On the modular decomposition of the spin representation of S_n indexed by the partition $(n-2,2)$ and the combinatorics of bar-core partitions

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Abstract

This thesis consists of two research projects on the spin representation theory of the symmetric group.

In Chapters 2 and 3, we determine the modular decomposition of the spin representation of S_n indexed by the partition $(n-2,2)$. Whilst James provided a characteristic-free construction of the linear representations of the symmetric group S_n, there is no analogous construction for the spin (or projective) representations of S_n, i.e. the linear representations of a double cover S_n^+ of S_n. The most crucial open problem in the spin representation theory of S_n is determining the number of times each prime characteristic irreducible appears in the decomposition of the modular reduction of a characteristic 0 irreducible. Inspired by James' description of the linear representations of S_n in terms of submodules and induced modules, recovering the Specht modules, Wales showed that inducing the basic representation from S_{n-1}^+ to S_n^+ provides an irreducible 2-modular representation other than the basic representation, leading to a description of the modular decomposition of the spin representations denoted by the partitions (n) and $(n-1,1)$. We extend this method to determine the decomposition of the spin representation corresponding to $(n-2,2)$.

In Chapter 4, we establish combinatorial results about bar-core partitions. When p and q are coprime odd integers no less than 3, Olsson proved that if λ is a p-bar-core partition, then the q-bar-core of λ is again a p-bar-core. We establish a generalisation of this theorem: that the p-bar-weight of the q-bar-core of any bar partition λ is at most the p-bar-weight of λ. We go on to study the set of bar partitions for which equality holds and show that it is a union of orbits for an action of a Coxeter group of type $\tilde{C}_{(p-1)/2} \times \tilde{C}_{(q-1)/2}$. We also provide an algorithm for constructing a bar partition in this set with a given p-bar-core and q-bar-core.

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Dedicated to my amazing partner Emma, without whom I could never have finished this thesis. I am also indebted to my supervisor, who went above and beyond to ensure we finished this project.

Contents

Chapter 1

Introduction

The modular representation theory of the symmetric group S_n and its double covers $S_n^{\pm}$ can be understood through the combinatorics of partitions. In the linear representation theory of S_n, the combinatorics of rim s-hooks and s-core partitions classifies the s-blocks of modular irreducible representations of S_n. In the projective representation theory of S_n, there is an analogous theory of p-bars first realised by Morris [19], and Humphreys [9] proved that p-bar-cores classify the p-blocks of modular irreducible spin representations of S_n. We will see that much of the linear representation theory of S_n is parallel to the spin representation theory.

Our first chapter will begin with James' construction of the Specht modules [11], providing a complete set of irreducible linear representations of S_n over a field of arbitrary characteristic. We then describe the theory of spin representations of S_n, i.e. the linear representations of a double cover S_n^+ of S_n sending the element $z \in S_n^+$ to -1, which are equivalent to projective representations of S_n. A complete set of ordinary irreducible spin representations is labelled by the set of bar partitions [25], i.e. partitions with distinct parts (or simply finite subsets of $\mathbb{N}$), and the combinatorics of removing bars determines the modular structure of the algebra arising from S_n^+. We will state the modular branching rules of Brundan and Kleshchev [2], who showed that the irreducible spin representations of S_n in characteristic p are labelled by restricted p-strict partitions, as well as the theorems of Wales [26] and Morotti [18], who determine the modular decomposition factors of the ordinary irreducible spin representations of S_n labelled by partitions of n into 1 or 2 parts. We will apply all of these theorems to obtain a new result: the precise decomposition of the ordinary irreducible spin representation labelled by $(n-2, 2)$ in arbitrary characteristic $p > 0$, with the exception when $p = 3$ and n is a multiple of 3. This exceptional case is the motivation for the second chapter.

Whilst spin decomposition numbers have been calculated for $n \leq 18$ by Maas [16], we are still a long way from understanding them in general. A construction for the spin representations of S_n over a field of arbitrary characteristic, analogous to James' construction of the Specht modules, is still out of reach. We can, however, emulate James' methods to provide a characteristic-free construction of the spin representation corresponding to $(n-2,2)$. In the second chapter, we will induce a basic spin representation of S^+_{n-1}, labelled by the partition $(n-1)$, and we will construct a copy of the ordinary irreducible spin representation $S((n-2,2))$ inside the induced module. Our construction will allow us to identify the p-modular irreducible spin representation $D((n)^R)$ as the quotient of two submodules of $S((n-2,2))$ when $p=3$ and n is a multiple of 3, hence showing that the multiplicity $[S((n-2,2)) : D((n)^R)]$ is positive.

In our final chapter, we continue to use the linear representation theory as a guide for the spin representation theory of S_n and establish analogues for bar partitions of the results in Fayers' 'A generalisation of core partitions' [6]. Fayers proved that for coprime positive integers s,t, the s-weight of the t-core of a partition α is at most the s-weight of α, generalising Olsson's result that the t-core of an s-core is again an s-core [24]. We generalise another result established by Olsson in [24], which says that for coprime odd integers $p,q \geq 3$, the q-bar-core of a p-bar-core is again a p-bar-core, by showing that the p-bar-weight of the q-bar-core of any bar partition λ is at most the p-bar-weight of λ. We further investigate the set $\overline{C}_{p,q}$ of bar partitions λ which have p-bar-weight equal to that of their q-bar-core, just as in [6], Fayers studies the set $C_{S:t}$ of partitions α which have s-weight equal to that of their t-core, and find that $\overline{C}_{p,q}$ has analogous properties to those of $C_{s:t}$. By considering a natural action of the affine Weyl group $\mathfrak{W}_p$ of type $\tilde{C}_{(p-1)/2}$ and its invariants, we prove that the two sets $\overline{C}_{p,q}$ and $\overline{C}_{q,p}$ are equal, that any bar partition in this set is a pq-bar-core, and that the p-bar-core of the q-bar-core of $\lambda \in \overline{C}_{p,q}$ is equal to the q-bar-core of the p-bar-core of λ. Finally, we provide an algorithm to determine the bar partition in $\overline{C}_{p,q}$ with a given p-bar-core μ and q-bar-core σ such that the q-bar-core of μ is equal to the p-bar-core of σ, and construct a bijection between the $\mathfrak{W}_p \times \mathfrak{W}_q$-orbit containing $\Upsilon_{p,q}$, the maximal bar partition which is both a p-bar-core and a q-bar-core, and the direct product of the power set of $\{1,\ldots,\frac{(p-1)}{2}\}$ with the set of p-bar-cores and the set of q-bar-cores.

Chapter 2

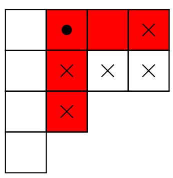
The projective representation theory of the symmetric group

2.1 Linear representations of S_n

Irreducible FS_n-modules, for a field F of characteristic $p > 0$, were classified by James [11], who did so by constructing the Specht modules in a characteristic-free way. For the benefit of readers unfamiliar with James and Kerber's work, we outline the theory of rim-hooks and cores here.

We may visually represent a partition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$, i.e. a decreasing sequence of (not necessarily distinct) positive integers $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_r$, by its **Young diagram** $[\alpha]$, which has α_i nodes in the i^{th} row, for each $i \in \{1, \ldots, r\}$, with each row starting in the first column. The **hook-length** h_{ij} of the (i, j)-node, in the i^{th} row and j^{th} column of $[\alpha]$, is found by adding the number of (k, j)-nodes with $k \geq i$ to the number of (i, l)-nodes with $l > j$. We refer to the (i, j)-nodes with $(i + 1, j + 1) \notin [\alpha]$ as the **rim** of $[\alpha]$. The h_{ij} pairwise adjacent nodes along the rim of $[\alpha]$ from the lowest node in the j^{th} column, i.e. the (k, j)-node with k maximal, to the (i, α_i)-node are collectively called a **rim** h_{ij}**-hook** and denoted by R_{ij}. Whenever a diagram $[\alpha]$ has an (i, j)-node with hook-length $s := h_{ij}$, we may remove a rim s-hook from $[\alpha]$ to obtain the Young diagram of a different partition $\alpha \backslash R_{ij}$ of S_{n-s}. If instead $[\alpha]$ has no nodes with hook-length s, then we say that the partition α is an s-**core**.

Example. Below is the Young diagram $[(4, 4, 2, 1)]$, which has just one 5-hook. The $(1, 2)$-node is highlighted with a $\bullet$, the $(1, 2)$-hook R_{12} is highlighted in red, and the removable nodes of the corresponding rim 5-hook are highlighted with $\times$'s:



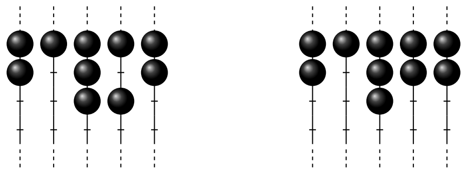
Adopting the convention that $\alpha_i = 0$ for each i greater than some fixed $r \in \mathbb{N}$, the strictly decreasing sequence of integers $\alpha_1 - 1 + k > \alpha_2 - 2 + k > \dots$, for some $k \in \mathbb{Z}$, is called a **beta-set** for the partition $\alpha = (\alpha_1, \alpha_2, \dots)$, and is denoted by $\mathcal{B}_k^\alpha$.

James' s-**abacus** has s runners extending infinitely in both directions, with the leftmost runner labelled by multiples of s, and the position directly to the right of i labelled by $i+1$. A bead configuration is associated with a partition α via the beta-set $\mathcal{B}^\alpha := \mathcal{B}_0^\alpha$ by placing a bead at the position labelled by $\alpha_i - i$ for each $i \in \mathbb{N}$.

Removing a rim s-hook from $[\alpha]$ then corresponds to removing an element $x \in \mathcal{B}^\alpha$ such that $x - s \notin \mathcal{B}^\alpha$ and replacing x with $x - s$. Thus we obtain the bead configuration for an s-core by moving the beads in the configuration for α on the s-abacus up their runners as far as possible. Since the order in which we move the beads is irrelevant, there is only one s-core which can be obtained from a partition α by removing rim s-hooks, and we denote **the s-core** of α by $\tilde{\alpha}_s$. The number of moves needed to reach the bead configuration of $\tilde{\alpha}_s$ from the configuration of α, or equivalently, the number of rim s-hooks which can be successively removed from the diagram $[\alpha]$, is the s-**weight** of α; we denote this quantity by $\mathrm{wt}_s(\alpha)$.

The s-**quotient** of α is the s-tuple of partitions corresponding to the bead configuration of each runner of the s-abacus as s separate 1-abaci. Each partition α is uniquely determined by its s-core $\tilde{\alpha}_s$ and its s-quotient.

Example. Below are the configurations of the partition $\alpha := (4,4,2,1)$ (on the left), and $\tilde{\alpha}_5 = (3,1,1,1)$ (on the right), on James' 5-abacus. As noted in the previous example, the 5-weight of α is 1. A beta-set for α is $\mathcal{B}_0^\alpha = \{3, 2, -1, -3, -5, -6, \dots\}$, we have $\mathcal{B}_0^{\tilde{\alpha}_5} = \{2, -1, -2, -3, -5, -6, \dots\}$, and the 5-quotient of α is $(\varnothing, \varnothing, \varnothing, (1), \varnothing)$ (where $\varnothing$ denotes the empty partition).



Removing rim s-hooks has a strong connection with the modular representation theory of the symmetric group, as the two ordinary irreducible representations corresponding to the partitions α and β belong to the same s-block of s-modular irreducible constituents if and only if $\tilde{\alpha}_s = \tilde{\beta}_s$. This important result was first conjectured by Nakayama, and should be referred to as the Brauer—Robinson Theorem after those who first proved it in 1947. There also exists a recursive formula to determine the values of the irreducible character $\langle \alpha \rangle$ of S_n on the conjugacy classes of S_n, both of which are indexed by partitions of n.

Theorem (The Murnaghan—Nakayama formula)**.** *For partitions α and σ of n, if σ has s as a part, and σ' is the partition obtained by removing s from σ, then*

$$\langle \alpha \rangle(\sigma) = \sum_{i,j} (-1)^{\alpha_i + j + k + 1} \langle \alpha \backslash R_{ij} \rangle(\sigma')$$

where the sum is taken over all nodes (i,j) *of* α *such that* $h_{ij} = s$ *(and* $\langle \varnothing \rangle = 1$*).*

Example.

$$\begin{aligned}\langle (4^2,1)\rangle((4,3,2)) =& -\langle (3^2,1)\rangle((4,3)) + \langle (4,2,1)\rangle((4,3)) \\ =& \langle (3)\rangle((3)) - \langle (1^3)\rangle((3)) \\ =& 1-1=0.\end{aligned}$$

Now we will outline James' construction of the Specht modules. For a partition $\alpha = (\alpha_1, \ldots, \alpha_r)$ of n, an α**-tableau** t is obtained by placing each of the numbers $1, \ldots, n$ in different boxes of the Young diagram $[\alpha]$. If t is an α-tableau, then $\overline{t}$ denotes an α**-tabloid**, an α-tableau with unordered row entries. We draw an α-tabloid as a Young diagram without vertical lines. There is a natural action of S_n on the set of α-tableaux, and therefore on the set of α-tabloids. The stabiliser of any α-tabloid is a conjugate of a *Young subgroup*

$$S_\alpha \cong S_{\alpha_1} \times \cdots \times S_{\alpha_r}.$$

The FS_n-module M^α spanned by α-tabloids is isomorphic to the permutation representation of S_n on the cosets of a Young subgroup S_α. An α**-polytabloid** is an element of M^α of the form

$$e_t := \sum_{\pi \in V(t)} \operatorname{sgn}(\pi)\overline{\pi t},$$

where t is an α-tableau and $V(t)$ is the column stabiliser of t.

Example. If

$$t = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 6 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}$$

then $V(t) = \{1, (1\,3), (2\,4), (1\,3)(2\,4)\}$ and

$$e_t = \begin{array}{c}\hline 1\ \ 2\ \ 5\ \ 6 \\ \hline 3\ \ 4 \\ \hline\end{array} - \begin{array}{c}\hline 3\ \ 2\ \ 5\ \ 6 \\ \hline 1\ \ 4 \\ \hline\end{array} - \begin{array}{c}\hline 1\ \ 4\ \ 5\ \ 6 \\ \hline 3\ \ 2 \\ \hline\end{array} + \begin{array}{c}\hline 3\ \ 4\ \ 5\ \ 6 \\ \hline 1\ \ 2 \\ \hline\end{array}.$$

We define the **Specht module** S^α to be the subspace of M^α spanned by polytabloids. Since $\pi e_t = e_{\pi t}$ for each α-tableau t and $\pi \in S_n$, the Specht module S^α is an FS_n-submodule of M^α, and it is generated by any one polytabloid. We say that an α-tableau t is **standard** if the numbers are strictly increasing from left to right along the rows and down the columns of t. Over any field, we can take $\{e_t | t \text{ is a standard } \alpha\text{-tableau}\}$ as a basis for S^α.

In characteristic 0, the Specht modules S^α, indexed by partitions α of n, give a complete set of (ordinary) irreducible representations of S_n. Moreover, since permutation modules have the same definition over any field, this construction makes it possible to determine the irreducible representations of S_n over a field F of arbitrary characteristic $p > 0$.

Define an S_n-invariant bilinear form on M^α: for α-tabloids $\overline{s}, \overline{t}$, let

$$f(\overline{s}, \overline{t}) := \begin{cases} 1 & \text{if } \overline{s} = \overline{t} \\ 0 & \text{if } \overline{s} \neq \overline{t} \end{cases}.$$

Then for every pair of α-polytaboids e_s, e_t, writing z_j for the number of parts of α equal to j, the integer $z_j!$ divides $f(e_s, e_t)$ for all $j > 0$. Hence, in positive characteristic p, the restriction of f to S^α is non-zero if and only if α is p-regular, i.e. $z_j < p$ for all j. It follows that the module

$$D^\alpha := S^\alpha/(S^\alpha \cap S^{\alpha\perp})$$

is non-zero if and only if α is p-regular, where

$$S^{\alpha\perp} = \{a \in M^\alpha | f(a, e_t) = 0 \text{ for all } \alpha\text{-tableaux } t\}.$$

Now by the Submodule Theorem [11, Theorem 7.1.7], for each partition α of n, every FS_n-submodule of M^α either contains S^α, or is contained in $S^{\alpha\perp}$. When α is p-regular, the radical $S^\alpha \cap S^{\alpha\perp}$ of f is a maximal proper submodule, and D^α is therefore an irreducible FS_n-module. In fact,

$$\{D^\alpha | \alpha \text{ is a } p\text{-regular partition of } n\}$$

is a complete set of irreducible FS_n-modules.

Example. When $\alpha = (n)$, the Specht module S^α is just the trivial FS_n-module, and when $\alpha = (1^n)$, M^α is isomorphic to the regular representation module, and S^α the alternating representation.

If $\alpha = (n-1, 1)$, then α-tabloids are uniquely defined by the number in their second row. The action of the standard generating transpositions $s_i := (i, i+1)$ of S_n on the α-tabloid t_k with k in the second row is given by

$$\begin{aligned} s_i t_i &= t_{i+1} \\ s_i t_{i+1} &= t_i \\ s_i t_j &= t_j && \text{if } j \neq i, i+1 \end{aligned}$$

so M^α is the natural module where elements of S_n act by permuting basis vectors, or equivalently, it is the trivial module induced from S_{n-1}. In this case, standard α-tableaux are also uniquely defined by the number in their second row, which must be one of $2, \ldots, n$. Thus $\{e_2, \ldots, e_n\}$ is a basis for S^α, where e_k is the polytabloid with the number k in the second row of the tabloid with a positive sign:

$$e_k = \begin{array}{l} \hline 1 \quad \cdots \quad k-1 \; k+1 \quad \cdots \quad n \\ \hline k \\ \hline \end{array} - \begin{array}{l} \hline 2 \quad \cdots \quad n \\ \hline 1 \\ \hline \end{array}.$$

Note that this is well-defined because we can always take standard tableaux as an index for the basis of S^α so that the tabloid with a negative sign in e_k will always have 1 in the second row.

Now suppose $\alpha = (n-2, 2)$. Then M^α can be thought of as the permutation module on 2-subsets of $\{1, \ldots, n\}$, or as the trivial module induced from the Young subgroup $S_{n-2} \times S_2$. We can take as a basis for S^α the set of polytabloids $\{e_{ij}\}$ where $e_{ij} = e_{ji}$ is indexed by the standard α-tableau with i, j

in its second row (so that $i < j \Rightarrow 2 \leq i \leq n-1$ and $4 \leq j \leq n$). If we write $\overline{t_{ij}}$ for the α-tabloid with i, j in its second row, then

$$e_{ij} = \overline{t_{ij}} - \overline{t_{1j}} - \overline{t_{ik}} + \overline{t_{1k}},$$

where $k = 3$ if $i = 2$, or $k = 2$ otherwise.

We will adopt similar methods in Chapter 3 in order to construct the spin representation of S_n corresponding to the partition $(n-2, 2)$.

2.2 Spin representations

A **projective representation** of a group G on a finite-dimensional vector space V over a field F is a function $\rho : G \to GL(V)$ such that $\rho(1) = 1$ and for each pair of elements $x, y \in G$, there exists a non-zero $\sigma(x, y) \in F$ with

$$\rho(x)\rho(y) = \sigma(x, y)\rho(xy).$$

Equivalently, we can think of projective representations as a homomorphism from G to $PGL(V)$, the projective general linear group acting on V. By the associative law in G and $GL(V)$, for all $x, y, z \in G$, the function σ must satisfy

$$\sigma(x, 1) = 1 = \sigma(1, x)$$
$$\sigma(x, yz)\sigma(y, z) = \sigma(x, y)\sigma(xy, z)$$

and σ is called a **2-cocycle** [8, p. 1]. Then a linear representation of G is a projective representation with trivial 2-cocycle: $\sigma(x, y) = 1$ for all $x, y \in G$.

Two 2-cocycles σ, θ are said to be **cohomologous** if there exists a function $f : G \to F$ such that $f(1) = 1$, $f(x) \neq 0$ for all $x \in G$, and for each pair $x, y \in G$,

$$\theta(x, y) = f(x)f(y)(f(xy))^{-1}\sigma(x, y)$$

and we call the set $[\sigma]$ of 2-cocycles related to σ in this way the **cohomology class** of σ. The set of cohomology classes $[\sigma]$ for 2-cocycles σ forms a finite abelian group $M(G)$ under the operation

$$[\sigma][\theta] = [\sigma\theta],$$

and is called the **Schur multiplier** of G. Now projective representations of G can be viewed as linear representations of a central extension of $M(G)$ by G which contains $M(G)$ in its commutator subgroup. We call this extension a **representation group** for G.

When $G = S_n$ is the symmetric group, the Schur multiplier $M(G)$ has order 2 if $n \geq 4$ and is trivial otherwise [8, Theorem 2.9]. We call a representation group of S_n a **double cover** of S_n. When $n \geq 4$, S_n has two proper double covers, often denoted by S_n^+ and S_n^-, and the linear representation theory of each group is equivalent to the projective representation theory of S_n. Each has a centre $\{e, z\}$ of order 2 contained in the commutator subgroup, making the exact sequence

$$1 \to \{e, z\} \to S_n^{\pm} \to S_n \to 1.$$

The two groups $S_n^{\pm}$ are distinguished by the order of the element corresponding to a transposition in S_n: order 4 in S_n^- and order 2 in S_n^+. We will restrict our attention to S_n^+, whose generators $t_1, \dots, t_{n-1}$ are lifted from the generating transpositions of S_n and satisfy the relations

$$t_i^2 = 1 = z^2, \qquad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1},$$

$$z t_i = t_i z, t_i t_j = z t_j t_i \text{ for } |i - j| > 1.$$

Note that $S_n \cong S_n^+/\{e, z\}$, and representations of S_n^+ that send the element z to the identity are precisely the linear representations of S_n. If z is not in the kernel of an irreducible representation ρ of S_n^+ then we say that ρ is a **spin representation** of S_n, and we necessarily have $\rho(z) = -1$.

Schur determined the irreducible spin characters of S_n [25], giving a method for finding character values with recursive functions that generate **shifted λ-tableau**, obtained by filling the nodes of the **shifted diagram** $\{(i, j)|1 \leq i \leq j < \lambda_i\}$ of λ with the numbers $1, \dots, n$. The ordinary irreducible spin characters of S_n are labelled by the set $\overline{\mathcal{P}}_n$ of **bar partitions** λ of n, decreasing sequences of distinct positive integers

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$$

that sum to n (often referred to as strict partitions).

Denoting by $l(\lambda) = r$ the length of λ, we say that λ is **even** if $n - l(\lambda)$ is even, and **odd** if $n - l(\lambda)$ is odd. Define the **sign representation** sgn on S_n^+ by $\text{sgn}(t_i) = -1$ for $i = 1, \dots, n-1$ and $\text{sgn}(z) = 1$. Then we say that two spin representations ρ, φ are **associate** when $\varphi = \text{sgn} \otimes \rho$.
If λ is even, then there is one self-associate ordinary irreducible spin character $\langle\lambda\rangle$ labelled by λ. If λ is odd, then there are two associate ordinary irreducible spin characters $\langle\lambda\rangle_0, \langle\lambda\rangle_1$ labelled by λ.

A conjugacy class of S_n corresponding to a partition α (giving the *cycle shape* of the permutations in that class) either lift to one conjugacy class in the double cover S_n^+, or split into two conjugacy classes of S_n^+. Schur showed that the elements in S_n^+ corresponding to a partition α split into two conjugacy classes if and only if α is either a bar partition with an odd number of even parts, or α has no even parts. Note that spin characters vanish on the non-split conjugacy classes.

Example. The conjugacy classes of S_4^+ are as follows:

- Two conjugacy classes of size 1: $\{1\}$ and $\{z\}$;
- One conjugacy class of size 12, consisting of elements corresponding to transpositions in S_4;
- One conjugacy class of size 6 corresponding to the partition (2^2);
- Two conjugacy classes of size 8, each corresponding to the partition $(3, 1)$;

- Two conjugacy classes of size 6 corresponding to (4).

There are three irreducible spin characters of S_4^+ with characters $(2,-2,0,0,-1,1,\pm\sqrt{-2},\mp\sqrt{-2})$, and $(4,-4,0,0,1,-1,0,0)$. The other irreducible representations of S_4^+ are lifted from linear representations of S_4.

Throughout this chapter, we will denote by F an algebraically closed field of odd positive characteristic p. Note that spin representations do not exist in characteristic 2, and if $p > n$, then FS_n^+ is semisimple, so we need only consider $p \leq n$.

A partition μ is p**-strict** if any repeated parts are divisible by p. When μ is p-strict we say that μ is **restricted** p**-strict** if for each $i = 1, \ldots, l(\mu) - 1$, the parts of μ satisfy $\mu_i - \mu_{i+1} < p$ when $p|\mu_i$, and $\mu_i - \mu_{i+1} \leq p$ when $p \nmid \mu_i$.

Over F, the irreducible spin characters of S_n are labelled by the set $\overline{\mathcal{P}^p}_n$ of restricted p-strict partitions μ. This labelling was established by Brundan and Kleshchev [2], who also provided branching rules for the induction and restriction of irreducible spin representations using the link between modular representation theory and the combinatorial theory of *crystal bases.* Thus the main problem in the spin representation theory of the symmetric group is determining the multiplicity of p-modular irreducible spin representations, labelled by restricted p-strict partitions μ, in the decomposition of ordinary irreducible spin representations, labelled by bar partitions λ, in odd positive characteristic p.

For a partition α, define the p**-residue** of the (r,c)-node of $[\alpha]$ to be the number in the r^{th} row and c^{th} column of the diagram of α when we fill each row with the repeating sequence of numbers

$$0, 1, \ldots, {}^{(p-3)}/{}_2, {}^{(p-1)}/{}_2, {}^{(p-3)}/{}_2, \ldots, 1, 0, 0, 1, \ldots$$

We say that $\mu \in \overline{\mathcal{P}^p}_n$ is p**-even/odd** if the number of non-zero residues in $[\mu]$ is even/odd. If μ is p-even, then there is one irreducible spin character of S_n labelled by μ; if μ is p-odd, there are two irreducible spin characters labelled by μ [2].

Example. The partition $\lambda := (9, 2)$ is 5-odd as the number of non-zero residues in the following diagram is odd

0	1	2	1	0	0	1	2	1
0	1							

so there are two associate irreducible spin characters labelled by λ.

As we saw in the previous section, the irreducible linear representations of S_n in characteristic p fall into blocks which are indexed by p-cores, and there is a rich, developed theory of p-hooks and p-cores of partitions revealing a great deal of information about the linear representation theory of S_n. The p-blocks of irreducible spin representations of S_n are indexed by *p-bar-cores*, and there is a theory based on the combinatorics of removing *p-bars* from bar partitions that is parallel to the linear theory of hooks.

For odd integers $p \geq 1$, removing a p**-bar** from a bar partition λ means either

(i) removing $x \in \lambda$ such that $0 \leq x - p \notin \lambda$, and replacing x with $x - p$ if $x \neq p$; or

(ii) removing two parts $x, p-x \in \lambda$ (where $0 < x < p$).
(p must be odd because of the incompatible possibility that a bar partition could have a $2p$-bar but not a p-bar, e.g. $p = 4$ and the partition $(6, 2)$.)

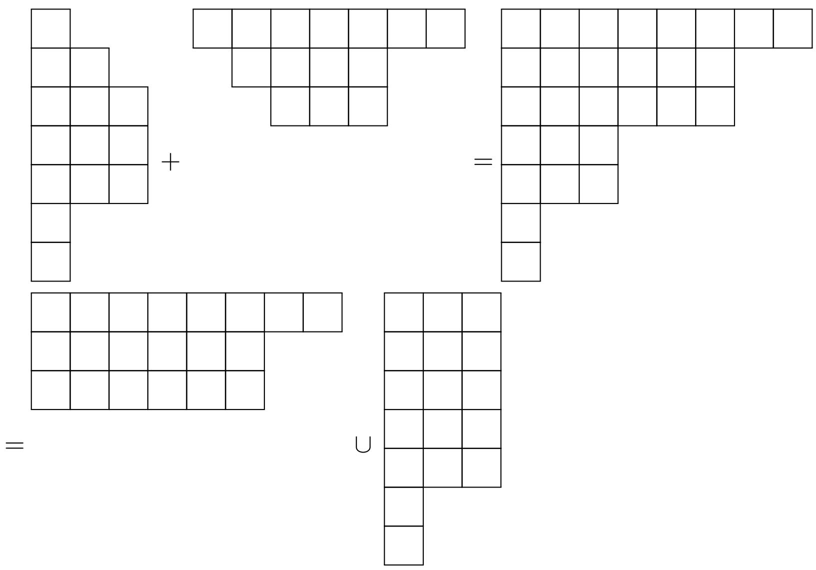
Morris was the first to realise that bars were the appropriate analogue to hooks for the spin representation theory of S_n, and established a spin analogue of the Murnaghan—Nakayama formula, giving a recursive method for calculating the values of the irreducible spin characters $\langle\lambda\rangle$ of S_n^+ [19]. In [25], Schur showed that for a bar partition λ and partition σ of n, if σ has at least one even part, then the irreducible character(s) labelled by λ vanish on the conjugacy class(es) of S_n^+ projecting to the conjugacy class of S_n indexed by σ except when λ is odd and $\sigma = \lambda$. In this exceptional case, the character value is

$$\pm i^{(n-l(\lambda)+3)/2}\sqrt{\frac{\lambda_1\lambda_2\cdots\lambda_l}{2}}.$$

Otherwise, when σ has only odd parts and λ is odd, we have $\langle\lambda\rangle_0(\sigma) = \langle\lambda\rangle_1(\sigma)$ and can therefore denote this value simply by $\langle\lambda\rangle(\sigma)$. In order to state Morris' recursive formula for $\langle\lambda\rangle(\sigma)$, we first need a couple of definitions.

The **double** of a bar partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ is the partition λ^+ whose Young diagram is obtained by amalgamating the shifted diagram of λ, which has λ_i nodes in the i^{th} row, with the left-most in the i^{th} column, and its reflection along the top left to bottom right diagonal. Equivalently, $[\lambda^+]$ is the union of $[(\lambda_1+1, \lambda_2+2, \ldots, \lambda_r+r)]$ and $[(\lambda_1, \lambda_2+1, \ldots, \lambda_r+r-1)']$ [17, p. 14].

Example. $(7,4,3)^+ = (8,6,6,3,3,1,1)$:



Now we define, for each (i,j)-node in the shifted diagram of a bar partition λ, the **bar-length** b_{ij} to be the hook-length of the corresponding node in the Young diagram of λ^+. Nodes (i,j) in the shifted diagram of λ with $b_{ij} = p$ correspond to p-bars removable from λ and to removable $2p$-hooks in the Young diagram of λ^+. This correspondence between p-bars of λ and p-hooks of λ^+ means that when λ is a p-bar-core, λ^+ must be a p-core. The **leg-length** $L(b_{ij})$ of the (i,j)-node is the number of nodes in the j^{th} column of $[\lambda^+]$ beneath row i.

Example. The bar-lengths of $(7,4,3)$ are obtained from the hook-lengths of its double $(8,6,6,3,3,1,1)$:

	11	10	7	6	5	2	1
		7	4	3	2		
			3	2	1		

The leg-lengths of the nodes in the first row of the shifted diagram of λ are:

$$L(b_{11}) = L(b_{12}) = 4, \qquad L(b_{13}) = L(b_{14}) = L(b_{15}) = 2, \qquad L(b_{16}) = L(b_{17}) = 0.$$

Now we can state Morris' spin analogue of the Murnaghan—Nakayama formula. Note that we must distinguish the two conjugacy classes of S_n^+ corresponding to a partition σ of n into odd parts, as the values of an irreducible spin character on these two conjugacy classes are additive inverses. We will essentially fix one of the two conjugacy classes of S_n^+ which project to cycle shape σ in S_n by choosing the conjugacy class on which the *basic spin character* (that of the basic spin representation, which we will define later) takes a positive value [8, p. 110]. Thus, for the following formula, we may identify conjugacy classes of S_n^+ with partitions σ of n into odd parts.

Theorem (Morris' recursion formula). *Let λ be a bar partition of n, σ a partition of n with the odd integer p as a part and with no even parts, and σ' the partition obtained by removing p from σ. Denote by $\lambda\backslash b$ the bar partition obtained by removing the p-bar b from λ. Then*

$$\langle\lambda\rangle(\sigma) = \sum_b (-1)^{L(b)} 2^{m(b)} \langle\lambda\backslash b\rangle(\sigma'),$$

where the sum is taken over all p-bars b removable from λ, and

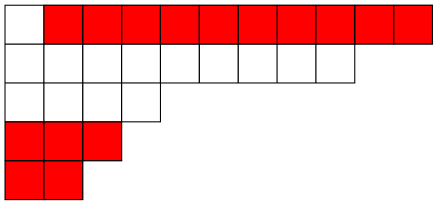
$$m(b) = \begin{cases} 1 & \text{if } \lambda \text{ is even and } \lambda\backslash b \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Example. The value of the characters labelled by the bar partition $(7,4,3)$ on the conjugacy class of S_{14}^+ projecting to cycle shape $(5^2,3,1)$ on which the basic spin character takes a positive value is

$$\begin{aligned} \langle(7,4,3)\rangle((5^2,3,1)) =& \langle(7,4)\rangle((5^2,1)) - \langle(7,3,1)\rangle((5^2,3,1)) \\ =& -\langle(4,2)\rangle((5,1)) + 2\langle(3,2,1)\rangle((5,1)) \\ =& 0 + 2\langle(1)\rangle((1)) \\ =& 2. \end{aligned}$$

A ***p*-bar-core** is a bar partition with no removable p-bars, and the (unique) p-bar-core obtained by successively removing p-bars from λ is called **the *p*-bar-core** of λ. Morris and Yaseen showed that two bar partitions λ, ν have the same p-bar-core if and only if they have the same multiset of p-residues [20, Theorem 5]. The number of p-bars removed from λ in order to obtain its p-bar-core is called the **p-bar-weight** of λ. Morris showed that the p-bar-weight and the p-bar-core of a bar partition are well-defined, and we will give a proof of this in Chapter 4.

Example. The bar partition $(11, 9, 4, 3, 2)$ has 5-bar-core $(9, 4, 1)$ and 5-bar-weight 3. The three removable 5-bars are highlighted in red in the diagram below: we can remove two 5 bars from the first row, and a split 5-bar from the third and fourth rows, but we cannot remove a 5-bar from the second row as this would leave us with two parts equal to 4.



The following theorem provides a spin version of the Brauer—Robinson Theorem. It was first conjectured by Morris [19], and later proved by Humphreys [9, Theorem 1.1].

Theorem 2.2.1. *The ordinary irreducible spin representations labelled by bar partitions λ and ν belong to the same p-block of modular irreducibles if and only if λ and ν have the same p-bar-core and λ is not an odd p-bar-core equal to ν.*

2.3 Induction and restriction of spin modules

Whilst there is a general construction of the ordinary irreducible spin representations of S_n due to Nazarov [22], a construction over a field of arbitrary characteristic analogous to the Specht modules does not yet exist. However, there do exist certain branching rules which can help us to determine the modular decomposition of ordinary irreducible spin representations in positive characteristic via combinatorial formulae for decomposition factors that depend on adding and removing nodes from the corresponding partitions. These branching rules, established by Brundan and Kleshchev [2], are based on the *crystal graph* of the basic representation for the Kac-Moody algebra of type $A^{(2)}_{p-1}$, and are analogous to Kleshchev's branching rules for S_n (which correspond to the Kac-Moody algebra of type $A^{(1)}_{p-1}$). This connection between the modular branching graph of S^+_n and the Kac-Moody algebra of type $A^{(2)}_{p-1}$ was first suggested by Leclerc and Thibon [14].

Due to the splitting pairs of associate irreducible spin modules, Brundan and Kleshchev work in the setting of a **superalgebra**, i.e. a $\mathbb{Z}/2\mathbb{Z}$-graded algebra $A = A_0 \oplus A_1$ where $a_i \in A_i$, $a_j \in A_j$ $\Rightarrow a_i a_j \in A_{i+j \pmod 2}$. Since $z \in S^+_n$ acts as 1 on representations of S^+_n that are equivalent to linear representations of S_n, and z acts as -1 on spin representations, the group algebra FS^+_n decomposes as the direct sum ${}^{(1+z)}/2 FS^+_n \oplus {}^{(1-z)}/2 FS^+_n$, where ${}^{(1+z)}/2 FS^+_n \cong FS_n$ and ${}^{(1-z)}/2 FS^+_n$ is the **twisted group algebra** $\mathcal{T}_n$ with generators $t_1, \ldots, t_{n-1}$ satisfying the relations

$$t_i^2 = 1, \qquad\qquad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1},$$

$$t_i t_j = -t_j t_i \text{ for } |i-j| > 1.$$

The spin representation theory of S_n is equivalent to the linear representation theory of $\mathcal{T}_n$, which we can view as a superalgebra where the generators $t_1, \dots t_{n-1}$ all lie in $(\mathcal{T}_n)_1$.

There are two types of irreducible supermodules:

(*i*) type **M**: the underlying module $\underline{M}$ is irreducible;

(*ii*) type **Q**: $\underline{M}$ is the direct sum of two irreducible modules.

Note that the underlying module $\underline{M}$ of an irreducible supermodule M is self-associate if M is of type **M**, or the direct sum of two associate irreducible modules if M is of type **Q**.

We write $S(\lambda)$ for the ordinary irreducible spin supermodule corresponding to the bar partition λ, so from Schur's classification [25] of the ordinary irreducible spin characters of S_n^+, we get a complete set of irreducible and pairwise non-isomorphic spin supermodules of S_n in characteristic 0:

$$\{S(\lambda) | \lambda \in \overline{\mathcal{P}}_n\}.$$

Over the field F of characteristic $p > 2$, we write $D(\mu)$ for the irreducible spin supermodule corresponding to the restricted p-strict partitions μ, so that

$$\{D(\mu) | \mu \in \overline{\mathcal{P}^p}_n\}$$

is a complete set of irreducible and pairwise non-isomorphic $\mathcal{T}_n$-supermodules [2, Corollary 3.13].
If $\lambda \in \overline{\mathcal{P}}_n$ is even (i.e. $n - l(\lambda)$ is even), then the ordinary irreducible supermodule $S(\lambda)$ is of type **M** and we denote the underlying irreducible module by $S(\lambda, 0)$. If λ is odd, then $S(\lambda)$ is of type **Q** and the underlying module decomposes into two irreducible modules $S(\lambda, +)$ and $S(\lambda, -)$.
Similarly, if $\mu \in \overline{\mathcal{P}^p}_n$ is p-even (i.e. the number of non-zero residues in $[\mu]$ is even), then the p-modular irreducible supermodule $D(\mu)$ is of type **M** and we denote the underlying irreducible module by $D(\mu, 0)$. If μ is p-odd, then $D(\mu)$ is of type **Q** and the underlying module decomposes into two irreducible modules $D(\mu, +)$ and $D(\mu, -)$. Since p-blocks of $\mathcal{T}_n$ are indexed by multisets of p-residues, irreducible supermodules belonging to the same block have the same type, so we may refer to the type of a block.
Hence a complete set of ordinary irreducible and pairwise non-isomorphic spin modules of S_n is given by

$$\{S(\lambda, 0) | \lambda \in \overline{\mathcal{P}}_n \text{ is even}\} \cup \{S(\lambda, +), S(\lambda, -) | \lambda \in \overline{\mathcal{P}}_n \text{ is odd}\}.$$

Over F, a complete set of irreducible and pairwise non-isomorphic spin modules of S_n is given by

$$\{D(\mu, 0) | \mu \in \overline{\mathcal{P}^p}_n \text{ is } p\text{-even}\} \cup \{D(\mu, +), D(\mu, -) | \mu \in \overline{\mathcal{P}^p}_n \text{ is } p\text{-odd}\}.$$

We need a few more combinatorial definitions before we can state Brundan and Kleshchev's modular branching rules for the superalgebra $\mathcal{T}_n$. A node $(r, c) \in [\mu]$ is called ***i*-removable** if it has residue i and either:

- removing the (r, c)-node from $[\mu]$ gives the diagram of a p-strict partition of $n-1$; or
- $(r, c+1) \in [\mu]$ also has residue i and both the diagram obtained by removing the $(r, c+1)$-node and the diagram obtained by removing the $(r, c+1)$- and (r, c)-nodes correspond to p-strict partitions of $n-1$ and $n-2$, respectively.

Note that the second condition can only occur when $i = 0$.

A node $(r, c) \notin [\mu]$ is called **i-addable** if it has residue i and either:

- adding (r, c) to $[\mu]$ gives the diagram of a p-strict partition of $n+1$; or
- $(r, c-1) \notin [\mu]$ also has residue i and both the diagrams $[\mu] \cup \{(r, c-1)\}$ and $[\mu] \cup \{(r, c-1), (r, c)\}$ correspond to p-strict partitions (of $n+1$ and $n+2$, respectively).

Again, the second case can only occur when $i = 0$.

If we label all of the i-addable nodes of $[\mu]$ by $+$ and all of the i-removable nodes by $-$, then the **i-signature** of μ is obtained by reading all of these signs along the rim of $[\mu]$ from bottom left to top right.

The **reduced i-signature** of μ is obtained by deleting all neighbouring pairs of the form $+-$ from the i-signature of μ. This will always be a sequence of $-$s followed by $+$s.

The nodes corresponding to a $-$ in the reduced i-signature are called **i-normal**, and the rightmost i-normal node is called **i-good**.

Nodes corresponding to a $+$ in the reduced i-signature are called **i-conormal**, and the leftmost i-conormal node is called **i-cogood**.

Example. The diagram below shows the 5-residues of $\mu := (6, 5^2, 3)$. There are two 0-removable nodes in red and no i-removable nodes for $i = 1, 2$. The 0-signature of μ is $+--$ and therefore the reduced 0-signature is $-$, with the rightmost 0-removable node 0-good. In addition to the 0-addable node, highlighted in blue below, there are two 1-addable nodes, and these are also blue in the diagram below.

0	1	2	1	0	0	1
0	1	2	1	0		
0	1	2	1	0		
0	1	2	1			
0						

Now adding i-cogood nodes to, or removing i-good nodes from, a restricted p-strict partition results in another restricted p-strict partition. These operations turn the set $\overline{\mathcal{P}^p}_n$ into a *crystal* [12] which coincides with the induction and restriction of irreducible supermodules [2], a connection first suggested in [14].

When we restrict a $\mathcal{T}_n$-supermodule M belonging to the block corresponding to the multiset of residues R down to $\mathcal{T}_{n-1}$, we can write

$$M \downarrow_{\mathcal{T}_{n-1}} = \mathrm{res}_0 M \oplus \mathrm{res}_1 M \oplus \cdots \oplus \mathrm{res}_{(p-1)/2} M$$

where $\mathrm{res}_i M$ is the component of $M \downarrow_{\mathcal{T}_{n-1}}$ lying in the block with multiset of residues $R \backslash \{i\}$ if such a block exists, or $\mathrm{res}_i M = 0$ otherwise. This process is called **i-restriction**, and **i-induction** is defined similarly [2].

Theorem 2.3.1. *If $M = S(\lambda)$ is an ordinary irreducible spin supermodule, then $res_i S(\lambda)$ has a filtration with factors $S(\nu)$ for each bar partition ν obtained by removing a node of residue i from λ. The factor $S(\nu)$ occurs twice in $res_i S(\lambda)$ if ν is even and λ is odd, and occurs once otherwise.*

Theorem 2.3.2. *Suppose $D := D(\mu)$ is a p-modular irreducible spin supermodule, and write*

$$D \downarrow_{\mathcal{T}_{n-1}} = res_0 D \oplus res_1 D \oplus \cdots \oplus res_{(p-1)/2} D.$$

Then there exist supermodules $e_0 D, \dots, e_{(p-1)/2} D$ such that

$$res_i D = \begin{cases} e_i D & \text{if } i = 0 \text{ or } \mu \text{ is } p\text{-even} \\ e_i D \oplus e_i D & \text{otherwise.} \end{cases}$$

If μ has no i-good nodes, then $e_i D = 0$. Otherwise, let $\nu \in \overline{\mathcal{P}^p}_n$ be the partition obtained by removing the i-good node from μ. Then $e_i D$ is a self-associate indecomposable supermodule with irreducible socle and head isomorphic to $D(\nu)$. Moreover, the multiplicity of $D(\nu)$ in $e_i D$ is equal to the number $\epsilon_i(\mu)$ of i-normal nodes in μ. For all other composition factors $D(\kappa)$ of $e_i D$, the restricted p-strict partition κ has fewer than $\epsilon_i(\mu) - 1$ i-normal nodes, and $e_i D$ is irreducible if and only if $\epsilon_i(\mu) = 1$.

We will apply these branching rules later on to calculate the p-modular decomposition of $S((n-2, 2))$.

Example. Considering the (5-odd) restricted 5-strict partition $\mu = (6, 5^2, 3)$ from the previous example, the 5-modular irreducible spin supermodule $D := D(\mu)$ satisfies $e_i D = 0$ for $i = 1, 2$ and $e_0 D = D((5^3, 3))$.

For linear representations of S_n in characteristic $p > 0$, James described the leading composition factor D^β of the Specht module S^α, in terms of a dominance order $\leq$ on the set of partitions, by defining the *regularisation* α^R of a partition α and the *shadow* $\mathrm{sh}(\beta)$ of a p-regular partition β, obtained by removing the leftmost node of the *outer ladder* of β [10].

Theorem 2.3.3. *(i) For all partitions α of n, D^{α^R} appears as a composition factor of S^α with multiplicity 1, and for any other composition factor D^β of S_α we have $\beta > \alpha^R$.*

(ii) For all p-regular partitions β of n, $D^{sh(\beta)}$ appears as a composition factor of the restriction $D^\beta \downarrow_{S_{n-1}}$ with multiplicity equal to the size of the outer ladder of β. For any other composition factor D^γ of $D^\beta \downarrow_{S_{n-1}}$ we have $\gamma > sh(\beta)$.

The spin version of this result was established by Brundan and Kleshchev [3], giving the leading composition factor of any ordinary irreducible spin supermodule over F.

Define a **dominance ordering** on the set of partitions of n by $\alpha = (\alpha_1, \dots) \geq \beta = (\beta_1, \dots)$ if and only if $\sum_{i=1}^r \alpha_i \geq \sum_{i=1}^r \beta_i$ for all $r \geq 1$.

Define the residue of $j \geq 1$ to be the residue of the nodes in the j^{th} column of the diagram of a partition. The j^{th} **ladder** L_j is defined as follows. If j has non-zero residue, then

$$L_j = \{(i, j - (i-1)p) | 1 \leq i \leq \lceil j/p \rceil\}.$$

If j has residue 0, then $j = mp$ or $mp + 1$ for some $m \in \mathbb{Z}$, and L_j is defined by

$$L_j = \{(i, mp - (i-1)p) | 1 \leq i \leq m\} \cup \{(i, mp + 1 - (i-1)p) | 1 \leq i \leq m+1\}.$$

Note that all nodes in the ladder L_j have the same residue as j. The **outer ladder** of λ is the ladder L_j with j maximal such that $\lambda \cap L_j \neq \emptyset$. In this case, the rightmost node on $\lambda \cap L_j$ is called the **shadow node** of λ. If $\lambda \in \overline{\mathcal{P}^p}_n$, then the partition ν obtained by removing the shadow node of λ is also restricted p-strict.

For each p-strict partition λ, the **regularisation** λ^R of λ is the set of nodes such that for all j, $\lambda^R \cap L_j$ consists of the leftmost $|\lambda \cap L_j|$ nodes on the ladder L_j. Thus we obtain λ^R by moving the nodes of λ along the ladders to the left as far as possible.

Example. When $p = 5$, the regularisation of the bar partition $(18, 1)$ is the partition $\mu = (6, 5^2, 3) = (18, 1)^R$ from our earlier example:

0	1	2	1	0	0	1	2	1	0	0	1	2	1	0	0	1	2
0																	

$$\downarrow$$

0	1	2	1	0	0
0	1	2	1	0	
0	1	2	1	0	
0	1	2			

For any p-strict partition λ of n, we have $\lambda^R \in \overline{\mathcal{P}^p}_n$, and $\lambda = \lambda^R$ if and only if $\lambda \in \overline{\mathcal{P}^p}_n$.

Now we can state the regularisation theorem of Brundan and Kleshschev [3].

Theorem 2.3.4. *(i) For any p-strict partition λ of n, $D(\lambda^R)$ appears as a composition factor of $S(\lambda)$ with multiplicity $2^{(l_p(\lambda)+x-y)/2}$, where $l_p(\lambda)$ denotes the number of parts of λ divisible by p,*

$$x = \begin{cases} 0 & \text{if } \lambda \text{ is even,} \\ 1 & \text{if } \lambda \text{ is odd,} \end{cases} \qquad \text{and} \qquad y = \begin{cases} 0 & \text{if } \lambda \text{ is } p\text{-even,} \\ 1 & \text{if } \lambda \text{ is } p\text{-odd.} \end{cases}$$

For any other composition factor $D(\mu)$ of $S(\lambda)$ we have $\mu < \lambda^R$.

(ii) For a restricted p-strict partition μ of n, $D(sh(\mu))$ appears as a composition factor of the restriction $D(\mu)\downarrow_{\mathcal{T}_{n-1}}$, and for any other composition factor $D(\nu)$ of $D(\mu)\downarrow_{\mathcal{T}_{n-1}}$ we have $\nu < sh(\mu)$.

Let $n = bp + c$ with $0 \leq c < p$. Define $\beta^0 := \varnothing$ and for $n > 0$,

$$\beta^n := \begin{cases} (p^b, c) & \text{if } c > 0 \\ (p^{b-1}, p-1, 1) & \text{if } c = 0. \end{cases}$$

Then $\beta^n = (n)^R$ is a restricted p-strict partition and labels the **basic spin representation** $S((n))$ in characteristic p [13, Lemma 22.3.3]. Wales [26] determined the modular decomposition of the basic spin representation and the ordinary irreducible spin representation $S((n-1,1))$ by inducing from $S((n-1))$, a technique we will use in the next chapter to find the decomposition of $S((n-2,2))$ when n is divisible by $p=3$.

Before stating Wales' theorems [26], we will introduce some notation. If we denote by $\mathcal{C}$ the module category, then the Grothendieck group of $\mathcal{C}$ is an abelian group with elements $[M]$, for $M \in \mathcal{C}$, where

$$[M] = [N] + [P] \qquad \Leftrightarrow \qquad 0 \to N \to M \to P \to 0 \text{ is exact.}$$

In particular,

$$[M] = [N] \qquad \Leftrightarrow \qquad M \text{ and } N \text{ have the same composition factors.}$$

Now we can define $[e_i M]$, for an arbitrary supermodule M, to be the sum $\sum [M:D][e_i D]$, summing over all (irreducible) composition factors D of M.

Theorem 2.3.5. *Suppose $\lambda = (n)$. If n is even, then $S(\lambda) = S(\lambda,+) \oplus S(\lambda,-)$ has dimension $2^{n/2}$ and decomposes over F with dimension $2^{n/2}$ as*

(i) $[D(\beta^n)] = [D(\beta^n,+) \oplus D(\beta^n,-)]$, *if $p \nmid n$;*

(ii) $2[D(\beta^n)] = 2[D(\beta^n,0)]$, *if $p|n$.*

If n is odd, then $S(\lambda) = S(\lambda,0)$ has dimension $2^{(n-1)/2}$ and decomposes over F with dimension $2^{(n-1)/2}$ as

(iii) $[D(\beta^n)] = [D(\beta^n,0)]$, *if $p \nmid n$;*

(iv) $[D(\beta^n)] = [D(\beta^n,+) \oplus D(\beta^n,-)]$, *if $p|n$.*

Thus, the basic spin supermodule $S((n))$ is irreducible over F unless n is even and $p|n$, while the underlying basic spin *module* $\underline{S((n))}$ is irreducible over F unless n is odd and $p|n$.

Theorem 2.3.6. *Suppose $\lambda = (n-1,1)$ and $n \geq 6$. If n is even, then $S(\lambda) = S(\lambda,0)$ has dimension $2^{(n-2)/2}(n-2)$ and decomposes over F with dimension $2^{(n-2)/2}(n-2)$ as*

(i) $[D(\lambda^R)] = [D(\lambda^R,0)]$, *if $p \nmid n(n-1)$;*

(ii) $[D(\lambda^R)] + [D(\beta^n)] = [D(\lambda^R,0)] + [D(\beta^n,0)]$, *if $p|n$;*

(iii) $[D(\lambda^R)] + [D(\beta^n)] = [D(\lambda^R,+) \oplus D(\lambda^R,-)] + [D(\beta^n,+) \oplus D(\beta^n,-)]$, *if $p|(n-1)$.*

If n is odd, then $S(\lambda) = S(\lambda,+) \oplus S(\lambda,-)$ has dimension $2^{(n-1)/2}(n-2)$ and decomposes over F with dimension $2^{(n-1)/2}(n-2)$ as

(iv) $[D(\lambda^R)] = [D(\lambda^R,+) \oplus D(\lambda^R,-)]$, *if $p \nmid n(n-1)$;*

(v) $[D(\lambda^R)] + [D(\beta^n)] = [D(\lambda^R, +) \oplus D(\lambda^R, -)] + [D(\beta^n, +) \oplus D(\beta^n, -)]$, *if* $p|n$;

(vi) $2[D(\lambda^R)] + 2[D(\beta^n)] = 2[D(\lambda^R, 0)] + 2[D(\beta^n, 0)]$, *if* $p|(n-1)$.

In the particular case of $\mathcal{T}_n$-supermodules indexed by partitions with two parts, Morotti [18] describes all of the possible composition factors of the ordinary irreducible $S((\lambda_1, \lambda_2))$, and also computes the multiplicities of the composition factors of $S((\lambda_1, \lambda_2))$ which are not composition factors of an ordinary irreducible $S((\mu_1, \mu_2))$ with $\mu_1 + \mu_2 = \lambda_1 + \lambda_2$ and $\mu_1 > \lambda_1$. The three cases $p = n$, $p = 3 < n$, and $5 \leq p < n$ are treated separately.

Theorem 2.3.7. *Suppose* $p = n \geq 3$. *Then* $[S((p-j, j))] = [D_j] + [D_{j-1}]$ *for* $0 \leq j \leq {}^{(p-1)}/_2$, *where* $D_{-1}, D_{(p-1)/2} := 0$, $D_0 := D((p-1, 1))$, *and* $D_j = D((p-j-1, j+1))$ *for* $1 \leq j \leq {}^{(p-3)}/_2$.

Theorem 2.3.8. *Suppose* $p = 3 < n$ *and set either* $m = \lfloor {}^{(n-1)}/_2 \rfloor - 2$, *if* $n \equiv 3 \bmod 6$, *or* $m = \lfloor {}^{(n-1)}/_2 \rfloor - 1$, *otherwise. For* $0 \leq j \leq m$, *define* $D_j := D(\beta^{n-j} + \beta^j)$. *If* $\lambda = (\lambda_1, \lambda_2)$ *with* $\lambda_1 > \lambda_2 \geq 0$, *then any composition factor of the reduction modulo 3 of* $S(\lambda)$ *is of the form* D_j *with* $0 \leq j \leq min\{\lambda_2, m\}$. *Further, if* $\lambda_2 \leq m$, *then* $[S(\lambda) : D_{\lambda_2}] = 2^a$ *with* $a = 1$ *if at least one of the following holds:*

(i) $\lambda_1, \lambda_2 > 0$ *are both divisible by 3;*

(ii) at least one of $\lambda_1, \lambda_2 > 0$ *is divisible by 3 and* n *is odd;*

(iii) $\lambda_2 = 0$ *and* n *is divisible by 6;*

or $a = 0$, *otherwise.*

For the case when $5 \leq p < n$, the following table defines the partitions μ^k for $1 \leq k \leq {}^{(p-1)}/_2$, depending on the integer $0 \leq c < p$ such that $n = bp + c$:

c	k	μ^k
0	1	$(p^{b-2}, p-1, p-2, 2, 1)$
0	$2 \leq k \leq {}^{(p-1)}/_2$	$(p^{b-1}, p-k, k)$
1	1	$(p^{b-1}, p-2, 2, 1)$
1	$2 \leq k \leq {}^{(p-1)}/_2$	$(p^{b-1}, p+1-k, k)$
$2 \leq c \leq p-2$	$1 \leq k \leq \lceil {}^c/_2 \rceil - 1$	$(p^b, \lfloor {}^c/_2 \rfloor + k, \lceil {}^c/_2 \rceil - k)$
$2 \leq c \leq p-2$	$\lceil {}^c/_2 \rceil$	$(p^{b-1}, p-1, c, 1)$
$2 \leq c \leq p-2$	$\lceil {}^c/_2 \rceil + 1 \leq k \leq {}^{(p-1)}/_2$	$(p^{b-1}, p + \lceil {}^c/_2 \rceil - k, \lfloor {}^c/_2 \rfloor + k)$
$p-1$	$1 \leq k \leq {}^{(p-3)}/_2$	$(p^b, {}^{(p-1+2k)}/_2, {}^{(p-1-2k)}/_2)$
$p-1$	${}^{(p-1)}/_2$	$(p^{b-1}, p-1, p-2, 2)$

When $n > p \geq 5$ and $1 \leq k \leq {}^{(p-1)}/_2$, or when $n = p \geq 5$ and $2 \leq k \leq {}^{(p-1)}/_2$, the partition μ^k is restricted p-strict.

Theorem 2.3.9. *Suppose $5 \leq p < n$, and set either $m = \lfloor (n-1)/2 \rfloor - 1$, if $n \equiv p \bmod 2p$, or $m = \lfloor (n-1)/2 \rfloor$, otherwise. Let $0 \leq c \leq p-1$ such that $n \equiv c \bmod p$, and suppose $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1 > \lambda_2 \geq 0$. Then any composition factor of the reduction modulo p of $S(\lambda)$ is of the form D_j with $0 \leq j \leq min\{\lambda_2, m\}$, where*

$$D_j := \begin{cases} D(\beta^{n-j} + \beta^j), & 0 \leq j \leq (2m-p+1)/2, \\ D(\mu^{m+1-j}), & (2m-p+1)/2 < j \leq m,\ 2|(n+c), \\ D(\mu^{(p-2m+2j-1)/2}), & (2m-p+1)/2 < j \leq m,\ 2 \nmid (n+c). \end{cases}$$

Further, if $\lambda_2 \leq m$, then $[S(\lambda) : D_{\lambda_2}] = 2^a$ with $a = 1$ if at least one of the following holds:

(i) *$\lambda_1, \lambda_2 > 0$ are both divisible by p;*

(ii) *at least one of $\lambda_1, \lambda_2 > 0$ is divisible by p and n is odd;*

(iii) *$\lambda_2 = 0$ and n is divisible by $2p$;*

(iv) *$\lambda = (n/2 + 1, n/2 - 1)$ and n is divisible by $2p$;*

or $a = 0$, otherwise.

Example. Suppose $p = 3$. Using all of the theorems in this section, we will try to find the decomposition of the ordinary irreducible $\mathcal{T}_n$-supermodules $S(\lambda)$ over F for $3 \leq n \leq 9$. These decomposition numbers have already been determined by Morris and Yaseen [21] using different techniques, but we re-derive them here to illustrate the use of Brundan and Kleshchev's modular branching rules [2] and the theorems of Wales [26] and Morotti [18].

$n = 3$: Using Theorem 2.3.4, we find

$$[S((3))] = [D((2,1))]$$
$$[S((2,1))] = [D((2,1))].$$

$n = 4$: Again, by Theorem 2.3.4,

$$[S((4))] = [D((3,1))]$$
$$[S((3,1))] = [D((3,1))].$$

$n = 5$: By Theorem 2.2.1, $S((4,1))$ shares no composition factors with $S((5))$ or $S((3,2))$. Using Theorem 2.3.8, we find

$$[S((5))] = [D((3,2))]$$
$$[S((4,1))] = [D((4,1))]$$
$$[S((3,2))] = 2[D((3,2))].$$

$n = 6$: $[S((6)) : D((6)^R)] = 2$ by Theorem 2.3.5, and we can find the decomposition of $S((5,1))$ using Theorem 2.3.6.

By the Regularisation Theorem 2.3.4, we know $[S((4,2))] = [D((4,2))] + m[D((3,2,1))]$ for some $m \geq 0$. The partition $(4,2)$ has one removable 0-node, so $[e_0 S((4,2))] = [S(3,2)]$, and $(3,2,1)$ has one 0-normal node, so $e_0 D((3,2,1)) = D(3,2)$. Since we know $[S((3,2)) : D((3,2))] = 2$, and since $e_0 D((4,2)) = 0$ (as $(4,2)$ has no 0-good nodes), we find $m = 2$.
Finally, $[S((3,2,1))] = 2[D((3,2,1))]$ by Theorem 2.3.4

$$\begin{aligned}
[S((6))] &= 2[D((3,2,1))] \\
[S((5,1))] &= [D((4,2))] + [D((3,2,1))] \\
[S((4,2))] &= [D((4,2))] + 2[D((3,2,1))] \\
[S((3,2,1))] &= 2[D((3,2,1))].
\end{aligned}$$

$n = 7$: Using the p-block classification, the Regularisation Theorem, Wales' and Morotti's theorems, we find

$$\begin{aligned}
[S((7))] &= [D((3^2,1))] \\
[S((6,1))] &= 2[D((4,2,1))] + 2[D((3^2,1))] \\
[S((5,2))] &= [D((5,2))] \\
[S((4,3))] &= 2[D((4,2,1))] + m_1[D((3^2,1))] \\
[S((4,2,1))] &= [D((4,2,1))] + m_2[D((3^2,1))].
\end{aligned}$$

Since $(4,3)$ is odd, and since $(4,2,1)$ has two 0-removable nodes, we have

$$\begin{aligned}
[e_0 S((4,3))] &= 2[S((4,2))], \text{ and} \\
[e_0 S((4,2,1))] &= [S((4,2))] + [S((3,2,1))].
\end{aligned}$$

Now $(3^2,1)$ has two 0-normal nodes, so $[e_0 D((3^2,1)) : D((3,2,1))] = 2$, while $e_0 D((4,2,1)) = D((4,2))$. Hence

$$\begin{aligned}
2m_1 &= [e_0 S((4,3)) : D((3,2,1))] = 2[S((4,2)) : D((3,2,1))] = 4, \\
2m_2 &= [e_0 S((4,2,1)) : D((3,2,1))] = [S((3,2,1)) : D((3,2,1))] + [S((4,2)) : D((3,2,1))] = 4,
\end{aligned}$$

and therefore $m_1 = m_2 = 2$.
$n = 8$: Without i-induction, we can compute

$$\begin{aligned}
[S((8))] &= [D((3^2,2))] \\
[S((7,1))] &= [D((4,3,1))] \\
[S((6,2))] &= [D((5,2,1))] + m_1[D((3^2,2))] \\
[S((5,3))] &= [D((5,2,1))] + m_2[D((3^2,2))] \\
[S((5,2,1))] &= [D((5,2,1))] + m_3[D((3^2,2))]
\end{aligned}$$

$$[S((4,3,1))] = 2[D((4,3,1))].$$

Since $(6,2)$ is even and $(3^2,2)$ is 3-odd, we find

$$[e_1 S((6,2))] = [S((6,1))],$$
$$e_1 D((3^2,2)) = D((3^2,1)) \oplus D((3^2,1)).$$

Since $e_1 D((5,2,1)) = D((4,2,1))$, we find $m_1 = 1$. Then

$$e_1 S((5,3)) = S((4,3))$$

gives $m_2 = 1$. Moreover, $(5,2,1)$ is odd and $(4,2,1)$ is even, so

$$e_1 S((5,2,1)) = 2S((4,2,1))$$

and $m_3 = 2$.

$n = 9$: Before using i-induction, we find

$$\begin{aligned}
[S((9))] &= [D((3^2,2,1))] \\
[S((8,1))] &= [D((4,3,2))] + [D((3^2,2,1))] \\
[S((7,2))] &= [D((5,3,1))] + m_1[D((4,3,2))] + m_2[D((3^2,2,1))] \\
[S((6,3))] &= 2[D((5,3,1))] + m_3[D((4,3,2))] + m_4[D((3^2,2,1))] \\
[S((5,4))] &= [D((5,3,1))] + m_5[D((4,3,2))] + m_6[D((3^2,2,1))] \\
[S((6,2,1))] &= [D((5,3,1))] + m_7[D((4,3,2))] + m_8[D((3^2,2,1))] \\
[S((5,3,1))] &= [D((5,3,1))] + m_9[D((4,3,2))] + m_{10}[D((3^2,2,1))] \\
[S((4,3,2))] &= [D((4,3,2))] + m_{11}[D((3^2,2,1))].
\end{aligned}$$

Using i-induction, we find

$$[e_1 S((7,2))] = 2[S((7,1))] = 2[D((4,3,1))] = [e_1 D((4,3,2))],$$

so $m_1 = [S((7,2)) : D((4,3,2))] = 1$.

In particular, we want to know the multiplicity m_2 of the basic spin supermodule in $S((7,2))$. Since

$$[e_0 S((7,2))] = 2[S((6,2))] = 2[(D((3^2,2))] + [D((5,2,1)))],$$
$$[e_0 D((3^2,2,1))] = [D((3^2,2))],$$

we know that $m_2 \leq 2$, but this is all we can say because $(5,3,1)$ has two 0-normal nodes, so we cannot compute $e_0 D((n-2,2)^R)$. Later we will see how an explicit construction of $S((7,2))$ will allow us to find the basic spin supermodule as a subquotient and conclude that $m_2 > 0$.

For the remainder of this chapter, we will turn our attention to the ordinary irreducible $\mathcal{T}_n$-supermodule $S((n-2,2))$, which is of type **M** when n is even, and of type **Q** when n is odd. By Morotti's theorems, we know that the only possible decomposition factors of $S((n-2,2))$ over F (when $p=3$ or $5 \leq p \leq n-4$) are $D((n-2,2)^R)$, $D((n-1,1)^R)$ and $D((n)^R)$, but the multiplicities are unknown in general. We will apply the theorems above to determine these multiplicities.

Theorem 2.3.10. *Suppose $p=3$, $n>6$ with $p \nmid n$, and $\lambda=(n-2,2)$. Then $S(\lambda)$ decomposes over F as*

$$[S(\lambda)] = \begin{cases} [D(\lambda^R)] & \text{if } n \equiv 1 \pmod 3, \\ [D(\lambda^R)] + [D((n)^R)] & \text{if } n \equiv 2 \pmod 6, \\ 2[D(\lambda^R)] + 2[D((n)^R)] & \text{if } n \equiv 5 \pmod 6. \end{cases}$$

Proof. The 3-bar-core of λ is $(5,2)$ if $n \equiv 1 \pmod 3$, or (2) if $n \equiv 2 \pmod 3$. In the first case, the 3-bar-core of both (n) and $(n-1,1)$ is (1), so by Humphrey's Theorem 2.2.1, the only possible decomposition factor of $S(\lambda)$ over F is $D(\lambda^R) = D((5, 3^{(n-7)/3}, 2))$. Moreover, the Regularisation Theorem 2.3.4 gives the multiplicity of this factor: When $n \equiv 1 \pmod 3$, we have $l_p(\lambda) = 0$ and λ is even if and only if it is 3-even, so $[S(\lambda) : D(\lambda^R)] = 1$.

If $n \equiv 2 \pmod 3$, then (n) also has 3-bar-core (2), but $(n-1,1)$ has 3-bar-core $(4,1)$, so by Theorem 2.2.1, $S(\lambda)$ has two possible decomposition factors: $D(\lambda^R) = D((5, 3^{(n-8)/3}, 2, 1))$ and $D((n)^R) = D(\beta^n) = D((3^{(n-2)/3}, 2))$. The multiplicity of the first factor is again given by the Regularisation Theorem 2.3.4: When $n \equiv 2 \pmod 3$, we have $l_p(\lambda) = 1$, and when n is even, λ is even and 3-odd, so $[S(\lambda) : D(\lambda^R)] = 1$; When n is odd, λ is odd and 3-even, so $[S(\lambda) : D(\lambda^R)] = 2$.

We can compute the multiplicity $[S(\lambda) : D(\beta^n)]$ using 1-restriction. There is a removable 1-node at the end of the second row of λ, and removing it gives the partition $(n-2,1)$. Since λ is even/odd when n is even/odd, by Theorem 2.3.1, we have

$$[\mathrm{res}_1 S(\lambda)] = \begin{cases} [S((n-2,1))] & \text{if } n \text{ is even} \\ 2[S((n-2,1))] & \text{if } n \text{ is odd,} \end{cases}$$

so $[e_1 S(\lambda)] = [S((n-2,1))]$. Now $\beta^n = (3^{(n-2)/3}, 2)$ has a 1-removable node at the end of the last row, and since β^n has no 1-addable nodes, it has 1-signature $-$, so the 1-removable node is 1-good. Thus, Theorem 2.3.2 tells us that

$$e_1 D(\beta^n) = D((3^{(n-2)/3}, 1)) = D((n-1)^R).$$

Since $3|(n-2)$, by Theorem 2.3.6, we have

$$[S((n-2,1)) : D((n-1)^R)] = \begin{cases} 2 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

Now $\lambda^R = (5, 3^{(n-8)/3}, 2, 1)$ when $n \equiv 2 \pmod 3$, so λ^R has only one 1-removable node and it is 1-good. Removing this node gives the partition $(n-2,1)^R$, so we find $[e_1 D(\lambda^R) : D((n-1)^R)] = 0$. Hence $[S(\lambda) : D((n)^R)] = 1$ if n is even, or 2 if n is odd. □

Note that for $p = 3$ and $n \leq 6$, we have $[S((3,2))] = 2[D((3,2))]$ and $[S((4,2))] = [D((4,2))] + 2[D((3,2,1))]$, as calculated in the example above.

Theorem 2.3.11. *Suppose $p = 3$, $n > 6$ is a multiple of 3, and $\lambda = (n-2,2)$. Then over F,*

$$[S(\lambda)] = [D(\lambda^R)] + [D((n-1,1)^R)] + m[D((n)^R)],$$

where the multiplicity $m = [S(\lambda) : D((n)^R)]$ is at most 2.

Proof. We know the possible composition factors of $S((n-2,2))$ by Morotti's Theorem 2.3.8, and all bar partitions of n have an empty 3-bar-core when 3 divides n. The multiplicity $[S(\lambda) : D(\lambda^R)]$ is 1 by the Regularisation Theorem 2.3.4, since $l_p(\lambda) = 0$ and λ is even if and only if it is 3-even.

When $3|n$, the 1st row of $(n-2,2)$ ends in two 0s and the second row is 0,1. Therefore we always have (two) removable 0-nodes and a removable 1-node. Removing the rightmost 0 gives $\gamma_0 := (n-3,2)$, and removing the 1 gives $\gamma_1 := (n-2,1)$, so we can find an upper bound for $[S(\lambda) : D((n)^R)]$ by considering the decomposition of $S((n-3,2))$ and $S((n-2,1))$. Note that for $i = 0,1$, we have

$$[\mathrm{res}_i S(\lambda)] = \begin{cases} [S(\gamma_i)] & \text{if } n \text{ is even} \\ 2[S(\gamma_i)] & \text{if } n \text{ is odd.} \end{cases}$$

Now since $3|n$, $(n)^R = \beta^n = (3^{(n-3)/3}, 2, 1)$ and $D(\beta^n)$ splits precisely when n is odd. The only removable node leaving a 3-strict partition is the last part, which has 3-residue 0. We can add a 0-node to obtain either $(3^{n/3}, 1)$ or $(4, 3^{(n-3)/3}, 2, 1)$, so the 0-signature of β^n is $-++$ and the node in the last row is 0-good. Hence

$$e_0 D(\beta^n) = D((3^{(n-3)/3}, 2)) = D(\beta^{n-1}).$$

By Theorem 2.3.10, when $n \equiv 2 \pmod 3$ we know that $[S(\gamma_0) : D(\beta^{n-1})] = 2$ if n is even, or 1 if n is odd. Hence $[e_0 S(\lambda) : D(\beta^{n-1})] = 2$. However, since the partition $(n-2,2)^R$ has two 0-normal nodes, we cannot compute $[e_0 D((n-2,2)^R)]$, so all we can say is that $0 \leq [S(\lambda) : D((n)^R)] \leq 2$.

Now it remains to compute the multiplicity $[S(\lambda) : D((n-1,1)^R)]$. Since $3|n$, $(n-1,1)^R = (4, 3^{(n-6)/3}, 2)$ has one 1-normal node, so $e_1 D((n-1,1)^R) = D((4, 3^{(n-6)/3}, 1)) = D((n-2,1)^R)$. Using Brundan and Kleshchev's branching rules, since neither of λ^R and $(n)^R$ has a 1-good node, we find $[S((n-2,2)) : D((n-1,1)^R)] = [S((n-2,1)) : D((n-2,1)^R)]$. Since $(n-2,1)$ has no parts divisible by 3 and is even if and only if it is 3-even, by the Regularisation Theorem, this multiplicity is 1. □

When $p = 3$ and $n > 6$ is a multiple of 3, we will see in the following chapter that $[S((n-2,2)) : D((n)^R)] > 0$. To prove this, we will explicitly construct $S((n-2,2))$ and its submodules. In fact, $[S(\lambda) : D((n)^R)] = 1$ for all $n \leq 18$ [16], so we expect that the multiplicity will always be 1.

We conclude this chapter by classifying the decomposition of $S((n-2,2))$ over a field of arbitrary characteristic $p \geq 5$. By Theorem 2.3.9, the only possible factors are $D((n-2,2)^R)$, $D((n-1,1)^R)$ and $D((n-2,2)^R)$ when $p \leq n-4$. Firstly, we will consider $n-3 \leq p \leq n$.

Theorem. *Suppose $p \geq 5$, $n-3 \leq p \leq n$ and $\lambda = (n-2,2)$. Then the decomposition of $S(\lambda)$ over F is*

$$[S(\lambda)] = \begin{cases} [D(\lambda)] & \text{if } n = p+1, p+3 \\ 2[D(\lambda)] & \text{if } n = p+2 \\ [D(\lambda)] + [D((n-3,3))] & \text{if } n = p. \end{cases}$$

Proof. If $n = p+1$, then λ is a p-bar-core and $\lambda^R = \lambda$, so by Theorems 2.2.1 and 2.3.4 we have

$$[S(\lambda)] = [D(\lambda)].$$

If $n = p+2$, then both (n) and λ have p-bar-core (2) and $(n)^R = \lambda = \lambda^R$, while $(n-1,1)$ is a p-bar-core, so by Theorems 2.2.1 and 2.3.4 we have

$$[S(\lambda)] = 2[D(\lambda)].$$

If $n = p+3$, then both $(n-1,1)$ and λ have p-bar-core $(2,1)$ and $(n-1,1)^R = \lambda = \lambda^R$, while (n) has p-bar-core (3), so by Theorems 2.2.1 and 2.3.4 we have

$$[S(\lambda)] = [D(\lambda)].$$

If $n = p$, then by Theorem 2.3.7 we have $[S((3,2))] = [D((3,2))]$, and for $n > 5$,

$$[S(\lambda)] = [D(\lambda)] + [D((n-3,3))].$$

□

Theorem 2.3.12. *Suppose $5 \leq p \leq n-4$ and $\lambda = (n-2,2)$. Then the decomposition of $S(\lambda)$ over F is*

$$[S(\lambda)] = \begin{cases} [D(\lambda^R)] & \text{if } n \not\equiv 0,2,3 \pmod p \\ [D(\lambda^R)] + [D((n)^R)] & \text{if } n \equiv 2 \pmod{2p} \\ 2[D(\lambda^R)] + 2[D((n)^R)] & \text{if } n \equiv p+2 \pmod{2p} \\ [D(\lambda^R)] + [D((n-1,1)^R)] & \text{if } n \equiv 0,3 \pmod p. \end{cases}$$

Proof. When $p \geq 5$, the possible composition factors of $S(\lambda)$ are given by Theorem 2.3.9. When $p \leq n-4$, these composition factors are $D((n-2,2)^R)$, $D((n-1,1)^R)$ and $D((n)^R)$. Writing $n = pb + c$ with $0 \leq c \leq p-1$, we will determine the multiplicities of these factors for each possible value of c.

First suppose $c = 0$, so that all bar partitions of n have empty p-bar-core. Then since $p \leq n-4$, the regularisation of $\lambda = (n-2,2)$ is $\lambda^R = (p+2, p^{(n-2p)/p}, p-2)$, which has one 2-normal node but no i-normal nodes for $i \neq 2$. The regularisation of $(n-1,1)$ is $(n-1,1)^R = (p+1, p^{(n-2p)/p}, p-1)$, which has one 1-normal node in the last row so that $e_1 D((n-1,1)^R) = D((p+1, p^{b-2}, p-2)) = D((n-2,1)^R)$. The regularisation of (n) is $(n)^R = \beta^n = (p^{(n-p)/p}, p-1, 1)$, which has one 0-normal node but no i-normal nodes for $i \neq 0$. We can remove a 2-node from the end of the first row of λ to obtain $\gamma_2 := (n-3,2)$, or we can remove the 1-node in the second row to obtain $\gamma_1 := (n-2,1)$, and by Theorem 2.3.1, we have

$$[\mathrm{res}_i S(\lambda)] = \begin{cases} [S(\gamma_i)] & \text{if } n \text{ is even} \\ 2[S(\gamma_i)] & \text{if } n \text{ is odd} \end{cases}$$

for $i = 1, 2$, and $\mathrm{res}_i S(\lambda) = 0$ for $i \neq 1, 2$. In particular, since λ has no removable 0-node (while $(n)^R = (p^{b-1}, p-1, 1)$ has no i-good node for $i \neq 0$), the branching rules tell us that $[S(\lambda) : D((n)^R)] = 0$. By the Regularisation Theorem 2.3.4, since $p|n$ we have

$$[S((n-2,1)) : D((n-2,1)^R)] = [S(\lambda) : D(\lambda^R)] = 1.$$

Since $D((n-1,1)^R)$ is of type $\mathbf{M}$ if and only if n is even, we find that $[S(\lambda) : D((n-1,1)^R)] = 1$.

