Profinite Galois descent in $K(h)$-local homotopy theory

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Declaration

I, Itamar Mor, confirm that the research included within this thesis is my own work or that where it has been carried out in collaboration with, or supported by others, that this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below.

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Itamar Mor

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Previously published work

- Except for Section 5.2, Part I appeared as [Mor23a].
- Part II, together with Section 5.2, forms [Mor23b].

Abstract

We investigate the category of $K(h)$-local spectra through the action of the Morava stabiliser group. Using condensed mathematics, we give a model for the continuous action of this profinite group on the ∞-category of $K(h)$-local modules over Morava E-theory E_h, and explain how this gives rise to descent spectral sequences computing the Picard and Brauer groups of $\mathcal{S}p_{K(h)}$. In the second part, we focus on the computation of these spectral sequences at height one, showing that they recover the Hopkins-Mahowald-Sadofsky computation of the Picard group, and giving a complete computation of the Brauer group relative to KU_p.

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Introduction

Chromatic homotopy theory

The computation of the stable homotopy groups of spheres is the central problem in modern homotopy theory, and the Adams spectral sequence the most successful tool in tackling it. Given a suitable ring spectrum E, this may be formed as the spectral sequence for the cobar complex

$$E \rightrightarrows E \otimes E \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots$$

of E, and reads

$$E_2 = \mathrm{Ext}^{*,*}_{E_*E}(E_*, E_*) \implies \pi_* \mathbb{S}^{\wedge}_E. \tag{1}$$

Its abutment is the *E-nilpotent completion* of $\mathbb{S}$, which under nilpotence assumptions on E may be identified with the Bousfield localisation $L_E\mathbb{S}$ [Bou79, Theorem 6.12]. A particularly successful strategy has been to take for E the complex cobordism spectrum MU, or its periodic variant MUP. In this case, it is a theorem of Quillen that the Hopf algebroid (MU_*, MU_*MU) presents the moduli stack $\mathcal{M}^{\mathrm{st}}$ of formal groups and strict isomorphisms; similarly, (MUP_*, MUP_*MUP) presents the moduli stack $\mathcal{M}$ of formal groups and *all* isomorphisms. As a consequence, the E_2-page (1) may be rewritten as

$$E_2^{s,2t} = H^s(\mathcal{M}, \omega^{\otimes t}),$$

where ω denotes the direct image of the sheaf $\pi_2 MUP$ on $\operatorname{Spec} \pi_0 MUP$ (see for example [Lur10, Lecture 11], [MM15, §2]). More generally, it was an insight of Morava that the geometry of the stack $\mathcal{M}$ should determine many structural properties of the stable homotopy category, and making a translation between the two is the focus of chromatic homotopy theory; this has resulted in a powerful organising principle for studying the stable homotopy category $\mathcal{S}p$, and in particular for the stable stems. At a prime p, the geometric points of $\mathcal{M}$ are indexed by $\mathbb{Z}_{\geq 0} \cup \{\infty\}$, the bijection given by the *height stratification*. In stable homotopy, this phenomenon is reflected in the following way. At fixed p there are spectra $K(h)$ for each $h \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, with $K(0) = H\mathbb{Q}$ and $K(\infty) = H\mathbb{F}_p$; in [HS98] Hopkins and Smith prove that these are in a precise sense the prime fields in $\mathcal{S}p_{(p)}$: for example, they represent those cohomology theories having a Künneth isomorphism.

These are the *Morava K-theories*, and using them one obtains a stratification

$$\mathcal{S}p_{\mathbb{Q}} = L_0\mathcal{S}p_{(p)} \subset \cdots \subset L_h\mathcal{S}p_{(p)} \subset \cdots \subset L_\infty\mathcal{S}p_{(p)} \subset \mathcal{S}p_{(p)}$$

where L_h is the Bousfield localisation with respect to $K(0)\oplus\cdots\oplus K(h)$; equivalently, L_h is localisation with *Morava E-theory*, constructed using Landweber exactness from the universal deformation of the unique formal group at height h. By construction, the localisation functor L_h therefore behaves as though it were "restriction to $\mathcal{M}_{\leq n}$": that is, to the open substack parameterising formal groups of height at most h. Moreover, the *Chromatic Convergence Theorem* of Hopkins-Ravenel says that any finite p-local spectrum is the homotopy limit of the resulting tower

$$X \to \cdots \to L_hX \to \cdots \to L_1X \to L_0X. \tag{2}$$

The localisations L_h thus provide a filtration of the stable homotopy category, and a reasonable first approximation to studying $\mathcal{S}p$ is to look at its associated graded, which is achieved by the localisation $L_{K(h)}(-)$. This functor behaves as if it were "completion along $\mathcal{M}_{=h} \coloneqq \mathcal{M}_{\leq h}\setminus\mathcal{M}_{\leq h-1}$".

The discussion can be summarised as follows: in order to understand finite spectra, one must (i) understand the categories $\mathcal{S}p_{K(h)}$ of $K(h)$-local spectra (at each p and h), and (ii) understand how to reconstruct the chromatic tower via fracture squares

$$\begin{tikzcd} L_h \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & L_{h-1} \arrow[d] \\ L_{K(h)} \arrow[r] & L_{h-1}L_{K(h)} \end{tikzcd}$$

In this thesis we focus on the first problem, harnessing tools of *Galois descent*. Specifically, a classical theorem of Lubin and Tate says that the formal group law of height h admits a universal deformation Γ_h over the Lubin-Tate ring $R_h \coloneqq \mathbb{W}(\mathbb{F}_{p^h})[[u_1,\ldots,u_{h-1}]]$. The *(extended) Morava stabiliser group* is the group of automorphisms of the pair $(\mathbb{F}_{p^h},\Gamma_h)$, and will be denoted $\mathbb{G}_h$; the completion of $\mathcal{M}_{\leq h}$ along the closed substack $\mathcal{M}_{=h}$ can be realised as the formal stack $\mathrm{Spf}(R_h)/\!\!/\mathbb{G}_h$. Morava E-theory lifts this construction to homotopy theory: by Landweber exactness, E_h has $\pi_0E_h = R_h$, has Γ_h as its formal group law, and has an (up to homotopy) action of $\mathbb{G}_h$. However, work of Goerss, Hopkins, Miller, and Lurie shows that the connection to homotopy theory is much stronger: E_h can be canonically realised as a commutative ring spectrum, and the action of $\mathbb{G}_h$ as an action of such. This gives enough structure to allow one to attack questions about $L_{K(h)}\mathbb{S}$ by means of descent along the $\mathbb{G}_h$-Galois extension to E_h: the most classical manifestation of this is in the $K(h)$-local Adams spectral sequence, whose E_2-page was shown by Morava to read

$$H^s(\mathbb{G}_h,\pi_tE_h) \implies \pi_{t-s}L_{K(h)}\mathbb{S}.$$

Much of the recent focus of chromatic homotopy theory has been on categorifying this to obtain structural statements about $\mathcal{S}p_{K(h)}$: for example, Mathew [Mat16] showed that the diagram of

∞-categories

$$\mathcal{S}p_{K(h)} \longrightarrow \mathrm{Mod}_{E_h}(\mathcal{S}p_{K(h)}) \rightrightarrows \mathrm{Mod}_{L_{K(h)}E_h \wedge E_h}(\mathcal{S}p_{K(h)}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots \tag{3}$$

is a limit, so that $\mathcal{S}p_{K(h)}$ can be recovered as the ∞-category of 'descent data' for the comonad $L_{K(h)}(E_h \otimes E_h \wedge_{E_h} -)$; since $L_{K(h)}\mathbb{S} \to E_h$ is Galois, this comonad is to be thought of informally as $\mathrm{Cont}(\mathbb{G}_h, -)$. Given this, one can ask about invariants of the ∞-category $\mathcal{S}p_{K(h)}$: for example, a famous conjecture of Ausoni and Rognes asks whether the assembly map

$$K(L_{K(h)}\mathbb{S}) \simeq K(E_h^{h\mathbb{G}_h}) \to K(E_h)^{h\mathbb{G}_h} \tag{4}$$

is an equivalence after localisation at a height-$(h+1)$ telescope. Burklund, Hahn, Levy and Schlank have recently announced a counterexample to this conjecture at all heights $h \geq 2$, which allowed them to disprove Ravenel's Telescope Conjecture. The disproof builds on work of Ben-Moshe, Carmeli, Yanovksi and Schlank [Ben+23] which shows that descent *does* hold for $K(h+1)$-local K-theory, at least for the maximal abelian p-extension $L_{K(h)}\mathbb{S} \to L_{K(h)}\mathbb{S}[\omega_{p^\infty}]$. This leaves open the question of $K(h+1)$-local descent for the full $\mathbb{G}_h$-extension (4).

It is therefore of central importance to understand the Galois action on the ∞-category $\mathrm{Mod}_{E_h}(\mathcal{S}p_{K(h)})$. Building on (3), we show in Theorem 3.1 that $\mathcal{S}p_{K(h)}$ may in a sense be thought of as $\mathrm{QCoh}(\mathrm{Spf}(E_h)/\!\!/\mathbb{G}_h) \coloneqq \mathrm{Mod}_{E_h}(\mathcal{S}p_{K(h)})^{h\mathbb{G}_h}$. Making sense of this the focus of Part I, and we now expand on our approach.

Condensed mathematics

To interpret the expression $\mathrm{Mod}_{E_h}(\mathcal{S}p_{K(h)})^{h\mathbb{G}_h}$, we need a suitable model of continuous actions of profinite groups on objects of ∞-categories. In this section we will explain how this is achieved using the proétale site. Given a profinite group G, we begin by listing some desiderata for theories $\mathcal{S}^G, \mathcal{S}p^G$ of continuous G-spaces or (naïve) G-spectra, inspired by the situation for finite or compact Lie groups.

(1) $\mathcal{S}^G$ should be a nice homotopy theory: ideally it should be an ∞-topos. Then $\mathcal{S}p^G \coloneqq \mathcal{S}p(\mathcal{S}^G)$ is a presentably symmetric monoidal ∞-category, a robust setting in which to do homotopy theory.

(2) there should be adjunctions

$$\mathcal{S}p \underset{\xleftarrow[\Delta_*]{}}{\overset{\xleftarrow{\Delta_!}}{\xrightarrow{\Delta^*}}} \mathcal{S}p^G \quad \text{and} \quad \mathcal{S}p \underset{\xrightarrow[\mathrm{coind}]{}}{\overset{\xrightarrow{\mathrm{ind}}}{\xleftarrow{\mathrm{res}}}} \mathcal{S}p^G\ ,$$

as well as a full subcategory $\mathcal{S}^{G,\delta}$ of *discrete* G-spaces[1].

[1]Recall [Ser97] that Serre calls a G-module *discrete* if all its stabilisers are open in G. Equivalently, M is the filtered colimit of its invariants M^U for open subgroups U, and the action on each of these factors through the *finite* quotient G/U. The latter definition also makes sense for G actions in an arbitrary ∞-category.

(3) given item (1), the heart $\mathcal{S}p^{G,\heartsuit}$ of the standard t-structure on $\mathcal{S}p^G$ should be a reasonably large category of continuous G-modules. Given also item (2), the functors

$$\mathcal{S}p^{G,\heartsuit} \hookrightarrow \mathcal{S}p^G \underset{\Delta_*}{\overset{\Delta_!}{\rightrightarrows}} \mathcal{S}p$$

should under reasonable conditions agree with continuous group (co)homology.

(4) Items (1) to (3) should be compatible with change of group in the obvious ways.

(5) Morava E-theory should naturally lift to an object $E_h \in \mathrm{CAlg}(\mathcal{S}p^{\mathbb{G}_h})$. Given items (1) and (2), one can form a 'homotopy fixed point' spectral sequence by applying the functor $(-)^{h\mathbb{G}_h}$ to a Postnikov tower for $E_h \in \mathcal{S}p^{\mathbb{G}_h}$; given item (3), the E_2-page would read

$$H^s(\mathbb{G}_h, \pi_t E_h) \implies \pi_{t-s} E_h^{h\mathbb{G}_h}.$$

This should agree with the $K(h)$-local E_h-Adams spectral sequence.

Having defined the ∞-categories $\mathcal{S}^G$ and $\mathcal{S}p^G$, we would next like to define an ∞-category Cat_∞^G of ∞-categories with continuous G-action, with similar properties. These should be related by a (monoidal) functor

$$\Theta^G : \mathrm{Alg}(\mathcal{S}p^G) \to \mathrm{Cat}_\infty^G$$

lifting the functor Θ which sends a ring spectrum to its module ∞-category [HA, §4.8], or more generally by a $\mathcal{C}$-linear variant $\Theta_\mathcal{C}^G$ for $\mathcal{C}$ such as $\mathcal{S}p_{K(h)}$.

We begin by searching for a good model for the $\mathbb{G}_h$-action on E_h. Profinite groups are rife in Galois theory, where they appear as Galois groups of maximal extensions. Indeed, the profinite nature of the Morava stabiliser group comes from its role as the automorphism group of a universal deformation of the height-h formal group in characteristic p. In the most classical number-theoretic applications, one is interested in discrete modules, and the cohomology of such modules may be defined as the limit of cohomology groups $H^*(G/U, M^U)$; as such, the resulting theory is not wildly different to cohomology of finite groups. In particular, the category of discrete G-modules is abelian, and is the heart of a nice ∞-category of *discrete G-spectra*, constructed by Jardine [Jar97] as a model category of spectral presheaves. Nevertheless, many Galois modules of interest are *not* discrete: most pertinently, this is the case for the homotopy groups $\pi_* E_h$. Indeed, even at height one the $\mathbb{G}_1 = \mathbb{Z}_p^\times$-module $\pi_{2t} KU_p$ is the Tate twist $\mathbb{Z}_p(t)$, and in particular free if $t \neq 0$. One would like to apply cohomological or homotopical techniques in this case too, but here an essential difficulty arises: the category of topological modules over a profinite group is not abelian. Even when the group is trivial, one has no reasonable notion of kernel or cokernel for maps such as

$$\mathbb{R}^\delta \to \mathbb{R},$$

where $(-)^\delta$ denotes the discrete topology, and in fact it is difficult to find an abelian category large enough to contain both discrete and profinite G-modules. Many solutions have been suggested to circumvent this problem [Tat76; Jan88; SW00], using relative notions of homological algebra. Inspired by Janssen's construction of profinite group cohomology and Jardine's construction of discrete G-spectra, Davis [Dav03; Dav06] defined continuous G-spectra as a suitable model category of towers of discrete G-spectra. On the other hand, Quick [Qui11; Qui08; Qui13] used an approach inspired by the étale homotopy theory of Artin and Mazur, defining a model category of simplicial profinite spaces, and spectrum objects therein. Both approaches suffice to obtain item (5), but there are formidable subtleties in the definitions. Most important for our purposes are the following issues:

(*i*) the iterated fixed point formula $X^{hG} \simeq (X^{hH})^{hG/H}$ for closed subgroups $H < G$ holds only under conditions on X and H [Dav09];

(*ii*) it was unclear to us how to equip either of the corresponding ∞-categories with symmetric monoidal structure (although see [DL14] for work in this direction);

(*iii*) it was unclear to us how to categorify either of these approaches (although see recent work by Li and Zhang [LZ23] for a model of Cat_∞^G of a similar flavour to Davis' category of G-spaces).

A drastically different construction of continuous cohomology was proposed by Bhatt and Scholze, which will be the starting point for us. In [BS14], they defined the *proétale site* of a scheme X; roughly speaking, this is the pro-category of X_{et}, equipped with the fpqc topology.

In fact we focus on the simplest case, in which case one can be very explicit. When X is the spectrum of a field k, one obtains the site $B\mathrm{Gal}(k)_{\mathrm{proet}}$: up to size issues, the underlying category is the category of profinite $\mathrm{Gal}(k)$-sets and continuous homomorphisms, and coverings are given by families $\{X_\alpha \to X\}$ for which some finite subfamily is jointly surjective. This definition admits an evident generalisation to general profinite groups G, giving a site BG_{proet} of continuous profinite G-sets. The essential features of BG_{proet} for our purposes, most of which may be found in [BS14], can be summarised as follows:

(*i*) it is a site, and as such $\mathrm{Ab}^G \coloneqq \mathrm{Ab}(BG_{\mathrm{proet}})$ is an abelian category. There is a full abelian subcategory $\mathrm{Ab}^{G,\delta}$, given by those sheaves that are left Kan extended from the site of *finite* G-sets, and this is equivalent to Serre's category of discrete G-modules. The construction is evidently functorial in G.

(*ii*) the restricted Yoneda functor
$$y : M \mapsto \mathrm{Cont}(-, M)$$
from the category of topological G-modules to Ab^G is fully faithful on the subcategory of compactly generated G-modules.

(*iii*) cohomology on Ab_G agrees with other definitions of continuous cohomology for a large class of topological G-modules: that is, there is a map
$$R\Gamma_{\mathrm{cont}}(G, M) \to R\Gamma(BG_{\mathrm{proet}}, yM),$$

constructed as the edge homomorphism in a Čech-to-sheaf cohomology spectral sequence, which is an equivalence for many topological modules M.

(*iv*) BG_{proet} is generated by either of the following full subcategories:

(a) the subcategory Free_G of free G-sets,

(b) the subcategory Proj_G of G-sets of the form

$$X = G \times T$$

for T an *extremally disconnected* profinite set. These are distinguished by the property that any covering $Y \twoheadrightarrow X$ splits.

Consequently, the topos $BG_{\text{proet}}^{\sim}$ is equivalent to the category $\mathcal{P}_\Sigma(\text{Proj}_G, \text{Set})$ of presheaves that send binary coproducts to products, and Ab^G satisfies the same Grothendieck AB axioms as the category of abelian groups. Likewise, the *hypercomplete* ∞-topos $\widehat{\mathcal{S}h}(BG_{\text{proet}})$ is the nonabelian derived category $\mathcal{P}_\Sigma(\text{Proj}_G)$.

(*v*) $BG_{\text{proet}}^{\sim}$ is *replete*, and so Postnikov completeness agrees with hypercompleteness in the associated ∞-topos [MR22].

As such, even the case $G = *$ is interesting: one obtains an abelian category $\text{Cond}(\text{Ab}) := \text{Ab}(B*_{\text{proet}})$ which contains the category of compactly generated topological abelian groups as a full subcategory. In fact, the analogous statement holds before passing to abelian groups: there is a fully faithful inclusion

$$\text{CG} \hookrightarrow \text{Cond}(\text{Set}) := (B*_{\text{proet}})^{\sim}$$

where the source is the category of compactly generated spaces. The topos on the right-hand side is the topos of *condensed sets*. The alternate description as $\mathcal{P}_\Sigma(\text{Proj})$ is extremely useful in practice: when restricting to the subcategory Proj of extremally disconnected sets, many presheaves occuring in nature are already sheaves. This gives an extremely efficient way to record the topology of spaces in CG; notably, this can be used to remember profinite topologies that are otherwise hard to distinguish homotopically.

The list of properties above makes the proétale formalism extremely appealing for our purposes. Starting with the proétale site, we can form the hypercomplete ∞-topos $\widehat{\mathcal{S}h}(BG_{\text{proet}})$, for which items (1) and (2) are more or less immediate. Moreover, item (3) follows from the original work of [BS14]. In fact, we can form $\widehat{\mathcal{S}h}(BG_{\text{proet}}, \mathcal{C})$ for any ∞-category $\mathcal{C}$, and this very formally acquires many desirable properties; taking $\mathcal{C} = \text{Cat}_\infty$ (or more generally $\mathcal{C} = \text{Cat}_{\mathcal{A}} := \text{Mod}_{\mathcal{A}}(\text{Pr}^L)$ for $\mathcal{A}$ such as $\mathcal{S}p_{K(h)}$) gives us our model for Cat_∞^G (respectively, $\text{Cat}_{\mathcal{A}}^G$). Details on these constructions are given in Sections 2.2 and 3.1.

What remains is to show that Morava E-theory fits well in this framework, which is the primary goal of Chapter 2. Here a few approaches are possible, and we choose to take one closely related to Galois descent theory in the sense of [Rog08; Mat16]. Namely, while E_h is not a discrete $\mathbb{G}_h$-object of $\mathcal{S}p$ (as noted, even its homotopy groups are not discrete), it *is* discrete in $\mathcal{S}p_{K(h)}$, essentially by Rognes' definition of profinite Galois extensions. We prove that any such Galois $A \to B$ extension which is *descendable* gives rise to a proétale sheaf by the formula

$$\mathcal{X} : G/H \mapsto B^{dhH},$$

where B^{dhH} for $H < G$ closed denotes the 'homotopy fixed point' spectrum defined by Devinatz and Hopkins [DH04]. Importantly, the descendability assumption implies that no sheafification is necessary, and so we maintain control of the fixed points $\mathcal{X}^{hH}$. While the descendability axiom is very strong—implying for example that the homotopy fixed point spectral sequences for all closed subgroups collapse with a horizontal vanishing line of uniform height—it is satisfied in the case of Morava E-theory by a classical theorem of Hopkins and Ravenel [Rav92]. Thus we obtain

$$\mathcal{E}_h \in \widehat{\mathcal{S}h}(B(\mathbb{G}_h)_{\text{proet}}, \mathcal{S}p_{K(h)}) \subset \widehat{\mathcal{S}h}(B(\mathbb{G}_h)_{\text{proet}}, \mathcal{S}p),$$

and hence

$$\mathrm{Mod}_{\mathcal{E}_h}(\mathcal{S}p_{K(h)}) \in \mathcal{S}h(B(\mathbb{G}_h)_{\text{proet}}, \mathrm{Cat}_\infty).$$

These turn out to be suitable models for the $\mathbb{G}_h$-action on E_h and on its module category.

Picard and Brauer groups via Galois descent

We now turn to applications of the abstract machinery of the previous section. Even for finite extensions, Galois descent techniques have been used to great effect. The main points is that E_h is even-periodic with $\pi_0 E_h$ being a regular complete noetherian local ring; the homotopy theory of such ring spectra often reduces to algebra at the level of π_0.

For example, the *Picard group* classifying invertible obects is a fundamental invariant of a symmetric monoidal category, and a problem first posed by Hopkins [HMS94] is to determine the groups $\mathrm{Pic}_h := \mathrm{Pic}(\mathcal{S}p_{K(h)})$. It is a theorem of Baker and Richter [BR05] that the Picard groups of even periodic ring spectra E satisfying the above assumptions on π_0 are given by $\mathbb{Z} \times \mathrm{Pic}(\pi_0 E)$, and this was used by Heard, Mathew and Stojanoska [HMS17] to give a descent theoretic computation of the Picard groups of the higher real K-theories $EO(h) := E_h^{hG}$ at height $h = p-1$, where $G \subset \mathbb{G}_h$ is a maximal finite subgroup. This is the first step in a programme to compute Pic_h by Galois descent from E_h, and our interest in the proétale site in Part I was motivated by the problem of extending these computations to the full profinite group $\mathbb{G}_h$. More precisely, the following is the main practical output of Part I:

Theorem. *There is a hypercomplete sheaf of connective spectra $\mathfrak{pic}(\mathcal{E})$ on the site $B(\mathbb{G}_h)_{\text{proet}}$ of profinite $\mathbb{G}_h$-sets, with*

$$\Gamma(\mathbb{G}_h/*, \mathfrak{pic}(\mathcal{E})) = \mathfrak{pic}(E_h) \quad \textit{and} \quad \tau_{\geq 0}\Gamma(\mathbb{G}_h/\mathbb{G}_h, \mathfrak{pic}(\mathcal{E})) = \mathfrak{pic}(\mathcal{S}p_{K(h)}).$$

The resulting descent spectral sequence takes the form

$$E_2^{s,t} = H^s(\mathbb{G}_h, \pi_t\mathfrak{pic}(E_h)) \implies \pi_{t-s}\mathfrak{pic}(\mathcal{S}p_{K(h)}). \tag{5}$$

It agrees in a large range with the $K(h)$-local E_h-based Adams spectral sequence, including differentials. On the boundary of this range there is an explicit correction term for d_r.

Evaluating the spectral sequence at height one (using the information in the theorem about differentials) gives a new proof of the computation in [HMS94] of Pic_1 at all primes; see Chapter 4. Our techniques also allow us to prove qualitative features of the spectral sequence: for example, we show that the exotic part of the Picard group is precisely the subgroup in filtration $s \geq 2$.

As already remarked, to prove the theorem we first prove a descent result for module ∞-categories. This in itself can be leveraged to glean information about the structure of $\mathcal{S}p_{K(h)}$, for example by using descent to compute invariants one categorical level up. More specifically, one can consider the problem of computing the Brauer group $\mathrm{Br}_h \coloneqq \mathrm{Br}(\mathcal{S}p_{K(h)})$, which classifies $K(h)$-local Azumaya algebras up to Morita equivalence; this admits an alternative description, as the group of twists of the $\mathbb{G}_h$-action on $\mathrm{Mod}_{E_h}(\mathcal{S}p_{K(h)})$. In Chapter 5 we show that the subgroup $\mathrm{Br}_h^0 \subset \mathrm{Br}_h$ of Azumaya algebras trivialised over E_h is computed by the (-1)-stem of spectral sequence (1.3), and in Part II we carry out this computation at height one:

Theorem. (*i*) *At odd primes,*

$$\mathrm{Br}_1^0 \cong \mu_{p-1}.$$

The isomorphism is given by the cyclic algebra construction for the μ_{p-1}-Galois extension $L_{K(1)}\mathbb{S} \to KU_p^{h(1+p\mathbb{Z}_p)}$.

(*ii*) *At the prime two,*

$$\mathrm{Br}_1^0 \cong \mathbb{Z}/4 \times \mathbb{Z}/8.$$

Notation and conventions

- Throughout, we will work at a fixed prime p and height h, mostly kept implicit. Also implicit is the choice of a height h formal group law Γ_h defined over $\mathbb{F}_{p^h}$; for concreteness we fix the Honda formal group, but this will not be used. For brevity, we will therefore write $\mathbf{E}, \mathbf{K}$ and $\mathbb{G}$ for Morava E-theory, Morava K-theory and the extended Morava stabiliser group, respectively. These will be our principal objects of study.

- We will freely use the language of ∞-categories (modeled as quasi-categories) as pioneered by Joyal and Lurie [HTT; HA; SAG]. In particular, all (co)limits are ∞-categorical. Following [Mat16], we use the phrase "stable homotopy theory" as a synonym for "presentably symmetric monoidal stable ∞-category". We will mostly be working internally to the $K(h)$-local category, and as such we stress that **the symbol $\otimes$ will denote the $K(h)$-local smash product throughout**; where we have a need for it, we will use the notation $\wedge$ for the smash product of spectra (despite this having become archaic in some circles). On the other hand, we will distinguish $K(h)$-local colimits by writing for example $L_{\mathbf{K}}\operatorname{colim} X$ or $L_{\mathbf{K}} \varinjlim X_i$, as we feel that not to do so would be unnecessarily confusing. We use the symbol $\varinjlim$ to denote a *filtered* colimit, and similarly for cofiltered limits. In particular, if T is a profinite set, we will use the expression '$T = \varprojlim T_i$' to refer to a presentation of T as a pro-object, leaving implicit that each T_i is finite. We will also assume throughout we have made a fixed choice of decreasing open subgroups $U_i \subset \mathbb{G}$ with trivial intersection; the symbols $\varprojlim_i$ and $\varinjlim_i$ will always refer to the (co)limit over such a family. Likewise, we will assume we have chosen a sequence of ideals $I \subset \pi_0 E_h$ generating the $\mathfrak{m}$-adic topology; without loss of generality, the ideals I will be chosen so that there exists a tower M_I of generalised Moore spectra, with $\pi_* M_I = (\pi_* E_h)/I$.

- When G is a profinite group, we will write $H^*(G, M)$ for *continuous* group cohomology with pro-p (or more generally profinite) coefficients, as defined for example in [SW00] (resp. [Jan88]). We write $H^*(BG_{\mathrm{proet}}, \mathcal{F})$ for cohomology on the proétale site.

- In Chapters 2 and 3 we form spectral sequences using the usual t-structure on spectral sheaves; this is useful for interpreting differentials and filtrations, for example in Theorem 4.4. To obtain familiar charts, we will declare that the spectral sequence associated to a filtered object starts at the E_2-page; in other words, this is the page given by homotopy groups of the associated graded object. Thus our spectral sequences run

$$E_2^{s,t} = H^s(G, \pi_t E) \implies \pi_{t-s} E^{hG},$$

with differentials d_r of $(s, t-s)$-bidegree $(r, -1)$, and this is what we display in all figures. However, we also make use of the Bousfield-Kan definition of the descent spectral sequence using the Čech complex of a covering, and we relate the two formulations by décalage (see Appendix A): there is an isomorphism between the two spectral sequences that reads

$$E_2^{s,t} \cong \check{E}_3^{2s-t,s}$$

if we use the same grading conventions for each of the underlying towers of spectra. We will always use s for filtration, t for internal degree, and $t-s$ for stem.

Our Lyndon-Hochschild-Serre spectral sequences are in cohomological Serre grading.

- When talking about the 'Adams spectral sequence', we always have in mind the $K(h)$-local E_h-based Adams spectral sequence; the classical Adams spectral sequence (based on $H\mathbb{F}_p$) makes no appearance in this document. We will freely use abbreviations such as 'ASS', 'HFPSS', 'BKSS'. We will also use the name 'descent spectral sequence' for either the t-structure or Čech complex definition, since in the cases of interest we show they agree up to reindexing; when we need to be more explicit, we refer to the latter as the 'Bousfield-Kan' or 'Čech' spectral sequence. The name 'Picard spectral sequence' will refer to the descent spectral sequence for the sheaf $\mathfrak{pic}(\mathcal{E})$.

- Finally, we largely ignore issues of set-theory, since these are discussed at length in [Cond] and [Pyk]. For example, one could for concreteness fix a hierarchy of strongly inaccessible cardinals $\delta_0 < \delta_1$ such that $|\mathbb{G}_h| < \delta_0$ and $L_{K(h)}\mathbb{S}$ is δ_0-compact, and work throughout over the 'δ_1-topos' of sheaves of δ_1-small spaces on profinite $\mathbb{G}_h$-sets of cardinality less than δ_0.

List of symbols

$\mathbf{1}_A$ A-local sphere spectrum.

$\mathbf{E}$ Morava E-theory, $E_h = E(\mathbb{F}_{p^h}, \Gamma_h)$.

$\mathbf{K}$ Morava K-theory, $K(h) = E_h/(p, v_1, \ldots, v_{h-1})$.

$\mathbb{G}$ Extended Morava stabiliser group $\mathbb{G}_h = \mathbb{S}_h \rtimes \mathrm{Gal}(\mathbb{F}_{p^h}/\mathbb{F}_p) = \mathrm{Aut}(\Gamma_h, \mathbb{F}_{p^h})$.

I_h Maximal ideal $(p, v_1, \ldots, v_{h-1}) \subset \pi_* E_h$.

$\mathbb{W}$ Witt vectors $\mathbb{W} = \mathbb{W}(\overline{\mathbb{F}}_p)$.

$\mathbb{SW}$ Spherical Witt vectors $\mathbb{SW} = \mathbb{SW}(\overline{\mathbb{F}}_p)$.

BG_{et} Étale classifying site of a profinite group G.

BG_{proet} Proétale classifying site of a profinite group G.

Free_G Subsite of free G-sets in BG_{proet}.

$\mathrm{Cont}_G(T, T')$ Set of continuous G-equivariant maps between profinite G-sets.

$\mathcal{E}^\delta$ Étale Morava E-theory sheaf.

$\mathcal{E}$ Proétale Morava E-theory sheaf.

$\mathrm{Mod}_{A,\mathbf{K}}$ $K(h)$-local module ∞-category, $\mathrm{Mod}_A(\mathcal{S}p_{K(h)}) = L_{K(h)}\mathrm{Mod}_A$.

$\mathrm{Pic}(-)\, (\mathfrak{Pic}(-), \mathfrak{pic}(-))$ Picard group (resp. space, spectrum).

$\mathfrak{pic}(\mathcal{E})$ Proétale Picard sheaf of Morava E-theory, $\mathfrak{pic}(\mathrm{Mod}_{\mathcal{E}_h}(\mathcal{S}p_{K(h)}))$.

$\mathrm{Pic}_h\, (\mathfrak{Pic}_h,\, \mathfrak{pic}_h)$ Picard group (resp. space, spectrum) of $K(h)$-local category, $\mathrm{Pic}(\mathcal{S}p_{K(h)})$.

$\mathrm{Pic}_h^{\mathrm{alg}}\, (\mathfrak{Pic}_h^{\mathrm{alg}})$ Algebraic Picard group (resp. groupoid) of $K(h)$-local category, $\mathrm{Pic}(\mathrm{Mod}_{\pi_* E_h}^{\mathbb{G}_h})$.

$\mathrm{Br}(-)\, (\mathfrak{Br}(-), \mathfrak{br}(-))$ Brauer group (resp. space, spectrum).

$\mathrm{Br}(- \mid A)$ Relative Brauer group, $\ker(\mathrm{Br}(\mathbf{1}) \to \mathrm{Br}(A))$.

$\mathfrak{Br}(- \mid A)$ Relative Brauer space of a G-Galois extension, $(B\mathfrak{Pic}(A))^{hG}$.

Br_h^0 Brauer group of $\mathcal{S}p_{K(h)}$ relative to E_h, $\mathrm{Br}(\mathcal{S}p_{K(h)} \mid E_h)$.

$\mathrm{Mod}_R^{\heartsuit\mathrm{cpl}}$ Category of L-complete (0-truncated) modules over a local ring R.

$\mathrm{Mod}_R^{\mathrm{cpl}}$ ∞-category of complete modules (in spectra) over a local ring R.

$\otimes$ $K(h)$-localised smash product of spectra, $L_{K(h)}(- \wedge -)$.

$\widehat{\otimes}$ Tensor product on $\mathrm{Mod}_R^{\heartsuit\mathrm{cpl}}$, $L_0(- \otimes_R -)$.

$\widehat{\otimes}^{\mathbb{L}}$ Tensor product on $\mathrm{Mod}_R^{\mathrm{cpl}}$, $L(- \otimes_R^{\mathbb{L}} -)$.

$\mathbf{E}_*^{\vee}(-)$ Completed E-homology functor $\pi_* L_{K(h)}(E_h \wedge -)$.

R^{nr} Maximal unramified extension of a commutative ring spectrum R.

$\mathcal{D}^{\mathrm{eg}}$ Full subcategory of $\mathcal{C}$-compact generators.

Part I

Profinite Galois descent

Chapter 1

Introduction

In [HMS94], Hopkins, Mahowald and Sadofsky study the *Picard group* of a symmetric monoidal category: by definition, this is the group of isomorphism classes of invertible objects with respect to the monoidal product. This is a notion that goes back much further, and gives a useful invariant of a ring or scheme. Its particular relevance to homotopy theory comes from the observation that if the category $\mathcal{C}$ is a Brown category (for example, $\mathcal{C}$ might be the homotopy category of a compactly-generated stable ∞-category), then the representability theorem applies and shows that the Picard group $\mathrm{Pic}(\mathcal{C})$ classifies homological automorphisms of $\mathcal{C}$, each of these being of the form $T \otimes (-)$ for some invertible object. The objective of *op. cit.* is to develop techniques for studying Picard groups in some examples coming from chromatic homotopy: the main theorem is the computation of the Picard group of $K(1)$-local spectra at all primes, where $K(1)$ is Morava K-theory at height one.

The aim of this project is to give a new proof of these computations using *Galois descent*, inspired by the formalism developed in [MS16]. We will write Pic_h for the Picard group of $K(h)$-local spectra. There are still many open questions regarding these groups: for example, it is unknown if they are finitely generated as modules over $\mathbb{Z}_p$. The question of computing Pic_2 has been studied by many authors (for example [KS04; Kar10; Goe+15]); using recent work at the prime 2 [Bea+22a], our results give a new potential approach to the computation of Pic_2 in [Bea+22b]. Following [GL21], we also extend these techniques to the *Brauer group* of $\mathcal{S}p_{K(h)}$, giving a cohomological approach to these, which allows us in Part II to compute the group of $K(1)$-local Azumaya algebras trivialised over $E_1 = KU_p$, at all primes.

The notion of Galois descent in algebra is very classical, and says that if $A \to B$ is a Galois extension of rings, then Mod_A can be recovered as the category of descent data in Mod_B: in particular, invertible A-modules can be recovered as invertible B-modules M equipped with isomorphisms $\psi_g : M \cong g^*M$ for each $g \in \mathrm{Gal}(B/A)$, subject to a cocycle condition. This gives an effective way to compute Picard groups and other invariants. One can try to play the same game in higher algebra, and use descent techniques to get a handle on the groups Pic_h. Fundamental to this approach is the notion of a Galois extension of commutative ring spectra, as set down in [Rog08]: this is a direct generalisation of the classical axioms in [AG60]. Given a finite G-Galois extension $A \to B$ which is *faithful*, the analogous descent statement (due to [Mei12; GL21]) is that the canonical functor

$$\mathrm{Mod}_A \to (\mathrm{Mod}_B)^{hG} := \lim \Big(\mathrm{Mod}_B \rightrightarrows \textstyle\prod_G \mathrm{Mod}_B \overset{\rightarrow}{\rightrightarrows} \cdots \Big) \tag{1.1}$$

is an equivalence of symmetric monoidal ∞-categories. Taking the Picard *spectrum* of a symmetric monoidal ∞-category preserves (homotopy) limits, and therefore any such Galois extension gives rise to an equivalence $\mathfrak{pic}(\mathrm{Mod}_A) \simeq \tau_{\geq 0}(\mathfrak{pic}(\mathrm{Mod}_B)^{hG})$. In particular one gets a homotopy fixed points spectral sequence (hereafter HFPSS), whose 0-stem converges to $\mathrm{Pic}(\mathrm{Mod}_A)$. This technique has proved very fruitful in Picard group computations: for example, the Picard groups of KO and Tmf are computed in [MS16], and the Picard group of the higher real K-theories $EO(h)$ in [HMS17]. In each case the starting point is a theorem of Baker and Richter [BR05], which says that the Picard group of an even-periodic ring spectrum E with $\pi_0 E$ regular Noetherian is a $\mathbb{Z}/2$-extension of the Picard group of the ring $\pi_0 E$. One can for example study the action of the Morava stabiliser group $\mathbb{G}_h$ on Morava E-theory:

Theorem 1.1 ([DH04; Rog08])**.** *Write E_h for height h Morava E-theory at the (implicit) prime p. The unit*

$$L_{K(h)}\mathbb{S} \to E_h$$

is a $K(h)$-local profinite Galois extension for the Goerss-Hopkins-Miller action of $\mathbb{G}_h$. That is, there are $K(h)$-local spectra E_h^{hU} for every open subgroup U of $\mathbb{G}_h$ such that the following hold:

(i) each E_h^{hU} is an $\mathbb{E}_\infty$-ring spectrum over which E_h is a commutative algebra,

(ii) choosing a cofinal sequence of open subgroups U yields $E_h \simeq L_{K(h)} \varinjlim_U E_h^{hU}$,

(iii) for any normal inclusion $V \lhd U$ of open subgroups, the map $E_h^{hU} \to E_h^{hV}$ is a faithful U/V-Galois extension of $K(h)$-local spectra.

In order to leverage this to compute the groups Pic_h it is necessary to understand not just the finite Galois descent technology mentioned above, but how this assembles over the entire system of extensions. Our main theorem is the following descent result for Picard groups:

Theorem A. *The unit $L_{K(h)}\mathbb{S} \to E_h$ induces an equivalence of spectra*

$$\mathfrak{pic}(\mathcal{S}p_{K(h)}) \simeq \tau_{\geq 0}\mathfrak{pic}(E_h)^{h\mathbb{G}_h}, \tag{1.2}$$

where the right-hand side denotes continuous *homotopy fixed points. The resulting spectral sequence takes the form*

$$E_2^{s,t} = H^s_{cont}(\mathbb{G}_h, \pi_t\mathfrak{pic}(E_h)) \implies \pi_{t-s}\mathfrak{pic}(\mathcal{S}p_{K(h)}). \tag{1.3}$$

In an explicit range, it agrees with the $K(h)$-local E_h-Adams spectral sequence, including differentials.

Here $\mathfrak{pic}(E_h)$ denotes the Picard spectrum of $K(h)$-*local* E_h-modules. One of the major tasks is to properly interpret the right-hand side of (1.2), in order to take into account the profinite topology on the Morava stabiliser group; to do so, we make use of the *proétale* (or *condensed/pyknotic*) formalism of [BS14; Cond; Pyk]. We elaborate on our approach later in the introduction, but will first mention some consequences.

Picard group computations

As a first corollary, we show how to recover the computation of Pic_1.

Theorem B ([HMS94]; Propositions 4.16 and 4.20)**.** *The Picard group of the* $K(1)$*-local category is as follows:*

(*i*) *At odd primes,* $\mathrm{Pic}_1 = \mathbb{Z}_p \times \mathbb{Z}/2(p-1)$,

(*ii*) *When* $p = 2$*,* $\mathrm{Pic}_1 = \mathbb{Z}_2 \times \mathbb{Z}/2 \times \mathbb{Z}/4$.

This is obtained by computing the spectral sequence (1.3) from knowledge of the $K(1)$-local E_1-Adams spectral sequence. In fact, we focus on computing the exotic part of Pic_1; in Theorem 4.4 we show that this is precisely the part detected in (1.3) in filtration at least two.

We also consider examples at height bigger than two. When combining the results of [HMS94; KS04; Kar10; Goe+15; Bea+22b; Pst22], the following gives an algebraic expression for the first undetermined Picard group:

Theorem C ([CZ22]; Corollary 4.12 and Example 4.14)**.** *If* $p = 5$*, there is an isomorphism*

$$\kappa_3 \simeq H_0(\mathbb{S}_3, \pi_8 E_3)^{\mathrm{Gal}(\mathbb{F}_{125}/\mathbb{F}_5)}.$$

At height two, spectral sequence (1.3) implies the (known) result that exotic elements exist only at the primes 2 and 3; the groups κ_2 at these primes were computed in [Bea+22b] and [Goe+15] respectively. Both cases used the filtation on Pic_2 defined in [HMS94, Prop. 7.6], and explicit constructions of Picard elements. On the other hand, Theorem A gives another possible approach to compute κ_2, by computing (1.3) via comparison with the $K(2)$-local E_2-Adams spectral sequence; the latter is studied for example in [HKM13; Bea+22a]. We hope to return to this in future.

Brauer group computations

In Chapter 5, we turn our attention to Galois descent for the Brauer group of $\mathcal{S}p_{K(h)}$. In [GL21], Gepner and Lawson use Galois descent to compute the relative Brauer group $\mathrm{Br}(KO \mid KU)$ from knowledge of the HFPSS for $\mathfrak{pic}(KO) \simeq \tau_{\geq 0}\mathfrak{pic}(KU)^{hC_2}$, and we show that their method extends to our context. Note that while the Picard group is amenable to computation by a variety of methods, the Brauer group is much more difficult to study in an *ad hoc* manner, and each of

[AG14; GL21; AMS22] uses some such form of descent to obtain an upper bound; this follows the classical picture, in which descent computations were pioneered by Grothendieck in [Gro68b]. The results of Chapter 5 strengthen existing descent technology, making the Brauer groups of the $K(h)$-local categories computationally tractable.

In analogy with the Picard case, we will write $\mathrm{Br}_h^0 \coloneqq \mathrm{Br}(\mathcal{S}p_{K(h)} \mid E_h)$ for the group of Brauer classes of $\mathcal{S}p_{K(h)}$ that become trivial (up to $K(h)$-local Morita equivalence) over Morava E-theory. We prove that this group is computed by the (-1)-stem of the Picard spectral sequence. In Part II, we use Galois descent to determine Br_1^0 at all primes.

Methods for profinite descent

We now summarise our approach to continuity of the $\mathbb{G}_h$-action on $\mathfrak{pic}(E_h)$. In the case of Morava E-theory itself, this was explored in the work of Davis [Dav03; Dav06] and Quick [Qui11], both of whom (using different methods) gave model-theoretic interpretations for continuous actions of profinite groups. In particular, both approaches recover the $K(h)$-local E_h-based Adams spectral sequence as a type of HFPSS for the action on E_h.

We will make crucial use of the *proétale classifying site* of the profinite group $\mathbb{G}_h$. This is closer in flavour to the approach of Davis, who uses the *étale* classifying site; his strategy is briefly recounted in Section 2, where we make the connection explicit. Namely, Davis makes use of the fact that viewed as an object of the $K(h)$-local category, Morava E-theory is a *discrete* $\mathbb{G}_h$-object, meaning that it is the filtered colimit of the objects E_h^{hU}. Such $\mathbb{G}_h$-actions can be effectively modelled using the site of *finite* $\mathbb{G}_h$-sets, and the requisite model category of simplicial sheaves was developed by Jardine [Jar97] as a way of formalising Thomason's work on descent for $K(1)$-localised algebraic K-theory. Of course, the action on E_h is *not* discrete when we view it as a plain spectrum, as can be seen on homotopy groups. Likewise, the induced action on its Picard spectrum is not discrete. As a model for more general continuous actions of a profinite group G, we therefore use the ∞-category of sheaves on the *proétale* site BG_{proet}, whose objects are the profinite G-sets. This was studied in [BS14]; as shown there, in many cases it gives a site-theoretic interpretation of continuous group cohomology. Even when the comparison fails, proétale sheaf cohomology exhibits many desirable properties absent in other definitions.

The equivalence in Theorem A is therefore interpreted as the existence of a sheaf of connective spectra $\mathfrak{pic}(\mathcal{E}_h)$ on the proétale site, having

$$\Gamma(\mathbb{G}_h/*, \mathfrak{pic}(\mathcal{E}_h)) \simeq \mathfrak{pic}(\mathrm{Mod}_{E_h}(\mathcal{S}p_{K(h)}))$$
$$\Gamma(\mathbb{G}_h/\mathbb{G}_h, \mathfrak{pic}(\mathcal{E}_h)) \simeq \mathfrak{pic}(\mathcal{S}p_{K(h)}).$$

To this end, we begin by proving a descent result for Morava E-theory itself.

Theorem A.I (Proposition 2.37 and Proposition 2.47)**.** *There is a hypercomplete sheaf of spectra*

$\mathcal{E}_h$ *on* $B(\mathbb{G}_h)_{\mathrm{proet}}$ *with*

$$\Gamma(\mathbb{G}_h/*, \mathcal{E}_h) \simeq E_h$$
$$\Gamma(\mathbb{G}_h/\mathbb{G}_h, \mathcal{E}_h) \simeq L_{K(h)}\mathbb{S}.$$

Its homotopy sheaves are given by $\pi_t\mathcal{E}_h \simeq \mathrm{Cont}_{\mathbb{G}}(-, \pi_t E_h)$*, and its descent spectral sequence agrees with the* $K(h)$*-local* E_h*-Adams spectral sequence (including differentials).*

This may be of independent interest as it gives a novel construction of the $K(h)$-local E_h-Adams spectral sequence, which may be extended to an arbitrary spectrum X; its E_2-page (given *a priori* in terms of sheaf cohomology on the proétale site) is continuous group cohomology for suitable X (Remark 2.46). Note moreover that proétale cohomology enjoys excellent functoriality properties, and that the category of L-complete abelian sheaves on $B\mathbb{G}_{\mathrm{proet}}$ is abelian, as opposed to L-complete $(E_h)^\vee_* E_h$-comodules.

Next, we deduce a descent result for ∞-categories of $K(h)$-local modules, which is really an extension to the condensed world of the following significant theorem:

Theorem 1.2 ([Mat16])**.** *The diagram of symmetric monoidal* ∞*-categories*

$$\mathcal{S}p_{K(h)} \longrightarrow \mathrm{Mod}_{E_h}(\mathcal{S}p_{K(h)}) \rightrightarrows \mathrm{Mod}_{L_{K(h)}E_h\wedge E_h}(\mathcal{S}p_{K(h)}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots$$

is a limit cone.

Namely, in Chapter 3 we prove the following profinite Galois descent result, which can be seen as the identification $\mathcal{S}p_{K(h)} \simeq \left(\mathrm{Mod}_{E_h}(\mathcal{S}p_{K(h)})\right)^{h\mathbb{G}_h}$ analogous to (1.1).

Theorem A.II (Theorem 3.1)**.** *There is a hypercomplete sheaf* $\mathrm{Mod}_{\mathcal{E}_h}(\mathcal{S}p_{K(h)})$ *of symmetric monoidal* ∞*-categories on* $B(\mathbb{G}_h)_{\mathrm{proet}}$ *with*

$$\Gamma(\mathbb{G}_h/*, \mathrm{Mod}_{\mathcal{E}_h}(\mathcal{S}p_{K(h)})) \simeq \mathrm{Mod}_{E_h}(\mathcal{S}p_{K(h)})$$
$$\Gamma(\mathbb{G}_h/\mathbb{G}_h, \mathrm{Mod}_{\mathcal{E}_h}(\mathcal{S}p_{K(h)})) \simeq \mathcal{S}p_{K(h)}.$$

One recovers the first part of Theorem A by taking Picard spectra pointwise. For the second part of that theorem, we must identify the E_2-page of the descent spectral sequence, which *a priori* begins which sheaf cohomology on the proétale site. The results of [BS14] allow us to deduce this, as a consquence of the following identification:

Theorem A.III (Theorem 3.11)**.** *There is a hypercomplete sheaf of connective spectra* $\mathfrak{pic}(\mathcal{E}_h)$ *on* $B(\mathbb{G}_h)_{\mathrm{proet}}$ *with*

$$\Gamma(\mathbb{G}_h/*, \mathfrak{pic}(\mathcal{E}_h)) \simeq \mathfrak{pic}(E_h)$$
$$\Gamma(\mathbb{G}_h/\mathbb{G}_h, \mathfrak{pic}(\mathcal{E}_h)) \simeq \mathfrak{pic}(\mathcal{S}p_{K(h)}).$$

The homotopy sheaves of $\mathfrak{pic}(\mathcal{E}_h)$ are

$$\pi_t \mathfrak{pic}(\mathcal{E}_h) = \mathrm{Cont}_{\mathbb{G}}(-, \pi_t \mathfrak{pic}(E_h)),$$

i.e. represented by the homotopy groups of $\mathfrak{pic}(E_h)$ (with their natural profinite topology).

The identification of homotopy groups is immediate for $t \geq 1$ (by comparing with $\pi_t \mathcal{E}_h$), but there is some work to do for $t = 0$; this is the same issue that accounts for the uncertainty in degree zero in the descent spectral sequence of [Hea22].

In fact we go a bit further, relating the spectral sequence of Theorem A to the $K(h)$-local E_h-Adams spectral sequence. In degrees $t \geq 2$, the homotopy groups of the Picard spectrum of an $\mathbb{E}_\infty$-ring are related by a shift to those of the ring itself. It is a result of [MS16] that this identification lifts to one between truncations of the two spectra, in a range that grows with t: that is, for every $t \geq 2$ there is an equivalence

$$\tau_{[t,2t-2]} \mathfrak{pic}(A) \simeq \tau_{[t,2t-2]} \Sigma A,$$

functorial in the ring spectrum A. Using the proétale model, it is quite straightforward to deduce the following comparison result, as proven in *op. cit.* for finite Galois extensions.

Theorem A.IV (Proposition 3.22 and Corollary 3.25; c.f. [MS16]). *(i) Suppose* $2 \leq r \leq t-1$. *Under the identification*

$$E_2^{s,t} = H^s(\mathbb{G}_h, \pi_t \mathfrak{pic}(E_h)) \cong H^s(\mathbb{G}_h, \pi_{t-1} E_h) = E_2^{s,t-1}(ASS),$$

the d_r-differential on the group $E_r^{s,t}$ in (1.3) agrees with the differential on $E_r^{s,t-1}(ASS)$ on classes that survive to E_r in both.

(ii) If $x \in H^t(\mathbb{G}_h, \pi_t \mathfrak{pic}(E_h)) \cong H^t(\mathbb{G}_h, \pi_{t-1} E_h)$ (and x survives to the t-th page in both spectral sequences), then the differential $d_t(x)$ is given by the following formula in the $K(h)$-local E_h-Adams spectral sequence:

$$d_t(x) = d_t^{ASS}(x) + x^2.$$

Comparison with Morava modules

Finally, we will says some words on how to derive Theorem C from the main result. Recall that a useful technique for computing Picard groups, originating already in [HMS94], is to use completed E-theory to compare the category of $K(h)$-local spectra to the category $\mathrm{Mod}_{\pi_* E_h}^{\mathbb{G}_h}$ of *Morava modules*, i.e. L-complete $\pi_* E_h$-modules equipped with a continuous action of the Morava stabiliser group $\mathbb{G}_h$:

$$(E_h)_*^\vee(-) \coloneqq \pi_* L_{K(h)}(E_h \wedge (-)) : \mathcal{S}p_{K(h)} \to \mathrm{Mod}_{\pi_* E_h}^{\mathbb{G}_h}. \tag{1.4}$$

This carries invertible $K(h)$-local spectra to invertible Morava modules, and hence induces a homomorphism on Picard groups. The category on the right-hand side is completely algebraic in nature, and its Picard group $\mathrm{Pic}_h^{\mathrm{alg}}$ can (at least in theory) be computed as a $\mathbb{Z}/2$-extension of $\mathrm{Pic}_h^{\mathrm{alg},0} = H^1(\mathbb{G}_h, (\pi_0 E_h)^\times)$; the strategy is therefore to understand the comparison map and how much of Pic_h it can see. A remarkable theorem of Pstrągowski says that the map $(E_h)_*^\vee : \mathrm{Pic}_h \to \mathrm{Pic}_h^{\mathrm{alg}}$ is an isomorphism if $p \gg n$ (more precisely, if $2p-2 > n^2+n$). This reflects the more general phenomenon that chromatic homotopy theory at large primes is well-approximated by algebra, as is made precise in [BSS20a; BSS20b].

As noted in [Pst22], the existence of a spectral sequence of the form (1.3) immediately yields an alternative proof, by sparseness of the $K(h)$-local E_h-Adams spectral sequence at large primes (in fact, this improves slightly the bound on p). Heard gave such a spectral sequence in [Hea22], and our results can be seen as a conceptual interpretation of that spectral sequence, analogous to the relation of [Dav06; Qui11] to [DH04]. Beyond the conceptual attractiveness, our derivation of the spectral sequence also clarifies certain phenomena: for example, we give a proof of the claim made in [Hea22] that the exotic part of the Picard group is given precisely by those elements in filtration at least 2:

Theorem E (Theorem 4.4)**.** *For any pair (n,p), the algebraic Picard group is computed by the truncation to filtration ≤ 1 of (1.3), and the exotic Picard group κ_h agrees with the subgroup of* Pic_h *in filtration at least 2 for (1.3).*

For example, when $h^2 = 2p-1$ this leads to the description of the exotic Picard group given in Theorem C.

