Quantum gravity and Cosmological models on finite space-times

Julio Narciso Argota Quiroz

Submitted in partial fulfillment of the requirements of the Degree of

Doctor of Philosophy

Queen Mary University of London

School of Mathematical Sciences

February 2023

Statement of originality

I, Julio Narciso Argota Quiroz, confirm that the research included within this thesis is my own work or that where it has been carried out in collaboration with, or supported by others, that this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below.

I attest that I have exercised reasonable care to ensure that the work is original, and does not to the best of my knowledge break any UK law, infringe any third party's copyright or other Intellectual Property Right, or contain any confidential material.

I accept that the College has the right to use plagiarism detection software to check the electronic version of the thesis.

I confirm that this thesis has not been previously submitted for the award of a degree by this or any other university.

Julio Narciso Argota Quiroz
Date: 8 February, 2023.

Details of collaboration and publications:

- J. N. Argota-Quiroz and S. Majid, *Quantum gravity on polygons and* $\mathbb{R}\times\mathbb{Z}_n$ *FLRW model*, Class. Quantum Grav. (2020) 245001 (43pp)
- J. N. Argota-Quiroz and S. Majid, *Fuzzy and discrete black hole models*, Class. Quantum Grav. 145020, (2021) 38pp.
- J. N. Argota-Quiroz and S. Majid , *Quantum gravity on finite spacetimes and dynamical mass*, PoS **CORFU2021** (2022), 210 doi:10.22323/1.406.0210
- J. N. Argota-Quiroz and S. Majid, *Quantum Riemannian geometry of the discrete interval and q-deformation*, in press J. Math. Phys. (Editor's Choice award).

To my mother, Laura Argota Quiroz,

and my sister, Laura Alejandra Gallardo Argota,

with all my love and gratitude.

Acknowledgments

I have no words to thank my supervisor Shahn Majid enough for all his patience and guidance during my Ph. D.

Also, I am very grateful to my sister and mother, whose support makes everything possible.

I cannot forget all my dear friends in Mexico, who never forget me and always take care of me. Fernando, Mario, Mariana, Moises, Mars, Felix, and Isabel among many others were in contact during the whole process, thank you so much.

Finally, I want to say thank you so much to the friends I made in London with whom I share a lot of happiness and sadness equally. Diego, Andrea, Oliver, Evanguelis, Marco and so many others. I am very grateful to all by everything. A special mention to my dear friend Ana Karen Mendoza Martinez, with whom I spent a lot of time and adventures during my Ph. D. She never stopped to be there when I most needed her. Also, I am very grateful to Erwing Rosas Valdez, who brings me his hand without hesitation. And last but not least, to the group of Quantum Algebras, thanks for everything. Their company has been invaluable, Francisco, Leo, and of course Evelin.

This thesis and my Ph.D. were founded partially by CONACyT Mexico.

Abstract

We study different quantum geometries using the Quantum Riemannian Geometry (QRG) formalism, constructing some quantum gravity and cosmological models over them. First, we fully solve the quantum geometry of $\mathbb{Z}_n$ as a polygon graph for a moduli of metrics with square-lengths on the edges. The classical limit for $n \to \infty$ is analysed and, correlation functions are numerically calculated for Euclidean quantum gravity for $3 \leq n \leq 6$. An FLRW model is analysed adding 'classical' time, finding the same expansion rate as for the classical flat FLRW model in 1+2 dimensions, i.e. a dimension jump. We apply the adiabatic particle creation method on $\mathbb{R} \times \mathbb{Z}_n$. Also, a Schwarzschild black hole model is proposed with classical time and radius where the Laplacian and the classical limit $\mathbb{Z}_n \to S^1$ are studied.

Using the quantum geometry of a fuzzy sphere as a base space, it is constructed an FLRW and a spherically-symmetric black hole adding classical coordinates of time and radius as appropriate. The Schwarzschild black hole model with static-spherical solutions for Ricci = 0 is developed. A dimension jump is also found in this model with solutions having the time and radial form of a classical 5D Tangherlini black hole.

Finally, we solve for quantum Riemannian geometries on the finite lattice interval $\bullet - \bullet - \cdots - \bullet$ with n nodes (the Dynkin graph of type A_n) and find that they are necessarily q-deformed with $q = e^{\frac{\imath\pi}{n+1}}$. Specifically, we discover a novel 'boundary effect' whereby, in order to admit a quantum-Levi Civita connection, the 'metric weight' at any edge is forced to be greater when pointing towards the bulk compared to towards the boundary. The Laplacian and QFT are studied under this geometry as quantum gravity for $n = 3$.

Although, the models are constructed using different geometries, the techniques used for constructing and solving them are analogous and show some similarities which apparently are always present. It is needed to construct more examples to identify which similarities are general.

Contents

Introduction

Quantum gravity (QG) has been one of the most important problems in theoretical physics for more than one hundred years. It would be the reconciliation of the theory of general relativity (GR) and the quantum mechanic, two of the most successful formalism in physics, that would explain the nature of space-time itself.

Historically speaking, Matvei Petrovich Bronstein was one of the first to try to apply quantum mechanics to the gravitational field [**57**]. He used the ideas of Bohr and Rosenfeld for quantizing the electromagnetic field [**58**]. After that, many formalisms have been created to achieve this purpose. Before to explain the generalities of the approach followed here, we will sketch the problem.

The most accepted formulation of the space-time is the General Relativity (GR) created by Einstein in 1915 which uses as mathematical language differential geometry. It says that mass curves space-time, producing the gravity that is experienced by the objects that are inside of this space-time. Under this approach, the metric is one of the most important elements, and it is required to obey by the Einstein's equations

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}$$

where G_N is the Newton's constant, $G_{\mu\nu}, T_{\mu,\nu}$ are the Einstein and mass-energy tensors respectively. One of the best-known consequences is the creation of black holes. This happens when enough quantity of mass is concentrated in a reduced volume of space. The black-hole gets bigger as more mass fall down into it, according to the relation

$$r_{BH} = \frac{G_N M}{c^2},$$

where c is the constant of light, M represents the total mass of the black hole and the scale of r_{BH} is twice the radius of Schwarzschild in the case of a spherical symmetric black hole.

On the other hand, one of the discoveries that provoked the creation of quantum mechanics was the fact that some waves, as light, are 'packed' in quanta of energy which is proportional to their frequency and inversely proportional to their wave length as follows

$$E = h\omega = \frac{hc}{\lambda},$$

where h is the Plank's constant. Using the celebrated relation of special relativity $E = mc^2$, it is possible to relate the energy E to the effective mass m of the particle. This energy

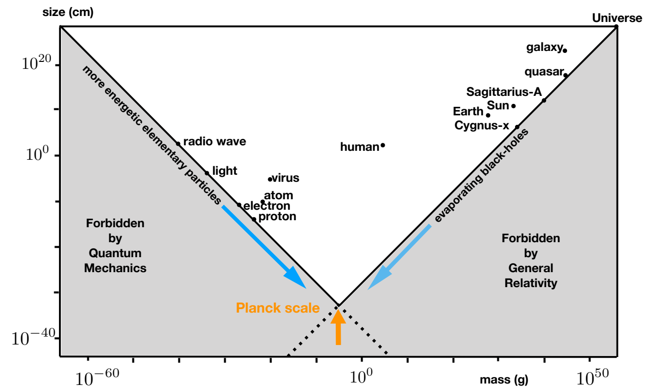

FIGURE 1. The big picture and the Planck scale from [**33, 14**].

also affects the space-time. Here we loosely understand m as energy-mass. Owing to de Broglie's work, it is known that all particles have some wave properties. This was used by Compton to develop a relation that relates the wavelength with the mass of a particle in the following way

$$\lambdabar = \frac{\hbar}{mc},$$

where $\lambdabar = \lambda/2\pi$ is the reduced wavelength and, $\hbar = h/2\pi$ is the reduced Plank's constant.

We have explained the relation between the mass and the geometry of the space-time through GR. But the way we actually see the geometry of the space-time is by using particles that travel along geodesics interacting with the elements of the geometry. In order to have more detail of the structure of the geometry, it is required finer wavelengths, which result in an increase in the resolution of the observation. However, according to Compton wavelength, this will require a bigger mass. Eventually, the mass will start to affect the structure of the space-time until the point a black hole is created. This point is pointed by the orange arrow in Figure 1 and happens when r_{BH} and $\lambdabar$ are of the same order. In the case when both are equal and $M = m$ we have the so-called Plank's scale

$$\lambda_P = \sqrt{\frac{G_N \hbar}{c^3}}.$$

Then resolution of space-time under this scale is intrinsically impossible. This implies that the continuum hypothesis which allows space-time to be infinite divisible can be wrong. Moreover, there are some problems that can be tracked back to the continuity principle like dark energy and the non-renormalizable infinities in QG.

The continuum hypothesis is deep inside current physics, because it is assumed in the differential calculus. However, this is just a special differential structure where the differentials commute with functions. A possible way to overcome this problem is to remove this feature.

Since the first days of quantum theory, a hypothesis was proposed that could be the key for solving QG:

Quantum spacetime hypothesis: *space-time could be better modelled due to quantum gravity effects by non-commutative coordinates or 'quantum space-time'.*

Nowadays it is widely accepted and many models have been constructed guided by it. An early specific model was by Snyder [**45**] and then in the 1990s emerged other models [**26**] such as θ-spacetime suggested by Snyder's model and by string theory, κ-Minkowski spacetime [**39**] motivated by quantum Poincaré symmetry [**32**] and q-Minkowski spacetime [**20**] from q-Lorentz symmetry. These models were largely flat and mainly constructed in an ad-hoc manner.

To truly address issues of the unification of quantum theory and gravity there was therefore a need for curved models and a systematic framework for Quantum Riemannian Geometry (QRG). That was developed in the last 10 years in a constructive form growing out of quantum groups, particularly by Beggs and Majid [**14**] using bimodule connections, a particularly nice type of connection first introduced by Dubois-Violette, Michor and Mourad [**27, 22**]. This is the approach used in our work.

While the mathematical picture here is well-developed, physical model building using it is still in its early stage. First of all, a bimodule connection ∇ on the 1-forms Ω^1 over a (possibly noncommutative) coordinate algebra acts on a metric $g \in \Omega^1 \otimes \Omega^1$ firstly on the first factor then on the second, but a braiding map σ that depends on ∇ is needed to make sense of this action on the second factor. As a result, the equation $\nabla g = 0$ is quadratic in the connection. That makes it hard to solve and much of our work has been to develop methods to solve for the quantum Levi-Civita connection (QLCs), which was done for the first time for significant moduli of metrics. Before our work only the square and the line $\mathbb{Z}$ could be solved for general metrics[**11, 12**]. The generalities about this approach are explained in chapter 1 from a practical point of view, and for a mathematically formal explanation of the theory, we refer to [**14**]. First, we start considering an algebra A on which we define a first-order calculus; this is the base for constructing geometrical structures such as metric, Quantum Levi-Civita connection (QLC), Riemann curvature tensor, etc. Afterward, we specialize the theory to the case of inner calculus and the particular case of the calculus of an algebra of $\mathbb{k}$-valued functions over a finite set or a finite group. The majority of the results and proofs of the background in this chapter are given in [**14, 15**].

We apply the machinery of inner calculus in chapter 2 to the group $\mathbb{Z}$ as a Cayley graph with generators $\{1, -1\}$ which is a polygon graph with arbitrary metric square-lengths on the edges, finding a unique $*$-preserving QLC for $n \neq 4$. Others non $*$-preserving solutions were found, but are never used in any of our models, hence they are reported for completeness in the Appendix. Next, it is considered the classical limit $n \to \infty$ of the polygon and this is identified as a central extension in the sense of [**36, 7**] of the classical calculus on a circle, with an extra 'normal' direction Θ_0. After, we make our first application of the geometry by calculating the correlation functions of Euclidean quantum gravity. It is found that for finite n, it is totally solvable for $n \leq 6$ depending only on computing power. The second main result in the chapter is then a detailed study of the FLRW model on $\mathbb{R} \times \mathbb{Z}_n$ with $\mathbb{R}$ classical, including cosmological particle creation following the approach of Parker [**63, 64, 65, 66**]. For this model, we find that quantum metrics on $\mathbb{R}\times\mathbb{Z}_n$ are forced to have the block form $g = \mu \mathrm{d}t \otimes \mathrm{d}t + h_{ab} e^a \otimes e^b$ (forced by the centrality of the metric) and, moreover, h_{ab} has to have a specific form where the time dependence enters uniformly in the spatial metric. In the case of the FLRW cosmology of a uniform metric on $\mathbb{Z}_n$ expanded by a time-dependent factor $R(t)$, we have

$$g = -\mathrm{d}t \otimes \mathrm{d}t - R^2(t)(e^+ \otimes e^- + e^- \otimes e^+).$$

The negative sign in the second term is required given that the inverse metric is negative definite in the sense that for a definite positive function a the inner product is $(e^+, e^{+*}) = -\frac{1}{a} < 0$, and we then find that the Friedmann equations for $R(t)$ in our discrete case actually come out the same as for the usual flat FLRW model in *two* spatial dimensions, which is in line with our cotangent bundle on $\mathbb{Z}_n$ being necessarily 2-dimensional, not 1-dimensional. Some elementary checks for QFT in the constant R case are provided, then we cover the cosmological particle creation for varying $R(t)$. Next, we consider the classical geometry case of $\mathbb{R} \times S^1$, which sets up the formalism, then the modifications for $\mathbb{R}\times\mathbb{Z}_n$. Of interest are the adiabatic no particle creation possibilities for $R(t)$ aside from the obvious constant R case; for $\mathbb{R} \times S^1$ there is a further possibility for the infinite mass limit $m \to \infty$, but for $\mathbb{R} \times \mathbb{Z}_n$ we find a second further possibility with $m \to 0$. The particle creation calculation itself is done only for 'in' and 'out' regimes of constant R, with results a little different in the $\mathbb{Z}_n$ case due to the periodic nature of the spatial momentum compared to the S^1 case. Next, we look for a 3D black hole model with the S^1 in polar coordinates replaced by $\mathbb{Z}_n$. The latter is not flat but Ricci flat (which can not happen classically in 3D) and has a naked singularity rather than a horizon. The chapter concludes with the $\mathbb{R}^2 \times \mathbb{Z}_n$ black hole-like model where $\beta = -r_H/r$ which is as for a usual black hole but without the constant term. This therefore approximates the metric *inside* a Schwarzschild black hole of infinite mass (so that the missing 1 factor is negligible). We also cover the case $\beta(r) = r_H/r$ of interest in its own right. This model has no horizon but a naked singularity. We describe the $\mathbb{Z}_n \to S^1$ limit where S^1 retains a 2D noncommutative differential structure, and the classical projection to the usual calculus on S^1 where the metric is no longer Ricci flat, i.e. this is a purely quantum-geometric solution of the vacuum Einstein equations.

In chapter 3 we use the same ideas of the previous chapter but using a fuzzy sphere as base space, constructing FLRW and black hole models. A dimension jump is also observed in the classical limit $\lambda_p \to 0$, an extra 'normal' direction θ' for the sphere embedded in $\mathbb{R}^3$.

This time the dimension jump means that the radial-time sector matches to the closed (positively curved) 4D FLRW model. For the black hole model, the dimension jump means we land on radial and time behaviour matching the 5D Tangherlini black hole[**46**] when we use the fuzzy sphere. Here the $\beta(r) = 1 - r_H/r$ factor in the familiar Schwarzschild metric case for horizon radius r_H is now a factor $\beta(r) = 1 - r_H^2/r^2$. This gets asymptotically flat faster than the Schwarzschild case and the effective gravity in the Newtonian limit is an inverse cubic force law. In chapters 2 and 3 a Klein-Gordon equation in the noncommutative background is studied with the corresponding geometry. We introduce the notion of a Schroedinger-like equation for an effective quantum theory relative to an exact solution in the same manner as usual quantum mechanics for a free particle can be obtained as a non-relativistic limit of the Klein-Gordon equations for solutions of the form $e^{-\imath mt}\psi$ with ψ slowly varying. The novel feature will be to replace $e^{-\imath mt}$ by an exact reference solution of the Klein-Gordon equation, and we explain first how this looks for a classical Schwarzschild black hole. This appears to be rather different from well-known methods of quantum field theory on a curved background [**54, 66, 67**] but fits with the general idea that a quantum geodesic flow is actually a Schroedinger-like evolution.

The previous noncommutative models are theoretical and we are not aware of an immediate application, but they do indicate an unusual phenomenon which has a purely quantum origin in an extra 'normal direction' θ' required for an associative differential calculus in our examples. We also began to explore some of the physics in our noncommutative backgrounds.

In chapter 4, we explore the quantum Riemannian geometry of the finite line graph $\bullet - \bullet - \cdots - \bullet$ with n nodes (the Dynkin graph of type A_n) as well as the half-line with nodes the natural numbers $\mathbb{N}$. Our result is that for the A_n graph with $n > 2$ and for $\mathbb{N}$ *there is no edge-symmetric QRG*. We are forced to introduce a 'direction coefficient' ϕ on edges to measure the ratio of the inbound arrow (towards the bulk) length compared to the outbound arrow length and find for $\mathbb{N}$ that these have to be a specific rational numbers as shown in Figure 4 in order to admit a quantum-Levi Civita connection. This ratio decays rapidly from 2 at the endpoint down to 1 in the bulk. As long as we keep these ratios, we are free to vary the actual metric coefficients or 'square-lengths' as we please, so the moduli of QRGs is the same as classically – a single 'square-length' on every link – but the new effect is that if we consider this as the outbound one then the inbound one is a multiple ϕ of it, namely twice at the first link from the end, 3/2 at the link which is one in from the end, etc.

The situation for A_n is similar and we again find a canonical choice of quantum-Levi Civita connection provided ϕ_i at edges $i = 0, \cdots, n-1$ are now given by q-integers

$$\phi_i = \frac{(i+1)_q}{(i)_q}, \quad q = e^{\frac{\imath\pi}{n+1}}, \quad (i)_q = \frac{q^i - q^{-i}}{q - q^{-1}}$$

deforming the canonical QRG for $\mathbb{N}$. The second result of the chapter is therefore that a finite-lattice interval is intrinsically q-deformed in its quantum Riemannian geometry, without a quantum group in sight. Next, we extrapolate from this to general formulae for A_n and $\mathbb{N}$ with a uniform solution for the QRG with the freely chosen metric weights $\{h_i\}$, sign parameter ϵ in the metric and modulus 1 parameter s in the quantum Levi-Civita connection. The physical case for all metric coefficients positive requires $\epsilon = 1$ and if we want the Christoffel symbols to also be real then we are forced to $s = \pm 1$. We only use these values for the rest of the chapter. After finding this canonical form for the QRGs, we then study scalar field theory on them. The effect of the direction dependence ϕ_i on $\mathbb{N}$ translates to a derivative term correction to the Laplacian which alternates with a $(-1)^i$ factor preventing a straightforward continuum limit. However, this is suppressed as $1/i$ so that as the lattice spacing tends to zero, this complication is pushed to the boundary at 0. A secondary effect of the ϕ_i factor is that the overall metric factor β^{-1} in front of the Laplacian has a correction compared to the same choices of h_i on $\mathbb{Z}$. We analyse this for the case of constant h_i as something like a $\frac{1}{x^2}$ force towards the origin. Some partition functions for scalar field theory on the A_3 graph are also computed as proof of concept with respect to a measure for integration on the A_3 graph.

We note that there is a popular approach to noncommutative geometry by Connes' [**25**], particularly with the starting point of the notion of spectral triplets, generalizing the properties of Dirac operators. Meanwhile, as mentioned before, QRG starts with a bimodule of differential forms over an algebra A and constructs the rest of the geometry based on this. However, it is also possible to construct spectral triplets in QRG with a bimodule connection and a Clifford action on a spinor bundle, see [**14**, Chap. 8.5].

We take $c, \hbar = 1$ throughout. The results of chapter 2 were published in [**2, 3**]. The results of chapter 3 were published in [**3**]. The results of chapter 4 were published in [**10**]. Also, we note a conference proceeding [**9**] that includes an overview of the models of chapters 2 and 3. This last work also reports a preliminary analysis of Kaluza-Klein theory for an algebra with a central basis not included in this thesis.

CHAPTER 1

Fundamentals

In terms of differential geometry, a differential structure on a manifold $\mathcal{M}$ with local coordinates x^i defines the derivates of real-valued functions on every chart by $\frac{\partial}{\partial x^i}$. The collection of all the possible linear combinations of these defines the tangent bundle $T\mathcal{M}$. Besides, a local dual base $\mathrm{d}x^i$ can be defined for every chart; thus, the cotangent bundle $T^*\mathcal{M}$ is defined analogously. Both structures work over the algebra of real-valued functions of the local coordinates of every manifold chart. The fact that the differential structure works over this algebra of functions suggests that it is possible to construct an algebraic version of differential geometry over a general algebra, not just an algebra of functions. The majority of the content of this chapter is from [**14, 15**] except the lemma 1.6 which was published in [**2**].

1. Quantum Riemannian Geometry generalities

In the QRG approach, instead of focusing on the manifold, we work over an algebra, the so-called coordinate algebra. However, there is no restriction about whether it is an algebra of functions. Even it is possible that there is no manifold related to the algebra. Of course, if we work over a commutative algebra of functions over $\mathcal{M}$, it is possible to recover the usual results of the standard differential geometry. Next, we describe the formalism.