Next, suppose $c \neq 0, 2, 3$. Then the p-bar-core of λ is $(p-1, 2)$ if $c = 1$, or $(c-2, 2)$ if $c > 1$, so by Theorem 2.2.1 we have $[S(\lambda) : D((n)^R)] = [S(\lambda) : D((n-1,1)^R)] = 0$. Hence, by the Regularisation Theorem 2.3.4, $[S(\lambda)] = [D(\lambda^R)]$.

If $c = 2$, then the p-bar-core of both λ and (n) is (2), while $(n-1,1)$ has p-bar-core $(p+1,1)$ so that $[S(\lambda) : D((n-1,1)^R)] = 0$. Using the Regularisation Theorem, we have

$$[S(\lambda) : D(\lambda^R)] = \begin{cases} 1 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd.} \end{cases}$$

Now λ has one removable 0-node in the first row and one removable 1-node in the second row. Set $\gamma_0 = (n-3, 2)$ and $\gamma_1 = (n-2, 1)$, so that again by Theorem 2.3.1, for $i = 0, 1$, we have

$$[\mathrm{res}_i S(\lambda)] = \begin{cases} [S(\gamma_i)] & \text{if } n \text{ is even} \\ 2[S(\gamma_i)] & \text{if } n \text{ is odd} \end{cases}$$

and $\mathrm{res}_i S(\lambda) = 0$ for $i \geq 2$. The restricted p-strict partition $(n)^R = \beta^n = (p^b, 2)$ has one 1-removable node in the last row and there are no 1-addable nodes so the 1-removable node is 1-good. Hence $e_1 D((n)^R) = D((n-1)^R)$. By Theorem 2.3.6, we have

$$[S((n-2,1)) : D((n-1)^R)] = \begin{cases} 2 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

Now $\lambda^R = (p+2, p^{b-2}, p-1, 1)$ has one 1-normal node at the end of the first row, so $e_1 D(\lambda^R) = D((p+1, p^{b-2}, p-1, 1)) = D((n-2,1)^R)$ and it follows that $[S(\lambda) : D((n)^R)] = 1$ if n is even, or 2 if n is odd.

If $c = 3$, then the p-bar-core of both λ and $(n-1,1)$ is $(2,1)$, while (n) has p-bar-core (3). We find $[S(\lambda) : D(\lambda^R)] = 1$ using the Regularisation Theorem, and $[S(\lambda) : D((n)^R)] = 0$ by Theorem 2.2.1. To compute $[S(\lambda) : D((n-1,1)^R)]$, we observe that the only nodes we can remove from λ have residue 1, so by Theorem 2.3.1,

$$[\mathrm{res}_1 S(\lambda)] = \begin{cases} [S((n-2,1))] + [S((n-3,2))] & \text{if } n \text{ is even} \\ 2[S((n-2,1))] + 2[S((n-3,2))] & \text{if } n \text{ is odd} \end{cases}$$

and $\mathrm{res}_i S(\lambda) = 0$ if $i \neq 1$. The partition $(n-1,1)^R = (p+1, p^{b-1}, 2)$ has one 1-normal node in the last row, so $e_1 D((n-1,1)^R) = D((p+1, p^{b-1}, 1)) = D((n-2,1)^R)$. Since $\lambda^R = (p+2, p^{b-1}, 1)$ has no 1-normal nodes and $[S((n-3,2)) : D((n-2,1)^R)] = 0$ by the proof of the case $c = 2$, using Theorem 2.3.2, we find $[S(\lambda) : D((n-1,1)^R)] = [S((n-2,1)) : D((n-2,1)^R)] = 1$. □

We have now calculated the decomposition of the ordinary irreducible $\mathcal{T}_n$-supermodule $S((n-2,2))$ over a field of arbitrary characteristic p, except when $p = 3$ and n is divisible by 3, where we only able to prove that the multiplicity $[S((n-2,2)) : D((n)^R)]$ is either 0, 1 or 2. In the next chapter we will explicitly construct $S((n-2,2))$ and show that the basic spin supermodule $D((n)^R)$ appears at least once as a composition factor when $p = 3$ and 3 divides n.

Chapter 3

Induction of the basic spin module

3.1 A construction of the basic spin module

In this chapter we will emulate James' technique of inducing modules to construct the $\mathcal{T}_n$-supermodule $S((n-2,2))$. Wales [26] used this method to determine the modular decomposition of the spin representation labelled by $(n-1,1)$ by inducing the basic spin representation labelled by (n). The dimension of the basic spin module is given by

$$d(n,p) := \begin{cases} 2^{(n-2)/2} & n \text{ is even} \\ 2^{(n-3)/2} & n \text{ is odd and } p|n \\ 2^{(n-1)/2} & n \text{ is odd and } p \nmid n, \end{cases}$$

and Wales showed that on the basic spin module the element $z \in S_n^+$ acts as -1 and the generators t_i satisfy

$$t_i t_{i+1} + t_{i+1} t_i + 1 = 0.$$

The basic spin representation was originally constructed by Schur [25], but we will instead use the construction of the basic spin representation of S_n^- by Maas [15], adapted to the double cover S_n^+ that we have been working with.

We recursively define matrices $T_1^{(n)}, \ldots, T_{n-1}^{(n)}$ giving the action of the generators $t_1, \ldots, t_{n-1}$ of S_n^+ on the basic spin module. For $n = 2$, set $T_1^{(2)} := 1$. For $n \geq 3$, we consider four separate cases.

Case 1: either n is odd and $p \nmid n$, or n is even and $p|(n-1)$

In this case, we have $d(n,p) = 2d(n-1,p)$. Denoting by I and 0 the $d(n-1,p) \times d(n-1,p)$ identity and zero matrices, we define $T_1^{(n)}, \ldots, T_{n-1}^{(n)}$ to be the block matrices

$$T_i^{(n)} := \begin{pmatrix} T_i^{(n-1)} & 0 \\ 0 & -T_i^{(n-1)} \end{pmatrix} \text{ for } i = 1, \ldots, n-3,$$

$$T_{n-2}^{(n)} := \begin{pmatrix} T_{n-2}^{(n-1)} & -I \\ 0 & -T_{n-2}^{(n-1)} \end{pmatrix},$$

$$T_{n-1}^{(n)} := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

In all of the remaining cases, the dimension $d(n,p)$ of the basic spin module is equal to $d(n-1,p)$. In these cases, we will set $T_i^{(n)} := T_i^{(n-1)}$ for $i = 1, \ldots, n-2$, and it remains to construct $T_{n-1}^{(n)}$.

Case 2: $p|n$

When n is divisible by p, we set

$$T_{n-1}^{(n)} := \sum_{i=1}^{n-2} iT_i^{(n-1)}.$$

Case 3: n is even and $p \nmid n(n-1)(n-2)$

Since $n-1$ is odd and $p \nmid (n-1)$, we can assume that the matrices $T_1^{(n-1)}, \ldots, T_{n-2}^{(n-1)}$ have been constructed as in Case 1 from the matrices $T_1^{(n-2)}, \ldots, T_{n-3}^{(n-2)}$ of size $d(n-2,p) = d(n,p)/2$. We define

$$a := \frac{-1+\sqrt{n/(2-n)}}{n-1}, \quad J := \sum_{i=1}^{n-3} iT_i^{(n-2)}$$

and set

$$T_{n-1}^{(n)} := \begin{pmatrix} aJ & (-1+(2-n)a)I \\ (n-2)aI & -aJ \end{pmatrix}.$$

Case 4: $n \equiv 2 \pmod{2p}$

Since $n-1 \equiv 1 \pmod{2p}$ (so that $n-1$ is odd and $p \nmid (n-1)$), $n-2$ is divisible by p, and $n-3 \equiv -1 \pmod{2p}$, here we can assume that $T_1^{(n-1)}, \ldots, T_{n-2}^{(n-1)}$ have been constructed from matrices $T_1^{(n-4)}, \ldots, T_{n-5}^{(n-4)}$ of size $d(n-4,p) = d(n,p)/4$ using Case 1, then Case 2, then Case 1 again. We define

$$J := \sqrt{-1} \begin{pmatrix} \sum_{i=1}^{n-5} iT_i^{(n-4)} & 2I \\ -2I & -\sum_{i=1}^{n-5} iT_i^{(n-4)} \end{pmatrix}$$

and set

$$T_{n-1}^{(n)} := \begin{pmatrix} J & -I \\ 0 & -J \end{pmatrix}.$$

Now we have an explicit construction of the basic spin module in arbitrary characteristic. In the next section we will induce this module to obtain a construction of the spin representation labelled by the partition $(n-2,2)$.

3.2 Constructing the spin supermodule $S((n-2,2))$

We want to show that when $p = 3$ and $n \geq 6$ is a multiple of 3, the irreducible basic spin supermodule $S((n))$ appears in the decomposition of $S((n-2,2))$ over F. To do this, we are going to induce from a subalgebra of $\mathcal{T}_n$ analogous to the Young subgroup $S_{n-2} \times S_2$ and find a copy of $S((n-2,2))$ in the induced module. This construction will be in arbitrary characteristic p and will allow us to work around the splitting irreducible spin supermodules, as the basic spin module M that we start with does not necessarily need to be irreducible, just a module satisfying the following relations adapted from Maas.

Therefore M will be a combination of copies of the irreducible basic spin module, as Wales showed in [26, Theorem 8.1]. We will refer to any module on which the generators $t_1, \ldots, t_{n-3}, t_{n-1}$ of $\mathcal{T}_n$ act as matrices $T_1, \ldots, T_{n-3}, T_{n-1}$ satisfying

$$T_i^2 = I \text{ for } i = 1, \ldots, n-1, \qquad T_iT_j = -T_jT_i \text{ for } |i-j| > 1,$$

$$T_iT_{i+1} + T_{i+1}T_i + I = 0 \text{ for } i = 1, \ldots, n-2.$$

as a **basic spin module**. We call these conditions the basic spin relations and define, for $i = 1, \ldots, n-1$, the operator

$$\delta_i := t_it_{i+1} + t_{i+1}t_i + 1$$

which annihilates basic spin modules. Note that in the basic spin module M, thanks to the third basic spin relation, we can write any combination of the matrices T_i as a linear combination of terms of the form $T_{i_0}T_{i_1}\cdots T_{i_k}$ with $i_0 < i_1 < \cdots < i_k$. We call this **standard form**. When the indices $i_0, \ldots, i_k$ are consecutive integers, we will write $T_{i_0}\cdots T_{i_k}$ to denote the product $T_{i_0}T_{i_0+1}\cdots T_{i_0+k}$, and set $T_i \cdots T_{i-1} = I$.

We are going to induce a basic spin module M of the subgroup of S_n^+ generated by $t_1, \ldots, t_{n-3}$ and t_{n-1}, and find a copy of $S((n-2,2))$ as a submodule of the induced module $M\uparrow^{\mathcal{T}_n}$.

When n is even, the dimension of $S((n)) = S((n),+) \oplus S((n),-)$ is $2^{n/2}$. In this case, we will construct a module which is a sum of copies of the module $S((n-2,2),0)$ of dimension $2^{(n-4)/2}(n-1)(n-4)$.

When n is odd, the dimension of $S((n)) = S((n),0)$ is $2^{(n-1)/2}$, and the module we construct will be a sum of copies of $S((n-2,2),+)$ and $S((n-2,2),-)$, where $S((n-2,2)) = S((n-2,2),+) \oplus S((n-2,2),-)$ has dimension $2^{(n-3)/2}(n-1)(n-4)$.

Before we construct $S((n-2,2))$, we'll begin with the more straightforward construction of $S((n-1,1))$. Let M be a (sum of copies of the irreducible) basic spin module $S((n-1))$ of $\mathcal{T}_{n-1}$. If $\{x_1, \ldots, x_r\}$ is a basis for M, then a basis for the induced module $M\uparrow^{\mathcal{T}_n}$ is given by the elements

$$(x_j)_i := t_it_{i+1}\cdots t_{n-1}x_j$$

for $1 \le j \le r$ and $i = 1, \ldots n$ (where $(b)_n := b$, for each $b \in M$). We can explicitly write down the action of S_n^+ on these elements:

$$\begin{aligned} t_i(b)_j &= (-1)^{n+j}(T_ib)_j \text{ if } i < j-1 \\ t_i(b)_{i+1} &= (b)_i \\ t_i(b)_i &= (b)_{i+1} \\ t_i(b)_j &= (-1)^{n+j}(T_{i-1}b)_j \text{ if } i > j. \end{aligned}$$

By considering this action, we can find a basis for a copy of the module labelled by $(n-1,1)$ inside the induced module by defining for each $b \in M$ and $1 \le i \le n-2$

$$P(b)_i := (b)_i + (-1)^{n+i}(T_ib)_{i+1} + (b)_{i+2}.$$

Theorem 3.2.1. *Suppose M is a basic spin representation of S_{n-1}^+ with a basis $\{x_1, \dots, x_r\}$. Then the copy of $S((n-1,1))$ inside the induced module $M \uparrow^{\mathcal{T}_n}$ has a basis*

$$\{P(x_j)_i | 1 \le i \le n-2,\ 1 \le j \le r\}.$$

For each $b \in M$, $j = 1, \dots, n-2$ and $i = 1, \dots, n-1$, we have

$$t_i P(b)_j = \begin{cases} (-1)^{n+j} P(T_i b)_j & i < j-1 \\ P(b)_i + (-1)^{n+i+1} P(T_i b)_{i+1} & i = j-1 \\ (-1)^{n+j} P(T_j b)_j & i = j, j+1 \\ (-1)^{n+i} P(T_{i-1} b)_{i-2} + P(b)_{i-1} & i = j+2 \\ (-1)^{n+j} P(T_{i-1} b)_j & i > j+2. \end{cases}$$

Proof. If $i < j-1$, then

$$t_i P(b)_j = (-1)^{n+j}(T_i b)_j + (T_j T_i b)_{j+1} + (-1)^{n+j}(T_i b)_{j+2};$$

$$\begin{aligned} t_i P(b)_{i+1} =& (b)_i + ((1 + T_{i+1} T_i) b)_{i+2} + (-1)^{n+i+1}(T_i b)_{i+3} \\ =& P(b)_i + (-1)^{n+i+1}(T_i b)_{i+1} + (T_{i+1} T_i b)_{i+2} + (-1)^{n+i+1}(T_i b)_{i+3} \\ =& P(b)_i + (-1)^{n+i+1} P(T_i b)_{i+1}; \end{aligned}$$

$$t_i P(b)_i = (b)_{i+1} + (-1)^{n+i}(T_i b)_i + (-1)^{n+i}(T_i b)_{i+2};$$

$$t_i P(b)_{i-1} = (-1)^{n+i+1}(T_{i-1} b)_{i-1} + (-1)^{n+i+1}(T_{i-1} b)_{i+1} + (b)_i;$$

$$\begin{aligned} t_i P(b)_{i-2} =& (-1)^{n+i}(T_{i-1} b)_{i-2} + (-T_{i-1} T_{i-2} b)_{i-1} + (b)_{i+1} \\ =& (-1)^{n+i} P(T_{i-1} b)_{i-2} + (b)_{i-1} + (-1)^{n+i+1}(T_{i-1} b)_i + (b)_{i+1}. \end{aligned}$$

Finally, when $i > j+2$, we have

$$t_i P(b)_j = (-1)^{n+j}(T_{i-1} b)_j + (T_j T_{i-1} b)_{j+1} + (-1)^{n+j}(T_{i-1} b)_{j+2}.$$

□

This shows that the elements $P(b)_i$ are analogous to the polytabloids e_k that span the Specht module $S^{(n-1,1)}$. Next we will construct a module N which will be a sum of copies of $S((n-2,2))$ (the number of copies will depend on the choice of the basic spin representation M we start with).

Let M be a basic spin representation of the subgroup of S_n^+ generated by $t_1, \dots, t_{n-3}$ and t_{n-1}: this means $t_1, \dots, t_{n-3}, t_{n-1}$ act via $T_1, \dots, T_{n-3}, T_{n-1}$ satisfying the basic spin relations. Given a basis $\{x_1, \dots, x_r\}$ for M, there is a basis for the induced module $M \uparrow^{\mathcal{T}_n}$ given by the elements

$$(x_k)_{ij} := t_j t_{j+1} \cdots t_{n-1} t_i t_{i+1} \cdots t_{n-2} x_k$$

for each $1 \le k \le r$ and $1 \le i < j \le n$. (There are ${}^{n(n-1)}/_2$ distinct $(b)_{ij}$ for each $b \in M$; by convention, $t_j t_{j+1} \cdots t_{n-1} = 1$ when $j = n$, and $(b)_{n-1,n} := b$.)

Lemma 3.2.2. *For $1 \le i < j \le n$, the action of the generators $t_1, \ldots, t_{n-1} \in \mathcal{T}_n$ on $(b)_{ij}$ is given by*

$$t_k(b)_{ij} = \begin{cases} (-1)^{i+j+1}(T_k b)_{ij} & k < i-1, \\ (-1)^{n+j}(b)_{i-1,j} & k = i-1, \\ (-1)^{n+j}(b)_{i+1,j} & k = i < j-1, \\ (T_{n-1}b)_{ij} & k = i = j-1, \\ (-1)^{i+j+1}(T_{k-1}b)_{ij} & i < k < j-1, \\ (b)_{i,j-1} & i < k = j-1, \\ (b)_{i,j+1} & k = j, \\ (-1)^{i+j+1}(T_{k-2}b)_{ij} & k > j. \end{cases}$$

Proof. When $k+1 < i < j$, we have

$$\begin{aligned} t_k(b)_{ij} &= (-1)^{n+j} t_j t_{j+1} \cdots t_{n-1} t_k t_i t_{i+1} \cdots t_{n-2} b \\ &= (-1)^{i+j+1} t_j t_{j+1} \cdots t_{n-1} t_i t_{i+1} \cdots t_{n-2} t_k b \\ &= (-1)^{i+j+1} (T_k b)_{ij}; \text{ when } k+1 = i < j, \\ t_{i-1}(b)_{ij} &= (-1)^{n+j} t_j t_{j+1} \cdots t_{n-1} t_{i-1} t_i t_{i+1} \cdots t_{n-2} b \\ &= (-1)^{n+j} (b)_{i-1,j}. \end{aligned}$$

Suppose $1 \le i \le n-3$ and $n+i$ is even. Then

$$\begin{aligned} t_i(b)_{i,i+1} &= t_i t_{i+1} \cdots t_{n-1} t_i t_{i+1} \cdots t_{n-2} b \\ &= -t_i t_{i+1} t_i t_{i+2} t_{i+1} t_{i+3} t_{i+4} \cdots t_{n-1} t_{i+2} t_{i+3} \cdots t_{n-2} b \\ &= t_{i+1} t_{i+2} t_i t_{i+1} t_{i+2} t_{i+3} t_{i+4} \cdots t_{n-1} t_{i+2} t_{i+3} \cdots t_{n-2} b \\ &= t_{i+1} t_{i+2} t_{i+3} t_{i+4} t_i t_{i+1} t_{i+2} t_{i+3} t_{i+4} t_{i+5} \cdots t_{n-1} t_{i+4} t_{i+5} \cdots t_{n-2} b \\ &= \cdots \\ &= t_{i+1} t_{i+2} \cdots t_{n-4} t_i t_{i+1} \cdots t_{n-1} t_{n-4} t_{n-3} t_{n-2} b \\ &= t_{i+1} t_{i+2} \cdots t_{n-4} t_{n-3} t_{n-2} t_i t_{i+1} \cdots t_{n-3} t_{n-2} t_{n-1} t_{n-2} b \\ &= t_{i+1} t_{i+2} \cdots t_{n-4} t_{n-3} t_{n-2} t_i t_{i+1} \cdots t_{n-3} t_{n-1} t_{n-2} t_{n-1} b \\ &= t_{i+1} t_{i+2} \cdots t_{n-1} t_i t_{i+1} \cdots t_{n-2} t_{n-1} b. \end{aligned}$$

If $1 \le i \le n-3$ and $n+i$ is odd, then

$$t_i(b)_{i,i+1} = t_i t_{i+1} \cdots t_{n-1} t_i t_{i+1} \cdots t_{n-2} b$$

$$
\begin{aligned}
&= t_{i+1}t_{i+2}t_it_{i+1}\cdots t_{n-1}t_{i+2}t_{i+3}\cdots t_{n-2}b \\
&= t_{i+1}t_{i+2}t_{i+3}t_{i+4}t_it_{i+1}\cdots t_{n-1}t_{i+4}t_{i+5}\cdots t_{n-2}b \\
&= \cdots \\
&= t_{i+1}t_{i+2}\cdots t_{n-3}t_it_{i+1}\cdots t_{n-1}t_{n-3}t_{n-2}b \\
&= t_{i+1}t_{i+2}\cdots t_{n-3}t_{n-2}t_{n-1}t_it_{i+1}\cdots t_{n-2}t_{n-1}b.
\end{aligned}
$$

Hence, for $i \in \{1, \ldots, n-3\} \cup \{n-1\}$, we have

$$
t_i(b)_{i,i+1} = (T_{n-1}b)_{i,i+1}
$$

(as by definition $(b)_{in} = t_it_{i+1}\cdots t_{n-2}b$, for $1 \leq i < n-1$, and $(b)_{n-1,n} = b$). Moreover, when $1 \leq i < j-1 < n$, we have

$$
\begin{aligned}
t_i(b)_{ij} &= t_it_jt_{j+1}\cdots t_{n-1}t_it_{i+1}\cdots t_{n-2}b \\
&= (-1)^{n+j}t_jt_{j+1}\cdots t_{n-1}t_i^2t_{i+1}t_{i+2}\cdots t_{n-2}b \\
&= (-1)^{n+j}(b)_{i+1,j}.
\end{aligned}
$$

When $1 \leq i < k < j-1 < n$, we have

$$
\begin{aligned}
t_k(b)_{ij} &= (-1)^{n+j}t_jt_{j+1}\cdots t_{n-1}t_kt_it_{i+1}\cdots t_{n-2}b \\
&= (-1)^{n+i+j+k+1}t_jt_{j+1}\cdots t_{n-1}t_it_{i+1}\cdots t_{k-2}t_kt_{k-1}t_kt_{k+1}\cdots t_{n-2}b \\
&= (-1)^{i+j+1}t_jt_{j+1}\cdots t_{n-1}t_it_{i+1}\cdots t_{k-2}t_{k-1}t_kt_{k+1}t_{k+2}\cdots t_{n-2}t_{k-1}b \\
&= (-1)^{i+j+1}(T_{k-1}b)_{ij}.
\end{aligned}
$$

If $1 \leq i < j-1 < n$, then

$$
\begin{aligned}
t_{j-1}(b)_{ij} &= t_{j-1}t_jt_{j+1}\cdots t_{n-1}t_it_{i+1}\cdots t_{n-2}b \\
&= (b)_{i,j-1}.
\end{aligned}
$$

(If $i = j-1$, then $t_{j-1}(b)_{ij} = t_i(b)_{i,i+1}$.)

When $1 \leq i < j < n$,

$$
\begin{aligned}
t_j(b)_{ij} &= t_j^2t_{j+1}t_{j+2}\cdots t_{n-1}t_it_{i+1}\cdots t_{n-2}b \\
&= t_{j+1}t_{j+2}\cdots t_{n-1}t_it_{i+1}\cdots t_{n-2}b \\
&= (b)_{i,j+1}.
\end{aligned}
$$

When $1 \leq i < j < k$, we have

$$
\begin{aligned}
t_k(b)_{ij} &= t_kt_jt_{j+1}\cdots t_{n-1}t_it_{i+1}\cdots t_{n-2}b \\
&= (-1)^{j+k+1}t_jt_{j+1}\cdots t_{k-2}t_kt_{k-1}t_kt_{k+1}\cdots t_{n-1}t_it_{i+1}\cdots t_{n-2}b \\
&= (-1)^{j+k+1}t_jt_{j+1}\cdots t_{k-2}t_{k-1}t_kt_{k-1}t_{k+1}t_{k+2}\cdots t_{n-1}t_it_{i+1}\cdots t_{n-2}b
\end{aligned}
$$

$$= (-1)^{j+k+1}(-1)^{n+k+1} t_j t_{j+1} \cdots t_{n-1} t_{k-1} t_i t_{i+1} \cdots t_{n-2} b$$

$$= (-1)^{n+j}(-1)^{i+k} t_j t_{j+1} \cdots t_{n-1} t_i t_{i+1} \cdots t_{k-3} t_{k-1} t_{k-2} t_{k-1} t_k \cdots t_{n-2} b$$

$$= (-1)^{n+i+j+k} t_j t_{j+1} \cdots t_{n-1} t_i t_{i+1} \cdots t_{k-3} t_{k-2} t_{k-1} t_{k-2} t_k t_{k+1} \cdots t_{n-2} b$$

$$= (-1)^{n+i+j+k}(-1)^{n+k+1} t_j t_{j+1} \cdots t_{n-1} t_i t_{i+1} \cdots t_{n-2} t_{k-2} b$$

$$= (-1)^{i+j+1}(T_{k-2}b)_{ij}.$$

□

We want to find (a sum of copies of) $S((n-2,2))$ as a submodule of this induced module. In order to find this submodule N, we will need a spin analogue of James' construction of the Specht module $S^{(n-2,2)}$. Recall that for a bar partition $\lambda = (\lambda_1, \dots)$ of n, we obtain a shifted λ-tableau by filling the nodes of the diagram $\{(i,j) | 1 \leq i \leq j < \lambda_i\}$ with the numbers $1, \dots, n$. We say that a shifted λ-tableau is **reverse standard** when the numbers are decreasing along each row and down each column. Clearly, the number of reverse standard shifted λ-tableaux is equal to the number of standard shifted λ-tableaux, for all bar partitions λ. We will show that for all $n \geq 5$, by inducing the basic spin module M with basis $\{x_1, \dots, x_r\}$, we can find a basis for N inside $M \uparrow^{\mathcal{T}_n}$ with size equal to r times ${}^{(n-1)(n-4)}/_2$, the number of standard shifted $(n-2,2)$-tableaux.

Example. The submodule labelled by $(4,2)$ is spanned by elements

$$P(b) := (b)_{14} + (T_3b)_{15} + (b)_{16} + (T_1b)_{24} + (T_1T_3b)_{25} + (T_1b)_{26} + (b)_{34} + (T_3b)_{35} + (b)_{36}$$

for each element b in the basic spin module M. For each $b \in M$, the action of the generators $t_1, \dots, t_5$ is given by

$$t_1P(b) = P(T_1b) = t_2P(b);$$

$$t_3P(b) = (b)_{13} - (T_2T_3b)_{15} + (T_2b)_{16} + (T_1b)_{23} - (T_2T_3T_1b)_{25} - (T_2T_1b)_{26} + (T_5b)_{34} - (T_3b)_{45} + (b)_{46};$$

$$t_4P(b) = P(T_3b) = t_5P(b);$$

$$t_2t_3P(b) = (b)_{12} + (T_1T_2T_3b)_{15} + (T_1T_2b)_{16} + (T_5T_1b)_{23} + (T_5b)_{24} + (T_2T_3T_1b)_{35} - (T_2T_1b)_{36} - (T_2T_3b)_{45} - (T_2b)_{46};$$

$$t_4t_3P(b) = -(T_2b)_{13} - (T_2T_3b)_{14} + (T_3T_2b)_{16} + (T_2T_1b)_{23} - (T_2T_3T_1b)_{24} + (T_3T_2T_1b)_{26} + (T_5b)_{35} - (T_5T_3b)_{45} + (b)_{56};$$

$$t_1t_2t_3P(b) = (T_5b)_{12} - (T_5T_1b)_{13} + (T_5b)_{14} - (T_1T_2T_3b)_{25} + (T_1T_2b)_{26} - (T_1T_2T_3T_1b)_{35} - (T_1T_2T_1b)_{36} - (T_1T_2T_3b)_{45} + (T_1T_2b)_{46};$$

$$t_4t_2t_3P(b) = (T_2b)_{12} + (T_1T_2T_3b)_{14} + (T_3T_1T_2b)_{16} - (T_5T_2T_1b)_{23} + (T_5b)_{25} + (T_2T_3T_1b)_{34} - (T_3T_2T_1b)_{36} - (T_5T_2T_3b)_{45} - (T_2b)_{56};$$

$$\begin{aligned}
t_5t_4t_3P(b) &= (T_3T_2b)_{13} - (T_2T_3T_2b)_{14} + (T_3T_2b)_{15} + (T_3T_2T_1b)_{23} + (T_2T_3T_2T_1b)_{24} \\
&\quad + (T_3T_2T_1b)_{25} + (T_5b)_{36} - (T_5T_3b)_{46} + (T_5b)_{56}; \\
t_4t_1t_2t_3P(b) &= -(T_5T_2b)_{12} - (T_5T_2T_1b)_{13} + (T_5b)_{15} - (T_1T_2T_3b)_{24} - (T_3T_1T_2b)_{26} \\
&\quad - (T_1T_2T_3T_1b)_{34} - (T_3T_1T_2T_1b)_{36} - (T_5T_1T_2T_3b)_{45} + (T_1T_2b)_{56}; \\
t_3t_4t_2t_3P(b) &= (T_1T_2b)_{12} + (T_1T_2T_3b)_{13} + (T_2T_3T_1T_2b)_{16} - (T_5T_2T_1b)_{24} - (T_5T_2b)_{25} \\
&\quad + (T_5T_2T_3T_1b)_{34} + (T_5T_2T_3b)_{35} - (T_3T_2T_1b)_{46} - (T_3T_2b)_{56} \\
&= -P(T_2T_1T_3T_2b) - t_3P(T_2T_1T_3b) - t_2t_3P(T_2T_1b) + t_4t_3P(T_2T_3b) - t_4t_2t_3P(T_2b); \\
t_5t_4t_2t_3P(b) &= (T_3T_2b)_{12} + (T_3T_1T_2T_3b)_{14} + (T_3T_1T_2b)_{15} + (T_5T_3T_2T_1b)_{23} + (T_5b)_{26} \\
&\quad + (T_2T_3T_2T_1b)_{34} - (T_3T_2T_1b)_{35} - (T_5T_2T_3b)_{46} - (T_5T_2b)_{56}; \\
t_3t_4t_1t_2t_3P(b) &= (T_5T_1T_2b)_{12} - (T_5T_2T_1b)_{14} + (T_5T_2b)_{15} - (T_1T_2T_3b)_{23} + (T_2T_3T_1T_2b)_{26} \\
&\quad - (T_5T_1T_2T_3T_1b)_{34} + (T_5T_1T_2T_3b)_{35} - (T_3T_1T_2T_1b)_{46} + (T_3T_1T_2b)_{56} \\
&= -t_5t_4t_2t_3P(T_2T_1T_3T_2T_5b); \\
t_5t_4t_1t_2t_3P(b) &= (T_5T_3T_2b)_{12} - (T_5T_3T_2T_1b)_{13} + (T_5b)_{16} + (T_3T_1T_2T_3b)_{24} - (T_3T_1T_2b)_{25} \\
&\quad - (T_3T_1T_2T_3T_1b)_{34} - (T_3T_1T_2T_1b)_{35} - (T_5T_1T_2T_3b)_{46} + (T_5T_1T_2b)_{56} \\
&= -t_3t_4t_2t_3P(T_2T_3T_1T_2T_5b); \\
t_2t_3t_4t_2t_3P(b) &= (T_1T_2T_3b)_{12} + (T_1T_2b)_{13} + (T_1T_2T_3T_1T_2b)_{16} + (T_5T_2T_3T_1b)_{24} \\
&\quad - (T_5T_2T_3b)_{25} - (T_5T_2T_1b)_{34} + (T_5T_2b)_{35} + (T_2T_3T_2T_1b)_{46} - (T_2T_3T_2b)_{56} \\
&= t_5t_4t_1t_2t_3P(T_1T_2T_1T_3T_2T_5b); \\
t_5t_3t_4t_2t_3P(b) &= (T_3T_1T_2b)_{12} - (T_3T_1T_2T_3b)_{13} + (T_2T_3T_1T_2b)_{15} - (T_5T_3T_2T_1b)_{24} - (T_5T_2b)_{26} \\
&\quad - (T_5T_2T_3T_2T_1b)_{34} + (T_5T_2T_3b)_{36} - (T_3T_2T_1b)_{45} - (T_5T_3T_2b)_{56} \\
&= -t_4t_1t_2t_3P(T_2T_1T_3T_2T_5b).
\end{aligned}$$

This shows that the action of any product of 5 or more generators t_i is equivalent to the action of a product of 4 or less generators on $P(b)$. Further, using Gaussian elimination, we find

$$\begin{aligned}
t_5t_4t_2t_3P((1+T_2T_5)b) =& P((1 - T_1T_2T_3T_2 + T_2T_1T_3T_5)b) + t_3P(T_2T_3T_2b) + t_2t_3P(T_3(T_2+T_5)b) \\
&+ t_4t_3P((T_3 - T_5)T_2b) - t_5t_4t_3P(T_2T_3T_5b); \\
t_4t_1t_2t_3P(b) =& P(-(T_1+T_2)(T_3+T_5)b) + t_3P(-(T_1+T_2)(1+T_3T_5)b) \\
&+ t_2t_3P((T_1 - T_5)T_2b) + t_4t_3P((1+T_3T_5)T_1T_2b) \\
&+ t_5t_4t_2t_3P(T_2T_1T_3T_2b).
\end{aligned}$$

Hence, starting with a basis $\{x_1, \ldots, x_r\}$ for the basic spin module M, we have a basis for $S((4,2))$:

$$\bigcup_{i=1}^{r} \{P(x_i), t_3P(x_i), t_2t_3P(x_i), t_4t_3P(x_i), t_1t_2t_3P(x_i), t_4t_2t_3P(x_i), t_5t_4t_3P(x_i)\}.$$

Now we can generalise this construction for $(4,2)$ to $n \geq 6$, but this will not include $n = 5$. However, we can find a linear combination of these 'spin polytabloids' in the span of the element $P(b)$ such that the terms $(b)_{1j}$, for $j = 2, \dots, 6$, are all zero:

$$\begin{aligned}
&P(T_2T_3T_2b) + t_3P(b) + t_4t_3P(T_2b) \\
&\quad = -(T_2b)_{23} - (T_3T_2b)_{24} - (T_3b)_{25} + ((1 - T_3T_2)b)_{26} + ((T_2T_3T_2 + T_5)b)_{34} \\
&\qquad + (T_2(T_3 - T_5)b)_{35} + (T_2T_3T_2b)_{36} - (T_3(1 - T_2T_5)b)_{45} + (b)_{46} + (T_2b)_{56}.
\end{aligned}$$

When we reduce all of the numbers above by 1, we obtain

$$\begin{aligned}
P_5(b) :=& -(T_1b)_{12} - (T_2T_1b)_{13} - (T_2b)_{14} + ((1 - T_2T_1)b)_{15} + ((T_4 - T_1 - T_2)b)_{23} \\
&+ ((T_1T_2 - T_1T_4)b)_{24} + ((-T_1 - T_2)b)_{25} + ((-T_2 - T_2T_1T_4)b)_{34} \\
&+ (b)_{35} + (T_1b)_{45}.
\end{aligned}$$

When $n = 5$, since $t_1P_5(b)$ and $P_5(b)$ are linearly independent,

$$\begin{aligned}
t_2P_5(b) &= -P_5(T_1b) - P_5(T_2b), \qquad t_3P_5(b) = P_5(T_1b), \\
2t_4P_5(b) &= P_5(T_1T_2T_4b) - P_5(T_2b) + t_1P_5(T_1T_2T_1T_4b) + t_1P_5(T_1T_2b) - t_1P_5(T_1T_4b), \\
\text{and } t_2t_1P_5(b) &= -P_5(T_2T_1b) - t_1P_5(T_1b),
\end{aligned}$$

so the submodule generated by elements $P_5(b)$ for $b \in M$ has dimension $2\dim M$ under the action of S_5^+ (the number of standard shifted $(3,2)$-tableaux is 2). This leads us to define

$$\begin{aligned}
P_n(b) :=& (T_{n-4}b)_{n-1,n} \\
&+ (b)_{n-2,n} \\
&+ ((-T_{n-3} + T_{n-1} + T_{n-4}T_{n-3}T_{n-1})b)_{n-2,n-1} \\
&+ ((-T_{n-4} - T_{n-3})b)_{n-3,n} \\
&+ ((T_{n-4}T_{n-3} - T_{n-4}T_{n-1})b)_{n-3,n-1} \\
&+ ((-T_{n-4} - T_{n-3} + T_{n-1})b)_{n-3,n-2} \\
&+ ((2 + T_{n-4}T_{n-3})b)_{n-4,n} \\
&+ (-T_{n-3}b)_{n-4,n-1} \\
&+ ((1 + T_{n-4}T_{n-3})b)_{n-4,n-2} \\
&+ (-T_{n-4}b)_{n-4,n-3},
\end{aligned}$$

for $n \geq 5$. The action of the generators $t_1, \dots, t_{n-1}$ on $P_n(b)$ is given by:

For $n > k + 5$,

$$\begin{aligned}
t_kP_n(b) =& (T_kT_{n-4}b)_{n-1,n} \\
&+ (-T_kb)_{n-2,n} \\
&+ (T_k(-T_{n-3} - T_{n-1} - T_{n-4}T_{n-3}T_{n-1})b)_{n-2,n-1}
\end{aligned}$$

$$
\begin{aligned}
&+ (T_k(-T_{n-4} - T_{n-3})b)_{n-3,n} \\
&+ (T_k T_{n-4}(-T_{n-3} + T_{n-1})b)_{n-3,n-1} \\
&+ (T_k(-T_{n-4} - T_{n-3} + T_{n-1})b)_{n-3,n-2} \\
&+ (T_k(-2 - T_{n-4}T_{n-3})b)_{n-4,n} \\
&+ (-T_k T_{n-3}b)_{n-4,n-1} \\
&+ (T_k(-1 - T_{n-4}T_{n-3})b)_{n-4,n-2} \\
&+ (-T_k T_{n-4}b)_{n-4,n-3} \\
&= P_n(-T_k b);
\end{aligned}
$$

For $n > 5$,

$$
\begin{aligned}
t_{n-5}P_n(b) =&(T_{n-5}T_{n-4}b)_{n-1,n} \\
&+ (-T_{n-5}b)_{n-2,n} \\
&+ (T_{n-5}(-T_{n-3} + T_{n-1} + T_{n-4}T_{n-3}T_{n-1})b)_{n-2,n-1} \\
&+ (T_{n-5}(-T_{n-4} - T_{n-3})b)_{n-3,n} \\
&+ (T_{n-5}T_{n-4}(-T_{n-3} + T_{n-1})b)_{n-3,n-1} \\
&+ (T_{n-5}(-T_{n-4} - T_{n-3} + T_{n-1})b)_{n-3,n-2} \\
&+ ((2 + T_{n-4}T_{n-3})b)_{n-5,n} \\
&+ (T_{n-3}b)_{n-5,n-1} \\
&+ ((1 + T_{n-4}T_{n-3})b)_{n-5,n-2} \\
&+ (T_{n-4}b)_{n-5,n-3};
\end{aligned}
$$

For $n > 4$,

$$
\begin{aligned}
t_{n-4}P_n(b) =&(b)_{n-1,n} \\
&+ (-T_{n-4}b)_{n-2,n} \\
&+ ((-T_{n-4}T_{n-3} + T_{n-4}T_{n-1} + T_{n-3}T_{n-1})b)_{n-2,n-1} \\
&+ ((2 + T_{n-4}T_{n-3})b)_{n-3,n} \\
&+ (T_{n-3}b)_{n-3,n-1} \\
&+ ((1 + T_{n-4}T_{n-3})b)_{n-3,n-2} \\
&+ ((-T_{n-4} - T_{n-3})b)_{n-4,n} \\
&+ (T_{n-4}(-T_{n-3} + T_{n-1})b)_{n-4,n-1} \\
&+ ((-T_{n-4} - T_{n-3} + T_{n-1})b)_{n-4,n-2} \\
&+ (T_{n-4}T_{n-1}b)_{n-4,n-3};
\end{aligned}
$$

$$
t_{n-3}P_n(b) =((-1 - T_{n-4}T_{n-3})b)_{n-1,n}
$$

$$
\begin{aligned}
&+ ((-T_{n-4} - T_{n-3})b)_{n-2,n} \\
&+ (T_{n-4}(-T_{n-3} + T_{n-1})b)_{n-2,n-1} \\
&+ (b)_{n-3,n} \\
&+ ((T_{n-3} - T_{n-1} - T_{n-4}T_{n-3}T_{n-1})b)_{n-3,n-1} \\
&+ ((1 + T_{n-4}T_{n-1} + T_{n-3}T_{n-1})b)_{n-3,n-2} \\
&+ ((-2T_{n-4} - T_{n-3})b)_{n-4,n} \\
&+ (-T_{n-4}T_{n-3}b)_{n-4,n-1} \\
&+ (-T_{n-4}b)_{n-4,n-2} \\
&+ ((1 + T_{n-4}T_{n-3})b)_{n-4,n-3} \\
&= P_n((-T_{n-4} - T_{n-3})b);
\end{aligned}
$$

$$
\begin{aligned}
t_{n-2}P_n(b) =&(b)_{n-1,n} \\
&+ (T_{n-4}b)_{n-2,n} \\
&+ ((1 + T_{n-4}T_{n-3} + T_{n-3}T_{n-1})b)_{n-2,n-1} \\
&+ (T_{n-4}T_{n-3}b)_{n-3,n} \\
&+ ((-T_{n-4} - T_{n-3} + T_{n-1})b)_{n-3,n-1} \\
&+ (T_{n-4}(T_{n-3} - T_{n-1})b)_{n-3,n-2} \\
&+ ((T_{n-4} - T_{n-3})b)_{n-4,n} \\
&+ ((1 + T_{n-4}T_{n-3})b)_{n-4,n-1} \\
&+ (-T_{n-3}b)_{n-4,n-2} \\
&+ (-b)_{n-4,n-3} \\
&= P_n(T_{n-4}b);
\end{aligned}
$$

$$
\begin{aligned}
t_{n-1}P_n(b) =&(-T_{n-4}T_{n-1}b)_{n-1,n} \\
&+ ((-T_{n-3} + T_{n-1} + T_{n-4}T_{n-3}T_{n-1})b)_{n-2,n} \\
&+ (b)_{n-2,n-1} \\
&+ (T_{n-4}(T_{n-3} - T_{n-1})b)_{n-3,n} \\
&+ ((-T_{n-4} - T_{n-3})b)_{n-3,n-1} \\
&+ ((T_{n-4}T_{n-3} + T_{n-3}T_{n-1})b)_{n-3,n-2} \\
&+ (-T_{n-3}b)_{n-4,n} \\
&+ ((2 + T_{n-4}T_{n-3})b)_{n-4,n-1} \\
&+ (T_{n-4}b)_{n-4,n-2} \\
&+ ((1 + T_{n-4}T_{n-3})b)_{n-4,n-3}
\end{aligned}
$$

$$
\begin{aligned}
&= P_n((-T_{n-3} + T_{n-4}T_{n-3}T_{n-1})b) - t_{n-1}P_n(b) \\
&\qquad + t_{n-4}P_n((T_{n-4}T_{n-3} - 2T_{n-4}T_{n-1} - T_{n-3}T_{n-1})b).
\end{aligned}
$$

Hence the set $\{P_n(b)|b \in M\}$ is fixed under left multiplication by t_i for $i \in \{1, \dots, n-1\} \backslash \{n-5, n-4\}$. Note that the following contains no $(b)_{n-4,n-3}$:

$$
\begin{aligned}
P_n(T_{n-1}b) + t_{n-4}P_n(b) =&((1 + T_{n-4}T_{n-1})b)_{n-1,n} \\
&+ ((-T_{n-4} + T_{n-1})b)_{n-2,n} \\
&+ ((1 + T_{n-4}T_{n-1})b)_{n-2,n-1} \\
&+ ((2 + T_{n-4}T_{n-3} - T_{n-4}T_{n-1} - T_{n-3}T_{n-1})b)_{n-3,n} \\
&+ ((-T_{n-4} + T_{n-3} + T_{n-4}T_{n-3}T_{n-1})b)_{n-3,n-1} \\
&+ ((2 + T_{n-4}T_{n-3} - T_{n-4}T_{n-1} - T_{n-3}T_{n-1})b)_{n-3,n-2} \\
&+ ((-T_{n-4} - T_{n-3} + 2T_{n-1} + T_{n-4}T_{n-3}T_{n-1})b)_{n-4,n} \\
&+ ((-T_{n-4}T_{n-3} + T_{n-4}T_{n-1} - T_{n-3}T_{n-1})b)_{n-4,n-1} \\
&+ ((-T_{n-4} - T_{n-3} + 2T_{n-1} + T_{n-4}T_{n-3}T_{n-1})b)_{n-4,n-2}.
\end{aligned}
$$

We have

$$
\begin{aligned}
P_n(T_{n-1}b) + t_{n-4}P_n(b) =&(T_{n-1}(-T_{n-4} + T_{n-1})b)_{n-1,n} \\
&+ ((-T_{n-4} + T_{n-1})b)_{n-2,n} \\
&+ (T_{n-1}(-T_{n-4} + T_{n-1})b)_{n-2,n-1} \\
&+ ((-T_{n-4} + T_{n-1} + T_{n-4}T_{n-3}T_{n-1})(-T_{n-4} + T_{n-1})b)_{n-3,n} \\
&+ ((1 + T_{n-4}(T_{n-3} - T_{n-1}))(-T_{n-4} + T_{n-1})b)_{n-3,n-1} \\
&+ ((-T_{n-4} + T_{n-1} + T_{n-4}T_{n-3}T_{n-1})(-T_{n-4} + T_{n-1})b)_{n-3,n-2} \\
&+ ((1 - T_{n-3}T_{n-1})(-T_{n-4} + T_{n-1})b)_{n-4,n} \\
&+ ((-T_{n-3} + T_{n-1})(-T_{n-4} + T_{n-1})b)_{n-4,n-1} \\
&+ ((1 - T_{n-3}T_{n-1})(-T_{n-4} + T_{n-1})b)_{n-4,n-2}.
\end{aligned}
$$

So that

$$
\begin{aligned}
P(b) :=& \tfrac{1}{2}(P_n((T_{n-4} + T_{n-1})b) + t_{n-4}P_n((1 - T_{n-4}T_{n-1})b)) \\
=&(b)_{n-1,n} \\
&+(T_{n-1}b)_{n-2,n} \\
&+(b)_{n-2,n-1} \\
&+((1 + T_{n-4}(T_{n-3} - T_{n-1}))b)_{n-3,n} \\
&+((-T_{n-4} + T_{n-1} + T_{n-4}T_{n-3}T_{n-1})b)_{n-3,n-1}
\end{aligned}
$$

$$+((1+T_{n-4}(T_{n-3}-T_{n-1}))b)_{n-3,n-2}$$
$$+((-T_{n-3}+T_{n-1})b)_{n-4,n}$$
$$+((1-T_{n-3}T_{n-1})b)_{n-4,n-1}$$
$$+((-T_{n-3}+T_{n-1})b)_{n-4,n-2}.$$

Then $t_kP(b) = P(T_kb)$ for all $k \in \{1,\ldots,n-1\}\backslash\{n-5,n-3,n-2\}$, $t_{n-2}P(b) = P(T_{n-1}b)$, and the two polytabloids

$$\begin{aligned} t_{n-5}P(b) =& (T_{n-5}b)_{n-1,n} \\ &+(-T_{n-5}T_{n-1}b)_{n-2,n} \\ &+(T_{n-5}b)_{n-2,n-1} \\ &+(T_{n-5}(1+T_{n-4}(T_{n-3}-T_{n-1}))b)_{n-3,n} \\ &+(T_{n-5}(T_{n-4}-T_{n-1}-T_{n-4}T_{n-3}T_{n-1})b)_{n-3,n-1} \\ &+(T_{n-5}(1+T_{n-4}(T_{n-3}-T_{n-1}))b)_{n-3,n-2} \\ &+((-T_{n-3}+T_{n-1})b)_{n-5,n} \\ &+((-1+T_{n-3}T_{n-1})b)_{n-5,n-1} \\ &+((-T_{n-3}+T_{n-1})b)_{n-5,n-2} \end{aligned}$$

and

$$\begin{aligned} t_{n-3}P(b) =& (T_{n-3}b)_{n-1,n} \\ &+((1+T_{n-4}(T_{n-3}-T_{n-1}))b)_{n-2,n} \\ &+((T_{n-4}-T_{n-1}-T_{n-4}T_{n-3}T_{n-1})b)_{n-2,n-1} \\ &+(T_{n-1}b)_{n-3,n} \\ &+(-b)_{n-3,n-1} \\ &+((T_{n-4}+T_{n-1}+T_{n-4}T_{n-3}T_{n-1})b)_{n-3,n-2} \\ &+(T_{n-4}(T_{n-3}-T_{n-1})b)_{n-4,n} \\ &+(T_{n-4}(1-T_{n-3}T_{n-1})b)_{n-4,n-1} \\ &+((-T_{n-3}+T_{n-1})b)_{n-4,n-3} \end{aligned}$$

are clearly linearly independent from each other and from $P(b)$.

We define the action of the generator t_i on a reverse standard shifted $(n-2,2)$-tableau to be the shifted tableau with the numbers i and $i-1$ transposed. We associate to $P(b)$ the reverse standard shifted $(n-2,2)$-tableau

n	$n-1$	$n-3$	$n-5$	$n-6$	$\ldots$	1
	$n-2$	$n-4$				

so that only the actions of t_3 and t_5 give reverse standard shifted $(n-2,2)$-tableaux.

Now $t_i t_j P(b)$ is linearly independent from $\{P(b), t_{n-5}P(b), t_{n-3}P(b)\}$ only when $(i,j) \in \{(6,5),(3,5),(5,3)\}$, as

$$\begin{aligned}
t_{n-6}t_{n-5}P(b) =&(T_{n-6}T_{n-5}b)_{n-1,n}\\
&+(T_{n-6}T_{n-5}T_{n-1}b)_{n-2,n}\\
&+T_{n-6}(T_{n-5}b)_{n-2,n-1}\\
&+(T_{n-6}T_{n-5}(1+T_{n-4}(T_{n-3}-T_{n-1}))b)_{n-3,n}\\
&+(T_{n-6}T_{n-5}(-T_{n-4}+T_{n-1}+T_{n-4}T_{n-3}T_{n-1})b)_{n-3,n-1}\\
&+(T_{n-6}T_{n-5}(1+T_{n-4}(T_{n-3}-T_{n-1}))b)_{n-3,n-2}\\
&+((-T_{n-3}+T_{n-1})b)_{n-6,n}\\
&+((1-T_{n-3}T_{n-1})b)_{n-6,n-1}\\
&+((-T_{n-3}+T_{n-1})b)_{n-6,n-2};
\end{aligned}$$

$$\begin{aligned}
t_{n-4}t_{n-5}P(b) =&((-1-T_{n-5}T_{n-4})b)_{n-1,n}\\
&+((-1-T_{n-5}T_{n-4})T_{n-1}b)_{n-2,n}\\
&+((-1-T_{n-5}T_{n-4})b)_{n-2,n-1}\\
&+(T_{n-5}(1+T_{n-4}(T_{n-3}-T_{n-1}))b)_{n-4,n}\\
&+(T_{n-5}(-T_{n-4}+T_{n-1}+T_{n-4}T_{n-3}T_{n-1})b)_{n-4,n-1}\\
&+(T_{n-5}(1+T_{n-4}(T_{n-3}-T_{n-1}))b)_{n-4,n-2}\\
&+(T_{n-5}(-T_{n-3}+T_{n-1})b)_{n-5,n}\\
&+(T_{n-5}(1-T_{n-3}T_{n-1})b)_{n-5,n-1}\\
&+(T_{n-5}(-T_{n-3}+T_{n-1})b)_{n-5,n-2}\\
&=-P(T_{n-5}T_{n-4}b)-t_{n-5}P(T_{n-5}b);
\end{aligned}$$

$$\begin{aligned}
t_{n-3}t_{n-5}P(b) =&(-T_{n-5}T_{n-3}b)_{n-1,n}\\
&+(T_{n-5}(1+T_{n-4}(T_{n-3}-T_{n-1}))b)_{n-2,n}\\
&+(T_{n-5}(-T_{n-4}+T_{n-1}+T_{n-4}T_{n-3}T_{n-1})b)_{n-2,n-1}\\
&+(-T_{n-5}T_{n-1}b)_{n-3,n}\\
&+(-T_{n-5}b)_{n-3,n-1}\\
&+(T_{n-5}(-T_{n-4}-T_{n-1}-T_{n-4}T_{n-3}T_{n-1})b)_{n-3,n-2}\\
&+(T_{n-4}(-T_{n-3}+T_{n-1})b)_{n-5,n}\\
&+(T_{n-4}(1-T_{n-3}T_{n-1})b)_{n-5,n-1}\\
&+((-T_{n-3}+T_{n-1})b)_{n-5,n-3};
\end{aligned}$$

$$
\begin{aligned}
t_{n-4}t_{n-3}P(b) =&(T_{n-4}T_{n-3}b)_{n-1,n}\\
&+((-T_{n-4}-T_{n-3}+T_{n-1})b)_{n-2,n}\\
&+((1-(T_{n-4}+T_{n-3})T_{n-1})b)_{n-2,n-1}\\
&+(T_{n-4}(T_{n-3}-T_{n-1})b)_{n-3,n}\\
&+(T_{n-4}(-1+T_{n-3}T_{n-1})b)_{n-3,n-1}\\
&+(T_{n-1}b)_{n-4,n}\\
&+(b)_{n-4,n-1}\\
&+((T_{n-4}+T_{n-1}+T_{n-4}T_{n-3}T_{n-1})b)_{n-4,n-2}\\
&+((1+T_{n-3}T_{n-1})b)_{n-4,n-3}\\
&= P((1+T_{n-4}T_{n-3})b)-t_{n-3}P(T_{n-3}b);
\end{aligned}
$$

$$
\begin{aligned}
t_{n-2}t_{n-3}P(b) =&((1+T_{n-4}(T_{n-3}-T_{n-1})b)_{n-1,n}\\
&+(T_{n-3}b)_{n-2,n}\\
&+((-1-T_{n-4}(T_{n-3}+T_{n-1}))b)_{n-2,n-1}\\
&+(T_{n-3}T_{n-1}b)_{n-3,n}\\
&+((T_{n-4}+T_{n-1}+T_{n-4}T_{n-3}T_{n-1})b)_{n-3,n-1}\\
&+(-b)_{n-3,n-2}\\
&+((T_{n-4}+T_{n-3}-(1+T_{n-4}T_{n-3})T_{n-1})b)_{n-4,n}\\
&+(T_{n-4}(1-T_{n-3}T_{n-1})b)_{n-4,n-2}\\
&+(T_{n-4}(-T_{n-3}+T_{n-1})b)_{n-4,n-3}\\
&= \tfrac{1}{2}(P((1-2T_{n-4}T_{n-1}-T_{n-3}T_{n-1})b)+t_{n-3}P((-2T_{n-4}-T_{n-3}+T_{n-1})b)).
\end{aligned}
$$

By looking at the second row of each tableau, there is a bijection between the set of reverse standard shifted $(n-2,2)$-tableaux and the set

$$\{(x,y)\in\mathbb{Z}^2 : 1\le y\le n-4,\ y<x\le n-2\}.$$

Using this bijection, we have a bijection between the sets

$$\{(n-2,y) : y\in\mathbb{Z},\ 1\le y\le n-4\}\leftrightarrow\{t_it_{i+1}\cdots t_{n-5}P(b) : 1\le i\le n-4\}.$$

This leads to a bijection

$$
\begin{aligned}
&\{(x,y)\in\mathbb{Z}^2 : 1\le y<x\le n-2\}\backslash\{(n-2,n-3)\}\leftrightarrow\\
&\{t_jt_{j+1}\cdots t_{n-3}t_it_{i+1}\cdots t_{n-5}P(b) : 1\le i<j\le n-2,\ (j,i)\ne(n-2,n-3)\}.
\end{aligned}
$$

Now all that is left is to inductively prove that this set of spin polytabloids is linearly independent. Suppose $b\in M$ and i,j are integers satisfying $1\le i\le n-4$, $i<j\le n-2$. Then we define

$$P_{ij} := t_jt_{j+1}\cdots t_{n-3}t_it_{i+1}\cdots t_{n-5}P(b),$$

and set $P_{n-3,n-2}(b) = 0$. We have

$$\begin{aligned}
P_{i,n-2}(b) = &t_i t_{i+1} \cdots t_{n-5} P(b) \\
=&(T_i \cdots T_{n-5} b)_{n-1,n} \\
&+((-1)^{n+i} T_i \cdots T_{n-5} T_{n-1} b)_{n-2,n} \\
&+(T_i \cdots T_{n-5} b)_{n-2,n-1} \\
&+(T_i \cdots T_{n-5}(1 + T_{n-4}T_{n-3} - T_{n-4}T_{n-1})b)_{n-3,n} \\
&+((-1)^{n+i} T_i \cdots T_{n-5}(-T_{n-4} + T_{n-1} + T_{n-4}T_{n-3}T_{n-1})b)_{n-3,n-1} \\
&+(T_i \cdots T_{n-5}(1 + T_{n-4}T_{n-3} - T_{n-4}T_{n-1})b)_{n-3,n-2} \\
&+((-T_{n-3} + T_{n-1})b)_{i,n} \\
&+((-1)^{n+i}(1 - T_{n-3}T_{n-1})b)_{i,n-1} \\
&+((-T_{n-3} + T_{n-1})b)_{i,n-2},
\end{aligned}$$

and for $1 \leq i < j \leq n-3$,

$$\begin{aligned}
P_{ij}(b) := &t_j t_{j+1} \cdots t_{n-3} t_i t_{i+1} \cdots t_{n-5} P(b) \\
=&(T_j \cdots T_{n-3} T_i \cdots T_{n-5} b)_{n-1,n} \\
&+((-1)^{n+j+1} T_j \cdots T_{n-4} T_i \cdots T_{n-5}(1 + T_{n-4}T_{n-3} - T_{n-4}T_{n-1})b)_{n-2,n} \\
&+((-1)^{n+i} T_j \cdots T_{n-4} T_i \cdots T_{n-5}(T_{n-4} - T_{n-1} - T_{n-4}T_{n-3}T_{n-1})b)_{n-2,n-1} \\
&+((-1)^{n+i} T_i \cdots T_{n-5} T_{n-1} b)_{j,n} \\
&+((-1)^{n+j} T_i \cdots T_{n-5} b)_{j,n-1} \\
&+((-1)^{n+i} T_i \cdots T_{n-5}(T_{n-4} + T_{n-1} + T_{n-4}T_{n-3}T_{n-1})b)_{j,n-2} \\
&+((-1)^{(n+i+1)(n+j)} T_{j-1} \cdots T_{n-4}(-T_{n-3} + T_{n-1})b)_{i,n} \\
&+((-1)^{(n+i)(n+j+1)} T_{j-1} \cdots T_{n-4}(1 - T_{n-3}T_{n-1})b)_{i,n-1} \\
&+((-T_{n-3} + T_{n-1})b)_{i,j}.
\end{aligned}$$

Remark. For all $i, j \in \{1, 2, \ldots, n-3\} \cup \{n-1\}$,

$$(T_i \pm T_j)^2 = 2I \pm (T_iT_j + T_jT_i) = \begin{cases} 2I & |i-j| > 1; \\ 2I \mp I & |i-j| = 1; \\ 2I \pm 2I & i = j. \end{cases}$$

Since we are not considering fields of characteristic 2, and since $(-T_{n-3} + T_{n-1})^2 = 2I$, we see that we have indeed found a linearly independent set of spin polytabloids in the orbit of $P(b)$ that has the same size as the set of standard shifted $(n-2, 2)$-tableaux. Thus we have a basis for the submodule N.

Theorem 3.2.3. *Suppose M is a basic spin representation of $\mathcal{S}_{n-1}^+$ with a given basis $\{x_1, \ldots, x_r\}$. Then the set*

$$\{P_{ij}(x_k) | 1 \leq i \leq n-4,\ i < j \leq n-2,\ 1 \leq k \leq r\}$$

is a basis for the submodule N corresponding to a sum of copies of $S((n-2,2))$ in the induced module $M\uparrow^{\mathcal{T}_n}$ over F. For each $b\in M$ and $k=1,\dots,n-1$, the action of the generator t_k of S_n^+ on the element $P_{ij}(b)$ in the induced module is given by the values of $t_kP_{ij}(b)$ in the following table:

Case		$t_kP_{ij}(b)$
1	$k<i-1$	$(-1)^{i+j}P_{ij}(T_kb)$
2	$k=i=n-4=j-2$	
3	$k=n-1$	
4	$k=i-1$	$(-1)^{n+j}P_{i-1,j}(b)$
5	$k=i<n-4,\ i<j-1$	$(-1)^{n+j}P_{i+1,j}(b)$
6	$k=i=j-1$	$P_{i,i+1}(-T_{n-3}b)+P_{i,n-2}(-T_i\cdots T_{n-5}T_{n-3}T_{n-4}b)$ $+P_{i+1,n-2}(-T_i\cdots T_{n-3}T_{n-4}b)$
7	$i<k<j-1,\ k<n-4$	$(-1)^{i+j}P_{ij}(T_{k-1}b)$
8	$i<k=n-4=j-2$	$(-1)^{n+i}(P_{i,n-2}(T_{n-5}b)+P(T_i\cdots T_{n-4}b))$
9	$i<k=j-1$	$P_{i,j-1}(b)$
10	$k=j<n-2$	$P_{i,j+1}(b)$
11	$k=j=n-2$	$(-1)^{n+i}P_{i,n-2}(T_{n-1}b)$
12	$j<k<n-3$	$(-1)^{i+j}P_{ij}(T_{k-2}b)$
13	$j<k=n-3$	$(-1)^{i+j}P_{ij}(T_{n-5}b)+P_{j,n-3}(-T_i\cdots T_{n-5}T_{n-3}T_{n-4}b)$ $+(-1)^{i+j+1}P_{j,n-2}(T_i\cdots T_{n-6}(1+(T_{n-5}+T_{n-4})T_{n-3})b)$
14	$j<k=n-2$	$\frac{(-1)^{i+j}}{2}(P_{ij}((2T_{n-4}+T_{n-3}-T_{n-1})b)$ $+(-1)^{(n+i)(n+j)}P_{i,n-2}(T_{j-1}\cdots T_{n-5}(-1+(2T_{n-4}+T_{n-3})T_{n-1})b)$ $+P_{j,n-2}(T_i\cdots T_{n-5}(-T_{n-4}+2T_{n-1}+T_{n-4}T_{n-3}T_{n-1})b))$

Proof. Case 1-3: If $0<k<i-1$ and $b\in M$, then by definition of $P_{ij}(b)$, we have

$$t_kP_{ij}(b)=(-1)^{i+j}P_{ij}(T_kb);$$

$$t_{n-4}P_{n-4,n-2}(b)=P(T_{n-4}b);$$

and

$$t_{n-1}P_{ij}(b)=(-1)^{i+j}P_{ij}(T_{n-1}b).$$

Case 4:

$$t_{i-1}P_{ij}(b)=(-1)^{n+j}P_{i-1,j}(b).$$

Case 5: If $i<j-1$ and $i<n-4$, then

$$\begin{aligned}t_iP_{ij}(b)&=(-1)^{n+j}t_jt_{j+1}\cdots t_{n-3}t_{i+1}t_{i+2}\cdots t_{n-5}P(b)\\&=(-1)^{n+j}P_{i+1,j}(b).\end{aligned}$$

Case 6: For $1 \le i \le n-4$, we have

$$\begin{aligned}
P_{i,i+1}(b) &= t_{i+1}\cdots t_{n-3}t_i t_{i+1}\cdots t_{n-5}P(b)\\
&= (T_{i+1}\cdots T_{n-3}T_i\cdots T_{n-5}b)_{n-1,n}\\
&+((-1)^{n+i}T_{i+1}\cdots T_{n-4}T_i\cdots T_{n-5}(1+T_{n-4}T_{n-3}-T_{n-4}T_{n-1})b)_{n-2,n}\\
&+((-1)^{n+i}T_{i+1}\cdots T_{n-4}T_i\cdots T_{n-5}(T_{n-4}-T_{n-1}-T_{n-4}T_{n-3}T_{n-1})b)_{n-2,n-1}\\
&+((-1)^{n+i}T_i\cdots T_{n-5}T_{n-1}b)_{i+1,n}\\
&+((-1)^{n+i+1}T_i\cdots T_{n-5}b)_{i+1,n-1}\\
&+((-1)^{n+i}T_i\cdots T_{n-5}(T_{n-4}+T_{n-1}+T_{n-4}T_{n-3}T_{n-1})b)_{i+1,n-2}\\
&+((-1)^{n+i}T_i\cdots T_{n-4}(T_{n-3}-T_{n-1})b)_{i,n}\\
&+((-1)^{n+i}T_i\cdots T_{n-4}(1-T_{n-3}T_{n-1})b)_{i,n-1}\\
&+((-T_{n-3}+T_{n-1})b)_{i,i+1};
\end{aligned}$$

Thus, the action of t_i on $P_{i,i+1}(b)$ is given by

$$\begin{aligned}
t_iP_{i,i+1}(b) &= t_it_{i+1}\cdots t_{n-3}t_it_{i+1}\cdots t_{n-5}P(b)\\
&= (T_i\cdots T_{n-3}T_i\cdots T_{n-5}b)_{n-1,n}\\
&+((-1)^{n+i}T_i\cdots T_{n-4}T_i\cdots T_{n-5}(-1-T_{n-4}T_{n-3}+T_{n-4}T_{n-1})b)_{n-2,n}\\
&+((-1)^{n+i}T_i\cdots T_{n-4}T_i\cdots T_{n-5}(T_{n-4}-T_{n-1}-T_{n-4}T_{n-3}T_{n-1})b)_{n-2,n-1}\\
&+((-1)^{n+i}T_i\cdots T_{n-4}(T_{n-3}-T_{n-1})b)_{i+1,n}\\
&+((-1)^{n+i}T_i\cdots T_{n-4}(-1+T_{n-3}T_{n-1})b)_{i+1,n-1}\\
&+((-1)^{n+i}T_i\cdots T_{n-5}T_{n-1}b)_{i,n}\\
&+((-1)^{n+i}T_i\cdots T_{n-5}b)_{i,n-1}\\
&+((-1)^{n+i}T_i\cdots T_{n-5}(T_{n-4}+T_{n-1}+T_{n-4}T_{n-3}T_{n-1})b)_{i,n-2}\\
&+((1+T_{n-3}T_{n-1})b)_{i,i+1};
\end{aligned}$$

Adding $P_{i,i+1}(T_{n-3}b)$, we get

$$\begin{aligned}
&t_iP_{i,i+1}(b)+P_{i,i+1}(T_{n-3}b)\\
&\quad= ((T_i\cdots T_{n-3}T_i\cdots T_{n-5}+T_{i+1}\cdots T_{n-3}T_i\cdots T_{n-5}T_{n-3})b)_{n-1,n}\\
&\quad+((-1)^{n+i}(T_i\cdots T_{n-4}T_i\cdots T_{n-5}(-1-T_{n-4}T_{n-3}+T_{n-4}T_{n-1})\\
&\qquad\qquad+T_{i+1}\cdots T_{n-4}T_i\cdots T_{n-5}(T_{n-4}+T_{n-3}+T_{n-4}T_{n-3}T_{n-1}))b)_{n-2,n}\\
&\quad+((-1)^{n+i}(T_i\cdots T_{n-4}T_i\cdots T_{n-5}(T_{n-4}-T_{n-1}-T_{n-4}T_{n-3}T_{n-1})\\
&\qquad\qquad+T_{i+1}\cdots T_{n-4}T_i\cdots T_{n-5}(T_{n-4}T_{n-3}+T_{n-4}T_{n-1}+T_{n-3}T_{n-1}))b)_{n-2,n-1}\\
&\quad+((-1)^{n+i}T_i\cdots T_{n-5}(T_{n-4}T_{n-3}-T_{n-4}T_{n-1}-T_{n-3}T_{n-1})b)_{i+1,n}\\
&\quad+((-1)^{n+i}T_i\cdots T_{n-5}(-T_{n-4}-T_{n-3}+T_{n-4}T_{n-3}T_{n-1})b)_{i+1,n-1}
\end{aligned}$$

$$+((-1)^{n+i}T_i\cdots T_{n-5}(T_{n-4}T_{n-3}-T_{n-4}T_{n-1}-T_{n-3}T_{n-1})b)_{i+1,n-2}$$

$$+((-1)^{n+i}T_i\cdots T_{n-5}(T_{n-4}+T_{n-1}+T_{n-4}T_{n-3}T_{n-1})b)_{i,n}$$

$$+((-1)^{n+i}T_i\cdots T_{n-5}(1+T_{n-4}T_{n-3}+T_{n-4}T_{n-1})b)_{i,n-1}$$

$$+((-1)^{n+i}T_i\cdots T_{n-5}(T_{n-4}+T_{n-1}+T_{n-4}T_{n-3}T_{n-1})b)_{i,n-2};$$

For each $i=1,\ldots,n-4$, we have

$$P_{i,n-2}(T_i\cdots T_{n-5}(-1-T_{n-4}T_{n-3})b)$$

$$=(T_i\cdots T_{n-5}T_i\cdots T_{n-5}(-1-T_{n-4}T_{n-3})b)_{n-1,n}$$

$$+(T_i\cdots T_{n-5}T_i\cdots T_{n-5}(-T_{n-1}-T_{n-4}T_{n-3}T_{n-1})b)_{n-2,n}$$

$$+(T_i\cdots T_{n-5}T_i\cdots T_{n-5}(-1-T_{n-4}T_{n-3})b)_{n-2,n-1}$$

$$+(T_i\cdots T_{n-5}(-1-T_{n-4}T_{n-3}+T_{n-4}T_{n-1})T_i\cdots T_{n-5}(1+T_{n-4}T_{n-3})b)_{n-3,n}$$

$$+((-1)^{n+i}T_i\cdots T_{n-5}(T_{n-4}-T_{n-1}-T_{n-4}T_{n-3}T_{n-1})T_i\cdots T_{n-5}(1+T_{n-4}T_{n-3})b)_{n-3,n-1}$$

$$+(T_i\cdots T_{n-5}(-1-T_{n-4}T_{n-3}+T_{n-4}T_{n-1})T_i\cdots T_{n-5}(1+T_{n-4}T_{n-3})b)_{n-3,n-2}$$

$$+((-1)^{n+i}T_i\cdots T_{n-5}(-T_{n-4}-T_{n-1}-T_{n-4}T_{n-3}T_{n-1})b)_{i,n}$$

$$+((-1)^{n+i}T_i\cdots T_{n-5}(-1-T_{n-4}T_{n-3}-T_{n-4}T_{n-1})b)_{i,n-1}$$

$$+((-1)^{n+i}T_i\cdots T_{n-5}(-T_{n-4}-T_{n-1}-T_{n-4}T_{n-3}T_{n-1})b)_{i,n-2};$$

Adding this, we get

$$t_iP_{i,i+1}(b)+P_{i,i+1}(T_{n-3}b)-P_{i,n-2}(T_i\cdots T_{n-5}(1+T_{n-4}T_{n-3})b)$$

$$=((T_i\cdots T_{n-3}T_i\cdots T_{n-5}+T_{i+1}\cdots T_{n-3}T_i\cdots T_{n-5}T_{n-3}$$

$$+T_i\cdots T_{n-5}T_i\cdots T_{n-5}(-1-T_{n-4}T_{n-3}))b)_{n-1,n}$$

$$+(((-1)^{n+i}T_i\cdots T_{n-4}T_i\cdots T_{n-5}(-1-T_{n-4}T_{n-3}+T_{n-4}T_{n-1})$$

$$+(-1)^{n+i}T_{i+1}\cdots T_{n-4}T_i\cdots T_{n-5}(T_{n-4}+T_{n-3}+T_{n-4}T_{n-3}T_{n-1})$$

$$+T_i\cdots T_{n-5}T_i\cdots T_{n-5}(-T_{n-1}-T_{n-4}T_{n-3}T_{n-1}))b)_{n-2,n}$$

$$+(((-1)^{n+i}T_i\cdots T_{n-4}T_i\cdots T_{n-5}(T_{n-4}-T_{n-1}-T_{n-4}T_{n-3}T_{n-1})$$

$$+(-1)^{n+i}T_{i+1}\cdots T_{n-4}T_i\cdots T_{n-5}(T_{n-4}T_{n-3}+T_{n-4}T_{n-1}+T_{n-3}T_{n-1})$$

$$+T_i\cdots T_{n-5}T_i\cdots T_{n-5}(-1-T_{n-4}T_{n-3}))b)_{n-2,n-1}$$

$$+(T_i\cdots T_{n-5}(-1-T_{n-4}T_{n-3}+T_{n-4}T_{n-1})T_i\cdots T_{n-5}(1+T_{n-4}T_{n-3})b)_{n-3,n}$$

$$+((-1)^{n+i}T_i\cdots T_{n-5}(T_{n-4}-T_{n-1}-T_{n-4}T_{n-3}T_{n-1})T_i\cdots T_{n-5}(1+T_{n-4}T_{n-3})b)_{n-3,n-1}$$

$$+(T_i\cdots T_{n-5}(-1-T_{n-4}T_{n-3}+T_{n-4}T_{n-1})T_i\cdots T_{n-5}(1+T_{n-4}T_{n-3})b)_{n-3,n-2}$$

$$+((-1)^{n+i}T_i\cdots T_{n-5}(T_{n-4}T_{n-3}-T_{n-4}T_{n-1}-T_{n-3}T_{n-1})b)_{i+1,n}$$

$$+((-1)^{n+i}T_i\cdots T_{n-5}(-T_{n-4}-T_{n-3}+T_{n-4}T_{n-3}T_{n-1})b)_{i+1,n-1}$$

$$+((-1)^{n+i}T_i\cdots T_{n-5}(T_{n-4}T_{n-3}-T_{n-4}T_{n-1}-T_{n-3}T_{n-1})b)_{i+1,n-2};$$

Now when $i = n-4$, we get

$$t_{n-4}P_{n-4,n-3}(b) = P((1+T_{n-4}T_{n-3})b) - P_{n-4,n-3}(T_{n-3}b);$$

For $1 \le i \le n-5$, we have

$$(T_i \cdots T_{n-5})^2 = T_{i+1} \cdots T_{n-5} T_i \cdots T_{n-6},$$

and using this relation, we get

$$\begin{aligned}
&t_i P_{i,i+1}(b) + P_{i,i+1}(T_{n-3}b) - P_{i,n-2}(T_i \cdots T_{n-5}(1+T_{n-4}T_{n-3})b)\\
&\quad = ((T_i \cdots T_{n-3} T_i \cdots T_{n-5} + (-1)^{n+i} T_{i+1} \cdots T_{n-4} T_i \cdots T_{n-5}\\
&\qquad\quad + T_i \cdots T_{n-5} T_i \cdots T_{n-5}(-1 - T_{n-4}T_{n-3}))b)_{n-1,n}\\
&\quad +(((-1)^{n+i} T_i \cdots T_{n-4} T_i \cdots T_{n-5}(-1 - T_{n-4}T_{n-3} + T_{n-4}T_{n-1})\\
&\qquad\quad + (-1)^{n+i} T_{i+1} \cdots T_{n-4} T_i \cdots T_{n-5}(T_{n-4} + T_{n-3} + T_{n-4}T_{n-3}T_{n-1})\\
&\qquad\qquad + T_i \cdots T_{n-5} T_i \cdots T_{n-5}(-T_{n-1} - T_{n-4}T_{n-3}T_{n-1}))b)_{n-2,n}\\
&\quad +(((-1)^{n+i} T_i \cdots T_{n-4} T_i \cdots T_{n-5}(T_{n-4} - T_{n-1} - T_{n-4}T_{n-3}T_{n-1})\\
&\qquad\quad + (-1)^{n+i} T_{i+1} \cdots T_{n-4} T_i \cdots T_{n-5}(T_{n-4}T_{n-3} + T_{n-4}T_{n-1} + T_{n-3}T_{n-1})\\
&\qquad\qquad + T_i \cdots T_{n-5} T_i \cdots T_{n-5}(-1 - T_{n-4}T_{n-3}))b)_{n-2,n-1}\\
&\quad +(T_i \cdots T_{n-5} T_i \cdots T_{n-6}(T_{n-4} + T_{n-1} - T_{n-5}T_{n-4}T_{n-3} + T_{n-5}T_{n-4}T_{n-1}\\
&\qquad\qquad\qquad + T_{n-5}T_{n-3}T_{n-1} + T_{n-4}T_{n-3}T_{n-1})b)_{n-3,n}\\
&\quad +(T_i \cdots T_{n-5} T_i \cdots T_{n-6}(-1 - T_{n-5}T_{n-4} - T_{n-5}T_{n-3} - T_{n-4}T_{n-3}\\
&\qquad\qquad\qquad - T_{n-4}T_{n-1} - T_{n-5}T_{n-4}T_{n-3}T_{n-1})b)_{n-3,n-1}\\
&\quad +(T_i \cdots T_{n-5} T_i \cdots T_{n-6}(T_{n-4} + T_{n-1} - T_{n-5}T_{n-4}T_{n-3} + T_{n-5}T_{n-4}T_{n-1}\\
&\qquad\qquad\qquad + T_{n-5}T_{n-3}T_{n-1} + T_{n-4}T_{n-3}T_{n-1})b)_{n-3,n-2}\\
&\quad +((-1)^{n+i} T_i \cdots T_{n-5}(T_{n-4}T_{n-3} - T_{n-4}T_{n-1} - T_{n-3}T_{n-1})b)_{i+1,n}\\
&\quad +((-1)^{n+i} T_i \cdots T_{n-5}(-T_{n-4} - T_{n-3} + T_{n-4}T_{n-3}T_{n-1})b)_{i+1,n-1}\\
&\quad +((-1)^{n+i} T_i \cdots T_{n-5}(T_{n-4}T_{n-3} - T_{n-4}T_{n-1} - T_{n-3}T_{n-1})b)_{i+1,n-2}\\
&\quad = P_{i+1,n-2}(T_i \cdots T_{n-5}(T_{n-4} + T_{n-3})b).
\end{aligned}$$

Case 7: If $i < k < j-1$ and $k < n-4$, then

$$\begin{aligned}
t_k P_{ij}(b) &= (-1)^{n+j} t_j t_{j+1} \cdots t_{n-3} t_k t_i t_{i+1} \cdots t_{n-5} P(b)\\
&= (-1)^{n+i+j+k+1} t_j t_{j+1} \cdots t_{n-3} t_i t_{i+1} \cdots t_{k-2} t_{k-1} t_k t_{k-1} t_{k+1} \cdots t_{n-5} P(b)\\
&= (-1)^{i+j} t_j t_{j+1} \cdots t_{n-3} t_i t_{i+1} \cdots t_{n-5} t_{k-1} P(b)\\
&= (-1)^{i+j} P_{ij}(T_{k-1}b).
\end{aligned}$$

Case 8: For $i < n-4$,

$$t_{n-4}P_{i,n-2}(b) = t_{n-4} t_i t_{i+1} \cdots t_{n-5} P(b)$$

$$
\begin{aligned}
&= (-1)^{n+i+1} t_i t_{i+1} \cdots t_{n-6} t_{n-4} t_{n-5} P(b) \\
&= (-1)^{n+i} t_i t_{i+1} \cdots t_{n-6} (t_{n-5} P(T_{n-5} b) + P(T_{n-5} T_{n-4} b)) \\
&= (-1)^{n+i} (P_{i,n-2}(T_{n-5} b) + P(T_i \cdots T_{n-4} b)).
\end{aligned}
$$

Case 9: If $i < j-1$, then

$$t_{j-1} P_{ij}(b) = P_{i,j-1}(b).$$

Case 10: If $j < n-2$, then

$$
\begin{aligned}
t_j P_{ij}(b) &= t_{j+1} \cdots t_{n-3} t_i t_{i+1} \cdots t_{n-5} P(b) \\
&= P_{i,j+1}(b).
\end{aligned}
$$

Case 11:

$$t_{n-2} P_{i,n-2}(b) = (-1)^{n+i} P_{i,n-2}(T_{n-1} b).$$

Case 12: If $j < k < n-3$, then

$$
\begin{aligned}
t_k P_{ij}(b) &= t_k t_j t_{j+1} \cdots t_{n-3} t_i t_{i+1} \cdots t_{n-5} P(b) \\
&= (-1)^{j+k+1} t_j t_{j+1} \cdots t_{k-2} t_k t_{k-1} t_k t_{k+1} \cdots t_{n-3} t_i t_{i+1} \cdots t_{n-5} P(b) \\
&= (-1)^{n+j} t_j t_{j+1} \cdots t_{n-3} t_{k-1} t_i t_{i+1} \cdots t_{n-5} P(b) \\
&= (-1)^{i+j} P_{ij}(T_{k-2} b).
\end{aligned}
$$

Case 13: If $j < n-3$, then

$$
\begin{aligned}
t_{n-3} P_{i,j}(b) &= (-1)^{n+j} t_j t_{j+1} \cdots t_{n-4} t_{n-3} t_{n-4} t_i t_{i+1} \cdots t_{n-5} P(b) \\
&= (-1)^{i+j+1} t_j t_{j+1} \cdots t_{n-3} t_i t_{i+1} \cdots t_{n-6} t_{n-4} t_{n-5} P(b) \\
&= (-1)^{i+j} t_j t_{j+1} \cdots t_{n-3} t_i t_{i+1} \cdots t_{n-6} (P(T_{n-5} T_{n-4} b) + t_{n-5} P(T_{n-5} b)) \\
&= (-1)^{i+j} (P_{ij}(T_{n-5} b) + t_j t_{j+1} \cdots t_{n-3} P(T_i \cdots T_{n-4} b)) \\
&= (-1)^{i+j} (P_{ij}(T_{n-5} b) + t_j t_{j+1} \cdots t_{n-5} (P((1 + T_{n-4} T_{n-3}) T_i \cdots T_{n-4} b) \\
&\qquad - t_{n-3} P(T_{n-3} T_i \cdots T_{n-4} b))) \\
&= (-1)^{i+j} P_{j,n-2}(T_i \cdots T_{n-6}(-1 - T_{n-5} T_{n-3} - T_{n-4} T_{n-3}) b) \\
&\qquad + P_{j,n-3}(T_i \cdots T_{n-5}(1 + T_{n-4} T_{n-3}) b) + (-1)^{i+j} P_{ij}(T_{n-5} b).
\end{aligned}
$$

Case 14: If $j < n-2$, then

$$
\begin{aligned}
t_{n-2} P_{ij}(b) &= (-1)^{n+j+1} t_j t_{j+1} \cdots t_{n-4} t_{n-2} t_{n-3} t_i t_{i+1} \cdots t_{n-5} P(b) \\
&= (-1)^{n+j+1} t_j t_{j+1} \cdots t_{n-4} t_i t_{i+1} \cdots t_{n-5} t_{n-2} t_{n-3} P(b) \\
&= \tfrac{1}{2} (-1)^{n+j+1} t_j t_{j+1} \cdots t_{n-4} t_i t_{i+1} \cdots t_{n-5} (P((1 - 2T_{n-4} T_{n-1} - T_{n-3} T_{n-1}) b)
\end{aligned}
$$

$$+ t_{n-3}P((-2T_{n-4} - T_{n-3} + T_{n-1})b))$$

$$= \tfrac{1}{2}((-1)^{n+j} t_j t_{j+1} \cdots t_{n-4} P_{i,n-2}((-1 + 2T_{n-4}T_{n-1} + T_{n-3}T_{n-1})b)$$
$$+ (-1)^{i+j} P_{ij}((2T_{n-4} + T_{n-3} - T_{n-1})b));$$

When $j = n-3$, we are done. For $j < n-3$, the action of $t_j t_{j+1} \cdots t_{n-4}$ on $P_{i,n-2}$ is given by

$$\begin{aligned}
&t_j t_{j+1} \cdots t_{n-4} P_{i,n-2}(b) \\
&\quad = (T_j \cdots T_{n-4} T_i \cdots T_{n-5} b)_{n-1,n} \\
&+((-1)^{i+j+1} T_j \cdots T_{n-4} T_i \cdots T_{n-5} T_{n-1} b)_{n-2,n} \\
&+(T_j \cdots T_{n-4} T_i \cdots T_{n-5} b)_{n-2,n-1} \\
&+(T_i \cdots T_{n-5}(1 + T_{n-4}T_{n-3} - T_{n-4}T_{n-1})b)_{j,n} \\
&+((-1)^{i+j} T_i \cdots T_{n-5}(T_{n-4} - T_{n-1} - T_{n-4}T_{n-3}T_{n-1})b)_{j,n-1} \\
&+(T_i \cdots T_{n-5}(1 + T_{n-4}T_{n-3} - T_{n-4}T_{n-1})b)_{j,n-2} \\
&+((-1)^{(n+i+1)(n+j+1)} T_{j-1} \cdots T_{n-5}(-T_{n-3} + T_{n-1})b)_{i,n} \\
&+((-1)^{(n+i)(n+j)} T_{j-1} \cdots T_{n-5}(1 - T_{n-3}T_{n-1})b)_{i,n-1} \\
&+((-1)^{(n+i+1)(n+j+1)} T_{j-1} \cdots T_{n-5}(-T_{n-3} + T_{n-1})b)_{i,n-2};
\end{aligned}$$

Now

$$\begin{aligned}
&(-1)^{(n+i)(n+j+1)+1} P_{i,n-2}(T_{j-1} \cdots T_{n-5} b) \\
&\quad = (T_j \cdots T_{n-5} T_i \cdots T_{n-6} b)_{n-1,n} \\
&+((-1)^{n+j} T_j \cdots T_{n-5} T_i \cdots T_{n-6} T_{n-1} b)_{n-2,n} \\
&+((-1)^{n+i+1} T_j \cdots T_{n-5} T_i \cdots T_{n-6} b)_{n-2,n-1} \\
&+((-1)^{n+i} T_j \cdots T_{n-5} T_i \cdots T_{n-6}(1 + (T_{n-5} + T_{n-4})(T_{n-3} - T_{n-1}))b)_{n-3,n} \\
&+((-1)^{n+j} T_j \cdots T_{n-5} \\
&+((-1)^{(n+i+1)(n+j+1)} T_{j-1} \cdots T_{n-5}(T_{n-3} - T_{n-1})b)_{i,n-2};
\end{aligned}$$

Adding these together, we get

$$\begin{aligned}
&t_j t_{j+1} \cdots t_{n-4} P_{i,n-2}(b) + (-1)^{(n+i)(n+j+1)+1} P_{i,n-2}(T_{j-1} \cdots T_{n-5} b) \\
&\quad = ((-1)^{n+i} T_j \cdots T_{n-5} T_i \cdots T_{n-4} b)_{n-1,n} \\
&+((-1)^{n+j+1} T_j \cdots T_{n-5} T_i \cdots T_{n-4} T_{n-1} b)_{n-2,n} \\
&+((-1)^{n+i} T_j \cdots T_{n-5} T_i \cdots T_{n-4} b)_{n-2,n-1} \\
&+((-1)^{n+i} T_j \cdots T_{n-5} T_i \cdots T_{n-6}(-1 + (T_{n-5} + T_{n-4})(-T_{n-3} + T_{n-1}))b)_{n-3,n} \\
&+((-1)^{n+j} T_j \cdots T_{n-5} T_i \cdots T_{n-6}(-T_{n-5} - T_{n-4} + T_{n-1} + T_{n-5}T_{n-3}T_{n-1} + T_{n-4}T_{n-3}T_{n-1})b)_{n-3,n-1} \\
&+((-1)^{n+i} T_j \cdots T_{n-5} T_i \cdots T_{n-6}(-1 + (T_{n-5} + T_{n-4})(-T_{n-3} + T_{n-1}))b)_{n-3,n-2}
\end{aligned}$$

$$+(T_i \cdots T_{n-5}(1 + T_{n-4}T_{n-3} - T_{n-4}T_{n-1})b)_{j,n}$$

$$+((-1)^{i+j}T_i \cdots T_{n-5}(T_{n-4} - T_{n-1} - T_{n-4}T_{n-3}T_{n-1})b)_{j,n-1}$$

$$+(T_i \cdots T_{n-5}(1 + T_{n-4}T_{n-3} - T_{n-4}T_{n-1})b)_{j,n-2}$$

$$= (-1)^{n+i}P_{j,n-2}(T_i \cdots T_{n-4}b).$$

□

Whilst we are still a long way off a construction of $S(\lambda)$ for arbitrary bar partitions λ, analogous to the Specht modules in the linear case, the methods we used in our construction of $S((n-2,2))$ could potentially be extended further to apply to all two-part bar partitions.

