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Hardy–Sobolev–Rellich, Hardy–Littlewood–Sobolev and Caffarelli–Kohn–Nirenberg Inequalities on General Lie Groups

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Abstract

In this paper, we establish a number of geometrical inequalities such as Hardy, Sobolev, Rellich, Hardy–Littlewood–Sobolev, Caffarelli–Kohn–Nirenberg, Gagliardo-Nirenberg inequalities and their critical versions for an ample class of sub-elliptic differential operators on general connected Lie groups, which include both unimodular and non-unimodular cases in compact and noncompact settings. We also obtain the corresponding uncertainty type principles.

Keywords Sobolev spaces · Sobolev embeddings · Hardy inequality · Rellich inequality · Hardy–Littlewood–Sobolev inequality · Caffarelli–Kohn–Nirenberg inequality · Lie groups

Mathematics Subject Classification 46E35 · 22E30 · 43A15

Luigi Rodino on the occasion of his 75th birthday

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1 Introduction

Let $\mathbb{G}$ be a connected Lie group, let ρ and λ denote a right and left Haar measure on $\mathbb{G}$, δ the modular function on $\mathbb{G}$ so that $d\lambda(x) = \delta(x)d\rho(x)$, $x \in \mathbb{G}$. Let $X = (X_1, \ldots, X_n)$ be a collection of left-invariant vector fields on $\mathbb{G}$ satisfying Hörmander's condition.

Let χ denote a positive continuous character and e the identity element on $\mathbb{G}$, and define $\mu_\chi = \chi\rho$ and let Δ_χ be the differential operator

$$-\sum_{j=1}^{n}(X_j^2 + X_j(\chi)(e)X_j). \tag{1.1}$$

It was shown in [11] that Δ_χ, when initially defined on $C_0^\infty(\mathbb{G})$, is essentially self-adjoint in $L^2(\mu_\chi)$ and conversely, if μ is a positive Borel measure on $\mathbb{G}$ such that $\Delta_1 - X$ is essentially self-adjoint in $L^2(\mu)$, where X is a left-invariant vector field, then there exists a positive continuous character χ on $\mathbb{G}$ such that $X = \Delta_1 - \Delta_\chi$ and $\mu = \mu_\chi$. By a common abuse of notation, from now on we will denote the unique self-adjoint realization of Δ_χ by the same symbol. Thus, the family of sub-Laplacians with drift Δ_χ turns out to be "the" natural family of second-order differential operators for which it is reasonable to apply functional calculus results and methods to define and study function spaces, and regularity of solutions of differential equations. In particular, when $\chi = \delta$, so that $\mu_\chi = \lambda$ is the left Haar measure, then Δ_δ is the *intrinsic* sub-Laplacian, introduced by Agrachev, Boscain, Gauthier and Rossi [1].

Sobolev spaces defined in terms of Δ_χ were introduced and studied in [3], where various embedding and algebra properties were proved. When $1 < p < \infty$ and $\alpha \geq 0$, the Sobolev spaces $L_\alpha^p(\mu_\chi)$ were defined as the completion of $C_0^\infty(\mathbb{G})$ with respect to the norm

$$\|f\|_{L_\alpha^p(\mu_\chi)} := \|f\|_{L^p(\mu_\chi)} + \left\|\Delta_\chi^{\alpha/2} f\right\|_{L^p(\mu_\chi)}.$$

These spaces, in particular these measures, appear naturally in embeddings and algebra properties, where the case $\chi = \delta$, that is, $\mu_\chi = \lambda$, plays a fundamental role. In the case when $\mathbb{G}$ is a unimodular group and with $\chi = 1$, we note that the spaces $L_\alpha^p(\mu_\chi)$ coincide with the Sobolev spaces defined by $\Delta_1 = -\sum_{j=1}^{n} X_j^2$ (see [6]). In the case $\chi \neq 1$, we note that this operator Δ_1 is not symmetric on $L^2(\mu_\chi)$, so that a Sobolev space adapted to μ_χ when $\chi \neq 1$ cannot be defined by means of fractional powers of Δ_1. For more details, we refer to [11, 14] and [3].

It became natural to study geometric inequalities for the scale of Sobolev spaces $L_\alpha^p(\mu_\chi)$, at least in the cases $1 < p < \infty$ and $\alpha > 0$.

In the unimodular case and with $\chi = 1$, for embedding theorems for these Sobolev spaces, we refer to [7] on stratified group, and to [8] and [9] on graded groups, as well as to [21] for the weighted versions. On general homogeneous groups, we refer to [15] and [16].

In the non-unimodular case, we refer to [23] for the first-order Sobolev spaces when χ is a power of δ, and to [3] for the higher order case.

For algebra properties of the Sobolev spaces, we refer to [6] on unimodular groups, and refer to [14] and to [3] on non-unimodular groups.

In this paper, we obtain a number of (versions of) classical geometric inequalities on Sobolev spaces in a unified way, in the setting of general Lie groups, for an ample class of sub-elliptic differential operators.

As usual, in this paper, $A \lesssim B$ means that there exists a positive constant c such that $A \leq cB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$. In these notations, if the left and right-hand sides feature some functions f, the constant (using this notation) does not depend on f.

Let us begin with the following **Hardy–Sobolev–Rellich** inequality on general connected Lie groups:

Theorem 1.1 *Let $\mathbb{G}$ be a connected Lie group. Let e be the identity element of $\mathbb{G}$, and let χ be a positive character of $\mathbb{G}$. Let $|x| := d_C(e, x)$ denote the Carnot-Carathéodory distance from e to x. Let d be the local dimension of $\mathbb{G}$ as recalled in (2.3) and $\alpha > 0$, $0 \leq \beta < d$, $1 < p, q < \infty$. Then, we have*

$$\left\| \frac{f}{|x|^{\frac{\beta}{q}}} \right\|_{L^q(\mu_{\chi^{q/p}\delta^{1-q/p}})} \lesssim \|f\|_{L^p_\alpha(\mu_\chi)} \tag{1.2}$$

for all $q \geq p$ such that $1/p - 1/q \leq \alpha/d - \beta/(dq)$.

Remark 1.2 Note that in the case $\alpha \geq d/p$ the condition $1/p - 1/q \leq \alpha/d - \beta/(dq)$ automatically holds true since

$$\frac{\alpha}{d} - \frac{\beta}{dq} \geq \frac{1}{p} - \frac{\beta}{dq} > \frac{1}{p} - \frac{1}{q},$$

which means the inequality (1.2) holds for all $q \geq p$ when $\alpha \geq d/p$.

Note that Theorem 1.1 when $\beta = 0$ implies the **Sobolev embedding** on general connected Lie group

$$L^p_\alpha(\mu_\chi) \hookrightarrow L^q(\mu_{\chi^{q/p}\delta^{1-q/p}}). \tag{1.3}$$

The Sobolev embedding (1.3) was proved in [3] in the noncompact case. We also refer to the very recent work [4] for the investigation of the behaviour of the Sobolev embedding constant on a general connected Lie group, endowed with a left Haar measure. On nilpotent Lie groups, we refer to [20] and [18] for the best constants in Sobolev, Gagliardo-Nirenberg and their critical cases for general left-invariant homogeneous hypoelliptic differential operators.

Furthermore, for $q = p$ and $\beta/q = \alpha$, the inequality (1.2) gives the following inhomogeneous **Hardy inequality** on general connected Lie groups:

$$\left\| \frac{f}{|x|^{\alpha}} \right\|_{L^p(\mu_\chi)} \lesssim \|f\|_{L^p_\alpha(\mu_\chi)}, \tag{1.4}$$

where $0 \leq \alpha < d/p$. In particular, for $\chi = 1$ and $\chi = \delta$ (respectively, with $\mu_1 = \rho$ and $\mu_\delta = \lambda$), this gives both right and left versions of Hardy inequalities, respectively. We note that the inequality (1.4) was obtained for $\chi = 1$ on stratified (hence also, in particular, nilpotent and unimodular) Lie groups in [5]. Note that when $\alpha = 2$ the inequality (1.4) yields the following inhomogeneous **Rellich inequality** on general connected Lie groups:

$$\left\| \frac{f}{|x|^2} \right\|_{L^p(\mu_\chi)} \lesssim \|f\|_{L_2^p(\mu_\chi)} \approx \|f\|_{L^p(\mu_\chi)} + \left\| \Delta_\chi f \right\|_{L^p(\mu_\chi)}, \; 0 \leq 2p < d. \tag{1.5}$$

Since the inequality (1.2) contains the Hardy, Rellich and Sobolev inequalities, we call this inequality Hardy–Sobolev–Rellich inequality. The remaining cases of the inequality (1.2) can be thought of as Sobolev embeddings in weighted L^q-spaces.

Moreover, we establish the Hardy–Sobolev–Rellich inequality (1.2) in the **critical case** $\beta = d$ that involves a logarithmic factor in the weight:

Theorem 1.3 *Let $\mathbb{G}$ be a connected Lie group. Let χ be a positive character of $\mathbb{G}$. Let $1 < p < r < \infty$ and $1/p + 1/p' = 1$. Then, we have*

$$\left\| \frac{f}{\left(\log\left(\mathrm{e} + \frac{1}{|x|}\right)\right)^{\frac{r}{q}} |x|^{\frac{d}{q}}} \right\|_{L^q(\mu_{\chi^{q/p}\delta^{1-q/p}})} \lesssim \|f\|_{L^p_{d/p}(\mu_\chi)} \tag{1.6}$$

for every $q \in [p, (r-1)p')$.

When $\chi = \delta$, Theorems 1.1 and 1.3 give the weighted embeddings with the same measure, namely with the left Haar measure (see Theorem 3.2 and 3.4), which is the unique case when such embeddings hold true as in the unweighted case. Moreover, for $q = p$, this gives the following **critical Hardy inequality** on general connected Lie groups:

$$\left\| \frac{f}{\left(\log\left(\mathrm{e} + \frac{1}{|x|}\right)\right)^{\frac{r}{p}} |x|^{\frac{d}{p}}} \right\|_{L^p(\mu_\chi)} \lesssim \|f\|_{L^p_{d/p}(\mu_\chi)}, \tag{1.7}$$

where $1 < p < r < \infty$, which is a critical case $\alpha = d/p$ of the Hardy inequality given in (1.4).

Remark 1.4 First, we will prove the above theorems in the noncompact case. Consequently, in Sect. 4, we show that these Theorems 1.1 and 1.3 with $\delta = 1$ hold on compact Lie groups (which are automatically unimodular) as well.

