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Wideband DOA Estimation with Frequency Decomposition via a Unified GS-WSpSF Framework

***Abstract*—A unified group sparsity based framework for wideband sparse spectrum fitting (GS-WSpSF) is proposed for wideband direction-of-arrival (DOA) estimation, which is capable of handling both uncorrelated and correlated sources. Then, by making four different assumptions on *a priori* knowledge about the sources, four variants under the proposed framework are formulated as solutions to the underdetermined DOA estimation problem without the need of employing sparse arrays. As verified by simulations, improved estimation performance can be achieved by the wideband methods compared with narrowband ones, and adopting more *a priori* information leads to better performance in terms of resolution capacity and estimation accuracy.**

***Index Terms*— Direction-of-arrival estimation, wideband, subband model, underdetermined, sparse spectrum fitting**

I. INTRODUCTION

WIDEBAND direction-of-arrival (DOA) estimation based on sensor arrays has been extensively studied over the decades. With an N-sensor uniform linear array (ULA), most of the conventional wideband DOA estimation methods can only resolve up to $N-1$ sources [1], [2]. As one of the important topics in wideband DOA estimation, the underdetermined wideband direction finding problem (where the number of impinging sources is larger than that of the physical sensors) has attracted significant interests [3]–[5].

In the narrowband case, it is necessary to employ sparse array structures for underdetermined DOA estimation, and accordingly nested arrays [6], co-prime arrays [7], [8] and their extensions [9]–[16] have been proposed. In order to exploit the increased degrees of freedom (DOFs) provided by sparse arrays, many effective methods including spatial smoothing (SS) based subspace methods [6], compressive sensing (CS) based methods [17], [18], and maximum likelihood (ML) methods [19], have been employed to resolve more sources than the number of physical sensors. To be specific, the original sparse spectrum fitting (SpSF) method recovers all the entries of the covariance matrix, while its simplified version for uncorrelated sources only considers the diagonal entries of the covariance matrix as unknown parameters to be recovered, which can be applied to sparse

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arrays for underdetermined narrowband DOA estimation as proved in [20], [21].

Similarly, sparse arrays are usually needed to ensure the feasibility of underdetermined wideband DOA estimation. Based on the difference co-array of a sparse array, focusing [22], [23] and group sparsity [24]–[26] can be adopted in the wideband case, leading to improved resolution capacity. In [27]–[29], the correlation between different frequencies is considered to generate the difference co-array in the spatio-spectral domain with further increased DOFs, and the corresponding Cramér-Rao bound (CRB) is derived in [30]. All these wideband DOA estimation methods assume that the sources are uncorrelated.

Recently, the wideband Cramér-Rao bounds under the subband model via frequency decomposition have been systematically studied [31]. The closed-form CRB expressions have been derived for four cases with different prior knowledge, where the sources are known *a priori* to 1) have flat spectra ($\mathcal{P}_{\mathrm{f}}$), 2) be uncorrelated to each other ($\mathcal{P}_{\mathrm{u}}$), 3) be uncorrelated to each other and have proportional spectra up to unknown factors ($\mathcal{P}_{\mathrm{up}}$), 4) be uncorrelated to each other and have flat spectra ($\mathcal{P}_{\mathrm{uf}}$). In practice, these cases have been employed by a number of DOA estimation methods and performance studies in literature. $\mathcal{P}_{\mathrm{f}}$ is available when all the lagged data correlation matrices contain approximately the same information, and was considered in underwater acoustics, radar, and also communications systems where signals with flat spectra are transmitted [32], [33]. It has also been adopted in practical design [34], [35] and theoretical analysis [36], [37]. When the propagation channel is unbounded, the sources are spatially uncorrelated [37], [38] and thus $\mathcal{P}_{\mathrm{u}}$ can be adopted. Most wideband and narrowband underdetermined DOA estimation methods employ $\mathcal{P}_{\mathrm{u}}$ to increase resolution capacity [3], [22], [25], [26], [39], [40]. $\mathcal{P}_{\mathrm{up}}$ exists in wireless communications and satellite communications with sources of the same modulation format and pulse shaping functions [41]–[43]. Since $\mathcal{P}_{\mathrm{uf}}$ comprises $\mathcal{P}_{\mathrm{u}}$ and $\mathcal{P}_{\mathrm{f}}$, it is suitable for a combination of the aforementioned cases [42], [44] with a number of studies related to DOA estimation and source localization reported [37], [45]–[47]. By examining the existing condition of the CRBs, the upper bounds on the resolution capacities in the above four cases are derived [31], indicating that under different *a priori* knowledge on the source spectra, it is possible to resolve more sources than the sensor number in the underdetermined wideband case even if a sparse array is not employed. However, to the best of our knowledge, only the *a priori* knowledge of uncorrelated sources has been considered in underdetermined estimation, while other *a priori* information related to the source spectra has not been fully exploited in the wideband case.

To fill in the gaps, a unified group sparsity based framework for wideband sparse spectrum fitting (GS-WSpSF) is proposed in this paper. After obtaining the subband model [48] via frequency decomposition, i.e., discrete Fourier transform (DFT) or a series of filter banks, the general GS-WSpSF method is derived with better performance compared with narrowband ones, capable of handling all kinds of sources including uncorrelated and correlated ones. Then, four variants under this GS-WSpSF framework with the four different kinds of *a priori* knowledge are presented. We show that underdetermined DOA estimation can be achieved without the assistance of sparse arrays, verifying the theoretical results proved in [31]. More specifically, for uncorrelated sources, the number of resolvable sources in the wideband case can be larger than $\frac{|\mathbb{D}|-1}{2}$ with $|\mathbb{D}|$ being the cardinality of the difference co-array generated by the physical array ($\frac{|\mathbb{D}|-1}{2}$ was considered as the maximum number of resolvable sources in existing literature). Adopting more *a priori* information leads to better performance in terms of both resolution capacity and estimation accuracy.