3.3 Constructing submodules

Now we have a construction of the submodule N of the induced basic spin representation which is spanned by the elements $P_{ij}(b)$. Next we want to consider its 3-modular decomposition, so we will set $p = 3$ and assume $n \geq 6$ is a multiple of 3 for the rest of this chapter. We will give a general construction for the $(n-3)r$- and $(n-2)r$-dimensional submodules of N, where r is the dimension of the basic spin module M. Thus, we will prove the following theorem.

Theorem 3.3.1. *Suppose $p = 3$, $n \geq 6$ is a multiple of 3, M is a basic spin representation of S^+_{n-1} with basis $\{x_1, \ldots, x_r\}$, and N is the submodule of dimension ${}^{(n-1)(n-4)r}/_2$ in the induced module $M \uparrow^{\mathcal{T}_n}$ spanned by the elements $P_{ij}(b)$. Then N has submodules L and K of dimensions $(n-2)r$ and $(n-3)r$. Moreover, the larger submodule L contains the smaller one K, and the quotient of the two submodules is a basic spin module.*

Recall that when $p|n$, the basic spin module $S((n))$ has dimension

$$d(n,3) := \begin{cases} 2^{(n-2)/2} & n \text{ is even} \\ 2^{(n-3)/2} & n \text{ is odd} \end{cases}$$

but in general, our basic spin module M will be a sum of copies of $S((n))$, and dim$M$ will be a multiple of $d(n,3)$.

To construct a basis for the $(n-3)d$-dimensional submodule K of N, we define the matrix $\zeta_n := \sum_{k=1}^{n-4}(k-1)T_k$ satisfying the following Lemma.

Lemma 3.3.2. *For $0 \leq j \leq n-5$,*

$$\zeta_n T_1 \cdots T_j = \sum_{k=1}^{j}(-1)^k T_1 \cdots T_{k-1}T_{k+1} \cdots T_j + (-1)^j \sum_{k=j+1}^{n-4}(k-1)T_1 \cdots T_j T_k.$$

The proof of this lemma is in the appendix.

Now set

$$B_{n-3}(b) := P_{1,n-2}(b) + \sum_{j=2}^{n-4}(-1)^{(n+1)(j+1)}P_{j,n-2}(\zeta_n T_1 \cdots T_{j-2}b)$$

for each $b \in M$, and for $i = 2, \ldots, n-3$, inductively define $B_{i-1}(b) := t_i B_i(b)$. Then in general, for all $i = 1, \ldots, n-3$, we have

$$\begin{aligned}
B_i(b) &:= t_{i+1}t_{i+2}\cdots t_{n-3}B_{n-3}(b) \\
&= P_{1,i+1}(b) + \sum_{j=2}^{n-4}(-1)^{(n+1)(j+1)}t_{i+1}t_{i+2}\cdots t_{n-3}P_{j,n-2}(\zeta_n T_1 \cdots T_{j-2}b) \\
&= P_{1,i+1}(b) + \sum_{j=2}^{i}(-1)^{(n+1)(j+1)}P_{j,i+1}(\zeta_n T_1 \cdots T_{j-2}b) \\
&\qquad + \sum_{j=i+1}^{n-4}(-1)^{(n+1)(j+1)}t_{i+1}t_{i+2}\cdots t_{n-3}P_{j,n-2}(\zeta_n T_1 \cdots T_{j-2}b) \\
&= P_{1,i+1}(b) + \sum_{j=2}^{i}(-1)^{(n+1)(j+1)}P_{j,i+1}(\zeta_n T_1 \cdots T_{j-2}b) \\
&\qquad + \sum_{j=i+2}^{n-3}(-1)^{(n+1)j}t_{i+1}t_{i+2}\cdots t_{n-3}P_{j-1,n-2}(\zeta_n T_1 \cdots T_{j-3}b) \\
&= P_{1,i+1}(b) + \sum_{j=2}^{i}(-1)^{(n+1)(j+1)}P_{j,i+1}(\zeta_n T_1 \cdots T_{j-2}b) \\
&\qquad + \sum_{j=i+2}^{n-3}(-1)^{(n+1)j}t_{i+1}t_{i+2}\cdots t_{j-1}P_{j-1,j}(\zeta_n T_1 \cdots T_{j-3}b) \\
&= P_{1,i+1}(b) + \sum_{j=2}^{i}(-1)^{(n+1)(j+1)}P_{j,i+1}(\zeta_n T_1 \cdots T_{j-2}b) \\
&\qquad + \sum_{j=i+2}^{n-3}(-1)^{(n+1)j}t_{i+1}t_{i+2}\cdots t_{j-2}(P_{j-1,j}(-T_{n-3}\zeta_n T_1 \cdots T_{j-3}b) \\
&\qquad\qquad + P_{j-1,n-2}(-T_{j-1}\cdots T_{n-5}T_{n-3}T_{n-4}\zeta_n T_1 \cdots T_{j-3}b) \\
&\qquad\qquad + P_{j,n-2}(T_{j-1}\cdots T_{n-5}(T_{n-4}+T_{n-3})\zeta_n T_1 \cdots T_{j-3}b)) \\
&= P_{1,i+1}(b) + \sum_{j=2}^{i}(-1)^{(n+1)(j+1)}P_{j,i+1}(\zeta_n T_1 \cdots T_{j-2}b) \\
&\qquad + \sum_{j=i+2}^{n-3}((-1)^{(n+1)j+(n+j)(i+j)+1}P_{i+1,j}(T_{n-3}\zeta_n T_1 \cdots T_{j-3}b) \\
&\qquad\qquad + (-1)^{(n+1)j}t_{i+1}t_{i+2}\cdots t_{j-2}(P_{j-1,n-2}(-T_{j-1}\cdots T_{n-5}T_{n-3}T_{n-4}\zeta_n T_1 \cdots T_{j-3}b) \\
&\qquad\qquad\qquad + P_{j,n-2}(T_{j-1}\cdots T_{n-5}(T_{n-4}+T_{n-3})\zeta_n T_1 \cdots T_{j-3}b))) \\
&= P_{1,i+1}(b) + \sum_{j=2}^{i}(-1)^{(n+1)(j+1)}P_{j,i+1}(\zeta_n T_1 \cdots T_{j-2}b) \\
&\qquad + \sum_{j=i+2}^{n-3}((-1)^{i(n+j)+1}P_{i+1,j}(T_{n-3}\zeta_n T_1 \cdots T_{j-3}b) \\
&\qquad\qquad + (-1)^{(n+1)j+1}P_{i+1,n-2}(T_{j-1}\cdots T_{n-5}T_{n-3}T_{n-4}\zeta_n T_1 \cdots T_{j-3}b)
\end{aligned}$$

$$+ (-1)^{(n+1)j+1} t_{i+1} t_{i+2} \cdots t_{j-2} P_{j,n-2}(T_{j-1} \cdots T_{n-5} T_{n-4} T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{j-3} b))$$

$$= P_{1,i+1}(b) + \sum_{j=2}^{i} (-1)^{(n+1)(j+1)} P_{j,i+1}(\zeta_n T_1 \cdots T_{j-2} b)$$

$$- \sum_{j=i+2}^{n-3} ((-1)^{i(n+j)} P_{i+1,j}(T_{n-3} \zeta_n T_1 \cdots T_{j-3} b)$$

$$+ (-1)^{(n+1)j} P_{i+1,n-2}(T_{j-1} \cdots T_{n-5} T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{j-3} b)$$

$$+ (-1)^{i(n+j)} P_{j,n-2}(T_{i+1} \cdots T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{j-3} b)).$$

Theorem 3.3.3. *Suppose $\{x_1, \ldots, x_r\}$ is a basis for a basic spin representation M of S^+_{n-1}, and N is the ${}^{(n-1)(n-4)r}/_2$-dimensional submodule of $M \uparrow^{\mathcal{T}_n}$ spanned by the elements $P_{ij}(b)$. Then there is a submodule K of N with basis $\{B_i(x_j) | 1 \le i \le n-3,\ 1 \le j \le r\}$ and the action of S^+_n on $B_i(b)$ is given by the values of $t_k B_i(b)$ in the following table:*

Case		$t_k B_i(b)$
1	$k = i = 1$	$B_1(-T_{n-3}b) + B_{n-3}(-T_1 \cdots T_{n-5} T_{n-3} T_{n-4} b)$ $+ \sum_{j=2}^{n-4} (B_j(-T_{n-3} \zeta_n T_1 \cdots T_{j-2} b)$ $+(-1)^{(n+1)(j+1)} B_{n-3}(-T_j \cdots T_{n-5} T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{j-2} b))$
2	$k = 1 < i$	$(-1)^i B_i(\zeta_n b)$
3	$1 < k < i$	$(-1)^i B_i(T_{k-1} b)$
4	$k = i$	$B_{i-1}(b)$
5	$k = i+1 < n-2$	$B_{i+1}(b)$
6	$i+1 < k \le n-3$	$(-1)^i B_i(T_{k-2} b)$
7	$i+1 < k = n-2$	$(-1)^i B_i((T_{n-4} - T_{n-3} + T_{n-1}) b)$ $+(-1)^{n(i+1)+1} B_{n-3}(T_i \cdots T_{n-5}(1 + (T_{n-4} - T_{n-3}) T_{n-1}) b)$
8	$k = i+1 = n-2$	$(-1)^i B_i(T_{n-1} b)$
9	$k = n-1$	

The proof of this result is in the appendix.

Now that we have a basis for the $(n-3)r$-dimensional submodule K of N in characteristic 3 (where r is the dimension of the basic spin module M and n is divisible by 3), we will extend this to a basis for the $(n-2)r$-dimensional submodule L of N.

Set $\chi_n := \zeta_n - T_{n-3}$, which satisfies the following Lemma.

Lemma 3.3.4. *For $j = 1, \ldots, n-3$, $2 \le i \le j+1$,*

$$T_i \cdots T_j \chi_n = \sum_{k=2}^{i-1} (-1)^{i+j+1} (k-1) T_k T_i \cdots T_j + \sum_{k=i}^{j} (-1)^{j+k} T_i \cdots T_{k-1} T_{k+1} \cdots T_j$$

$$+\sum_{k=j+1}^{n-3}(k-1)T_i\cdots T_jT_k.$$

The proof of this lemma is in the appendix.

We define

$$\begin{aligned}
B_{n-1}(b) :=& \sum_{j=4}^{n-2}(-1)^{n+1}P_{2j}(T_2\cdots T_{j-3}\chi_n b)\\
&+\sum_{j=4}^{n-2}P_{3j}(-T_{n-3}T_2\cdots T_{j-3}\chi_n b)\\
&+\sum_{i=4}^{n-4}\sum_{j=i+1}^{n-2}(-1)^{n(i+1)}P_{ij}((1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{j-3}\chi_n b)\\
&+\sum_{i=4}^{n-4}\sum_{x=4}^{i}P_{i,n-2}(((-1)^{ni+x+1}T_3\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-3}T_{n-4}\\
&\qquad +(-1)^{(n+1)(i+1)+x}T_2\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_n b),
\end{aligned}$$

and $B_{n-2}(b) := t_3 B_{n-1}$. Then

$$\begin{aligned}
B_{n-2}(b) =&(-1)^{n+1}P_{23}(\chi_n b)+\sum_{j=5}^{n-2}(-1)^{n+j+1}P_{2j}(T_3\cdots T_{j-3}\chi_n b)\\
&+P_{34}(\chi_n b)+\sum_{j=5}^{n-2}(-1)^{j}P_{3j}((1+T_2T_{n-3})T_2\cdots T_{j-3}\chi_n b)\\
&+(-1)^{n+1}P_{3,n-2}(T_2\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n b)\\
&+\sum_{j=5}^{n-2}(-1)^{n+j+1}P_{4j}(T_{n-3}T_2\cdots T_{j-3}\chi_n b)\\
&+P_{4,n-2}(T_3\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n b)\\
&+\sum_{i=5}^{n-4}\sum_{j=i+1}^{n-2}(-1)^{n+(n+1)i+j}P_{ij}((1+(T_2+T_3)T_{n-3})T_4\cdots T_{i-2}T_2\cdots T_{j-3}\chi_n b)\\
&+\sum_{i=5}^{n-4}\sum_{x=4}^{i}P_{i,n-2}(((-1)^{(n+1)(i+1)+x}T_4\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-3}T_{n-4}\\
&\qquad +(-1)^{ni+x+1}T_3T_2\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_n b).
\end{aligned}$$

Adding an r-dimensional basis for the span of $B_{n-2}(x_1),\ldots,B_{n-2}(x_r)$ and $B_{n-1}(x_1),\ldots,B_{n-1}(x_r)$ to our basis

$$\{B_i(x_j)|1\le i\le n-3,\ 1\le j\le r\}$$

for the $(n-3)r$-dimensional submodule K of N, we obtain a basis for the $(n-2)r$-dimensional submodule L of N.

Theorem 3.3.5. *Suppose $\{x_1,\ldots,x_r\}$ is a basis for a basic spin representation M of S^+_{n-1} and N is the ${}^{(n-1)(n-4)r}/_2$-dimensional submodule of $M\uparrow^{\mathcal{T}_n}$ spanned by the elements $P_{ij}(b)$. Then there is an $(n-2)r$-dimensional submodule L of N spanned by the set*

$$\{B_i(x_j)|1\le i\le n-1,\ 1\le j\le r\}.$$

The action of S_n^+ on $B_{n-1}(b)$ and $B_{n-2}(b)$, for $b \in M$, is given in the following table:

Case	$1 \leq k \leq n-1$	$t_k B_{n-1}(b)$	$t_k B_{n-2}(b)$
1	$k=1$	$B_{n-1}(-T_{n-3}b)$ $+\sum_{j=3}^{n-3}(-1)^j B_j(T_2 \cdots T_{j-2}\chi_n b)$	$B_{n-2}(T_{n-3}b)$ $+\sum_{j=3}^{n-3}(-1)^{j+1} t_3 B_j(T_2 \cdots T_{j-2}\chi_n b)$
2	$k=2$	$B_{n-1}(-T_{n-3}b)$	$B_{n-2}(T_{n-3}b) + B_{n-1}(-b)$
3	$k=3$	$B_{n-2}(b)$	$B_{n-1}(b)$
4	$k=4$	$B_{n-1}(-T_2 b)$	$B_{n-2}(T_2 b) + B_{n-1}(-b)$
5	$5 \leq k \leq n-3$	$B_{n-1}(-T_{k-2}b)$	$B_{n-2}(T_{k-2}b)$
6	$k=n-2$	$B_{n-1}((-T_{n-4}+T_{n-3}-T_{n-1})b)$	$B_{n-2}((T_{n-4}-T_{n-3}+T_{n-1})b)$
7	$k=n-1$	$B_{n-1}(-T_{n-1}b)$	$B_{n-2}(T_{n-1}b)$

The proof of this theorem is in the appendix.

Now we can find a basic spin representation as a quotient of the submodules L and K of N by using Theorems 3.3.3 and 3.3.5 to compute the action of the operators $\delta_i := t_i t_{i+1} + t_{i+1} t_i + 1$, for $i = 1, \ldots, n-2$, on $B_{n-1}(b)$, thus proving Theorem 3.3.1.

Proof of Theorem 3.3.1:

Proof. The action of δ_i on $B_{n-2}(b)$ and $B_{n-1}(b)$, for $b \in M$ and $i = 1, \ldots, n-2$, is given by:

$$\begin{aligned}
\delta_1 B_{n-1}(b) =& t_1 B_{n-1}(-T_{n-3}b) + t_2 B_{n-1}(-T_{n-3}b) + \sum_{j=3}^{n-3}(-1)^j t_2 B_j(T_2 \cdots T_{j-2}\chi_n b) + B_{n-1}(b) \\
=& \sum_{j=3}^{n-3}(-1)^j (B_j(-T_2 \cdots T_{j-2}\chi_n T_{n-3}b) + t_2 B_j(T_2 \cdots T_{j-2}\chi_n b); \\
\delta_2 B_{n-1}(b) =& t_2 B_{n-2}(b) + t_3 B_{n-1}(-T_{n-3}b) + B_{n-1}(b) = 0; \\
\delta_3 B_{n-1}(b) =& t_3 B_{n-1}(-T_2 b) + t_4 B_{n-2}(b) + B_{n-1}(b) = 0;
\end{aligned}$$

If $4 \leq k \leq n-4$ then

$$\begin{aligned}
\delta_k B_{n-1}(b) =& t_k B_{n-1}(-T_{k-1}b) + t_{k+1} B_{n-1}(-T_{k-2}b) + B_{n-1}(b) \\
=& B_{n-1}((1 + T_{k-2}T_{k-1} + T_{k-1}T_{k-2})b) = 0; \\
\delta_{n-3} B_{n-1}(b) =& t_{n-3} B_{n-1}((-T_{n-4} + T_{n-3} - T_{n-1})b) + t_{n-2} B_{n-1}(-T_{n-5}b) + B_{n-1}(b) \\
=& B_{n-1}((1 + T_{n-5}(T_{n-4} - T_{n-3} + T_{n-1}) + (T_{n-4} - T_{n-3} + T_{n-1})T_{n-5})b) = 0; \\
\delta_{n-2} B_{n-1}(b) =& t_{n-2} B_{n-1}(-T_{n-1}b) + t_{n-1} B_{n-1}((-T_{n-4} + T_{n-3} - T_{n-1})b) + B_{n-1}(b) \\
=& B_{n-1}((1 + (T_{n-4} - T_{n-3} + T_{n-1})T_{n-1} + T_{n-1}(T_{n-4} - T_{n-3} + T_{n-1}))b) = 0; \\
\delta_1 B_{n-2}(b) =& B_{n-2}(b) + t_1 B_{n-2}(T_{n-3}b) + t_1 B_{n-1}(-b) + t_2 B_{n-2}(T_{n-3}b) \\
&+ \sum_{j=3}^{n-3}(-1)^{j+1} t_2 t_3 B_j(T_2 \cdots T_{j-2}\chi_n b) \\
=& \sum_{j=3}^{n-3}(-1)^{j+1}(t_3 B_j(T_2 \cdots T_{j-2}\chi_n T_{n-3}b) + B_j(T_2 \cdots T_{j-2}\chi_n b));
\end{aligned}$$

$$\begin{aligned}
\delta_2 B_{n-2}(b) =& t_2 B_{n-1} + t_3 B_{n-2}(T_{n-3}b) + t_3 B_{n-1}(-b) + B_{n-2}(b) = 0;\\
\delta_3 B_{n-2}(b) =& t_3 B_{n-2}(T_2 b) + t_3 B_{n-1}(-b) + t_4 B_{n-1}(b) + B_{n-2}(b) = 0;\\
\delta_4 B_{n-2}(b) =& (-t_4 t_3 t_5 + t_5 t_4 t_3 + t_3) B_{n-1}(b)\\
=& t_4 B_{n-2}(T_3 b) + B_{n-2}(T_3 T_2 b) + B_{n-1}(T_3 b) + B_{n-2}(b) = 0;
\end{aligned}$$

If $k \geq 5$ then

$$\delta_k B_{n-2}(b) = t_3 \delta_k B_{n-1}(b) = 0.$$

Since δ_i annihilates $B_{n-2}(b)$ and $B_{n-1}(b)$ for $i > 1$, and $\delta_1 B_{n-2}(b), \delta_1 B_{n-1}(b)$ belong to the smaller subsupermodule K, it follows that the quotient of L and K is a basic spin supermodule. □

Chapter 4

Bar-core partitions

4.1 Properties of bar partitions

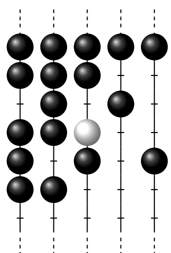
We have already seen the importance of the combinatorics of bars for the spin representation of the symmetric group. In this chapter we will provide further results on bar partitions that serve as spin analogues to all of Fayers' results in [6], and may help us to uncover more of the modular structure of $\mathcal{T}_n$.

Note that while our results on simultaneous p- and q-bar-cores give spin analogues of Fayers' results on simultaneous s- and t-core partitions, called (s,t)-*cores*, there also exist results on $(s,t,u,\dots)$-*cores* which currently have no spin analogue, providing a potential avenue for further research.

Recall that a bar partition $\lambda \in \overline{\mathcal{P}}_n$ is a decreasing sequence of distinct positive integers summing to n. For odd integers $p \geq 3$, the p**-runner abacus** [4] has p infinite vertical runners numbered from left to right ${}^{(p+1)}/_2, {}^{(p+3)}/_2, \dots, p-1, 0, 1, \dots, {}^{(p-1)}/_2$, with the positions on runner i labelled with the integers with p residue i, increasing down the runner, so that position $x+1$ appears directly to the right of position x (for $x \not\equiv {}^{(p-1)}/_2 \bmod p$). We obtain a visual representation of λ on the p-runner abacus by placing a bead on position x for each $x \in \lambda$ and each integer $x < 0$ such that $-x \notin \lambda$; position 0 remains empty.

This differs from the way that Olsson, for example, represents bar partitions and bar-cores, but this p-runner abacus will be more useful for our purposes.

Example. The bar partition $(9,8,7,5,3)$ has the following bead configuration on the 5-runner abacus (we indicate the zero position with a white bead):



Let $\mathcal{A}(\lambda)$ denote the set containing all integers that label positions occupied by beads in the bead configuration for $\lambda \in \overline{\mathcal{P}}_n$ on the p-runner abacus:

$$x \in \mathcal{A}(\lambda) \Leftrightarrow \begin{cases} x \in \lambda, & x > 0, \text{ or} \\ -x \notin \lambda, & x < 0. \end{cases}$$

Note that this is independent from the choice of p.
Recall that for odd integers $p \geq 3$, removing a p-bar from $\lambda \in \overline{\mathcal{P}}_n$ means either
(i) removing $x \in \lambda$ such that $0 \leq x - p \notin \lambda$, and replacing x with $x - p$ if $x \neq p$; or
(ii) removing two parts $x, p - x \in \lambda$ (where $0 < x < p$).
In terms of the abacus, removing a p-bar from λ corresponds to moving a bead at position x to position $x - p$ (replacing $x \in \mathcal{A}(\lambda)$ with $x - p$), then moving the bead at position $p - x$ to position $-x$ (replacing $p - x \in \mathcal{A}(\lambda)$ with $-x$).
When it is not possible to remove any p-bars from λ, i.e. when $x - p \in \mathcal{A}(\lambda)$ for all $x \in \mathcal{A}(\lambda)$, we say that λ is a p-bar-core, and we denote the set of p-bar-cores by $\overline{C}_p$.
Since removing a p-bar always corresponds to moving beads up on their runners to unoccupied positions, we have reached the bead configuration of a p-bar-core when all beads are moved up their runners as far as possible. The order in which these moves are made is irrelevant; we always end up at the same bead configuration. Hence we may define **the p-bar-core** of a bar partition λ, which we denote by $\overline{\lambda}_p$. The number of p-bars which can be successively removed from λ is the p-bar-weight of λ, and we denote this quantity by $\overline{\mathrm{wt}}_p(\lambda)$; denoting by $|\mu|$ the sum of the parts of the bar partition μ,

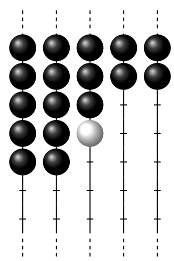
$$\overline{\mathrm{wt}}_p(\lambda) := \tfrac{|\lambda| - |\overline{\lambda}_p|}{p}.$$

The number of bead moves needed to reach the bead configuration for $\overline{\lambda}_p$ from the bead configuration for λ is equal to twice the p-bar-weight of λ, because removing a p-bar corresponds to two moves of the beads. The p-bar-weight of λ is therefore equal to half the number of pairs $(x, a) \in \mathcal{A}(\lambda) \times \mathbb{N}$ such that $x - ap \notin \mathcal{A}(\lambda)$.

Example. The 5-bar-core of the bar partition from the previous example is

$$\overline{(9, 8, 7, 5, 3)}_5 = (4, 3),$$

and has the following bead configuration on the 5-runner abacus:



We are now equipped with tools analogous to those James and Kerber used in their seminal book on the representation theory of the symmetric group via the combinatorics of (not necessarily strict) partitions [11]. Next, we introduce a useful way to encode bar partitions, just as the s-core and s-quotient encode partitions [11].

Define the **p-set** [5] of a bar partition λ to be the set $\{\Delta_{i \bmod p}\lambda | i \equiv 0, 1, \ldots, p-1\}$, where $\Delta_{i \bmod p}\lambda$ is the smallest integer $x \equiv i$ modulo p such that $x \notin \mathcal{A}(\overline{\lambda}_p)$. Since $x \in \mathcal{A}(\overline{\lambda}_p) \Leftrightarrow -x \notin \mathcal{A}(\overline{\lambda}_p)$, for any bar partition λ and $k \not\equiv 0 \pmod p$ we have $\Delta_{k \bmod p}\lambda + \Delta_{-k \bmod p}\lambda = p$, so all of the elements in the p-set of any bar partition sum to $p(p-1)/2$.

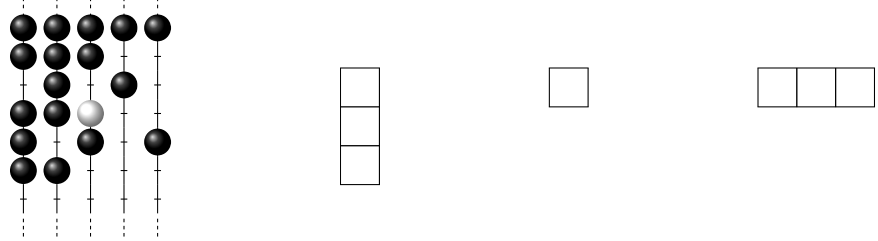
Next we will introduce the **p-quotient** of $\lambda \in \overline{\mathcal{P}}_n$ [23], which is the p-tuple

$$\mathcal{Q}_p(\lambda) := (\lambda^{(0 \bmod p)}, \ldots, \lambda^{(p-1 \bmod p)}).$$

(We will drop the 'mod p' in our notation for both the p-set and p-quotient when it is clear which p we are referring to.) Note that this differs from the s-quotient of a partition we defined in Chapter 2.
The parts of the bar partition $\lambda^{(0 \bmod p)}$ are the elements of the set $\{x/p | x \in \lambda, x \in p\mathbb{Z}\}$.
For $j \not\equiv 0 \pmod p$, the i^{th} part of the (not necessarily strict) partition $\lambda^{(j \bmod p)}$ is equal to the number of empty spaces above the i^{th} lowest bead on runner j in the bead configuration for λ on the p-runner abacus.
It follows from the definition of the p-runner abacus that for each $j \not\equiv 0 \pmod p$, the partition $\lambda^{(-j \bmod p)}$ is equal to $(\lambda^{(j \bmod p)})'$, the **conjugate** of the partition $\lambda^{(j \bmod p)}$, the parts of which are the lengths of the columns in the Young diagram for $\lambda^{(j \bmod p)}$.

Each p-set corresponds to a unique p-bar-core, and Olsson proved that every bar partition is uniquely determined by its p-bar-core and p-quotient [23, Proposition 2.2].

Example. The bar partition $(9, 8, 7, 5, 3)$ has 5-set $\{0, -4, -3, 8, 9\}$ as its 5-bar-core is $(4, 3)$ (see earlier example). While the 5-quotient $\mathcal{Q}_5((4, 3))$ contains only empty partitions (as all p-quotients of p-bar-cores do), the 5-quotient of $\lambda := (9, 8, 7, 5, 3)$ is $((1), (1), (3), (1, 1, 1), (1))$. As remarked above, we have $\lambda^{(j \bmod p)} = (\lambda^{(-j \bmod p)})'$ for $j \not\equiv 0 \pmod p$, and this is illustrated below with the Young diagrams of the conjugate partitions $\lambda^{(3 \bmod 5)} = (1, 1, 1)$ and $\lambda^{(2 \bmod 5)} = (3) = (1, 1, 1)'$, and the self-conjugate partition $\lambda^{(1 \bmod 5)} = (1) = (1)' = \lambda^{(4 \bmod 5)}$, which also happens to be the bar partition $\lambda^{(0 \bmod 5)}$:



Recall that the s-weight of a partition α, i.e. the number of rim s-hooks which can be successively removed from α, is denoted by $\mathrm{wt}_s(\alpha)$, an s-core is a partition with s-weight 0, and the beta-set $\mathcal{B}_k^\alpha$ is the sequence of integers $(\alpha_i - i + k)_{i \in \mathbb{N}}$. Later we will utilise the following properties of $\mathcal{Q}_p(\lambda)$ which are analogous to properties of the s-quotient in the non-spin case [6, Lemma 2.5].

Lemma 4.1.1. *Suppose $\lambda \in \overline{\mathcal{P}}_n$ and c, p are odd postive integers, with $p \geq 3$.*

(i) $\overline{wt}_{cp}(\lambda) = \overline{wt}_c(\lambda^{(0 \ mod\ p)}) + \frac{1}{2}\sum_{j=1}^{p-1} wt_c(\lambda^{(j \ mod\ p)})$*; in particular, λ is a cp-bar-core if and only if $\lambda^{(0 \ mod\ p)}$ is a c-bar-core and every other component of the p-quotient of λ is a c-core.*

(ii) $\overline{(\lambda^{(0 \ mod\ p)})}_c = (\overline{\lambda}_{cp})^{(0 \ mod\ p)}$*, and for $j \not\equiv 0 (mod\ p)$, the c-core of $\lambda^{(j \ mod\ p)}$ is equal to $(\overline{\lambda}_{cp})^{(j \ mod\ p)}$.*

Proof. Removing a cp-bar from λ means replacing an element $x \in \lambda$ with $x - cp$ (when $x - cp \notin \mathcal{A}(\lambda)$), then replacing $cp - x \in \mathcal{A}(\lambda)$ with $-x$.
If $x \equiv 0 \pmod p$, then this is equivalent to replacing $x/p \in \mathcal{A}(\lambda^{(0)})$ with $(x-cp)/p = x/p - c$, and replacing $(cp-x)/p \in \mathcal{A}(\lambda^{(0)})$ with $-x/p = (cp-x)/p - c$, i.e. removing a c-bar from $\lambda^{(0)}$.
If $x \equiv j \pmod p$, then this is equivalent to replacing $(x-j)/p \in \mathcal{B}_r^{\lambda^{(j)}}$ with $((x-cp)-j)/p = (x-j)/p - c$, and then replacing $((cp-x)-(p-j))/p \in \mathcal{B}_r^{\lambda^{(p-j)}}$ with $(-x-(p-j))/p = ((cp-x)-(p-j))/p - c$, thus removing a rim c-hook from both $\lambda^{(j)}$ and $\lambda^{(p-j)}$. □

Remark. Note that since $\lambda^{(-j \text{ mod } p)} = (\lambda^{(j \text{ mod } p)})'$ for $j \not\equiv 0 \pmod p$, we can rewrite the first part of the previous lemma as

$$\overline{\text{wt}}_{cp}(\lambda) = \overline{\text{wt}}_c(\lambda^{(0 \text{ mod } p)}) + \sum_{j=1}^{(p-1)/2} \text{wt}_c(\lambda^{(j \text{ mod } p)}).$$

Now we will consider an action of $\mathfrak{W}_p$, the affine Coxeter group of type $\tilde{C}_{(p-1)/2}$, with generators $g_0, \ldots, g_{(p-1)/2}$ and relations

$$\begin{aligned}
g_i^2 &= 1 && \text{for } 0 \leq i \leq \tfrac{p-1}{2},\\
g_i g_j &= g_j g_i && \text{for } 0 \leq i < j-1 \leq \tfrac{p-3}{2},\\
g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} && \text{for } 1 \leq i \leq \tfrac{p-5}{2},\\
g_0 g_1 g_0 g_1 &= g_1 g_0 g_1 g_0 && \text{if } p > 3,\\
g_{(p-3)/2} g_{(p-1)/2} g_{(p-3)/2} g_{(p-1)/2} &= g_{(p-1)/2} g_{(p-3)/2} g_{(p-1)/2} g_{(p-3)/2} && \text{if } p > 3.
\end{aligned}$$

For coprime odd integers $p, q \geq 3$, we define a *level q action* of $\mathfrak{W}_p$ on $\mathbb{Z}$ [5]:

$$g_0 x = \begin{cases} x - 2q & \text{if } x \equiv q \pmod p,\\ x + 2q & \text{if } x \equiv -q \pmod p,\\ x & \text{otherwise;} \end{cases}$$

$$g_i x = \begin{cases} x - q & \text{if } x \equiv (i+1)q, -iq \pmod p,\\ x + q & \text{if } x \equiv iq, -(i+1)q \pmod p,\\ x & \text{otherwise,} \end{cases} \qquad \text{for } 1 \leq i \leq \tfrac{p-1}{2}.$$

This natural action of $\mathfrak{W}_p$ will serve a similar purpose to the natural action of the Coxeter group W_s of type $\tilde{A}_{s-1}$ that Fayers exploits in [6].

For the rest of this paper, we will assume that p and q are coprime odd integers no less than 3.

Lemma 4.1.2. *The above defines a group action of $\mathfrak{W}_p$ on $\mathbb{Z}$, and this can be extended to an action on $\overline{\mathcal{P}}_n$.*

Proof. We must have $p > 3$ for the fourth and fifth relations of the generators of $\mathfrak{W}_p$ to hold for the group action. The first relation $g_i^2 = 1$ is clear for all i. Moreover, the generators g_i, g_j commute when $0 \leq i < j - 1 \leq {}^{(p-3)}/_2$ because they act on distinct congruence classes of integers modulo p. For the third relation, when $1 \leq i \leq {}^{(p-5)}/_2$ and $x \in \mathbb{Z}$ we have

$$g_i g_{i+1} g_i x = \left\{ \begin{array}{l|l} g_i(x-2q) & x \equiv -iq \\ g_i(x+2q) & x \equiv iq \\ g_i(x-q) & x \equiv (i+1)q, (i+2)q \\ g_i(x+q) & x \equiv -(i+1)q, -(i+2)q \\ g_i x & \text{otherwise} \end{array} \right\} = \left\{ \begin{array}{l|l} x-2q & x \equiv (i+2)q, -iq \\ x+2q & x \equiv iq, -(i+2)q \\ x & \text{otherwise} \end{array} \right.$$

$$= \left\{ \begin{array}{l|l} g_{i+1}(x-2q) & x \equiv (i+2)q \\ g_{i+1}(x+2q) & x \equiv -(i+2)q \\ g_{i+1}(x-q) & x \equiv -iq, -(i+1)q \\ g_{i+1}(x+q) & x \equiv iq, (i+1)q \\ g_{i+1} x & \text{otherwise} \end{array} \right\} = g_{i+1} g_i g_{i+1} x.$$

For the fourth and fifth relations, assuming $p > 3$,

$$(g_0 g_1)^2 x = \left\{ \begin{array}{l|l} g_0 g_1(x-3q) & x \equiv 2q \\ g_0 g_1(x+3q) & x \equiv -2q \\ g_0 g_1(x-q) & x \equiv -q \\ g_0 g_1(x+q) & x \equiv q \\ g_0 g_1 x & \text{otherwise} \end{array} \right\} = \left\{ \begin{array}{l|l} x-4q & x \equiv 2q \\ x+4q & x \equiv -2q \\ x-2q & x \equiv q \\ x+2q & x \equiv -q \\ x & \text{otherwise} \end{array} \right.$$

$$= \left\{ \begin{array}{l|l} g_1 g_0(x-3q) & x \equiv q \\ g_1 g_0(x+3q) & x \equiv -q \\ g_1 g_0(x-q) & x \equiv 2q \\ g_1 g_0(x+q) & x \equiv -2q \\ g_1 g_0 x & \text{otherwise} \end{array} \right\} = (g_1 g_0)^2 x,$$

$$(g_{(p-3)/2} g_{(p-1)/2})^2 x = \left\{ \begin{array}{l|l} g_{(p-3)/2} g_{(p-1)/2}(x-2q) & x \equiv \frac{(p+1)q}{2} \\ g_{(p-3)/2} g_{(p-1)/2}(x+2q) & x \equiv \frac{(p-1)q}{2} \\ g_{(p-3)/2} g_{(p-1)/2}(x-q) & x \equiv \frac{(p+3)q}{2} \\ g_{(p-3)/2} g_{(p-1)/2}(x+q) & x \equiv \frac{(p-3)q}{2} \\ g_{(p-3)/2} g_{(p-1/2} x & \text{otherwise} \end{array} \right\} = \left\{ \begin{array}{l|l} x-3q & x \equiv \frac{(p+3)q}{2} \\ x+3q & x \equiv \frac{(p-3)q}{2} \\ x-q & x \equiv \frac{(p+1)q}{2} \\ x+q & x \equiv \frac{(p-1)q}{2} \\ x & \text{otherwise} \end{array} \right.$$

$$= \left\{ \begin{array}{l|l} g_{(p-1)/2}g_{(p-3)/2}(x-2q) & x \equiv \frac{(p+3)q}{2} \\ g_{(p-1)/2}g_{(p-3)/2}(x+2q) & x \equiv \frac{(p-3)q}{2} \\ g_{(p-1)/2}g_{(p-3)/2}(x-q) & x \equiv \frac{(p-1)q}{2} \\ g_{(p-1)/2}g_{(p-3)/2}(x+q) & x \equiv \frac{(p+1)q}{2} \\ g_{(p-1)/2}g_{(p-3)/2}x & \text{otherwise} \end{array} \right\} \qquad = (g_{(p-1)/2}g_{(p-3)/2})^2 x.$$

If X is a subset of $\mathbb{Z}\backslash\{0\}$ that is bounded above, and its complement in $\mathbb{Z}$ is bounded below, then it is easy to see that the same is true for $aX := \{ax|x \in X\}$, for any $a \in \mathfrak{W}_p$. Moreover, when $x \in X \Leftrightarrow -x \notin X$, for all $x \in \mathbb{Z}\backslash\{0\}$, then the set aX also satisfies this rule (since if $x, y \in X$ and $g_i x = y$, then $g_i y = x$). Hence, this action can be extended to an action on bar partitions λ by defining $g_i\lambda$ to be the bar partition with $\mathcal{A}(g_i\lambda) = g_i\mathcal{A}(\lambda)$. $\square$

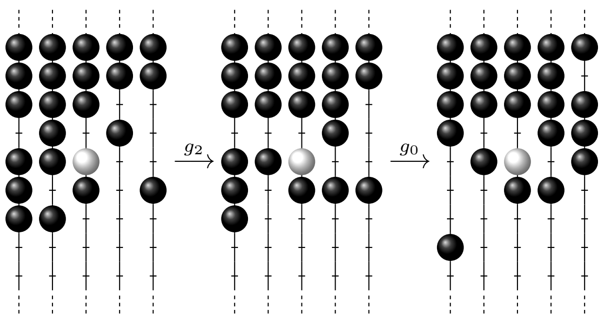
Example. In the first example of this chapter, with $p = 5$, we illustrated the bead configuration

$$\mathcal{A}((9,8,7,5,3)) = \{9,8,7,5,3,-1,-2,-4,-6,-10,-11,-12,\dots\}.$$

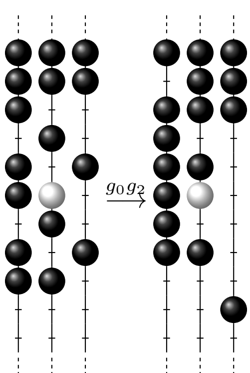
We obtain the bead configuration of the bar partition $(13,6,5,2) = (g_0g_2)(9,8,7,5,3)$ by first subtracting $q := 3$ from each element in $\mathcal{A}((9,8,7,5,3))$ that is congruent to 4 (mod 5), and adding 3 to each element congruent to 1 (mod 5), then subtracting 6 from each element congruent to 3 (mod 5), and adding 6 to each element congruent to 2 (mod 5). We thus obtain the set

$$\{13,6,5,2,-1,-3,-4,-7,-8,-9,-10,-11,-12,-14,-15,\dots\} = \mathcal{A}((13,6,5,2))$$

illustrated on the abacus below:



Notice that both bar partitions have the same q-bar-core; this will always be the case, as the level q action defined on the generators of $\mathfrak{W}_p$ always corresponds to adding or removing q-bars. On the 3-runner abacus, the action of $g_0g_2 \in \mathfrak{W}_5$ has the following effect on the bead configuration of $(9,8,7,5,3)$.



We now give some invariants of the level q action of $\mathfrak{W}_p$ which we will later use to give an explicit criterion for when two bar partitions lie in the same orbit under the level q action, which we refer to as a **level** q **orbit**.

Lemma 4.1.3. *Suppose $\lambda \in \overline{\mathcal{P}}_n$ and $a \in \mathfrak{W}_p$, and define $a\lambda$ using the level q action.*

(i) $\overline{(a\lambda)}_q = \overline{\lambda}_q$;

(ii) $\mathcal{Q}_p(a\lambda)$ *is the same as* $\mathcal{Q}_p(\lambda)$ *with the components reordered;*

(iii) $\overline{wt}_p(a\lambda) = \overline{wt}_p(\lambda)$;

(iv) $\overline{(a\lambda)}_p = a(\overline{\lambda}_p)$.

Proof. The relations occurring in all four parts are transitive, so we need only prove them in the case where a is simply a generator g_i of $\mathfrak{W}_p$.

(i) An element $x \in \mathcal{A}(\lambda)$ is fixed by the level q action of g_0 if and only if

$$x \not\equiv q, -q \pmod{p};\ x \equiv q \pmod{p},\ x - 2q \in \mathcal{A}(\lambda);\ \text{or } x \equiv -q \pmod{p},\ x + 2q \in \mathcal{A}(\lambda).$$

If $x \in \mathcal{A}(\lambda)$, $x \equiv q \pmod{p}$ and $x - 2q \notin \mathcal{A}(\lambda)$, then g_0 sends x to $x - 2q$, and sends $2q - x \in \mathcal{A}(\lambda)$ to $-x$. If $x \in \mathcal{A}(\lambda)$, $x \equiv -q \pmod{p}$ and $x + 2q \notin \mathcal{A}(\lambda)$, then g_0 sends x to $x + 2q$, and sends $-x - 2q \in \mathcal{A}(\lambda)$ to $-x$. Thus the action of g_0 on $\mathcal{A}(\lambda)$ corresponds to removing $2q$-bars from or adding $2q$-bars to λ.

For $i \in \{1, \ldots, {}^{(p-1)}/_{2}\}$, the level q action of g_i fixes $x \in \mathcal{A}(\lambda)$ if and only if

$$x \not\equiv (i+1)q, -iq, iq, -(i+1)q \pmod{p};\ x \equiv (i+1)q, -iq \pmod{p},\ x - q \in \mathcal{A}(\lambda);$$
$$\text{or } x \equiv iq, -(i+1)q \pmod{p},\ x + q \in \mathcal{A}(\lambda).$$

If $x \in \mathcal{A}(\lambda)$, $x \equiv (i+1)q$ or $-iq \pmod{m}$ and $x - q \notin \mathcal{A}(\lambda)$, then g_i sends x to $x - q$, and sends $q - x \in \mathcal{A}(\lambda)$ to $-x$. If $x \in \mathcal{A}(\lambda)$, $x \equiv iq$ or $-(i+1)q \pmod{p}$ and $x + q \notin \mathcal{A}(\lambda)$, then g_i sends x to $x + q$, and sends $-x - q \in \mathcal{A}(\lambda)$ to $-x$. Thus the action of g_i on $\mathcal{A}(\lambda)$ corresponds to removing q-bars from or adding q-bars to λ.

Moreover, since there are only finitely many $x \in \mathcal{A}(\lambda)$ such that at least one of $x - 2q$, $x + 2q$, $x - q$ or $x + q$ is not in $\mathcal{A}(\lambda)$, we obtain $\mathcal{A}(g_i\lambda) := g_i\mathcal{A}(\lambda)$ from $\mathcal{A}(\lambda)$ via a finite number of changes. Hence $\overline{(g_i\lambda)}_q = \overline{\lambda}_q$.

(ii) When $x \not\equiv \pm q \pmod{p}$, we have $x \in \mathcal{A}(g_0\lambda) \Leftrightarrow x \in \mathcal{A}(\lambda)$, so $(g_0\lambda)^{(j)} = \lambda^{(j)}$ for all $j \not\equiv \pm q \pmod{p}$. If instead $x \equiv q \pmod{p}$, then $x \in \mathcal{A}(g_0\lambda) \Leftrightarrow x + 2q \in \mathcal{A}(\lambda)$, or if $x \equiv -q \pmod{p}$, then $x \in \mathcal{A}(g_0\lambda) \Leftrightarrow x - 2q \in \mathcal{A}(\lambda)$; hence $(g_0\lambda)^{(q)} = \lambda^{(-q)} = (\lambda^{(q)})'$ and $(g_0\lambda)^{(-q)} = \lambda^{(q)} = (\lambda^{(-q)})'$.

If $i \in \{1, \ldots, {}^{(p-1)}/_{2}\}$ and $x \not\equiv (i+1)q, -iq, iq, -(i+1)q \pmod{p}$, then $x \in \mathcal{A}(g_i\lambda) \Leftrightarrow x \in \mathcal{A}(\lambda)$, so $(g_i\lambda)^{(j)} = \lambda^{(j)}$ for all $j \not\equiv (i+1)q, -iq, iq, -(i+1)q \pmod{p}$. If instead $x \equiv (i+1)q$ or $-iq \pmod{p}$, then $x \in \mathcal{A}(g_i\lambda) \Leftrightarrow x + q \in \mathcal{A}(\lambda)$, and if $x \equiv iq$ or $-(i+1)q \pmod{p}$, then $x \in \mathcal{A}(g_i\lambda) \Leftrightarrow x - q \in \mathcal{A}(\lambda)$; hence if $j \equiv (i+1)q$ or $-iq \pmod{p}$, then $(g_i\lambda)^{(j)} = \lambda^{(j-q)}$ and $(g_i\lambda)^{(-j)} = \lambda^{(j+q)}$.

(iii) This follows from (ii) and Lemma 4.1.1(i) (taking $c = 1$): the bead configuration for $g_i\lambda$ on the p-runner abacus is the same as that of λ but with the runners reordered, so the p-bar-weights of the two bar partitions are equal.

(iv) We need to show that $\Delta_{j \bmod p}(g_i\lambda) = g_i(\Delta_{k \bmod p}\lambda)$, for each $i \in \{0, \dots, {}^{(p-1)}\!/_2\}$ and $j \equiv g_i k \pmod p$. We suppose the contrary.
When $i = 0$, we may assume that $j \equiv \pm q \equiv -k \pmod p$, as otherwise $j = k$ and

$$\Delta_j(g_0\lambda) = \Delta_j\lambda = \Delta_k\lambda = g_0(\Delta_k\lambda).$$

But

$$j \equiv \pm q \equiv -k \Rightarrow \Delta_j(g_0\lambda) = \Delta_k\lambda \pm 2q = g_0(\Delta_k\lambda),$$

so we must have $i > 0$, and we may also assume that $j \equiv \pm iq$ or $\pm(i+1)q$, and $j \equiv k \pm q \pmod p$. But then

$$\Delta_j(g_i\lambda) = \Delta_{g_i(j)}\lambda = g_i(\Delta_j\lambda),$$

so in fact, the p-set of $g_i\lambda$ is equal to the image of the p-set of λ under the action of g_i. Hence $\overline{(g_i\lambda)}_p = g_i(\overline{\lambda}_p)$, for all $i \in \{0, \dots, {}^{(p-1)}\!/_2\}$. □

Next we will give a criterion for when two bar partitions lie in the same level q orbit; to this end, we will first establish a condition for two p-bar-cores to lie in the same level q orbit.

Proposition 4.1.4. *Suppose $\lambda, \mu \in \overline{C}_p$, and that the multisets*

$$[\Delta_i\lambda \ (mod\ q) | i \equiv 0, \dots, p-1 \ (mod\ p)],\ [\Delta_i\mu \ (mod\ q) | i \equiv 0, \dots, p-1 \ (mod\ p)]$$

are equal. Then $\overline{\lambda}_q = \overline{\mu}_q$, and λ and μ lie in the same level q orbit (of p-bar-cores).

Proof. The fact that λ and μ have the same q-bar-core is established by Fayers' [5, Proposition 4.1], and by Lemma 4.1.3(i), the level q action of $\mathfrak{W}_p$ on the set of p-bar-cores preserves the q-bar-core of a bar partition as by definition g_i does not change the multiset of residues modulo q of the elements of the p-set. Therefore each orbit of the level q action on $\overline{C}_p$ can contain at most one q-bar-core.

In the same paper [5], Fayers proves the following result:

Suppose $\mathcal{O}$ is a level q orbit. Let ν be an element of $\mathcal{O}$ for which the sum $\sum_{i=0}^{p-1}(\Delta_{i \bmod p}\nu - {}^{p}\!/_2)^2$ is minimised. Then ν is a q-bar-core.

This ν is uniquely defined as each level q orbit contains no more than one q-bar-core. Thus letting ν be the q-bar-core of both λ and μ, it must be contained in both the level q orbit containing λ and the level q orbit containing μ; so these orbits coincide. □

For the more general result, we define the **q-weighted p-quotient** of $\lambda \in \overline{\mathcal{P}}_n$ with p-set $\{\Delta_0\lambda, \dots, \Delta_{p-1}\lambda\}$ and p-quotient $\mathcal{Q}_p(\lambda) = (\lambda^{(0)}, \dots, \lambda^{(p-1)})$ to be the multiset

$$\mathcal{Q}_p^q(\lambda) := [(\Delta_i\lambda \ (\text{mod}\ q), \lambda^{(i)}) | i \equiv 0, 1, \dots, p-1].$$

Example. We have already seen that the bar partition $(9, 8, 7, 5, 3)$ has 5-set $\{0, -4, -3, 8, 9\}$ and 5-quotient $((1), (1), (3), (1^3), (1))$, so $\mathcal{Q}_5^3((9, 8, 7, 5, 3))$ is the multiset

$$[(0 \pmod 3), (1)), (2 \pmod 3), (1)), (0 \pmod 3), (3)), (2 \pmod 3), (1^3)), (0 \pmod 3), (1))].$$

Proposition 4.1.5. *Suppose $\lambda, \mu \in \overline{\mathcal{P}}_n$. Then λ and μ lie in the same level q orbit of $\mathfrak{W}_p$ if and only if they have the same q-weighted p-quotient.*

Proof. Firstly suppose that λ and μ lie in the same level q orbit; we may assume that $\mu = g_i\lambda$ for some $i \in \{0, \ldots, {}^{(p-1)}/_2\}$. Then we get $\mathcal{Q}_p^q(\lambda) = \mathcal{Q}_p^q(\mu)$ from the proof of Lemma 4.1.3: when $\mu = g_0\lambda$, we have

$$(\Delta_j\mu, \mu^{(j)}) = \begin{cases} (\Delta_{-j}\lambda + 2q, \lambda^{(-j)}) & \text{for } j \equiv q \pmod p, \\ (\Delta_{-j}\lambda - 2q, \lambda^{(-j)}) & \text{for } j \equiv -q \pmod p, \\ (\Delta_j\lambda, \lambda^{(j)}) & \text{otherwise,} \end{cases}$$

and when $\mu = g_i\lambda$ for some $i \in \{1, \ldots, {}^{(p-1)}/_2\}$, we have

$$(\Delta_j\mu, \mu^{(j)}) = \begin{cases} (\Delta_{j-q}\lambda + q, \lambda^{(j-q)}) & \text{for } j \equiv (i+1)q \text{ or } -iq \pmod p, \\ (\Delta_{j+q}\lambda - q, \lambda^{(j+q)}) & \text{for } j \equiv iq \text{ or } -(i+1)q \pmod p, \\ (\Delta_j\lambda, \lambda^{(j)}) & \text{otherwise.} \end{cases}$$

For the other direction, suppose that λ and μ share the q-weighted p-quotient $\mathcal{Q}_p^q(\lambda) = \mathcal{Q}_p^q(\mu)$. By the definition of the p-set, and since all components of the p-quotient of a p-bar-core are equal to the empty bar partition, the p-bar-cores of λ and μ must have the same q-weighted p-quotient $\mathcal{Q}_p^q(\overline{\lambda}_p) = \mathcal{Q}_p^q(\overline{\mu}_p)$. Thus, by Proposition 4.1.4 we may find $a, b \in \mathfrak{W}_p$ such that $a(\overline{\lambda}_p) = b(\overline{\mu}_p) = \sigma$, where σ is the q-bar-core of both $\overline{\lambda}_p$ and $\overline{\mu}_p$. Then by Lemma 4.1.3(iv) we have $\overline{(a\lambda)}_p = \overline{(b\mu)}_p = \sigma$, so using Lemma 4.1.3(i) we see that σ is the p-bar-core and the q-bar-core of both $a\lambda$ and $b\mu$; in particular, $a\lambda$ and $b\mu$ have the same p-set. Moreover, by our assumption and the proof of the only 'only if' part of the proposition above both $a\lambda$ and $b\mu$ have q-weighted p-quotient $\mathcal{Q}_p^q(\lambda)$.

Since $\sigma \in \overline{C}_p \cap \overline{C}_q$, by [5, Lemma 4.2] the elements $x_i := \Delta_{iq}\sigma$ in the p-set (of σ, $a\lambda$ and $b\mu$) satisfy $x_{i+1} \leq x_i + q$ for each $i = 0, \ldots, p-2$: $x_1 \leq q$, as otherwise $q \in \sigma$, and if $x_{i+1} > x_i + q$ for some $i = 1, \ldots, p-2$, then it can be shown that σ must have a removable q-bar. Since $x_i = p - x_{p-i}$ for $i = 1, \ldots, p-1$, and therefore $x_{p-1} + q \geq p$, we may write a sequence S of $p + q - 1$ integers as follows:

$$\begin{array}{lllll} q, & q-p, & q-2p, & \ldots, & x_1, \\ x_1+q, & x_1+q-p, & x_1+q-2p, & \ldots, & x_2, \\ x_2+q, & x_2+q-p, & x_2+q-2p, & \ldots, & x_3, \\ & & \vdots & & \\ x_{p-2}+q, & x_{p-2}+q-p, & x_{p-2}+q-2p, & \ldots, & x_{p-1}, \\ x_{p-1}+q, & x_{p-1}+q-p, & x_{p-1}+q-2p, & \ldots, & p. \end{array}$$

As in the proof of [5, Proposition 4.1], for each $k \in \{0, \dots, q-1\}$ the integers in S congruent to k modulo q must be consecutive in S as there are only $q-1$ steps equal to $-p$. It follows that the elements in the p-set of σ that are congruent to k modulo q form an arithmetic progression with common difference q.

By the first paragraph of this proof we can therefore apply the level q action to $a\lambda$ and arbitrarily reorder the elements $(a\lambda)^{(j)} \in \mathcal{Q}_p(a\lambda)$ such that $j \equiv k \pmod q$, for each k, without affecting the q-weighted p-quotient.

Thus we can apply elements of $\mathfrak{W}_p$ to transform $a\lambda$ to $b\mu$, as $\mathcal{Q}_p^q(a\lambda) = \mathcal{Q}_p^q(b\mu)$ if and only if the multisets $[(a\lambda)^{(j)} | \Delta_j\sigma \equiv k \pmod q]$ and $[(b\mu)^{(j)} | \Delta_j\sigma \equiv k \pmod q]$ are equal for each $k \in \{0, \dots, q-1\}$, and it follows that λ and μ lie in the same level q orbit. □

Remark. The invariants of the natural action of $\mathfrak{W}_p$ on bar partitions are spin analogues of the invariants of the Coxeter group W_s of type $\tilde{A}_{s-1}$ on partitions established by Fayers in [6], providing further motivation for applying the techniques used in the linear theory to the spin representation theory of S_n.

4.2 Results on bar-cores

Now that we have covered all of the necessary definitions and basic results relating to the action of $\mathfrak{W}_p$, we arrive at the first of our main results on bar partitions. The following theorem is a generalisation of a theorem by Olsson [24, Theorem 4] which states that the q-bar-core of a p-bar-core is again a p-bar-core, or in the notation used above,

$$\overline{\mathrm{wt}}_p(\lambda) = 0 \Rightarrow \overline{\mathrm{wt}}_p(\overline{\lambda}_q) = 0.$$

Note that Gramain and Nath proved that the condition that p and q are coprime in Olsson's theorem is unnecessary [7, Theorem 2.4]. We show that the p-bar-weight of any bar partition is at least the p-bar-weight of its q-bar-core, providing a spin analogue of Fayers' result which says that the s-weight of any partition is at least the s-weight of its t-core [6, Theorem 4.2].

Theorem 4.2.1. *For all bar partitions λ,*

$$\overline{wt}_p(\overline{\lambda}_q) \leq \overline{wt}_p(\lambda).$$

Proof. We use induction on $\overline{\mathrm{wt}}_q(\lambda)$, with the trivial case being that λ is a q-bar-core. Assuming that this is not the case, we may find a removable q-bar: $y \in \lambda$ such that $y - q \notin \mathcal{A}(\lambda)$. We will describe how to remove q-bars from λ to obtain a new partition with the same q-bar-core as λ, with q-bar-weight strictly less than $\overline{\mathrm{wt}}_q(\lambda)$, and with p-bar-weight no more than $\overline{\mathrm{wt}}_p(\lambda)$.

Let y be any part of λ such that $y - q \notin \mathcal{A}(\lambda)$. For any $x \in \mathcal{A}(\lambda)$ congruent to y modulo p such that $x - q \notin \mathcal{A}(\lambda)$, replace x with $x - q$, then replace $q - x \in \mathcal{A}(\lambda)$ with $-x$. We keep repeating this process until there are no more such x (the process will terminate because λ has finitely many removable q-bars), then we name our new bar partition ν. Since each action corresponds to removing a q-bar from λ, and since we have removed at least one q-bar (replacing y with $y-q$, and $q-y$ with $-y$, in $\mathcal{A}(\lambda)$), we have

$$\overline{\nu}_q = \overline{\lambda}_q \text{ and } \overline{\mathrm{wt}}_q(\nu) < \overline{\mathrm{wt}}_q(\lambda).$$

We remarked earlier that the p-bar-weight of a bar partition λ is equal to half the number of pairs $(x,a) \in \mathcal{A}(\lambda) \times \mathbb{N}$ such that $x - ap \notin \mathcal{A}(\lambda)$. We will call such a pair (x,a) a **p-bar-weight pair for λ**. It follows from our construction of ν that for any $x \not\equiv y, y-q, q-y, -y \pmod p$ and $a \in \mathbb{N}$, (x,a) is a p-bar-weight pair for ν if and only if it is a p-bar-weight pair for λ. We will consider the remaining possibilities for the residue of y modulo p and show that in each case ν has no more p-bar-weight pairs than λ, hence $\overline{\mathrm{wt}}_p(\nu) \leq \overline{\mathrm{wt}}_p(\lambda)$.

First suppose that $y \equiv q \pmod p$, so that we obtain $\mathcal{A}(\nu)$ by repeatedly replacing each $x \in \mathcal{A}(\lambda)$ such that $x \equiv q \pmod p$ and $q - x \in \mathcal{A}(\lambda)$ with $x - q$, then replacing $q - x$ with $-x$, until there are no more such x. Since in this case $y - q \equiv 0 \equiv q - y \pmod p$, we may compare the p-bar-weights of λ and ν by counting how many of the three pairs (x,a), $(x-q,a)$, $(x-2q,a)$ are p-bar-weight pairs for each of the two bar partitions when $x \equiv q \pmod p$ and $a \in \mathbb{N}$. We will do this by considering each of the four possibilities for the size of $X := \mathcal{A}(\lambda) \cap \{x, x-q, x-2q\}$.

$|X| = 3$: If $x, x-q, x-2q \in \mathcal{A}(\lambda)$, then $x, x-q, x-2q \in \mathcal{A}(\nu)$, so the number of p-bar-weight pairs for λ, and for ν, amongst the three pairs (x,a), $(x-q,a)$, and $(x-2q,a)$ is equal to
$3 - |\mathcal{A}(\lambda) \cap \{x-ap, x-ap-q, x-ap-2q\}|$.

$|X| = 2$: We have $\mathcal{A}(\nu) \cap \{x, x-q, x-2q\} = \{x-q, x-2q\}$, so clearly (x,a) is not a p-bar-weight pair for ν. If only one of the three pairs is a p-bar-weight pair for ν, it must be $(x-q,a)$ as necessarily
$\mathcal{A}(\nu) \cap \{x-ap, x-ap-q, x-ap-2q\} = \{x-ap-2q\}$. If $(x-q,a)$ and $(x-2q,a)$ are both
p-bar-weight pairs for ν, then we must have $\mathcal{A}(\lambda) \cap \{x-ap, x-ap-q, x-ap-2q\} = \emptyset$, so λ also has two p-bar-weight pairs out of the three.

$|X| = 1$: We have $\mathcal{A}(\nu) \cap \{x, x-q, x-2q\} = \{x-2q\}$, so neither of (x,a), $(x-q,a)$ can be
p-bar-weight pairs for ν. If $\mathcal{A}(\lambda) \cap \{x-ap, x-ap-q, x-ap-2q\} = \emptyset$, then exactly one of (x,a),
$(x-q,a)$, $(x-2q,a)$ is a p-bar-weight pair for λ. If $|\mathcal{A}(\lambda) \cap \{x-ap, x-ap-q, x-ap-2q\}| \geq 1$, then none of (x,a), $(x-q,a)$, $(x-2q,a)$ can be p-bar-weight pairs for ν as $x-ap-2q \in \mathcal{A}(\nu)$.

$|X| = 0$: Since $\mathcal{A}(\nu) \cap \{x, x-q, x-2q\} = \emptyset$, none of (x,a), $(x-q,a)$, $(x-2q,a)$ are p-bar-weight pairs for ν.

Next suppose that $y \equiv 0 \pmod p$, so that we obtain $\mathcal{A}(\nu)$ by repeatedly replacing each $x \in \mathcal{A}(\lambda)$ such that $p|x$ and $q - x \in \mathcal{A}(\lambda)$ with $x - q$, then replacing $q - x$ with $-x$, until there are no much such x. Since $y \equiv -y \pmod p$, we can apply the same argument as above, when $y \equiv q \pmod p$, and conclude that ν has no more p-bar-weight pairs than λ amongst $(x+q,a)$, (x,a), $(x-q,a)$, and thus $\overline{\mathrm{wt}}_p(\nu) \leq \overline{\mathrm{wt}}_p(\lambda)$.

Finally, suppose that $y \not\equiv q, 0 \pmod p$, so that $y \not\equiv -y$ and $y - q \not\equiv q - y$. In this case, we need only consider replacing all pairs $x, q-x \in \mathcal{A}(\lambda)$ such that $x \equiv y \pmod p$ with $x-q, -x$, so we are in a simpler situation; $\overline{\mathrm{wt}}_p(\nu) \leq \overline{\mathrm{wt}}_p(\lambda)$ since for any $x \equiv y \pmod p$ and $a \in \mathbb{N}$, ν has no more p-bar-weight pairs than λ amongst (x, a) and $(x - q, a)$.

Hence ν has no more p-bar-weight pairs than λ, and therefore has p-bar-weight no more than the p-bar-weight of λ. The result follows by induction. □

From this purely combinatorial result we obtain an interesting algebraic corollary.

Corollary 4.2.2. *If w is the weight of the p-block containing $S(\mu)$, a spin representation of the symmetric group S_n, then any spin representation of S_{n+iq}, for $i \in \mathbb{N}$, corresponding to a bar partition λ obtained by adding q-bars to μ belongs to a p-block of weight $\geq w$.*
In particular, if $S(\mu)$ belongs to a p-block of weight $w > 0$, then $S(\lambda)$ belongs to a block of positive weight.

Next we will consider the set $\overline{C}_{p,q}$ containing all bar partitions λ which satisfy

$$\overline{\mathrm{wt}}_p(\lambda) = \overline{\mathrm{wt}}_p(\overline{\lambda}_q).$$

We find that $\overline{C}_{p,q}$ behaves very similarly to the set $C_{s:t}$ of partitions α such that $\mathrm{wt}_s(\alpha) = \mathrm{wt}(\tilde{\alpha}_t)$ that Fayers studies in [6].

Lemma 4.2.3. *For all $\lambda \in \overline{\mathcal{P}}_n$, the equality $\overline{wt}_p(\overline{\lambda}_q) = \overline{wt}_p(\lambda)$ holds if and only if there do not exist integers a, b, c such that:*

$$a \equiv b \, (mod \; p);$$

$$a \equiv c \, (mod \; q);$$

$$a, b + c - a \in \mathcal{A}(\lambda);$$

$$b, c \notin \mathcal{A}(\lambda).$$

Proof. Say that (a, b, c) is a bad triple for λ if a, b, c satisfy the conditions above. When (a, b, c) is bad, either $a > c$ or $b + c - a > b$; either way, since $a \equiv c \pmod q$ we find that λ has a removable q-bar and is thus not a q-bar-core. Hence the lemma is true when $\lambda \in \overline{C}_q$.

Now we assume λ is not a q-bar-core, and choose $y \in \lambda$ such that $y - q \notin \mathcal{A}(\lambda)$. We define a new bar partition ν as in the proof of Theorem 4.2.1: by repeatedly replacing pairs $x, q - x \in \mathcal{A}(\lambda)$ with $x - q$ and $-x$, respectively, when $x \equiv y \pmod p$.
By induction it suffices to show that either:
$\overline{\mathrm{wt}}_p(\nu) = \overline{\mathrm{wt}}_p(\lambda)$, and there is a bad triple for ν if and only if there is a bad triple for λ; or
$\overline{\mathrm{wt}}_p(\nu) < \overline{\mathrm{wt}}_p(\lambda)$, and there is a bad triple for λ.

Suppose first that there are no pairs $x, q - x$ satisfying

$$x, q - x \notin \mathcal{A}(\lambda) \text{ and } x \equiv y \pmod p. \tag{4.1}$$

We first assume that $y \equiv q \pmod p$, and let $x \equiv q \pmod p$. Then there are eight different possibilities for the intersection of $\mathcal{A}(\lambda)$ and $\{x, x - q, x - 2q\}$:

$$x, x - q, x - 2q \in \mathcal{A}(\lambda);$$

$$
\begin{aligned}
x, x-q \in &\mathcal{A}(\lambda) \not\ni x-2q;\\
x \in &\mathcal{A}(\lambda) \not\ni x-q, x-2q;\\
&\mathcal{A}(\lambda) \not\ni x, x-q, x-2q;\\
x, x-2q \in &\mathcal{A}(\lambda) \not\ni x-q;\\
x-q, x-2q \in &\mathcal{A}(\lambda) \not\ni x;\\
x-q \in &\mathcal{A}(\lambda) \not\ni x, x-2q;\\
x-2q \in &\mathcal{A}(\lambda) \not\ni x, x-q.
\end{aligned}
$$

However, the last four possibilities are all excluded by our assumption that there are no pairs $x, q-x$ satisfying (4.1), so we find that $\nu = g_0\lambda$. Therefore $\overline{\mathrm{wt}}_p(\nu) = \overline{\mathrm{wt}}_p(\lambda)$ by Lemma 4.1.3(iii), and (a, b, c) is bad for λ exactly when (g_0a, g_0b, g_0c) is bad for ν, since $a \equiv b \pmod{p} \Rightarrow g_0(b+c-a) = g_0b + g_0c - g_0a$.

If $y \equiv 0 \pmod{p}$ we are in an identical situation to the above: $\nu = g_0\lambda$.

When $y \not\equiv q, 0 \pmod{p}$, we have a similar situation: since there are no pairs $x, q-x$ satisfying (4.1), we have $\nu = g_i\lambda$, where $(i+1)q \equiv y \pmod{p}$. Hence $\overline{\mathrm{wt}}_p(\nu) = \overline{\mathrm{wt}}_p(\lambda)$, and (a, b, c) is bad for $\lambda \Leftrightarrow (g_ia, g_ib, g_ic)$ is bad for ν.

