration P_*. Their is an isomorphism of κ-algebras

$$\mathrm{gr}(U_\kappa) \to \mathrm{Sym}(\mathfrak{g}/\pi\mathfrak{g}) \ ; \ e + P_0(U_\kappa) \mapsto \tilde{e} \ ; \ h + P_0(U_\kappa) \mapsto \tilde{h} \ ; \ f + P_0(U_\kappa) \mapsto \tilde{f},$$

where $\tilde{e} = E + \pi\mathfrak{g}$, $\tilde{f} = F + \pi\mathfrak{g}$ and $\tilde{h} = H + \pi\mathfrak{g}$. Set $\overline{\Omega} = \Omega + \pi U$ and set $\tilde{\Omega} = \overline{\Omega} + P_1(U)$. Then the above isomorphism sends $\tilde{\Omega}$ to $4\tilde{f}\tilde{e} + \tilde{h}^2$, and the surjection $\mathrm{gr}(U_\kappa) \to \mathrm{gr}(U_{\lambda,\kappa})$ has kernel $\tilde{\Omega}\mathrm{gr}(U_\kappa)$.

Set $\widehat{U} = \varprojlim_{n\in\mathbb{N}} U/\pi^n U$, and for $\lambda \in R$, $\widehat{U_\lambda} = \varprojlim_{n\in\mathbb{N}} U_\lambda/\pi^n U_\lambda$. The diagonal homomorphism $U \to \widehat{U_\lambda}$ lifts to a surjection $\widehat{U} \to \widehat{U_\lambda}$ with kernel $\widehat{U}(\Omega - \lambda^2 - 2\lambda)$ [2, Proposition 6.10]. The graded ring of $\widehat{U}$ with respect to the π-adic filtration is isomorphic to $U_\kappa(\mathfrak{g})[s]$, a left and right Noetherian ring, so by proposition 2.2.7(2), $\widehat{U}$ is left and right Zariskian, so that for each $\lambda \in R$, $\widehat{U_\lambda}$ is complete with respect to its π-adic filtration.

Fix $n \in \mathbb{Z}$. Write e_n, f_n, h_n for the images of E, F, and H in $\widehat{U}_n$ respectively. If no confusion will arise, we simply write e, f and h for e_n, f_n, and h_n respectively.

Throughout this section we will be using various graded arguments, so set $\overline{\mathcal{D}} = \mathcal{D}/\pi\mathcal{D}$. If U is an open subset of Y and $\alpha \in \mathcal{D}(U)$, then we will write $\overline{\alpha}$ for $\alpha + \pi\mathcal{D}(U) \in \overline{\mathcal{D}}(U)$, or just α when no confusion will arise.

$\overline{\mathcal{D}}$ is a filtered $\mathcal{O}_Y$-ring, where for an open $U \subset Y_0$ the filtration F_* on $\overline{\mathcal{D}}(U)$ is defined so that $F_i(\overline{\mathcal{D}}(U))$ is the set of elements of $\overline{\mathcal{D}}(U)$ of ∂_τ-degree less than or equal to i. For a general open $U \subset Y$ we define the filtration on $\overline{\mathcal{D}}(U)$ to be the subspace filtration induced by the embedding $\mathrm{res}^U_{U\cap Y_0} : \overline{\mathcal{D}}(U) \to \overline{\mathcal{D}}(U \cap Y_0)$. Let $\mathcal{G}$ be the sheaf of graded rings associated to F_* on Y. Then $\mathcal{G}|_{Y_0}$ is generated over $\mathcal{O}_Y/\pi\mathcal{O}_Y|_{Y_0}$ by $\tilde{\partial}_\tau = \partial_\tau + F_0(\mathcal{D}(Y_0))$, subject only to the relation $\tilde{\partial}_\tau\tau = \tau\tilde{\partial}_\tau$. Similarly $\mathcal{G}|_{Y_\infty}$ is generated over $\mathcal{O}_Y/\pi\mathcal{O}_Y|_{Y_\infty}$ by $\tilde{\partial}_{\tau^{-1}} = \partial_{\tau^{-1}} + F_0(\mathcal{D}(Y_\infty))$. Of course $\tilde{\partial}_{\tau^{-1}}|_{Y_{0\infty}} = -\tau^2\tilde{\partial}_\tau|_{Y_{0\infty}}$.

Lemma 6.2.2 *1. Let X be a copy of $\mathbb{A}^1_R$ and choose a coordinate t for X. Let ∂_t be the operator dual to $dt \in \Omega^1_X(X)$. Then there is an R-algebra embedding*

$$\gamma_X(t,n) : \widehat{U_n} \to \widehat{\mathcal{D}^{[0]}}(X) \ ; \ e \mapsto \partial_t \ ; \ h \mapsto t^n(-2t\partial_t)t^{-n} - n \ ; \ f \mapsto t^n(-t^2\partial_t)t^{-n}$$

2. Set $\gamma_0 = \gamma_{Y_0}(\tau, n)$ and set $\gamma_\infty = \gamma_{Y_\infty}(\tau^{-1}, n)$. Let $\omega : \widehat{U_n} \to \widehat{U_n}$ by the R-algebra *isomorphism that sends $e \mapsto f$, $f \mapsto e$, and $h \mapsto -h$. Define*

$$\psi : \widehat{U_n} \to \mathcal{D}(Y_0) \oplus \mathcal{D}(Y_\infty) \ ; u \mapsto (\gamma_0(u), \gamma_\infty(\omega(u))$$

and

$$\phi : \mathcal{D}(Y_0) \oplus \mathcal{D}(Y_\infty) \to \mathcal{D}(Y_{0\infty}) \ ; (\alpha, \beta) \mapsto \alpha|_{Y_{0\infty}} - \tau^n \beta|_{Y_{0\infty}} \tau^{-n}.$$

Then $\psi(\widehat{U_n}) \subset \ker(\phi)$.