This paper is organized as follows. Section II provides basics about the subband model for wideband sources via frequency decomposition. The unified GS-WSpSF framework is proposed in Section III, while its four variants adopting different *a priori* knowledge are presented in Section IV. Simulation results are given in Section V, and conclusions are drawn in Section VI.

II. Signal Model

Consider a general linear array consisting of N physical sensors with a unit spacing of d. The set of sensor positions is given by $\mathbb{A} = \{\hbar_1, \hbar_2, \ldots, \hbar_N\}d$. Assume that there are K wideband signals $\{s_k[i]\}_{k=1}^{K}$ with the same bandwidth impinging from incident angles $\{\theta_k\}_{k=1}^{K}$, respectively, where θ_k is measured from the broadside of the array.

For the wideband DOA estimation problem, the subband signal model via frequency decomposition is adopted, where the received signal at each sensor is first decomposed into subbands by applying an L-point DFT to every non-overlapping L samples.

The discrete array output signal model of the p-th non-overlapping group after DFT is expressed as

$$\mathbf{X}[l,p] = \mathbf{A}(l,\boldsymbol{\theta})\mathbf{S}[l,p] + \overline{\mathbf{N}}[l,p], \tag{1}$$

where $\mathbf{X}[l,p]$ is the observed signal vector at frequency f_l corresponding to the l-th frequency bin (subband), $\mathbf{A}(l,\boldsymbol{\theta}) = [\mathbf{a}(l,\theta_1), ..., \mathbf{a}(l,\theta_K)] \in \mathbb{C}^{N\times K}$ is the steering matrix, with

$$\mathbf{a}(l,\theta_k) = [e^{-j2\pi\hbar_1 d\sin(\theta_1)/\lambda_l}, \ldots, e^{-j2\pi\hbar_N d\sin(\theta_N)/\lambda_l}] \tag{2}$$

representing the steering vector at frequency f_l and angle θ_k. Here $\lambda_l = \frac{c}{f_l}$ and c is the signal propagation speed. $\mathbf{S}[l,p] = [S_1[l,p], ..., S_K[l,p]]^T$ is a column vector holding K source signals at the l-th subband. $\overline{\mathbf{N}}[l,p] = [\overline{N}_1[l,p], ..., \overline{N}_N[l,p]]^T$ is the corresponding column noise vector in the frequency domain, whose elements are assumed to be zero-mean Gaussian white and uncorrelated with the sources.

The covariance matrix of the observed signal at the l-th subband is

$$\begin{aligned}\mathbf{R_{xx}}[l] &= \mathrm{E}\left\{\mathbf{X}[l,p]\cdot\mathbf{X}^H[l,p]\right\}\\ &= \mathbf{A}(l,\boldsymbol{\theta})\mathbf{R_s}[l]\mathbf{A}^H(l,\theta)+\sigma_n^2[l]\mathbf{I}_N,\end{aligned}\tag{3}$$

where $\mathrm{E}\{\cdot\}$ denotes the expectation operator, and $\{\cdot\}^H$ is the Hermitian transpose. $\mathbf{R_s}[l]=\mathbf{E}\left\{\mathbf{S}[l,p]\cdot\mathbf{S}^H[l,p]\right\}$, $\sigma_n^2[l]$ is the noise power at the l-th subband, and $\mathbf{I}_N$ is the $N\times N$ identity matrix.

III. A Unified Group Sparsity Framework for Wideband Sparse Spectrum Fitting

The core idea for narrowband sparse spectrum fitting (SpSF) method is to fit the sparse source covariance matrix (i.e., $\mathbf{R_s}[l]$ for each subband) to the observed spatial covariance matrix ($\mathbf{R_{xx}}[l]$) based on ℓ_1-norm penalization [49].

For the l-th subband, we construct an overcomplete array steering matrix from $K_g \gg K$ possible incident angles $\boldsymbol{\theta}_g=\{\theta_{g,1},\ldots,\theta_{g,K_g}\}$ by $\mathbf{A}(l,\boldsymbol{\theta}_g)=[\mathbf{a}(l,\theta_{g,1}),\ldots,\mathbf{a}(l,\theta_{g,K_g})]$.

According to (3) and sparse signal recovery theory, the covariance matrix under the CS framework is equivalent to

$$\mathbf{R_{xx}}[l]\triangleq\mathbf{A}(l,\boldsymbol{\theta}_g)\mathbf{R_{s_g}}[l]\mathbf{A}^H(l,\boldsymbol{\theta}_g)+\sigma_n^2[l]\mathbf{I}_N,\tag{4}$$

where $\mathbf{R_{s_g}}[l]$ is defined as the $K_g\times K_g$ source covariance matrix with its each diagonal element representing the power of a potential source at the corresponding incident angle.

Then, the SpSF method [49] for narrowband DOA estimation can be applied to the l-th frequency bin, formulated as

$$\min_{\mathbf{R_{s_g}}[l]}\left\|\mathbf{R_{xx}}[l]-\mathbf{A}(l,\boldsymbol{\theta}_g)\mathbf{R_{s_g}}[l]\mathbf{A}^H(l,\boldsymbol{\theta}_g)\right\|_F^2+\beta\left\|\widetilde{\mathbf{r}}_{\mathbf{s_g}}[l]\right\|_1,$$
$$\text{subject to}\quad \mathbf{R_{s_g}}[l]=\mathbf{R}^H_{\mathbf{s_g}}[l],\ \mathrm{diag}\left(\mathbf{R_{s_g}}[l]\right)\succeq\mathbf{0},\tag{5}$$

where $\|\cdot\|_F$ is the Frobenius norm, $\|\cdot\|_1$ is the ℓ_1-norm, and β is a regularization parameter to balance the fitting error and the ℓ_1-norm, which can be chosen to give the best estimation result through an automatic selector [49]. $\widetilde{\mathbf{r}}_{\mathbf{s_g}}[l]=\mathrm{vec}(\mathbf{R_{s_g}}[l])$ with $\mathrm{vec}(\cdot)$ being the vectorization operator, and $\mathrm{diag}(\cdot)$ returns the set of diagonal elements of the input matrix. The vector $\mathrm{diag}\left(\mathbf{R_{s_g}}[l]\right)$ reflects the signal powers over K_g search grids in $\boldsymbol{\theta}_g$, and DOAs can be estimated from $\mathrm{diag}\left(\mathbf{R_{s_g}}[l]\right)$. $\succeq$ denotes element-wise $\geq$.