Finally, we assume that there is a pair $x, q-x$ satisfying (4.1), so that $(y, x, y-q)$ is a bad triple for λ. We argue that $\overline{\mathrm{wt}}_p(\nu) < \overline{\mathrm{wt}}_p(\lambda)$, as in the proof of Theorem 4.2.1:

If $y \equiv q \pmod{p}$ and we let $z := \max\{x, y\}$, $l := {}^{|x-y|}/_{p}$, then exactly one of (z, l), $(z-q, l)$ is a p-bar-weight pair for λ ((z, l) if $x < y$, or $(z-q, l)$ if $x > y$). If $(z-2q, l)$ is a p-bar-weight pair for λ, then $(z-q, l)$ is a p-bar-weight pair for ν but neither of (z, l), $(z-2q, l)$ are; and if $(z-2q, l)$ is not a p-bar-weight pair for λ, then none of (z, l), $(z-q, l)$, $(z-2q, l)$ are p-bar-weight pairs for ν.

If instead we have $y \equiv 0 \pmod{p}$, and again let $z := \max\{x, y\}$ and $l := {}^{|x-y|}/_{p}$, then exactly one of (z, l), $(z-q, l)$ is a p-bar-weight pair for λ. Now if $(z+q, l)$ is a p-bar-weight pair for λ, then (z, l) is a p-bar-weight pair for ν but neither of $(z+q, l)$, $(z-q, l)$ are; and if $(z+q, l)$ is not a p-bar-weight pair for λ, then none of $(z+q, l)$, (z, l), $(z-q, l)$ are p-bar-weight pairs for ν.

If $y \not\equiv q, 0 \pmod{p}$ then we define z and l as above so that exactly one of $(z, l), (z-q, l)$ is a p-bar-weight pair for λ and neither is a p-bar-weight pair for ν.

It follows that there are fewer p-bar-weight pairs for ν than there are p-bar-weight pairs for λ, hence $\overline{\mathrm{wt}}_p(\nu) < \overline{\mathrm{wt}}_p(\lambda)$. □

Theorem 4.2.4. $\overline{C}_{p,q} = \overline{C}_{q,p}$.

Proof. The condition in Lemma 4.2.3 is symmetric in p and q. □

While the last result may seem surprising given the definition of $\overline{C}_{p,q}$, this symmetry is the motivation behind the study of this set. Furthermore, the next result shows that $\overline{C}_{p,q}$ is closed under the level q action of $\mathfrak{W}_p$.

Proposition 4.2.5. *For any $\lambda \in \overline{\mathcal{P}}_n$ and $a \in \mathfrak{W}_p$, if $\lambda \in \overline{C}_{p,q}$, then $a\lambda \in \overline{C}_{p,q}$.*

Proof. Using Lemma 4.1.3(1, 3) and the fact that $\lambda \in \overline{C}_{p,q}$, we have

$$\overline{\mathrm{wt}}_p(\overline{(a\lambda)}_q) = \overline{\mathrm{wt}}_p(\overline{\lambda}_q) = \overline{\mathrm{wt}}_p(\lambda) = \overline{\mathrm{wt}}_p(a\lambda).$$

□

Interchanging p and q and appealing to Theorem 4.2.4, we see that $\overline{C}_{p,q}$ is also a union of orbits for the level p action of $\mathfrak{W}_q$. The actions of $\mathfrak{W}_p$ and $\mathfrak{W}_q$ clearly commute because the action of $\mathfrak{W}_p$ on an integer does not change its residue modulo q, and the action of $\mathfrak{W}_q$ does not change its residue modulo p. Hence $\overline{C}_{p,q}$ is a union of orbits for the action of $\mathfrak{W}_p \times \mathfrak{W}_q$. We will look at these orbits in more detail, first by considering just the level q action of $\mathfrak{W}_p$.

Proposition 4.2.6. *Suppose $\lambda \in \overline{\mathcal{P}}_n$, and let $\mathcal{O}$ be the orbit containing λ under the level q action of $\mathfrak{W}_p$. Then the following are equivalent:*

(i) $\lambda \in \overline{C}_{p,q}$*;*

(ii) $\mathcal{O}$ *contains a q-bar-core;*

(iii) $\mathcal{O}$ *contains* $\overline{\lambda}_q$*.*

Proof. Since $\overline{C}_q \subset \overline{C}_{p,q}$, Proposition 4.2.5 shows that if $\mathcal{O}$ contains a q-bar-core, then $\lambda \in \overline{C}_{p,q}$. Hence the second statement implies the first. Trivially the third statement implies the second, so it remains to show that the first implies the third. So suppose that $\lambda \in \overline{C}_{p,q}$, and we can assume that λ is not a q-bar-core or the third statement is trivial. Thus we may find a pair $y, q-y \in \mathcal{A}(\lambda)$. By the proof of Lemma 4.2.3, there are no pairs $x, q-x \notin \mathcal{A}(\lambda)$ with either x or $q-x \equiv y \pmod{p}$, and if we take $i \in \{0, \dots, (p-1)/2\}$ such that $iq \equiv y \pmod{p}$, then the bar partition $\nu = g_i\lambda$ satisfies $\overline{\nu}_q = \overline{\lambda}_q$ and $\overline{\mathrm{wt}}_q(\nu) < \overline{\mathrm{wt}}_q(\lambda)$. Since $\overline{\mathrm{wt}}_p(\nu) = \overline{\mathrm{wt}}_p(g_i\lambda) = \overline{\mathrm{wt}}_p(\lambda)$, ν is also in $\overline{C}_{p,q}$, and by induction the orbit containing ν contains $\overline{\nu}_q$. □

The next result demonstrates that the connection between $\overline{C}_{p,q}$ and simultaneous p- and q-bar-cores mirrors the connection between $C_{s:t}$ and simultaneous s- and t-core partitions [6, Corollary 4.7].

Corollary 4.2.7. *Let $\mathcal{O}$ be an orbit of $\mathfrak{W}_p \times \mathfrak{W}_q$ consisting of bar partitions in $\overline{C}_{p,q}$. Then $\mathcal{O}$ contains exactly one bar partition that is both a p-bar-core and a q-bar-core.*

Proof. Let λ be a bar partition in $\mathcal{O}$. Then by Proposition 4.2.6, $\overline{\lambda}_q \in \mathcal{O}$, and by the same result with p and q interchanged, the bar partition $\nu = \overline{(\overline{\lambda}_q)}_p$ lies in $\mathcal{O}$. Obviously ν is a p-bar-core, and by Theorem 4.2.1, it is also a q-bar-core.

Now suppose that there is another bar partition in $\mathcal{O}$ that is both a p-bar-core and a q-bar-core. We can write this as $ba\nu$, with $a \in \mathfrak{W}_p$ and $b \in \mathfrak{W}_q$. Since $\overline{\mathrm{wt}}_q(g_j\lambda) = \overline{\mathrm{wt}}_q(\lambda)$ for any $j \in \{0, \dots, (q-1)/2\}$ (by interchanging p and q in the proof of Proposition 4.2.5), it follows that

$$\overline{\mathrm{wt}}_q(a\nu) = \overline{\mathrm{wt}}_q(ba\nu) = 0,$$

hence it follows from $\overline{(g_i\lambda)}_q = \overline{\lambda}_q$ (for any $i \in \{0, \dots, (p-1)/2\}$) that

$$a\nu = \overline{a\nu}_q = \overline{\nu}_q = \nu.$$

Similarly $b\nu = \nu$, and thus $ba\nu = \nu$. □

Remark. From Proposition 4.2.6 and Corollary 4.2.7, we see that two bar partitions $\lambda, \mu \in \overline{C}_{p,q}$ lie in the same orbit of $\mathfrak{W}_p \times \mathfrak{W}_q$ if and only if the p-bar-cores of $(\overline{\lambda}_q)$ and $(\overline{\nu}_q)$ are equal. However, it does not seem to be easy to tell when two arbitrary bar partitions lie in the same orbit.

Corollary 4.2.8. *Suppose $\lambda \in \overline{C}_{p,q}$. Then λ is a pq-bar-core, and $\overline{(\overline{\lambda}_q)}_p = \overline{(\overline{\lambda}_p)}_q$.*

Proof. If we can remove a pq-bar from λ to obtain a new bar partition ν, then we can also remove q successive p-bars, or p successive q-bars, to obtain ν from λ. Thus $\overline{\nu}_q = \overline{\lambda}_q$ and $\overline{\mathrm{wt}}_p(\nu) \leq \overline{\mathrm{wt}}_p(\lambda) - q$, so we have

$$\overline{\mathrm{wt}}_p(\overline{\lambda}_q) = \overline{\mathrm{wt}}_p(\overline{\nu}_q) \leq \overline{\mathrm{wt}}_p(\nu) < \overline{\mathrm{wt}}_p(\lambda);$$

hence $\overline{\mathrm{wt}}_{pq}(\lambda) > 0 \Rightarrow \lambda \notin \overline{C}_{p,q}$.

It follows from Theorem 4.2.1 that $\overline{(\overline{\lambda}_q)}_p$ and $\overline{(\overline{\lambda}_p)}_q$ are both p-bar-cores and q-bar-cores, and by Proposition 4.2.6 they both lie in the same orbit as λ under the action of $\mathfrak{W}_p \times \mathfrak{W}_q$. Hence the result follows from Corollary 4.2.7. □

4.3 The sum of a p-bar-core and a q-bar-core

In the present section we will give a constructive method for determining the bar partition in $\overline{C}_{p,q}$ with a given p-bar-core μ and q-bar-core σ such that $\overline{\mu}_q = \overline{\sigma}_p$. The resulting bar partition can be interpreted as the 'sum' of μ and σ, analogous to the partition in $C_{s:t}$ with a specified s-core and t-core that is constructed in [6].

Proposition 4.3.1. *Suppose $\mu \in \overline{C}_p$ and $\sigma \in \overline{C}_q$, and that $\overline{\mu}_q = \overline{\sigma}_p$. Then there is a unique bar partition $\lambda \in \overline{C}_{p,q}$ with $\overline{\lambda}_p = \mu$ and $\overline{\lambda}_q = \sigma$. Moreover,*

$$|\lambda| = |\mu| + |\sigma| - |\overline{\sigma}_p|,$$

and λ is the unique smallest bar partition with p-bar-core μ and q-bar-core σ.

Proof. Let $\tau = \overline{\mu}_q$, and consider the action of $\mathfrak{W}_p \times \mathfrak{W}_q$ on $\overline{\mathcal{P}}_n$. By Proposition 4.2.6 we can find $a \in \mathfrak{W}_p$ and $b \in \mathfrak{W}_q$ such that $a\tau = \mu$ and $b\tau = \sigma$, and we let $\lambda = a\sigma$, so that $\lambda \in \overline{C}_{p,q}$ (as it lies in the same orbit as the q-bar-core σ). Then we have $\overline{\lambda}_q = \overline{\sigma}_q = \sigma$, and by the proof of Proposition 4.2.5, we have

$$\overline{\lambda}_p = \overline{(ab\tau)}_p = \overline{(ba\tau)}_p = \overline{(b\mu)}_p = \overline{\mu}_p = \mu.$$

Moreover, we have

$$|\lambda| = |\overline{\lambda}_p| + p \cdot \overline{\mathrm{wt}}_p(\lambda)$$

$$= |\mu| + p \cdot \overline{\mathrm{wt}}_p(a\sigma)$$
$$= |\mu| + p \cdot \overline{\mathrm{wt}}_p(\sigma)$$
$$= |\mu| + |\sigma| - |\overline{\sigma}_p|.$$

Now suppose ν is a bar partition distinct from λ with $\overline{\nu}_p = \mu$ and $\overline{\nu}_q = \sigma$, and let a, b be as above. Then we have $\overline{(a^{-1}\nu)}_q = \overline{\nu}_q = \sigma$, but $a^{-1}\nu \neq a^{-1}\lambda = \sigma$, so $|a^{-1}\nu| > |\sigma|$. Hence, again using the proof of Proposition 4.2.5, we have

$$\overline{\mathrm{wt}}_p(\nu) = \overline{\mathrm{wt}}_p(a^{-1}\nu) = \tfrac{|a^{-1}\nu| - |\tau|}{p} > \tfrac{|\sigma| - |\tau|}{p} = \overline{\mathrm{wt}}_p(\sigma)$$

which means that $\nu \notin \overline{C}_{p,q}$. Furthermore, we see that $\overline{\mathrm{wt}}_p(\nu) > \overline{\mathrm{wt}}_p(\lambda)$, so $|\nu| > |\lambda|$. Hence λ is the unique smallest bar partition with p-bar-core μ and q-bar-core σ. □

Remark. For $\mu, \sigma \in \overline{\mathcal{P}}_n$ with $\overline{\mu}_q = \overline{\sigma}_p$, we denote by $\mu \boxplus \sigma$ the unique bar partition λ in $\overline{C}_{p,q}$ with $\overline{\lambda}_p = \mu$ and $\overline{\lambda}_q = \sigma$. We have shown that λ is the smallest bar partition with p-bar-core μ and q-bar-core σ in terms of the sum of its parts. However, it is not the case that any other bar partition ν with $\overline{\nu}_p = \mu$ and $\overline{\nu}_q = \sigma$ has a Young diagram containing the diagram of λ, i.e. that $\lambda := \{\lambda_1, \lambda_2, \dots\}$ and $\nu = \{\lambda_1 + a_1, \lambda_2 + a_2, \dots\}$ for some $a_1, a_2, \dots \in \mathbb{Z}_{\geq 0}$; If we take $\mu = (4, 1)$ and $\sigma = (3)$, then $\overline{\mu}_5 = \overline{\sigma}_3$, $\lambda = (4, 3, 1) \in \overline{C}_{3,5}$, $\overline{\lambda}_3 = \mu$, and $\overline{\lambda}_5 = \sigma$, but $\nu = (13, 10)$ also has 3-bar-core μ and 5-bar-core σ, and ν does not contain λ.

Proposition 4.3.2. *Suppose $\lambda \in \overline{\mathcal{P}}_n$ is such that $|\lambda| = N$. Then $\lambda \in \overline{C}_{p,q}$ if and only if there is no $\nu \in \overline{\mathcal{P}}_n \backslash \{\lambda\}$ with $|\nu| = N$, $\overline{\nu}_p = \overline{\lambda}_p$ and $\overline{\nu}_q = \overline{\lambda}_q$.*

Proof. Suppose $\lambda \in \overline{C}_{p,q}$ and let $\mu = \overline{\lambda}_p$, $\sigma = \overline{\lambda}_q$. Then by Proposition 4.3.1, λ is the unique smallest bar partition in $\overline{C}_{p,q}$ with p-bar-core μ and q-bar-core σ, and therefore the only one whose parts sum to N.

Conversely, suppose $\lambda \notin \overline{C}_{p,q}$. Then we can find integers a, b, c such that $a \equiv b \pmod p$, $a \equiv c \pmod q$, $a, b + c - a \in \mathcal{A}(\lambda)$ and $b, c \notin \mathcal{A}(\lambda)$ (by Lemma 4.2.3).

We first assume that $a, b, c, b + c - a, -a, -b, -c$ and $a - b - c$ are distinct integers. Define a new bar partition ν by its bead configuration on the p-runner abacus,

$$\mathcal{A}(\nu) = \{-a, b, c, a - b - c\} \cup \mathcal{A}(\lambda) \backslash \{a, -b, -c, b + c - a\}.$$

Then ν can be obtained from λ by removing a $|b - a|$-bar and adding a $|b - a|$-bar, so $|\nu| = |\lambda|$ and since $p | (a - b)$, we have $\overline{\nu}_p = \overline{\lambda}_p$. Alternatively, we can obtain ν from λ by removing a $|c - a|$-bar and adding a $|c - a|$-bar, so it follows from the divisibility of $a - c$ by q that $\overline{\nu}_q = \overline{\lambda}_q$. Hence λ is not the only partition with p-bar-core μ and q-bar-core σ whose parts sum to N.

If instead the integers a, b, c and $b + c - a$ are not distinct, i.e. if $a = b + c - a$ or $b = c$, then they must all be congruent modulo pq, and λ therefore cannot be a pq-bar-core. By adding and removing the same number of pq-bars, we can obtain a new partition ν from λ with $\overline{\nu}_{pq} = \overline{\lambda}_{pq}$, $|\nu| = |\lambda|$, and $\nu \neq \lambda$.

Then it is easy to see that $\overline{\nu}_p = \mu$ and $\overline{\nu}_q = \sigma$, as removing a pq-bar is the same as removing p q-bars or q p-bars.

Now we may assume that $a, b, c, b+c-a$ are distinct but

$$\{a, b, c, b+c-a\} \cap \{-a, -b, -c, a-b-c\} \neq \emptyset.$$

However, $-a = a \Rightarrow a = 0 \notin \mathcal{A}(\lambda)$, a contradiction;
$a-b-c = a \Rightarrow b = -c \in \mathcal{A}(\lambda)$, or $b = c = 0$ and $b+c-a = -a \in \mathcal{A}(\lambda)$, both contradictions;
and $a-b-c = b+c-a \Rightarrow b+c-a = 0 \notin \mathcal{A}(\lambda)$, a contradiction;
so we need to consider six separate cases (or three, up to symmetry):

(i) $a = -b$;
then $-b = a \equiv b \pmod p \Rightarrow p|2b \Rightarrow p|b$ and $p|a$, and $b+c-a = c-2a$. If $c-a \in \mathcal{A}(\lambda)$, then we may define $\mathcal{A}(\nu) = \{-a, c, a-c\} \cup \mathcal{A}(\lambda)\backslash\{a, -c, c-a\}$ so that we can obtain ν from λ by removing and adding $|a|$-bars, or by removing and adding $|c-a|$-bars, and thus ν has the same size, p-bar-core, and q-bar-core as λ, since $p|a$ and $q|(a-c)$, but is distinct from λ. If instead $a-c \in \mathcal{A}(\lambda)$, defining $\mathcal{A}(\nu) = \{-a, c-a, 2a-c\} \cup \mathcal{A}(\lambda)\backslash\{a, a-c, c-2a\}$, we also have $|\nu| = |\lambda|$, and we can obtain ν from λ by adding and removing $|a|$-bars, or by adding and removing $|c-a|$-bars.

(ii) $a = -c \equiv 0 \bmod q$;

$$\text{let } \mathcal{A}(\nu) = \begin{cases} \{-a, b, a-b\} \cup \mathcal{A}(\lambda)\backslash\{a, -b, b-a\}, & \text{if } b-a \in \mathcal{A}(\lambda); \\ \{-a, b-a, 2a-b\} \cup \mathcal{A}(\lambda)\backslash\{a, a-b, b-2a\}, & \text{if } a-b \in \mathcal{A}(\lambda). \end{cases}$$

(iii) $b = -b$;
if we let $\mathcal{A}(\nu) = \{-a, c, a-c\} \cup \mathcal{A}(\lambda)\backslash\{a, -c, c-a\}$, then we can obtain ν from λ by removing and adding $|a|$-bars, or by removing and adding $|c-a|$-bars, and thus ν meets our criteria since $0 = b \equiv a \bmod p$.

(iv) $c = -c$;
let $\mathcal{A}(\nu) = \{-a, b, a-b\} \cup \mathcal{A}(\lambda)\backslash\{a, -b, b-a\}$.

(*v*) $b = a-b-c$;
we have $q|(a-c) = 2b$, so $q|b = \frac{a-c}{2}$, and $p|(a-b) = \frac{a+c}{2}$. If $a - \frac{a-c}{2} = \frac{a+c}{2} \in \mathcal{A}(\lambda)$, then letting $\mathcal{A}(\nu) = \{\frac{a-c}{2}, c, -\frac{a+c}{2}\} \cup \mathcal{A}(\lambda)\backslash\{\frac{c-a}{2}, -c, \frac{a+c}{2}\}$, we have a bar partition ν that can be obtained from λ by adding and removing $\frac{a-c}{2}$-bars, or by adding and removing $\frac{a+c}{2}$-bars. If $-\frac{a+c}{2} \in \mathcal{A}(\lambda)$, and we let $\mathcal{A}(\nu) = \{-a, \frac{a-c}{2}, \frac{a+c}{2}\} \cup \mathcal{A}(\lambda)\backslash\{a, \frac{c-a}{2}, -\frac{a+c}{2}\}$, then ν can be obtained from λ by adding and removing $\frac{a-c}{2}$-bars, or by adding and removing $\frac{a+c}{2}$-bars.

(*vi*) $c = \frac{a-b}{2}$;

$$\text{let } \mathcal{A}(\nu) = \begin{cases} \{\frac{a-b}{2}, b, -\frac{a+b}{2}\} \cup \mathcal{A}(\lambda)\backslash\{\frac{b-a}{2}, -b, \frac{a+b}{2}\}, & \text{if } \frac{a+b}{2} \in \mathcal{A}(\lambda); \\ \{-a, \frac{a-b}{2}, \frac{a+b}{2}\} \cup \mathcal{A}(\lambda)\backslash\{a, \frac{b-a}{2}, -\frac{a+b}{2}\}, & \text{if } -\frac{a+b}{2} \in \mathcal{A}(\lambda). \end{cases}$$

Hence, we have proved that a bar partition λ of N has p-bar-weight equal to the p-bar-weight of its q-bar-core precisely when there is no other bar partition of N with p-bar-core $\overline{\lambda}_p$ and q-bar-core $\overline{\lambda}_q$. □

We will now give a method for constructing $\mu \boxplus \sigma$ for a given p-bar-core μ and q-bar-core σ with $\overline{\mu}_q = \overline{\sigma}_p$. In theory one can do this as in the proof of Proposition 4.3.1: find $a \in \mathfrak{W}_p$ such that $a\overline{\mu}_q = \mu$, and then compute $a\sigma$. But in practice, the method we give here will prove much more efficient. We will need the following lemma.

Lemma 4.3.3. *Suppose that* $\lambda \in \overline{C}_q$ *and* $j, k \not\equiv 0\ (mod\ p)$*. Then*

$$\Delta_{j\ mod\ p}\lambda \equiv \Delta_{k\ mod\ p}\lambda\ (mod\ q) \Rightarrow \lambda^{(j\ mod\ p)} = \lambda^{(k\ mod\ p)}.$$

Proof. Without loss of generality, suppose $\Delta_k\lambda > \Delta_j\lambda$. We have that

$$\{x \in \mathcal{A}(\lambda) | x \equiv j \pmod{p}\} = \{p(\lambda_i^{(j)} - i) + \Delta_j\lambda | i \in \mathbb{N}\},$$

where $\lambda_i^{(j)} = 0$ for $i > l$, if l is the number of (positive) parts of $\lambda^{(j)}$, since

$$r := |\{x \in \lambda | x \equiv j \pmod{p}\}| - |\{x \in \lambda | x \equiv -j \pmod{p}\}| = \tfrac{\Delta_j\lambda - j}{p}.$$

If $\lambda^{(j)} \neq \lambda^{(k)}$, then $\mathcal{B}_0^{\lambda^{(k)}} \not\subseteq \mathcal{B}_0^{\lambda^{(j)}}$, by the following result [6, Lemma 2.1]:

Suppose τ and ρ are partitions and $r \in \mathbb{Z}$. If $\mathcal{B}_r^\tau \subseteq \mathcal{B}_r^\rho$, then $\tau = \rho$.

The proof of this is simple: Choose N sufficiently large that $\tau_N = \rho_N = 0$. Then $\mathcal{B}_r^\tau$ and $\mathcal{B}_r^\rho$ both contain all integers less than or equal to $r - N$. Moreover, $\mathcal{B}_r^\tau$ contains exactly $N - 1$ elements greater than $r - N$, namely $\tau_1 - 1 + r, \ldots, \tau_{N-1} - (N-1) + r$, and similarly $\mathcal{B}_r^\rho$ contains exactly $N - 1$ elements greater than $r - N$. Since $\mathcal{B}_r^\tau \subseteq \mathcal{B}_r^\rho$, we get $\mathcal{B}_r^\tau = \mathcal{B}_r^\rho$, and hence $\tau = \rho$.

Thus there is an i with $\lambda_i^{(k)} - i \notin \mathcal{B}_0^{\lambda^{(j)}}$, or equivalently, $\lambda_i^{(k)} - i + r \notin \mathcal{B}_r^{\lambda^{(j)}}$, and it follows that $m(\lambda_i^{(k)} - i) + \Delta_j\lambda \notin \mathcal{A}(\lambda)$. Since $m(\lambda_i^{(k)} - i) + \Delta_k\lambda \in \mathcal{A}(\lambda)$, λ cannot be a $(\Delta_k\lambda - \Delta_j\lambda)$-bar-core and is therefore not a q-bar-core; a contradiction. □

Note that since $\Delta_0\lambda = 0$ for all bar partitions λ, it is not necessary that $\lambda^{(0)}$ and $\lambda^{(j)}$ are equal when $\Delta_j\lambda \equiv 0 \pmod{q}$.

Proposition 4.3.4. *Suppose* $\mu \in \overline{C}_p$ *and* $\sigma \in \overline{C}_q$ *are such that* $\overline{\mu}_q = \overline{\sigma}_p$*. Then there is a unique* $\lambda \in \overline{\mathcal{P}}_n$ *with* p*-bar-core* μ *and with the same* q*-weighted* p*-quotient as* σ*.*

Proof. Let $\nu = \overline{\mu}_q$. Then by Proposition 4.2.6 μ and ν lie in the same level q orbit of $\mathfrak{W}_p$, so there is a permutation ϕ on $\{0, \ldots, p-1\}$ such that $\Delta_j\mu \equiv \Delta_{\phi(j)}\nu \pmod{n}$ for each $j \in \{0, \ldots, p-1\}$. Thus by Lemma 4.3.3 ν must have the same q-weighted p-quotient as μ. Since $\nu = \overline{\sigma}_p$, we may construct λ by taking the bar partition with p-bar-core μ and $\lambda^{(j)} = \sigma^{(\phi(j))}$ for each $j \in \{0, \ldots, p-1\}$.

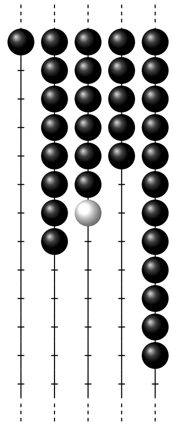
By Lemma 4.3.3 we have $\sigma^{(j)} = \sigma^{(k)}$ whenever $\Delta_j\sigma \equiv \Delta_k\sigma \pmod{q}$, i.e. whenever $\Delta_{\phi^{-1}(j)}\mu \equiv \Delta_{\phi^{-1}(k)}\mu \pmod{q}$, so the λ we construct is unique (as it is uniquely determined by its p-bar-core and p-quotient). □

Corollary 4.3.5. *Suppose* $\mu \in \overline{C}_p$ *and* $\sigma \in \overline{C}_q$ *are such that* $\overline{\mu}_q = \overline{\sigma}_p$*, and let* λ *be the bar partition with* p*-bar-core* μ *and with the same* q*-weighted* p*-quotient as* σ*. Then* $\lambda = \mu \boxplus \sigma$*.*

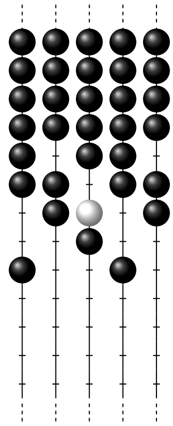
Proof. By Proposition 4.1.5 λ and σ lie in the same level q orbit of $\mathfrak{W}_p$. Hence it follows from Proposition 4.2.6 that $\lambda \in \overline{C}_{p,q}$ and $\overline{\lambda}_q = \sigma$. □

Example.

$\mu = (27,22,17,12,7,4,2) = \overline{\mu}_5$



$\sigma = (11,8,5,2) = \overline{\sigma}_3$

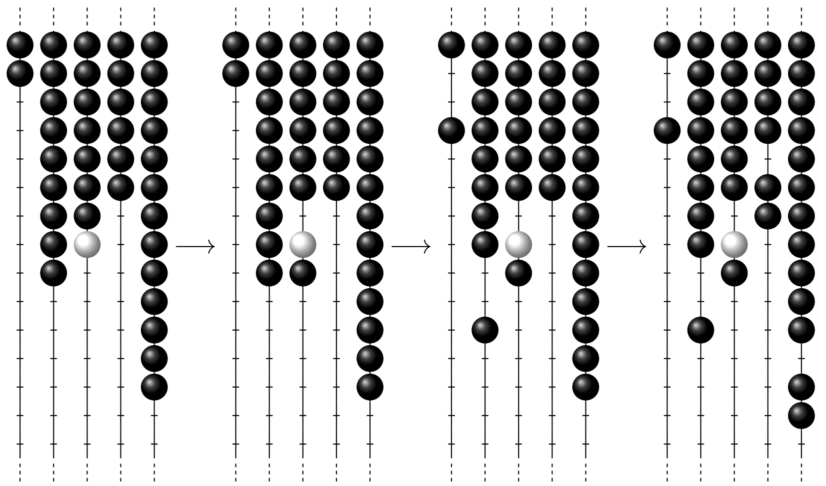


$\overline{\mu}_3 = (1) = \overline{\sigma}_5$

$\mathcal{Q}_5^3(\mu) = [(0,\varnothing),(2,\varnothing),(2,\varnothing),(0,\varnothing),(0,\varnothing)]$

$Q_5^3(\sigma) = [(0,(1)),(0,(2)),(2,(1^2)),(0,(2)),(2,(1^2))]$

Using our algorithm, we will obtain the bead configuration for $\mu \boxplus \sigma$:
We first replace the 0-runner with the 0-runner from the configuration for σ. Next, we have $\Delta_{1 \bmod 5}\sigma \equiv 0 \pmod 3$ so for each runner $j \in \{1,\dots,4\}$ such that $\Delta_{j \bmod 5}\mu \equiv 0 \pmod 3$, in this case runners 3 and 4, since $\sigma^{(1 \bmod 5)} = (2)$, we move the lowest bead on the runner upwards 2 spaces. Finally, $\Delta_{2 \bmod 5}\sigma \equiv 2 \equiv \Delta_{1 \bmod 5}\mu \equiv \Delta_{2 \bmod 5}\mu \pmod 3$ and $\sigma^{(2 \bmod 5)} = (1^2)$, so we move the second lowest beads up by two spaces on runners 1 and 2.



Thus we obtain the bar partition $\mu \boxplus \sigma = (32,27,17,14,12,7,5,2)$.

This method for constructing $\mu \boxplus \sigma$ is very similar to the algorithm given in [6, Proposition 5.5] for constructing the partition in the set $C_{s:t}$ with a given s-core and t-core, which is done in terms of the s-quotient of a partition from the linear theory.

In [1], Bessenrodt & Olsson showed that there is a maximal bar partition $\Upsilon_{\min\{p,q\},\max\{p,q\}}$ which is both a p-bar-core and a q-bar-core, where $\Upsilon_{p,q}$ is the **Yin/Yang partition**, with parts

$$(\tfrac{p-1}{2} - k)q - (l+1)p, \text{ for } k, l \in \mathbb{Z}_{\geq 0}.$$

It is maximal in the sense that whenever $\lambda := \{\lambda_1, \lambda_2, \dots\}$ is both a p-bar-core and a q-bar-core, we may write $\Upsilon_{\min\{p,q\},\max\{p,q\}} = \{\lambda_1 + a_1, \lambda_2 + a_2, \dots\}$ with $a_1, a_2, \dots \in \mathbb{Z}_{\geq 0}$. When $p < q$, $\Upsilon_{p,q}$ is called the Yin partition, and $\Upsilon_{q,p}$ the Yang partition.

Since $\Upsilon_{p,q} \in \overline{C}_p \cap \overline{C}_q$, its p-set is $\{0, q, \dots, q(p-1)/2\} \cup \{p-q, p-2q, \dots, p - q(p-1)/2\}$, and its q-set is $\{0\} \cup \{q(p+1)/2 - kp | k = 1, \dots, q-1\}$. Thus, the p-set of $\Upsilon_{p,q}$ consists of $(p+1)/2$ elements divisible by q and $(p-1)/2$ elements congruent to p modulo q, while the q-set of $\Upsilon_{p,q}$ consists of 0 and $q-1$ integers congruent to $q(p+1)/2$ modulo p.

Example. The bar partition

$$\Upsilon_{5,11} = (17, 12, 7, 6, 2, 1)$$

has 5-set $\{0, 11, 22, -17, -6\}$, and 11-set $\{0, 23, 13, 3, -7, -17, 28, 18, 8, -2, -12\}$, while

$$\Upsilon_{11,5} = (14, 9, 4, 3)$$

has 5-set $\{0, -14, -3, 8, 19\}$, and 11-set $\{0, 1, -9, 25, 15, 5, 6, -4, -14, 20, 10\}$.

To conclude, we will consider $\overline{C}^{\Upsilon}_{p,q}$, the **$\Upsilon$-orbit** of the set $\overline{C}_{p,q}$, which we define to be the $\mathfrak{W}_p \times \mathfrak{W}_q$-orbit containing $\Upsilon_{p,q}$. Our final result will establish a bijection between $\overline{C}^{\Upsilon}_{p,q}$ and the direct product of the power set $2^{\{1,\dots,(p-1)/2\}}$ and $\overline{C}_p \times \overline{C}_q$. This result provides a spin analogue of [6, Proposition 6.5], which gives a bijection between the direct product of the set of s-cores and the set of t-cores and an orbit of an action of the group $W_s \times W_t$ (where W_s denotes the Coxeter group of type $\tilde{A}_{s-1}$) containing the unique largest partition which is simultaneously an s-core and a t-core.

Lemma 4.3.6. *Suppose that $\sigma \in \overline{C}_q$. If $\overline{\sigma}_p = \Upsilon_{p,q}$, then $\sigma^{(0 \text{ mod } p)}$ is a q-bar-core and there is a q-core g such that*

$$\sigma^{(j \text{ mod } p)} = \begin{cases} g, & \text{for } j \equiv kq,\ k \in \{1, \dots, \frac{p-1}{2}\}, \\ g', & \text{for } j \equiv kq,\ k \in \{\frac{p+1}{2}, \dots, p-1\}. \end{cases}$$

If instead $\overline{\sigma}_p = \Upsilon_{q,p}$, then $\sigma^{(0 \text{ mod } p)}$ is a q-bar-core and there is a self-conjugate q-core γ such that $\sigma^{(j \text{ mod } p)} = \gamma = \gamma'$, for all $j \not\equiv 0 \ (\text{mod } p)$.

Proof. When $\overline{\sigma}_p = \Upsilon_{p,q}$, the non-zero elements of the shared p-set of $\Upsilon_{p,q}$ and σ belong to two congruence classes modulo p, so by Lemmas 4.1.1(i) and 4.3.3, $\mathcal{Q}_p(\sigma)$ consists of a q-bar-core $\sigma^{(0 \text{ mod } p)}$ and at most two distinct q-cores. Moreover, since $(\sigma^{(j \text{ mod } p)})' = \sigma^{(-j \text{ mod } p)}$ for each $j \not\equiv 0 \ (\text{mod } p)$, the p-quotient of σ consists of $\sigma^{(0 \text{ mod } p)}$ (the parts of σ which are multiples of p, divided by p) and either $p-1$ other empty bar partitions (when $\sigma = \Upsilon_{p,q}$), or $(p-1)/2$ copies each of two conjugate partitions.

When $\overline{\sigma}_p = \Upsilon_{q,p}$, all of the non-zero elements in the p-set of σ are congruent modulo p, so again by Lemmas 4.1.1(i) and 4.3.3, the q-quotient $\mathcal{Q}_q(\sigma)$ simply consists of a q-bar-core $\sigma^{(0 \bmod p)}$ and $p-1$ copies of a self-conjugate q-core. □

The above lemma means that the construction of $\mu \boxplus \sigma$ becomes even more straightforward when μ and σ are contained in the Υ-orbit.

Proposition 4.3.7. *Suppose $\mu \in \overline{C}_p$ and $\sigma \in \overline{C}_q$ are such that $\overline{\mu}_q = \overline{\sigma}_p = \Upsilon_{p,q}$. Then $\mu \boxplus \sigma$ is the bar partition λ with $\overline{\lambda}_p = \mu$, $\lambda^{(0 \ mod \ p)} = \sigma^{(0 \ mod \ p)}$, and*

$$\lambda^{(j \ mod \ p)} = \begin{cases} \sigma^{(j \ mod \ p)} & \text{if } \Delta_{j \ mod \ p}\mu \equiv \Delta_{j \ mod \ p}\Upsilon_{p,q} \ (mod \ q), \\ (\sigma^{(j \ mod \ p)})' & \text{otherwise.} \end{cases}$$

Moreover, $\mu \boxplus \sigma$ is also the bar partition with q-bar-core σ and the same q-quotient as μ.

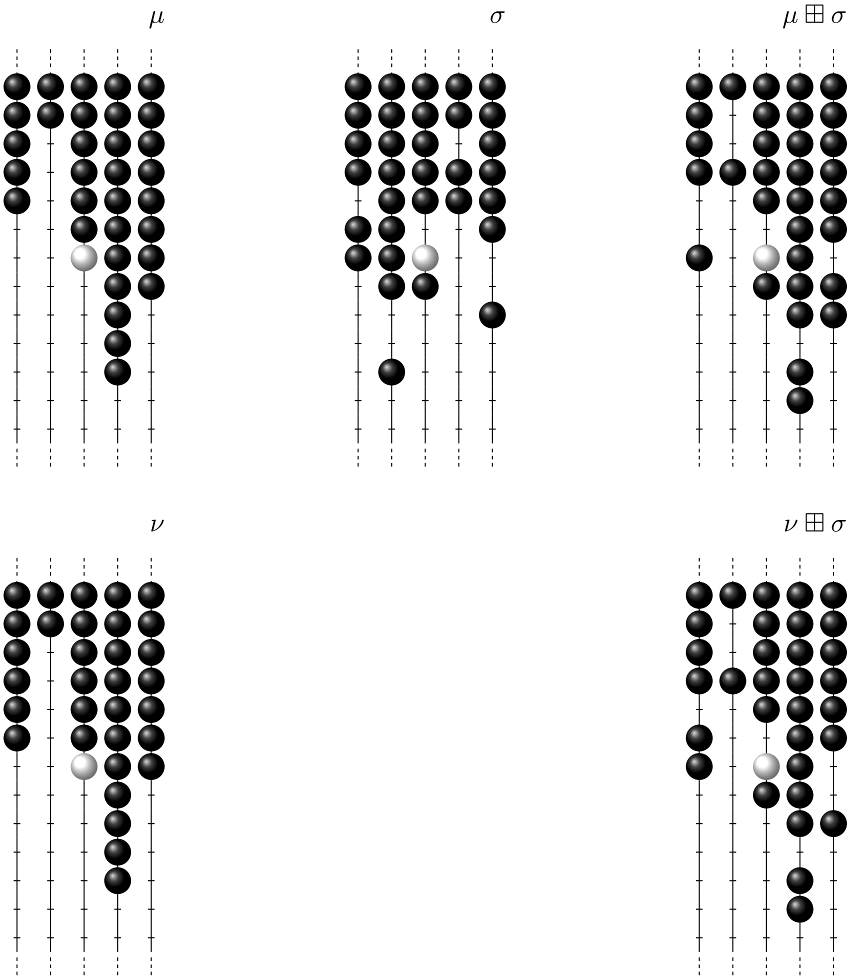
Proof. There are $(p+1)/2$ elements in the p-set of both $\Upsilon_{p,q} = \overline{\sigma}_p$ and σ that are divisible by q, and the other $(p-1)/2$ elements are congruent to p modulo q. Hence, it follows from Proposition 4.3.4 that $\mu \boxplus \sigma = \lambda$. The elements $\Delta_{1 \bmod q}\sigma, \dots, \Delta_{p-1 \bmod q}\sigma$ of the q-set of σ are all congruent modulo p, so the p-quotients of the two bar partitions σ and $\mu \boxplus \sigma$ are exactly the same. □

Example. When $\mu = (21, 16, 11, 7, 6, 2, 1) \in \overline{C}_5$ and $\sigma = (19, 12, 5, 4) \in \overline{C}_7$, so that $\overline{\mu}_7 = \overline{\sigma}_5 = (9, 4, 2) = \Upsilon_{5,7}$, we find that the 5-set of μ and $\mu \boxplus \sigma$ is $\{0, 26, 12, -7, -21\}$, the 5-set of $\Upsilon_{5,7}$ and σ is $\{0, -9, 7, -2, 14\}$,
$\mathcal{Q}_5^7(\Upsilon_{5,7}) = [(0, \varnothing), (5, \varnothing), (0, \varnothing), (5, \varnothing), (0, \varnothing)]$,
$\mathcal{Q}_5^7(\mu) = [(0, \varnothing), (5, \varnothing), (5, \varnothing), (0, \varnothing), (0, \varnothing)]$,
$\mathcal{Q}_5^7(\sigma) = [(0, (1)), (5, (1^2)), (0, (2)), (5, (1^2)), (0, (2))]$, and
$\mathcal{Q}_5^7(\mu \boxplus \sigma) = [(0, (1)), (5, (1^2)), (5, (1^2)), (0, (2)), (0, (2))]$;
using our algorithm we obtain

$$\mu \boxplus \sigma = (26, 21, 12, 11, 7, 6, 5, 1).$$

But with $\nu = (21, 16, 11, 6, 2, 1) = \overline{\nu}_5$, so that $\overline{\nu}_7 = \Upsilon_{5,7}$, and $\sigma = (19, 12, 5, 4)$ again, we find that the 5-set of ν and $\nu \boxplus \sigma$ is $\{0, 26, 7, -2, -21\}$,
$\mathcal{Q}_5^7(\nu) = [(0, \varnothing), (5, \varnothing), (0, \varnothing), (5, \varnothing), (0, \varnothing)] = \mathcal{Q}_5^7(\Upsilon_{5,7})$, and
$\mathcal{Q}_5^7(\sigma) = [(0, (1)), (5, (1^2)), (0, (2)), (5, (1^2)), (0, (2))] = \mathcal{Q}_5^7(\nu \boxplus \sigma)$, so we instead get

$$\nu \boxplus \sigma = (26, 21, 12, 11, 6, 5, 1).$$



Next we will consider the 5-weighted 7-quotient, so whichever 5-bar-core μ and 7-bar-core σ with $\overline{\mu}_7 = \overline{\sigma}_5 = \Upsilon_{5,7}$ we choose, we will find that the 7-set of σ contains $\Delta_0\sigma = 0$ and 6 elements in the same congruence class modulo 5. This is because $\overline{\mu}_7 = \Upsilon_{5,7}$ so that the corresponding elements in the 7-sets of the two bar partitions can only differ by multiples of 5, and as discussed above, $\Upsilon_{5,7}$ has a 7-set of this form.

Let $\mu = (16, 11, 7, 6, 2, 1) = \overline{\mu}_5$ and $\sigma = (19, 12, 5, 4) = \overline{\sigma}_7$, so that $\overline{\mu}_7 = \overline{\sigma}_5 = (9, 4, 2) = \Upsilon_{5,7}$. The 7-set of $\Upsilon_{m,n}$ and μ is $\{0, 1, 16, -4, 11, -9, 6\}$, the 7-set of σ and $\mu \boxplus \sigma$ is $\{0, 1, -19, -4, 11, 26, 6\}$,

$\mathcal{Q}_7^5(\Upsilon_{5,7}) = [(0, \varnothing), (1, \varnothing), (1, \varnothing), (1, \varnothing), (1, \varnothing), (1, \varnothing), (1, \varnothing)] = \mathcal{Q}_7^5(\sigma)$, and

$\mathcal{Q}_7^5(\mu) = [(0, (1)), (1, (1)), (1, (1)), (1, (1)), (1, (1)), (1, (1)), (1, (1))] = \mathcal{Q}_7^5(\mu \boxplus \sigma)$,

so we obtain

$$\mu \boxplus \sigma = (26, 12, 11, 7, 6, 5, 1).$$

With $\sigma = (19, 12, 5, 4)$ again and $\nu = (23, 18, 13, 9, 8, 4, 3) = \overline{\nu}_5$, so that $\overline{\nu}_7 = \Upsilon_{5,7}$, we find the 7-set of $\Upsilon_{m,n}$ and ν is $\{0, 1, 16, -4, 11, -9, 6\}$,

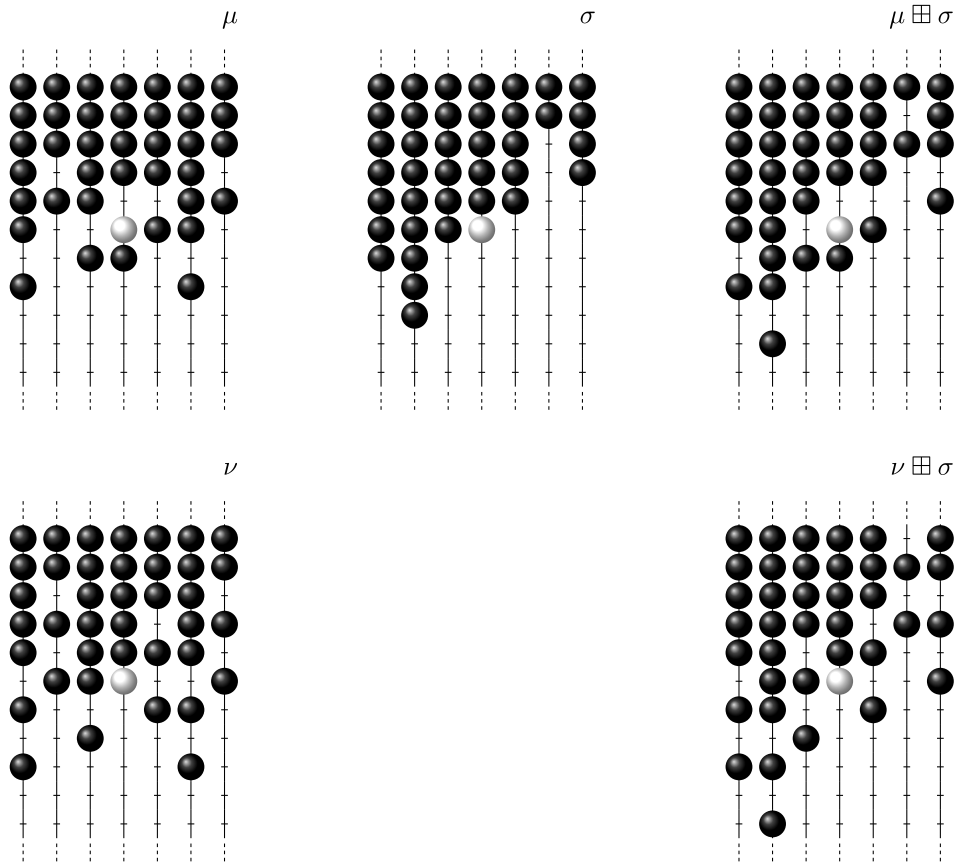
the 7-set of $\nu \boxplus \sigma$ is again $\{0, 1, -19, -4, 11, 26, 6\}$,

$\mathcal{Q}_7^5(\sigma) = [(0, \varnothing), (1, \varnothing), (1, \varnothing), (1, \varnothing), (1, \varnothing), (1, \varnothing), (1, \varnothing)]$, and

$\mathcal{Q}_7^5(\nu \boxplus \sigma) = \mathcal{Q}_7^5(\nu) = [(0,(2,1)),(1,(1,(2,1)),(1,(2,1)),(1,(2,1)),(1,(2,1)),(1,(2,1)),(1,(2,1))]$,

so we obtain

$$\nu \boxplus \sigma = (33, 19, 18, 13, 8, 5, 4, 3).$$



The next result gives the converse to Lemma 4.3.6, establishes that q-bar-cores σ are uniquely determined by $\sigma^{(0 \bmod p)}$ when $\overline{\sigma}_p = \Upsilon_{p,q}$, and gives the number of q-bar-cores μ with p-bar-core $\Upsilon_{q,p}$ when $\mu^{(0 \bmod p)}$ is fixed. First it is necessary to recall the definition of the beta-set $\mathcal{B}_r^\alpha := \{\alpha_i - i + r | i \in \mathbb{Z}_{>0}\}$ of a partition α, and that the double λ^+ of a bar partition λ is the partition whose Young diagram is obtained by amalgamating the shifted diagram of λ, which has λ_i nodes in the i^{th} row, with the left-most in the i^{th} column, and its reflection along the top left to bottom right diagonal. Also recall the correspondence between p-bars of λ and p-hooks of λ^+ which means that λ^+ is a p-core when λ is a p-bar-core.

Proposition 4.3.8. *(i) Suppose $\mu \in \overline{\mathcal{P}}_n$ and $\overline{\mu}_q = \Upsilon_{p,q}$. Then $\mu \in \overline{C}_p$ if and only if $\alpha := \mu^{(0 \bmod q)}$ is a p-bar-core, there is a self-conjugate p-core γ with $\gamma = \mu^{(j \bmod q)}$ for all $j \not\equiv 0 \pmod q$, and*

$$\mathcal{B}^\gamma_{(1-p)/2} \subseteq \mathcal{A}(\alpha) = \{\tfrac{x}{q} | x \in \mathcal{A}(\mu) \cap q\mathbb{Z}\} \subseteq \mathcal{B}^\gamma_{(p+1)/2}.$$

Moreover, for each $\alpha \in \overline{\mathcal{P}}_n$, there are $2^{(p-1)/2}$ p-bar-cores μ with $\overline{\mu}_q = \Upsilon_{p,q}$ and $\mu^{(0 \bmod q)} = \alpha$.

(ii) Suppose $\sigma \in \overline{\mathcal{P}}_n$ and $\overline{\sigma}_p = \Upsilon_{p,q}$. Then

$$\sigma \in \overline{C}_q \Leftrightarrow \exists \beta \in \overline{C}_q,\ \sigma^{(j \ mod \ p)} = \begin{cases} \beta & \text{if } j \equiv 0\,(mod\ p), \\ \beta^+ & \text{if } j \equiv iq\,(mod\ p) \text{ for some } 1 \le i \le \frac{p-1}{2}, \\ (\beta^+)' & \text{otherwise.} \end{cases}$$

Proof. (i) First suppose that $\mathcal{Q}_q(\mu)$ consists of a p-bar-core $\alpha := \mu^{(0 \bmod q)}$ and $q-1$ copies of a self-conjugate p-core γ such that

$$\mathcal{B}^{\gamma}_{(1-p)/2} \subseteq \mathcal{A}(\alpha) \subseteq \mathcal{B}^{\gamma}_{(p+1)/2}.$$

Since the q-set of $\Upsilon_{p,q}$ and μ is $\{0\} \cup \{q(p+1)/2 - np | n = 1, 2, \ldots, q-1\}$, for each $n \in \{1, \ldots, q-1\}$ we have

$$\{x \in \mathcal{A}(\mu) | x \equiv np \pmod q\} = \{(\gamma_i - i - \tfrac{p-1}{2})q + np | i \in \mathbb{N}\},$$

so clearly $x - p \in \mathcal{A}(\mu)$ whenever $n \in \{2, \ldots, q-1\}$, $x \in \mathcal{A}(\mu)$, $x \equiv np \pmod q$. Furthermore, from the assumed inclusions involving $\mathcal{A}(\alpha)$ and beta-sets for γ,

$$\begin{aligned} x \in \mathcal{A}(\mu) \cap q\mathbb{Z} \Rightarrow \exists j, x - p &= (\gamma_j - j + \tfrac{p+1}{2})q - p \\ &= (\gamma_j - j - \tfrac{p-1}{2})q + (q-1)p \in \mathcal{A}(\mu); \\ x \in \mathcal{A}(\mu), x \equiv p \pmod q \Rightarrow \exists i, x - p &= (\gamma_i - i - \tfrac{p-1}{2})q + p - p \\ &= (\gamma_i - i + \tfrac{1-p}{2})q \in \mathcal{A}(\mu); \end{aligned}$$

hence $\overline{\mu}_p = \mu$.

Now suppose $\mu \in \overline{C}_p$. Then, again by Lemma 4.3.6, $\mathcal{Q}_q(\mu)$ consists of a p-bar-core $\alpha = \mu^{(0 \bmod q)}$ and a self-conjugate p-core γ, so

$$\{x \in \mathcal{A}(\mu) | x \equiv np \pmod q\} = \{(\gamma_i - i - (p-1)/2)q + np | i \in \mathbb{N}\}$$

for each $n \in \{1, \ldots, q-1\}$. Thus

$$\begin{aligned} &\forall k \in \mathbb{N}, \alpha_k q - p \in \mathcal{A}(\mu) \Rightarrow \exists j, \alpha_k = \gamma_j - j + \tfrac{p+1}{2}; \\ &\forall i \in \mathbb{N}, (\gamma_i - i - \tfrac{p-1}{2})q \in \mathcal{A}(\mu) \Rightarrow \exists h, \gamma_i - i + \tfrac{1-p}{2} = \alpha_h; \end{aligned}$$

hence

$$\mathcal{B}^{\gamma}_{(1-p)/2} \subseteq \mathcal{A}(\alpha) \subseteq \mathcal{B}^{\gamma}_{(p+1)/2}.$$

Let z be the largest element of $\mathcal{A}(\alpha)$ (i.e. $z := -1$ if $\alpha = \varnothing$, or $z := \alpha_1$ is the largest part of α, otherwise). If $\mu \in \overline{C}_p$, then for each $m \in \mathbb{N}$ and $n \in \{0, 1, \ldots, q-1\}$ we have $(z+m)q + np \notin \mathcal{A}(\mu)$ (and $-(z+m)q - np \in \mathcal{A}(\mu)$). Since $\overline{\mu}_q = \Upsilon_{p,q}$, $\Delta_{np \bmod q}\mu = q(p+1)/2 - (q-n)p$ for $n \in \{1, \ldots, q-1\}$, so the number of integers $x \in \mathcal{A}(\mu)$ such that $x \equiv np \pmod q$ and $-zq - (q-n)p < x < (z+1)q + np$ is $\frac{1}{q}(q(p+1)/2 - (q-n)p - (-zq - (q-n)p)) = z + (p+1)/2$. The partition γ must be self-conjugate, so for each $m \in \mathbb{Z}$ and $n \in \{1, \ldots, q-1\}$, we have

$$(z+1)q + np - (z + \tfrac{p+1}{2})q + m = np - \tfrac{p-1}{2}q + m \in \mathcal{A}(\mu) \Leftrightarrow np - \tfrac{p+1}{2}q - m \notin \mathcal{A}(\mu).$$

Moreover, we must have $yq - p \in \mathcal{A}(\mu)$ (and $p - yq \notin \mathcal{A}(\mu)$) for all $y \in \mathcal{A}(\alpha) \cup \{0\}$, so for each $n \in \{1, \dots, q-1\}$, there are $z+1$ elements in $\{x \in \mathcal{A}(\mu) | x \equiv np \pmod q\}$ which are fixed by α, and $z + {}^{(p+1)}/_2 - (z+1) = {}^{(p-1)}/_2$ integers which are free to either belong in this set or not. Hence the number of possible partitions γ, and therefore the number of p-bar-cores μ with q-bar-core $\Upsilon_{p,q}$ and $\mu^{(0 \bmod q)} = \alpha$, is $2^{(p-1)/2}$.

(ii) Next, suppose $\beta := \sigma^{(0 \bmod p)}$ is a q-bar-core, and for $j \not\equiv 0 \pmod p$,

$$\sigma^{(j \bmod p)} = \begin{cases} \beta^+ & \text{if } j \equiv iq \pmod p \text{ for some } 1 \leq i \leq \frac{p-1}{2}, \\ (\beta^+)' & \text{otherwise.} \end{cases}$$

Since the p-set of $\Upsilon_{p,q}$ and σ is $\{nq | n = 0, 1, \dots, {}^{(p-1)}/_2\} \cup \{p - nq | n = 1, 2, \dots, {}^{(p-1)}/_2\}$, for each $n \in \{1, \dots, {}^{(p-1)}/_2\}$, (adopting the convention that for a partition λ, we put $\lambda_i = 0$ for all $i > \lambda_1'$) we have

$$\{x \in \mathcal{A}(\sigma) | x \equiv nq \pmod p\} = \{((\beta^+)_i - i)p + nq | i \in \mathbb{N}\};$$
$$\{x \in \mathcal{A}(\sigma) | x \equiv -nq \pmod p\} = \{((\beta^+)_j' - j + 1)p - nq | j \in \mathbb{N}\}.$$

It follows that $x - q \in \mathcal{A}(\sigma)$ whenever $x \in \mathcal{A}(\sigma)$, $n \in \{2, \dots, p-1\} \backslash \{{}^{(p+1)}/_2\}$ and $x \equiv nq \pmod p$. We also have $x - q \in \mathcal{A}(\sigma)$ for each $x \equiv 0, q \pmod p$ by the definition of β^+: for each part β_k of $\beta = (\beta_1, \dots, \beta_r)$, $(\beta^+)_k' - k + 1 = \beta_k = \beta_k^+ - k$. Since β is a q-bar-core, β^+ is a q-core, so when $1 \leq k \leq r$ we have

$$(\beta^+)_k' - k + 1 - q = \beta_k - q = \beta_k^+ - k - q \in \mathcal{B}^{(\beta^+)}.$$

It therefore follows from the symmetry of $\mathcal{A}(\sigma)$ that

$$\begin{aligned} x \equiv \tfrac{p+1}{2} q \pmod p \Rightarrow \exists j, x - q &= ((\beta^+)_j' - j + 1)p - \tfrac{p+1}{2} q \\ &= ((\beta^+)_j' - j + 1 - q)p + \tfrac{p-1}{2} q \\ \Rightarrow \exists i, x - q &= (\beta_i^+ - i)p + \tfrac{p-1}{2} q \in \mathcal{A}(\sigma); \end{aligned}$$

hence $\overline{\sigma}_q = \sigma$.

Conversely, suppose $\sigma \in \overline{C}_q$. Then since σ shares the p-set of $\Upsilon_{p,q}$, by Lemma 4.3.6 the p-quotient $\mathcal{Q}_p(\sigma)$ must consist of a q-bar-core $\beta = \sigma^{(0 \bmod p)}$ and two q-cores g and g', where

$$\sigma^{(j \bmod p)} = \begin{cases} g & j \equiv iq \bmod p, \ i \in \{1, \dots, \frac{p-1}{2}\}, \\ g' & j \equiv -iq \bmod p, \ i \in \{1, \dots, \frac{p-1}{2}\}. \end{cases}$$

For each $n \in \{1, \dots, {}^{(p-1)}/_2\}$, we therefore have

$$\{x \in \mathcal{A}(\sigma) | x \equiv nq \pmod p\} = \{(g_i - i)p + nq | i \in \mathbb{N}\}$$
$$\{x \in \mathcal{A}(\sigma) | x \equiv -nq \pmod p\} = \{(g_j' - j + 1)p - nq | j \in \mathbb{N}\}.$$

Since $\overline{\sigma}_q = \sigma$,

$$\forall k \in \mathbb{N}, \beta_k p - q \in \mathcal{A}(\sigma) \Rightarrow \exists j, \beta_k = g_j' - j + 1;$$

$$\forall i \in \mathbb{N}, (g_i - i)p \in \mathcal{A}(\sigma) \Rightarrow \exists h, g_i - i = \beta_h;$$

thus $\mathcal{B}^g \subseteq \mathcal{A}(\beta) \subseteq \mathcal{B}^{g'}$. Since $yp - q \in \mathcal{A}(\sigma)$ for each $y \in \mathcal{A}(\beta) \cup \{0\}$, and since $\Delta_{-q \bmod p}\sigma = p - q$, it follows that

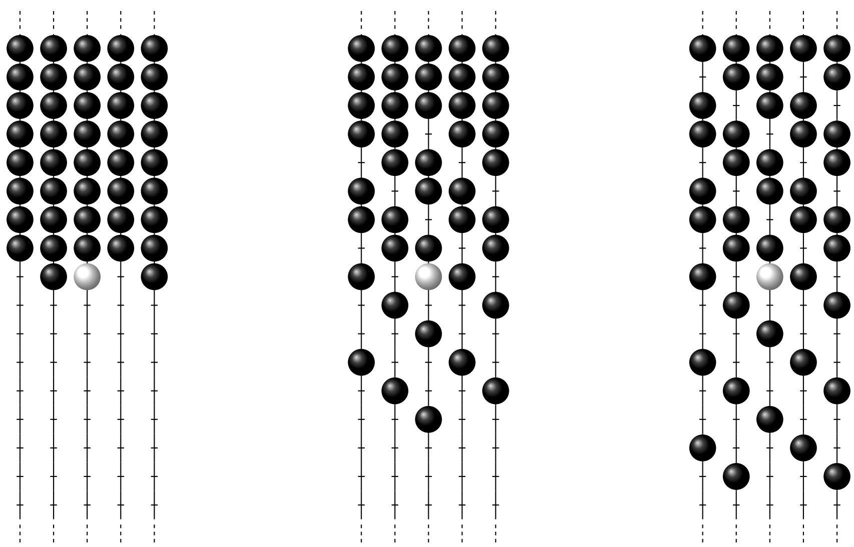
$$\{x \in \mathcal{A}(\sigma) | x \equiv -q \pmod{p}\} = \{yp - q | y \in \mathcal{A}(\beta)\} \cup \{-q\} = \{(g'_j - j + 1)p - q | j \in \mathbb{N}\};$$

hence $\sigma^{(-q \bmod p)} = (\beta^+)'$. □

Example. There are 2 possibilities for $\mu \in \overline{C}_3$ when $\overline{\mu}_5 = \Upsilon_{3,5} = (2)$ and $\mu^{(0 \bmod 5)} = (5, 2)$:

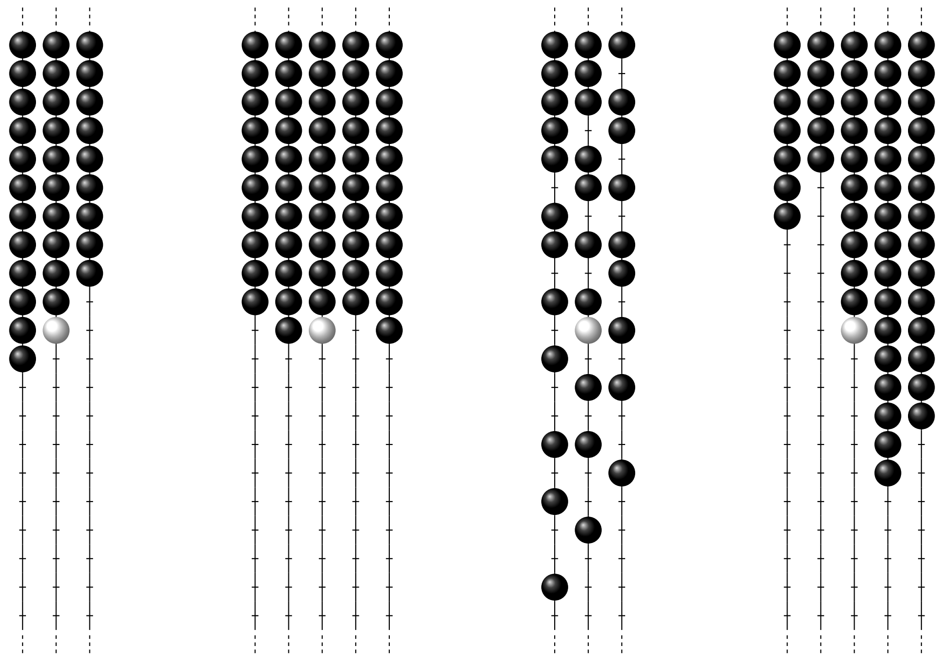
$$\mu = (25, 22, 19, 16, 13, 10, 7, 4, 1) \text{ and } \mu = (37, 34, 31, 28, 25, 22, 19, 16, 13, 10, 7, 4, 1),$$

shown in the second and third configurations below, respectively.



Example. If $\sigma \in \overline{C}_5$, $\overline{\sigma}_3 = \Upsilon_{3,5} = (2)$, and $\sigma^{(0 \bmod 3)} = (7, 4, 2) =: \beta$, then $\sigma^{(5 \bmod 3)} = \beta^+ = (8, 6, 5, 3, 2, 1, 1)$, so $\sigma^{(-5 \bmod 3)} = (\beta^+)' = (7, 5, 4, 3, 3, 2, 1, 1)$.

The configurations for $\Upsilon_{3,5} = (2)$ and $\sigma = (26, 21, 17, 16, 12, 11, 7, 6, 2, 1)$:



Our final result establishes that as a $\mathfrak{W}_p \times \mathfrak{W}_q$ set, $\overline{C}^{\Upsilon}_{p,q}$ is isomorphic to $(2^{\{1,\ldots,{(p-1)}/2\}} \times \overline{C}_p) \times \overline{C}_q$, where $\mathfrak{W}_p$ and $\mathfrak{W}_q$ act at level 1 on $\overline{C}_p$ and $\overline{C}_q$.

Theorem 4.3.9. *(i) There is a bijection*

$$\begin{aligned}\overline{C}^{\Upsilon}_{p,q} &\to 2^{\{1,\ldots,{(p-1)}/2\}} \times \overline{C}_p \times \overline{C}_q \\ \lambda &\mapsto (\{i \in \{1,\ldots,{(p-1)}/2\} | \lambda_i^{(0 \ mod \ q)} q + p \in \mathcal{A}(\lambda)\}, \lambda^{(0 \ mod \ q)}, \lambda^{(0 \ mod \ p)}).\end{aligned}$$

(ii) Suppose $\lambda \in \overline{\mathcal{P}}_n$ and $a \in \mathfrak{W}_p$. Then $(a\lambda)^{(0 \ mod \ q)} = a(\lambda^{(0 \ mod \ q)})$, where a acts at level q on $\overline{\mathcal{P}}_n$ and at level 1 on $\overline{C}_p$.

Proof. (i) Denote the map by Φ. If λ belongs to the orbit of $\Upsilon_{p,q}$ under the action of $\mathfrak{W}_p \times \mathfrak{W}_q$, then by Corollary 4.2.8, λ is a pq-bar-core. Thus, by Lemma 4.1.1(i), $\lambda^{(0 \bmod q)}$ is a p-bar-core, and $\lambda^{(0 \bmod p)}$ is a q-bar-core, so Φ is well-defined. To show that Φ is a bijection, we construct an inverse.

When $X \subseteq \{1,\ldots,{(p-1)}/2\}$, $\alpha \in \overline{C}_p$, and $\beta \in \overline{C}_q$, we let μ be the bar partition with $\overline{\mu}_q = \Upsilon_{p,q}$, $\mu^{(0 \bmod q)} = \alpha$, and $\mu^{(j \bmod q)} = \gamma$ when $j \not\equiv 0 \pmod q$, for some self-conjugate p-core partition γ such that

$$\mathcal{B}^{\gamma}_{(1-p)/2} \subseteq \mathcal{A}(\alpha) \subseteq \mathcal{B}^{\gamma}_{(p+1)/2} \text{ and } \alpha_i q + p \in \mathcal{A}(\mu), i \in \{1,\ldots,{(p-1)}/2\} \Leftrightarrow i \in X.$$

We let σ be the bar partition with $\overline{\sigma}_p = \Upsilon_{p,q}$ and

$$\sigma^{(j \bmod p)} = \begin{cases} \beta & \text{if } j \equiv 0 \pmod p, \\ \beta^+ & \text{if } j \equiv iq \pmod p \text{ for some } 1 \leq i \leq \frac{p-1}{2}, \\ (\beta^+)' & \text{otherwise.} \end{cases}$$

Then $\mu \in \overline{C}_p$ and $\sigma \in \overline{C}_q$ by Proposition 4.3.8, so we can define $\Psi(X, \alpha, \beta)$ to be the bar partition $\mu \boxplus \sigma$, which is contained in the orbit of $\Upsilon_{p,q}$ under the action of $\mathfrak{W}_p \times \mathfrak{W}_q$ by Proposition 4.1.5.

Suppose $\lambda \in \overline{C}^{\Upsilon}_{p,q}$, and let $\Phi(\lambda) = (X, \alpha, \beta)$. We need to show that $\Psi((X, \alpha, \beta)) = \lambda$. By Corollary 4.2.7, $\overline{\lambda}_q$ has p-bar-core $\Upsilon_{p,q}$. Since λ and $\overline{\lambda}_q$ lie in the same level q orbit of $\mathfrak{W}_p$, they have the same p-quotient up to reordering. Thus $(\overline{\lambda}_q)^{(0 \bmod p)} = \beta = \lambda^{(0 \bmod p)}$, and $\overline{\lambda}_q$ is the unique q-bar-core σ with p-bar-core $\Upsilon_{p,q}$ and p-quotient consisting of β, β^+, and $(\beta^+)'$ (by Proposition 4.3.8). Similarly, $\overline{\lambda}_p$ is one of $2^{(p-1)/2}$ bar partitions with q-bar-core $\Upsilon_{p,q}$ and q-quotient $\mathcal{Q}_q(\lambda)$ consisting of $\alpha \in \overline{C}_p$ and a self-conjugate p-core γ such that $\mathcal{B}^{\gamma}_{(1-p)/2} \subseteq \mathcal{A}(\alpha) \subseteq \mathcal{B}^{\gamma}_{(p+1)/2}$. If we denote by μ the unique such bar partition with $\alpha_i q + p \in \mathcal{A}(\mu), i \in \{1, \ldots, (p-1)/2\} \Leftrightarrow i \in X$, then $\Psi((X, \alpha, \beta)) = \mu \boxplus \sigma = \lambda$.

Finally, let $(X, \alpha, \beta) \in 2^{\{1,\ldots,(p-1)/2\}} \times \overline{C}_p \times \overline{C}_q$, so that $\Psi(X, \alpha, \beta) = \mu \boxplus \sigma$, where μ is the bar partition with $\overline{\mu}_q = \Upsilon_{p,q}$, $\mu^{(0 \bmod q)} = \alpha$, and $\mu^{(j \bmod q)} = \gamma$ when $j \not\equiv 0 \pmod q$, for some self-conjugate p-core partition γ such that

$$\mathcal{B}^{\gamma}_{(1-p)/2} \subseteq \mathcal{A}(\alpha) \subseteq \mathcal{B}^{\gamma}_{(p+1)/2} \text{ and } \alpha_i q + p \in \mathcal{A}(\mu), i \in \{1, \ldots, (p-1)/2\} \Leftrightarrow i \in X,$$

and σ is the bar partition with $\overline{\sigma}_p = \Upsilon_{p,q}$ and

$$\sigma^{(j \bmod p)} = \begin{cases} \beta & \text{if } j \equiv 0 \pmod p, \\ \beta^+ & \text{if } j \equiv iq \pmod p \text{ for some } 1 \leq i \leq \frac{p-1}{2}, \\ (\beta^+)' & \text{otherwise.} \end{cases}$$

Then since $\mu \boxplus \sigma$ shares a q-quotient with μ and has the same p-quotient as σ up to reordering, we find that $\Phi(\mu \boxplus \sigma) = (X, \alpha, \beta)$. Hence Φ and Ψ are mutual inverses, and thus bijections.

(ii) Let $\rho = \lambda^{(0 \bmod q)}$, so that $\mathcal{A}(\lambda) \cap q\mathbb{Z} = \{xq | x \in \mathcal{A}(\rho)\}$. Since $xq \equiv iq \pmod p \Leftrightarrow x \equiv i \pmod p$, for each $i \in \{0, \ldots, (p-1)/2\}$, by the definition of the $\mathfrak{W}_p$-action we have

$$\mathcal{A}(g_i\lambda) \cap q\mathbb{Z} = g_i\mathcal{A}(\lambda) \cap q\mathbb{Z} = \{(g_i x)q | x \in \mathcal{A}(\rho)\} = \{xq | x \in \mathcal{A}(g_i\rho)\},$$

thus $(g_i\lambda)^{(0 \bmod q)} = g_i(\lambda^{(0 \bmod q)})$ for each i, where g_i acts at level q on λ, and at level 1 on ρ. The result follows for all $a \in \mathfrak{W}_p$. □

Example. We calculate part of $\overline{C}^{\Upsilon}_{3,5}$, the orbit of $\Upsilon_{3,5} = (2)$ under the action of $\mathfrak{W}_3 \times \mathfrak{W}_5$, to illustrate the bijection between this orbit and the set $2^{\{1\}} \times \overline{C}_3 \times \overline{C}_5$. The top part of the diagram Figure 4.1 shows the level 1 action of $\mathfrak{W}_5 = \langle g_0, g_1, g_2 \rangle$ on $\overline{C}_5$, and the left part of the diagram shows the level 1 action of $\mathfrak{W}_3 = \langle h_0, h_1 \rangle$ on $\overline{C}_3$:

$$g_0 x = \begin{cases} x - 2 & x \equiv 1 \pmod 5, \\ x + 2 & x \equiv 4 \pmod 5, \\ x & \text{otherwise;} \end{cases} \qquad g_1 x = \begin{cases} x - 1 & x \equiv 2, 4 \pmod 5, \\ x + 1 & x \equiv 1, 3 \pmod 5, \\ x & \text{otherwise;} \end{cases}$$

$$g_2 x = \begin{cases} x - 1 & x \equiv 3 \pmod 5, \\ x + 1 & x \equiv 2 \pmod 5, \\ x & \text{otherwise;} \end{cases}$$

$$h_0 x = \begin{cases} x-2 & x \equiv 1 \pmod 3, \\ x+2 & x \equiv 2 \pmod 3, \\ x & \text{otherwise}; \end{cases} \qquad h_1 x = \begin{cases} x-1 & x \equiv 2 \pmod 3, \\ x+1 & x \equiv 1 \pmod 3, \\ x & \text{otherwise}. \end{cases}$$

The edges in the main part of the diagram represent the level 3 action of the generators g_0, g_1, g_2 of $\mathfrak{W}_5$, and the level 5 action of the generators h_0, h_1 of $\mathfrak{W}_3$:

$$g_0 x = \begin{cases} x-6 & x \equiv 3 \pmod 5, \\ x+6 & x \equiv 2 \pmod 5, \\ x & \text{otherwise}; \end{cases} \qquad g_1 x = \begin{cases} x-3 & x \equiv 1,2 \pmod 5, \\ x+3 & x \equiv 3,4 \pmod 5, \\ x & \text{otherwise}; \end{cases}$$

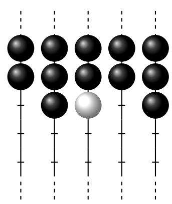
$$g_2 x = \begin{cases} x-3 & x \equiv 4 \pmod 5, \\ x+3 & x \equiv 1 \pmod 5, \\ x & \text{otherwise}; \end{cases}$$

$$h_0 x = \begin{cases} x-10 & x \equiv 2 \pmod 3, \\ x+10 & x \equiv 1 \pmod 3, \\ x & \text{otherwise}; \end{cases} \qquad h_1 x = \begin{cases} x-5 & x \equiv 1 \pmod 3, \\ x+5 & x \equiv 2 \pmod 3, \\ x & \text{otherwise}. \end{cases}$$

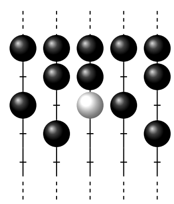
Below are some abacus displays to illustrate the bijection $\overline{C}^{\Upsilon}_{3,5} \to 2^{\{1\}} \times \overline{C}_3 \times \overline{C}_5$:

$(X \in 2^{\{1\}}, \varnothing, \varnothing)$:

$\Upsilon_{3,5} = (2)$

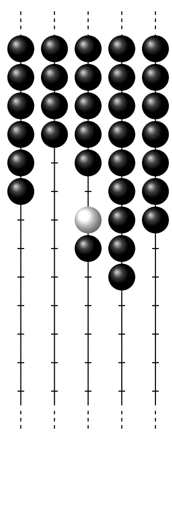


$(7,4,1)$

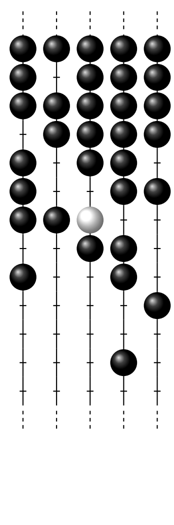


$(X \in 2^{\{1\}}, (1), (2))$:

$(11,6,5,2,1)$



$(26,17,11,8,6,5)$



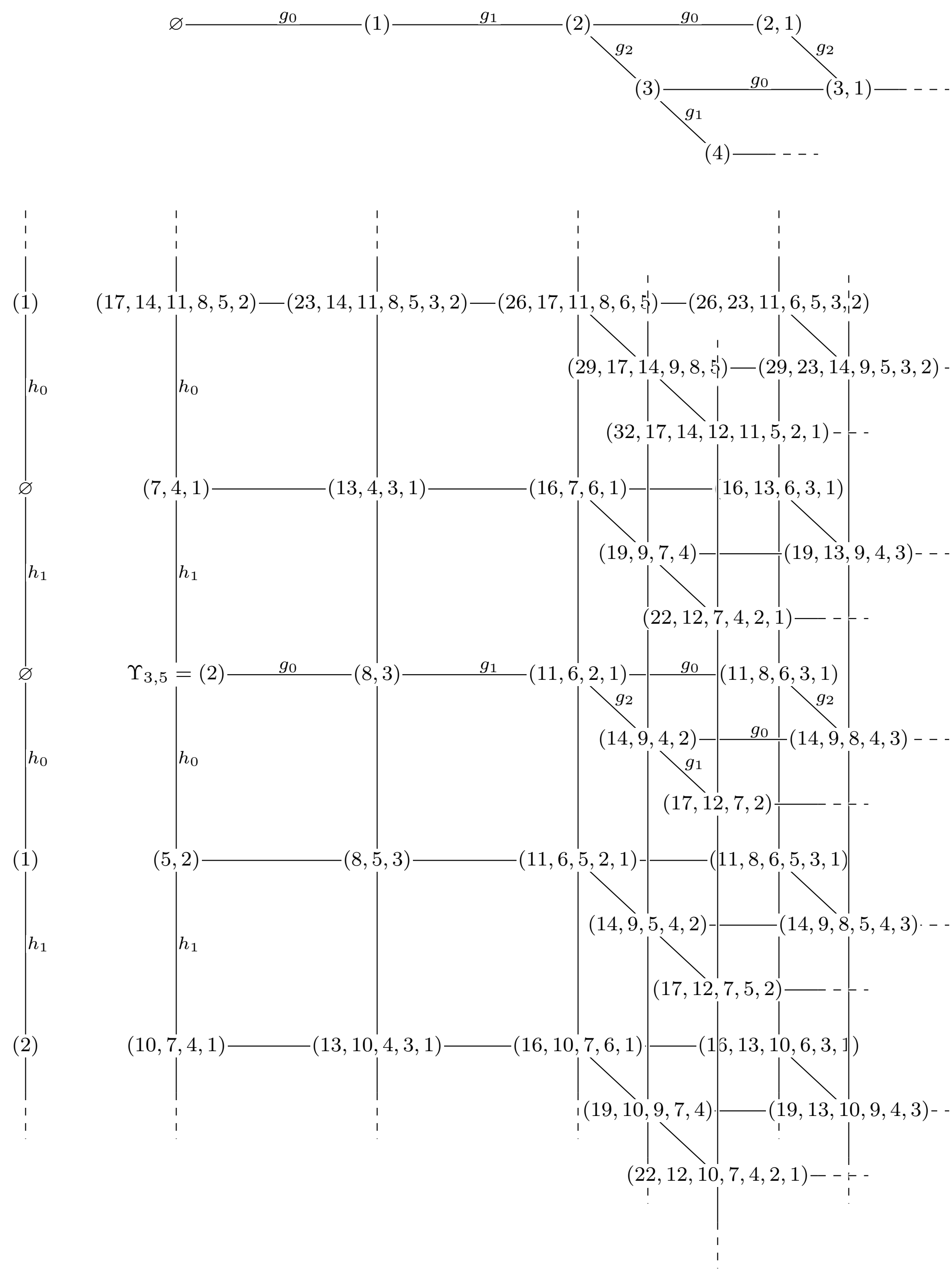
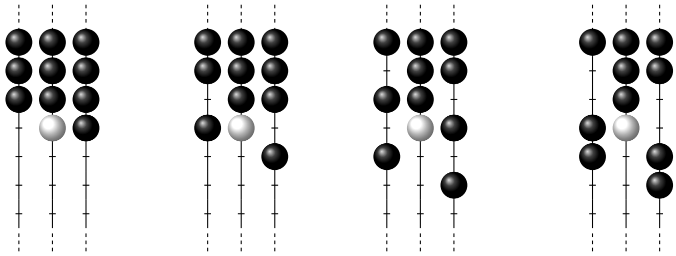


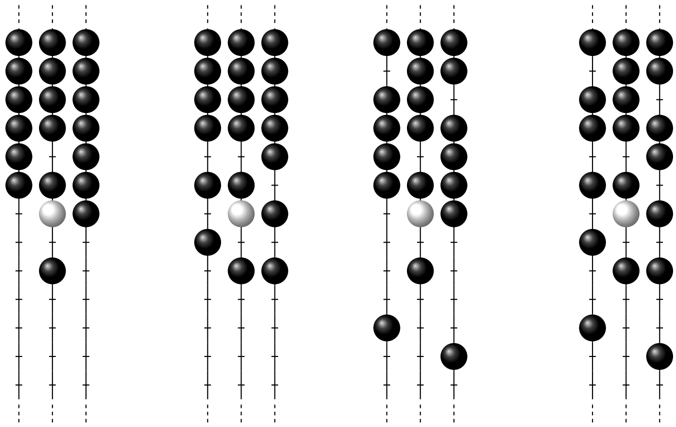
Figure 4.1: The bijection between $\overline{C}^{\Upsilon}_{3,5}$ and $2^{\{1\}} \times \overline{C}_3 \times \overline{C}_5$

The following abacus displays illustrate the bijection $\overline{C}^{\Upsilon}_{5,3} \to 2^{\{1,2\}} \times \overline{C}_5 \times \overline{C}_3$:

$(X \in 2^{\{1,2\}}, \varnothing, \varnothing)$: $\quad \Upsilon_{5,3} = (1) \quad (4) \quad (7,2,1) \quad (7,4,2)$



$(X \in 2^{\{1,2\}}, (2), \varnothing)$: $\quad (6,1) \quad (7,6,2,1) \quad (16,11,6,1) \quad (16,11,7,6,2,1)$



The orbit of $\Upsilon_{5,3} = (1)$ under the action of $\mathfrak{W}_3 \times \mathfrak{W}_5$ is in bijection with the set $2^{\{1,2\}} \times \overline{C}_3 \times \overline{C}_5$. In Figure 4.2, we illustrate part of this orbit; the edges in this diagram represent the level 3 action of $\mathfrak{W}_5 = \langle g_0, g_1, g_2 \rangle$, and for each $\lambda \in \overline{\mathcal{P}}_n$ in the diagram, $\lambda^{(0 \bmod 5)} = \varnothing$.