Proof:

1. R-algebra homomorphism $\gamma : U_n \to \mathcal{D}^{[0]}(X)$ to be the restriction of $\gamma_X(t,n)$ to U_n. Using the fact that $t^n(-2t\partial_t)t^{-n} - n = -2t\partial_t + n$ and $t^n(-t^2\partial_t)t^{-n} = -t^2(\partial_t - n)$, we can calculate that

   (a)
   $$[\gamma(e), \gamma(f)] = [\partial_t, -t(t\partial_t - n)] = -2t\partial_t + n = \gamma(h) = \gamma([e,f])$$

   (b)
   $$[\gamma(h), \gamma(e)] = [-2t\partial_t + n, \partial_t] = 2\partial_t = 2e = \gamma([h,e]).$$

   (c)
   $$[\gamma(h), \gamma(f)] = [-2t\partial_t + n, -t(t\partial_t - n)]$$
   $$= 2\left(t\left[\partial_t, t^2\partial_t - nt\right] + \left[t, t^2\partial_t - nt\right]\partial_t\right)$$
   $$= 2\left(2t^2\partial_t - nt - t^2\partial_t\right) = -2f = \gamma([h,f])$$

   so that γ in an R-algebra homomorpism, and we have an R-algebra homomorphism

   $$\widehat{\gamma} : \widehat{U_n} \to \widehat{\mathcal{D}^{[0]}}(X).$$

To see that it is an embedding, by proposition 2.2.6(2) it will be enough to show that the associated sequence

$$0 \to \widehat{U_n}/\pi\widehat{U_n} \xrightarrow{\overline{\gamma}} \widehat{\mathcal{D}^{[0]}}(X)/\pi\widehat{\mathcal{D}^{[0]}}(X) = \overline{\mathcal{D}}(X)$$

is exact. $\widehat{U_n}/\pi\widehat{U_n}$ is isomorphic to $U_n/\pi U_n = U_{n,\kappa}$, and $\overline{\gamma}$ is filtered if $U_{n,\kappa}$ is equipped with the filtration P_* and $\overline{\mathcal{D}}(X)$ is equipped with the filtration F_*. Then to show the above sequence is exact it will be enough to show that the associated graded sequence

$$0 \to \mathrm{gr}(U_{n,\kappa}) \to \mathrm{gr}(\overline{\mathcal{D}}(X))$$

is exact. $\mathrm{gr}(\widehat{\mathcal{D}^{[0]}}(X))$ is generated over κ by t and $\tilde{\partial}_t = \partial_t + F_0(\overline{\mathcal{D}}(X))$, subject only to the relation $\left[\tilde{\partial}_t, t\right] = 0$. We identify $\mathrm{gr}(U_\kappa)$ with $A = \mathrm{Sym}(\mathfrak{g}/\pi\mathfrak{g})$ and let $\tilde{e}$, $\tilde{f}$ and $\tilde{h}$ be defined as above, so that the natural surjection $A \to \mathrm{gr}(U_{n,\kappa})$ has kernel $\tilde{\Omega}A = (4\tilde{f}\tilde{e} + \tilde{h}^2)A$, so we just need to show that the κ-algebra homomorphism

$$\tilde{\gamma} : A \to \kappa\left[t, \tilde{\partial}_t\right] \ ; \tilde{e} \mapsto \tilde{\partial}_t \ ; \tilde{h} \mapsto -2t\tilde{\partial}_t \ ; f \mapsto -t^2\tilde{\partial}_t$$

has kernel $(4\tilde{f}\tilde{e} + \tilde{h}^2)A$. If we localize A at $\tilde{e}$ and $\kappa\left[t, \tilde{\partial}_t\right]$ at $\tilde{\partial}_t$ then since $\tilde{\gamma}(\tilde{e}) = \tilde{\partial}_t$ we can extend $\tilde{\gamma}$ to a homomorphism $A_{\tilde{e}} \to \kappa\left[t, \tilde{\partial}_t\right]_{\tilde{\partial}_t}$. Now, since $(\tilde{f} + 4\tilde{h}^2\tilde{e}^{-1})A = (4\tilde{f}\tilde{e}+\tilde{h}^2)A$, we have an isomorphism $\kappa\left[\tilde{e}, \tilde{h}\right]_{\tilde{e}} \to A_{\tilde{e}}/(4\tilde{f}\tilde{e}+\tilde{h}^2)A$, so it will be enough to show that the induced map $\tilde{\gamma} : \kappa\left[\tilde{e}, \tilde{h}\right]_{\tilde{e}} \to \kappa\left[t, \tilde{\partial}_t\right]_{\tilde{\partial}_t}$; $\tilde{e} \mapsto \tilde{\partial}_\tau$; $\tilde{h} \mapsto -2\tau\tilde{\partial}_\tau$ is an embedding. In fact, it is an isomorphism with inverse sending $t \mapsto -\frac{1}{2}\tilde{h}\tilde{e}^{-1}$ and $\tilde{\partial}_t \mapsto e$.