Traditional incoherent signal subspace method (ISM) [1] and coherent signal subspace method (CSM) [2] can be applied for wideband DOA estimation; however, by fusing the DOA results or focusing signal models to a reference frequency, the prior information across frequency bins cannot be employed to resolve more sources and improve the estimation accuracy. Inspired by the group sparsity concept [24], [25], a more effective wideband DOA estimation method, referred to as group sparsity based wideband sparse spectrum fitting (GS-WSpSF), is proposed by exploiting all subband information simultaneously.

We first consider the general scenario without any *a priori* knowledge. By vectorizing $\mathbf{R_{xx}}[l]$, the l-th subband signal model is changed to

$$\mathbf{z}[l]=\mathrm{vec}\left\{\mathbf{R_{xx}}[l]\right\}=\widetilde{\mathbf{A}}[l]\widetilde{\mathbf{s}}[l]+\sigma_n^2[l]\widetilde{\mathbf{i}}_{N^2},\tag{6}$$

where $\widetilde{\mathbf{s}}[l]=\mathrm{vec}(\mathbf{R_s}[l])$, and $\widetilde{\mathbf{i}}_{N^2}=\mathrm{vec}(\mathbf{I}_N)$ is an $N^2\times 1$ column vector, and the equivalent steering matrix

$$\widetilde{\mathbf{A}}[l]=[\widetilde{\mathbf{a}}_l(\theta_1,\theta_1),\ldots,\widetilde{\mathbf{a}}_l(\theta_K,\theta_1),\widetilde{\mathbf{a}}_l(\theta_1,\theta_2),\ldots,\widetilde{\mathbf{a}}_l(\theta_K,\theta_K)]$$

with $\widetilde{\mathbf{a}}_l(\theta_i,\theta_k)=\mathrm{vec}(\mathbf{a}(l,\theta_i)\mathbf{a}^H(l,\theta_k))$.

Consider Q ($Q\leq L$) subbands of interest indexed by $\{l_q\}_{q=1}^Q$. We construct a block diagonal matrix $\widetilde{\mathbf{B}}=\mathrm{blkdiag}\{\widetilde{\mathbf{A}}[l_1],\widetilde{\mathbf{A}}[l_2],\ldots,\widetilde{\mathbf{A}}[l_Q]\}$, and a $K^2\times Q$ matrix $\mathbf{R}=\{[\widetilde{\mathbf{s}}[l_1],\widetilde{\mathbf{s}}[l_2],\ldots,\widetilde{\mathbf{s}}[l_Q]]\}$. Then, a general wideband virtual array fitting model is obtained by

$$\widetilde{\mathbf{z}}=\widetilde{\mathbf{B}}\widetilde{\mathbf{r}}+\mathbf{E},\tag{7}$$

where $\widetilde{\mathbf{z}}=[\mathbf{z}^T[l_1],\ldots,\mathbf{z}^T[l_Q]]^T$, $\widetilde{\mathbf{r}}=\mathrm{vec}(\mathbf{R})$, and $\mathbf{E}=[\sigma_n^2[l_1]\widetilde{\mathbf{i}}_{N^2},\ldots,\sigma_n^2[l_Q]\widetilde{\mathbf{i}}_{N^2}]^T$.

Remark: It is noted that $\{\widetilde{\mathbf{A}}[l_q]\}_{q=1}^Q$ are combined to form a block diagonal matrix $\widetilde{\mathbf{B}}$ with increased dimension. Therefore, the number of distinct rows in $\widetilde{\mathbf{B}}$ is usually larger than that of each subband due to the different λ_l, implying that it is possible to resolve more sources under this framework compared with narrowband DOA estimation methods according to the widely adopted number-of-equations condition [50], [51], as will be further discussed later with different *a priori* information considered.

Assume that the same search grid with K_g potential incident angles are employed for all subbands. Under the CS framework, we construct

$$\begin{aligned}\widetilde{\mathbf{B}}_{\mathbf{g}}&=\mathrm{blkdiag}\{\widetilde{\mathbf{A}}_{\mathbf{g}}[l_1],\widetilde{\mathbf{A}}_{\mathbf{g}}[l_2],\ldots,\widetilde{\mathbf{A}}_{\mathbf{g}}[l_Q]\},\\ \mathbf{R_g}&=\left\{\widetilde{\mathbf{r}}_{\mathbf{s_g}}[l_1],\widetilde{\mathbf{r}}_{\mathbf{s_g}}[l_2],\ldots,\widetilde{\mathbf{r}}_{\mathbf{s_g}}[l_Q]\right\},\end{aligned}\tag{8}$$

where the overcomplete representation of the equivalent steering matrix at the l_q-th subband is $\widetilde{\mathbf{A}}_{\mathbf{g}}[l_q]=[\widetilde{\mathbf{a}}_{l_q}(\theta_1,\theta_1),\ldots,\widetilde{\mathbf{a}}_{l_q}(\theta_{K_g},\theta_1),\widetilde{\mathbf{a}}_{l_q}(\theta_1,\theta_2),\ldots,\widetilde{\mathbf{a}}_{l_q}(\theta_{K_g},\theta_{K_g})]$.