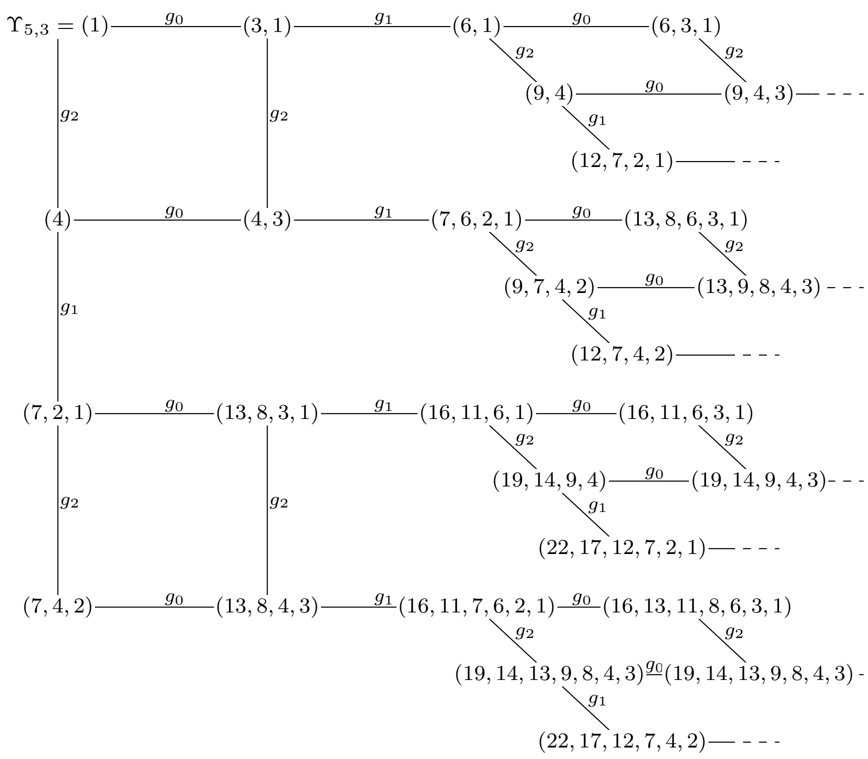


Figure 4.2: The branch of $\overline{C}^{\Upsilon}_{5,3}$ corresponding to the 3-bar-core $\varnothing$

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Appendix A

Proofs

To prove Theorem 3.3.3, we will need the identity involving $\zeta_n := \sum_{k=1}^{n-4}(k-1)T_k$ given in Lemma 3.3.2: For $0 \le j \le n-5$,

$$\zeta_n T_1 \cdots T_j = \sum_{k=1}^{j}(-1)^k T_1 \cdots T_{k-1}T_{k+1}\cdots T_j + (-1)^j \sum_{k=j+1}^{n-4}(k-1)T_1\cdots T_j T_k.$$

Note that if $n \ge 6$ is a multiple of 3, then in characteristic 3, the matrix ζ_n also satisfies the following:

$$\begin{aligned} \zeta_n^2 &= \sum_{k=1}^{n-5}(k^2 I + k(k-1)(T_kT_{k-1} + T_{k-1}T_k) \\ &= \sum_{k=1}^{n-5} kI = -2I = I. \\ \zeta_n T_1 + T_1\zeta_n &= T_1T_2 + T_2T_1 = -I. \end{aligned}$$

If $1 < i < n-4$, then

$$\begin{aligned} \zeta_n T_i + T_i \zeta_n &= \sum_{k=1}^{n-4}(k-1)(T_kT_i + T_iT_k) \\ &= (i-2)(T_{i-1}T_i + T_iT_{i-1}) + 2(i-1)I + i(T_{i+1}T_i + T_iT_{i+1}) \\ &= (2 - i + 2i - 2 - i)I = 0. \\ \zeta_n T_{n-4} + T_{n-4}\zeta_n &= 2I = -I. \\ \zeta_n T_{n-3} + T_{n-3}\zeta_n &= T_{n-4}T_{n-3} + T_{n-3}T_{n-4} = -I. \\ \zeta_n T_{n-2} + T_{n-2}\zeta_n &= \zeta_n T_{n-1} + T_{n-1}\zeta_n = 0. \end{aligned}$$

Hence $T_j\zeta_nT_j = -\zeta_n - T_j = \zeta_nT_j\zeta_n$ for $j = 1, n-4, n-3$.

Proof of Lemma 3.3.2:

Proof. Trivially, when $j = 0$ we have

$$\zeta_n T_1 \cdots T_0 = \zeta_n = \sum_{k=1}^{n-4}(k-1)T_k = \sum_{k=1}^{0}(-1)^k T_1 \cdots T_{k-1}T_{k+1}\cdots T_0 + \sum_{k=1}^{n-4}(k-1)T_1 \cdots T_0 T_k.$$

Now assume that the relation holds for all some $j < n-5$. Then

$$\begin{aligned}
\zeta_n T_1 \cdots T_{j+1} &= \sum_{k=1}^{j} (-1)^k T_1 \cdots T_{k-1} T_{k+1} \cdots T_j T_{j+1} + (-1)^j \sum_{k=j+1}^{n-4} (k-1) T_1 \cdots T_j T_k T_{j+1} \\
&= \sum_{k=1}^{j+1} (-1)^k T_1 \cdots T_{k-1} T_{k+1} \cdots T_{j+1} \\
&+ (-1)^j T_1 \cdots T_j \\
&+ (-1)^j j T_1 \cdots T_j \\
&+ (-1)^j (j+1) T_1 \cdots T_j T_{j+2} T_{j+1} \\
&+ (-1)^j (j+1) T_1 \cdots T_{j+2} \\
&+ (-1)^{j+1} \sum_{k=j+2}^{n-4} (k-1) T_1 \cdots T_{j+1} T_k \\
&= \sum_{k=1}^{j+1} (-1)^k T_1 \cdots T_{k-1} T_{k+1} \cdots T_{j+1} \\
&+ (-1)^j (j+1) T_1 \cdots T_j (1 + T_{j+1} T_{j+2} + T_{j+2} T_{j+1}) \\
&+ (-1)^{j+1} \sum_{k=j+2}^{n-4} (k-1) T_1 \cdots T_{j+1} T_k \\
&= \sum_{k=1}^{j+1} (-1)^k T_1 \cdots T_{k-1} T_{k+1} \cdots T_{j+1} + (-1)^{j+1} \sum_{k=j+2}^{n-4} (k-1) T_1 \cdots T_{j+1} T_k.
\end{aligned}$$

□

Proof of Theorem 3.3.3:

Proof. We first verify that for $i = 1, \ldots, n-3$, we have

$$\begin{aligned}
B_i(b) = P_{1,i+1}(b) &+ \sum_{j=2}^{i} (-1)^{(n+1)(j+1)} P_{j,i+1}(\zeta_n T_1 \cdots T_{j-2} b) \\
&- \sum_{j=i+2}^{n-3} ((-1)^{i(n+j)} P_{i+1,j}(T_{n-3} \zeta_n T_1 \cdots T_{j-3} b) \\
&\quad + (-1)^{(n+1)j} P_{i+1,n-2}(T_{j-1} \cdots T_{n-5} T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{j-3} b) \\
&\quad + (-1)^{i(n+j)} P_{j,n-2}(T_{i+1} \cdots T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{j-3} b)).
\end{aligned}$$

Case 4: For $1 < i \leq n-3$,

$$\begin{aligned}
t_i B_i(b) = t_i P_{1,i+1}(b) &+ \sum_{j=2}^{i} (-1)^{(n+1)(j+1)} t_i P_{j,i+1}(\zeta_n T_1 \cdots T_{j-2} b) \\
&- \sum_{j=i+2}^{n-3} ((-1)^{i(n+j)} t_i P_{i+1,j}(T_{n-3} \zeta_n T_1 \cdots T_{j-3} b) \\
&\quad + (-1)^{(n+1)j} t_i P_{i+1,n-2}(T_{j-1} \cdots T_{n-5} T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{j-3} b) \\
&\quad + (-1)^{i(n+j)} t_i P_{j,n-2}(T_{i+1} \cdots T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{j-3} b))
\end{aligned}$$

$$
\begin{aligned}
&= P_{1,i}(b) + \sum_{j=2}^{i-1}(-1)^{(n+1)(j+1)}P_{j,i}(\zeta_n T_1\cdots T_{j-2}b)\\
&\qquad + (-1)^{n(i+1)+i}P_{i,i+1}(T_{n-3}\zeta_n T_1\cdots T_{i-2}b)\\
&\qquad + (-1)^{n(i+1)+i}P_{i,n-2}(T_i\cdots T_{n-5}T_{n-3}T_{n-4}\zeta_n T_1\cdots T_{i-2}b)\\
&\qquad + (-1)^{n(i+1)+i}P_{i+1,n-2}(T_i\cdots T_{n-5}T_{n-3}T_{n-4}T_{n-3}\zeta_n T_1\cdots T_{i-2}b)\\
&\qquad - \sum_{j=i+2}^{n-3}((-1)^{(i+1)(n+j)}P_{i,j}(T_{n-3}\zeta_n T_1\cdots T_{j-3}b)\\
&\qquad\qquad + (-1)^{(n+1)j}P_{i,n-2}(T_{j-1}\cdots T_{n-5}T_{n-3}T_{n-4}\zeta_n T_1\cdots T_{j-3}b)\\
&\qquad\qquad + (-1)^{(i+1)(n+j)}P_{j,n-2}(T_i\cdots T_{n-5}T_{n-4}T_{n-3}T_{n-4}\zeta_n T_1\cdots T_{j-3}b))\\
&= P_{1,i}(b) + \sum_{j=2}^{i-1}(-1)^{(n+1)(j+1)}P_{j,i}(\zeta_n T_1\cdots T_{j-2}b)\\
&\qquad - \sum_{j=i+1}^{n-3}((-1)^{(i+1)(n+j)}P_{i,j}(T_{n-3}\zeta_n T_1\cdots T_{j-3}b)\\
&\qquad\qquad + (-1)^{(n+1)j}P_{i,n-2}(T_{j-1}\cdots T_{n-5}T_{n-3}T_{n-4}\zeta_n T_1\cdots T_{j-3}b)\\
&\qquad\qquad + (-1)^{(i+1)(n+j)}P_{j,n-2}(T_i\cdots T_{n-5}T_{n-4}T_{n-3}T_{n-4}\zeta_n T_1\cdots T_{j-3}b))\\
&= B_{i-1}(b).
\end{aligned}
$$

Case 5: It follows that for $1 \le i < n-4$,

$$t_{i+1}B_i(b) = t_{i+1}^2 B_{i+1}(b) = B_{i+1}(b).$$

Case 1:

$$
\begin{aligned}
t_1B_1(b) =& t_1P_{12}(b)\\
&- \sum_{j=3}^{n-3}((-1)^{(n+j)}t_1P_{2j}(T_{n-3}\zeta_n T_1\cdots T_{j-3}b)\\
&\qquad + (-1)^{(n+1)j}t_1P_{2,n-2}(T_{j-1}\cdots T_{n-5}T_{n-3}T_{n-4}\zeta_n T_1\cdots T_{j-3}b)\\
&\qquad + (-1)^{(n+j)}t_1P_{j,n-2}(T_2\cdots T_{n-3}T_{n-4}\zeta_n T_1\cdots T_{j-3}b))\\
=& P_{12}(-T_{n-3}b) + P_{1,n-2}(-T_1\cdots T_{n-5}T_{n-3}T_{n-4}b) + P_{2,n-2}(-T_1\cdots T_{n-3}T_{n-4})\\
&- \sum_{j=3}^{n-3}(P_{1j}(T_{n-3}\zeta_n T_1\cdots T_{j-3}b)\\
&\qquad + (-1)^{(n+1)j}P_{1,n-2}(T_{j-1}\cdots T_{n-5}T_{n-3}T_{n-4}\zeta_n T_1\cdots T_{j-3}b)\\
&\qquad + P_{j,n-2}(T_1\cdots T_{n-3}T_{n-4}\zeta_n T_1\cdots T_{j-3}b))\\
=& P_{12}(-T_{n-3}b) + P_{1,n-2}(-T_1\cdots T_{n-5}T_{n-3}T_{n-4}b)\\
&+ \sum_{j=2}^{n-4}(P_{1,j+1}(-T_{n-3}\zeta_n T_1\cdots T_{j-2}b)\\
&\qquad + (-1)^{(n+1)(j+1)}P_{1,n-2}(-T_j\cdots T_{n-5}T_{n-3}T_{n-4}\zeta_n T_1\cdots T_{j-2}b)
\end{aligned}
$$

$$+ P_{j+1,n-2}(-T_1 \cdots T_{n-3}T_{n-4}\zeta_n T_1 \cdots T_{j-2}b))$$

$$+ P_{2,n-2}(-T_1 \cdots T_{n-3}T_{n-4}b).$$

Looking at the coefficients of the terms P_{1j}, it is clear that this expression should be equal to the combination of

$$\begin{aligned}
&B_1(-T_{n-3}b) + B_{n-3}(-T_1 \cdots T_{n-5}T_{n-3}T_{n-4}b) \\
&\quad + \sum_{j=2}^{n-4}(B_j(-T_{n-3}\zeta_n T_1 \cdots T_{j-2}b) \\
&\qquad + (-1)^{(n+1)(j+1)}B_{n-3}(-T_j \cdots T_{n-5}T_{n-3}T_{n-4}\zeta_n T_1 \cdots T_{j-2}b)) \\
=&P_{12}(-T_{n-3}b) \\
&\quad + \sum_{j=3}^{n-3}((-1)^{(n+j)}P_{2j}(T_{n-3}\zeta_n T_1 \cdots T_{j-3}T_{n-3}b) \\
&\qquad + (-1)^{(n+1)j}P_{2,n-2}(T_{j-1} \cdots T_{n-5}T_{n-3}T_{n-4}\zeta_n T_1 \cdots T_{j-3}T_{n-3}b) \\
&\qquad + (-1)^{(n+j)}P_{j,n-2}(T_2 \cdots T_{n-3}T_{n-4}\zeta_n T_1 \cdots T_{j-3}T_{n-3}b)) \\
&\quad + P_{1,n-2}(-T_1 \cdots T_{n-5}T_{n-3}T_{n-4}b) \\
&\quad + \sum_{j=2}^{n-4}(-1)^{(n+1)(j+1)}P_{j,n-2}(-\zeta_n T_1 \cdots T_{j-2}T_1 \cdots T_{n-5}T_{n-3}T_{n-4}b) \\
&\quad + \sum_{j=2}^{n-4}(P_{1,j+1}(-T_{n-3}\zeta_n T_1 \cdots T_{j-2}b) \\
&\qquad + \sum_{k=2}^{j}(-1)^{(n+1)(k+1)}P_{k,j+1}(-\zeta_n T_1 \cdots T_{k-2}T_{n-3}\zeta_n T_1 \cdots T_{j-2}b) \\
&\qquad + \sum_{k=j+2}^{n-3}((-1)^{j(n+k)}P_{j+1,k}(T_{n-3}\zeta_n T_1 \cdots T_{k-3}T_{n-3}\zeta_n T_1 \cdots T_{j-2}b) \\
&\qquad\quad + (-1)^{(n+1)k}P_{j+1,n-2}(T_{k-1} \cdots T_{n-5}T_{n-3}T_{n-4}\zeta_n T_1 \cdots T_{k-3}T_{n-3}\zeta_n T_1 \cdots T_{j-2}b) \\
&\qquad\quad + (-1)^{j(n+k)}P_{k,n-2}(T_{j+1} \cdots T_{n-3}T_{n-4}\zeta_n T_1 \cdots T_{k-3}T_{n-3}\zeta_n T_1 \cdots T_{j-2}b)) \\
&\qquad + (-1)^{(n+1)(j+1)}P_{1,n-2}(-T_j \cdots T_{n-5}T_{n-3}T_{n-4}\zeta_n T_1 \cdots T_{j-2}b) \\
&\qquad + \sum_{k=2}^{n-4}(-1)^{(n+1)(j+k)}P_{k,n-2}(-\zeta_n T_1 \cdots T_{k-2}T_j \cdots T_{n-5}T_{n-3}T_{n-4}\zeta_n T_1 \cdots T_{j-2}b)) \\
=&P_{12}(-T_{n-3}b) + P_{1,n-2}(-T_1 \cdots T_{n-5}T_{n-3}T_{n-4}b) \\
&\quad + \sum_{j=2}^{n-4}(P_{1,j+1}(-T_{n-3}\zeta_n T_1 \cdots T_{j-2}b) \\
&\qquad + (-1)^{(n+1)(j+1)}P_{1,n-2}(-T_j \cdots T_{n-5}T_{n-3}T_{n-4}\zeta_n T_1 \cdots T_{j-2}b) \\
&\qquad + (-1)^{n+j+1}P_{2,j+1}(T_{n-3}\zeta_n T_1 \cdots T_{j-2}T_{n-3}b) \\
&\qquad + (-1)^{(n+1)(j+1)}P_{2,n-2}(T_j \cdots T_{n-5}T_{n-3}T_{n-4}\zeta_n T_1 \cdots T_{j-2}T_{n-3}b) \\
&\qquad + (-1)^{(n+1)(j+1)}P_{j,n-2}(-\zeta_n T_1 \cdots T_{j-2}T_1 \cdots T_{n-5}T_{n-3}T_{n-4}b)
\end{aligned}$$

$$+ (-1)^{n+j+1} P_{j+1,n-2}(T_2 \cdots T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{j-2} T_{n-3} b)$$
$$+ \sum_{k=j+2}^{n-3} ((-1)^{(n+1)k} P_{j+1,n-2}(T_{k-1} \cdots T_{n-5} T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{k-3} T_{n-3} \zeta_n T_1 \cdots T_{j-2} b)$$
$$+ (-1)^{j(n+k)} P_{j+1,k}(T_{n-3} \zeta_n T_1 \cdots T_{k-3} T_{n-3} \zeta_n T_1 \cdots T_{j-2} b)$$
$$+ (-1)^{j(n+k)} P_{k,n-2}(T_{j+1} \cdots T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{k-3} T_{n-3} \zeta_n T_1 \cdots T_{j-2} b))$$
$$+ \sum_{k=2}^{n-4} (-1)^{(n+1)(j+k)} P_{k,n-2}(-\zeta_n T_1 \cdots T_{k-2} T_j \cdots T_{n-5} T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{j-2} b)$$
$$+ \sum_{k=2}^{j} (-1)^{(n+1)(k+1)} P_{k,j+1}(-\zeta_n T_1 \cdots T_{k-2} T_{n-3} \zeta_n T_1 \cdots T_{j-2} b)).$$

Subtracting $t_1 B_1(b)$, we get

$$P_{2,n-2}(T_1 \cdots T_{n-3} T_{n-4} b)$$
$$+ \sum_{j=2}^{n-4} ((-1)^{n+j+1} P_{2,j+1}(T_{n-3} \zeta_n T_1 \cdots T_{j-2} T_{n-3} b)$$
$$+ (-1)^{(n+1)(j+1)} P_{2,n-2}(T_j \cdots T_{n-5} T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{j-2} T_{n-3} b)$$
$$+ (-1)^{(n+1)(j+1)} P_{j,n-2}(-\zeta_n T_1 \cdots T_{j-2} T_1 \cdots T_{n-5} T_{n-3} T_{n-4} b)$$
$$+ (-1)^{n+j+1} P_{j+1,n-2}(T_2 \cdots T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{j-2} T_{n-3} b)$$
$$+ P_{j+1,n-2}(T_1 \cdots T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{j-2} b)$$
$$+ \sum_{k=j+2}^{n-3} ((-1)^{(n+1)k} P_{j+1,n-2}(T_{k-1} \cdots T_{n-5} T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{k-3} T_{n-3} \zeta_n T_1 \cdots T_{j-2} b)$$
$$+ (-1)^{j(n+k)} P_{j+1,k}(T_{n-3} \zeta_n T_1 \cdots T_{k-3} T_{n-3} \zeta_n T_1 \cdots T_{j-2} b)$$
$$+ (-1)^{j(n+k)} P_{k,n-2}(T_{j+1} \cdots T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{k-3} T_{n-3} \zeta_n T_1 \cdots T_{j-2} b))$$
$$+ \sum_{k=2}^{n-4} (-1)^{(n+1)(j+k)} P_{k,n-2}(-\zeta_n T_1 \cdots T_{k-2} T_j \cdots T_{n-5} T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{j-2} b)$$
$$+ \sum_{k=2}^{j} (-1)^{(n+1)(k+1)} P_{k,j+1}(-\zeta_n T_1 \cdots T_{k-2} T_{n-3} \zeta_n T_1 \cdots T_{j-2} b)).$$

Now this expression should be zero, so we need to rearrange the terms so that they all cancel. As the first step, we change summation variables so that all related terms are written in the same way, and we collect all related terms.

$$P_{2,n-2}((T_1 \cdots T_{n-3} T_{n-4} + (-1)^n \zeta_n T_1 \cdots T_{n-5} T_{n-3} T_{n-4}) b)$$
$$+ \sum_{j=2}^{n-4} ((-1)^{(n+1)(j+1)} P_{2,n-2}((T_j \cdots T_{n-5} T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{j-2} T_{n-3}$$
$$+ (-1)^n \zeta_n T_j \cdots T_{n-5} T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{j-2}) b)$$
$$+ (-1)^n P_{2,j+1}(((-1)^{j+1} T_{n-3} \zeta_n T_1 \cdots T_{j-2} T_{n-3}$$
$$+ \zeta_n T_{n-3} \zeta_n T_1 \cdots T_{j-2}) b)$$
$$+ P_{j+1,n-2}(((-1)^{(n+1)j+1} \zeta_n T_1 \cdots T_{j-1} T_1 \cdots T_{n-5} T_{n-3} T_{n-4}$$

$$
\begin{aligned}
&+ (-1)^{n+j+1} T_2 \cdots T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{j-2} T_{n-3} \\
&+ T_1 \cdots T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{j-2} \\
&+ \sum_{k=j+2}^{n-3} (-1)^{(n+1)k} T_{k-1} \cdots T_{n-5} T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{k-3} T_{n-3} \zeta_n T_1 \cdots T_{j-2} \\
&+ \sum_{k=2}^{n-4} (-1)^{n(j+k+1)+j+k} \zeta_n T_1 \cdots T_{j-1} T_k \cdots T_{n-5} T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{k-2}) b) \\
+ \sum_{k=3}^{j} &(-1)^{(n+j+1)(k+1)} (P_{j+1,n-2}(T_k \cdots T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{j-2} T_{n-3} \zeta_n T_1 \cdots T_{k-3} b) \\
&+ P_{k,j+1}((T_{n-3} \zeta_n T_1 \cdots T_{j-2} T_{n-3} \zeta_n T_1 \cdots T_{k-3} \\
&\qquad + (-1)^{j(k+1)+1} \zeta_n T_1 \cdots T_{k-2} T_{n-3} \zeta_n T_1 \cdots T_{j-2}) b))).
\end{aligned}
$$

Now we just need to cancel the coefficients of $P_{ij}(b)$, for each i, j. Using the identity, for $3 \leq k \leq j \leq n-4$,

$$T_2 \ldots T_{k-2} T_1 \cdots T_{j-2} = (-1)^{j(k+1)} T_1 \cdots T_{j-2} T_1 \cdots T_{k-3},$$

we see that the $P_{k,j+1}$ terms in our expression cancel, and we are left with

$$
\begin{aligned}
&P_{2,n-2}((T_1 \cdots T_{n-3} T_{n-4} + (-1)^n \zeta_n T_1 \cdots T_{n-5} T_{n-3} T_{n-4}) b) \\
&+ \sum_{j=2}^{n-4} ((-1)^{n(j+1)} P_{2,n-2}(T_j \cdots T_{n-5} T_{n-3} \zeta_n T_1 \cdots T_{j-2} b) \\
&\quad + P_{j+1,n-2}(((-1)^{(n+1)j+1} \zeta_n T_1 \cdots T_{j-1} T_1 \cdots T_{n-5} T_{n-3} T_{n-4} \\
&\qquad + (-1)^{n+j+1} T_2 \cdots T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{j-2} T_{n-3} \\
&\qquad + T_1 \cdots T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{j-2} \\
&\qquad + \sum_{k=j+2}^{n-3} (-1)^{(n+1)k} T_{k-1} \cdots T_{n-5} T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{k-3} T_{n-3} \zeta_n T_1 \cdots T_{j-2} \\
&\qquad + \sum_{k=2}^{n-4} (-1)^{n(j+k+1)+j+k} \zeta_n T_1 \cdots T_{j-1} T_k \cdots T_{n-5} T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{k-2} \\
&\qquad + \sum_{k=3}^{j} (-1)^{(n+j+1)(k+1)} T_k \cdots T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{j-2} T_{n-3} \zeta_n T_1 \cdots T_{k-3}) b)).
\end{aligned}
$$

For $2 \leq j < k-1 < n-3$, we have

$$T_1 \cdots T_{k-3} T_1 \cdots T_{j-2} = (-1)^{j(k+1)} T_2 \cdots T_{j-1} T_1 \cdots T_{k-3},$$

and for $2 \leq k \leq j - q \leq n-5$, we have

$$T_{k+1} \cdots T_{n-5} T_2 \cdots T_{j-2} = (-1)^{n(j+1)+jk} T_2 \cdots T_{j-1} T_k \cdots T_{n-5}.$$

Hence, we can rewrite our expression as

$$
\begin{aligned}
&P_{2,n-2}((T_1 \cdots T_{n-3} T_{n-4} + (-1)^n \zeta_n T_1 \cdots T_{n-5} T_{n-3} T_{n-4}) b) \\
&+ \sum_{j=2}^{n-4} ((-1)^{n(j+1)} P_{2,n-2}(T_j \cdots T_{n-5} T_{n-3} \zeta_n T_1 \cdots T_{j-2} b)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + P_{j+1,n-2}(((-1)^{(n+1)j+1}\zeta_n T_1\cdots T_{j-1}T_1\cdots T_{n-5}T_{n-3}T_{n-4}\\
&\qquad + (-1)^{n+j+1}T_2\cdots T_{n-3}T_{n-4}\zeta_n T_1\cdots T_{j-2}T_{n-3}\\
&\qquad + T_1\cdots T_{n-3}T_{n-4}\zeta_n T_1\cdots T_{j-2}\\
&\qquad + \sum_{k=3}^{j}(-1)^{(n+j+1)(k+1)+j}T_k\cdots T_{n-5}T_{n-4}T_{n-3}T_{n-4}\zeta_n T_{n-3}T_1\cdots T_{j-2}\zeta_n T_1\cdots T_{k-3}\\
&\qquad + \sum_{k=j+2}^{n-3}(-1)^{(n+1)k+jk+1}T_{k-1}\cdots T_{n-5}T_{n-3}T_{n-4}T_{n-3}\zeta_n T_{n-3}T_1\cdots T_{j-1}\zeta_n T_1\cdots T_{k-3}\\
&\qquad + \sum_{k=2}^{n-4}(-1)^{n(j+k+1)+j+k}\zeta_n T_1\cdots T_{j-1}T_k\cdots T_{n-5}T_{n-3}T_{n-4}\zeta_n T_1\cdots T_{k-2})b))\\
=&P_{2,n-2}((T_1\cdots T_{n-3}T_{n-4} + (-1)^n\zeta_n T_1\cdots T_{n-5}T_{n-3}T_{n-4})b)\\
&+ \sum_{j=2}^{n-4}((-1)^{n(j+1)}P_{2,n-2}(T_j\cdots T_{n-5}T_{n-3}\zeta_n T_1\cdots T_{j-2}b)\\
&\quad + P_{j+1,n-2}(((-1)^{(n+1)j+1}\zeta_n T_1\cdots T_{j-1}T_1\cdots T_{n-5}T_{n-3}T_{n-4}\\
&\qquad + (-1)^{n+j+1}T_2\cdots T_{n-3}T_{n-4}\zeta_n T_1\cdots T_{j-2}T_{n-3}\\
&\qquad + T_1\cdots T_{n-3}T_{n-4}\zeta_n T_1\cdots T_{j-2}\\
&\qquad + \sum_{k=2}^{j-1}(-1)^{n(j+k)}T_1\cdots T_{j-1}T_k\cdots T_{n-5}T_{n-3}\zeta_n T_1\cdots T_{k-2}\\
&\qquad + \sum_{k=j+1}^{n-4}(-1)^{n(j+k)}T_1\cdots T_{j-1}T_k\cdots T_{n-5}T_{n-3}\zeta_n T_1\cdots T_{k-2}\\
&\qquad + \sum_{k=2}^{j-1}(-1)^{(n+1)(j+k+1)}\zeta_n T_1\cdots T_{j-1}T_k\cdots T_{n-5}T_{n-3}T_{n-4}\zeta_n T_1\cdots T_{k-2}\\
&\qquad + \sum_{k=j+1}^{n-4}(-1)^{(n+1)(j+k+1)}\zeta_n T_1\cdots T_{j-1}T_k\cdots T_{n-5}T_{n-3}T_{n-4}\zeta_n T_1\cdots T_{k-2}\\
&\qquad + \sum_{k=2}^{n-4}(-1)^{(n+1)(j+k+1)+1}\zeta_n T_1\cdots T_{j-1}T_k\cdots T_{n-5}T_{n-3}T_{n-4}\zeta_n T_1\cdots T_{k-2})b))\\
=&P_{2,n-2}((T_1\cdots T_{n-3}T_{n-4} + (-1)^n\zeta_n T_1\cdots T_{n-5}T_{n-3}T_{n-4})b)\\
&+ \sum_{j=2}^{n-4}((-1)^{n(j+1)}P_{2,n-2}(T_j\cdots T_{n-5}T_{n-3}\zeta_n T_1\cdots T_{j-2}b)\\
&\quad + P_{j+1,n-2}(((-1)^{(n+1)j+1}\zeta_n T_1\cdots T_{j-1}T_1\cdots T_{n-5}T_{n-3}T_{n-4}\\
&\qquad + (-1)^{n+j+1}T_2\cdots T_{n-3}T_{n-4}\zeta_n T_1\cdots T_{j-2}T_{n-3}\\
&\qquad + T_1\cdots T_{n-5}T_{n-4}T_{n-3}T_{n-4}\zeta_n T_1\cdots T_{j-2}\\
&\qquad + \sum_{k=2}^{n-4}(-1)^{n(j+k)}T_1\cdots T_{j-1}T_k\cdots T_{n-5}T_{n-3}\zeta_n T_1\cdots T_{k-2}\\
&\qquad - T_1\cdots T_{n-5}T_{n-3}\zeta_n T_1\cdots T_{j-2}\\
&\qquad + (-1)^n\zeta_n T_1\cdots T_{n-5}T_{n-3}T_{n-4}\zeta_n T_1\cdots T_{j-2})b))
\end{aligned}
$$

$$
\begin{aligned}
=&P_{2,n-2}((T_1\cdots T_{n-3}T_{n-4}+(-1)^n\zeta_nT_1\cdots T_{n-5}T_{n-3}T_{n-4})b)\\
&+\sum_{j=2}^{n-4}((-1)^{n(j+1)}P_{2,n-2}(T_j\cdots T_{n-5}T_{n-3}\zeta_nT_1\cdots T_{j-2}b)\\
&\qquad+P_{j+1,n-2}((((-1)^{(n+1)j+1}\zeta_nT_1\cdots T_{j-1}T_1\cdots T_{n-5}T_{n-3}T_{n-4}\\
&\qquad\qquad+(-1)^{n+1}T_2\cdots T_{n-3}T_{n-4}\zeta_nT_{n-3}T_1\cdots T_{j-2}\\
&\qquad\qquad+T_1\cdots T_{n-5}(T_{n-3}-T_{n-4})\zeta_nT_1\cdots T_{j-2}\\
&\qquad\qquad+(-1)^n\zeta_nT_1\cdots T_{n-5}T_{n-3}T_{n-4}\zeta_nT_1\cdots T_{j-2}\\
&\qquad\qquad+\sum_{k=2}^{n-4}(-1)^{n(j+k)}T_1\cdots T_{j-1}T_k\cdots T_{n-5}T_{n-3}\zeta_nT_1\cdots T_{k-2})b)).
\end{aligned}
$$

Since $T_1\cdots T_{j-1}T_1\cdots T_{n-5}=(-1)^{(n+1)j}T_2\cdots T_{n-5}T_1\cdots T_{j-2}$, we can express this element as

$$
\begin{aligned}
&P_{2,n-2}((T_1\cdots T_{n-3}T_{n-4}+(-1)^n\zeta_nT_1\cdots T_{n-5}T_{n-3}T_{n-4})b)\\
&+\sum_{j=2}^{n-4}((-1)^{n(j+1)}P_{2,n-2}(T_j\cdots T_{n-5}T_{n-3}\zeta_nT_1\cdots T_{j-2}b)\\
&\qquad+P_{j+1,n-2}((((-1)^nT_2\cdots T_{n-5}T_{n-3}T_1\cdots T_{j-2}\\
&\qquad\qquad+T_1\cdots T_{n-5}(T_{n-3}-T_{n-4})\zeta_nT_1\cdots T_{j-2}\\
&\qquad\qquad+(-1)^n\zeta_nT_1\cdots T_{n-5}T_{n-3}T_{n-4}\zeta_nT_1\cdots T_{j-2}\\
&\qquad\qquad+\sum_{k=2}^{n-4}(-1)^{n(j+k)}T_1\cdots T_{j-1}T_k\cdots T_{n-5}T_{n-3}\zeta_nT_1\cdots T_{k-2})b))\\
=&P_{2,n-2}((T_1\cdots T_{n-3}T_{n-4}+(-1)^n\zeta_nT_1\cdots T_{n-5}T_{n-3}T_{n-4})b)\\
&+\sum_{j=2}^{n-4}((-1)^{n+j}P_{2,n-2}(T_{n-3}\zeta_nT_1\cdots T_{j-2}T_j\cdots T_{n-5}b)\\
&\qquad+P_{j+1,n-2}((((-1)^nT_2\cdots T_{n-5}T_{n-3}(1-T_{n-4}\zeta_n)T_1\cdots T_{j-2}\\
&\qquad\qquad-T_1\cdots T_{n-5}T_{n-3}T_{n-4}T_1\cdots T_{j-2}\\
&\qquad\qquad+\sum_{k=2}^{n-4}(-1)^{nj+k}T_1\cdots T_{j-1}T_{n-3}\zeta_nT_1\cdots T_{k-2}T_k\cdots T_{n-5})b)).
\end{aligned}
$$

Now considering only the $P_{2,n-2}$ term, applying Lemma 3.3.2, we have

$$
\begin{aligned}
&T_1\cdots T_{n-3}T_{n-4}+(-1)^n\zeta_nT_1\cdots T_{n-5}T_{n-3}T_{n-4}\\
&+\sum_{j=2}^{n-4}(-1)^{n(j+1)}T_j\cdots T_{n-5}T_{n-3}\zeta_nT_1\cdots T_{j-2}\\
=&T_1\cdots T_{n-3}T_{n-4}\\
&+\sum_{k=1}^{n-5}(-1)^{n+k}T_1\cdots T_{k-1}T_{k+1}\cdots T_{n-5}T_{n-3}T_{n-4}-T_1\cdots T_{n-5}T_{n-4}T_{n-3}T_{n-4}\\
&+\sum_{j=2}^{n-4}(\sum_{k=1}^{j-2}(-1)^{n(j+1)+k}T_j\cdots T_{n-5}T_{n-3}T_1\cdots T_{k-1}T_{k+1}\cdots T_{j-2}
\end{aligned}
$$

$$+\sum_{k=j-1}^{n-4}(-1)^{n(j+1)+j}(k-1)T_j\cdots T_{n-5}T_{n-3}T_1\cdots T_{j-2}T_k)$$

$$=\sum_{k=1}^{n-5}(-1)^{n+k}T_1\cdots T_{k-1}T_{k+1}\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{j=2}^{n-4}(\sum_{k=1}^{j-2}(-1)^{(n+1)(j+1)+k}T_j\cdots T_{n-5}T_1\cdots T_{k-1}T_{k+1}\cdots T_{j-2}T_{n-3}$$

$$+\sum_{k=j-1}^{n-4}(-1)^{n+j+1}(k-1)T_1\cdots T_{j-2}T_j\cdots T_{n-5}T_kT_{n-3})$$

$$=\sum_{j=2}^{n-4}(-1)^{n+j+1}T_1\cdots T_{j-2}T_j\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{j=2}^{n-4}((-1)^{n+j}T_1\cdots T_{j-2}T_j\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{k=1}^{j-2}(-1)^{j+k+1}T_1\cdots T_{k-1}T_{k+1}\cdots T_{j-2}T_j\cdots T_{n-5}T_{n-3}$$

$$+\sum_{k=j-1}^{n-5}(-1)^{n+j+1}(k-1)T_1\cdots T_{j-2}T_j\cdots T_{n-5}T_kT_{n-3})$$

$$=\sum_{j=2}^{n-4}(\sum_{k=1}^{j-2}(-1)^{j+k+1}T_1\cdots T_{k-1}T_{k+1}\cdots T_{j-2}T_j\cdots T_{n-5}T_{n-3}$$

$$+\sum_{k=j-1}^{n-6}(-1)^{n+j+1}(k-1)T_1\cdots T_{j-2}T_j\cdots T_{n-5}T_kT_{n-3})$$

$$=\sum_{j=2}^{n-4}(\sum_{k=1}^{j-2}(-1)^{j+k+1}T_1\cdots T_{k-1}T_{k+1}\cdots T_{j-2}T_j\cdots T_{n-5}T_{n-3}$$

$$+\sum_{k=j-1}^{n-6}(-1)^{j+k}(k-1)T_1\cdots T_{j-2}T_j\cdots T_kT_{k+2}\cdots T_{n-5}T_{n-3}$$

$$+\sum_{k=j-1}^{n-6}(-1)^{j+k}(k-1)T_1\cdots T_{j-2}T_j\cdots T_kT_k\cdots T_{n-5}T_{n-3})$$

$$=\sum_{j=2}^{n-4}(\sum_{k=1}^{j-2}(-1)^{j+k+1}T_1\cdots T_{k-1}T_{k+1}\cdots T_{j-2}T_j\cdots T_{n-5}T_{n-3}$$

$$+\sum_{k=j-1}^{n-6}(-1)^{j+k}(k-1)T_1\cdots T_{j-2}T_j\cdots T_kT_{k+2}\cdots T_{n-5}T_{n-3}$$

$$+\sum_{k=j}^{n-6}(-1)^{j+k}(k-1)T_1\cdots T_{j-2}T_j\cdots T_{k-1}T_{k+1}\cdots T_{n-5}T_{n-3}$$

$$-(j+1)T_1\cdots T_{n-5}T_{n-3})$$

$$=\sum_{j=2}^{n-4}(\sum_{k=1}^{j-2}(-1)^{j+k+1}T_1\cdots T_{k-1}T_{k+1}\cdots T_{j-2}T_j\cdots T_{n-5}T_{n-3}$$

$$+\sum_{k=j}^{n-5}(-1)^{j+k+1}(k+1)T_1\cdots T_{j-2}T_j\cdots T_{k-1}T_{k+1}\cdots T_{n-5}T_{n-3}$$

$$
\begin{aligned}
&+\sum_{k=j}^{n-5}(-1)^{j+k}(k-1)T_1\cdots T_{j-2}T_j\cdots T_{k-1}T_{k+1}\cdots T_{n-5}T_{n-3}\\
&-(j+1)T_1\cdots T_{n-5}T_{n-3})\\
=&\sum_{j=2}^{n-4}(\sum_{k=1}^{j-2}(-1)^{j+k+1}T_1\cdots T_{k-1}T_{k+1}\cdots T_{j-2}T_j\cdots T_{n-5}T_{n-3}\\
&+\sum_{k=j}^{n-5}(-1)^{j+k}T_1\cdots T_{j-2}T_j\cdots T_{k-1}T_{k+1}\cdots T_{n-5}T_{n-3}\\
&-(j+1)T_1\cdots T_{n-5}T_{n-3})\\
=&\sum_{j=2}^{n-4}(\sum_{k=j}^{n-5}(-1)^{j+k+1}T_1\cdots T_{j-2}T_j\cdots T_{k-1}T_{k+1}\cdots T_{n-5}T_{n-3}\\
&+\sum_{k=j}^{n-5}(-1)^{j+k}T_1\cdots T_{j-2}T_j\cdots T_{k-1}T_{k+1}\cdots T_{n-5}T_{n-3}\\
&-(j+1)T_1\cdots T_{n-5}T_{n-3}).
\end{aligned}
$$

Since $\sum_{j=2}^{n-4}(j+1)=0$ when 3 divides n (since we are in characteristic 3), we conclude that the coefficient of $P_{2,n-2}(b)$ is 0.

Now we will again use Lemma 3.3.2 to show that the coefficient of $P_{i,n-2}(b)$, for $3\le i\le n-4$, is 0. We will need the following identities:

$$T_{k+1}\cdots T_{n-5}T_k=(-1)^{n+k+1}T_k\cdots T_{n-5}+(-1)^{n+k+1}T_{k+2}\cdots T_{n-5};$$

$$(-1)^{(n+1)(i+1)}T_2\cdots T_{n-5}T_1\cdots T_{i-3}=T_1\cdots T_{i-2}T_1\cdots T_{n-5}=T_2\cdots T_{i-2}T_1\cdots T_{i-3}T_{i-1}\cdots T_{n-5};$$

for $2\le j\le i-2$, we have

$$
\begin{aligned}
T_1\cdots T_{i-2}T_1\cdots T_{j-2}T_j\cdots T_{n-5}&=(-1)^{n(i+1)+j+1}\sum_{k=2}^{j}T_kT_2\cdots T_{n-5}T_1\cdots T_{i-3};\\
T_2\cdots T_{n-5}T_1\cdots T_{j-2}T_j\cdots T_{i-3}&=(-1)^{n+j}\sum_{k=1}^{j}T_kT_2\cdots T_{n-5}T_1\cdots T_{i-3}.
\end{aligned}
$$

Thus, the coefficient of $P_{i,n-2}(b)$, for $3\le i\le n-4$, is

$$
\begin{aligned}
&(-1)^nT_2\cdots T_{n-5}T_{n-3}T_1\cdots T_{i-3}\\
&-T_1\cdots T_{n-5}T_{n-3}T_{n-4}T_1\cdots T_{i-3}\\
&+(-1)^{n+1}T_2\cdots T_{n-5}T_{n-3}T_{n-4}\zeta_nT_1\cdots T_{i-3}\\
&+\sum_{j=2}^{n-4}(-1)^{n(i+j)+j}T_1\cdots T_{i-2}T_{n-3}T_j\cdots T_{n-5}\zeta_nT_1\cdots T_{j-2}\\
=&(-1)^nT_2\cdots T_{n-5}T_{n-3}T_1\cdots T_{i-3}\\
&-T_1\cdots T_{n-5}T_{n-3}T_{n-4}T_1\cdots T_{i-3}\\
&+\sum_{k=1}^{i-3}(-1)^{n+k+1}T_2\cdots T_{n-5}T_{n-3}T_{n-4}T_1\cdots T_{k-1}T_{k+1}\cdots T_{i-3}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=i-2}^{n-4}(-1)^{n+i}(k-1)T_2\cdots T_{n-5}T_{n-3}T_{n-4}T_1\cdots T_{i-3}T_k\\
&+\sum_{j=2}^{n-4}(\sum_{k=1}^{j-2}(-1)^{n(i+j)+j+k}T_1\cdots T_{i-2}T_{n-3}T_j\cdots T_{n-5}T_1\cdots T_{k-1}T_{k+1}\cdots T_{j-2}\\
&\qquad+\sum_{k=j-1}^{n-4}(-1)^{n(i+j)}(k-1)T_1\cdots T_{i-2}T_{n-3}T_j\cdots T_{n-5}T_1\cdots T_{j-2}T_k)\\
=&-T_1\cdots T_{n-5}T_1\cdots T_{i-3}T_{n-3}T_{n-4}\\
&+\sum_{k=1}^{i-3}(-1)^{n+k+1}T_2\cdots T_{n-5}T_1\cdots T_{k-1}T_{k+1}\cdots T_{i-3}T_{n-3}T_{n-4}\\
&+\sum_{k=i-2}^{n-6}(-1)^{ni+1}(k-1)T_1\cdots T_{i-2}T_1\cdots T_{n-5}T_kT_{n-3}T_{n-4}\\
&+\sum_{j=2}^{n-4}(\sum_{k=1}^{j-2}(-1)^{ni+j+k+1}T_1\cdots T_{i-2}T_1\cdots T_{k-1}T_{k+1}\cdots T_{j-2}T_j\cdots T_{n-5}T_{n-3}\\
&\qquad+\sum_{k=j-1}^{n-6}(-1)^{n(i+1)+j+1}(k-1)T_1\cdots T_{i-2}T_1\cdots T_{j-2}T_j\cdots T_{n-5}T_kT_{n-3}\\
&\qquad+(-1)^{n(i+j+1)}T_1\cdots T_{i-2}T_j\cdots T_{n-5}T_1\cdots T_{j-2}T_{n-3}T_{n-4})\\
=&-T_1\cdots T_{n-5}T_1\cdots T_{i-3}T_{n-3}T_{n-4}\\
&+\sum_{k=1}^{i-3}(-1)^{n+k+1}T_2\cdots T_{n-5}T_1\cdots T_{k-1}T_{k+1}\cdots T_{i-3}T_{n-3}T_{n-4}\\
&+\sum_{k=i-2}^{n-6}(-1)^{n(i+1)+k}(k-1)T_1\cdots T_{i-2}T_1\cdots T_k(1+T_kT_{k+1})T_{k+2}\cdots T_{n-5}T_{n-3}T_{n-4}\\
&+\sum_{j=2}^{n-4}(\sum_{k=1}^{j-2}(-1)^{ni+j+k+1}T_1\cdots T_{i-2}T_1\cdots T_{k-1}T_{k+1}\cdots T_{j-2}T_j\cdots T_{n-5}T_{n-3}\\
&\qquad+\sum_{k=j-1}^{n-6}(-1)^{ni+j+k}(k-1)T_1\cdots T_{i-2}T_1\cdots T_{j-2}T_j\cdots T_k(1+T_kT_{k+1})T_{k+2}\cdots T_{n-5}T_{n-3}\\
&\qquad+(-1)^{n(i+j+1)}T_1\cdots T_{i-2}T_j\cdots T_{n-5}T_1\cdots T_{j-2}T_{n-3}T_{n-4})\\
=&-T_1\cdots T_{n-5}T_1\cdots T_{i-3}T_{n-3}T_{n-4}\\
&+\sum_{k=1}^{i-3}(-1)^{n+k+1}T_2\cdots T_{n-5}T_1\cdots T_{k-1}T_{k+1}\cdots T_{i-3}T_{n-3}T_{n-4}\\
&+\sum_{k=i-2}^{n-6}(-1)^{n(i+1)+k+1}T_1\cdots T_{i-2}T_1\cdots T_kT_{k+2}\cdots T_{n-5}T_{n-3}T_{n-4}\\
&+(-1)^{n(i+1)+i}iT_1\cdots T_{i-2}T_1\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\\
&+\sum_{j=2}^{n-4}(\sum_{k=1}^{j-2}(-1)^{ni+j+k+1}T_1\cdots T_{i-2}T_1\cdots T_{k-1}T_{k+1}\cdots T_{j-2}T_j\cdots T_{n-5}T_{n-3}\\
&\qquad+\sum_{k=j-1}^{n-6}(-1)^{ni+j+k+1}T_1\cdots T_{i-2}T_1\cdots T_{j-2}T_j\cdots T_kT_{k+2}\cdots T_{n-5}T_{n-3}\\
&\qquad+(-1)^{ni+1}(j+1)T_1\cdots T_{i-2}T_1\cdots T_{n-5}T_{n-3}
\end{aligned}
$$

$$+ (-1)^{n(i+j+1)}T_1\cdots T_{i-2}T_j\cdots T_{n-5}T_1\cdots T_{j-2}T_{n-3}T_{n-4})$$

$$=(-1)^{n(i+1)+i}iT_2\cdots T_{i-2}T_1\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+ (-1)^{ni+1}T_1\cdots T_{i-2}T_1\cdots T_{n-5}T_{n-3}(\sum_{j=2}^{n-4}(j+1))$$

$$+\sum_{k=2}^{i-2}(-1)^{n+k}T_2\cdots T_{n-5}T_1\cdots T_{k-2}T_k\cdots T_{i-3}T_{n-3}T_{n-4}$$

$$+\sum_{k=i}^{n-4}(-1)^{n(i+1)+k+1}T_1\cdots T_{i-2}T_1\cdots T_{k-2}T_k\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{j=2}^{i-2}(-1)^{n(i+1)+j}T_1\cdots T_{i-2}T_1\cdots T_{j-2}T_j\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{j=i}^{n-4}(-1)^{n(i+1)+j}T_1\cdots T_{i-2}T_1\cdots T_{j-2}T_j\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{j=2}^{n-4}\sum_{k=1}^{j-2}(-1)^{ni+j+k+1}T_1\cdots T_{i-2}T_1\cdots T_{k-1}T_{k+1}\cdots T_{j-2}T_j\cdots T_{n-5}T_{n-3}$$

$$+\sum_{k=2}^{n-4}\sum_{j=1}^{k-2}(-1)^{ni+j+k}T_1\cdots T_{i-2}T_1\cdots T_{j-1}T_{j+1}\cdots T_{k-2}T_k\cdots T_{n-5}T_{n-3}$$

$$=(-1)^{n(i+1)+i}iT_2\cdots T_{i-2}T_1\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{k=2}^{i-2}(-1)^{n+k}T_2\cdots T_{n-5}T_1\cdots T_{k-2}T_k\cdots T_{i-3}T_{n-3}T_{n-4}$$

$$+\sum_{j=2}^{i-2}(-1)^{n(i+1)+j}T_1\cdots T_{i-2}T_1\cdots T_{j-2}T_j\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$=(-1)^{n(i+1)+i}iT_2\cdots T_{i-2}T_1\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{k=2}^{i-2}\sum_{j=1}^{k}T_jT_2\cdots T_{n-5}T_1\cdots T_{i-3}T_{n-3}T_{n-4}$$

$$-\sum_{j=2}^{i-2}\sum_{k=2}^{j}T_kT_2\cdots T_{n-5}T_1\cdots T_{i-3}T_{n-3}T_{n-4}$$

$$=(-1)^{n(i+1)+i}iT_2\cdots T_{i-2}T_1\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+ (-1)^{(n+1)(i+1)}\sum_{k=2}^{i-2}T_2\cdots T_{i-2}T_1\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$=(-1)^{n(i+1)+i}iT_2\cdots T_{i-2}T_1\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+ (-1)^{(n+1)(i+1)}iT_2\cdots T_{i-2}T_1\cdots T_{n-5}T_{n-3}T_{n-4}=0.$$

Case 2: For $1<i\le n-3$, since $T_1\zeta_nT_1=\zeta_nT_1\zeta_n$,

$$t_1B_i(b)=t_1P_{1,i+1}(b)+\sum_{j=2}^{i}(-1)^{(n+1)(j+1)}t_1P_{j,i+1}(\zeta_nT_1\cdots T_{j-2}b)$$

$$-\sum_{j=i+2}^{n-3}((-1)^{i(n+j)}t_1P_{i+1,j}(T_{n-3}\zeta_nT_1\cdots T_{j-3}b)$$

$$
\begin{aligned}
&\qquad + (-1)^{(n+1)j} t_1 P_{i+1,n-2}(T_{j-1}\cdots T_{n-5}T_{n-3}T_{n-4}\zeta_n T_1\cdots T_{j-3}b)\\
&\qquad + (-1)^{i(n+j)} t_1 P_{j,n-2}(T_{i+1}\cdots T_{n-3}T_{n-4}\zeta_n T_1\cdots T_{j-3}b))\\
&= (-1)^{n+i+1}P_{2,i+1}(b) + (-1)^i P_{1,i+1}(\zeta_n b)\\
&\quad + \sum_{j=3}^{i}(-1)^{n(j+1)+i}P_{j,i+1}(T_1\zeta_n T_1\cdots T_{j-2}b)\\
&\quad - \sum_{j=i+2}^{n-3}((-1)^{i(n+j)+i+j+1}P_{i+1,j}(T_1T_{n-3}\zeta_n T_1\cdots T_{j-3}b)\\
&\qquad + (-1)^{(n+1)j+n+i+1}P_{i+1,n-2}(T_1T_{j-1}\cdots T_{n-5}T_{n-3}T_{n-4}\zeta_n T_1\cdots T_{j-3}b)\\
&\qquad + (-1)^{(i+1)(n+j)}P_{j,n-2}(T_1T_{i+1}\cdots T_{n-5}T_{n-4}T_{n-3}T_{n-4}\zeta_n T_1\cdots T_{j-3}b))\\
&= (-1)^i P_{1,i+1}(\zeta_n b) + \sum_{j=2}^{i}(-1)^{(n+1)(j+1)+i}P_{j,i+1}(\zeta_n T_1\cdots T_{j-2}\zeta_n b)\\
&\quad - \sum_{j=i+2}^{n-3}((-1)^{i(n+j)+i}P_{i+1,j}(T_{n-3}\zeta_n T_1\cdots T_{j-3}\zeta_n b)\\
&\qquad + (-1)^{(n+1)j+i}P_{i+1,n-2}(T_{j-1}\cdots T_{n-5}T_{n-3}T_{n-4}\zeta_n T_1\cdots T_{j-3}\zeta_n b)\\
&\qquad + (-1)^{i(n+j)+i}P_{j,n-2}(T_{i+1}\cdots T_{n-5}T_{n-4}T_{n-3}T_{n-4}\zeta_n T_1\cdots T_{j-3}\zeta_n b))\\
&= (-1)^i B_i(\zeta_n b).
\end{aligned}
$$

Case 3: For $1 < k < i \leq n-3$,

$$
\begin{aligned}
t_kB_i(b) &= t_kP_{1,i+1}(b) + \sum_{j=2}^{i}(-1)^{(n+1)(j+1)}t_kP_{j,i+1}(\zeta_n T_1\cdots T_{j-2}b)\\
&\quad - \sum_{j=i+2}^{n-3}((-1)^{i(n+j)}t_kP_{i+1,j}(T_{n-3}\zeta_n T_1\cdots T_{j-3}b)\\
&\qquad + (-1)^{(n+1)j}t_kP_{i+1,n-2}(T_{j-1}\cdots T_{n-5}T_{n-3}T_{n-4}\zeta_n T_1\cdots T_{j-3}b)\\
&\qquad + (-1)^{i(n+j)}t_kP_{j,n-2}(T_{i+1}\cdots T_{n-3}T_{n-4}\zeta_n T_1\cdots T_{j-3}b))\\
&= (-1)^i P_{1,i+1}(T_{k-1}b) + \sum_{j=2}^{k-1}(-1)^{n(j+1)+i}P_{j,i+1}(T_{k-1}\zeta_n T_1\cdots T_{j-2}b)\\
&\quad + (-1)^{(n+1)k+i}P_{k+1,i+1}(\zeta_n T_1\cdots T_{k-2}b)\\
&\quad + (-1)^{(n+1)(k+1)+i}P_{k,i+1}(\zeta_n T_1\cdots T_{k-1}b)\\
&\quad + \sum_{j=k+2}^{i}(-1)^{n(j+1)+i}P_{j,i+1}(T_k\zeta_n T_1\cdots T_{j-2}b)\\
&\quad - \sum_{j=i+2}^{n-3}((-1)^{i(n+j)+i+j+1}P_{i+1,j}(T_kT_{n-3}\zeta_n T_1\cdots T_{j-3}b)\\
&\qquad + (-1)^{(n+1)(j+1)+i}P_{i+1,n-2}(T_kT_{j-1}\cdots T_{n-5}T_{n-3}T_{n-4}\zeta_n T_1\cdots T_{j-3}b)\\
&\qquad + (-1)^{(i+1)(n+j)}P_{j,n-2}(T_kT_{i+1}\cdots T_{n-5}T_{n-4}T_{n-3}T_{n-4}\zeta_n T_1\cdots T_{j-3}b))\\
&= (-1)^i P_{1,i+1}(T_{k-1}b) + \sum_{j=2}^{i}(-1)^{(n+1)(j+1)+i}P_{j,i+1}(\zeta_n T_1\cdots T_{j-2}T_{k-1}b)
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{j=i+2}^{n-3} ((-1)^{i(n+j)+i} P_{i+1,j}(T_{n-3}\zeta_n T_1 \cdots T_{j-3}T_{k-1}b) \\
&\qquad + (-1)^{(n+1)j+i} P_{i+1,n-2}(T_{j-1} \cdots T_{n-5}T_{n-3}T_{n-4}\zeta_n T_1 \cdots T_{j-3}T_{k-1}b) \\
&\qquad + (-1)^{i(n+j)+i} P_{j,n-2}(T_{i+1} \cdots T_{n-5}T_{n-4}T_{n-3}T_{n-4}\zeta_n T_1 \cdots T_{j-3}T_{k-1}b)) \\
&= (-1)^i B_i(T_{k-1}b).
\end{aligned}
$$

Case 6: For $i+1 < k < n-4$, since $t_{k-1}B_{n-3}(b) = (-1)^{n+1}B_{n-3}(T_{k-2}b)$,

$$
\begin{aligned}
t_k B_i(b) =& t_k t_{i+1} t_{i+2} \cdots t_{n-3} B_{n-3}(b) \\
=& (-1)^{i+k} t_{i+1} t_{i+2} \cdots t_{k-2} t_k t_{k-1} t_k t_{k+1} t_{k+2} \cdots t_{n-3} B_{n-3}(b) \\
=& (-1)^{i+k} t_{i+1} t_{i+2} \cdots t_{k-2} t_{k-1} t_k t_{k-1} t_{k+1} t_{k+2} \cdots t_{n-3} B_{n-3}(b) \\
=& (-1)^{n+i+1} t_{i+1} t_{i+2} \cdots t_{n-3} t_{k-1} B_{n-3}(b) \\
=& (-1)^i t_{i+1} t_{i+2} \cdots t_{n-3} B_{n-3}(T_{k-2}b) \\
=& (-1)^i B_i(T_{k-2}b).
\end{aligned}
$$

Case 7:

$$
\begin{aligned}
t_{n-2}B_{n-4}(b) =& t_{n-2}P_{1,n-3}(b) + \sum_{j=2}^{n-4} (-1)^{(n+1)(j+1)} t_{n-2} P_{j,n-3}(\zeta_n T_1 \cdots T_{j-2}b) \\
=& (-1)^n P_{1,n-3}((T_{n-4} - T_{n-3} + T_{n-1})b) \\
&- P_{1,n-2}((1 + (T_{n-4} - T_{n-3})T_{n-1})b) \\
&+ \sum_{j=2}^{n-4} (-1)^{(n+1)(j+1)} ((-1)^{n+j+1} P_{j,n-3}((T_{n-4} - T_{n-3} + T_{n-1})\zeta_n T_1 \cdots T_{j-2}b) \\
&\qquad - P_{j,n-2}((1 + (T_{n-4} - T_{n-3})T_{n-1})\zeta_n T_1 \cdots T_{j-2}b)) \\
=& (-1)^n P_{1,n-3}((T_{n-4} - T_{n-3} + T_{n-1})b) \\
&- P_{1,n-2}((1 + (T_{n-4} - T_{n-3})T_{n-1})b) \\
&+ \sum_{j=2}^{n-4} (-1)^{(n+1)(j+1)} ((-1)^n P_{j,n-3}(\zeta_n T_1 \cdots T_{j-2}(T_{n-4} - T_{n-3} + T_{n-1})b) \\
&\qquad - P_{j,n-2}(\zeta_n T_1 \cdots T_{j-2}(1 + (T_{n-4} - T_{n-3})T_{n-1})b) \\
=& (-1)^n B_{n-4}((T_{n-4} - T_{n-3} + T_{n-1})b) - B_{n-3}((1 + (T_{n-4} - T_{n-3})T_{n-1})b).
\end{aligned}
$$

It follows that, for $1 \leq i < n-4$,

$$
\begin{aligned}
t_{n-2}B_i(b) =& t_{n-2} t_{i+1} t_{i+2} \cdots t_{n-3} B_{n-3}(b) \\
=& (-1)^{n+i} t_{i+1} t_{i+2} \cdots t_{n-4} t_{n-2} t_{n-3} B_{n-3}(b) \\
=& (-1)^{n+i} t_{i+1} t_{i+2} \cdots t_{n-4} t_{n-2} B_{n-4}(b) \\
=& (-1)^i t_{i+1} t_{i+2} \cdots t_{n-4} B_{n-4}((T_{n-4} - T_{n-3} + T_{n-1})b) \\
&+ (-1)^{n+i+1} t_{i+1} t_{i+2} \cdots t_{n-4} B_{n-3}((1 + (T_{n-4} - T_{n-3})T_{n-1})b)
\end{aligned}
$$

$$
\begin{aligned}
&=(-1)^i B_i((T_{n-4} - T_{n-3} + T_{n-1})b) \\
&\quad + (-1)^i t_{i+1} t_{i+2} \cdots t_{n-5} B_{n-3}(T_{n-5}(1 + (T_{n-4} - T_{n-3})T_{n-1})b) \\
&= \dots \\
&=(-1)^i B_i((T_{n-4} - T_{n-3} + T_{n-1})b) \\
&\quad + (-1)^{n+i+1+(n+1)(n+i)} B_{n-3}(T_i \cdots T_{n-5}(1 + (T_{n-4} - T_{n-3})T_{n-1})b) \\
&=(-1)^i B_i((T_{n-4} - T_{n-3} + T_{n-1})b) \\
&\quad + (-1)^{n(i+1)+1} B_{n-3}(T_i \cdots T_{n-5}(1 + (T_{n-4} - T_{n-3})T_{n-1})b).
\end{aligned}
$$

Case 8:

$$
\begin{aligned}
t_{n-2} B_{n-3}(b) &= (-1)^{n+1} P_{1,n-2}(T_{n-1} b) + \sum_{j=2}^{n-4} (-1)^{nj+1} P_{j,n-2}(T_{n-1} \zeta_n T_1 \cdots T_{j-2} b) \\
&= (-1)^{n+1} P_{1,n-2}(T_{n-1} b) + \sum_{j=2}^{n-4} (-1)^{(n+1)j} P_{j,n-2}(\zeta_n T_1 \cdots T_{j-2} T_{n-1} b) \\
&= (-1)^{n+1} B_{n-3}(T_{n-1} b).
\end{aligned}
$$

Case 9: For $1 \leq i \leq n-3$,

$$
\begin{aligned}
t_{n-1} B_i(b) &= (-1)^i P_{1,i+1}(T_{n-1} b) + \sum_{j=2}^{i} (-1)^{(n+1)(j+1)+i+j+1} P_{j,i+1}(T_{n-1} \zeta_n T_1 \cdots T_{j-2} b) \\
&\quad - \sum_{j=i+2}^{n-3} ((-1)^{i(n+j)+i+j+1} P_{i+1,j}(T_{n-1} T_{n-3} \zeta_n T_1 \cdots T_{j-3} b) \\
&\qquad + (-1)^{(n+1)j+n+i+1} P_{i+1,n-2}(T_{n-1} T_{j-1} \cdots T_{n-5} T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{j-3} b) \\
&\qquad + (-1)^{(i+1)(n+j)} P_{j,n-2}(T_{n-1} T_{i+1} \cdots T_{n-5} T_{n-4} T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{j-3} b)) \\
&= (-1)^i P_{1,i+1}(T_{n-1} b) + \sum_{j=2}^{i} (-1)^{(n+1)(j+1)+i} P_{j,i+1}(\zeta_n T_1 \cdots T_{j-2} T_{n-1} b) \\
&\quad - \sum_{j=i+2}^{n-3} ((-1)^{i(n+j+1)} P_{i+1,j}(T_{n-3} \zeta_n T_1 \cdots T_{j-3} T_{n-1} b) \\
&\qquad + (-1)^{(n+1)j+i} P_{i+1,n-2}(T_{j-1} \cdots T_{n-5} T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{j-3} T_{n-1} b) \\
&\qquad + (-1)^{i(n+j)+i} P_{j,n-2}(T_{i+1} \cdots T_{n-5} T_{n-4} T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{j-3} T_{n-1} b)) \\
&= (-1)^i B_i(T_{n-1} b).
\end{aligned}
$$

□

To prove Theorem 3.3.5, we will need the identity involving $\chi_n := \zeta_n - T_{n-3}$ given in Lemma 3.3.4: For $1 \leq i-1 \leq j \leq n-3$,

$$
\begin{aligned}
T_i \cdots T_j \chi_n =& \sum_{k=2}^{i-1} (-1)^{i+j+1} (k-1) T_k T_i \cdots T_j \\
&+ \sum_{k=i}^{j} (-1)^{j+k} T_i \cdots T_{k-1} T_{k+1} \cdots T_j
\end{aligned}
$$

$$+\sum_{k=j+1}^{n-3}(k-1)T_i\cdots T_jT_k.$$

Note that χ_n anticommutes with T_j for all $j\in\{2,\dots,n-1\}\backslash\{n-2\}$ and $\chi_n^2=0$. Moreover,

$$\begin{aligned}\chi_nT_1\chi_n=&(\zeta_n-T_{n-3})T_1(\zeta_n-T_{n-3})\\ =&\zeta_nT_1\zeta_n-\zeta_nT_1T_{n-3}-T_{n-3}T_1\zeta_n-T_1\\ =&-\zeta_n+T_{n-3}=-\chi_n.\end{aligned}$$

Proof of Lemma 3.3.4:

Proof. When $i=2$, $j=1$, we have

$$\begin{aligned}T_2\cdots T_j\chi_n=&\sum_{k=2}^{n-3}(k-1)T_k\\ =&\sum_{k=2}^{j}(-1)^{j+k}T_2\cdots T_{k-1}T_{k+1}\cdots T_j\\ &+\sum_{k=j+1}^{n-3}(k-1)T_2\cdots T_jT_k.\end{aligned}$$

Now if the equality holds for $1\le i-1\le j\le n-4$, then

$$\begin{aligned}T_i\cdots T_{j+1}\chi_n=&\sum_{k=2}^{i-1}(-1)^{i+j}(k-1)T_kT_i\cdots T_{j+1}\\ &+\sum_{k=i}^{j}(-1)^{j+k+1}T_i\cdots T_{k-1}T_{k+1}\cdots T_{j+1}\\ &+\sum_{k=j+1}^{n-3}(1-k)T_i\cdots T_jT_kT_{j+1}\\ =&\sum_{k=2}^{i-1}(-1)^{i+j}(k-1)T_kT_i\cdots T_{j+1}\\ &+\sum_{k=i}^{j+1}(-1)^{j+k+1}T_i\cdots T_{k-1}T_{k+1}\cdots T_{j+1}\\ &-(j+1)T_i\cdots T_j\\ &+\sum_{k=j+2}^{n-3}(1-k)T_i\cdots T_jT_kT_{j+1}.\end{aligned}$$

When $j=n-4$, since $3|n$ (and we are in characteristic 3), we have

$$\begin{aligned}T_i\cdots T_{n-3}\chi_n=&\sum_{k=2}^{i-1}(-1)^{n+i}(k-1)T_kT_i\cdots T_{n-3}\\ &+\sum_{k=i}^{n-3}(-1)^{n+k+1}T_i\cdots T_{k-1}T_{k+1}\cdots T_{n-3}.\end{aligned}$$

If $j<n-4$, then

$$T_i\cdots T_{j+1}\chi_n=\sum_{k=2}^{i-1}(-1)^{i+j}(k-1)T_kT_i\cdots T_{j+1}$$

$$
\begin{aligned}
&+\sum_{k=i}^{j+1}(-1)^{j+k+1}T_i\cdots T_{k-1}T_{k+1}\cdots T_{j+1}\\
&-(j+1)T_i\cdots T_j\\
&-(j+1)T_i\cdots T_jT_{j+2}T_{j+1}\\
&+\sum_{k=j+3}^{n-3}(k-1)T_i\cdots T_{j+1}T_k\\
=&\sum_{k=2}^{i-1}(-1)^{i+j}(k-1)T_kT_i\cdots T_{j+1}\\
&+\sum_{k=i}^{j+1}(-1)^{j+k+1}T_i\cdots T_{k-1}T_{k+1}\cdots T_{j+1}\\
&+\sum_{k=j+2}^{n-3}(k-1)T_i\cdots T_{j+1}T_k.
\end{aligned}
$$

□

We will also use the fact that when $j \le k$, we have

$$(T_j\cdots T_k)^2 = (-1)^{\binom{k-j+1}{2}}\sum_I T_I$$

where I ranges over all evenly sized subsets $\{i_1,\ldots,i_m\} \subseteq \{j,\ldots,k\}$ and T_I is the product $T_{i_1}T_{i_2}\cdots T_{i_m}$ with $i_1 < i_2 < \cdots < i_m$, and therefore

$$T_j(T_j\cdots T_k)^2 = (T_j\cdots T_k)^2T_k.$$

We can prove this identity by induction on k: when $j = k$, both sides are equal since the only evenly sized subset of $\{j,\ldots,k\}$ is $\emptyset$ and $T_\emptyset = T_j^2$ is the identity matrix. Now suppose $j < k$ and $(T_j\cdots T_{k-1})^2 = (-1)^{\binom{k-j}{2}}\sum_J T_J$, where J is the set of all evenly sized subsets of $\{j,\ldots,k-1\}$. Then we have

$$
\begin{aligned}
(T_j\cdots T_k)^2 =&(-1)^{j+k}(T_j\cdots T_{k-1})^2(1+T_{k-1}T_k)\\
=&(-1)^{\binom{k-j+1}{2}}\sum_I T_I
\end{aligned}
$$

as desired. Alternatively, we can write

$$(T_j\cdots T_k)^2 = (-1)^{\binom{k-j+1}{2}}\prod_{i=j}^{k-1}(I+T_iT_{i+1}).$$

Proof of Theorem 3.3.5:

Proof. Case 1: Recall that for $j = 1,\ldots,n-3$, we have

$$
\begin{aligned}
B_j(b) = P_{1,j+1}(b) &+ \sum_{k=2}^{j}(-1)^{(n+1)(k+1)}P_{k,j+1}(\zeta_nT_1\cdots T_{k-2}b)\\
&-\sum_{k=j+2}^{n-3}((-1)^{j(n+k)}P_{j+1,k}(T_{n-3}\zeta_nT_1\cdots T_{k-3}b)
\end{aligned}
$$

$$+ (-1)^{(n+1)k} P_{j+1,n-2}(T_{k-1} \cdots T_{n-5} T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{k-3} b)$$

$$+ (-1)^{j(n+k)} P_{k,n-2}(T_{j+1} \cdots T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{k-3} b)).$$

The action of t_1 on $B_{n-1}(b)$, for $b \in M$, is given by

$$\begin{aligned}
t_1 B_{n-1}(b) = &\sum_{j=4}^{n-2} (-1)^{j+1} P_{1j}(T_2 \cdots T_{j-3} \chi_n b) \\
&+ \sum_{j=4}^{n-2} (-1)^j P_{3j}(T_1 T_{n-3} T_2 \cdots T_{j-3} \chi_n b) \\
&+ \sum_{i=4}^{n-4} \sum_{j=i+1}^{n-2} (-1)^{n(i+1)+i+j} P_{ij}(T_1(1 + T_2 T_{n-3}) T_3 \cdots T_{i-2} T_2 \cdots T_{j-3} \chi_n b) \\
&+ \sum_{i=4}^{n-4} \sum_{x=4}^{i} P_{i,n-2}(((-1)^{(n+1)(i+1)+x} T_1 T_3 \cdots T_{i-2} T_2 \cdots T_{x-3} T_{x-1} \cdots T_{n-3} T_{n-4} \\
&\qquad + (-1)^{ni+x+1} T_1 \cdots T_{i-2} T_2 \cdots T_{x-3} T_{x-1} \cdots T_{n-5} T_{n-3} T_{n-4}) \chi_n b) \\
= &\sum_{j=3}^{n-3} ((-1)^j B_j(T_2 \cdots T_{j-2} \chi_n b) \\
&\quad + \sum_{k=2}^{j} (-1)^{n+(n+1)k+j} P_{k,j+1}(\zeta_n T_1 \cdots T_{k-2} T_2 \cdots T_{j-2} \chi_n b) \\
&\quad + \sum_{k=j+2}^{n-3} ((-1)^{(n+k+1)j} P_{j+1,k}(T_{n-3} \zeta_n T_1 \cdots T_{k-3} T_2 \cdots T_{j-2} \chi_n b) \\
&\qquad + (-1)^{(n+1)k+j} P_{j+1,n-2}(T_{k-1} \cdots T_{n-5} T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{k-3} T_2 \cdots T_{j-2} \chi_n b) \\
&\qquad + (-1)^{(n+k+1)j} P_{k,n-2}(T_{j+1} \cdots T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{k-3} T_2 \cdots T_{j-2} \chi_n b))) \\
&+ \sum_{j=4}^{n-2} (-1)^j P_{3j}(T_1 T_{n-3} T_2 \cdots T_{j-3} \chi_n b) \\
&+ \sum_{i=4}^{n-4} \sum_{j=i+1}^{n-2} (-1)^{n(i+1)+i+j} P_{ij}(T_1(1 + T_2 T_{n-3}) T_3 \cdots T_{i-2} T_2 \cdots T_{j-3} \chi_n b) \\
&+ \sum_{i=4}^{n-4} \sum_{x=4}^{i} P_{i,n-2}(((-1)^{(n+1)(i+1)+x} T_1 T_3 \cdots T_{i-2} T_2 \cdots T_{x-3} T_{x-1} \cdots T_{n-3} T_{n-4} \\
&\qquad + (-1)^{ni+x+1} T_1 \cdots T_{i-2} T_2 \cdots T_{x-3} T_{x-1} \cdots T_{n-5} T_{n-3} T_{n-4}) \chi_n b).
\end{aligned}$$