Note also that the group in bidegree $(s,t) = (1,0)$ of Heard's Picard spectral sequence is undetermined; as discussed in Section 3.2, the relevant group in (1.3) really is $H^1(\mathbb{G}_h, \mathrm{Pic}(E_h))$. As expected, the computation simplifies at sufficiently large primes, and this should give rise to an algebraicity statement for the group Br_h^0. We intend to explore the algebraic analogue $\mathrm{Br}_h^{0,alg}$ in future work.

1.1 Outline

In Chapter 2, we collect the results we need on the Devinatz-Hopkins action and the proétale site, showing how to define the sheaf of spectra $\mathcal{E}_h$. One can in fact deduce it is a sheaf from Theorem 3.1. Nevertheless, we wanted to give a self-contained proof of the spectrum-level hyperdescent; we also explain how this compares with Davis' construction of the continuous action on E_h. In the second half of Chapter 2 we compute the homotopy sheaves of $\mathcal{E}_h$, and explain how this leads to the identification with the $K(h)$-local E_h-Adams spectral sequence; the requisite décalage results are collected in Appendix A.

In Chapter 3 we categorify, obtaining descent results for the presheaf of $K(h)$-local module ∞-categories over $\mathcal{E}_h$. We discuss how the Picard spectrum functor yields a sheaf of connective spectra exhibiting the identification of fixed points $\mathfrak{pic}(\mathcal{S}p_{K(h)}) \simeq \tau_{\geq 0}\mathfrak{pic}(E_h)^{h\mathbb{G}_h}$, and investigate the resulting descent spectral sequence.

In Chapter 4 we use the previous results to compute Picard groups. We first identify the algebraic and exotic Picard groups in the descent spectral sequence. Combining this with the well-known form of the $K(1)$-local E_1-Adams spectral sequence allows us to reprove the results of [HMS94]: we are particularly interested in computing the exotic Picard group at the prime 2. We also consider Picard groups in the boundary case $h^2 = 2p-1$. In Appendix B we give a method to compute the height one Adams spectral sequence at $p = 2$ using the Postnikov tower for the sheaf $\mathcal{E}_h$.

Finally, in Chapter 5 we show how to use our results in Brauer group computations. We show that the (-1)-stem in the Picard spectral sequence gives an upper bound for the relative Brauer group, and compute this bound at height one.

1.2 Relation to other work

As already mentioned, Heard has obtained a spectral sequence similar to that of Theorem A, and one of our objectives in this work was to understand how to view that spectral sequence as a HFPSS for the Goerss-Hopkins-Miller action. Recently, a result close to Theorem A was proven by Guchuan Li and Ningchuan Zhang [LZ23]. Their approach differs somewhat from ours, using Burklund's result on multiplicative towers of generalised Moore spectra to produce pro-object presentations of $\mathrm{Mod}_{E_h}(\mathcal{S}p_{K(h)})$ and $\mathfrak{pic}_h$; a detailed comparison between the two would certainly be of interest.

Chapter 2

The continuous action on Morava E-theory

2.1 Discrete Morava E-theory

Let $\mathbf{E} \coloneqq E(\mathbb{F}_{p^h}, \Gamma_h)$ be Morava E-theory based on the Honda formal group at height h and prime p; h and p will henceforth be fixed, and kept implicit to ease notation. Let $\mathbf{K} \coloneqq K(\mathbb{F}_{p^h}, \Gamma_h)$ be Morava K-theory, its residue field. Recall that $\mathbf{E}$ is the $\mathbf{K}$-local Landweber exact spectrum whose formal group is the universal deformation of Γ_h to the Lubin-Tate ring $\pi_*\mathbf{E} = \mathbb{W}(\mathbb{F}_{p^h})[[u_1, \dots, u_{n-1}]][u_h^{\pm 1}]$. Functoriality yields an action of the extended Morava stabiliser group $\mathbb{G} = \mathbb{G}_h \coloneqq \mathrm{Aut}(\mathbb{F}_{p^h}, \Gamma_h)$ on the homotopy ring spectrum $\mathbf{E}$, and celebrated work of Goerss, Hopkins, Miller and Lurie [GH04; Lur18] promotes $\mathbf{E}$ to an $\mathbb{E}_\infty$-ring and the action to one by $\mathbb{E}_\infty$ maps. This action controls much of the structure of the $\mathbf{K}$-local category, and is the central object of study in this document. In this section, we formulate the action of $\mathbb{G}$ on $\mathbf{E}$ in a sufficiently robust way for our applications; to do so, we will present $\mathbf{E}$ as a sheaf of spectra on the *proétale classifying site* of $\mathbb{G}$. Descriptions of the $\mathbf{K}$-local $\mathbf{E}$-Adams spectral sequence have been previously given, notably in work of Davis and of Quick [Dav03; Dav06; Qui11; BD10; DQ16], who described a number of formulations of this action as the *continuous* action of the *profinite* group $\mathbb{G}$.

Recall that continuous actions and continuous cohomology of a topological group G are generally much more straightforward when we assume our modules to have *discrete* topology. There are notable categorical benefits in this case: for example, it is classical that the category of discrete G-modules is abelian with enough injectives, which is not true of the full category of topological modules. Further, in the discrete context we can understand actions completely by looking at the induced actions of all finite quotients of G. This was pioneered by Thomason in his study of $K(1)$-local descent for algebraic K-theory, and formalised in a model-theoretic sense by Jardine.

Any profinite group G has an *étale classifying site*, denoted BG_{et}, whose objects are the (discrete) finite G-sets and whose coverings are surjections; as shown in [Jar97, §6], the category of sheaves of abelian groups on BG_{et} gives a category equivalent to the category Ab_G^δ of discrete G-modules in the sense of [Ser97]. As noted below, sheaf cohomology on BG_{et} corresponds to continuous group cohomology with discrete coefficients, again in the sense of Serre. Motivated by this, Jardine defines a model structure on presheaves of *spectra* on BG_{et}, which models the category of 'discrete' continuous G-spectra, i.e. those that can be obtained as the filtered colimit of their fixed points at open subgroups. We will more generally refer to objects of $\widehat{\mathcal{S}h}(BG_{\mathrm{et}}, \mathcal{C})$ as *discrete G-objects* of an arbitrary (cocomplete) ∞-category $\mathcal{C}$. Davis uses this as his starting point, and we observe below that in this formalism it is easy to pass to the ∞-categorical setting. Namely, we begin by collecting the following facts:

Theorem 2.1 (Devinatz-Hopkins, Davis, Rognes, Dugger-Hollander-Isaksen). *There is a hypercomplete sheaf of spectra $\mathcal{E}^\delta$ on $B\mathbb{G}_{\mathrm{et}}$, such that*

(*i*) *Any ordered, cofinal sequence (U_i) of open subgroups of $\mathbb{G}$ induces*

$$L_{\mathbf{K}} \varinjlim_i \mathcal{E}^\delta(G/U_i) \simeq \mathbf{E},$$

(*ii*) *On global sections, $\Gamma\mathcal{E}^\delta := \Gamma(\mathbb{G}/\mathbb{G}, \mathcal{E}^\delta) \simeq \mathbf{1}_{\mathbf{K}}$ is the $\mathbf{K}$-local sphere spectrum,*

(*iii*) *$\mathcal{E}^\delta$ lifts to $\mathrm{CAlg}(\mathcal{S}p_{\mathbf{K}})$,*

(*iv*) *For any normal inclusion of open subgroups $V \subset U \subset \mathbb{G}$, the map $\mathcal{E}^\delta(\mathbb{G}/U) \to \mathcal{E}^\delta(\mathbb{G}/V)$ is a faithful U/V-Galois extension.*

Proof. The presheaf of spectra $\mathcal{E}^\delta$ is constructed in [DH04, §4], with (*ii*) being part of Theorem 1 therein and the identification (*i*) being the trivial case of Theorem 3; see also [Bar+22, §2] for a nice summary. Devinatz and Hopkins construct $\mathcal{E}^\delta$ by hand (by taking the limit of the *a priori* form of its Amitsur resolution in $K(h)$-local E_h-modules); they denote $\Gamma(\mathbb{G}/U, \mathcal{E}^\delta)$ by E_h^{hU}, but we copy [BD10] and write E_h^{dhU}. By [DH04, Theorem 4], $\mathcal{E}^\delta$ is a sheaf on $B\mathbb{G}_{\mathrm{et}}$; in fact, Devinatz and Hopkins already define $\mathcal{E}^\delta$ to land in $\mathbf{K}$-local $\mathbb{E}_\infty$-rings, and so we may consider it as an object of $\mathcal{S}h(B\mathbb{G}_{\mathrm{et}}, \mathrm{CAlg}(\mathcal{S}p_{\mathbf{K}}))$ since limits in $\mathrm{CAlg}(\mathcal{S}p_{\mathbf{K}})$ are computed at the level of underlying spectra [HA, Corollary 3.2.2.5]. Item (*iv*) is [Rog08, Theorem 5.4.4], so it remains to show that $\mathcal{E}$ is hypercomplete; this we will deduce from Davis' work.

More specifically, Davis utilises the Jardine model structure on the category of presheaves of spectra on $B\mathbb{G}_{\mathrm{et}}$, denoted $\mathrm{Spt}_{\mathbb{G}}$[1]; this is defined in such a way that there is a Quillen adjunction

$$\mathrm{Spt} \underset{(-)^{h\mathbb{G}}}{\overset{\mathrm{const}}{\rightleftarrows}} \mathrm{Spt}_{\mathbb{G}}\,.$$

Recall that the main result of [DHI04] says that the fibrant objects of $\mathrm{Spt}_{\mathbb{G}}$ are precisely those projectively fibrant presheaves that satisfy (*i*) the (1-categorical) sheaf condition for coverings in

[1]We have taken slight notational liberties: in [Dav03], $\mathrm{Spt}_{\mathbb{G}}$ denotes the category of spectra based on discrete $\mathbb{G}$-sets, which is equipped with a model structure lifted from the Jardine model structure on the (equivalent) category **ShvSpt** of sheaves of spectra.

$B\mathbb{G}_{\mathrm{et}}$, and (*ii*) descent for all hypercovers, and so the ∞-category associated to $\mathrm{Spt}_{\mathbb{G}}$ is a full subcategory of $\widehat{\mathcal{S}h}(B\mathbb{G}_{\mathrm{et}}, \mathcal{S}p)$.

In this setting, Davis shows that the spectrum $F_h := \varinjlim E_h^{dhU}$ (the colimit taken in plain spectra) defines a fibrant object of $\mathrm{Spt}_{\mathbb{G}}$, $\mathcal{F}\colon \mathbb{G}/U \mapsto F_h^{hU}$ [Dav06, Corollary 3.14]; Behrens and Davis show that $E_h^{hU} \simeq L_{\mathbf{K}} F_h^{hU} \simeq E_h^{dhU}$ for any open subgroup $U \subset \mathbb{G}$ [BD10, Discussion above Prop. 8.1.2 and Lemma 6.3.6 respectively]. In fact, [BD10, Theorem 8.2.1] proves the same equivalence for any *closed* subgroup H, but restricting our attention to open subgroups cuts out some of the complexity and makes clear the equivalences happen naturally in U (we remark that they also appeared in Chapter 7 of Davis' thesis [Dav03]). This provides a projective equivalence between $\mathcal{E}^\delta$ and $L_{\mathbf{K}}\mathcal{F}$, and hence an equivalence of *presheaves* between $\mathcal{E}^\delta$ and a hypercomplete sheaf of spectra. $\square$

If we pick a cofinal sequence of open normal subgroups (U_i) we can identify the starting page of the descent spectral sequence for $\mathcal{E}^\delta$:

Lemma 2.2. *Let G be a profinite group and $\mathcal{F} \in \widehat{\mathcal{S}h}(BG_{\mathrm{et}}, \mathcal{C})$, where $\mathcal{C} = \mathcal{S}p$ or $\mathcal{S}p_{\geq 0}$. There is a spectral sequence with starting page*

$$E_2^{s,t} = H^s(G, \pi_t \varinjlim \mathcal{F}(G/U_i)),$$

and converging conditionally to $\pi_{t-s}\Gamma \lim_t \tau_{\leq t}\mathcal{F}$.

Proof. This is the spectral sequence for the Postnikov tower of $\mathcal{F}$, formed as in [HA, §1.2.2]. Its starting page is given by sheaf cohomology of the graded abelian sheaf $\pi_*\mathcal{F}$ on BG_{et}. Explicitly, form the Postnikov tower in sheaves of spectra

$$\begin{tikzcd} \mathcal{F} \arrow[r] & \cdots \arrow[r] & \tau_{\leq 1}\mathcal{F} \arrow[r] \arrow[d] & \tau_{\leq 0}\mathcal{F} \arrow[r] \arrow[d] & \cdots \\ & & \Sigma\pi_1\mathcal{F} \arrow[ul] & \pi_0\mathcal{F} \arrow[ul] & \end{tikzcd}$$

Applying global sections and taking homotopy groups gives an exact couple, and we obtain a spectral sequence with

$$E_2^{s,t} = \pi_{t-s}\Gamma\Sigma^t\pi_t\mathcal{F} = R^s\Gamma\pi_t\mathcal{F} = H^s(B\mathbb{G}_{\mathrm{et}}, \pi_t\mathcal{F}),$$

with abutment $\pi_{t-s}\Gamma \lim \tau_{\leq t}\mathcal{F}$. To identify this with continuous group cohomology, we make use of the equivalence

$$\begin{aligned} \mathrm{Ab}_G^\delta &\to \mathcal{S}h(BG_{\mathrm{et}}, \mathrm{Ab}) \\ M &\mapsto \left(\coprod G/U \mapsto \prod M^U\right) \end{aligned}$$

of [Jar97], whose inverse sends $\mathcal{F} \mapsto \varinjlim_i \mathcal{F}(G/U_i)$. Under this equivalence the fixed points functor on Ab_G^δ corresponds to global sections, and so taking derived functors identifies sheaf cohomology on the right-hand side with derived fixed points on the left; as in [Ser97, §2.2], when we take discrete coefficients this agrees with the definition in terms of continuous cochains. □

Writing $\mathrm{Mod}_{(-),\mathbf{K}} := \mathrm{Mod}_{L_{\mathbf{K}}(-)}(\mathcal{S}p_{\mathbf{K}}) = L_{\mathbf{K}}\,\mathrm{Mod}_{(-)}$, our strategy is is to apply the functor $\mathfrak{pic} \circ \mathrm{Mod}_{(-),\mathbf{K}} : \mathrm{CAlg} \to \mathcal{S}p_{\geq 0}$ pointwise to the sheaf $\mathcal{E}^{\delta}$, in order to try to obtain a sheaf $\mathfrak{pic}_{\mathbf{K}}(\mathcal{E}^{\delta}) \in \widehat{\mathcal{S}h}(B\mathbb{G}_{\mathrm{et}}, \mathcal{S}p_{\geq 0})$; we'd then like to apply the above lemma to deduce the existence of the descent spectral sequence. This does not quite work for the same reason that the lemma applied to $\mathcal{E}^{\delta}$ does not recover the **K**-local **E**-Adams spectral sequence: while **E** is **K**-locally discrete, it is certainly not discrete as a $\mathbb{G}$-spectrum (for example, even the action on $\pi_2\mathbf{E}$ is not discrete). Nevertheless, it is worth remarking that the first step of this approach does work: since **E** is a discrete $\mathbb{G}$-object of ***K**-local spectra*, $\mathrm{Mod}_{\mathbf{E},\mathbf{K}}$ is discrete as a presentable ∞-category with $\mathbb{G}$-action. This is a consequence of the following two results.

Lemma 2.3. *The composition* $\mathrm{Mod}_{\mathcal{E}^{\delta},\mathbf{K}} : {B\mathbb{G}_{\mathrm{et}}}^{op} \to \mathrm{CAlg}(\mathcal{S}p_{\mathbf{K}}) \to \mathrm{Pr}^{L,\mathrm{smon}}$ *is a sheaf.*

Proof. To check the sheaf condition for $\mathcal{F} : B\mathbb{G}_{\mathrm{et}}^{op} \to \mathcal{C}$ we need to show that finite coproducts are sent to coproducts, and that for any inclusion $U \subset U'$ of open subgroups the canonical map $\mathcal{F}(\mathbb{G}/U') \to \mathrm{Tot}\,\mathcal{F}(\mathbb{G}/U^{\times_{\mathbb{G}/U'}\bullet+1})$ is an equivalence. For the presheaf $\mathrm{Mod}_{\mathcal{E}^{\delta},\mathbf{K}}$, the first is obvious (using the usual idempotent splitting), while the second is *finite* Galois descent [Mei12, Proposition 6.2.6] or [GL21, Theorem 6.10], at least after refining U to a normal open subgroup of U'. □

Fixing again a cofinal sequence (U_i), we write $F_{ij} \dashv R_{ij} : \mathrm{Mod}_{\mathcal{E}^{\delta}(\mathbb{G}/U_i),\mathbf{K}} \rightleftarrows \mathrm{Mod}_{\mathcal{E}^{\delta}(\mathbb{G}/U_j),\mathbf{K}}$ and $F_j \dashv R_j$ for the composite adjunction $\mathrm{Mod}_{\mathcal{E}^{\delta}(\mathbb{G}/U_i),\mathbf{K}} \rightleftarrows \mathrm{Mod}_{\mathbf{E},\mathbf{K}}$, and

$$\varinjlim \mathrm{Mod}_{\mathcal{E}^{\delta}(\mathbb{G}/U_i),\mathbf{K}} \underset{R}{\overset{F}{\rightleftarrows}} \mathrm{Mod}_{\mathbf{E},\mathbf{K}}$$

for the colimit (along the functors F_{ij}) in $\mathrm{Pr}^{L,\mathrm{smon}}$.

Proposition 2.4. *The functors F and R define an adjoint equivalence*

$$\varinjlim \mathrm{Mod}_{\mathcal{E}^{\delta}(\mathbb{G}/U_i),\mathbf{K}} \simeq \mathrm{Mod}_{\mathbf{E},\mathbf{K}}.$$

More generally:

Proposition 2.5. *Let $\mathcal{C}$ be a presentably symmetric monoidal stable ∞-category. Suppose $A_{(-)} : I \to \mathrm{CAlg}(\mathcal{C})$ is a filtered diagram, and write A for a colimit (formed equivalently in $\mathcal{C}$ or $\mathrm{CAlg}(\mathcal{C})$). Then the induced adjunction*

$$\varinjlim \mathrm{Mod}_{A_i}(\mathcal{C}) \underset{R}{\overset{F}{\rightleftarrows}} \mathrm{Mod}_A(\mathcal{C})$$

is an equivalence of presentable *symmetric-monoidal ∞-categories.*

Proof. This is implied by [HA, Corollary 4.8.5.13], since filtered categories are weakly contractible and filtered colimits in $\mathrm{CAlg}_{\mathcal{C}}(\mathrm{Pr}^L)$ computed in Pr^L. Alternatively, one can give an explicit description of the unit and counit and verify that both are natural equivalences. □

Warning 2.6. The result above *fails* if we consider the same diagram in $\mathrm{Cat}_\infty^{\mathrm{smon}}$ (after restricting to κ-compact objects for κ a regular cardinal chosen such that $\mathbf{1}_{\mathbf{K}} \in \mathcal{S}p_{\mathbf{K}}$ is κ-compact, say). Indeed, one can see that the homotopy type of the mapping spectra $\mathrm{map}(\mathbf{1}, X)$ out of the unit in this colimit would be different from that in $\mathrm{Mod}_{\mathbf{E},\mathbf{K}}$, an artefact of the failure of $\mathbf{K}$-localisation to be smashing.

Applying $\mathfrak{pic} : \mathrm{Pr}^{L,\mathrm{smon}} \to \mathcal{S}p_{\geq 0}$ to the coefficients, one obtains a sheaf of connective spectra on $B\mathbb{G}_{\mathrm{et}}$ given by

$$G/U_i \mapsto \mathfrak{pic}(\mathrm{Mod}_{\mathcal{E}^\delta(G/U_i),\mathbf{K}}).$$

In particular, $\Gamma\mathfrak{pic}(\mathrm{Mod}_{\mathcal{E}^\delta,\mathbf{K}}) = \mathfrak{pic}_h := \mathfrak{pic}(\mathcal{S}p_{\mathbf{K}})$. Unfortunately, this sheaf is unsuitable for the spectral sequence we would like to construct: the E_1-page will be group cohomology with coefficients in the homotopy of $\varinjlim \mathfrak{pic}(\mathrm{Mod}_{\mathcal{E}^\delta(G/U_i),\mathbf{K}})$, which by [MS16, Proposition 2.2.3] is the Picard spectrum of the colimit of module categories computed in $\mathrm{Cat}_\infty^{\mathrm{smon}}$; as noted above, this need *not* agree with $\mathfrak{pic}(\mathrm{Mod}_{\mathbf{E},\mathbf{K}})$.

2.2 Proétale homotopy theory

In Section 2.1, we showed how the work of Devinatz-Hopkins and Davis defines Morava E-theory as a discrete $\mathbb{G}$-object in the category of $\mathbf{K}$-local $\mathbb{E}_\infty$-rings, and that using this it is straightforward to construct a discrete $\mathbb{G}$-action on the presentable ∞-category $\mathrm{Mod}_{\mathbf{E},\mathbf{K}}$, having $\mathcal{S}p_{\mathbf{K}}$ as fixed points. As noted in Warning 2.6, this does not suffice to prove our desired descent result for Picard spectra. Likewise, the descent spectral sequence for the sheaf $\mathcal{E}^\delta$ on $B\mathbb{G}_{\mathrm{et}}$ is *not* the $\mathbf{K}$-local $\mathbf{E}$-Adams spectral sequence: the issue is that the action on the (unlocalised) spectrum $\mathbf{E}$ is not discrete. We are led to the following solution: rather than work with the site of *finite* $\mathbb{G}$-sets, we think of Morava E-theory as a sheaf $\mathcal{E}$ on *profinite* $\mathbb{G}$-sets.

Definition 2.7. Let G be a δ_0-small profinite group. We denote by BG_{proet} the *proétale site*, whose underlying category consists of all δ_0-small continuous profinite G-sets and continuous equivariant maps. BG_{proet} is equipped with the topology whose coverings are collections $\{S_\alpha \to S\}$ for which there is a *finite* subset A with $\coprod_{\alpha \in A} S_\alpha \twoheadrightarrow A$.

The proétale site and resulting 1-topos were extensively studied by Bhatt and Scholze in [BS14]; when G is trivial, one recovers the *condensed/pyknotic* formalism of [Cond; Pyk], up to a choice of set-theoretic foundations. One key feature for our purposes is that sheaf cohomology on BG_{proet} recovers continuous group cohomology for a wide range of coefficient modules; this will allow us to recover the desired homotopy sheaves, which we saw we could not do using $B\mathbb{G}_{\mathrm{et}}$—in particular, we will recover the $\mathbf{K}$-local $\mathbf{E}$-Adams spectral sequence as a descent spectral sequence. Moreover,

$B\mathbb{G}_{\mathrm{proet}}$ gives a site-theoretic definition of continuous group cohomology, which makes the passage to actions on module categories (and hence Picard spectra) transparent.

To begin, we will discuss some generalities of proétale homotopy theory. Most important for the rest of the document will be Section 2.2.3, in which we give criteria for a sheaf on BG_{et} to extend without sheafification to one on BG_{proet}; the other sections will be used only cursorily, but are included for completeness.

2.2.1 Free G-sets

A significant difference between the étale and proétale sites is that the latter contains G-torsors. In fact, for any profinite set T we have an object $G \times T \in BG_{\mathrm{proet}}$. It will often be useful to restrict to such free objects.

Lemma 2.8. *If S is a continous G-set, the projection $S \to S/G$ is split: that is, there is a continuous section*

$$S/G \to S.$$

In particular, if the G-action on S is free then $S \cong S/G \times G$.

Proof. This is proven as part of [Sch13, Prop. 3.7]. □

Definition 2.9. The subsite $\mathrm{Free}_G \subset BG_{\mathrm{proet}}$ is given by the full subcategory of G-sets with *free* G-action; equivalently, these are G-sets isomorphic to ones of the form $T \times G$ for T a profinite set with trivial G-action.

Lemma 2.10. *The subsite* Free_G *generates BG_{proet}: any $T \in BG_{\mathrm{proet}}$ admits a covering by a free G-set. Consequently, restriction induces an equivalence*

$$\mathcal{S}h(BG_{\mathrm{proet}}) \xrightarrow{\sim} \mathcal{S}h(\mathrm{Free}_G).$$

Proof. The map $S \times G \to S$ is surjective, and the (diagonal) G-action on the domain is clearly free. □

Remark 2.11. In fact, one can restrict further to the category $\mathrm{Proj}_G = BG^{\mathrm{wc}}_{\mathrm{proet}}$ of *weakly contractible* G-sets: these are the free G-sets of the form $G \times T$, where T is extremally disconnected. While this is not a site (it does not have pullbacks), Proj_G does generate the *hypercomplete* proétale topos, in the sense that restriction induces an equivalence

$$\widehat{\mathcal{S}h}(BG_{\mathrm{proet}}) \xrightarrow{\sim} \mathcal{P}_\Sigma(\mathrm{Proj}_G),$$

where the codomain is the full subcategory of $\mathcal{P}(\mathrm{Proj}_G)$ spanned by multiplicative presheaves—i.e., those that send binary coproducts to products.

2.2.2 Descent spectral sequence

Suppose that G is a profinite group. Taking $\mathcal{X} = \mathcal{S}h(BG_{\mathrm{proet}})$ or $\widehat{\mathcal{S}h}(BG_{\mathrm{proet}})$ in [SAG, Proposition 1.3.2.7] endows $\mathcal{S}h(BG_{\mathrm{proet}}, \mathcal{S}p)$ and $\widehat{\mathcal{S}h}(BG_{\mathrm{proet}}, \mathcal{S}p)$ with t-structures having the following properties:

(i) $\mathcal{F}$ is coconnective if and only if it is pointwise coconnective,

(ii) $\mathcal{F}$ is connective if and only if its homotopy sheaves $\pi_t\mathcal{F}$ vanish for $t < 0$,

(iii) both t-structures are compatible with colimits and right complete,

(iv) in either case, π_0 is an equivalence from the heart of the t-structure to $\mathrm{Ab}(BG_{\mathrm{proet}})$.

In particular, any $\mathcal{F} \in \mathcal{S}h(BG_{\mathrm{proet}}, \mathcal{S}p)$ has a Postnikov tower

$$\mathcal{F} \to \cdots \to \mathcal{F}_{\leq n} \to \cdots .$$

If $\mathcal{F}$ is hypercomplete, the Postnikov tower converges by [BS14, Proposition 3.2.3]. If H is any subgroup, we may evaluate the Postnikov tower at the G-set G/H, obtaining a tower of spectra and hence a spectral sequence

$$E_2^{s,t} = \pi_{-s}\Gamma(G/H, \pi_t\mathcal{F}) \implies \pi_{t-s}\Gamma(G/H, \mathcal{F}).$$

Since $\mathcal{S}h(BG_{\mathrm{proet}}, \mathcal{S}p)^{\heartsuit} \simeq \mathrm{Ab}(BG_{\mathrm{proet}})$, the universal property of the derived category allows us to identify the E_2-page may be identified with sheaf cohomology:

Corollary 2.12. *If $\mathcal{F} \in \mathcal{S}h(BG_{\mathrm{proet}}, \mathcal{S}p)$, there is a spectral sequence*

$$E_2^{s,t} = H^s(BG_{\mathrm{proet}}, \pi_t\mathcal{F}) \implies \pi_{t-s}\Gamma\mathcal{F}.$$

It is conditionally convergent when $\mathcal{F}$ is hypercomplete.

Proof. Since the t-structures are right-complete and $\mathrm{Ab}(BG_{\mathrm{proet}})$ has enough injectives, the square

$$\begin{tikzcd}
\mathrm{Ab}(BG_{\mathrm{proet}}) \arrow[r, "\sim"] \arrow[d, "H^0"'] & \mathcal{S}h(BG_{\mathrm{proet}}, \mathcal{S}p)^{\heartsuit} \arrow[d, "\Gamma^{\heartsuit}"] \\
\mathrm{Ab} \arrow[r, "\sim"'] & \mathcal{S}p^{\heartsuit}
\end{tikzcd}$$

gives rise to a commutative square of ∞-categories

$$\begin{tikzcd}
\mathcal{D}^+(BG_{\mathrm{proet}}) \arrow[r] \arrow[d, "\Gamma"'] & \mathcal{S}h(BG_{\mathrm{proet}}, \mathcal{S}p) \arrow[d, "\Gamma"] \\
\mathcal{D}^+(\mathrm{Ab}) \arrow[r] & \mathcal{S}p
\end{tikzcd}$$

by [HA, Proposition 1.3.3.2] (applied to opposites). The bottom horizontal functor is t-exact, and so gives rise to isomorphisms

$$\pi_{-s}\Gamma\mathcal{F} \cong H^s(BG_{\text{proet}}, \pi_0\mathcal{F})$$

for any $\mathcal{F} \in \mathcal{S}h(BG_{\text{proet}}, \mathcal{S}p)^{\heartsuit}$. □

Remark 2.13. The fully faithful left adjoint $\mathcal{S}p_{\geq 0} \hookrightarrow \mathcal{S}p$ gives rise to a left adjoint $(-)^+ : \mathcal{S}h(BG_{\text{proet}}, \mathcal{S}p_{\geq 0}) \to \mathcal{S}h(BG_{\text{proet}}, \mathcal{S}p)$ with right adjoint $(-)_{\geq 0}$. Given $\mathcal{F} \in \mathcal{S}h(BG_{\text{proet}}, \mathcal{S}p_{\geq 0})$, we will consider the spectral sequence above for its image $\mathcal{F}^+ \in \mathcal{S}h(BG_{\text{proet}}, \mathcal{S}p)$, noting that

$$\Gamma(\mathcal{F}^+)_{\geq 0} \simeq \Gamma((\mathcal{F}^+)_{\geq 0}) \simeq \Gamma\mathcal{F}.$$

The first equivalence follows from the natural equivalence $(-)_{\geq 0} \circ \Gamma \simeq \Gamma \circ (-)_{\geq 0}$, which in turn follows from the fact that both functors are right adjoint to the constant sheaf functor $\mathcal{S}p_{\geq 0} \to \mathcal{S}h(BG_{\text{proet}}, \mathcal{S}p)$.

When $\mathcal{F} = \text{Cont}_G(-, M) \in \mathcal{S}h(BG_{\text{proet}}, \text{Ab})$ is represented by some topological G-module, there is a comparison map

$$\Phi : H^s_{\text{cont}}(G, M) \to H^s(BG_{\text{proet}}, \mathcal{F})$$

from continuous group cohomology, which arises as the edge map in a Čech-to-derived functor spectral sequence. In this context, [BS14, Lemma 4.3.9] gives conditions for Φ to be an isomorphism.

2.2.3 Extending discrete objects

For a profinite group G, the proétale site BG_{proet} is related to the site BG_{et} of finite G-sets by a map of sites $\nu : BG_{\text{et}} \hookrightarrow BG_{\text{proet}}$, which induces a geometric morphism at the level of topoi. More generally, if $\mathcal{C}$ is any complete and cocomplete ∞-category we write

$$\nu^* : \mathcal{S}h(BG_{\text{et}}, \mathcal{C}) \rightleftarrows \mathcal{S}h(BG_{\text{proet}}, \mathcal{C}) : \nu_*$$

for the resulting adjunction; the left adjoint is the sheafification of the presheaf extension, given in turn by

$$\nu^p\mathcal{F} : S = \varprojlim_i S_i \mapsto \varinjlim_i \mathcal{F}(S_i) \tag{2.1}$$

when each S_i is a finite G-set. When $\mathcal{C} = \text{Set}$, it is a basic result of [BS14] that $\nu^* = \nu^p$: essentially, this is because

(*i*) the sheaf condition is a finite limit,

(*ii*) it therefore commutes with the filtered colimit in (2.1).

This fails for sheaves valued in an arbitrary ∞-category $\mathcal{C}$ with limits and filtered colimits, where both properties might fail; inspired by the machinery of [Mat16], we will circumvent this by introducing further finiteness assumptions.

Definition 2.14. (*i*) Let $Y : \mathbb{Z}_{\geq 0}^{\mathrm{op}} \to \mathcal{C}$ be a pretower in a stable ∞-category $\mathcal{C}$. If $\mathcal{C}$ has sequential limits, we can form the map

$$f : \{\lim Y\} \to \{Y_n\}$$

in the ∞-category $\mathrm{Fun}(\mathbb{Z}_{\geq 0}^{\mathrm{op}}, \mathcal{C})$ (in which the source is a constant tower). Recall that Y is *d-rapidly convergent to its limit* if each d-fold composite in $\mathrm{fib}(f) \in \mathrm{Fun}(\mathbb{Z}_{\geq 0}^{\mathrm{op}}, \mathcal{C})$ is null (c.f. [CM21, Definition 4.8]; see also [HPS99]). More generally, if $Y : (\mathbb{Z}_{\geq 0} \cup \{\infty\})^{\mathrm{op}} \to \mathcal{C}$ is a tower, we say Y is *d-rapidly convergent* if the same condition holds for the map

$$f : \{Y_\infty\} \to \{Y_n\}.$$

In particular, this implies that $Y_\infty \simeq \lim Y$; in fact, $F(Y_\infty) \simeq \lim F(Y)$ for any exact functor $F : \mathcal{C} \to \mathcal{D}$.

(*ii*) Now suppose $\mathcal{C}$ also has finite limits, and $X^\bullet : \Delta_+ \to \mathcal{C}$ is an augmented cosimplicial object. Recall that X is *d-rapidly convergent* if the Tot-tower $\{\mathrm{Tot}_n X^\bullet\}$ is d-rapidly convergent. In particular $X^{-1} \simeq \mathrm{Tot}\, X^\bullet$, and the same is true after applying any exact functor F.

Example 2.15. Given an $\mathbb{E}_1$-algebra A in any presentably symmetric monoidal stable ∞-category $\mathcal{C}$, the Adams tower $T(A, M)$ for $M \in \mathcal{C}$ over A is defined by the property that every map in $A \otimes T(A, M)$ is null. It is well-known to agree with the dual tower to the Tot-tower for the Amitsur complex for M over A, in the sense that

$$T_n(A, M) \simeq \mathrm{fib}(M \to \mathrm{Tot}_n(A^{\otimes \bullet + 1} \otimes M)).$$

For example, this is worked out in detail in [MNN17, §2.1]. In particular the cosimplicial object $A^{\otimes \bullet + 2} \otimes M$, given by smashing the Amitsur complex with a further copy of A, is always 1-rapidly convergent.

Remark 2.16. Let G be a profinite group and $\mathcal{F}^\delta$ a presheaf on BG_{proet}, valued in a stable ∞-category $\mathcal{C}$. Inspired by [CM21], the following will be the key finiteness assumption we invoke:

($\star$) There exists $d \geq 0$ such that for any normal inclusion $V \subset U$ of open subgroups of G, the Čech complex

$$\mathcal{F}^\delta(G/U) \longrightarrow \mathcal{F}^\delta(G/V) \rightrightarrows \mathcal{F}^\delta(G/V \times_{G/U} G/V) \mathrel{\substack{\rightarrow\\[-1ex]\rightarrow\\[-1ex]\rightarrow}} \cdots$$

is d-rapidly convergent.

When $\mathcal{C} = \mathcal{S}p$ or $\mathcal{S}p_{\mathbf{K}}$, we will show that assumption ($\star$) implies that the left extension of $\mathcal{F}^\delta$ is a hypercomplete sheaf on BG_{proet} (Proposition 2.18). Recall that a map $f \colon X \to Y$ in $\mathcal{S}p_{\mathbf{K}}$ is *phantom* if any composite $C \to X \xrightarrow{f} Y$ with C compact is null. I'm grateful to Neil Strickland for suggesting the following argument:

Lemma 2.17. *Let I be a filtered category, and suppose that $\mathcal{C}$ is a stable ∞-category with $h\mathcal{C}$ a Brown category in the sense of [HPS97], or $\mathcal{C} = \mathcal{S}p_{\mathbf{K}}$. Let $p\colon I \to \mathrm{Fun}(\Delta, \mathcal{C})$ be a diagram of d-rapidly convergent cosimplicial objects. Then $\varinjlim p\colon \Delta \to \mathcal{S}p$ is $2d$-rapidly convergent.*

Proof. Each $p(i)^\bullet$ is d-rapidly convergent, and in particular for each i, each d-fold composite in the fibre of

$$\{p(i)^{-1}\} \to \{\mathrm{Tot}_n\, p(i)^\bullet\}$$

is phantom. The colimit of a filtered diagram of phantom maps is phantom (since its precomposition with any map from a compact object factors through some finite stage), and hence any d-fold composite in the fibre of

$$\{\varinjlim_i p(i)^{-1}\} \to \{\mathrm{Tot}_n \varinjlim_i p(i)^\bullet\}$$

is phantom too. By [HPS97, Theorem 4.18] or [HS99, Theorem 9.5] respectively, the composite of any two phantoms in $\mathcal{S}p_{\mathbf{K}}$ is null. Thus $\varinjlim p(i)^\bullet$ is $2d$-rapidly convergent. □

Proposition 2.18. *Let G be a profinite group, and $\mathcal{A}^\delta \in \mathcal{S}h(BG_{\mathrm{et}}, \mathcal{S}p_{\mathbf{K}})$. If $\mathcal{A}^\delta$ satisfies $(\star)$ then $\nu^p\mathcal{A}^\delta \in \mathcal{P}(BG_{\mathrm{proet}}, \mathcal{S}p_{\mathbf{K}})$ is a sheaf.*

Proof. As above, $\nu^p\mathcal{A}^\delta$ is given by the formula

$$S = \varprojlim S_i \mapsto L_{\mathbf{K}} \varinjlim \mathcal{A}^\delta(S_i).$$

Suppose we are given a surjection $S' = \lim_{j'\in J'} S'_{j'} \xrightarrow{\alpha} S = \lim_{j\in J} S_j$ of profinite G-sets. By [HTT, Proposition 5.3.5.15] (applied to $A = \Delta^1$ and $\mathcal{C} = \mathrm{Fin}^{\mathrm{op}}$), one can present this as the limit of finite covers $S'_i \to S_i$. We therefore want to show that the following is a limit diagram:

$$L_{\mathbf{K}} \varinjlim \mathcal{A}^\delta(S_i) \longrightarrow L_{\mathbf{K}} \varinjlim \mathcal{A}^\delta(S'_i) \rightrightarrows L_{\mathbf{K}} \varinjlim \mathcal{A}^\delta(S'_i \times_{S_i} S'_i) \overrightarrow{\rightrightarrows} \cdots . \tag{2.2}$$

To show that $\mathcal{A} = \nu^p\mathcal{A}^\delta$ is a sheaf it will suffice to show that there is some $d < \infty$ such that

$$\mathcal{A}(S) \longrightarrow \mathcal{A}(S') \rightrightarrows \mathcal{A}(S' \times_S S') \overrightarrow{\rightrightarrows} \cdots$$

is d-rapidly convergent for *all* finite coverings $S' \to S$. Indeed, in that case 2.17 implies that the Čech complex for an arbitrary covering in BG_{proet} is $2d$-rapidly convergent, and in particular converges.

In fact, one can further reduce to the case of a covering of orbits $G/V \to G/U$, in which case the Čech complex is d-rapidly convergent by assumption $(\star)$.

Indeed, suppose that $S' \to S$ is a finite covering. Decomposing S into transitive G-sets splits (2.2) into a product of the Čech complexes for $S'_H = S' \times_S G/H \to G/H$, and further writing $S'_H :=$

$\coprod_i G/K_i$ we will reduce to the case $G/K \to G/H$, where $K \subset H$ are open subgroups. For this last point, note that if $S_1', S_2' \to S$ are coverings then $S_1' \sqcup S_2' \to S$ factors as $S_1' \sqcup S_2' \to S \sqcup S_2' \to S$. For the first map we have $(S_1' \sqcup S_2')^{\times_{S \sqcup S_2'} \bullet} \simeq S_1'^{\times_S \bullet} \sqcup S_2'$, so d-rapid convergence for $S \to S_1'$ implies the same for $S_1' \sqcup S_2' \to S \sqcup S_2'$. On the other hand, the second is split, and so the Čech complex for $S_1' \sqcup S_2' \to S$ is a retract of that for $S_1' \sqcup S_2' \to S \sqcup S_2'$. In particular, d-rapid convergence for the covering $S_1' \to S$ implies the same for $S_1' \sqcup S_2' \to S$. This allows us to reduce to covers by a single finite G-orbit. This completes the proof that $\nu^p \mathcal{A}^\delta$ is a sheaf. $\square$

In our key example of interest, the starting point will be a *descendable* Galois extension in the sense of [Mat16]. We will therefore show that if $\mathbf{1} \to A$ is a descendable Galois extension in $\mathcal{S}p_{\mathbf{K}}$, then the presheaf $\mathcal{A}^\delta$ satisfies $(\star)$. Applying Proposition 2.18, this implies that its left Kan extension to BG_{proet} is a sheaf. This relies on some preliminary lemmas.

Definition 2.19. (i) Suppose $(\mathcal{C}, \otimes, \mathbf{1})$ is symmetric monoidal, and $Z \in \mathcal{C}$. Then $\mathrm{Thick}^{\otimes}(Z)$ is the smallest full subcategory of $\mathcal{C}$ containing Z and closed under extensions, retracts, and $(-) \otimes X$ for every $X \in \mathcal{C}$. Note that $\mathrm{Thick}^{\otimes}(Z)$ is the union of full subcategories $\mathrm{Thick}^{\otimes}_r(Z)$, for $r \geq 1$, spanned by retracts of those objects that can be obtained by at most r-many extensions of objects $Z \otimes X$; each of these is a $\otimes$-ideal closed under retracts, but not thick.

(ii) Suppose now that $A \in \mathrm{Alg}(\mathcal{C})$. Recall that $M \in \mathcal{C}$ is *A-nilpotent* ([Rav92, Definition 7.1.6]) if $M \in \mathrm{Thick}^{\otimes}(A)$, and that A is *descendable* ([Mat16, Definition 3.18]) if $\mathrm{Thick}^{\otimes}(A) = \mathcal{C}$. Thus A is descendable if and only if $\mathbf{1}$ is A-nilpotent.

In [Mat16, Proposition 3.20] it is shown that $A \in \mathrm{Alg}(\mathcal{C})$ is descendable if and only if the Tot-tower of the Amitsur complex

$$\mathbf{1} \longrightarrow A \rightrightarrows A \otimes A \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots$$

defines a constant pro-object converging to $\mathbf{1}$. It will be useful to have the following quantitative refinement of this result, which explicates some of the relations between various results in *op. cit.* with [CM21; MNN17; Mat15]:

Lemma 2.20. *Let $\mathcal{C}$ be stable and symmetric monoidal. Consider the following conditions:*

$(1)_d$ *The Amitsur complex for A is d-rapidly convergent to $\mathbf{1}$.*

$(2)_d$ *The canonical map $\mathbf{1} \to \mathrm{Tot}_d A^{\otimes \bullet+1}$ admits a retraction.*

$(3)_d$ *The canonical map $\mathrm{Tot}^d A^{\otimes \bullet+1} := \mathrm{fib}(\mathbf{1} \to \mathrm{Tot}_d A^{\otimes \bullet+1}) \to \mathbf{1}$ is null. In the notation of [MNN17, §4], this says that $\exp_A(\mathbf{1}) = d$.*

$(4)_d$ *For any $X \in \mathcal{C}$, the spectral sequence for the tower of mapping spectra*

$$\cdots \to \mathrm{map}_{\mathcal{C}}(X, \mathrm{Tot}_n A^{\otimes \bullet}) \to \cdots \to \mathrm{map}_{\mathcal{C}}(X, \mathrm{Tot}_0 A^{\otimes \bullet}) \tag{2.3}$$

collapses at a finite page, with a horizontal vanishing line at height d.

Then we have implications $(1)_d \Leftrightarrow (4)_d$, $(2)_d \Leftrightarrow (3)_d$, *and* $(1)_d \Rightarrow (2)_d \Rightarrow (1)_{d+1}$.

Proof. We begin with the first three conditions. The implication $(1)_d \Rightarrow (2)_d$ is immediate from the diagram

$$\begin{tikzcd}
\mathrm{Tot}^d A^{\otimes \bullet+1} \arrow[r] \arrow[d, "0"'] & \mathbf{1} \arrow[r] \arrow[d, equal] & \mathrm{Tot}_d A^{\otimes \bullet+1} \arrow[d] \arrow[dl, dashed, "\exists"] \\
\mathrm{Tot}^0 A^{\otimes \bullet+1} \arrow[r] & \mathbf{1} \arrow[r] & \mathrm{Tot}_0 A^{\otimes \bullet+1}
\end{tikzcd}$$

in which the rows are cofibre sequences. The equivalence $(2)_d \Leftrightarrow (3)_d$ is clear, so we now prove $(2)_d \Rightarrow (1)_{d+1}$.

Recall from the proof of [Mat16, Proposition 3.20] that the full subcategory

$$\{X : X \otimes A^{\otimes \bullet+1} \text{ is a constant pro-object}\} \subset \mathcal{C}$$

is a thick $\otimes$-ideal containing A (since $A \otimes A^{\otimes \bullet+1}$ is split), and therefore contains $\mathbf{1}$. We can in fact define full subcategories

$$\mathcal{C}_r := \{X : X \otimes A^{\otimes \bullet+1} \text{ is } r\text{-rapidly convergent}\} \subset \mathcal{C},$$

which are closed under retracts and $(-) \otimes X$ for any $X \in \mathcal{C}$. Each of these is *not* thick, but if $X \to Y \to Z$ is a cofibre sequence with $X \in \mathcal{C}_r$ and $Z \in \mathcal{C}_{r'}$, then $Y \in \mathcal{C}_{r+r'}$ (this follows by contemplating the diagram

$$\begin{tikzcd}
X \otimes \mathrm{Tot}_n A^{\otimes \bullet+1} \arrow[r] \arrow[d] & Y \otimes \mathrm{Tot}_n A^{\otimes \bullet+1} \arrow[r] \arrow[d] & Z \otimes \mathrm{Tot}_n A^{\otimes \bullet+1} \arrow[d, "0"] \\
X \otimes \mathrm{Tot}_{n+r'} A^{\otimes \bullet+1} \arrow[r] \arrow[d, "0"'] & Y \otimes \mathrm{Tot}_{n+r'} A^{\otimes \bullet+1} \arrow[r] \arrow[d] & Z \otimes \mathrm{Tot}_{n+r'} A^{\otimes \bullet+1} \arrow[d] \\
X \otimes \mathrm{Tot}_{n+r+r'} A^{\otimes \bullet+1} \arrow[r] & Y \otimes \mathrm{Tot}_{n+r+r'} A^{\otimes \bullet+1} \arrow[r] & Z \otimes \mathrm{Tot}_{n+r+r'} A^{\otimes \bullet+1}
\end{tikzcd}$$

as in [HPS99, Lemma 2.1] or [Mat15, Proposition 3.5]). In particular, $\mathrm{Tot}_r A^{\otimes \bullet+1} \in \mathcal{C}_{r+1}$ since this can be constructed iteratively by taking $r+1$-many extensions by free A-modules (while $A \in \mathcal{C}_1$ by Example 2.15). Now assumption $(2)_d$ implies that $\mathbf{1} \in \mathcal{C}_{d+1}$, that is, $(1)_{d+1}$ holds.

The implication $(1)_d \Leftrightarrow (4)_d$ is proven in [Mat15, Proposition 3.12] (in the case $\mathfrak{U} = \mathcal{C}$), by keeping track of the index d (called N therein). ☐

Remark 2.21. The implications in this lemma are special to the Amitsur/cobar complex. For an arbitrary cosimplicial object (even in spectra), condition $(2)_d$ does not in general imply $(1)_{d'}$ for any d' (although $(1)_d \Rightarrow (2)_d$ still holds).

Lemma 2.22. *Let G be a profinite group, $\mathcal{C}$ a stable homotopy theory, and $\mathbf{1} \to A$ a faithful G-Galois extension in $\mathcal{C}$ corresponding to $\mathcal{A}^\delta \in \widehat{\mathcal{S}h}(BG_{\mathrm{et}}, \mathrm{CAlg}(\mathcal{C}))$. Then $\mathcal{A}^\delta$ sends pullbacks of finite G-sets to pushouts in* $\mathrm{CAlg}(\mathcal{C})$.

Proof. Let $S_1 \to S_0 \leftarrow S_2$ be a cospan in BG_{et}, and assume first that each S_i is of the form G/U_i where U_i is an open subgroup.

We can write $S_1 \times_{S_0} S_2 = \coprod_{g \in U_1 \backslash U_0 / U_2} G/U_1 \cap g^{-1}U_2 g$, so that

$$\nu^p \mathcal{A}^\delta(S_1 \times_{S_0} S_2) = \bigoplus_{U_1 \backslash U_0 / U_2} \mathcal{A}^\delta(U_1 \cap g^{-1}U_2 g).$$

This admits an algebra map from the pushout $\mathcal{A}^\delta(S_1) \otimes_{\mathcal{A}^\delta(S_0)} \mathcal{A}^\delta(S_2)$, and we can check this map is an equivalence after base-change to A, a faithful algebra over $\mathcal{A}^\delta(S_0)$. But

$$\begin{aligned}
A \otimes_{\mathcal{A}^\delta(S_0)} \mathcal{A}^\delta(S_1) \otimes_{\mathcal{A}^\delta(S_0)} \mathcal{A}^\delta(S_2) &\simeq \bigoplus_{U_0/U_1} A \otimes_{\mathcal{A}^\delta(S_0)} \mathcal{A}^\delta(S_2) \\
&\simeq \bigoplus_{U_0/U_1} \bigoplus_{U_0/U_2} A \\
&\simeq \bigoplus_{U_1 \backslash U_0 / U_2} \bigoplus_{U_0/U_1 \cap g^{-1}U_2 g} A \\
&\simeq A \otimes_{\mathcal{A}^\delta(S_0)} \bigoplus_{U_1 \backslash U_0 / U_2} \mathcal{A}^\delta(U_0/U_1 \cap g^{-1}U_2 g),
\end{aligned}$$

using the isomorphism of U_0-sets

$$U_0/U_1 \times U_0/U_2 \simeq \coprod_{g \in U_1 \backslash U_0 / U_2} U_0/U_1 \cap g^{-1}U_2 g.$$

The result for finite G-sets follows from the above by taking coproducts. □

Corollary 2.23. *Let G be a profinite group, $\mathcal{C}$ a stable homotopy theory, and $\mathbf{1} \to A$ a faithful G-Galois extension in $\mathcal{C}$ corresponding to $\mathcal{A}^\delta \in \widehat{\mathcal{S}h}(BG_{\mathrm{et}}, \mathrm{CAlg}(\mathcal{C}))$. The presheaf $\nu^p\mathcal{A}^\delta$ sends pullbacks in BG_{proet} to pushouts in* $\mathrm{CAlg}(\mathcal{C})$*.*

Proof. This follows from the previous lemma by passing to limits. □

Finally, we prove the desired descent result.

Proposition 2.24. *Let G be a profinite group, and $\mathbf{1} \to A$ a G-Galois extension in $\mathcal{S}p_{\mathbf{K}}$ corresponding to $\mathcal{A}^\delta \in \widehat{\mathcal{S}h}(BG_{\mathrm{et}}, \mathrm{CAlg}(\mathcal{S}p_{\mathbf{K}}))$. Suppose moreover that $\mathbf{1} \to A$ is descendable. Then $\nu^p\mathcal{A}^\delta \in \mathcal{P}(BG_{\mathrm{proet}}, \mathcal{S}p_{\mathbf{K}})$ is a hypersheaf.*

Proof. By Proposition 2.18 it is enough to prove that $\mathcal{A}^\delta$ satisfies $(\star)$. By Lemma 2.22, the Čech complex in question is the cobar complex

$$A^{hU} \longrightarrow A^{hV} \rightrightarrows A^{hV} \otimes_{A^{hU}} A^{hV} \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots . \tag{2.4}$$

For every $r \geq 1$, we shall consider the following full subcategory of $\mathcal{D} := \mathrm{Mod}_{A^{hU}}(\mathcal{C})$:

$$\mathcal{D}_r = \mathcal{D}_r(U,V) := \{X : X \otimes_{A^{hU}} (A^{hV})^{\otimes_{A^{hU}} \bullet+1} \text{ is } r\text{-rapidly convergent}\}.$$

Our aim is to show that $A^{hU} \in \mathcal{D}_d$. To this end, we observe the following properties of $\mathcal{D}_r$:

(i) $\mathcal{D}_r$ is closed under retracts.

(ii) $\mathcal{D}_r$ is a $\otimes$-ideal: if $X \in \mathcal{D}_r$ then $X \otimes Y \in \mathcal{D}_r$ for any Y.

(iii) $\mathcal{D}_r$ is not thick. However, if $X \to Y \to Z$ is a cofibre sequence such that $X \in \mathcal{D}_r$ and $Z \in \mathcal{D}_{r'}$, then $Y \in \mathcal{D}_{r+r'}$.

But $A \in \mathcal{D}_1$ by Example 2.15. Since A^{hU} splits over A (so that $A \otimes A^{hU}$ contains A as a retract), we have

$$\exp_A(A^{hU}) = \exp_{A\otimes A^{hU}}(A^{hU}) \leq \exp_A(\mathbf{1}) = d,$$

with the inequality following from [MNN17, Corollary 4.13]. Thus $A^{hU} \in \mathcal{D}_d$ as desired. This proves that $\mathcal{A}^\delta$ satisfies $(\star)$, and hence that $\nu^p\mathcal{A}^\delta$ is a sheaf.

For hyperdescent, we consider the unlocalised version

$$\mathcal{A}^{\mathrm{un}} \colon S = \varprojlim S_i \mapsto \varinjlim \mathcal{A}^\delta(S_i) \in \mathrm{CAlg},$$

which is the left Kan extension along $BG_{\mathrm{et}} \hookrightarrow BG_{\mathrm{proet}}$ formed in *spectra* (equivalently CAlg, since filtered colimits in CAlg are computed on the underlying spectrum). Since $\mathcal{A}^{\mathrm{un}}$ is pointwise $\mathbf{E}$-local, we have $\mathcal{A} = \mathrm{map}(M\mathbb{S}, \mathcal{A}^{\mathrm{un}})$ for $M\mathbb{S}$ the monochromatic sphere at height h [HS99, Theorem 7.10(e)]; thus hyperdescent for $\mathcal{A}^{\mathrm{un}}$ will imply it for $\mathcal{A}$. This is the content of Lemma 2.25, proven immediately below. □

Lemma 2.25. *Let G be a profinite group, and $\mathcal{F}^\delta \in \mathcal{S}h(BG_{\mathrm{et}}, \mathcal{S}p)$ a sheaf satisfying condition $(\star)$. Then its left Kan extension $\mathcal{F} \in \mathcal{P}(BG_{\mathrm{proet}}, \mathcal{S}p)$ is a hypercomplete sheaf.*

Proof. The same proof as Proposition 2.18 shows that $\mathcal{F}$ is a sheaf of spectra on BG_{proet}, since $h\mathcal{S}p$ is Brown and hence Lemma 2.17 still applies. We will show that $\mathcal{F}$ is Postnikov complete, and hence hypercomplete: indeed, any truncated sheaf of spectra is hypercomplete, and the class of hypercomplete sheaves is closed under limits in $\mathcal{P}(BG_{\mathrm{proet}}, \mathcal{S}p)$. To prove Postnikov completeness we can restrict to the subsite $\mathrm{Free}_G \subset BG_{\mathrm{proet}}$, since this generates the proétale ∞-topos; to prevent the notation from becoming too cluttered, we leave the restriction implicit below.