By performing ℓ_2-norm to each row of $\mathbf{R_g}$, a new column vector $\hat{\mathbf{r}}=[\|\mathbf{r}_1\|_2,\ldots,\|\mathbf{r}_{K_g^2}\|_2]^T$ is formed with $\mathbf{r}_k$ $\left(1\leq k\leq K_g^2\right)$ representing the k-th row vector in $\mathbf{R_g}$. Finally, the GS-WSpSF method is formulated as follows

$$\begin{aligned}&\min_{\mathbf{R_g}}\quad \left\|\widetilde{\mathbf{z}}-\widetilde{\mathbf{B}}_{\mathbf{g}}\widetilde{\mathbf{r}}_{\mathbf{g}}\right\|_2^2+\beta\|\hat{\mathbf{r}}\|_1,\\ &\text{subject to}\quad \mathbf{R_{s_g}}[l_q]=\mathbf{R}^H_{\mathbf{s_g}}[l_q],\ \mathrm{diag}\left(\mathbf{R_{s_g}}[l_q]\right)\succeq\mathbf{0},\end{aligned}\tag{9}$$

where $\widetilde{\mathbf{r}}_{\mathbf{g}}=\mathrm{vec}(\mathbf{R_g})$, and $q=1,\cdots,Q$.

The optimization problem in (9) can be solved in a second-order cone programming (SOCP) framework using CVX software package [52]. After fitting all subband covariance matrices of interest, the recovered $\hat{\mathbf{r}}$ represents the DOAs over K_g^2 grids.

The GS-WSpSF can be considered as a unified framework for wideband DOA estimation, and can be applied to all kinds of source signals (including correlated ones).

IV. Variants of GS-WSpSF Framework with Different *a Priori* Knowledge

As proved in *Propositions 1* and *2* in [31], exploiting different *a priori* knowledge is beneficial in resolving more sources than sensor number without the assistance of a sparse structure, and upper bounds on resolution capacities have been analyzed. However, underdetermined wideband DOA estimation methods based on different *a priori* knowledge have not been well considered. Motivated by these interesting results, variants of GS-WSpSF are studied, overcoming the limitation in resolution capacity imposed by narrowband DOA estimation methods and traditional wideband ones.

In this section, the following four types of a *priori* knowledge will be considered:

- $\mathcal{P}_{\rm f}$: The spectra are *flat* across subbands of interest.
- $\mathcal{P}_{\rm u}$: The sources are spatially *uncorrelated*.
- $\mathcal{P}_{\rm up}$: The sources are spatially *uncorrelated*, and their spectra are *proportional* up to a series of *unknown* factors across subbands of interest.
- $\mathcal{P}_{\rm uf}$: The sources are spatially *uncorrelated*, and their spectra are *flat* or *proportional* up to a series of *known* factors across subbands of interest.

When different *a priori* knowledge is employed, the unknown parameters to be estimated can be reduced or combined, leading to different optimization forms under the GS-WSpSF framework.

A. GS-WSpSF under $\mathcal{P}_{\rm f}$

If $\mathcal{P}_{\rm f}$ is employed, flat spectra indicate that the equivalent source signals $\{\widetilde{\mathbf{s}}_{\mathbf{g}_{\rm f}}[l_q]=\text{vec}(\mathbf{R}_{\mathbf{s}_{\mathbf{g}}}[l_q])\}_{q=1}^{Q}$ are independent of subband index l_q, i.e., $\hat{\mathbf{r}}_{\mathbf{g}_{\rm f}}=\widetilde{\mathbf{s}}_{\mathbf{g}_{\rm f}}[l_q]$ and $\mathbf{R}_{\mathbf{s}_{\mathbf{g}\rm f}}=\mathbf{R}_{\mathbf{s}_{\mathbf{g}}}[l_q]$, $\forall q=1,2,\ldots,Q$.

Stacking steering matrices $\{\widetilde{\mathbf{A}}_{\mathbf{g}}[l_q]\}_{q=1}^{Q}$ following the column direction instead of the block diagonal direction leads to $\widetilde{\mathbf{B}}_{\mathbf{g}_{\rm f}}=[\widetilde{\mathbf{A}}_{\mathbf{g}}^{T}[l_1],\widetilde{\mathbf{A}}_{\mathbf{g}}^{T}[l_2],\ldots,\widetilde{\mathbf{A}}_{\mathbf{g}}^{T}[l_Q]]^{T}$. Then, GS-WSpSF under $\mathcal{P}_{\rm f}$, referred to as GS-WSpSF ($\mathcal{P}_{\rm f}$), is formulated as

$$\begin{aligned}&\min_{\hat{\mathbf{r}}_{\mathbf{g}_{\rm f}}}\quad \left\|\widetilde{\mathbf{z}}-\widetilde{\mathbf{B}}_{\mathbf{g}_{\rm f}}\hat{\mathbf{r}}_{\mathbf{g}_{\rm f}}\right\|_2^2+\beta\|\hat{\mathbf{r}}_{\mathbf{g}_{\rm f}}\|_1,\\&\text{subject to}\quad \mathbf{R}_{\mathbf{s}_{\mathbf{g}\rm f}}=\mathbf{R}_{\mathbf{s}_{\mathbf{g}\rm f}}^{H},\ \text{diag}\left(\mathbf{R}_{\mathbf{s}_{\mathbf{g}\rm f}}\right)\succeq\mathbf{0}.\end{aligned}\tag{10}$$

B. GS-WSpSF under $\mathcal{P}_{\rm u}$

Under $\mathcal{P}_{\rm u}$ with spatially uncorrelated sources, $\{\mathbf{R}_{\mathbf{s}}[l_q]\}_{q=1}^{Q}$ becomes a real-valued diagonal matrix, and the covariance matrix of the observed signal at the l_q-th subband in (3) is reformulated as

$$\begin{aligned}\mathbf{R}_{\mathbf{xx}}[l_q]&=\text{E}\left\{\mathbf{X}[l_q,p]\cdot\mathbf{X}^{H}[l_q,p]\right\}\\&=\sum\nolimits_{k=1}^{K}\sigma_k^2[l_q]\mathbf{a}\left(l_q,\theta_k\right)\mathbf{a}^{H}\left(l_q,\theta_k\right)+\sigma_n^2[l]\mathbf{I}_N,\end{aligned}\tag{11}$$

where $\sigma_k^2[l_q]$ is the power of the k-th source signal at the l_q-th subband. Then, a virtual array corresponding to the difference co-array [25] is generated by vectorizing $\mathbf{R}_{\mathbf{xx}}[l_q]$, leading to