Adding $B_{n-1}(T_{n-3} b) + \sum_{j=3}^{n-3} (-1)^{j+1} B_j(T_2 \cdots T_{j-2} \chi_n b)$, we get

$$\begin{aligned}
&\sum_{j=4}^{n-2} (-1)^{n+1} P_{2j}(T_2 \cdots T_{j-3} \chi_n T_{n-3} b) \\
&+ \sum_{j=4}^{n-2} P_{3j}(-T_{n-3} T_2 \cdots T_{j-3} \chi_n T_{n-3} b) \\
&+ \sum_{i=4}^{n-4} \sum_{j=i+1}^{n-2} (-1)^{n(i+1)} P_{ij}((1 + T_2 T_{n-3}) T_3 \cdots T_{i-2} T_2 \cdots T_{j-3} \chi_n T_{n-3} b) \\
&+ \sum_{i=4}^{n-4} \sum_{x=4}^{i} P_{i,n-2}(((-1)^{ni+x+1} T_3 \cdots T_{i-2} T_2 \cdots T_{x-3} T_{x-1} \cdots T_{n-3} T_{n-4}
\end{aligned}$$

$$+ (-1)^{(n+1)(i+1)+x} T_2 \cdots T_{i-2} T_2 \cdots T_{x-3} T_{x-1} \cdots T_{n-5} T_{n-3} T_{n-4}) \chi_n T_{n-3} b)$$

$$+ \sum_{j=3}^{n-3} (\sum_{k=2}^{j} (-1)^{n+(n+1)k+j} P_{k,j+1}(\zeta_n T_1 \cdots T_{k-2} T_2 \cdots T_{j-2} \chi_n b)$$

$$+ \sum_{k=j+2}^{n-3} ((-1)^{(n+k+1)j} P_{j+1,k}(T_{n-3} \zeta_n T_1 \cdots T_{k-3} T_2 \cdots T_{j-2} \chi_n b)$$

$$+ (-1)^{(n+1)k+j} P_{j+1,n-2}(T_{k-1} \cdots T_{n-5} T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{k-3} T_2 \cdots T_{j-2} \chi_n b)$$

$$+ (-1)^{(n+k+1)j} P_{k,n-2}(T_{j+1} \cdots T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{k-3} T_2 \cdots T_{j-2} \chi_n b)))$$

$$+ \sum_{j=4}^{n-2} (-1)^j P_{3j}(T_1 T_{n-3} T_2 \cdots T_{j-3} \chi_n b)$$

$$+ \sum_{i=4}^{n-4} \sum_{j=i+1}^{n-2} (-1)^{n(i+1)+i+j} P_{ij}(T_1(1 + T_2 T_{n-3}) T_3 \cdots T_{i-2} T_2 \cdots T_{j-3} \chi_n b)$$

$$+ \sum_{i=4}^{n-4} \sum_{x=4}^{i} P_{i,n-2}(((-1)^{(n+1)(i+1)+x} T_1 T_3 \cdots T_{i-2} T_2 \cdots T_{x-3} T_{x-1} \cdots T_{n-3} T_{n-4}$$

$$+ (-1)^{ni+x+1} T_1 \cdots T_{i-2} T_2 \cdots T_{x-3} T_{x-1} \cdots T_{n-5} T_{n-3} T_{n-4}) \chi_n b).$$

To show that this expression is equal to zero, we change summation variables and rearrange inside M. The relations $\zeta_n T_1 \chi_n = -(1 + T_1 T_{n-3})\chi_n$ and $\zeta_n \chi_n = T_{n-3}\chi_n$ are enough to cancel the $P_{2j}(b)$ and the $P_{3j}(b)$ terms. We also need the identities $T_1 \cdots T_{j-3} T_2 \cdots T_{i-3} = (-1)^{i(j+1)} T_3 \cdots T_{i-2} T_1 \cdots T_{j-3}$, for $4 \le i < j \le n-3$, and $\chi_n T_1 = -T_1 \chi_n - I$ to cancel the $P_{ij}(b)$ terms. Then we have

$$\sum_{j=4}^{n-2} (-1)^{n+1} P_{2j}((\chi_n T_{n-3} + \zeta_n \chi_n) T_2 \cdots T_{j-3} b) + \sum_{j=4}^{n-2} P_{3j}((1 + \zeta_n T_1 + T_1 T_{n-3}) \chi_n T_2 \cdots T_{j-3} b)$$

$$+ \sum_{i=4}^{n-4} \sum_{j=i+1}^{n-2} (-1)^{n(i+1)+i+j} P_{ij}((-T_1(T_2 + T_{n-3})\chi_n) T_3 \cdots T_{i-2} T_2 \cdots T_{j-3} \chi_n b)$$

$$+ \sum_{i=4}^{n-4} (-1)^i P_{i,n-2}(T_{n-3} \zeta_n T_1 \cdots T_{n-5} T_2 \cdots T_{i-3} \chi_n b)$$

$$+ \sum_{i=4}^{n-4} \sum_{j=i+1}^{n-3} (-1)^{(n+1)j+i+1} P_{i,n-2}(T_{j-1} \cdots T_{n-5} T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{j-3} T_2 \cdots T_{i-3} \chi_n b)$$

$$+ \sum_{i=4}^{n-4} \sum_{j=3}^{i-2} (-1)^{(n+i+1)j} P_{i,n-2}(T_{j+1} \cdots T_{n-3} T_{n-4} \zeta_n T_1 \cdots T_{i-3} T_2 \cdots T_{j-2} \chi_n b)$$

$$+ \sum_{i=4}^{n-4} \sum_{x=4}^{i} P_{i,n-2}(((-1)^{n+(n+1)i+x} (-T_1 + T_2) T_3 \cdots T_{i-2} T_2 \cdots T_{x-3} T_{x-1} \cdots T_{n-3} T_{n-4}$$

$$+ (-1)^{ni+x} (1 - T_1 T_2) T_3 \cdots T_{i-2} T_2 \cdots T_{x-3} T_{x-1} \cdots T_{n-5} T_{n-3} T_{n-4}) \chi_n b).$$

Since $\chi_n^2 = 0$, we are left with only $P_{i,n-2}(b)$ terms. Replacing each ζ_n with $\chi_n + T_{n-3}$, using the relation $\chi_n T_1 \chi_n = -\chi_n$, and then rearranging the matrices T_i that anticommute, we get

$$\sum_{i=4}^{n-4} P_{i,n-2}((-1)^i T_{n-3} \chi_n T_1 \cdots T_{n-5} T_2 \cdots T_{i-3}$$

$$+ (-1)^i T_1 \cdots T_{n-5} T_2 \cdots T_{i-3}$$

$$
\begin{aligned}
&+ \sum_{j=i+1}^{n-3} ((-1)^{(n+1)j+i+1} T_{j-1} \cdots T_{n-5} T_{n-3} T_{n-4} \chi_n T_1 \cdots T_{j-3} T_2 \cdots T_{i-3} \\
&\qquad + (-1)^{(n+1)j+i+1} T_{j-1} \cdots T_{n-3} T_{n-4} T_1 \cdots T_{j-3} T_2 \cdots T_{i-3}) \\
&+ \sum_{j=3}^{i-2} ((-1)^{(n+1)j+i} T_{j+1} \cdots T_{n-3} T_{n-4} \chi_n T_1 T_3 \cdots T_{j-1} T_2 \cdots T_{i-3} \\
&\qquad + (-1)^{(n+1)j+i} T_{j+1} \cdots T_{n-5} T_{n-3} T_{n-4} T_1 T_3 \cdots T_{j-1} T_2 \cdots T_{i-3}) \\
&+ \sum_{j=4}^{i} ((-1)^{n+(n+1)i+j} (-T_1 + T_2) T_3 \cdots T_{i-2} T_2 \cdots T_{j-3} T_{j-1} \cdots T_{n-3} T_{n-4} \\
&\qquad + (-1)^{ni+j} (1 - T_1 T_2) T_3 \cdots T_{i-2} T_2 \cdots T_{j-3} T_{j-1} \cdots T_{n-5} T_{n-3} T_{n-4}) \chi_n b) \\
= &\sum_{i=4}^{n-4} P_{i,n-2} ((-1)^{i+1} T_{n-3} T_2 \cdots T_{n-5} T_2 \cdots T_{i-3} \\
&+ (-1)^i T_1 \cdots T_{n-5} T_2 \cdots T_{i-3} \\
&+ \sum_{j=i}^{n-4} ((-1)^{n+i+j} T_1 \cdots T_{j-2} T_j \cdots T_{n-3} T_{n-4} T_2 \cdots T_{i-3} \\
&\qquad + (-1)^{i+j+1} T_2 \cdots T_{j-2} T_j \cdots T_{n-5} T_{n-3} T_{n-4} T_2 \cdots T_{i-3}) \\
&+ \sum_{j=4}^{i-1} ((-1)^{n+i+j} T_3 \cdots T_{j-2} T_j \cdots T_{n-3} T_{n-4} T_2 \cdots T_{i-3} \\
&\qquad + (-1)^{i+j+1} T_1 T_3 \cdots T_{j-2} T_j \cdots T_{n-5} T_{n-3} T_{n-4} T_2 \cdots T_{i-3}) \\
&+ \sum_{j=4}^{i} ((-1)^{n+(n+1)i+j} (-T_1 + T_2) T_3 \cdots T_{i-2} T_2 \cdots T_{j-3} T_{j-1} \cdots T_{n-3} T_{n-4} \\
&\qquad + (-1)^{ni+j} (1 - T_1 T_2) T_3 \cdots T_{i-2} T_2 \cdots T_{j-3} T_{j-1} \cdots T_{n-5} T_{n-3} T_{n-4}) \chi_n b).
\end{aligned}
$$

For $4 \leq j \leq i \leq n-4$, the identity $T_1 \cdots T_{i-2} T_2 \cdots T_{j-3} = (-1)^{ij} T_3 \cdots T_{j-2} T_1 \cdots T_{i-2}$ used above implies that $T_3 \cdots T_{i-2} T_2 \cdots T_{j-3} T_{j-1} \cdots T_{n-5} = (-1)^{ij} T_2 T_1 T_3 \cdots T_{j-2} T_1 \cdots T_{i-2} T_{j-1} \cdots T_{n-5}$. Then if $i = j$, we have $T_1 \cdots T_{i-2} T_{j-1} \cdots T_{n-5} = T_1 \cdots T_{n-5}$ so that

$$
T_2 T_1 T_3 \cdots T_{i-2} T_1 \cdots T_{i-2} T_{i-1} \cdots T_{n-5} = (-1)^{(n+1)i} T_3 \cdots T_{n-5} T_2 \cdots T_{i-3}.
$$

If $i > j$, then

$$
\begin{aligned}
T_1 \cdots T_{i-2} T_{j-1} \cdots T_{n-5} &= (-1)^{(n+j+1)(i+j+1)} T_1 \cdots T_{j-2} T_j \cdots T_{n-5} T_{j-1} \cdots T_{i-3} \\
&= (-1)^{ni+(i+1)(j+1)} T_1 T_j \cdots T_{n-5} T_2 \cdots T_{i-3}.
\end{aligned}
$$

Hence, we are left to show that for each $i = 4, \ldots, n-4$, the following element of M is zero:

$$
\begin{aligned}
&((-1)^i (T_1 - T_{n-3}) T_2 \cdots T_{n-5} \\
&+ (-1)^{n+i} (-T_1 + T_2) T_3 \cdots T_{n-3} T_{n-4} \\
&+ (-1)^i (1 - T_1 T_2) T_3 \cdots T_{n-5} T_{n-3} T_{n-4} \\
&+ \sum_{j=4}^{n-4} ((-1)^{n+i+j} T_1 \cdots T_{j-2} T_j \cdots T_{n-3} T_{n-4}
\end{aligned}
$$

$$+ (-1)^{i+j+1}T_2 \cdots T_{j-2}T_j \cdots T_{n-5}T_{n-3}T_{n-4}))T_2 \cdots T_{i-3}\chi_n.$$

If $i = 4$, using Lemma 3.3.4, we have

$$\begin{aligned}
&\sum_{k=2}^{n-5}(-1)^{n+k+1}(T_1 - T_{n-3})T_2 \cdots T_{k-1}T_{k+1} \cdots T_{n-5} \\
&+ (T_1 - T_{n-3})T_2 \cdots T_{n-5}(T_{n-4} - T_{n-3}) \\
&+ (1 - T_1T_2)T_3 \cdots T_{n-3}T_{n-4} \\
&+ \sum_{k=3}^{n-3}(-1)^{k+1}(T_1 - T_2)T_3 \cdots T_{k-1}T_{k+1} \cdots T_{n-3}T_{n-4} \\
&+ (-1)^n(T_1 - T_2)T_3 \cdots T_{n-5}T_{n-3}T_{n-4} \\
&+ \sum_{k=3}^{n-5}(-1)^{n+k+1}(1 - T_1T_2)T_3 \cdots T_{k-1}T_{k+1} \cdots T_{n-5}T_{n-3}T_{n-4} \\
&+ (1 - T_1T_2)T_3 \cdots T_{n-5}(T_{n-4} - T_{n-3}) \\
&+ \sum_{j=4}^{n-4}\sum_{k=2}^{j-1}(1 - k)T_1 \cdots T_{j-2}T_kT_j \cdots T_{n-3}T_{n-4} \\
&+ \sum_{j=4}^{n-4}\sum_{k=j}^{n-3}(-1)^{j+k}T_1 \cdots T_{j-2}T_j \cdots T_{k-1}T_{k+1} \cdots T_{n-3}T_{n-4} \\
&+ \sum_{j=4}^{n-4}\sum_{k=2}^{j-1}(-1)^{n+1}(k-1)T_2 \cdots T_{j-2}T_kT_j \cdots T_{n-5}T_{n-3}T_{n-4} \\
&+ \sum_{j=4}^{n-4}\sum_{k=j}^{n-5}(-1)^{n+j+k}T_2 \cdots T_{j-2}T_j \cdots T_{k-1}T_{k+1} \cdots T_{n-5}T_{n-3}T_{n-4} \\
&+ \sum_{j=4}^{n-4}(-1)^{j+1}T_2 \cdots T_{j-2}T_j \cdots T_{n-5}(T_{n-4} - T_{n-3}).
\end{aligned}$$

Now we rearrange each term into standard form using the basic spin relations, and get

$$\begin{aligned}
&\sum_{k=2}^{n-5}(-1)^{n+k+1}T_1 \cdots T_{k-1}T_{k+1} \cdots T_{n-5} \\
&+ \sum_{k=2}^{n-5}(-1)^{k+1}T_2 \cdots T_{k-1}T_{k+1} \cdots T_{n-5}T_{n-3} \\
&+ T_1 \cdots T_{n-5}(T_{n-4} - T_{n-3}) \\
&+ (-1)^{n+1}T_2 \cdots T_{n-5}(1 - T_{n-4}T_{n-3}) \\
&+ (-1 + T_1T_2)T_3 \cdots T_{n-5}(T_{n-4} + T_{n-3}) \\
&+ \sum_{k=3}^{n-5}(-1)^k(T_1 - T_2)T_3 \cdots T_{k-1}T_{k+1} \cdots T_{n-5}(T_{n-4} + T_{n-3}) \\
&+ (-1)^n(T_1 - T_2)T_3 \cdots T_{n-5} \\
&+ \sum_{k=3}^{n-5}(-1)^{n+k}(1 - T_1T_2)T_3 \cdots T_{k-1}T_{k+1} \cdots T_{n-5}(1 + T_{n-4}T_{n-3}) \\
&+ (1 - T_1T_2)T_3 \cdots T_{n-5}(T_{n-4} - T_{n-3})
\end{aligned}$$

$$+\sum_{j=4}^{n-4}\sum_{k=2}^{j-3}(-1)^{j+k}(k-1)T_1\cdots T_kT_{k+2}\cdots T_{j-2}T_j\cdots T_{n-5}(T_{n-4}+T_{n-3})$$
$$+\sum_{j=4}^{n-4}\sum_{k=2}^{j-3}(-1)^{j+k}(k-1)T_1\cdots T_{k-1}T_{k+1}\cdots T_{j-2}T_j\cdots T_{n-5}(T_{n-4}+T_{n-3})$$
$$+\sum_{j=4}^{n-4}jT_1\cdots T_{j-3}T_j\cdots T_{n-5}(T_{n-4}+T_{n-3})$$
$$+T_1\cdots T_{n-5}(-T_{n-4}-T_{n-3})$$
$$+\sum_{j=4}^{n-4}\sum_{k=j}^{n-5}(-1)^{j+k+1}T_1\cdots T_{j-2}T_j\cdots T_{k-1}T_{k+1}\cdots T_{n-5}(T_{n-4}+T_{n-3})$$
$$+\sum_{j=4}^{n-4}(-1)^{n+j+1}T_1\cdots T_{j-2}T_j\cdots T_{n-5}(-1+T_{n-4}T_{n-3})$$
$$+\sum_{j=4}^{n-4}\sum_{k=2}^{j-3}(-1)^{n+j+k}(k-1)T_2\cdots T_kT_{k+2}\cdots T_{j-2}T_j\cdots T_{n-5}(1+T_{n-4}T_{n-3})$$
$$+\sum_{j=4}^{n-4}\sum_{k=2}^{j-3}(-1)^{n+j+k}(k-1)T_2\cdots T_{k-1}T_{k+1}\cdots T_{j-2}T_j\cdots T_{n-5}(1+T_{n-4}T_{n-3})$$
$$+\sum_{j=4}^{n-4}(-1)^{n}jT_2\cdots T_{j-3}T_j\cdots T_{n-5}(1+T_{n-4}T_{n-3})$$
$$+(-1)^{n+1}T_2\cdots T_{n-5}(1+T_{n-4}T_{n-3})$$
$$+\sum_{j=4}^{n-4}\sum_{k=j}^{n-5}(-1)^{n+j+k+1}T_2\cdots T_{j-2}T_j\cdots T_{k-1}T_{k+1}\cdots T_{n-5}(1+T_{n-4}T_{n-3})$$
$$+\sum_{j=4}^{n-4}(-1)^{j+1}T_2\cdots T_{j-2}T_j\cdots T_{n-5}(T_{n-4}-T_{n-3}).$$

Cancelling terms and reordering, we get

$$\sum_{j=4}^{n-4}jT_1\cdots T_{j-3}T_j\cdots T_{n-5}(T_{n-4}+T_{n-3})$$
$$+\sum_{k=3}^{n-5}(-1)^{k}T_1T_3\cdots T_{k-1}T_{k+1}\cdots T_{n-5}(T_{n-4}+T_{n-3})$$
$$+\sum_{j=4}^{n-4}(-1)^{n}jT_2\cdots T_{j-3}T_j\cdots T_{n-5}(1+T_{n-4}T_{n-3})$$
$$+\sum_{k=3}^{n-5}(-1)^{n+k}T_3\cdots T_{k-1}T_{k+1}\cdots T_{n-5}(1+T_{n-4}T_{n-3})$$
$$+\sum_{j=4}^{n-4}\sum_{k=2}^{j-3}(-1)^{j+k}(k-1)T_1\cdots T_kT_{k+2}\cdots T_{j-2}T_j\cdots T_{n-5}(T_{n-4}+T_{n-3})$$
$$+\sum_{j=4}^{n-4}\sum_{k=2}^{j-3}(-1)^{j+k}(k-1)T_1\cdots T_{k-1}T_{k+1}\cdots T_{j-2}T_j\cdots T_{n-5}(T_{n-4}+T_{n-3})$$
$$+\sum_{j=4}^{n-4}\sum_{k=j}^{n-5}(-1)^{j+k+1}T_1\cdots T_{j-2}T_j\cdots T_{k-1}T_{k+1}\cdots T_{n-5}(T_{n-4}+T_{n-3})$$

$$+\sum_{j=4}^{n-4}\sum_{k=2}^{j-3}(-1)^{n+j+k}(k-1)T_2\cdots T_kT_{k+2}\cdots T_{j-2}T_j\cdots T_{n-5}(1+T_{n-4}T_{n-3})$$
$$+\sum_{j=4}^{n-4}\sum_{k=2}^{j-3}(-1)^{n+j+k}(k-1)T_2\cdots T_{k-1}T_{k+1}\cdots T_{j-2}T_j\cdots T_{n-5}(1+T_{n-4}T_{n-3})$$
$$+\sum_{j=4}^{n-4}\sum_{k=j}^{n-5}(-1)^{n+j+k+1}T_2\cdots T_{j-2}T_j\cdots T_{k-1}T_{k+1}\cdots T_{n-5}(1+T_{n-4}T_{n-3}).$$

Rearranging sums, we get

$$\begin{aligned}
&\sum_{j=4}^{n-4}(-1)^n jT_2\cdots T_{j-3}T_j\cdots T_{n-5}(1+T_{n-4}T_{n-3})\\
&+\sum_{k=3}^{n-5}(-1)^{n+k}T_3\cdots T_{k-1}T_{k+1}\cdots T_{n-5}(1+T_{n-4}T_{n-3})\\
&+\sum_{j=5}^{n-4}\sum_{k=2}^{j-3}(-1)^{n+j+k}(k-1)T_2\cdots T_kT_{k+2}\cdots T_{j-2}T_j\cdots T_{n-5}(1+T_{n-4}T_{n-3})\\
&+\sum_{j=5}^{n-4}\sum_{k=2}^{j-3}(-1)^{n+j+k}(k-1)T_2\cdots T_{k-1}T_{k+1}\cdots T_{j-2}T_j\cdots T_{n-5}(1+T_{n-4}T_{n-3})\\
&+\sum_{j=4}^{n-5}\sum_{k=j}^{n-5}(-1)^{n+j+k+1}T_2\cdots T_{j-2}T_j\cdots T_{k-1}T_{k+1}\cdots T_{n-5}(1+T_{n-4}T_{n-3})\\
=&(-1)^nT_4\cdots T_{n-5}(1+T_{n-4}T_{n-3})\\
&+\sum_{k=3}^{n-5}(-1)^{n+k}T_3\cdots T_{k-1}T_{k+1}\cdots T_{n-5}(1+T_{n-4}T_{n-3})\\
&+\sum_{j=4}^{n-5}(-1)^{n+j+1}T_3\cdots T_{j-1}T_{j+1}\cdots T_{n-5}(1+T_{n-4}T_{n-3})=0.
\end{aligned}$$

Now we use induction on i. If the following element is 0, for $4<i\leq n-4$,

$$\begin{aligned}
&((-1)^{i+1}(T_1-T_{n-3})T_2\cdots T_{n-5}\\
&+(-1)^{n+i+1}(-T_1+T_2)T_3\cdots T_{n-3}T_{n-4}\\
&+(-1)^{i+1}(1-T_1T_2)T_3\cdots T_{n-5}T_{n-3}T_{n-4}\\
&+\sum_{j=4}^{n-4}((-1)^{n+i+j+1}T_1\cdots T_{j-2}T_j\cdots T_{n-3}T_{n-4}\\
&\qquad+(-1)^{i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-5}T_{n-3}T_{n-4}))T_2\cdots T_{i-4}\chi_n,
\end{aligned}$$

then

$$\begin{aligned}
&((-1)^{i}(T_1-T_{n-3})T_2\cdots T_{n-5}\\
&+(-1)^{n+i}(-T_1+T_2)T_3\cdots T_{n-3}T_{n-4}\\
&+(-1)^{i}(1-T_1T_2)T_3\cdots T_{n-5}T_{n-3}T_{n-4}\\
&+\sum_{j=4}^{n-4}((-1)^{n+i+j}T_1\cdots T_{j-2}T_j\cdots T_{n-3}T_{n-4}
\end{aligned}$$

$$
\begin{aligned}
&\qquad + (-1)^{i+j+1} T_2 \cdots T_{j-2} T_j \cdots T_{n-5} T_{n-3} T_{n-4})) T_2 \cdots T_{i-3} \chi_n \\
=& ((-1)^{i+1} (T_1 - T_{n-3}) T_2 \cdots T_{n-5} \\
& + (-1)^{n+i+1} (-T_1 + T_2) T_3 \cdots T_{n-3} T_{n-4} \\
& + (-1)^{i+1} (1 - T_1 T_2) T_3 \cdots T_{n-5} T_{n-3} T_{n-4} \\
& + \sum_{j=4}^{n-4} ((-1)^{n+i+j+1} T_1 \cdots T_{j-2} T_j \cdots T_{n-3} T_{n-4} \\
&\qquad + (-1)^{i+j} T_2 \cdots T_{j-2} T_j \cdots T_{n-5} T_{n-3} T_{n-4})) T_2 \cdots T_{i-4} \chi_n T_{i-3} \\
=& 0.
\end{aligned}
$$

For all $k \neq 2, 3, 4$ we have

$$
t_k B_{n-2}(b) = t_k t_3 B_{n-1}(b) = t_3 t_k B_{n-1}(-b)
$$

so we find

$$
\begin{aligned}
t_1 B_{n-2}(b) =& t_3 t_1 B_{n-1}(-b) \\
=& B_{n-2}(T_{n-3} b) + \sum_{j=3}^{n-3} (-1)^{j+1} t_3 B_j (T_2 \cdots T_{j-2} \chi_n b).
\end{aligned}
$$

Case 2:

$$
\begin{aligned}
t_2 B_{n-1}(b) =& \sum_{j=4}^{n-2} (-1)^{j+1} P_{3j} (T_2 \cdots T_{j-3} \chi_n b) \\
& + \sum_{j=4}^{n-2} (-1)^{n+j+1} P_{2j} (T_{n-3} T_2 \cdots T_{j-3} \chi_n b) \\
& + \sum_{i=4}^{n-4} \sum_{j=i+1}^{n-2} (-1)^{n+(n+1)i+j} P_{ij} (T_2 (1 + T_2 T_{n-3}) T_3 \cdots T_{i-2} T_2 \cdots T_{j-3} \chi_n b) \\
& + \sum_{i=4}^{n-4} \sum_{x=4}^{i} P_{i,n-2} (((-1)^{n+(n+1)i+x+1} T_2 \cdots T_{i-2} T_2 \cdots T_{x-3} T_{x-1} \cdots T_{n-3} T_{n-4} \\
&\qquad + (-1)^{ni+x+1} T_3 \cdots T_{i-2} T_2 \cdots T_{x-3} T_{x-1} \cdots T_{n-5} T_{n-3} T_{n-4}) \chi_n b) \\
=& \sum_{j=4}^{n-2} (-1)^n P_{2j} (T_2 \cdots T_{j-3} \chi_n T_{n-3} b) \\
& + \sum_{j=4}^{n-2} P_{3j} (T_{n-3} T_2 \cdots T_{j-3} \chi_n T_{n-3} b) \\
& + \sum_{i=4}^{n-4} \sum_{j=i+1}^{n-2} (-1)^{n(i+1)+1} P_{ij} ((1 + T_2 T_{n-3}) T_3 \cdots T_{i-2} T_2 \cdots T_{j-3} \chi_n T_{n-3} b) \\
& + \sum_{i=4}^{n-4} \sum_{x=4}^{i} P_{i,n-2} (((-1)^{ni+x} T_3 \cdots T_{i-2} T_2 \cdots T_{x-3} T_{x-1} \cdots T_{n-3} T_{n-4} \\
&\qquad + (-1)^{n+(n+1)i+x} T_2 \cdots T_{i-2} T_2 \cdots T_{x-3} T_{x-1} \cdots T_{n-5} T_{n-3} T_{n-4}) \chi_n T_{n-3} b) \\
=& B_{n-1}(-T_{n-3} b).
\end{aligned}
$$

$$
\begin{aligned}
t_2B_{n-2}(b) =&(-1)^nP_{23}(T_{n-3}\chi_nb)\\
&+(-1)^nP_{2,n-2}(T_2\cdots T_{n-5}T_{n-3}T_{n-4}\chi_nb)\\
&+(-1)^nP_{3,n-2}(T_2\cdots T_{n-3}T_{n-4}\chi_nb)\\
&+\sum_{j=5}^{n-2}P_{3j}(-T_3\cdots T_{j-3}\chi_nb)\\
&+(-1)^nP_{24}(\chi_nb)+\sum_{j=5}^{n-2}(-1)^nP_{2j}((1+T_2T_{n-3})T_2\cdots T_{j-3}\chi_nb)\\
&+(-1)^{n+1}P_{2,n-2}(T_2\cdots T_{n-5}T_{n-3}T_{n-4}\chi_nb)\\
&+\sum_{j=5}^{n-2}(-1)^nP_{4j}(T_{n-3}T_3\cdots T_{j-3}\chi_nb)\\
&+(-1)^nP_{4,n-2}(T_2\cdots T_{n-5}T_{n-3}T_{n-4}\chi_nb)\\
&+\sum_{i=5}^{n-4}\sum_{j=i+1}^{n-2}(-1)^{n(i+1)}P_{ij}((T_2+(1+T_2T_3)T_{n-3})T_4\cdots T_{i-2}T_2\cdots T_{j-3}\chi_nb)\\
&+\sum_{i=5}^{n-4}\sum_{x=4}^{i}P_{i,n-2}(((-1)^{ni+x+1}T_2T_4\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-3}T_{n-4}\\
&\qquad+(-1)^{(n+1)(i+1)+x}T_3T_2T_4\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_nb)\\
=&(-1)^{n+1}P_{23}(\chi_nT_{n-3}b)+\sum_{j=5}^{n-2}(-1)^{n+j+1}P_{2j}(T_3\cdots T_{j-3}\chi_nT_{n-3}b)\\
&+P_{34}(\chi_nT_{n-3}b)+\sum_{j=5}^{n-2}(-1)^jP_{3j}((1+T_2T_{n-3})T_2\cdots T_{j-3}\chi_nT_{n-3}b)\\
&+(-1)^{n+1}P_{3,n-2}(T_2\cdots T_{n-5}T_{n-3}T_{n-4}\chi_nT_{n-3}b)\\
&+\sum_{j=5}^{n-2}(-1)^{n+j+1}P_{4j}(T_{n-3}T_2\cdots T_{j-3}\chi_nT_{n-3}b)\\
&+P_{4,n-2}(T_3\cdots T_{n-5}T_{n-3}T_{n-4}\chi_nT_{n-3}b)\\
&+\sum_{i=5}^{n-4}\sum_{j=i+1}^{n-2}(-1)^{n+(n+1)i+j}P_{ij}((1+(T_2+T_3)T_{n-3})T_4\cdots T_{i-2}T_2\cdots T_{j-3}\chi_nT_{n-3}b)\\
&+\sum_{i=5}^{n-4}\sum_{x=4}^{i}P_{i,n-2}(((-1)^{(n+1)(i+1)+x}T_4\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-3}T_{n-4}\\
&\qquad+(-1)^{ni+x+1}T_3T_2\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_nT_{n-3}b)\\
&+\sum_{j=4}^{n-2}(-1)^nP_{2j}(T_2\cdots T_{j-3}\chi_nb)\\
&+\sum_{j=4}^{n-2}P_{3j}(T_{n-3}T_2\cdots T_{j-3}\chi_nb)\\
&+\sum_{i=4}^{n-4}\sum_{j=i+1}^{n-2}(-1)^{n(i+1)+1}P_{ij}((1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{j-3}\chi_nb)
\end{aligned}
$$

$$+\sum_{i=4}^{n-4}\sum_{x=4}^{i} P_{i,n-2}(((-1)^{ni+x}T_3\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-3}T_{n-4}$$

$$+(-1)^{n+(n+1)i+x}T_2\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_n b)$$

$$=B_{n-2}(T_{n-3}b)+B_{n-1}(-b).$$

Case 3: $t_3B_{n-1}(b) =: B_{n-2}(b)$ by definition and therefore $t_3B_{n-2}(b)=t_3^2B_{n-1}(b)=B_{n-1}(b)$.

Case 4:

$$t_4B_{n-2}(b) =(-1)^nP_{23}(T_2\chi_n b)+(-1)^nP_{24}(\chi_n b)$$

$$+\sum_{j=6}^{n-2}(-1)^{n+1}P_{2j}(T_4\cdots T_{j-3}\chi_n b)$$

$$+P_{35}(\chi_n b)+P_{34}(-(1+T_2T_{n-3})T_2\chi_n b)$$

$$+\sum_{j=6}^{n-2}P_{3j}(-T_3(1+T_2T_{n-3})T_2\cdots T_{j-3}\chi_n b)$$

$$+P_{3,n-2}(T_3T_2\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n b)$$

$$+(-1)^nt_4P_{45}(T_{n-3}T_2\chi_n b)$$

$$+\sum_{j=6}^{n-2}P_{5j}(-T_{n-3}T_2\cdots T_{j-3}\chi_n b)$$

$$+P_{5,n-2}(T_3\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n b)$$

$$+\sum_{j=6}^{n-2}(-1)^{n+1}P_{4j}((1+(T_2+T_3)T_{n-3})T_2\cdots T_{j-3}\chi_n b)$$

$$+\sum_{i=6}^{n-4}\sum_{j=i+1}^{n-2}(-1)^{n(i+1)}P_{ij}(T_4(1+(T_2+T_3)T_{n-3})T_4\cdots T_{i-2}T_2\cdots T_{j-3}\chi_n b)$$

$$+\sum_{x=4}^{5}P_{4,n-2}(((-1)^xT_2\cdots T_{x-3}T_{x-1}\cdots T_{n-3}T_{n-4}$$

$$+(-1)^{n+x+1}T_3T_2T_3T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_n b)$$

$$+\sum_{i=6}^{n-4}\sum_{x=4}^{i}P_{i,n-2}(((-1)^{ni+x+1}T_5\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-3}T_{n-4}$$

$$+(-1)^{(n+1)(i+1)+x}T_4T_3T_2\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_n b)$$

$$=(-1)^{n+1}P_{23}(\chi_nT_2b)+\sum_{j=5}^{n-2}(-1)^{n+j+1}P_{2j}(T_3\cdots T_{j-3}\chi_nT_2b)$$

$$+P_{34}(\chi_nT_2b)+\sum_{j=5}^{n-2}(-1)^jP_{3j}((1+T_2T_{n-3})T_2\cdots T_{j-3}\chi_nT_2b)$$

$$+(-1)^{n+1}P_{3,n-2}(T_2\cdots T_{n-5}T_{n-3}T_{n-4}\chi_nT_2b)$$

$$+\sum_{j=5}^{n-2}(-1)^{n+j+1}P_{4j}(T_{n-3}T_2\cdots T_{j-3}\chi_nT_2b)$$

$$+P_{4,n-2}(T_3\cdots T_{n-5}T_{n-3}T_{n-4}\chi_nT_2b)$$

$$+\sum_{i=5}^{n-4}\sum_{j=i+1}^{n-2}(-1)^{n+(n+1)i+j}P_{ij}((1+(T_2+T_3)T_{n-3})T_4\cdots T_{i-2}T_2\cdots T_{j-3}\chi_nT_2b)$$

$$+\sum_{i=5}^{n-4}\sum_{x=4}^{i}P_{i,n-2}(((-1)^{(n+1)(i+1)+x}T_4\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-3}T_{n-4}$$

$$+(-1)^{ni+x+1}T_3T_2\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_nT_2b)$$

$$+\sum_{j=4}^{n-2}(-1)^nP_{2j}(T_2\cdots T_{j-3}\chi_nb)$$

$$+\sum_{j=4}^{n-2}P_{3j}(T_{n-3}T_2\cdots T_{j-3}\chi_nb)$$

$$+\sum_{i=4}^{n-4}\sum_{j=i+1}^{n-2}(-1)^{n(i+1)+1}P_{ij}((1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{j-3}\chi_nb)$$

$$+\sum_{i=4}^{n-4}\sum_{x=4}^{i}P_{i,n-2}(((-1)^{ni+x}T_3\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-3}T_{n-4}$$

$$+(-1)^{n+(n+1)i+x}T_2\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_nb)$$

$$=B_{n-2}(T_2b)+B_{n-1}(-b).$$

Case 5: Suppose $4\le k\le n-5$. Then we separate out the calculation $t_kB_{n-1}(b)$. Firstly, the action of t_k on the P_{2j} and P_{3j} terms of B_{n-1}:

$$\sum_{j=4}^{n-2}(-1)^{n+1}t_kP_{2j}(T_2\cdots T_{j-3}\chi_nb)$$

$$+\sum_{j=4}^{n-2}t_kP_{3j}(-T_{n-3}T_2\cdots T_{j-3}\chi_nb)$$

$$=\sum_{j=4}^{k-1}(-1)^{n+j+1}P_{2j}(T_{k-2}T_2\cdots T_{j-3}\chi_nb)$$

$$+(-1)^{n+1}P_{2,k+1}(T_2\cdots T_{k-3}\chi_nb)$$

$$+(-1)^{n+1}P_{2k}(T_2\cdots T_{k-2}\chi_nb)$$

$$+\sum_{j=k+2}^{n-2}(-1)^{n+j+1}P_{2j}(T_{k-1}T_2\cdots T_{j-3}\chi_nb)$$

$$+\sum_{j=4}^{k-1}(-1)^{j}P_{3j}(T_{k-2}T_{n-3}T_2\cdots T_{j-3}\chi_nb)$$

$$+P_{3,k+1}(-T_{n-3}T_2\cdots T_{k-3}\chi_nb)$$

$$+P_{3k}(-T_{n-3}T_2\cdots T_{k-2}\chi_nb)$$

$$+\sum_{j=k+2}^{n-2}(-1)^{j}P_{3j}(T_{k-1}T_{n-3}T_2\cdots T_{j-3}\chi_nb)$$

$$=\sum_{j=4}^{n-2}(-1)^nP_{2j}(T_2\cdots T_{j-3}\chi_nT_{k-2}b)$$

$$+\sum_{j=4}^{n-2}P_{3j}(T_{n-3}T_2\cdots T_{j-3}\chi_nT_{k-2}b).$$

Next we calculate the action of t_k on the sum

$$\sum_{i=4}^{n-4}\sum_{j=i+1}^{n-2}(-1)^{n(i+1)}P_{ij}((1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{j-3}\chi_n b)$$

using the fact that for all $k>3$

$$T_2\cdots T_{k-2}T_2\cdots T_{k-3}=T_3\cdots T_{k-2}T_2\cdots T_{k-2}$$

to rearrange the $P_{k,k+1}$ term:

$$\begin{aligned}
&\sum_{i=4}^{k-1}\sum_{j=i+1}^{k-1}(-1)^{n(i+1)}t_kP_{ij}((1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{j-3}\chi_n b)\\
&+\sum_{i=4}^{k-1}(-1)^{n(i+1)}t_kP_{ik}((1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{k-3}\chi_n b)\\
&+\sum_{i=4}^{k-1}(-1)^{n(i+1)}t_kP_{i,k+1}((1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{k-2}\chi_n b)\\
&+\sum_{i=4}^{k-1}\sum_{j=k+2}^{n-2}(-1)^{n(i+1)}t_kP_{ij}((1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{j-3}\chi_n b)\\
&+(-1)^{n(k+1)}t_kP_{k,k+1}((1+T_2T_{n-3})T_3\cdots T_{k-2}T_2\cdots T_{k-2}\chi_n b)\\
&+\sum_{j=k+2}^{n-2}(-1)^{n(k+1)}t_kP_{kj}((1+T_2T_{n-3})T_3\cdots T_{k-2}T_2\cdots T_{j-3}\chi_n b)\\
&+\sum_{j=k+2}^{n-2}(-1)^{nk}t_kP_{k+1,j}((1+T_2T_{n-3})T_3\cdots T_{k-1}T_2\cdots T_{j-3}\chi_n b)\\
&+\sum_{i=k+2}^{n-4}\sum_{j=i+1}^{n-2}(-1)^{n(i+1)}t_kP_{ij}((1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{j-3}\chi_n b)\\
=&\sum_{i=4}^{n-4}\sum_{j=i+1}^{n-2}(-1)^{n(i+1)+1}P_{ij}((1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{j-3}\chi_nT_{k-2}b)\\
&+(-1)^{n(k+1)+1}P_{k,n-2}(T_k\cdots T_{n-5}T_{n-3}T_{n-4}(1+T_2T_{n-3})T_3\cdots T_{k-2}T_2\cdots T_{k-2}\chi_n b)\\
&+(-1)^{n(k+1)+1}P_{k+1,n-2}(T_k\cdots T_{n-3}T_{n-4}(1+T_2T_{n-3})T_3\cdots T_{k-2}T_2\cdots T_{k-2}\chi_n b).
\end{aligned}$$

Finally, we calculate the action of t_k on the sum

$$\begin{aligned}
\sum_{i=4}^{n-4}\sum_{x=4}^{i}P_{i,n-2}(((-1)^{ni+x+1}&T_3\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-3}T_{n-4}\\
&+(-1)^{(n+1)(i+1)+x}T_2\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_n b)
\end{aligned}$$

and we get

$$\begin{aligned}
&\sum_{i=4}^{k-1}\sum_{x=4}^{i}t_kP_{i,n-2}(((-1)^{ni+x+1}T_3\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-3}T_{n-4}\\
&\qquad+(-1)^{(n+1)(i+1)+x}T_2\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_n b)\\
&+\sum_{x=4}^{k}t_kP_{k,n-2}(((-1)^{nk+x+1}T_3\cdots T_{k-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-3}T_{n-4}
\end{aligned}$$

$$+ (-1)^{(n+1)(k+1)+x} T_2 \cdots T_{k-2} T_2 \cdots T_{x-3} T_{x-1} \cdots T_{n-5} T_{n-3} T_{n-4}) \chi_n b)$$

$$+ \sum_{x=4}^{k+1} t_k P_{k+1,n-2}(((-1)^{n(k+1)+x+1} T_3 \cdots T_{k-1} T_2 \cdots T_{x-3} T_{x-1} \cdots T_{n-3} T_{n-4}$$

$$+ (-1)^{(n+1)k+x} T_2 \cdots T_{k-1} T_2 \cdots T_{x-3} T_{x-1} \cdots T_{n-5} T_{n-3} T_{n-4}) \chi_n b)$$

$$+ \sum_{i=k+2}^{n-4} \sum_{x=4}^{i} t_k P_{i,n-2}(((-1)^{ni+x+1} T_3 \cdots T_{i-2} T_2 \cdots T_{x-3} T_{x-1} \cdots T_{n-3} T_{n-4}$$

$$+ (-1)^{(n+1)(i+1)+x} T_2 \cdots T_{i-2} T_2 \cdots T_{x-3} T_{x-1} \cdots T_{n-5} T_{n-3} T_{n-4}) \chi_n b)$$

$$= \sum_{i=4}^{k-1} \sum_{x=4}^{i} P_{i,n-2}(((-1)^{(n+1)(i+1)+x} T_{k-1} T_3 \cdots T_{i-2} T_2 \cdots T_{x-3} T_{x-1} \cdots T_{n-3} T_{n-4}$$

$$+ (-1)^{ni+x+1} T_{k-1} T_2 \cdots T_{i-2} T_2 \cdots T_{x-3} T_{x-1} \cdots T_{n-5} T_{n-3} T_{n-4}) \chi_n b)$$

$$+ \sum_{x=4}^{k} P_{k+1,n-2}(((-1)^{nk+x+1} T_3 \cdots T_{k-2} T_2 \cdots T_{x-3} T_{x-1} \cdots T_{n-3} T_{n-4}$$

$$+ (-1)^{(n+1)(k+1)+x} T_2 \cdots T_{k-2} T_2 \cdots T_{x-3} T_{x-1} \cdots T_{n-5} T_{n-3} T_{n-4}) \chi_n b)$$

$$+ \sum_{x=4}^{k+1} P_{k,n-2}(((-1)^{n(k+1)+x+1} T_3 \cdots T_{k-1} T_2 \cdots T_{x-3} T_{x-1} \cdots T_{n-3} T_{n-4}$$

$$+ (-1)^{(n+1)k+x} T_2 \cdots T_{k-1} T_2 \cdots T_{x-3} T_{x-1} \cdots T_{n-5} T_{n-3} T_{n-4}) \chi_n b)$$

$$+ \sum_{i=k+2}^{n-4} \sum_{x=4}^{i} P_{i,n-2}(((-1)^{(n+1)(i+1)+x} T_k T_3 \cdots T_{i-2} T_2 \cdots T_{x-3} T_{x-1} \cdots T_{n-3} T_{n-4}$$

$$+ (-1)^{ni+x+1} T_k T_2 \cdots T_{i-2} T_2 \cdots T_{x-3} T_{x-1} \cdots T_{n-5} T_{n-3} T_{n-4}) \chi_n b).$$

To rearrange these terms, we use the identity, for $4 \le x \le n-4$:

$$T_{k-1} T_2 \cdots T_{x-3} T_{x-1} \cdots T_{n-5} = \begin{cases} (-1)^{n+1} T_2 \cdots T_{x-3} T_{x-1} \cdots T_{n-5} T_{k-2} & x \neq k, k+1 \\ (-1)^{n} T_2 \cdots T_{k-2} T_k \cdots T_{n-5} T_{k-2} & x = k \\ (-1)^{n+1} T_2 \cdots T_{k-2} T_k \cdots T_{n-5} T_{k-2} & \\ +(-1)^{k+1} T_2 \cdots T_{n-5} & x = k+1 \end{cases}$$

and we get

$$\sum_{i=4}^{n-4} \sum_{x=4}^{i} P_{i,n-2}(((-1)^{ni+x} T_3 \cdots T_{i-2} T_2 \cdots T_{x-3} T_{x-1} \cdots T_{n-3} T_{n-4}$$

$$+ (-1)^{n+(n+1)i+x} T_2 \cdots T_{i-2} T_2 \cdots T_{x-3} T_{x-1} \cdots T_{n-5} T_{n-3} T_{n-4}) \chi_n T_{k-2} b)$$

$$+ P_{k,n-2}(((-1)^{n(k+1)+1} T_3 \cdots T_{k-2} T_2 \cdots T_{k-3} T_k \cdots T_{n-3} T_{n-4}$$

$$+ (-1)^{(n+1)k} T_2 \cdots T_{k-2} T_2 \cdots T_{k-3} T_k \cdots T_{n-5} T_{n-3} T_{n-4}) \chi_n b)$$

$$+ P_{k+1,n-2}(((-1)^{(n+1)k+1} T_3 \cdots T_{k-2} T_2 \cdots T_{k-2} T_k \cdots T_{n-3} T_{n-4}$$

$$+ (-1)^{n(k+1)+1} T_2 \cdots T_{k-2} T_2 \cdots T_{k-2} T_k \cdots T_{n-5} T_{n-3} T_{n-4}) \chi_n b).$$

Adding all of the terms of $t_k B_{n-1}(b)$ and again using the identity

$$T_2 \cdots T_{k-2} T_2 \cdots T_{k-3} = T_3 \cdots T_{k-2} T_2 \cdots T_{k-2},$$

we get

$$
\begin{aligned}
&\sum_{j=4}^{n-2}(-1)^n P_{2j}(T_2\cdots T_{j-3}\chi_n T_{k-2}b)\\
&+\sum_{j=4}^{n-2} P_{3j}(T_{n-3}T_2\cdots T_{j-3}\chi_n T_{k-2}b)\\
&+\sum_{i=4}^{n-4}\sum_{j=i+1}^{n-2}(-1)^{n(i+1)+1}P_{ij}((1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{j-3}\chi_n T_{k-2}b)\\
&+\sum_{i=4}^{n-4}\sum_{x=4}^{i} P_{i,n-2}(((-1)^{ni+x}T_3\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-3}T_{n-4}\\
&\qquad\qquad +(-1)^{n+(n+1)i+x}T_2\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_n T_{k-2}b)\\
&+(-1)^{n(k+1)+1}P_{k,n-2}(T_k\cdots T_{n-5}T_{n-3}T_{n-4}(1+T_2T_{n-3})T_3\cdots T_{k-2}T_2\cdots T_{k-2}\chi_n b)\\
&+P_{k,n-2}(((-1)^{n(k+1)+1}T_3\cdots T_{k-2}T_2\cdots T_{k-3}T_k\cdots T_{n-3}T_{n-4}\\
&\qquad\qquad +(-1)^{(n+1)k}T_2\cdots T_{k-2}T_2\cdots T_{k-3}T_k\cdots T_{n-5}T_{n-3}T_{n-4})\chi_n b)\\
&+(-1)^{n(k+1)+1}P_{k+1,n-2}(T_k\cdots T_{n-3}T_{n-4}(1+T_2T_{n-3})T_3\cdots T_{k-2}T_2\cdots T_{k-2}\chi_n b)\\
&+P_{k+1,n-2}(((-1)^{(n+1)k+1}T_3\cdots T_{k-2}T_2\cdots T_{k-2}T_k\cdots T_{n-3}T_{n-4}\\
&\qquad\qquad +(-1)^{n(k+1)+1}T_2\cdots T_{k-2}T_2\cdots T_{k-2}T_k\cdots T_{n-5}T_{n-3}T_{n-4})\chi_n b)\\
=&B_{n-1}(-T_{k-2}b).
\end{aligned}
$$

Recall that $P_{n-4,n-2}(b) := P(b)$ and $P_{n-3,n-2}(b) := 0$. When $k = n-4$, we have

$$
\begin{aligned}
t_{n-4}B_{n-1}(b) =&\sum_{j=4}^{n-5}(-1)^{n+1}t_{n-4}P_{2j}(T_2\cdots T_{j-3}\chi_n b)\\
&+(-1)^{n+1}t_{n-4}P_{2,n-4}(T_2\cdots T_{n-7}\chi_n b)\\
&+(-1)^{n+1}t_{n-4}P_{2,n-3}(T_2\cdots T_{n-6}\chi_n b)\\
&+(-1)^{n+1}t_{n-4}P_{2,n-2}(T_2\cdots T_{n-5}\chi_n b)\\
&+\sum_{j=4}^{n-5} t_{n-4}P_{3j}(-T_{n-3}T_2\cdots T_{j-3}\chi_n b)\\
&+t_{n-4}P_{3,n-4}(-T_{n-3}T_2\cdots T_{n-7}\chi_n b)\\
&+t_{n-4}P_{3,n-3}(-T_{n-3}T_2\cdots T_{n-6}\chi_n b)\\
&+t_{n-4}P_{3,n-2}(-T_{n-3}T_2\cdots T_{n-5}\chi_n b)\\
&+\sum_{i=4}^{n-6}\sum_{j=i+1}^{n-5}(-1)^{n(i+1)}t_{n-4}P_{ij}((1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{j-3}\chi_n b)\\
&+\sum_{i=4}^{n-5}(-1)^{n(i+1)}t_{n-4}P_{i,n-4}((1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{n-7}\chi_n b)\\
&+\sum_{i=4}^{n-5}(-1)^{n(i+1)}t_{n-4}P_{i,n-3}((1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{n-6}\chi_n b)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=4}^{n-5}(-1)^{n(i+1)}t_{n-4}P_{i,n-2}((1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{n-5}\chi_n b)\\
&+t_{n-4}P_{n-4,n-3}((1+T_2T_{n-3})T_3\cdots T_{n-6}T_2\cdots T_{n-6}\chi_n b)\\
&+t_{n-4}P_{n-4,n-2}((1+T_2T_{n-3})T_3\cdots T_{n-6}T_2\cdots T_{n-5}\chi_n b)\\
&+\sum_{i=4}^{n-5}\sum_{x=4}^{i}t_{n-4}P_{i,n-2}(((-1)^{ni+x+1}T_3\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-3}T_{n-4}\\
&\qquad\qquad+(-1)^{(n+1)(i+1)+x}T_2\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_n b)\\
&+\sum_{x=4}^{n-4}t_{n-4}P_{n-4,n-2}(((-1)^{n+x+1}T_3\cdots T_{n-6}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-3}T_{n-4}\\
&\qquad\qquad+(-1)^{n+x+1}T_2\cdots T_{n-6}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_n b)\\
=&\sum_{j=4}^{n-5}(-1)^{n+j+1}P_{2j}(T_{n-6}T_2\cdots T_{j-3}\chi_n b)\\
&+(-1)^{n+1}P_{2,n-3}(T_2\cdots T_{n-7}\chi_n b)\\
&+(-1)^{n+1}P_{2,n-4}(T_2\cdots T_{n-6}\chi_n b)\\
&+P_{2,n-2}(-T_{n-5}T_2\cdots T_{n-5}\chi_n b)\\
&+P(-T_2\cdots T_{n-4}T_2\cdots T_{n-5}\chi_n b)\\
&+\sum_{j=4}^{n-5}(-1)^{j}P_{3j}(T_{n-6}T_{n-3}T_2\cdots T_{j-3}\chi_n b)\\
&+P_{3,n-3}(-T_{n-3}T_2\cdots T_{n-7}\chi_n b)\\
&+P_{3,n-4}(-T_{n-3}T_2\cdots T_{n-6}\chi_n b)\\
&+(-1)^{n}P_{3,n-2}(T_{n-5}T_{n-3}T_2\cdots T_{n-5}\chi_n b)\\
&+(-1)^{n}P(T_3\cdots T_{n-3}T_2\cdots T_{n-5}\chi_n b)\\
&+\sum_{i=4}^{n-6}\sum_{j=i+1}^{n-5}(-1)^{n(i+1)+i+j}P_{ij}(T_{n-6}(1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{j-3}\chi_n b)\\
&+\sum_{i=4}^{n-5}(-1)^{n(i+1)}P_{i,n-3}((1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{n-7}\chi_n b)\\
&+\sum_{i=4}^{n-5}(-1)^{n(i+1)}P_{i,n-4}((1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{n-6}\chi_n b)\\
&+\sum_{i=4}^{n-5}(-1)^{(n+1)i}P_{i,n-2}(T_{n-5}(1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{n-5}\chi_n b)\\
&+\sum_{i=4}^{n-5}(-1)^{(n+1)i}P(T_i\cdots T_{n-4}(1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{n-5}\chi_n b)\\
&+P_{n-4,n-3}(-T_{n-3}(1+T_2T_{n-3})T_3\cdots T_{n-6}T_2\cdots T_{n-6}\chi_n b)\\
&+P(-T_{n-3}T_{n-4}(1+T_2T_{n-3})T_3\cdots T_{n-6}T_2\cdots T_{n-6}\chi_n b)\\
&+P(T_{n-4}(1+T_2T_{n-3})T_3\cdots T_{n-6}T_2\cdots T_{n-5}\chi_n b)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=4}^{n-5}\sum_{x=4}^{i} P_{i,n-2}(((-1)^{(n+1)(i+1)+x}T_{n-5}T_3\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-3}T_{n-4}\\
&\qquad\qquad + (-1)^{ni+x+1}T_{n-5}T_2\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_n b)\\
&+\sum_{i=4}^{n-5}\sum_{x=4}^{i} P(((-1)^{(n+1)(i+1)+x}T_i\cdots T_{n-4}T_3\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-3}T_{n-4}\\
&\qquad\qquad + (-1)^{ni+x+1}T_i\cdots T_{n-4}T_2\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_n b)\\
&+\sum_{x=4}^{n-4} P(((-1)^{n+x+1}T_{n-4}T_3\cdots T_{n-6}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-3}T_{n-4}\\
&\qquad\qquad + (-1)^{n+x+1}T_{n-4}T_2\cdots T_{n-6}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_n b)\\
=&\sum_{j=4}^{n-2}(-1)^n P_{2j}(T_2\cdots T_{j-3}\chi_n T_{n-6}b)\\
&+\sum_{j=4}^{n-2} P_{3j}(T_{n-3}T_2\cdots T_{j-3}\chi_n T_{n-6}b)\\
&+\sum_{i=4}^{n-5}\sum_{j=i+1}^{n-2}(-1)^{n(i+1)+1}P_{ij}((1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{j-3}\chi_n T_{n-6}b)\\
&+P_{n-4,n-3}(-(1+T_2T_{n-3})T_3\cdots T_{n-6}T_2\cdots T_{n-6}\chi_n T_{n-6}b)\\
&+\sum_{i=4}^{n-5}\sum_{x=4}^{i} P_{i,n-2}(((-1)^{ni+x}T_3\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-3}T_{n-4}\\
&\qquad\qquad + (-1)^{n+(n+1)i+x}T_2\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_n T_{n-6}b)\\
&+P(-T_2\cdots T_{n-4}T_2\cdots T_{n-5}\chi_n b)\\
&+(-1)^n P(T_3\cdots T_{n-3}T_2\cdots T_{n-5}\chi_n b)\\
&+\sum_{i=4}^{n-5}(-1)^{(n+1)i}P(T_i\cdots T_{n-4}(1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{n-5}\chi_n b)\\
&+P(-T_{n-3}T_{n-4}(1+T_2T_{n-3})T_3\cdots T_{n-6}T_2\cdots T_{n-6}\chi_n b)\\
&+P(T_{n-4}(1+T_2T_{n-3})T_3\cdots T_{n-6}T_2\cdots T_{n-5}\chi_n b)\\
&+\sum_{i=4}^{n-5}\sum_{x=4}^{i} P(((-1)^{(n+1)(i+1)+x}T_i\cdots T_{n-4}T_3\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-3}T_{n-4}\\
&\qquad\qquad + (-1)^{ni+x+1}T_i\cdots T_{n-4}T_2\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_n b)\\
&+\sum_{x=4}^{n-4} P(((-1)^{n+x+1}T_{n-4}T_3\cdots T_{n-6}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-3}T_{n-4}\\
&\qquad\qquad + (-1)^{n+x+1}T_{n-4}T_2\cdots T_{n-6}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_n b)
\end{aligned}
$$

where we have used the fact that $T_j(T_j\cdots T_k)^2 = (T_j\cdots T_k)^2T_k$ for $j\le k$ to show that

$$P_{n-4,n-3}(-T_{n-3}(1+T_2T_{n-3})T_3\cdots T_{n-6}T_2\cdots T_{n-6}\chi_n b) = P_{n-4,n-3}(-(1+T_2T_{n-3})T_3\cdots T_{n-6}T_2\cdots T_{n-6}\chi_n T_{n-6}b).$$

Now we observe that the coefficient of $P_{ij}(b)$ in $t_{n-4}B_{n-1}(b)$ is equal to the coefficient of $P_{ij}(b)$ in $B_{n-1}(-T_{n-6}b)$ when $(i,j)\neq(n-4,n-2)$. Thus the element $\phi\in M$ satisfying $P(\phi b) = t_{n-4}B_{n-1}(b)+$

$B_{n-1}(T_{n-6}b)$ can be written as

$$\begin{aligned}\phi :=& -T_2\cdots T_{n-4}T_2\cdots T_{n-5}\chi_n\\ &+(-1)^nT_3\cdots T_{n-3}T_2\cdots T_{n-5}\chi_n\\ &-T_{n-3}T_{n-4}(1+T_2T_{n-3})T_3\cdots T_{n-6}T_2\cdots T_{n-6}\chi_n\\ &+T_{n-4}(1+T_2T_{n-3})T_3\cdots T_{n-6}T_2\cdots T_{n-5}\chi_n\\ &+(-1)^{n+1}(1+T_2T_{n-3})T_3\cdots T_{n-5}T_2\cdots T_{n-5}\chi_n\\ &+\sum_{i=4}^{n-5}(-1)^{(n+1)i}T_i\cdots T_{n-4}(1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{n-5}\chi_n\\ &+\sum_{x=4}^{n-4}((-1)^{n+x+1}T_{n-4}T_3\cdots T_{n-6}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-3}T_{n-4}\\ &\qquad+(-1)^{n+x+1}T_{n-4}T_2\cdots T_{n-6}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_n\\ &+\sum_{i=4}^{n-4}(-1)^{n+i}(T_3\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-3}T_{n-4}T_{n-6}\\ &\qquad+T_2\cdots T_{n-6}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4}T_{n-6})\chi_n\\ &+\sum_{i=4}^{n-5}\sum_{x=4}^{i}(-1)^{ni+x+1}((-1)^{n+i}T_i\cdots T_{n-4}T_3\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-3}T_{n-4}\\ &\qquad\qquad+T_i\cdots T_{n-4}T_2\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_n.\end{aligned}$$

First we will rearrange to write this element in the simplest way with χ_n still factored on the right, then we will use Lemma 3.3.4 to show that ϕ, and therefore $t_{n-4}B_{n-1}(b)+B_{n-1}(T_{n-6}b)$, is equal to zero. First, we arrange the double sums:

$$\begin{aligned}&\sum_{i=4}^{n-5}\sum_{x=4}^{i}(-1)^{ni+x+1}((-1)^{n+i}T_i\cdots T_{n-4}T_3\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-3}T_{n-4}\\ &\qquad\qquad+T_i\cdots T_{n-4}T_2\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_n\\ =&\sum_{i=4}^{n-5}\sum_{x=4}^{i}(-1)^{n+i+x}(-T_3\cdots T_{i-2}T_i\cdots T_{n-4}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-4}T_{n-3}T_{n-4}\\ &\qquad\qquad+T_2\cdots T_{i-2}T_i\cdots T_{n-4}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_n.\end{aligned}$$

Using the relation for $x=3,\dots,n-5$, $3\le k\le n-4$,

$$T_kT_2\cdots T_{x-2}T_x\cdots T_{n-4}=\begin{cases}(-1)^nT_2\cdots T_{x-2}T_x\cdots T_{n-4}T_{x-2}+(-1)^{x+1}T_2\cdots T_{n-4} & k=x-1\\ (-1)^nT_2\cdots T_{x-2}T_x\cdots T_{n-4}T_{x-1}+(-1)^{x}T_2\cdots T_{n-4} & k=x\\ (-1)^nT_2\cdots T_{x-2}T_x\cdots T_{n-4}T_{k-1} & k\neq,x-1,x,\end{cases}$$

we have

$$\begin{aligned}&\sum_{i=4}^{n-5}\sum_{x=4}^{i}(-1)^{n+i+x+1}T_3\cdots T_{i-2}T_i\cdots T_{n-4}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-4}T_{n-3}T_{n-4}\chi_n\\ =&\sum_{i=4}^{n-5}\sum_{x=3}^{i-1}(-1)^{n+i+x}T_3\cdots T_{i-2}T_i\cdots T_{n-4}T_2\cdots T_{x-2}T_x\cdots T_{n-4}T_{n-3}T_{n-4}\chi_n\end{aligned}$$

$$= \sum_{i=4}^{n-5}\sum_{x=3}^{i-1}(-1)^{n+(n+1)i+x}T_3\cdots T_{i-2}T_2\cdots T_{x-2}T_x\cdots T_{n-4}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n$$

$$= \sum_{i=5}^{n-5}\sum_{x=3}^{i-2}(-1)^{n+(n+1)i+x}T_3\cdots T_{i-2}T_2\cdots T_{x-2}T_x\cdots T_{n-4}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n$$

$$+ \sum_{i=4}^{n-5}(-1)^{n(i+1)+1}T_3\cdots T_{i-2}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-4}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n$$

$$= \sum_{i=5}^{n-5}\sum_{x=3}^{i-2}(-1)^{n+i+(n+1)x}T_3\cdots T_xT_2\cdots T_{x-2}T_x\cdots T_{n-4}T_x\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n$$

$$+ \sum_{i=4}^{n-5}(-1)^{n(i+1)+1}T_2\cdots T_{i-2}T_2\cdots T_{n-4}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n$$

$$= \sum_{i=5}^{n-5}\sum_{x=3}^{i-2}(-1)^{i+(n+1)x}T_3\cdots T_{x-1}T_2\cdots T_{x-2}T_x\cdots T_{n-4}T_{x-1}\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n$$

$$+ \sum_{i=5}^{n-5}\sum_{x=3}^{i-2}(-1)^{n+i+nx}T_3\cdots T_{x-1}T_2\cdots T_{n-4}T_x\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n$$

$$+ \sum_{i=4}^{n-5}(-1)^{n+i+1}T_3\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n$$

$$= \sum_{i=6}^{n-5}\sum_{x=4}^{i-2}(-1)^{i+(n+1)x}T_3\cdots T_{x-1}T_2\cdots T_{x-2}T_x\cdots T_{n-4}T_{x-1}\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n$$

$$+ \sum_{i=5}^{n-5}(-1)^{n+i+1}T_3\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n$$

$$+ \sum_{i=5}^{n-5}\sum_{x=3}^{i-2}(-1)^{i+x+1}T_2\cdots T_{n-4}T_2\cdots T_{x-2}T_x\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n$$

$$+ \sum_{i=4}^{n-5}(-1)^{n+i+1}T_3\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n$$

$$= \sum_{i=6}^{n-5}\sum_{x=4}^{i-2}(-1)^{n+i+(n+1)x}T_3\cdots T_{x-2}T_2\cdots T_{x-2}T_x\cdots T_{n-4}T_{x-2}\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n$$

$$+ \sum_{i=6}^{n-5}\sum_{x=4}^{i-2}(-1)^{i+nx+1}T_3\cdots T_{x-2}T_2\cdots T_{n-4}T_{x-1}\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n$$

$$+ \sum_{i=5}^{n-5}(-1)^{n+i}T_3\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n$$

$$+ (-1)^{n+1}T_3\cdots T_{n-4}T_3\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n$$

$$+ \sum_{i=5}^{n-5}\sum_{x=3}^{i-2}(-1)^{i+x+1}T_2\cdots T_{n-4}T_2\cdots T_{x-2}T_x\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n$$

$$= \sum_{i=6}^{n-5}\sum_{x=4}^{i-2}(-1)^{n+i+(n+1)x}T_3\cdots T_{x-2}T_2\cdots T_{x-2}T_x\cdots T_{n-4}T_{x-2}\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n$$

$$+ \sum_{i=6}^{n-5}\sum_{x=4}^{i-2}(-1)^{i+x+1}T_2\cdots T_{n-4}T_2\cdots T_{x-3}T_{x-1}\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n$$

$$+ \sum_{i=5}^{n-5}\sum_{x=4}^{i-1}(-1)^{i+x}T_2\cdots T_{n-4}T_2\cdots T_{x-3}T_{x-1}\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n$$

$$
\begin{aligned}
&+\sum_{i=5}^{n-5}(-1)^{n+i}T_3\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n\\
&+(-1)^{n+1}T_3\cdots T_{n-4}T_3\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n\\
=&\sum_{i=6}^{n-5}\sum_{x=4}^{i-2}(-1)^{n+i+x}T_2\cdots T_{x-2}T_x\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n\\
&-\sum_{i=5}^{n-5}T_2\cdots T_{n-4}T_2\cdots T_{i-4}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n\\
&+\sum_{i=5}^{n-5}(-1)^{n+i}T_3\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n\\
&+(-1)^{n+1}T_3\cdots T_{n-4}T_3\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n,
\end{aligned}
$$

where we have also used the fact that $T_j\cdots T_kT_2\cdots T_{n-4}=(-1)^{(n+1)(j+k+1)}T_2\cdots T_{n-4}T_{j-1}\cdots T_{k-1}$, for $3\le j\le k\le n-4$. Hence

$$
\begin{aligned}
&\sum_{i=4}^{n-5}\sum_{x=4}^{i}(-1)^{n+i+x}(-T_3\cdots T_{i-2}T_i\cdots T_{n-4}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-4}T_{n-3}T_{n-4}\\
&\qquad\qquad+T_2\cdots T_{i-2}T_i\cdots T_{n-4}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_n\\
=&\sum_{i=6}^{n-5}\sum_{x=4}^{i-2}(-1)^{n+i+x}T_2\cdots T_{x-2}T_x\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n\\
&+\sum_{i=4}^{n-5}\sum_{x=4}^{i}(-1)^{n+i+x}T_2\cdots T_{i-2}T_i\cdots T_{n-4}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n\\
&-\sum_{i=5}^{n-5}T_2\cdots T_{n-4}T_2\cdots T_{i-4}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n\\
&+\sum_{i=5}^{n-5}(-1)^{n+i}T_3\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n\\
&+(-1)^{n+1}T_3\cdots T_{n-4}T_3\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n\\
=&\sum_{i=6}^{n-5}\sum_{x=4}^{i-2}(-1)^{n+i+x}T_2\cdots T_{x-2}T_x\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n\\
&+\sum_{x=4}^{n-5}\sum_{i=x}^{n-5}(-1)^{n+i+x}T_2\cdots T_{i-2}T_i\cdots T_{n-4}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n\\
&-\sum_{i=5}^{n-5}T_2\cdots T_{n-4}T_2\cdots T_{i-4}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n\\
&+\sum_{i=5}^{n-5}(-1)^{n+i}T_3\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n\\
&+(-1)^{n+1}T_3\cdots T_{n-4}T_3\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n\\
=&\sum_{i=6}^{n-5}\sum_{x=4}^{i-2}(-1)^{n+i+x}T_2\cdots T_{x-2}T_x\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n\\
&+\sum_{x=4}^{n-5}\sum_{i=x-1}^{n-5}(-1)^{n+i+x}T_2\cdots T_{i-2}T_i\cdots T_{n-4}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{i=5}^{n-5}(-1)^n T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n \\
&- \sum_{i=5}^{n-5} T_2\cdots T_{n-4}T_2\cdots T_{i-4}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n \\
&+ \sum_{i=5}^{n-5}(-1)^{n+i} T_3\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n \\
=& \sum_{i=4}^{n-5}\sum_{x=4}^{n-5}(-1)^{n+i+x} T_2\cdots T_{x-2}T_x\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n \\
&+ (-1)^{n+1}T_3\cdots T_{n-4}T_3\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n \\
&+ \sum_{i=5}^{n-5}((-1)^n T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-4}T_2\cdots T_{i-3} \\
&\qquad - T_2\cdots T_{n-4}T_2\cdots T_{i-4} \\
&\qquad + (-1)^{n+i}T_3\cdots T_{n-4}T_2\cdots T_{i-3})T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n.
\end{aligned}
$$

Substituting this into ϕ, we get

$$
\begin{aligned}
\phi =& - T_2\cdots T_{n-4}T_2\cdots T_{n-5}\chi_n \\
&+ (-1)^n T_3\cdots T_{n-3}T_2\cdots T_{n-5}\chi_n \\
&- T_{n-3}T_{n-4}(1+T_2T_{n-3})T_3\cdots T_{n-6}T_2\cdots T_{n-6}\chi_n \\
&+ T_{n-4}(1+T_2T_{n-3})T_3\cdots T_{n-6}T_2\cdots T_{n-5}\chi_n \\
&+ (-1)^{n+1}(1+T_2T_{n-3})T_3\cdots T_{n-5}T_2\cdots T_{n-5}\chi_n \\
&+ (-1)^{n+1}T_3\cdots T_{n-4}T_3\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n \\
&+ \sum_{i=4}^{n-5}(-1)^{(n+1)i}T_i\cdots T_{n-4}(1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{n-5}\chi_n \\
&+ \sum_{x=4}^{n-4}((-1)^{n+x+1}T_{n-4}T_3\cdots T_{n-6}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-3}T_{n-4} \\
&\qquad + (-1)^{n+x+1}T_{n-4}T_2\cdots T_{n-6}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_n \\
&+ \sum_{i=4}^{n-4}(-1)^{n+i}(T_3\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-3}T_{n-4}T_{n-6} \\
&\qquad + T_2\cdots T_{n-6}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4}T_{n-6})\chi_n \\
&+ \sum_{i=5}^{n-5}((-1)^n T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-4}T_2\cdots T_{i-3} \\
&\qquad - T_2\cdots T_{n-4}T_2\cdots T_{i-4} \\
&\qquad + (-1)^{n+i}T_3\cdots T_{n-4}T_2\cdots T_{i-3})T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n \\
&+ \sum_{i=4}^{n-5}\sum_{x=4}^{n-5}(-1)^{n+i+x}T_2\cdots T_{x-2}T_x\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n.
\end{aligned}
$$

Next we want to simplify the single sums. Rearranging, we have

$$\begin{aligned}
&\sum_{i=4}^{n-5}(-1)^{(n+1)i}T_i\cdots T_{n-4}(1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{n-5}\chi_n\\
&\quad+\sum_{i=4}^{n-4}((-1)^{n+i+1}T_{n-4}T_3\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-3}T_{n-4}\\
&\qquad\quad+(-1)^{n+i+1}T_{n-4}T_2\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_n\\
&\quad+\sum_{i=4}^{n-4}(-1)^{n+i}(T_3\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-3}T_{n-4}T_{n-6}\\
&\qquad\quad+T_2\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}T_{n-6})\chi_n\\
&\quad+\sum_{i=5}^{n-5}((-1)^nT_2\cdots T_{i-3}T_{i-1}\cdots T_{n-4}T_2\cdots T_{i-3}\\
&\qquad\quad-T_2\cdots T_{n-4}T_2\cdots T_{i-4}\\
&\qquad\quad+(-1)^{n+i}T_3\cdots T_{n-4}T_2\cdots T_{i-3})T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n\\
=&\sum_{i=4}^{n-5}(-1)^{(n+1)i}T_i\cdots T_{n-4}T_3\cdots T_{i-2}T_2\cdots T_{n-5}\chi_n\\
&\quad+\sum_{i=4}^{n-5}(-1)^{(n+1)i}T_i\cdots T_{n-4}T_2T_{n-3}T_3\cdots T_{i-2}T_2\cdots T_{n-5}\chi_n\\
&\quad+\sum_{i=4}^{n-4}(-1)^{n+i+1}T_{n-4}T_3\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-3}T_{n-4}\chi_n\\
&\quad+\sum_{i=4}^{n-4}(-1)^{n+i+1}T_{n-4}T_2\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n\\
&\quad+\sum_{i=4}^{n-4}(-1)^{n+i}T_3\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-3}T_{n-4}T_{n-6}\chi_n\\
&\quad+\sum_{i=4}^{n-4}(-1)^{n+i}T_2\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}T_{n-6}\chi_n\\
&\quad+\sum_{i=5}^{n-5}(-1)^nT_2\cdots T_{i-3}T_{i-1}\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n\\
&\quad-\sum_{i=5}^{n-5}T_2\cdots T_{n-4}T_2\cdots T_{i-4}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n\\
&\quad+\sum_{i=5}^{n-5}(-1)^{n+i}T_3\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n\\
=&\sum_{i=4}^{n-5}(-1)^iT_3\cdots T_{i-2}T_i\cdots T_{n-4}T_2\cdots T_{n-5}\chi_n\\
&\quad+\sum_{i=4}^{n-5}(-1)^{i+1}T_2\cdots T_{i-2}T_i\cdots T_{n-4}T_2\cdots T_{n-5}T_{n-3}\chi_n\\
&\quad+\sum_{i=4}^{n-4}(-1)^{n+i+1}T_3\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-4}T_{n-5}T_{n-3}T_{n-4}\chi_n\\
&\quad+\sum_{i=4}^{n-4}(-1)^{n+i}T_2\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-6}T_{n-4}T_{n-5}T_{n-3}T_{n-4}\chi_n
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=4}^{n-4}(-1)^{n+i+1}T_3\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-6}T_{n-4}T_{n-3}T_{n-4}\chi_n\\
&+\sum_{i=4}^{n-4}(-1)^{n+i}T_2\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-6}T_{n-3}T_{n-4}\chi_n\\
&+\sum_{i=5}^{n-5}(-1)^{(n+1)(i+1)}(T_2\cdots T_{i-3})^2(T_{i-1}\cdots T_{n-4})^2(T_{n-4}+T_{n-3})\chi_n\\
&-\sum_{i=5}^{n-5}T_2\cdots T_{n-4}T_2\cdots T_{i-4}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n\\
&+\sum_{i=5}^{n-5}(-1)^{n+i}T_3\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n\\
=&\sum_{i=4}^{n-5}(-1)^{i}T_3\cdots T_{i-2}T_i\cdots T_{n-4}T_2\cdots T_{n-5}\chi_n\\
&+\sum_{i=4}^{n-4}(-1)^{n+i}T_2\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-6}(T_{n-4}T_{n-5}+T_{n-5}T_{n-6})T_{n-3}T_{n-4}\chi_n\\
&+\sum_{i=4}^{n-5}(-1)^{i+1}T_2\cdots T_{i-2}T_i\cdots T_{n-4}T_2\cdots T_{n-5}T_{n-3}\chi_n\\
&+\sum_{i=4}^{n-4}(-1)^{n+i+1}T_3\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}(T_{n-4}T_{n-5}-T_{n-6}T_{n-5})T_{n-3}T_{n-4}\chi_n\\
&+\sum_{i=5}^{n-5}(-1)^{(n+1)(i+1)}(T_2\cdots T_{i-3})^2(T_{i-1}\cdots T_{n-4})^2(T_{n-4}+T_{n-3})\chi_n\\
&-\sum_{i=5}^{n-5}T_2\cdots T_{n-4}T_2\cdots T_{i-4}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n\\
&+(-1)^nT_3\cdots T_{n-6}T_3\cdots T_{n-5}T_{n-6}T_{n-4}T_{n-5}T_{n-4}T_{n-3}T_{n-4}\chi_n\\
&+T_3\cdots T_{n-6}T_2\cdots T_{n-7}T_{n-5}T_{n-6}T_{n-4}T_{n-5}T_{n-4}T_{n-3}T_{n-4}\chi_n.
\end{aligned}
$$