We consider an associative unital $*$-algebra A, which we call a coordinate algebra. This will eventually define space-time. From now on, every time we refer to an algebra, we mean this setup. It does not matter if the algebra is commutative or not; the formalism works anyway. In order to construct a geometry over A, a differential structure is needed. A bimodule Ω^1 of 1-forms over A fulfills this role, given the fact that, in general, the space of 1-forms has a natural bimodule structure. The bimodule structure says for $\omega \in \Omega^1$ and $a, b \in A$, that

$$(a\omega)b = a(\omega b).$$

Definition 1.1. A first-order *differential calculus* (Ω^1, d) over an algebra A is an $A-A$-bimodule Ω^1 with a linear map $\mathrm{d} : A \to \Omega^1$ that acts as an exterior derivative which must respect the Leibniz rule of the product, i.e., $\mathrm{d}(ab) = (\mathrm{d}a)b + a\mathrm{d}b$ for all $a, b \in A$. Moreover, the bimodule must have a surjectivity behavior such that $\Omega^1 = \mathrm{span}\{a\mathrm{d}b \mid a, b \in A\}$.

Optionally, the calculus can be connected i.e., ker d $= \mathbb{k}.1$, where 1 denotes the unity in A. Also, it is possible to call this structure a differential algebra $(A, \Omega^1, \mathrm{d})$.

In general, the previous definition is the minimum for defining a differential structure over an algebra. Nevertheless, it is usual to require to extend Ω^1 to a *differential graded algebra* (DGA) of forms $\Omega = \oplus_{i=0}^{n}\Omega^i$ (where $\Omega^0 = A$) with an exterior product $\wedge$. Using the exterior derivative as $\mathrm{d} : \Omega^i \to \Omega^{i+1}$ where it has to accomplish $\mathrm{d}^2 = 0$ and the graded Leibniz rule with respect to the graded product as

$$\mathrm{d}(\omega\gamma) = \mathrm{d}\omega \wedge \gamma + (-1)^i \omega \wedge \mathrm{d}\gamma,$$

for $\omega \in \Omega^i$ and $\gamma \in \Omega$. Extending the surjective property, Ω has to be generated by $A, \mathrm{d}A$ as it would be in the classical case. If there is a top degree, it is called the volume dimension. In general Ω is a quotient of the tensor algebra over A with a graded product $\wedge$.

Generally speaking, there is not a unique calculus given an algebra A and, it is the first choice that has to be made for constructing a model. However, it is known that every algebra A has always a first-order connected differential calculus, called the universal calculus, given by

$$\Omega^1_{\mathrm{uni}} = ker(\cdot) \subseteq A \otimes A, \quad \mathrm{d}_{\mathrm{uni}} a = 1_A \otimes a - a \otimes 1_A,$$

where 1_A is the unit of the algebra and $\cdot : A \otimes_A A \to A$ denotes the product in A. Then, the elements of the universal calculus, have the form $a \otimes_A b$ for all $a, b \in A$ such that $ab = 0$. Thus, the left action of an element $c \in A$ over Ω^1 is $c(a \otimes_A b) = (ca) \otimes_A b$, meanwhile the right action is $(a \otimes_A b)c = a \otimes_A (bc)$, showing explicitly that the actions by the left and right are in general distinct. A last remark about the universal calculus is that any other calculus over the algebra A is isomorphic to a quotient of the universal one, i.e for any differential calculus, there is a sub-bimodule $\mathcal{N} \subseteq \Omega^1_{\mathrm{uni}}$, such that Ω^1 is isomorphic to $\Omega^1_{\mathrm{uni}}/\mathcal{N}$. The proofs of these statements can be consulted in [**14**].

Regardless, in order to construct physical applications, we need to extend the anti-linear $*$-map of the algebra to the calculus.

Definition 1.2. A first degree $*$-differential calculus is a differential calculus (Ω^1, d) plus an extension of the anti-linear map of the algebra to the bimodule $* : \Omega^1 \to \Omega^1$, which is compatible with the bimodule structure in the sense that $(a\omega)^* = \omega^* a^*$ for all $a \in A, \omega \in \Omega^1$. Also, the map has to commute with the exterior derivative, i.e $*(\mathrm{d}a) = \mathrm{d}(*a)$. A natural consequence is $(a\mathrm{d}b)^* = (\mathrm{d}b)^* a$.

For an algebra of operators in a Hilbert space, the $*$-map indicates which operators are hermitian, which are the observables in quantum mechanics. In our case, it plays a similar role. We want our geometrical structures to be compatible with this map to distinguish 'real-valued' elements of the algebra as criteria for 'reality'.

Now, we are ready to define the concepts related to *distance* in geometry, the metric and the inverse metric (inner product). Consider a bimodule map that acts as inner product

$(\,,\,) : \Omega^1 \otimes \Omega^1 \to A$, which is a well defined map on $\Omega^1 \otimes_A \Omega^1$, this means

$$(\omega a, \eta) = (\omega, a\eta), \quad (a\omega, \eta) = a(\omega, \eta), \quad (\omega, \eta a) = (\omega, \eta)a,$$

For this map to be compatible with the $*$-algebra, it is imposed the following relation

$$(\omega, \eta)^* = (\eta^*, \omega^*).$$

It is desired that this inner product be nondegenerate or, in the best of cases, to be an inverse metric of a tensor in the following way.

Definition 1.3. Let $(A, \Omega^1, \mathrm{d})$ be a differential calculus, a *generalized quantum metric* is a tensor $g \in \Omega^1 \otimes_A \Omega^1$, which is invertible in the sense that exists an inner product such that for all $\omega \in \Omega^1$ holds

$$((\omega,\,) \otimes \mathrm{id})g = (\mathrm{id} \otimes (\,,\omega))g.$$

A consequence of these conditions is that a metric g has to be central in the algebra A, i.e., $ag = ga$ for all $a \in A$ (see Chapter 1.3, [**14**]).

At this point, it is useful to define the antilinear map $\dagger : \Omega^1 \otimes_A \Omega^1 \to \Omega^1 \otimes_A \Omega^1$, which works as follows

$$(\omega \otimes a\gamma)^\dagger = \gamma^* a^* \otimes \omega^* = \gamma^* \otimes a^* \omega^* = (\omega a \otimes \gamma)^\dagger,$$

for any $\omega, \gamma \in \Omega^1$ and $a \in A$. Now, a generalized metric is called *real* if $g^\dagger = g$.

Our next step is to introduce connections or covariant derivatives under our approach. For this purpose, it is noticed that a section of a bundle can be multiplied by elements of the algebra from the left, behaving as a (left) module over the algebra. Thus, we have a left connection $\nabla_E : E \to \Omega^1 \otimes_A E$ on the left A-module E. This is the general setup (see Chapter 3, [**14**]). However, for the rest of this work, we are interested in the case when E is the bimodule Ω^1.

Now consider a differential algebra (A, Ω, d), where Ω is well defined at least to degree two. A *left connection* over the bimodule Ω is a map $\nabla : \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ which obeys the Leibniz rule

$$\nabla(a\omega) = \mathrm{d}a \otimes \omega + a\nabla\omega, \quad a \in A, \quad \omega \in \Omega^1$$

A *bimodule connection* is a left connection that obeys the twisted right Leibniz rule $\nabla(\omega a) = (\nabla\omega)a + \sigma(\omega \otimes \mathrm{d}a)$ for some bimodule map $\sigma : \Omega^1 \otimes \Omega^1 \to \Omega^1 \otimes \Omega^1$ called *generalized braiding* which is unique, in case it exists. The braiding map is totally defined for the connection

$$\sigma(\omega \otimes \mathrm{d}a) = \mathrm{d}a \otimes \omega + \nabla[\omega, a] - [\nabla\omega, a],$$

and the other way around. Then, it is possible we refer to the braiding mapping as the connection.

One of the first and most useful operators we can construct now is the Laplacian

$$\square = (,)\nabla \mathrm{d}$$

which just needs the inverse metric and the connection. In analogy with the classical case, the connection has curvature and torsion defined as

$$\mathrm{R}_\nabla = \Omega^1 \to \Omega^2 \otimes_A \Omega^1, \quad \mathrm{R}_\nabla = (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla)\nabla$$

$$\mathrm{T}_\nabla : \Omega^1 \to \Omega^2, \quad \mathrm{T}_\nabla = \wedge\nabla - \mathrm{d}$$

where $\wedge : \Omega^1 \otimes \Omega^1 \to \Omega^2$ is the exterior product between forms. Remember that Ω^n denotes the space of n-forms of a DGA, (see the annotation after definition 1.1). The connection just needs that the calculus is defined until first order, the definition of the curvature and the torsion need order two, which is also the minimum order for constructing the Laplacian operator, which is very important for the physical applications over the geometrical models that we are interested.

Another tensor that can be constructed from the connection and the metric g is the cotorsion

$$\mathrm{coT}_\nabla \in \Omega^2 \otimes_A \Omega^1, \quad \mathrm{coT}_\nabla = (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla)g.$$

A *weak Levi-Civita connection* is a connection which is torsion-free and cotorsion-free, i.e., $T_\nabla = 0, coT_\nabla = 0$. The models presented do not use this type of connection. Instead, we use another type of connection, which is more restrictive. Before defining it, we need to extend the application of the connection to $\Omega \otimes \Omega$.

Using the braiding map, it is possible canonically extend the connection to act as $\nabla : \Omega^1 \otimes_A \Omega^1 \to \Omega^1 \otimes_A \Omega^1 \otimes_A \Omega^1$ where the explicit way is

$$\nabla(\omega \otimes \eta) = (\nabla\omega) \otimes \eta + (\sigma \otimes \mathrm{id})(\omega \otimes \nabla\eta), \quad \omega, \eta \in \Omega^1$$

Then it makes sense to ask for a connection that is metric-compatible in the sense of $\nabla g = 0$ for some metric $g \in \Omega^1 \otimes \Omega^1$, in analogy with the classical case.

Definition 1.4. A *Quantum Levi-Civita connection* (QLC) is a connection that is metric compatible and torsion-free, i.e. $\nabla g = 0,\ T_\nabla = 0$, respectively.

This is the connection that we are interested in for constructing our models.

Definition 1.5. A $*$*-preserving connection* is a bimodule connection that accomplishes

$$\sigma \circ \dagger \circ \nabla = \nabla \circ *$$

This condition may be too strong; then we have a weaker condition in which case, we call it $*$*-compatible connection*

$$\dagger \circ \sigma = \sigma^{-1} \circ \dagger.$$

As is expected, the first condition implies the second one.

The Ricci tensor is constructed using the quantum metric, the Riemannian curvature and introducing a lift bimodule map $i : \Omega^2 \to \Omega \otimes_A \Omega$ as additional data. It has the following form

$$\mathrm{Ricci} = ((\,,\,) \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes i \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{R}_\nabla)g.$$

In general, it does not exist the notion of anti-symmetry in non-commutative geometry as in the classical case. That is why, a lift map i, that obeys $\wedge \circ i = \mathrm{id}$, is introduced as a generalization of the map of the 2-forms to the anti-symmetric tensor product of 1-forms. In the case of the $*$-calculus, it can be required the map i to be "real" in the sense $\dagger \circ i = -i \circ *$. There exists the possibility that this definition does not give a unique Ricci tensor, but a moduli space of them. A possible condition for overcoming this situation is imposing the condition $\wedge\mathrm{Ricci} = 0$. The reality condition will be $\mathrm{Ricci}^\dagger = \mathrm{Ricci}$. An important remark is that this construction produces a factor of $-1/2$ compared to the classical Ricci tensor.

The construction of the Ricci tensor presented above is not considered the final formalism for constructing it, a more general approach could replace this one with more experience and information. However, it gives a good point to start and so far has produced good results for the models.

The Ricci scalar is defined analogously to the classical case using the inner product as $S = (\,,\,)\mathrm{Ricci}$. It also inherits the dependence of the choice of i and, in the case the Ricci tensor accomplishes the reality condition, satisfies $S = S^*$.

Finally, we make some remarks about the notion of integration $\int : A \to \mathbb{C}$ over the 'manifold' underlying A. Even though there are a lot of open problems related to this topic, making some assumptions about the measure, we can achieve some good results. Classically, it would be given in a local coordinate chart by the Lebesgue measure times a covariant factor $\sqrt{\det(g)}$ but how this is defined for a quantum metric is unclear. Considering the reality conditions we could require

$$\overline{\int a} = \int a^*, \quad \int a^* a \geq 0 \tag{1.1}$$

with equality if and only if $a = 0$. This is a non-degenerate positive linear functional in the sense of $*$-algebras, typically a maximally impure state used to define integration on the algebra. We also want compatibility with the metric and classically this can be done via the divergence of vector fields. The corresponding analog to the divergence in QRG approach then will be

$$\mathrm{div}(\omega) = (\ ,\ )\nabla\omega.$$

In that case a natural divergence condition on $\int$ motivated by [**18**] is

$$\int a\,\mathrm{div}(\omega) = -\int (\mathrm{d}a, \omega) \tag{1.2}$$

for all $a \in A$ and $\omega \in \Omega^1$. We note that this is compatible with the Leibniz rule:

$$\begin{aligned}\int (ab)\mathrm{div}(\omega) &= \int a(\ , \ )(b\nabla\omega) = \int a(\ , \ )\nabla(b\omega) - \int a(\mathrm{d}b, \omega)\\ &= -\int (\mathrm{d}a, b\omega) - \int a(\mathrm{d}b, \omega) = -\int (\mathrm{d}(ab), \omega)\end{aligned}$$

but not necessarily with $*$. For that, it would be natural to impose a further condition

$$\int (\ , \ )(\mathrm{id} - \sigma) = 0. \tag{1.3}$$

In fact (1.2)-(1.3) are too strong in most cases and this is an area for further development, e.g., in connection with quantum geodesics[**18**].

2. Calculus Over Finite Sets

Here, the material of the previous section is put into context of the finite set and graphs, which turns out to be an inner calculus. Starting with the well-known fact that given a finite set V, a differential structures over the algebra $A = \mathbb{k}(V)$, for some field $\mathbb{k}$, are in one-to-one correspondence with digraphs which have the vertex set V. Thus, a digraph represents a differential structure over an algebra A of functions. What we mean with a digraph is $\mathbb{G} = (V, E)$ where $E \subset V \times V \backslash$diagonal is the set of edges, where the ones that start and end in the same vertex are not allowed. The Kronecker delta-functions δ_x, with value 1_A at $x \in V$ and zero elsewhere, is used as a central basis of A, which satisfies $1_A = \sum_x \delta_x$, where the index x runs over all the elements of V. Thus, an arbitrary element $f \in A$ has the form $f = \sum_x f(x)\delta_x$.

We denote $x \to y$ for $(x, y) \in E$ and $x, y \in V$. Then the differential structure has the form

$$f.\omega_{x\to y} = f(x)\omega_{x\to y}, \quad \omega_{x\to y}.f = \omega_{x\to y}f(y), \quad \mathrm{d}f = \sum_{x\to y}(f(y) - f(x))\omega_{x\to y}$$

for all $f \in A$ and $\omega_{x\to y} \in \Omega^1$. The basis elements of Ω^1 have the form $\delta_x \otimes \delta_y$ for any $x, y \in V$ such that $x \to y \in E$, reason why the arrows in E define the differential structure and the other way around.

An undirected graph can be seen as a "bi-directed" graph where for each edge $(x, y) \in E$ exists the inverse $(y, x) \in E$, this implies they are arrows in both directions for each edge. This characteristic produces a *symmetric* calculus.

A calculus over A is called left\right *parallelisable* with cotangent dimension $m\backslash n$ if and only if the graph is m-left\n-right regular in the sense that the number of going out\in arrows are the same for each vertex. In fact choosing a colouring of the outgoing arrows from a fixed pallet of colours $i = 1, 2, \ldots m$ means choosing a left-parallelisation, which allows to define partial derivatives as $\partial_i f(x) = f(y) - f(x)$ where y represents the vertex obtained for moving along the arrow coloured i starting from x.

Any bimodule inner product for a directed graph ($\Omega^1(V)$) calculus takes the form $(\omega_{x\to y}, \omega_{y'\to x'}) = \lambda_{x\to y}\delta_{x,x'}\delta_{y,y'}\delta_x$ for some arrow weights $\{\lambda_{x\to y}\}$ and where $\delta_{x,x'}$ is 1 when x and x' represent the same vertex and zero otherwise. The elements $\{\delta_x\}$ are projectors of the central basis of A which is 1 in x and zero elsewhere. These projectors sum to unity. If and only if the calculus is symmetric and all the elements $\lambda_{x\to y}$ are non zero there is a generalized quantum metric

$$g = \sum_{x\to y} g_{x\to y}\omega_{x\to y} \otimes \omega_{y\to x}, \quad g_{x\to y} = 1/\lambda_{y\to x}.$$

The case when $g_{x\to y} = g_{y\to x}$ is called *edge symmetric* and, even though it is not mandatory for the models or the formalism, brings some physical meaning.

If the calculus over the finite set is symmetric and ${\omega_{x\to y}}^* = -\omega_{y\to x}$ then there is a $*$-differential calculus. Besides, the reality conditions for the inner product are accomplished, if and only if, the coefficients $\lambda_{y\to x}$ are real.

When we want to find a QLC for a metric, we depend on Ω^2 and here there are four canonical choices for Ω in the sense that they are defined for any graph. They are all quotients of the path algebra which in degree n consists of the n-step paths $\omega_{x_0\to x_1} \otimes \cdots \otimes \omega_{x_{n-1}\to x_n} \in \Omega^1 \otimes_A \cdots \otimes_A \Omega^1$ (this is the tensor algebra of Ω^1 over A). We quotient this by the quadratic relations[**14**, Prop. 1.40]

$$\sum_{y:p\to y\to q} \omega_{p\to y} \wedge \omega_{y\to q} = 0$$

for all fixed p, q that obey one of the four conditions below. This leads to the four exterior algebras forming a diamond:

$$\begin{array}{ccc} & \Omega_{max} & \\ \swarrow & & \searrow \\ \Omega_{med} & & \Omega_{med'} \\ \searrow & & \swarrow \\ & \Omega_{min} & \end{array}$$

where the conditions are all p, q such that

$$\begin{aligned} \Omega_{min} &: \quad \text{all } p, q \\ \Omega_{med} &: \quad p \neq q \\ \Omega_{med'} &: \quad p \not\leftrightarrow q \\ \Omega_{max} &: \quad p \neq q, \quad p \not\leftrightarrow q. \end{aligned}$$

Three of these were explicitly discussed in [**14, 47**] while $\Omega_{med'}$ was used in [**19**]. The chosen definition of the external derivative, which extends Ω^1 to high ranks, has to be consistent with the selected quotient. This consistency depends on the graph itself.

One of the most useful characteristics of the first-order calculus on graphs is that they are inner; this means there exists a 1-form Θ such that $\mathrm{d}f = [\Theta, f]$, for any $f \in A$. Sometimes, it is possible to extend this to forms of higher degree $\eta \in \Omega^n$ using the graded commutator as $\mathrm{d}\eta = [\Theta, \eta\} = \Theta \wedge \eta - (-1)^n \eta \wedge \Theta$, where n is the degree of η. Then for the differential algebra $(A, \Omega^1, \mathrm{d})$ of finite sets there always exist the inner form $\Theta = \sum_{x\to y} \omega_{x\to y}$ where the sum includes all the arrows of the graph. In the case of the $*$-differential calculus, if the calculus is also symmetric, the inner form satisfies $\Theta^* = -\Theta$.

When the calculus is inner, some of the geometrical structures take a specific form, see [**14**] . For example, the connection is

$$\nabla = \Theta \otimes () + \alpha - \sigma_\Theta; \quad \sigma_\Theta = \sigma(() \otimes \Theta), \tag{1.4}$$

for some bimodules maps $\alpha : \Omega^1 \to \Omega^1 \otimes \Omega^1$, $\sigma : \Omega^1 \otimes \Omega^1 \to \Omega^1 \otimes \Omega^1$ and $\omega \in \Omega^1$. The curvature is

$$R_\nabla \omega = \Theta \wedge \Theta \otimes \omega - (\wedge \otimes \mathrm{id})(\mathrm{id} \otimes (\alpha - \sigma_\Theta))(\alpha - \sigma_\Theta)\omega. \tag{1.5}$$

It is important to point out that the QLC has a one-to-one relation with the maps α and σ, which means those maps completely define the connection. The torsion condition is

$$\mathrm{T}_\nabla = - \wedge (() \otimes \Theta + \sigma_\Theta - \alpha),$$

and a connection is torsion-free if and only if $\wedge\alpha = 0$ and $\wedge\sigma = -\wedge$. Finally, the metric compatible condition is

$$\theta \otimes g + (\alpha \otimes g) + (\sigma \otimes \mathrm{id})(\mathrm{id} \otimes (\alpha - \sigma_\theta))g = 0 \tag{1.6}$$

for the metric $g \in \Omega^1 \otimes \Omega^1$. In the case the $*$-algebra is over the field of complex numbers, the connection is $*$-preserving if and only if

$$\dagger \circ \sigma = \sigma^{-1} \circ \dagger, \quad \sigma \circ \dagger \circ \alpha = \alpha \circ * \tag{1.7}$$

2.1. Calculus over finite groups. Finally, we specialize to the case when a finite set G has a group structure. First, we select a subset $C \subset G \backslash e$ of so-called generators that do not include the group identity e. The reason for excluding the identity e is does not have arrows that start and end in the same vertex. Thus for a group G the vertexes are the elements of the group and the edges are defined as $E = \{x \to xa | x \in G, a \in C\}$. The resulting Cayley graph is left regular with C as the colouring pallet. A direct consequence is that the first-order calculus is parallelisable and with the form $\Omega^1 = \Bbbk(G)\Lambda^1$ as a free module over a vector space Λ^1 of left-invariant 1-forms with basis

$$e^a = \sum_{x\in G} \omega_{x\to xa}. \tag{1.8}$$

for each $a \in C$. The commutation relation and the exterior derivative then are

$$e^a f = R_a(f)e^a, \quad \mathrm{d}f = \sum_{a\in C}(R_a(f) - f)e^a, \tag{1.9}$$

where $f \in \Bbbk(G)$ and $R_a(f) = f(()a)$. A quantum metric exists if and only if C has inverses, to be central, and has the form

$$g = \sum_{a\in C} c_a e^a \otimes e^{a^{-1}}, \quad (e^a, e^b) = \frac{\delta_{a^{-1},b}}{R_a(c_{a^{-1}})}, \tag{1.10}$$

where $c_a \in \Bbbk(G)$ are nowhere zero. The edge-symmetry condition is accomplished if $R_a(c_{a^{-1}}) = c_a$ and the inner form is $\Theta = \sum_{a\in C} e^a$. For the $*$-differential calculus, the elements of the basis of Λ^1 obey $e^{a*} = -e^{a^{-1}}$. Meanwhile, when $\Bbbk = \mathbb{C}$, the reality condition for the metric imposes that the coefficients c_a have to be real.