$$\mathbf{z}[l_q]=\text{vec}\{\mathbf{R}_{\mathbf{xx}}[l_q]\}=\widehat{\mathbf{A}}[l_q]\hat{\mathbf{s}}[l_q]+\sigma_n^2[l_q]\widetilde{\mathbf{i}}_{N^2},\tag{12}$$

where the equivalent steering matrix of the virtual array $\widehat{\mathbf{A}}[l_q]=[\widetilde{\mathbf{a}}_{l_q}(\theta_1,\theta_1),\ldots,\widetilde{\mathbf{a}}_{l_q}(\theta_K,\theta_K)]$ with the k-th column vector $\widetilde{\mathbf{a}}_{l_q}(\theta_k,\theta_k)=\text{vec}(\mathbf{a}(l_q,\theta_k)\mathbf{a}^{H}(l_q,\theta_k))$, and the equivalent source signal vector $\hat{\mathbf{s}}[l_q]=[\sigma_1^2[l_q],\ldots,\sigma_K^2[l_q]]^{T}$.

With the same search grid of K_g potential incident angles for all Q subbands, we construct

$$\begin{aligned}\widetilde{\mathbf{B}}_{\mathbf{g}_{\rm u}}&=\text{blkdiag}\{\widehat{\mathbf{A}}_{\mathbf{g}}\left[l_1\right],\widehat{\mathbf{A}}_{\mathbf{g}}\left[l_2\right],\ldots,\widehat{\mathbf{A}}_{\mathbf{g}}\left[l_Q\right]\},\\\mathbf{R}_{\mathbf{g}_{\rm u}}&=[\hat{\mathbf{s}}_{\mathbf{g}}\left[l_1\right],\hat{\mathbf{s}}_{\mathbf{g}}\left[l_2\right],\ldots,\hat{\mathbf{s}}_{\mathbf{g}}\left[l_Q\right]],\end{aligned}\tag{13}$$

where $\widehat{\mathbf{A}}_{\mathbf{g}}[l_q]=[\widetilde{\mathbf{a}}_{l_q}(\theta_{g,1},\theta_{g,1}),\ldots,\widetilde{\mathbf{a}}_{l_q}(\theta_{g,K_g},\theta_{g,K_g})]$ is the overcomplete representation of the equivalent steering matrix, and the $K_g\times 1$ column vector $\hat{\mathbf{s}}_{\mathbf{g}}[l_q]$ contains all potential source signal powers at corresponding incident angles.

Denote $\widetilde{\mathbf{r}}_{\mathbf{g}_{\rm u}}=\left[\|\mathbf{r}_{\mathbf{g}_{\rm u},1}\|_2,\ldots,\|\mathbf{r}_{\mathbf{g}_{\rm u},K_g}\|_2\right]^{T}$ and $\hat{\mathbf{r}}_{\mathbf{g}_{\rm u}}=\text{vec}(\mathbf{R}_{\mathbf{g}_{\rm u}})$ with $\mathbf{r}_{\mathbf{g}_{\rm u},k}$ $(1\le k\le K_g)$ representing the k-th row of $\mathbf{R}_{\mathbf{g}_{\rm u}}$. Then, GS-WSpSF under $\mathcal{P}_{\rm u}$, referred to as GS-WSpSF ($\mathcal{P}_{\rm u}$), is formulated as

$$\begin{aligned}&\min_{\widetilde{\mathbf{r}}_{\mathbf{g}_{\rm u}}}\quad \left\|\widetilde{\mathbf{z}}-\widetilde{\mathbf{B}}_{\mathbf{g}_{\rm u}}\hat{\mathbf{r}}_{\mathbf{g}_{\rm u}}\right\|_2^2+\beta\|\widetilde{\mathbf{r}}_{\mathbf{g}_{\rm u}}\|_1,\\&\text{subject to}\quad \hat{\mathbf{r}}_{\mathbf{g}_{\rm u}}\succeq\mathbf{0}.\end{aligned}\tag{14}$$

Note that GS-WSpSF under $\mathcal{P}_{\rm u}$ is equivalent to the wideband method proposed in [25], where uncorrelated source assumption is adopted for underdetermined DOA estimation based on sparse arrays.

C. GS-WSpSF under $\mathcal{P}_{\rm up}$

Based on (12) and (13) where uncorrelated sources are considered, we set $\hat{\mathbf{s}}_{\mathbf{g}}\left[l_q\right]=\xi_q\hat{\mathbf{s}}_{\mathbf{g}}\left[l_1\right]=\xi_q\hat{\mathbf{r}}_{\mathbf{g}_{\rm up}}$ if $\mathcal{P}_{\rm up}$ is employed with $\{\xi_q\}_{q=1}^{Q}$ being a series of unknown positive real-valued proportional factors to be estimated.

Then, we construct

$$\begin{aligned}\widetilde{\mathbf{B}}_{\mathbf{g}_{\rm up}}&=\widetilde{\mathbf{B}}_{\mathbf{g}_{\rm u}}=\text{blkdiag}\{\widehat{\mathbf{A}}_{\mathbf{g}}\left[l_1\right],\widehat{\mathbf{A}}_{\mathbf{g}}\left[l_2\right],\ldots,\widehat{\mathbf{A}}_{\mathbf{g}}\left[l_Q\right]\},\\\mathbf{R}_{\mathbf{g}_{\rm up}}&=[\xi_1\hat{\mathbf{r}}_{\mathbf{g}_{\rm up}},\xi_2\hat{\mathbf{r}}_{\mathbf{g}_{\rm up}},\ldots,\xi_Q\hat{\mathbf{r}}_{\mathbf{g}_{\rm up}}].\end{aligned}\tag{15}$$

The optimization problem for GS-WSpSF under $\mathcal{P}_{\rm up}$, referred to as GS-WSpSF ($\mathcal{P}_{\rm up}$), is expressed as

$$\begin{aligned}&\min_{\hat{\mathbf{r}}_{\mathbf{g}_{\rm up}},\{\xi_q\}_{q=1}^{Q}}\quad \left\|\widetilde{\mathbf{z}}-\widetilde{\mathbf{B}}_{\mathbf{g}_{\rm up}}\widetilde{\mathbf{r}}_{\mathbf{g}_{\rm up}}\right\|_2^2+\beta\left\|\hat{\mathbf{r}}_{\mathbf{g}_{\rm up}}\right\|_1,\\&\text{subject to}\quad \hat{\mathbf{r}}_{\mathbf{g}_{\rm up}}\succeq\mathbf{0},\ \{\xi_q\}_{q=1}^{Q}\ge 0,\end{aligned}\tag{16}$$

where $\widetilde{\mathbf{r}}_{\mathbf{g}_{\rm up}}=\text{vec}(\mathbf{R}_{\mathbf{g}_{\rm up}})$.