Using the relation for $i=3,\dots,n-5$, $k=2,\dots,n-5$,

$$
T_2\cdots T_{i-2}T_i\cdots T_{n-4}T_k=\begin{cases}(-1)^nT_{k+1}T_2\cdots T_{i-2}T_i\cdots T_{n-4}+(-1)^{n+i}T_2\cdots T_{n-4} & k=i-2\\ (-1)^nT_{k+1}T_2\cdots T_{i-2}T_i\cdots T_{n-4}+(-1)^{n+i+1}T_2\cdots T_{n-4} & k=i-1\\ (-1)^nT_{k+1}T_2\cdots T_{i-2}T_i\cdots T_{n-4} & k\neq i-2,i-1,\end{cases}
$$

we have

$$
\begin{aligned}
&\sum_{i=4}^{n-5}(-1)^{i}T_2\cdots T_{i-2}T_i\cdots T_{n-4}T_2\cdots T_{n-5}\chi_n\\
=&\sum_{i=4}^{n-5}(-1)^{(n+1)i}T_3\cdots T_{i-2}T_2\cdots T_{i-2}T_i\cdots T_{n-4}T_{i-2}\cdots T_{n-5}\chi_n\\
=&\sum_{i=4}^{n-5}(-1)^{n+(n+1)i}T_3\cdots T_{i-1}T_2\cdots T_{i-2}T_i\cdots T_{n-4}T_{i-1}\cdots T_{n-5}\chi_n\\
&+\sum_{i=4}^{n-5}(-1)^{n(i+1)}T_3\cdots T_{i-2}T_2\cdots T_{n-4}T_{i-1}\cdots T_{n-5}\chi_n
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=4}^{n-5} (-1)^{(n+1)i} T_3 \cdots T_i T_2 \cdots T_{i-2} T_i \cdots T_{n-4} T_i \cdots T_{n-5} \chi_n \\
&\quad + \sum_{i=4}^{n-5} (-1)^{ni+1} T_3 \cdots T_{i-1} T_2 \cdots T_{n-4} T_i \cdots T_{n-5} \chi_n \\
&\quad + \sum_{i=4}^{n-5} (-1)^{n(i+1)} T_3 \cdots T_{i-2} T_2 \cdots T_{n-4} T_{i-1} \cdots T_{n-5} \chi_n \\
&= \sum_{i=4}^{n-5} (-1)^{n+i} T_3 \cdots T_{n-4} T_2 \cdots T_{i-2} T_i \cdots T_{n-4} \chi_n \\
&\quad + \sum_{i=4}^{n-5} (-1)^{n+i} T_2 \cdots T_{n-4} T_2 \cdots T_{i-2} T_i \cdots T_{n-5} \chi_n \\
&\quad + \sum_{i=4}^{n-5} (-1)^{n+i} T_2 \cdots T_{n-4} T_2 \cdots T_{i-3} T_{i-1} \cdots T_{n-5} \chi_n \\
&= \sum_{i=4}^{n-5} (-1)^{n+i} T_3 \cdots T_{n-4} T_2 \cdots T_{i-2} T_i \cdots T_{n-4} \chi_n \\
&\quad + \sum_{i=4}^{n-5} (-1)^{n+i} T_2 \cdots T_{n-4} T_2 \cdots T_{i-2} T_i \cdots T_{n-5} \chi_n \\
&\quad + \sum_{i=3}^{n-6} (-1)^{n+i+1} T_2 \cdots T_{n-4} T_2 \cdots T_{i-2} T_i \cdots T_{n-5} \chi_n \\
&= \sum_{i=4}^{n-5} (-1)^{n+i} T_3 \cdots T_{n-4} T_2 \cdots T_{i-2} T_i \cdots T_{n-4} \chi_n \\
&\quad - T_2 \cdots T_{n-4} T_2 \cdots T_{n-7} T_{n-5} \chi_n \\
&\quad + (-1)^n T_2 \cdots T_{n-4} T_3 \cdots T_{n-5} \chi_n .
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
&\sum_{i=5}^{n-5} (-1)^{(n+1)(i+1)} (T_2 \cdots T_{i-3})^2 (T_{i-1} \cdots T_{n-4})^2 (T_{n-4} + T_{n-3}) \chi_n \\
&= \sum_{i=5}^{n-5} (-1)^{n+(n+1)i} T_2 \cdots T_{i-3} T_2 \cdots T_{i-3} T_{i-1} \cdots T_{n-4} T_{i-1} \cdots T_{n-5} T_{n-3} T_{n-4} \chi_n \\
&= \sum_{i=5}^{n-5} (-1)^{n+ni+1} T_2 \cdots T_{i-2} T_2 \cdots T_{i-4} T_{i-2} T_{i-3} T_{i-1} \cdots T_{n-4} T_{i-1} \cdots T_{n-5} T_{n-3} T_{n-4} \chi_n \\
&= \sum_{i=5}^{n-5} (-1)^{n+ni} T_2 \cdots T_{i-2} T_2 \cdots T_{i-4} T_{i-1} \cdots T_{n-4} T_{i-1} \cdots T_{n-5} T_{n-3} T_{n-4} \chi_n \\
&\quad + \sum_{i=5}^{n-5} (-1)^{n+ni} T_2 \cdots T_{i-2} T_2 \cdots T_{n-4} T_{i-1} \cdots T_{n-5} T_{n-3} T_{n-4} \chi_n \\
&= \sum_{i=5}^{n-5} T_2 \cdots T_{n-4} T_2 \cdots T_{i-4} T_{i-1} \cdots T_{n-5} T_{n-3} T_{n-4} \chi_n \\
&\quad + \sum_{i=5}^{n-5} (-1)^{n+i} T_3 \cdots T_{n-4} T_2 \cdots T_{i-3} T_{i-1} \cdots T_{n-5} T_{n-3} T_{n-4} \chi_n .
\end{aligned}
$$

Hence we can express ϕ as

$$
\phi = - T_2 \cdots T_{n-4} T_2 \cdots T_{n-5} \chi_n
$$

$$+ (-1)^n T_3 \cdots T_{n-3} T_2 \cdots T_{n-5} \chi_n$$

$$- T_{n-3} T_{n-4} (1 + T_2 T_{n-3}) T_3 \cdots T_{n-6} T_2 \cdots T_{n-6} \chi_n$$

$$+ T_{n-4} (1 + T_2 T_{n-3}) T_3 \cdots T_{n-6} T_2 \cdots T_{n-5} \chi_n$$

$$+ (-1)^{n+1} (1 + T_2 T_{n-3}) T_3 \cdots T_{n-5} T_2 \cdots T_{n-5} \chi_n$$

$$+ (-1)^{n+1} T_3 \cdots T_{n-4} T_3 \cdots T_{n-5} T_{n-3} T_{n-4} \chi_n$$

$$+ (-1)^n T_3 \cdots T_{n-6} T_3 \cdots T_{n-5} T_{n-6} T_{n-4} T_{n-5} T_{n-4} T_{n-3} T_{n-4} \chi_n$$

$$+ T_3 \cdots T_{n-6} T_2 \cdots T_{n-7} T_{n-5} T_{n-6} T_{n-4} T_{n-5} T_{n-4} T_{n-3} T_{n-4} \chi_n$$

$$- T_3 \cdots T_{n-4} T_2 \cdots T_{n-7} T_{n-5} \chi_n$$

$$+ (-1)^n T_3 \cdots T_{n-4} T_3 \cdots T_{n-5} \chi_n$$

$$+ T_2 \cdots T_{n-4} T_2 \cdots T_{n-7} T_{n-5} T_{n-3} \chi_n$$

$$+ (-1)^{n+1} T_2 \cdots T_{n-4} T_3 \cdots T_{n-5} T_{n-3} \chi_n$$

$$+ \sum_{i=4}^{n-5} (-1)^{n+i} T_2 \cdots T_{n-4} T_2 \cdots T_{i-2} T_i \cdots T_{n-4} \chi_n$$

$$+ \sum_{i=4}^{n-4} (-1)^{n+i} T_2 \cdots T_{n-6} T_2 \cdots T_{i-3} T_{i-1} \cdots T_{n-6} (T_{n-4} T_{n-5} + T_{n-5} T_{n-6}) T_{n-3} T_{n-4} \chi_n$$

$$+ \sum_{i=4}^{n-5} (-1)^{n+i+1} T_3 \cdots T_{n-4} T_2 \cdots T_{i-2} T_i \cdots T_{n-3} \chi_n$$

$$+ \sum_{i=4}^{n-4} (-1)^{n+i+1} T_3 \cdots T_{n-6} T_2 \cdots T_{i-3} T_{i-1} \cdots T_{n-5} (T_{n-4} T_{n-5} - T_{n-6} T_{n-5}) T_{n-3} T_{n-4} \chi_n$$

$$+ \sum_{i=5}^{n-5} (-1)^{n+i} T_3 \cdots T_{n-4} T_2 \cdots T_{i-3} T_{i-1} \cdots T_{n-5} T_{n-3} T_{n-4} \chi_n$$

$$+ \sum_{i=4}^{n-5} \sum_{j=4}^{n-5} (-1)^{n+i+j} T_2 \cdots T_{j-2} T_j \cdots T_{n-4} T_2 \cdots T_{i-3} T_{i-1} \cdots T_{n-5} T_{n-3} T_{n-4} \chi_n$$

and rearranging using the identity $T_j (T_j \cdots T_k)^2 = (T_j \cdots T_k)^2 T_k$, we have

$$\phi = - T_2 \cdots T_{n-4} T_2 \cdots T_{n-5} \chi_n$$

$$+ (-1)^n T_3 \cdots T_{n-3} T_2 \cdots T_{n-5} \chi_n$$

$$- T_{n-3} T_{n-4} (1 + T_2 T_{n-3}) T_3 \cdots T_{n-6} T_2 \cdots T_{n-6} \chi_n$$

$$+ T_{n-4} (1 + T_2 T_{n-3}) T_3 \cdots T_{n-6} T_2 \cdots T_{n-5} \chi_n$$

$$+ (-1)^{n+1} (1 + T_2 T_{n-3}) T_3 \cdots T_{n-5} T_2 \cdots T_{n-5} \chi_n$$

$$+ (-1)^{n+1} T_3 \cdots T_{n-4} T_3 \cdots T_{n-5} T_{n-3} T_{n-4} \chi_n$$

$$+ (-1)^n T_3 \cdots T_{n-6} T_3 \cdots T_{n-5} T_{n-6} T_{n-4} T_{n-5} T_{n-4} T_{n-3} T_{n-4} \chi_n$$

$$+ T_3 \cdots T_{n-6} T_2 \cdots T_{n-7} T_{n-5} T_{n-6} T_{n-4} T_{n-5} T_{n-4} T_{n-3} T_{n-4} \chi_n$$

$$- T_3 \cdots T_{n-4} T_2 \cdots T_{n-7} T_{n-5} \chi_n$$

$$+ (-1)^n T_3 \cdots T_{n-4} T_3 \cdots T_{n-5} \chi_n$$

$$
\begin{aligned}
&+T_2\cdots T_{n-4}T_2\cdots T_{n-7}T_{n-5}T_{n-3}\chi_n\\
&+(-1)^{n+1}T_2\cdots T_{n-4}T_3\cdots T_{n-5}T_{n-3}\chi_n\\
&+\sum_{i=5}^{n-5}(-1)^{n+i}T_2\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-6}((T_{n-4}T_{n-5}+T_{n-5}T_{n-6})T_{n-3}T_{n-4}+T_{n-5}T_{n-6}T_{n-4}T_{n-5})\chi_n\\
&-T_2\cdots T_{n-4}T_2\cdots T_{n-7}T_{n-5}T_{n-4}\chi_n\\
&+(-1)^nT_2\cdots T_{n-6}T_3\cdots T_{n-6}(T_{n-4}T_{n-5}+T_{n-5}T_{n-6})T_{n-3}T_{n-4}\chi_n\\
&+T_2\cdots T_{n-6}T_2\cdots T_{n-7}(T_{n-4}T_{n-5}+T_{n-5}T_{n-6})T_{n-3}T_{n-4}\chi_n\\
&+\sum_{i=5}^{n-5}(-1)^{n+i+1}T_3\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}(T_{n-6}T_{n-4}T_{n-5}T_{n-3}+(T_{n-4}T_{n-5}-T_{n-6}T_{n-5})T_{n-3}T_{n-4}\\
&\qquad\qquad+T_{n-6}T_{n-4}T_{n-5}T_{n-3}T_{n-4}T_{n-3})\chi_n\\
&+T_3\cdots T_{n-4}T_2\cdots T_{n-7}T_{n-5}T_{n-4}T_{n-3}\chi_n\\
&+(-1)^{n+1}T_3\cdots T_{n-6}T_3\cdots T_{n-5}(T_{n-4}T_{n-5}-T_{n-6}T_{n-5})T_{n-3}T_{n-4}\chi_n\\
&-T_3\cdots T_{n-6}T_2\cdots T_{n-7}T_{n-5}(T_{n-4}T_{n-5}-T_{n-6}T_{n-5})T_{n-3}T_{n-4}\chi_n\\
&+\sum_{i=4}^{n-5}\sum_{j=4}^{n-5}(-1)^{n+i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n\\
=&(T_2\cdots T_{n-6})^2(T_{n-6}(1+T_{n-4}T_{n-3})-T_{n-4}-T_{n-3})\chi_n\\
&+(-1)^n(T_3\cdots T_{n-6})^2(T_{n-6}(1+(T_{n-5}-T_{n-4})T_{n-3})+T_{n-5}T_{n-4}T_{n-3}-T_{n-4})\chi_n\\
&+(-1)^nT_2(T_3\cdots T_{n-6})^2(-1+T_{n-6}(T_{n-5}(1-T_{n-4}T_{n-3})+T_{n-3})+T_{n-5}(T_{n-4}-T_{n-3}))\chi_n\\
&+\sum_{i=5}^{n-5}(-1)^{n+i}T_2\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-6}(T_{n-6}(T_{n-5}(-1+T_{n-4}T_{n-3})+T_{n-4})\\
&\qquad\qquad+T_{n-5}(-T_{n-4}+T_{n-3})-T_{n-4}T_{n-3})\chi_n\\
&+\sum_{i=5}^{n-5}(-1)^{n+i}T_3\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-6}(T_{n-6}(-1+T_{n-5}T_{n-4}+T_{n-4}T_{n-3})+T_{n-5}-T_{n-3})\chi_n\\
&+\sum_{i=4}^{n-5}\sum_{j=4}^{n-5}(-1)^{n+i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n.
\end{aligned}
$$

Next we want to substitute for χ_n on the right of each term. We start with the double sum. Using Lemma 3.3.4, when $n\geq 12$ we have

$$
\begin{aligned}
&\sum_{i=4}^{n-5}\sum_{j=4}^{n-5}(-1)^{n+i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n\\
=&\sum_{i=4}^{n-5}\sum_{j=4}^{n-5}\sum_{k=2}^{i-2}(-1)^{j+1}(k-1)T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-3}T_kT_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\\
&+\sum_{i=4}^{n-5}\sum_{j=4}^{n-5}\sum_{k=i-1}^{n-5}(-1)^{i+j+k+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{k-1}T_{k+1}\cdots T_{n-5}T_{n-3}T_{n-4}\\
&+\sum_{i=4}^{n-5}\sum_{j=4}^{n-5}(-1)^{n+i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}(T_{n-4}-T_{n-3})T_{n-3}T_{n-4}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=5}^{n-5}\sum_{j=4}^{n-5}\sum_{k=2}^{i-3}(-1)^{j+1}(k-1)T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-3}T_kT_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\\
&\quad+\sum_{i=4}^{n-5}\sum_{j=4}^{n-5}(-1)^{j+1}iT_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{n-5}T_{n-3}T_{n-4}\\
&\quad+\sum_{i=4}^{n-5}\sum_{j=4}^{n-5}\sum_{k=i-1}^{n-5}(-1)^{i+j+k+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{k-1}T_{k+1}\cdots T_{n-5}T_{n-3}T_{n-4}\\
&\quad+\sum_{i=4}^{n-5}\sum_{j=4}^{n-5}(-1)^{n+i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}(T_{n-4}-T_{n-3})\\
&= \sum_{i=6}^{n-5}\sum_{j=4}^{n-5}\sum_{k=2}^{i-4}(-1)^{j+1}(k-1)T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-3}T_kT_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\\
&\quad+\sum_{i=5}^{n-5}\sum_{j=4}^{n-5}(-1)^{j+1}(i-1)T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-4}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\\
&\quad+\sum_{i=4}^{n-5}\sum_{j=4}^{n-5}(-1)^{j+1}iT_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{n-5}T_{n-3}T_{n-4}\\
&\quad+\sum_{i=4}^{n-5}\sum_{j=4}^{n-5}\sum_{k=i-1}^{n-5}(-1)^{i+j+k+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{k-1}T_{k+1}\cdots T_{n-5}T_{n-3}T_{n-4}\\
&\quad+\sum_{i=4}^{n-5}\sum_{j=4}^{n-5}(-1)^{n+i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}(T_{n-4}-T_{n-3})\\
&= \sum_{i=6}^{n-5}\sum_{j=4}^{n-5}\sum_{k=2}^{i-4}(-1)^{i+j+k}(k-1)T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{k-1}(T_k+T_{k+1})T_{k+2}\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\\
&\quad+\sum_{i=5}^{n-5}\sum_{j=4}^{n-5}(-1)^{j+1}(i-1)T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-4}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\\
&\quad+\sum_{j=4}^{n-5}(-1)^{j+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{n-5}T_{n-3}T_{n-4}\\
&\quad+\sum_{i=4}^{n-5}\sum_{j=4}^{n-5}\sum_{k=i-1}^{n-5}(-1)^{i+j+k+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{k-1}T_{k+1}\cdots T_{n-5}T_{n-3}T_{n-4}\\
&\quad+\sum_{i=4}^{n-5}\sum_{j=4}^{n-5}(-1)^{n+i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}(T_{n-4}-T_{n-3})\\
&= \sum_{i=6}^{n-5}\sum_{j=4}^{n-5}\sum_{k=2}^{i-4}(-1)^{i+j+k}(k-1)T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_kT_{k+2}\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\\
&\quad+\sum_{i=6}^{n-5}\sum_{j=4}^{n-5}\sum_{k=2}^{i-4}(-1)^{i+j+k}(k-1)T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{k-1}T_{k+1}\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\\
&\quad+\sum_{i=5}^{n-5}\sum_{j=4}^{n-5}(-1)^{j+1}(i-1)T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-4}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\\
&\quad+\sum_{j=4}^{n-5}(-1)^{j+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{n-5}T_{n-3}T_{n-4}\\
&\quad+\sum_{i=4}^{n-5}\sum_{j=4}^{n-5}\sum_{k=i-1}^{n-5}(-1)^{i+j+k+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{k-1}T_{k+1}\cdots T_{n-5}T_{n-3}T_{n-4}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=4}^{n-5}\sum_{j=4}^{n-5}(-1)^{n+i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}(T_{n-4}-T_{n-3})\\
=&\sum_{i=6}^{n-5}\sum_{j=4}^{n-5}\sum_{k=3}^{i-3}(-1)^{i+j+k+1}(k+1)T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{k-1}T_{k+1}\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\\
&+\sum_{i=7}^{n-5}\sum_{j=4}^{n-5}\sum_{k=3}^{i-4}(-1)^{i+j+k}(k-1)T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{k-1}T_{k+1}\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\\
&+\sum_{i=6}^{n-5}\sum_{j=4}^{n-5}(-1)^{i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_3\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\\
&+\sum_{i=5}^{n-5}\sum_{j=4}^{n-5}(-1)^{j+1}(i-1)T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-4}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\\
&+\sum_{j=4}^{n-5}(-1)^{j+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{n-5}T_{n-3}T_{n-4}\\
&+\sum_{i=4}^{n-5}\sum_{j=4}^{n-5}\sum_{k=i-1}^{n-5}(-1)^{i+j+k+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{k-1}T_{k+1}\cdots T_{n-5}T_{n-3}T_{n-4}\\
&+\sum_{i=4}^{n-5}\sum_{j=4}^{n-5}(-1)^{n+i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}(T_{n-4}-T_{n-3})\\
=&\sum_{i=7}^{n-5}\sum_{j=4}^{n-5}\sum_{k=3}^{i-4}(-1)^{i+j+k+1}(k+1)T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{k-1}T_{k+1}\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\\
&+\sum_{i=6}^{n-5}\sum_{j=4}^{n-5}(-1)^{j}(i+1)T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-4}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\\
&+\sum_{i=7}^{n-5}\sum_{j=4}^{n-5}\sum_{k=3}^{i-4}(-1)^{i+j+k}(k-1)T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{k-1}T_{k+1}\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\\
&+\sum_{i=6}^{n-5}\sum_{j=4}^{n-5}(-1)^{i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_3\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\\
&+\sum_{i=5}^{n-5}\sum_{j=4}^{n-5}(-1)^{j+1}(i-1)T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-4}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\\
&+\sum_{j=4}^{n-5}(-1)^{j+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{n-5}T_{n-3}T_{n-4}\\
&+\sum_{i=4}^{n-5}\sum_{j=4}^{n-5}\sum_{k=i-1}^{n-5}(-1)^{i+j+k+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{k-1}T_{k+1}\cdots T_{n-5}T_{n-3}T_{n-4}\\
&+\sum_{i=4}^{n-5}\sum_{j=4}^{n-5}(-1)^{n+i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}(T_{n-4}-T_{n-3})\\
=&\sum_{i=7}^{n-5}\sum_{j=4}^{n-5}\sum_{k=3}^{i-4}(-1)^{i+j+k}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{k-1}T_{k+1}\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\\
&+\sum_{i=6}^{n-5}\sum_{j=4}^{n-5}(-1)^{j}(i+1)T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-4}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\\
&+\sum_{i=6}^{n-5}\sum_{j=4}^{n-5}(-1)^{i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_3\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}
\end{aligned}
$$

$$+\sum_{i=6}^{n-5}\sum_{j=4}^{n-5}(-1)^{j+1}(i-1)T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-4}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{j=4}^{n-5}(-1)^{j+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_4\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{j=4}^{n-5}(-1)^{j+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{i=4}^{n-5}\sum_{j=4}^{n-5}\sum_{k=i-1}^{n-5}(-1)^{i+j+k+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{k-1}T_{k+1}\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{i=4}^{n-5}\sum_{j=4}^{n-5}(-1)^{n+i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}(T_{n-4}-T_{n-3})$$

$$=\sum_{i=7}^{n-5}\sum_{j=4}^{n-5}\sum_{k=3}^{i-4}(-1)^{i+j+k}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{k-1}T_{k+1}\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{i=6}^{n-5}\sum_{j=4}^{n-5}(-1)^{j+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-4}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{i=6}^{n-5}\sum_{j=4}^{n-5}(-1)^{i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_3\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{j=4}^{n-5}(-1)^{j+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_4\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{j=4}^{n-5}(-1)^{j+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{i=4}^{n-5}\sum_{j=4}^{n-5}\sum_{k=i-1}^{n-5}(-1)^{i+j+k+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{k-1}T_{k+1}\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{i=4}^{n-5}\sum_{j=4}^{n-5}(-1)^{n+i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}(T_{n-4}-T_{n-3}).$$

Now we will change the order of summation in the triple sums so that they cancel. Reordering, we get

$$\sum_{i=4}^{n-5}\sum_{j=4}^{n-5}(-1)^{n+i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n$$

$$=\sum_{j=4}^{n-5}(-1)^{j+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{j=4}^{n-5}(-1)^{j+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_4\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{j=4}^{n-5}\sum_{i=4}^{n-5}(-1)^{n+i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}(T_{n-4}-T_{n-3})$$

$$+\sum_{j=4}^{n-5}\sum_{i=6}^{n-5}(-1)^{j+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-4}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{j=4}^{n-5}\sum_{i=6}^{n-5}(-1)^{i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_3\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{j=4}^{n-5}\sum_{i=5}^{n-7}\sum_{k=3}^{i-2}(-1)^{i+j+k}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{k-1}T_{k+1}\cdots T_{i-1}T_{i+1}\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{j=4}^{n-5}\sum_{i=2}^{n-7}\sum_{k=i+1}^{n-5}(-1)^{i+j+k+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-1}T_{i+1}\cdots T_{k-1}T_{k+1}\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$=\sum_{j=4}^{n-5}(-1)^{j+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{j=4}^{n-5}(-1)^{j+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_4\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{j=4}^{n-5}\sum_{i=4}^{n-5}(-1)^{n+i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}(T_{n-4}-T_{n-3})$$

$$+\sum_{j=4}^{n-5}\sum_{i=6}^{n-5}(-1)^{j+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-4}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{j=4}^{n-5}\sum_{i=6}^{n-5}(-1)^{i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_3\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{j=4}^{n-5}\sum_{i=3}^{n-9}\sum_{k=i+2}^{n-7}(-1)^{i+j+k}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-1}T_{i+1}\cdots T_{k-1}T_{k+1}\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{j=4}^{n-5}\sum_{i=2}^{n-7}\sum_{k=i+1}^{n-5}(-1)^{i+j+k+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-1}T_{i+1}\cdots T_{k-1}T_{k+1}\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$=\sum_{j=4}^{n-5}(-1)^{j+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{j=4}^{n-5}(-1)^{j+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_4\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{j=4}^{n-5}\sum_{i=4}^{n-5}(-1)^{n+i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}(T_{n-4}-T_{n-3})$$

$$+\sum_{j=4}^{n-5}\sum_{i=6}^{n-5}(-1)^{j+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-4}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{j=4}^{n-5}\sum_{i=6}^{n-5}(-1)^{i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_3\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{j=4}^{n-5}\sum_{k=4}^{n-7}(-1)^{j+k+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_3\cdots T_{k-1}T_{k+1}\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{j=4}^{n-5}\sum_{i=2}^{n-7}(-1)^{j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-1}T_{i+2}\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{j=4}^{n-5}\sum_{i=2}^{n-8}(-1)^{n+i+j+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-1}T_{i+1}\cdots T_{n-7}T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{j=4}^{n-5}\sum_{i=2}^{n-7}(-1)^{n+i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-1}T_{i+1}\cdots T_{n-6}T_{n-3}T_{n-4}$$

$$=\sum_{j=4}^{n-5}(-1)^{j+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$+\sum_{j=4}^{n-5}(-1)^{j+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_4\cdots T_{n-5}T_{n-3}T_{n-4}$$

$$
\begin{aligned}
&+\sum_{j=4}^{n-5}\sum_{i=2}^{n-7}(-1)^{n+i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-1}T_{i+1}\cdots T_{n-5}(T_{n-4}-T_{n-3})\\
&+\sum_{j=4}^{n-5}\sum_{i=2}^{n-8}(-1)^{n+i+j+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-1}T_{i+1}\cdots T_{n-7}T_{n-5}T_{n-3}T_{n-4}\\
&+\sum_{j=4}^{n-5}\sum_{i=2}^{n-7}(-1)^{n+i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-1}T_{i+1}\cdots T_{n-6}T_{n-3}T_{n-4}\\
&+\sum_{j=4}^{n-5}\sum_{i=3}^{n-8}(-1)^{j+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-1}T_{i+2}\cdots T_{n-5}T_{n-3}T_{n-4}\\
&+\sum_{j=4}^{n-5}\sum_{i=2}^{n-7}(-1)^{j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-1}T_{i+2}\cdots T_{n-5}T_{n-3}T_{n-4}\\
&+\sum_{j=4}^{n-5}\sum_{i=4}^{n-7}(-1)^{i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_3\cdots T_{i-1}T_{i+1}\cdots T_{n-5}T_{n-3}T_{n-4}\\
&+\sum_{j=4}^{n-5}\sum_{k=4}^{n-7}(-1)^{j+k+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_3\cdots T_{k-1}T_{k+1}\cdots T_{n-5}T_{n-3}T_{n-4}\\
=&\sum_{j=4}^{n-5}(-1)^{j+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{n-5}T_{n-3}T_{n-4}\\
&+\sum_{j=4}^{n-5}(-1)^{j+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_4\cdots T_{n-5}T_{n-3}T_{n-4}\\
&+\sum_{j=4}^{n-5}(-1)^{j+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{n-8}T_{n-6}T_{n-5}(T_{n-4}-T_{n-3})\\
&+\sum_{j=4}^{n-5}(-1)^{j+1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{n-8}T_{n-6}T_{n-3}T_{n-4}\\
&+\sum_{j=4}^{n-5}(-1)^{j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_4\cdots T_{n-5}T_{n-3}T_{n-4}\\
&+\sum_{j=4}^{n-5}(-1)^{j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{n-8}T_{n-5}T_{n-3}T_{n-4}\\
&+\sum_{j=4}^{n-5}\sum_{i=2}^{n-8}(-1)^{n+i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-1}T_{i+1}\cdots T_{n-5}((-T_{n-6}+T_{n-5})(1+T_{n-4}T_{n-3})+T_{n-4}-T_{n-3})\\
=&\sum_{j=4}^{n-5}(-1)^{j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}(\\
&\qquad T_2\cdots T_{n-8}(T_{n-7}T_{n-6}T_{n-5}(1+T_{n-4}T_{n-3})+T_{n-6}T_{n-5}(-T_{n-4}+T_{n-3})+(T_{n-6}-T_{n-5})(1+T_{n-4}T_{n-3}))\\
&\qquad+\sum_{i=2}^{n-8}(-1)^{n+i}T_2\cdots T_{i-1}T_{i+1}\cdots T_{n-5}((-T_{n-6}+T_{n-5})(1+T_{n-4}T_{n-3})+T_{n-4}-T_{n-3})).
\end{aligned}
$$

When $n=9$ we have

$$
\begin{aligned}
&\sum_{j=4}^{n-5}(-1)^{j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}(\\
&\qquad T_2\cdots T_{n-8}(T_{n-7}T_{n-6}T_{n-5}(1+T_{n-4}T_{n-3})+T_{n-6}T_{n-5}(-T_{n-4}+T_{n-3})+(T_{n-6}-T_{n-5})(1+T_{n-4}T_{n-3}))
\end{aligned}
$$

$$+\sum_{i=2}^{n-8}(-1)^{n+i}T_2\cdots T_{i-1}T_{i+1}\cdots T_{n-5}((-T_{n-6}+T_{n-5})(1+T_{n-4}T_{n-3})+T_{n-4}-T_{n-3}))$$

$$=T_2T_4T_5(T_2T_3T_4(1+T_5T_6)+T_3T_4(-T_5+T_6)+(T_3-T_4)(1+T_5T_6))$$

and

$$\sum_{i=4}^{n-5}\sum_{j=4}^{n-5}(-1)^{n+i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}\chi_n$$
$$=-T_2T_4T_5T_3T_4T_6T_5(T_2-T_3+T_5-T_6)$$
$$=T_2T_4T_5(T_2T_3T_4(1+T_5T_6)+T_3T_4(-T_5+T_6)+(T_3-T_4)(1+T_5T_6)).$$

Hence we can make the substitution for all $n\geq 9$ and ϕ becomes

$$\begin{aligned}
\phi =&(T_2\cdots T_{n-6})^2(T_{n-6}(1+T_{n-4}T_{n-3})-T_{n-4}-T_{n-3})\chi_n\\
&+(-1)^n(T_3\cdots T_{n-6})^2(T_{n-6}(1+(T_{n-5}-T_{n-4})T_{n-3})+T_{n-5}T_{n-4}T_{n-3}-T_{n-4})\chi_n\\
&+(-1)^nT_2(T_3\cdots T_{n-6})^2(-1+T_{n-6}(T_{n-5}(1-T_{n-4}T_{n-3})+T_{n-3})+T_{n-5}(T_{n-4}-T_{n-3}))\chi_n\\
&+\sum_{i=5}^{n-5}(-1)^{n+i}T_2\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-6}(T_{n-6}(T_{n-5}(-1+T_{n-4}T_{n-3})+T_{n-4})\\
&\qquad+T_{n-5}(-T_{n-4}+T_{n-3})-T_{n-4}T_{n-3})\chi_n\\
&+\sum_{i=5}^{n-5}(-1)^{n+i}T_3\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-6}(T_{n-6}(-1+T_{n-5}T_{n-4}+T_{n-4}T_{n-3})+T_{n-5}-T_{n-3})\chi_n\\
&+\sum_{j=4}^{n-5}(-1)^jT_2\cdots T_{j-2}T_j\cdots T_{n-4}(\\
&\quad T_2\cdots T_{n-8}(T_{n-7}T_{n-6}T_{n-5}(1+T_{n-4}T_{n-3})+T_{n-6}T_{n-5}(-T_{n-4}+T_{n-3})+(T_{n-6}-T_{n-5})(1+T_{n-4}T_{n-3}))\\
&\quad+\sum_{i=2}^{n-8}(-1)^{n+i}T_2\cdots T_{i-1}T_{i+1}\cdots T_{n-5}((-T_{n-6}+T_{n-5})(1+T_{n-4}T_{n-3})+T_{n-4}-T_{n-3})).
\end{aligned}$$

Next we need to expand χ_n in the single sums. Using Lemma 3.3.4, we have

$$\begin{aligned}
&\sum_{i=5}^{n-5}(-1)^iT_3\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-6}\chi_n\\
=&\sum_{i=5}^{n-5}\sum_{k=2}^{i-2}(-1)^n(k-1)T_3\cdots T_{n-6}T_2\cdots T_{i-3}T_kT_{i-1}\cdots T_{n-6}\\
&+\sum_{i=5}^{n-5}\sum_{k=i-1}^{n-6}(-1)^{n+i+k}T_3\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{k-1}T_{k+1}\cdots T_{n-6}\\
&+\sum_{i=5}^{n-5}(-1)^iT_3\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-6}(T_{n-4}-T_{n-3})\\
=&\sum_{i=6}^{n-5}\sum_{k=2}^{i-4}(-1)^{n+i+k+1}(k-1)T_3\cdots T_{n-6}T_2\cdots T_{k-1}(T_k+T_{k+1})T_{k+2}\cdots T_{i-3}T_{i-1}\cdots T_{n-6}\\
&+\sum_{i=5}^{n-5}(-1)^n(i-1)T_3\cdots T_{n-6}T_2\cdots T_{i-4}T_{i-1}\cdots T_{n-6}\\
&+\sum_{i=5}^{n-5}(-1)^niT_3\cdots T_{n-6}T_2\cdots T_{n-6}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=5}^{n-5}\sum_{k=i-1}^{n-6}(-1)^{n+i+k}T_3\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{k-1}T_{k+1}\cdots T_{n-6}\\
&+\sum_{i=5}^{n-5}(-1)^{i}T_3\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-6}(T_{n-4}-T_{n-3})\\
=&\sum_{i=6}^{n-5}\sum_{k=3}^{i-3}(-1)^{n+i+k}(k+1)T_3\cdots T_{n-6}T_2\cdots T_{k-1}T_{k+1}\cdots T_{i-3}T_{i-1}\cdots T_{n-6}\\
&+\sum_{i=6}^{n-5}\sum_{k=2}^{i-4}(-1)^{n+i+k+1}(k-1)T_3\cdots T_{n-6}T_2\cdots T_{k-1}T_{k+1}\cdots T_{i-3}T_{i-1}\cdots T_{n-6}\\
&+\sum_{i=5}^{n-5}(-1)^{n}(i-1)T_3\cdots T_{n-6}T_2\cdots T_{i-4}T_{i-1}\cdots T_{n-6}\\
&+\sum_{i=5}^{n-5}(-1)^{n}iT_3\cdots T_{n-6}T_2\cdots T_{n-6}\\
&+\sum_{i=5}^{n-5}\sum_{k=i-1}^{n-6}(-1)^{n+i+k}T_3\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{k-1}T_{k+1}\cdots T_{n-6}\\
&+\sum_{i=5}^{n-5}(-1)^{i}T_3\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-6}(T_{n-4}-T_{n-3})\\
=&\sum_{i=7}^{n-5}\sum_{k=3}^{i-4}(-1)^{n+i+k+1}T_3\cdots T_{n-6}T_2\cdots T_{k-1}T_{k+1}\cdots T_{i-3}T_{i-1}\cdots T_{n-6}\\
&+\sum_{i=6}^{n-5}(-1)^{n+1}(i+1)T_3\cdots T_{n-6}T_2\cdots T_{i-4}T_{i-1}\cdots T_{n-6}\\
&+\sum_{i=6}^{n-5}(-1)^{n+i+1}T_3\cdots T_{n-6}T_3\cdots T_{i-3}T_{i-1}\cdots T_{n-6}\\
&+\sum_{i=5}^{n-5}(-1)^{n}(i-1)T_3\cdots T_{n-6}T_2\cdots T_{i-4}T_{i-1}\cdots T_{n-6}\\
&+\sum_{i=5}^{n-5}\sum_{k=i-1}^{n-6}(-1)^{n+i+k}T_3\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{k-1}T_{k+1}\cdots T_{n-6}\\
&+\sum_{i=5}^{n-5}(-1)^{i}T_3\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-6}(T_{n-4}-T_{n-3})\\
=&\sum_{i=7}^{n-5}\sum_{k=3}^{i-4}(-1)^{n+i+k+1}T_3\cdots T_{n-6}T_2\cdots T_{k-1}T_{k+1}\cdots T_{i-3}T_{i-1}\cdots T_{n-6}\\
&+\sum_{i=6}^{n-5}(-1)^{n+i+1}T_3\cdots T_{n-6}T_3\cdots T_{i-3}T_{i-1}\cdots T_{n-6}\\
&+\sum_{i=6}^{n-5}(-1)^{n}T_3\cdots T_{n-6}T_2\cdots T_{i-4}T_{i-1}\cdots T_{n-6}\\
&+(-1)^{n}T_3\cdots T_{n-6}T_4\cdots T_{n-6}\\
&+\sum_{i=5}^{n-5}\sum_{k=i-1}^{n-6}(-1)^{n+i+k}T_3\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{k-1}T_{k+1}\cdots T_{n-6}\\
&+\sum_{i=5}^{n-5}(-1)^{i}T_3\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-6}(T_{n-4}-T_{n-3})
\end{aligned}
$$

$$
\begin{aligned}
=&(-1)^n T_3 \cdots T_{n-6} T_4 \cdots T_{n-6} \\
&+ \sum_{i=5}^{n-5} (-1)^i T_3 \cdots T_{n-6} T_2 \cdots T_{i-3} T_{i-1} \cdots T_{n-6} (T_{n-4} - T_{n-3}) \\
&+ \sum_{i=6}^{n-5} (-1)^n T_3 \cdots T_{n-6} T_2 \cdots T_{i-4} T_{i-1} \cdots T_{n-6} \\
&+ \sum_{i=6}^{n-5} (-1)^{n+i+1} T_3 \cdots T_{n-6} T_3 \cdots T_{i-3} T_{i-1} \cdots T_{n-6} \\
&+ \sum_{k=3}^{n-9} \sum_{i=k+2}^{n-7} (-1)^{n+i+k+1} T_3 \cdots T_{n-6} T_2 \cdots T_{k-1} T_{k+1} \cdots T_{i-1} T_{i+1} \cdots T_{n-6} \\
&+ \sum_{i=3}^{n-7} \sum_{k=i+1}^{n-6} (-1)^{n+i+k} T_3 \cdots T_{n-6} T_2 \cdots T_{i-1} T_{i+1} \cdots T_{k-1} T_{k+1} \cdots T_{n-6} \\
=&(-1)^n T_3 \cdots T_{n-6} T_4 \cdots T_{n-6} \\
&+ \sum_{i=5}^{n-5} (-1)^i T_3 \cdots T_{n-6} T_2 \cdots T_{i-3} T_{i-1} \cdots T_{n-6} (T_{n-4} - T_{n-3}) \\
&+ \sum_{i=6}^{n-5} (-1)^n T_3 \cdots T_{n-6} T_2 \cdots T_{i-4} T_{i-1} \cdots T_{n-6} \\
&+ \sum_{i=6}^{n-5} (-1)^{n+i+1} T_3 \cdots T_{n-6} T_3 \cdots T_{i-3} T_{i-1} \cdots T_{n-6} \\
&+ \sum_{i=3}^{n-7} (-1)^{n+1} T_3 \cdots T_{n-6} T_2 \cdots T_{i-1} T_{i+2} \cdots T_{n-6} \\
&+ \sum_{i=3}^{n-8} (-1)^i T_3 \cdots T_{n-6} T_2 \cdots T_{i-1} T_{i+1} \cdots T_{n-7} \\
&+ \sum_{k=3}^{n-9} \sum_{i=k+2}^{n-7} (-1)^{n+i+k+1} T_3 \cdots T_{n-6} T_2 \cdots T_{k-1} T_{k+1} \cdots T_{i-1} T_{i+1} \cdots T_{n-6} \\
&+ \sum_{i=3}^{n-9} \sum_{k=i+2}^{n-7} (-1)^{n+i+k} T_3 \cdots T_{n-6} T_2 \cdots T_{i-1} T_{i+1} \cdots T_{k-1} T_{k+1} \cdots T_{n-6} \\
=&(-1)^n T_3 \cdots T_{n-6} T_4 \cdots T_{n-6} \\
&+ \sum_{i=5}^{n-5} (-1)^i T_3 \cdots T_{n-6} T_2 \cdots T_{i-3} T_{i-1} \cdots T_{n-6} (T_{n-4} - T_{n-3}) \\
&+ \sum_{i=5}^{n-6} (-1)^i T_3 \cdots T_{n-6} T_2 \cdots T_{i-3} T_{i-1} \cdots T_{n-7} \\
&+ \sum_{i=6}^{n-5} (-1)^n T_3 \cdots T_{n-6} T_2 \cdots T_{i-4} T_{i-1} \cdots T_{n-6} \\
&+ \sum_{i=6}^{n-4} (-1)^{n+1} T_3 \cdots T_{n-6} T_2 \cdots T_{i-4} T_{i-1} \cdots T_{n-6} \\
&+ \sum_{i=6}^{n-5} (-1)^{n+i+1} T_3 \cdots T_{n-6} T_3 \cdots T_{i-3} T_{i-1} \cdots T_{n-6} \\
=&(-1)^n T_3 \cdots T_{n-6} T_4 \cdots T_{n-6}
\end{aligned}
$$

$$
\begin{aligned}
&+(-1)^{n+1}T_3\cdots T_{n-6}T_2\cdots T_{n-8}T_{n-6}(T_{n-4}-T_{n-3})\\
&+(-1)^{n+1}T_3\cdots T_{n-6}T_2\cdots T_{n-8}\\
&+\sum_{i=5}^{n-6}(-1)^iT_3\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-7}(1+T_{n-6}(T_{n-4}-T_{n-3}))\\
&+\sum_{i=6}^{n-5}(-1)^{n+i+1}T_3\cdots T_{n-6}T_3\cdots T_{i-3}T_{i-1}\cdots T_{n-6}\\
=&\sum_{i=5}^{n-5}(-1)^iT_3\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-6}(T_{n-6}+T_{n-4}-T_{n-3})\\
&+\sum_{i=5}^{n-5}(-1)^{n+i+1}T_3\cdots T_{n-6}T_3\cdots T_{i-3}T_{i-1}\cdots T_{n-6}.
\end{aligned}
$$

Hence, again using Lemma 3.3.4 to expand the remaining terms with χ_n on the right, we can write

$$
\begin{aligned}
\phi =&(T_2\cdots T_{n-6})^2(T_{n-6}(1+T_{n-4}T_{n-3})-T_{n-4}-T_{n-3})\chi_n\\
&+(-1)^n(T_3\cdots T_{n-6})^2(T_{n-6}(1+(T_{n-5}-T_{n-4})T_{n-3})+T_{n-5}T_{n-4}T_{n-3}-T_{n-4})\chi_n\\
&+(-1)^nT_2(T_3\cdots T_{n-6})^2(-1+T_{n-6}(T_{n-5}(1-T_{n-4}T_{n-3})+T_{n-3})+T_{n-5}(T_{n-4}-T_{n-3}))\chi_n\\
&+\sum_{i=5}^{n-5}(-1)^{n+i}T_2\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-6}(T_{n-6}+T_{n-4}-T_{n-3})(T_{n-6}(T_{n-5}(-1+T_{n-4}T_{n-3})+T_{n-4})\\
&\qquad\qquad+T_{n-5}(-T_{n-4}+T_{n-3})-T_{n-4}T_{n-3})\\
&+\sum_{i=5}^{n-5}(-1)^{i+1}T_2\cdots T_{n-6}T_3\cdots T_{i-3}T_{i-1}\cdots T_{n-6}(T_{n-6}(T_{n-5}(-1+T_{n-4}T_{n-3})+T_{n-4})\\
&\qquad\qquad+T_{n-5}(-T_{n-4}+T_{n-3})-T_{n-4}T_{n-3})\\
&+\sum_{i=5}^{n-5}(-1)^{n+i+1}T_3\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-6}(T_{n-6}+T_{n-4}-T_{n-3})\\
&\qquad\qquad\cdot(T_{n-6}(-1+T_{n-5}T_{n-4}+T_{n-4}T_{n-3})+T_{n-5}-T_{n-3})\\
&+\sum_{i=5}^{n-5}(-1)^iT_3\cdots T_{n-6}T_3\cdots T_{i-3}T_{i-1}\cdots T_{n-6}(T_{n-6}(-1+T_{n-5}T_{n-4}+T_{n-4}T_{n-3})+T_{n-5}-T_{n-3})\\
&+\sum_{j=4}^{n-5}(-1)^jT_2\cdots T_{j-2}T_j\cdots T_{n-4}(\\
&\qquad T_2\cdots T_{n-8}(T_{n-7}T_{n-6}T_{n-5}(1+T_{n-4}T_{n-3})+T_{n-6}T_{n-5}(-T_{n-4}+T_{n-3})+(T_{n-6}-T_{n-5})(1+T_{n-4}T_{n-3}))\\
&\qquad+\sum_{i=2}^{n-8}(-1)^{n+i}T_2\cdots T_{i-1}T_{i+1}\cdots T_{n-5}((-T_{n-6}+T_{n-5})(1+T_{n-4}T_{n-3})+T_{n-4}-T_{n-3}))\\
=&\sum_{k=2}^{n-6}(-1)^{n+k}T_2\cdots T_{n-6}T_2\cdots T_{k-1}T_{k+1}\cdots T_{n-6}(T_{n-6}(-1-T_{n-4}T_{n-3})+T_{n-4}+T_{n-3})\\
&+T_2\cdots T_{n-6}T_2\cdots T_{n-6}(T_{n-4}-T_{n-3})(T_{n-6}(-1-T_{n-4}T_{n-3})+T_{n-4}+T_{n-3})\\
&-T_3\cdots T_{n-6}T_2\cdots T_{n-6}(T_{n-6}(1+(T_{n-5}-T_{n-4})T_{n-3})+T_{n-5}T_{n-4}T_{n-3}-T_{n-4})\\
&+\sum_{k=3}^{n-6}(-1)^{k+1}T_3\cdots T_{n-6}T_3\cdots T_{k-1}T_{k+1}\cdots T_{n-6}(T_{n-6}(1+(T_{n-5}-T_{n-4})T_{n-3})+T_{n-5}T_{n-4}T_{n-3}-T_{n-4})
\end{aligned}
$$

$$
\begin{aligned}
&+(-1)^{n+1}T_3\cdots T_{n-6}T_3\cdots T_{n-6}(T_{n-4}-T_{n-3})(T_{n-6}(1+(T_{n-5}-T_{n-4})T_{n-3})+T_{n-5}T_{n-4}T_{n-3}-T_{n-4})\\
&+T_2\cdots T_{n-6}T_2\cdots T_{n-6}(-1+T_{n-6}(T_{n-5}(1-T_{n-4}T_{n-3})+T_{n-3})+T_{n-5}(T_{n-4}-T_{n-3}))\\
&+\sum_{k=3}^{n-6}(-1)^kT_2\cdots T_{n-6}T_3\cdots T_{k-1}T_{k+1}\cdots T_{n-6}(-1+T_{n-6}(T_{n-5}(1-T_{n-4}T_{n-3})+T_{n-3})+T_{n-5}(T_{n-4}-T_{n-3}))\\
&+(-1)^nT_2\cdots T_{n-6}T_3\cdots T_{n-6}(T_{n-4}-T_{n-3})(-1+T_{n-6}(T_{n-5}(1-T_{n-4}T_{n-3})+T_{n-3})+T_{n-5}(T_{n-4}-T_{n-3}))\\
&+\sum_{i=5}^{n-5}(-1)^{n+i}T_2\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-6}(T_{n-6}+T_{n-4}-T_{n-3})(T_{n-6}(T_{n-5}(-1+T_{n-4}T_{n-3})+T_{n-4})\\
&\qquad\qquad\qquad\qquad\qquad\qquad+T_{n-5}(-T_{n-4}+T_{n-3})-T_{n-4}T_{n-3})\\
&+\sum_{i=5}^{n-5}(-1)^{i+1}T_2\cdots T_{n-6}T_3\cdots T_{i-3}T_{i-1}\cdots T_{n-6}(T_{n-6}(T_{n-5}(-1+T_{n-4}T_{n-3})+T_{n-4})\\
&\qquad\qquad\qquad\qquad\qquad\qquad+T_{n-5}(-T_{n-4}+T_{n-3})-T_{n-4}T_{n-3})\\
&+\sum_{i=5}^{n-5}(-1)^{n+i+1}T_3\cdots T_{n-6}T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-6}(T_{n-6}+T_{n-4}-T_{n-3})\\
&\qquad\qquad\qquad\qquad\cdot(T_{n-6}(-1+T_{n-5}T_{n-4}+T_{n-4}T_{n-3})+T_{n-5}-T_{n-3})\\
&+\sum_{i=5}^{n-5}(-1)^iT_3\cdots T_{n-6}T_3\cdots T_{i-3}T_{i-1}\cdots T_{n-6}(T_{n-6}(-1+T_{n-5}T_{n-4}+T_{n-4}T_{n-3})+T_{n-5}-T_{n-3})\\
&+\sum_{j=4}^{n-5}(-1)^jT_2\cdots T_{j-2}T_j\cdots T_{n-4}(\\
&\qquad T_2\cdots T_{n-8}(T_{n-7}T_{n-6}T_{n-5}(1+T_{n-4}T_{n-3})+T_{n-6}T_{n-5}(-T_{n-4}+T_{n-3})+(T_{n-6}-T_{n-5})(1+T_{n-4}T_{n-3}))\\
&\qquad+\sum_{i=2}^{n-8}(-1)^{n+i}T_2\cdots T_{i-1}T_{i+1}\cdots T_{n-5}((-T_{n-6}+T_{n-5})(1+T_{n-4}T_{n-3})+T_{n-4}-T_{n-3}))\\
=&(T_2\cdots T_{n-6})^2(-1+T_{n-6}(T_{n-5}(1+T_{n-4}T_{n-3})-T_{n-3})+T_{n-5}(T_{n-4}+T_{n-3}))\\
&+(-1)^nT_2(T_3\cdots T_{n-6})^2(T_{n-6}(-1+T_{n-4}T_{n-3})+T_{n-4}-T_{n-3})\\
&+(-1)^{n+1}(T_3\cdots T_{n-6})^2(1+T_{n-6}(T_{n-5}(-1+T_{n-4}T_{n-3})+T_{n-3})+T_{n-5}(-T_{n-4}+T_{n-3})+T_{n-4}T_{n-3})\\
&+\sum_{k=2}^{n-6}(-1)^{n+k}T_2\cdots T_{n-6}T_2\cdots T_{k-1}T_{k+1}\cdots T_{n-6}(T_{n-6}(-1-T_{n-4}T_{n-3})+T_{n-4}+T_{n-3})\\
&+\sum_{i=3}^{n-7}(-1)^{n+i}T_2\cdots T_{n-6}T_2\cdots T_{i-1}T_{i+1}\cdots T_{n-6}(T_{n-6}(T_{n-5}(-T_{n-4}+T_{n-3})-T_{n-4}T_{n-3})\\
&\qquad\qquad\qquad\qquad\qquad\qquad+T_{n-5}(-1+T_{n-4}T_{n-3})+T_{n-4})\\
&+\sum_{k=3}^{n-6}(-1)^kT_2\cdots T_{n-6}T_3\cdots T_{k-1}T_{k+1}\cdots T_{n-6}(-1+T_{n-6}(T_{n-5}(1-T_{n-4}T_{n-3})+T_{n-3})+T_{n-5}(T_{n-4}-T_{n-3}))\\
&+\sum_{i=3}^{n-7}(-1)^{i+1}T_2\cdots T_{n-6}T_3\cdots T_{i-1}T_{i+1}\cdots T_{n-6}(T_{n-6}(T_{n-5}(-1+T_{n-4}T_{n-3})+T_{n-4})\\
&\qquad\qquad\qquad\qquad\qquad\qquad+T_{n-5}(-T_{n-4}+T_{n-3})-T_{n-4}T_{n-3})\\
&+\sum_{i=3}^{n-7}(-1)^{n+i+1}T_3\cdots T_{n-6}T_2\cdots T_{i-1}T_{i+1}\cdots T_{n-6}(-1+T_{n-6}(T_{n-5}T_{n-4}T_{n-3}+T_{n-4}-T_{n-3})+T_{n-5}T_{n-3})
\end{aligned}
$$

$$+\sum_{k=3}^{n-6}(-1)^{k+1}T_3\cdots T_{n-6}T_3\cdots T_{k-1}T_{k+1}\cdots T_{n-6}(T_{n-6}(1+(T_{n-5}-T_{n-4})T_{n-3})+T_{n-5}T_{n-4}T_{n-3}-T_{n-4})$$
$$+\sum_{i=3}^{n-7}(-1)^{i}T_3\cdots T_{n-6}T_3\cdots T_{i-1}T_{i+1}\cdots T_{n-6}(T_{n-6}(-1+T_{n-5}T_{n-4}+T_{n-4}T_{n-3})+T_{n-5}-T_{n-3})$$
$$+\sum_{j=4}^{n-5}(-1)^{j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{n-8}$$
$$\cdot(T_{n-7}T_{n-6}T_{n-5}(1+T_{n-4}T_{n-3})+T_{n-6}T_{n-5}(-T_{n-4}+T_{n-3})+(T_{n-6}-T_{n-5})(1+T_{n-4}T_{n-3}))$$
$$+\sum_{j=4}^{n-5}\sum_{i=2}^{n-8}(-1)^{n+i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-1}T_{i+1}\cdots T_{n-5}((-T_{n-6}+T_{n-5})(1+T_{n-4}T_{n-3})+T_{n-4}-T_{n-3}).$$

When $n=9$, we have

$$\begin{aligned}\phi =&T_3T_2(-1+T_3(T_4(1+T_5T_6)-T_6)+T_4(T_5+T_6))\\
&-T_2(T_3(-1+T_5T_6)+T_5-T_6)\\
&+(1+T_3(T_4(-1+T_5T_6)+T_6)+T_4(-T_5+T_6)+T_5T_6)\\
&-T_2(T_3(-1-T_5T_6)+T_5+T_6)\\
&+T_2T_3T_2(T_3(-1-T_5T_6)+T_5+T_6)\\
&-T_2T_3(-1+T_3(T_4(1-T_5T_6)+T_6)+T_4(T_5-T_6))\\
&+T_3(T_3(1+(T_4-T_5)T_6)+T_4T_5T_6-T_5)\\
&+T_2T_4T_5(T_2T_3T_4(1+T_5T_6)+T_3T_4(-T_5+T_6)+(T_3-T_4)(1+T_5T_6))\\
=&0.\end{aligned}$$

Now we assume that $n\geq 12$. Then

$$\begin{aligned}\phi =&(T_2\cdots T_{n-6})^2(-1+T_{n-6}(T_{n-5}(1+T_{n-4}T_{n-3})-T_{n-3})+T_{n-5}(T_{n-4}+T_{n-3}))\\
&+(-1)^nT_2(T_3\cdots T_{n-6})^2(T_{n-6}(-1+T_{n-4}T_{n-3})+T_{n-4}-T_{n-3})\\
&+(-1)^{n+1}(T_3\cdots T_{n-6})^2(1+T_{n-6}(T_{n-5}(-1+T_{n-4}T_{n-3})+T_{n-3})+T_{n-5}(-T_{n-4}+T_{n-3})+T_{n-4}T_{n-3})\\
&+(-1)^nT_2\cdots T_{n-6}T_3\cdots T_{n-6}(T_{n-6}(-1-T_{n-4}T_{n-3})+T_{n-4}+T_{n-3})\\
&+T_2\cdots T_{n-6}T_2\cdots T_{n-7}(T_{n-6}(-1-T_{n-4}T_{n-3})+T_{n-4}+T_{n-3})\\
&+(-1)^nT_2\cdots T_{n-6}T_3\cdots T_{n-7}(-1+T_{n-6}(T_{n-5}(1-T_{n-4}T_{n-3})+T_{n-3})+T_{n-5}(T_{n-4}-T_{n-3}))\\
&+(-1)^{n+1}T_3\cdots T_{n-6}T_3\cdots T_{n-7}(T_{n-6}(1+(T_{n-5}-T_{n-4})T_{n-3})+T_{n-5}T_{n-4}T_{n-3}-T_{n-4})\\
&+\sum_{i=3}^{n-7}(-1)^{n+i}T_2\cdots T_{n-6}T_2\cdots T_{i-1}T_{i+1}\cdots T_{n-6}(T_{n-6}(-1+T_{n-5}(-T_{n-4}+T_{n-3})+T_{n-4}T_{n-3})\\
&\qquad+T_{n-5}(-1+T_{n-4}T_{n-3})-T_{n-4}+T_{n-3})\\
&+\sum_{i=3}^{n-7}(-1)^{i+1}T_2\cdots T_{n-6}T_3\cdots T_{i-1}T_{i+1}\cdots T_{n-6}(1+T_{n-6}(T_{n-5}(1-T_{n-4}T_{n-3})+T_{n-4}-T_{n-3})\\
&\qquad+T_{n-5}(T_{n-4}-T_{n-3})-T_{n-4}T_{n-3})\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=3}^{n-7}(-1)^{n+i+1}T_3\cdots T_{n-6}T_2\cdots T_{i-1}T_{i+1}\cdots T_{n-6}(-1+T_{n-6}(T_{n-5}T_{n-4}T_{n-3}+T_{n-4}-T_{n-3})+T_{n-5}T_{n-3})\\
&+\sum_{i=3}^{n-7}(-1)^{i}T_3\cdots T_{n-6}T_3\cdots T_{i-1}T_{i+1}\cdots T_{n-6}(T_{n-6}(1+T_{n-5}(T_{n-4}-T_{n-3})-T_{n-4}T_{n-3})\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+T_{n-5}(1-T_{n-4}T_{n-3})+T_{n-4}-T_{n-3})\\
&+\sum_{j=4}^{n-5}(-1)^{j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{n-8}\\
&\qquad\cdot(T_{n-7}T_{n-6}T_{n-5}(1+T_{n-4}T_{n-3})+T_{n-6}T_{n-5}(-T_{n-4}+T_{n-3})+(T_{n-6}-T_{n-5})(1+T_{n-4}T_{n-3}))\\
&+\sum_{j=4}^{n-5}\sum_{i=2}^{n-8}(-1)^{n+i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-1}T_{i+1}\cdots T_{n-5}((-T_{n-6}+T_{n-5})(1+T_{n-4}T_{n-3})+T_{n-4}-T_{n-3})\\
=&(T_2\cdots T_{n-6})^2(1+T_{n-6}(T_{n-5}(1+T_{n-4}T_{n-3})+T_{n-4})+T_{n-5}(T_{n-4}+T_{n-3})-T_{n-4}T_{n-3})\\
&+(-1)^nT_2(T_3\cdots T_{n-6})^2(T_{n-6}T_{n-5}(T_{n-4}-T_{n-3})+T_{n-5}(1-T_{n-4}T_{n-3})-T_{n-4}+T_{n-3})\\
&+(-1)^{n+1}(T_3\cdots T_{n-6})^2(-1+T_{n-6}(T_{n-5}(-1-T_{n-4}T_{n-3})-T_{n-4}+T_{n-3})+T_{n-5}(-T_{n-4}-T_{n-3}))\\
&+\sum_{i=3}^{n-7}(-1)^{n+i}T_2\cdots T_{n-6}T_2\cdots T_{i-1}T_{i+1}\cdots T_{n-6}(T_{n-6}(-1+T_{n-5}(-T_{n-4}+T_{n-3})+T_{n-4}T_{n-3})\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+T_{n-5}(-1+T_{n-4}T_{n-3})-T_{n-4}+T_{n-3})\\
&+\sum_{i=3}^{n-7}(-1)^{i+1}T_2\cdots T_{n-6}T_3\cdots T_{i-1}T_{i+1}\cdots T_{n-6}(1+T_{n-6}(T_{n-5}(1-T_{n-4}T_{n-3})+T_{n-4}-T_{n-3})\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+T_{n-5}(T_{n-4}-T_{n-3})-T_{n-4}T_{n-3})\\
&+\sum_{i=3}^{n-7}(-1)^{n+i+1}T_3\cdots T_{n-6}T_2\cdots T_{i-1}T_{i+1}\cdots T_{n-6}(-1+T_{n-6}(T_{n-5}T_{n-4}T_{n-3}+T_{n-4}-T_{n-3})+T_{n-5}T_{n-3})\\
&+\sum_{i=3}^{n-7}(-1)^{i}T_3\cdots T_{n-6}T_3\cdots T_{i-1}T_{i+1}\cdots T_{n-6}(T_{n-6}(1+T_{n-5}(T_{n-4}-T_{n-3})-T_{n-4}T_{n-3})\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+T_{n-5}(1-T_{n-4}T_{n-3})+T_{n-4}-T_{n-3})\\
&+\sum_{j=4}^{n-5}(-1)^{j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{n-8}\\
&\qquad\cdot(T_{n-7}T_{n-6}T_{n-5}(1+T_{n-4}T_{n-3})+T_{n-6}T_{n-5}(-T_{n-4}+T_{n-3})+(T_{n-6}-T_{n-5})(1+T_{n-4}T_{n-3}))\\
&+\sum_{j=4}^{n-5}\sum_{i=2}^{n-8}(-1)^{n+i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-1}T_{i+1}\cdots T_{n-5}((-T_{n-6}+T_{n-5})(1+T_{n-4}T_{n-3})+T_{n-4}-T_{n-3}).
\end{aligned}
$$

Now

$$
\begin{aligned}
&\sum_{j=4}^{n-5}(-1)^{j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{n-8}\\
=&\sum_{j=4}^{n-5}(-1)^{(n+1)j}T_3\cdots T_{j-2}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_{j-2}\cdots T_{n-8}\\
=&(-1)^{n+1}T_3\cdots T_{n-7}T_2\cdots T_{n-7}T_{n-5}T_{n-4}\\
&+\sum_{j=4}^{n-6}(-1)^{n+(n+1)j}T_3\cdots T_{j-1}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_{j-1}\cdots T_{n-8}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=4}^{n-6}(-1)^{n(j+1)}T_3\cdots T_{j-2}T_2\cdots T_{n-4}T_{j-1}\cdots T_{n-8}\\
=&(-1)^{n+1}T_3\cdots T_{n-7}T_2\cdots T_{n-7}T_{n-5}T_{n-4}\\
&+(-1)^{n}T_3\cdots T_{n-7}T_2\cdots T_{n-8}T_{n-6}T_{n-5}T_{n-4}\\
&+\sum_{j=4}^{n-7}(-1)^{(n+1)j}T_3\cdots T_jT_2\cdots T_{j-2}T_j\cdots T_{n-4}T_j\cdots T_{n-8}\\
&+\sum_{j=4}^{n-7}(-1)^{nj+1}T_3\cdots T_{j-1}T_2\cdots T_{n-4}T_j\cdots T_{n-8}\\
&+\sum_{j=4}^{n-6}(-1)^{n+j}T_2\cdots T_{n-4}T_2\cdots T_{j-3}T_{j-1}\cdots T_{n-8}\\
=&(T_2\cdots T_{n-6})^2(T_{n-7}-T_{n-6})T_{n-5}T_{n-4}\\
&+\sum_{j=4}^{n-7}(-1)^{j}T_3\cdots T_{n-7}T_2\cdots T_{j-2}T_j\cdots T_{n-4}\\
&+\sum_{j=4}^{n-7}(-1)^{n+j}T_2\cdots T_{n-4}T_2\cdots T_{j-2}T_j\cdots T_{n-8}\\
&+\sum_{j=4}^{n-6}(-1)^{n+j}T_2\cdots T_{n-4}T_2\cdots T_{j-3}T_{j-1}\cdots T_{n-8}\\
=&-(T_2\cdots T_{n-6})^2T_{n-7}T_{n-5}T_{n-4}\\
&+(-1)^{n+1}T_2(T_3\cdots T_{n-6})^2(1+T_{n-7}T_{n-6})T_{n-5}T_{n-4}\\
&+\sum_{j=3}^{n-7}(-1)^{n+j+1}T_3\cdots T_{n-6}T_2\cdots T_{j-1}T_{j+1}\cdots T_{n-6}T_{n-7}T_{n-5}T_{n-4}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\phi =&(T_2\cdots T_{n-6})^2(1+T_{n-6}(T_{n-5}(1+T_{n-4}T_{n-3})+T_{n-4})+T_{n-5}(T_{n-4}+T_{n-3})-T_{n-4}T_{n-3})\\
&-(T_2\cdots T_{n-6})^2T_{n-7}T_{n-5}T_{n-4}\\
&\qquad\cdot(T_{n-7}T_{n-6}T_{n-5}(1+T_{n-4}T_{n-3})+T_{n-6}T_{n-5}(-T_{n-4}+T_{n-3})+(T_{n-6}-T_{n-5})(1+T_{n-4}T_{n-3}))\\
&+(-1)^nT_2(T_3\cdots T_{n-6})^2(T_{n-6}T_{n-5}(T_{n-4}-T_{n-3})+T_{n-5}(1-T_{n-4}T_{n-3})-T_{n-4}+T_{n-3})\\
&+(-1)^{n+1}T_2(T_3\cdots T_{n-6})^2(1+T_{n-7}T_{n-6})T_{n-5}T_{n-4}\\
&\qquad\cdot(T_{n-7}T_{n-6}T_{n-5}(1+T_{n-4}T_{n-3})+T_{n-6}T_{n-5}(-T_{n-4}+T_{n-3})+(T_{n-6}-T_{n-5})(1+T_{n-4}T_{n-3}))\\
&+(-1)^{n+1}(T_3\cdots T_{n-6})^2(-1+T_{n-6}(T_{n-5}(-1-T_{n-4}T_{n-3})-T_{n-4}+T_{n-3})+T_{n-5}(-T_{n-4}-T_{n-3}))\\
&+\sum_{i=3}^{n-7}(-1)^{n+i}T_2\cdots T_{n-6}T_2\cdots T_{i-1}T_{i+1}\cdots T_{n-6}(T_{n-6}(-1+T_{n-5}(-T_{n-4}+T_{n-3})+T_{n-4}T_{n-3})\\
&\qquad\qquad+T_{n-5}(-1+T_{n-4}T_{n-3})-T_{n-4}+T_{n-3})\\
&+\sum_{i=3}^{n-7}(-1)^{i+1}T_2\cdots T_{n-6}T_3\cdots T_{i-1}T_{i+1}\cdots T_{n-6}(1+T_{n-6}(T_{n-5}(1-T_{n-4}T_{n-3})+T_{n-4}-T_{n-3})\\
&\qquad\qquad+T_{n-5}(T_{n-4}-T_{n-3})-T_{n-4}T_{n-3})
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=3}^{n-7}(-1)^{n+i+1}T_3\cdots T_{n-6}T_2\cdots T_{i-1}T_{i+1}\cdots T_{n-6}(-1+T_{n-6}(T_{n-5}T_{n-4}T_{n-3}+T_{n-4}-T_{n-3})+T_{n-5}T_{n-3})\\
&+\sum_{j=3}^{n-7}(-1)^{n+j+1}T_3\cdots T_{n-6}T_2\cdots T_{j-1}T_{j+1}\cdots T_{n-6}T_{n-7}T_{n-5}T_{n-4}\\
&\qquad\cdot(T_{n-7}T_{n-6}T_{n-5}(1+T_{n-4}T_{n-3})+T_{n-6}T_{n-5}(-T_{n-4}+T_{n-3})+(T_{n-6}-T_{n-5})(1+T_{n-4}T_{n-3}))\\
&+\sum_{i=3}^{n-7}(-1)^{i}T_3\cdots T_{n-6}T_3\cdots T_{i-1}T_{i+1}\cdots T_{n-6}(T_{n-6}(1+T_{n-5}(T_{n-4}-T_{n-3})-T_{n-4}T_{n-3})\\
&\qquad+T_{n-5}(1-T_{n-4}T_{n-3})+T_{n-4}-T_{n-3})\\
&+\sum_{j=4}^{n-5}\sum_{i=2}^{n-8}(-1)^{n+i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-4}T_2\cdots T_{i-1}T_{i+1}\cdots T_{n-5}((-T_{n-6}+T_{n-5})(1+T_{n-4}T_{n-3})+T_{n-4}-T_{n-3})\\
=&(T_2\cdots T_{n-6})^2(-1+T_{n-7}(T_{n-6}(-1+T_{n-5}T_{n-4}+T_{n-4}T_{n-3})+T_{n-5}-T_{n-3})\\
&\qquad+T_{n-6}(T_{n-5}(-1-T_{n-4}T_{n-3})-T_{n-4}+T_{n-3})+T_{n-5}(-T_{n-4}-T_{n-3}))\\
&+(-1)^nT_2(T_3\cdots T_{n-6})^2(T_{n-7}(-1+T_{n-6}(T_{n-5}-T_{n-3})+T_{n-5}T_{n-4}+T_{n-4}T_{n-3})+T_{n-6}(1+T_{n-5}(T_{n-4}+T_{n-3}))\\
&\qquad+T_{n-5}(1+T_{n-4}T_{n-3})+T_{n-4}-T_{n-3})\\
&+(-1)^{n+1}(T_3\cdots T_{n-6})^2(-1+T_{n-6}(T_{n-5}(-1-T_{n-4}T_{n-3})-T_{n-4}+T_{n-3})+T_{n-5}(-T_{n-4}-T_{n-3}))\\
&+\sum_{i=3}^{n-7}(-1)^{n+i}T_2\cdots T_{n-6}T_2\cdots T_{i-1}T_{i+1}\cdots T_{n-6}(T_{n-6}(-1+T_{n-5}(-T_{n-4}+T_{n-3})+T_{n-4}T_{n-3})\\
&\qquad+T_{n-5}(-1+T_{n-4}T_{n-3})-T_{n-4}+T_{n-3})\\
&+\sum_{i=3}^{n-7}(-1)^{i+1}T_2\cdots T_{n-6}T_3\cdots T_{i-1}T_{i+1}\cdots T_{n-6}(1+T_{n-6}(T_{n-5}(1-T_{n-4}T_{n-3})+T_{n-4}-T_{n-3})\\
&\qquad+T_{n-5}(T_{n-4}-T_{n-3})-T_{n-4}T_{n-3})\\
&+\sum_{i=3}^{n-7}(-1)^{n+i+1}T_3\cdots T_{n-6}T_2\cdots T_{i-1}T_{i+1}\cdots T_{n-6}(1+T_{n-7}(T_{n-6}(1-T_{n-5}T_{n-4}-T_{n-4}T_{n-3})-T_{n-5}+T_{n-3})\\
&\qquad+T_{n-6}(-T_{n-5}+T_{n-3})-T_{n-5}T_{n-4}-T_{n-4}T_{n-3})\\
&+\sum_{i=3}^{n-7}(-1)^{i}T_3\cdots T_{n-6}T_3\cdots T_{i-1}T_{i+1}\cdots T_{n-6}(T_{n-6}(1+T_{n-5}(T_{n-4}-T_{n-3})-T_{n-4}T_{n-3})\\
&\qquad+T_{n-5}(1-T_{n-4}T_{n-3})+T_{n-4}-T_{n-3})\\
&+\sum_{j=3}^{n-6}\sum_{i=2}^{n-8}(-1)^{n+i+j}T_2\cdots T_{j-1}T_{j+1}\cdots T_{n-6}T_2\cdots T_{i-1}T_{i+1}\cdots T_{n-6}\\
&\qquad\cdot(1+T_{n-6}(-T_{n-5}+T_{n-3})-T_{n-5}T_{n-4}-T_{n-4}T_{n-3}).
\end{aligned}
$$

It is still unclear how the remaining terms cancel, so we will have to write each term in standard form.