The QLC for this case is also completely defined for the bimodule maps σ and α, which need to accomplish $\sigma^{a,b}{}_{m,n} = 0$ unless $ab = mn$ in the group and $\alpha(e^a) = \alpha^a{}_{m,n}e^m\otimes e^n$ needs $\alpha^a{}_{m,n} = 0$ unless $a = mn$ in the group, see [**14**, Chap. 8.2.2][**15**]. The indices here range over elements of the generating set C of the calculus and are *not* being multiplied in the 4-index and 3-index tensors $\sigma^{a,b}{}_{m,n}, \alpha^a{}_{m,n}$. A more explicit form is given in the following lemma.

Lemma 1.6. *Let $\Omega(G)$ be a Cayley graph calculus and (cf.* [**14, 15**]*), write a bimodule connection on Ω^1 in the form*

$$\sigma^{a,b}{}_{m,n} = \delta^a{}_n\delta^b{}_m + \delta^b{}_{a^{-1}mn}\tau^a{}_{m,n}, \quad \Gamma^a{}_{b,c} = \tau^a{}_{b,c} - \delta^a{}_{bc}\alpha_{b,c}$$

for coefficient functions $\tau^a{}_{b,c} = 0$ unless $a^{-1}bc \in C$ and $\alpha_{b,c} = 0$ unless $bc \in C$.

(1) For G abelian, the condition for torsion freeness is that the indexes b, c in $\tau^a{}_{b,c}$ and $\alpha_{b,c}$ must be symmetric.

(2) The conditions for 'reality' of the connection (to be $*$-preserving) are

$$\alpha_{b,c} + R_{bc}(\overline{\alpha_{c^{-1},b^{-1}}}) + \sum_n R_{nbcn^{-1}}(\overline{\alpha_{c^{-1}b^{-1}n^{-1},n}})\tau^{n^{-1}}{}_{b,c} = 0,$$

$$\tau^{a^{-1}}{}_{c,d} + R_{cd}(\overline{\tau^a{}_{c^{-1},d^{-1}}}) + \sum_n R_{cd}(\overline{\tau^a{}_{c^{-1}d^{-1}n,n^{-1}}})\tau^n{}_{c,d} = 0$$

for all a, b, c, d.

(3) The conditions for metric compatibility with an edge-symmetric metric are

$$h_{m,n}\alpha_{m,n} + R_n(h_{n^{-1}}\alpha_{m,n^{-1}m^{-1}}) - \sum_a R_{a^{-1}}(h_a\alpha_{amn,n^{-1}m^{-1}}) - R_n(h_{n^{-1}}\tau^{n^{-1}}{}_{m,n^{-1}m^{-1}}) = 0,$$

$$\delta^p{}_{n^{-1}}\partial_m h_n = h_{p^{-1}}\tau^{p^{-1}}{}_{m,n} - \sum_a R_{a^{-1}}(h_a\tau^a{}_{amn,p})\tau^{a^{-1}}{}_{m,n}$$

for all m, n, p.

Proof. (1) The first formula displayed is basically in [**15**] (or see [**14**, Chap. 8.2.2]) in the inner case with $\Theta = \sum_a e^a$, merely put in terms of the components of Γ and after subtracting off the flip map from σ and imposing the bimodule properties of the maps α, σ (hence written in terms of τ). It is easy to see that $\wedge\alpha = 0$ and $\wedge(\mathrm{id} + \sigma) = 0$ for the Grassmann algebra case reduce to symmetry in the lower indices (this technique is used in [**14**] but is in any case straightforward). Note that $e \notin C$ so $\Gamma^a{}_{b,c}$ has value $-\alpha_{b,c} := -\alpha_{b,c}$ when $a = bc$ and $\tau^a{}_{b,c} := \tau^a{}_{b,c}$ otherwise. We omit the commas when there are only two elements not being multiplied.

(2) The condition for α is immediate from $\sigma \circ \dagger \circ \alpha = \alpha \circ *$ evaluated on e^a with $e^{a*} = -e^{a^{-1}}$. The condition $\sigma \circ \dagger \circ \sigma = \dagger$ is easily seen (as in the proof of [**14**, Lemma 8.17] for $\alpha = 0$) to be

$$\sum_{m,n} R_{n^{-1}m^{-1}}(\overline{\sigma^{a,b}{}_{m,n}})\sigma^{n^{-1},m^{-1}}{}_{c,d} = \delta^{b^{-1}}{}_c \delta^{a^{-1}}{}_d, \tag{1.11}$$

which we now evaluate for the stated form of σ.

(3) Metric compatibility is

$$\nabla(h_{ab}e^a) \otimes e^b - \sigma(h_{ab}e^a \otimes \Gamma^b{}_{cd}e^c) \otimes e^d = 0, \tag{1.12}$$

which expands out using the Leibniz rules and the form of the metric to

$$\delta_{p,n^{-1}}\partial_m h_n - h_{p^{-1}}\Gamma^{p^{-1}}{}_{m,n} - h_a R_a(\Gamma^{a^{-1}}{}_{b,p})\sigma^{a,b}{}_{m,n} = 0 \tag{1.13}$$

In the edge-symmetric case, this becomes

$$\delta_{p,n^{-1}}\partial_m h_n - h_{p^{-1}}\Gamma^{p^{-1}}{}_{m,n} - R_a(h_{a^{-1}}\Gamma^{a^{-1}}{}_{b,p})\sigma^{a,b}{}_{m,n} = 0. \tag{1.14}$$

We now insert the form of Γ, σ to obtain the condition stated in the mutually exclusive cases $p = n^{-1}m^{-1}$ and $p \neq n^{-1}m^{-1}$ (where the terms shown do not contribute when $p = n^{-1}m^{-1}$ due to the conditions on τ and $e \notin C$, so we do not need to write that this p is excluded). $\square$

In the Cayley graph case of Lemma 1.6, there is a canonical Ω with e^a as Grassmann algebra generators and with a canonical i in [**14**, Lem. 8.18], which for an abelian group is just

$$i(e^a \wedge e^b) = \frac{1}{2}(e^a \otimes e^b - e^b \otimes e^a) \tag{1.15}$$

extended as a bimodule map. The rest of the geometric structures are the same as developed before.

CHAPTER 2

QRG of Polygons and Application to Cosmological Models and Quantum Gravity

The results of the first two sections of this chapter were published in [**2**] and the ones of the last section were published in [**3**].

The standard concepts of General relativity and cosmology as Einstein equations, dust pressure, stress-energy tensor, etc., can be consulted in [**23**]. We refer to the specific material when it is considered adequate.

1. Quantization of $\mathbb{Z}_n$

Here we consider the general theory of chapter 1 for the case of an n-gon for $n \geq 3$. A metric is a free assignment of a 'square-length' to each edge and Section 1.1 solves the quantum Riemannian geometry to find the quantum Levi-Civita connection. Section 5 then constructs Euclidean quantum gravity on the polygon.

1.1. Quantum Riemannian geometry on $\mathbb{Z}_n$. Just as it is useful in classical geometry to use local coordinates where the differential structure is the standard one for $\mathbb{R}^n$, it is similarly useful to regard the n-gon as the group $\mathbb{G} = \mathbb{Z}_n$ for its differential structure as explained in Section 1. Here the calculus $\Omega^1(\mathbb{Z}_n)$ for generators $C = \{1, -1\}$ has corresponding left-invariant basis e^+, e^- given by

$$e^+ = \sum_{i=0}^{n-1} \omega_{i\to i+1}; \quad e^- = \sum_{i=0}^{n-1} \omega_{i\to i-1}, \tag{2.1}$$

where $i \in \mathbb{Z}_n$ runs over the vertices.

Since the $e^\pm$ are a basis over the algebra, a bimodule invertible quantum metric must take the central form

$$g = ae^+ \otimes e^- + be^- \otimes e^+ \tag{2.2}$$

for non-vanishing functions $a, b \in \mathbb{R}(\mathbb{Z}_n)$, with inverse metric

$$(e^+, e^+) = (e^-, e^-) = 0, \quad (e^+, e^-) = 1/R_+(b), \quad (e^-, e^+) = 1/R_-(a). \tag{2.3}$$

We write $R_\pm = R_{\pm 1}$ for the shift operators. We also have an inner element $\Theta = e^+ + e^-$ and the canonical $*$-structure $(e^+)^* = -e^-; (e^-)^* = -e^+$. On the other hand, from the graph perspective, the relevant Cayley graph of $\mathbb{Z}_n$ with the above generators is a polygon

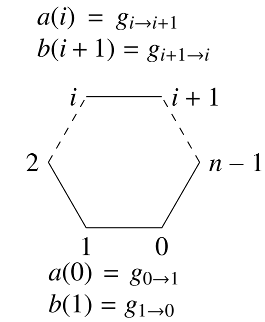

FIGURE 1. A quantum metric on $\mathbb{Z}_n$ is given by metric coefficient functions a, b or equivalently by directed edge weights $g_{i\to i\pm 1}$. Figure as in [**2**].

of n sides where the values of the functions a, b are directed edge weights according to Figure 1. From this, it is clear that the edge-symmetric case, where each side of the polygon has weight independent of the direction, requires $b = R_-a$. Proceeding in this case, the quantum metric is therefore

$$g = ae^+ \otimes e^- + R_-(a)e^- \otimes e^+, \quad (e^+, e^-) = \frac{1}{a}, \quad (e^-, e^+) = \frac{1}{R_-a} \tag{2.4}$$

as governed by one nonzero function a. For convenience, we define functions on $\mathbb{Z}_n$,

$$\rho = \frac{R_+(a)}{a}. \tag{2.5}$$

PROPOSITION 2.1. *For $n \geq 3$, there is a $*$-preserving QLC for any given edge-symmetric metric (2.4) on $\Omega^1(\mathbb{Z}_n)$. This is the unique for $n \neq 4$ and coincides with the restriction to periodic metrics mod n of the unique such connection on $\mathbb{Z}$ in* [**38**]*, namely*

$$\sigma(e^+ \otimes e^+) = \rho e^+ \otimes e^+, \quad \sigma(e^+ \otimes e^-) = e^- \otimes e^+,$$
$$\sigma(e^- \otimes e^+) = e^+ \otimes e^-, \quad \sigma(e^- \otimes e^-) = R_-^2\rho^{-1} e^- \otimes e^-$$

with the geometric structures

$$\nabla e^+ = (1 - \rho)e^+ \otimes e^+, \quad \nabla e^- = (1 - R_-^2\rho^{-1})e^- \otimes e^-,$$
$$R_\nabla e^+ = \partial_-\rho e^+ \wedge e^- \otimes e^+, \quad R_\nabla e^- = -\partial_+(R_-^2\rho^{-1})e^+ \wedge e^- \otimes e^-,$$
$$\mathrm{Ricci} = \frac{1}{2}\left(\partial_-(R_-\rho)e^- \otimes e^+ - \partial_-\rho^{-1} e^+ \otimes e^-\right),$$
$$S = \frac{1}{2}\left(-\frac{\partial_-\rho^{-1}}{a} + \frac{\partial_-(R_-\rho)}{R_-a}\right), \quad \Box f = -\frac{R_-\rho + 1}{a}(\partial_+ + \partial_-)f.$$

(For $n = 4$, there is a second $*$–preserving QLC given below.)

Proof. Due to the grading restrictions for a bimodule map, the most general σ for $n \neq 4$ has the form

$$\sigma(e^+\otimes e^+) = \sigma_0 e^+\otimes e^+,\quad \sigma(e^+\otimes e^-) = \sigma_1 e^+\otimes e^- + \sigma_2 e^-\otimes e^+,$$
$$\sigma(e^-\otimes e^+) = \sigma_3 e^+\otimes e^- + \sigma_4 e^-\otimes e^+,\quad \sigma(e^-\otimes e^-) = \sigma_5 e^-\otimes e^- \tag{2.6}$$

(where the σ_i are functional parameters) while for $n = 4$ we can have additional terms leading to another solution (given below). Similarly, for $n \neq 3$ we can only have the map $\alpha = 0$ while for $n = 3$ we may have additional terms leading to non $*$-preserving solutions in the Appendix. Taking the displayed main form of σ and $\alpha = 0$, torsion freeness $\wedge(\mathrm{id} + \sigma) = 0$ amounts to

$$\sigma_2 = \sigma_1 + 1,\quad \sigma_3 = \sigma_4 + 1, \tag{2.7}$$

while metric compatibility is

$$R_+(a) = aR_+(\sigma_3)\sigma_0,\quad a = aR_+(\sigma_4)\sigma_1 + R_-(a)R_-(\sigma_0)\sigma_3,$$
$$R_-(a) = aR_+(\sigma_5)\sigma_2 + R_-(a)R_-(\sigma_1)\sigma_4,\quad R_-^2(a) = R_-(a)R_-(\sigma_2)\sigma_5,$$
$$0 = aR_1(\sigma_5)\sigma_1 + R_-(a)R_-(\sigma_1)\sigma_3,\quad 0 = aR_+(\sigma_4)\sigma_2 + R_-(a)R_-(\sigma_0)\sigma_4. \tag{2.8}$$

It is then a matter of solving these, which was done using SAGE[**55**]. Among the solutions, we find a unique one that is $*$-preserving. The others are described for completeness in the Appendix. Note that the form of $\square$ in comparison to the usual lattice Laplacian makes it clear that a has units of length^2 [**38, 11**]. □

That the restriction of the unique $*$-preserving QLC on $\mathbb{Z}$ in [**38**] to periodic metrics gives a $*$-preserving QLC is not surprising, but that this gives all $*$-preserving solutions for $n \neq 4$ is a non-trivial result of solving the equations as described. For $n = 4$, similar methods lead to a further 2-parameter moduli of $*$-preserving connections of the form

$$\sigma(e^+\otimes e^+) = \gamma e^-\otimes e^-,\quad \sigma(e^+\otimes e^-) = -e^+\otimes e^-,$$
$$\sigma(e^-\otimes e^+) = -e^-\otimes e^+,\quad \sigma(e^-\otimes e^-) = \frac{R_+a}{R_-(a\gamma)}e^+\otimes e^+,$$

where $\gamma = (\gamma_0, \gamma_1, \bar\gamma_0^{-1}, \bar\gamma_1^{-1})$ specifies a function on the four points of $\mathbb{Z}_4$ (in order) in terms of two complex parameters γ_0, γ_1, such that $R_+^2\gamma = \bar\gamma^{-1}$. The associated quantum geometric structures are

$$\nabla e^+ = e^+\otimes e^+ + e^-\otimes e^+ + e^+\otimes e^- - \gamma e^-\otimes e^-,$$
$$\nabla e^- = e^-\otimes e^- + e^+\otimes e^- + e^-\otimes e^+ - re^+\otimes e^+,$$
$$R_\nabla e^+ = (R_- r - 1)\, e^+\wedge e^-\otimes e^+,\quad R_\nabla e^- = (1 - r)\, e^+\wedge e^-\otimes e^-,$$
$$\mathrm{Ricci} = \frac{1}{2}(R_+ r - 1)\, e^+\otimes e^- + \frac{1}{2}\left(R_+^2 r - 1\right) e^-\otimes e^+,$$

$$S = \frac{1}{2a}\Big((R_-\rho)(R_+^2 r-1)+R_+ r-1\Big),$$
$$\Box f = 0,$$

where we use the shorthand

$$r := \frac{R_+(a)}{R_-(a)}|\gamma|^2. \tag{2.9}$$

This is the $*$-preserving case of the general $n=4$ solution (i) in the Appendix.

1.2. The circle limit of the $\mathbb{Z}_n$ quantum geometry. We now turn to the matter of the classical limit $n\to\infty$ of the quantum geometry on $\mathbb{Z}_n$. Given that $\Omega^1(\mathbb{Z}_n)$ is 2-dimensional, we can not expect exactly a classical circle in the limit.

To put the quantum geometry in a more convenient form, we first use the Fourier transform to change a new variable s, where $s\in\mathbb{C}(\mathbb{Z}_n)$ is defined by

$$s(i) = q^i,\quad q = e^{\frac{2\pi\imath}{n}},\quad \mathbb{C}(\mathbb{Z}_n)\cong\mathbb{C}\mathbb{Z}_n := \mathbb{C}[s]/(s^n-1) \tag{2.10}$$

We will see in proposition 2.2 how the differential calculus is re-defined using these new variables. In this new description, our same algebra A is generated by s with the relation $s^n=1$. Also note that

$$\mathrm{d}s^{-1} = -s^{-1}(\mathrm{d}s)s^{-1} \tag{2.11}$$

depends on the commutation relations of $\mathrm{d}s$ with s. We thus define two left-invariant 1-forms

$$f^+ := s^{-1}\mathrm{d}s,\quad f^- := s\mathrm{d}s^{-1}. \tag{2.12}$$

For the $n\to\infty$ limit, we can now just drop the $s^n=1$ relation so that $A=\mathbb{C}[s,s^{-1}]$, the algebraic circle with $s^*=s^{-1}$. One can think of this as $s=e^{\imath\theta}$ in terms of an angle coordinate θ. Its *classical* differential calculus has $\mathrm{d}s$ central and hence one left-invariant 1-form $\bar f^+ = \imath\mathrm{d}\theta = -\bar f^-$, and the standard constant metric is

$$\mathrm{d}\theta\otimes\mathrm{d}\theta = -\bar f^+\otimes\bar f^+. \tag{2.13}$$

Comparing this metric with the one in proposition 2.2, it is clear that we are not in this classical case. We set $[m]_q := (1-q^m)/(1-q)$ as the usual q-deformed integer.

Proposition 2.2. *In these new coordinates, the $f^\pm$ form a Grassmann algebra and*

$$f^- s = -sf^+,\quad f^+s = s(f^-+(q+q^{-1})f^+),\quad \mathrm{d}s^m = -\frac{q[m]_q s^m}{(q+1)}\Big(q[-1-m]_q f^+ + [1-m]_q f^-\Big),$$

while the $*$-operation and the element that makes the calculus inner are

$$f^{\pm *} = -f^\pm,\quad \Theta = \frac{q}{(q-1)^2}\Theta_0;\quad \Theta_0 = f^+ + f^-$$

and the constant $a = 1$ metric $g = e^+ \otimes e^- + e^- \otimes e^+$ is

$$g = \frac{g_0}{(q-q^{-1})^2}; \quad g_0 = -2f^+ \otimes f^+ + \Theta_0 \otimes f^+ + f^+ \otimes \Theta_0 + \frac{2q}{(q-1)^2}\Theta_0 \otimes \Theta_0.$$

Moreover, the above does not require $s^n = 1$, i.e. applies equally well to the algebraic circle $\mathbb{C}[s, s^{-1}]$ with q a real or modulus 1 free parameter.

Proof. Working in our original calculus $\Omega(\mathbb{Z}_n)$ and s, q the function and the root of unity specified in (2.10), we compute that

$$f^- = s\mathrm{d}s^{-1} = (q^{-1} - 1)e^+ + (q-1)e^-, \quad f^+ = s^{-1}\mathrm{d}s = (q-1)e^+ + (q^{-1}-1)e^- \tag{2.14}$$

which inverts for $n > 2$ as

$$e^\pm = \frac{qf^\pm + f^\mp}{(q-q^{-1})(q-1)}. \tag{2.15}$$

As they are linear combinations, the $f^\pm$ are closed and form a Grassmann algebra since the $e^\pm$ do. We have $e^\pm s = R_\pm(s)e^\pm = q^{\pm 1}se^\pm$ which implies the relations shown for $f^\pm$. Finally, $\mathrm{d}s^m = (\partial_+ s^m)e^+ + (\partial_- s^m)e^- = (q^m - 1)e^+ + (q^{-m} - 1)e^-$ which translates to the formula shown in terms of $f^\pm$. The $*$ structure also matches but is in any case required by $f^{+*} = (s^{-1}\mathrm{d}s)^* = (\mathrm{d}s^{-1})s = s^{-1}f^- s = -f^+$ and similarly for f^-. We also have $\Theta = e^+ + e^-$ and g as stated when written as above in terms of $f^\pm$. The quantum Levi-Civita connection now appears equivalently as $\nabla f^\pm = 0$.