D. GS-WSpSF under $\mathcal{P}_{\rm uf}$

Under $\mathcal{P}_{\rm uf}$, the sources are mutually uncorrelated and their spectra are flat or proportional up to a series of known factors across subbands of interest. $\mathcal{P}_{\rm uf}$ is a combined *a priori* knowledge of $\mathcal{P}_{\rm u}$ and $\mathcal{P}_{\rm f}$, and it is the basic assumption in applications such as communications where the spectrum of the transmitted signal is known in advance.

Similarly, (12) still holds true due to the uncorrelated property. According to the spectral property, all column vectors in $\mathbf{R}_{\mathbf{g}_\text{u}}$ in (13) are equal or proportional with a series of known factors $\{\eta_q\}_{q=1}^{Q}$ to each other, i.e., $\hat{\mathbf{s}}_\mathbf{g}[l_q] = \eta_q \hat{\mathbf{s}}_\mathbf{g}[l_1] = \eta_q \hat{\mathbf{r}}_{\mathbf{g}_\text{uf}}, \forall q = 1, 2, \ldots, Q$. For a special case of $\eta_q = 1, \ \forall q = 1, 2, \ldots, Q$, the spectra of the sources become flat.

By stacking $\{\eta_q \hat{\mathbf{A}}_\mathbf{g}[l_q]\}_{q=1}^{Q}$ following the column direction, we construct

$$\widetilde{\mathbf{B}}_{\mathbf{g}_\text{uf}} = \widetilde{\mathbf{B}}_{\mathbf{g}_\text{f}} = [\eta_1 \hat{\mathbf{A}}_\mathbf{g}^T [l_1], \eta_2 \hat{\mathbf{A}}_\mathbf{g}^T [l_2], \ldots, \eta_Q \hat{\mathbf{A}}_\mathbf{g}^T [l_Q]]^T.$$

Then, the proposed GS-WSpSF under $\mathcal{P}_\text{uf}$, referred to as GS-WSpSF ($\mathcal{P}_\text{uf}$), is formulated as

$$\begin{aligned} \min_{\hat{\mathbf{r}}_{\mathbf{g}_\text{uf}}} \quad & \left\|\widetilde{\mathbf{z}} - \widetilde{\mathbf{B}}_{\mathbf{g}_\text{uf}} \hat{\mathbf{r}}_{\mathbf{g}_\text{uf}}\right\|_2^2 + \beta \|\hat{\mathbf{r}}_{\mathbf{g}_\text{uf}}\|_1, \\ \text{subject to} \quad & \hat{\mathbf{r}}_{\mathbf{g}_\text{uf}} \succeq \mathbf{0}. \end{aligned} \tag{17}$$

Remark: Without *a priori* knowledge, the number of parameters to be estimated in $\mathbf{R}_\mathbf{g}$ in (9) is $K_g^2 Q$. Under different *a priori* knowledge $\mathcal{P}_\text{f}$, $\mathcal{P}_\text{u}$, $\mathcal{P}_\text{up}$, and $\mathcal{P}_\text{uf}$, the number of unknown parameters to be optimized in $\hat{\mathbf{r}}_{\mathbf{g}_\text{f}}$, $\hat{\mathbf{r}}_{\mathbf{g}_\text{u}}$, $\hat{\mathbf{r}}_{\mathbf{g}_\text{up}}$, and $\hat{\mathbf{r}}_{\mathbf{g}_\text{uf}}$ is reduced to K_g^2, $K_g Q$, K_g, and K_g, respectively. The computational complexity is reduced if *a priori* knowledge is available.

V. Simulation Results

Throughout this section, a ULA is employed and the capability of resolving more sources than the sensor number in the underdetermined case is verified. The wideband sources with flat spectra are considered; however, this information is not always known *a priori*.

Consider a ULA with $N = 7$ sensors, and the unit spacing between adjacent sensors is $d = \frac{c}{f_\text{max}}$, where f_max is the maximum frequency of interest. The number of samples in the time domain is 32768, and DFT of $L = 64$ points is applied. As a result, the number of non-overlapping groups (equal to the number of samples in the frequency domain) is $P = 512$. The center frequency of the l-th subband is $f_l = (l-1) f_s / L$. K_g search grids cover the angle range from $-90°$ to $90°$ are generated with a step size of $0.1°$, and the subbands of interest indexed from 54 to 64 are exploited by all proposed wideband methods. The regularization parameter β is chosen to give the best estimation results through trial-and-error in every experiment.

A. Comparisons of Resolution Capacity

Based on the existing analysis of CRBs, the lower bounds of resolution capacity (i.e., the maximum number of resolvable wideband sources) under different *a priori* knowledge have been derived in *Propositions 2* in [31]. As summarized in Table I, we have $K_\text{f} \geq N$, $K_\text{u} > \frac{|\mathbb{D}|-1}{2}$, $K_\text{up} \geq |\mathbb{D}| > N$, and $K_\text{uf} \geq |\mathbb{D}| > N$, where K_f, K_u, K_up, and K_uf denote the lower bounds of resolution capacity under $\mathcal{P}_\text{f}$, $\mathcal{P}_\text{u}$, $\mathcal{P}_\text{up}$, and $\mathcal{P}_\text{uf}$, respectively. $|\cdot|$ returns the cardinality of the input set. N is the number of physical sensors, and $\mathbb{D}$ is the difference co-array of the physical array. In this example, the number of senors of the ULA is $N = 7$, and the number of unique co-arrays provided by the difference co-array $\mathbb{D}$ is $2N - 1 = 13$.