We also show that Theorem 1.1 gives the following fractional **Caffarelli–Kohn–Nirenberg type inequality**:

Theorem 1.5 *Let $\mathbb{G}$ be a connected Lie group. Let e be the identity element of $\mathbb{G}$, and let χ be a positive character of $\mathbb{G}$. Let $|x| := d_C(e, x)$ denote the Carnot-Carathéodory* *distance from* e *to* x. *Let* $1 < p < \infty$, $0 < q, r < \infty$ *and* $0 < \theta \le 1$ *be such that* $\theta > (r-q)/r$ *and* $p \le q\theta r/(q-(1-\theta)r)$. *Let* a *and* b *be real numbers and* $\alpha > 0$ *such that* $0 \le qr(b(1-\theta)-a)/(q-(1-\theta)r) < d$ *and* $1/p-(q-(1-\theta)r)/(qr\theta) \le \alpha/d-(b(1-\theta)-a)/(\theta d)$. *Then, we have*

$$\||x|^a f\|_{L^r(\mu_{\chi^{\widetilde{q}/p}\delta^{1-\widetilde{q}/p}})} \lesssim \|f\|^{\theta}_{L^p_\alpha(\mu_\chi)} \||x|^b f\|^{1-\theta}_{L^q(\mu_{\chi^{\widetilde{q}/p}\delta^{1-\widetilde{q}/p}})}, \tag{1.8}$$

where $\widetilde{q} := \frac{qr\theta}{q-(1-\theta)r}$.

Remark 1.6 Note that when $\theta = 1$ then $\theta > (r-q)/r$ automatically holds, $\widetilde{q} = r$, $p \le q\theta r/(q-(1-\theta)r)$ gives $p \le r$, while conditions $0 \le qr(b(1-\theta)-a)/(q-(1-\theta)r) < d$ and $1/p-(q-(1-\theta)r)/(qr\theta) \le \alpha/d-(b(1-\theta)-a)/(\theta d)$ are equivalent to $0 \le -ar < d$ and $1/p-1/r \le \alpha/d+a/d$, respectively. Then, in this case, the inequality (1.8) has the following form

$$\||x|^a f\|_{L^r(\mu_{\chi^{r/p}\delta^{1-r/p}})} \lesssim \|f\|_{L^p_\alpha(\mu_\chi)},$$

which is (1.2).

Remark 1.7 We note that if we take $a = b = 0$ in (1.8), then it gives the Gagliardo-Nirenberg type inequality on general connected Lie groups: Let $1 < p < \infty$, $0 < q, r < \infty$, $0 < \theta \le 1$ and $\alpha > 0$ be such that $\theta > (r-q)/r$, $p \le q\theta r/(q-(1-\theta)r)$ and $1/p-(q-(1-\theta)r)/(qr\theta) \le \alpha/d$. Then, we have the following **Gagliardo-Nirenberg inequality**:

$$\|f\|_{L^r(\mu_{\chi^{\widetilde{q}/p}\delta^{1-\widetilde{q}/p}})} \lesssim \|f\|^{\theta}_{L^p_\alpha(\mu_\chi)} \|f\|^{1-\theta}_{L^q(\mu_{\chi^{\widetilde{q}/p}\delta^{1-\widetilde{q}/p}})}, \tag{1.9}$$

where $\widetilde{q} := \frac{qr\theta}{q-(1-\theta)r}$.

Similarly, from (1.6), we can obtain the inequality (1.8) in the **critical case** $a = b(1-\theta)-d(q-(1-\theta)r)/qr$:

Theorem 1.8 *Let* $\mathbb{G}$ *be a connected Lie group. Let* χ *be a positive character of* $\mathbb{G}$. *Let* $b \in \mathbb{R}$, $1 < p < r < \infty$, $0 < q, r < \infty$ *and* $0 < \theta \le 1$ *be such that* $\theta > (r-q)/r$ *and* $p \le \widetilde{q} < (r-1)p'$ *with* $p' = p/(p-1)$ *and* $\widetilde{q} := \frac{qr\theta}{q-(1-\theta)r}$. *Then, we have*

$$\|\omega_r^{b(1-\theta)-d\theta/\widetilde{q}} f\|_{L^r(\mu_{\chi^{\widetilde{q}/p}\delta^{1-\widetilde{q}/p}})} \lesssim \|f\|^{\theta}_{L^p_{d/p}(\mu_\chi)} \|\omega_r^b f\|^{1-\theta}_{L^q(\mu_{\chi^{\widetilde{q}/p}\delta^{1-\widetilde{q}/p}})}, \tag{1.10}$$

where $\omega_r := (\log(\mathrm{e}+1/|x|))^{\frac{r}{d}}|x|$.

We also introduce the following **Hardy-Littlewood-Sobolev inequality**:

Theorem 1.9 *Let* $\mathbb{G}$ *be a connected Lie group. Let* e *be the identity element of* $\mathbb{G}$, *and let* χ *be a positive character of* $\mathbb{G}$. *Let* $|x| := d_C(e, x)$ *denote the Carnot-Carathéodory distance from* e *to* x. *Let* $1 < p, q < \infty$, $\alpha \ge 0$ *and* $0 \le \beta < d/q$. *Let* $0 \le a_1 <$

$dp/(p+q)$, $a_2>0$ with $0\le 1/p-q/(p+q)\le \alpha/d$ and $1/q-p/(p+q)\le (a_2-a_1)/d$. Then, we have

$$\left|\int_{\mathbb{G}}\int_{\mathbb{G}}\frac{\overline{f(x)}g(y)G^{c}_{a_2,\chi}(y^{-1}x)}{|x|^{a_1}|y|^{\beta}}d\mu_{\chi^{(p+q)/pq}\delta^{1-(p+q)/pq}}(x)d\rho(y)\right|\lesssim \|f\|_{L^{p}_{\alpha}(\mu_\chi)}\|g\|_{L^{q}_{\beta}(\mu_\chi)}, \tag{1.11}$$

where $G^{c}_{a_2,\chi}$ is defined in (2.8). In particular, $G^{c}_{a_2,\chi}$ is the convolution kernel of the operator $(\Delta_\chi+cI)^{-a_2/2}$, i.e. $(\Delta_\chi+cI)^{-a_2/2}f=f*G^{c}_{a_2,\chi}$.

Moreover, we show that Theorems 1.1 and 1.3 imply the following **uncertainty type principles**:

Corollary 1.10 *Let $\mathbb{G}$ be a connected Lie group. Let χ be a positive character of $\mathbb{G}$. Let $1/p+1/p'=1$ and $1/q+1/q'=1$.*

- *If $0\le\beta<d$, $1<p,q<\infty$ and $\alpha>0$, then we have*

$$\|f\|_{L^{p}_{\alpha}(\mu_\chi)}\||x|^{\frac{\beta}{q}}f\|_{L^{q'}(\mu_{\chi^{q/p}\delta^{1-q/p}})}\gtrsim \|f\|^{2}_{L^{2}(\mu_{\chi^{q/p}\delta^{1-q/p}})} \tag{1.12}$$

  for all $q\ge p$ such that $1/p-1/q\le\alpha/d-\beta/(dq)$, where $|x|:=d_C(e,x)$ is the Carnot-Carathéodory distance from the identity element e to x;
- *If $1<p<r<\infty$, then we have*

$$\|f\|_{L^{p}_{d/p}(\mu_\chi)}\left\|\left(\log\left(\mathrm{e}+\frac{1}{|x|}\right)\right)^{\frac{r}{p}}|x|^{\frac{d}{p}}f\right\|_{L^{q'}(B_1,\mu_{\chi^{q/p}\delta^{1-q/p}})}\gtrsim \|f\|^{2}_{L^{2}(\mu_{\chi^{q/p}\delta^{1-q/p}})} \tag{1.13}$$

  for all $q\in[p,(r-1)p')$.

On compact Lie groups, note that all the above results still hold true with $\delta=1$, which we will discuss in Sect. 4.

The main novelty of the paper is the extension to the case of general connected Lie groups and to the case of sub-Laplacians with drift, of results proved by various people, including the authors of this paper, in the cases of stratified, and general nilpotent Lie groups, and also compact groups, in the case of sub-Laplacians. , The organisation of the paper is as follows. In Sect. 2, we briefly recall some known properties of Sobolev spaces on connected Lie groups. Then, in Sect. 3, we prove Theorems 1.1, 1.3, 1.5, 1.8, 1.9 and Corollary 1.10. Finally, in Sect. 4 we discuss the obtained results of Sect. 3 on compact Lie groups.

2 Preliminaries

In this section, we very briefly recall some known properties of Sobolev spaces on connected Lie groups.

Let $\mathbb{G}$ be a noncompact connected Lie group with identity e. Let us denote the right and left Haar measure by ρ and λ, respectively. Let δ be the modular function, i.e. the function on $\mathbb{G}$ such that

$$d\lambda = \delta d\rho. \tag{2.1}$$

Then, recall that δ is a smooth positive character of $\mathbb{G}$, i.e. a smooth homomorphism of $\mathbb{G}$ into the multiplicative group $\mathbb{R}^+$. Let χ be a continuous positive character of $\mathbb{G}$, which is then automatically smooth. Let μ_χ be a measure with density χ with respect to ρ,

$$d\mu_\chi = \chi d\rho. \tag{2.2}$$

Then, by above (2.1) and (2.2), we see that $\mu_\delta = \lambda$ and $\mu_1 = \rho$.

Let $X = \{X_1, \dots, X_n\}$ be a family of left-invariant, linearly independent vector fields which satisfy Hörmander's condition. We recall that these vector fields induce the Carnot-Carathéodory distance $d_C(\cdot,\cdot)$. Let $B = B(c_B, r_B)$ be a ball with respect to such distance, where c_B and r_B are its centre and radius, respectively. We write $|x| := d_C(e, x)$. If $V(r) = \rho(B_r)$ is the volume of the ball $B(e, r) =: B_r$ with respect to the right Haar measure ρ, then it is well-known (see e.g. [10] or [23]) that there exist two constants $d \in \mathbb{N}^*$ and $D > 0$ such that

$$V(r) \approx r^d \quad \forall r \in (0, 1], \tag{2.3}$$

$$V(r) \lesssim \mathrm{e}^{Dr} \quad \forall r \in (1, \infty). \tag{2.4}$$

We say that d and D are local and global dimensions of the metric measure space $(\mathbb{G}, d_C, \rho)$, respectively. Recall that d is uniquely determined by $\mathbb{G}$ and X, while the set of $D > 0$ such that (2.4) holds is independent of X but does not have a minimum in general, see e.g. [6, p. 285], [24, Chapter 4] or [4]. We fix a $D > 0$ such that (2.4) holds. One can observe that the metric measure space $(\mathbb{G}, d_C, \rho)$ is locally doubling, but not doubling in general.