2. It will be enough to show that $\psi(e)$, $\psi(f)$ and $\psi(h) \in \ker(\phi)$. Noting that $\partial_\tau|_{Y_{0\infty}} = -\tau^{-1}\partial_{\tau^{-1}}|_{Y_{0\infty}}$, we can see that

$$\phi\psi(e) = \phi(\gamma_0(e), \gamma_\infty(f)) = \phi(\partial_\tau, \tau^{-n}(-\tau^{-2}\partial_{\tau^{-1}})\tau^n) = \partial_\tau|_{Y_{0\infty}} - (-\tau^{-2}\partial_{\tau^{-1}}|_{Y_{0\infty}}) = 0,$$

so $\psi(e) \in \ker(\phi)$. Similarly

$$\phi\psi(f) = \phi(\gamma_0(f), \gamma_\infty(e)) = \phi(\tau^n(-\tau^2\partial_\tau)\tau^{-n}, \partial_{\tau^{-1}})$$

$$= \tau^n(-\tau^2\partial_\tau|_{Y_{0\infty}} - \partial_{\tau^{-1}}|_{Y_{0\infty}})\tau^{-n} = 0,$$

and finally, noting that $\tau^n(-2\tau\partial_\tau)\tau^{-n} - n = -2\tau\partial_\tau + n$, we have

$$\phi\psi(h) = \phi(\gamma_0(h), \gamma_\infty(-h)) = \phi(-2\tau\partial_\tau + n, \tau^{-n}(2\tau^{-1}\partial_{\tau^{-1}})\tau^n + n) =$$

$$-2\tau\partial_\tau|_{Y_{0\infty}} - 2\tau^{-1}\partial_{\tau^{-1}}|_{Y_{0\infty}} = 0.$$

Hence $\psi(\widehat{U_n}) \subset \ker(\phi)$.

□

Theorem 6.2.2 *The image of the R-algebra embedding* $\theta_n = \gamma_{Y_0}(\tau, n) : \widehat{U_n} \to \mathcal{D}(Y_0)$ *is* D^n.

Proof: Let ϕ and ψ be defined as in lemma 6.2.2(2). Let $p : \mathcal{D}(Y_0) \oplus \mathcal{D}(Y_\infty) \to \mathcal{D}(Y_0)$ be the projection map $(\alpha, \beta) \mapsto \alpha$. Since the restriction maps $\mathcal{D}(Y_\dagger) \to \mathcal{D}(Y_{0\infty})$ are embeddings for $\dagger \in \{0, \infty\}$, we can see that the restriction of p to $\ker(\phi)$ is an isomorphism onto D^n. By lemma 6.2.2(1) we know that ψ is an embedding, so if we can show that $\psi(\widehat{U_n}) = \ker(\phi)$ then $p\psi = \theta_n$ is an isomorphism from $\widehat{U_n}$ to D^n. By lemma 6.2.2(2), $\psi(\widehat{U_n}) \subset \ker(\phi)$, so it will be enough to prove that $\ker(\phi) \subset \psi(\widehat{U_n})$.

By lemma 2.2.6(2) it will be enough to show that the induced sequence

$$U_{n,\kappa} \xrightarrow{\overline{\psi}} \overline{\mathcal{D}}(Y_0) \oplus \overline{\mathcal{D}}(Y_\infty) \xrightarrow{\overline{\phi}} \overline{\mathcal{D}}(Y_{0\infty})$$

is exact. Equivalently, we can show that the sequence

$$U_\kappa \xrightarrow{\overline{\psi}r} \overline{\mathcal{D}}(Y_0) \oplus \overline{\mathcal{D}}(Y_\infty) \xrightarrow{\overline{\phi}} \overline{\mathcal{D}}(Y_{0\infty})$$

is exact, where r is the projection $U_\kappa \to U_{n,\kappa}$. Set $A = \kappa\left[\tilde{e}, \tilde{f}, \tilde{h}\right]$. Then using lemma 2.2.6(1) it will be enough to show that the associated graded sequence

$$\kappa\left[\tilde{e}, \tilde{f}, \tilde{h}\right] \xrightarrow{\tilde{\psi}} \mathcal{G}(Y_0) \oplus \mathcal{G}(Y_\infty) \xrightarrow{\mathrm{gr}(\overline{\phi})} \mathcal{G}(Y_{0\infty})$$

is exact.

The restriction maps $\mathcal{G}(Y_\dagger) \to \mathcal{G}(Y_{0\infty})$ are embeddings for $\dagger \in \{0, \infty\}$, so if $l : \mathcal{G}(Y_0) \oplus \mathcal{G}(Y_\infty) \to \mathcal{G}(Y_0)$ is the projection map $(\alpha, \beta) \mapsto \alpha$, then $\ker(\mathrm{gr}(\overline{\phi})) = \tilde{\psi}(A)$ if and only if $l(\ker(\mathrm{gr}(\overline{\phi}))) = l\tilde{\psi}(A)$.