Considering the double sum, we have

$$
\begin{aligned}
&\sum_{j=3}^{n-6}\sum_{i=2}^{n-8}(-1)^{n+i+j}T_2\cdots T_{j-1}T_{j+1}\cdots T_{n-6}T_2\cdots T_{i-1}T_{i+1}\cdots T_{n-6}\\
=&\sum_{j=3}^{n-6}(\sum_{i=2}^{j-1}(-1)^{n+i+j}T_2\cdots T_{j-1}T_{j+1}\cdots T_{n-6}T_2\cdots T_{i-1}T_{i+1}\cdots T_{n-6}
\end{aligned}
$$

$$+\sum_{i=j}^{n-6}(-1)^{n+i+j}T_2\cdots T_{j-1}T_{j+1}\cdots T_{n-6}T_2\cdots T_{i-1}T_{i+1}\cdots T_{n-6}$$

$$+(-1)^{j+1}T_2\cdots T_{j-1}T_{j+1}\cdots T_{n-6}T_2\cdots T_{n-6}(T_{n-7}+T_{n-6})$$

$$+(-1)^{j+1}T_2\cdots T_{j-1}T_{j+1}\cdots T_{n-6}T_2\cdots T_{n-7})$$

$$=\sum_{j=3}^{n-6}(\sum_{i=2}^{j-1}(-1)^{i+(n+i+1)j}(T_2\cdots T_i)^2T_i(T_{i+1}\cdots T_{j-1})^2(T_j\cdots T_{n-6})^2T_{n-6}$$

$$+\sum_{i=j}^{n-6}(-1)^{n+(n+j)i}(T_2\cdots T_{j-1})^2(T_j\cdots T_i)^2(T_{i+1}\cdots T_{n-6})^2$$

$$+(-1)^{nj+1}(T_2\cdots T_{j-1})^2(T_j\cdots T_{n-6})^2(1-T_{n-7}T_{n-6}))$$

$$=\sum_{j=3}^{n-6}(\sum_{i=2}^{j-1}(-1)^{i+(n+i+1)j}T_2(T_2\cdots T_i)^2(T_{i+1}\cdots T_{j-1})^2T_j(T_j\cdots T_{n-6})^2$$

$$+\sum_{i=2}^{j-1}(-1)^{n+(n+i+1)j}(T_2\cdots T_i)^2(T_{i+1}\cdots T_j)^2(T_{j+1}\cdots T_{n-6})^2$$

$$+(-1)^{nj+1}(T_2\cdots T_{j-1})^2(T_j\cdots T_{n-6})^2(1-T_{n-7}T_{n-6})).$$

Now using the relation $(T_j\cdots T_k)^2=(-1)^{(k-j+1)(k-j)/2}\prod_{x=j}^{k-1}(1+T_xT_{x+1})$ for $j\le k+1$ (where we interpret the empty product as the identity matrix), we have

$$\sum_{j=3}^{n-6}\sum_{i=2}^{n-8}(-1)^{n+i+j}T_2\cdots T_{j-1}T_{j+1}\cdots T_{n-6}T_2\cdots T_{i-1}T_{i+1}\cdots T_{n-6}$$

$$=(-1)^{(n-1)(n-2)/2}\sum_{j=3}^{n-6}\left(\sum_{i=2}^{j-1}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)T_i\left(\prod_{y=i+1}^{j-2}(1+T_yT_{y+1})\right)\left(\prod_{z=j}^{n-7}(1+T_zT_{z+1})\right)T_{n-6}\right.$$

$$-\sum_{i=2}^{j-1}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)\left(\prod_{y=i+1}^{j-1}(1+T_yT_{y+1})\right)\left(\prod_{z=j+1}^{n-7}(1+T_zT_{z+1})\right)$$

$$\left.+\left(\prod_{x=2}^{j-2}(1+T_xT_{x+1})\right)\left(\prod_{y=j}^{n-7}(1+T_yT_{y+1})\right)(1-T_{n-7}T_{n-6})\right)$$

$$=(-1)^{(n-1)(n-2)/2}\left(\sum_{i=2}^{n-7}\sum_{j=i+1}^{n-6}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)T_i\left(\prod_{y=i+1}^{j-2}(1+T_yT_{y+1})\right)\left(\prod_{z=j}^{n-7}(1+T_zT_{z+1})\right)T_{n-6}\right.$$

$$-\sum_{i=2}^{n-7}\sum_{j=i+1}^{n-6}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)\left(\prod_{y=i+1}^{j-1}(1+T_yT_{y+1})\right)\left(\prod_{z=j+1}^{n-7}(1+T_zT_{z+1})\right)$$

$$\left.+\sum_{i=2}^{n-7}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)\left(\prod_{y=i+1}^{n-7}(1+T_yT_{y+1})\right)(1-T_{n-7}T_{n-6})\right).$$

Applying the same technique to the other terms of ϕ, we have

$$\phi=(T_2\cdots T_{n-6})^2(-1+T_{n-7}(T_{n-6}(-1+T_{n-5}T_{n-4}+T_{n-4}T_{n-3})+T_{n-5}-T_{n-3})$$

$$+T_{n-6}(T_{n-5}(-1-T_{n-4}T_{n-3})-T_{n-4}+T_{n-3})+T_{n-5}(-T_{n-4}-T_{n-3}))$$

$$+(-1)^nT_2(T_3\cdots T_{n-6})^2(T_{n-7}(-1+T_{n-6}(T_{n-5}-T_{n-3})+T_{n-5}T_{n-4}+T_{n-4}T_{n-3})+T_{n-6}(1+T_{n-5}(T_{n-4}+T_{n-3}))$$

$$+T_{n-5}(1+T_{n-4}T_{n-3})+T_{n-4}-T_{n-3})$$

$$+(-1)^{n+1}(T_3\cdots T_{n-6})^2(-1+T_{n-6}(T_{n-5}(-1-T_{n-4}T_{n-3})-T_{n-4}+T_{n-3})+T_{n-5}(-T_{n-4}-T_{n-3}))$$

$$+\sum_{i=3}^{n-7}(-1)^{n(i+1)}(T_2\cdots T_i)^2T_i(T_{i+1}\cdots T_{n-6})^2$$

$$\cdot(T_{n-6}(-1+T_{n-5}(-T_{n-4}+T_{n-3})+T_{n-4}T_{n-3})+T_{n-5}(-1+T_{n-4}T_{n-3})-T_{n-4}+T_{n-3})$$

$$+\sum_{i=3}^{n-7}(-1)^{(n+1)(i+1)}T_2(T_3\cdots T_i)^2T_i(T_{i+1}\cdots T_{n-6})^2$$

$$\cdot(1+T_{n-6}(T_{n-5}(1-T_{n-4}T_{n-3})+T_{n-4}-T_{n-3})+T_{n-5}(T_{n-4}-T_{n-3})-T_{n-4}T_{n-3})$$

$$+\sum_{i=3}^{n-7}(-1)^{n(i+1)+1}(T_2\cdots T_i)^2(T_{i+1}\cdots T_{n-6})^2$$

$$\cdot(1+T_{n-7}(T_{n-6}(1-T_{n-5}T_{n-4}-T_{n-4}T_{n-3})-T_{n-5}+T_{n-3})+T_{n-6}(-T_{n-5}+T_{n-3})-T_{n-5}T_{n-4}-T_{n-4}T_{n-3})$$

$$+\sum_{i=3}^{n-7}(-1)^{n+(n+1)i}(T_3\cdots T_i)^2T_i(T_{i+1}\cdots T_{n-6})^2$$

$$\cdot(T_{n-6}(1+T_{n-5}(T_{n-4}-T_{n-3})-T_{n-4}T_{n-3})+T_{n-5}(1-T_{n-4}T_{n-3})+T_{n-4}-T_{n-3})$$

$$+\sum_{j=3}^{n-6}\sum_{i=2}^{n-8}(-1)^{n+i+j}T_2\cdots T_{j-1}T_{j+1}\cdots T_{n-6}T_2\cdots T_{i-1}T_{i+1}\cdots T_{n-6}$$

$$\cdot(1-T_{n-7}T_{n-6})(1+T_{n-6}(-T_{n-5}+T_{n-3})-T_{n-5}T_{n-4}-T_{n-4}T_{n-3})$$

$$=(-1)^{n(n+1)/2}\left(\prod_{x=2}^{n-7}(1+T_xT_{x+1})\right)(-1+T_{n-7}(T_{n-6}(-1+T_{n-5}T_{n-4}+T_{n-4}T_{n-3})+T_{n-5}-T_{n-3})$$

$$+T_{n-6}(T_{n-5}(-1-T_{n-4}T_{n-3})-T_{n-4}+T_{n-3})+T_{n-5}(-T_{n-4}-T_{n-3}))$$

$$+(-1)^{n(n+1)/2}T_2\left(\prod_{x=3}^{n-7}(1+T_xT_{x+1})\right)(T_{n-7}(-1+T_{n-6}(T_{n-5}-T_{n-3})+T_{n-5}T_{n-4}+T_{n-4}T_{n-3})$$

$$+T_{n-6}(1+T_{n-5}(T_{n-4}+T_{n-3}))+T_{n-5}(1+T_{n-4}T_{n-3})+T_{n-4}-T_{n-3})$$

$$+(-1)^{(n-2)(n-1)/2}\left(\prod_{x=3}^{n-7}(1+T_xT_{x+1})\right)(-1+T_{n-6}(T_{n-5}(-1-T_{n-4}T_{n-3})-T_{n-4}+T_{n-3})+T_{n-5}(-T_{n-4}-T_{n-3}))$$

$$+\sum_{i=3}^{n-7}(-1)^{n(n+1)/2}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)T_i\left(\prod_{y=i+1}^{n-7}(1+T_yT_{y+1})\right)$$

$$\cdot(T_{n-6}(-1+T_{n-5}(-T_{n-4}+T_{n-3})+T_{n-4}T_{n-3})+T_{n-5}(-1+T_{n-4}T_{n-3})-T_{n-4}+T_{n-3})$$

$$+\sum_{i=3}^{n-7}(-1)^{(n-2)(n-1)/2}T_2\left(\prod_{x=3}^{i-1}(1+T_xT_{x+1})\right)T_i\left(\prod_{y=i+1}^{n-7}(1+T_yT_{y+1})\right)$$

$$\cdot(1+T_{n-6}(T_{n-5}(1-T_{n-4}T_{n-3})+T_{n-4}-T_{n-3})+T_{n-5}(T_{n-4}-T_{n-3})-T_{n-4}T_{n-3})$$

$$+\sum_{i=3}^{n-7}(-1)^{(n-2)(n-1)/2}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)\left(\prod_{y=i+1}^{n-7}(1+T_yT_{y+1})\right)$$

$$\cdot(1+T_{n-7}(T_{n-6}(1-T_{n-5}T_{n-4}-T_{n-4}T_{n-3})-T_{n-5}+T_{n-3})+T_{n-6}(-T_{n-5}+T_{n-3})-T_{n-5}T_{n-4}-T_{n-4}T_{n-3})$$

$$+\sum_{i=3}^{n-7}(-1)^{n(n+1)/2}\left(\prod_{x=3}^{i-1}(1+T_xT_{x+1})\right)T_i\left(\prod_{y=i+1}^{n-7}(1+T_yT_{y+1})\right)$$

$$\cdot(T_{n-6}(1+T_{n-5}(T_{n-4}-T_{n-3})-T_{n-4}T_{n-3})+T_{n-5}(1-T_{n-4}T_{n-3})+T_{n-4}-T_{n-3})$$

$$+(-1)^{(n-1)(n-2)/2}\sum_{i=2}^{n-7}\sum_{j=i+1}^{n-6}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)T_i\left(\prod_{y=i+1}^{j-2}(1+T_yT_{y+1})\right)\left(\prod_{z=j}^{n-7}(1+T_zT_{z+1})\right)$$
$$\cdot T_{n-6}(1+T_{n-6}(-T_{n-5}+T_{n-3})-T_{n-5}T_{n-4}-T_{n-4}T_{n-3})$$
$$-(-1)^{(n-1)(n-2)/2}\sum_{i=2}^{n-7}\sum_{j=i+1}^{n-6}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)\left(\prod_{y=i+1}^{j-1}(1+T_yT_{y+1})\right)\left(\prod_{z=j+1}^{n-7}(1+T_zT_{z+1})\right)$$
$$\cdot(1+T_{n-6}(-T_{n-5}+T_{n-3})-T_{n-5}T_{n-4}-T_{n-4}T_{n-3})$$
$$+(-1)^{(n-1)(n-2)/2}\sum_{i=2}^{n-7}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)\left(\prod_{y=i+1}^{n-7}(1+T_yT_{y+1})\right)$$
$$\cdot(1-T_{n-7}T_{n-6})(1+T_{n-6}(-T_{n-5}+T_{n-3})-T_{n-5}T_{n-4}-T_{n-4}T_{n-3}).$$

Rearranging and multiplying by $(-1)^{(n-2)(n-1)/2}$, we get

$$(1-T_2T_3)\left(\prod_{x=3}^{n-8}(1+T_xT_{x+1})\right)(1+T_{n-7}(T_{n-6}(1-T_{n-5}T_{n-4}-T_{n-4}T_{n-3})-T_{n-5}+T_{n-3})$$
$$+T_{n-6}(-T_{n-5}+T_{n-3})-T_{n-5}T_{n-4}-T_{n-4}T_{n-3})$$
$$+\sum_{i=3}^{n-7}\left(\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)\left(\prod_{y=i+1}^{n-7}(1+T_yT_{y+1})\right)\right.$$
$$\cdot(-1+T_{n-6}(T_{n-5}-T_{n-3})+T_{n-5}T_{n-4}+T_{n-4}T_{n-3})$$
$$+T_2\left(\prod_{x=3}^{n-7}(1+T_xT_{x+1})\right)$$
$$\left.\cdot(T_{n-6}(1+T_{n-5}(T_{n-4}-T_{n-3})-T_{n-4}T_{n-3})+T_{n-5}(1-T_{n-4}T_{n-3})+T_{n-4}-T_{n-3})\right)$$
$$+\sum_{i=2}^{n-7}\sum_{j=i+1}^{n-6}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)\left(T_i\left(\prod_{y=i+1}^{j-2}(1+T_yT_{y+1})\right)T_j\left(\prod_{z=j}^{n-7}(1+T_zT_{z+1})\right)\right.$$
$$\cdot(1+T_{n-6}(-T_{n-5}+T_{n-3})-T_{n-5}T_{n-4}-T_{n-4}T_{n-3})$$
$$-\left(\prod_{y=i+1}^{j-1}(1+T_yT_{y+1})\right)\left(\prod_{z=j+1}^{n-7}(1+T_zT_{z+1})\right)$$
$$\left.\cdot(1+T_{n-6}(-T_{n-5}+T_{n-3})-T_{n-5}T_{n-4}-T_{n-4}T_{n-3})\right).$$
$$=(1-T_2T_3)\left(\prod_{x=3}^{n-7}(1+T_xT_{x+1})\right)(1+T_{n-6}(-T_{n-5}+T_{n-3})-T_{n-5}T_{n-4}-T_{n-4}T_{n-3})$$
$$+\sum_{i=3}^{n-7}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)\left(\prod_{y=i+1}^{n-7}(1+T_yT_{y+1})\right)$$
$$\cdot(-1+T_{n-6}(T_{n-5}-T_{n-3})+T_{n-5}T_{n-4}+T_{n-4}T_{n-3})$$
$$+\sum_{i=2}^{n-7}\sum_{j=i+1}^{n-6}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)\left(T_i\left(\prod_{y=i+1}^{j-2}(1+T_yT_{y+1})\right)T_j\left(\prod_{z=j}^{n-7}(1+T_zT_{z+1})\right)\right.$$
$$\cdot(1+T_{n-6}(-T_{n-5}+T_{n-3})-T_{n-5}T_{n-4}-T_{n-4}T_{n-3})$$

$$- \left(\prod_{y=i+1}^{j-1} (1 + T_y T_{y+1}) \right) \left(\prod_{z=j+1}^{n-7} (1 + T_z T_{z+1}) \right)$$

$$\cdot (1 + T_{n-6}(-T_{n-5} + T_{n-3}) - T_{n-5}T_{n-4} - T_{n-4}T_{n-3}) \Bigg).$$

Since the matrix $(1 + T_{n-6}(-T_{n-5} + T_{n-3}) - T_{n-5}T_{n-4} - T_{n-4}T_{n-3})$ is invertible, we are left to show that the following basic spin element ψ is equal to 0:

$$\begin{aligned}
\psi :=& (-1)^{(n-2)(n-1)/2} \phi (T_{n-6} - T_{n-4})(T_{n-5} - T_{n-3}) \\
=& (1 - T_2 T_3) \left(\prod_{x=3}^{n-7} (1 + T_x T_{x+1}) \right) - \sum_{i=3}^{n-7} \left(\prod_{x=2}^{i-1} (1 + T_x T_{x+1}) \right) \left(\prod_{y=i+1}^{n-7} (1 + T_y T_{y+1}) \right) \\
&+ \sum_{i=2}^{n-7} \sum_{j=i}^{n-7} \left(\prod_{x=2}^{i-1} (1 + T_x T_{x+1}) \right) \left(T_i \left(\prod_{y=i+1}^{j-1} (1 + T_y T_{y+1}) \right) T_{j+1} - \left(\prod_{y=i+1}^{j-1} (1 + T_y T_{y+1}) \right) \right) \left(\prod_{z=j+1}^{n-7} (1 + T_z T_{z+1}) \right).
\end{aligned}$$

The coefficient of the identity matrix in the standard form of ψ is

$$1 - \sum_{i=3}^{n-7} 1 - \sum_{i=2}^{n-7} \sum_{j=i}^{n-7} 1 = 1 + \sum_{i=2}^{n-7} i = 0.$$

We need to show that the for each set $I = \{i_1, \ldots, i_m\}$ with $2 \leq i_1 < \cdots < i_m \leq n-6$, the term $T_I := T_{i_1} T_{i_2} \cdots T_{i_m}$ has coefficient 0. We can assume that m is even (and ≥ 2) since all of the terms in ψ have even length.

If $i_m = 3$, then $T_I = T_2 T_3$ has coefficient

$$-1 - \sum_{i=3}^{n-7} 1 + 1 - \sum_{i=3}^{n-7} \sum_{j=i}^{n-7} 1 = \sum_{i=3}^{n-7} (i-1) = 0.$$

For $k = 2, \ldots n-6$, define ψ_k to be the basic spin element equal to ψ without any terms T_I with $|I \cap \{k+1, \ldots, n-6\}| > 0$. Then $\psi = 0$ if and only if $\psi_k = 0$ for all k. We have already shown that $\psi_2 = \psi_3 = 0$. For $k > 3$, we can write

$$\begin{aligned}
\psi_k =& (1 - T_2 T_3) \left(\prod_{x=3}^{k-1} (1 + T_x T_{x+1}) \right) \\
&- \sum_{i=3}^{k-1} \left(\prod_{x=2}^{i-1} (1 + T_x T_{x+1}) \right) \left(\prod_{y=i+1}^{k-1} (1 + T_y T_{y+1}) \right) \\
&- \sum_{i=k}^{n-7} \left(\prod_{x=2}^{k-1} (1 + T_x T_{x+1}) \right) \\
&+ \sum_{i=2}^{k-1} \sum_{j=i}^{k-1} \left(\prod_{x=2}^{i-1} (1 + T_x T_{x+1}) \right) \left(T_i \left(\prod_{y=i+1}^{j-1} (1 + T_y T_{y+1}) \right) T_{j+1} - \left(\prod_{y=i+1}^{j-1} (1 + T_y T_{y+1}) \right) \right) \left(\prod_{z=j+1}^{k-1} (1 + T_z T_{z+1}) \right) \\
&- \sum_{i=2}^{k-1} \sum_{j=k}^{n-7} \left(\prod_{x=2}^{i-1} (1 + T_x T_{x+1}) \right) \left(\prod_{y=i+1}^{k-1} (1 + T_y T_{y+1}) \right) \\
&- \sum_{i=k}^{n-7} \sum_{j=i}^{n-7} \left(\prod_{x=2}^{k-1} (1 + T_x T_{x+1}) \right) \\
=& (1 - T_2 T_3) \left(\prod_{x=3}^{k-1} (1 + T_x T_{x+1}) \right)
\end{aligned}$$

$$
\begin{aligned}
&+k^2\left(\prod_{x=2}^{k-1}(1+T_xT_{x+1})\right)\\
&-\sum_{i=3}^{k-1}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)\left(\prod_{y=i+1}^{k-1}(1+T_yT_{y+1})\right)\\
&+k\sum_{i=2}^{k-1}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)\left(\prod_{y=i+1}^{k-1}(1+T_yT_{y+1})\right)\\
&+\sum_{i=2}^{k-1}\sum_{j=i}^{k-1}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)\left(T_i\left(\prod_{y=i+1}^{j-1}(1+T_yT_{y+1})\right)T_{j+1}-\left(\prod_{y=i+1}^{j-1}(1+T_yT_{y+1})\right)\right)\left(\prod_{z=j+1}^{k-1}(1+T_zT_{z+1})\right)\\
=&(k-1)(k+1)\left(\prod_{x=2}^{k-1}(1+T_xT_{x+1})\right)\\
&+(k-1)\sum_{i=2}^{k-1}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)\left(\prod_{y=i+1}^{k-1}(1+T_yT_{y+1})\right)\\
&+\sum_{i=2}^{k-1}\sum_{j=i}^{k-1}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)\left(T_i\left(\prod_{y=i+1}^{j-1}(1+T_yT_{y+1})\right)T_{j+1}-\left(\prod_{y=i+1}^{j-1}(1+T_yT_{y+1})\right)\right)\left(\prod_{z=j+1}^{k-1}(1+T_zT_{z+1})\right)
\end{aligned}
$$

where we have used the fact that $\sum_{i=k}^{n-7}\sum_{j=i}^{n-7}1=\sum_{i=k}^{n-7}(-i)=\sum_{i=1}^{k-1}i=k(1-k)$. Then the terms T_I in ψ with $i_m=k$ are given by

$$
\begin{aligned}
\psi_k-\psi_{k-1}=&(k-1)(k+1)\left(\prod_{x=2}^{k-1}(1+T_xT_{x+1})\right)\\
&+(k-1)\sum_{i=2}^{k-1}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)\left(\prod_{y=i+1}^{k-1}(1+T_yT_{y+1})\right)\\
&+\sum_{i=2}^{k-1}\sum_{j=i}^{k-1}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)T_i\left(\prod_{y=i+1}^{j-1}(1+T_yT_{y+1})\right)T_{j+1}\left(\prod_{z=j+1}^{k-1}(1+T_zT_{z+1})\right)\\
&-\sum_{i=2}^{k-1}\sum_{j=i}^{k-1}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)\left(\prod_{y=i+1}^{j-1}(1+T_yT_{y+1})\right)\left(\prod_{z=j+1}^{k-1}(1+T_zT_{z+1})\right)\\
&-k(k+1)\left(\prod_{x=2}^{k-2}(1+T_xT_{x+1})\right)\\
&-(k+1)\sum_{i=2}^{k-2}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)\left(\prod_{y=i+1}^{k-2}(1+T_yT_{y+1})\right)\\
&-\sum_{i=2}^{k-2}\sum_{j=i}^{k-2}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)T_i\left(\prod_{y=i+1}^{j-1}(1+T_yT_{y+1})\right)T_{j+1}\left(\prod_{z=j+1}^{k-2}(1+T_zT_{z+1})\right)\\
&+\sum_{i=2}^{k-2}\sum_{j=i}^{k-2}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)\left(\prod_{y=i+1}^{j-1}(1+T_yT_{y+1})\right)\left(\prod_{z=j+1}^{k-2}(1+T_zT_{z+1})\right)\\
=&k(k+1)\left(\prod_{x=2}^{k-2}(1+T_xT_{x+1})\right)T_{k-1}T_k\\
&+(k+1)\sum_{i=2}^{k-2}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)\left(\prod_{y=i+1}^{k-2}(1+T_yT_{y+1})\right)T_{k-1}T_k\\
&+\sum_{i=2}^{k-2}\sum_{j=i}^{k-2}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)T_i\left(\prod_{y=i+1}^{j-1}(1+T_yT_{y+1})\right)\left(\prod_{z=j+1}^{k-2}(1+T_zT_{z+1})\right)T_k
\end{aligned}
$$

$$-\sum_{i=2}^{k-2}\sum_{j=i}^{k-2}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)\left(\prod_{y=i+1}^{j-1}(1+T_yT_{y+1})\right)\left(\prod_{z=j+1}^{k-2}(1+T_zT_{z+1})\right)T_{k-1}T_k.$$

Then we have

$$\begin{aligned}
(\psi_{k+1}-\psi_k)T_{k+1} =&(k-1)(k+1)\left(\prod_{x=2}^{k-1}(1+T_xT_{x+1})\right)T_k \\
&+(k-1)\sum_{i=2}^{k-1}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)\left(\prod_{y=i+1}^{k-1}(1+T_yT_{y+1})\right)T_k \\
&+\sum_{i=2}^{k-1}\sum_{j=i}^{k-1}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)T_i\left(\prod_{y=i+1}^{j-1}(1+T_yT_{y+1})\right)\left(\prod_{z=j+1}^{k-1}(1+T_zT_{z+1})\right) \\
&-\sum_{i=2}^{k-1}\sum_{j=i}^{k-1}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)\left(\prod_{y=i+1}^{j-1}(1+T_yT_{y+1})\right)\left(\prod_{z=j+1}^{k-1}(1+T_zT_{z+1})\right)T_k \\
=&(k-1)(k+1)\left(\prod_{x=2}^{k-2}(1+T_xT_{x+1})\right)(T_{k-1}+T_k) \\
&+(k-1)\left(\prod_{x=2}^{k-2}(1+T_xT_{x+1})\right)T_k \\
&+(k-1)\sum_{i=2}^{k-2}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)\left(\prod_{y=i+1}^{k-2}(1+T_yT_{y+1})\right)(T_{k-1}+T_k) \\
&+\sum_{i=2}^{k-1}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)T_i\left(\prod_{y=i+1}^{k-2}(1+T_yT_{y+1})\right) \\
&-\sum_{i=2}^{k-1}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)\left(\prod_{y=i+1}^{k-2}(1+T_yT_{y+1})\right)T_k \\
&+\sum_{i=2}^{k-2}\sum_{j=i}^{k-2}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)T_i\left(\prod_{y=i+1}^{j-1}(1+T_yT_{y+1})\right)\left(\prod_{z=j+1}^{k-2}(1+T_zT_{z+1})\right)(1+T_{k-1}T_k) \\
&-\sum_{i=2}^{k-2}\sum_{j=i}^{k-2}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)\left(\prod_{y=i+1}^{j-1}(1+T_yT_{y+1})\right)\left(\prod_{z=j+1}^{k-2}(1+T_zT_{z+1})\right)(T_{k-1}+T_k) \\
=&k^2\left(\prod_{x=2}^{k-2}(1+T_xT_{x+1})\right)T_{k-1}+k(k+1)\left(\prod_{x=2}^{k-2}(1+T_xT_{x+1})\right)T_k \\
&+(k+1)\sum_{i=2}^{k-2}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)\left(\prod_{y=i+1}^{k-2}(1+T_yT_{y+1})\right)(T_{k-1}+T_k) \\
&+\sum_{i=2}^{k-2}\left(\prod_{x=2}^{k-2}(1+T_xT_{x+1})\right)T_{k-1} \\
&+\sum_{i=2}^{k-2}\sum_{j=i}^{k-2}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)T_i\left(\prod_{y=i+1}^{j-1}(1+T_yT_{y+1})\right)T_{j+1}\left(\prod_{z=j+1}^{k-2}(1+T_zT_{z+1})\right)(T_{k-1}+T_k) \\
&-\sum_{i=2}^{k-2}\sum_{j=i}^{k-2}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)\left(\prod_{y=i+1}^{j-1}(1+T_yT_{y+1})\right)\left(\prod_{z=j+1}^{k-2}(1+T_zT_{z+1})\right)(T_{k-1}+T_k) \\
=&k(k+1)\left(\prod_{x=2}^{k-2}(1+T_xT_{x+1})\right)(T_{k-1}+T_k) \\
&+(k+1)\sum_{i=2}^{k-2}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)\left(\prod_{y=i+1}^{k-2}(1+T_yT_{y+1})\right)(T_{k-1}+T_k)
\end{aligned}$$

$$+\sum_{i=2}^{k-2}\sum_{j=i}^{k-2}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)T_i\left(\prod_{y=i+1}^{j-1}(1+T_yT_{y+1})\right)T_{j+1}\left(\prod_{z=j+1}^{k-2}(1+T_zT_{z+1})\right)(T_{k-1}+T_k)$$
$$-\sum_{i=2}^{k-2}\sum_{j=i}^{k-2}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)\left(\prod_{y=i+1}^{j-1}(1+T_yT_{y+1})\right)\left(\prod_{z=j+1}^{k-2}(1+T_zT_{z+1})\right)(T_{k-1}+T_k)$$
$$=k(k+1)\left(\prod_{x=2}^{k-2}(1+T_xT_{x+1})\right)(-T_{k-1}T_kT_{k-1})$$
$$+(k+1)\sum_{i=2}^{k-2}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)\left(\prod_{y=i+1}^{k-2}(1+T_yT_{y+1})\right)(-T_{k-1}T_kT_{k-1})$$
$$+\sum_{i=2}^{k-2}\sum_{j=i}^{k-2}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)T_i\left(\prod_{y=i+1}^{j-1}(1+T_yT_{y+1})\right)T_{j+1}\left(\prod_{z=j+1}^{k-2}(1+T_zT_{z+1})\right)(-T_{k-1}T_kT_{k-1})$$
$$-\sum_{i=2}^{k-2}\sum_{j=i}^{k-2}\left(\prod_{x=2}^{i-1}(1+T_xT_{x+1})\right)\left(\prod_{y=i+1}^{j-1}(1+T_yT_{y+1})\right)\left(\prod_{z=j+1}^{k-2}(1+T_zT_{z+1})\right)(-T_{k-1}T_kT_{k-1})$$
$$=(-\psi_k+\psi_{k-1})T_{k-1}.$$

Hence $\psi_{k+1}-\psi_k=(\psi_k-\psi_{k-1})(-T_{k-1}T_{k+1})$ for all $2\le k\le n-7$. Since $\psi_3-\psi_2=0$, by induction all terms T_I have coefficient 0 in ψ. Hence $t_{n-4}B_{n-1}(b)=B_{n-1}(-T_{n-6}b)$.

For $k=n-3$, we have

$$t_{n-3}B_{n-1}(b)=\sum_{j=4}^{n-4}(-1)^{n+j+1}P_{2j}(T_{n-5}T_2\cdots T_{j-3}\chi_nb)$$
$$+\sum_{j=4}^{n-4}(-1)^nP_{j,n-3}(T_2\cdots T_{n-5}T_{n-3}T_{n-4}T_2\cdots T_{j-3}\chi_nb)$$
$$+\sum_{j=4}^{n-4}(-1)^{n+i+j}P_{j,n-2}(T_2\cdots T_{n-6}(1+(T_{n-5}+T_{n-4})T_{n-3})T_2\cdots T_{j-3}\chi_nb)$$
$$+(-1)^{n+1}P_{2,n-2}(T_2\cdots T_{n-6}\chi_nb)+(-1)^{n+1}P_{2,n-3}(T_2\cdots T_{n-5}\chi_nb)$$
$$+\sum_{j=4}^{n-4}(-1)^jP_{3j}(T_{n-5}T_{n-3}T_2\cdots T_{j-3}\chi_nb)$$
$$+\sum_{j=4}^{n-4}P_{j,n-3}(T_3\cdots T_{n-5}T_{n-3}T_{n-4}T_{n-3}T_2\cdots T_{j-3}\chi_nb)$$
$$+\sum_{j=4}^{n-4}(-1)^{j+1}P_{j,n-2}(T_3\cdots T_{n-6}(1+(T_{n-5}+T_{n-4})T_{n-3})T_{n-3}T_2\cdots T_{j-3}\chi_nb)$$
$$+P_{3,n-2}(-T_{n-3}T_2\cdots T_{n-6}\chi_nb)+P_{3,n-3}(-T_{n-3}T_2\cdots T_{n-5}\chi_nb)$$
$$+\sum_{i=4}^{n-4}\sum_{j=i+1}^{n-4}(-1)^{n(i+1)+i+j}P_{ij}(T_{n-5}(1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{j-3}\chi_nb)$$
$$+\sum_{i=4}^{n-4}\sum_{j=i+1}^{n-4}(-1)^{n(i+1)+1}P_{j,n-3}(T_i\cdots T_{n-5}T_{n-3}T_{n-4}(1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{j-3}\chi_nb)$$
$$+\sum_{i=4}^{n-4}\sum_{j=i+1}^{n-4}(-1)^{(n+1)(i+1)+j}P_{j,n-2}(T_i\cdots T_{n-6}(1+(T_{n-5}+T_{n-4})T_{n-3})$$
$$\cdot(1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{j-3}\chi_nb)$$

$$+\sum_{i=4}^{n-4}(-1)^{n(i+1)}P_{i,n-2}((1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{n-6}\chi_n b)$$
$$+\sum_{i=4}^{n-4}(-1)^{n(i+1)}P_{i,n-3}((1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{n-5}\chi_n b)$$
$$+\sum_{i=4}^{n-4}\sum_{x=4}^{i}P_{i,n-3}(((-1)^{ni+x+1}T_3\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-3}T_{n-4}$$
$$+(-1)^{(n+1)(i+1)+x}T_2\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_n b).$$

Rearranging, we get

$$\sum_{j=4}^{n-2}(-1)^nP_{2j}(T_2\cdots T_{j-3}\chi_nT_{n-5}b)$$
$$+\sum_{j=4}^{n-2}P_{3j}(T_{n-3}T_2\cdots T_{j-3}\chi_nT_{n-5}b)$$
$$+\sum_{i=4}^{n-2}\sum_{j=i+1}^{n-4}(-1)^{n(i+1)+1}P_{ij}((1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{j-3}\chi_nT_{n-5}b)$$
$$+\sum_{i=4}^{n-4}((-1)^nP_{i,n-3}(T_2\cdots T_{n-5}T_{n-3}T_{n-4}T_2\cdots T_{i-3}\chi_n b)$$
$$+P_{i,n-3}(T_3\cdots T_{n-5}T_{n-3}T_{n-4}T_{n-3}T_2\cdots T_{i-3}\chi_n b)$$
$$+\sum_{j=4}^{i-1}(-1)^{n(j+1)+1}P_{i,n-3}(T_j\cdots T_{n-5}T_{n-3}T_{n-4}(1+T_2T_{n-3})T_3\cdots T_{j-2}T_2\cdots T_{i-3}\chi_n b)$$
$$+\sum_{j=4}^{i}P_{i,n-3}(((-1)^{ni+j+1}T_3\cdots T_{i-2}T_2\cdots T_{j-3}T_{j-1}\cdots T_{n-3}T_{n-4}$$
$$+(-1)^{(n+1)(i+1)+j}T_2\cdots T_{i-2}T_2\cdots T_{j-3}T_{j-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_n b)$$
$$+(-1)^{n+i}P_{i,n-2}(T_2\cdots T_{n-6}(1+(T_{n-5}+T_{n-4})T_{n-3})T_2\cdots T_{i-3}\chi_n b)$$
$$+(-1)^{i+1}P_{i,n-2}(T_3\cdots T_{n-6}(1+(T_{n-5}+T_{n-4})T_{n-3})T_{n-3}T_2\cdots T_{i-3}\chi_n b)$$
$$+\sum_{j=4}^{i-1}(-1)^{(n+1)(j+1)+i}P_{i,n-2}(T_j\cdots T_{n-6}(1+(T_{n-5}+T_{n-4})T_{n-3})$$
$$\cdot(1+T_2T_{n-3})T_3\cdots T_{j-2}T_2\cdots T_{i-3}\chi_n b)).$$

Adding $B_{n-1}(T_{n-5}b)$, we have

$$\sum_{i=4}^{n-4}((-1)^nP_{i,n-3}(T_2\cdots T_{n-5}T_{n-3}T_{n-4}T_2\cdots T_{i-3}\chi_n b)$$
$$+P_{i,n-3}(T_3\cdots T_{n-5}T_{n-3}T_{n-4}T_{n-3}T_2\cdots T_{i-3}\chi_n b)$$
$$+\sum_{j=4}^{i-1}(-1)^{n(j+1)+1}P_{i,n-3}(T_j\cdots T_{n-5}T_{n-3}T_{n-4}(1+T_2T_{n-3})T_3\cdots T_{j-2}T_2\cdots T_{i-3}\chi_n b)$$
$$+\sum_{j=4}^{i}P_{i,n-3}(((-1)^{ni+j+1}T_3\cdots T_{i-2}T_2\cdots T_{j-3}T_{j-1}\cdots T_{n-3}T_{n-4}$$
$$+(-1)^{(n+1)(i+1)+j}T_2\cdots T_{i-2}T_2\cdots T_{j-3}T_{j-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_n b)$$

$$
\begin{aligned}
&+ (-1)^{n+i} P_{i,n-2}(T_2 \cdots T_{n-6}(1 + (T_{n-5} + T_{n-4})T_{n-3})T_2 \cdots T_{i-3}\chi_n b) \\
&+ (-1)^{i+1} P_{i,n-2}(T_3 \cdots T_{n-6}(1 + (T_{n-5} + T_{n-4})T_{n-3})T_{n-3}T_2 \cdots T_{i-3}\chi_n b) \\
&+ \sum_{j=4}^{i-1} (-1)^{(n+1)(j+1)+i} P_{i,n-2}(T_j \cdots T_{n-6}(1 + (T_{n-5} + T_{n-4})T_{n-3}) \\
&\qquad\qquad\qquad\qquad \cdot (1 + T_2T_{n-3})T_3 \cdots T_{j-2}T_2 \cdots T_{i-3}\chi_n b) \\
&+ \sum_{j=4}^{i} P_{i,n-2}(((-1)^{ni+j+1} T_3 \cdots T_{i-2}T_2 \cdots T_{j-3}T_{j-1} \cdots T_{n-3}T_{n-4} \\
&\qquad + (-1)^{(n+1)(i+1)+j} T_2 \cdots T_{i-2}T_2 \cdots T_{j-3}T_{j-1} \cdots T_{n-5}T_{n-3}T_{n-4})\chi_n T_{n-5} b)).
\end{aligned}
$$

Hence, we are left to prove that for each $i = 4, \ldots, n-4$, the following two elements of M are both equal to 0: the coefficient of $P_{i,n-3}(b)$

$$
\begin{aligned}
&((-1)^n T_2 \cdots T_{n-5}T_{n-3}T_{n-4}T_2 \cdots T_{i-3} \\
&+ T_3 \cdots T_{n-5}T_{n-3}T_{n-4}T_{n-3}T_2 \cdots T_{i-3} \\
&+ \sum_{j=4}^{i-1} (-1)^{n(j+1)+1} T_j \cdots T_{n-5}T_{n-3}T_{n-4}(1 + T_2T_{n-3})T_3 \cdots T_{j-2}T_2 \cdots T_{i-3} \\
&+ \sum_{j=4}^{i} ((-1)^{ni+j+1} T_3 \cdots T_{i-2}T_2 \cdots T_{j-3}T_{j-1} \cdots T_{n-3}T_{n-4} \\
&\qquad + (-1)^{(n+1)(i+1)+j} T_2 \cdots T_{i-2}T_2 \cdots T_{j-3}T_{j-1} \cdots T_{n-5}T_{n-3}T_{n-4}))\chi_n
\end{aligned}
$$

and the coefficient of $P_{i,n-2}(b)$

$$
\begin{aligned}
&((-1)^{n+i} T_2 \cdots T_{n-6}(1 + (T_{n-5} + T_{n-4})T_{n-3})T_2 \cdots T_{i-3} \\
&+ (-1)^{i+1} T_3 \cdots T_{n-6}(1 + (T_{n-5} + T_{n-4})T_{n-3})T_{n-3}T_2 \cdots T_{i-3} \\
&+ \sum_{j=4}^{i-1} (-1)^{(n+1)(j+1)+i} T_j \cdots T_{n-6}(1 + (T_{n-5} + T_{n-4})T_{n-3})(1 + T_2T_{n-3})T_3 \cdots T_{j-2}T_2 \cdots T_{i-3} \\
&+ \sum_{j=4}^{i} ((-1)^{ni+j} T_3 \cdots T_{i-2}T_2 \cdots T_{j-3}T_{j-1} \cdots T_{n-3}T_{n-4}T_{n-5} \\
&\qquad + (-1)^{n+(n+1)i+j} T_2 \cdots T_{i-2}T_2 \cdots T_{j-3}T_{j-1} \cdots T_{n-5}T_{n-3}T_{n-4}T_{n-5}))\chi_n.
\end{aligned}
$$

The coefficient of $P_{4,n-3}(b)$ is

$$
\begin{aligned}
&((-1)^n T_2 \cdots T_{n-5}T_{n-3}T_{n-4} \\
&+ T_3 \cdots T_{n-5}T_{n-3}T_{n-4}T_{n-3} \\
&- T_3 \cdots T_{n-3}T_{n-4} \\
&+ (-1)^{n+1} T_2 \cdots T_{n-5}T_{n-3}T_{n-4})\chi_n = 0.
\end{aligned}
$$

Since $T_3 \cdots T_{i-2}T_2 \cdots T_{j-3} = (-1)^{(i+1)j} T_2 \cdots T_{j-2}T_2 \cdots T_{i-2}$, for $4 \leq j \leq i \leq n-4$, the coefficient of $P_{i,n-3}(b)$ is

$$
((-1)^n T_2 \cdots T_{n-5}T_{n-3}T_{n-4}T_2 \cdots T_{i-3}
$$

$$
\begin{aligned}
&+ T_3 \cdots T_{n-5} T_{n-3} T_{n-4} T_{n-3} T_2 \cdots T_{i-3} \\
&+ \sum_{j=4}^{i-1} (-1)^{n(j+1)+1} T_j \cdots T_{n-5} T_{n-3} T_{n-4} (1 + T_2 T_{n-3}) T_3 \cdots T_{j-2} T_2 \cdots T_{i-3} \\
&+ \sum_{j=4}^{i} ((-1)^{(n+j)i+1} T_2 \cdots T_{j-2} T_2 \cdots T_{i-2} T_{j-1} \cdots T_{n-5} T_{n-4} T_{n-3} T_{n-4} \\
&\qquad + (-1)^{(n+1)(i+1)+ij} T_3 \cdots T_{j-2} T_2 \cdots T_{i-2} T_{j-1} \cdots T_{n-5} T_{n-3} T_{n-4})) \chi_n \\
=&((-1)^n T_2 \cdots T_{n-5} T_{n-3} T_{n-4} \\
&+ T_3 \cdots T_{n-5} T_{n-3} T_{n-4} T_{n-3} \\
&+ \sum_{j=4}^{i-1} (-1)^{n(j+1)+1} T_j \cdots T_{n-5} T_{n-3} T_{n-4} (1 + T_2 T_{n-3}) T_3 \cdots T_{j-2} \\
&- T_3 \cdots T_{n-3} T_{n-4} \\
&+ (-1)^{n+1} T_2 \cdots T_{n-5} T_{n-3} T_{n-4} \\
&+ \sum_{j=4}^{i-1} ((-1)^j T_2 \cdots T_{j-2} T_j \cdots T_{n-3} T_{n-4} \\
&\qquad + (-1)^{n+j} T_3 \cdots T_{j-2} T_j \cdots T_{n-5} T_{n-3} T_{n-4})) T_2 \cdots T_{i-3} \chi_n.
\end{aligned}
$$

Suppose $i > 4$ and the coefficient of $P_{i-1,n-3}(b)$ is 0. Then the coefficient of $P_{i,n-3}(b)$ is

$$
\begin{aligned}
&((-1)^{n+1} T_2 \cdots T_{n-5} T_{n-3} T_{n-4} \\
&- T_3 \cdots T_{n-5} T_{n-3} T_{n-4} T_{n-3} \\
&+ \sum_{j=4}^{i-2} (-1)^{n(j+1)} T_j \cdots T_{n-5} T_{n-3} T_{n-4} (1 + T_2 T_{n-3}) T_3 \cdots T_{j-2} \\
&+ T_3 \cdots T_{n-3} T_{n-4} \\
&+ (-1)^n T_2 \cdots T_{n-5} T_{n-3} T_{n-4} \\
&+ \sum_{j=4}^{i-2} ((-1)^{j+1} T_2 \cdots T_{j-2} T_j \cdots T_{n-3} T_{n-4} \\
&\qquad + (-1)^{n+j+1} T_3 \cdots T_{j-2} T_j \cdots T_{n-5} T_{n-3} T_{n-4}) \\
&+ (-1)^{i+1} T_2 \cdots T_{i-3} T_{i-1} \cdots T_{n-5} T_{n-3} T_{n-4} T_{n-3} \\
&+ (-1)^i T_2 \cdots T_{i-3} T_{i-1} \cdots T_{n-3} T_{n-4} \\
&+ (-1)^{n+i+1} T_3 \cdots T_{i-3} T_{i-1} \cdots T_{n-5} T_{n-3} T_{n-4} \\
&+ (-1)^{n+i} T_3 \cdots T_{i-3} T_{i-1} \cdots T_{n-5} T_{n-3} T_{n-4}) T_2 \cdots T_{i-4} \chi_n T_{i-3}.
\end{aligned}
$$

The last four terms cancel and the remaining terms correspond to the coefficient of $P_{i-1,n-3}(b)$ multiplied by $-T_{i-3}$ on the right, and therefore 0.

Similarly, the coefficient of $P_{4,n-2}(b)$ is

$$
((-1)^n T_2 \cdots T_{n-6} (1 + (T_{n-5} + T_{n-4}) T_{n-3})
$$

$$
\begin{aligned}
&- T_3 \cdots T_{n-6}(T_{n-5} + T_{n-4} + T_{n-3}) \\
&+ T_3 \cdots T_{n-6} T_{n-5} T_{n-4} T_{n-3} T_{n-4} T_{n-5} \\
&+ (-1)^n T_2 \cdots T_{n-6} T_{n-5} T_{n-3} T_{n-4} T_{n-5}) \chi_n = 0
\end{aligned}
$$

and the coefficient of $P_{i,n-2}(b)$ can be written as

$$
\begin{aligned}
&((-1)^{n+i} T_2 \cdots T_{n-6}(1 + (T_{n-5} + T_{n-4}) T_{n-3}) \\
&+ (-1)^{i+1} T_3 \cdots T_{n-6}(1 + (T_{n-5} + T_{n-4}) T_{n-3}) T_{n-3} \\
&+ \sum_{j=4}^{i-1} ((-1)^{(n+1)(j+1)+i} T_j \cdots T_{n-6}(1 + (T_{n-5} + T_{n-4}) T_{n-3})(1 + T_2 T_{n-3}) T_3 \cdots T_{j-2} \\
&\qquad + (-1)^{i+j+1} T_2 \cdots T_{j-2} T_j \cdots T_{n-3} T_{n-4} T_{n-5} \\
&\qquad + (-1)^{n+i+j+1} T_3 \cdots T_{j-2} T_j \cdots T_{n-5} T_{n-3} T_{n-4} T_{n-5}) \\
&+ (-1)^i T_3 \cdots T_{n-3} T_{n-4} T_{n-5} \\
&+ (-1)^{n+i} T_2 \cdots T_{n-5} T_{n-3} T_{n-4} T_{n-5}) T_2 \cdots T_{i-3} \chi_n.
\end{aligned}
$$

If $i > 4$ and the coefficient of $P_{i-1,n-2}(b)$ is 0, then the coefficient of $P_{i,n-2}(b)$ is

$$
\begin{aligned}
&((-1)^{n+i+1} T_2 \cdots T_{n-6}(1 + (T_{n-5} + T_{n-4}) T_{n-3}) \\
&+ (-1)^i T_3 \cdots T_{n-6}(1 + (T_{n-5} + T_{n-4}) T_{n-3}) T_{n-3} \\
&+ \sum_{j=4}^{i-2} ((-1)^{(n+1)(j+1)+i+1} T_j \cdots T_{n-6}(1 + (T_{n-5} + T_{n-4}) T_{n-3})(1 + T_2 T_{n-3}) T_3 \cdots T_{j-2} \\
&\qquad + (-1)^{i+j} T_2 \cdots T_{j-2} T_j \cdots T_{n-3} T_{n-4} T_{n-5} \\
&\qquad + (-1)^{n+i+j} T_3 \cdots T_{j-2} T_j \cdots T_{n-5} T_{n-3} T_{n-4} T_{n-5}) \\
&+ (-1)^{i+1} T_3 \cdots T_{n-3} T_{n-4} T_{n-5} \\
&+ (-1)^{n+i+1} T_2 \cdots T_{n-5} T_{n-3} T_{n-4} T_{n-5}) T_2 \cdots T_{i-4} \chi_n T_{i-3} \\
&+ ((-1)^{ni} T_{i-1} \cdots T_{n-6}(1 + (T_{n-5} + T_{n-4}) T_{n-3})(1 + T_2 T_{n-3}) T_3 \cdots T_{i-3} \\
&+ T_2 \cdots T_{i-3} T_{i-1} \cdots T_{n-3} T_{n-4} T_{n-5} \\
&+ (-1)^n T_3 \cdots T_{i-3} T_{i-1} \cdots T_{n-5} T_{n-3} T_{n-4} T_{n-5}) T_2 \cdots T_{i-3} \chi_n \\
=&((-1)^{ni} T_{i-1} \cdots T_{n-6}(1 + (T_{n-5} + T_{n-4}) T_{n-3}) T_3 \cdots T_{i-3} \\
&+ (-1)^{ni} T_{i-1} \cdots T_{n-6} T_{n-5} T_{n-3} T_{n-4} T_{n-5} T_3 \cdots T_{i-3} \\
&+ (-1)^{ni+1} T_{i-1} \cdots T_{n-6}(T_{n-5} + T_{n-4} + T_{n-3}) T_2 \cdots T_{i-3} \\
&+ (-1)^{ni} T_{i-1} \cdots T_{n-6} T_{n-5} T_{n-4} T_{n-3} T_{n-4} T_{n-5} T_2 \cdots T_{i-3}) T_2 \cdots T_{i-3} \chi_n = 0.
\end{aligned}
$$

For $5 \leq k \leq n-3$,

$$
t_k B_{n-2}(b) = t_3 t_k B_{n-1}(-b) = t_3 B_{n-1}(T_{k-2} b) = B_{n-2}(T_{k-2} b).
$$

Case 6: In characteristic 3, when $2 \leq i \leq n-4$, $i < j \leq n-3$, we have

$$\begin{aligned} t_{n-2}P_{ij}(b) =& (-1)^{i+j}P_{ij}((T_{n-4}-T_{n-3}+T_{n-1})b) \\ &+ (-1)^{n+(n+1)(i+j)+ij}P_{i,n-2}(T_{j-1}\cdots T_{n-5}(1+(T_{n-4}-T_{n-3})T_{n-1})b) \\ &+ (-1)^{i+j}P_{j,n-2}(T_i\cdots T_{n-5}(T_{n-4}(1-T_{n-3}T_{n-1})+T_{n-1})b). \end{aligned}$$

Thus the action of t_{n-2} on $B_{n-1}(b)$ is given by

$$\begin{aligned} &t_{n-2}B_{n-1}(b) = \\ &\quad \sum_{j=4}^{n-3}(-1)^{n+j+1}P_{2j}((T_{n-4}-T_{n-3}+T_{n-1})T_2\cdots T_{j-3}\chi_n b) \\ &\quad + \sum_{j=4}^{n-3}(-1)^{(n+1)j+1}P_{2,n-2}(T_{j-1}\cdots T_{n-5}(1+(T_{n-4}-T_{n-3})T_{n-1})T_2\cdots T_{j-3}\chi_n b) \\ &\quad + \sum_{j=4}^{n-3}(-1)^{n+j+1}P_{j,n-2}(T_2\cdots T_{n-5}(T_{n-4}(1-T_{n-3}T_{n-1})+T_{n-1})T_2\cdots T_{j-3}\chi_n b) \\ &\quad + P_{2,n-2}(-T_{n-1}T_2\cdots T_{n-5}\chi_n b) \\ &\quad + \sum_{j=4}^{n-3}(-1)^{j}P_{3j}((T_{n-4}-T_{n-3}+T_{n-1})T_{n-3}T_2\cdots T_{j-3}\chi_n b) \\ &\quad + \sum_{j=4}^{n-3}(-1)^{nj}P_{3,n-2}(T_{j-1}\cdots T_{n-5}(1+(T_{n-4}-T_{n-3})T_{n-1})T_{n-3}T_2\cdots T_{j-3}\chi_n b) \\ &\quad + \sum_{j=4}^{n-3}(-1)^{j}P_{j,n-2}(T_3\cdots T_{n-5}(T_{n-4}(1-T_{n-3}T_{n-1})+T_{n-1})T_{n-3}T_2\cdots T_{j-3}\chi_n b) \\ &\quad + (-1)^{n}P_{3,n-2}(T_{n-1}T_{n-3}T_2\cdots T_{n-5}\chi_n b) \\ &\quad + \sum_{i=4}^{n-4}\sum_{j=i+1}^{n-3}((-1)^{n+(n+1)i+j}P_{ij}((T_{n-4}-T_{n-3}+T_{n-1})(1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{j-3}\chi_n b) \\ &\quad + (-1)^{i+(n+i+1)j}P_{i,n-2}(T_{j-1}\cdots T_{n-5}(1+(T_{n-4}-T_{n-3})T_{n-1})(1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{j-3}\chi_n b) \\ &\quad + (-1)^{n+(n+1)i+j}P_{j,n-2}(T_i\cdots T_{n-5}(T_{n-4}(1-T_{n-3}T_{n-1})+T_{n-1})(1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{j-3}\chi_n b)) \\ &\quad + \sum_{i=4}^{n-4}(-1)^{(n+1)i}P_{i,n-2}(T_{n-1}(1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{n-5}\chi_n b) \\ &\quad + \sum_{i=4}^{n-4}\sum_{x=4}^{i}P_{i,n-2}((-1)^{(n+1)(i+1)+x}T_{n-1}T_3\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-3}T_{n-4} \\ &\qquad\qquad + (-1)^{ni+x+1}T_{n-1}T_2\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_n b). \end{aligned}$$

Rearranging, we get

$$\begin{aligned} t_{n-2}B_{n-1}(b) =& \sum_{j=4}^{n-2}(-1)^{n}P_{2j}(T_2\cdots T_{j-3}\chi_n(T_{n-4}-T_{n-3}+T_{n-1})b) \\ &+ (-1)^{n}P_{2,n-2}(T_2\cdots T_{n-5}\chi_n(-T_{n-4}+T_{n-3})b) \\ &+ \sum_{j=4}^{n-3}(-1)^{j+1}P_{2,n-2}(T_2\cdots T_{j-3}T_{j-1}\cdots T_{n-5}(1+(T_{n-4}-T_{n-3})T_{n-1})\chi_n b) \end{aligned}$$

$$+\sum_{j=4}^{n-2} P_{3j}(T_{n-3}T_2\cdots T_{j-3}\chi_n(T_{n-4}-T_{n-3}+T_{n-1})b)$$

$$+P_{3,n-2}(T_{n-3}T_2\cdots T_{n-5}\chi_n(-T_{n-4}+T_{n-3})b)$$

$$+\sum_{j=4}^{n-3}(-1)^j P_{3,n-2}(T_2\cdots T_{j-3}T_{j-1}\cdots T_{n-5}(-T_{n-4}T_{n-3}T_{n-1}+T_{n-3}+T_{n-1})\chi_n b)$$

$$+\sum_{i=4}^{n-4}\sum_{j=i+1}^{n-2}(-1)^{n(i+1)+1}P_{ij}((1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{j-3}\chi_n(T_{n-4}-T_{n-3}+T_{n-1})b)$$

$$+\sum_{i=4}^{n-4}(-1)^{n(i+1)+1}P_{i,n-2}((1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{n-5}\chi_n(-T_{n-4}+T_{n-3})b)$$

$$+\sum_{j=4}^{n-4}(-1)^{n+1}P_{j,n-2}(T_2\cdots T_{n-5}T_2\cdots T_{j-3}(T_{n-4}(1-T_{n-3}T_{n-1})+T_{n-1})\chi_n b)$$

$$+\sum_{j=4}^{n-4}(-1)^j P_{j,n-2}(T_3\cdots T_{n-5}T_2\cdots T_{j-3}(T_{n-4}(T_{n-3}+T_{n-1})-T_{n-3}T_{n-1})\chi_n b)$$

$$+\sum_{i=4}^{n-4}\sum_{j=i+1}^{n-3}(-1)^{i+(n+i+1)j}P_{i,n-2}(T_{j-1}\cdots T_{n-5}T_3\cdots T_{i-2}T_2\cdots T_{j-3}(1+(T_{n-4}-T_{n-3})T_{n-1})\chi_n b)$$

$$+\sum_{i=4}^{n-4}\sum_{j=i+1}^{n-3}(-1)^{(n+i)j}P_{i,n-2}(T_{j-1}\cdots T_{n-5}T_2\cdots T_{i-2}T_2\cdots T_{j-3}(-T_{n-4}T_{n-3}T_{n-1}+T_{n-3}+T_{n-1})\chi_n b)$$

$$+\sum_{i=4}^{n-4}\sum_{j=i+1}^{n-4}(-1)^{n+i+j}P_{j,n-2}(T_3\cdots T_{i-2}T_i\cdots T_{n-5}(T_{n-4}(1-T_{n-3}T_{n-1})+T_{n-1})T_2\cdots T_{j-3}\chi_n b)$$

$$+\sum_{i=4}^{n-4}\sum_{j=i+1}^{n-4}(-1)^{i+j+1}P_{j,n-2}(T_2\cdots T_{i-2}T_i\cdots T_{n-5}(T_{n-4}(T_{n-3}+T_{n-1})-T_{n-3}T_{n-1})T_2\cdots T_{j-3}\chi_n b)$$

$$+\sum_{i=4}^{n-4}\sum_{x=4}^{i}P_{i,n-2}((-1)^{ni+x+1}T_3\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-3}T_{n-4}T_{n-1}$$

$$+(-1)^{(n+1)(i+1)+x}T_2\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4}T_{n-1})\chi_n b).$$

Adding $B_{n-1}((T_{n-4}-T_{n-3}+T_{n-1})b)$, and changing summation variables, we get

$$t_{n-4}B_{n-1}(b)+B_{n-1}((T_{n-4}-T_{n-3}+T_{n-1})b)$$

$$=(-1)^n P_{2,n-2}(T_2\cdots T_{n-5}\chi_n(-T_{n-4}+T_{n-3})b)$$

$$+\sum_{j=2}^{n-5}(-1)^{j+1}P_{2,n-2}(T_2\cdots T_{j-1}T_{j+1}\cdots T_{n-5}(1+(T_{n-4}-T_{n-3})T_{n-1})\chi_n b)$$

$$+P_{3,n-2}(T_{n-3}T_2\cdots T_{n-5}\chi_n(-T_{n-4}+T_{n-3})b)$$

$$+\sum_{j=2}^{n-5}(-1)^j P_{3,n-2}(T_2\cdots T_{j-1}T_{j+1}\cdots T_{n-5}(-T_{n-4}T_{n-3}T_{n-1}+T_{n-3}+T_{n-1})\chi_n b)$$

$$+\sum_{i=4}^{n-4}(-1)^{n(i+1)+1}P_{i,n-2}((1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{n-5}\chi_n(-T_{n-4}+T_{n-3})b)$$

$$+\sum_{i=4}^{n-4}(-1)^{n+1}P_{i,n-2}(T_2\cdots T_{n-5}T_2\cdots T_{i-3}(T_{n-4}(1-T_{n-3}T_{n-1})+T_{n-1})\chi_n b)$$

$$+\sum_{i=4}^{n-4}(-1)^{i}P_{i,n-2}(T_3\cdots T_{n-5}T_2\cdots T_{i-3}(T_{n-4}(T_{n-3}+T_{n-1})-T_{n-3}T_{n-1})\chi_n b)$$
$$+\sum_{i=4}^{n-4}\sum_{j=i+1}^{n-3}(-1)^{i+(n+i+1)j}P_{i,n-2}(T_{j-1}\cdots T_{n-5}T_3\cdots T_{i-2}T_2\cdots T_{j-3}(1+(T_{n-4}-T_{n-3})T_{n-1})\chi_n b)$$
$$+\sum_{i=4}^{n-4}\sum_{j=i+1}^{n-3}(-1)^{(n+i)j}P_{i,n-2}(T_{j-1}\cdots T_{n-5}T_2\cdots T_{i-2}T_2\cdots T_{j-3}(-T_{n-4}T_{n-3}T_{n-1}+T_{n-3}+T_{n-1})\chi_n b)$$
$$+\sum_{i=5}^{n-4}\sum_{j=4}^{i-1}(-1)^{n+i+j}P_{i,n-2}(T_3\cdots T_{j-2}T_j\cdots T_{n-5}(T_{n-4}(1-T_{n-3}T_{n-1})+T_{n-1})T_2\cdots T_{i-3}\chi_n b)$$
$$+\sum_{i=5}^{n-4}\sum_{j=4}^{i-1}(-1)^{i+j+1}P_{i,n-2}(T_2\cdots T_{j-2}T_j\cdots T_{n-5}(T_{n-4}(T_{n-3}+T_{n-1})-T_{n-3}T_{n-1})T_2\cdots T_{i-3}\chi_n b)$$
$$+\sum_{i=4}^{n-4}\sum_{j=4}^{i}P_{i,n-2}((-1)^{ni+j+1}T_3\cdots T_{i-2}T_2\cdots T_{j-3}T_{j-1}\cdots T_{n-3}T_{n-4}(-T_{n-4}+T_{n-3})$$
$$+(-1)^{(n+1)(i+1)+j}T_2\cdots T_{i-2}T_2\cdots T_{j-3}T_{j-1}\cdots T_{n-5}T_{n-3}T_{n-4}(-T_{n-4}+T_{n-3}))\chi_n b).$$

First we will show that the coefficient of $P_{2,n-2}(b)$ in $t_{n-2}B_{n-1}(b)+B_{n-1}((T_{n-4}-T_{n-3}+T_{n-1})b)$ is 0. Using Lemma 3.3.4 and the relation

$$\sum_{j=2}^{n-5}(-1)^{j}T_2\cdots T_{j-1}T_{j+1}\cdots T_{n-5}\chi_n=\sum_{j=2}^{n-5}(-1)^{j}T_2\cdots T_{j-1}T_{j+1}\cdots T_{n-5}(T_{n-4}-T_{n-3})$$

(and $(T_{n-4}-T_{n-3})^2=0$), we can write the coefficient of $P_{2,n-2}(b)$ as

$$(-1)^{n}T_2\cdots T_{n-5}\chi_n(-T_{n-4}+T_{n-3})+\sum_{j=2}^{n-5}(-1)^{j+1}T_2\cdots T_{j-1}T_{j+1}\cdots T_{n-5}\chi_n(1+(T_{n-4}-T_{n-3})T_{n-1})$$
$$=\sum_{k=2}^{n-5}(-1)^{k+1}T_2\cdots T_{k-1}T_{k+1}\cdots T_{n-5}(-T_{n-4}+T_{n-3})$$
$$+\sum_{k=n-4}^{n-3}(-1)^{n}(k-1)T_2\cdots T_{n-5}T_k(-T_{n-4}+T_{n-3})$$
$$+\sum_{j=2}^{n-5}(-1)^{j}T_2\cdots T_{j-1}T_{j+1}\cdots T_{n-5}(-T_{n-4}+T_{n-3})(1+(T_{n-4}-T_{n-3})T_{n-1})$$
$$=\sum_{k=2}^{n-5}(-1)^{k+1}T_2\cdots T_{k-1}T_{k+1}\cdots T_{n-5}(-T_{n-4}+T_{n-3})$$
$$+(-1)^{n}T_2\cdots T_{n-5}(T_{n-4}-T_{n-3})(-T_{n-4}+T_{n-3})$$
$$+\sum_{j=2}^{n-5}(-1)^{j}T_2\cdots T_{j-1}T_{j+1}\cdots T_{n-5}(-T_{n-4}+T_{n-3})$$
$$=0.$$

Similarly, we can write the coefficient of $P_{3,n-2}(b)$ in $t_{n-2}B_{n-1}(b)+B_{n-1}((T_{n-4}-T_{n-3}+T_{n-1})b)$ as

$$T_{n-3}T_2\cdots T_{n-5}\chi_n(-T_{n-4}+T_{n-3})+\sum_{j=2}^{n-5}(-1)^{j}T_2\cdots T_{j-1}T_{j+1}\cdots T_{n-5}\chi_n(T_{n-4}T_{n-3}T_{n-1}-T_{n-3}-T_{n-1})$$
$$=\sum_{k=2}^{n-5}(-1)^{n+k+1}T_{n-3}T_2\cdots T_{k-1}T_{k+1}\cdots T_{n-5}(-T_{n-4}+T_{n-3})$$

$$+T_{n-3}T_2\cdots T_{n-5}(T_{n-4}-T_{n-3})(-T_{n-4}+T_{n-3})$$

$$+\sum_{j=2}^{n-5}(-1)^jT_2\cdots T_{j-1}T_{j+1}\cdots T_{n-5}(T_{n-4}-T_{n-3})(T_{n-4}T_{n-3}T_{n-1}-T_{n-3}-T_{n-1})$$

$$=\sum_{k=2}^{n-5}(-1)^kT_2\cdots T_{k-1}T_{k+1}\cdots T_{n-5}(-1+T_{n-4}T_{n-3})$$

$$+\sum_{j=2}^{n-5}(-1)^jT_2\cdots T_{j-1}T_{j+1}\cdots T_{n-5}(1-T_{n-4}T_{n-3})$$

$$=0.$$

Hence we are left with

$$t_{n-2}B_{n-1}(b)+B_{n-1}((T_{n-4}-T_{n-3}+T_{n-1})b)$$

$$=\sum_{i=4}^{n-4}(-1)^{n(i+1)+1}P_{i,n-2}((1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{n-5}\chi_n(-T_{n-4}+T_{n-3})b)$$

$$+\sum_{i=4}^{n-4}(-1)^{n+1}P_{i,n-2}(T_2\cdots T_{n-5}T_2\cdots T_{i-3}(T_{n-4}(1-T_{n-3}T_{n-1})+T_{n-1})\chi_nb)$$

$$+\sum_{i=4}^{n-4}(-1)^{i}P_{i,n-2}(T_3\cdots T_{n-5}T_2\cdots T_{i-3}(T_{n-4}(T_{n-3}+T_{n-1})-T_{n-3}T_{n-1})\chi_nb)$$

$$+\sum_{i=4}^{n-4}\sum_{j=i+1}^{n-3}(-1)^{i+(n+i+1)j}P_{i,n-2}(T_{j-1}\cdots T_{n-5}T_3\cdots T_{i-2}T_2\cdots T_{j-3}(1+(T_{n-4}-T_{n-3})T_{n-1})\chi_nb)$$

$$+\sum_{i=4}^{n-4}\sum_{j=i+1}^{n-3}(-1)^{(n+i)j}P_{i,n-2}(T_{j-1}\cdots T_{n-5}T_2\cdots T_{i-2}T_2\cdots T_{j-3}(-T_{n-4}T_{n-3}T_{n-1}+T_{n-3}+T_{n-1})\chi_nb)$$

$$+\sum_{i=4}^{n-4}\sum_{j=4}^{i-1}(-1)^{n+i+j}P_{i,n-2}(T_3\cdots T_{j-2}T_j\cdots T_{n-5}(T_{n-4}(1-T_{n-3}T_{n-1})+T_{n-1})T_2\cdots T_{i-3}\chi_nb)$$

$$+\sum_{i=4}^{n-4}\sum_{j=4}^{i-1}(-1)^{i+j+1}P_{i,n-2}(T_2\cdots T_{j-2}T_j\cdots T_{n-5}(T_{n-4}(T_{n-3}+T_{n-1})-T_{n-3}T_{n-1})T_2\cdots T_{i-3}\chi_nb)$$

$$+\sum_{i=4}^{n-4}\sum_{j=4}^{i}P_{i,n-2}((-1)^{ni+j+1}T_3\cdots T_{i-2}T_2\cdots T_{j-3}T_{j-1}\cdots T_{n-3}T_{n-4}(-T_{n-4}+T_{n-3})$$

$$+(-1)^{(n+1)(i+1)+j}T_2\cdots T_{i-2}T_2\cdots T_{j-3}T_{j-1}\cdots T_{n-5}T_{n-3}T_{n-4}(-T_{n-4}+T_{n-3}))\chi_nb)$$

$$=\sum_{i=4}^{n-4}P_{i,n-2}(((-1)^{n(i+1)}(1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{n-5}(-T_{n-4}+T_{n-3})$$

$$+(-1)^{n+i+1}T_2\cdots T_{n-5}(T_{n-4}(1-T_{n-3}T_{n-1})+T_{n-1})T_2\cdots T_{i-3}$$

$$+(-1)^{i}T_3\cdots T_{n-5}(T_{n-4}(T_{n-3}+T_{n-1})-T_{n-3}T_{n-1})T_2\cdots T_{i-3}$$

$$+\sum_{j=i+1}^{n-3}(-1)^{i+(n+i+1)j}T_{j-1}\cdots T_{n-5}T_3\cdots T_{i-2}T_2\cdots T_{j-3}(1+(T_{n-4}-T_{n-3})T_{n-1})$$

$$+\sum_{j=i+1}^{n-3}(-1)^{(n+i)j}T_{j-1}\cdots T_{n-5}T_2\cdots T_{i-2}T_2\cdots T_{j-3}(-T_{n-4}T_{n-3}T_{n-1}+T_{n-3}+T_{n-1})$$

$$+\sum_{j=4}^{i-1}(-1)^{n+i+j}T_3\cdots T_{j-2}T_j\cdots T_{n-5}(T_{n-4}(1-T_{n-3}T_{n-1})+T_{n-1})T_2\cdots T_{i-3}$$

$$+\sum_{j=4}^{i-1}(-1)^{i+j+1}T_2\cdots T_{j-2}T_j\cdots T_{n-5}(T_{n-4}(T_{n-3}+T_{n-1})-T_{n-3}T_{n-1})T_2\cdots T_{i-3}$$
$$+\sum_{j=4}^{i}(-1)^{ni+j+1}T_3\cdots T_{i-2}T_2\cdots T_{j-3}T_{j-1}\cdots T_{n-3}T_{n-4}(-T_{n-4}+T_{n-3})$$
$$+\sum_{j=4}^{i}(-1)^{(n+1)(i+1)+j}T_2\cdots T_{i-2}T_2\cdots T_{j-3}T_{j-1}\cdots T_{n-5}T_{n-3}T_{n-4}(-T_{n-4}+T_{n-3}))\chi_n b).$$

The coefficient of $P_{4,n-2}(b)$ is

$$=((-1)^n(1+T_2T_{n-3})T_2\cdots T_{n-5}(-T_{n-4}+T_{n-3})$$
$$+(-1)^{n+1}T_2\cdots T_{n-5}(T_{n-4}(1-T_{n-3}T_{n-1})+T_{n-1})$$
$$+T_3\cdots T_{n-5}(T_{n-4}(T_{n-3}+T_{n-1})-T_{n-3}T_{n-1})$$
$$+\sum_{j=5}^{n-3}(-1)^{(n+1)j}T_{j-1}\cdots T_{n-5}T_2\cdots T_{j-3}(1+(T_{n-4}-T_{n-3})T_{n-1})$$
$$+\sum_{j=5}^{n-3}(-1)^{nj}T_{j-1}\cdots T_{n-5}T_3\cdots T_{j-3}(-T_{n-4}T_{n-3}T_{n-1}+T_{n-3}+T_{n-1})$$
$$+T_3\cdots T_{n-3}T_{n-4}(T_{n-4}-T_{n-3})$$
$$+(-1)^{n+1}T_2\cdots T_{n-5}T_{n-3}T_{n-4}(-T_{n-4}+T_{n-3}))\chi_n$$
$$=(-1)^nT_2\cdots T_{n-5}\chi_n(T_{n-4}-T_{n-3})$$
$$+T_3\cdots T_{n-5}\chi_n(-1+T_{n-4}T_{n-3})$$
$$+\sum_{j=2}^{n-5}(-1)^jT_2\cdots T_{j-1}T_{j+1}\cdots T_{n-5}\chi_n(1+(T_{n-4}-T_{n-3})T_{n-1})$$
$$+\sum_{j=2}^{n-5}(-1)^{n+j}T_2T_2\cdots T_{j-1}T_{j+1}\cdots T_{n-5}\chi_n(-T_{n-4}T_{n-3}T_{n-1}+T_{n-3}+T_{n-1})$$
$$=\sum_{k=2}^{n-5}(-1)^{k+1}T_2\cdots T_{k-1}T_{k+1}\cdots T_{n-5}(T_{n-4}-T_{n-3})$$
$$+(-1)^nT_2\cdots T_{n-5}(T_{n-4}-T_{n-3})(T_{n-4}-T_{n-3})$$
$$+(-1)^{n+1}T_2\cdots T_{n-5}(-1+T_{n-4}T_{n-3})$$
$$+\sum_{k=3}^{n-5}(-1)^{n+k+1}T_3\cdots T_{k-1}T_{k+1}\cdots T_{n-5}(-1+T_{n-4}T_{n-3})$$
$$+T_3\cdots T_{n-5}(T_{n-4}-T_{n-3})(-1+T_{n-4}T_{n-3})$$
$$+\sum_{j=2}^{n-5}(-1)^jT_2\cdots T_{j-1}T_{j+1}\cdots T_{n-5}(T_{n-4}-T_{n-3})(1+(T_{n-4}-T_{n-3})T_{n-1})$$
$$+\sum_{j=2}^{n-5}(-1)^{n+j}T_2T_2\cdots T_{j-1}T_{j+1}\cdots T_{n-5}(T_{n-4}-T_{n-3})(-T_{n-4}T_{n-3}T_{n-1}+T_{n-3}+T_{n-1})$$
$$=\sum_{k=2}^{n-5}(-1)^{k+1}T_2\cdots T_{k-1}T_{k+1}\cdots T_{n-5}(T_{n-4}-T_{n-3})$$

$$+\sum_{k=3}^{n-5}(-1)^{n+k+1}T_3\cdots T_{k-1}T_{k+1}\cdots T_{n-5}(-1+T_{n-4}T_{n-3})$$
$$+\sum_{j=2}^{n-5}(-1)^{j}T_2\cdots T_{j-1}T_{j+1}\cdots T_{n-5}(T_{n-4}-T_{n-3})$$
$$+\sum_{j=3}^{n-5}(-1)^{n+j}T_3\cdots T_{j-1}T_{j+1}\cdots T_{n-5}(-1+T_{n-4}T_{n-3})$$
$$=0.$$

To prove that the coefficient of $P_{i,n-2}(b)$ in $t_{n-2}B_{n-1}(b)+B_{n-1}((T_{n-4}-T_{n-3}+T_{n-1})b)$ is 0 for $i=5,\dots n-4$, we will use induction on i. We can write the coefficient of $P_{i,n-2}(b)$ for $4\le i\le n-4$ as

$$((-1)^{n(i+1)}(1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{n-5}(-T_{n-4}+T_{n-3})$$
$$+(-1)^{n+i+1}T_2\cdots T_{n-5}(T_{n-4}(1-T_{n-3}T_{n-1})+T_{n-1})T_2\cdots T_{i-3}$$
$$+(-1)^{i}T_3\cdots T_{n-5}(T_{n-4}(T_{n-3}+T_{n-1})-T_{n-3}T_{n-1})T_2\cdots T_{i-3}$$
$$+\sum_{j=i+1}^{n-3}(-1)^{i+(n+i+1)j}T_{j-1}\cdots T_{n-5}T_3\cdots T_{i-2}T_2\cdots T_{j-3}(1+(T_{n-4}-T_{n-3})T_{n-1})$$
$$+\sum_{j=i+1}^{n-3}(-1)^{(n+i)j}T_{j-1}\cdots T_{n-5}T_2\cdots T_{i-2}T_2\cdots T_{j-3}(-T_{n-4}T_{n-3}T_{n-1}+T_{n-3}+T_{n-1})$$
$$+\sum_{j=4}^{i-1}(-1)^{n+i+j}T_3\cdots T_{j-2}T_j\cdots T_{n-5}(T_{n-4}(1-T_{n-3}T_{n-1})+T_{n-1})T_2\cdots T_{i-3}$$
$$+\sum_{j=4}^{i-1}(-1)^{i+j+1}T_2\cdots T_{j-2}T_j\cdots T_{n-5}(T_{n-4}(T_{n-3}+T_{n-1})-T_{n-3}T_{n-1})T_2\cdots T_{i-3}$$
$$+\sum_{j=2}^{i-2}(-1)^{ni+j+1}T_3\cdots T_{i-2}T_2\cdots T_{j-1}T_{j+1}\cdots T_{n-3}T_{n-4}(-T_{n-4}+T_{n-3})$$
$$+\sum_{j=2}^{i-2}(-1)^{(n+1)(i+1)+j}T_2\cdots T_{i-2}T_2\cdots T_{j-1}T_{j+1}\cdots T_{n-5}T_{n-3}T_{n-4}(-T_{n-4}+T_{n-3}))\chi_n$$
$$=((-1)^{n+i}(1+T_2T_{n-3})T_2\cdots T_{n-5}(-T_{n-4}+T_{n-3})$$
$$+(-1)^{n+i+1}T_2\cdots T_{n-5}(T_{n-4}(1-T_{n-3}T_{n-1})+T_{n-1})$$
$$+(-1)^{i}T_3\cdots T_{n-5}(T_{n-4}(T_{n-3}+T_{n-1})-T_{n-3}T_{n-1})$$
$$+\sum_{j=i+1}^{n-3}(-1)^{i+(n+1)j}T_{j-1}\cdots T_{n-5}T_2\cdots T_{j-3}(1+(T_{n-4}-T_{n-3})T_{n-1})$$
$$+\sum_{j=i+1}^{n-3}(-1)^{nj+i}T_{j-1}\cdots T_{n-5}T_2T_2\cdots T_{j-3}(-T_{n-4}T_{n-3}T_{n-1}+T_{n-3}+T_{n-1})$$
$$+\sum_{j=4}^{i-1}(-1)^{n+i+j}T_3\cdots T_{j-2}T_j\cdots T_{n-5}(T_{n-4}(1-T_{n-3}T_{n-1})+T_{n-1})$$
$$+\sum_{j=4}^{i-1}(-1)^{i+j+1}T_2\cdots T_{j-2}T_j\cdots T_{n-5}(T_{n-4}(T_{n-3}+T_{n-1})-T_{n-3}T_{n-1})$$
$$+\sum_{j=2}^{i-2}(-1)^{i+j+1}T_2\cdots T_{j-1}T_{j+1}\cdots T_{n-3}T_{n-4}(-T_{n-4}+T_{n-3})$$

$$+\sum_{j=2}^{i-2}(-1)^{n+i+j+1}T_2T_2\cdots T_{j-1}T_{j+1}\cdots T_{n-5}T_{n-3}T_{n-4}(-T_{n-4}+T_{n-3}))T_2\cdots T_{i-3}\chi_n.$$

Now suppose $5 \le i \le n-4$ and the coefficient of $P_{i-1,n-2}(b)$ in $t_{n-2}B_{n-1}(b)+B_{n-1}((T_{n-4}-T_{n-3}+T_{n-1})b)$ is 0. Then the coefficient of $P_{i,n-2}(b)$ is

$$\begin{aligned}
&((-1)^{n+i+1}(1+T_2T_{n-3})T_2\cdots T_{n-5}(-T_{n-4}+T_{n-3})\\
&+(-1)^{n+i}T_2\cdots T_{n-5}(T_{n-4}(1-T_{n-3}T_{n-1})+T_{n-1})\\
&+(-1)^{i+1}T_3\cdots T_{n-5}(T_{n-4}(T_{n-3}+T_{n-1})-T_{n-3}T_{n-1})\\
&+\sum_{j=i}^{n-3}(-1)^{i+(n+1)j+1}T_{j-1}\cdots T_{n-5}T_2\cdots T_{j-3}(1+(T_{n-4}-T_{n-3})T_{n-1})\\
&+\sum_{j=i}^{n-3}(-1)^{nj+i+1}T_{j-1}\cdots T_{n-5}T_2T_2\cdots T_{j-3}(-T_{n-4}T_{n-3}T_{n-1}+T_{n-3}+T_{n-1})\\
&+\sum_{j=4}^{i-2}(-1)^{n+i+j+1}T_3\cdots T_{j-2}T_j\cdots T_{n-5}(T_{n-4}(1-T_{n-3}T_{n-1})+T_{n-1})\\
&+\sum_{j=4}^{i-2}(-1)^{i+j}T_2\cdots T_{j-2}T_j\cdots T_{n-5}(T_{n-4}(T_{n-3}+T_{n-1})-T_{n-3}T_{n-1})\\
&+\sum_{j=2}^{i-3}(-1)^{i+j}T_2\cdots T_{j-1}T_{j+1}\cdots T_{n-3}T_{n-4}(-T_{n-4}+T_{n-3})\\
&+\sum_{j=2}^{i-3}(-1)^{n+i+j}T_2T_2\cdots T_{j-1}T_{j+1}\cdots T_{n-5}T_{n-3}T_{n-4}(-T_{n-4}+T_{n-3}))T_2\cdots T_{i-4}\chi_nT_{i-3}\\
&((-1)^{ni+1}T_{i-1}\cdots T_{n-5}T_2\cdots T_{i-3}(1+(T_{n-4}-T_{n-3})T_{n-1})\\
&+(-1)^{(n+1)i+1}T_{i-1}\cdots T_{n-5}T_3\cdots T_{i-3}(-T_{n-4}T_{n-3}T_{n-1}+T_{n-3}+T_{n-1})\\
&+(-1)^{n+1}T_3\cdots T_{i-3}T_{i-1}\cdots T_{n-5}(T_{n-4}(1-T_{n-3}T_{n-1})+T_{n-1})\\
&+T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}(T_{n-4}(T_{n-3}+T_{n-1})-T_{n-3}T_{n-1})\\
&-T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-3}T_{n-4}(-T_{n-4}+T_{n-3})\\
&+(-1)^{n+1}T_3\cdots T_{i-3}T_{i-1}\cdots T_{n-5}T_{n-3}T_{n-4}(-T_{n-4}+T_{n-3}))T_2\cdots T_{i-3}\chi_n\\
=&(-T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}(1+(T_{n-4}-T_{n-3})T_{n-1})\\
&+(-1)^nT_3\cdots T_{i-3}T_{i-1}\cdots T_{n-5}(-T_{n-4}T_{n-3}T_{n-1}+T_{n-3}+T_{n-1})\\
&+(-1)^{n+1}T_3\cdots T_{i-3}T_{i-1}\cdots T_{n-5}(-T_{n-4}T_{n-3}T_{n-1})+T_{n-3}+T_{n-1})\\
&+T_2\cdots T_{i-3}T_{i-1}\cdots T_{n-5}(1+T_{n-4}T_{n-1}-T_{n-3}T_{n-1}))T_2\cdots T_{i-3}\chi_n\\
=&0.
\end{aligned}$$

We have

$$t_{n-2}B_{n-2}(b)=t_3t_{n-2}B_{n-1}(-b)=t_3B_{n-1}((T_{n-4}-T_{n-3}+T_{n-1})b)=B_{n-2}((T_{n-4}-T_{n-3}+T_{n-1})b).$$

Case 7:

$$
\begin{aligned}
t_{n-1}B_{n-1}(b) =& \sum_{j=4}^{n-2}(-1)^{n+j+1}P_{2j}(T_{n-1}T_2\cdots T_{j-3}\chi_n b)\\
&+\sum_{j=4}^{n-2}(-1)^{j}P_{3j}(T_{n-1}T_{n-3}T_2\cdots T_{j-3}\chi_n b)\\
&+\sum_{i=4}^{n-4}\sum_{j=i+1}^{n-2}(-1)^{n+(n+1)i+j}P_{ij}(T_{n-1}(1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{j-3}\chi_n b)\\
&+\sum_{i=4}^{n-4}\sum_{j=4}^{i}P_{i,n-2}(((-1)^{(n+1)(i+1)+j}T_{n-1}T_3\cdots T_{i-2}T_2\cdots T_{j-3}T_{j-1}\cdots T_{n-3}T_{n-4}\\
&\qquad\qquad + (-1)^{ni+j+1}T_{n-1}T_2\cdots T_{i-2}T_2\cdots T_{j-3}T_{j-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_n b)\\
=&\sum_{j=4}^{n-2}(-1)^{n}P_{2j}(T_2\cdots T_{j-3}\chi_n T_{n-1}b)\\
&+\sum_{j=4}^{n-2}P_{3j}(T_{n-3}T_2\cdots T_{j-3}\chi_n T_{n-1}b)\\
&+\sum_{i=4}^{n-4}\sum_{j=i+1}^{n-2}(-1)^{n(i+1)+1}P_{ij}((1+T_2T_{n-3})T_3\cdots T_{i-2}T_2\cdots T_{j-3}\chi_n T_{n-1}b)\\
&+\sum_{i=4}^{n-4}\sum_{x=4}^{i}P_{i,n-2}(((-1)^{ni+x}T_3\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-3}T_{n-4}\\
&\qquad\qquad + (-1)^{n+(n+1)i+x}T_2\cdots T_{i-2}T_2\cdots T_{x-3}T_{x-1}\cdots T_{n-5}T_{n-3}T_{n-4})\chi_n T_{n-1}b)\\
=&B_{n-1}(-T_{n-1}b).
\end{aligned}
$$

$$t_{n-1}B_{n-2}(b) = t_3t_{n-1}B_{n-1}(-b) = t_3B_{n-1}(T_{n-1}b) = B_{n-2}(T_{n-1}b).$$

□