Moreover, these formulae do not directly reference n and one can check directly that they give a $*$-differential calculus even without the relation $s^n = 1$, i.e. on the algebraic circle. Now q is a free parameter but a check shows that we still need it real or modulus one for a $*$-calculus. □

The end result of Proposition 2.2 is a novel, 2-dimensional, q-deformed calculus on the algebraic circle. In the q real case, we can quotient it by a relation such as $f^+ = -qf^-$, which is equivalent to the relation $e^- = 0$ and gives the standard 1-dimensional q-deformed calculus on the circle [**14**, Ex 1.11] with $\mathrm{d}s.s = qs\mathrm{d}s$ or $\mathrm{d}s^m = [m]_q s^{m-1}\mathrm{d}s$. In this quotient, we would have $g = 0$ (this quotient calculus in fact admits no quantum metric due to the centrality requirement, making it unsuitable for our purposes).

Corollary 2.3. *In the limit $q \to 1$, the above q-deformed calculus on the circle algebra $\mathbb{C}[s, s^{-1}]$ tends to a noncommutative 2D calculus with*

$$f^- s = -sf^+, \quad f^+ s = s(f^- + 2f^+), \quad \mathrm{d}s^m = \frac{ms^m}{2}\left((m+1)f^+ + (m-1)f^-\right), \quad f^{\pm *} = -f^\pm$$

In this limit, the 1-form Θ_0 is closed and graded-central and the classical calculus on S^1 is then given by the quotient where we set $\Theta_0 = 0$. Conversely, this 2D calculus is a central extension in the sense of [**36, 7**] *of the classical calculus on S^1 by Θ_0.*

Proof. Most of this is immediate. For the last sentence, note that if f is a function of s then we can write the differential in the corollary equivalently as

$$\mathrm{d}f = s\frac{\mathrm{d}f}{\mathrm{d}s}f^+ + \frac{s^2}{2}\frac{\mathrm{d}^2 f}{\mathrm{d}s^2}\Theta_0, \tag{2.16}$$

where the first term is the expected left-invariant derivative associated to f^+ and the second is a higher-order derivative associated to an 'extra direction' Θ_0. This has the structure of a central extension of the classical calculus on S^1 in the sense of [**14**, Prop. 1.22][**36, 7**] according to the canonical Riemannian structure of S^1 and a second-order operator with respect to it. The central extension here is defined by a deformed $\bullet$ product where $s \bullet \bar{f}^+ = s\bar{f}^+$ is undeformed for left multiplication on the classical left-invariant 1-form $\bar{f}^+ = s^{-1}\mathrm{d}s = \imath \mathrm{d}\theta$. From the other side, we set $\bar{f}^+ \bullet s = \bar{f}^+ s + (\bar{f}^+, \mathrm{d}s)\Theta_0 = s\bar{f}^+ + s(\bar{f}^+, \bar{f}^+)\Theta_0 = s \bullet \bar{f}^+ + s\Theta_0$, which is the stated commutation relation for f^+ if we take the classical constant metric on S^1 with normalisation $(\bar{f}^+, \bar{f}^+) = 1$. As Θ_0 commutes with functions, this determines the correct commutation relation for f^- also. The second order operator defines d and here is $s^2\frac{\mathrm{d}^2}{\mathrm{d}s^2}$, which is the Laplacian plus a vector field as an example of the general set up [**14**, Thm 8.23][**36**]. □

Next, we note that the rescaled metric g_0 in Proposition 2.2 has a part with a $q \to 1$ limit plus a singular term proportional to $\Theta_0 \otimes \Theta_0$. Hence, if π_{class} denotes taking $q \to 1$ and simultaneously projecting to the classical calculus, we have

$$\pi_{class}(g_0) = -2\bar{f}^+ \otimes \bar{f}^+ = 2\mathrm{d}\theta \otimes \mathrm{d}\theta, \tag{2.17}$$

provided that in this process, the killing of Θ_0 takes precedence over setting $q \to 1$ in the singular term. It is not clear how to make this precise (one cannot simply set $\Theta_0 = 0$ first without destroying the structure of the q-deformed calculus). Aside from this technical detail, we still have the trivial flat QLC $\nabla f^\pm = 0$ and the projection is covariantly constant with respect to this and the usual classical connection. We have focussed here on the limit of the constant metric on $\mathbb{Z}_n$, but one can similarly analyse general metrics. Also, in the $q \to 1$ limit as in Corollary 2.3, one can directly analyse the possible generalised (not necessarily quantum-symmetric) metrics, e.g. the ones with constant coefficients have the form

$$g = \mathrm{Re}(z)(f^+ \otimes f^+ + f^- \otimes f^-) + zf^+ \otimes f^- + \bar{z}f^- \otimes f^+ \tag{2.18}$$

for a complex parameter z in order to be central and obey the reality condition. If we then impose quantum symmetry, we are forced to a real multiple of $\Theta_0 \otimes \Theta_0$, which is indeed the only component of the flat metric g if we fully scale out the singularity visible in Proposition 2.2 and then set $q \to 1$. This is a 'purely quantum' metric in the 2D calculus in Corollary 2.3, in that it projects to zero in the classical calculus on S^1.

We have already seen that the extra direction Θ_0 of the calculus in Corollary 2.3 arises as the residue of the element Θ that makes the q-deformed calculus on the circle inner, which is a purely quantum phenomenon. It can also be viewed as defining a central extension of the classical calculus on S^1 with the associated 'partial derivative' Θ_0, the second-order operator in (2.16). A third point of view is given by moving to 'cartesian coordinates'

$$x = \frac{1}{2}(s + s^{-1}), \quad y = \frac{1}{2\imath}(s - s^{-1}); \quad x^2 + y^2 = 1 \tag{2.19}$$

from which we compute

$$2(x\mathrm{d}x + y\mathrm{d}y) = s^{-1}\mathrm{d}s + s\mathrm{d}s^{-1} = \Theta_0. \tag{2.20}$$

Thus, Θ_0 can be thought of as something like the normal to the circle viewed in the plane, similar to the picture for the extra direction for the 3D calculus on the fuzzy sphere in [**31**].

Finally, in cohomological terms, one can check that the noncommutative de Rham cohomology ring $H_{\mathrm{dR}}(\mathbb{Z}_n)$ is the Grassmann algebra generated by $e^{\pm}$ i.e. dimensions 1 : 2 : 1 and spanned by $e^{\pm}$ in degree 1. The same is true in terms of the $f^{\pm}$ for finite n, which latter description holds also for $n \to \infty$; $H_{\mathrm{dR}}(\mathbb{Z})$ is generated by $f^{\pm}$ in the case of the corollary. This is the same as the cohomology of a torus, so it is tempting to think of the quantum geometry as a circle thickened into a torus, at least in a cohomological sense. The geometric picture, as we have seen, is a little like this with an extra direction related to the normal to the circle (rather than an actual torus).

1.3. Euclidean quantum gravity on $\mathbb{Z}_n$. As for the integer line graph [**38**], the two-dimensional cotangent bundle on $\mathbb{Z}_n$ required by the quantum geometry now admits the possibility of curvature. We envision that there could be various applications of such curved discrete geometries, but here we focus on just one, namely Euclidean quantum gravity on $\mathbb{Z}_n$. The approach used in this section is to construct the Hilbert-Einstein action, which is the integral of the Ricci scalar with a certain measure, that in the commutative case is $\sqrt{|\det g|}$. The action is quantized using the path integral approach, see [**59**].

In our case, the Hilbert-Einstein action is the sum over $\mathbb{Z}_n$ with the Ricci scalar given in proposition 2.1. In general, there is no way to choose a measure as in the commutative case, then we use a , which has the merit that produce the next action

$$S_g = aS = \frac{1}{2}\sum_{\mathbb{Z}_n}(R_-\rho\partial_-R_-\rho) = \frac{1}{2}\sum_{\mathbb{Z}_n}\rho\partial_-\rho = \frac{1}{2}\sum_{\mathbb{Z}_n}\rho\partial_+\rho = \frac{1}{4}\sum_{\mathbb{Z}_n}\rho(\partial_+ + \partial_-)\rho, \tag{2.21}$$

where $\partial_+ + \partial_-$ is the usual lattice double-differential on $\mathbb{Z}_n$ and ρ is defined in equation 2.5. This has the same form as for a scalar field except that ρ is a positive function, as already observed for $\mathbb{Z}$ in [**38**]. To quantize, i.e. solve the path integral, we consider two approaches, depending on what we regard as our underlying field, and in both cases maintaining $\mathbb{Z}_n$ symmetry in the result.

(i) **First Approach**. The action 2.21 only depends on ρ, which suggests we can take

$$\rho_0 = \frac{a(1)}{a(0)}, \quad \cdots, \quad \rho_{n-2} = \frac{a(n-1)}{a(n-2)}, \quad \rho_{n-1} = \frac{a(0)}{a(n-1)} \tag{2.22}$$

as n dynamical variables subject to the constraint $\rho_0 \cdots \rho_{n-1} = 1$. We think of the constraint as a hypersurface of positive real numbers in $\mathbb{R}^n_{>0}$, which induces a metric $\mathfrak{g}_\rho$ on the hypersurface, and we use the Riemannian measure in this. Thus, we can take $\rho_0, \cdots, \rho_{n-2}$ as local coordinates and measure $\mathcal{D}\rho = (\prod_{i=0}^{n-2} \mathrm{d}\rho_i)\sqrt{\det\big(\mathfrak{g}_\rho\big)}$ for the path integral. The measure here maintains the $\mathbb{Z}_n$ symmetry as ultimately independent of the choice of coordinates.

Explicitly, for $n = 3$, we take ρ_0, ρ_1 as coordinates and the constrained surface in $\mathbb{R}^3_{>0}$ is $\rho_2 = 1/(\rho_0\rho_1)$. The coordinate tangent vectors and induced metric are

$$v_0 = (1, 0, -\frac{1}{\rho_0^2\rho_1}), \quad v_1 = (0, 1, -\frac{1}{\rho_0\rho_1^2}); \tag{2.23}$$

$$\mathfrak{g}_\rho = (v_i \cdot v_j) = \begin{pmatrix} 1 + \frac{1}{\rho_0^4\rho_2} & \frac{1}{\rho_0^3\rho_1^3} \\ \frac{1}{\rho_0^3\rho_1^3} & 1 + \frac{1}{\rho_0^2\rho_1^4} \end{pmatrix}, \quad \det\big(\mathfrak{g}_\rho\big) = 1 + \frac{1}{\rho_0^4\rho_1^2} + \frac{1}{\rho_0^2\rho_1^4}. \tag{2.24}$$

Hence the partition function is

$$Z = \int_0^\infty \mathrm{d}\rho_0 \int_0^\infty \mathrm{d}\rho_1 \sqrt{\det\big(\mathfrak{g}_\rho\big)}\, e^{-\frac{1}{2G}(\rho_0^2+\rho_1^2+\rho_2^2-\rho_0\rho_1-\rho_1\rho_2-\rho_2\rho_0)}; \quad \rho_2 := \frac{1}{\rho_0\rho_1} \tag{2.25}$$

These integrals can be done numerically and appear to converge for all values $G > 0$ of the coupling constant (the numerical results need G not too small for working precision but this case can be analysed separately). We are interested in expectation values $\langle \rho_{i_1} \cdots \rho_{i_m} \rangle$, where we insert $\rho_{i_1} \cdots \rho_{i_m}$ in the integrand and take the ratio with Z.

Some results obtained from this theory for $n = 3$ are plotted in Figure 2. Numerical evidence is limited due to numerical convergence accuracy issues, but it seems clear that expectation values of products of ρ_i tend to 1 and hence $\Delta\rho_i = (\rho_{i+1} - \rho_i) \to 0$ as $G \to 0$, as might be expected. As in [**11**], this should be thought of as the weak gravity limit given that fluctuations expressed in ρ enter the action relative to G. Meanwhile, it appears as $G \to \infty$ that

$$\frac{\Delta\rho_i}{\langle \rho_i \rangle} \sim 1.11, \quad \frac{\langle \rho_i^2 \rangle}{\langle \rho_i \rangle^2} \sim 2.23, \quad \frac{\langle \rho_i \rho_j \rangle}{\langle \rho_i \rangle \langle \rho_j \rangle} \sim 0.845 \tag{2.26}$$

for $i \neq j$. The asymptotic values here are from plotting out to $G = 500$, but would need to be confirmed analytically due to potential numerical convergence issues. The first of these limits, if confirmed, would be a similar phenomenon of the uniform relative metric uncertainty in [**11**] in the 'strong gravity' limit. The correlations are real and relative correlation between two distinct vertices of the triangle is lower than the relative self-correlation, which is in line with the $n = 3$ case of the relative quantisation in Figure 3.

(ii) **Second approach**. We can take (as more usual) the metric coefficients as the underlying field, so in our case the edge 'square-lengths' $a = (a_0, \cdots, a_{n-1}) \in \mathbb{R}^n_{>0}$. Assuming

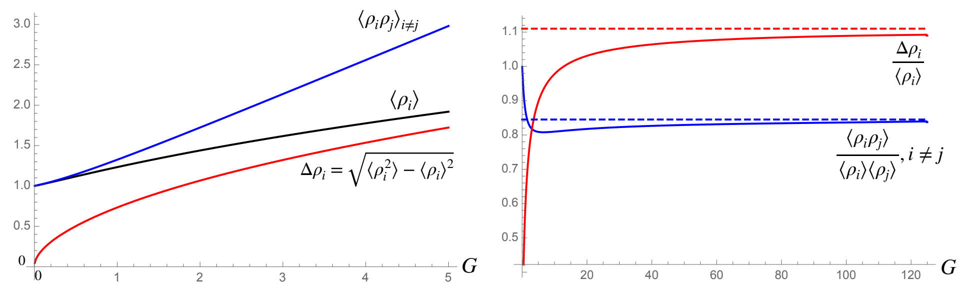

FIGURE 2. Euclidean quantum gravity on $\mathbb{Z}_3$ for variables ρ. Figure as in [**2**]

Lebesgue measure, the partition function is

$$Z = \int_0^\infty (\prod_i \mathrm{d}a_i) e^{\frac{S_g}{G}} = \int_0^L (\prod_i \mathrm{d}a_i) e^{\frac{1}{2G}\sum_{\mathbb{Z}_n}\rho\partial_+\rho} \tag{2.27}$$

and we introduce a field strength upper bound L to control divergences as in [**11**]. One can then look at ratios independent of L or indeed consider a formal renormalisation process.

On the other hand, the divergences of the path integral come from the global scaling symmetry $a_i \mapsto \lambda a_i$ for $\lambda \in \mathbb{R}_{>0}$ of the action (since this depends only on the ratios ρ) and therefore another approach would be to 'factor out' the geometric mean as a new variable which we do not integrate over, keeping only the ratios relative to this as the dynamic degrees of freedom. This is again in the spirit of [**11**], except that we proceed multiplicatively. Thus, we let $A = (\prod_i a_i)^{\frac{1}{n}}$ be the geometric mean and $b_i := a_i/A$, which by construction obeys $b_0 \cdots b_{n-1} = 1$. These are similar to the ρ_i variables in forming the corresponding hypersurface in $\mathbb{R}^n_{>0}$, but the action is different and the measure is also different since it is inherited from the Lebesgue measure on the a_i.

Again, we will look at this explicitly for $n = 3$. Then the action is

$$S_g = \frac{1}{2}\left(\frac{b_0}{b_1} + \frac{b_1}{b_2} + \frac{b_2}{b_0} - (\frac{b_1}{b_0})^2 - (\frac{b_2}{b_1})^2 - (\frac{b_0}{b_2})^2\right); \quad b_2 = \frac{1}{b_0 b_1}, \tag{2.28}$$

while the Jacobean for the change of variables from a_0, a_1, a_2 to b_0, b_1, A gives us

$$\mathrm{d}a_0\,\mathrm{d}a_1\,\mathrm{d}a_2 = \frac{3A^2}{b_0 b_1}\mathrm{d}b_0\,\mathrm{d}b_1\,\mathrm{d}A. \tag{2.29}$$

Omitting the now decoupled integration over A as an (infinite) constant, we have effectively

$$Z = \int_0^\infty \mathrm{d}b_0 \int_0^\infty \mathrm{d}b_1 \frac{1}{b_0 b_1} e^{\frac{1}{2Gb_0^2 b_1^4}(-1+(1+b_0^3)b_1^3+(-1+b_0^3-b_0^6)b_1^6)}. \tag{2.30}$$

The graphical expectation values against G look qualitatively similar to those of ρ_i in Figure 2, but one also has $\langle b_i\rangle = \langle b_i b_j\rangle$ for $i \neq j$, albeit this is specific to $n = 3$.

Larger $n > 3$ can proceed entirely similarly and one has $1 < \langle b_i\rangle < \langle b_i b_{i+1}\rangle$. One can also then see that the i-step correlations $\langle b_0 b_i\rangle$ (or between any two points differing by i)

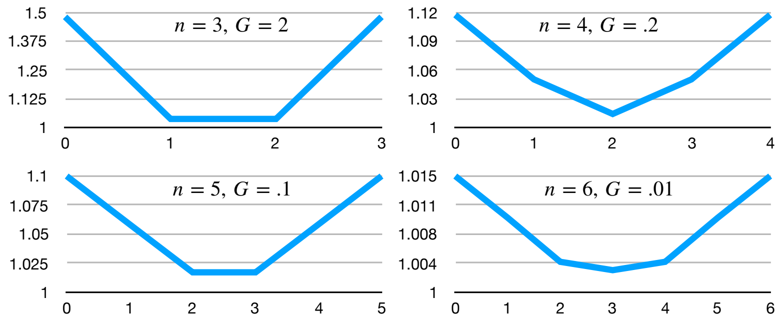

FIGURE 3. Euclidean quantum gravity correlations $\langle b_0 b_i\rangle$ plotted against i for $3 \le n \le 6$ and suitable G. Figure as in [**2**]

decrease as i increases from $i = 0$ to reach a minimum (as expected) half way around the polygon. This is based on numerical data for small n as shown in Figure 3. The data for $n = 6$ are already noisy due to numerical convergence issues but suggest that for large n the $\langle b_0 b_i\rangle$ may be approximated by $\alpha - \beta\sin\left(\frac{\pi i}{n}\right)$ for positive $\alpha > \beta$ depending on G and n. This is broadly similar to the form of correlation functions for a scalar field $\langle\phi_0\phi_i\rangle$ in a lattice box in [**38**], but without the overall $\imath$ there.

The results in Figure 3, are somewhat similar to correlations for a scalar field lattice box in [**38**], but now in a real positive version, which both reassures us that the model is giving reasonable answers and gives a flavour of what to expect for quantum gravity in our approach. Clearly, more baby models should be computed to develop our intuition further. As discussed in [**11**], our approach is not immediately comparable with other computable approaches such as [**60, 4, 61, 62**].

2. Quantum geometric cosmological models on $\mathbb{R}\times\mathbb{Z}_n$

In this section, we first start with an analysis of quantum metrics and QLCs on $\mathbb{R}\times\mathbb{Z}_n$, where $\mathbb{R}$ is a classical time and $\mathbb{Z}_n$ is the background space given for the polygon geometry. We find that the full 'strongly tensorial' bimodule properties for an invertible quantum metric force us to the block diagonal case, without taking this as an assumption. The existence of a QLC further dictates its form, again without taking this as an assumption, and we then find a unique $*$-preserving one. We then focus on the case where the $\mathbb{Z}_n$ geometry is flat (modeling an actual geometric circle) but possibly time-dependent as in FLRW cosmology.

2.1. Quantum metric and QLC on $\mathbb{R}\times\mathbb{Z}_n$. We consider a general metric on the product $\mathbb{R}\times\mathbb{G}$ where $\mathbb{R}$ has a variable t and we are interested in the finite group $\mathbb{G} = \mathbb{Z}_n$ with $e^a = e^\pm$, but we do not need to specialize at this stage. We consider the most general metric, which has the form

$$g=\mu \mathrm{d}t\otimes \mathrm{d}t+h_{ab}e^a\otimes e^b+n_a(e^a\otimes \mathrm{d}t+\mathrm{d}t\otimes e^a) \tag{2.31}$$

for μ, h_{ab}, n_a in $A=\mathbb{C}^\infty(\mathbb{R})\otimes\mathbb{C}(\mathbb{G})$ but note right away that if we take the tensor product calculus where the continuous variable and its differential $t, \mathrm{d}t$ graded commute with functions and forms on $\mathbb{G}$ then centrality of the metric needed for a bimodule inverse dictates that $n_a=0$. To see this is enough notice

$$(\mathrm{d}t\otimes e^a).f(i)=f(i+a)\mathrm{d}t\otimes e^a\neq f(i).(\mathrm{d}t\otimes e^a)=f(i)\mathrm{d}t\otimes e^a,$$

and similarly for $(e^a\otimes \mathrm{d}t)$. In general, the term $\mathrm{d}t\otimes e^a$ is different of zero reason why $n_a=0$. We therefore proceed in this case.

Similarly, we look for general the most general QLCs

$$\nabla \mathrm{d}t=-\Gamma \mathrm{d}t\otimes \mathrm{d}t+c_a(e^a\otimes \mathrm{d}t+\mathrm{d}t\otimes e^a)+d_{ab}e^a\otimes e^b, \tag{2.32}$$

$$\nabla e^a=-\Gamma^a{}_{bc}e^b\otimes e^c+\gamma^a{}_b(e^b\otimes \mathrm{d}t+\mathrm{d}t\otimes e^b)+f^a\mathrm{d}t\otimes \mathrm{d}t \tag{2.33}$$

and note that for the tensor form of calculus along with the natural choice where $\sigma(\mathrm{d}t\otimes\ ), \sigma(\ \otimes \mathrm{d}t)$ are the flip on the basic 1-forms $\mathrm{d}t, e^a$, requiring the above to be a bimodule connection compatible with the relations of each algebra, like Leibniz rule and $\nabla(e^\pm f)=\nabla(R_\pm(f)e^\pm)$, forces us to

$$c_a=0,\quad f^a=0,\quad \gamma^a{}_b=\gamma_a\delta_{a,b},\quad d_{a,b}=d_a\delta_{a,b^{-1}} \tag{2.34}$$

for some functions γ_a. We therefore proceed in this case.