Since $\mathcal{G}(Y_{0\infty})$ is a commutative ring we have that

$$\mathrm{gr}(\overline{\phi})(\alpha, \beta) = \alpha|_{Y_{0\infty}} - \tau^n \beta|_{Y_{0\infty}} \tau^{-n} = \alpha|_{Y_{0\infty}} - \beta|_{Y_{0\infty}}$$

for all $\alpha \in \mathcal{G}(Y_0)$, $\beta \in \mathcal{G}(Y_\infty)$. Hence, if $\alpha = \sum_{i,j \in \mathbb{N}} \alpha_{ij} \tau^i \tilde{\partial}_\tau^j \in \mathcal{G}(Y_0)$, we have $\alpha \in l(\ker(\mathrm{gr}(\overline{\phi})))$ if and only if $\alpha|_{Y_{0\infty}} \in \mathrm{res}^{Y_\infty}_{Y_{0\infty}} \mathcal{G}(Y_\infty)$. Since $\mathrm{res}^{Y_\infty}_{Y_{0\infty}} \mathcal{G}(Y_\infty)$ is generated by τ^{-1} and $\tilde{\partial}_{\tau^{-1}}|_{Y_{0\infty}} = -\tau^2 \tilde{\partial}_\tau$, we can see that this is the case if and only if $\alpha_{ij} = 0$ whenever $2i > j$. Now, if $i, j \in \mathbb{N}$ and $2i \leq j$, then it is a trivial fact that we can find $k_1, k_2, k_3 \in \mathbb{N}$ such that $k_1 + k_2 + k_3 = j$ and $k_2 + 2k_3 = i$. Then $\tau^i \tilde{\partial}_\tau^j = \tilde{\partial}_\tau^{k_1} (\tau \tilde{\partial}_\tau)^{k_2} (\tau^2 \tilde{\partial}_\tau)^{k_3}$. Hence $l(\ker(\mathrm{gr}(\overline{\phi})))$ is generated as a κ-algebra by $\tilde{\partial}_\tau$, $\tau \tilde{\partial}_\tau$, and $\tau^2 \tilde{\partial}_\tau$.

Now, $l\tilde{\psi}(A)$ is generated over κ by $l\tilde{\psi}(\tilde{e}) = \tilde{\partial}_\tau$, $l\tilde{\psi}(\tilde{h}) = -\tau\tau\tilde{\partial}_\tau - n$ and $l\tilde{\psi}(\tilde{f}) = -\tau^2 \tilde{\partial}_\tau$, so clearly $l(A)$ coincides with $l(\ker(\mathrm{gr}(\overline{\phi})))$, proving the theorem. □

6.2.3 Construction of a Morita context

Let $\widehat{P_n^{(r)}}$ be the right ideal

$$\widehat{P_n^{(r)}} = (h+n)\widehat{U_n} + e\widehat{U_n} \subset \widehat{U_n}$$

and let $\widehat{P_n^{(l)}}$ be the left ideal

$$\widehat{P_n^{(l)}} = \widehat{U_n}(h+n+2) + \widehat{U_n}e \subset \widehat{U_n}.$$

Let $\widehat{I_n^{(l)}}$ be the left $\widehat{U_n}$-module

$$\widehat{I_n^{(l)}} = \widehat{U_n} + \widehat{U_n}((h-n)e^{-1}) \subset Q(\widehat{U_n})$$

(where $Q(\widehat{U_n})$ is the skew-field of fractions of $\widehat{U_n}$,) and let $\widehat{I_n^{(r)}}$ be the right $\widehat{U_n}$-module

$$\widehat{I_n^{(r)}} = \widehat{U_n} + ((h-n)e^{-1})\widehat{U_n} \subset Q(\widehat{U_n}).$$

In this section we will construct a Morita context

$$B = \begin{bmatrix} M_{p-1}(\widehat{U_{n-1}}) & (\widehat{P_n^{(r)}})^{p-1} \\ (\widehat{I_n^{(l)}})^{p-1} & \widehat{U_n} \end{bmatrix},$$

and show that there is an isomorphism of R-algebras $B \to A$, where

$$A = \begin{bmatrix} M_{p-1}(D^{n-1}) & (R^n)^{p-1} \\ (L^n)^{p-1} & D^n \end{bmatrix} \subset M_p(\mathcal{D}(Y_0)).$$

A priori $\widehat{P_n^{(r)}}$ is not a $\widehat{U_{n-1}}$-$\widehat{U_n}$-bimodule and $\widehat{I_n^{(l)}}$ is not a $\widehat{U_n}$-$\widehat{U_{n-1}}$-bimodule, but in proposition 6.2.3(1) we will show that there are isomorphisms of Abelian groups $\widehat{P_n^{(r)}} \to \widehat{P_{n-1}^{(l)}}$ and $\widehat{I_n^{(l)}} \to \widehat{I_{n-1}^{(r)}}$ which give $\widehat{P_n^{(r)}}$ the structure of a $\widehat{U_{n-1}}$-$\widehat{U_n}$-bimodule and $\widehat{I_n^{(l)}}$ the structure of a $\widehat{U_n}$-$\widehat{U_{n-1}}$-bimodule. We will then show that multiplication in $Q(\widehat{U_n})$ defines a homomorphism of $\widehat{U_n}$-bimodules

$$\widehat{I_n^{(l)}} \otimes_{\widehat{U_{n-1}}} \widehat{P_n^{(r)}} \to \widehat{U_n}$$

and the multiplication in $Q(\widehat{U_{n-1}})$ defines a homomorphism of $\widehat{U_{n-1}}$-bimodules

$$\widehat{P_n^{(r)}} \otimes_{\widehat{U_n}} \widehat{I_n^{(l)}} \to \widehat{U_{n-1}}$$

which satisfy the necessary compatibility conditions to define a Morita context.