Since Postnikov towers in presheaves always converge, it suffices to prove that the truncations $\tau_{\leq t}\mathcal{F} \in \mathcal{S}h(\mathrm{Free}_G, \mathcal{S}p)$ are given by pointwise truncation. Given a covering of free G-sets $S' \to S$, we claim that the Čech complex $\tau_{\leq t}\mathcal{F}(S) \to \tau_{\leq t}\mathcal{F}(S'^{\times_S \bullet+1})$ is d-rapidly convergent for some $d < \infty$, and in particular converges. Indeed, since

$$\tau_{\leq t}\mathcal{F}(T \times G) \simeq \tau_{\leq t}\left(\varinjlim_i \prod_{T_i} \mathcal{F}(G)\right) \simeq \left(\varinjlim_i \prod_{T_i} \mathbb{S}\right) \otimes \tau_{\leq t}\mathcal{F}(G),$$

it is enough to verify that the cosimplicial object

$$\varinjlim_i \textstyle\prod_{T_i} \mathbb{S} \longrightarrow \varinjlim_i \prod_{T'_i} \mathbb{S} \rightrightarrows \varinjlim_i \prod_{T'_i \times_{T_i} T'_i} \mathbb{S} \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \cdots \tag{2.5}$$

is rapidly convergent, where $S \cong T \times G$ and $S' \cong T' \times G$. Again, it is enough to find $d < \infty$ such that each d-fold composite in the fibre tower is phantom; by [Mat15, Proposition 3.12], this is equivalent to the existence of a horizontal vanishing line at a finite page in the descent spectral sequence for $\mathrm{Map}(F, \varinjlim_i \prod_{T_i} \mathbb{S})$ for any finite spectrum, at height independent of F. But

$$\pi_t \mathrm{Map}(F, \varinjlim_i \prod_{T_i} \mathbb{S}) \simeq \varinjlim_i \prod_{T_i} \pi_t \mathrm{Map}(F, \mathbb{S}) \simeq \mathrm{Cont}(T, \pi_t \mathbb{D}F),$$

since each group $\pi_t \mathbb{D}F$ is discrete (this is the case for $F = \mathbb{S}$, and follows in general by taking extensions). The E_2-page of the corresponding spectral sequence is thus the cohomology of the Moore complex of

$$\mathrm{Cont}(T', \pi_t \mathbb{D}F) \rightrightarrows \mathrm{Cont}(T' \times_T T', \pi_t \mathbb{D}F) \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \cdots \;\; \cdots .$$

When T and T' are both finite this complex is exact (by choosing a splitting of $T' \twoheadrightarrow T$), and has

$$\begin{aligned} H^0 &= \mathrm{Eq}\left(\mathrm{Cont}(T', \pi_t \mathbb{D}F) \rightrightarrows \mathrm{Cont}(T' \times_T T', \pi_t \mathbb{D}F)\right) \\ &\cong \mathrm{Cont}(T, \pi_t \mathbb{D}F). \end{aligned}$$

Thus both properties hold for a general covering by passing to colimits. □

2.2.4 Functoriality in G

Let $f \colon H \to G$ be a continuous homomorphism between profinite groups. In this section, we will consider the functoriality of the construction

$$G \mapsto \widehat{\mathcal{S}h}(BG_{\mathrm{proet}}, \mathcal{C}).$$

For example, taking $p \colon G \to *$ we obtain (homotopy) invariants and coinvariants functors; functoriality will imply the iterated fixed points formula

$$(X^{hH})^{hG/H} \simeq X^{hG}$$

for $H \subset G$ a normal subgroup.

Remark 2.26. For any presentable $\mathcal{C}$ we have $\mathcal{S}h(BG_{\mathrm{proet}}, \mathcal{C}) \simeq \mathcal{S}h(BG_{\mathrm{proet}}) \otimes \mathcal{C}$ by [SAG, Remark 1.3.1.6], and so we restrict ourselves to sheaves of spaces.

Proposition 2.27. *(i) Any map of profinite groups $f\colon H \to G$ gives rise to a geometric morphism*

$$f^* : \mathcal{S}h(BG_{\mathrm{proet}}) \rightleftarrows \mathcal{S}h(BH_{\mathrm{proet}}) : f_*$$

This defines a functor $\mathrm{Grp}(\mathrm{Profin}) \to \mathcal{L}\mathcal{T}\mathrm{op}_\infty$ *to the ∞-category of ∞-topoi and left adjoints.*

(ii) The left adjoint $f^*\colon \mathcal{S}h(BG_{\mathrm{proet}}) \to \mathcal{S}h(BH_{\mathrm{proet}})$ admits a further left adjoint $f_!$.

Proof. Given $f\colon H \to G$, restriction determines a morphism of sites

$$\mathrm{res}_f : BG_{\mathrm{proet}} \to BH_{\mathrm{proet}},$$

hence a geometric morphism

$$f^* : \mathcal{S}h(BG_{\mathrm{proet}}) \rightleftarrows \mathcal{S}h(BH_{\mathrm{proet}}) : f_*$$

whose right adjoint is given by precomposition with res_f.

For (ii), we apply Lemma 2.28 to the adjunction

$$(-) \times_H G : BH_{\mathrm{proet}} \rightleftarrows BG_{\mathrm{proet}} : \mathrm{res}_f,$$

noting that if $S \to S'$ is a surjection of H-sets then $S \times_H G \to S' \times_H G$ is also surjective. □

Lemma 2.28. *Given a morphism of sites $f\colon \mathcal{C} \to \mathcal{C}'$, if f admits a left adjoint g then f^* is given by the sheafification of g_*,*

$$f^*\mathcal{F} = (g_*\mathcal{F})^+.$$

If g is itself a morphism of sites, then $f^* = g_*$ admits a further left exact left adjoint $f_! = g^*$.

Proof. In general, f^* is the sheafification of the presheaf extension $f^p = \mathrm{Lan}_f(-)$. Since $f^p\colon \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{C}')$ is left adjoint to f_* [HTT, Proposition 4.3.3.7], precomposition with the counit η of the $g \dashv f$ adjunction is a map

$$g_* f_* \simeq (gf)_* \xrightarrow{\eta_*} id_{\mathcal{P}(\mathcal{C}')},$$

and hence induces a natural transformation $\alpha\colon g_* \to f^p$ [RV22, Lemma 2.3.7]. On vertices,

$$f^p\mathcal{F} : C' \mapsto \varinjlim_{\mathcal{C}\downarrow C'} \mathcal{F}(C),$$

and the above natural transformation is the map $\mathcal{F}(g(C')) \to f^p\mathcal{F}(C)$ induced by the counit $gf(C') \to C'$, viewed as an object of $\mathcal{C} \downarrow C'$. But the counit is terminal in $\mathcal{C} \downarrow C'$, so

$$\alpha_{C'} : g_*\mathcal{F}(C') \xrightarrow{\sim} f^p\mathcal{F}(C'),$$

which gives the first claim after sheafifying. If moreover g preserves coverings then $g_*\mathcal{F}$ is already a sheaf, and so $f^* \simeq g_*$ admits g^* as a left adjoint. □

Lemma 2.29. *Suppose that $i\colon H \subset G$ is a normal subgroup with quotient $p\colon G \to G/H$, and $X \in \mathcal{S}h(BG_{\mathrm{proet}}, \mathcal{C})$. Then*
$$(X^{hH})^{hG/H} \simeq X^{hG}.$$

Proof. Writing Γ_G for evaluation on $G/G \in BG_{\mathrm{proet}}$ and likewise for G/H, the claim is that

$$\Gamma_{G/H} p_* X \simeq \Gamma_G X.$$

This is immediate: if $\pi^G\colon G \to *$ denotes the projection, then Proposition 2.27 implies the middle equivalence in
$$\Gamma_G \simeq \Gamma_* \pi^G_* \simeq \Gamma_* \pi^{G/H}_* p_* \simeq \Gamma_{G/H} p_*.$$ □

2.2.5 Comparison with pyknotic G-objects

Given a profinite group G and a base ∞-category $\mathcal{C}$, we have described the ∞-category $\widehat{\mathcal{S}h}(BG_{\mathrm{proet}})$ as a good model for the ∞-topos of continuous G-spaces. Using the theory of pyknotic objects, there are two other natural candidates:

(*i*) G has image $\underline{G} \in \mathrm{Pyk}(\mathrm{Set})$ under the embedding of compactly generated spaces in pyknotic sets, and $\underline{G}$ is a pyknotic group. Since $\mathcal{S}$ is tensored over sets (and hence $\mathrm{Pyk}(\mathcal{S})$ over $\mathrm{Pyk}(\mathrm{Set})$), we can form a category of $\underline{G}$-modules in $\mathrm{Pyk}(\mathcal{S})$.

(*ii*) G has a *pyknotic classifying space* $BG \in \mathrm{Pyk}(\mathcal{S})$, given by the limit in $\mathrm{Pyk}(\mathcal{S})$ of the classifying spaces $BG_i \in \mathcal{S} \hookrightarrow \mathrm{Pyk}(\mathcal{S})$, where $G = \lim G_i$ is a presentation of G as a profinite group. We can therefore consider the ∞-topos $\mathrm{Pyk}(\mathcal{S})_{/BG}$.

In this section we show that all these notions agree. See [SAG, Theorem A.5.6.1] for a related result in the context of *pro-π-finite spaces*.

Remark 2.30. We have chosen to use the ∞-topos $\mathrm{Pyk}(\mathcal{S})$ in this section, as some of our results depend on [Wol22]. Note however that a translation of these results to $\mathrm{Cond}(\mathcal{S})$ is given in [Wol22, Remark 4.19].

Proposition 2.31. *For any profinite group G, the adjunction*

$$i^* : \widehat{\mathcal{S}h}(BG_{\mathrm{proet}}) \rightleftarrows \widehat{\mathcal{S}h}(B*_{\mathrm{proet}}) = \mathrm{Pyk}(\mathcal{S}) : i_*$$

induced by $i\colon * \to G$ is comonadic, and induces an equivalence

$$\widehat{\mathcal{S}h}(BG_{\mathrm{proet}}) \simeq \mathrm{Mod}_{\underline{G}}(\mathrm{Pyk}(\mathcal{S})).$$

In particular, for any presentable $\mathcal{C}$ we have

$$\widehat{\mathcal{S}h}(BG_{\mathrm{proet}}, \mathcal{C}) \simeq \mathrm{Mod}_{\underline{G}}(\mathrm{Pyk}(\mathcal{C})).$$

Proof. To see that the adjunction is monadic, we must verify the conditions of the Barr-Beck-Lurie theorem [HA, Theorem 4.7.3.5]. Since both sides are presentable, this amounts to checking:

(i) i^* is conservative,

(ii) i^* preserves totalisations of i^*-split simplicial objects.

But i^* admits a further left adjoint, which implies (ii). On the other hand, $i^*\mathcal{F} = i^p\mathcal{F} \colon T \mapsto \mathcal{F}(T \times G)$ by Lemma 2.28, so that i^* is conservative by Lemma 2.10.

As a result, we see that $\widehat{\mathcal{S}h}(BG_{\mathrm{proet}}) \simeq \mathrm{Mod}_{\mathbb{T}}(\mathrm{Pyk}(\mathcal{S}))$, where $\mathbb{T}$ is the comonad i^*i_*. To identify this with the coaction comonad for $\underline{G}$ it suffices to do so for compact generators of $\widehat{\mathcal{S}h}(B{*}_{\mathrm{proet}})$, since both comonads preserve colimits. But if $T \in B{*}_{\mathrm{proet}}$ then

$$\begin{aligned}\mathbb{T}(\underline{T})(T') &= \mathrm{Cont}(\mathrm{res}_i(T' \times G), T)\\ &\cong \mathrm{Cont}(\mathrm{res}_i(T'), \mathrm{Cont}(\mathrm{res}_i(G), T))\\ &= \mathrm{Map}(\underline{G}, \underline{T})(T')\end{aligned}$$

since $B{*}_{\mathrm{proet}}$ is Cartesian closed; that is, $\mathbb{T} = \mathrm{Map}(\underline{G}, -)$ so

$$\widehat{\mathcal{S}h}(BG_{\mathrm{proet}}) \simeq \mathrm{Mod}_{\underline{G}}(\widehat{\mathcal{S}h}(B{*}_{\mathrm{proet}})).$$

For general $\mathcal{C}$, this implies the equivalences

$$\begin{aligned}\widehat{\mathcal{S}h}(BG_{\mathrm{proet}}, \mathcal{C}) &\simeq \widehat{\mathcal{S}h}(BG_{\mathrm{proet}}) \otimes \mathcal{C}\\ &\simeq \mathrm{Mod}_{\underline{G}}(\mathrm{Pyk}(\mathcal{S})) \otimes \mathcal{C}\\ &\simeq \mathrm{Mod}_{\underline{G}}(\mathrm{Pyk}(\mathcal{S})) \otimes_{\mathrm{Pyk}(\mathcal{S})} \mathrm{Pyk}(\mathcal{S}) \otimes \mathcal{C}\\ &\simeq \mathrm{Mod}_{\underline{G}}(\mathrm{Pyk}(\mathcal{S})) \otimes_{\mathrm{Pyk}(\mathcal{S})} \mathrm{Pyk}(\mathcal{C})\\ &\simeq \mathrm{Mod}_{\underline{G}}(\mathrm{Pyk}(\mathcal{C})).\end{aligned}$$

Here we have used [Pyk, Remark 2.3.5] for the fourth equivalence, and [HA, Theorem 4.8.4.6] for the final equivalence. □

Remark 2.32. In this case we could have also proceeded before hypercompletion to give an equivalence $\mathcal{S}h(BG_{\mathrm{proet}}, \mathcal{C}) \simeq \mathrm{Mod}_{\underline{G}}(\mathcal{S}h(B{*}_{\mathrm{proet}}, \mathcal{C}))$. Nevertheless, restricting to hypercomplete sheaves is generally greatly simplifying since it allows us to restrict to projective G-sets, for which descent is often easier to prove.

Proposition 2.33. *For any profinite group G, there is a natural equivalence*

$$\widehat{\mathcal{S}h}(BG_{\mathrm{proet}}) \simeq \mathrm{Pyk}(\mathcal{S})_{/BG}.$$

Proof. This is a very mild modification of [Wol22, Corollary 1.2]. In the trivially stratified case, the main theorem of *op. cit.* provides a natural *exodromy* equivalence

$$\mathcal{X}^{\mathrm{Pyk}} := \widehat{\mathcal{S}h}_{\mathrm{eff}}(\mathrm{Pro}(\mathcal{X}^{\mathrm{coh}}_{<\infty})) \xrightarrow[\mathrm{ex}]{\sim} \mathrm{Fun}^{\mathrm{cts}}(\widehat{\Pi}_\infty\mathcal{X}, \mathrm{Pyk}(\mathcal{S})) \tag{2.6}$$

between the *pyknotification* of an ∞-topos $\mathcal{X}$ with the ∞-category of *pyknotic presheaves* on its profinite shape, viewed as a pyknotic space[2]. Any $X \in \mathrm{Pro}(\mathcal{S}h(BG_{\mathrm{et}})^{\mathrm{coh}}_{<\infty})$ may be covered by a profinite G-set: this follows just as in the case $G = *$, which is [BGH20, Proposition 13.4.9]. Thus the subcategory $j\colon BG_{\mathrm{proet}} = \mathrm{Pro}(BG_{\mathrm{et}}) \hookrightarrow \mathrm{Pro}(\mathcal{S}h(BG_{\mathrm{et}})^{\mathrm{coh}}_{<\infty})$ generates the same topos, and we obtain natural equivalences

$$\begin{aligned}\widehat{\mathcal{S}h}(BG_{\mathrm{proet}}) &\xleftarrow[j_*]{\sim} \widehat{\mathcal{S}h}(\mathrm{Pro}(\mathcal{S}h(BG_{\mathrm{et}})^{\mathrm{coh}}_{<\infty})) \\ &= \mathcal{S}h(BG_{\mathrm{et}})^{\mathrm{Pyk}} \xrightarrow[\mathrm{ex}]{\sim} \mathrm{Fun}^{\mathrm{cts}}(\widehat{\Pi}_\infty\mathcal{S}h(BG_{\mathrm{et}}), \mathrm{Pyk}(\mathcal{S})),\end{aligned}$$

where j_* is restriction and ex denotes the pyknotic exodromy equivalence (2.6). We now identify $\widehat{\Pi}_\infty\mathcal{S}h(BG_{\mathrm{et}}) \in \mathcal{S}^\wedge_\pi = \mathrm{Pro}(\mathcal{S}_\pi)$: if we write $G = \lim G_i$, then by [CM21, Construction 4.5] there is an equivalence

$$\mathcal{S}h(BG_{\mathrm{et}}) \simeq \varprojlim \mathcal{S}_{/BG_i}.$$

By [SAG, Theorem E.2.4.1], the pro-extension $\Psi\colon \mathcal{S}^\wedge_\pi \to \mathcal{L}\mathcal{T}\mathrm{op}_\infty$ of $X \mapsto \mathcal{S}_{/X}$ is a fully faithful right adjoint to $\widehat{\Pi}_\infty$, so

$$\widehat{\Pi}_\infty\mathcal{S}h(BG_{\mathrm{et}}) \simeq \widehat{\Pi}_\infty \varprojlim \Psi(BG_i) \simeq \widehat{\Pi}_\infty\Psi(\{BG_i\}) \simeq \{BG_i\} \in \mathcal{S}^\wedge_\pi.$$

By definition, BG is image of $\{BG_i\}$ in $\mathrm{Pyk}(\mathcal{S})$, so continuous unstraightening [Wol22, Corollary 3.20] yields the desired equivalence

$$\widehat{\mathcal{S}h}(BG_{\mathrm{proet}}) \simeq \mathrm{Fun}^{\mathrm{cts}}(BG, \mathrm{Pyk}(\mathcal{S})) \simeq \mathrm{Pyk}(\mathcal{S})_{/BG}. \qquad \square$$

Remark 2.34. The results of [SAG, Appendix E.5] imply that the profinite group G, as a 0-truncated object of $\mathrm{Grp}(\mathcal{S}^\wedge_\pi)$, has a profinite classifying space; that is, a delooping in $\mathcal{S}^\wedge_\pi$. In fact, in the proof of [SAG, Theorem E.5.0.4] ([SAG, E.5.6]) Lurie observes that this is $\{BG_i\}$; passing to $\mathrm{Pyk}(\mathcal{S})$ we see that BG is a delooping of $\underline{G}$. In the hypercomplete case, one can use this to deduce Proposition 2.31 from Proposition 2.33.

Corollary 2.35. *For any presentable $\mathcal{C}$, the assignment $G \mapsto \widehat{\mathcal{S}h}(BG_{\mathrm{proet}}, \mathcal{C})$ extends to a Wirthmüller context on profinite groups and continuous homomorphisms.*

Remark 2.36. Recall [FHM03] that this says that there is a 6-functor formalism, with $f^! \simeq f^*$ for any f and f^* strong symmetric monoidal. It does not assert the existence of a Wirthmüller isomorphism, but produces a candidate map in the presence of a dualising object.

[2]Recall [BGH20, Remark 13.4.4] that the inclusion $よ\colon \mathcal{S}^\wedge_\pi \hookrightarrow \widehat{\mathcal{S}h}_{\mathrm{eff}}(\mathcal{S}^\wedge_\pi) \simeq \mathrm{Pyk}(\mathcal{S})$ preserves all small limits. Combining this with [BGH20, p. 13.4.11] we see that $よ\{X_i\} = \lim よX_i = \lim \Gamma^* X_i$ is the limit in $\mathrm{Pyk}(\mathcal{S})$ of the π-finite spaces X_i, viewed as discrete pyknotic spaces.

Proof. By [Man22, Prop. A.5.10], it remains to check:

(*i*) For every map $f\colon H \to G$ of profinite groups, the functor f^* admits a left adjoint $f_!\colon \widehat{\mathcal{S}h}(BH_{\mathrm{proet}}) \to \widehat{\mathcal{S}h}(BG_{\mathrm{proet}})$.

(*ii*) If moreover $g\colon G' \to G$ and $H' \coloneqq H \times_G G'$, then the following square is left adjointable:

$$\begin{array}{ccc} \widehat{\mathcal{S}h}(BG_{\mathrm{proet}}) & \xrightarrow{f^*} & \widehat{\mathcal{S}h}(BH_{\mathrm{proet}}) \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle g'^*} \\ \widehat{\mathcal{S}h}(BG'_{\mathrm{proet}}) & \xrightarrow[f'^*]{} & \widehat{\mathcal{S}h}(BH'_{\mathrm{proet}}) \end{array}$$

That is, the push-pull transformation $f'_!g'^* \to g^*f_!$ is an equivalence.

(*iii*) If $M \in \widehat{\mathcal{S}h}(BG_{\mathrm{proet}})$ and $N \in \widehat{\mathcal{S}h}(BH_{\mathrm{proet}})$, the natural map

$$f_!(N \times f^*M) \to f_!N \times M$$

is an equivalence.

In fact, these properties will all follow formally from Proposition 2.33. The functor $G \mapsto \widehat{\mathcal{S}h}(BG_{\mathrm{proet}})$ is the restriction to classifying spaces of profinite groups of the functor $X \mapsto \mathrm{Pyk}(\mathcal{S})_{/X}$, which in the language of [Mar22] is the *universe* $\Omega_{\mathrm{Pyk}(\mathcal{S})}$. But the universe Ω of any ∞-topos $\mathcal{B}$ satisfies properties (*i*) to (*iii*). Explicitly:

(*i*) For any $f\colon A \to B$ in $\mathcal{B}$, the étale geometric morphism $f^* : \mathcal{B}_{/B} \to \mathcal{B}_{/A}$ is given by basechange along f, and admits a left adjoint $f_! = f \circ -$.

(*ii*) For a pullback square

$$\begin{array}{ccc} A & \xrightarrow{g'} & B \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow[g]{} & D \end{array}$$

in $\mathcal{B}$, the evaluation of the push-pull transformation $f'_!g'^* \to g^*f_!$ on $X \to B$ may be identified with the left-hand vertical map in the extended diagram

$$\begin{array}{ccccc} X \times_B A & \longrightarrow & A & \xrightarrow{g'} & B \\ \downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X \times_D C & \longrightarrow & C & \xrightarrow[g]{} & D \end{array}$$

and is therefore an equivalence.

(*iii*) For $f\colon A \to B$ in $\mathcal{B}$, the projection formula is given in [HTT, Remark 6.3.5.12]. $\square$

2.3 Morava E-theory as a proétale spectrum

The aim of this section is to show that the proétale site allows us to capture the continuous action on $\mathbf{E}$ as a sheaf of spectra on $B\mathbb{G}_{\mathrm{proet}}$. Our first task is to exhibit Morava E-theory itself as a proétale sheaf of spectra, so as to recover the $\mathbf{K}$-local $\mathbf{E}$-Adams spectral sequence as a descent spectral sequence. This will allow us to compare it to the descent spectral sequence for the Picard spectrum.

Proposition 2.37. *The presheaf of* $\mathbf{K}$*-local spectra*

$$\mathcal{E} := \nu^p \mathcal{E}^\delta : S = \varprojlim_i S_i \mapsto L_{\mathbf{K}} \varinjlim_i \mathcal{E}^\delta(S_i)$$

is a sheaf on $B\mathbb{G}_{\mathrm{proet}}$.

Remark 2.38. Since $\mathcal{S}p_{\mathbf{K}} \subset \mathcal{S}p$ is a right adjoint, the same formula defines a sheaf of spectra (or even of $\mathbb{E}_\infty$-rings). We stress however that ν^p **will always refer to the left Kan extension internal to** $\mathcal{S}p_{\mathbf{K}}$**, or equivalently the K-localisation of the Kan extension in spectra**.

Proof. This is immediate from Proposition 2.24: by [Mat16, Proposition 10.10], Morava E-theory $\mathbf{E} \in \mathrm{CAlg}(\mathcal{S}p_{\mathbf{K}})$ is descendable. This is a consequence of descendability of $L_h\mathbb{S} \to \mathbf{E}$, which is proven in [Rav92] (the proof crucially uses the fact that $\mathbb{G}$ has finite *virtual* cohomological dimension). In fact, when p is sufficiently large (or more generally, when $p-1$ does not divide h), Proposition 2.37 is a consequence of the descent results of [CM21]. □

As an aside, note that the proof of Proposition 2.24 allows us likewise to consider $\mathbf{E}$-homology of any spectrum X:

Corollary 2.39. *If X is any* $\mathbf{K}$*-local spectrum, then the presheaf $\mathcal{E} \otimes X$ is a hypercomplete sheaf.*

Proof. Given a covering $S' \to S$ in $B\mathbb{G}_{\mathrm{proet}}$, the proof of Proposition 2.24 showed that the Amitsur complex for $S' \to S$ is d-rapidly convergent. Since the functor $(-) \otimes X : \mathcal{S}p_{\mathbf{K}} \to \mathcal{S}p_{\mathbf{K}}$ preserves finite limits, the same is true for the augmented cosimplicial object given by tensoring everywhere by X. □

We can now define a descent spectral sequence for the sheaf $\mathcal{E}$.

Remark 2.40. Proposition 2.37 gives us a sheaf $\mathcal{E} \in \widehat{\mathcal{S}h}(B\mathbb{G}_{\mathrm{proet}}, \mathcal{S}p_{\mathbf{K}})$. On applying the forgetful functor $\mathcal{S}p_{\mathbf{K}} \hookrightarrow \mathcal{S}p$, we obtain $\mathcal{E} \in \widehat{\mathcal{S}h}(B\mathbb{G}_{\mathrm{proet}}, \mathcal{S}p)$; note that this sheaf is *not* left Kan extended from $B\mathbb{G}_{\mathrm{et}}$. Nevertheless, applying Corollary 2.12, we obtain the following result.

Corollary 2.41. *There is a conditionally convergent spectral sequence of the form*

$$E_{2,+}^{s,t} = H^s(B\mathbb{G}_{\mathrm{proet}}, \pi_t\mathcal{E}) \implies \pi_{t-s}\Gamma \lim \tau_{\leq j}\mathcal{E}. \tag{2.7}$$

To identify the E_2-page and the abutment, we need to identify the homotopy sheaves of $\mathcal{E}$.

Lemma 2.42. *The homotopy sheaves of $\mathcal{E}$ are given by*

$$\pi_t\mathcal{E} : S \mapsto \mathrm{Cont}_{\mathbb{G}}(S, \pi_t\mathbf{E}), \tag{2.8}$$

continuous equivariant maps from S to the homotopy groups of Morava E-theory (equipped with their profinite topology).

Proof. To prove the lemma, it is enough to prove that the homotopy *presheaves* $\pi_t^p\mathcal{E}$ take the form $\mathrm{Cont}_{\mathbb{G}}(-, \pi_t\mathbf{E})$, since the topology is subcanonical. Moreover, since $\mathrm{Free}_{\mathbb{G}}$ generates the proétale topos, it is enough to show that $(\pi_t^p\mathcal{E})|_{\mathrm{Free}_{\mathbb{G}}} : S \mapsto \mathrm{Cont}_{\mathbb{G}}(S, \pi_t\mathbf{E}) = \mathrm{Cont}(S/\mathbb{G}, \pi_t\mathbf{E})$.

Using Lemma 2.22, we have for any free $\mathbb{G}$-set of the form $S = T \times \mathbb{G}$ (with trivial action on T),

$$\mathcal{E}(S) \simeq \mathcal{E}(T) \otimes \mathcal{E}(\mathbb{G}) \simeq L_{\mathbf{K}} \varinjlim_j \prod_{T_j} \mathbf{E}. \tag{2.9}$$

Since $\mathbf{E}$-localisation is smashing, the spectrum $\varinjlim_j \prod_{T_j} \mathbf{E}$ is $\mathbf{E}$-local, and so its $\mathbf{K}$-localisation can be computed by smashing with a tower of generalised Moore spectra M_I; see for example [HS99]. Thus

$$\pi_t\left(\mathcal{E}(S)\right) = \pi_t \lim_I (\varinjlim_j \prod_{T_j} \mathbf{E} \otimes M_I),$$

and we obtain a Milnor sequence

$$0 \to {\lim_I}^1 \pi_t \varinjlim_j \prod_{T_j} \mathbf{E} \otimes M_I \to \pi_t\left(\mathcal{E}(S)\right) \to \lim_I \pi_t \varinjlim_j \prod_{T_j} \mathbf{E} \otimes M_I \to 0.$$

Now observe that

$$\begin{aligned}
\pi_t \varinjlim_j \prod_{T_j} \mathbf{E} \otimes M_I &= \varinjlim_j \prod_{T_j} \pi_t\mathbf{E} \otimes M_I \\
&= \varinjlim_j \prod_{T_j} (\pi_t\mathbf{E})/I \\
&= \varinjlim_j \mathrm{Cont}(T_j, (\pi_t\mathbf{E})/I) \\
&= \mathrm{Cont}(T, (\pi_t\mathbf{E})/I),
\end{aligned}$$

using for the last equality that the target is finite. In particular, each of the maps $\mathrm{Cont}(T, (\pi_t\mathbf{E})/I') \to \mathrm{Cont}(T, (\pi_t\mathbf{E})/I)$ is surjective, since the inclusion $I' \subset I$ induces a surjection of finite sets $(\pi_t\mathbf{E})/I' \twoheadrightarrow (\pi_t\mathbf{E})/I$, and so admits a (set-theoretic) splitting. Thus $\lim^1$ vanishes and

$$\pi_t\left(\mathcal{E}(S)\right) = \lim_I \mathrm{Cont}(T, (\pi_t\mathbf{E})/I) = \mathrm{Cont}(T, \pi_t\mathbf{E}).$$

□

Corollary 2.43. *The Postnikov tower of the sheaf $\mathcal{E}$ converges. Thus* (2.7) *converges conditionally* *to* $\pi_*\Gamma\mathcal{E} = \pi_*\mathbf{1}_{\mathbf{K}}$.

Proof. This follows from Corollary 2.43, since Posntnikov towers converge in the hypercomplete ∞-topos of BG_{proet} [BS14, Theorem 3.2.3]. □

Remark 2.44. In fact, we will compare this spectral sequence to the **K**-local **E**-Adams spectral sequence (Proposition 2.48) to show existence of a horizontal vanishing line (at least for $t \geq 2$); thus the spectral sequence converges completely in this region.

Corollary 2.45. *The starting page of the spectral sequence (2.7) is given by continuous group cohomology:*

$$E_{2,+}^{s,t} = H^s(\mathbb{G}, \pi_t\mathbf{E}). \tag{2.10}$$

Proof. Using the identification in Lemma 2.42 of the homotopy sheaves, this follows from [BS14, Lemma 4.3.9(4)], which implies that the canonical map

$$\Phi_M : H^*(\mathbb{G}, M) \to H^*(B\mathbb{G}_{\mathrm{proet}}, \mathrm{Cont}_G(-, M))$$

for a topological $\mathbb{G}$-module M is an isomorphism whenever M can be presented as the limit of a countable tower of finite $\mathbb{G}$-modules. □

Remark 2.46 (c.f. [BH16])**.** By virtue of Corollary 2.39, one obtains for any **K**-local spectrum X a conditionally convergent spectral sequence

$$E_2^{s,t} = H^s(B\mathbb{G}_{\mathrm{proet}}, \mathcal{E}_t^\vee X) \implies \pi_{t-s}X,$$

where $\mathcal{E}_t^\vee X$ denotes the sheaf $\pi_t(\mathcal{E} \otimes X)$. We briefly comment on the E_2-page:

(*i*) To compare with group cohomology, one would first like to assert that

$$H^s(B\mathbb{G}_{\mathrm{proet}}, \mathbf{E}_t^\vee X) \cong H^s\left(B\mathbb{G}_{\mathrm{proet}}, \mathrm{Cont}_{\mathbb{G}}(-, \mathbf{E}_t^\vee X)\right), \tag{2.11}$$

i.e. that $\pi_t\mathcal{E} \otimes X$ is represented by the topological $\mathbb{G}$-module $\mathbf{E}_t^\vee X$. As in Lemma 2.42, there is an exact sequence

$$0 \to \lim_I{}^1 \mathrm{Cont}_{\mathbb{G}}(S, \mathbf{E}_t(X/I)) \to \pi_t\Gamma(S, \mathcal{E} \otimes X) \to \mathrm{Cont}_{\mathbb{G}}(S, \lim_I \mathbf{E}_t(X/I)) \to 0$$

for any free $\mathbb{G}$-set S, and this implies for example that (2.11) holds if $\mathbf{E}_*(X/I) \simeq (\mathbf{E}_*X)/I$ for each of the ideals I (i.e., $\mathbf{E}_*X$ is Landweber exact). We expect that a more general comparison can be made.

(*ii*) If moreover each of the $\mathbb{G}$-modules $\mathbf{E}_t^\vee X$ satisfies one of the assumptions of [BS14, Lemma 4.3.9], then the E_2-page is given by the continuous group cohomology $H^s(\mathbb{G}, \mathbf{E}_t^\vee X)$. For example, this happens if $\mathbf{E}_*^\vee X$ is degreewise profinite; it also holds for $\mathbf{E}_*^\vee X[p^{-1}]$ whenever it holds for X.

We have thus identified the E_2-page of the descent spectral sequence for $\mathcal{E}$ with the E_2-page of the $\mathbf{K}$-local $\mathbf{E}$-Adams spectral sequence. The next step is to show this extends to an identification of the spectral sequences. We do this by using the décalage technique originally due to Deligne [Del71]; the following theorem is standard, but for the sake of completeness (and to fix indexing conventions, one of the great difficulties in the subject) we include the argument in Appendix A.

Proposition 2.47 (Lemma A.3)**.** *Let $\mathcal{F}$ be a sheaf of spectra on a site $\mathcal{C}$, and let $X \twoheadrightarrow *$ be a covering of the terminal object. Suppose that for every t and $q > 0$ we have $\Gamma(X^q, \tau_t \mathcal{F}) = \tau_t \Gamma(X^q, \mathcal{F})$. Then there is an isomorphism between the descent and Bousfield-Kan spectral sequences, up to reindexing: for all r,*

$$E_r^{s,t} \cong \check{E}_{r+1}^{2s-t,s}.$$

Proposition 2.48. *Décalage of the Postnikov filtration induces an isomorphism between the following spectral sequences:*

$$\begin{aligned} E_{2,+}^{s,t} = \pi_s \Gamma \tau_t \mathcal{E} &= H^s(\mathbb{G}, \pi_t \mathbf{E}) \implies \pi_{t-s} \mathbf{1}_{\mathbf{K}}, \\ \check{E}_{3,+}^{2s-t,s} = \pi^s(\pi_t \mathbf{E}^{\otimes \bullet + 1}) &= H^s(\mathbb{G}, \pi_t \mathbf{E}) \implies \pi_{t-s} \mathbf{1}_{\mathbf{K}}. \end{aligned}$$

The first is the descent spectral sequence for the sheaf $\mathcal{E}$, and the second is the $\mathbf{K}$*-local* $\mathbf{E}$*-Adams spectral sequence.*

Proof. By Lemma 2.22, the Tot-filtration associated to the cosimplicial spectrum $\Gamma(\mathbb{G}^\bullet, \mathcal{E})$ is precisely the Adams tower for the Amitsur complex of $\mathbf{1}_{\mathbf{K}} \to \mathbf{E}$. The resulting spectral sequence is the $\mathbf{K}$-local $\mathbf{E}$-Adams spectral sequence by definition.

According to Lemma 2.47, all that remains to check is that each spectrum

$$\Gamma(\mathbb{G}^q, \tau_t \mathcal{E}), \qquad q > 0$$

is Eilenberg-Mac Lane. Note that when $q = 1$ this is immediate since any cover of $\mathbb{G}$ in $B\mathbb{G}_{\mathrm{proet}}$ is split; when $q > 1$ the profinite set $\mathbb{G}^{q-1}$ is not extremally disconnected, and we will deduce this from Lemma 2.42. Indeed, we know that $\Gamma(\mathbb{G}^q, \tau_t \mathcal{E})$ is t-truncated, while for $s \geq t$ we have

$$\begin{aligned} \pi_s \Gamma(\mathbb{G}^q, \tau_t \mathcal{E}) &= H^{t-s}(B\mathbb{G}_{\mathrm{proet}/\mathbb{G}^q}, \pi_t \mathcal{E}) \\ &= H^{t-s}(\mathrm{Profin}_{/\mathbb{G}^{q-1}}, \mathrm{Cont}(-, \pi_t \mathbf{E})) \\ &= H^{t-s}_{\mathrm{cond}}(\mathbb{G}^{q-1}, \mathrm{Cont}(-, \pi_t \mathbf{E})), \end{aligned}$$

where H^*_{cond} denotes condensed cohomology. Here we have used the equivalence $B\mathbb{G}_{\mathrm{proet}/\mathbb{G}^q} \simeq \mathrm{Profin}_{/\mathbb{G}^{q-1}}$ sending $S \mapsto S/\mathbb{G}$. We now argue that the higher cohomology groups vanish, essentially as in the first part of [Cond, Theorem 3.2]. Namely, the sheaves $\mathrm{Cont}(-, \pi_t \mathbf{E})$ on $B\mathbb{G}_{\mathrm{proet}/\mathbb{G}^q}$ satisfy the conditions of [BS14, Lemma 4.3.9(4)], and so the cohomology groups in question can be computed by Čech cohomology: the Čech-to-derived spectral sequence collapses, since the higher direct images of $\mathrm{Cont}(-, \pi_t \mathbf{E})$ vanish. As a result, to check they vanish it will be enough to check that the Čech complex

$$\mathrm{Cont}_{\mathbb{G}}(\mathbb{G}^q, \pi_t \mathbf{E}) \to \mathrm{Cont}_{\mathbb{G}}(S, \pi_t \mathbf{E}) \to \mathrm{Cont}_{\mathbb{G}}(S \times_{\mathbb{G}^q} S, \pi_t \mathbf{E}) \to \cdots \tag{2.12}$$

is exact for any surjection $S \twoheadrightarrow \mathbb{G}^q$. Writing this as a limit of surjections of finite $\mathbb{G}$-sets $S_i \twoheadrightarrow S'_i$ (with $\lim S'_i = \mathbb{G}^q$ and $\lim S_i = S$), and writing $A^j_{i,I} := \mathrm{Cont}_{\mathbb{G}}(S_i^{\times_{S'_i} j}, \pi_t \mathbf{E}/I)$ for brevity, (2.12) is the complex

$$\lim_I \operatorname{colim}_i A^0_{i,I} \to \lim_I \operatorname{colim}_i A^1_{i,I} \to \lim_I \operatorname{colim}_i A^2_{i,I} \to \cdots .$$

Its cohomology therefore fits in a Milnor sequence[3]

$$0 \to {\lim_I}^1 H^{j-1}(\operatorname{colim}_i A^*_{i,I}) \to H^j(\lim_I \operatorname{colim}_i A^*_{i,I}) \to \lim_I H^j(\operatorname{colim}_i A^*_{i,I}) \to 0. \tag{2.13}$$

But for each fixed pair (i, I), the complex $A^*_{i,I}$ is split; the same is therefore also true of the colimit, which thus has zero cohomology. Now (2.13) shows that (2.12) is exact, and so

$$\pi_s \Gamma(\mathbb{G}^q, \tau_t \mathcal{E}) = \begin{cases} \pi_t \Gamma(\mathbb{G}^q, \mathcal{E}) & s = t \\ 0 & s < t \end{cases}$$

That is, $\Gamma(\mathbb{G}^q, \tau_t \mathcal{E}) = \Sigma^t \pi_t \Gamma(\mathbb{G}^q, \mathcal{E})$ as required. □

Remark 2.49. The same proof shows that $H^*_{\mathrm{cond}}(T, \mathrm{Cont}(-, M)) = 0$ in positive degrees, for any profinite set S and profinite abelian group M with a presentation as a directed colimit $M = \lim_{\mathbb{N}^{\mathrm{op}}} M_n$ satisfying the Mittag-Leffler condition.

Remark 2.50. There are alternative ways to obtain a proétale or condensed object from $\mathbf{E}$, and it is sometimes useful to make use of these. Since we have proven that $\mathcal{E} = \nu^p \mathcal{E}^\delta$ is Kan extended from the étale classifying site, it is uniquely determined as an object of $\widehat{\mathcal{S}h}(BG_{\mathrm{proet}}, \mathcal{S}p_{\mathbf{K}})$ by the following properties:

(i) it has underlying $\mathbf{K}$-local spectrum $\Gamma(\mathbb{G}/*, \mathcal{E}) = \mathbf{E}$.

(ii) its homotopy sheaves are $\mathrm{Cont}_{\mathbb{G}}(-, \pi_t \mathbf{E})$, where $\pi_t \mathbf{E}$ is viewed as a topological $\mathbb{G}$-module with its adic topology.

[3] According to [Wei94, Prop. 3.5.8], this is true whenever the system $A^j_{i,*} \to A^j_{i,*}$ is Mittag-Leffler for each fixed j. But we have already noted that any inclusion $J \subset I$ induces a surjection $A^j_{i,J} \twoheadrightarrow A^j_{i,I}$.

Chapter 3

Descent for modules and the Picard spectrum

In the previous section, we showed that Morava E-theory defines a hypercomplete sheaf of spectra $\mathcal{E}$ on the proétale classifying site of $\mathbb{G}$. Our next aim is to improve this to a statement about its module ∞-category and therefore its Picard spectrum. The main result of this section is the construction of a hypercomplete sheaf of connective spectra $\mathfrak{pic}(\mathcal{E})$, with global sections $\Gamma\mathfrak{pic}(\mathcal{E}) = \mathfrak{pic}_h := \mathfrak{pic}(\mathcal{S}p_{\mathbf{K}})$. The functoriality of the construction via the proétale site will allow us to compare the resulting descent spectral sequence to the $\mathbf{K}$-local $\mathbf{E}$-Adams spectral sequence, including differentials.

3.1 Descent for $K(h)$-local module categories

A similar strategy to that of Proposition 2.37 fails for the sheaf of module categories, since it is not clear that filtered colimits in $\mathrm{Pr}^{L,\mathrm{smon}}$ are exact; moreover, we would need an analogue of [CM21, Theorem 4.16]. This section is devoted to the following result, which builds on the descent results of [MS16]:

Theorem 3.1. *The presheaf $\nu^p\mathrm{Mod}_{\mathcal{E}^\delta,\mathbf{K}} : B\mathbb{G}^{op}_{\mathrm{proet}} \to \mathrm{Pr}^{L,\mathrm{smon}}$ satisfies hyperdescent.*

Remark 3.2. By Proposition 2.5, $\nu^p\mathrm{Mod}_{\mathcal{E}^\delta,\mathbf{K}} = \mathrm{Mod}_{\nu^p\mathcal{E}^\delta,\mathbf{K}} = \mathrm{Mod}_{\mathcal{E},\mathbf{K}}$. Thus taking endomorphisms of the unit gives an alternative proof of Proposition 2.37 as an immediate corollary.

We will deduce Theorem 3.1 from a more general result, true for any descendable Galois extension in a sufficiently nice stable homotopy theory:

Theorem 3.3. *Let $\mathcal{C}$ be a compactly assembled stable homotopy theory, and G a profinite group. Suppose that $\mathbf{1} \to A$ is a descendable G-Galois extension in $\mathcal{C}$, corresponding to $\mathcal{A}^\delta \in \mathcal{S}h(BG_{\mathrm{et}}, \mathcal{C})$. Then the presheaf*

$$\nu^p\mathrm{Mod}_{\mathcal{A}^\delta}(\mathcal{C}) \in \mathcal{P}(BG_{\mathrm{proet}}, \mathrm{Pr}^{L,\mathrm{smon}})$$

satisfies hyperdescent.

Before giving the proof, let us make a few remarks.

Remark 3.4. (*i*) By [Mat16, Proposition 6.15], descendable Galois extensions are faithful.

(*ii*) Recall [SAG, §21.1.2] that a presentable ∞-category is said to be *compactly assembled* if it is a retract in Pr^L of a compactly generated presentable ∞-category; if $\mathcal{C}$ is stable, this is equivalent to being dualisable in the symmetric monoidal structure on $\mathrm{Pr}^L_{\mathrm{st}}$.

(*iii*) For the proof, we do *not* require that $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_\omega)$. In particular, $\mathcal{S}p_{\mathbf{K}}$ is compactly generated (though not by the unit) and so is an example of such an ∞-category, and Theorem 3.1 follows since $\mathbf{1}_{\mathbf{K}} \to \mathbf{E}$ is descendable.

The functor $\mathfrak{pic}$ preserves limits of symmetric monoidal ∞-categories by [MS16, Proposition 2.2.3], and so taking the composition $\mathfrak{pic}(\mathcal{E}) := \mathfrak{pic} \circ \mathrm{Mod}_{\mathcal{E},\mathbf{K}}$ we obtain the first part of our main result:

Corollary 3.5. *There is a hypercomplete sheaf* $\mathfrak{pic}(\mathcal{E}) \in \widehat{\mathcal{S}h}(B\mathbb{G}_{\mathrm{proet}}, \mathcal{S}p_{\geq 0})$ *having*

$$\Gamma(\mathbb{G}, \mathfrak{pic}(\mathcal{E})) \simeq \mathfrak{pic}(\mathrm{Mod}_{\mathbf{E},\mathbf{K}}) \qquad \textit{and} \qquad \Gamma(*, \mathfrak{pic}(\mathcal{E})) \simeq \mathfrak{pic}(\mathcal{S}p_{\mathbf{K}}) = \mathfrak{pic}_h.$$

In particular, we get a conditionally convergent spectral sequence

$$E_2^{s,t} = H^s(B\mathbb{G}_{\mathrm{proet}}, \pi_t \mathfrak{pic}(\mathcal{E})) \implies \pi_{t-s}\mathfrak{pic}_h. \tag{3.1}$$

Remark 3.6. In Section 3.2, we will evaluate the homotopy sheaves $\pi_t\mathfrak{pic}(\mathcal{E})$ and hence identify the E_2-page with continuous cohomology $H^s(\mathbb{G}, \pi_t\mathfrak{pic}(\mathbf{E}))$.

We begin the proof of Theorem 3.1 with the following observation, stated for later use in slightly greater generality than is needed for this section.

Lemma 3.7. *Let* $\mathcal{C}^\otimes$ *be a symmetric monoidal* ∞*-category with geometric realisations, and* $1 \leq k \leq \infty$. *Suppose that* $A \in \mathcal{C}$ *is an* $\mathbb{E}_k$*-algebra, and that* $B \in \mathrm{CAlg}(\mathcal{C})$ *is such that*

$$\mathcal{C} \simeq \lim\Big(\mathrm{Mod}_B(\mathcal{C}) \rightrightarrows \mathrm{Mod}_{B\otimes B}(\mathcal{C}) \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} \cdots \Big). \tag{3.2}$$

Then

$$\mathrm{RMod}_A(\mathcal{C}) \simeq \lim\Big(\mathrm{RMod}_{A\otimes B}(\mathcal{C}) \rightrightarrows \mathrm{RMod}_{A\otimes B\otimes B}(\mathcal{C}) \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} \cdots \Big).$$

Remark 3.8. We call $B \in \mathrm{CAlg}(\mathcal{C})$ a *descent algebra* if (3.2) holds.

Proof. Write $B' := A \otimes B$; this is itself an $\mathbb{E}_k$-algebra. We will verify the hypotheses of the Barr-Beck-Lurie theorem. Namely,

(*i*) $(-) \otimes_A B' \simeq (-) \otimes B$ is conservative, by virtue of the equivalence (3.2).

(*ii*) $\mathrm{RMod}_A(\mathcal{C})$ has limits of B'-split cosimplicial objects: given $M^\bullet : \mathbf{\Delta} \to \mathrm{RMod}_A(\mathcal{C})$ with $M^\bullet \otimes_A B'$ split, we can form the limit M in $\mathcal{C}$, by the descendability assumption. Since $\mathrm{RMod}_A(\mathcal{C}) \subset \mathcal{C}$ is closed under cosifted (in fact, all) limits, M is also a limit in $\mathrm{RMod}_A(\mathcal{C})$. This limit is clearly preserved by $(-) \otimes_A B'$. □

Lemma 3.9. *Let $\mathcal{C}$ be a compactly assembled stable homotopy theory and $A \in \mathrm{Alg}(\mathcal{C})$. Then $\mathrm{Mod}_A(\mathcal{C})$ is compactly assembled.*

Proof. We need to exhibit $\mathrm{Mod}_A(\mathcal{C})$ as a retract in Pr^L of some compactly generated ∞-category. To this end, recall [SAG, Theorem 21.1.2.10] that $\mathcal{C}$ is a retract in Pr^L of $\mathrm{Ind}(\mathcal{C}) = \mathrm{Ind}_\omega(\mathcal{C})$. More precisely, write $\varinjlim\colon \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ for the Ind-extension of the identity; this is left adjoint to the Yoneda embedding y, and admits a further left adjoint $\widehat{y}\colon \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ under the assumption that $\mathcal{C}$ is compactly assembled. In principle, $\varinjlim$ can be made symmetric monoidal with respect to Day convolution on $\mathrm{Ind}(\mathcal{C})$, which implies that $\mathrm{Mod}_A(\mathcal{C})$ is also compactly assembled. To make this precise we must proceed with some care, as $\mathcal{C}$ is not small.

Let κ be a regular cardinal such that $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_\kappa)$. In particular $\mathbf{1} \in \mathcal{C}^\kappa$, and $\mathcal{C} = \mathrm{Ind}_\kappa(\mathcal{C}^\kappa)$. At the expense of enlarging κ, we can moreover assume that $\mathcal{C}^\kappa$ is closed under the tensor product on $\mathcal{C}$. Indeed, for this it suffices to choose $\kappa' > \kappa$ large enough that each object $C_1 \otimes C_2$, where $C_1, C_2 \in \mathcal{C}^\kappa$, is κ'-compact: then $C_1' \otimes C_2' \in \mathcal{C}^{\kappa'}$ for each $C_1', C_2' \in \mathcal{C}^{\kappa'}$, since this may be written as a κ'-small colimit of κ'-compact objects. Enlarging κ further, we assume that $A \in \mathcal{C}^\kappa$.

Since $\mathcal{C}$ is compactly assembled, the left adjoint $\varinjlim\colon \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint $\widehat{y}$. Since $\widehat{y}$ preserves colimits and $\mathcal{C} = \mathrm{Ind}_\kappa(\mathcal{C}^\kappa)$, this factors through an adjunction

$$\widehat{y} : \mathcal{C} \rightleftarrows \mathrm{Ind}(\mathcal{C}^\kappa) : \varinjlim .$$

Moreover, the right adjoint is the Ind-extension of the inclusion $\mathcal{C}^\kappa \hookrightarrow \mathcal{C}$, and can therefore be canonically made symmetric monoidal for Day convolution on the source by [HA, Corollary 4.8.1.14]. Since $\varinjlim yA = A$, we obtain a functor

$$\varinjlim : \mathrm{Mod}_{yA}(\mathrm{Ind}(\mathcal{C}^\kappa)) \to \mathrm{Mod}_A(\mathcal{C}),$$

which preserves limits since these are computed on underlying objects. Both source and target are presentable, and so we obtain a left adjoint $\widehat{y}$ at the level of modules. This gives the desired retraction in Pr^L. □

Equipped with this and Lemma 2.22, we can now prove the main theorem of this section. I am grateful to Dustin Clausen for the strategy of the following result, which forms the key complement to the results of [Mat16].

Proposition 3.10. *Let $\mathcal{C}$ be a compactly assembled stable homotopy theory, and G a profinite group. Suppose that $\mathbf{1} \to A$ is a descendable G-Galois extension in $\mathcal{C}$, corresponding to $\mathcal{A}^\delta \in \mathcal{S}h(BG_{\mathrm{et}}, \mathcal{C})$. Then the restriction*

$$\nu^p \mathrm{Mod}_{\mathcal{A}^\delta}(\mathcal{C})|_{\mathrm{Free}_G} \in \mathcal{P}(\mathrm{Free}_G, \mathrm{Pr}^L)$$

is a hypercomplete sheaf.

Proof. We do this in a number of steps.

(1) As noted in Section 2.2.1, any G-set is covered by a free one and every free G-set is split. A consequence is that for any free G-set S, the functor $S' \mapsto S'/G$ is an equivalence $(BG_{\mathrm{proet}})_{/S} \simeq \{G\} \times \mathrm{Profin}_{/(S/G)}$, since any G-set over S is itself free. Thus suppose $T = \lim_i T_i$ is a profinite set over S/G, and choose a convergent sequence of neighbourhoods $U_j \subset G$ of the identity; applying Lemmas 2.5 and 2.22 we deduce the equivalences

$$\begin{aligned}
\nu^p \mathrm{Mod}_{\mathcal{A}^\delta}(T \times G) &= \varinjlim_{i,j} \mathrm{Mod}_{\mathcal{A}^\delta(T_i \times G/U_j)} \\
&\simeq \varinjlim_{i,j} \mathrm{Mod}_{\mathcal{A}^\delta(T_i) \otimes \mathcal{A}^\delta(G/U_j)} \\
&\simeq \varinjlim_{i} \mathrm{Mod}_{\varinjlim_j \mathcal{A}^\delta(T_i) \otimes \mathcal{A}^\delta(G/U_j)} \\
&\simeq \varinjlim_{i} \mathrm{Mod}_{\mathcal{A}^\delta(T_i) \otimes A} \\
&\simeq \varinjlim_{i} \mathrm{Mod}_{\prod_{T_i} A} \\
&\simeq \varinjlim_{i} \prod_{T_i} \mathrm{Mod}_A .
\end{aligned}$$

Under the aforementioned equivalence we think of this as a presheaf on $\mathrm{Profin}_{/(S/G)}$: that is, if T is a profinite set over S/G, then

$$T = \lim_i T_i \mapsto \varinjlim_i \prod_{T_i} \mathrm{Mod}_A .$$

(2) Since T_i is a finite set, $\nu^p \mathrm{Mod}_{\mathcal{A}}(T_i \times G) \simeq \prod_{T_i} \mathrm{Mod}_A \simeq \mathcal{S}h(T_i, \mathrm{Mod}_A)$; we claim that the same formula holds for arbitrary T. Writing $T = \lim_i T_i$, it will suffice to prove that the adjunction

$$q^* : \varinjlim \mathcal{S}h(T_i, \mathrm{Mod}_A) \leftrightarrows \mathcal{S}h(T, \mathrm{Mod}_A) : q_*,$$

induced by the adjunctions $(q_i)^* \dashv (q_i)_*$ for each projection $q_i : T \to T_i$, is an equivalence; then the claim follows by passing to a limit of finite T_i. In fact, since the adjunction is obtained by tensoring the adjunction

$$q^* : \varinjlim \mathcal{S}h(T_i) \leftrightarrows \mathcal{S}h(T) : q_*, \tag{3.3}$$

with Mod_A (combine [SAG, Remark 1.3.1.6 and Prop. 1.3.1.7]), it will suffice to prove that (3.3) is an equivalence.