Next, for zero torsion, we need that

$$d_{ab}=d_{ba},\quad \Gamma^a{}_{bc}=\Gamma^a{}_{cb},\quad \wedge(\mathrm{id}+\sigma)(e^a\otimes e^b)=0 \tag{2.35}$$

(which means σ restricted to the $\{e^a\}$ has the form studied before for a torsion-free bimodule connection on an inner calculus, but note the calculus as a whole is *not* inner). And for $\nabla g=0$, we obtain 8 equations which we compute under our assumptions above for a central metric and bimodule connection, with $\dot{\mu} = \frac{\partial}{\partial t}\mu$,

$$\mathrm{d}t^{\otimes 3}: \quad \frac{\dot{\mu}}{2} - \mu\Gamma = 0, \tag{2.36}$$

$$\mathrm{d}t \otimes \mathrm{d}t \otimes e^a: \quad 0 = 0, \tag{2.37}$$

$$\mathrm{d}t \otimes e^a \otimes \mathrm{d}t: \quad 0 = 0, \tag{2.38}$$

$$e^a \otimes \mathrm{d}t \otimes \mathrm{d}t: \quad \partial_a \mu = 0, \tag{2.39}$$

$$\mathrm{d}t \otimes e^a \otimes e^b: \quad h_{cb}\gamma^c{}_a + h_{ac}R_a(\gamma^c{}_b) + \dot{h}_{ab} = 0, \tag{2.40}$$

$$e^a \otimes \mathrm{d}t \otimes e^b: \quad h_{cb}\gamma^c{}_a + \mu d_{ab} = 0, \tag{2.41}$$

$$e^a \otimes e^b \otimes \mathrm{d}t: \quad \mu d_{ab} + h_{mp}R_m(\gamma^p{}_n)\sigma^{mn}{}_{ab} = 0, \tag{2.42}$$

$$e^m \otimes e^n \otimes e^p: \quad \partial_m h_{np} - h_{ap}\Gamma^a{}_{mn} - h_{ac}R_a(\Gamma^c{}_{bp})\sigma^{ab}{}_{mn} = 0. \tag{2.43}$$

The first and last of the 8 equations are just that Γ is a QLC on the line and $\sigma, \Gamma^a{}_{bc}$ a QLC on $\mathbb{G}$. The 4th equation tells us that μ is constant on $\mathbb{G}$. If we write the metric as $h_{ab} = h_a\delta_{a,b^{-1}}$ for functions h_a etc., then the 6th equation tells us

$$d_a = -\frac{h_a\gamma_a}{\mu} \tag{2.44}$$

and the 5th and 7th equations reduce to

$$\dot{h}_a + h_a\gamma_a + R_a(h_{a^{-1}}\gamma_{a^{-1}}) = 0, \quad \sum_p R_{p^{-1}}(h_p\gamma_p)\sigma^{p^{-1},p}{}_{a,b} = h_a\gamma_a\delta_{a,b^{-1}}. \tag{2.45}$$

Finally, we impose $*$-structure $\mathrm{d}t^* = \mathrm{d}t$ and suppose that the connection on $\mathbb{G}$ is also $*$-preserving for $e^{a*} = -e^{a^{-1}}$ as usual. The extended metric then obeys the quantum reality condition if μ is real, which we suppose henceforth, and the metric on $\mathbb{G}$ is 'real' in the required sense (which amounts to h_a real-valued). Then the additional condition for our extended ∇ to be $*$-preserving comes down to Γ real and

$$\bar{\gamma}_a = R_a\gamma_{a^{-1}}, \quad \sum_a \bar{d}_a\sigma(e^a \otimes e^{a^{-1}}) = \sum_a d_{a^{-1}}e^{a^{-1}} \otimes e^a, \tag{2.46}$$

where the 1st part comes from ∇e^{a*} and the 2nd from $\nabla \mathrm{d}t^*$. Next, we use (2.44) and that h_a are real and edge-symmetric to deduce from the 1st part that $\bar{d}_a = R_a d_{a^{-1}}$. Then since d_a are constant on $\mathbb{G}$, we have $\bar{d}_a = d_{a^{-1}}$ and our condition to be $*$-preserving is

$$\bar{\gamma}_a = R_a\gamma_{a^{-1}}, \quad \sum_a d_{a^{-1}}(\sigma(e^a \otimes e^{a^{-1}}) - e^{a^{-1}} \otimes e^a) = 0. \tag{2.47}$$

Since μ has to be a constant on $\mathbb{G}$, it is some function of t alone. Generically, we can absorb this in a change of the variable t, so we proceed for simplicity with $\mu = -1$ for a FLRW type solution (see [**23**, Chap. 8.2]). In the next theorem, the quantities with a super-index $^{\mathbb{Z}_n}$ denote quantities of the polygon geometry.

THEOREM 2.4. *For $\sigma, \nabla^{\mathbb{Z}_n}$ the $*$-preserving QLC on $\mathbb{Z}_n$ in Proposition 2.1, a quantum metric on $\mathbb{R}\times\mathbb{Z}_n$ admitting a $*$-preserving QLC has the form*

$$g = -\mathrm{d}t\otimes\mathrm{d}t - ae^+\otimes e^- - R_-ae^-\otimes e^+$$

up to a choice of the t parametrization, such that $\partial_-\dot a = 0$, i.e., a has the form

$$a(t,i) = \alpha(t) + \beta(i)$$

for some functions α,β with $\sum_i\beta(i) = 0$. In these terms, there is a unique $*$-preserving QLC with scalar curvature and Laplacian

$$\begin{aligned}2S =& -\ddot\alpha\left(\frac{1}{\alpha+\beta}+\frac{1}{\alpha+R_-\beta}\right)+\frac{\dot\alpha^2}{4}\left(\frac{1}{(\alpha+\beta)^2}+\frac{1}{(\alpha+R_-\beta)^2}\right)\\&+\frac{s}{(\alpha+\beta)^2(\alpha+R_+\beta)}+R_-\left(\frac{s}{(\alpha+\beta)^2(\alpha+R_-\beta)}\right),\\ \Box f =& -\partial_t^2+\left(\frac{1}{\alpha+\beta}+\frac{1}{\alpha+R_-\beta}\right)(-\frac{\dot\alpha}{2}\partial_t f+\Box_{\mathbb{Z}_n}f),\end{aligned}$$

where

$$s := (\alpha+R_+\beta)(\alpha+R_-\beta)-(\alpha+\beta)^2 = \alpha(\Box_{\mathbb{Z}_n}\beta)+(\partial_+\beta)\partial_-\beta-\beta^2$$

in terms of the usual Laplacian $\Box_{\mathbb{Z}_n}\beta = (\partial_+ + \partial_-)\beta = R_+\beta + R_-\beta - 2\beta$ on $\mathbb{Z}_n$.

PROOF. We use the general analysis above applied in the specific case of $\mathbb{Z}_n$. Also, for the purpose of the proof, it is convenient to have a shorthand notation $a_+ = a$ and $a_- = R_-a$, so that $h_\pm = a_\pm$ for our particular metric. Then the 2nd of (2.45) holds automatically as $\sigma(e^\pm\otimes e^\mp) = e^\mp\otimes e^\pm$ and $a_\pm\gamma_\pm = d_\pm(t)$ are constants on $\mathbb{Z}_n$ for a solution, while the 1st of (2.45) is that $\dot a_\pm = -d_+ - d_-$, which requires $\partial_-\dot a = 0$ as stated. We assume the QLC on $\mathbb{Z}_n$ at each t for the metric functions $a = a(t,i)$. The flip form of $\sigma(e^\pm\otimes e^\mp)$ for this also means that the 2nd part of (2.47) is automatic and we just need $\bar\gamma_\pm = R_\pm\gamma_\mp$, or equivalently $\bar d_\pm = d_\mp$, for a $*$-preserving connection. This means that

$$d_+ = -\frac{\dot a}{2}+\imath b,\quad d_- = \bar d_+ = -\frac{\dot a}{2}-\imath b;\quad \gamma_\pm = -\frac{\dot a}{2a_\pm}\pm\frac{\imath b}{a_\pm} \tag{2.48}$$

for any real-valued function $b(t)$. The unique solution with real coefficients for ∇ in our basis is $b = 0$ and gives the $*$-preserving QLC

$$\nabla\mathrm{d}t = \frac{\dot a}{2}(e^+\otimes e^- + e^-\otimes e^+),\quad \nabla e^\pm = \nabla^{\mathbb{Z}_n}e^\pm - \frac{\dot a}{2a_\pm}(e^\pm\otimes\mathrm{d}t+\mathrm{d}t\otimes e^\pm). \tag{2.49}$$

The σ for this when one argument is dt is the flip. We then proceed to compute the curvature of this QLC,

$$
\begin{aligned}
R_\nabla e^\pm &= R_\nabla^{\mathbb{Z}_n} e^\pm - \left(\dot{\Gamma}^\pm{}_{ab} - \Gamma^\pm{}_{ab} R_a(\frac{\dot{a}}{2a_b}) + \frac{\dot{a}}{2a_\pm}\Gamma^\pm{}_{ab}\right) \mathrm{d}t \wedge e^a \otimes e^b - \Gamma^\pm{}_{ab} R_a(\frac{\dot{a}}{2a_b}) e^a \wedge e^b \otimes \mathrm{d}t \\
&\quad \pm (\frac{\dot{a}}{2a_\pm})^2 a_\pm e^+ \wedge e^- \otimes e^\pm - \frac{\dot{a}}{2}\partial_b(\frac{1}{a_\pm}) e^b \wedge e^\pm \otimes \mathrm{d}t + \frac{\dot{a}}{2}\partial_b(\frac{1}{a_\pm}) \mathrm{d}t \wedge e^b \otimes e^\pm \\
&\quad - \left(\frac{\partial}{\partial t}(\frac{\dot{a}}{2a_\pm}) + (\frac{\dot{a}}{2a_\pm})^2\right) \mathrm{d}t \wedge e^\pm \otimes \mathrm{d}t,
\end{aligned}
$$

$$
\begin{aligned}
R_\nabla \mathrm{d}t &= \frac{\ddot{a}}{2}\mathrm{d}t \wedge (e^+ \otimes e^- + e^- \otimes e^+) + \frac{\dot{a}}{2} e^+ \wedge \Gamma^-{}_{-b} e^- \otimes e^b + \frac{\dot{a}}{2} e^- \wedge \Gamma^+{}_{+b} e^+ \otimes e^b \\
&\quad + \sum_\pm (\frac{\dot{a}}{2a_\pm})^2 a_\pm e^\pm \wedge (e^\mp \otimes \mathrm{d}t + \mathrm{d}t \otimes e^\mp),
\end{aligned}
$$

in terms of the Christoffel symbols on $\mathbb{Z}_n$. The Ricci tensor and the Ricci scalar S are then

$$
\begin{aligned}
\mathrm{Ricci} &= \mathrm{Ricci}^{\mathbb{Z}_n} + \frac{\ddot{a}}{4}(e^+ \otimes e^- + e^- \otimes e^+) + \frac{1}{2}\left(R_+(\dot{\Gamma}^-_{--}) - \frac{\dot{a}}{2}(R_+(\Gamma^-_{--}) + 1)\partial_-\left(\frac{1}{a}\right)\right) \mathrm{d}t \otimes e^- \\
&\quad + \frac{1}{2}\left(R_-(\dot{\Gamma}^+_{++}) - \frac{\dot{a}}{2}(R_-(\Gamma^+_{++}) + 1)\partial_+\left(\frac{1}{a_-}\right)\right) \mathrm{d}t \otimes e^+ + \frac{\dot{a}}{4}\left((R_-(\Gamma^+_{+-}) + 1)\partial_-\left(\frac{1}{a_-}\right)\right) e^- \otimes \mathrm{d}t \\
&\quad - \frac{\dot{a}}{4}\left((R_+(\Gamma^-_{+-}) + 1)\partial_-\left(\frac{1}{R_+(a)}\right)\right) e^+ \otimes \mathrm{d}t + \frac{1}{2}\left(\partial_t\left(\frac{\dot{a}}{2a} + \frac{\dot{a}}{2a_-}\right) + \left(\frac{\dot{a}}{2a}\right)^2 + \left(\frac{\dot{a}}{2a_-}\right)^2\right) \mathrm{d}t \otimes \mathrm{d}t, \\
S &= -S^{\mathbb{Z}_n} - \frac{\ddot{a}}{2}\left(\frac{1}{a} + \frac{1}{a_-}\right) + \frac{1}{2}\left(\frac{\dot{a}}{2a}\right)^2 + \frac{1}{2}\left(\frac{\dot{a}}{2a_-}\right)^2
\end{aligned}
$$

(where we have used that $\Gamma^\pm_{+-} = \Gamma^\pm_{-+}$). We now insert values for the QLC in Proposition 2.1 to obtain

$$
\begin{aligned}
R_\nabla e^\pm &= \pm\left(-\partial_\pm\left(\frac{a_\pm}{a_\mp}\right) + \left(\frac{\dot{a}}{2a_\pm}\right)^2 a_\pm\right) e^+ \wedge e^- \otimes e^\pm + \frac{\dot{a}}{2a_\pm^2}\partial_\pm(a_\pm)\,\mathrm{d}t \wedge e^\pm \otimes e^\pm \\
&\quad + \frac{\dot{a}}{2}\partial_\mp\left(\frac{1}{a_\pm}\right)(e^\pm \wedge e^\mp \otimes \mathrm{d}t + \mathrm{d}t \wedge e^\mp \otimes e^\pm) \\
&\quad + \left(-\frac{\ddot{a}}{2a_\pm} + \left(\frac{\dot{a}}{2a_\pm}\right)^2\right) \mathrm{d}t \wedge e^\pm \otimes \mathrm{d}t,
\end{aligned}
\tag{2.50}
$$

$$
\begin{aligned}
R_\nabla \mathrm{d}t &= \sum_\pm\left(\frac{\ddot{a}}{2a_\pm} - \left(\frac{\dot{a}}{2a_\pm}\right)^2\right) a_\pm \mathrm{d}t \wedge e^\pm \otimes e^\mp + \sum_\pm \frac{\dot{a}}{2a_\pm}\partial_-(a) e^+ \wedge e^- \otimes e^\mp \\
&\quad + \frac{\dot{a}^2}{4}\partial_-\left(\frac{1}{a^2}\right) e^+ \wedge e^- \otimes \mathrm{d}t
\end{aligned}
\tag{2.51}
$$

and as a result,

$$\mathrm{Ricci} = \frac{1}{2}\sum_{\pm}\Big((\frac{\ddot a}{2}+\partial_\pm(\frac{a_\mp}{a_\pm}))e^\pm\otimes e^\mp - \frac{\dot a}{2a_\mp^2}\partial_\pm(a_\mp)\mathrm{d}t\otimes e^\pm + \frac{\dot a}{2}\partial_\pm(\frac{1}{a_\pm})e^\pm\otimes \mathrm{d}t\Big)$$
$$-\frac{1}{2}\left(-\frac{\ddot a}{2}\left(\frac{1}{a}+\frac{1}{a_-}\right)+\left(\frac{\dot a}{2a}\right)^2+\left(\frac{\dot a}{2a_-}\right)^2\right)\mathrm{d}t\otimes\mathrm{d}t, \tag{2.52}$$

$$S = \frac{1}{2}\left(-\ddot a\left(\frac{1}{a}+\frac{1}{a_-}\right)+\left(\frac{\dot a}{2a}\right)^2+\left(\frac{\dot a}{2a_-}\right)^2-\frac{1}{a}\partial_+\left(\frac{a_-}{a}\right)-\frac{1}{a_-}\partial_-\left(\frac{a}{a_-}\right)\right). \tag{2.53}$$

We now note that the requirement $\partial_-\dot a = 0$ is equivalent to a being of the form stated. Clearly, such a form obeys this condition as $\dot a = \alpha$ is constant on $\mathbb{Z}_n$. Conversely, given $a(t,i)$ obeying the condition, we let $\alpha(t) = \frac{1}{n}\sum_i a(t,i)$ be the average value and $\beta = a-\alpha$. The latter averages to zero and has zero time derivative by the assumption on a, hence depends only on i. We now insert this specific form into the curvature calculations to obtain

$$\mathrm{Ricci} = \left(\frac{\ddot\alpha}{4}-\frac{s}{(\alpha+\beta)(\alpha+R_+\beta)}\right)e^+\otimes e^- + \left(\frac{\ddot\alpha}{4}-R_-\left(\frac{s}{(\alpha+\beta)(\alpha+R_-\beta)}\right)\right)e^-\otimes e^+$$
$$-\frac{\dot\alpha}{4}R_-\left(\frac{\partial_+\beta}{(\alpha+\beta)^2}\right)\mathrm{d}t\otimes e^+ - \frac{\partial_+\beta}{(\alpha+\beta)(\alpha+R_+\beta)}e^+\otimes\mathrm{d}t$$
$$-\frac{\dot\alpha}{4}\frac{\partial_-\beta}{(\alpha+\beta)^2}\mathrm{d}t\otimes e^- - R_-\left(\frac{\partial_-\beta}{(\alpha+\beta)(\alpha+R_-\beta)}\right)e^-\otimes\mathrm{d}t$$
$$+\left(\frac{\ddot\alpha}{4}\left(\frac{2\alpha+\beta+R_-\beta}{(\alpha+\beta)(\alpha+R_-\beta)}\right)+\frac{\dot\alpha^2}{4}\left(\frac{(\alpha+\beta+R_-\beta)^2-(\alpha^2+2\beta R_-\beta)}{(\alpha+\beta)^2(\alpha+R_-\beta)^2}\right)\right)\mathrm{d}t\otimes\mathrm{d}t \tag{2.54}$$

and the scalar curvature as stated. Without loss of generality, we have fixed $\sum_i\beta(i) = 0$ since this could be shifted into the value of α. We also have the geometric Laplacian

$$\square f = -\square^{\mathbb{Z}_n}f-\left(\frac{1}{a}+\frac{1}{a_-}\right)\frac{\dot a}{2}\partial_t f-\partial_t^2 f = -\left(\frac{1}{a}+\frac{1}{a_-}\right)(\frac{\dot a}{2}\partial_t f-\square_{\mathbb{Z}_n}f)-\partial_t^2 f, \tag{2.55}$$

which simplifies as stated. We are using $\square^{\mathbb{Z}_n}$ for the Laplacian in Proposition 2.1 and $\square_{\mathbb{Z}_n}$ with lower label for the standard finite difference Laplacian. □

The coefficient $a(t,i) = \alpha(t)+\beta(i)$ usually plays the role of the 'radius' that depends on time. However, in this theorem, $\alpha(t) > 0$ is the average 'radius' of the $\mathbb{Z}_n$ geometry, evolving with time, while $\beta(i)$ as a fluctuation as we go around $\mathbb{Z}_n$ and we see that this has to be 'frozen' (does not depend on time) in order for the metric to admit a quantum geometry. It is striking that this includes the FLRW-type models studied in the remaining section in the class forced by the quantum geometry. Note that we also need to restrict to

$$\min_i\beta(i) > -\inf_t\alpha(t) \tag{2.56}$$

so that $a(t,i)$ is everywhere positive.

2.2. Equations of state in FLRW model on $\mathbb{R}\times\mathbb{Z}_n$. Here, we focus on the cosmological FLRW model case. For details of the standard FLRW model see [**23**, Chap. 8]. We use the result of the theorem 2.4 where $a = R^2(t)$ with no fluctuation $\beta(i)$ over $\mathbb{Z}_n$ and hence

$$g = -\mathrm{d}t\otimes\mathrm{d}t - R^2(t)e^+\otimes_s e^-, \tag{2.57}$$

where $e^+\otimes_s e^- = e^+\otimes e^- + e^-\otimes e^+$. In this case, the results above simplify to

$$\nabla\mathrm{d}t = R\dot{R}e^+\otimes_s e^-,\quad \nabla e^\pm = -\frac{\dot{R}}{R}e^\pm\otimes_s \mathrm{d}t, \tag{2.58}$$

$$R_\nabla e^\pm = -\frac{\ddot{R}}{R}\mathrm{d}t\wedge e^\pm\otimes\mathrm{d}t \pm \left(\frac{\dot{R}}{R}\right)^2 R^2 e^+\wedge e^-\otimes e^\pm,\quad R_\nabla\mathrm{d}t = \ddot{R}R\mathrm{d}t\wedge e^+\otimes_s e^-, \tag{2.59}$$

$$\mathrm{Ricci} = \frac{\ddot{R}}{R}\mathrm{d}t\otimes\mathrm{d}t + \frac{1}{2}\left(\frac{\dot{R}^2}{R^2}+\frac{\ddot{R}}{R}\right)R^2e^+\otimes_s e^-,\quad S = -2\frac{\ddot{R}}{R} - \left(\frac{\dot{R}}{R}\right)^2. \tag{2.60}$$

Although a general scheme for a noncommutative Einstein tensor is not known, we define it as

$$\mathrm{Eins} = \mathrm{Ricci} - \frac{1}{2}S\,g = -\frac{1}{2}\left(\frac{\dot{R}}{R}\right)^2\mathrm{d}t\otimes\mathrm{d}t - \frac{R\ddot{R}}{2}e^+\otimes_s e^-. \tag{2.61}$$

In the present model, it seems sufficient to define it in the usual way because it is conserved, i.e. it has divergence zero. This is proved in the next lemma.