Proposition 6.2.3 *1.* $\theta_n(\widehat{P_n^{(r)}}) = R^n = \theta_{n-1}(\widehat{P_{n-1}^{(l)}})$.

2. *If we extend θ_n to a homomorphism of R-algebras $Q(\widehat{U_n}) \to Q(D^n)$ then*

$$\theta_n(\widehat{I_n^{(l)}}) = L^n = \theta_{n-1}(\widehat{I_{n-1}^{(r)}}).$$

Proof:

1. From the definitions if $\alpha \in \mathcal{D}(Y_0)$, then $\alpha \in R^n$ if and only if $\tau^{-n}\alpha|_{Y_{0\infty}}\tau^n \in \tau^{-1}\mathrm{res}_{Y_{0\infty}}^{Y_\infty}\mathcal{D}(Y_\infty)$ if and only if $\tau\alpha \in D^n$, and $\alpha \in L^n$ if and only if $\tau^{-n}\alpha|_{Y_{0\infty}}\tau^n \in \mathrm{res}_{Y_{0\infty}}^{Y_\infty}\mathcal{D}(Y_\infty)\tau$. Since $\tau^{-1} \in \mathrm{res}_{Y_{0\infty}}^{Y_\infty}\mathcal{D}(Y_\infty)$, R^n is a right ideal of D^n. Similarly, we can say that $\alpha \in R^n$ if and only if $\alpha\tau \in D^{n-1}$, and $\alpha \in L^n$ if and only if $\tau^{-(n-1)}\alpha|_{Y_{0\infty}}\tau^{n-1} \in \tau\mathrm{res}_{Y_{0\infty}}^{Y_\infty}\mathcal{D}(Y_\infty)$.

   First we will show that R^n contains $\theta_n(\widehat{P_n^{(r)}})$, i.e. we need to show that $\tau\theta_n(e)$ and $\tau\theta_n(h+n) \in D^n$ as then $\tau\theta_n(e\alpha + (h+n)\beta) \in D^n$ for all α, $\beta \in \widehat{U_n}$. First $\tau\theta_n(e) = \tau\partial_\tau = -\frac{1}{2}(\theta_n(h) - n) \in D^n$ and second $\tau\theta_n(h+n) = \tau(\tau^n(-2\tau\partial_\tau)\tau^{-n}) = 2\theta_n(f) \in D^n$, so $\theta_n(\widehat{P_n^{(r)}}) \subset R^n$.

   Now we will show that R^n contains $\theta_{n-1}(\widehat{P_{n-1}^{(l)}})$. For this it will be enough to show that $\theta_{n-1}(e)\tau \in D^{n-1}$ and $\theta_{n-1}(h+n+1)\tau \in D^{n-1}$ as then $\theta_{n-1}(\alpha e+\beta(h+n+1))\tau \in D^{n-1}$ for all α, $\beta \in D^{n-1}$. Clearly $\theta_{n-1}(e)\tau = \tau\partial_\tau + 1 \in D^{n-1}$, and

   $$\theta_{n-1}(h+(n+1)) = \tau^{n-1}(-2\tau\partial_\tau)\tau^{-(n-1)} + 2 = \tau^{n-1}(-2(\tau\partial_\tau - 1))\tau^{-(n-1)},$$

   hence

   $$\theta_{n-1}(h+(n+1))\tau = \tau^{n-1}(-2(\tau\partial_\tau - 1))\tau^{-(n-1)}\tau =$$
   $$\tau(\tau^{n-1}(-2\tau\partial_\tau)\tau^{-(n-1)}) = 2\theta_{n-1}(f).$$

   Hence $\theta_{n-1}(\widehat{P_{n-1}^{(l)}}) \subset R^n$.

   Now we need to show $\theta_n(\widehat{P_n^{(r)}})$ contains R^n. Let $M = D^n/\theta_n(\widehat{P_n^{(r)}})$. Then the map $\eta : D^n \to M$ sends $\theta_n(e) \mapsto 0$ and $\theta_n(h) \mapsto -n$, so the restriction of η to $R\langle\theta_n(f)\rangle$, the closed R-subalgebra of D^n generated by $\theta_n(f)$, is a surjection.

For a contradiction, suppose that R^n strictly contains $\theta_n(\widehat{P_n^{(r)}})$. Then some non-zero element $F(\theta_n(f)) \in R\langle\theta_n(f)\rangle$ must belong to R^n. Now, $\tau^{-n}\theta_n(f)|_{Y_{0\infty}}\tau^n = \partial_{\tau^{-1}}|_{Y_{0\infty}}$, so from the assumptions we get that $\tau^{-n}F(\theta_n(f))|_{Y_{0\infty}}\tau^n = F(\partial_{\tau^{-1}}|_{Y_{0\infty}}) \in \tau^{-1}\mathrm{res}^{Y_\infty}_{Y_{0\infty}}\mathcal{D}(Y_\infty)$. But every element $\lambda \in \mathrm{res}^{Y_\infty}_{Y_{0\infty}}\mathcal{D}(Y_\infty)$ can be written uniquely in the form

$$\lambda = \sum_{i,j\in\mathbb{N}} \lambda_{ij}\tau^{-i}\partial_{\tau^{-1}}|^j_{Y_{0\infty}}$$

with $\lambda_{ij} \in R$, $\lambda_{ij} \to 0$ as $i+j \to \infty$, a contradiction. Hence $R^n = \theta_n(\widehat{P_n^{(r)}})$. A similar proof shows that $R^n = \theta_{n-1}(\widehat{P_{n-1}^{(l)}})$.