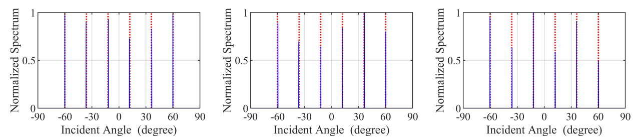
TABLE I

Resolution capacities under different *a priori* knowledge

Prior knowledge	$\mathcal{P}_\text{f}$	$\mathcal{P}_\text{u}$	$\mathcal{P}_\text{up}$	$\mathcal{P}_\text{uf}$
Capacity	$\geq N$ †	$\geq \frac{\lvert\mathbb{D}\rvert-1}{2}$ ‡	$\geq \lvert\mathbb{D}\rvert$	$\geq \lvert\mathbb{D}\rvert$
Examples with a 7-sensor ULA				
Capacity	≥ 7	≥ 7	≥ 13	≥ 13

† N is the number of physical sensors.

‡ $\mathbb{D}$ is the difference co-array of the physical array.



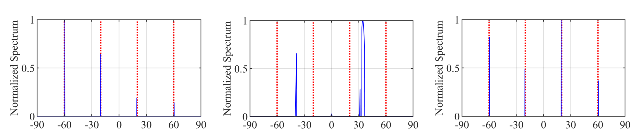
(a) SpSF-C (b) SpSF-U (c) GS-WSpSF

Fig. 1. Overdetermined DOA estimation results for 6 uncorrelated sources with a 7-sensors ULA, obtained by (a) SpSF-C, (b) SpSF-U, and (c) GS-WSpSF for uncorrelated sources.

For the first set of simulations, there are $K = 6$ ($K < N$) wideband signals impinging on this 7-sensor ULA with incident angles uniformly distributed between $-60°$ and $60°$, and the signal-to-noise ratio (SNR) is 10 dB. For a single subband indexed by 31, the SpSF based methods [49], referred to as SpSF-C for the conventional one and SpSF-U for the one dealing with uncorrelated sources, are employed for narrow/subband DOA estimation. The DOA estimation results for uncorrelated sources are shown in Fig. 1, where blue solid lines represent the estimated results, while red dotted ones are the true DOAs. Clearly, all the aforementioned methods are capable of resolving the 6 uncorrelated sources.

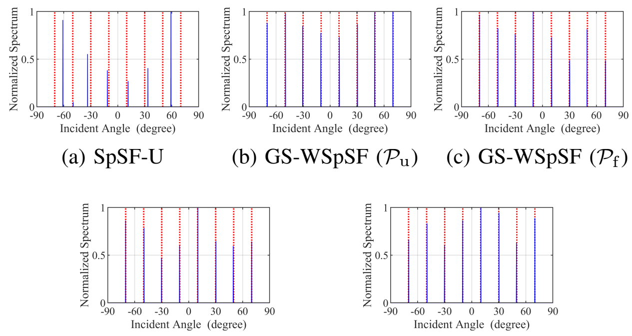
For coherent sources, we set $K = 4$, and the corresponding results are given in Fig. 2, where we can see that both SpSF-C and GS-WSpSF can resolve coherent sources, while SpSF-U fails.

Then, we focus on the underdetermined case with $K = 8$ ($N < K < |\mathbb{D}|$). The input SNR is fixed at 10 dB, and the sources are uniformly distributed between $-70°$ and $70°$. The DOA estimation results of the narrowband SpSF-U method employing the 31-th subband is shown in Fig. 3(a). Clearly, the SpSF-U



(a) SpSF-C (b) SpSF-U (c) GS-WSpSF

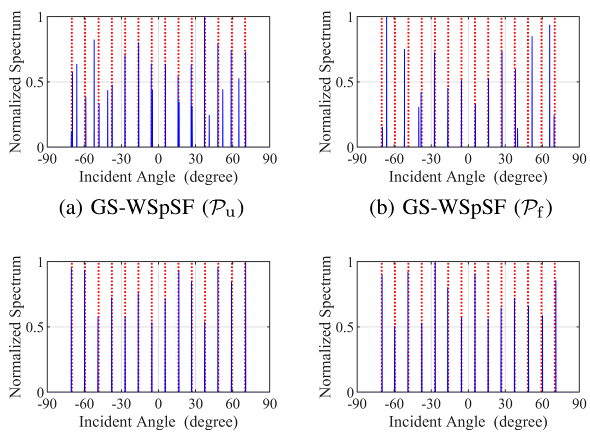
Fig. 2. Overdetermined DOA estimation results for 4 coherent sources with a 7-sensors ULA, obtained by (a) SpSF-C, (b) SpSF-U, and (c) GS-WSpSF.



(a) SpSF-U (b) GS-WSpSF ($\mathcal{P}_\mathrm{u}$) (c) GS-WSpSF ($\mathcal{P}_\mathrm{f}$)

(d) GS-WSpSF ($\mathcal{P}_\mathrm{up}$) (e) GS-WSpSF ($\mathcal{P}_\mathrm{uf}$)

Fig. 3. Underdetermined DOA estimation results of 8 uncorrelated sources based on a 7-sensor ULA obtained by different methods.