Let us assume for notational convenience that the diagram T_i is indexed over a filtered poset J, which we may do without loss of generality. Then the claim is a consequence of the fact that the topology on T is generated by subsets $q_i^{-1}(x_i)$, where $q_i \colon T \to T_i$ is a finite quotient. Indeed, let

$\mathcal{O} \coloneqq \mathrm{Open}(T)$; subsets of the form $q_i^{-1}(x_i)$ form a clopen basis, and we will write $\mathcal{B} \subset \mathcal{O}$ for the full subcategory spanned by such. Note that

$$\mathcal{S}h(T) \simeq \mathcal{P}_\Sigma(\mathcal{B}),$$

where the right-hand side denotes the full subcategory of presheaves that send binary coproducts to products. On the other hand, an object of $\varinjlim \mathcal{S}h(T_i)$ is a Cartesian section of the fibration determined by $i \mapsto \mathcal{S}h(T_i)$; abusively, we will denote such an object by $(\mathcal{F}_i)$, where $\mathcal{F}_i \in \mathcal{S}h(T_i)$, leaving the coherence data implicit. Write also

$$(q_{j,\infty})^* : \mathcal{S}h(T_j) \rightleftarrows \varinjlim \mathcal{S}h(T_i) : (q_{j,\infty})_*$$

for the colimit adjunction. By applying Yoneda, one verifies that the map

$$\varinjlim (q_i)^*\mathcal{F}_i \to q^*((\mathcal{F}_i)),$$

obtained by adjunction from the maps

$$(q_{j,\infty})_*(\eta)\colon \mathcal{F}_j = (q_{j,\infty})_*((\mathcal{F}_i)) \to (q_{j,\infty})_* q_* q^*((\mathcal{F}_j)) = (q_j)_* q^*((\mathcal{F}_i))$$

as j varies, is an equivalence. Likewise, adjunct to $q^*((q_i)_*\mathcal{F}) \simeq \varinjlim (q_i)^*(q_i)_*\mathcal{F} \to \mathcal{F}$ is an equivalence

$$((q_i)_*\mathcal{F}) \xrightarrow{\sim} q_*\mathcal{F}.$$

Restricting to $\mathcal{B}$ (where no sheafification is required for forming the left adjoint), we will show that the unit and counit of the adjunction are equivalences. For the counit $q^*q_*\mathcal{F} \to \mathcal{F}$, this is clear:

$$\begin{aligned}
[q^*q_*\mathcal{F}]\,(q_i^{-1}(x_i)) &\simeq \left[\varinjlim_{j\geq i} (q_j)^*(q_j)_*\mathcal{F}\right](q_i^{-1}(x_i)) \\
&\simeq \varinjlim_{j\geq i} \left[(q_j)^*(q_j)_*\mathcal{F}(q_i^{-1}(x_i))\right] \\
&\simeq \varinjlim_{j\geq i} \left[(q_j)_*\mathcal{F}(q_{ij}^{-1}(x_i))\right] \\
&\simeq \varinjlim_{j\geq i} \left[\mathcal{F}(q_i^{-1}(x_i)\right] \simeq \mathcal{F}(q_i^{-1}(x_i)).
\end{aligned}$$

For the unit $(\mathcal{F}_j) \to q_*q^*(\mathcal{F}_j)$, it will suffice to prove for each i that the canonical map

$$\mathcal{F}_i \to (q_i)_* \varinjlim_{j\geq i} (q_j)^*\mathcal{F}_j \tag{3.4}$$

is an equivalence. But

$$\begin{aligned}\left[(q_i)_* \varinjlim_{j\geq i}(q_j)^*\mathcal{F}_j\right](x_i) &\simeq \left[\varinjlim_{j\geq i}(q_j)^*\mathcal{F}_j\right](q_i^{-1}(x_i))\\ &\simeq \varinjlim_{j\geq i}\left[(q_j)^*\mathcal{F}_j(q_i^{-1}x_i)\right]\\ &\simeq \varinjlim_{j\geq i}\left[\mathcal{F}_j(q_{ij}^{-1}x_i)\right]\\ &\simeq \varinjlim_{j\geq i}\left[(q_{ij})_*\mathcal{F}_j(x_i)\right],\end{aligned}$$

with respect to which (3.4) is the structure map for $j = i$. This is an equivalence: since each of the coherence maps

$$\mathcal{F}_i \to (q_{ij})_*\mathcal{F}_j$$

defining the colimit is an equivalence by definition of $\varprojlim \mathcal{S}h(T_i)$, the diagram $j \mapsto (q_{ij})_*\mathcal{F}_j(x_i)$ factors through the groupoid completion $J_{i/}^{\mathrm{gpd}}$. But since $J_{i/}$ is filtered, both inclusions

$$J_{i/} \hookrightarrow J_{i/}^{\mathrm{gpd}} \hookleftarrow \{i\}$$

are cofinal by [HTT, Corollary 4.1.2.6].

(3) We are left to prove that $T \mapsto \mathcal{S}h(T, \mathrm{Mod}_A) \in \mathrm{Pr}^{L,\mathrm{smon}}$ is a hypercomplete sheaf on $\mathrm{Profin}_{/(S/G)}$. This is precisely the content of [Hai22, Theorem 0.5], noting that

- (i) limits in $\mathrm{Pr}^{L,\mathrm{smon}}$ are computed in Cat_∞;
- (ii) $\mathcal{C}$, and so $\mathrm{Mod}_A(\mathcal{C})$, is compactly assembled;
- (iii) any profinite set T has homotopy dimension zero by [HTT, Theorem 7.2.3.6 and Remark 7.2.3.3], so that Postnikov towers in $\mathcal{S}h(T)$ converge: $\mathcal{S}h(T) \simeq \lim_n \mathcal{S}h(T, \mathcal{S}_{\leq n})$ by [HTT, Theorem 7.2.1.10]. $\square$

The proof of Theorem 3.3 follows by combining the previous proposition with [Mat16, Proposition 3.22]:

Proof (Prop. 3.3). Let $T_\bullet \twoheadrightarrow T_{-1} = S$ be a hypercovering in BG_{proet}, and form the diagram

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\downarrow\!\downarrow\!\downarrow & & \downarrow\!\downarrow\!\downarrow & & \downarrow\!\downarrow\!\downarrow & & \\
S\times G\times G & \longleftarrow & T_0\times G\times G & \leftleftarrows & T_1\times G\times G & \Lleftarrow & \cdots\\
\downarrow\!\downarrow & & \downarrow\!\downarrow & & \downarrow\!\downarrow & & \\
S\times G & \longleftarrow & T_0\times G & \leftleftarrows & T_1\times G & \Lleftarrow & \cdots\\
\downarrow & & \downarrow & & \downarrow & & \\
S & \longleftarrow & T_0 & \leftleftarrows & T_1 & \Lleftarrow & \cdots
\end{array}$$

To prove descent for the hypercover, it is enough to show descent for each column and for each non-negative row. For each such row we are in the context of Proposition 3.10, and so obtain a limit diagram after applying $\nu^p\mathrm{Mod}_{\mathcal{A}^\delta}$. On the other hand, writing $T_j = \lim T_{ij}$ and applying $\nu^p\mathrm{Mod}_{\mathcal{A}^\delta}$ to a column we obtain the complex

$$\varinjlim_i \mathrm{Mod}_{\mathcal{A}^\delta(T_{ij})} \longrightarrow \varinjlim_{i,k} \mathrm{Mod}_{\mathcal{A}^\delta(T_{ij}\times G/U_k)} \rightrightarrows \varinjlim_{i,k} \mathrm{Mod}_{\mathcal{A}^\delta(T_{ij}\times G/U_k\times G/U_k)} \substack{\to\\[-1em] \to \\[-1em] \to} \cdots$$

Using Lemma 2.5 and the equivalences

$$\varinjlim_{i,k} \mathcal{A}^\delta(T_{ij}\times (G/U_k)^n) \simeq \mathcal{A}(T_j)\otimes A^{\otimes n}$$

one identifies this with the complex

$$\mathrm{Mod}_{\mathcal{A}(T_j)} \longrightarrow \mathrm{Mod}_{\mathcal{A}(T_j)\otimes A} \rightrightarrows \mathrm{Mod}_{\mathcal{A}(T_j)\otimes A\otimes A} \substack{\to\\[-1em] \to \\[-1em] \to} \cdots . \tag{3.5}$$

When $T_j = *$, this is the complex

$$\mathcal{C} \longrightarrow \mathrm{Mod}_A \rightrightarrows \mathrm{Mod}_{A\otimes A} \substack{\to\\[-1em] \to \\[-1em] \to} \cdots$$

which is a limit diagram according to [Mat16]. For general T, (3.5) is a limit diagram by combining the $T = *$ case with Lemma 3.7. $\square$

3.2 The Picard spectrum as a proétale spectrum

In Section 3.1 we showed that $\mathbf{K}$-local modules determine a sheaf of symmetric-monoidal ∞-categories on the site $B\mathbb{G}_{\mathrm{proet}}$. Since limits in $\mathrm{Pr}^{L,\mathrm{smon}}$ are computed in $\mathrm{Cat}_\infty^{\mathrm{smon}}$, the functor

$$\mathfrak{pic} : \mathrm{Pr}^{L,\mathrm{smon}} \to \mathcal{S}p_{\geq 0},$$

preserves them [MS16, Prop. 2.2.3], and so the composite $\mathfrak{pic}(\mathcal{E}) = \mathfrak{pic}\circ \mathrm{Mod}_{\mathcal{E},\mathbf{K}}$ is immediately seen to be a sheaf of connective spectra. As a result, the 0-stem in its descent spectral sequence (3.1) converges conditionally to Pic_h. Its E_2-page consists of cohomology of the homotopy sheaves $\pi_*\mathfrak{pic}(\mathcal{E})$, and as in Corollary 2.45 we'd like to identify this with group cohomology with coefficients in the continuous $\mathbb{G}$-module $\pi_*\mathfrak{pic}(\mathbf{E})$. In order to deduce this once again from [BS14, Lemma 4.3.9], we need to show that

$$\pi_t\mathfrak{pic}(\mathcal{E}) = \mathrm{Cont}_{\mathbb{G}}(-, \pi_t\mathfrak{pic}(\mathbf{E})). \tag{3.6}$$

for $t = 0, 1$; for $t \geq 2$ the result follows from Lemma 2.42 and the isomorphism $\pi_t\mathfrak{pic}(A) \simeq \pi_{t-1}A$, natural in the ring spectrum A. The first aim of this section is to prove (3.6). Having done this, we will evaluate the resulting spectral sequence.

Theorem 3.11. *The homotopy sheaves of the Picard sheaf are given by*

$$\pi_t\mathfrak{pic}(\mathcal{E}) = \mathrm{Cont}_{\mathbb{G}}\left(-, \pi_t\mathfrak{pic}(\mathbf{E})\right) = \begin{cases} \mathrm{Cont}_{\mathbb{G}}\left(-, \mathrm{Pic}(\mathbf{E})\right) & t=0 \\ \mathrm{Cont}_{\mathbb{G}}\left(-, (\pi_0\mathbf{E})^{\times}\right) & t=1 \\ \mathrm{Cont}_{\mathbb{G}}\left(-, \pi_{t-1}\mathbf{E}\right) & t\geq 2 \end{cases} \tag{3.7}$$

Before proving this, we give the desired identification of the E_2-page in (3.1):

Corollary 3.12. *The starting page of the descent spectral sequence for the Picard spectrum $\mathfrak{pic}(\mathcal{E})$ is given by continuous group cohomology:*

$$E_2^{s,t} = H^s(\mathbb{G}, \pi_t\mathfrak{pic}(\mathbf{E})). \tag{3.8}$$

We now focus on the proof Theorem 3.11.

Proof. As in Lemma 2.42 we appeal to [BS14, Lemma 4.3.9(4)]. We have already noted that the requisite conditions hold for the sheaves $\pi_t\mathfrak{pic}(\mathcal{E}) \simeq \pi_{t-1}\mathcal{E}$ when $t\geq 2$; all that remains to justify is what happens at $t=0$ and 1. But $\pi_0\mathfrak{pic}(\mathcal{E}) \simeq \mathrm{Cont}_{\mathbb{G}}(-, \mathbb{Z}/2)$ certainly satisfies the hypotheses of [BS14, Lemma 4.3.9], while $\pi_1\mathfrak{pic}(\mathcal{E}) = \mathrm{Cont}_{\mathbb{G}}(-, (\pi_0\mathbf{E})^{\times})$, and $(\pi_0\mathbf{E})^{\times}$ is the limit of finite $\mathbb{G}$-modules $(\pi_0\mathbf{E}/I)^{\times}$. □

Remark 3.13. As in Chapter 2, it will suffice to prove that the homotopy sheaves take this form on $\mathrm{Free}_{\mathbb{G}}$. If T is a finite set, then $\mathcal{E}(T\times\mathbb{G}) \simeq \bigoplus_T \mathbf{E}$, and so

$$\mathrm{Cont}(T, \mathrm{Pic}(\mathbf{E})) \cong \bigoplus_T \mathrm{Pic}(\mathbf{E}) \cong \mathrm{Pic}(\mathrm{Mod}_{\mathcal{E}(T\times\mathbb{G}),\mathbf{K}}).$$

If T is an arbitrary profinite set, the isomorphisms above induce a canonical map

$$\chi : \mathrm{Cont}(T, \mathrm{Pic}(\mathbf{E})) \to \mathrm{Pic}(\mathrm{Mod}_{\mathcal{E}(T\times\mathbb{G}),\mathbf{K}}),$$

which may be described explicitly as follows: any continuous map $f\colon T\to \mathrm{Pic}(\mathbf{E}) = \mathbb{Z}/2\{\Sigma\mathbf{E}\}$ defines a clopen decomposition $T = T^0 \sqcup T^1$, with $T^i = f^{-1}(\Sigma^i\mathbf{E})$. Projecting to finite quotients gives $T_i = T_i^0 \sqcup T_i^1$ for i sufficiently large, and since $\mathcal{E}(T\times\mathbb{G}) = L_{\mathbf{K}} \varinjlim_i \bigoplus_{T_i}\mathbf{E}$ we set

$$\chi(f) = L_{\mathbf{K}} \varinjlim_i \left(\bigoplus_{T_i^0} \mathbf{E} \oplus \bigoplus_{T_i^1} \Sigma\mathbf{E} \right) \in \mathrm{Pic}(\mathrm{Mod}_{\mathcal{E}(T\times\mathbb{G}),\mathbf{K}}). \tag{3.9}$$

We will deduce Theorem 3.11 by showing that the map χ is an isomorphism. The key point will be the following:

Proposition 3.14. *Let T be a profinite set and $X \in \mathrm{Pic}(\mathrm{Mod}_{\mathcal{E}(T\times\mathbb{G}),\mathbf{K}})$. Then X is in the image of χ.*

Given this, it is straightforward to prove the main result of the section:

Proof (Theorem 3.11). For $t \geq 1$, this follows from the equivalence

$$\Omega\mathfrak{Pic}\left(\mathrm{Mod}_{\mathcal{E},\mathbf{K}}\right) \simeq \mathrm{GL}(\mathcal{E}).$$

and the identification of the homotopy sheaves $\pi_t\mathcal{E}$ in Lemma 2.42.

For $t = 0$, Proposition 3.14 implies that χ is surjective, while injectivity is clear from (3.9). This yields isomorphisms of sheaves on $\mathrm{Free}_{\mathbb{G}}$,

$$\mathrm{Cont}_{\mathbb{G}}(-, \mathrm{Pic}(\mathbf{E})) \simeq \mathrm{Cont}((-)/\mathbb{G}, \mathrm{Pic}(\mathbf{E})) \simeq \pi_0\mathfrak{pic}(\mathcal{E}). \qquad \square$$

To prove Proposition 3.14 it will be convenient to work in the context of sheaves of **E**-modules, using the equivalence

$$\mathrm{Mod}_{\mathcal{E}(T\times\mathbb{G}),\mathbf{K}} \simeq \mathcal{S}h(T, \mathrm{Mod}_{\mathbf{E},\mathbf{K}})$$

from the proof of Theorem 3.3. We begin by recording some basic lemmas.

Lemma 3.15 ([Stacks], Tag 0081)**.** *Let T be a topological space, and A a set. The constant sheaf A_T on the T takes the form*

$$U \mapsto \mathrm{LC}\left(U, A\right), \tag{3.10}$$

that is, locally constant functions $U \to A$.

Lemma 3.16. *For any $X \in \mathcal{S}h(T, \mathrm{Mod}_{\mathbf{E},\mathbf{K}})$,*

$$[\mathbf{E}_T, X] \simeq \mathrm{Hom}(\pi_*\mathbf{E}_T, \pi_*X)$$

Proof. We have isomorphisms

$$\begin{aligned}
[\mathbf{E}_T, X] &\simeq \pi_0\mathrm{Map}(\mathbf{E}_T, X) \\
&\simeq \pi_0\Gamma X \\
&\simeq \Gamma\pi_0 X \\
&\simeq \mathrm{Hom}(\pi_*\mathbf{E}_T, \pi_*X),
\end{aligned}$$

because the descent spectral sequence for ΓX collapses immediately to the 0-line. Indeed, profinite sets have homotopy dimension zero, and therefore cohomological dimension zero [HTT, Corollary 7.2.2.30]. $\square$

Remark 3.17. After sheafification, postcomposition with the functor $\mathbf{K}_*^{\mathbf{E}}(-) := \pi_*(\mathbf{K}\otimes_{\mathbf{E}}-)\colon \mathrm{Mod}_{\mathbf{E},\mathbf{K}} \to \mathrm{Mod}_{\pi_*\mathbf{K}}$ defines a functor

$$\mathbf{K}_*^{\mathbf{E}} : \mathcal{S}h(T, \mathrm{Mod}_{\mathbf{E},\mathbf{K}}) \to \mathcal{S}h(T, \mathrm{Mod}_{\pi_*\mathbf{K}}).$$

Similarly to [HMS94; BR05], our strategy will be to use monoidality of this functor to deduce results about invertible objects in $\mathcal{S}h(T, \mathrm{Mod}_{\mathbf{E},\mathbf{K}})$ by first proving them at the level of $\mathbf{K}_*^{\mathbf{E}}(-)$.

Lemma 3.18. *Let T be a profinite set, and $X \in \mathrm{Pic}(\mathcal{S}h(T, \mathrm{Mod}_{\mathbf{E},\mathbf{K}}))$. Then $\mathbf{K}^{\mathbf{E}}_* X \in \mathrm{Pic}(\mathcal{S}h(T, \mathrm{Mod}_{\pi_*\mathbf{K}}))$.*

Proof. The functor $\mathbf{K}^{\mathbf{E}}_*(-)$ on spectra is (strict) monoidal: this is [BR05, Corollary 33]. It follows that the induced functor

$$\mathbf{K}^{\mathbf{E}}_* : \mathcal{S}h(T, \mathrm{Mod}_{\mathbf{E},\mathbf{K}}) \to \mathcal{S}h(T, \mathrm{Mod}_{\pi_*\mathbf{K}})$$

is also strictly monoidal, and hence preserves invertible objects. □

Lemma 3.19. *Let T be a profinite set and k a field graded by an abelian group A. Then any $X \in \mathrm{Pic}(\mathcal{S}h(T, \mathrm{Mod}_k))$ is locally free of rank one.*

Proof. Since $\mathcal{S}h(T, \mathrm{Mod}_k) = \mathrm{Mod}_{k_T}$ for k_T the constant sheaf, the result follows from [Stacks, Tag 0B8M] in the ungraded case. In the graded case, [Stacks, Tag 0B8K] still shows that X is locally a summand of a finite free module (though not necessarily one in degree zero); since k is a field it follows that X is locally a shift of k_T by some $a \in A$. □

Proof (Proposition 3.14). Note that under the equivalence

$$\mathrm{Mod}_{\mathcal{E}(T\times\mathbb{G}),\mathbf{K}} \simeq \mathcal{S}h(T, \mathrm{Mod}_{\mathbf{E},\mathbf{K}}),$$

the image of χ in $\mathcal{S}h(T, \mathrm{Mod}_{\mathbf{E},\mathbf{K}})$ consists of those invertible sheaves that are locally free of rank one: indeed, T has a basis of finite clopen covers $T = \bigsqcup_{x\in T_i} U_x$, and $\chi(\mathrm{Cont}(T_i, \mathrm{Pic}(\mathbf{E})))$ is the subgroup of invertible sheaves that are constant along $\bigsqcup_{x\in T_i} U_x$.

We will deduce that any $X \in \mathrm{Pic}(\mathcal{S}h(T, \mathrm{Mod}_{\mathbf{E},\mathbf{K}}))$ is locally free of rank one by proving in turn each of the following statements:

(i) $\mathbf{K}^{\mathbf{E}}_* X \in \mathcal{S}h(T, \mathrm{Mod}_{\pi_*\mathbf{K}})$ is locally free of rank one,

(ii) for every $i_0, \dots, i_{n-1} \geq 1$, the sheaf $\pi_*(X/(p^{i_0}, \dots, u_{n-1}^{i_{n-1}})) \in \mathcal{S}h(T, \mathrm{Mod}_{\pi_*\mathbf{E}})$ is locally constant with value $\Sigma^{\varepsilon}\pi_*\mathbf{E}/(p^{i_0}, \dots, u_{n-1}^{i_{n-1}})$ [1],

(iii) $X \in \mathcal{S}h(T, \mathrm{Mod}_{\mathbf{E},\mathbf{K}})$ is locally free of rank one.

Essentially, this follows the proof of [BR05, Theorem 3.7].

(i) This follows immediately by combining Lemmas 3.18 and 3.19. Since all our claims are local, we will assume for simplicity that $\mathbf{K}^{\mathbf{E}}_* X \simeq \pi_*\mathbf{K}_T$.

(ii) For $i_0, \dots, i_{n-1} \geq 1$, we first show that that $\pi_*(X/(p^{i_0}, \dots, u_{n-1}^{i_{n-1}}))$ admits a surjection by $\pi_*\mathbf{E}_T$. This follows by induction on $\sum_j i_j$, with the base case being (i). For the induction step, note that $\mathcal{S}h(T, \mathrm{Mod}_{\mathbf{E},\mathbf{K}})$ is tensored over $\mathrm{Mod}_{\mathbf{E},\mathbf{K}}$, and so the cofibre sequences in [BR05,

[1]Here $\varepsilon \in \{0, 1\}$ may vary on T.

Lemma 34] give rise to cofibre sequences

$$\begin{aligned} X/(p^{i_0}, \ldots, u_{n-1}^{i_{n-1}}) &\xrightarrow{u_j} X/(p^{i_0}, \ldots, u_{n-1}^{i_{n-1}}) \\ &\to X/(p^{i_0}, \ldots, u_j, \ldots u_{n-1}^{i_{n-1}}) \oplus \Sigma X/(p^{i_0}, \ldots, u_j, \ldots u_{n-1}^{i_{n-1}}) \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} X/(p^{i_0}, \ldots, u_j^{i_j-1}, \ldots, u_{n-1}^{i_{n-1}}) &\xrightarrow{u_j} X/(p^{i_0}, \ldots, u_j^{i_j}, \ldots, u_{n-1}^{i_{n-1}}) \\ &\to X/(p^{i_0}, \ldots, u_j, \ldots u_{n-1}^{i_{n-1}}) \end{aligned} \tag{3.12}$$

for $i_j > 1$; in particular, using (3.12) and the inductive hypothesis, we can assume that $\pi_*(X/(p^{i_0}, \ldots, u_{n-1}^{i_{n-1}}))$ is concentrated in even degrees. Thus (3.11) implies we have an exact sequence

$$\begin{aligned} 0 \to u_j \pi_{2t}(X/(p^{i_0}, \ldots, u_j^{i_j}, \ldots, u_{n-1}^{i_{n-1}})) &\xrightarrow{u_j} \pi_{2t}(X/(p^{i_0}, \ldots, u_j^{i_j}, \ldots, u_{n-1}^{i_{n-1}})) \\ &\to \pi_{2t}(X/(p^{i_0}, \ldots, u_j, \ldots, u_{n-1}^{i_{n-1}})) \to 0 \end{aligned} \tag{3.13}$$

for any t. Since $\pi_* \mathbf{E}_T$ is free, we can choose a lift α below:

$$\begin{tikzcd} & \pi_*(X/(p^{i_0}, \ldots, u_j^{i_j}, \ldots, u_{n-1}^{i_{n-1}})) \arrow[d] \\ \pi_* \mathbf{E}_T \arrow[ur, dashed, "\alpha"] \arrow[r, two heads] & \pi_*(X/(p^{i_0}, \ldots, u_j, \ldots, u_{n-1}^{i_{n-1}})) \end{tikzcd}$$

Now (3.13) implies that α is a surjection, since Nakayama's lemma implies that it is so on stalks. As in [BR05, Lemma 36], this implies (ii) by induction on $\sum i_j$.

(iii) Since X is invertible (and in particular dualisable),

$$\begin{aligned} \lim_{i_0, \ldots, i_{n-1}} (\mathbf{E}/(p^{i_0}, \ldots, u_{n-1}^{i_{n-1}}) \otimes_{\mathbf{E}} X) &\simeq \lim_{i_0 \ldots, i_{n-1}} (\mathbf{E}/(p^{i_0}, \ldots, u_{n-1}^{i_{n-1}})) \otimes_{\mathbf{E}} X \\ &\simeq X. \end{aligned}$$

Now (ii) implies that all $\lim^1$-terms vanish and that

$$\begin{aligned} \pi_* X &\cong \pi_* \lim_{i_0, \ldots, i_{n-1}} (\mathbf{E}/(p^{i_0}, \ldots, u_{n-1}^{i_{n-1}}) \otimes_{\mathbf{E}} X) \\ &\cong \lim_{i_0, \ldots, i_{n-1}} \pi_* \mathbf{E}_T/(p^{i_0}, \ldots, u_{n-1}^{i_{n-1}}) \cong \pi_* \mathbf{E}_T. \end{aligned}$$

By Lemma 3.16 we can lift the inverse to a map $\mathbf{E}_T \to X$, which is necessarily an equivalence. □

This completes the construction of the Picard sheaf $\mathfrak{pic}(\mathcal{E})$, and the proof that its homotopy sheaves take the desired form; Corollary 3.12 follows. We will now study the resulting descent spectral sequence, as we did for the descent spectral sequence of $\mathcal{E}$ in Chapter 2. The construction of the descent spectral sequence using a Postnikov-style filtration is useful in making the comparison with the Picard spectral sequence (3.8). Namely, we make the following observation:

Lemma 3.20. *Suppose that $\mathcal{C}$ is a site, and $\mathcal{D}$ a presentable ∞-category. Let $X \in \mathcal{S}h(\mathcal{C}, \mathcal{D})$, and $p, q\colon \mathcal{D} \to \mathcal{S}p_{\geq 0}$ two limit-preserving functors related by a natural equivalence $\tau_{[a,b]}p \simeq \tau_{[a,b]}q$. Then the descent spectral sequences for pX and qX' satisfy the following:*

(i) The E_2-pages agree in a range: $E_{2,pX}^{s,t} \simeq E_{2,qX}^{s,t}$ if $t \in [a,b]$.

(ii) Under the isomorphism induced by (i), we have $d_{r,pX}^{s,t} = d_{r,qX}^{s,t}$ if $2 \leq r \leq b - t + 1$.

Proof. This can be seen directly by comparing the Postnikov towers, since both claims depend only on their $[a,b]$-truncations. □

Remark 3.21. We are not asserting that $x \in E_{2,pX}$ survives to E_r if and only if the corresponding class in $E_{2,qX}$ does; this should really be taken as another assumption on the class x.

In our setting, the equivalence

$$\tau_{[t,2t-2]}\mathfrak{pic}(A) \simeq \tau_{[t,2t-2]}\Sigma A \tag{3.14}$$

of [MS16, Corollary 5.2.3], valid for $t \geq 3$ and *functorial* in the ring spectrum A, implies immediately the following corollary:

Proposition 3.22. *Let $A \in \mathcal{S}h(B\mathbb{G}_{\mathrm{proet}}, \mathrm{CAlg})$, and consider the two spectral sequences (2.7) and (3.8). Then*

(i) $E_2^{s,t} \simeq E_{2,+}^{s,t-1}$ if $t \geq 2$.

(ii) The differentials d_r and $d_{r,+}$ on $E_r^{s,t} \simeq E_{r,+}^{s,t-1}$ agree as long as $r \leq t-1$ (whenever both are defined).

Finally, we want to prove décalage for the sheaf $\mathfrak{pic}(\mathcal{E})$, as we did for the sheaf $\mathcal{E}$ itself. This will allow us to determine the differential d_t on classes in $E_2^{s,t}$ too.

Proposition 3.23. *Décalage of the Postnikov filtration induces an isomorphism between the following spectral sequences:*

$$\begin{aligned} E_2^{s,t} = \pi_{t-s}\Gamma\tau_t\mathcal{E} = H^s(\mathbb{G}, \pi_t\mathfrak{pic}(\mathbf{E})) &\implies \pi_{t-s}\mathfrak{pic}_h, \\ \check{E}_3^{2s-t,s} = \pi^s\pi_t\mathfrak{pic}(\mathbf{E}^{\otimes\bullet+1}) &\implies \pi_s\mathfrak{pic}_h. \end{aligned}$$

The first is the Picard spectral sequence (3.8), and the second is the Bousfield-Kan spectral sequence for the cosimplicial spectrum $\Gamma(\mathbb{G}^{\bullet+1}, \mathfrak{pic}(\mathcal{E})) = \mathfrak{pic}(\mathrm{Mod}_{\mathbf{E}^{\otimes\bullet+1},\mathbf{K}})$.

Remark 3.24. The Bousfield-Kan spectral sequence of Proposition 3.23 is by definition Heard's spectral sequence [Hea22, Theorem 6.13].

Proof. As in Proposition 2.48, this follows from Proposition 2.47 once we prove the following equivalences:

$$\Gamma(\mathbb{G}^q, \tau_0\mathfrak{pic}(\mathcal{E})) \simeq \pi_0\Gamma(\mathbb{G}^q, \mathfrak{pic}(\mathcal{E})),$$
$$\Gamma(\mathbb{G}^q, \Sigma\pi_1\mathfrak{pic}(\mathcal{E})) \simeq \tau_1\Gamma(\mathbb{G}^q, \mathfrak{pic}(\mathcal{E})).$$

Using Theorem 3.11, we compute that

$$H^{t-s}(\mathbb{G}^q, \mathrm{Pic}(\mathbf{E})) \simeq H^{t-s}(\mathbb{G}^q, (\pi_0\mathbf{E})^\times) = 0$$

for $t - s > 0$, since condensed cohomology with profinite coefficients vanishes (Remark 2.49). $\square$

Following the method of [MS16], we can now identify the first new differential in the Picard spectral sequence:

Corollary 3.25. *Suppose that $t \geq 2$ and $x \in E_2^{t,t}$. We abuse notation to identify x with its image in the starting page of the descent spectral sequence for $\mathcal{E}$, and assume that both classes survive to the respective E_t-pages. The first nonadditive differential on x in the Picard spectral sequence is*

$$d_t x = d_t^{ASS} x + x^2 \tag{3.15}$$

The ring structure in the right hand side is that of the **K***-local* **E***-Adams spectral sequence (2.7).*

Proof. The same proof that appears in [MS16] goes through: namely, this formula holds for the universal cosimplicial $\mathbb{E}_\infty$-ring having a class in $E_2^{t,t}$ of its Bousfield-Kan spectral sequence, and so for the cosimplicial spectrum $\mathcal{E}(\mathbb{G}^{\bullet+1}) = \mathbf{E}^{\otimes \bullet+1}$ too. $\square$

Chapter 4

Picard group computations

In the previous parts we constructed proétale models for the continuous action of $\mathbb{G}$ on Morava E-theory, its $\mathbf{K}$-local module ∞-category and its Picard spectrum. Respectively, these are $\mathcal{E}$ (constructed in Chapter 2), $\mathrm{Mod}_{\mathcal{E},\mathbf{K}}$ (in Section 3.1), and $\mathfrak{pic}(\mathcal{E})$ (in Section 3.2). We also described the resulting spectral sequences. In this section, we compute the Picard spectral sequence in the height one case and use this to give a new proof of the results of [HMS94] at all primes. As is common at height one, this splits into two cases: the case of odd primes, and the case $p = 2$. In both cases, the strategy is first to compute the descent spectral sequence for $\mathcal{E}$ (which by Proposition 2.48 is the $\mathbf{K}$-local $\mathbf{E}$-Adams spectral sequence) and then to use this to compute the Picard spectral sequence. However, the spectral sequences look somewhat different in the two cases. We start with some generalities, which are true uniformly in h and p.

4.1 Morava modules and the algebraic Picard group

A productive strategy for studying Pic_h is to compare it to a certain algebraic variant $\mathrm{Pic}_h^{\mathrm{alg}}$ first defined in [HMS94, §7]; we recall its definition below. In [Pst22], Pstrągowski shows that the algebraic approximation is precise if $p \gg h$, a consequence of the fact that in this case the vanishing line in the Adams spectral sequence occurs at the starting page. While there is always a vanishing line, when $p - 1 \mid h$ it appears only at a later page, since $\mathbb{G}$ no longer has finite cohomological dimension mod p. In this section we will show that this leaves a possibility for *exotic* Picard elements, and more importantly explain how to identify these in the Picard spectral sequence (Theorem 4.4).

To do so, we recall that the *completed E-homology* of a $\mathbf{K}$-local spectrum X is

$$\mathbf{E}_*^\vee X \coloneqq \pi_*(\mathbf{E} \otimes X) = \pi_* L_{\mathbf{K}}(\mathbf{E} \wedge X).$$

This is naturally a $\pi_*\mathbf{E}$-module: indeed $\mathbf{K}$-localisation is symmetric monoidal, and therefore sends the $\mathbf{E}$-module $\mathbf{E} \wedge X$ to a module over $L_{\mathbf{K}}\mathbf{E} = \mathbf{E}$. As we discuss below, the abelian group $\mathbf{E}^\vee_* X$ has significantly more structure, and is a crucial tool in understanding the $\mathbf{K}$-local category. It was first studied by Hopkins, Mahowald and Sadofsky in [HMS94], where it is denoted $\mathcal{K}_{h,*}(-)$; it is almost a homology theory, but fails to preserve infinite coproducts as a result of the failure of $\mathbf{K}$-localisation to be smashing. It is nevertheless an extremely effective invariant for Picard group computations, by virtue of the following theorem:

Theorem 4.1 ([HMS94], Theorem 1.3). *$M \in \mathcal{S}p_{\mathbf{K}}$ is invertible if and only if $\mathbf{E}^\vee_* M$ is a free $\pi_*\mathbf{E}$-module of rank one.*

In particular, Theorem 4.1 implies that completed E-homology is not a useful invariant of invertible $\mathbf{K}$-local spectra when we only remember its structure as a $\pi_*\mathbf{E}$-module: since $\pi_0\mathbf{E}$ is a Noetherian local ring, its Picard group is trivial. To get a more interesting invariant, we should remember the equivariant structure coming from the Morava action on $\mathbf{E}$. That is, if X is a $\mathbf{K}$-local spectrum then $\mathbb{G}$ acts on $\mathbf{E} \otimes X$ by acting on the first factor, and therefore acts on $\mathbf{E}^\vee_* X$. This action makes $\mathbf{E}^\vee_* X$ into a *twisted* $\mathbb{G}$-$\pi_*\mathbf{E}$*-module*, which means by definition that

$$g(a \cdot x) = ga \cdot gx$$

for $x \in \mathbf{E}^\vee_* X$, $a \in \pi_*\mathbf{E}$ and $g \in \mathbb{G}$. We will write $\mathrm{Mod}^{\mathbb{G}}_{\pi_*\mathbf{E}}$ for the category of twisted $\mathbb{G}$-$\pi_*\mathbf{E}$-modules.

Remark 4.2. For any twisted $\mathbb{G}$-$\pi_*\mathbf{E}$-module M, the $\mathbb{G}$-action is continuous for the I_h-adic topology: if $gx = y \in M$, then Section 4.1 implies that

$$g(x + ax') = y + ga \cdot gx' \in y + I_h^k M$$

for $a \in I_h^k$ and $x' \in M$, since the action of $\mathbb{G}$ on $\pi_*\mathbf{E}$ fixes the I_h-adic filtration; that is, $g^{-1}(y + I_h^k)$ contains the open neighbourhood $x + I_h^k M$ (compare [BH16, Lemma 5.2]).

Definition 4.3. The *algebraic Picard group* of $\mathcal{S}p_{\mathbf{K}}$ is

$$\mathrm{Pic}_h^{\mathrm{alg}} \coloneqq \mathrm{Pic}\left(\mathrm{Mod}^{\mathbb{G}}_{\pi_*\mathbf{E}}\right).$$

The *exotic Picard group* of $\mathcal{S}p_{\mathbf{K}}$ is defined by the exact sequence of abelian groups

$$1 \to \kappa_h \to \mathrm{Pic}_h \xrightarrow{\mathbf{E}^\vee_*} \mathrm{Pic}_h^{\mathrm{alg}},$$

whose existence follows from Theorem 4.1. Restricting both Picard groups to their subgroups of elements concentrated in even degrees, one can equally obtain κ_h as the kernel of the map

$$\mathrm{Pic}_h^0 \xrightarrow{\mathbf{E}^\vee_*} \mathrm{Pic}_h^{\mathrm{alg},0} \simeq \mathrm{Pic}(\mathrm{Mod}^{\mathbb{G}}_{\pi_0\mathbf{E}}).$$

One of the main theorems of [HMS94] is the computation $\kappa_1 \simeq \mathbb{Z}/2$ at the prime 2 (this is Theorem 3.3 therein). We want to show how the descent spectral sequence for $\mathfrak{pic}(\mathcal{E})$ recovers this computation, and we begin by identifying the algebraic Picard group in the spectral sequence. The aim of this section is therefore to prove the following result:

Theorem 4.4. *At arbitrary height h, the 1-line of the descent spectral sequence for $\mathfrak{pic}(\mathcal{E})$ computes the image of Pic_h^0 in $\mathrm{Pic}_h^{\mathrm{alg},0}$. The exotic Picard group κ_h is computed by the subgroup in filtration ≥ 2.*

Proving Theorem 4.4 will require a short discussion of derived complete modules. Firstly, recall that $\pi_0\mathbf{E}$ is a regular Noetherian local ring, with maximal ideal $I_h = (p, v_1, \ldots, v_h)$. If R is any such (classical) ring and $\mathfrak{m}$ its maximal ideal, the $\mathfrak{m}$-adic completion functor $(-)^\wedge_{\mathfrak{m}}$ has left-derived functors L_i, defined for example in [GM92]; this is in spite of $(-)^\wedge_{\mathfrak{m}}$ not being right-exact. For any R-module M the completion map $M \to M^\wedge_{\mathfrak{m}}$ factors through the zero-th derived functor

$$M \xrightarrow{\eta_M} L_0 M \xrightarrow{\epsilon_M} M^\wedge_{\mathfrak{m}},$$

and one says that M is *L-complete* or *derived $\mathfrak{m}$-complete* if η_M is an isomorphism. Hovey and Strickland prove the following facts about L-completion:

(1) For any R, the full subcategory spanned by the L-complete modules is a thick abelian subcategory of $\mathrm{Mod}_R^\heartsuit$, with enough projective generators [HS99, Theorem A.6 and Corollary A.12]. This is denoted $\widehat{\mathcal{M}}$ in *op. cit.*, but we will write $\mathrm{Mod}_R^{\heartsuit\mathrm{cpl}}$. The projective objects are precisely those R-modules which are *pro-free*, that is $M = L_0 \bigoplus_S R$ for some (possibly infinite) set S.

(2) The functor L_0 from modules to L-complete modules is a localisation. In particular, colimits in L-complete are computed as
$$L_0 \operatorname{colim} M,$$
where $\operatorname{colim} M$ denotes the colimit at the level of modules. Thus $\mathrm{Mod}_R^{\heartsuit\mathrm{cpl}}$ is still generated under colimits by the L-completion of the unit, and in particular κ-presentable where κ is chosen so that $L_0 R$ is κ-compact (note that we may have to take $\kappa > \omega$).

(3) Any $\mathfrak{m}$-adically complete module is L-complete [HS99, Theorem A.6]. In particular, $\pi_*\mathbf{E}$ is an L-complete module over itself.

(4) The category $\mathrm{Mod}_R^{\heartsuit\mathrm{cpl}}$ admits a unique symmetric monoidal product making L_0 a monoidal functor. This is given by the formula
$$M \widehat{\otimes}_{L_0 R} N = L_0(M \otimes_R N)$$
for M and N L-complete. For $\mathfrak{m}$-adically complete modules, one can also define the $\mathfrak{m}$-complete module
$$(M \otimes_R N)^\wedge_{\mathfrak{m}} \simeq (M \widehat{\otimes}_{L_0 R} N)^\wedge_{\mathfrak{m}},$$
but we will have no use for this.

(5) Derived completion agrees with ordinary completion on finitely generated modules, and on projective modules: in other words, ϵ_M is an isomorphism in either of these cases. Moreover, for any N the composition

$$M \otimes_R L_0N \to L_0M \otimes_R L_0N \to L_0(M \otimes_R N) = L_0M \widehat{\otimes}_{L_0R} L_0N$$

is an isomorphism when M is finitely generated [HS99, Proposition A.4]. In particular, if R is itself L-complete then finitely generated modules are complete, i.e. $M = L_0M = M_{\mathfrak{m}}^{\wedge}$ and $L_iM = 0$ for $i > 0$.

If A is an L-complete R-algebra (not necessarily Noetherian), we can define a category of A-modules which are L-complete with respect to $\mathfrak{m} \subset R$, $\mathrm{Mod}_A^{\heartsuit\mathrm{cpl}} := \mathrm{Mod}_A(\mathrm{Mod}_R^{\heartsuit\mathrm{cpl}})$.

We now specialise to the case $(R, \mathfrak{m}) = (\pi_0\mathbf{E}, I_h)$, and work towards the proof of Theorem 4.4. It is shown in [HS99, Proposition 8.4] that for any $\mathbf{K}$-local spectrum X, the $\pi_0\mathbf{E}$-module $\mathbf{E}_0^{\vee}X$ is L-complete, and that $\mathbf{E}_*^{\vee}X$ is finitely generated over $\pi_*\mathbf{E}$ if and only if X is $\mathbf{K}$-locally dualisable [HS99, Theorem 8.6]. Our first task is to prove that the presheaf of 1-categories

$$S \mapsto \mathrm{Mod}_{\pi_*\mathcal{E}(S)}^{\heartsuit\mathrm{cpl}}$$

is a stack on $\mathrm{Free}_{\mathbb{G}}$.

Warning 4.5. We would like to proceed as in Section 3.1, but there is a small subtlety: namely, to deduce descent from the results of [Hai22] (as in part (3) of the proof of Proposition 3.10), we would need to show that (the nerve of) $\mathrm{Mod}_{\pi_*\mathbf{E}}^{\heartsuit\mathrm{cpl}}$ is compactly generated; in fact, it would suffice to show it is compactly assembled. This is not clear: for example, Barthel and Frankland observe [BF15, Appendix A] that the unit in $\mathrm{Mod}_R^{\heartsuit\mathrm{cpl}}$ (which *is* a generator) cannot be compact. For our purposes it is enough to find *any* (small) set of compact generators: by comparison, the $\mathbf{K}$-local category is compactly generated by the $\mathbf{K}$-localisation of any finite type-h spectrum, even though its unit is not compact.

At this point it will be useful to pass to the ∞-category of complete modules over the discrete rings $\pi_0\mathcal{E}(S)$, as defined in [SAG, §7.3] or [BS14, §3.4]. This *can* be seen to be compactly generated by virtue of local (=Greenlees-May) duality; we will make use of this observation in the proof of Proposition 4.8. Given a discrete commutative ring R complete with respect to a finitely generated maximal ideal $\mathfrak{m}$, we will view R as a (connective) $\mathbb{E}_\infty$-ring and write $\mathrm{Mod}_R^{\mathrm{cpl}} \subset \mathrm{Mod}_R$ for the sub-∞-category of complete objects [SAG, Definition 7.3.1.1], which is a localisation of Mod_R. There is a unique symmetric monoidal product on $\mathrm{Mod}_R^{\mathrm{cpl}}$ for which the localisation is a monoidal functor, and to avoid confusion we denote this by $\widehat{\otimes}_R^{\mathbb{L}}$. Moreover, the abelian category $\mathrm{Mod}_R^{\heartsuit\mathrm{cpl}}$ of L-complete discrete R-modules includes as the heart of $\mathrm{Mod}_R^{\mathrm{cpl}}$ for a t-structure constructed in [SAG, Proposition 7.3.4.4], and the ∞-categorical localisation functor $L : \mathrm{Mod}_R \to \mathrm{Mod}_R^{\mathrm{cpl}}$ agrees upon restriction with the (total) left derived functor of L-completion [SAG, Corollary 7.3.7.5]; in particular, $L_0 \simeq \pi_0 L$.

Example 4.6. To give an example which makes the difference between 1-categories and ∞-

categories apparent, one can consider the colimit along the multiplication-by-p maps

$$\mathbb{Z}/p \to \mathbb{Z}/p^2 \to \cdots$$

over $\mathbb{Z}_p$. The colimit in $\mathrm{Mod}_{\mathbb{Z}_p}^{\heartsuit\mathrm{cpl}}$ is $L_0(\mathbb{Z}/p^\infty) = 0$; on the other hand, in the ∞-categorical setting one has $L \varinjlim \mathbb{Z}/p = \Sigma L_1(\mathbb{Z}/p^\infty) = \Sigma\mathbb{Z}_p$. We do not know if $\mathbb{Z}_p$ can be written as the filtered colimit of finite p-groups in $\mathrm{Mod}_{\mathbb{Z}_p}^{\heartsuit\mathrm{cpl}}$ in some other way.

In particular, this example shows that the t-structure on $\mathrm{Mod}_R^{\mathrm{cpl}}$ is generally *not* compatible with colimits in the sense of [HA, Definition 1.2.2.12].

Lemma 4.7. *For any profinite set T, we have an equivalence*

$$\mathrm{Mod}^{\mathrm{cpl}}_{\pi_0\mathcal{E}(\mathbb{G}\times T)} \simeq \mathcal{S}h(T, \mathrm{Mod}^{\mathrm{cpl}}_{\pi_0\mathbf{E}}), \tag{4.1}$$

Proof. According to [Hov08, Corollary 2.5], if $i \mapsto X_i \in \mathcal{S}p_{\mathbf{K}}$ is a filtered diagram such that $\mathbf{E}_*^\vee X_i$ is *pro-free* for each $i \in I$, then the natural map

$$L_0 \varinjlim_i \mathbf{E}_0^\vee X_i \to \mathbf{E}_0^\vee \varinjlim_i X_i$$

is an isomorphism. In particular this applies to give the middle equivalence in

$$\pi_0\mathcal{E}(\mathbb{G}\times T) \simeq \mathbf{E}_0^\vee \left(\varinjlim_i \bigoplus_{T_i} \mathbf{1}_{\mathbf{K}} \right) \simeq L_0 \varinjlim_i \bigoplus_{T_i} \pi_0\mathbf{E} = L_0 \varinjlim_i \pi_0\mathcal{E}(\mathbb{G}\times T_i), \tag{4.2}$$

since $\mathbf{E}_0^\vee \left(\bigoplus_{T_i} \mathbf{1}_{\mathbf{K}}\right) = \bigoplus_{T_i} \mathbf{E}_0^\vee \mathbf{1}_{\mathbf{K}} = \bigoplus_{T_i} \pi_0\mathbf{E}$ is certainly pro-free, each T_i being finite. As a result, item (2) implies that $\pi_0\mathcal{E}(\mathbb{G}\times T)$ is the colimit in $\mathrm{Mod}^{\heartsuit\mathrm{cpl}}_{\pi_0\mathbf{E}}$ of the algebras $\pi_0\mathcal{E}(\mathbb{G}\times T_i) = \bigoplus_{T_i} \pi_0\mathbf{E}$. In fact this particular example *is* also the limit in $\mathrm{Mod}^{\mathrm{cpl}}_{\pi_0\mathbf{E}}$: indeed, for $s > 0$ we see that $L_s \varinjlim \pi_0\mathcal{E}(\mathbb{G}\times T_i) = 0$ by [HS99, Theorem A.2(b)], since $\varinjlim \bigoplus_{T_i} \pi_0\mathbf{E}$ is projective in $\mathrm{Mod}^{\heartsuit}_{\pi_0\mathbf{E}}$.

This yields the first of the following equivalences of (presentably symmetric monoidal) ∞-categories:

$$\mathrm{Mod}^{\mathrm{cpl}}_{\pi_0\mathcal{E}(\mathbb{G}\times T)} \simeq \mathrm{Mod}^{\mathrm{cpl}}_{L\varinjlim \pi_0\mathcal{E}(\mathbb{G}\times T_i)} \simeq \varinjlim \mathrm{Mod}^{\mathrm{cpl}}_{\pi_0\mathcal{E}(\mathbb{G}\times T_i)} \simeq \mathcal{S}h(T, \mathrm{Mod}^{\mathrm{cpl}}_{\pi_0\mathbf{E}}).$$

The second equivalence follows from Proposition 2.5, and the second can be proved identically to part (2) of the proof of Proposition 3.10 (replacing $\mathcal{E}$ there by $\pi_0\mathcal{E}$). $\square$

Proposition 4.8. *The presheaf*

$$S \mapsto \mathrm{Mod}^{\heartsuit\mathrm{cpl}}_{\pi_0\mathcal{E}(S)}$$

it is a stack of 1-categories; in other words, it is a sheaf of 1-truncated symmetric monoidal ∞-categories on $\mathrm{Free}_{\mathbb{G}}$.

Proof. We will proceed in a few steps: we will begin by showing that $S \mapsto \mathrm{Mod}^{\mathrm{cpl}}_{\pi_0\mathcal{E}(S)}$ is a sheaf of ∞-categories, and then deduce the desired result at the level of 1-categories.

(1) Since any covering in $\mathrm{Free}_{\mathbb{G}}$ is of the form $p \times \mathbb{G}\colon T' \times \mathbb{G} \to T \times \mathbb{G}$ for $p : T' \to T$ a covering of profinite sets, we can restrict attention to the presheaf

$$T \mapsto \mathrm{Mod}^{\mathrm{cpl}}_{\pi_0 \mathcal{E}(T \times \mathbb{G})} \tag{4.3}$$

on $\mathrm{Profin} \simeq B\mathbb{G}_{\mathrm{proet}/\mathbb{G}}$. The local duality equivalence

$$\mathrm{Mod}^{\mathrm{tors}}_{\pi_0 \mathbf{E}} \simeq \mathrm{Mod}^{\mathrm{cpl}}_{\pi_0 \mathbf{E}}$$

of [BHV18, Theorem 3.7] or [SAG, Proposition 7.3.1.3] implies that $\mathrm{Mod}^{\mathrm{cpl}}_{\pi_0 \mathbf{E}}$ is compactly generated, and hence dualisable in Pr^L. Thus the presheaf

$$T \mapsto \mathcal{S}h(T, \mathrm{Mod}^{\mathrm{cpl}}_{\pi_0 \mathbf{E}}) \simeq \mathcal{S}h(T) \otimes \mathrm{Mod}^{\mathrm{cpl}}_{\pi_0 \mathbf{E}}$$

is a sheaf on Profin by the main theorem of [Hai22], and so Lemma 4.7 implies that (4.3) is a sheaf too.

(2) Next, we deduce descent for the presheaf of 1-categories $S \mapsto \mathrm{Mod}^{\heartsuit\mathrm{cpl}}_{\pi_0 \mathcal{E}(S)}$. Given a covering $p : T' \to T$ of profinite sets, we form the diagram

$$\begin{tikzcd}
\mathrm{Mod}^{\heartsuit\mathrm{cpl}}_{\pi_0\mathcal{E}(S)} \arrow[r, "\theta"] \arrow[d, hook] & \lim \arrow[r] \arrow[d, hook] & \mathrm{Mod}^{\heartsuit\mathrm{cpl}}_{\pi_0\mathcal{E}(S')} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook] & \mathrm{Mod}^{\heartsuit\mathrm{cpl}}_{\pi_0\mathcal{E}(S' \times_S 2)} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, hook] & \mathrm{Mod}^{\heartsuit\mathrm{cpl}}_{\pi_0\mathcal{E}(S' \times_S 3)} \arrow[r] \arrow[d, hook] & \cdots \\
\mathrm{Mod}^{\mathrm{cpl}}_{\pi_0\mathcal{E}(S)} \arrow[r, "\sim"] & \lim \arrow[r] & \mathrm{Mod}^{\mathrm{cpl}}_{\pi_0\mathcal{E}(S')} \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Mod}^{\mathrm{cpl}}_{\pi_0\mathcal{E}(S' \times_S 2)} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathrm{Mod}^{\mathrm{cpl}}_{\pi_0\mathcal{E}(S' \times_S 3)} \arrow[r] & \cdots
\end{tikzcd}$$

where $S^{(i)} = T^{(i)} \times \mathbb{G}$ as usual. Note that the limit of the top row can be computed as the limit of the truncated diagram $\Delta_{\leq 3} \hookrightarrow \Delta \to \mathrm{Cat}_\infty$, since each term is a 1-category. Moreover, the diagram shows that θ is fully faithful, and so to prove descent it remains to show that θ is essentially surjective. That is, given $M \in \mathrm{Mod}^{\mathrm{cpl}}_{\pi_0\mathcal{E}(S)}$ with $M \widehat{\otimes}^{\mathbb{L}}_{\pi_0\mathcal{E}(S)} \pi_0\mathcal{E}(S')$ discrete, we must show that M was discrete to begin with.