2. It remains to show that $L^n = \theta_n(\widehat{I_n^{(l)}})$. We will first show that L^n contains $\theta_n(\widehat{I_n^{(l)}})$. Since $L^n = \{\alpha \in \mathcal{D}(Y_0) : \tau^{-n}\alpha|_{Y_{0\infty}}\tau^n \in \mathrm{res}^{Y_\infty}_{Y_{0\infty}}\mathcal{D}(Y_\infty)\tau\}$, and $\mathrm{res}^{Y_\infty}_{Y_{0\infty}}\mathcal{D}(Y_\infty) \subset \mathrm{res}^{Y_\infty}_{Y_{0\infty}}\mathcal{D}(Y_\infty)\tau$ we have that $D^n \subset L^n$. Of course $\tau \in L^n$, since $\tau \in \mathrm{res}^{Y_\infty}_{Y_{0\infty}}\mathcal{D}(Y_\infty)\tau$.

   Let $\mu = \frac{-1}{2}(h-n)e^{-1} \in Q(\widehat{U_n})$. Then

$$\theta_n(\mu) = \frac{-1}{2}(-2\tau\partial_\tau)\partial_\tau^{-1} = \tau \in L^n.$$

   Of course $\widehat{I_n^{(l)}} = \widehat{U_n} + \widehat{U_n}\mu$, so $\theta_n(\widehat{I_n^{(l)}}) = D^n + D^n\tau$. Now L^n is a left D^n module under multiplication, so we find that $\theta_n(\widehat{I_n^{(l)}}) \subset L^n$. A similar argument shows that $\theta_{n-1}(\widehat{I_{n-1}^{(r)}}) \subset L^n$.

   Now, set $A = \tau^n\mathrm{res}^{Y_\infty}_{Y_{0\infty}}\mathcal{D}(Y_\infty)\tau^{-n}$. From the definitions, if $\alpha \in \mathcal{D}(Y_0)$ then $\alpha|_{Y_{0\infty}} \in A$ if and only if $\alpha \in D^n$ and $\alpha|_{Y_{0\infty}} \in A\tau$ if and only if $\alpha \in L^n$. For the rest of the proof, we will abuse notation and identify D^n with $\mathrm{res}^{Y_0}_{Y_{0\infty}}(D^n)$ and L^n with $\mathrm{res}^{Y_0}_{Y_{0\infty}}(L^n)$. Then $A \cap L^n = D^n$, so

$$L^n/D^n = L^n/(L^n \cap A) \subset A\tau/A.$$

   Let $\sigma = \tau^n\partial_{\tau^{-1}}|_{Y_{0\infty}}\tau^{-n}$. Then every $\lambda \in A$ can be written uniquely in the form

$\lambda = \sum_{i,j\in\mathbb{N}} \lambda_{ij}\sigma^i\tau^{-j}$ with $\lambda_{ij} \in R$ and $\lambda_{ij} \to 0$ as $i+j \to \infty$, so

$$\lambda\tau - (\sum_{i\in\mathbb{N}} \lambda_{i0}(\sigma)^i)\tau \in A.$$

Hence the embedding $R\langle\sigma\rangle\tau \to A\tau/A$ is an isomorphism.

Now, $\theta_n(\mu) = \tau$, so $\theta_n(\widehat{I_n^{(l)}}) = D^n\tau + D^n$. In the proof of part 1 we have seen that $\theta_n(e)\tau$ and $\theta_n(h+n+2)\tau \in D^n$, so we have a surjection

$$R\langle\theta_n(f)\rangle\tau = R\langle\sigma\rangle\tau \to \theta_n(\widehat{I_n^{(l)}})/D^n$$

Which gives us the following commutative diagram

$$\begin{array}{ccccc} & & R\langle\sigma\rangle\tau & & \\ & \swarrow & & \searrow & \\ \theta_n(\widehat{I_n^{(l)}})/D^n & \longrightarrow & L^n/D^n & \longrightarrow & A\tau/A \end{array}$$

where the left diagonal arrow is a surjection, the right diagonal arrow is an isomorphism, and the other maps are inclusions. It follows that all of the maps are isomorphisms, so $L^n \subset \theta_n(\widehat{I_n^{(l)}})$. Hence $L^n = \theta_n(\widehat{I_n^{(l)}})$, and a similar argument shows that $L^n = \theta_{n-1}(\widehat{I_{n-1}^{(r)}})$.

□

Now we can see that the isomorphisms of Abelian groups $\widehat{P_n^{(r)}} \to \widehat{P_{n-1}^{(l)}}$; $\mu \mapsto \theta_{n-1}^{-1}\theta_n(\mu)$ and $\widehat{I_n^{(l)}} \to \widehat{I_{n-1}^{(r)}}$; $\nu \mapsto \theta_{n-1}^{-1}\theta_n(\nu)$ satisfy the properties we described at the beginning of this section. If we let $\beta\cdot\mu$ be the unique solution to the equation

$$\theta_n(\beta\cdot\mu) = \theta_{n-1}(\beta)\theta_n(\mu)$$

and

$$\theta_n(\nu\cdot\beta) = \theta_n(\nu)\theta_{n-1}(\beta)$$

for all $\mu \in \widehat{P_n^{(r)}}$, $\nu \in \widehat{I_n^{(l)}}$, and $\beta \in \widehat{U_{n-1}}$ then for $\alpha \in \widehat{U_n}$ we can calculate that