We shall denote by Δ_1 the smallest self-adjoint extension on $L^2(\rho)$ of the "sum-of-squares" operator

$$\Delta_1 = -\sum_{j=1}^{n} X_j^2$$

on $C_0^\infty(\mathbb{G})$. We shall denote with $P_t(\cdot,\cdot)$ and p_t the smooth integral kernel of $\mathrm{e}^{-t\Delta_1}$ and its smooth convolution kernel (i.e. $\mathrm{e}^{-t\Delta_1} f = f * p_t$) with respect to the measure ρ, respectively, where $*$ is the convolution between two functions f and g (when it exists), i.e.

$$f * g(x) = \int_{\mathbb{G}} f(xy^{-1}) g(y) d\rho(y).$$

Recall the following relation

$$P_t(x, y) = p_t(y^{-1}x)\delta(y) \ \ \forall x, y \in \mathbb{G}.$$

It is also known that the generated semigroup $e^{-t\Delta_\chi}$ on $L^2(\mu_\chi)$ admits an integral kernel $P_t^\chi \in \mathcal{D}'(\mathbb{G}\times\mathbb{G})$ with respect to the measure μ_χ

$$e^{-t\Delta_\chi} f(x) = \int_{\mathbb{G}} P_t^\chi(x,y) f(y) d\mu_\chi(y),$$

and admits a convolution kernel $p_t^\chi \in \mathcal{D}'(\mathbb{G})$

$$e^{-t\Delta_\chi} f(x) = f * p_t^\chi(x) = \int_{\mathbb{G}} f(xy^{-1}) p_t^\chi(y) d\rho(y).$$

For P_t^χ and p_t^χ, we have

$$P_t^\chi(x,y) = p_t^\chi(y^{-1}x)\chi^{-1}(y)\delta(y).$$

Denoting $b_X := \frac{1}{2}\left(\sum_{i=1}^n c_i^2\right)^{1/2}$ with $c_i = (X_i\chi)(e)$, we also have

$$p_t^\chi(x) = e^{-tb_X^2} p_t(x)\chi^{-1/2}(x), \tag{2.5}$$

so that P_t^χ and p_t^χ are smooth on $\mathbb{G}\times\mathbb{G}$ and $\mathbb{G}$, respectively.

According to [3], we now recall some useful properties of $L_\alpha^p(\mu_\chi)$. For every $1 < p < \infty$, $\alpha \geq 0$ and $c > 0$, we have

$$\|f\|_{L_\alpha^p(\mu_\chi)} \approx \|(\Delta_\chi + cI)^{\alpha/2} f\|_{L^p(\mu_\chi)} \tag{2.6}$$

and

$$\|(\Delta_\chi + cI)^{\alpha_2/2} f\|_{L^p(\mu_\chi)} \leq \|(\Delta_\chi + cI)^{\alpha_1/2} f\|_{L^p(\mu_\chi)}$$

when $\alpha_1 > \alpha_2$, i.e. $L_{\alpha_1}^p(\mu_\chi) \hookrightarrow L_{\alpha_2}^p(\mu_\chi)$.

Denote by $\mathfrak{I}$ the set $\{1,\ldots,n\}$. Let $\mathfrak{I}^m$ be the set of multi-indices $J = (j_1,\ldots,j_m)$ such that $j_i \in \mathfrak{I}$ for every $m, i \in \mathbb{N}$, and let X_J be the left differential operator $X_J = X_{j_1}\ldots X_{j_m}$ for $J \in \mathfrak{I}^m$.

Proposition 2.1 [3, Propositions 3.3 and 3.4]

- *Let $k \in \mathbb{N}$ and $1 < p < \infty$. Then, we have*

$$\|f\|_{L_k^p(\mu_\chi)} \approx \sum_{J\in\mathfrak{I}^m, m\leq k} \|X_J f\|_{L^p(\mu_\chi)}.$$

- *For every $\alpha \geq 0$ and $1 < p < \infty$, we have*

$$f \in L_{\alpha+1}^p(\mu_\chi) \Leftrightarrow f \in L_\alpha^p(\mu_\chi) \;\; \textit{and} \;\; X_i f \in L_\alpha^p(\mu_\chi)$$

for every $i \in \mathfrak{I}$*. In particular,*

$$\|f\|_{L^p_{\alpha+1}(\mu_\chi)} \approx \|f\|_{L^p_\alpha(\mu_\chi)} + \sum_{i=1}^n \|X_i f\|_{L^p_\alpha(\mu_\chi)}.$$

Proposition 2.2 [11, Proposition 5.7 (ii)] *Let* $\mathbb{G}$ *be a noncompact connected Lie group. Let* $\|X\| = \left(\sum_{i=1}^n c_i^2\right)^{1/2}$ *with* $c_i = (X_i\chi)(e)$*,* $i \in \mathfrak{I}$*. Then, for every* $r \in \mathbb{R}^+$*, we have*

$$\sup_{x\in B_r} \chi(x) = \mathrm{e}^{\|X\|r}.$$

We also recall that for every character χ and $R > 0$, there exists a constant $c = c(\chi, R)$ such that

$$c^{-1}\chi(x) \le \chi(y) \le c\chi(x) \quad \forall x, y \in \mathbb{G} \text{ s.t. } d_C(x, y) \le R, \tag{2.7}$$

which means that the metric measure space $(\mathbb{G}, d_C, \mu_\chi)$ is locally doubling.

Lemma 2.3 [3, Lemma 2.3] *Let* $\mathbb{G}$ *be a noncompact connected Lie group. Then, we have*

(i) $\mathrm{e}^{-t\Delta_\chi}$ *is a diffusion semigroup on* $(\mathbb{G}, \mu_\chi)$*;*
(ii) *Let* χ *be a positive character of* $\mathbb{G}$*. Then, we have* $\int_{B_r} \chi d\rho \le \mathrm{e}^{(\|X\|+D)r}$ *for every* $r > 1$*, where* $\|X\| = \left(\sum_{i=1}^n c_i^2\right)^{1/2}$ *with* $c_i = (X_i\chi)(e)$*,* $i \in \mathfrak{I}$*.*
(iii) *Furthermore, there exist two positive constants* ω *and* b *such that, for every* $m \in \mathbb{N}$ *and* $J \in \mathfrak{I}^m$*, we have* $|X_J p_t^\chi(x)| \lesssim \chi^{-1/2}(x) t^{-(d+m)/2} \mathrm{e}^{\omega t} \mathrm{e}^{-b|x|^2/t}$*, for all* $t > 0$ *and* $x \in \mathbb{G}$*.*

By virtue of the next proposition, proofs of Theorems 1.1 and 1.3 can be reduced to proofs of Theorems 3.2 and 3.4, respectively:

Proposition 2.4 [3, Proposition 3.5] *Let* $p \in (1, \infty)$ *and* $\alpha \ge 0$*. Then, we have*

$$\|f\|_{L^p_\alpha(\mu_\chi)} \approx \|\chi^{1/p} f\|_{L^p_\alpha(\rho)}.$$

We note (see also [3]) that the function

$$G^c_{\alpha,\chi}(x) = C(\alpha) \int_0^\infty t^{\alpha/2-1} \mathrm{e}^{-ct} p_t^\chi(x) dt \tag{2.8}$$

is the convolution kernel of the operator $(\Delta_\chi + cI)^{-\alpha/2}$, i.e.

$$(\Delta_\chi + cI)^{-\alpha/2} f = f * G^c_{\alpha,\chi}, \tag{2.9}$$

where $c > \omega$ and ω is from Lemma 2.3.

Lemma 2.5 [3, Lemma 4.1] *Let b and ω be as in Lemma 2.3. Let $c > \omega$ and $c' = \frac{1}{2}\sqrt{b(c-\omega)}$. Then, we have*

$$|G^{c}_{\alpha,\chi}| \leq C \begin{cases} |x|^{\alpha-d} \ \ \textit{if} \ \ 0 < \alpha < d \ \ \textit{and} \ \ |x| \leq 1, \\ \chi^{-1/2}(x) \mathrm{e}^{-c'|x|} \ \ \textit{when} \ \ |x| > 1 \end{cases} \tag{2.10}$$

for some positive constant C.

We will also use Young's inequalities in the following form:

Lemma 2.6 [3, Lemma 4.3] *Let $1 < p \leq q < \infty$ and $r \geq 1$ be such that $1/p + 1/r = 1 + 1/q$. Then, we have*

$$\|f * g\|_{L^q(\lambda)} \leq \|f\|_{L^p(\lambda)} (\|\check{g}\|^{r/p'}_{L^r(\lambda)} \|g\|^{r/q}_{L^r(\lambda)}), \tag{2.11}$$

where $\check{g}(x) = g(x^{-1})$.

Remark 2.7 For a simpler version of Young's inequality on general locally compact groups, we refer to [13, cf. Lemma 2.1].

The following integral Hardy inequalities on general metric measure spaces, which are the special cases of [12, Theorems 2.1 and 3.1 (a)] (see also [22]), play important role in the proof of the main results:

Theorem 2.8 *Let $\mathbb{X}$ be a metric measure space with a σ-finite measure. Let 0 be a fixed element of $\mathbb{X}$ and $|x| = d(0,x)$. Let $1 < p \leq q < \infty$. Let $\{\phi_i\}_{i=1}^2$ and $\{\psi_i\}_{i=1}^2$ be positive functions on $\mathbb{X}$. Then, the inequalities*

$$\left(\int_{\mathbb{X}} \left(\int_{B(0,|x|/2)} f(z)dz\right)^q \phi_1(x)dx\right)^{\frac{1}{q}} \leq A_1 \left(\int_{\mathbb{X}} (f(x))^p \psi_1(x)dx\right)^{\frac{1}{p}} \tag{2.12}$$

and

$$\left(\int_{\mathbb{X}} \left(\int_{\mathbb{X}\backslash B(0,2|x|)} f(z)dz\right)^q \phi_2(x)dx\right)^{\frac{1}{q}} \leq A_2 \left(\int_{\mathbb{X}} (f(x))^p \psi_2(x)dx\right)^{\frac{1}{p}} \tag{2.13}$$

hold for all $f \geq 0$ a.e. on $\mathbb{X}$ if we have

$$B_1 := \sup_{R>0} \left(\int_{\{|x|\geq R\}} \phi_1(x)dx\right)^{\frac{1}{q}} \left(\int_{\{|x|<R\}} (\psi_1(x))^{1-p'}dx\right)^{\frac{1}{p'}} < \infty \tag{2.14}$$

and

$$B_2 := \sup_{R>0} \left(\int_{\{|x|\leq R\}} \phi_2(x)dx\right)^{\frac{1}{q}} \left(\int_{\{|x|\geq R\}} (\psi_2(x))^{1-p'}dx\right)^{\frac{1}{p'}} < \infty, \tag{2.15}$$

respectively. Moreover, if $\{A_i\}_{i=1}^2$ *are the smallest constants for which (2.12) and (2.13) hold, then*

$$c_1^i B_i \leq A_i \leq c_2^i B_i, \quad i = 1, 2. \tag{2.16}$$

Remark 2.9 In the setting of metric measure spaces, first Theorem 2.8 was proved in [19] on metric measure spaces possessing polar decompositions. We can also refer to [2] for the analysis of polar decompositions in metric measure spaces. To avoid such technicalities, we will be using this result as it follows from [12].

3 Proof of Main Results

3.1 Proof of Theorem 1.1

In this section, we prove Theorems 1.1, 1.3, 1.5, 1.8 and Corollary 1.10 when $\mathbb{G}$ is noncompact, and in the case when $\mathbb{G}$ is compact we refer to Sect. 4 for the differences in the argument in this setting.

Before starting the proof, we need to prove the following lemma:

Lemma 3.1 *Let* $a, s \in \mathbb{R}$ *and* $r > 0$. *If* $c' > 0$ *is sufficiently large, then we have*

$$\int_{B_1^c} |\delta^a \chi^s e^{-c'|x|}|^r d\rho < \infty. \tag{3.1}$$

Actually, the proof of this lemma follows from the proof of [3, Corollary 4.2], but to be more precise, let us give it.