Lemma 2.5. *The divergence $\nabla\cdot = ((\ ,\ )\otimes\mathrm{id})\nabla$ of a 1-1 tensor of the form*

$$T = f\mathrm{d}t\otimes\mathrm{d}t - pR^2e^+\otimes_s e^-$$

defined by functions f, p on $\mathbb{R}\times\mathbb{Z}_n$, and for metric defined as above by $R(t)$, is

$$\nabla\cdot T = -\left(\dot{f} + 2\frac{\dot{R}}{R}(f+p)\right)\mathrm{d}t + \partial_b p e^b.$$

In particular, the Einstein tensor (2.61) is conserved in the sense $\nabla\cdot\mathrm{Eins} = 0$.

Proof. The Leibniz rule for the action of the connection produces

$$\begin{aligned}
&\nabla(f\mathrm{d}t\otimes\mathrm{d}t - pR^2e^+\otimes_s e^-)\\
&\quad= \mathrm{d}f\otimes\mathrm{d}t\otimes\mathrm{d}t - \mathrm{d}p\otimes R^2e^+\otimes_s e^- + f\nabla(\mathrm{d}t\otimes\mathrm{d}t) - p\nabla(R^2e^+\otimes_s e^-)\\
&\quad= \mathrm{d}f\otimes\mathrm{d}t\otimes\mathrm{d}t - \mathrm{d}p\otimes R^2e^+\otimes_s e^- + (f+p)\nabla(\mathrm{d}t\otimes\mathrm{d}t)\\
&\quad= \dot{f}\mathrm{d}t\otimes\mathrm{d}t\otimes\mathrm{d}t - \dot{p}\mathrm{d}t\otimes R^2e^+\otimes_s e^- + \partial_b f e^b\otimes\mathrm{d}t\otimes\mathrm{d}t + \partial_b p e^b\otimes R^2e^+\otimes_s e^-\\
&\qquad+ R\dot{R}(f+p)\,(e^+\otimes_s e^-\otimes\mathrm{d}t + e^-\otimes\mathrm{d}t\otimes e^+ + e^+\otimes\mathrm{d}t\otimes e^-)
\end{aligned} \tag{2.62}$$

on using metric compatibility whereby $\nabla(\mathrm{d}t\otimes\mathrm{d}t) = -\nabla(R^2e^+\otimes_s e^-)$ and then evaluating the former with σ =flip on $\mathrm{d}t$. Now applying $(,)\otimes\mathrm{id}$ with the inverse metric, we arrive at the stated result for the divergence.

For Eins in (2.61), the coefficients just depend on time, and then they are constant on $\mathbb{Z}_n$, so there is no $e^\pm$ term in $\nabla\cdot\mathrm{Eins}$. For the $\mathrm{d}t$ term it is easy to verify that $\dot{f}+2\frac{\dot{R}}{R}(f+p) = 0$

automatically for the effective values of the specific coefficients f, p in (2.61) defined by $R(t)$. □

Next, recall from chapter 1 that our formulation of Ricci is -1/2 of the usual value, hence Einstein's equation for us should be written as

$$\mathrm{Eins} + 4\pi G T = 0 \tag{2.63}$$

and from (2.61) we see that this holds if T has the form for dust of pressure p and density f (see [**23**, Chap. 8.3]), namely

$$T = pg + (f + p)\mathrm{d}t \otimes \mathrm{d}t = f\mathrm{d}t \otimes \mathrm{d}t - pR^2 e_+ \otimes_s e_- \tag{2.64}$$

for pressure and density

$$p = -\frac{1}{8\pi G}\left(\frac{\ddot{R}}{R}\right), \quad f = \frac{1}{8\pi G}\left(\frac{\dot{R}}{R}\right)^2. \tag{2.65}$$

Note that T is automatically conserved by the same calculation as for the Einstein tensor and this does not give any constraint on $R(t)$. Setting

$$H := \frac{\dot{R}}{R}, \tag{2.66}$$

conservation is equivalent to the continuity equation

$$\dot{f} = -2H(f + p), \tag{2.67}$$

which also holds automatically. The standard consideration in cosmology at this point is to assume an equation of state $p = \omega f$ for a real parameter ω, in which case the continuity equation becomes $\frac{\mathrm{d}f}{\mathrm{d}R} = -2f(1 + \omega)$ so that $f \propto R^{-2(1+\omega)}$. Given this form of the density f, our assumption $p = \omega f$ can be solved for $\omega \neq -1$ to give

$$R(t) = R_0\left(1 + \sqrt{8\pi G f_0}(1 + \omega)t\right)^{\frac{1}{1+\omega}} \tag{2.68}$$

for initial radius and pressure R_0, f_0. Here $\omega > -1$ leads to an expanding universe. Recall that one usually takes $\omega = 0, 1/3$ for cold dust and radiation respectively, see [**23**, Chap. 8.3 and 8.4].

If we add a cosmological constant so that $\mathrm{Eins} - \frac{1}{2}g\Lambda + 4\pi G T = 0$, this is equivalent to a modified stress-energy tensor given as before but with modified

$$f^\Lambda = f + \frac{\Lambda}{8\pi G}, \quad p^\Lambda = p - \frac{\Lambda}{8\pi G} = \omega f^\Lambda - \frac{1 + \omega}{8\pi G}\Lambda. \tag{2.69}$$

The effective equation of state now leads to

$$R(t) = R_0\left(\frac{\cosh\left(\mathrm{arccosh}(\sqrt{-\frac{\Lambda}{8\pi G f_0}}) + \sqrt{\Lambda}(1 + \omega)t\right)}{\sqrt{-\frac{\Lambda}{8\pi G f_0}}}\right)^{\frac{1}{1+\omega}} \tag{2.70}$$

with reasonable behaviour for $f_0 > 0$ (with f remaining positive) and real Λ but a limited range of t when $\Lambda < 0$.

For comparison, note that the classical Einstein tensor on $\mathbb{R}\times S^1$ with $g = -\mathrm{d}t\otimes\mathrm{d}t + R^2(t)\mathrm{d}x\otimes\mathrm{d}x$ vanishes as for any 2-manifold and $T = f\mathrm{d}t\otimes\mathrm{d}t + pR^2(t)\mathrm{d}x\otimes\mathrm{d}x = pg + (f+p)\mathrm{d}t\otimes\mathrm{d}t$ admits only zero pressure and density if we want Einstein's equation. One can also add a cosmological constant, in which case we need $p = -\frac{\Lambda}{8\pi G}$ and $f = \frac{\Lambda}{8\pi G}$ and $\omega = -1$. This is therefore not the right comparison.

Proposition 2.6. *The results (2.68)-(2.70) for $R(t)$ (as well as for $f(t)$) for the FLRW model on $\mathbb{R}\times\mathbb{Z}_n$ are the same as for the classical flat FLRW-model on $\mathbb{R}\times\mathbb{R}^2$.*

Proof. The flat FLRW model in 1+2 dimensions is an easy exercise starting with the metric $g = -\mathrm{d}t\otimes\mathrm{d}t + R^2(t)(\mathrm{d}x\otimes\mathrm{d}x + \mathrm{d}y\otimes\mathrm{d}y)$ to compute the Ricci tensor (in our conventions, which is $-\frac{1}{2}$ of the usual values) as

$$\mathrm{Ricci} = \frac{\ddot{R}}{R}\mathrm{d}t\otimes\mathrm{d}t - \frac{1}{2}\left(\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2}\right)R^2(\mathrm{d}x\otimes\mathrm{d}x + \mathrm{d}y\otimes\mathrm{d}y) \tag{2.71}$$

and the same scalar curvature S as in (2.60). The Einstein tensor is therefore

$$\mathrm{Eins} = -\frac{1}{2}\left(\frac{\dot{R}}{R}\right)^2\mathrm{d}t\otimes\mathrm{d}t + \frac{R\ddot{R}}{2}(\mathrm{d}x\otimes\mathrm{d}x + \mathrm{d}y\otimes\mathrm{d}y) \tag{2.72}$$

by a similar calculation as for (2.61). The stress tensor for dust being similarly $f\mathrm{d}t\otimes\mathrm{d}t + pR^2(\mathrm{d}x\otimes\mathrm{d}x + \mathrm{d}y\otimes\mathrm{d}y)$ means that the Einstein equations give p, f by the same expressions (2.65) as before. The Friedmann equations are therefore the same as we solved. □

This is perhaps not too surprising given that Ω^1 on $\mathbb{Z}_n$ is 2-dimensional, indeed $-e^+\otimes_s e^-$ plays the same role as the classical spatial metric $\mathrm{d}x\otimes\mathrm{d}x + \mathrm{d}y\otimes\mathrm{d}y$. We also recall by way of comparison that the standard $k = 0$ Friedmann equations for the FLRW model $\mathbb{R}\times\mathbb{R}^3$ have the well-known solution,

$$R(t) = R_0(1 + \sqrt{6\pi G f_0}(w+1)t)^{\frac{2}{3(w+1)}} \tag{2.73}$$

without cosmological constant and can also be solved with it, as

$$R(t) = R_0\left(\frac{\cosh\left(\operatorname{arccosh}\left(\sqrt{-\frac{\Lambda}{8\pi G f_0}}\right) + \sqrt{\frac{3\Lambda}{4}}(w+1)t\right)}{\sqrt{-\frac{\Lambda}{8\pi G f_0}}}\right)^{\frac{2}{3(w+1)}}. \tag{2.74}$$

See [**23**, Chap. 8.3]. As usual, the case of $R(t)$ independent of time is a solution for the Einstein vacuum equation with Ricci = 0. It is easy to see that there are no other solutions of interest with Ricci $\propto g$ or Eins $\propto g$. On the other hand, we do have the following.

Proposition 2.7. *The equation* Ricci $- \lambda S\, g = 0$ *with time-varying* $R(t)$ *and constant* λ *has a unique solution of the form*

$$\lambda = \frac{1}{3}, \quad R(t) = R_0 e^{\mu t}$$

for some growth constant $\mu \neq 0$ *and initial* $R_0 > 0$.

Proof. Considering the equation Ricci $= \lambda g S$, where λ is an arbitrary real constant, we have two equations; one related to $e^{\pm} \otimes e^{\mp}$ is

$$\frac{\ddot{R}}{R} + \left(\frac{2\lambda}{1-4\lambda} + 1\right)\left(\frac{\dot{R}}{R}\right)^2 = 0 \tag{2.75}$$

and other related to $\mathrm{d}t \otimes \mathrm{d}t$ is

$$\frac{\ddot{R}}{R} + \left(\frac{\lambda-1}{1-2\lambda} + 1\right)\left(\frac{\dot{R}}{R}\right)^2 = 0. \tag{2.76}$$

This requires $\lambda = \frac{1}{3}$ and $\frac{\ddot{R}}{R} = \left(\frac{\dot{R}}{R}\right)^2$, which has the solution claimed. □

2.3. Quantum field theory on $\mathbb{R} \times \mathbb{Z}_n$**.** Here we consider quantum field theory in the flat case where R is a constant. To construct the Klein-Gordon equation, we need the corresponding Laplacian operator. In this case, is obtained taking $\alpha = R^2(t)$ and $\beta = 0$ from the theorem 2.4, which are the same consideration for the metric as the previous section. Thus we have

$$\Box = \frac{2}{R^2}(\partial_+ + \partial_-) - \partial_t^2; \quad (-\Box + m^2)\phi = 0. \tag{2.77}$$

We write $q = e^{\frac{2\pi\imath}{n}}$, where $\imath$ denotes the imaginary unit, and Fourier transform on $\mathbb{Z}_n$ by considering solutions of the form $\phi(t,i) = q^{ik} e^{-\imath w_k t}$, where i denotes the position in $\mathbb{Z}_n$. This is labelled by a discrete momentum $k = 0, \cdots, n-1$ with associated 'mass on-shell' expression

$$w_k^2 = \frac{8}{R^2}\sin^2\left(\frac{\pi}{n}k\right) + m^2. \tag{2.78}$$

We then consider the corresponding operator-valued fields starting with

$$\phi_i = \sum_{k=0}^{n-1} \frac{1}{\sqrt{2w_k}}(q^{ik} a_k + q^{-ik} a_k^\dagger), \tag{2.79}$$

where now $a_k, a_k^\dagger$ are self-adjoint operators and $a_k |0\rangle = 0$, with $|k\rangle$ eigenvectors of the corresponding Hamiltonian

$$H = \sum_{k=0}^{n-1} w_k (a_k a_k^\dagger + \frac{n}{2}). \tag{2.80}$$

In our formalism we do not have a way to calculate the Hamiltonian, then we assume the Hamiltonian form in the same way as the usual QFT approach.

From the commutators $[H, a_k] = -w_k a_k$ and $[H, a_k^\dagger] = w_k a_k^\dagger$, and using the Heisenberg representation for the time evolution of the field, we obtain

$$\phi_i(t) = e^{\imath Ht}\phi_i e^{-\imath Ht} = \sum_{k=0}^{n-1} \frac{1}{\sqrt{2w_k}}(q^{ik-\imath w_k t}a_k + q^{-ik+\imath w_k t}a_k^\dagger) \tag{2.81}$$

with the time-ordered correlation function

$$\langle 0|\, T[\phi_i(t_a)\phi_j(t_b)]\, |0\rangle = \sum_{k=0}^{n-1} \frac{1}{w_k} \cos\left(\frac{2\pi}{n}k(i-j)\right) e^{-\imath w_k |t_a - t_b|}. \tag{2.82}$$

Next, we check that we obtain the same correlation function via a formal path integral approach with the $\imath\epsilon$-prescription. The partition functional integral $Z[J]$ with source J is defined as

$$Z[J] = \frac{\int \mathcal{D}\phi\, e^{\frac{1}{\beta}S[\phi] + \frac{1}{\beta}\int \sum_{i=0}^{n-1} J_i(t)\phi_i(t)}}{\int \mathcal{D}\phi\, e^{\frac{1}{\beta}S[\phi]}} = \frac{\int \mathcal{D}\phi\, e^{\frac{1}{2\beta}\int dt \sum_{i=0}^{n-1}\left(\phi_i(t)(\Box - m^2 + \imath\epsilon)\phi_i(t) + 2J_i(t)\phi_i(t)\right)}}{\int \mathcal{D}\phi\, e^{\frac{1}{2\beta}\int dt \sum_{i=0}^{n-1}\left(\phi_i(t)(\Box - m^2 + \imath\epsilon)\phi_i(t)\right)}}, \tag{2.83}$$

where β is a dimensionless coupling constant. We diagonalize the action $S[\phi]$ using Fourier transform to write

$$\phi_i(t) = \sum_{k=0}^{n-1}\int_{-\infty}^{\infty} \frac{dw}{2\pi}\tilde{\phi}_k(w) q^{ik} e^{\imath wt}; \quad J_i(t) = \sum_{k=0}^{n-1}\int_{-\infty}^{\infty} \frac{dw}{2\pi}\tilde{J}_k(w) q^{ik} e^{\imath wt}, \tag{2.84}$$

which produces the action

$$S[\tilde{\phi}] = \int_{-\infty}^{\infty} \frac{dw}{2\pi}\frac{1}{2\beta}\sum_{k=0}^{n-1}\left(\tilde{\phi}'_{-k}(-w)(-w^2 + w_k^2)\tilde{\phi}'_k(w) + \tilde{J}_{-k}(-w)\frac{1}{-w^2 + w_k^2}\tilde{J}_k(w)\right), \tag{2.85}$$

where $\tilde{\phi}'_k(w) = \tilde{\phi}_k(w) - (-w^2 + w_k^2)^{-1}\tilde{J}_k(w)$. The first term in terms of the new variables gives a Gaussian integral, which we ignore as an overall factor independent of the source. Using

$$\tilde{J}_k(w) = \frac{1}{n}\int dt \sum_{i=0}^{n-1} J_i(t) q^{-ik} e^{\imath wt}, \tag{2.86}$$

the functional integral becomes

$$Z[J] = e^{\frac{1}{\beta}\int dt' dt'' J_i(t')\imath\Delta_f(i,t';j,t'')J_j(t'')}, \tag{2.87}$$

where the Feynman propagator is

$$\begin{aligned}\Delta_f(i,t';j,t'') &= \sum_{k=0}^{n-1} q^{k(i-j)} \int \frac{dw}{2\pi}\frac{e^{-\imath w(t'-t'')}}{(-w + w_k - \imath\epsilon)(w + w_k + \imath\epsilon)} \\ &= \sum_{k=0}^{n-1}\frac{1}{w_k}\cos\left(\frac{2\pi}{n}k(i-j)\right)e^{-\imath w_k|t_a - t_b|}.\end{aligned} \tag{2.88}$$

Finally, by construction, we have

$$\langle 0|\, T[\phi_i(t_a)\phi_j(t_b)]\, |0\rangle = \frac{\beta^2}{\imath^2}\frac{\partial}{\partial J_i(t_a)}\frac{\partial}{\partial J_j(t_b)}Z[J] = \Delta_f(i,t';j,t''), \tag{2.89}$$

which therefore gives the same result as obtained by Hamiltonian quantisation. This is as expected, but provides a useful check that our methodology makes sense at least in the flat case of constant R.

2.4. Particle creation in FLRW model on $\mathbb{R}\times\mathbb{Z}_n$. Here we follow the procedure developed by Parker [**63, 64, 65, 66**] to study cosmological particle creation, adapted now to an FLRW model on $\mathbb{R}\times\mathbb{Z}_n$ with an expanding quantum metric (2.57).

In general, the assumptions that we make in this section are the standard ones in the particle creation procedure. A steady universe that goes through a process of expansion, and ends in another steady state. This justifies the form of 2.126. A number operator $|N_k\rangle$ which 'counts' the particles of frequency k before and after the expansion is defined and finally, we use the approximation that all the time-derivatives of the metric smoothly go to zero.

2.4.1. *Model case of* $\mathbb{R}\times S^1$. We start with the classical background geometry case of $\mathbb{R}\times S^1$, which is presumably known but sets up the procedure and our notations. Here the metric has the usual 2D FLRW form

$$g = -\mathrm{d}t\otimes\mathrm{d}t + R^2(t)\mathrm{d}x\otimes\mathrm{d}x, \tag{2.90}$$

where $R(t)$ is an arbitrary positive function. Thus the Klein-Gordon equation for the field ϕ is

$$\left(g^{\mu\nu}\nabla_\mu\nabla_\nu - m^2\right)\phi = 0 \tag{2.91}$$

or in explicit form

$$\ddot{\phi} + \frac{\dot{R}}{R}\dot{\phi} - \frac{1}{R^2}\partial_x^2\phi + m^2\phi = 0. \tag{2.92}$$

We impose the periodic boundary condition $\phi(t, x+L) = \phi(t, x)$, where L is a dimensionless parameter for the normalisation of the box geometry. We then expand the field in terms of a Fourier series

$$\phi(t,x) = \sum_k (A_k f_k(t,x) + A_k^* f_k^*(t,x)), \tag{2.93}$$

where

$$f_k(t,x) = \frac{1}{\sqrt{LR}}e^{\imath xk}h_k(t) \tag{2.94}$$

and $k = 2l\pi/L$ for l an integer. Here k/R is the physical momentum and l the corresponding 'integer momentum' on a circle. Then ϕ obeys (2.92) provided

$$\ddot{h}_k(t) + \left(\frac{k^2}{R^2} + m^2\right) h_k(t) + \left(\frac{1}{4}\left(\frac{\dot{R}}{R}\right)^2 - \frac{1}{2}\frac{\ddot{R}}{R}\right) h_k(t) = 0 \tag{2.95}$$

for each momentum mode. We will be particularly interested in the adiabatic limit, where R varies slowly with respect to the time in such way that $\dot{R}/R \to 0, \ddot{R}/R \to 0$. The solutions to (2.95) in this approximation are

$$h_k(t) \sim (w_k)^{-\frac{1}{2}} \left(\alpha_k e^{\imath \int^t w_k(t')\mathrm{d}t'} + \beta_k e^{-\imath \int^t w_k(t')\mathrm{d}t'}\right), \tag{2.96}$$

where α_k and β_k are complex constants that satisfy

$$|\alpha_k|^2 - |\beta_k|^2 = 1 \tag{2.97}$$

and

$$w_k(t) = \sqrt{m^2 + \frac{k^2}{R^2(t)}}. \tag{2.98}$$

In order to have an exact solution, we now let α_k and β_k be functions of time such that

$$h_k(t) = (w_k(t))^{-\frac{1}{2}} \left(\alpha_k(t) e^{\imath \int^t w_k(t')\mathrm{d}t'} + \beta_k(t) e^{-\imath \int^t w_k(t')\mathrm{d}t'}\right) \tag{2.99}$$

and

$$|\alpha_k(t)|^2 - |\beta_k(t)|^2 = 1 \tag{2.100}$$

for all t. Equivalently, we can rewrite the expansion of the field as

$$\phi(t, x) = \sum_k (a_k(t) g_k(t, x) + a_k^*(t) g_k^*(t, x)), \tag{2.101}$$

where now

$$g_k(t, x) = \frac{R^{-\frac{1}{2}}}{\sqrt{L w_k}} e^{\imath(xk - \int^t w_k(t')\mathrm{d}t')} \tag{2.102}$$

and

$$a_k(t) = \alpha_k(t)^* A_k + \beta_k(t) A_k^*. \tag{2.103}$$

In order to follow the usual procedure of canonical quantisation, we next define the conjugate momentum as

$$\pi(t, x) = R\dot{\phi}(t, x), \tag{2.104}$$

promote the field $\phi(t, x)$ and the momentum $\pi(t, x)$ to operators $\hat{\phi}(t, x), \hat{\pi}(t, x)$ respectively, and impose the commutators relations

$$[\hat{\phi}(t, x), \hat{\phi}(t, x')] = [\hat{\pi}(t, x), \hat{\pi}(t, x')] = 0, \quad [\hat{\phi}(t, x), \hat{\pi}(t, x')] = \imath\delta(x - x'). \tag{2.105}$$

This requires that A_k and A_k^* in (2.103) are promoted to operators A_k and $A_k^\dagger$ with the usual commutation relations

$$[A_{k'}, A_k] = [A_k^\dagger, A_{k'}^\dagger] = 0, \quad [A_{k'}, A_k^\dagger] = \delta_{k,k'}. \tag{2.106}$$

It then follows from these and a conserved quantity (see [**63**]), that the operator versions of (2.103) obey

$$[a_k(t), a_{k'}(t)] = [a_k^\dagger(t), a_{k'}^\dagger(t)] = 0, \quad [a_k(t), a_{k'}^\dagger(t)] = \delta_{k,k'}. \tag{2.107}$$

Now note that for any function $W_k(t)$ with at least derivatives to second order, the function

$$H(t) := W_k(t)^{-\frac{1}{2}}(\alpha_k e^{\imath \int^t dt' W_k(t')} + \beta_k e^{-\imath \int^t dt' W_k(t')}) \tag{2.108}$$

for any constants α_k, β_k is an exact solution of the equation

$$\ddot{H}(t) + \left[W_k^2 - W_k^{\frac{1}{2}} \frac{d^2}{dt^2} W_k^{-\frac{1}{2}}\right] H(t) = 0. \tag{2.109}$$

Hence, if we can solve for $W_k(t)$ such that

$$W_k^2 = W_k^{\frac{1}{2}} \frac{d^2}{dt^2} W_k^{-\frac{1}{2}} + w_k^2 + \sigma \tag{2.110}$$

holds, where

$$\sigma = \frac{1}{4}\left(\frac{\dot{R}}{R}\right)^2 - \frac{1}{2}\frac{\ddot{R}}{R}, \tag{2.111}$$

then $H(t)$ provides exact solutions $h_k(t)$ of (2.95) for each k.