$$\theta_n(\beta \cdot (\mu\alpha)) = \theta_{n-1}(\beta)\theta_n(\mu\alpha) = \theta_{n-1}(\beta)\theta_n(\mu)\theta_n(\alpha) = \theta_n((\beta \cdot \mu)\alpha),$$

so that $\widehat{P_n^{(r)}}$ has the structure of a $\widehat{U_{n-1}}$-$\widehat{U_n}$-bimodule, and a similar calculation shows that $\widehat{I_n^{(l)}}$ has the structure of a $\widehat{U_n}$-$\widehat{U_{n-1}}$-bimodule. Furthermore we can calculate that

$$\theta_n((\nu \cdot \beta)\mu) = (\theta_n(\mu)\theta_{n-1}(\beta))\theta_n(\nu) = \theta_n(\mu)(\theta_{n-1}(\beta)\theta_n(\nu)) = \theta_n(\nu(\beta \cdot \mu))$$

so that we have a homomorphism

$$\omega : \widehat{I_n^{(l)}} \otimes_{\widehat{U_{n-1}}} \widehat{P_n^{(r)}} \to \widehat{U_n} \ ; \nu \otimes \mu \mapsto \nu\mu.$$

Similarly, the multiplication in $Q(\widehat{U_n})$ defines a homomorphism of $\widehat{U_{n-1}}$-bimodules

$$\varepsilon : \widehat{P_n^{(r)}} \otimes_{\widehat{U_n}} \widehat{I_n^{(l)}} \to \widehat{U_{n-1}} \ ; \mu \otimes \nu \mapsto \theta_{n-1}^{-1}\theta_n(\mu\nu).$$

Since θ_n and θ_{n-1} are ring embeddings, we can calculate that for all $\mu, \mu' \in \widehat{P_n^{(r)}}$ and $\nu \in \widehat{I_n^{(l)}}$

$$\mu\omega(\nu \otimes \mu') = \mu\nu\mu' = \theta_n^{-1}\theta_{n-1}(\theta_{n-1}^{-1}\theta_n(\mu\nu))\mu' = \varepsilon(\mu \otimes \nu) \cdot \mu',$$

and a similar calculation shows that for all $\nu' \in \widehat{I_n^{(l)}}$,

$$\nu\varepsilon(\mu \otimes \nu') = \omega(\nu \otimes \mu)\nu'.$$

Hence, we have constructed a well defined Morita context

$$B = \begin{bmatrix} M_{p-1}(\widehat{U_{n-1}}) & (\widehat{P_n^{(r)}})^{p-1} \\ (\widehat{I_n^{(l)}})^{p-1} & \widehat{U_n} \end{bmatrix}.$$

We are now ready to prove corollary 1.6.

Corollary 6.2.3 *1. Let*

$$A = \begin{bmatrix} M_{p-1}(D^{n-1}) & (R^n)^{p-1} \\ (L^n)^{p-1} & D^n \end{bmatrix} \subset M_p(\mathcal{D}(Y_0)),$$

and let B be defined as above. Then the map $\Theta : B \to A$ which acts by θ_n on $\widehat{U_n}$, $(\widehat{I_n^{(l)}})^{p-1}$, and $(\widehat{P_n^{(r)}})^{p-1}$ and by θ_{n-1} on $M_{p-1}(\widehat{U_{n-1}})$ is an isomorphism of R-algebras.

2. *If $n \in R^\times$ then $L^nR^n = D^n$ and if $n+1 \in R^\times$ then $R^nL^n = D^{n-1}$.*

Proof:

1. That Θ is an isomorphism on the level of R-modules follows from theorem 6.2.2 and proposition 6.2.3(1) and (2), so we only need to check multiplicativity. Set $\gamma = \theta_{n-1}^{-1}\theta_n$. Then, since θ_n and θ_{n-1} are multiplicative, we can calculate that

$$\begin{aligned}&\Theta\left(\begin{bmatrix} u & a \\ b & v \end{bmatrix}\begin{bmatrix} u' & a' \\ b' & v' \end{bmatrix}\right)\\ =&\Theta\left(\begin{bmatrix} uu' + \gamma(a)^T\gamma(b') & \gamma^{-1}(u\gamma(a')) + av' \\ \gamma^{-1}(\gamma(b)u') + vb' & b(a')^T + vv' \end{bmatrix}\right)\\ =&\begin{bmatrix} \theta_{n-1}(uu') + \theta_n(a)^T\theta_n(b') & \theta_{n-1}(u)\theta_n(a') + \theta_n(av') \\ \theta_n(b)\theta_{n-1}(u') + \theta_n(vb') & \theta_n(b)\theta_n(a')^T + \theta_n(vv') \end{bmatrix}\\ =&\Theta\left(\begin{bmatrix} u & a \\ b & v \end{bmatrix}\right)\Theta\left(\begin{bmatrix} u' & a' \\ b' & v' \end{bmatrix}\right),\end{aligned}$$

proving the statement.