Proof of Lemma 3.1 Taking into account Lemma 2.3 (ii), a direct calculation gives that

$$\begin{aligned}\int_{B_1^c} |\delta^a \chi^s e^{-c'|x|}|^r d\rho &= \sum_{k=0}^{\infty} e^{-rc'2^k} \int_{\{2^k \leq |x| < 2^{k+1}\}} (\delta(x))^{ra} (\chi(x))^{rs} d\rho(x) \\ &\lesssim \sum_{k=0}^{\infty} e^{-rc'2^k} e^{C\cdot 2^k} < \infty,\end{aligned}$$

since c' is large enough. □

Once we prove the special case $\chi = \delta$ of Theorem 1.1, then we can immediately obtain Theorem 1.1 by Proposition 2.4. Therefore, let us prove the following theorem:

Theorem 3.2 *Let* $\alpha > 0$, $0 \leq \beta < d$ *and* $1 < p, q < \infty$. *Then, we have*

$$\left\| \frac{f}{|x|^{\frac{\beta}{q}}} \right\|_{L^q(\lambda)} \lesssim \|f\|_{L^p_\alpha(\lambda)} \tag{3.2}$$

for all $q \geq p$ *such that* $1/p - 1/q \leq \alpha/d - \beta/(dq)$.

Remark 3.3 When $\beta = 0$, in [3, Sect. 4], it was shown that an embedding of the form

$$L^p_\alpha(\mu_\chi) \hookrightarrow L^q(\mu_\chi),\ 1 < p < \infty,\ \alpha \geq 0,\ q \in (1, \infty]\backslash\{p\}$$

for some positive character χ may hold only if $\mu_\chi = \lambda$ is the left Haar measure of $\mathbb{G}$. In the exactly same way, one can show that the same statement is true for the weighted Sobolev embedding case. However, this is different for $q = p$, see (1.4).

Proof of Theorem 3.2 Notice that we may reduce to prove Theorem 3.2 when $0 < \alpha < d$, since when $\alpha \geq d$ we may find $0 < \alpha' < d$ such that

$$\frac{\alpha'}{d} - \frac{\beta}{dq} \geq 1 - \frac{1}{q} > \frac{1}{p} - \frac{1}{q},$$

so that the condition $1/p - 1/q \leq \alpha'/d - \beta/(dq)$ holds. Then, we apply Theorem 3.2 to $\alpha' \in (0, d)$ and use the Sobolev embedding $L^p_\alpha(\lambda) \subseteq L^p_{\alpha'}(\lambda)$ (see (1.3) when $q = p$ and $\mu_\chi = \lambda$). Therefore, it is enough to prove Theorem 3.2 when $0 < \alpha < d$.

By (2.6) and (2.9), we note that to obtain (3.2) it is enough to prove the following

$$\int_{\mathbb{G}} |(f * G^c_{\alpha,\chi})(x)|^q \frac{d\lambda(x)}{|x|^\beta} \lesssim \|f\|^q_{L^p(\lambda)}.$$

For this, let us split the left-hand side of above inequality into three parts as follows

$$\int_{\mathbb{G}} |(f * G^c_{\alpha,\chi})(x)|^q \frac{d\lambda(x)}{|x|^\beta} \leq 3^q(M_1 + M_2 + M_3), \tag{3.3}$$

where

$$M_1 := \int_{\mathbb{G}} \left(\int_{\{2|y|<|x|\}} |G^c_{\alpha,\chi}(y^{-1}x) f(y)| d\lambda(y) \right)^q \frac{d\lambda(x)}{|x|^\beta},$$
$$M_2 := \int_{\mathbb{G}} \left(\int_{\{|x|\leq 2|y|<4|x|\}} |G^c_{\alpha,\chi}(y^{-1}x) f(y)| d\lambda(y) \right)^q \frac{d\lambda(x)}{|x|^\beta}$$

and

$$M_3 := \int_{\mathbb{G}} \left(\int_{\{|y|\geq 2|x|\}} |G^c_{\alpha,\chi}(y^{-1}x) f(y)| d\lambda(y) \right)^q \frac{d\lambda(x)}{|x|^\beta}.$$

Let us start by estimating the first term M_1. By using the reverse triangle inequality and $2|y| < |x|$, we have

$$\begin{aligned} &|y^{-1}x| \geq |x| - |y| > |x| - \frac{|x|}{2} = \frac{|x|}{2}, \\ &|y^{-1}x| \leq |x| + |y| < \frac{3|x|}{2}. \end{aligned} \tag{3.4}$$

Taking into account this, for M_1, we write

$$M_1 \leq \int_{\mathbb{G}} \left(\int_{\{2|y|<|x|\}} |f(y)| d\lambda(y) \right)^q \left(\sup_{\{|x|<2|z|<3|x|\}} |G^c_{\alpha,\chi}(z)| \right)^q \frac{d\lambda(x)}{|x|^\beta}.$$

In order to apply the integral Hardy inequality (2.12), let us check the following condition (2.14):

$$\left(\int_{\{2r_0 \leq |x|\}} \left(\sup_{\{|x|<2|z|<3|x|\}} |G^c_{\alpha,\chi}(z)| \right)^q \frac{d\lambda(x)}{|x|^\beta} \right)^{\frac{1}{q}} \left(\int_{\{|x|<2r_0\}} d\lambda(x) \right)^{\frac{1}{p'}} < \infty \quad (3.5)$$

for all $r_0 > 0$. Indeed, once (3.5) has been established, the integral Hardy inequality (2.12) implies

$$M_1^{\frac{1}{q}} \leq C \|f\|_{L^p(\lambda)}, \quad (3.6)$$

where C does not depend on f.

Now, let us check (3.5). By Lemma 2.5, we have

$$\sup_{\{|x|<2|z|<3|x|\}} |G^c_{\alpha,\chi}(z)| \leq C_1 \begin{cases} |x|^{\alpha-d} & \text{if } 0 < \alpha < d \text{ and } |z| \leq 1, \\ e^{C_2|x|} e^{-c'|x|/2} & \text{if } |z| > 1 \end{cases} \quad (3.7)$$

where we have used $\sup_{\{|x|<2|z|<3|x|\}} \chi^{-1/2}(z) \leq e^{C_2|x|}$ by Proposition 2.2.

For this, we consider the following cases: $r_0 > 1$ and $0 < r_0 \leq 1$.

In the case $r_0 > 1$, we have $2 < 2r_0 \leq |x| < 2|z|$. Then, using (3.7) one has

$$\begin{aligned} &\int_{\{2r_0 \leq |x|\}} \left(\sup_{\{|x|<2|z|<3|x|\}} |G^c_{\alpha,\chi}(z)| \right)^q \frac{d\lambda(x)}{|x|^\beta} \leq C_1 \frac{e^{-qc'\frac{r_0}{4}}}{r_0^\beta} \int_{\{2r_0 \leq |x|\}} e^{-c'q\frac{|x|}{4}} e^{C_2 q|x|} d\lambda(x) \\ &= C_1 \frac{e^{-qc'\frac{r_0}{4}}}{r_0^\beta} \sum_{k=0}^{\infty} \int_{\{2^{k+1}r_0 \leq |x| \leq 2^{k+2}r_0\}} e^{-c'q\frac{|x|}{4}} e^{C_2 q|x|} d\lambda(x) \\ &\leq C_1 \frac{e^{-qc'\frac{r_0}{4}}}{r_0^\beta} \sum_{k=0}^{\infty} e^{-c'q2^{k-1}r_0} e^{C_2 2^{k+2} r_0 q} \int_{\{2^{k+1}r_0 \leq |x| \leq 2^{k+2}r_0\}} d\lambda(x) \lesssim r_0^{-\beta} e^{-qc'\frac{r_0}{4}}, \end{aligned} \quad (3.8)$$

where the sum is finite since c (hence c') is large enough.

By Part (ii) of Lemma 2.3, for $r_0 > 1$, we also have

$$\int_{\{|x|<2r_0\}} d\lambda(x) = \int_{\{|x|<2r_0\}} \delta d\rho(x) \leq e^{2Cr_0}. \quad (3.9)$$

Then, plugging (3.8) and (3.9) into (3.5), we obtain

$$\left(\int_{\{2r_0\le|x|\}}\left(\sup_{\{|x|<2|z|<3|x|\}}|G^c_{\alpha,\chi}(z)|\right)^q\frac{d\lambda(x)}{|x|^\beta}\right)^{\frac{1}{q}}\left(\int_{\{|x|<2r_0\}}d\lambda(x)\right)^{\frac{1}{p'}}$$
$$\lesssim r_0^{-\frac{\beta}{q}}\mathrm{e}^{-c'\frac{r_0}{4}}\mathrm{e}^{2C\frac{r_0}{p'}}<\infty, \quad (3.10)$$

since c (hence c') is large enough.

Now, we check the condition (3.5) for $0<r_0\le 1$. As noticed above, when $|z|>1$ due to the exponential decay $G^c_{\alpha,\chi}(z)$ from (3.7), we can easily obtain (3.5). So, let us discuss the case $|z|\le 1$. In this case, taking into account $|x|<2|z|\le 2$, we write

$$\int_{\{2r_0\le|x|\}}\left(\sup_{\{|x|<2|z|<3|x|\}}|G^c_{\alpha,\chi}(z)|\right)^q\frac{d\lambda(x)}{|x|^\beta}$$
$$=\int_{\{2r_0\le|x|\le1\}}\left(\sup_{\{|x|<2|z|<3|x|\}}|G^c_{\alpha,\chi}(z)|\right)^q\frac{d\lambda(x)}{|x|^\beta}$$
$$+\int_{\{1<|x|<2\}}\left(\sup_{\{|x|<2|z|<3|x|\}}|G^c_{\alpha,\chi}(z)|\right)^q\frac{d\lambda(x)}{|x|^\beta}. \quad (3.11)$$

Using estimate (3.7), one can observe that the last integral is finite. To estimate the first integral on the right-hand side of (3.11), we split it into two cases: $(\alpha-d)q-\beta+d\neq 0$ and $(\alpha-d)q-\beta+d=0$. In the first case, taking into account (3.7), we calculate

$$\int_{\{2r_0\le|x|\le1\}}\left(\sup_{\{|x|<2|z|<3|x|\}}|G^c_{\alpha,\chi}(z)|\right)^q\frac{d\lambda(x)}{|x|^\beta}\lesssim\int_{\{2r_0\le|x|\le1\}}|x|^{(\alpha-d)q-\beta}d\lambda(x)$$
$$\lesssim\int_{2r_0}^1 u^{(\alpha-d)q-\beta}u^{d-1}du$$
$$\lesssim 1+r_0^{(\alpha-d)q-\beta+d}. \quad (3.12)$$

For the inequality of passing from the integral with respect to $d\lambda$ to the one with respect to u, we first observe that the left Haar measure is absolutely continuous with respect to the Riemannian measure. Indeed, if one considers a full form on the Lie algebra of $\mathbb{G}$, it can be moved around by the group action to yield the left Haar measure on $\mathbb{G}$. By the uniqueness of the left Haar measure, one gets the absolute continuity as above, with the left Haar measure being a smooth multiple of the volume measure. Consequently, $d\lambda$ is absolutely continuous with respect to the radial measure, see e.g. [2, Corollary 2, p. 81], but the question of the Jacobian remains. However, for an estimate (as opposed to the exact equality), we can give a short direct argument.