We can then expand W_k as a sum of terms

$$W_k = w^{(0)} + w^{(1)} + w^{(2)} + \ldots, \tag{2.112}$$

where the superfix denotes the adiabatic order. Putting this into (2.110) and just keeping the elements of order zero, we have $w^{(0)} = w_k$. Just keeping the elements of first order tell us that $w^{(1)} = 0$, while for elements of second adiabatic order we require

$$w^{(2)} = \frac{(w^{(0)})^{-\frac{1}{2}}}{2} \frac{d^2}{dt^2}\left((w^{(0)})^{-\frac{1}{2}}\right) + \frac{\sigma}{2w^{(0)}}. \tag{2.113}$$

We can continue this procedure to any desired order to find odd $w^{(i)} = 0$ and even $w^{(i)}$ determined from lower even ones. The form of the functions $\alpha_k(t)$ and $\beta_k(t)$ can be obtained when we impose (2.100). From its temporal derivative, one is led to the ansatz

$$\alpha_k(t) = -\dot{\beta}_k(t) e^{-2\imath \int^t dt' W_k(t')}, \quad \beta_k(t) = -\dot{\alpha}_k(t) e^{2\imath \int^t dt' W_k(t')} \tag{2.114}$$

as justified by consistency with (2.95), given (2.110). For a more explicit form of these coefficients, see [**48**].

A special case of interest here is when the $w_k^{(i)}$ vanish for all the orders bigger than zero (and all k). In this case, the operator $a_k(t)$ defined in (2.103) is independent of time, the number of particles is constant and there is no particle creation. From the above remarks, it is sufficient that $w_k^{(2)} = 0$, which amounts to

$$\frac{1}{4}\frac{m^2\left(4\frac{k^2}{R^2}-m^2\right)}{\left(\frac{k^2}{R^2}+m^2\right)^2}\left(\frac{\dot{R}}{R}\right)^2+\frac{1}{2}\frac{m^2}{(\frac{k^2}{R^2}+m^2)}\frac{\ddot{R}}{R}=0. \tag{2.115}$$

The only way that this can hold for all time and k is in the infinite mass limit $m \to \infty$ (cf. [**63**]), where it reduces to an FLRW-like equation

$$\frac{1}{2}\frac{\ddot{R}}{R}=\frac{1}{4}\left(\frac{\dot{R}}{R}\right)^2 \tag{2.116}$$

with solution $R \propto t^2$. As well as the obvious flat Minkowski case of constant R, this represents a further possibility for no particle creation.

For an actual particle creation computation, it is convenient to move to a new time variable η such that

$$\mathrm{d}\eta = \frac{\mathrm{d}t}{R(t)}, \tag{2.117}$$

in which case our metric becomes conformally flat as

$$g = C(\eta)(-\mathrm{d}\eta \otimes \mathrm{d}\eta + \mathrm{d}x \otimes \mathrm{d}x), \tag{2.118}$$

where $C(\eta) = R^2(t)$ is now regarded as a function of η. Following the same steps as before but using this metric puts the wave equation (2.95) on spatial momentum modes in the simpler form

$$\frac{\mathrm{d}^2 h_k(\eta)}{\mathrm{d}\eta^2} + w_k(\eta)h_k(\eta) = 0, \tag{2.119}$$

where

$$w_k(\eta) = \sqrt{C(\eta)m^2 + k^2} \tag{2.120}$$

as a modification of (2.98).

We now consider particle creation under the assumption that R and hence C has a constant constant value $C(\eta) = R_{in}^2$ for early times $\eta < \eta_{in}$, say, and a constant value $C(\eta) = R_{out}^2$ for late times $\eta > \eta_{out}$, with $\eta_{in} < \eta_{out}$. For these early and late times, we let

$$w_k^{\mathrm{in}} = \sqrt{R_{in}^2 m^2 + k^2}; \quad w_k^{\mathrm{out}} = \sqrt{R_{out}^2 m^2 + k^2} \tag{2.121}$$

as functions of k. The fields at early and late times behave exactly as flat Minkowski space-time with the corresponding frequency or effective mass, with solutions of (2.119) at early and late times provided by

$$h_k^{\mathrm{in}}(\eta) = (w_k^{\mathrm{in}})^{-\frac{1}{2}} e^{\imath w_k^{\mathrm{in}}\eta}, \quad h_k^{\mathrm{out}}(\eta) = (w_k^{\mathrm{out}})^{-\frac{1}{2}} e^{\imath w_k^{\mathrm{out}}\eta}. \tag{2.122}$$

Now suppose that we start with $h_k^{\rm in}(\eta)$ at early times, i.e. $h_k(\eta)$ for $\alpha_k(\eta_{in}) = 1$ and $\beta_k(\eta_{in}) = 0$ in the analogue of (2.99), and extend this by solving (2.119) to late times. There we expand it as the Bogolyubov transformation

$$h_k^{\rm in} = \alpha_k h_k^{\rm out} + \beta_k h_k^{\rm out *} \tag{2.123}$$

valid at late times and for some complex constants α_k, β_k. Comparing with the analogue of (2.99) at late times, these constants up to phases are just the evolved values $\alpha_k(\eta_{out}), \beta_k(\eta_{out})$ in the general scheme. (The phases come from $e^{\imath\int_{\eta_{in}}^{\eta_{out}} w_k(\eta){\rm d}\eta}$ and are not relevant in what follows.)

Finally, we fix a vacuum $|0\rangle$ as characterised by $A_k|0\rangle = 0$ and consider the number operator $N_k(\eta) = a_k^\dagger(\eta)a_k(\eta)$, which evolves in time, where we use the analogue of (2.103) as our solution evolves. Starting now with $\alpha_k(\eta_{in}) = 1, \beta_k(\eta_{in}) = 0$ in defining $a_k, a_k^\dagger$, we have of course

$$\langle 0| N_k(\eta_{in})|0\rangle = 0 \tag{2.124}$$

at early times, but in this same state at late times we have the possibility of particle creation according to

$$\langle N_k\rangle := \langle 0| N_k(\eta_{out})|0\rangle = |\beta_k(\eta_{out})|^2 = |\beta_k|^2. \tag{2.125}$$

This completes the general scheme, which is also well-known from several other points of view. To proceed further we need to fix a particular $C(\eta)$, and the standard choice for purposes of calculation is to interpolate the initial and final values as

$$C(\eta) = \frac{R_{in}^2 + R_{out}^2}{2} + \frac{R_{out}^2 - R_{in}^2}{2}\tanh(\mu\eta), \tag{2.126}$$

where μ is a positive constant parameter. Equation (2.119) can then be solved with hypergeometric functions that have the correct asymptotic limit for late and early times. Comparison with (2.123) gives (see [**54**]),

$$\alpha_k = \left(\frac{w_k^{\rm out}}{w_k^{\rm in}}\right)^{1/2} \frac{\Gamma(1-\imath\frac{w_k^{\rm in}}{\mu})\Gamma(-\imath\frac{w_k^{\rm out}}{\mu})}{\Gamma(-\imath\frac{w_k^{+}}{\mu})\Gamma(1-\imath\frac{w_k^{+}}{\mu})}, \tag{2.127}$$

$$\beta_k = \left(\frac{w_k^{\rm out}}{w_k^{\rm in}}\right)^{1/2} \frac{\Gamma(1-\imath\frac{w_k^{\rm in}}{\mu})\Gamma(\imath\frac{w_k^{\rm out}}{\mu})}{\Gamma(\imath\frac{w_k^{-}}{\mu})\Gamma(1+\imath\frac{w_k^{-}}{\mu})}, \tag{2.128}$$

where

$$w_k^{\pm} = \frac{1}{2}(w_k^{out} \pm w_k^{in}). \tag{2.129}$$

These values result in

$$|\alpha_k|^2 = \frac{\sinh^2\left(\pi\frac{w_k^+}{\mu}\right)}{\sinh(\pi\frac{w_k^{in}}{\mu})\sinh(\pi\frac{w_k^{out}}{\mu})},\quad |\beta_k|^2 = \frac{\sinh^2\left(\pi\frac{w_k^-}{\mu}\right)}{\sinh(\pi\frac{w_k^{in}}{\mu})\sinh(\pi\frac{w_k^{out}}{\mu})}, \tag{2.130}$$

which, as one can check, obeys the unitarity condition (2.100). Figure 4 includes a plot of $\langle N_k|N_k\rangle = |\beta_k|^2$ as a function of k, or rather of the associated integer momentum l.

2.4.2. *Adaptation to* $\mathbb{R}\times\mathbb{Z}_n$. We now repeat the previous analysis for the polygon case with n sides and time-varying metric (2.57). We have the Laplacian

$$\Box = -\partial_t^2 - 2\frac{\dot{R}}{R}\partial_t + \frac{2}{R^2}(\partial_+ + \partial_-) \tag{2.131}$$

from Theorem 2.4 with $\beta = 0$. The Klein-Gordon equation $(-\Box + m^2)\phi = 0$ is

$$\left(-\frac{2}{R^2}(\partial_+ + \partial_-) + \frac{1}{R^2}\partial_t(R^2\partial_t) + m^2\right)\phi = 0. \tag{2.132}$$

Next, we expand the field in terms of a Fourier series

$$\phi(t,i) = \sum_k (A_k f_k(t,i) + A_k^* f_k^*(t,i)) \tag{2.133}$$

in place of (2.93), where now

$$f_k(t,i) = \frac{1}{R(t)} q^{ik} h_k(t) \tag{2.134}$$

and k is an integer mod n. For the modes f_k to obey (2.132), the h_k have to solve

$$\ddot{h}_k(t) + \left(m^2 + \frac{8}{R^2}\sin^2\left(\frac{\pi}{n}k\right)\right)h_k(t) - \frac{\ddot{R}}{R}h_k(t) = 0. \tag{2.135}$$

The corresponding on-shell frequency is therefore

$$w_k(t) = \sqrt{m^2 + \frac{8}{R^2(t)}\sin^2\left(\frac{\pi}{n}k\right)} \tag{2.136}$$

instead of (2.98). We again consider an exact solution of the form

$$h_k(t) = (w_k(t))^{-\frac{1}{2}}\left(\alpha_k(t)e^{\imath\int^t w_k(t')\mathrm{d}t'} + \beta_k(t)e^{-\imath\int^t w_k(t')\mathrm{d}t'}\right). \tag{2.137}$$

Analogously to the previous case, we can re-write the expansion of the field as

$$\phi(t,i) = \sum_k (a_k(t)g_k(t,i) + a_k^*(t)g_k^*(t,i)), \tag{2.138}$$

where

$$g_k(t,i) = \frac{R^{-1}}{\sqrt{w_k}} q^{ik} e^{-\imath\int^t w_k(t')\mathrm{d}t'} \tag{2.139}$$

and the operator $a_k(t)$ has the same form as (2.103). The quantisation procedure and analysis then proceed as before. Our previous expressions for $W_k(t), \alpha_k(t), \alpha_k$ are still valid, but we have to take into account that the zero adiabatic order term w_k is different and that now

$$\sigma = -\frac{\ddot{R}}{R} \tag{2.140}$$

as the factor in (2.135).

For our first result, we look at when the $w_k^{(2)}$ correction vanishes so that there is no particle creation. In place of (2.115), we now require

$$\frac{\frac{4}{R^2}\sin^2\left(\frac{\pi}{n}k\right)(\frac{4}{R^2}\sin^2\left(\frac{\pi}{n}k\right)+3m^2)}{\left(\frac{8}{R^2}\sin^2\left(\frac{\pi}{n}k\right)+m^2\right)^2}\left(\frac{\dot{R}}{R}\right)^2+\frac{\frac{4}{R^2}\sin^2\left(\frac{\pi}{n}k\right)+m^2}{\left(\frac{8}{R^2}\sin^2\left(\frac{\pi}{n}k\right)+m^2\right)}\frac{\ddot{R}}{R}=0. \tag{2.141}$$

This can happen for all time and all k in the infinite mass limit $m\to\infty$ if

$$\ddot{R}=0 \tag{2.142}$$

with solution $R\propto t$. However, we *also* have a new possibility when $m\to 0$, with

$$\frac{\ddot{R}}{R}=-\frac{1}{2}\left(\frac{\dot{R}}{R}\right)^2 \tag{2.143}$$

and solution $R\propto t^{\frac{2}{3}}$. Thus we have not one but two additional possibilities for no particle creation beyond the constant Minkowski metric case.

For our second result, we want to analyse particle creation for the $\mathbb{R}\times\mathbb{Z}_n$ model in an analogous way to the case when space is a circle. Thus, we make the same change of variable (2.117) in the metric (2.57) to write

$$g=C(\eta)(-\mathrm{d}\eta\otimes\mathrm{d}\eta-e^+\otimes_s e^-), \tag{2.144}$$

where $C(\eta)=R^2(t)$, and the corresponding connection is

$$\nabla\mathrm{d}\eta=\frac{\dot{R}}{R}(-\mathrm{d}\eta\otimes\mathrm{d}\eta+e^+\otimes_s e^-),\quad \nabla e^{\pm}=-\frac{\dot{R}}{R}e^{\pm}\otimes_s\mathrm{d}\eta. \tag{2.145}$$

Using the quantum geometric Laplacian for this connection, we require

$$\frac{\mathrm{d}^2h_k(\eta)}{\mathrm{d}\eta^2}+\left(C(\eta)m^2+8\sin^2\left(\frac{\pi}{n}k\right)\right)h_k(\eta)=0 \tag{2.146}$$

analogously to (2.119), but now in place (2.120) we have

$$w_k(\eta)=\sqrt{C(\eta)m^2+8\sin^2\left(\frac{\pi}{n}k\right)}. \tag{2.147}$$

The rest of the procedure follows in the same way with the same considerations, and in particular (2.130) is still valid but with (2.147) instead of (2.120). Figure 4 shows the expected value of the number operator $\langle N_k|N_k\rangle$ as a function of k as well as comparing to the circle case. The big difference of course is that the $\mathbb{Z}_n$ has to be periodic in k since this is only defined mod n. As last remark here, we pointing out that a usual procedure of quantization leads to the equations (2.130), however, we would not to be able to determine the no particle creation conditions.

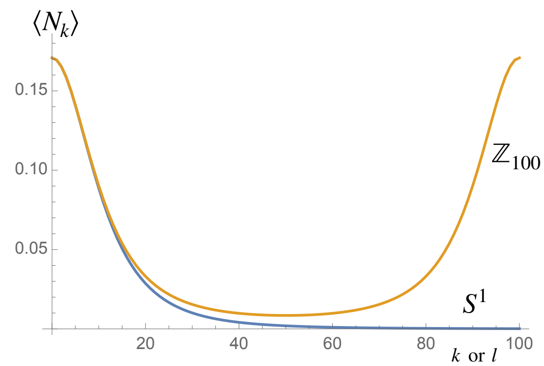

FIGURE 4. Number operator for $\mathbb{Z}_{100}$ against k compared to S^1 with length scale factor $L = 100/\sqrt{2}$, plotted against integer momentum l where $k = 2\pi l/L$. In both cases, $R_{\rm in}\, m = 1$, $R_{\rm out}\, m = \sqrt{5}$ and $\mu = 100$ for the interpolation parameter. Figure as in [**2**]

3. Black hole with the discrete circle

We now consider the spacetime metric of a symmetric static black hole with S^1 in polar coordinate replaced by the discrete group $\mathbb{Z}_n$. Now the spacetime coordinate algebra is $A = C^\infty(\mathbb{R}\times\mathbb{R}_{>0})\otimes\mathbb{C}(\mathbb{Z}_n)$ with t, r for the time and radial classical variables, and we consider a static Schwarzschild-like metric of the form

$$g = -\beta(r)\mathrm{d}t\otimes\mathrm{d}t + H(r)\mathrm{d}r\otimes\mathrm{d}r - \alpha_{ab}(r,i)e^a\otimes e^b. \tag{2.148}$$

Invertibility of the metric requires centrality, which dictates $\alpha_{ab}(r,i) = \alpha_a(r,i)\delta_{ab^{-1}}$ for some real-valued functions α_a. We also require edge-symmetry $\alpha_a = R_a(\alpha_{a^{-1}})$ so that the length of each edge $\bullet_i - \bullet_{i+1}$ for the $\mathbb{Z}_n$ at radius r is the same in either direction, namely given by some real function $a(r,i)$ according to

$$\alpha_+(r,i) = a(r,i),\quad \alpha_-(r,i) = R_-a(r,i). \tag{2.149}$$

We limit attention to this form of metric.