To this end, we claim first that $\pi_0\mathcal{E}(S')$ is projective over $\pi_0\mathcal{E}(S)$; since the graph of p exhibits $\pi_0\mathcal{E}(S') = \mathrm{Cont}(T', \pi_0\mathbf{E})$ as a retract of $\mathrm{Cont}(T' \times T, \pi_0\mathbf{E}) = \mathrm{Cont}(T', \mathrm{Cont}(T, \pi_0\mathbf{E}))$, it will suffice for this part to show that the latter is projective. But $\mathrm{Cont}(T, \pi_0\mathbf{E})$ is pro-discrete, which implies that

$$\begin{aligned}
\mathrm{Cont}(T', \mathrm{Cont}(T, \pi_0\mathbf{E})) &\cong \lim_I \varinjlim_i \bigoplus_{T'_i} \mathrm{Cont}(T, \pi_0\mathbf{E}/I) \\
&\cong \lim_I \varinjlim_i \bigoplus_{T'_i} \mathrm{Cont}(T, \pi_0\mathbf{E}) \otimes_{\pi_0\mathbf{E}} \pi_0\mathbf{E}/I \\
&\cong \lim_I \left[\varinjlim_i \bigoplus_{T'_i} \mathrm{Cont}(T, \pi_0\mathbf{E}) \right] \otimes_{\pi_0\mathbf{E}} \pi_0\mathbf{E}/I \\
&\cong L_0 \varinjlim_i \bigoplus_{T'_i} \mathrm{Cont}(T, \pi_0\mathbf{E})
\end{aligned}$$

is pro-free. To obtain the final isomorphism, we've used the fact that each term in the colimit is free over $\mathrm{Cont}(T, \pi_0\mathbf{E})$, so that the (uncompleted) colimit is projective. As a result, for any complete $\pi_0\mathcal{E}(S)$-module spectrum M we have

$$\pi_* \left(M \widehat{\otimes}^{\mathbb{L}}_{\pi_0\mathcal{E}(S)} \pi_0\mathcal{E}(S') \right) = (\pi_* M) \widehat{\otimes}_{\pi_0\mathcal{E}(S)} \pi_0\mathcal{E}(S').$$

Since $\pi_0\mathcal{E}(S')$ is faithful over $\pi_0\mathcal{E}(S)$, we deduce that M is discrete whenever its basechange is. $\square$

Remark 4.9. For any graded ring $\bigoplus_{a\in A} R_a$ with R_0 a complete noetherian local ring, invertible objects in $\mathrm{Mod}_R^{\heartsuit\mathrm{cpl}}$ are locally free: this follows from Lemma 3.19 and Nakayama's lemma.

It is convenient at this stage to work with Picard spaces, which as usual we denote $\mathfrak{Pic}$.

Corollary 4.10. *The presheaf*

$$S \mapsto \mathfrak{Pic}\left(\mathrm{Mod}^{\heartsuit\mathrm{cpl}}_{\pi_*\mathcal{E}(S)}\right)$$

is a sheaf of groupoids on $\mathrm{Free}_{\mathbb{G}}$*.*

Proof. By Proposition 4.8, the assignment

$$S \mapsto \mathfrak{Pic}\left(\mathrm{Mod}^{\heartsuit\mathrm{cpl}}_{\pi_0\mathcal{E}(S)}\right) \tag{4.4}$$

is a sheaf. Since invertible objects in $\mathrm{Mod}^{\heartsuit\mathrm{cpl}}_{\pi_0\mathcal{E}(S)}$ are locally free, this extends to the graded case. $\square$

We are now equipped to prove the promised result, identifying the algebraic elements in the Picard spectral sequence.

Proof. (Theorem 4.4). In the 0-stem of the descent spectral sequence, the bottom two lines compute the image of the map $\pi_0\Gamma\mathfrak{pic}(\mathcal{E}) \to \pi_0\Gamma\tau_{\leq 1}\mathfrak{pic}(\mathcal{E})$. We will argue by computing the target, identifying it with the algebraic Picard group.

First recall that by definition, $\mathfrak{pic}(\mathcal{E}) = \mathfrak{pic}(\mathrm{Mod}_{\mathcal{E},\mathbf{K}})$. Observe that

$$\tau_{\leq 1}\mathfrak{Pic}(\mathrm{Mod}_{\mathcal{E}(S),\mathbf{K}}) = \mathfrak{Pic}(h\mathrm{Mod}_{\mathcal{E}(S),\mathbf{K}}),$$

and so $\tau_{\leq 1}\mathfrak{Pic}(\mathcal{E})$ is the sheafification of

$$S \mapsto \mathfrak{Pic}\left(h\mathrm{Mod}_{\mathcal{E}(S),\mathbf{K}}\right).$$

Given $\mathcal{E}(S)$-modules M and M' with $M \in \mathrm{Pic}(\mathrm{Mod}_{\mathcal{E}(S),\mathbf{K}})$, we saw in Proposition 3.14 that M and so $\pi_* M$ is locally free, and therefore the latter is projective over $\pi_*\mathcal{E}(S)$. The universal coefficient spectral sequence [Elm+97, Theorem 4.1] over $\mathcal{E}(S)$ therefore collapses. Thus

$$[M, M']_{\mathcal{E}(S)} \simeq \mathrm{Hom}_{\pi_*\mathcal{E}(S)}(\pi_* M, \pi_* M'),$$

and we see that the functor

$$\pi_* : \mathfrak{Pic}\left(h\mathrm{Mod}_{\mathcal{E}(S),\mathbf{K}}\right) \to \mathfrak{Pic}\left(\mathrm{Mod}^{\heartsuit\mathrm{cpl}}_{\pi_*\mathcal{E}(S)}\right) \tag{4.5}$$

is fully faithful. It is in fact an equivalence: any invertible L-complete module over $\pi_*\mathcal{E}(S)$ is locally free, and so projective, and in particular lifts to $\mathrm{Mod}_{\mathcal{E}(S),\mathbf{K}}$ [Wol98, Theorem 3].

By Corollary 4.10, no sheafification is therefore required when we restrict $\tau_{\leq 1}\mathfrak{Pic}\left(\mathrm{Mod}_{\mathcal{E},\mathbf{K}}\right)$ to $\mathrm{Free}_{\mathbb{G}}$ and so we obtain

$$\begin{aligned}
\Gamma\tau_{\leq 1}\mathfrak{Pic}(\mathcal{E}) &\simeq \mathrm{Tot}\left[\mathfrak{Pic}\left(h\mathrm{Mod}_{\mathcal{E}(\mathbb{G}^{\bullet+1}),\mathbf{K}}\right)\right] \\
&\simeq \mathrm{Tot}\left[\mathfrak{Pic}\left(\mathrm{Mod}^{\heartsuit\mathrm{cpl}}_{\mathrm{Cont}(\mathbb{G}^{\bullet},\pi_* E)}\right)\right] \\
&\simeq \mathrm{Tot}\left[\mathfrak{Pic}\left(\mathrm{Mod}_{\mathrm{Cont}(\mathbb{G}^{\bullet},\pi_* E)}\right)\right] \\
&\simeq \mathrm{Tot}_3\left[\mathfrak{Pic}\left(\mathrm{Mod}_{\mathrm{Cont}(\mathbb{G}^{\bullet},\pi_* E)}\right)\right]
\end{aligned}$$

This is the groupoid classifying twisted $\mathbb{G}$-$\pi_*\mathbf{E}$-modules with invertible underlying module; in particular, $\pi_0\Gamma\tau_{\leq 1}\mathfrak{Pic}(\mathcal{E}) \simeq \mathrm{Pic}^{\mathrm{alg}}_h$. On free $\mathbb{G}$-sets, the truncation map

$$\mathfrak{Pic}\left(\mathrm{Mod}_{\mathcal{E}(S),\mathbf{K}}\right) \to \tau_{\leq 1}\mathfrak{Pic}\left(\mathrm{Mod}^{\heartsuit\mathrm{cpl}}_{\mathcal{E}(S),\mathbf{K}}\right) \simeq \mathfrak{Pic}\left(\mathrm{Mod}^{\heartsuit\mathrm{cpl}}_{\pi_*\mathcal{E}(S)}\right)$$

is just $M \mapsto \pi_* M$. On global sections, it therefore sends M to the homotopy groups of the associated descent datum for the covering $\mathbb{G} \to *$; this is its Morava module $\mathbf{E}^\vee_* M = \pi_*\mathbf{E} \otimes M$, by definition. $\square$

Remark 4.11. As a consequence, the map $\mathbf{E}^\vee_* : \mathrm{Pic}_h \to \mathrm{Pic}^{\mathrm{alg}}_h$ is:

- (i) injective if and only if the zero stem in the E_∞ term of the Picard spectral sequence is concentrated in filtration ≤ 1;
- (ii) surjective if and only if there are no differentials in the Picard spectral sequence having source $(0,1)$ (the generator of the $\mathbb{Z}/2$ in bidegree $(0,0)$ certainly survives, and represents $\Sigma\mathbf{1}_{\mathbf{K}}$).

This refines the algebraicity results in [Pst22], although in practise it is hard to verify either assertion without assuming a horizontal vanishing line at E_2 in the ASS (which is what makes the results of *op. cit.* go through).

Corollary 4.12 ([CZ22], Proposition 1.25)**.** *If $p > 2, p-1 \nmid h$ and $h^2 \le 4p-4$, there is an isomorphism*

$$\kappa_h \cong H^{2p-1}(\mathbb{G}, \pi_{2p-2}\mathbf{E}).$$

Proof. This follows from sparsity in the Adams spectral sequence. By [Hea14, Prop. 4.2.1], the lowest-filtration contribution to the exotic Picard group comes from $E_\infty^{2p-1,2p-1}$. If $p-1 \nmid h$, the vanishing line in the Adams spectral sequence occurs at the starting page, and if moreover $h^2 \le 4p-4$ then the group $E_\infty^{2p-1,2p-1}$ is the only possibly nonzero entry in this region, and hence fits in the exact sequence

$$1 \to (\pi_0\mathbf{E})^\times \xrightarrow{d_{2p-1}} H^{2p-1}(\mathbb{G}, \pi_{2p-2}\mathbf{E}) \to E_\infty^{2p-1,2p-1} \to 0.$$

In fact, the differential must vanish. Indeed, a weak form of chromatic vanishing proven in [BG18, Lemma 1.33] shows that

$$H^0(\mathbb{G}, \pi_0\mathbf{E}) \simeq H^0(\mathbb{G}, \mathbb{W}(\mathbb{F}_{p^h})) \simeq H^0(\mathrm{Gal}(\mathbb{F}_{p^h}/\mathbb{F}_p), \mathbb{W}(\mathbb{F}_{p^h})) \simeq \mathbb{Z}_p.$$

This isomorphism is inverse to a component of the map on the E_2-pages of Adams spectral sequences associated to the diagram

$$\begin{tikzcd} \mathbf{1}_p \arrow[r] \arrow[d] & \mathbf{1}_{\mathbf{K}} \arrow[d] \\ \mathbb{SW}_h \arrow[r] & \mathbf{E} \end{tikzcd}$$

Here $\mathbf{1}_p$ denotes the p-complete sphere, and $\mathbb{SW}_h$ the spherical Witt vectors of $\mathbb{F}_{p^h}$; the bottom map is defined under the universal property of $\mathbb{SW}_h$ [Lur18, Definition 5.2.1] by the inclusion $\mathbb{F}_{p^h} \hookrightarrow (\pi_0\mathbf{E})/p$. Since $(\pi_0\mathbb{SW}^{h\mathrm{Gal}})^\times = (\pi_0\mathbf{1}_p)^\times = \mathbb{Z}_p^\times$, the map on Picard spectral sequences induced by the above square implies that the group $E_2^{0,1} \simeq \mathbb{Z}_p^\times$ in the Picard spectral sequence consists of permanent cycles. □

Example 4.13. At the prime three this gives $\kappa_3 \cong H^5(\mathbb{G}, \pi_4\mathbf{E})$. In this case, the Morava stabiliser group has cohomological dimension nine.

Example 4.14. In the boundary case $2p-1 = \mathrm{cd}_p(\mathbb{G}) = h^2$, we can use Poincaré duality to simplify the relevant cohomology group: this gives

$$\kappa_h \cong H^{2p-1}(\mathbb{G}, \pi_{2p-2}\mathbf{E}) \cong H_0(\mathbb{S}, \pi_{2p-2}\mathbf{E})^{\mathrm{Gal}}. \tag{4.6}$$

Examples of such pairs (h, p) are $(3, 5)$, $(5, 13)$, $(9, 41)$ and $(11, 61)$; in each case, this is the first prime for which [Pst22, Remark 2.6] leaves open the possibility of exotic Picard elements. The case $(h, p) = (3, 5)$ case was considered by Culver and Zhang, using different methods; however, they show as above that Heard's spectral sequence combined with the conjectural vanishing

$$H_0(\mathbb{S}, \pi_{2p-2}\mathbf{E}) = 0,$$

would imply that $\kappa_h = 0$ [CZ22, Corollary 1.27].

To our knowledge, it is not known if there are infinitely primes p for which $2p-1$ is a perfect square; this is closely tied to Landau's (unsolved) fourth problem, which asks if there are infinitely many primes of the form h^2+1.

4.2 Picard groups at height one

It is well-known that Morava E-theory at height one (and a fixed prime p) is p-completed complex K-theory KU_p, and as such its homotopy is given by

$$\pi_*\mathbf{E} = \mathbb{Z}_p[u^{\pm 1}], \tag{4.7}$$

with $u \in \pi_2\mathbf{E}$ the Bott element. In this case, the Morava stabiliser group is isomorphic to the p-adic units $\mathbb{Z}_p^\times$, acting on KU_p by Adams operations

$$\psi^a : u \mapsto au.$$

The **K**-local **E**-Adams spectral sequence therefore reads

$$E_{2,+}^{s,t} = H^s(\mathbb{Z}_p^\times, \mathbb{Z}_p(t/2)) \implies \pi_{t-s}L_{\mathbf{K}}S, \tag{4.8}$$

where $\mathbb{Z}_p(t/2)$ denotes the representation

$$\mathbb{Z}_p^\times \xrightarrow{t/2} \mathbb{Z}_p^\times \to \mathbb{Z}_p$$

when t is even, and zero when t is odd. Note that these are never discrete $\mathbb{Z}_p^\times$-modules, except at $t=0$. Nevertheless, cohomology of continuous pro-p modules is sensible for profinite groups G of type p-FP_∞ [SW00, §4.2]. The *small* Morava stabiliser groups $\mathbb{S}$ are in general p-adic Lie groups, and so satisfy this assumption; this implies that $\mathbb{G}$ is type p-FP_∞, since $\mathbb{S}$ is a finite index subgroup. In this case cohomology is continuous, in the sense that its value on a pro-p module is determined by its value on finite quotients:

$$H^*(G, \varprojlim M_i) = \varprojlim H^*(G, M_i).$$

Under the same assumption on G, there is also a Lyndon-Hochschild-Serre spectral sequence for

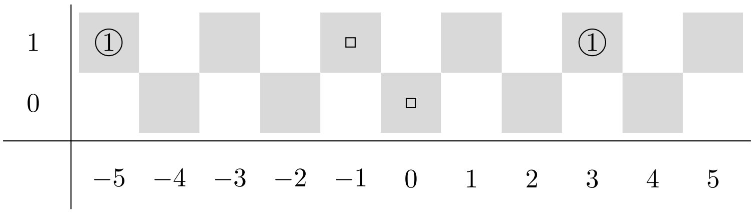


Figure 4.1: The E_2-page of the descent spectral sequence for $\mathbf{E}$ at odd primes (implicitly at $p = 3$). Squares denote $\mathbb{Z}_p$-summands, and circles are p-torsion summands (labelled by the degree of the torsion). This recovers the well-known computation of $\pi_* \mathbf{1}_{\mathbf{K}}$ at height 1 and odd primes.

any closed normal subgroup $N <_o G$ [SW00, Theorem 4.2.6]: that is,

$$E_2^{i,j} = H^i(G/N, H^j(N, M)) \implies H^{i+j}(G, M). \tag{4.9}$$

See also [Jan88, Theorem 3.3] for a similar result for profinite coefficients that are not necessarily pro-p: one obtains a Lyndon-Hochschild-Serre spectral sequence by replacing the Galois covering $X' \to X$ therein by a map of sites $BG_{\mathrm{et}} \to BN_{\mathrm{et}}$.

The descent spectral sequence for $\mathfrak{pic}(\mathcal{E})$ at height one (and all primes) therefore has starting page

$$E_2^{s,t} = \begin{cases} H^s(\mathbb{Z}_p^\times, \mathbb{Z}/2) & t = 0 \\ H^s(\mathbb{Z}_p^\times, \mathbb{Z}_p(0)^\times) & t = 1 \\ H^s(\mathbb{Z}_p^\times, \mathbb{Z}_p(\frac{t-1}{2})) & t \geq 2 \end{cases} \tag{4.10}$$

The results of Section 3.2 also tell us how to discern many differentials in the Picard spectral sequence from those in the Adams spectral sequence: in particular, we will make use of Proposition 3.22 and Corollary 3.25. Our input is the well-known computation of the $\mathbf{K}$-local $\mathbf{E}$-Adams spectral sequence at height one. A convenient reference is [BGH22, §4], but for completeness a different argument is presented in Appendix B.

4.2.1 Odd primes

When $p > 2$, the Adams spectral sequence collapses immediately:

Lemma 4.15 (Lemma B.1)**.** *The starting page of the descent spectral sequence for $\mathcal{E}$ is given by*

$$E_{2,+}^{s,t} = H^s(\mathbb{Z}_p^\times, \pi_t \mathbf{E}) = \begin{cases} \mathbb{Z}_p & t = 0 \text{ and } s = 0,1 \\ \mathbb{Z}/p^{\nu_p(t)+1} & t = 2(p-1)t' \neq 0 \text{ and } s = 1 \end{cases} \tag{4.11}$$

and zero otherwise. The result is displayed in Fig. 4.1.

As a result of the vanishing line, the computation of $\mathrm{Pic}(\mathcal{S}p_{\mathbf{K}})$ in this case depends *only* on $H^*(\mathbb{Z}_p^\times, \mathrm{Pic}(\mathbf{E}))$ and $H^*(\mathbb{Z}_p^\times, (\pi_0 \mathbf{E})^\times)$. This recovers the computation in [HMS94, Proposition 2.7].

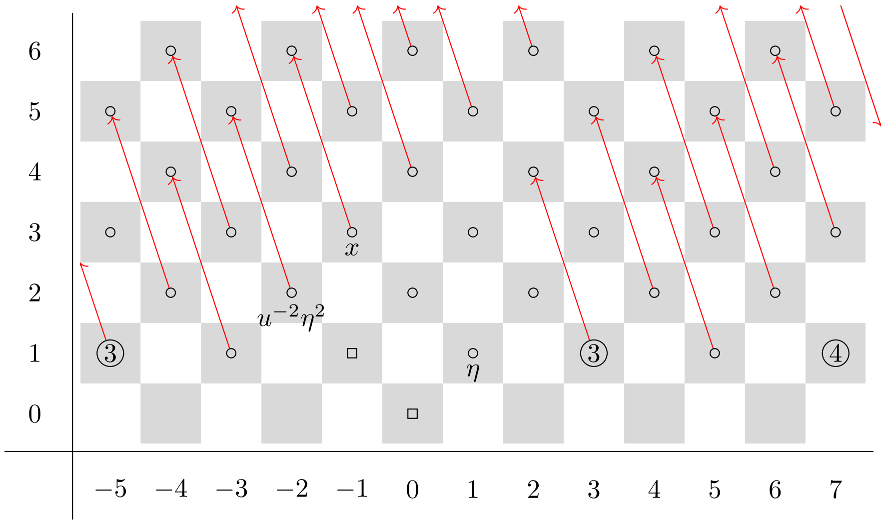


Figure 4.2: The E_3-page of the descent spectral sequence for $\mathbf{E}$ at $p = 2$.

Proposition 4.16. *The height one Picard group is algebraic at odd primes:*

$$\mathrm{Pic}_1 \cong \mathrm{Pic}_1^{\mathrm{alg}} . \tag{4.12}$$

4.2.2 The case $p = 2$

At the even prime, the Morava stabiliser group contains 2-torsion, and therefore its cohomology with 2-complete coefficients is periodic.

Lemma 4.17. *The starting page of the descent spectral sequence for $\mathcal{E}$ is given by*

$$E_{2,+}^{s,t} = H^s(\mathbb{Z}_2^\times, \pi_t \mathbf{E}) = \begin{cases} \mathbb{Z}_2 & t = 0 \text{ and } s = 0, 1 \\ \mathbb{Z}/2 & t \equiv_4 2 \text{ and } s \geq 1 \\ \mathbb{Z}/2^{\nu_2(t)+1} & 0 \neq t \equiv_4 0 \text{ and } s = 1 \\ \mathbb{Z}/2 & t \equiv_4 0 \text{ and } s > 1 \end{cases} \tag{4.13}$$

and zero otherwise.

This time we see that the spectral sequence can support many differentials. These can be computed by various methods, as for example in [BGH22]. We give another proof, more closely related to our methods, in Appendix B.

Proposition 4.18. *The descent spectral sequence collapses at E_4 with a horizontal vanishing line. The differentials on the third page are displayed in Fig. 4.2.*

As a result of the previous section, we can compute the groups of exotic and algebraic Picard elements at the prime 2. We will need one piece of the multiplicative structure: write η for the generator in bidegree $(s,t) = (1,2)$, and $u^{-2}\eta^2$ for the generator in bidegree $(s,t) = (2,0)$.

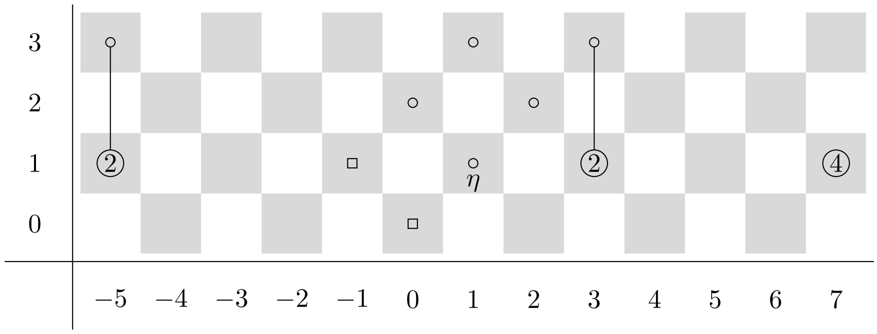


Figure 4.3: The $E_4 = E_\infty$-page of the descent spectral sequence for $\mathbf{E}$ at $p = 2$.

Lemma 4.19. *In the descent spectral sequence for $\mathcal{E}$, the class*

$$x \coloneqq u^{-2}\eta^2 \cdot \eta \in E_2^{3,2}$$

is non-nilpotent. In particular, x^j generates the group in bidegree $(s, t) = (3j, 2j)$ of the descent spectral sequence for $\mathcal{E}$.

Proof. The classes $u^{-2}\eta^2$ and η are detected by elements of the same name in the HFPSS for the conjugation action on KU_2, under the map of spectral sequences induced by the square of Galois extensions

$$\begin{tikzcd} \mathbf{1_K} \arrow[r] \arrow[d] & KO_2 \arrow[d] \\ KU_2 \arrow[r, equal] & KU_2 \end{tikzcd}$$

To see this, one can trace through the computations of Appendix B: indeed, the proof of Proposition B.3 identifies the map of spectral sequences induced by the map of C_2-Galois extensions

$$\begin{tikzcd} \mathbf{1_K} \arrow[r] \arrow[d] & KO_2 \arrow[d] \\ KU_2^{h(1+4\mathbb{Z}_2)} \arrow[r] & KU_2 \end{tikzcd}$$

and Lemma B.5 identifies the descent spectral sequence for $\mathcal{E}$ with the HFPSS for $KU_2^{h(1+4\mathbb{Z}_2)}$, up to a filtration shift. But the starting page of the HFPSS for KU_2 is

$$H^*(C_2, \pi_*KU_2) = \mathbb{Z}_2[\eta, u^{\pm 2}]/2\eta,$$

and here in particular $(u^{-2}\eta^2 \cdot \eta)^j = u^{-2j}\eta^{3j} \neq 0$. $\square$

Proposition 4.20. *At the prime 2, the exotic Picard group κ_1 is $\mathbb{Z}/2$.*

Proof. We will deduce this from Proposition 4.18, which implies that the descent spectral sequence for $\mathfrak{pic}(\mathcal{E})$ takes the form displayed in Figure 4.4. According to Theorem 4.4, the only differential

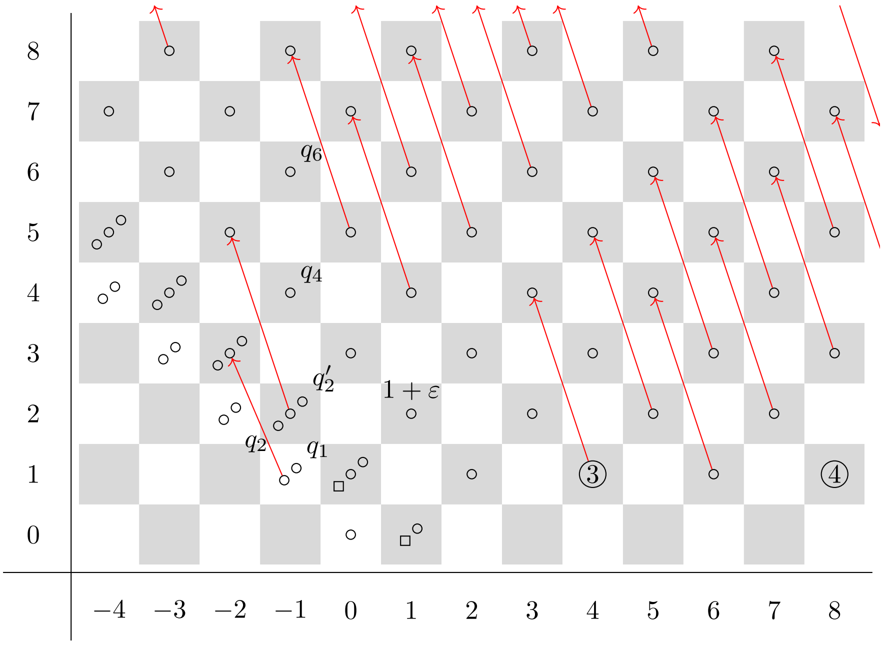


Figure 4.4: The E_3-page of the Picard spectral sequence at $p = 2$. For details on the negative stems, see Part II

remaining for the computation of κ_1 is that on the class in bidegree $(s,t) = (3,0)$, which corresponds to the class $x \in E_{2,+}^{3,2}$ of the Adams spectral sequence. Applying the formula from Lemma 3.25,

$$d_3(x) = d_3^{ASS}(x) + x^2 = 2x^2 = 0.$$

After this there is no space for further differentials on x, so $\kappa_1 = \mathbb{Z}/2$. $\square$

Chapter 5

Brauer groups

In the previous sections, we considered Galois descent for the Picard spectrum of the **K**-local category. Recall that the Picard spectrum deloops to the *Brauer spectrum*, which classifies derived Azumaya algebras. In this section we consider Galois descent for the **K**-local Brauer spectrum; see [BRS12; AG14; GL21; AMS22] for related work. Unlike most of these sources, the unit in our context is *not* compact, and this makes the corresponding descent statements slightly more delicate. We begin with the basic definitions:

Definition 5.1 ([HL17], Definition 2.2.1)**.** Suppose $\mathcal{C}$ is a stable homotopy theory, and $R \in \mathrm{CAlg}(\mathcal{C})$.

(i) Two R-algebras $A, B \in \mathrm{Alg}_R(\mathcal{C})$ are *Morita equivalent* if there is an R-linear equivalence $\mathrm{LMod}_A(\mathcal{C}) \simeq \mathrm{LMod}_B(\mathcal{C})$. We will write $A \sim B$.

(ii) An algebra $A \in \mathrm{Alg}_R(\mathcal{C})$ is an *Azumaya R-algebra* if there exists $B \in \mathrm{Alg}_R(\mathcal{C})$ such that $A \otimes_R B \sim R$.

We will mostly be interested in the case $\mathcal{C} = \mathcal{S}p_{\mathbf{K}}$ and $R = \mathbf{1}_{\mathbf{K}}$.

Remark 5.2. Hopkins and Lurie show [HL17, Corollary 2.2.3] that this definition is equivalent to a more familiar definition in terms of intrinsic properties of A. Using [HL17, Prop. 2.9.6] together with faithfulness of the map $\mathbf{1}_{\mathbf{K}} \to \mathbf{E}$, one sees that $A \in \mathcal{S}p_{\mathbf{K}}$ is Azumaya if and only if

(i) A is nonzero;

(ii) A is dualisable in $\mathrm{Alg}_{R,\mathbf{K}}$;

(iii) The left/right-multiplication action $A \otimes_R A^{op} \to \mathrm{End}_R(A)$ is an equivalence.

We will define the space $\mathfrak{Az}(R) \subset \iota\mathrm{Alg}_{R,\mathbf{K}}$ to be the subgroupoid spanned by **K**-local Azumaya algebras over R, and $\mathrm{Az}(R) \coloneqq \pi_0\mathfrak{Az}(R)$.

Definition 5.3. The $\mathbf{K}$-*local Brauer group* of R is the set

$$\mathrm{Br}(R) \coloneqq \mathrm{Az}(R)/\sim,$$

equipped with the abelian group structure $[A] + [B] = [A \otimes_R B]$.

Remark 5.4. This notation should not be confused with the group of Azumaya R-algebras in spectra, which is potentially different.

In analogy with Picard groups we will write $\mathrm{Br}_h \coloneqq \mathrm{Br}(\mathbf{1}_{\mathbf{K}})$. Our objective is to show how the Picard spectral sequence (3.1) can be used to compute this. First recall the main theorem of *op. cit.*:

Theorem 5.5 ([HL17], Theorem 1.0.11)**.** *There is an isomorphism*

$$\mathrm{Br}(\mathbf{E}) \simeq \mathrm{BW}(\kappa) \times \mathrm{Br}'(\mathbf{E}),$$

where $\mathrm{BW}(\kappa)$ *is the* Brauer-Wall *group classifying* $\mathbb{Z}/2$*-graded Azumaya algebras over* $\kappa \coloneqq \pi_0\mathbf{E}/I_h$*, and* $\mathrm{Br}'(\mathbf{E})$ *admits a filtration with associated graded* $\bigoplus_{k\geq 2} I_h^k/I_h^{k+1}$*.*

We will not discuss this result, and instead focus on the orthogonal problem of computing the group of $\mathbf{K}$-local Brauer algebras over the sphere which become Morita trivial over Morava E-theory; in the terminology of [GL21] this is the *relative* Brauer group, and will be denoted by Br_h^0. The full Brauer group can then (at least in theory) be obtained from the exact sequence

$$1 \to \mathrm{Br}_h^0 \to \mathrm{Br}_h \to \mathrm{Br}(\mathbf{E})^{\mathbb{G}}.$$

While interesting, the problem of understanding the action of $\mathbb{G}$ on $\mathrm{Br}(\mathbf{E})$ is somewhat separate, and again we do not attempt to tackle this.

5.1 Brauer groups and descent

In this subsection we show that the Picard spectral sequence gives an upper bound on the size of the relative Brauer group, as proven in [GL21, Theorem 6.32] for finite Galois extensions of unlocalised ring spectra. To this end, we define a 'cohomological' Brauer group; this might also be called a 'Brauer-Grothendieck group' of the $\mathbf{K}$-local category, as opposed to the 'Brauer-Azumaya' group discussed above. The ideas described in this section goes back to work of Toën on Brauer groups in derived algebraic geometry [Toë12].

Given $R \in \mathrm{CAlg}(\mathcal{S}p_{\mathbf{K}})$, the ∞-category $\mathrm{Mod}_{R,\mathbf{K}}$ is symmetric monoidal, and therefore defines an object of $\mathrm{CAlg}_{\mathcal{S}p_{\mathbf{K}}}(\mathrm{Pr}^L)$. We will consider the symmetric monoidal ∞-category

$$\mathrm{Cat}_{R,\mathbf{K}} \coloneqq \mathrm{Mod}_{\mathrm{Mod}_{R,\mathbf{K}}}(\mathrm{Pr}^L_\kappa),$$

where κ is chosen to be large enough that $\mathrm{Mod}_{R,\mathbf{K}} \in \mathrm{CAlg}(\mathrm{Pr}^L_\kappa)$.

Definition 5.6. The *cohomological K-local Brauer space* of R is the Picard space

$$\mathfrak{Br}^{\mathrm{coh}}(R) := \mathfrak{Pic}(\mathrm{Cat}_{R,\mathbf{K}}).$$

Since Pr^L_κ is presentable [HA, Lemma 5.3.2.9(2)], this is once again a small space. In analogy with the Picard case, we also write $\mathfrak{Br}_h^{0,\mathrm{coh}}$ for the full subspace of $\mathfrak{Br}_h^{\mathrm{coh}} := \mathfrak{Br}^{\mathrm{coh}}(\mathbf{1}_{\mathbf{K}})$ spanned by invertible $\mathcal{S}p_{\mathbf{K}}$-modules $\mathcal{C}$ for which there is an R-linear equivalence $\mathcal{C} \otimes_{\mathcal{S}p_{\mathbf{K}}} \mathrm{Mod}_{\mathbf{E},\mathbf{K}} \simeq \mathrm{Mod}_{\mathbf{E},\mathbf{K}}$.

Definition 5.7. Passing to ∞-categories of left modules defines a functor $\mathrm{Alg}(\mathrm{Mod}_{R,\mathbf{K}}) \to \mathrm{Cat}_{R,\mathbf{K}}$, with algebra maps acting by extension of scalars. This restricts by the Azumaya condition to

$$\mathfrak{Az}(R) \to \mathfrak{Br}^{\mathrm{coh}}(R) \tag{5.1}$$

We define $\mathfrak{Br}(R) \subset \mathfrak{Br}^{\mathrm{coh}}(R)$ to be the full subgroupoid spanned by the essential image of $\mathfrak{Az}(R)$. We moreover define $\mathfrak{Br}_h := \mathfrak{Br}(\mathbf{1}_{\mathbf{K}})$ and $\mathfrak{Br}_h^0 := \mathfrak{Br}_h \cap \mathfrak{Br}_h^{0,coh}$.

Warning 5.8. When working with plain $\mathbb{E}_\infty$-rings, one can take $\kappa = \omega$ in the definition of the cohomological Brauer group, and so the two groups agree by Schwede-Shipley theory. Indeed, a cohomological Brauer class is then an invertible compactly generated R-linear ∞-category $\mathcal{C}$, and in particular admits a *finite set* $\{C_1, \ldots, C_n\}$ of compact generators [AG14, Lemma 3.9]. Thus $\mathcal{C} \simeq \mathrm{Mod}_A$ for $A := \mathrm{End}_{\mathcal{C}}(\bigoplus_i C_i)$ an Azumaya algebra. This argument fails in Pr^L_κ for $\kappa > \omega$; on the other hand, $\mathcal{S}p_{\mathbf{K}} \notin \mathrm{CAlg}(\mathrm{Pr}^L_\omega)$ since the unit is not compact. For *relative* Brauer classes, we refer to Section 5.2 for an alternative solution.

We now provide a descent formalism suitable for our context, based on the approach of [GL21]. If $\mathcal{C}$ is a presentably symmetric-monoidal ∞-category with κ-compact unit, we will write $\mathrm{Mod}_{\mathcal{C}} := \mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^L_\kappa)$. If $R \in \mathrm{CAlg}(\mathcal{C})$, we will write $\mathrm{Cat}_R := \mathrm{Mod}_{\mathrm{Mod}_R(\mathcal{C})}\left(\mathrm{Pr}^L_\kappa\right)$. We will be interested in descent properties of the functor $\mathrm{Cat}_{(-)} \colon \mathrm{CAlg}(\mathcal{C}) \to \mathrm{Pr}^{L,\mathrm{smon}}$: that is, if $R \to R'$ is a map of commutative algebras, we would like to know how close the functor θ below is to an equivalence:

$$\mathrm{Cat}_R \xrightarrow{\theta} \lim \Big[\ \mathrm{Cat}_{R'} \rightrightarrows \mathrm{Cat}_{R' \otimes_R R'} \substack{\rightarrow\\[-1ex]\rightarrow\\[-1ex]\rightarrow} \cdots\ \Big]. \tag{5.2}$$

Lemma 5.9. *If R' is a descent R-algebra, then θ is fully faithful when restricted to the full subcategory spanned by ∞-categories of left modules.*

Proof. Let $A, A' \in \mathrm{Alg}_R(\mathcal{C})$. Writing $\mathrm{LMod}_A = \mathrm{LMod}_A(\mathcal{C})$, we have equivalences of ∞-categories

$$\begin{aligned} \mathrm{Fun}_R(\mathrm{LMod}_A, \mathrm{LMod}_{A'}) &\simeq \mathrm{RMod}_{A \otimes_R A'^{\mathrm{op}}} \\ &\simeq \lim \mathrm{RMod}_{A \otimes_R A'^{\mathrm{op}} \otimes_R R'^{\bullet}} \\ &\simeq \lim \mathrm{RMod}_{(A \otimes_R R'^{\bullet}) \otimes_{R'^{\bullet}} (A' \otimes_R R'^{\bullet})^{\mathrm{op}}} \\ &\simeq \lim \mathrm{Fun}_{R'^{\bullet}}(\mathrm{LMod}_{A \otimes_R R'^{\bullet}}, \mathrm{LMod}_{A' \otimes_R R'^{\bullet}}). \end{aligned} \tag{5.3}$$

Here we have twice appealed to [HA, Theorem 4.8.4.1] (applied to A^{op} and A'^{op}), and to Lemma 3.7 for the second equivalence. Passing to cores, we obtain

$$\begin{aligned}\mathrm{Map}_{\mathrm{Cat}_R}(\mathrm{LMod}_A, \mathrm{LMod}_{A'}) &\simeq \lim \mathrm{Map}_{\mathrm{Cat}_{R'^\bullet}}(\mathrm{LMod}_{A\otimes_R R'^\bullet}, \mathrm{LMod}_{A'\otimes_R R'^\bullet}) \\ &\simeq \mathrm{Map}_{\lim \mathrm{Cat}_{R'^\bullet}}(\mathrm{LMod}_{A\otimes_R R'^\bullet}, \mathrm{LMod}_{A'\otimes_R R'^\bullet}).\end{aligned}$$ □

Corollary 5.10. *For any covering $S' \to S$ in $B\mathbb{G}_{\mathrm{proet}}$, the functor*

$$\theta : \mathrm{Cat}_{\mathcal{E}(S),\mathbf{K}} \to \lim \mathrm{Cat}_{\mathcal{E}(S'\times_S \bullet),\mathbf{K}}$$

is fully faithful when restricted to left module ∞-categories. □

Using this, we can give a bound on the size of Br_h^0.

Proposition 5.11. *The group Br_h^0 is a subgroup of $\pi_0 \lim B\mathfrak{Pic}(\mathbf{E}^{\bullet+1})$.*

Proof. If $S' \to S$ is a covering in $B\mathbb{G}_{\mathrm{proet}}$, we will write $R \to R'$ for the extension $\mathcal{E}(S) \to \mathcal{E}(S')$. Writing $\mathfrak{Br}(R \mid R')$ for the full subgroupoid of $\mathfrak{Br}(R)$ spanned by objects $\mathrm{LMod}_{A,\mathbf{K}}$ such that $\mathrm{LMod}_{A,\mathbf{K}} \otimes_{\mathrm{Mod}_{R,\mathbf{K}}} \mathrm{Mod}_{R',\mathbf{K}} \simeq \mathrm{Mod}_{R',\mathbf{K}}$, we will exhibit $\mathfrak{Br}(R \mid R')$ as a full subspace of $\lim B\mathfrak{Pic}(R'^\bullet)$. Taking the covering $\mathbb{G} \to *$ gives the desired result on π_0.

By definition, $\mathfrak{Br}(R)$ is the full subcategory of $\mathfrak{Br}^{\mathrm{coh}}(R)$ spanned by module categories. Using the inclusion of $B\mathfrak{Pic}(R')$ as the component of the unit in $\mathfrak{Br}(R')$ we can form the diagram

$$\begin{tikzcd}
\mathfrak{Br}(R \mid R') \arrow[dd, hook] \arrow[dr, dashed] \arrow[drr, bend left] \arrow[drrr, bend left] & & & & \\
& \lim B\mathfrak{Pic}(R'^\bullet) \arrow[r] \arrow[d, hook] & B\mathfrak{Pic}(R') \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook] & B\mathfrak{Pic}(R' \otimes_R R') \arrow[r] \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, hook] & \cdots \\
\mathfrak{Br}(R) \arrow[r, hook, "(\star)"] \arrow[d, hook] & \lim \mathfrak{Br}(R'^\bullet) \arrow[r] \arrow[d, hook] & \mathfrak{Br}(R') \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook] & \mathfrak{Br}(R' \otimes_R R') \arrow[r] \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, hook] & \cdots \\
\iota\mathrm{Cat}_{R,\mathbf{K}} \arrow[r, "\theta"] & \lim \iota\mathrm{Cat}_{R'^\bullet,\mathbf{K}} \arrow[r] & \iota\mathrm{Cat}_{R',\mathbf{K}} \arrow[r, shift left] \arrow[r, shift right] & \iota\mathrm{Cat}_{R'\otimes_R R',\mathbf{K}} \arrow[r] \arrow[r, shift left=2] \arrow[r, shift right=2] & \cdots
\end{tikzcd} \tag{5.4}$$

Here hooked arrows denote fully faithful functors: this is essentially by definition in most cases, with the starred functor being fully faithful by virtue of Corollary 5.10 (and the fact that passing to the core preserves limits). The dashed arrow, which clearly exists, is fully faithful by 2-out-of-3. □

In particular, the (-1)-stem in the descent spectral sequence for the Picard sheaf $\mathfrak{pic}(\mathcal{E})$ gives an upper bound on the size of the relative Brauer group. We will draw consequences from this in the next subsection.

As we now discuss, the cohomological Brauer space also admits a description in terms of the proétale site. We will not use this, but include a proof for completeness.

Lemma 5.12. *The restriction of the presheaf $B\mathfrak{Pic}(\mathcal{E})$ to $\mathrm{Free}_{\mathbb{G}}$ is a hypercomplete sheaf of connective spectra.*

Proof. We will prove descent of the Čech nerve for a covering $T' \times \mathbb{G} \to T \times \mathbb{G}$. That is, we would like to show that the following is a limit diagram:

$$B\mathfrak{Pic}(\mathcal{E}(T)) \longrightarrow B\mathfrak{Pic}(\mathcal{E}(T')) \rightrightarrows B\mathfrak{Pic}(\mathcal{E}(T' \times_T T')) \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots$$

We will consider the Bousfield-Kan spectral sequence for the limit of the Čech complex, which reads

$$E_2^{s,t} = \pi^s \pi_t B\mathfrak{Pic}(\mathcal{E}(T'^{\times_T \bullet+1})) \implies \pi_{t-s} \lim B\mathfrak{Pic}(\mathcal{E}(T'^{\times_T \bullet+1})) \tag{5.5}$$

By Theorem 3.11, the homotopy presheaves of $B\mathfrak{Pic}(\mathcal{E})$ are

$$\pi_t B\mathfrak{Pic}(\mathcal{E}) = \mathrm{Cont}_{\mathbb{G}}(-, \pi_t B\mathfrak{Pic}(\mathbf{E})).$$

We claim that $E_2^{s,s-1} = 0$ for $s > 0$, so that the map

$$B\mathfrak{Pic}(\mathcal{E}(T)) \to \lim B\mathfrak{Pic}(\mathcal{E}(T'^{\times_T \bullet+1}))$$

is an equivalence; note that the only possible difference is on π_0, since $\tau_{\geq 1} B\mathfrak{Pic}(\mathcal{E})$ is a sheaf of connected spaces. We are therefore interested in the cosimplicial abelian groups

$$\mathrm{Cont}(T', \pi_{s-1} B\mathfrak{Pic}(\mathbf{E})) \rightrightarrows \mathrm{Cont}(T' \times_T T', \pi_{s-1} B\mathfrak{Pic}(\mathbf{E})) \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots . \tag{5.6}$$

This is split when $T' \to T$ is a covering of finite sets, since there is a section at the level of coverings. As a result, by passing to a limit of finite coverings we see that (5.6) is also split. In particular, the descent spectral sequence (5.5) collapses immediately to the 0-line, which implies hyperdescent by [CM21, Proposition 2.25]. $\square$

Definition 5.13. Write $\mathfrak{Br}(\mathcal{E} \mid \mathbf{E}) \in \mathcal{S}h(B\mathbb{G}_{\mathrm{proet}}, \mathcal{S})$ for the sheafification of $S \mapsto B\,\mathrm{Pic}(\mathcal{E}(S))$ on $B\mathbb{G}_{\mathrm{proet}}$. By Lemma 5.12, no sheafification is required on the subsite $\mathrm{Free}_{\mathbb{G}}$.

Corollary 5.14. *For any closed subgroup $U \subset \mathbb{G}$, we have*

$$\mathfrak{Br}^{\mathrm{coh}}(\mathbf{E}^{hU} \mid \mathbf{E}) \simeq \Gamma(\mathbb{G}/U, \mathfrak{Br}(\mathcal{E} \mid \mathbf{E})).$$

In particular, $\mathfrak{Br}^{\mathrm{coh}}(\mathbf{1} \mid \mathbf{E}) \simeq \Gamma\mathfrak{Br}(\mathcal{E} \mid \mathbf{E})$. $\square$

5.2 Compact generators

Given a Galois extension $\mathbf{1} \to A$ in a stable homotopy theory $\mathcal{C}$, we showed in Section 5.1 that the Picard spectral sequence computes the subgroup

$$\mathrm{Br}^{\mathrm{coh}}(\mathbf{1} \mid A) \coloneqq \pi_0 B\mathfrak{Pic}(A)^{hG} \subset \mathrm{Pic}(\mathrm{Mod}_{\mathcal{C}})$$

of the Brauer-Grothendieck group. To relate this to the Brauer-Azumaya group classifying Azumaya algebras in $\mathcal{C}$, we prove a descent result for compact generators in the $K(h)$-local setting. This is entirely analogous to the theory of [AG14, §6.3] and [GL21, §6.4].

Remark 5.15. Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category, and $\mathcal{D} \in \mathrm{Mod}_{\mathcal{C}}$. Then $\mathcal{D}$ is $\mathcal{C}$-tensored, and for any $D \in \mathcal{D}$ the functor

$$- \otimes D : \mathcal{C} \to \mathcal{D}$$

preserves colimits. By presentability it admits a right adjoint, which we shall denote

$$\underline{\mathrm{Map}}_{\mathcal{D}}(D, -) : \mathcal{D} \to \mathcal{C}.$$

Given $D' \in \mathcal{D}$, the object $\underline{\mathrm{Map}}_{\mathcal{D}}(D, D')$ is a mapping object; this endows $\mathcal{D}$ with a canonical $\mathcal{C}$-enrichment [GH15, Example 7.2.12].

Definition 5.16. Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category, and $\mathcal{D} \in \mathrm{Mod}_{\mathcal{C}}$. An object $D \in \mathcal{D}$ is $\mathcal{C}$-*compact* if the functor

$$\underline{\mathrm{Map}}_{\mathcal{D}}(D, -) : \mathcal{D} \to \mathcal{C} \tag{5.7}$$

preserves filtered colimits. D is an $\mathcal{C}$-*compact generator* if it is $\mathcal{C}$-compact. We say that D is a $\mathcal{C}$-*generator* if the functor (5.7) is conservative; when $\mathcal{C}$ is stable, it is equivalent that (5.7) detects zero objects. A $\mathcal{C}$-*compact generator* of $\mathcal{D}$ is an object $D \in \mathcal{D}$ that is both $\mathcal{C}$-compact and a $\mathcal{C}$-generator, and we shall write $\mathcal{D}^{\mathrm{eg}} \subset \mathcal{D}$ [1] for the full subcategory of such D.

Example 5.17. The $K(h)$-local sphere is an $\mathcal{S}p_{K(h)}$-compact generator of $\mathcal{S}p_{K(h)}$. More generally, we always have $\mathbf{1} \in \mathcal{C}^{\mathrm{eg}}$ since in this case (5.7) is the identity functor.

Our first objective is to show that Schwede-Shipley theory goes through in the presence of a $\mathcal{C}$-compact generator.

Definition 5.18. Let $\mathcal{C}$ be a stable homotopy theory. We say $\mathcal{C}$ is *rigidly generated* if it is generated under colimits by dualisable objects. That is, the localising category generated by $\mathcal{C}^{\mathrm{dbl}}$ is $\mathcal{C}$ itself.

Example 5.19. *(i)* $\mathcal{S}p$ is generated under colimits by shifts of $\mathbf{1}$, and so rigidly generated.

[1] For "enriched generators".

(*ii*) If $\mathcal{C}$ is a rigidly generated stable homotopy theory and $L\colon \mathcal{C} \to \mathcal{C}'$ a monoidal localisation, then $\mathcal{C}'$ is rigidly generated. Thus $\mathcal{S}p_{K(h)}$ is rigidly generated.

(*iii*) For a compact lie group G, the ∞-category $\mathcal{S}^G$ of G-spaces is generated under colimits by orbits G/H (e.g. [MM02, Theorem 1.8]). Its stabilisation $\mathcal{S}p^G_{\mathscr{U}}$ at any G-universe $\mathscr{U}$ (as defined in [GM23, Corollary C.7]) is generated under colimits by shifts $\Sigma^{-V}\Sigma^\infty_{\mathscr{U}} G/H_+$ as V ranges over representations in $\mathscr{U}$, and if $\mathscr{U}$ is complete then these are dualisable by virtue of the Wirthmüller isomorphism [GM95, Theorem 4.17]. Thus the ∞-category $\mathcal{S}p^G$ of genuine G-spectra is rigidly generated.

Proposition 5.20 (Enriched Schwede-Shipley)**.** *Let $\mathcal{C}$ be a rigidly generated stable homotopy theory and $\mathcal{D} \in \mathrm{Mod}_{\mathcal{C}}$. Suppose that $D \in \mathcal{D}^{\mathrm{eg}}$, and write $A := \underline{\mathrm{End}}_{\mathcal{D}}(D) \in \mathrm{Alg}(\mathcal{C})$. Then there is an $\mathcal{C}$-linear equivalence*

$$\mathcal{D} \simeq \mathrm{LMod}_A(\mathcal{C}).$$

Proof. The object $D \in \mathcal{D}$ determines canonically a $\mathcal{C}$-linear left adjoint $F\colon \mathcal{C} \to \mathcal{D}$, with right adjoint $G := \underline{\mathrm{Map}}_{\mathcal{D}}(D, -)$. According to [HA, Proposition 4.8.5.8], it is enough to check the following:

(*i*) G preserves colimits of simplicial objects: in fact G preserves all colimits. Indeed, G preserves filtered colimits since D is $\mathcal{C}$-compact, and finite colimits as it is a right adjoint.

(*ii*) G is conservative: this is by definition of $\mathcal{C}$-compact generators.

(*iii*) for every $D' \in \mathcal{D}$ and $C \in \mathcal{C}$, the map

$$C \otimes FG(D') = C \otimes G(D') \otimes D \to C \otimes D'$$

is adjoint to an equivalence

$$C \otimes G(D') \xrightarrow{\sim} G(C \otimes D'). \tag{5.8}$$

But by (*i*), the functor G preserves all colimits, so by rigid generation we reduce to C dualisable. In this case, (5.8) is the composite equivalence

$$\begin{aligned} C \otimes \underline{\mathrm{Map}}_{\mathcal{D}}(D, D') &\simeq \underline{\mathrm{Map}}_{\mathcal{C}}(C^\vee, \underline{\mathrm{Map}}_{\mathcal{D}}(D, D')) \\ &\simeq \underline{\mathrm{Map}}_{\mathcal{D}}(C^\vee \otimes D, D') \\ &\simeq \underline{\mathrm{Map}}_{\mathcal{D}}(D, C \otimes D'). \end{aligned}$$ □

To use this to produce Azumaya algebras, we first need to be able to produce $\mathcal{C}$-compact generators. For this we will prove the following result:

Proposition 5.21 (Descent for $\mathcal{C}$-compact generators)**.** *Let $\mathbf{1} \to A$ be a descendable extension in a rigidly generated stable homotopy theory $\mathcal{C}$, and suppose that A is dualisable. Then $\mathcal{D} \in \mathrm{Mod}_{\mathcal{C}}$ admits a $\mathcal{C}$-compact generator if and only if $\mathrm{Mod}_A(\mathcal{D}) := \mathcal{D} \otimes_{\mathcal{C}} \mathrm{Mod}_A(\mathcal{C})$ admits one.*

The proof relies on the following basic lemma:

Lemma 5.22. *Suppose $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L)$ and $A \in \mathcal{C}$ is dualisable, then A is faithful if and only if $A^\vee$ is.*

Proof. Assume that A is faithful, and that $A^\vee \otimes X = 0$; the converse is given by taking duals. Then the identity on $A \otimes X$ factors as

$$A \otimes X \to A^\vee \otimes A \otimes A^\vee \otimes X \to A \otimes X,$$

and in particular $A \otimes X = 0$. By faithfulness of A, this implies $X = 0$. $\square$

Proof (Proposition 5.21). As in [GL21], we will make use of the functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left=2ex, "i_!"] \arrow[r, shift right=2ex, "i_*"'] & \mathrm{Mod}_A(\mathcal{C}) \arrow[l, "i^*" description] \end{tikzcd}$$

and the adjunctions (denoted by the same symbols) between $\mathcal{D}$ and $\mathrm{Mod}_A(\mathcal{D})$. In fact, we claim these adjunctions are $\mathcal{C}$-linear.

Assuming this, we give the proof of the proposition. If $\mathcal{D}$ admits a $\mathcal{C}$-compact generator D, it is straightforward to check that $i_!D \in \mathrm{Mod}_A(\mathcal{D})^{\mathrm{eg}}$: indeed, $i_!D$ is $\mathcal{C}$-compact because D is so and i^* preserves colimits, while $i_!D$ generates because D does and i^* is conservative. Conversely, suppose we have $D \in \mathrm{Mod}_A(\mathcal{D})^{\mathrm{eg}}$, and consider $i^*D \in \mathcal{D}$. By dualisability of A, the right adjoint

$$i_* = \underline{\mathrm{Map}}_{\mathcal{C}}(A, -) \simeq A^\vee \otimes - : \mathcal{C} \to \mathrm{Mod}_A(\mathcal{C})$$

preserves colimits, and hence the right adjoint $i_* \colon \mathrm{Mod}_A(\mathcal{D}) \to \mathcal{D}$ does too. As a result, i^*D is $\mathcal{C}$-compact. On the other hand, if $X \in \mathcal{D}$ and $\underline{\mathrm{Map}}_{\mathcal{D}}(i^*D, X) = 0$, then $\underline{\mathrm{Map}}_{\mathrm{Mod}_A(\mathcal{D})}(D, i_*X) = 0$ and so

$$i_*X = A^\vee \otimes X = 0.$$

Now faithfulness of $A^\vee$ implies that $X = 0$.