Let B_r denote the ball, centred at a fixed point, of radius r with respect to the Carnot-Carathéodory distance, that is, $x\in B_r$ if $|x|<r$. Let us introduce the function $s=s(r)$ given by $s(r):=\lambda(B_r)^{1/d}$, where $d\lambda$ is the left Haar measure on $\mathbb{G}$, and d is

the local dimension of $\mathbb{G}$. Let us identify the balls with radii given by $s(r)$ and r, by writing $\tilde{B}_s = B_r$. Then, we have $\lambda(\tilde{B}_s) = \lambda(B_r) = s^d$. Since $\lambda(B_r) \leq cr^d$, we have that

$$s^d = \lambda(\tilde{B}_s) = \lambda(B_r) \leq cr^d,$$

that is, $s \leq cr$ for some $c > 0$. Consequently, for any $\gamma > 0$, we have $r^{-\gamma} \leq cs^{-\gamma}$ for some $c > 0$, and $B_{r_0} \subset B_{s_0/c}$, for $s_0 = s(r_0)$. Consequently, we can estimate, with $r = |x|$, and using that now we have the equality $\lambda(\tilde{B}_s) = s^d$,

$$\int_{B_{r_0}} r^{-\gamma} d\lambda(x) \leq C \int_{B_{s_0/c}} s^{-\gamma} d\lambda(x) \leq C \int_0^{s_0/c} s^{-\gamma} s^{d-1} ds < \infty, \tag{3.13}$$

provided that $\gamma < d$. Applying and adapting arguments of this type here and in the sequel justify local estimates like the one in (3.12).

Taking into account (3.11) and (3.12) in (3.5), we have for any $0 < r_0 \leq 1$ that

$$\left(\int_{\{2r_0 \leq |x|\}} \left(\sup_{\{|x|<2|z|<3|x|\}} |G^c_{\alpha,\chi}(z)|\right)^q \frac{d\lambda(x)}{|x|^\beta}\right)^{\frac{1}{q}} \left(\int_{\{|x|<2r_0\}} d\lambda(x)\right)^{\frac{1}{p'}}$$
$$\leq Cr_0^{\frac{d}{p'}}(1 + r_0^{\frac{(\alpha-d)q-\beta+d}{q}}) < \infty \tag{3.14}$$

since $1/p - 1/q \leq \alpha/d - \beta/(dq)$.

Now, in the case $(\alpha - d)q - \beta + d = 0$, from (3.12) and noting the fact that $r_0^{\frac{d}{p'}} |\log r_0|^{\frac{1}{q}} \to 0$ as $r_0 \to 0$ we have

$$\left(\int_{\{2r_0 \leq |x|\}} \left(\sup_{\{|x|<2|z|<3|x|\}} |G^c_{\alpha,\chi}(z)|\right)^q \frac{d\lambda(x)}{|x|^\beta}\right)^{\frac{1}{q}} \left(\int_{\{|x|<2r_0\}} d\lambda(x)\right)^{\frac{1}{p'}}$$
$$\leq Cr_0^{\frac{d}{p'}} |\log r_0|^{\frac{1}{q}} < \infty \tag{3.15}$$

for all $0 < r_0 \leq 1$.

Now let us estimate M_3. Similarly to (3.4), it is easy to see that the condition $2|x| < |y|$ implies $|y| < 2|y^{-1}x| < 3|y|$. Then, taking into account this and (3.7), we obtain for M_3 that

$$M_3 \leq C \int_{\mathbb{G}} \left(\int_{\{|y|\geq 2|x|\}} |f(y)| \sup_{\{|y|\leq 2|z|\leq 3|y|\}} |G^c_{\alpha,\chi}(z)| d\lambda(y)\right)^q \frac{d\lambda(x)}{|x|^\beta}.$$

Here, we now apply the conjugate integral Hardy inequality (2.13) for M_3, for which we need to check the following condition (2.15):

$$\left(\int_{\{|x|\leq 2r_0\}} \frac{d\lambda(x)}{|x|^\beta}\right)^{\frac{1}{q}} \left(\int_{\{2r_0\leq|y|\}} \left(\sup_{\{|y|\leq 2|z|\leq 3|y|\}} |G^c_{\alpha,\chi}(z)|\right)^{p'} d\lambda(y)\right)^{\frac{1}{p'}} < \infty \tag{3.16}$$

for all $r_0 > 0$. Indeed, once (3.16) has been established, the conjugate integral Hardy inequality (2.13) yields

$$M_3^{\frac{1}{q}} \leq C\|f\|_{L^p(\lambda)}, \tag{3.17}$$

where C does not depend on f.

For this, we again consider two cases: $r_0 > 1$ and $0 < r_0 \leq 1$. When $r_0 > 1$, hence $|z| > 1$, then as in (3.8), we have

$$\int_{\{2r_0\leq|y|\}} \left(\sup_{\{|y|\leq 2|z|\leq 3|y|\}} |G^c_{\alpha,\chi}(z)|\right)^{p'} d\lambda(y) \leq C\mathrm{e}^{-p'c'\frac{r_0}{4}}. \tag{3.18}$$

Applying Part (ii) of Lemma 2.3, one gets for $r_0 > 1$ that

$$\begin{aligned}\int_{\{|x|\leq 2r_0\}} \frac{d\lambda(x)}{|x|^\beta} &\leq \int_{\{|x|\leq 1\}} \frac{d\lambda(x)}{|x|^\beta} + \int_{\{1<|x|\leq 2r_0\}} d\lambda(x)\\ &\leq C\int_0^1 u^{d-1-\beta}du + \int_{\{|x|\leq 2r_0\}} \delta d\rho(x) \leq C_3 + \mathrm{e}^{C_4 r_0}\end{aligned} \tag{3.19}$$

for some positive constants C_3 and C_4. Then, putting (3.18) and (3.19) in (3.16), we obtain

$$\begin{aligned}&\left(\int_{\{|x|\leq 2r_0\}} \frac{d\lambda(x)}{|x|^\beta}\right)^{\frac{1}{q}} \left(\int_{\{2r_0\leq|y|\}} \left(\sup_{\{|y|\leq 2|z|\leq 3|y|\}} |G^c_{\alpha,\chi}(z)|\right)^{p'} d\lambda(y)\right)^{\frac{1}{p'}}\\ &\quad\leq C(C_3 + \mathrm{e}^{C_4 r_0})^{\frac{1}{q}}\mathrm{e}^{-c'\frac{r_0}{4}} < \infty\end{aligned} \tag{3.20}$$

since c (hence c') is large enough.

Now, let us check the condition (3.16) for $0 < r_0 \leq 1$. When $|z| > 1$, we readily obtain (3.16) because of the estimate (3.7). In the case $|z| \leq 1$, as in (3.11) and (3.12), we obtain for $(\alpha - d)p' + d \neq 0$ that

$$\int_{\{2r_0\leq|y|\}} \left(\sup_{\{|y|\leq 2|z|\leq 3|y|\}} |G^c_{\alpha,\chi}(z)|\right)^{p'} d\lambda(x) \leq C(1 + r_0^{(\alpha-d)p'+d}). \tag{3.21}$$

Using this in (3.16) implies for $0 < r_0 \le 1$ that

$$\left(\int_{\{|x|\le 2r_0\}} \frac{d\lambda(x)}{|x|^{\beta}}\right)^{\frac{1}{q}} \left(\int_{\{2r_0\le|y|\}} \left(\sup_{\{|y|\le 2|z|\le 3|y|\}} |G^c_{\alpha,\chi}(z)|\right)^{p'} d\lambda(y)\right)^{\frac{1}{p'}}$$
$$\le C(1+r_0^{(\alpha-d)p'+d})^{\frac{1}{p'}} r_0^{\frac{d-\beta}{q}} < \infty \tag{3.22}$$

since $d > \beta$ and $1/p - 1/q \le \alpha/d - \beta/(dq)$.

Note that (3.22) is still finite for $(\alpha - d)p' + d = 0$ and $0 < r_0 \le 1$, since as in (3.12) and (3.15), we have

$$\left(\int_{\{|x|\le 2r_0\}} \frac{d\lambda(x)}{|x|^{\beta}}\right)^{\frac{1}{q}} \left(\int_{\{2r_0\le|y|\}} \left(\sup_{\{|y|\le 2|z|\le 3|y|\}} |G^c_{\alpha,\chi}(z)|\right)^{p'} d\lambda(y)\right)^{\frac{1}{p'}}$$
$$\le C|\log r_0|^{\frac{1}{p'}} r_0^{\frac{d-\beta}{q}} < \infty \tag{3.23}$$

since $|\log r_0|^{\frac{1}{p'}} r_0^{\frac{d-\beta}{q}} \to 0$ as $r_0 \to 0$ when $d > \beta$.

Now, it remains to estimate M_2. We rewrite M_2 as

$$M_2 = \sum_{k\in\mathbb{Z}} \int_{\{2^k\le|x|<2^{k+1}\}} \left(\int_{\{|x|\le 2|y|\le 4|x|\}} |G^c_{\alpha,\chi}(y^{-1}x)f(y)|d\lambda(y)\right)^q \frac{d\lambda(x)}{|x|^{\beta}}.$$

We obtain that $2^{k-1} \le |y| < 2^{k+2}$ from $|x| \le 2|y| \le 4|x|$ and $2^k \le |x| < 2^{k+1}$. Let us show that $G^c_{\alpha,\chi} \in L^r(\lambda)$ for $r \in [1,\infty]$ such that $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p}$, which is useful in the rest of proof. Indeed, by Lemmata 2.5 and 3.1, we see that

$$\int_{\mathbb{G}} |G^c_{\alpha,\chi}(x)|^r d\lambda(x) = \int_{\{|x|<1\}} |G^c_{\alpha,\chi}(x)|^r d\lambda(x) + \int_{\{|x|\ge 1\}} |G^c_{\alpha,\chi}(x)|^r d\lambda(x)$$
$$\le \mathfrak{C}_1 \int_0^1 u^{(\alpha-d)r} u^{d-1} du + \mathfrak{C}_2 \int_{\{|x|\ge 1\}} (\chi(x))^{-r/2} \mathrm{e}^{-c'r|x|} \delta d\rho(x) < \infty \tag{3.24}$$

for some positive $\mathfrak{C}_1$ and $\mathfrak{C}_2$, since $\alpha > d(1/p - 1/q)$ and c (hence c') is large enough. Similarly, one can show that

$$\|\check{G}^c_{\alpha,\chi}\|_{L^r(\lambda)} < \infty, \tag{3.25}$$

where $\check{G}^c_{\alpha,\chi}(x) = G^c_{\alpha,\chi}(x^{-1})$.