(a) GS-WSpSF ($\mathcal{P}_\mathrm{u}$) (b) GS-WSpSF ($\mathcal{P}_\mathrm{f}$)

(c) GS-WSpSF ($\mathcal{P}_\mathrm{up}$) (d) GS-WSpSF ($\mathcal{P}_\mathrm{uf}$)

Fig. 4. Underdetermined DOA estimation results of 14 uncorrelated sources based on a 7-sensor ULA obtained by different methods.

method fails in distinguishing all 8 uncorrelated sources, indicating that for a ULA with the assistance of $\mathcal{P}_\mathrm{u}$, the maximum resolvable sources by the narrowband method cannot exceed the number of physical sensors. On the other hand, the proposed GS-WSpSF ($\mathcal{P}_\mathrm{u}$), GS-WSpSF ($\mathcal{P}_\mathrm{f}$), GS-WSpSF ($\mathcal{P}_\mathrm{up}$), and GS-WSpSF ($\mathcal{P}_\mathrm{uf}$) exploiting different *a priori* knowledge have resolved the 8 sources successfully based on a 7-sensor ULA, verifying their superior resolution performance in the underdetermined case.

Then, we consider the case of $K = 14$ ($K > |\mathbb{D}| > N$). Fig. 4 shows the DOA estimation results of 14 uncorrelated sources based on a 7-sensor ULA. It is obvious that the proposed GS-WSpSF ($\mathcal{P}_\mathrm{u}$) and GS-WSpSF ($\mathcal{P}_\mathrm{f}$) cannot resolve so many sources, while both GS-WSpSF ($\mathcal{P}_\mathrm{up}$) and GS-WSpSF ($\mathcal{P}_\mathrm{uf}$) have succeeded. Therefore, in the wideband case, it is feasible to resolve more wideband sources than the cardinality of difference co-array by employing *a priori* knowledge on sources. The potential maximum number of DOFs in the wideband case can be far more than that in the narrowband case.

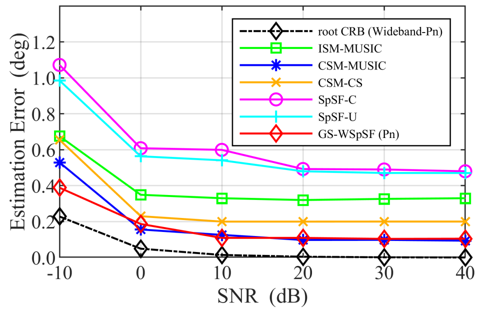
B. Comparisons of Estimation Performance

Finally, we compare the estimation accuracy of the proposed methods in both the overdetermined ($K = 4$) and underdetermined ($K = 8$) cases. The root mean square error (RMSE) is adopted as a metric for comparison, calculated by 100 Monte Carlo trials with respect to each varied input SNR. The wideband root CRB and the proposed method for the general case without *a priori* knowledge are represented by root CRB (Wideband-$\mathcal{P}_\mathrm{n}$) and GS-WSpSF ($\mathcal{P}_\mathrm{n}$), respectively. The RMSE results versus input SNR for the overdetermined case with $K = 4$ are given in Fig. 5(a), where ISM-MUSIC [1], CSM-MUSIC (a method focusing on the co-array instead of physical array to reduce focusing errors, followed by MUSIC for DOA estimation) [22], and CSM-CS (a method focusing on the co-array instead of physical array, followed by a CS-based approach for DOA estimation) [22], [23] are involved in comparisons. Clearly, the narrowband methods SpSF-C and SpSF-U exploiting only one subband perform worse than those wideband methods, showing that better estimation performance can be achieved by jointly exploiting more subband information. Similar RMSE results are achieved by the proposed GS-WSpSF ($\mathcal{P}_\mathrm{n}$) and CSM-MUSIC, both outperforming other methods consistently with a big margin, while GS-WSpSF ($\mathcal{P}_\mathrm{n}$) yields the best performance for the smallest input SNR. Note that the proposed GS-WSpSF ($\mathcal{P}_\mathrm{n}$) is capable of handling all kinds of sources including correlated and uncorrelated ones, while CSM-MUSIC and CSM-CS are only suitable to deal with uncorrelated sources.

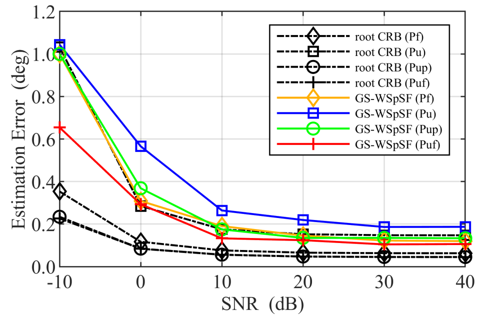
Fig. 5(b) gives the RMSE results of different methods versus SNR in the underdetermined case with $K = 8$, where we can see that the GS-WSpSF ($\mathcal{P}_\mathrm{uf}$) performs the best among all considered wideband methods. More importantly, we see that $R(\mathcal{P}_\mathrm{uf}) < R(\mathcal{P}_\mathrm{up}) < R(\mathcal{P}_\mathrm{u})$ and $R(\mathcal{P}_\mathrm{uf}) < R(\mathcal{P}_\mathrm{f})$ for a fixed SNR, where $R(\cdot)$ is the RMSE of the method under given *a priori* knowledge. These results are consistent with the order relationship among CRBs (proved in Lemma 1 in [31], i.e., employing the *a priori* knowledge that removes part of the nuisance parameters yields a lower CRB for DOAs). With more *a priori* information adopted, better results can be achieved.

VI. Conclusion

In this paper, by filling the gaps in the study of underdetermined wideband DOA estimation with the assistance of different *a priori* knowledge, a unified group sparsity framework for wideband sparse spectrum fitting (GS-WSpSF) was proposed by simultaneously exploiting the information at different subbands. Then, four variants with different *a priori* knowledge were presented, allowing underdetermined wideband DOA estimation without requiring a sparse array. It has been shown by simulations that improved performance can be achieved by the general GS-WSpSF compared with narrowband methods and GS-WSpSF ($\mathcal{P}_\mathrm{u}$) is capable of resolving more uncorrelated



(a) RMSE in the overdetermined case with $K = 4$.



(b) RMSE in the underdetermined case with $K = 8$.

Fig. 5. RMSE results of different methods with respect to input SNR.

sources than narrowband methods. It has also been shown by simulations that both increased resolution capacity and improved estimation accuracy can be achieved if more *a priori* information is considered.

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