It remains to prove that the adjunctions $i_! \dashv i^* \dashv i_*$ are $\mathcal{C}$-linear. For $i_!$ we observed $\mathcal{C}$-linearity in the proof of Proposition 5.20. To see $\mathcal{C}$-linearity for i^* it is enough to prove that the canonical map

$$\theta : C \otimes i^*M \to i^*(C \otimes M)$$

is an equivalence for every C and M, and by rigid generation we reduce to C dualisable. As in

[SAG, Remark D.7.4.4] one checks that $\mathrm{Map}_{\mathcal{C}}(C', \theta)$ is the composite equivalence

$$\begin{aligned}\mathrm{Map}(C', C \otimes i^* M) &\simeq \mathrm{Map}(C' \otimes C^\vee, i^* M)\\ &\simeq \mathrm{Map}(i_!(C' \otimes C^\vee), M)\\ &\simeq \mathrm{Map}(i_! C' \otimes C^\vee, M)\\ &\simeq \mathrm{Map}(i_!(C'), C \otimes M)\\ &\simeq \mathrm{Map}(C', i^*(C \otimes M))\end{aligned}$$

for any $C' \in \mathcal{C}$, which gives the claim. □

Example 5.23. If $A \to B$ is an E-local Galois extension of ring spectra with stably dualisable Galois group G, then Rognes [Rog08, Proposition 6.2.1] shows that B is dualisable over A. For example, this covers the following cases:

(*i*) $E = \mathbb{S}$ and G is finite or compact Lie.

(*ii*) $E = \mathbb{F}_p$ and G is p-compact.

(*iii*) $E = \mathbf{K}$ and $G = K(\pi, m)$ for π a finite p-group and $m \leq h$.

Corollary 5.24 ($\mathrm{Br} = \mathrm{Br}'$)**.** *Let* $\mathbf{1} \to A$ *be a faithful dualisable Galois extension in a rigidly generated stable homotopy theory* $\mathcal{C}$, *and* $\mathcal{Q} \in \pi_0 B\mathfrak{Pic}(A)^{hG}$ *a relative Brauer-Grothendieck class. Then* $\mathcal{Q}$ *is represented by some Azumaya algebra* Q *whose basechange to* A *is (Morita) trivial: that is,*

$$\mathcal{Q} \simeq \mathrm{Mod}_Q(\mathcal{C}) \in \mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^L),$$

and $\mathrm{Mod}_{A \otimes Q}(\mathcal{C}) \simeq \mathrm{Mod}_A(\mathcal{C})$.

Proof. We claim that $\mathcal{Q}$ is κ-compactly generated, so that $\mathcal{Q} \in \mathrm{Pic}(\mathrm{Mod}_{\mathcal{C}})$. Given this, the result follows from Proposition 5.21: by assumption, $\mathrm{Mod}_A(\mathcal{Q}) \simeq \mathrm{Mod}_A(\mathcal{C})$, and so

$$A \in \mathrm{Mod}_A(\mathcal{C})^{\mathrm{eg}} \simeq \mathrm{Mod}_A(\mathcal{Q})^{\mathrm{eg}}.$$

By descent for compact generators we obtain $D \in \mathcal{Q}^{\mathrm{eg}}$, and so Schwede-Shipley theory yields a $\mathcal{C}$-linear equivalence $\mathcal{Q} \simeq \mathrm{Mod}_Q(\mathcal{C})$, where $Q = \underline{\mathrm{End}}_{\mathcal{Q}}(D)$.

For the claim, note that as in [Mat16, Corollary 3.42], $\mathcal{Q}$ is the limit of the cosimplicial diagam

$$\mathcal{Q} \otimes_{\mathcal{C}} \mathrm{Mod}_A(\mathcal{C}) \rightrightarrows \mathcal{Q} \otimes_{\mathcal{C}} \mathrm{Mod}_{A^{\otimes 2}}(\mathcal{C}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots,$$

and for $j \geq 1$ we have $\mathcal{Q} \otimes \mathrm{Mod}_{A^{\otimes j}}(\mathcal{C}) \simeq \mathrm{Mod}_{A^{\otimes j}}(\mathcal{C}) \in \mathrm{Pr}^L_\kappa$. Thus $\mathcal{Q} \in \mathrm{Pr}^L_\kappa$. □

Remark 5.25. While $\mathbf{E}$ is Spanier-Whitehead self-dual [Str00; Bea+22c] and hence *reflexive*, it is not dualisable: for example, $\mathbf{K}_*\mathbf{E}$ would otherwise be finite by [HS99, Theorem 8.6]. We will bypass this issue at height one by showing that all generators of $\mathrm{Br}'(\mathcal{S}p_{\mathbf{K}} \mid \mathbf{E})$ are in fact trivialised in a *finite* Galois extension of the sphere, and hence lift to Azumaya algebras by Example 5.23 and Corollary 5.24; we do not know if $\mathrm{Br}_h^0 \cong \mathrm{Br}'(\mathcal{S}p_{\mathbf{K}} \mid \mathbf{E})$ at arbitrary height.

Part II

The relative Brauer group at height one

Chapter 6

Introduction

The classification of central simple algebras over a field is a classical question in number theory, and by the Wedderburn theorem any such algebra is a matrix algebra over some division algebra, determined up to isomorphism. If one identifies those algebras which arise as matrix algebras over the same division ring, then tensor product defines a group structure on the resulting set of equivalence classes. This relation is *Morita equivalence*, and the resulting group is the *Brauer group* $\mathrm{Br}(K)$. One formulation of class field theory is the determination of $\mathrm{Br}(K)$ in the case that K is a number field.

One consequence of Wedderburn's theorem is that every central simple algebra is split by some extension L/K; in fact, one can take L to be Galois. This opens the door to cohomological descriptions of the Brauer group: by Galois descent, one obtains the first isomorphism in

$$\mathrm{Br}(K) \cong H^1(K, \mathrm{PGL}_\infty) \cong H^2(K, \mathbb{G}_m), \tag{6.1}$$

where PGL_∞ denotes the Galois module $\varinjlim \mathrm{PGL}_k(K^{\mathrm{s}})$. The second isomorphism is [Ser79, Proposition X.9], and follows from Hilbert 90. This presents $\mathrm{Br}(K)$ as an étale cohomology group, and allows the use of cohomological techniques in its determination. Conversely, it gives a concrete interpretation of 2-cocycles, analogous to the relation between 1-cocycles and Picard elements.

Globalising this picture was a deep problem in algebraic geometric, initiated by the work of Azumaya, Auslander and Goldman, and the Grothendieck school. There are two ways to proceed: on one hand, one can generalise the notion of central simple algebra, obtaining that of an *Azumaya algebra* on X, and the group $\mathrm{Br}(X)$ classifying such algebras (again, up to Morita equivalence). On the other, one can use the right-hand side of (6.1), which globalises in an evident way; this yields the more computable group $\mathrm{Br}'(X) \coloneqq H^2(X, \mathbb{G}_m)_{\mathrm{tors}}$. A famous problem posed by Grothendieck asks if these groups agree in general. While there is always an injective map $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}'(X)$, surjectivity is known to fail for arbitrary schemes (e.g. [Edi+01, Corollary 3.11]). The fundamental insight of Toën was that it is key to pass to *derived* Azumaya algebras, in which case one *can* represent any class in $H^2(X, \mathbb{G}_m)$ as a derived Azumaya algebra, including nontorsion classes [Toë12].

Toën's work shows that even the Brauer groups of classical rings are most naturally studied in the context of derived or homotopical algebraic geometry. This initiated a study of Brauer groups through the techniques of higher algebra, and in recent years they have become objects of intense study in homotopy theory.

In this context, the basic definitions appear in [BRS12]: as in the classical case, the Brauer group of a ring spectrum R classifies Morita equivalence classes of Azumaya algebras, defined by entirely analogous axioms. This gives a useful class of noncommutative algebras over an $\mathbb{E}_\infty$-ring: for example, Hopkins and Lurie [HL17] computed Brauer groups as part of a program to classify E_h-algebra structures on Morava K-theory, where E_h denotes Lubin-Tate theory. Categorifying, the Azumaya condition on an R-algebra A is equivalent to the requirement that the ∞-category Mod_A be R-linearly invertible, and so the Brauer group is also the Picard group of the $\mathbb{E}_\infty$-algebra $\mathrm{Mod}_R \in \mathrm{Pr}^L$; in this guise the Brauer space and its higher-categorical analogues are the targets for (invertible) factorisation homology, as an extended quantum field theory [Lur09; Sch14; AF17; Hau17; FH21]. Computations of Brauer groups of ring spectra of particular interest appear in [AG14; GL21; AMS22].

In this document we study Brauer groups in the monochromatic setting. Recall that Hopkins and Lurie focus on the $K(h)$-local Brauer group of E_h. Our objective is to extend this to the Brauer group of the entire $K(h)$-local category: explicitly, this group classifies of $K(h)$-local Azumaya algebras up to Morita equivalence. The computations of *op. cit.* show that the groups $\mathrm{Br}(E_h)$ are highly nontrivial, unlike the analogous Picard groups. We therefore complement the this by computing the *relative* Brauer group, which classifies those $K(h)$-local Azumaya algebras which become trivial after basechange to E_h. In analogy with the established notation for Picard groups (e.g. [Goe+15, p. 5]), this group will be denoted Br_h^0. It fits in an exact sequence

$$1 \to \mathrm{Br}_h^0 \to \mathrm{Br}_h \to \mathrm{Br}(E_h)^{\mathbb{G}_h},$$

where $\mathbb{G}_h$ denotes the Morava stabiliser group, and this gives access (at least in theory) to the group $\mathrm{Br}_h \coloneqq \mathrm{Br}(\mathcal{S}p_{K(h)})$. The group Br_h^0 also has a concrete interpretation in terms of chromatic homotopy theory: it classifies twists of the $\mathbb{G}_h$-action on the ∞-category $\mathrm{Mod}_{E_h}(\mathcal{S}p_{K(h)})$. For the standard action (by basechange along the Goerss-Hopkins-Miller action on E_h), one has $\mathcal{S}p_{K(h)} \simeq \mathrm{Mod}_{E_h}(\mathcal{S}p_{K(h)})^{h\mathbb{G}_h}$ (see [Mat16, Proposition 10.10] and (1.1) for two formulations). Taking fixed points for a twisted action therefore gives a twisted version of the $K(h)$-local category.

Our main theorems give the computation of the relative Brauer group at height one:

Theorem F (Theorem 9.13)**.** *At the prime two,*

$$\mathrm{Br}_1^0 \cong \mathbb{Z}/8\{Q_1\} \oplus \mathbb{Z}/4\{Q_2\}.$$

(i) The $\mathbb{Z}/4$-factor is mapped injectively to $\mathrm{Br}(KO_2)$, and $Q_2^{\otimes 2} \otimes KO_2$ is the image of the generator under

$$\mathbb{Z}/2 \cong \mathrm{Br}(KO \mid KU) \to \mathrm{Br}(KO_2 \mid KU_2).$$

The $\mathbb{Z}/8$ factor is the relative Brauer group $\mathrm{Br}(\mathcal{S}p_{K(1)} \mid KO_2)$.

(ii) $Q_4 := Q_1^{\otimes 4}$ *is the cyclic algebra formed using the C_2-Galois extension* $\mathbb{S}_{K(1)} \to KU_2^{h(1+4\mathbb{Z}_2)}$ *and the* strict *unit*

$$1 + \varepsilon \in \pi_0 \mathrm{GL}_1(\mathbb{S}_{K(1)}) = (\mathbb{Z}_p[\varepsilon]/(2\varepsilon, \varepsilon^2))^\times.$$

We have indexed generators on the filtration in which they are detected in the descent spectral sequence, which we recall later in the introduction. For the final part, note [BRS12, §4] that cyclic algebras are defined using *strict* units: that is, maps of spectra

$$u : \mathbb{Z} \to \mathfrak{gl}_1(E).$$

We give an alternative construction of cyclic algebras from strict units in Chapter 8, and using this we show that they are detected in the HFPSS by a *symbol* in the sense of [Ser79, Chapter XIV]. This allows us to deduce when they give rise to nontrivial Brauer classes.

Any strict unit has an underlying unit, and we abusively denote these by the same symbol. The unit $1 + \varepsilon$ was shown to be strict in [CY23], which is what is what gives rise to the claimed representative for the class Q_4 at the prime two. Likewise, at odd primes the roots of unity are strict, which leads to our second main computation:

Theorem G (Theorem 9.1)**.** *At odd primes,*

$$\mathrm{Br}_1^0 \cong \mathbb{Z}/(p-1).$$

A generator is given by the cyclic algebra $(KU_p^{h(1+p\mathbb{Z}_p)}, \chi, \omega)$*, where* $\chi : \mu_{p-1} \cong \mathbb{Z}/(p-1)$ *is a character and* $\omega \in (\pi_0 \mathbb{S}_{K(1)})^\times \cong \mathbb{Z}_p^\times$ *is a primitive* $(p-1)$*st root of unity.*

We now give an outline of the computation. Since Grothendieck, the main approach to computing Brauer groups has been *étale* or *Galois descent*, and this is the case for us too. Namely, recall that at any height h, Morava E-theory defines a $K(h)$-local Galois extension

$$\mathbb{S}_{K(h)} \to E_h,$$

with profinite Galois group $\mathbb{G}_h$. In Part I, we used condensed mathematics to prove a Galois descent statement of the form

$$\mathcal{S}p_{K(h)} \simeq \mathrm{Mod}_{E_h}(\mathcal{S}p_{K(h)})^{h\mathbb{G}_h},$$

and deduced from this a homotopy fixed point spectral sequence for Picard and Brauer groups, extending the Galois descent results of Mathew and Stojanoska [MS16] and Gepner and Lawson [GL21]. For our purposes, the main computational upshot of that paper is:

Theorem 6.1. *There is a descent spectral sequence*

$$E_2^{s,t} = H^s(\mathbb{G}_h, \pi_t \mathfrak{pic}(E_h)) \implies \pi_{t-s} \mathfrak{pic}(\mathcal{S}p_{K(h)}).$$

whose (-1)-stem gives an upper bound on Br_h^0 as a subgroup. There is an explicit comparison of differentials with the $K(h)$-local E_h-Adams spectral sequence.

A more precise form of the theorem is recalled in Chapter 7. In the present paper, we determine this spectral sequence completely at height one, at least for $t-s \geq -1$. Unsurprisingly, the computation looks very different in the cases $p=2$ and $p>2$, and the former represents the majority of our work. To prove a lower bound on Br_1^0, we also prove a realisation result in the spirit of Toën's theorem. Namely, we show in a very general context that all classes on the E_∞-page may be represented by Azumaya algebras:

Theorem H (Corollary 5.24)**.** *Let $\mathcal{C}$ be a nice symmetric monoidal ∞-category and $A \in \mathrm{CAlg}(\mathcal{C})$. Then the map sending an Azumaya algebra to its module ∞-category,*

$$\mathrm{Br}(\mathcal{C} \mid A) \to \mathrm{Br}'(\mathcal{C} \mid A) := \{\mathcal{D} : \mathrm{Mod}_A(\mathcal{D}) \simeq \mathrm{Mod}_A(\mathcal{C})\} \subset \mathrm{Pic}(\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^L))$$

is an isomorphism.

See also [AG14, §6.3] and [GL21, §6.4] for related results. We refer the reader to Section 5.2 for the exact conditions in the theorem; most importantly, the chromatic localisations of spectra give examples of such ∞-categories. This accounts immediately for most of the classes in Br_1^0, by sparsity in the descent spectral sequence. At the prime two there is one final computation necessary, which is the relative Brauer group of KO_2. Recall that the group $\mathrm{Br}(KO \mid KU)$ was computed by Gepner and Lawson [GL21]. In Section 9.2, we prove:

Theorem I (Theorem 9.3)**.** *The relative Brauer group of KO_2 is*

$$\mathrm{Br}(KO_2 \mid KU_2) \cong \mathbb{Z}/4,$$

and the basechange map from $\mathrm{Br}(KO \mid KU) \cong \mathbb{Z}/2$ is injective.

The generator, which we denote P_2, may be thought of as a cyclic algebra for the unit $1+4\zeta$, where ζ is a topological generator of $\mathbb{Z}_2$ (see Chapter 8). We remark that this unit is *not* strict, and so the cyclic algebra cannot be constructed by hand; instead, we invoke Theorem H to prove that the possible obstructions vanish. We then show that P_2 survives the descent spectral sequence for the extension $\mathbb{S}_{K(1)} \to KO_2$, and therefore gives rise to the class $Q_2 \in \mathrm{Br}_1^0$.

As an aside, we observe that Theorem I implies the following result, which may be of independent interest:

Theorem J. *There is no C_2-equivariant splitting*

$$\mathfrak{gl}_1 KU_2 \simeq \tau_{\leq 3}\mathfrak{gl}_1 KU_2 \oplus \tau_{\geq 4}\mathfrak{gl}_1 KU_2.$$

We do not know if such an equivariant splitting exists for $\mathfrak{gl}_1 KU$. It seems conceivable that one would arise from a discrete model for K-theory, and this is an interesting problem.

6.1 Outline

In Chapter 7, we compute the E_2-page of the descent spectral sequences and most differentials, giving an upper bound on the relative Brauer groups. The objective of the rest of the document is to show this bound is achieved. In Section 5.2 we proved that the property of "admitting a compact generator" satisfies $\mathcal{S}p_{K(h)}$-linear Galois descent, which reduces the problem to computing the E_∞-page and solving extension problems. In Chapter 8 we give a construction of cyclic algebras which makes clear where they are detected in the descent spectral sequence; this allows us to assert that the cyclic algebras we form are distinct, and completes the odd-prime computation. We also give a construction of Brauer classes from 1-cocycles, which we use at the prime two. In Chapter 9 we compute the relative Brauer group $\mathrm{Br}(KO_2 \mid KU_2)$, and use this to complete the computation of Br_1^0 at the prime two.

Chapter 7

The descent spectral sequence

In this short section, we record the descent spectral sequence that will be the starting point for our computations. At any characteristic (p, h), this arises as the descent spectral sequence for the sheaf

$$\mathfrak{pic}(\mathcal{E}) \in \mathcal{S}h(B\mathbb{G}_{\mathrm{proet}}, \mathcal{S}p)$$

of Section 3.2. The main input from is the following:

Theorem 7.1 (Theorem A and Corollary 5.24)**.** *(i) There is a strongly convergent spectral sequence*

$$E_2^{s,t} = H^s(\mathbb{G}, \pi_t \mathfrak{pic}(\mathbf{E})) \implies \pi_{t-s}\mathfrak{pic}(\mathcal{S}p_{\mathbf{K}}).$$

(ii) Its (-1)-stem converges to Br_h^1.

(iii) Differentials on the E_r page agree with those in the $\mathbf{K}$-local $\mathbf{E}$-Adams spectral sequence in the region $t \geq r+1$, and for classes $x \in E_2^{r,r}$ we have

$$d_r(x) = d_r^{ASS}(x) + x^2.$$

In Chapter 4, we used this spectral sequence to recover the computation of $\mathrm{Pic}_1 \coloneqq \mathrm{Pic}(\mathcal{S}p_{K(1)})$ (due to [HMS94]). In this case, Morava E-theory is the p-completed complex K-theory spectrum KU_p, acted upon by $\mathbb{G} \cong \mathbb{Z}_p^\times$ via Adams operations ψ^a.

In the rest of the section, we draw some immediate consequences. These suffice to give us an upper bound, which we show in Chapter 9 to be tight.

7.1 Odd primes

We first consider the case $p > 2$. The starting page of the Picard spectral sequence is recorded below:

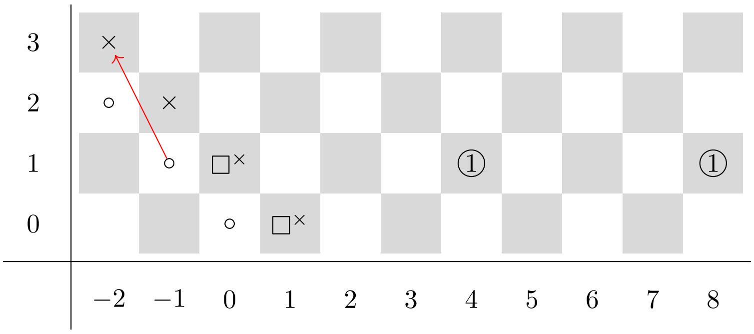


Figure 7.1: The height one Picard spectral sequence for odd primes (implicitly at $p = 3$). Classes are labelled as follows: $\circ = \mathbb{Z}/2$, $\square^\times = \mathbb{Z}_p^\times$, $\times = \mu_{p-1}$, and circles denote p-power torsion (labelled by the torsion degree). Since $\mathrm{Pic}_1 \cong \mathrm{Pic}_1^{\mathrm{alg}} \cong \mathbb{Z}_p^\times$, no differentials can hit the (-1)-stem. Differentials with source in stem $t - s \leq -2$ have been omitted.

Lemma 7.2 (Lemma B.1)**.** *At odd primes, the starting page of the descent spectral sequence is given by*

$$E_2^{s,t} = H^s(\mathbb{Z}_p^\times, \pi_t \mathfrak{pic}(KU_p)) = \begin{cases} \mathbb{Z}/2 & t = 0 \text{ and } s \geq 0 \\ \mathbb{Z}_p^\times & t = 1 \text{ and } s = 0, 1 \\ \mu_{p-1} & t = 1 \text{ and } s \geq 2 \\ \mathbb{Z}/p^{\nu_p(t)+1} & t = 2(p-1)t' + 1 \neq 1 \text{ and } s = 1 \end{cases} \tag{7.1}$$

This is displayed in Fig. 7.1. In particular, the spectral sequence collapses for degree reasons at the E_3-page.

Proposition 7.3. *At odd primes, Br_1^0 is isomorphic to a subgroup of μ_{p-1}.*

Proof. The only possible differentials are d_2-differentials on classes in the (-1)-stem; note that there are no differentials *into* the (-1)-stem, since every E_2-class in the 0-stem is a permanent cycle. The generator in $E_2^{1,0}$ supports a d_2, since this is the case for the class in $E_2^{1,0}$ of the descent spectral sequence for the C_2-action on KU [GL21, Prop. 7.15] (this is displayed in Fig. 7.3), and the span of Galois extensions

$$\begin{tikzcd} \mathbf{1_K} \arrow[r] \arrow[d] & KO_p \arrow[d] & KO \arrow[l] \arrow[d] \\ KU_p \arrow[r] & KU_p & KU \arrow[l] \end{tikzcd}$$

allows us to transport this differential (see also Fig. 9.1). Note that the induced span on E_2-pages is

$$\begin{tikzcd} \mathbb{Z}/2 \arrow[r, equal] \arrow[d, "d_2"] & \mathbb{Z}/2 \arrow[r, equal] \arrow[d, "d_2"] & \mathbb{Z}/2 \arrow[d, "d_2", "="'] \\ \mu_{p-1} \arrow[r, equal] & \mu_{p-1} & \mathbb{Z}/2 \arrow[l, hook'] \end{tikzcd}$$

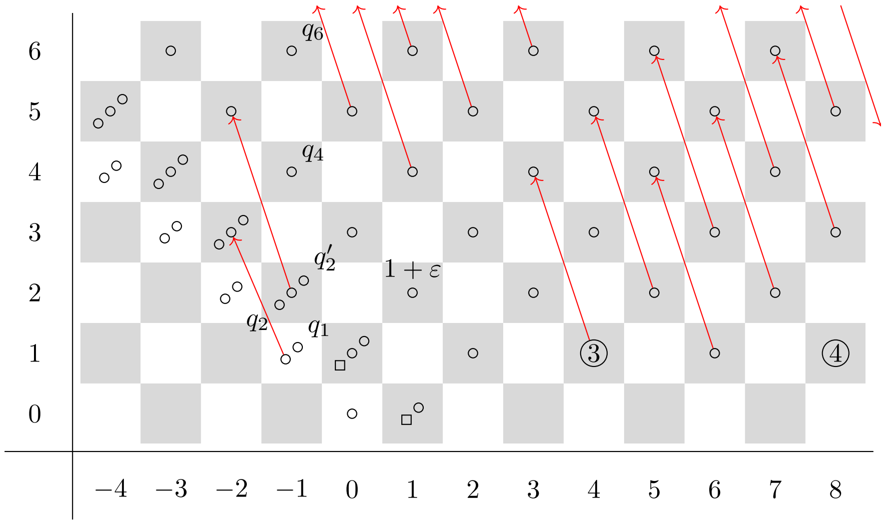


Figure 7.2: The Picard spectral sequence for the Galois extension $\mathbf{1_K} \to \mathbf{E} = KU_2$ at $p = 2$. We know that all remaining classes in the 0-stem survive, by comparing to the algebraic Picard group. Thus the only differentials that remain to compute are those *out* of the (-1)-stem; those displayed can be transported from the descent spectral sequence for $\mathfrak{Pic}(KO)^{hC_2}$—see Figs. 7.3 and 7.4. We have not displayed possible differentials out of stem ≤ -2.

in bidegrees $(s,t) = (1,0)$ and $(3,1)$ respectively. Thus

$$\pi_0(B\mathfrak{Pic}(KU_p))^{h\mathbb{Z}_p^\times} \cong \mu_{p-1}. \qquad \square$$

In Chapter 8 we will show that this bound is achieved using the cyclic algebra construction of [BRS12]; abstractly, this also follows from the detection result of Section 5.2.

7.2 The case $p = 2$

We now proceed with the computation of the (-1)-stem for the even prime.

Lemma 7.4 (Lemma B.2)**.** *We have*

$$H^s(\mathbb{Z}_2^\times, \mathrm{Pic}(KU_2)) = \begin{cases} \mathbb{Z}/2 & s = 0 \\ (\mathbb{Z}/2)^2 & s \geq 1 \end{cases}$$

$$H^s(\mathbb{Z}_2^\times, (\pi_0 KU_2)^\times) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}/2 & s = 0 \\ \mathbb{Z}_2 \oplus (\mathbb{Z}/2)^2 & s = 1 \\ (\mathbb{Z}/2)^3 & s \geq 2 \end{cases}$$

The resulting spectral sequence is displayed in Figure 7.2.

Proposition 7.5. *At the prime two,*

$$|\,\mathrm{Br}_1^0\,| \leq 32.$$

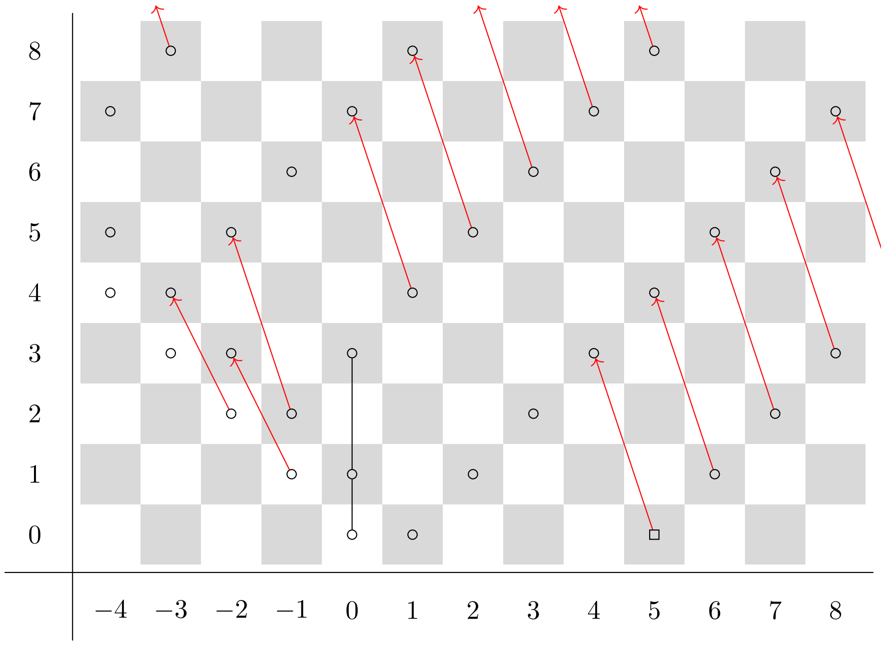


Figure 7.3: The E_3-page of the Picard spectral sequence for KO, as in [GL21, Figure 7.2].

Proof. In Chapter 4, we determined the following differentials:

- in degrees $t - s \geq 3$, differentials agree with the well-known pattern of Adams differentials (e.g. [BGH22, Figure 3]).
- the class in bidegree $(s, t) = (3, 3)$, which supports a d_3 in the Adams spectral sequence, is a permanent cycle.

By comparing with the Adams spectral sequence, any classes in the (-1)-stem that survive to E_∞ are in filtration at most six; on the E_2-page, there are seven such generators. By comparing to the HFPSS for $\mathfrak{Br}(KO_2 \mid KU_2) := (B\mathfrak{Pic}(KU_2))^{hC_2}$ as in Section 7.1, we obtain the following differentials:

- a d_2 on the class in $H^1(C_2, \mathrm{Pic}(KU_2)) \subset H^1(\mathbb{Z}_2^\times, \mathrm{Pic}(KU_2))$,
- a d_3 on the class in $H^2(C_2, (\pi_0 KU_2)^\times) \subset H^2(\mathbb{Z}_2^\times, (\pi_0 KU_2)^\times)$.

This gives the claimed upper bound. $\square$

In Chapter 9 we will show that this bound is *also* achieved.

Remark 7.6. For later reference, we name the following generators:

(i) $q_1 \in E_2^{1,0}$ is the generator of $H^1(1 + 4\mathbb{Z}_2, \mathbb{Z}/2) \subset H^1(\mathbb{Z}_2^\times, \mathbb{Z}/2)$,

(ii) $q_2 \in E_2^{2,1}$ is the generator of $H^2(C_2, 1 + 4\mathbb{Z}_2) \subset H^2(\mathbb{Z}_2^\times, \mathbb{Z}_2^\times)$,

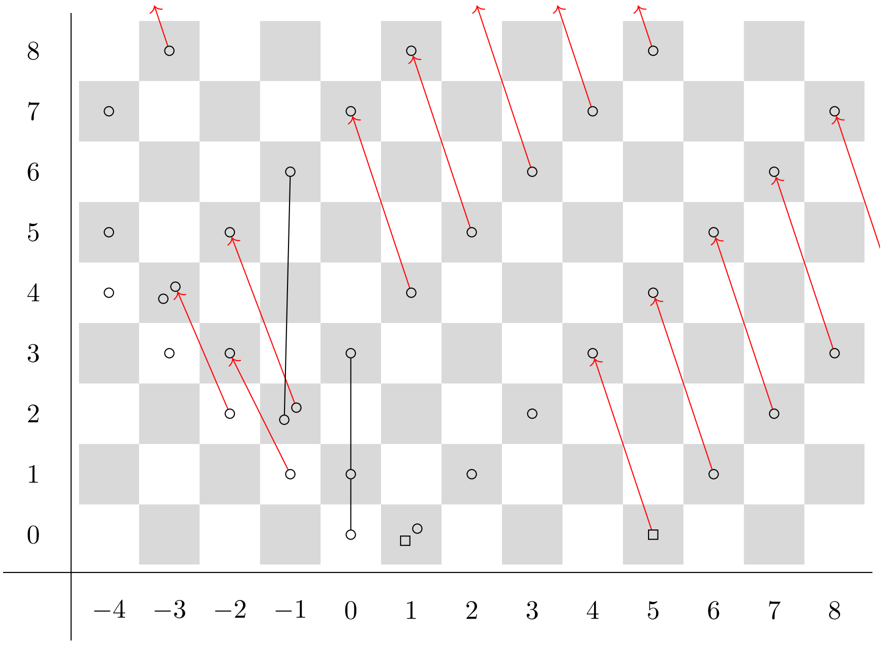


Figure 7.4: The Picard spectral sequence for KO_2. Differentials come from the comparison with KO, and the resulting map of spectral sequences. See Chapter 9 for the extension in the (-1)-stem.

(*iii*) $q_2' \in E_2^{2,1}$ is the generator of $H^1(C_2, \mathbb{Z}_2^\times) \otimes H^1(\mathbb{Z}_2, \mathbb{Z}_2^\times) \subset H^2(\mathbb{Z}_2^\times, \mathbb{Z}_2^\times)$,

(*iv*) q_4 is the unique class in $E_2^{4,3}$,

(*v*) q_6 is the unique class in $E_2^{6,5}$.

While q_6 survives to E_∞ by Fig. 7.2, the other classes are sources of possible differentials. We will show that in fact all are permanent cycles.

Chapter 8

Explicit generators

In this section, we give some explicit constructions of Azumaya algebras from Galois extensions. Most of this section works in an arbitrary stable homotopy theory $\mathcal{C}$. We will use these constructions in Chapter 9 to describe generators of the group Br_1^0, and hence to solve extension problems.

8.1 $\mathbb{Z}_p$-extensions

We will begin with a straightforward construction for extensions with Galois group $\mathbb{Z}_p$, using the fact that $\mathrm{cd}(\mathbb{Z}_p) = 1$.

Remark 8.1. Let G be a profinite group, and $\mathcal{B} \in \mathcal{S}h(BG_{\mathrm{proet}}, \mathcal{S}_*)$. Under suitable assumptions on $\mathcal{B}$, décalage gives an isomorphism between the descent spectral sequence and the spectral sequence for the Čech nerve of $G \to *$ (Appendix A). Moreover, the homotopy groups of the latter can often be identified with the complex of continuous cochains with coefficients in $\pi_t B$, yielding an isomorphism on the E_2-pages

$$H^*(G, \pi_t B) \to H^*(BG_{\mathrm{proet}}, \pi_t \mathcal{B}). \tag{8.1}$$

For example, this is the case for the sheaf $\mathfrak{pic}(\mathcal{E})$ for any Morava E-theory $\mathbf{E} = E(k, \Gamma)$.

Lemma 8.2. *Let $G = \mathbb{Z}_p$ or $\widehat{\mathbb{Z}}$, and write $\zeta \in G$ for a topological generator. Suppose $\mathcal{C}$ is a stable homotopy and $\mathcal{B} \in \mathcal{S}h(BG_{\mathrm{proet}}, \mathcal{S}_*)$, with $B \coloneqq \mathcal{B}(G/*)$ and $B^{hG} \coloneqq \Gamma\mathcal{B}$. If the canonical map (8.1) is an isomorphism, then*

$$B^{hG} \simeq \mathrm{Eq}\left(id, \zeta : B \rightrightarrows B\right). \tag{8.2}$$

Proof. Write B' for the equaliser in (8.2). The G-map $(id, \zeta)\colon G \to G \times G$ gives rise to maps

$$B \rightrightarrows \mathcal{B}(G \times G) \to B \tag{8.3}$$

factoring $id, \zeta\colon B \rightrightarrows B$, and the identification $B^{hG} \simeq \lim \mathcal{B}(G^{\bullet+1})$ gives a distinguished null-homotopy in (8.3) after precomposing with the coaugmentation $\eta\colon B^{hG} \to B$. Thus η factors through $\theta\colon B^{hG} \to B'$. Taking fibres, the descent spectral sequence for $\mathcal{B}$ implies that $\pi_*\theta$ fits in a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (\pi_{t+1}B)_G \arrow[r] \arrow[d, equal] & \pi_t B^{hG} \arrow[r] \arrow[d, "\pi_t\theta"] & (\pi_t B)^G \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (\pi_{t+1}B)_G \arrow[r] & \pi_t B' \arrow[r] & (\pi_t B)^G \arrow[r] & 0
\end{tikzcd}$$

and is therefore an equivalence. $\square$

Construction 8.3. Suppose $\mathbf{1} \to A$ is a descendable Galois extension in a stable homotopy theory $\mathcal{C}$, with group $G = \mathbb{Z}_p$ or $\widehat{\mathbb{Z}}$. Then $\mathcal{B} := B\mathfrak{Pic}(A) \in \mathcal{S}h(BG_{\mathrm{proet}}, \mathcal{S}_*)$, and in good cases $\mathcal{B}$ satisfies the assumption that (8.1) be an isomorphism: for example, this is the case whenever each $\pi_t\mathfrak{pic}(A)$ is profinite by [BS14, Lemma 4.3.9]. Thus

$$\mathfrak{Br}(\mathbf{1} \mid A) \simeq \mathrm{Eq}\left(id, \zeta^* : B\mathfrak{Pic}(A) \rightrightarrows B\mathfrak{Pic}(A)\right),$$

and so a relative Brauer class is given by the data of an A-linear equivalence

$$\xi : \mathrm{Mod}_A(\mathcal{C}) \xrightarrow{\sim} \zeta^*\mathrm{Mod}_A(\mathcal{C}).$$

In fact, if $X \in \mathrm{Pic}(A)$, then we may form the A-linear composite

$$\zeta_!^X : \mathrm{Mod}_A \xrightarrow{X\otimes_A -} \mathrm{Mod}_A \xrightarrow{\zeta_!} \zeta^*\mathrm{Mod}_A.$$

This gives an isomorphism

$$(A, -) := \zeta_!^{(-)} : \mathrm{Pic}(A)_G \cong H^1(G, \mathrm{Pic}(A)) \to \mathrm{Br}(\mathbf{1} \mid A).$$

Example 8.4. At the prime 2, the extension $\mathbf{1}_{\mathbf{K}} \to KO_2$ is a descendable $\mathbb{Z}_2$-Galois extension. As a result, Construction 8.3 applies, and we can form the Brauer class (KO_2, X) associated to any $X \in \mathrm{Pic}(KO_2)$ as above. For example, since KO_2 is 8-periodic one can form the cohomological Brauer class

$$(KO_2, \Sigma KO_2) \in \mathrm{Br}'(\mathbf{1}_{\mathbf{K}} \mid KO_2).$$

Example 8.5. Let $KO_2^{\mathrm{nr}} := \varinjlim_n (KO_2)_{\mathbb{W}(\mathbb{F}_{2^n})}$ be the ind-étale KO_2-algebra given by the maximal unramified extension of $\pi_0 KO_2 = \mathbb{Z}_2$; since étale extensions are uniquely determined by their π_0, one can also describe this as

$$KO_2^{\mathrm{nr}} = KO_2 \otimes_{\mathbb{S}} \mathbb{SW},$$

where $\mathbb{SW} = \mathbb{SW}(\overline{\mathbb{F}}_2)$ denotes the spherical Witt vectors [Lur18, §5.2]. The extension $KO_2 \to KO_2^{\mathrm{nr}}$ is a descendable Galois extension: indeed, $KO_2 \otimes_{\mathbb{S}} (-)$ preserves finite limits, and $\mathbb{S} \to \mathbb{SW}$ is descendably $\widehat{\mathbb{Z}}$-Galois. As a result, Construction 8.3 applies, and we can form the Brauer class

(KO_2^{nr}, X) asssociated to any $X \in \mathrm{Pic}(KO_2^{\mathrm{nr}})$ as above. For example, since KO_2^{nr} is 8-periodic one can form the Azumaya algebra

$$(KO_2^{\mathrm{nr}}, \Sigma KO_2^{\mathrm{nr}}) \in \mathrm{Br}(KO_2).$$

In fact, $(KO_2^{\mathrm{nr}}, \Sigma KO_2^{\mathrm{nr}})$ is an element of the *étale locally trivial* Brauer group $\mathrm{LBr}(KO_2)$ of [AMS22]; we discuss this in Chapter 9.

8.2 Cyclic algebras

Suppose that $\mathcal{C}$ is a stable homotopy theory and $\mathbf{1} \to A$ a finite Galois extension in $\mathcal{C}$ with group G. Suppose also given the following data:

(*i*) an isomorphism $\chi : G \cong \mathbb{Z}/k$.

(*ii*) a *strict* unit $u \in \pi_0 \mathbb{G}_m(\mathbf{1})$.

In this section, we will use this to define a relative Azumaya algebra $(A, \chi, u) \in \mathrm{Br}(\mathcal{C} \mid A)$.

Let us first recall the construction when $\mathcal{C}$ is the category of modules over a classical ring. Then we begin with a G-Galois extension $R \to A$ of rings, and define a G-action on the matrix algebra $M_k(A)$ as follows: we let $\sigma \coloneqq \chi^{-1}(1)$ act as conjugation by the matrix

$$\widetilde{u} \coloneqq \begin{bmatrix} 0 & & & u \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \tag{8.4}$$

Since $\widetilde{u}^k = uI_k \in \mathrm{GL}_k A$ is central, this gives a well-defined action on $M_k A$. We can use this to twist the usual action of G on the matrix algebra; passing to fixed points, we obtain the cyclic algebra

$$(A, \chi, u) \in \mathrm{Br}(R \mid A). \tag{8.5}$$

The construction for general $\mathcal{C}$ will be directly analogous. Moreover, when R is a field it is well-known (see for example [CS21]) that under the isomorphism

$$\mathrm{Br}(R \mid A) \cong H^2(G, A^\times),$$

the cyclic algebra (A, χ, u) maps to the cup-product $\beta(\chi) \cup u$, where β denotes the Bockstein homomorphism

$$\chi \in \mathrm{Hom}(G, \mathbb{Z}/k) = H^1(G, \mathbb{Z}/k) \xrightarrow{\beta} H^2(G, \mathbb{Z}),$$

and we use the $\mathbb{Z}$-module structure on $A^\times$. As a result, we obtain an isomorphism

$$\widehat{H}^0(G, A^\times) = A^\times / N_e^G A^\times \to \mathrm{Br}(R \mid A)$$

sending $u \mapsto (A, \chi, u)$. We will prove an analogous result for cyclic algebras in arbitrary stable homotopy theories, which will allow us to detect permanent cycles in the descent spectral sequence; conversely, this will allow us to assert that the cyclic algebras we construct are nontrivial.

We now begin the construction of (A, χ, u) for arbitrary $\mathcal{C}$.

Definition 8.6. Let $\mathcal{C}$ be a stable homotopy theory. Given $A \in \mathrm{CAlg}(\mathcal{C})$ and $k \geq 1$, we will write

- $M_k(A) := \mathrm{End}_{\mathrm{Mod}_A(\mathcal{C})}(A^{\oplus k})$,
- $\mathrm{GL}_k(A) := \mathrm{Aut}_{\mathrm{Mod}_A(\mathcal{C})}(A^{\oplus k})$,
- $\mathrm{PGL}_k(A) := \mathrm{Aut}_{\mathrm{Alg}_A(\mathcal{C})}(M_k(A))$.

These are all $\mathbb{E}_1$-monoids under composition, and we have $\mathbb{E}_1$ maps

$$\mathrm{GL}_k(A) \to M_k(A) \qquad \text{and} \qquad \mathrm{GL}_k(A) \to \mathrm{PGL}_k(A),$$

the first of which is by definition. The second will be defined immediately below.

Remark 8.7. If R is an $\mathbb{E}_\infty$-ring, then Gepner and Lawson [GL21, Corollary 5.19] show there is a fibre sequence

$$\coprod_{\pi_0 \mathrm{Mod}_R^{\mathrm{cg}}} B\mathrm{Aut}_A(M) \to \coprod_{\pi_0 \mathfrak{Az}_R^{\mathrm{triv}}} B\mathrm{Aut}_{\mathrm{Alg}_R}(A) \to B\mathfrak{Pic}(R). \tag{8.6}$$

Here $\mathrm{Mod}_R^{\mathrm{cg}} \subset \iota\mathrm{Mod}_R$ denotes the classifying space of *compact generators*, and $\mathfrak{Az}_R^{\mathrm{triv}} \subset \iota\mathrm{Alg}_R$ the space of *Morita trivial* Azumaya algebras. To see this, we consider the components over $\mathrm{Mod}_R \in \mathrm{Br}(R)$ in the commutative square

$$\begin{tikzcd}
{}_A\mathrm{BMod}_A^{\mathrm{cg}} \arrow[r] \arrow[d] & \{\mathrm{LMod}_A\} \arrow[d] \\
\mathfrak{Az}_R \arrow[r] & \mathfrak{Br}_R
\end{tikzcd}$$

noting that this diagram is Cartesian [GL21, Proposition 5.17]: this follows by identifying those objects in

$$\mathrm{Fun}_R(\mathrm{LMod}_A, \mathrm{LMod}_B) \simeq {}_A\mathrm{BMod}_B$$

which correspond to the R-linear equivalences. By [HL17, Corollary 2.1.3], $\mathcal{C}$-linear equivalences between module categories in an arbitrary $\mathcal{C}$ correspond to bimodules M that are *full* and *dualis-*

able, and so in this context there is an analogous pullback diagram

$$\begin{tikzcd} {}_A\mathrm{BMod}_A^{\mathrm{fd}}(\mathcal{C}) \arrow[r] \arrow[d] & \{\mathrm{LMod}_A(\mathcal{C})\} \arrow[d] \\ \mathfrak{Az}(\mathcal{C}) \arrow[r] & \mathfrak{Br}(\mathcal{C}) \end{tikzcd}$$

for any Azumaya algebra A. Restricting again to the unit component in $\mathfrak{Br}(\mathcal{C})$, we obtain a fibre sequence

$$\coprod_{M \in \pi_0 \mathcal{C}^{\mathrm{fd}}} B\mathrm{Aut}_A(M) \to \coprod_{A \in \pi_0 \mathfrak{Az}(\mathcal{C})^{\mathrm{triv}}} B\mathrm{Aut}_{\mathrm{Alg}(\mathcal{C})}(A) \to B\mathfrak{Pic}(\mathcal{C}).$$

In particular, taking $A = M_k(\mathbf{1})$ yields the fibre sequence

$$\coprod_{\mathrm{End}(M)=A} B\mathrm{Aut}_A(M) \to B\mathrm{PGL}_k(\mathbf{1}) \to B\mathfrak{Pic}(\mathcal{C}), \tag{8.7}$$

and restricting to $M = \mathbf{1}^{\oplus k} \in \mathcal{C}^{\mathrm{fd}}$ gives the desired map $\mathrm{GL}_k(\mathbf{1}) \to \mathrm{PGL}_k(\mathbf{1})$ after taking loops.

Remark 8.8. Suppose that $\mathcal{C}$ is a stable homotopy theory and $u \in \pi_0\mathrm{GL}_1(\mathbf{1}) = \pi_1 \mathrm{Pic}(\mathcal{C})$. We define a map $\hat{u} : \mathbb{Z}/k \to \mathrm{PGL}_k(\mathbf{1})$ by virtue of the following commutative diagram, whose bottom row is a shift of the fibre sequence (8.7):

$$\begin{tikzcd} \mathbb{Z}/k \arrow[r] \arrow[d, dashed, "\widehat{u}"'] & S^1 \arrow[r, "k"] \arrow[d, "Bu"] & S^1 \arrow[d, "B\widetilde{u}"] \\ \mathrm{PGL}_k(\mathbf{1}) \arrow[r] & \mathfrak{Pic}(\mathcal{C}) \arrow[r] & B\mathrm{GL}_k(\mathbf{1}) \end{tikzcd} \tag{8.8}$$

The map

$$\widetilde{u} \in \pi_0\mathrm{Map}_{\mathbb{E}_1}(\mathbb{Z}, \mathrm{GL}_k(\mathbf{1})) \cong \pi_0\mathrm{GL}_k(\mathbf{1}) \cong \mathrm{GL}_k([\mathbf{1},\mathbf{1}])$$

is given by the matrix in (8.4)—commutativity of the right-hand square above is implied by the computation $\widetilde{u}^k = uI_k \in \mathrm{GL}_k([\mathbf{1},\mathbf{1}])$. In fact, when the chosen unit is *strict*, we will prove at the end of the section that $\widehat{u}$ deloops:

Proposition 8.9. *If $u \in \pi_0\mathbb{G}_m(A)$, the map $\widehat{u}$ lifts naturally to an $\mathbb{E}_1$ map $\widehat{u}\colon \mathbb{Z}/k \to \mathrm{PGL}_k(A)$. More precisely, there is a natural transformation $\widehat{u}\colon \mathbb{G}_m \to \mathrm{Map}_{\mathbb{E}_1}(\mathbb{Z}/k, \mathrm{PGL}_k)$ which lifts*

$$\widehat{u} \in \pi_0\mathrm{Map}(\mathbb{Z}/k, \mathrm{PGL}_k(\mathbf{1}[u^{\pm 1}])) \cong \pi_0\mathrm{Map}_{\mathcal{P}(\mathrm{CAlg}(\mathcal{C})^{\mathrm{op}})}(\mathbb{G}_m, \mathrm{Map}(\mathbb{Z}/k, \mathrm{PGL}_k)).$$

Remark 8.10. The choice of lift $\widehat{u}$ is only determined up to homotopy in $\mathrm{Map}_{\mathcal{P}(\mathrm{CAlg}(\mathcal{C})^{\mathrm{op}})}(\mathbb{G}_m, \mathrm{Map}_{\mathbb{E}_1}(\mathbb{Z}/k, \mathrm{PGL}_k))$.

Construction 8.11. Suppose that $\mathbf{1} \to A$ is a finite G-Galois extension in $\mathcal{C}$, and $u \in \pi_0\mathbb{G}_m(\mathbf{1})$.

We obtain a commutative square of spaces with G-action

$$\begin{tikzcd} B\mathbb{Z}/k \arrow[r, "\beta"] \arrow[d, "B\widehat{u}"'] & B^2\mathbb{Z} \arrow[d, "B^2u"] \\ B\mathrm{PGL}_k(A) \arrow[r, "\delta"'] & B\mathfrak{Pic}(A) \end{tikzcd} \tag{8.9}$$

where the action on the top row is trivial. On homotopy fixed points, we obtain the square

$$\begin{tikzcd} B\mathbb{Z}/k^{BG} \arrow[r] \arrow[d] & B^2\mathbb{Z}^{BG} \arrow[d] \\ (B\mathrm{PGL}_k(A))^{hG} \arrow[r] & (B\mathfrak{Pic}(A))^{hG} \end{tikzcd} \tag{8.10}$$

whose bottom right term is the relative Brauer space $\mathfrak{Br}(\mathbf{1} \mid A)$.

Definition 8.12. Given a Galois extension $\mathbf{1} \to A$ with group G in a stable homotopy theory $\mathcal{C}$, and given $\chi : G \cong \mathbb{Z}/k$ and $u \in \pi_0\mathbb{G}_m(\mathbf{1})$, define the relative Brauer class $(A, \chi, u) \in \mathrm{Br}(\mathbf{1} \mid A)$ to be the image of χ under the composite map in (8.10),

$$\chi \in \mathrm{Hom}(G, \mathbb{Z}/k) = \pi_0 B\mathbb{Z}/k^{BG} \to \pi_0(B\mathfrak{Pic}(A))^{hG} \cong \mathrm{Br}(\mathbf{1} \mid A).$$

The final isomorphism is inverse to the map $\mathrm{Br}(\mathbf{1} \mid A) \xrightarrow{\sim} \mathrm{Br}'(\mathbf{1} \mid A)$ considered in Section 5.2.

Proposition 8.13. *The Brauer class (A, χ, u) agrees with the class of the cyclic algebra $A(A, \chi, u)$ defined in [BRS12, §4].*

Proof. Going down and left in (8.10), we see that (A, χ, u) is described as follows: it is the A-semilinear action on $\mathrm{Mod}_A(\mathcal{C}) \simeq \mathrm{Mod}_{M_k(A)}(\mathcal{C})$ given by the G-equivariant map

$$BG \xrightarrow{\chi} B\mathbb{Z}/k \xrightarrow{\widehat{u}} B\mathrm{PGL}_k(A) = B\mathrm{Aut}_A(M_k(A)) = B\mathrm{Aut}_A(\mathrm{Mod}_A(\mathcal{C})).$$

This is the same action that Baker, Richter and Szymik define at the level of $M_k(A)$. $\square$

Theorem 8.14. *Suppose given a strict unit $u \neq 1 \in \pi_0\mathbb{G}_m(\mathbf{1})$. Its image in $\pi_0\mathrm{GL}_1(\mathbf{1})$ is detected in the HFPSS for $\mathfrak{pic}(\mathcal{C}) \simeq \mathfrak{pic}(A)^{hG}$ by a class $v \in E_2^{s,s+1}$, and we assume that one of the following holds:*

(i) v is in positive filtration;

(ii) v is in filtration zero, and has nonzero image in $\widehat{H}^0(G, (\pi_0 A)^\times)$.

Then the cyclic algebra (A, χ, u) is detected by the symbol

$$\beta(\chi) \cup v \in H^{s+2}(G, \pi_{s+1}\mathfrak{pic}(A)) = E_2^{s+2,s+1}.$$

In particular, $\beta(\chi) \cup v$ is a permanent cycle. If it survives to E_∞ then $(A, \chi, u) \neq \mathbf{1} \in \mathrm{Br}(\mathbf{1} \mid A)$.

Proof. The square of G-spaces (8.9) gives rise to a commutative square of HFPSS as below:

$$\begin{tikzcd}
H^*(G,\mathbb{Z}/k) \arrow[r, "\beta_*"] \arrow[d, "\widehat{u}_*"'] & H^*(G,\mathbb{Z}) \arrow[d, "u_*"] \\
H^*(G,\pi_*\mathrm{PGL}_k A) \arrow[r, "\delta_*"'] & H^*(G,\pi_*\mathfrak{Pic}(A)) \\
\arrow[u, phantom, "\Downarrow"] & \\
\pi_*(B\mathbb{Z}/k)^{BG} \arrow[r, "\beta^{BG}"] \arrow[d, "(B\widehat{u})^{hG}"'] & \pi_*(B^2\mathbb{Z})^{BG} \arrow[d, "(B^2u)^{hG}"] \\
\pi_*(B\mathrm{PGL}_k A)^{hG} \arrow[r, "\delta^{hG}"'] & \pi_*(B\mathfrak{Pic}(A))^{hG}
\end{tikzcd} \tag{8.11}$$

Note that the maps on E_2 need not preserve filtration. By definition, (A,χ,u) is the image of $\chi \in \pi_0(B\mathbb{Z}/k)^{BG}$ under the composite to $\pi_0(B\mathfrak{Pic}(A))^{hG}$, and is therefore detected on the E_2-page for $(B\mathfrak{Pic}(A))^{hG}$ by $u_*\beta_*(\chi)$, as long as this class is nonzero. It is standard that the map β_* is indeed the Bockstein, and we claim that the map

$$u_* : H^2(G,\mathbb{Z}) \to H^s(G,\pi_{s-1}\mathfrak{Pic}(A))$$

induced on E_2-pages by $B^2u\colon B^2\mathbb{Z} \to B\mathfrak{Pic}(A)$ agrees with $v \cup -$. Indeed, this map can be identified with the composition

$$\begin{aligned}
\mathrm{Map}_G(EG_+, B^2\mathbb{Z}) &\simeq * \times \mathrm{Map}_G(EG_+, B^2\mathbb{Z}) \\
&\xrightarrow{B^2u\times id} \mathrm{Map}_G(B^2\mathbb{Z}, B\mathfrak{Pic}(A)) \times \mathrm{Map}_G(EG_+, B^2\mathbb{Z}) \\
&\xrightarrow{\circ} \mathrm{Map}_G(EG_+, B\mathfrak{Pic}(A)),
\end{aligned}$$

and the induced map on HFPSS shows that

$$u_*\beta(\chi) = [v \circ \beta(\chi)] = [\beta(\chi) \cup v]$$

by compatibility of the cup and composition products. The class $\beta(\chi) \cup v$ is nonzero since Tate cohomology is $\beta(\chi)$-periodic, which gives the result. ☐

8.3 Proof of Proposition 8.9

We will write $\mathrm{CMon} \subset \mathrm{Fun}(\mathrm{Fin}_*, \mathcal{S})$ for the ∞-category of special Γ-spaces, which admits a unique symmetric monoidal structure making the free commutative monoid functor $\mathcal{S} \to \mathrm{CMon}$ symmetric monoidal [GGN16]; its unit is the nerve of the category of finite pointed sets and isomorphisms,

$$\mathcal{F} := |N\mathrm{Fin}_*^{\sim}| \simeq \coprod_{\mathbb{Z}_{\geq 0}} B\Sigma_n.$$

Definition 8.15. We will denote by $\mathrm{Rig} := \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{CMon})$ the ∞-category of *associative semirings*[1]. Likewise, we will write $\mathrm{CRig} := \mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{CMon})$ for the ∞-category of *commutative semirings*. Since the adjunction $\mathcal{S} \rightleftarrows \mathrm{CMon}$ is symmetric monoidal, it passes to algebras to yield monoid semiring functors,

$$\mathcal{F}[-] : \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{S}) \rightleftarrows \mathrm{Rig} \qquad \text{and} \qquad \mathcal{F}[-] : \mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{S}) \rightleftarrows \mathrm{CRig},$$

which factor the respective monoid algebra functors to spectra.