Then, taking into account (3.24) and (3.25), and applying Young's inequality (2.11) for $1+\frac{1}{q}=\frac{1}{r}+\frac{1}{p}$ with $r\in[1,\infty]$, we calculate

$$\begin{aligned} M_2 &\le C\sum_{k\in\mathbb{Z}}\int_{\{2^k\le|x|<2^{k+1}\}}\left(\int_{\{|x|\le 2|y|\le 4|x|\}}|G^c_{\alpha,\chi}(y^{-1}x)f(y)|d\lambda(y)\right)^q d\lambda(x)\\ &\le C\sum_{k\in\mathbb{Z}}\|[f\cdot\chi_{\{2^{k-1}\le|\cdot|<2^{k+2}\}}]*G^c_{\alpha,\chi}\|^q_{L^q(\lambda)}\\ &\le C\|\check{G}^c_{\alpha,\chi}\|^{qr/p'}_{L^r(\lambda)}\|G^c_{\alpha,\chi}\|^r_{L^r(\lambda)}\sum_{k\in\mathbb{Z}}\|f\cdot\chi_{\{2^{k-1}\le|\cdot|<2^{k+2}\}}\|^q_{L^p(\lambda)}\\ &= C\sum_{k\in\mathbb{Z}}\left(\int_{\{2^k\le|x|<2^{k+1}\}}|f(x)|^p d\lambda(x)\right)^{\frac{q}{p}}\\ &= C\|f\|^q_{L^p(\lambda)}. \end{aligned} \tag{3.26}$$

Thus, (3.6), (3.17), (3.26) and (3.3) complete the proof of Theorem 3.2. □

3.2 Proof of Theorem 1.3

We now prove the critical case $\beta=d$ of Theorem 1.1 on B_1.

As in Sect. 3.1, we first show the special case $\chi=\delta$ of Theorem 1.3, that is,

Theorem 3.4 *Let* $1<p<r<\infty$ *and* $1/p+1/p'=1$. *Then, we have*

$$\left\|\frac{f}{\left(\log\left(e+\frac{1}{|x|}\right)\right)^{\frac{r}{q}}|x|^{\frac{d}{q}}}\right\|_{L^q(\lambda)}\lesssim\|f\|_{L^p_{d/p}(\lambda)} \tag{3.27}$$

for every $q\in[p,(r-1)p')$.

Once we prove Theorem 3.4, then by Proposition 2.4 we obtain immediately Theorem 1.3. Therefore, we only prove Theorem 3.4.

Proof of Theorem 3.4 As in the proof of Theorem 3.2, we split the integral into three parts

$$\int_{\mathbb{G}}|(f*G^c_{d/p,\chi})(x)|^q\frac{d\lambda(x)}{\left|\log\left(e+\frac{1}{|x|}\right)\right|^r|x|^d}\le 3^q(N_1+N_2+N_3), \tag{3.28}$$

where

$$N_1:=\int_{\mathbb{G}}\left(\int_{\{2|y|<|x|\}}|G^c_{d/p,\chi}(y^{-1}x)f(y)|d\lambda(y)\right)^q\frac{d\lambda(x)}{\left|\log\left(e+\frac{1}{|x|}\right)\right|^r|x|^d},$$

$$N_2 := \int_{\mathbb{G}} \left(\int_{\{|x| \le 2|y| < 4|x|\}} |G^c_{d/p,\chi}(y^{-1}x) f(y)| d\lambda(y) \right)^q \frac{d\lambda(x)}{\left|\log\left(\mathrm{e} + \frac{1}{|x|}\right)\right|^r |x|^d}$$

and

$$N_3 := \int_{\mathbb{G}} \left(\int_{\{|y| \ge 2|x|\}} |G^c_{d/p,\chi}(y^{-1}x) f(y)| d\lambda(y) \right)^q \frac{d\lambda(x)}{\left|\log\left(\mathrm{e} + \frac{1}{|x|}\right)\right|^r |x|^d}.$$

First, we estimate N_1. As in the case of M_1, taking into account (3.4) and (3.7), we have

$$N_1 \le \int_{\mathbb{G}} \left(\int_{\{2|y| < |x|\}} |f(y)| d\lambda(y) \right)^q \left(\sup_{\{|x| < 2|z| < 3|x|\}} |G^c_{d/p,\chi}(z)| \right)^q \frac{d\lambda(x)}{\left|\log\left(\mathrm{e} + \frac{1}{|x|}\right)\right|^r |x|^d}. \tag{3.29}$$

Here, we will apply the integral Hardy inequality (2.12), for which we need to check the following condition (2.14):

$$\left(\int_{\{2r_0 \le |x|\}} \left(\sup_{\{|x| < 2|z| < 3|x|\}} |G^c_{d/p,\chi}(z)| \right)^q \frac{d\lambda(x)}{\left|\log\left(\mathrm{e} + \frac{1}{|x|}\right)\right|^r |x|^d} \right)^{\frac{1}{q}} \left(\int_{\{|x| < 2r_0\}} d\lambda(x) \right)^{\frac{1}{p'}} < \infty \tag{3.30}$$

holds for all $r_0 > 0$. Indeed, once (3.30) has been established, the integral Hardy inequality (2.12) gives

$$N_1^{\frac{1}{q}} \le C \|f\|_{L^p(\lambda)}, \tag{3.31}$$

where C does not depend on f.

Let us now verify the condition (3.30). For this, we consider the cases: $r_0 > 1$ and $0 < r_0 \le 1$.

In the case $r_0 > 1$, we have $2 < 2r_0 \le |x| < 2|z|$. For $r_0 > 1$, by (3.7) and (3.8), one has

$$\begin{aligned} &\int_{\{2r_0 \le |x|\}} \left(\sup_{\{|x| < 2|z| < 3|x|\}} |G^c_{d/p,\chi}(z)| \right)^q \frac{d\lambda(x)}{\left|\log\left(\mathrm{e} + \frac{1}{|x|}\right)\right|^r |x|^d} \\ &\le \int_{\{2r_0 \le |x|\}} \left(\sup_{\{|x| < 2|z| < 3|x|\}} |G^c_{d/p,\chi}(z)| \right)^q \frac{d\lambda(x)}{|x|^d} \\ &\lesssim r_0^{-d} \mathrm{e}^{-qc' \frac{r_0}{4}}. \end{aligned} \tag{3.32}$$

Then, as in (3.10), plugging (3.32) and (3.9) into (3.30), we obtain

$$\left(\int_{\{r_0\le|x|\}}\left(\sup_{\{|x|<2|z|<3|x|\}}|G^c_{d/p,\chi}(z)|\right)^q\frac{d\lambda(x)}{|x|^d}\right)^{\frac{1}{q}}\left(\int_{\{|x|<r_0\}}d\lambda(x)\right)^{\frac{1}{p'}}$$
$$\lesssim r_0^{-\frac{d}{q}}\mathrm{e}^{-c'\frac{r_0}{4}}\mathrm{e}^{C\frac{r_0}{p'}}<\infty \tag{3.33}$$

since c (hence c') is large enough.

Let us now check (3.30) for $0<r_0\le 1$. When $|z|>1$ using the exponential decay estimate of $G^c_{d/p,\chi}(z)$ from (3.7), it is easy to verify (3.30). So, let us show the case $|z|\le 1$. In this case, taking into account $|x|<2|z|\le 2$, we write

$$\int_{\{2r_0\le|x|\}}\left(\sup_{\{|x|<2|z|<3|x|\}}|G^c_{d/p,\chi}(z)|\right)^q\frac{d\lambda(x)}{\left|\log\left(\mathrm{e}+\frac{1}{|x|}\right)\right|^r|x|^d}$$
$$=\int_{\{2r_0\le|x|\le1\}}\left(\sup_{\{|x|<2|z|<3|x|\}}|G^c_{d/p,\chi}(z)|\right)^q\frac{d\lambda(x)}{\left|\log\left(\mathrm{e}+\frac{1}{|x|}\right)\right|^r|x|^d}$$
$$+\int_{\{1<|x|<2\}}\left(\sup_{\{|x|<2|z|<3|x|\}}|G^c_{d/p,\chi}(z)|\right)^q\frac{d\lambda(x)}{\left|\log\left(\mathrm{e}+\frac{1}{|x|}\right)\right|^r|x|^d}. \tag{3.34}$$

We see from (3.7) that the second integral on the right-hand side of (3.34) is finite. For the first integral on the right-hand side, noting (3.7) we deduce that

$$\int_{\{2r_0\le|x|\le1\}}\left(\sup_{\{|x|<2|z|<3|x|\}}|G^c_{d/p,\chi}(z)|\right)^q\frac{d\lambda(x)}{\left|\log\left(\mathrm{e}+\frac{1}{|x|}\right)\right|^r|x|^d}$$
$$\le\int_{\{2r_0\le|x|\le1\}}\left(\sup_{\{|x|<2|z|<3|x|\}}|G^c_{d/p,\chi}(z)|\right)^q\frac{d\lambda(x)}{|x|^d}$$
$$\lesssim\int_{\{2r_0\le|x|\le1\}}|x|^{-dq/p'-d}d\lambda(x)$$
$$\lesssim r_0^{-dq/p'},$$

which implies with (3.34) that

$$\left(\int_{\{2r_0\le|x|\}}\left(\sup_{\{|x|<2|z|<3|x|\}}|G^c_{d/p,\chi}(z)|\right)^q\frac{d\lambda(x)}{\left|\log\left(\mathrm{e}+\frac{1}{|x|}\right)\right|^r|x|^d}\right)^{\frac{1}{q}}\left(\int_{\{|x|<2r_0\}}d\lambda(x)\right)^{\frac{1}{p'}}$$
$$\le C(r_0^{-d/p'}+1)r_0^{d/p'}\le C$$

for any $0<r_0\le 1$.

Now we estimate N_3. As in the case for M_3, we have for N_3 that

$$N_3 \leq \int_{\mathbb{G}} \left(\int_{\{|y| \geq 2|x|\}} |f(y)| \left(\sup_{\{|y| \leq 2|z| \leq 3|y|\}} |G^c_{d/p,\chi}(z)| \right) d\lambda(y) \right)^q \frac{d\lambda(x)}{\left| \log\left(\mathrm{e} + \frac{1}{|x|}\right) \right|^r |x|^d}.$$

For N_3, we apply the conjugate integral Hardy inequality (2.13), for which we need to check the following condition (2.15):

$$\left(\int_{\{|x| \leq 2r_0\}} \frac{d\lambda(x)}{\left| \log\left(\mathrm{e} + \frac{1}{|x|}\right) \right|^r |x|^d} \right)^{\frac{1}{q}} \times \left(\int_{\{2r_0 \leq |y|\}} \left(\sup_{\{|y| \leq 2|z| \leq 3|y|\}} |G^c_{d/p,\chi}(z)| \right)^{p'} d\lambda(y) \right)^{\frac{1}{p'}} < \infty \tag{3.35}$$

for all $r_0 > 0$. Indeed, once (3.35) has been established, the conjugate integral Hardy inequality (2.13) yields

$$N_3^{\frac{1}{q}} \leq C \|f\|_{L^p(\lambda)}, \tag{3.36}$$

where C does not depend on f.