2. By proposition 6.2.1 L^nR^n is a two-sided ideal of D^n and R^nL^n is a two-sided ideal of D^{n-1} so we just need to show that $1 \in L^nR^n$ and $1 \in R^nL^n$. From proposition 6.2.3 we have that $L^n = \theta_n(\widehat{I_n^{(l)}}) = D^n + D^n\tau$ and $R^n = \theta_n(\widehat{P_n^{(r)}}) = \partial_\tau D^n + (\tau\partial_\tau - n)D^n$. Hence $\tau\partial_\tau \in L^nR^n$ and $\tau\partial_\tau - n \in L^nR^n$, so $n \in L^nR^n$, and if $n \in R^\times$, it follows that $1 \in L^nR^n$. On the other hand $\partial_\tau\tau = \tau\partial_\tau + 1 \in R^nL^n$ and $\tau\partial_\tau - n \in R^nL^n$, so $n+1 \in R^nL^n$, and if $n+1 \in R^\times$ it follows that $1 \in R^nL^n$.

□

6.2.4 Morita Equivalence of Global Sections

Theorem 6.2.4 *$\widehat{\mathcal{D}_X^{[1]}}(X)$ is Morita equivalent to D^0.*

Proof: Let $W = \widehat{\mathcal{D}_X^{[1]}}(X)$. Let e be the global idempotent $e_{(p-1)(p-1)}$ of W defined in section 6.1. Let $\mathcal{C}$ be defined as in section 6.2.1 and let $\mathcal{M} = \mathcal{C}(\mathcal{L}^{\otimes -1}, \mathcal{O}_Y)$. Let $\Phi : F_*\widehat{\mathcal{D}_X^{[1]}} \to \mathcal{M}$ be the isomorphism of $\mathcal{O}_Y$-rings provided by theorem 6.1.8. Let $\Psi : \mathcal{M}(Y) \to \begin{bmatrix} M_{p-1}(D^{-1}) & (R^0)^{p-1} \\ (L^0)^{p-1} & D^0 \end{bmatrix}$ be the isomorphism of R-algebras defined in proposition 6.2.1. By corollary 6.2.3(2) $R^0L^0 = D^{-1}$, so

$$\Psi\Phi(Y)(WeW) = \begin{bmatrix} 0 & (R^0)^{p-1} \\ 0 & D^0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ (L^0)^{p-1} & D^0 \end{bmatrix}$$
$$= \begin{bmatrix} M_{p-1}(R^0L^0) & (R^0)^{p-1} \\ (L^0)^{p-1} & D^0 \end{bmatrix} = \begin{bmatrix} M_{p-1}(D^{-1}) & (R^0)^{p-1} \\ (L^0)^{p-1} & D^0 \end{bmatrix}.$$

Hence $WeW = W$, and we of course know that eWe is isomorphic as an R-algebra to D^0, so by [10, Proposition 3.5.6] D^0 is Morita equivalent to W. □

Bibliography

[1] Konstantin Ardakov and Simon Wadsley. $\widetilde{D}$-modules for rigid analytic spaces i. *preprint?*, 2014.

[2] Konstantin Ardakov and SJ Wadsley. On irreducible representations of compact p-adic analytic groups. *Annals of Mathematics*, 178:453–557, 2011.

[3] Pierre Berthelot. Cohomologie rigide et théorie des D-modules. In *p-adic Analysis*, pages 80–124. Springer, 1990.

[4] Pierre Berthelot. D-modules arithmétiques i. opérateurs différentiels de niveau fini. *Ann. Sci. École Norm. Sup.(4)*, 29(2):185–272, 1996.

[5] Pierre Berthelot. $\mathcal{D}$-modules arithmétiques. ii: Descente par Frobenius. *Mémoires de la Société Mathématique de France*, 81:1–138, 2000.

[6] Pierre Berthelot. Introduction à la théorie arithmétique des D-modules. *Astérisque*, pages 1–80, 2002.

[7] Jacques Dixmier. *Enveloping algebras*, volume 14. Newnes, 1977.

[8] R. Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics. Springer, 1977.

[9] Li Huishi and Freddy Van Oystaeyen. *Zariskian filtrations*. Springer, 1996.

[10] J John C McConnell and J James Christopher Robson. *Noncommutative Noetherian rings*, volume 30. AMS Bookstore, 2001.

[11] Christine Noot-Huyghe. A Beilinson-Bernstein theorem for arithmetic D-modules. *Bulletin de la Societe Mathematique de France*, 137(2):159–183, 2009.

[12] Alexis Pangalos. *p-adic Weyl algebras*. PhD thesis, PhD thesis, University of Münster, 2007.

[13] Daniel Quillen. On the endomorphism ring of a simple module over an enveloping algebra. In *Proc. Amer. Math. Soc*, volume 21, pages 171–172, 1969.

[14] L.H. Rowen. *Polynomial identities in ring theory*. Pure and Applied Mathematics. Elsevier Science, 1980.

[15] Tobias Schmidt. Analytic vectors in continuous p-adic representations. *Compositio Mathematica*, 145(01):247–270, 2009.

[16] Tobias Schmidt. Stable flatness of nonarchimedean hyperenveloping algebras. *Journal of Algebra*, 323(3):757–765, 2010.

[17] Tobias Schmidt. Verma modules over p-adic arens–michael envelopes of reductive lie algebras. *Journal of Algebra*, 390:160–180, 2013.

[18] Peter Schneider and Jeremy Teitelbaum. Algebras of p-adic distributions and admissible representations. *Inventiones mathematicae*, 153(1):145–196, 2003.

[19] Jean-Pierre Serre. Lie groups and lie algebras. *Lecture Notes in Math*, 1500, 1965.

[20] JT Stafford. Homological properties of the enveloping algebra $U(\mathfrak{sl}_2)$. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 91, page 29, 1982.