Remark 8.16. The key observation for proving Proposition 8.9, pointed out to us by Maxime Ramzi, is that for $R \in \mathrm{CRig}$ and $u \in \mathbb{G}_m(R)$, the map $\widehat{u} \in \pi_0\mathrm{Map}(\mathbb{Z}/k, \mathrm{PGL}_k(R^{\mathrm{gp}}))$ exists in $\pi_0\mathrm{Map}(\mathbb{Z}/k, \mathrm{PGL}_k(R))$ before group completion. This is because the matrix $\widetilde{u}$ in (8.4) is defined without subtraction, as are all its powers. This allows us to make use of the fact that the representing object $\mathcal{F}[u^{\pm 1}] := \mathcal{F}[\mathbb{Z}]$ for $\mathbb{G}_m$ is substantially simpler than $\mathbb{S}[u^{\pm 1}]$, as the following remark shows.

Remark 8.17. Consider the square

$$\begin{tikzcd}
\mathcal{S} \arrow[r, shift left, "L"] \arrow[d, shift right, "F"'] & \mathrm{CMon} \arrow[l, shift left, "R"] \arrow[d, shift right, "F'"'] \\
\mathrm{Alg}_{\mathcal{O}} \arrow[u, shift right, "U"'] \arrow[r, shift left, "L'=\mathcal{F}[-]"] & \mathrm{Alg}_{\mathcal{O}}(\mathrm{CMon}) \arrow[l, shift left, "R'"] \arrow[u, shift right, "U'"']
\end{tikzcd}$$

of [GGN16, §7]. By Lemma 8.18 below, we have an equivalence

$$RU'\mathcal{F}[u^{\pm 1}] = RU'L'\mathbb{Z} \simeq RLU\mathbb{Z} \simeq RL \varinjlim_n [-n, n] \in \mathcal{S}.$$

Since CMon is preadditive and $R\colon \mathrm{CMon} \to \mathcal{S}$ preserves sifted colimits, the underlying space of $\mathcal{F}[u^{\pm 1}]$ is $\varinjlim_n \prod_{[-n,n]} \mathcal{F}$. One therefore has a pullback square

$$\begin{tikzcd}
\prod_i B\Sigma_{n_i} \arrow[r] \arrow[d] & \mathcal{F}[u^{\pm 1}] \arrow[d, "\pi_0"] \\
\{\sum_i n_i u^{t_i}\} \arrow[r] & \mathbb{N}[u^{\pm 1}]
\end{tikzcd} \tag{8.12}$$

for any basepoint of $\mathcal{F}[u^{\pm 1}]$.

Lemma 8.18. *Let $(L \dashv R)\colon \mathcal{C} \rightleftarrows \mathcal{D}$ be a symmetric monoidal adjunction between presentably symmetric monoidal ∞-categories. Let λ be an uncountable regular cardinal, and suppose that $\mathcal{O}^{\otimes} \to \mathrm{Sym}^{\otimes}$ is a fibration of ∞-operads compatible with colimits. Then the square*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "L"] \arrow[d, "F"] & \mathcal{D} \arrow[d, "F'"] \\
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \arrow[r, "L'"'] & \mathrm{Alg}_{\mathcal{O}}(\mathcal{D})
\end{tikzcd}$$

[1] that is, rings without *n*egatives

is right adjointable.

Proof. In the notation of Remark 8.17, we want to show that the mate

$$LU \to U'F'LU \simeq U'L'FU \to U'L' \tag{8.13}$$

is an equivalence, and since $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ is generated under sifted colimits by free algebras (as noted in the proof of [HA, Corollary 3.2.3.3]) and U, U' preserve sifted colimits, it is enough to check this on free algebras FC, $C \in \mathcal{C}$. But $L'F \simeq F'L$, and using the description of F as an operadic Kan extension [HA, Proposition 3.1.3.13] we factor (8.13) as an equivalence

$$\begin{aligned} LUFC =& L\left(\coprod_{\mathbb{N}} \mathcal{O}(n) \otimes_{\Sigma_n} C^{\otimes n}\right) \\ \xrightarrow{\sim}& \coprod_{\mathbb{N}} \mathcal{O}(n) \otimes_{\Sigma_n} (LC)^{\otimes n} \\ =& U'F'LC \simeq U'L'FC. \end{aligned}$$

□

Every nonzero coefficient in the matrix (8.4) is 1, so it is a consequence of Remark 8.17 that all relevant components of $\mathcal{F}[u^{\pm 1}]$ are contractible (which is certainly not true of $\mathbb{S}[\mathbb{Z}] = \mathbb{S}[u^{\pm 1}]$). This will allow us to prove that there is an essentially unique lift of semiring maps

$$\begin{tikzcd} \mathbb{Z}/k \arrow[r, dashed, "\widehat{u}"] \arrow[dr, "\widehat{u}"'] & \mathrm{PGL}_k(R) \arrow[d] \\ & \pi_0\mathrm{PGL}_k(R) \end{tikzcd}$$

which is functorial in u. In other words, we will deduce Proposition 8.9 from the following result:

Proposition 8.19. *The fibre over $u \mapsto \widehat{u}$ of*

$$\mathrm{Map}_{\mathcal{P}(\mathrm{CRig}^{\mathrm{op}})}(\mathbb{G}_m, \mathrm{Map}_{\mathbb{E}_1}(\mathbb{Z}/k, \mathrm{PGL}_k)) \to \mathrm{Map}_{\mathcal{P}(\mathrm{CRig}^{\mathrm{op}})}(\mathbb{G}_m, \mathrm{Map}_{\mathbb{E}_1}(\mathbb{Z}/k, \pi_0\mathrm{PGL}_k))$$

is contractible.

Proof. To prove the claim, note that $\mathbb{G}_m \in \mathcal{P}(\mathrm{CRig}^{\mathrm{op}})$ is corepresented by $\mathcal{F}[u^{\pm 1}]$, and so we are interested in the fibre of

$$\mathrm{Map}_{\mathbb{E}_1}(\mathbb{Z}/k, \mathrm{PGL}_k(\mathcal{F}[u^{\pm 1}])) \to \mathrm{Map}_{\mathbb{E}_1}(\mathbb{Z}/k, \pi_0\mathrm{PGL}_k(\mathcal{F}[u^{\pm 1}])).$$

Using the inclusion $\mathrm{Alg}_{\mathbb{E}_1}(\mathcal{S}) \subset \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{S})$ [HA, Prop. 4.1.2.10] we have that

$$\mathrm{Map}_{\mathbb{E}_1}(\mathbb{Z}/k, \mathrm{PGL}_k(\mathcal{F}[u^{\pm 1}])) \simeq \lim_{[n]\in\Delta^{\mathrm{op}}} \mathrm{Map}_{\mathcal{S}}(\mathbb{Z}/k^{\times n}, \mathrm{PGL}_k(\mathcal{F}[u^{\pm 1}])^{\times n}),$$

and it therefore suffices to prove that each of the maps

$$\mathrm{Map}_{\mathcal{S}}(\mathbb{Z}/k^{\times n}, \mathrm{PGL}_k(\mathcal{F}[u^{\pm 1}])) \to \mathrm{Map}_{\mathcal{S}}(\mathbb{Z}/k^{\times n}, \pi_0\mathrm{PGL}_k(\mathcal{F}[u^{\pm 1}]))$$

has contractible fibre over $(\widehat{u}^{j_1}, \dots, \widehat{u}^{j_k})$; for this, it is enough to show that

$$\mathrm{PGL}_k(\mathcal{F}[u^{\pm 1}]) \times_{\pi_0\mathrm{PGL}_k(\mathcal{F}[u^{\pm 1}])} \{\widehat{u}^j\} \simeq *$$

for $j \geq 0$. But

$$\begin{aligned} \mathrm{PGL}_k(\mathcal{F}[u^{\pm 1}]) &= \mathrm{Aut}_{\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Mod}_{\mathcal{F}[u^{\pm 1}]})}(M_k(\mathcal{F}[u^{\pm 1}])) \\ &\subset \mathrm{End}_{\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Mod}_{\mathcal{F}[u^{\pm 1}]})}(M_k(\mathcal{F}[u^{\pm 1}])) \\ &\simeq \lim_{n \in \Delta^{\mathrm{op}}} \mathrm{End}_{\mathcal{F}[u^{\pm 1}]}(M_k(\mathcal{F}[u^{\pm 1}])^{\times n}), \end{aligned}$$

appealing this time to [HA, Prop. 4.1.2.11 and Theorem 2.3.3.23] for the final equivalence. Since $\mathrm{Mod}_{\mathcal{F}[u^{\pm 1}]}(\mathrm{CMon})$ is semiadditive, we reduce to showing that the fibre

$$\mathrm{End}_{\mathcal{F}[u^{\pm 1}]}(M_k(\mathcal{F}[u^{\pm 1}])) \to \pi_0\mathrm{End}_{\mathcal{F}[u^{\pm 1}]}(M_k(\mathcal{F}[u^{\pm 1}]))$$

over conjugation by $\widehat{u}^j$ is contractible. But any element in the right-hand side can be expressed as a sum of elementary matrix operations,

$$\sum_{1 \leq s,t,s',t' \leq k} a(s,t,s',t')(e_{st} \mapsto e_{s't'}),$$

where $a(s,t,s',t') = n(s,t,s',t')\, u^{r(s,t,s',t')} \in \pi_0\mathcal{F}[u^{\pm 1}] = \mathbb{N}[u^{\pm 1}]$; the component of $\mathrm{End}_{\mathcal{F}[u^{\pm 1}]}(M_k(\mathcal{F}[u^{\pm 1}]))$ at this basepoint is

$$\prod_{s,t,s',t'} B\Sigma_{n(s,t,s',t')}$$

by (8.12). In particular, basic matrix algebra shows that the coefficients $n(s,t,s',t')$ for $\widehat{u}^j$ are all of the form 0 or 1, and so

$$\begin{aligned} \mathrm{End}_{\mathcal{F}[u^{\pm 1}]}(M_k(\mathcal{F}[u^{\pm 1}])) \times_{\pi_0\mathrm{End}_{\mathcal{F}[u^{\pm 1}]}(M_k(\mathcal{F}[u^{\pm 1}]))} \{\widehat{u}^j\} &\simeq \prod B\Sigma_0 \times \prod B\Sigma_1 \\ &\simeq *. \end{aligned}$$

□

Proof (Proposition 8.9). We defined $\widehat{u}$ as an element of

$$\pi_0\mathrm{Map}_{\mathcal{P}(\mathrm{CRig}^{\mathrm{op}})}(\mathbb{G}_m, \mathrm{Map}(\mathbb{Z}/k, \mathrm{PGL}_k)) = \pi_0\mathrm{Map}(\mathbb{Z}/k, \mathrm{PGL}_k(\mathcal{F}[u^{\pm 1}])).$$

Write $P_k \coloneqq \mathrm{PGL}_k(\mathcal{F}[u^{\pm 1}])$. Forming the commutative square

$$\begin{tikzcd}
\pi_0\mathrm{Map}_{\mathbb{E}_1}(\mathbb{Z}/k, P_k) \arrow[r] \arrow[d] & \pi_0\mathrm{Map}_{\mathbb{E}_1}(\mathbb{Z}/k, \pi_0 P_k) \arrow[d] \\
\pi_0\mathrm{Map}(\mathbb{Z}/k, P_k) \arrow[r, equal] & \pi_0\mathrm{Map}(\mathbb{Z}/k, \pi_0 P_k)
\end{tikzcd}$$

we observe that $\widehat{u}$ lifts to $\pi_0\mathrm{Map}_{\mathbb{E}_1}(\mathbb{Z}/k, \pi_0 P_k) = \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}/k, \pi_0 P_k)$, by its definition. We thus obtain from Proposition 8.19 a lift to $\pi_0\mathrm{Map}_{\mathbb{E}_1}(\mathbb{Z}/k, P_k)$, and hence

$$(u \mapsto \widehat{u}) \in \mathrm{Map}_{\mathcal{P}(\mathrm{CAlg}^{\mathrm{op}})}(\mathbb{G}_m, \mathrm{Map}_{\mathbb{E}_1}(\mathbb{Z}/k, \mathrm{PGL}_k)). \tag{8.14}$$

after passing to group completions. Now suppose $\mathcal{C}$ is any stable homotopy theory; then $\mathcal{C}$ is tensored over $\mathcal{S}p$, and $\mathbb{S}[u^{\pm 1}] \otimes \mathbf{1} \simeq \mathbf{1}[u^{\pm 1}]$. One therefore gets an $\mathbb{E}_1$ map

$$\mathrm{PGL}_k(\mathbb{S}[u^{\pm 1}]) = \mathrm{Aut}_{\mathrm{Alg}_{\mathbb{E}_1}}(M_k(\mathbb{S}[u^{\pm 1}])) \to \mathrm{PGL}_k(\mathbf{1}[u^{\pm 1}]) = \mathrm{Aut}_{\mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C})}(M_k(\mathbf{1}[u^{\pm 1}])). \tag{8.15}$$

Since $\mathbf{1}[u^{\pm 1}] \in \mathrm{CAlg}(\mathcal{C})$ corepresents $\mathbb{G}_m$, one obtains the desired map

$$(u \mapsto \widehat{u}) \in \mathrm{Map}_{\mathbb{E}_1}(\mathbb{Z}/k, \mathrm{PGL}_k(\mathbf{1}[u^{\pm 1}])) = \mathrm{Map}_{\mathcal{P}(\mathrm{CAlg}(\mathcal{C})^{\mathrm{op}})}(\mathbb{G}_m, \mathrm{Map}_{\mathbb{E}_1}(\mathbb{Z}/k, \mathrm{PGL}_k))$$

as the image of (8.14) on applying $\mathrm{Map}_{\mathbb{E}_1}(\mathbb{Z}/k, -)$ to (8.15). $\square$

Chapter 9

Computing Br_1^0

We are now ready to complete the proofs of Theorems F and G. At odd primes, we will see that the cyclic algebra construction of Chapter 8 gives all possible Brauer classes. On the other hand, when $p = 2$ not all classes on the E_∞-page of the descent spectral sequence will be detected in this way. In this case, we will first compute the relative Brauer group $\mathrm{Br}(KO_2 \mid KU_2)$.

9.1 Odd primes

When $p \geq 3$, Fig. 7.1 shows that $\mathrm{Br}_1^0 \subset \mu_{p-1}$. In fact, we can deduce from Theorem 8.14 that this inclusion is an equality:

Theorem 9.1. *Let p be an odd prime, and choose $\chi\colon \mu_{p-1} \cong \mathbb{Z}/p-1$. There is an isomorphism*

$$\mu_{p-1} \xrightarrow{\sim} \mathrm{Br}_1^0,$$

given by the cyclic algebra construction $\omega \mapsto (KU_p^{h(1+p\mathbb{Z}_p)}, \chi, \omega)$.

Proof. By Proposition 7.3 there is an inclusion $\mathrm{Br}_1^0 \subset \mathbb{Z}/(p-1)$. Moreover, there are no differentials out of the 0-stem in Fig. 7.1 by the computation of Pic_1 ([Mor23a, §4]). Since $H^2(\mu_{p-1}, (\pi_0 B)^\times) \cong \mu_{p-1}\langle \beta(\chi) \cup \omega \rangle$, Theorem 8.14 implies it is enough to show that the roots of unity in $(\pi_0 \mathbf{1_K})^\times = \mathbb{Z}_p^\times$ are strict. But we have a commuting square

$$\begin{tikzcd}
\pi_0 \mathbb{G}_m(\mathbb{S}) \arrow[r] \arrow[d] & \pi_0 \mathbb{G}_m(\mathbf{1_K}) \arrow[d] \\
\pi_0 GL_1(\mathbb{S}) \arrow[r, equal] & \pi_0 GL_1(\mathbf{1_K})
\end{tikzcd}$$

and at odd primes the roots of unity are strict in $\mathbb{S}$ by [Car23, Theorem A]. $\square$

Remark 9.2. In fact, Theorem 9.1 also follows from Corollary 5.24. Indeed, Fig. 7.1 shows that

$$\mathrm{Br}'(\mathcal{S}p_{\mathbf{K}} \mid KU_p) \cong H^2(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times) \cong H^2(\mu_{p-1}, \mathbb{Z}_p^\times).$$

In particular, this group is killed in the μ_{p-1}-Galois extension $\mathbf{1}_{\mathbf{K}} \to KU_p^{h(1+p\mathbb{Z}_p)}$, since the group

$$\mathrm{Br}'(KU_p^{h(1+p\mathbb{Z}_p)} \mid KU_p) = \pi_0 B\mathfrak{Pic}(KU_2)^{h(1+p\mathbb{Z}_p)}$$

is concentrated in filtration $s \leq 1$ of the Picard spectral sequence for the $(1+p\mathbb{Z}_p)$-action. Since the extension $\mathbf{1}_{\mathbf{K}} \to KU_p^{h(1+p\mathbb{Z}_p)}$ is finite, Corollary 5.24 yields the second isomorphism below:

$$\begin{aligned} \mathrm{Br}'(\mathcal{S}p_{\mathbf{K}} \mid KU_p) &\cong \mathrm{Br}'(\mathcal{S}p_{\mathbf{K}} \mid KU_p^{h(1+p\mathbb{Z}_p)}) \\ &\cong \mathrm{Br}(\mathcal{S}p_{\mathbf{K}} \mid KU_p^{h(1+p\mathbb{Z}_p)}) \subset \mathrm{Br}_1^0 . \end{aligned}$$

9.2 Completed K-theory

In this section we use Galois descent to compute the Brauer group $\mathrm{Br}(KO_p \mid KU_p)$. This builds on the integral case computed by Gepner and Lawson [GL21], and we will therefore also determine the maps

$$\mathrm{Br}(KO \mid KU) \to \mathrm{Br}(KO_p \mid KU_p)$$

induced by completion. The computation for $p = 2$ will be important for our main computation: we will show that the relative Brauer classes of KO_2 descend to $\mathbf{1}_{\mathbf{K}}$, which will help us determine the group Br_1^0. Therefore, we start with the computation in this case:

Theorem 9.3. *At the prime two we have*

$$\mathrm{Br}(KO_2 \mid KU_2) \simeq \mathbb{Z}/4,$$

and the completion map from $\mathrm{Br}(KO \mid KU)$ *is injective.*

It follows by combining Fig. 7.4 with Corollary 5.24 that $|\mathrm{Br}(KO_2 \mid KU_2)| = 4$. To prove the theorem, we need to prove there is an extension, and we do this by reducing to computations of étale cohomology.

Definition 9.4. Recall the *étale locally trivial Brauer group* $\mathrm{LBr}(KO_2) \subset \mathrm{Br}(KO_2)$ of [AMS22]; more generally, Antieau, Meier and Stojanoska define

$$\mathrm{LBr}(R) \coloneqq \pi_0 \Gamma B\mathfrak{Pic}_{\mathcal{O}_R}$$

for any commutative ring spectrum R, where $B\mathfrak{Pic}_{\mathcal{O}_R}$ is the sheafification of $B\mathfrak{Pic}(\mathcal{O}_R)$ on the étale site of $\mathrm{Spec}\, R$ [1]. Explicitly ([AMS22, Lemma 2.17]) this is the group of Brauer classes that

[1] Note that this is equivalent to the étale site of $\mathrm{Spec}\, \pi_0 R$.

are trivialised in some faithful étale extension $R \to R'$ in the sense of [SAG, Definition 7.5.0.4]. Likewise, for an extension $R \to S$ of commutative ring spectra we write

$$\mathrm{LBr}(R \mid S) := \ker(\mathrm{LBr}(R) \to \mathrm{LBr}(S)) \subset \mathrm{Br}(R \mid S).$$

The group $\mathrm{LBr}(R)$ is sometimes more computationally tractable than $\mathrm{Br}(R)$: for example, one can often reduce to étale cohomology of $\operatorname{Spec} \pi_0 R$, which gives access to the standard cohomological toolkit. When $R = KO_p$, this allows us to use Gabber-Huber rigidity in the proof of Theorem 9.3.

Remark 9.5. One may (rightly) worry about the difference between the groups of unlocalised and of $K(1)$-local Brauer classes, since the results of [AMS22] pertain to the former. While we do not know if the two groups agree in general (even for nice even-periodic rings), in our applications this is taken care of by restricting to *relative* Brauer classes. Indeed, in that case we are computing the space

$$\mathfrak{Br}(\mathbf{1} \mid A) := B\mathfrak{Pic}(\mathrm{Mod}_A(\mathcal{C}))^{hG},$$

and by [HMS17, Remark 3.7] the canonical map

$$\iota_A : \mathfrak{Pic}(\mathrm{Mod}_A) \to \mathfrak{Pic}(\mathrm{Mod}_A(\mathcal{S}p_{\mathbf{K}}))$$

is an equivalence of infinite loop-spaces in the following cases:

- $A = E(k, \Gamma)$ is any Morava E-theory,
- A admits a descendable extension $A \to B$ for which ι_B is an equivalence.

For example, this means that there is no ambiguity in writing $\mathrm{LBr}(KO_2 \mid KU_2)$ below.

We begin with a preliminary computation; the following is essentially [AMS22, Proposition 3.8]:

Proposition 9.6. *The étale sheaf $\pi_0\mathfrak{pic}(\mathcal{O}_{KO_2})$ fits in a nonsplit extension*

$$0 \to i_*\mathbb{Z}/2 \to \pi_0\mathfrak{pic}(\mathcal{O}_{KO_2}) \to \mathbb{Z}/4 \to 0,$$

where $i\colon \operatorname{Spec} \mathbb{F}_2 \to \operatorname{Spec} \mathbb{Z}_2$ is the inclusion of the closed point.

Proof. We specify the necessary adjustments from the case of integral K-theory KO. Recall that Antieau, Meier and Stojanoska compute the sheaf $\pi_0\mathfrak{pic}(\mathcal{O}_{KO})$ on $\operatorname{Spec} KO = \operatorname{Spec} \mathbb{Z}$, using the sheaf-valued HFPSS for

$$\mathfrak{pic}(\mathcal{O}_{KO}) \simeq \mathfrak{pic}(\mathcal{O}_{KU})^{hC_2},$$

which is [AMS22, Figure 1]. The same figure gives the HFPSS for

$$\mathfrak{pic}(\mathcal{O}_{KO_2}) \simeq \mathfrak{pic}(\mathcal{O}_{KU_2})^{hC_2},$$

as long as one correctly interprets the symbols as in [AMS22, Table 1], replacing $\mathcal{O} = \mathcal{O}_{\mathbb{Z}}$ with $\mathcal{O}_{\mathbb{Z}_2}$. The proofs of [AMS22, Lemmas 3.5 and 3.6] go through verbatim to give the 0-stem in the E_∞-page, so that $\pi_0\mathfrak{pic}(\mathcal{O}_{KO_2})$ admits a filtration

$$\begin{tikzcd} F^2 \arrow[r, hook] \arrow[d, equal] & F^1 \arrow[r, hook] \arrow[d, two heads] & \pi_0\mathfrak{pic}(\mathcal{O}_{KO_2}) \arrow[d, two heads] \\ i_*\mathbb{Z}/2 & \mathbb{Z}/2 & \mathbb{Z}/2 \end{tikzcd}$$

Now the determination of the extensions follows as in [AMS22, Proposition 3.8], by using the exact sequence

$$H^1(\operatorname{Spec}\mathbb{Z}_2, \mathbb{G}_m) = \operatorname{Pic}(\mathbb{Z}_2) \to \operatorname{Pic}(KO_2) \to H^0(\operatorname{Spec}\mathbb{Z}_2, \pi_0\mathfrak{pic}(\mathcal{O}_{KO_2}))$$

of [AMS22, Proposition 2.25], and the fact that $\mathbb{Z}_2$ is local so has trivial Picard group. $\square$

Proof (Theorem 9.3). By inspection of the HFPSS for the C_2 action on $\mathfrak{Pic}(KU_2)$ (Fig. 7.4), the Brauer group is of order four. It remains to prove there is an extension relating these two classes. In fact, we will prove that

$$\operatorname{Br}(KO_2 \mid KU_2) \supset \operatorname{LBr}(KO_2 \mid KU_2) \cong \mathbb{Z}/4,$$

When $\pi_0 R$ is a regular complete local ring with finite residue field, the exact sequence [AMS22, Proposition 2.25] simplifies to an isomorphism

$$\operatorname{LBr}(R) \cong H^1(\operatorname{Spec}\pi_0 R, \pi_0\mathfrak{pic}(\mathcal{O}_R)),$$

since the cohomology of $\mathbb{G}_m$ vanishes [Maz73, 1.7(a)]. One has that $\pi_0\mathfrak{pic}(\mathcal{O}_{KU_2}) \simeq \mathbb{Z}/2$ is constant since KU_2 is even periodic with regular Noetherian π_0, while $\pi_0\mathfrak{pic}(\mathcal{O}_{KO_2})$ is torsion by Proposition 9.6. We can therefore use Gabber-Huber rigidity [Gab94; Hub93] to compute

$$\begin{aligned} \operatorname{LBr}(KO_2) &\cong H^1(\operatorname{Spec}\mathbb{Z}_2, \pi_0\mathfrak{pic}(\mathcal{O}_{KO_2})) \cong H^1(\operatorname{Spec}\mathbb{F}_2, i^*\pi_0\mathfrak{pic}(\mathcal{O}_{KO_2})), \\ \operatorname{LBr}(KU_2) &\cong H^1(\operatorname{Spec}\mathbb{Z}_2, \pi_0\mathfrak{pic}(\mathcal{O}_{KU_2})) \cong H^1(\operatorname{Spec}\mathbb{F}_2, i^*\pi_0\mathfrak{pic}(\mathcal{O}_{KU_2})). \end{aligned}$$

Since $i^*\pi_0\mathfrak{pic}(\mathcal{O}_{KO_2}) \simeq \mathbb{Z}/8$, we obtain

$$\operatorname{LBr}(KO_2) \cong \mathbb{Z}/8 \qquad \text{and} \qquad \operatorname{LBr}(KU_2) \cong \mathbb{Z}/2,$$

which implies that $\operatorname{LBr}(KO_2 \mid KU_2) \cong \mathbb{Z}/4$ or $\mathbb{Z}/8$; but $|\operatorname{LBr}(KO_2 \mid KU_2)| \leq 4$, so we are done. $\square$

Remark 9.7. The generator of $\operatorname{LBr}(KO_2)$ is the Azumaya algebra $(KO_2^{\mathrm{nr}}, \Sigma KO_2^{\mathrm{nr}})$ constructed in Example 8.5; the generator of $\operatorname{LBr}(KO_2 \mid KU_2)$ is $(KO_2^{\mathrm{nr}}, \Sigma^2 KO_2^{\mathrm{nr}})$.

Remark 9.8. In [May77, Lemma V.3.1], May proves a splitting of infinite loop-spaces

$$BU^{\otimes} \simeq \tau_{\leq 3}BU^{\otimes} \times \tau_{\geq 4}BU^{\otimes},$$

where the monoidal structure is given by tensor product of vector spaces. The map $BU(1) = \tau_{\leq 3}BU \to BU$ is induced from a map at the level of bipermutative categories, and classifies the canonical line bundle. One can ask if this extends to a splitting of $\mathfrak{gl}_1 KU$, and if this splitting happens C_2-equivariantly.

As an aside to Theorem 9.3, we deduce that the completed, equivariant analogue of this splitting fails:

Corollary 9.9. *There is no C_2-equivariant splitting*

$$\mathfrak{gl}_1 KU_2 \simeq \tau_{\leq 3}\mathfrak{gl}_1 KU_2 \oplus \tau_{\geq 4}\mathfrak{gl}_1 KU_2.$$

Proof. The generators in the (-1)-stem of the E_∞-page of Fig. 7.4 are in filtrations two and six. If such a splitting did exist, there could be no extension between them in $\mathrm{Br}(KO_2 \mid KU_2)$. □

We do not know if $\mathfrak{gl}_1 KU$ (or its 2-completion) splits equivariantly: note that the obstruction in the case of KU_2 comes in the form of an extension on a class originating in $H^2(\mathbb{Z}_2^\times, 1 + 4\mathbb{Z}_2) \subset H^2(\mathbb{Z}_2^\times, \mathbb{Z}_2^\times)$.

Let us briefly also mention the case when p is odd; this will not be necessary for the computation of Br_1^0 at odd primes.

Proposition 9.10. *When p is odd, we have*

$$\mathrm{Br}(KO_p \mid KU_p) \cong \mu_{p-1}/\mu_{p-1}^2,$$

and the map $\mathrm{Br}(KO \mid KU) \to \mathrm{Br}(KO_p \mid KU_p)$ is zero.

Proof. Since $\mathbb{Z}_p$ is local away from 2, the E_2-page takes the form in Fig. 9.1, from which $\mathrm{Br}(KO_p \mid KU_2) \cong \mu_{p-1}/\mu_{p-1}^2$ follows by Corollary 5.24. To describe the generators, note that the roots of unity $\mu_{p-1} \subset \pi_0 KO_p^\times$ are strict, since they are so in the p-complete sphere [Car23]. Choosing $\chi : \mu_{p-1} \cong \mathbb{Z}/(p-1)$, Theorem 8.14 implies that the cyclic algebra construction

$$\omega \mapsto (KU_p, \chi, \omega)$$

yields $\beta(\chi) \cup \mu_{p-1} = H^2(\mu_2, \mu_{p-1}) \cong \mathrm{Br}(KO_2 \mid KU_2)$. The map from $\mathrm{Br}(KO \mid KU)$ is zero, since these classes are all detected in filtration 2 while the generator of $\mathrm{Br}(KO \mid KU)$ is detected in filtration 6. □

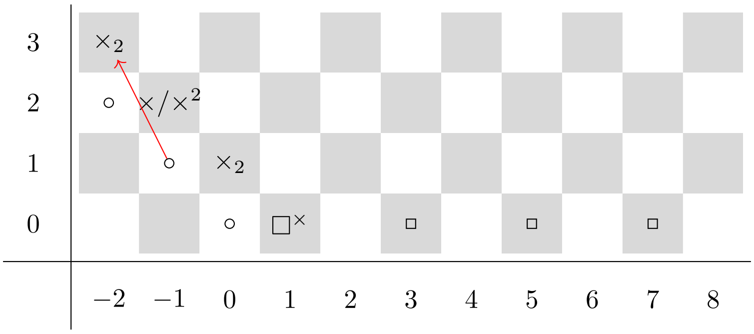


Figure 9.1: The HFPSS for $\mathfrak{pic}(KO_p) \simeq \mathfrak{pic}(KU_p)^{C_2}$ at odd primes. Here $\circ = \mathbb{Z}/2$, $\times = \mu_{p-1}$, $\times_2 = \mu_{p-1}[2] = C_2$ and $\square = \mathbb{Z}_p$.

9.3 The case $p = 2$

Putting together the work of the previous sections, we complete the computation of Br_1^0 at the prime two.

9.3.1 Descent from KO_2

Theorem 9.3 yields

$$\pi_t \mathfrak{Br}(KO_2 \mid KU_2) = \begin{cases} \mathbb{Z}/4\{(KO_2^{\mathrm{nr}}, \Sigma^2 KO_2^{\mathrm{nr}})\} & t = 0 \\ \mathbb{Z}/8\{\Sigma KO_2\} & t = 1 \\ \mathbb{Z}_2^\times & t = 2 \\ \pi_{t-2} KO_2 & t \geq 3 \end{cases}$$

We will use this to compute the group $\mathrm{Br}(\mathbf{1_K} \mid KU_2)$ by Galois descent along the $\mathbb{Z}_2$-Galois extension $\mathbf{1_K} \to KO_2$. Namely, we use the iterated fixed points formula

$$\mathfrak{Br}(\mathbf{1_K} \mid KU_2) \simeq (B\mathfrak{Pic}(KU_2)^{hC_2})^{h(1+4\mathbb{Z}_2)} = \mathfrak{Br}(KO_2 \mid KU_2)^{h(1+4\mathbb{Z}_2)}.$$

to form the descent spectral sequence

$$E_2^{s,t} = H^s(\mathbb{Z}_2, \pi_t \mathfrak{Br}(KO_2 \mid KU_2)) \implies \pi_{t-s} \mathfrak{Br}(\mathbf{1_K} \mid KU_2),$$

which is displayed in Fig. 9.2. Note that we have shifted degrees by one to match other figures, so that the relative Brauer group is still computed by the (-1)-stem. Since $\mathbb{Z}_2$ has cohomological dimension one, there is no room for differentials and the spectral sequence collapses immediately. What remains to compute is the following:

- The group $E_2^{0,1} = \mathrm{Br}(KO_2 \mid KU_2)^{1+4\mathbb{Z}_2}$,
- The extension between the groups $E_\infty^{0,-1} = E_2^{0,-1} \cong \mathrm{Br}(KO_2 \mid KU_2)^{1+4\mathbb{Z}_2}$ and $E_\infty^{1,0} = E_2^{1,0} \cong$

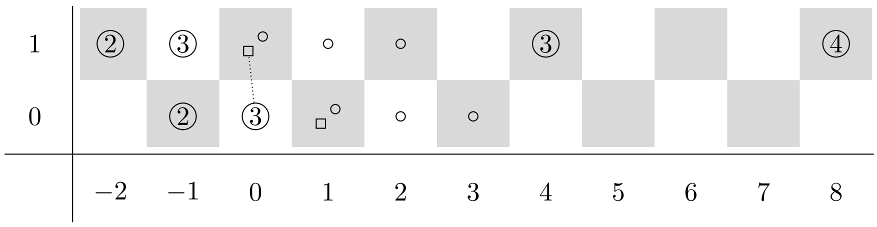


Figure 9.2: The descent spectral sequence for $\mathfrak{Br}(\mathbf{1_K} \mid KU_2) \simeq \mathfrak{Br}(KO_2 \mid KU_2)^{h(1+4\mathbb{Z}_2)}$. To match other figures, we have shifted everything in degree by one (so one may think of this as the spectral sequence for $\Sigma^{-1}\mathfrak{br}$). The extension in the 0-stem is $4 \in \mathrm{Ext}(\mathbb{Z}/8, \mathbb{Z}_2) \simeq \mathbb{Z}/8$, which gives $\pi_0\Sigma^{-1}\mathfrak{br}(\mathbf{1_K} \mid KU_2) = \mathrm{Pic}_1 = \mathbb{Z}_2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2$.

$$\mathbb{Z}/8\{(KO_2, \Sigma KO_2)\}.$$

This is achieved in the next couple of results.

Proposition 9.11. *We have* $\mathrm{Br}(KO_2 \mid KU_2)^{1+4\mathbb{Z}_2} = \mathbb{Z}/4$*, so the map*

$$\mathrm{Br}'(\mathbf{1_K} \mid KU_2) \to \mathrm{Br}(KO_2 \mid KU_2)$$

is surjective.

Proof. It suffices to prove that $\psi^*(KO_2^{\mathrm{nr}}, \Sigma^2 KO_2^{\mathrm{nr}}) \simeq (KO_2^{\mathrm{nr}}, \Sigma^2 KO_2^{\mathrm{nr}})$, where $\psi = \psi^\ell$ is the Adams operation for a topological generator $\ell \in 1 + 4\mathbb{Z}_2$. By Construction 8.3, this class is given by

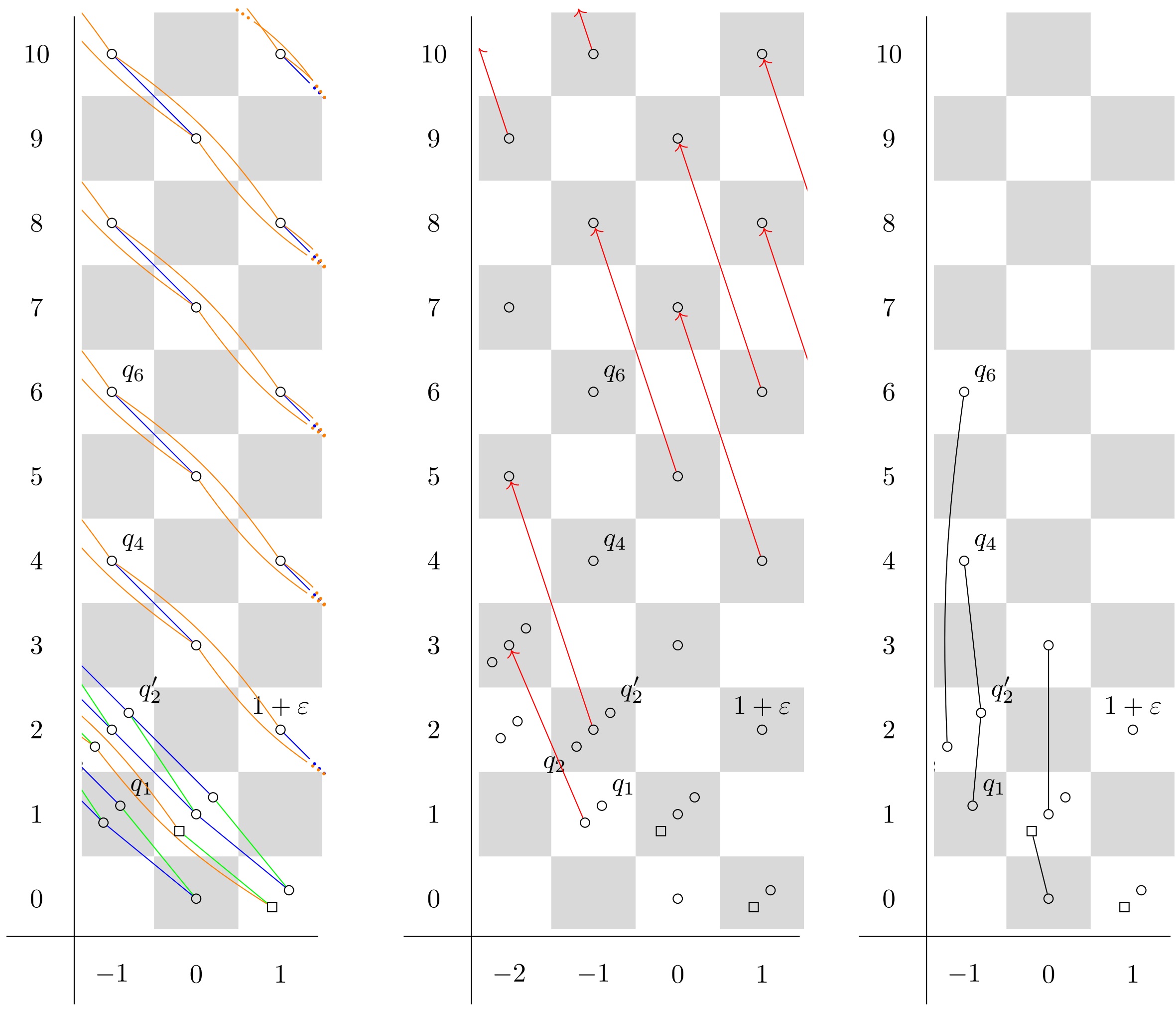
$$\left(\varphi_!^{\Sigma KO_2^{\mathrm{nr}}} : \mathrm{Mod}_{KO_2^{\mathrm{nr}}} \to \varphi^*\mathrm{Mod}_{KO_2^{\mathrm{nr}}}\right) \in \mathrm{Eq}\left(id, \varphi^* : B\mathfrak{Pic}(KO_2^{\mathrm{nr}}) \rightrightarrows B\mathfrak{Pic}(KO_2^{\mathrm{nr}})\right),$$

where $\varphi = KO_2 \otimes \varphi_2$ is the Frobenius on $KO_2^{\mathrm{nr}} = KO_2 \otimes_{\mathbb{S}} \mathbb{SW}$. In particular, note that $\psi \otimes \mathbb{SW}$ commutes with the φ. Thus the proposition follows from the square

$$\begin{tikzcd}
\mathrm{Mod}_{KO_2^{\mathrm{nr}}} \arrow[r, "\Sigma^2"] \arrow[dd, "\psi_!"'] & \mathrm{Mod}_{KO_2^{\mathrm{nr}}} \arrow[r, "\varphi_!"] & \varphi^*\mathrm{Mod}_{KO_2^{\mathrm{nr}}} \arrow[d, "\psi_!"] \\
 & & \psi^*\varphi^*\mathrm{Mod}_{KO_2^{\mathrm{nr}}} \arrow[d, "\sim"] \\
\psi^*\mathrm{Mod}_{KO_2^{\mathrm{nr}}} \arrow[r, "\Sigma^2"'] & \psi^*\mathrm{Mod}_{KO_2^{\mathrm{nr}}} \arrow[r, "\varphi_!"'] & \varphi^*\psi^*\mathrm{Mod}_{KO_2^{\mathrm{nr}}}
\end{tikzcd}$$

whose commutativity is witnessed by the natural equivalence

$$\psi_!\varphi_!\Sigma^2 \simeq \varphi_!\psi_!\Sigma^2 \simeq \varphi_!\Sigma^2\psi_!. \qquad \square$$



(a) Multiplications by χ (in blue), $\beta(\chi)$ (orange), and π (green).

(b) Differentials out of the (-1)-stem.

(c) Extensions in the E_∞-page.

Figure 9.3: Detailed view of the Picard spectral sequence (Fig. 7.2) in low degrees.

Proposition 9.12. *The relative cohomological Brauer group at the prime two is*

$$\mathrm{Br}'(\mathcal{S}p_{\mathbf{K}} \mid KU_2) \cong \mathbb{Z}/8 \oplus \mathbb{Z}/4.$$

Proof. Based on Fig. 9.2, what remains is to compute the extension from $\mathrm{Br}'(\mathbf{1_K} \mid KO_2) \cong \mathbb{Z}/8\{(KO_2, \Sigma KO_2)\}$ to $\mathrm{Br}(KO_2 \mid KU_2)^{1+4\mathbb{Z}_2} \cong \mathbb{Z}/4\{(KO_2^{\mathrm{nr}}, \Sigma^2 KO_2^{\mathrm{nr}})\}$. We will conclude by showing that the extension is split.

To prove the claim, note that both $(KO_2, \Sigma KO_2)$ and $(KO_2^{\mathrm{nr}}, \Sigma^2 KO_2^{\mathrm{nr}})$ split over KO_2^{nr}, so that the inclusion of $\mathrm{Br}'(\mathbf{1_K} \mid KU_2)$ in $\mathrm{Br}'(\mathbf{1_K})$ factors as

$$\begin{tikzcd}
 & \mathrm{Br}'(\mathbf{1_K} \mid KO_2^{\mathrm{nr}}) \arrow[d, hook] \\
\mathrm{Br}'(\mathbf{1_K} \mid KU_2) \arrow[r, hook] \arrow[ur, dashed] & \mathrm{Br}'(\mathbf{1_K})
\end{tikzcd}$$

We will compute the relative Brauer group $\mathrm{Br}'(\mathbf{1_K} \mid KO_2^{\mathrm{nr}})$ by means of the descent spectral sequence

$$H^s(\mathbb{Z}_2 \times \widehat{\mathbb{Z}}, \pi_t B\mathfrak{Pic}(KO_2^{\mathrm{nr}})) \implies \pi_{t-s}\mathfrak{Br}'(\mathbf{1_K} \mid KO_2^{\mathrm{nr}}),$$

which collapses at the E_3 page since $\mathbb{Z}_2 \times \widehat{\mathbb{Z}}$ has cohomological dimension two for profinite modules. To compute the E_2-page, note that

$$\mathrm{Pic}(KO_2^{\mathrm{nr}}) = \mathbb{Z}/8\{\Sigma KO_2^{\mathrm{nr}}\} \quad \text{and} \quad \pi_0 \mathrm{GL}_1(KO_2^{\mathrm{nr}}) = \mathbb{W}^\times,$$

by C_2-Galois descent from $KU_2^{\mathrm{nr}} := E(\overline{\mathbb{F}}_2, \widehat{\mathbb{G}}_m) \simeq KU_2 \otimes_{\mathbb{S}} \mathbb{SW}$. The action on $\mathrm{Pic}(KO_2^{\mathrm{nr}})$ is trivial, while the action on $\pi_0\mathrm{GL}_1(KO_2^{\mathrm{nr}})$ is trivial for the $\mathbb{Z}_2$-factor, and Frobenius for the $\widehat{\mathbb{Z}}$-factor. In particular, note that $H^0(\widehat{\mathbb{Z}}, \mathbb{W}^\times) = \mathbb{Z}_2^\times$; using the multiplicative splitting $\mathbb{W}^\times \simeq \overline{\mathbb{F}}_2^\times \times U_2$ (where $U_2 = \{x : \nu(x-1) \geq 2\} \cong \mathbb{W}$), we see that

$$\begin{aligned}
H^1(\widehat{\mathbb{Z}}, \mathbb{W}^\times) &\cong H^1(\widehat{\mathbb{Z}}, \overline{\mathbb{F}}_2^\times) \oplus H^1(\widehat{\mathbb{Z}}, \mathbb{W}) \\
&\cong 0 \oplus \varinjlim H^1(\mathbb{Z}/n, \mathbb{W}(\mathbb{F}_{2^n})) \\
&= 0
\end{aligned}$$

by Hilbert 90. This implies that $H^2(\mathbb{Z}_2 \times \widehat{\mathbb{Z}}, \pi_0\mathrm{GL}_1(KO_2^{\mathrm{nr}})) = 0$, since $\mathrm{cd}(\mathbb{Z}_2) = \mathrm{cd}(\widehat{\mathbb{Z}}) = 1$ for profinite coefficients. The (-1)-stem of the E_2-page is hence concentrated in filtration one, and hence agrees with the E_∞-page in this range. Thus

$$\mathrm{Br}'(\mathbf{1_K} \mid KO_2^{\mathrm{nr}}) = H^1(\mathbb{Z}_2 \times \widehat{\mathbb{Z}}, \mathbb{Z}/8) = \mathbb{Z}/8 \oplus \mathbb{Z}/8$$

and $\mathrm{Br}'(\mathbf{1_K} \mid KU_2) \hookrightarrow \mathbb{Z}/8 \oplus \mathbb{Z}/8$, which implies the claim. ∎

9.3.2 Generators at the prime two

Theorem 9.13. *The relative Brauer group at the prime two is*

$$\mathrm{Br}_0^1 = \mathbb{Z}/8 \oplus \mathbb{Z}/4.$$

Proof. To lift the cohomological Brauer classes generating $\mathrm{Br}'(\mathcal{S}p_{\mathbf{K}} \mid KU_2) \cong \mathbb{Z}/8 \oplus \mathbb{Z}/4$ to Azumaya algebras, it is enough by Corollary 5.24 to prove that they are trivialised in some finite extension of $\mathbf{1}_{\mathbf{K}}$. Recall from the previous subsection that:

(*i*) The generator of the $\mathbb{Z}/4$-factor is $(KO_2, \Sigma KO_2) \in \mathrm{Br}'(\mathcal{S}p_{\mathbf{K}} \mid KO_2)$ (Example 8.4), and detected by $2 \in \mathbb{Z}/8 \cong H^1(\mathbb{Z}_2, \mathrm{Pic}(KO_2))$. Since 2 is in the kernel of

$$H^1(\mathbb{Z}_2, \mathrm{Pic}(KO_2)) \to H^1(4\mathbb{Z}_2, \mathrm{Pic}(KO_2)),$$

this cohomological Brauer class is trivialised in the $\mathbb{Z}/4$-Galois extension $KO_2^{h(1+16\mathbb{Z}_2)}$, so

$$(KO_2, \Sigma^2 KO_2) \in \mathrm{Br}'(\mathbf{1}_{\mathbf{K}} \mid KO_2^{h(1+16\mathbb{Z}_2)}) \cong \mathrm{Br}(\mathbf{1}_{\mathbf{K}} \mid KO_2^{h(1+16\mathbb{Z}_2)}) \subset \mathrm{Br}_1^0\,.$$

(*ii*) Similarly, the generator of the $\mathbb{Z}/8$-factor is detected in the descent spectral sequence for $\mathbf{1}_{\mathbf{K}} \to KO_2^{\mathrm{nr}}$ by $(1,0) \in \mathbb{Z}/8 \oplus \mathbb{Z}/8 \cong H^1(\widehat{\mathbb{Z}} \times \mathbb{Z}_2, \mathbb{Z}/8)$. We claim that this generator is trivialised in the extension $(KO_2^{\mathrm{nr}})^{h(8\widehat{\mathbb{Z}} \times \mathbb{Z}_2)} \simeq \mathbf{1}_{\mathbf{K}} \otimes_{\mathbb{S}} \mathbb{SW}_8$, where $\mathbb{SW}_8 := W^+(\mathbb{F}_{2^8})$. Indeed, since

$$(1,0) \in \ker\Big(H^1(\widehat{\mathbb{Z}} \times \mathbb{Z}_2, \mathbb{Z}/8) \to H^1(8\widehat{\mathbb{Z}} \times \mathbb{Z}_2, \mathbb{Z}/8)\Big)$$

and the relative cohomological Brauer group $\mathrm{Br}'(\mathbf{1}_K \otimes_{\mathbb{S}} \mathbb{SW}_8 \mid KO_2^{\mathrm{nr}})$ is concentrated in filtration $s \leq 1$ of the Picard spectral sequence for $\mathbf{1}_{\mathbf{K}} \otimes_{\mathbb{S}} \mathbb{SW}_8 \to KO_2^{\mathrm{nr}}$, we see that

$$[(1,0)] \in \mathrm{Br}'(\mathbf{1}_{\mathbf{K}} \mid \mathbf{1}_{\mathbf{K}} \otimes_{\mathbb{S}} \mathbb{SW}_8) \cong \mathrm{Br}(\mathbf{1}_{\mathbf{K}} \mid \mathbf{1}_{\mathbf{K}} \otimes_{\mathbb{S}} \mathbb{SW}_8) \subset \mathrm{Br}_1^0\,. \qquad \square$$

Using Theorem 9.13, we can completely determine the behaviour of the Picard spectral sequence Fig. 7.2. Recall the E_2-generators specified in Remark 7.6.

- Under the base-change to KO_2, the generators q_2 and q_6 map to the E_2-classes representing $P_2, P_6 = P_2^2 \in \mathrm{Br}(KO_2 \mid KU_2)$. The splitting in Theorem 9.13 of the surjection $\mathrm{Br}_1^0 \twoheadrightarrow \mathrm{Br}(KO_2 \mid KU_2)$ of Proposition 9.11 gives a canonical choice of classes $Q_2, Q_6 = Q_2^2 \in \mathrm{Br}_1^0$ lifting these. In particular, q_2 must also be a permanent cycle.
- Since $\mathrm{Br}_1^0 \supset \mathrm{Br}(\mathbf{1}_{\mathbf{K}} \mid KO_2) \cong \mathbb{Z}/8$, the classes q_1, q_2' and q_4 must also survive, and detect Brauer classes Q_1, Q_2' and Q_4 trivialised over KO_2. We have $Q_2' = Q_1^2$ and $Q_4 = Q_1^4$ for this choice.

In Fig. 9.3 we have displayed the E_∞-page of the descent spectral sequence, including extensions. We have also included the module structure over $H^*(\mathbb{Z}_2^\times, \mathbb{Z}/2)$ and $H^*(C_2, \mathbb{Z})$, as appropriate; in particular, in Fig. 9.3a we display multiplications by the generators

- $\chi \in H^1(C_2, \mathbb{Z}/2) \cong \mathrm{Hom}(C_2, \mathbb{Z}/2)$,
- $\pi \in H^1(1 + 4\mathbb{Z}_2, \mathbb{Z}/2) \cong \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}/2)$,
- $\beta(\chi) \in H^2(C_2, \mathbb{Z})$.

From the form of the spectral sequence, it follows that the class in $E_2^{7,5}$ survives to E_∞—this should have implications for the nonconnective Brauer spectrum of $\mathcal{S}p_{\mathbf{K}}$ in the sense of [Hau17].

Finally, we consider the consequences of Theorem 8.14 at the prime two. In this case we do not know if the units $\pm i \in \mathbb{Z}_2^\times \subset \pi_0 \mathbf{1}_{\mathbf{K}}^\times$ are strict; for example, they are not strict in Morava

E-theory by [BSY22, Theorem 8.17]. In fact, we expect that the descent spectral sequence for $\mathbb{G}_m(\mathbf{1_K}) \simeq \mathbb{G}_m(E(\bar{\mathbb{F}}_2, \Gamma_1))^{h(\widehat{\mathbb{Z}}\times\mathbb{Z}_2^\times)}$ will yield

$$\pi_0\mathbb{G}_m(\mathbf{1_K}) \cong \mathbb{Z}/2\,\{1+\varepsilon\} \subset (\pi_0\mathbf{1_K})^\times = (\mathbb{Z}_2[\varepsilon]/(2\varepsilon, \varepsilon^2))^\times.$$

This will be discussed in future work. Nevertheless, we have the following corollary of Theorem 8.14:

Corollary 9.14. *For any $\chi\colon C_2 \cong \mathbb{Z}/2$, we have*

$$Q_1^4 = Q_4 \coloneqq (KU^{h(1+4\mathbb{Z}_2)}, \chi, 1+\varepsilon) \in \mathrm{Br}_1^0\,.$$

Proof. The unit $1+\varepsilon$ is strict by [CY23, Corollary 5.5.5], so the result follows from Theorem 8.14 since $q_4 = \beta(\chi) \cup (1+\varepsilon)$. □

Appendices

Appendix A

Décalage results

We make the derivation of the descent spectral sequence a little more explicit, and relate it to the spectral sequence obtained from the covering $\mathbb{G} \twoheadrightarrow *$. For clarity most of this section is written in a general context, but the main result is Lemma A.3, which will be used to relate the descent spectral sequence to the **K**-local **E**-Adams spectral sequence through décalage.