In order to check this, we consider the cases: $r_0 > 1$ and $0 < r_0 \leq 1$. If we write

$$\int_{\{|x| \leq 2r_0\}} \frac{dx}{\left| \log\left(\mathrm{e} + \frac{1}{|x|}\right) \right|^r |x|^d} = \int_{\left\{|x| < \frac{1}{2}\right\}} \frac{dx}{\left| \log\left(\mathrm{e} + \frac{1}{|x|}\right) \right|^r |x|^d} + \int_{\left\{\frac{1}{2} \leq |x| \leq 2r_0\right\}} \frac{dx}{\left| \log\left(\mathrm{e} + \frac{1}{|x|}\right) \right|^r |x|^d},$$

then we see that the first summand on the right-hand side of above is finite since $r > 1$. For the second term, using (3.9) we have

$$\int_{\left\{\frac{1}{2} \leq |x| \leq 2r_0\right\}} \frac{d\lambda(x)}{\left| \log\left(\mathrm{e} + \frac{1}{|x|}\right) \right|^r |x|^d} \leq \int_{\left\{\frac{1}{2} \leq |x| \leq 2r_0\right\}} \frac{d\lambda(x)}{|x|^d} \leq 2^d \mathrm{e}^{C_5 r_0} \tag{3.37}$$

for some positive constant C_5. Combining (3.18) and (3.37), one obtains for $r_0 > 1$ that

$$\left(\int_{\{|x| \leq 2r_0\}} \frac{d\lambda(x)}{\left| \log\left(\mathrm{e} + \frac{1}{|x|}\right) \right|^r |x|^d} \right)^{\frac{1}{q}} \left(\int_{\{2r_0 \leq |y|\}} \left(\sup_{\{|y| \leq 2|z| \leq 3|y|\}} |G^c_{d/p,\chi}(z)| \right)^{p'} d\lambda(y) \right)^{\frac{1}{p'}}$$
$$\lesssim (1 + 2^d \mathrm{e}^{C_5 r_0})^{\frac{1}{q}} \mathrm{e}^{-c' \frac{r_0}{4}} < \infty,$$

since c (hence c') is large enough.

Now we check the condition (3.35) for $0 < r_0 \leq 1$. As above, for $|z| > 1$ using (3.7), it is straightforward to get (3.35). So, for $|z| \leq 1$, we write

$$\begin{aligned}
&\int_{\{2r_0\leq |y|\}} \left(\sup_{\{|y|\leq 2|z|\leq 3|y|\}} |G^c_{d/p,\chi}(z)| \right)^{p'} d\lambda(y) \\
&= \int_{\{2r_0\leq |y|\leq 1\}} \left(\sup_{\{|y|\leq 2|z|\leq 3|y|\}} |G^c_{d/p,\chi}(z)| \right)^{p'} d\lambda(y) \\
&\quad + \int_{\{|y|>1\}} \left(\sup_{\{|y|\leq 2|z|\leq 3|y|\}} |G^c_{d/p,\chi}(z)| \right)^{p'} d\lambda(y).
\end{aligned} \tag{3.38}$$

We note from (3.18) that the second integral on the right-hand side of above is finite. Then, by (3.7), we get for the first integral that

$$\int_{\{2r_0\leq |y|\leq 1\}} \left(\sup_{\{|y|\leq 2|z|\leq 3|y|\}} |G^c_{d/p,\chi}(z)| \right)^{p'} d\lambda(y) \leq C \int_{\{2r_0\leq |y|\leq 1\}} |y|^{-d} d\lambda(y) \leq C \log\left(\frac{1}{r_0}\right).$$

It follows with (3.38) that

$$\int_{\{2r_0\leq |y|\}} \left(\sup_{\{|y|\leq 2|z|\leq 3|y|\}} |G^c_{d/p,\chi}(z)| \right)^{p'} d\lambda(y) \leq C \left(1 + \log\left(\frac{1}{r_0}\right)\right). \tag{3.39}$$

Since we have

$$\int_{\{|x|\leq 2r_0\}} \frac{dx}{\left|\log\left(e + \frac{1}{|x|}\right)\right|^r |x|^d} \leq C \left(\log\left(e + \frac{1}{r_0}\right)\right)^{-(r-1)},$$

and (3.39), then taking into account $r > 1$ and $q < (r-1)p'$, we obtain that

$$\begin{aligned}
&\left(\int_{\{|x|\leq 2r_0\}} \frac{d\lambda(x)}{\left|\log\left(e + \frac{1}{|x|}\right)\right|^r |x|^d} \right)^{\frac{1}{q}} \left(\int_{\{2r_0\leq |y|\}} \left(\sup_{\{|y|\leq 2|z|\leq 3|y|\}} |G^c_{d/p,\chi}(z)| \right)^{p'} d\lambda(y) \right)^{\frac{1}{p'}} \\
&\qquad \leq C \left(\log\left(e + \frac{1}{r_0}\right)\right)^{-\frac{r-1}{q}} \left(1 + \left(\log\left(\frac{1}{r_0}\right)\right)^{\frac{1}{p'}}\right) \\
&\qquad \leq C.
\end{aligned} \tag{3.40}$$

Now it remains to estimate N_2. We rewrite N_2 as

$N_2 =$

$$\sum_{k\in\mathbb{Z}}\int_{\{2^k\le|x|<2^{k+1}\}}\left(\int_{\{|x|\le 2|y|\le 4|x|\}}|G^c_{d/p}(y^{-1}x)f(y)|d\lambda(y)\right)^q\frac{d\lambda(x)}{\left|\log\left(\mathrm{e}+\frac{1}{|x|}\right)\right|^r|x|^d}.$$

Since the function $\left(\log\left(\frac{1}{|x|}\right)\right)^r|x|^d$ is non-decreasing with respect to $|x|$ near the origin, then we can say that there exists an integer $k_0\in\mathbb{Z}$ with $k_0\le -3$ such that this function is non-decreasing in $|x|\in(0,2^{k_0+1})$. We decompose N_2 with this k_0 as follows

$$N_2 = N_{21}+N_{22}, \tag{3.41}$$

where

$$N_{21} := \sum_{k=-\infty}^{k_0}\int_{\{2^k\le|x|<2^{k+1}\}}\left(\int_{\{|x|\le 2|y|\le 4|x|\}}|G^c_{d/p}(y^{-1}x)f(y)|d\lambda(y)\right)^q\frac{d\lambda(x)}{\left|\log\left(\mathrm{e}+\frac{1}{|x|}\right)\right|^r|x|^d}$$

and

$$N_{22} := \sum_{k=k_0+1}^{\infty}\int_{\{2^k\le|x|<2^{k+1}\}}\left(\int_{\{|x|\le 2|y|\le 4|x|\}}|G^c_{d/p}(y^{-1}x)f(y)|d\lambda(y)\right)^q\frac{d\lambda(x)}{\left|\log\left(\mathrm{e}+\frac{1}{|x|}\right)\right|^r|x|^d}.$$

Let us first estimate N_{22}. Using (3.26), we obtain the following estimate for N_{22}

$$N_{22}\le C\sum_{k=k_0+1}^{\infty}\int_{\{2^k\le|x|<2^{k+1}\}}\left(\int_{\{|x|\le 2|y|\le 4|x|\}}|G^c_{d/p}(y^{-1}x)f(y)|dy\right)^q dx\le C\|f\|^q_{L^p(\lambda)}. \tag{3.42}$$

To complete the proof of Theorem 3.4, it is left to estimate N_{21}. Note that the condition $|y|\le 2|x|$ implies

$$3|x| = |x|+2|x|\ge|x|+|y|\ge|y^{-1}x|. \tag{3.43}$$

Since $\left(\log\left(\frac{1}{|x|}\right)\right)^r|x|^d$ is non-decreasing in $|x|\in(0,2^{k_0+1})$ and $3|x|\ge|y^{-1}x|$, we get

$$\left(\log\left(\frac{1}{|x|}\right)\right)^r|x|^d\ge\left(\log\left(\frac{1}{\left|\frac{y^{-1}x}{3}\right|}\right)\right)^r\left|\frac{y^{-1}x}{3}\right|^d.$$

Then, these and (3.7) give

$$N_{21}\le C\sum_{k=-\infty}^{k_0}\int_{\{2^k\le|x|<2^{k+1}\}}\left(\int_{\{|x|\le 2|y|\le 4|x|\}}|y^{-1}x|^{-\frac{d}{p'}}|f(y)|d\lambda(y)\right)^q\frac{d\lambda(x)}{\left(\log\left(\frac{1}{|x|}\right)\right)^r|x|^d}$$

$$= \quad C\sum_{k=-\infty}^{k_0}\int_{\{2^k\le|x|<2^{k+1}\}}\left(\int_{\{|x|\le 2|y|\le 4|x|\}}\frac{|y^{-1}x|^{-\frac{d}{p'}}|f(y)|}{\left(\left(\log\left(\frac{1}{|x|}\right)\right)^r|x|^d\right)^{\frac{1}{q}}}d\lambda(y)\right)^q d\lambda(x)$$

$$\le C\sum_{k=-\infty}^{k_0}\int_{\{2^k\le|x|<2^{k+1}\}}\left(\int_{\{|x|\le 2|y|\le 4|x|\}}\frac{|y^{-1}x|^{-\frac{d}{p'}}|f(y)|d\lambda(y)}{\left(\left(\log\left(\frac{3}{|y^{-1}x|}\right)\right)^r\left|\frac{y^{-1}x}{3}\right|^d\right)^{\frac{1}{q}}}\right)^q d\lambda(x).$$

Since the conditions $|x|\le 2|y|\le 4|x|$ and $2^k\le|x|<2^{k+1}$ with $k\le k_0$ imply $2^{k-1}\le|y|<2^{k+2}$, while (3.43) and $k_0\le -3$ yield $|y^{-1}x|\le 3|x|<3\cdot 2^{k_0+1}\le 3/4$. By these and setting

$$g_2(x):=\frac{\chi_{B_{\frac{3}{4}}(0)}(x)}{\left(\log\left(\frac{1}{|x|}\right)\right)^{\frac{r}{q}}|x|^{\frac{d}{q}+\frac{d}{p'}}},$$

we obtain for N_{21} that

$$N_{21}$$

$$\le C\sum_{k=-\infty}^{k_0}\int_{\{2^k\le|x|<2^{k+1}\}}\left(\int_{\{|x|\le 2|y|\le 4|x|\}}\frac{|f(y)|d\lambda(y)}{\left(\log\left(\frac{1}{|y^{-1}x|}\right)\right)^{\frac{r}{q}}|y^{-1}x|^{\frac{d}{q}+\frac{d}{p'}}}\right)^q d\lambda(x)$$

$$\le C\sum_{k=-\infty}^{k_0}\|[f\cdot\chi_{\{2^{k-1}\le|\cdot|<2^{k+2}\}}]*g\|^q_{L^q(\lambda)}.$$

Since $p\le q<(r-1)p'$, we apply Young's inequality (2.11) for $1+\frac{1}{q}=\frac{1}{\tilde{r}}+\frac{1}{p}$ with $\tilde{r}\in[1,\infty)$ to get

$$N_{21}\le C\|g_2\|^q_{L^{\tilde{r}}(\lambda)}\sum_{k=-\infty}^{k_0}\|f\cdot\chi_{\{2^{k-1}\le|\cdot|<2^{k+2}\}}\|^q_{L^p(\lambda)}\le C\|f\|^q_{L^p(\lambda)}, \tag{3.44}$$

provided that $g_2\in L^{\tilde{r}}(\lambda)$. Since $\left(\frac{d}{q}+\frac{d}{p'}\right)\tilde{r}=d$, $\frac{r\tilde{r}}{q}=\frac{rp'}{p'+q}$ and $q<(r-1)p'$, then the change of the variable $t=\log\left(\frac{1}{|x|}\right)$ gives

$$\|g_2\|^{\tilde{r}}_{L^{\tilde{r}}(\lambda)}=\int_{B(0,3/4)}\frac{d\lambda(x)}{\left(\log\left(\frac{1}{|x|}\right)\right)^{\frac{rp'}{p'+q}}|x|^d}\le C\int_{\log\left(\frac{4}{3}\right)}^{\infty}\frac{dt}{t^{\frac{rp'}{p'+q}}}<\infty.$$

Thus, (3.29), (3.36), (3.41), (3.42), (3.44) and (3.28) complete the proof of Theorem 3.4. □

3.3 Proof of Theorems 1.5, 1.8, 1.9 and Corollary 1.10

First, let us prove Theorem 1.5, using the Hardy–Sobolev–Rellich inequality (1.2).

Proof of Theorem 1.5 Since $\theta > (r-q)/r$, using Hölder's inequality for $\frac{q-(1-\theta)r}{q} + \frac{(1-\theta)r}{q} = 1$, we calculate

$$\begin{aligned}
\| |x|^a f\|_{L^r(\mu_{\chi^{\widetilde{q}/p}\delta^{1-\widetilde{q}/p}})} &= \left(\int_{\mathbb{G}} \frac{|f(x)|^{\theta r}}{|x|^{r(b(1-\theta)-a)}} \cdot \frac{|f(x)|^{(1-\theta)r}}{|x|^{-br(1-\theta)}} d\mu_{\chi^{\widetilde{q}/p}\delta^{1-\widetilde{q}/p}}(x)\right)^{\frac{1}{r}} \\
&\leq \left(\left(\int_{\mathbb{G}} \frac{|f(x)|^{\theta r \frac{q}{q-(1-\theta)r}}}{|x|^{r(b(1-\theta)-a)\frac{q}{q-(1-\theta)r}}} d\mu_{\chi^{\widetilde{q}/p}\delta^{1-\widetilde{q}/p}}(x)\right)^{\frac{q-(1-\theta)r}{q}} \right.\\
&\quad \left. \times \left(\int_{\mathbb{G}} \frac{|f(x)|^{(1-\theta)r\frac{q}{(1-\theta)r}}}{|x|^{-br(1-\theta)\frac{q}{(1-\theta)r}}} d\mu_{\chi^{\widetilde{q}/p}\delta^{1-\widetilde{q}/p}}(x)\right)^{\frac{(1-\theta)r}{q}}\right)^{\frac{1}{r}} \\
&= \left\| \frac{f}{|x|^{\frac{b(1-\theta)-a}{\theta}}} \right\|^{\theta}_{L^{\frac{qr\theta}{q-(1-\theta)r}}(\mu_{\chi^{\widetilde{q}/p}\delta^{1-\widetilde{q}/p}})} \| |x|^b f\|^{1-\theta}_{L^q(\mu_{\chi^{\widetilde{q}/p}\delta^{1-\widetilde{q}/p}})}.
\end{aligned}$$

Now, since we have $\alpha > 0$, $p \leq q\theta r/(q-(1-\theta)r)$, $0 \leq qr(b(1-\theta)-a)/(q-(1-\theta)r) < d$ and $1/p - (q-(1-\theta)r)/(qr\theta) \leq \alpha/d - (b(1-\theta)-a)/(\theta d)$, then applying (1.2) we obtain (1.8).

Similarly, one can obtain Theorem 1.8 from Theorem 1.3.

Now let us give the proof of Hardy-Littlewood-Sobolev inequality (1.11) on general Lie group:

Proof of Theorem 1.9 Using Hölder's inequality for $q/(p+q) + p/(p+q) = 1$, one has

$$\begin{aligned}
&\left|\int_{\mathbb{G}}\int_{\mathbb{G}} \frac{\overline{f(x)} g(y) G^c_{a_2,\chi}(y^{-1}x)}{|x|^{a_1}|y|^{\beta}} d\mu_{\chi^{(p+q)/pq}\delta^{1-(p+q)/pq}}(x) d\rho(y)\right| \\
&\quad = \left|\int_{\mathbb{G}} \overline{f(x)} \frac{\left(\frac{g}{|x|^{\beta}} * G^c_{a_2,\chi}\right)(x)}{|x|^{a_1}} d\mu_{\chi^{(p+q)/pq}\delta^{1-(p+q)/pq}}(x)\right| \\
&\quad \leq \|f\|_{L^{(p+q)/q}(\mu_{\chi^{(p+q)/pq}\delta^{1-(p+q)/pq}})} \left\| \frac{\frac{g}{|x|^{\beta}} * G^c_{a_2,\chi}}{|x|^{a_1}} \right\|_{L^{(p+q)/p}(\mu_{\chi^{(p+q)/pq}\delta^{1-(p+q)/pq}})}.
\end{aligned} \tag{3.45}$$

Since $\alpha \geq 0$, $1/p - q/(p+q) \leq \alpha/d$, and the fact that $0 \leq 1/p - q/(p+q)$ implies $(p+q)/q \geq p$, then applying unweighted version of the Hardy–Sobolev–Rellich inequality (1.2), we have

$$\|f\|_{L^{(p+q)/q}(\mu_{\chi^{(p+q)/pq}\delta^{1-(p+q)/pq}})} \lesssim \|f\|_{L^p_\alpha(\mu_\chi)}. \tag{3.46}$$

Since $0 \le a_1 < dp/(p+q)$, $a_2 > 0$ and $1/q - p/(p+q) \le (a_2 - a_1)/d$, then (1.2) implies

$$\left\| \frac{\frac{g}{|x|^\beta} * G^c_{a_2,\chi}}{|x|^{a_1}} \right\|_{L^{(p+q)/p}(\mu_{\chi^{(p+q)/pq}\delta^{1-(p+q)/pq}})} \lesssim \left\| \frac{g}{|x|^\beta} \right\|_{L^q(\mu_\chi)} \lesssim \|g\|_{L^q(\mu_\chi)}, \tag{3.47}$$

where we have used (1.4) in the last inequality since $0 \le \beta < d/q$. Thus, plugging (3.46) and (3.47) into (3.45), we obtain (1.11).

Now we prove Corollary 1.10.

Proof of Corollary 1.10 By (1.2) and Hölder's inequality for $1/q + 1/q' = 1$, we obtain

$$\begin{aligned} &\|f\|_{L^p_\alpha(\mu_\chi)} \| |x|^{\frac{\beta}{q}} f \|_{L^{q'}(\mu_{\chi^{q/p}\delta^{1-q/p}})} \\ &\gtrsim \left\| \frac{f}{|x|^{\frac{\beta}{q}}} \right\|_{L^q(\mu_{\chi^{q/p}\delta^{1-q/p}})} \| |x|^{\frac{\beta}{q}} f \|_{L^{q'}(\mu_{\chi^{q/p}\delta^{1-q/p}})} \\ &\ge \|f\|^2_{L^2(\mu_{\chi^{q/p}\delta^{1-q/p}})}, \end{aligned}$$

which is (1.12).

Similarly, Theorem 1.3 implies the second part of Corollary 1.10.

4 Appendix: The Case of Compact Lie Groups

In this section, we show that the obtained results on noncompact Lie groups actually hold also on compact Lie groups in a similar way. In the setting of compact Lie groups, we have $\delta = 1$ hence $d\lambda = d\rho$, and the continuous positive character χ must be identically equal to 1. We refer to [17] for the background material as well as the Fourier analysis on compact Lie groups.

Let us recall the following result:

Theorem 4.1 *[24, VIII.2.9 Theorem] If $\mathbb{G}$ has a polynomial growth, there exist two positive constants C_1 and C_2 such that*

$$C_1 V(\sqrt{t})^{-1} \exp(C_2|x|^2/t) \le p_t(x) \le C_2 V(\sqrt{t})^{-1} \exp(-C_1|x|^2/t) \tag{4.1}$$

for all $t > 0$ and $x \in \mathbb{G}$.

Now we give an analogue of Lemma 2.5 on compact Lie groups when $0 < \alpha < d$, since we have actually used only this case of Lemma 2.5 in the noncompact case:

Lemma 4.2 *Let $0 < \alpha < d$. If $c > 0$ is sufficiently large, then we have*

$$|G^c_{\alpha,\chi}| \leq C|x|^{\alpha-d} \tag{4.2}$$

for all $x \in \mathbb{G}$ and some positive constant C.

Proof of Lemma 4.2 Taking into account Theorem 4.1 with (2.3) and (2.4) as well as the relation (2.8), we have

$$\begin{aligned} |G^c_{\alpha,\chi}(x)| &= \left| C(\alpha) \int_0^\infty t^{\alpha/2-1} \mathrm{e}^{-ct} p_t(x) dt \right| \\ &\lesssim \int_0^1 t^{(\alpha-d)/2-1} \mathrm{e}^{-ct} \mathrm{e}^{-C|x|^2/t} dt \\ &+ \int_1^\infty t^{(\alpha-D)/2-1} \mathrm{e}^{-ct} \mathrm{e}^{-C|x|^2/t} dt =: G_1(x) + G_2(x). \end{aligned}$$

It is easy to see that $G_2(x) \lesssim 1$, since c is large enough.

In the exact same way as in the proof of Lemma 2.5 (see [3, Proof of Lemma 4.1]), using the change of variables $|x|^2/t = u$, we arrive at

$$G_1(x) \lesssim \int_0^1 t^{(\alpha-d)/2-1} \mathrm{e}^{-C|x|^2/t} dt = |x|^{\alpha-d} \int_{|x|^2}^\infty u^{\frac{d-\alpha}{2}} \mathrm{e}^{-Cu} \frac{du}{u},$$

which gives the estimate (4.2). ☐

Since we have Lemma 4.2, $\sup\limits_{x \in B_r} \chi(x) = \text{const}$ and $\int_{B_r} \chi d\rho = \text{const}$ for every $r \gg 1$, which play key roles in the proof of Theorems 1.1 and 1.3, then we also have these Theorems 1.1 and 1.3 with $\delta = 1$ on compact Lie group.

Note that in the proof of Theorems 1.5, 1.8 and 1.9, and Corollary 1.10, we only use Hölder's inequality, and Theorems 1.1 and 1.3. Therefore, since now we have Theorems 1.1 and 1.3 with $\delta = 1$ on compact Lie groups, then Theorems 1.5, 1.8 and 1.9, and Corollary 1.10 also hold on compact Lie group, with $\delta = 1$.

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