Let $\mathcal{C}$ be a site, and write $\mathcal{A} := \mathcal{S}h(\mathcal{C}, \mathcal{S}p)$. Given any object $\mathcal{F} \in \mathcal{A}$, there are two natural filtrations one can consider:

(*i*) The usual t-structure on $\mathcal{S}p$ induces one on $\mathcal{A}$, defined by the property that $\mathcal{F}' \in \mathcal{A}$ is t-truncated if and only if $\mathcal{F}'(X) \in \mathcal{S}p_{\leq t}$ for every $X \in \mathcal{C}$. One can therefore form the Postnikov tower

$$\mathcal{F} \simeq \lim(\cdots \to \tau_{\leq t}\mathcal{F} \to \cdots)$$

in $\mathcal{A}$, and obtain a tower of spectra on taking global sections.

(*ii*) Suppose $U \twoheadrightarrow *$ is a covering of the terminal object. Since A is a sheaf, we can recover $\Gamma\mathcal{F}$ as the limit of its Čech complex for the covering, and this has an associated tower. Explicitly, write $\mathrm{Tot}_q = \lim_{\mathbf{\Delta}_{\leq q}}$ so that

$$\Gamma\mathcal{F} \simeq \mathrm{Tot}\,\mathcal{F}(U^\bullet) \simeq \lim(\cdots \to \mathrm{Tot}_0\,\mathcal{F}(U^\bullet)).$$

Any tower of spectra $X = \lim(\cdots \to X_t \to \cdots)$ gives rise to a (conditionally convergent) spectral sequence, as in Lemma 2.2. Respectively, in the cases above these read

$$E_2^{s,t} = \pi_{t-s}\Gamma\tau_t\mathcal{F} \implies \pi_{t-s}\Gamma\mathcal{F}, \tag{A.1}$$

and

$$\check{E}_2^{p,q} = \pi_{q-p} f_q \mathcal{F}(U^\bullet) \implies \pi_{q-p}\Gamma\mathcal{F}, \tag{A.2}$$

where f_q denotes the fibre of the natural transformation $\mathrm{Tot}_q \to \mathrm{Tot}_{q-1}$. In each case, the d_r

differential has bidegree $(r+1, r)$ in the displayed grading.

For our purposes, the first spectral sequence, whose E_2-page and differentials are both defined at the level of truncations, is useful for interpreting the descent spectral sequence: for example, we use this in Theorem 4.4. On the other hand, the second spectral sequence is more easily evaluated once we know the value of a proétale sheaf on the free $\mathbb{G}$-sets. It will therefore be important to be able to compare the two, and this comparison is made using the *décalage* technique originally due to Deligne in [Del71]. The following material is well known (see for example [Lev15]), but we include an exposition for convenience and to fix indexing conventions. The décalage construction of [Hed21], which 'turns the page' of a spectral sequence by functorially associating to a filtered spectrum its *decalée* filtration, is closely related but not immediately equivalent.

Recall that any tower dualises to a filtration (this will be recounted below); the proof is cleanest in the slightly more general context of *bifiltered* spectra, and so we will make the connection between (A.1) and (A.2) explicit after proving a slightly more general result.

We suppose therefore that X is a spectrum equipped with a (complete and decreasing) bifiltration. That is, we have a diagram of spectra $X^{t,q} : \mathbb{Z}^{\mathrm{op}} \times \mathbb{Z}^{\mathrm{op}} \to \mathcal{S}p$ such that $X = \operatorname{colim}_{p,q} X^{t,q}$. We will write $X^{-\infty,q} := \operatorname{colim}_t X^{t,q}$ for any fixed q, and likewise $X^{t,-\infty} := \operatorname{colim}_q X^{t,q}$ for fixed t. Finally, write $X^{t/t',q/q'} := \operatorname{cofib}(X^{t',q'} \to X^{t,q})$.

Proposition A.1. *Let X be a bifiltered spectrum, and suppose that for all t and q we have $\pi_s X^{t/t+1,q/q+1} = 0$ unless $s = t - q$. Then there is an isomorphism*

$$ {}^1E_2^{s,t} \simeq {}^2E_3^{2s-t,s}, \tag{A.3}$$

where the left-hand side is the spectral sequence for the filtration $X = \operatorname{colim}_t X^{t,-\infty}$ and the right-hand for $X = \operatorname{colim}_q X^{-\infty,q}$.

This isomorphism is compatible with differentials, and so extends to isomorphisms

$$ {}^1E_r^{s,t} \simeq {}^2E_{r+1}^{2s-t,s} $$

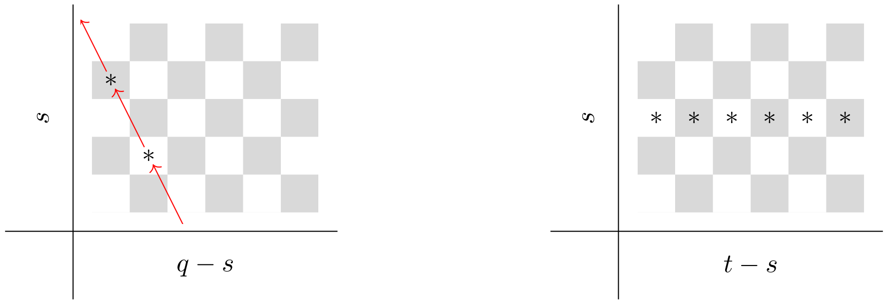
for $1 \leq r \leq \infty$.

Proof. We begin with the isomorphism (A.3): it is obtained by considering the following trigraded spectral sequences, which converge to the respective E_2-pages.

$$ E_2^{s,t,q} = \pi_{q-s} X^{t/t+1,q+1/q} \implies \pi_{q-s} X^{t/t+1,-\infty} = {}^1E_2^{s+t-q,t}, \tag{A.4}$$

$$ E_2^{s,t,q} = \pi_{t-s} X^{t+1/t,q+1/q} \implies \pi_{t-s} X^{-\infty,q+1/q} = {}^2E_2^{s+q-t,q}. \tag{A.5}$$

The d_r differentials have (s,t,q)-tridegrees $(r+1, 0, r)$ and $(r+1, r, 0)$ respectively. Both spectral sequences therefore take a very simple form, because we have assumed each object $X^{t+1/t,q+1/q}$ is Eilenberg-Mac Lane. They are displayed in Fig. A.1.



(a) Spectral sequence (A.4) at fixed t. The y-intercept is $s = t$.

(b) Spectral sequence (A.5) at fixed q. The y-intercept is $s = q$.

Figure A.1: The trigraded spectral sequences computing ${}^1E_2^{*,t}$ and ${}^2E_2^{*,q}$ respectively.

In particular, the first collapses after the E_2-page, so that ${}^1E_2^{*,t}$ is the cohomology of the complex

$$\cdots \to \pi_{t-q}X^{t/t+1,q/q+1} \to \pi_{t-q-1}X^{t/t+1,q+1/q+2} \to \cdots \tag{A.6}$$

whose differentials are induced by the composites

$$X^{t+1/t,q/q+1} \to \Sigma X^{t+1/t,q+1} \to \Sigma X^{t+1/t,q+1/q+2}.$$

More precisely, ${}^1E_2^{s,t} = \pi_{t-s}X^{t/t+1,-\infty} \cong H^s\left(\pi_{t-*}X^{t/t+1,*/*+1}\right)$.

The second spectral sequence is even simpler, collapsing immediately to give

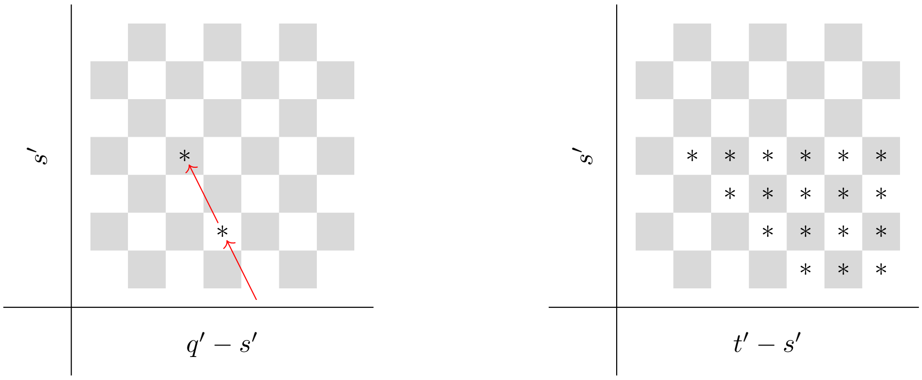
$${}^2E_2^{2s-t,s} = \pi_{t-s}X^{-\infty,s/s+1} \cong \pi_{t-s}X^{t/t+1,s/s+1}.$$

We can further identify the first differential on ${}^2E_2^{2s-t,s}$: it is induced by the map $X^{-\infty,2s-t/2s-t+1} \to \Sigma X^{-\infty,2s-t+1/2s-t+2}$, and so the identification ${}^1E_2^{s,t} \cong {}^2E_3^{2s-t,s}$ follows from the commutative diagram below, in which the top row is part of the complex (A.6).

$$\begin{tikzcd}
\pi_{t-s}X^{t/t+1,s/s+1} \arrow[r] \arrow[d, "\sim"'] & \pi_{t-s}\Sigma X^{t/t+1,s+1} \arrow[r] \arrow[d] & \pi_{t-s}\Sigma X^{t/t+1,s+1/s+2} \arrow[d, "\sim"] \\
\pi_{t-s}X^{-\infty,s/s+1} \arrow[r] & \pi_{t-s}\Sigma X^{-\infty,s+1} \arrow[r] & \pi_{t-s}\Sigma X^{-\infty,s+1/s+2}
\end{tikzcd}$$

We next argue that this extends to an isomorphism of spectral sequences ${}^1E_r^{s,t} \cong {}^2E_{r+1}^{2s-t,s}$. To do so we will give a map of exact couples as below:

$$\left(\begin{tikzcd}
{}^1D_2^{s,t} \arrow[rr] & & {}^1D_2^{s,t} \arrow[dl] \\
& {}^1E_2^{s,t} \arrow[ul] &
\end{tikzcd}\right) \to \left(\begin{tikzcd}
{}^2D_3^{2s-t,s} \arrow[rr] & & {}^2D_3^{2s-t,s} \arrow[dl] \\
& {}^2E_3^{2s-t,s} \arrow[ul] &
\end{tikzcd}\right)$$



(a) Spectral sequence (A.8). The leftmost nonzero entry is $(q' - s', s') = (t' - s + 2, 2s - t' - 4)$.

(b) The E_2-page of (A.7). The leftmost nonzero entry is $(t' - s', s') = (t - s + 2, s - 2)$.

Figure A.2: Truncated spectral sequences computing $\pi_* C$.

By definition of the derived couple on the right-hand side, this amounts to giving maps

$$\pi_{t-s} X^{t,-\infty} \to \operatorname{im}(\pi_{t-s} X^{-\infty,s} \to \pi_{t-s} X^{-\infty,s-1})$$

for all s and t, subject to appropriate naturality conditions. To this end we claim that the first map in each of the spans below induces an isomorphism on π_{t-s}:

$$X^{t,-\infty} \leftarrow X^{t,s-1} \to X^{-\infty,s-1}.$$

Indeed, writing $C := \operatorname{cofib}(X^{t,s-1} \to X^{t,-\infty})$, we have a filtration $C = \operatorname{colim} C^{t'}$, where $t' \geq t$ and $C^{t'} = \operatorname{cofib}(X^{t',q-1} \to X^{t',-\infty})$. The resulting spectral sequence reads

$$\pi_{t'-s'} C^{t'/t'+1} \implies \pi_{t'-s'} C \qquad (t' \geq t), \tag{A.7}$$

and its E_2-page is in turn computed by a trigraded spectral sequence

$$\pi_{q'-s'} X^{t'/t'+1,q'/q'+1} X \implies \pi_{q'-s'} C^{t'/t'+1} \qquad (q' \leq s-2). \tag{A.8}$$

Spectral sequence (A.8) can be thought of as a truncation of Fig. A.1a; it is in turn displayed in Fig. A.2a. the form of its E_2-page implies that $C^{t'/t'+1}$ is $(t' - s + 2)$-connected, so that (A.7) takes the form displayed in Fig. A.2b.

In particular, C is $t - s + 1$-connected and so the map $X^{t,s-1} \to X^{t,-\infty}$ is $(t - s + 1)$-connected. Applying an identical analysis to $X^{t,s} \to X^{t,-\infty}$ shows that this map has $(t - s)$-connected cofibre, and so induces a surjection on π_{t-s}; the diagram below therefore produces the requisite map

${}^1D_2^{s,t} \to {}^2D_3^{2s-t,s}$.

$$\begin{tikzcd}
\pi_{t-s}X^{t,s} \arrow[r, two heads] \arrow[d] & \pi_{t-s}X^{t,s-1} \arrow[r, "\sim"] \arrow[d] & \pi_{t-s}X^{t,-\infty} \arrow[dl, "\exists"] \\
\pi_{t-s}X^{-\infty,s} \arrow[r] & \pi_{t-s}X^{-\infty,s-1} &
\end{tikzcd}$$

We will not show that this indeed defines a map of exact couples, since this result is well-known: for example, see [Lev15, §6]. □

We now relate this to the context of Postnikov towers and cobar complexes. If $X \simeq \lim_t(\cdots \to X_t \to \cdots)$ is a convergent tower of spectra with $\operatorname{colim} X_t = 0$, we can form a *dual* filtered spectrum

$$X^t := \operatorname{fib}(X \to X_{t-1}).$$

Then $\operatorname{colim} X^t \xrightarrow{\sim} X$, and we get another spectral sequence

$$\pi_{t-s}f^tX \implies \pi_{t-s}X,$$

where once again $X^{t/t+1} := \operatorname{cofib}(X^{t+1} \to X^t)$. Observe that $X^{t/t+1} \simeq X_{t/t-1} := \operatorname{fib}(X_t \to X_{t-1})$, by the octahedral axiom:

$$\begin{tikzcd}
X^{t+1} \arrow[dr] \arrow[r] & X^t \arrow[d] \arrow[r] & X_{t/t-1} \arrow[dl] \\
& X \arrow[dl] \arrow[r] & X_t \arrow[dll] \\
X_{t-1} & &
\end{tikzcd}$$

Lemma A.2. *The spectral sequences for (X_t) and (X^t) agree.*

Proof. The observation above is that the E_2-pages agree. To show that the entire spectral sequences match it is enough to show that the differentials d_2 do, in other words that the outer diagram below commutes.

$$\begin{tikzcd}
X_{t/t-1} = X^{t/t+1} \arrow[r] \arrow[d] & \Sigma X^{t+1} \arrow[d] \\
X_t \arrow[r] \arrow[ur, dashed] & \Sigma X_{t+1/t} = \Sigma X^{t+1/t+2}
\end{tikzcd}$$

The dashed arrow is given by applying the 3×3-lemma [Nee01] to obtain the diagrams below; note that each triangle above only anticommutes, and so the outer square is commutative.

$$\begin{tikzcd}
\Sigma^{-1}X^{t+1} \arrow[r] \arrow[d] & \Sigma^{-1}X^{t} \arrow[r] \arrow[d] & \Sigma^{-1}X^{t/t+1} \arrow[r] \arrow[d] & X^{t+1} \arrow[d] \\
\Sigma^{-1}X \arrow[r, equal] \arrow[d] & \Sigma^{-1}X \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & X \arrow[d] \\
\Sigma^{-1}X_{t} \arrow[r] \arrow[d] & \Sigma^{-1}X_{t-1} \arrow[r] \arrow[d] & X_{t/t-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "(-1)"] & X_{t} \arrow[d, dotted] \\
X^{t+1} \arrow[r] & X^{t} \arrow[r] & X^{t/t+1} \arrow[r] & \Sigma X^{t+1}
\end{tikzcd}$$

$$\begin{tikzcd}
X_{t-1} \arrow[r] \arrow[d, dotted] \arrow[dr, phantom, "(-1)"] & \Sigma X_{t/t-1} \arrow[r] \arrow[d] & \Sigma X_{t} \arrow[r] \arrow[d] & \Sigma X_{t-1} \arrow[d] \\
\Sigma X^{t} \arrow[r] \arrow[d] & \Sigma X^{t/t+1} \arrow[r] \arrow[d] & \Sigma^{2}X^{t+1} \arrow[r] \arrow[d] & \Sigma^{2}X^{t} \arrow[d] \\
\Sigma X \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \Sigma^{2}X \arrow[r, equal] \arrow[d] & \Sigma^{2}X \arrow[d] \\
\Sigma X_{t-1} \arrow[r] & \Sigma^{2}X_{t/t-1} \arrow[r] & \Sigma^{2}X_{t} \arrow[r] & \Sigma^{2}X_{t-1}
\end{tikzcd}$$

□

By dualising the Postnikov and Tot-towers, it will therefore suffice to verify that the induced bifiltration satisfies the assumptions of Proposition A.1.

Lemma A.3. *Let $\mathcal{F}$ be a sheaf of spectra on a site $\mathcal{C}$, and let $X \twoheadrightarrow *$ be a covering of the terminal object. Suppose that for every t and every $q > 0$ we have $\Gamma(X^q, \tau_t\mathcal{F}) = \tau_t\Gamma(X^q, \mathcal{F})$. Then there is an isomorphism between the descent and Bousfield-Kan spectral sequences, up to reindexing: for all r,*

$$E_r^{s,t} \cong \check{E}_{r+1}^{2s-t,s}.$$

Proof. We form the bifiltrations $(\Gamma\mathcal{F})_{t,q} = \operatorname{Tot}_q \Gamma(U^\bullet, \tau_{\leq t}\mathcal{F})$. Then

$$(\Gamma\mathcal{F})_{t,-\infty} = \lim_q \operatorname{Tot}_q \Gamma(U^\bullet, \tau_{\leq t}\mathcal{F}) = \operatorname{Tot}\Gamma(U^\bullet, \tau_{\leq t}\mathcal{F}) = \Gamma\tau_{\leq t}\mathcal{F},$$

while

$$(\Gamma\mathcal{F})_{-\infty,q} = \lim_t \operatorname{Tot}_q \Gamma(U^\bullet, \tau_{\leq t}\mathcal{F}) = \operatorname{Tot}_q \Gamma(U^\bullet, \lim_t \tau_{\leq t}\mathcal{F}) = \operatorname{Tot}_q \Gamma(U^\bullet, \mathcal{F}).$$

Applying Proposition A.1 to the dual filtration, we need to verify that

$$\operatorname{Tot}^{q/q+1}\Gamma(X^\bullet, \tau_t\mathcal{F})$$

is Eilenberg-Mac Lane. But recall that for any cosimplicial spectrum $B^\bullet$ we have

$$\operatorname{fib}(\operatorname{Tot}_q B^\bullet \to \operatorname{Tot}_{q-1} B^\bullet) \simeq \Omega^q N^q B^\bullet,$$

where N^q denotes the fibre of the map from X^q to the q-th matching spectrum; $N^qB^\bullet$ is a pointed space with

$$\pi_j N^q B^\bullet = \pi_j B^q \cap \ker s^0 \cap \ker s^{q-1}.$$

In the case of pointed spaces, this fact is [BK87, Prop. X.6.3]; the proof, which appears also as [GJ09, Lemma VIII.1.8], works equally well for a cosimplicial spectrum[1]. By abuse of notation, we also denote this group by $N^q\pi_j B^\bullet$. Thus

$$\begin{aligned}
\pi_j \operatorname{Tot}^{q/q+1} \Gamma(X^\bullet, \tau_t \mathcal{F}) &\simeq \pi_j \Omega^q N^q \Gamma(X^\bullet, \tau_t\mathcal{F}) \\
&\simeq N^q \pi_{j+q} \Gamma(X^\bullet, \tau_t \mathcal{F}) \\
&\simeq N^q \pi_{j+q} \tau_t \Gamma(X^\bullet, \mathcal{F}) \\
&\subset \pi_{j+q}\tau_t\Gamma(X^\bullet, \mathcal{F}) = \begin{cases} \pi_t \Gamma(X^\bullet, \mathcal{F}) & j = t-q \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

□

Remark A.4. On the starting pages, one has

$$E_2^{s,t} = \pi_{t-s}\Gamma\tau_t\mathcal{F} \simeq H^s(\mathcal{C}, \pi_t\mathcal{F})$$

and

$$\check{E}_3^{2s-t,s} = H^s(\pi_{t-*}\Omega^* N^* \Gamma(X^\bullet, \mathcal{F})) = H^s(N^*\pi_t\Gamma(X^\bullet, \mathcal{F})) \simeq \check{H}^s(X \twoheadrightarrow *, \pi_t\mathcal{F}).$$

In particular, note that our assumption implies that the Čech-to-derived functor spectral sequence collapses.

[1] Note that the inductive argument there applies in the 'cosimplicial' direction, i.e., in the notation of *loc. cit.*one shows for any fixed t that $N^{n,k}\pi_t X = \ker(\pi_t X \to M^{n,k}\pi_t X)$ for $k, n \le s$.

Appendix B

The Adams spectral sequence at height one

In this appendix, we compute the height-one $\mathbf{K}$-local $\mathbf{E}$-Adams spectral sequences at all primes. The material is well-known, and was surely known at the time of [Rav84; MRW77]. At the prime two we give a different perspective to [BGH22] on the computation, which illustrates how one can make use of the Postnikov tower of a sheaf of spectra on $B\mathbb{G}_{\mathrm{proet}}$: this approach may be useful in higher height examples that are more computationally challenging. In particular, we use the first example of the *finite resolution* technique, and as such our only input is knowledge of the HFPSS for the conjugation action on KU, recalled in Fig. B.2.

B.1 Odd primes

At odd primes, the multiplicative lift gives a splitting

$$\mathbb{Z}_p^\times \simeq (1 + p\mathbb{Z}_p) \times \mu_{p-1} \simeq \mathbb{Z}_p \times \mu_{p-1},$$

with the second isomorphism given by the p-adic logarithm. A pair $(a, b) \in \mathbb{Z}_p \times \mu_{p-1}$ therefore acts on $\pi_t\mathbf{E}$ as $(b \exp_p(a))^t$.

Lemma B.1. *The starting page of the* $\mathbf{K}$*-local* $\mathbf{E}$*-Adams spectral sequence is*

$$E_2^{s,t} = H^s(\mathbb{Z}_p^\times, \pi_t\mathbf{E}) = \begin{cases} \mathbb{Z}_p & t = 0 \text{ and } s = 0, 1 \\ \mathbb{Z}/p^{\nu_p(t')+1} & t = 2(p-1)t' \neq 0 \text{ and } s = 1 \end{cases} \tag{B.1}$$

and zero otherwise. In particular, it collapses immediately to E_∞.

Proof. We will use the Lyndon-Hochschild-Serre spectral sequence [SW00, Theorem 4.2.6] for the inclusion $\mathbb{Z}_p \simeq 1 + p\mathbb{Z}_p \hookrightarrow \mathbb{Z}_p^\times$, which reads

$$H^i(\mu_{p-1}, H^j(\mathbb{Z}_p, \pi_t KU_p)) \implies H^{i+j}(\mathbb{Z}_p^\times, \pi_t KU_p).$$

Since everything is (p)-local, taking μ_{p-1} fixed-points is exact. The spectral sequence therefore collapses, and what remains is to compute $\mathbb{Z}_p$-cohomology.

By [SW00, §3.2], the trivial pro-p module $\mathbb{Z}_p$ admits a projective resolution

$$0 \to \mathbb{Z}_p[[\mathbb{Z}_p]] \xrightarrow{\zeta - 1} \mathbb{Z}_p[[\mathbb{Z}_p]] \to \mathbb{Z}_p \to 0$$

in the (abelian) category $\mathfrak{C}_p(\mathbb{Z}_p)$ of pro-p continuous $\mathbb{Z}_p^\times$-modules; here we write ζ for a topological generator, and $\mathbb{Z}_p[[G]] \coloneqq \lim_{U <_o G} \mathbb{Z}_p[G/U]$ for the *completed* group algebra of a profinite group G. In particular,

$$H^j(\mathbb{Z}_p, \pi_{2t} KU_p) = H^j\left(\mathbb{Z}_p \xrightarrow{\exp_p(\zeta)^t - 1} \mathbb{Z}_p\right) \simeq \begin{cases} \mathbb{Z}_p & t = 0 \text{ and } j = 0, 1 \\ \mathbb{Z}_p/p^{\nu_p(t)+1} & t \neq 0 \text{ and } j = 1 \end{cases}$$

where for the final isomorphism we have used the isomorphism

$$\exp_p : p^{j-1}\mathbb{Z}_p/p^j\mathbb{Z}_p \simeq 1 + p^j\mathbb{Z}_p^\times/1 + p^{j+1}\mathbb{Z}_p^\times$$

to obtain $\nu_p(\exp_p(\zeta)^t - 1) = \nu_p(t\zeta) + 1 = \nu_p(t) + 1$. The μ_{p-1}-action has fixed points precisely when it is trivial, i.e. when $p - 1 \mid t$, which gives the stated form. □

B.2 $p = 2$

What changes at even primes? In this case the multiplicative lift is instead defined on $(\mathbb{Z}/4)^\times$, and provides a splitting

$$\mathbb{Z}_2^\times \simeq (1 + 4\mathbb{Z}_2) \times C_2 \simeq \mathbb{Z}_2 \times C_2.$$

This implies the following more complicated form for the starting page, since $\mathrm{cd}_2(C_2) = \infty$.

Lemma B.2. *The starting page of the descent spectral sequence for the action of $\mathbb{G}$ on $\mathbf{E}$ is given by*

$$E_2^{s,t} = H^s(\mathbb{Z}_2^\times, \pi_t \mathbf{E}) = \begin{cases} \mathbb{Z}_2 & t = 0 \text{ and } s = 0, 1 \\ \mathbb{Z}/2 & t \equiv_4 2 \text{ and } s \text{ odd} \\ \mathbb{Z}/2 & t \equiv_4 0 \text{ and } s > 1 \text{ odd} \\ \mathbb{Z}/2^{\nu_2(t)+2} & 0 \neq t \equiv_4 0 \text{ and } s = 1 \end{cases} \tag{B.2}$$

and zero otherwise. The result is displayed in Fig. B.1, which is reproduced for convenience of the reader.

Proof. We will again use the Lyndon-Hochschild-Serre spectral sequence for the inclusion $\mathbb{Z}_2 \simeq$

$1 + 4\mathbb{Z}_2 \hookrightarrow \mathbb{Z}_2^\times$, which reads

$$H^i(C_2, H^j(\mathbb{Z}_2, \pi_t KU_2)) \implies H^{i+j}(\mathbb{Z}_2^\times, \pi_t KU_2).$$

Since C_2 is 2-torsion, we will have higher C_2-cohomology contributions. Nevertheless, the computation of $\mathbb{Z}_2$-cohomology is identical to the odd-prime case, except that now one has $\nu_2(\exp_2(\zeta)^t - 1) = \nu_2(t) + 2$. Thus

$$H^j(\mathbb{Z}_2, \pi_t KU_2) = H^j\left(\mathbb{Z}_2 \xrightarrow{\exp_p(\zeta)^t - 1} \mathbb{Z}_2\right) \simeq \begin{cases} \mathbb{Z}_2 & t = 0 \text{ and } j = 0, 1 \\ \mathbb{Z}_2/2^{\nu_2(t)+2} & t \neq 0 \text{ and } j = 1 \end{cases}$$

For $t \neq 0$, the E_2-page of the Lyndon-Hochschild-Serre is therefore concentrated in degrees $j = 1$, and so collapses. This yields

$$H^s(\mathbb{Z}_2^\times, \pi_t KU_2) \simeq H^{s-1}(C_2, \mathbb{Z}/2^{\nu_2(t)+2}),$$

which accounts for most of the groups in (B.2). For $t = 0$, it instead takes the form

$$E_2^{*,*} = \mathbb{Z}_2[x^{(0,1)}, y^{(2,0)}]/(2x, 2y, x^2) \implies H^{i+j}(\mathbb{Z}_2^\times, \pi_0 KU_2).$$

In particular, the entire spectral sequence is determined by the differential $d_2(x) = 0$, which implies that all other differentials vanish by multiplicativity. To deduce this differential, note that the edge map $H^1(\mathbb{Z}_2^\times, \pi_0 KU_2) \to H^0(C_2, H^1(\mathbb{Z}_2, \pi_0 KU_2))$ can be interpreted as the restriction map

$$\mathrm{Hom}\mathbb{Z}_2^\times, \mathbb{Z}_2 \to \mathrm{Hom}\mathbb{Z}_2, \mathbb{Z}_2,$$

along $\exp_2 : \mathbb{Z}_2 \hookrightarrow \mathbb{Z}_2^\times$. This is an isomorphism since $\mathrm{Hom}\mathbb{Z}/2, \mathbb{Z}_2 = 0$, so d_2 must act trivially on bidegree $(0, 1)$. $\square$

Our next task is to compute the differentials. In the rest of the appendix we will prove the following:

Proposition B.3. *The differentials on the third page are as displayed in Fig. B.1. The spectral sequence collapses at E_4 with a horizontal vanishing line.*

We will compute these differentials by comparing to the HFPSS for the conjugation action on KU_2 (Fig. B.2), which reads

$$E_2^{s,t} = H^s(C_2, \pi_* KU_2) = \mathbb{Z}_2[\eta, u^{\pm 2}]/2\eta \implies \pi_{t-s} KO_2.$$

The following result folklore:

Lemma B.4. *The $\mathbf{K}$-local sphere fits in a fibre sequence*

$$\mathbf{1}_{\mathbf{K}} \to KO_2 \xrightarrow{\psi^5 - 1} KO_2.$$

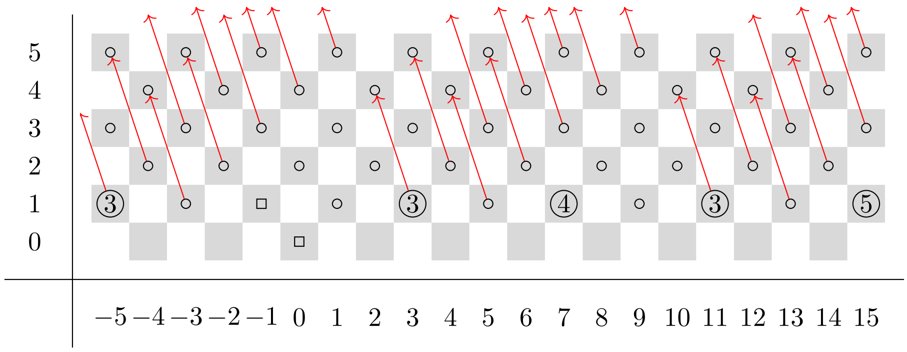


Figure B.1: The E_3-page of the descent spectral sequence for $\mathbf{E}$ at $p = 2$.

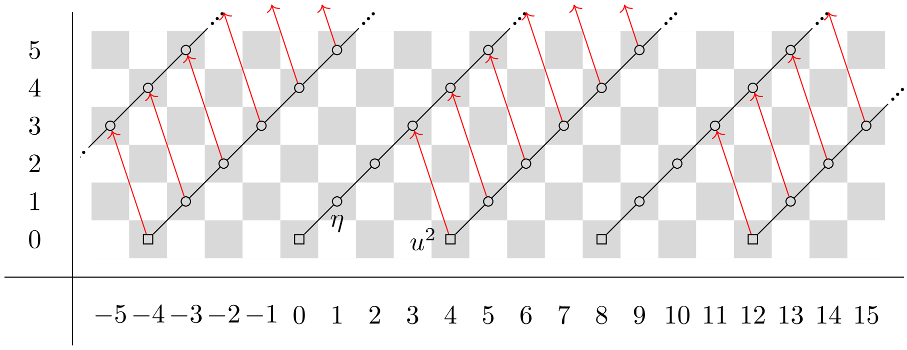


Figure B.2: The HFPSS for the C_2-Galois extension $KO \to KU$. The class η represents η in the Hurewicz image in $\pi_* KO$, and towers of slope one are related by η-multiplications. In particular, η^4 cannot survive to E_∞ since $\eta^4 = 0 \in \pi_* \mathbb{S}$. The only option is $d_3(u^2\eta) = \eta^4$, which implies the rest by multiplicativity.

Proof. We first consider the map $\psi^5 - 1 : KU_2 \to KU_2$. Certainly ψ^5 acts trivially on $KU_2^{1+4\mathbb{Z}_2} := \Gamma(\mathbb{Z}_2^\times/1+4\mathbb{Z}_2, \mathcal{E})$, so there is a map

$$KU_2^{h(1+4\mathbb{Z}_2)} \to \mathrm{fib}(KU_2 \xrightarrow{\psi^5-1} KU_2) \tag{B.3}$$

induced by the inclusion of fixed points $KU_2^{1+4\mathbb{Z}_2} \to KU_2$. As observed in Lemma B.2, the HFPSS computing $\pi_* KU_2^{h(1+4\mathbb{Z}_2)}$ collapses at E_2 with horizontal vanishing above filtration one, and one therefore sees that (B.3) must be an equivalence by computing homotopy groups of $\mathrm{fib}(\psi^5 - 1)$ using the exact sequence.

Taking fixed points for the C_2 action now yields the result:

$$\begin{aligned}
\mathbf{1}_{\mathbf{K}} &\simeq (KU_2^{h(1+4\mathbb{Z}_2)})^{C_2} \\
&\simeq \mathrm{fib}(KU_2 \xrightarrow{\psi^5-1} KU_2)^{C_2} \\
&\simeq \mathrm{fib}(KU_2^{C_2} \xrightarrow{\psi^5-1} KU_2^{C_2}) \\
&\simeq \mathrm{fib}(KO_2 \xrightarrow{\psi^5-1} KO_2).
\end{aligned}$$

□

Proof (Proposition B.3). The previous lemma gives the diagram

$$\begin{tikzcd}
\Sigma^{-1}KO_2 \arrow[r] \arrow[d] & \mathbf{1}_{\mathbf{K}} \arrow[r] \arrow[d] & KO_2 \arrow[d] \\
\Sigma^{-1}KU_2 \arrow[r] & KU_2^{h(1+4\mathbb{Z}_2)} \arrow[r] & KU_2
\end{tikzcd} \tag{B.4}$$

in which the top row is obtained as C_2-fixed points of the bottom. The HFPSS for the middle map,

$$E_2^{s,t} = H^s(C_2, \pi_t KU_2^{1+4\mathbb{Z}_2}) \implies \pi_{t-s}\mathbf{1}_{\mathbf{K}}, \tag{B.5}$$

is very closely to the descent spectral sequence; it is displayed in Fig. B.3. In fact, in Lemma B.5 we will show that the two spectral sequences are isomorphic (including differentials), up to a certain filtration shift. To infer the differentials in Fig. B.1, it is therefore enough to compute the differentials in Fig. B.3.

Next observe that Lemma B.4 implies existence of exact sequences

$$0 \to \pi_{2t} KU_2^{h(1+4\mathbb{Z}_2)} \to \mathbb{Z}_2 \xrightarrow{5^t-1} \mathbb{Z}_2 \to \pi_{2t-1} KU_2^{h(1+4\mathbb{Z}_2)} \to 0,$$

in which the leftmost term vanishes for $t \neq 0$ and the middle map is null for $t = 0$. Taking C_2-cohomology yields isomorphisms

$$H^*(C_2, \pi_0 KU_2^{1+4\mathbb{Z}_2}) \xrightarrow{\sim} H^*(C_2, \pi_0 KU_2) \tag{B.6}$$

$$H^*(C_2, \pi_0 KU_2) \xrightarrow{\sim} H^*(C_2, \pi_{-1} KU_2^{1+4\mathbb{Z}_2}), \tag{B.7}$$

and an exact sequence

$$\begin{aligned}0 &\to H^{s-1}(C_2, \pi_{2t-1}KU^{h(1+4\mathbb{Z}_2)}) \to H^s(C_2, \pi_{2t}KU_2)\\ &\to H^s(C_2, \pi_{2t}KU_2) \to H^s(C_2, \pi_{2t-1}KU^{h(1+4\mathbb{Z}_2)}) \to 0\end{aligned}$$

for $s \geq 1$ (and $t \neq 0$). The middle terms are either both zero or both $\mathbb{Z}/2$, and in the latter case we obtain the following further isomorphisms:

$$H^{s-1}(C_2, \pi_{2t-1}KU_2^{1+4\mathbb{Z}_2}) \xrightarrow{\sim} H^s(C_2, \pi_{2t}KU_2) \simeq \mathbb{Z}/2, \tag{B.8}$$

$$\mathbb{Z}/2 \simeq H^s(C_2, \pi_{2t}KU_2) \xrightarrow{\sim} H^s(C_2, \pi_{2t-1}KU_2^{1+4\mathbb{Z}_2}). \tag{B.9}$$

For $s = 0$ and $t \neq 0$ even, we instead have an exact sequence

$$0 \to \mathbb{Z}_2 \xrightarrow{5^t-1} \mathbb{Z}_2 \to H^0(C_2, \pi_{2t-1}KU_2^{h(1+4\mathbb{Z}_2)}) \to 0. \tag{B.10}$$

Equations (B.6) to (B.10) compute the effect of the maps in (B.4) on spectral sequences. Namely:

(i) The map $\Sigma^{-1}KU_2 \to KU_2^{1+4\mathbb{Z}_2}$ induces a (filtration preserving) surjection from $E_2(KU)$ onto the unfilled classes in Fig. B.3.

(ii) The map $KU_2^{1+4\mathbb{Z}_2} \to KU_2$ is injective on the solid classes in Fig. B.3. It induces a filtration-preserving isomorphism on the subalgebras in internal degree $t = 0$, and away from this increases filtration by one.

The differentials in Fig. B.3 follow almost immediately: on unfilled classes, they are images of differentials in the HFPSS for KU_2, and on most solid classes they are detected by differentials in the HFPSS for KU_2. We are left to determine a small number of differentials on classes with internal degree t close to zero.

(i) The exact sequence induced by Lemma B.4 shows that the map $\mathbf{1_K} \to KO_2$ induces an isomorphism on π_0. As a result, the unit in the HFPSS for $KU^{h(1+4\mathbb{Z}_2)}$ is a permanent cycle.

(ii) Write $u^{-2}\eta^2$ for the generator in bidegree $(s,t) = (2,0)$, which maps to a class of the same name in the HFPSS for KU_2. This cannot survive in the HFPSS for $KU_2^{h(1+4\mathbb{Z}_2)}$: one sees that $\pi_{-2}\mathbf{1_K} = 0$ using the fibre sequence of Lemma B.4 and the fact that $\pi_{-2}KO_2 = \pi_{-1}KO_2 = 0$. The only possibility is a nonzero d_2 since all other possible differentials occur at E_4 or later, when there are no possible targets left (by virtue of the known d_3-differentials). Likewise, this implies

$$d_2((u^{-2}\eta^2)^{2j+1}) \equiv_2 (u^{-2}\eta^2)^{2j} d_2(u^{-2}\eta^2) \neq 0$$

by the Leibniz rule.

(iii) Write z for the generator in bidegree $(s, t-s) = (0,-3)$, which is detected in filtration one by $u^{-2}\eta$. As above one computes that $\pi_{-3}\mathbf{1_K} = 0$, and so this class must die; the only option

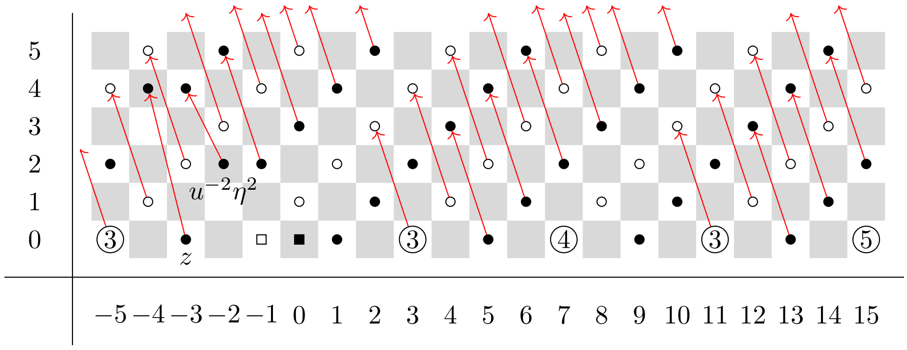


Figure B.3: The HFPSS for C_2 acting on $KU^{h(1+4\mathbb{Z}_2)}$. Solid classes are detected in the HFPSS for KU_2 by the map $KU_2^{h(1+4\mathbb{Z}_2)} \to KU_2$, with a filtration shift of one in internal degrees $t \neq 0$. Unfilled classes are in the image of the HFPSS for KU_2 under $\Sigma^{-1}KU_2 \to KU_2^{1+4\mathbb{Z}_2}$. The only differentials not immediately determined by this are the d_2 and d_4 differentials on classes in internal degree $t = 0$ and -3 respectively, which are treated at the end of Proposition B.3.

is a nonzero d_4 on z. In fact, when j is even the exact sequence

$$\pi_{4j-2}KO_2 \to \pi_{4j-3}\mathbf{1_K} \to \pi_{4j-3}KO_2$$

implies that $\pi_{4j-3}\mathbf{1_K} = 0$, so that $(u^{-2}\eta^2)^{2j}z$ supports a d_4 by the same argument. When j is odd the sequence only gives a bound $|\pi_{4j-3}\mathbf{1_K}| \leq 4$, but this is already populated by elements in filtration $s \leq 2$ (which survive by comparison to the HFPSS for KU_2). Thus $(u^{-2}\eta^2)^{2j}z$ supports a d_4 in these cases too.

After this, the spectral sequence collapses by sparsity. $\square$

To conclude, we must show that the two spectral sequences

$$H^*(\mathbb{G}, \pi_*KU_2) \implies \pi_*\mathbf{1_K} \tag{B.11}$$

$$H^*(C_2, \pi_*KU_2^{1+4\mathbb{Z}_2}) \implies \pi_*\mathbf{1_K} \tag{B.12}$$

are isomorphic, up to a shift in filtration.

Lemma B.5. *For each $s \geq 0$ and t there are isomorphisms*

$$H^{s+1}(\mathbb{G}, \pi_tKU_2) \simeq H^s(C_2, \pi_{t-1}KU_2^{1+4\mathbb{Z}_2}) \qquad (t \neq 1),$$

$$H^s(\mathbb{G}, \pi_0KU_2) \simeq H^s(C_2, \pi_0KU_2).$$

These are compatible with differentials, and together yield a (filtration-shifting) isomorphism of spectral sequences between (B.11) *and* (B.12)*.*

Remark B.6. In other words, when passing from the HFPSS for $KU_2^{1+4\mathbb{Z}_2}$ (B.12) to the descent spectral sequence (B.11), the filtration shift is precisely by one away from internal degree $t = 0$, and zero otherwise.

Proof. Note first that the E_1-pages are abstractly isomorphic:

$$H^{i+j}(\mathbb{G}, \pi_{2t} KU_2) \simeq H^i(C_2, H^j(H, \pi_{2t} KU_2)) \simeq H^i(C_2, \pi_{2t-j} KU_2^{1+4\mathbb{Z}_2}), \tag{B.13}$$

where $j = 1$ unless $t = 0$, in which case we also have $j = 0$. The first isomorphism is given by the Lyndon-Hochschild-Serre spectral sequence, which collapses with each degree of the abutment concentrated in a single filtration; the second isomorphism comes from the same fact about the homotopy fixed point spectral sequence for the action of $1 + 4\mathbb{Z}_2$ on KU_2. Note that in both cases the abutment is sometimes in positive filtration, and this will account for the shift.

Next recall that the descent spectral sequence comes from global sections of the Postnikov tower for $\mathcal{E} \in \widehat{\mathcal{S}h}(B\mathbb{G}_{\mathrm{proet}}, \mathcal{S}p)$; on the other hand, the C_2-fixed points spectral sequence is induced by global sections of the sheaf $j_*\mathcal{E} \in \widehat{\mathcal{S}h}(B(C_2)_{\mathrm{proet}}, \mathcal{S}p)$, where $j : \mathbb{G} \to C_2$ is the quotient by the open subgroup $1 + 4\mathbb{Z}_2$. Since each $j_*\tau_{\le t}\mathcal{E}(T) = \tau_{\le t}\mathcal{E}(\mathrm{res}^{C_2}_{\mathbb{G}} T)$ is t-truncated, we obtain a map of towers in $\widehat{\mathcal{S}h}(B(C_2)_{\mathrm{proet}}, \mathcal{S}p)$,

$$\begin{array}{ccccccccc}
j_*\mathcal{E} & \longrightarrow & \cdots & \longrightarrow & \tau_{\le t} j_*\mathcal{E} & \longrightarrow & \tau_{\le t-1} j_*\mathcal{E} & \longrightarrow & \cdots \\
\| & & & & \downarrow & & \downarrow & & \\
j_*\mathcal{E} & \longrightarrow & \cdots & \longrightarrow & j_*\tau_{\le t}\mathcal{E} & \longrightarrow & j_*\tau_{\le t-1}\mathcal{E} & \longrightarrow & \cdots
\end{array}$$

Taking global sections of the top row yields the HFPSS (B.12), while the bottom yields the descent spectral sequence (B.11) (bearing in mind that $\Gamma(C_2/C_2, j_*(-)) \simeq \Gamma(\mathbb{G}/\mathbb{G}, -)$).

To proceed, we compute the sections of these towers. Note that any cover of $C_2 \in B(C_2)_{\mathrm{proet}}$ must split, and so sheafification on $B(C_2)_{\mathrm{proet}}$ preserves any multiplicative presheaf when restricted to the generating sub-site Free_{C_2}, i.e. any $\mathcal{F}$ satisfying $\mathcal{F}(S \sqcup S') \simeq \mathcal{F}(S) \times \mathcal{F}(S')$. Thus

$$\tau_t j_*\mathcal{E} : C_2 \mapsto \tau_t\mathcal{E}(\mathbb{Z}_2^\times/1 + 4\mathbb{Z}_2) = \tau_t KU_2^{1+4\mathbb{Z}_2}$$

This sheaf is therefore zero unless t is odd or zero. On the other hand,

$$\pi_s\Gamma j_*\tau_t\mathcal{E} : C_2 \mapsto \pi_s\Gamma(C_2, j_*\tau_t\mathcal{E}) = \pi_s\Gamma(\mathbb{Z}_2^\times/1 + 4\mathbb{Z}_2, \tau_t\mathcal{E}) = H^{t-s}(1 + 4\mathbb{Z}_2, \pi_t KU_2).$$

This is zero unless t is even and $s = 1$, or $t = s = 0$. In particular, for $t \neq 0$ we have

$$\tau_{2t-1} j_*\mathcal{E} \simeq \tau_{2t-1} j_*\tau_{2t}\mathcal{E} \simeq j_*\tau_{2t}\mathcal{E},$$

while $j_*\tau_0\mathcal{E}$ has homotopy concentrated in degrees $\{-1, 0\}$. Note that this also implies that $\Gamma j_*\tau_{\le 2t}\mathcal{E}$ is $(2t-1)$-truncated for $t \neq 0$, since $\pi_{2t}\Gamma j_*\tau_{\le 2t}\mathcal{E} = \pi_{2t}\Gamma j_*\tau_{2t}\mathcal{E} = 0$.

The isomorphisms (B.13) can be interpreted as arising from the two trigraded spectral sequences

$$\begin{aligned}
H^i(C_2, H^j(1 + 4\mathbb{Z}_2, \pi_{2t} KU_2)) &\implies H^{i+j}(\mathbb{G}, \pi_{2t} KU_2), \\
H^i(C_2, H^j(1 + 4\mathbb{Z}_2, \pi_{2t} KU_2)) &\implies H^i(C_2, \pi_{2t-j} KU_2^{h(1+4\mathbb{Z}_2)})
\end{aligned} \tag{B.14}$$

coming from the bifiltration $\Gamma\mathcal{E} = \Gamma\tau_{\leq j} j_* \tau_{\leq 2t}\mathcal{E}$ (c.f. Proposition A.1). At a fixed t, the first is associated to the filtration $\Gamma\tau_{2t}\mathcal{E} = \Gamma j_*\tau_{2t}\mathcal{E} = \lim_j \Gamma\tau_{\leq j} j_*\tau_{2t}\mathcal{E}$, or equivalently to the resolution of $H^*(1+4\mathbb{Z}_2, \pi_{2t}KU_2)$ by acyclic C_2-modules, and is therefore the LHSSS. On the other hand, the second is C_2-cohomology applied pointwise to the HFPSS for the $1+4\mathbb{Z}_2$-action (at a fixed j); in particular, both collapse at E_1.

For $t \neq 0$, the towers therefore look as in the following diagram, in which we have identified in both rows those consecutive layers with zero fibre, i.e. we run the tower 'at double speed'.

$$\begin{tikzcd}
\cdots \arrow[r] & \tau_{\leq 2t-1} j_*\mathcal{E} \arrow[rr] \arrow[dd] & & \tau_{\leq 2t-3} j_*\mathcal{E} \arrow[r] \arrow[dl] \arrow[dd] & \cdots \\
& & \tau_{2t-1} j_*\mathcal{E} \arrow[ul] \arrow[dd] & & \\
\cdots \arrow[r] & j_*\tau_{\leq 2t}\mathcal{E} \arrow[rr] & & j_*\tau_{\leq 2t-2}\mathcal{E} \arrow[r] \arrow[dl] & \cdots \\
& & j_*\tau_{2t}\mathcal{E} \arrow[ul] & &
\end{tikzcd}$$

A large but routine diagram verifies that the map $\tau_{2t-1} j_*\mathcal{E} \to j_*\tau_{2t}\mathcal{E}$ agrees on homotopy with (B.13).

Near zero, we have instead the diagram

$$\begin{tikzcd}
\tau_{\leq 1} j_*\mathcal{E} \arrow[rr] \arrow[dd] & & \tau_{\leq 0} j_*\mathcal{E} \arrow[rr] \arrow[dl] \arrow[dd] & & \tau_{\leq -1} j_*\mathcal{E} \arrow[rr] \arrow[dl] \arrow[dd] & & \tau_{\leq -2} j_*\mathcal{E} \arrow[dl] \arrow[dd] \\
& \tau_1 j_*\mathcal{E} \arrow[ul] \arrow[dd] & & \tau_0 j_*\mathcal{E} \arrow[ul] \arrow[dd] & & \tau_{-1} j_*\mathcal{E} \arrow[ul] \arrow[dd] \arrow[dll, dashed, bend left, "\beta"] & \\
j_*\tau_{\leq 1}\mathcal{E} \arrow[rr, equal] & & j_*\tau_{\leq 0}\mathcal{E} \arrow[rr] \arrow[dl] & & j_*\tau_{\leq -1}\mathcal{E} \arrow[rr, equal] \arrow[dl] & & j_*\tau_{\leq -2}\mathcal{E} \arrow[dl] \\
& 0 \arrow[ul] & & j_*\tau_0\mathcal{E} \arrow[ul] & & 0 \arrow[ul] &
\end{tikzcd}$$

To see that the dashed arrow exists, note that

$$d_1 : \tau_{-1} j_*\mathcal{E} \to \Sigma\tau_0 j_*\mathcal{E}$$

is zero on homotopy: in the proof Proposition B.3 we computed the only nontrivial d_2 differentials, which have source in internal degree $t = 0$. As a map between Eilenberg-Mac Lane objects, it is in fact null, so we can lift as below:

$$\begin{tikzcd}
\tau_{-1} j_*\mathcal{E} \arrow[rr, "0", "d_1"'] \arrow[dr] \arrow[dd, dashed, "\exists\alpha"'] & & \Sigma\tau_0 j_*\mathcal{E} \\
& \tau_{\leq -1} j_*\mathcal{E} \arrow[ur] & \\
\tau_{\leq 0} j_*\mathcal{E} \arrow[ur] & &
\end{tikzcd}$$

$$\begin{tikzcd}
\tau_{-1}j_*\mathcal{E} \arrow[rrr] \arrow[dr, "\alpha"] \arrow[dd, dashed, "\exists\beta"'] & & & 0 \arrow[dd] \\
& \tau_{\leq 0}j_*\mathcal{E} \arrow[dr] & & \\
j_*\tau_0\mathcal{E} \arrow[rr] & & j_*\tau_{\leq 0}\mathcal{E} \arrow[r] & j_*\tau_{\leq -1}\mathcal{E}
\end{tikzcd}$$

We must show that evident map $\tau_0 j_*\mathcal{E} \to j_*\tau_0\mathcal{E}$ and the map β agree with the respective compositions of edge maps. To deduce this, it is enough to contemplate the diagrams below, in which the dashed arrow are equivalences and the dotted arrows admit right inverses.

$$\begin{tikzcd}
\tau_0 j_*\mathcal{E} \arrow[rrr] \arrow[dr] \arrow[dd, squiggly, bend right] & & & \tau_{\leq 0}j_*\mathcal{E} \arrow[dl] \arrow[dddd] \\
\tau_0 j_*\tau_{\geq 0}\mathcal{E} \arrow[u, dashed] \arrow[d] & \tau_0 j_*\tau_{\leq 0}\mathcal{E} \arrow[dr] & & \\
\tau_0 j_*\tau_0\mathcal{E} \arrow[ur] \arrow[dr] \arrow[dd, squiggly] & & \tau_{\leq 0}j_*\tau_{\leq 0}\mathcal{E} & \\
& \tau_{\leq 0}j_*\tau_p\mathcal{E} \arrow[ur] & & \\
j_*\tau_0\mathcal{E} \arrow[rrr] \arrow[ur, dashed] & & & j_*\tau_{\leq 0}\mathcal{E} \arrow[uul, dashed]
\end{tikzcd}$$

$$\begin{tikzcd}
\tau_{-1} j_*\mathcal{E} \arrow[rrr] \arrow[dr] \arrow[dd, squiggly, bend right] & & & \tau_{\leq 0}j_*\mathcal{E} \arrow[dl] \arrow[dddd] \\
\tau_{-1} j_*\tau_{\geq 0}\mathcal{E} \arrow[u, dashed] \arrow[d] & \tau_{-1} j_*\tau_{\leq 0}\mathcal{E} \arrow[dr] & & \\
\tau_{-1} j_*\tau_0\mathcal{E} \arrow[ur] \arrow[dr] \arrow[dd, squiggly] & & \tau_{\leq -1}j_*\tau_{\leq 0}\mathcal{E} & \\
& \tau_{\leq -1}j_*\tau_0\mathcal{E} \arrow[ur] & & \\
j_*\tau_0\mathcal{E} \arrow[rrr] \arrow[ur, dotted] & & & j_*\tau_{\leq 0}\mathcal{E} \arrow[uul, dotted]
\end{tikzcd}$$

The squiggly arrows are the edges maps from the trigraded spectral sequences, which are isomorphisms thanks to the collapse of the two trigraded spectral sequences. $\square$

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