We take analogous conditions on the tensor product calculus as in the previous section, in the sense that the functions of the time t, radius r as well as $\mathrm{d}t, \mathrm{d}r$ are classical and graded-commute with everything. In view of this, *we make the simplifying assumption that the connection braiding σ among the differentials* $\mathrm{d}r, \mathrm{d}t$ *and between them and* $e^\pm$ *is just the flip map*. In this case, the most general form of a potential bimodule connection, removing the terms that make the connection not compatible with the algebra (similar to the previous section), turns out to be

$$\nabla e^a = -\Gamma^a{}_{bc} e^b \otimes e^c + \nu^a{}_b \mathrm{d}t \otimes_s e^b + \gamma^a{}_b \mathrm{d}r \otimes_s e^b,$$
$$\nabla \mathrm{d}t = \xi_{ab} e^a \otimes e^b + b\mathrm{d}t \otimes \mathrm{d}t + c\mathrm{d}r \otimes \mathrm{d}r + h\mathrm{d}r \otimes_s \mathrm{d}t,$$
$$\nabla \mathrm{d}r = A_{ab} e^a \otimes e^b + B\mathrm{d}t \otimes \mathrm{d}t + C\mathrm{d}r \otimes \mathrm{d}r + D\mathrm{d}r \otimes_s \mathrm{d}t,$$

where the coefficients are elements of the algebra A and of the form

$$\nu^a{}_b = \nu_a \delta_{a,b^{-1}}, \quad \gamma^a{}_b = \gamma_a \delta_{a,b^{-1}}, \quad A_{ab} = A_a \delta_{a,b^{-1}}, \quad \xi_{ab} = \xi_a \delta_{a,b^{-1}}. \tag{2.150}$$

We now analyse when such a bimodule connection is a QLC. The requirement to be torsion-free comes down to

$$A_{ab} = A_{ba}, \quad \xi_{ab} = \xi_{ba}, \quad \Gamma^a{}_{bc} = \Gamma^a{}_{cb}, \quad \wedge(\mathrm{id} + \sigma)(e^a \otimes e^b) = 0, \tag{2.151}$$

while to be metric compatible comes down to the 13 equations:

$$\mathrm{d}r \otimes \mathrm{d}t \otimes \mathrm{d}t : \partial_r\beta + 2\beta h = 0,$$

$$\begin{aligned}
\mathrm{d}r^{\otimes 3} &: \partial_r H + 2HC = 0,\\
\mathrm{d}t^{\otimes 3} &: 2\beta b = 0,\\
\mathrm{d}r \otimes \mathrm{d}t \otimes \mathrm{d}r / \mathrm{d}r \otimes \mathrm{d}r \otimes \mathrm{d}t &: -\beta c + HD = 0,\\
\mathrm{d}t \otimes \mathrm{d}t \otimes \mathrm{d}r / \mathrm{d}t \otimes \mathrm{d}r \otimes \mathrm{d}t &: -\beta h + HB = 0,\\
\mathrm{d}t \otimes \mathrm{d}r \otimes \mathrm{d}r &: 2HD = 0,\\
\mathrm{d}r \otimes e^a \otimes e^b &: -\partial_r \alpha_{ab} - \alpha_{cb}\gamma^c_a - \alpha_{ac} R_a(\gamma^c_b) = 0,\\
e^a \otimes e^b \otimes \mathrm{d}t &: -\beta \xi_{ab} - \alpha_{cd} R_c(\nu^d{}_f)\sigma^{cf}{}_{ab} = 0,\\
e^a \otimes \mathrm{d}t \otimes e^b &: -\beta \xi_{ab} - \alpha_{cb}\nu^c{}_a = 0,\\
e^a \otimes e^b \otimes \mathrm{d}r &: HA_{ab} - \alpha_{cd} R_c(\gamma^d{}_f)\sigma^{cf}{}_{ab} = 0,\\
e^a \otimes \mathrm{d}r \otimes e^b &: HA_{ab} - \alpha_{cb}\gamma^c{}_a = 0,\\
e^a \otimes e^b \otimes e^c &: -\partial_a \alpha_{bc} - \alpha_{dc}\Gamma^d{}_{ab} - \alpha_{df} R_d(\Gamma^f{}_{gc})\sigma^{dg}{}_{ab} = 0,\\
\mathrm{d}t \otimes e^a \otimes e^b &: -\alpha_{cb}\nu^c{}_a - \alpha_{ac} R_a(\nu^c{}_b) = 0.
\end{aligned}$$

The 1st and 2nd equations give h, C respectively, and these together with the 5th equation give B, as

$$h = -\frac{1}{2\beta}\partial_r\beta, \quad C = -\frac{1}{2H}\partial_r H, \quad B = -\frac{1}{2H}\partial_r\beta. \tag{2.152}$$

The 3rd, 6th, and 4th equations imply that $c = b = D = 0$. Next, the 9th and 11th equations tell us that

$$\nu_a = -\frac{\beta\xi_a}{\alpha_a}, \quad \gamma_a = \frac{HA_a}{\alpha_a}, \tag{2.153}$$

while, given the edge-symmetry, the 13th and 7th equations reduce to

$$\nu_a + R_a(\nu_{a^{-1}}) = 0, \quad \gamma_a + R_a(\gamma_{a^{-1}}) = -\frac{\partial_r \alpha_a}{\alpha_a}. \tag{2.154}$$

Given that the 12th equation for metric compatibility and the torsion-freeness conditions are the same as for the polygon in the previous section, we are led to take $\Gamma^a{}_{bc}$ at each radius r the same as for the QLC $\nabla^{\mathbb{Z}_n}$ on the polygon found there. This has

$$\nabla^{\mathbb{Z}_n} e^+ = (1-\rho) e^+ \otimes e^+, \quad \nabla^{\mathbb{Z}_n} e^- = (1 - R_-^2 \rho^{-1}) e^- \otimes e^-, \quad \rho(r,i) = \frac{a(r,i+1)}{a(r,i)}$$

and its braiding obeys $\sigma(e^\pm \otimes e^\mp) = e^\mp \otimes e^\pm$, in which case the 8th and 10th metric compatibility equations become

$$R_a(\nu_{a^{-1}}) = -\frac{\beta}{\alpha_a}\xi_{a^{-1}}, \quad R_a(\gamma_{a^{-1}}) = \frac{HA_{a^{-1}}}{\alpha_a}. \tag{2.155}$$

Using the first of (2.153) and (2.155) in (2.154) leads us to $\xi_a = -\xi_{a^{-1}}$, which together with the second half of the torsion-freeness conditions (2.151) requires $\xi_a = 0$, and as consequence $\nu_a = 0$. Similarly, inserting the second half of (2.153) and (2.155) in (2.154) produces

$$-A_a - A_{a^{-1}} = \frac{\partial_r \alpha_a}{H}. \tag{2.156}$$

In summary, for a QLC, it only remains to solve for A_a, γ_a subject to such residual equations, with the other coefficients zero or determined. It also remains to impose reality in the form of ∇ $*$-preserving.

Proposition 2.8. *Assuming a static edge-symmetric central metric (2.148) and σ the flip on generators involving* $\mathrm{d}r, \mathrm{d}t$ *leads to a $*$-preserving QLC if and only if $\partial_- \partial_r \alpha_a = 0$ (which needs the underlying $a(r,i)$ to be the sum of a function of r and a function of i). The $*$-preserving QLC with real coefficients is then unique and given by*

$$\begin{aligned}
\nabla \mathrm{d}t &= -\frac{1}{2\beta}\partial_r \beta \mathrm{d}r \otimes_s \mathrm{d}t\\
\nabla \mathrm{d}r &= -\frac{1}{2H}\partial_r H \mathrm{d}r \otimes \mathrm{d}r - \frac{\partial_r \alpha_+}{2H} e^+ \otimes e^- - \frac{\partial_r \alpha_-}{2H} e^- \otimes e^+ - \frac{1}{2H}\partial_r \beta \mathrm{d}t \otimes \mathrm{d}t\\
\nabla e^\pm &= \nabla^{\mathbb{Z}_n} e^\pm - \frac{1}{r}\mathrm{d}r \otimes_s e^\pm.
\end{aligned}$$

Proof. The $*$-preserving conditions for ∇ include conditions on Γ which coincide at each r with those for a QLC on $\mathbb{Z}_n$, for which the solution is unique, so we are forced to this choice for Γ. The remaining $*$-preserving conditions require B, C, h to be real-valued, which already holds because they are functions of the metric coefficients, together with the conditions

$$\sum_a (\overline{A}_{a^{-1}}\sigma(e^{a^{-1}}\otimes e^a) - A_a e^a \otimes e^{a^{-1}}) = 0, \quad \overline{\gamma}_a = R_a(\gamma_{a^{-1}}), \tag{2.157}$$

$$\sum_a (\overline{\xi}_{a^{-1}}\sigma(e^{a^{-1}}\otimes e^a) - \xi_a e^a \otimes e^{a^{-1}}) = 0, \quad \overline{\nu}_a = R_a(\nu_{a^{-1}}). \tag{2.158}$$

The conditions (2.158) are trivially fulfilled, while the second half of (2.157) implies $\overline{A}_a = A_{a^{-1}}$, which together with the form of the braiding map σ solves the first half of (2.157). In this case, (2.156) takes the form

$$-A_a - \overline{A}_a = \frac{\partial_r \alpha_a}{H}. \tag{2.159}$$

The second halves of (2.153) and (2.155) together with the edge-symmetric condition, tell us that $A_a = R_{a^{-1}}(A_a)$ and hence that A_a is independent of the discrete variable, i.e., just function of r. In this case, we must have

$$A_\pm = -\frac{\partial_r \alpha_\pm}{2H} \pm \imath y(r), \quad \gamma_\pm = -\frac{\partial_r \alpha_\pm}{2\alpha_\pm} \pm \imath \frac{Hy(r)}{\alpha_\pm}$$

for some function real-valued function $y(r)$. It is natural at this point to set $y(r) = 0$ so as to keep coefficients real. Another consequence of $A_\pm$ being constant in the polygon is $\partial_\pm A_\pm = 0$, which leads us to $\partial_\pm \partial_r \alpha_a = 0$. This corresponds to restricting underlying metric function $a(r, i)$ in (2.149). □

This is a general result, but we now restrict attention to the $\mathbb{Z}_n$-invariant metric where $a(r, i)$ is independent of i and moreover of the expected radial form.

THEOREM 2.9. *The static $\mathbb{Z}_n$-invariant Schwarzschild-like metric*

$$g = -\beta(r)\mathrm{d}t \otimes \mathrm{d}t + H(r)\mathrm{d}r \otimes \mathrm{d}r - r^2 e^+ \otimes_s e^-$$

has a canonical $*$-preserving QLC,

$$\begin{aligned}
\nabla \mathrm{d}t &= -\frac{1}{2\beta}\partial_r\beta \mathrm{d}r \otimes_s \mathrm{d}t,\\
\nabla \mathrm{d}r &= -\frac{1}{2H}\partial_r H \mathrm{d}r \otimes \mathrm{d}r - \frac{r}{H} e^+ \otimes_s e^- - \frac{1}{2H}\partial_r\beta \mathrm{d}t \otimes \mathrm{d}t,\\
\nabla e^\pm &= -\frac{1}{r}\mathrm{d}r \otimes_s e^\pm
\end{aligned}$$

with the corresponding Ricci scalar and Laplacian

$$\begin{aligned}
S &= \frac{1}{2H\beta}\partial_r^2\beta - \frac{1}{4H\beta^2}(\partial_r\beta)^2 - \frac{1}{4H^2\beta}\partial_r H \partial_r\beta - \frac{1}{rH^2}\partial_r H + \frac{1}{rH\beta}\partial_r\beta + \frac{1}{r^2 H},\\
\square &= \frac{2}{r^2}(\partial_+ + \partial_-) - \frac{1}{\beta}\partial_t^2 + \frac{1}{H}\partial_r^2 + \left(\frac{2}{rH} - \frac{1}{2H^2}\partial_r H + \frac{1}{2H\beta}\partial_r\beta\right)\partial_r.
\end{aligned}$$

This is Ricci flat if and only if

$$H(r) = \frac{1}{\beta(r)}, \quad \beta(r) = \frac{r_H}{r} \tag{2.160}$$

for some constant r_H of length dimension.

Proof. Taking $\alpha_\pm = r^2$ in the preceding proposition immediately gives the canonical QLC stated. Its associated curvature comes out as

$$R_\nabla \mathrm{d}t = \frac{1}{2\beta}\left(\partial_r^2\beta - \frac{1}{2\beta}(\partial_r\beta)^2 - \frac{1}{2H}\partial_r H\partial_r\beta\right)\mathrm{d}t\wedge\mathrm{d}r\otimes\mathrm{d}r - \frac{r}{2H\beta}\partial_r\beta\mathrm{d}t\wedge e^+\otimes_s e^-,$$
$$R_\nabla e^\pm = -\frac{\partial_r H}{2rH}e^\pm\wedge\mathrm{d}r\otimes\mathrm{d}r - \frac{1}{2rH}\partial_r\beta e^\pm\wedge\mathrm{d}t\otimes\mathrm{d}t - \frac{1}{H}e^\pm\wedge e^\mp\otimes e^\pm,$$
$$R_\nabla \mathrm{d}r = \frac{1}{2H}\left(\frac{1}{2H}\partial_r\beta\partial_r H - \partial_r^2\beta + \frac{1}{2\beta}(\partial_r\beta)^2\right)\mathrm{d}r\wedge\mathrm{d}t\otimes\mathrm{d}t - \frac{1}{2}r\partial_r\beta\mathrm{d}r\wedge e^+\otimes_s e^-.$$

Taking the antisymmetric lift of products of basic 1-forms and tracing gives the associated Ricci tensor

$$\begin{aligned}2\mathrm{Ricci} = &\left(\frac{1}{2\beta}\partial_r^2\beta - \left(\frac{\partial_r\beta}{2\beta}\right)^2 - \frac{1}{4H\beta}\partial_r H\partial_r\beta - \frac{1}{rH}\partial_r H\right)\mathrm{d}r\otimes\mathrm{d}r\\ &\left(-\frac{r}{2H\beta}\partial_r\beta + \frac{r}{2H^2}\partial_r H - \frac{1}{H}\right)e^+\otimes_s e^-\\ &+\left(\frac{1}{4H^2}\partial_r\beta\partial_r H - \frac{1}{2H}\partial_r^2\beta + \frac{1}{4H\beta}(\partial_r\beta)^2 - \frac{\partial_r\beta}{rH}\right)\mathrm{d}t\otimes\mathrm{d}t\end{aligned}$$

The Ricci scalar and Laplacian follow on application of the inverse metric. We then solve for Ricci = 0. The calculations are straightforward and are omitted. □

The quantum geometric structures in the 'discrete black hole' Ricci-flat case are

$$g = -\frac{r_H}{r}\mathrm{d}t\otimes\mathrm{d}t + \frac{r}{r_H}\mathrm{d}r\otimes\mathrm{d}r - r^2 e^+\otimes_s e^-, \tag{2.161}$$

$$(\mathrm{d}t,\mathrm{d}t) = -\frac{r_H}{r},\quad (\mathrm{d}r,\mathrm{d}r) = \frac{r}{r_H},\quad (e^\pm,e^\mp) = -\frac{1}{r^2}, \tag{2.162}$$

$$\nabla\mathrm{d}t = \frac{1}{2r}\mathrm{d}r\otimes_s\mathrm{d}t, \tag{2.163}$$

$$\nabla\mathrm{d}r = -\frac{1}{2r}\mathrm{d}r\otimes\mathrm{d}r - r_H e^+\otimes_s e^- + \frac{r_H^2}{2r^3}\mathrm{d}t\otimes\mathrm{d}t, \tag{2.164}$$

$$\nabla e^\pm = -\frac{1}{r}\mathrm{d}r\otimes_s e^\pm, \tag{2.165}$$

$$\mathrm{R}_\nabla\mathrm{d}t = \frac{1}{r^2}\mathrm{d}t\wedge\mathrm{d}r\otimes\mathrm{d}r + \frac{r_H}{2r}\mathrm{d}t\wedge e^+\otimes_s e^-, \tag{2.166}$$

$$\mathrm{R}_\nabla\mathrm{d}r = -\frac{r_H^2}{r^4}\mathrm{d}r\wedge\mathrm{d}t\otimes\mathrm{d}t + \frac{r_H}{2r}\mathrm{d}r\wedge e^+\otimes_s e^-, \tag{2.167}$$

$$\mathrm{R}_\nabla e^\pm = -\frac{1}{2r^2}e^\pm\wedge\mathrm{d}r\otimes\mathrm{d}r + \frac{r_H^2}{2r^4}e^\pm\wedge\mathrm{d}t\otimes\mathrm{d}t \mp \frac{r_H}{r}e^+\wedge e^-\otimes e^\pm, \tag{2.168}$$

$$\square = -\frac{r}{r_H}\partial_t^2 + \frac{r_H}{r}\partial_r^2 + \frac{r_H}{r^2}\partial_r + \frac{2}{r^2}(\partial_+ + \partial_-). \tag{2.169}$$

To keep the signature, we can take $r_H > 0$ and we will analyse this case first. However, to approximately match the inside of a black hole, we should also analyse the case $r_H = -2GM < 0$ with the physical roles of r, t interchanged.

We also note that $\beta = H = 1$ leads to

$$g = -\mathrm{d}t \otimes \mathrm{d}t + \mathrm{d}r \otimes \mathrm{d}r - r^2 e_+ \otimes_s e_-, \quad \mathrm{Ricci} = -\frac{1}{2} e^+ \otimes_s e^-, \quad S = \frac{1}{r^2},$$
$$\Box = -\partial_t^2 + \partial_r^2 + \frac{2}{r}\partial_r + \frac{2}{r^2}(\partial_+ + \partial_-),$$

which is more like the spacetime Laplacian in 3 spatial dimensions, again showing the dimension jump and the constant curvature at each fixed radius and time. Here S^1 behaves more like S^2 in polar coordinates, just with $2(\partial_+ + \partial_-)$ in the role of the angular Laplacian.

3.1. Klein-Gordon equation on the discrete-circle black hole for $\beta(r) > 0$. Here, we analyse the case of the length scale $r_H > 0$ in the Laplacian (2.169) found for the discrete black hole above in 'polar coordinates' form. The eigenvalues of the angular Laplacian $\partial_+ + \partial_-$ are labelled by $l \in \mathbb{Z}_n$ and given by

$$\lambda_l = q^l + q^{-l} - 2 = 2(\cos\left(\frac{2\pi l}{n}\right) - 1) = -4\sin^2(\frac{\pi l}{n}); \quad q = e^{\frac{2\pi \imath}{n}}$$

with eigenfunctions $q^{\imath l}$. We first consider the 'quantum mechanical' solutions of Klein-Gordon equations $\Box\phi = m^2\phi$ of the form

$$\phi = e^{-\imath m t}\psi_l(t, r)$$

of orbital angular momentum l and slowly varying in t. The idea here is described more in chapter 3 including how it works for a usual black hole. Such a form of ϕ is not particularly justified from the form of the metric but leads to

$$\imath\dot{\psi} = -\frac{r_H}{2mr}\left(\Box_r + \frac{2\lambda_l}{r^2}\right)\psi_l + (1 - \frac{r_H}{r})\frac{m}{2}\psi_l; \quad \Box_r = \frac{r_H}{r^2}\partial_r(r\partial_r).$$

The mass term has not been canceled from the Klein-Gordon equation due to the r_H/r factor in the $\mathrm{d}t \otimes \mathrm{d}t$ term in the metric, except in the vicinity of $r \approx r_H$.

Here it makes more sense to look in the 'comoving' case where we start with an $l = 0$ solution of the Klein-Gordon equation of the form

$$\phi = e^{-\imath\omega t}\phi_\omega; \quad \phi_\omega'' + \frac{1}{r}\phi_\omega' + (\frac{r^2}{r_H^2}\omega^2 - \frac{r}{r_H}m^2)\phi_\omega = 0.$$

A generic solution for $\omega = m = r_H = 1$ is shown in Figure 5, which illustrates that we can have an extended region where ϕ_ω is approximately constant, here with boundary condition

$$\phi_\omega'(r_0) = 0, \quad \phi_\omega(r_0) = 1; \quad r_0 := r_H\frac{m^2}{\omega^2}.$$

This results in

$$\left|\frac{\phi_\omega'(r)}{\phi_\omega(r)}\right| < \frac{m}{|\omega| r_H}, \quad r \approx r_0$$

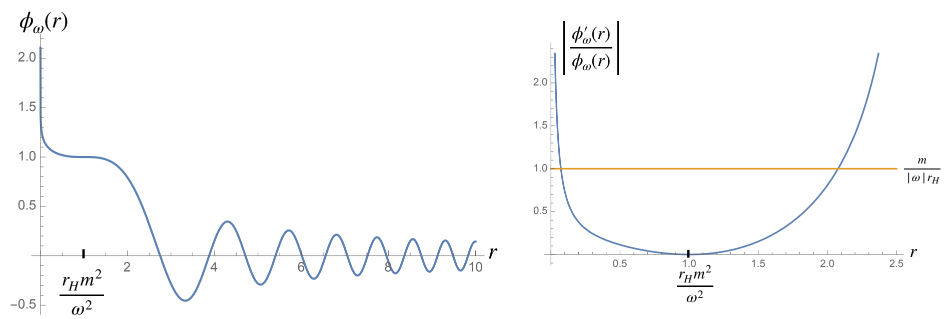

FIGURE 5. Solution of Klein-Gordon equation for $l = 0$ and $\omega = m = r_H = 1$, with Cauchy boundary condition at $r_0 = r_H \frac{m^2}{\omega^2}$.. Image as in [**3**]

for a reasonable range around the central value, as illustrated in the second half of the figure 5. An obvious choice would be $\omega = m$ and hence $r_0 = r_H$, but we can choose other ω to have other central values r_0.

Next, we use this as a reference and look for solutions of the Klein-Gordon equations of the form $\phi = e^{-\imath\omega t}\phi_\omega(r)\psi_l(t, r)$ with ψ_l in the λ_l eigenspace and slowly varying in t. Discarding $\ddot{\psi}_l$ terms, we have

$$\imath\dot{\psi}_l = -\frac{r_H}{2\omega r}\left(\Box_r + \frac{2\phi'_\omega}{\phi_\omega}\frac{r_H}{r}\partial_r + \frac{2\lambda_l}{r^2}\right)\psi_l$$

and hence in any regime where the ϕ'_ω/ϕ_ω term can be neglected, we have approximately

$$\imath\dot{\psi}_l \approx -\frac{r_H}{2\omega r}\left(\Box_r + \frac{2\lambda_l}{r^2}\right)\psi_l$$

as an effective Schroedinger-like equation. We still have an expected scale factor out front, but now the unwanted mass terms are absent, i.e. this looks more like free motion as expected.

We can go further and replace r by a new variable

$$\rho(r) = \frac{r^2}{2r_H}, \quad \frac{\partial}{\partial r} = \frac{r}{r_H}\frac{\partial}{\partial\rho}, \quad \frac{\partial^2}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{r}{r_H}\frac{\partial}{\partial\rho}\right) = \frac{r^2}{r_H^2}\frac{\partial^2}{\partial\rho^2} + \frac{1}{r_H}\frac{\partial}{\partial\rho},$$

in which case

$$\imath\dot{\psi}_l \approx -\frac{1}{2\omega}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{2\lambda_l}{(2\rho)^{\frac{3}{2}} r_H^{\frac{1}{2}}}\right)\psi_l.$$

This absorbs the $\beta^2 = r_H^2/r^2$ factor in front of the radial double derivative so as to look more like flat space quantum mechanics, but has an unusual radial power for the angular contribution. Here ω plays the role of the effective mass and determines the central value

$$\rho_0 = \frac{r_H}{2}\left(\frac{m}{\omega}\right)^4$$

around which we wish our approximation to hold.

3.2. Continuum limit of the discrete black hole. Here we use the classical limit of section 1.2 of this chapter for having S^1 instead of $\mathbb{Z}_n$. We now work on $A = C^\infty(\mathbb{R}\times\mathbb{R}_{>0})\otimes\mathbb{C}[s, s^{-1}]$ with $t, r, \mathrm{d}t, \mathrm{d}r$ classical and graded-commuting with the $s, f^\pm$. We take the metric

$$g = -\frac{r_H}{r}\mathrm{d}t\otimes\mathrm{d}t + \frac{r}{r_H}\mathrm{d}r\otimes\mathrm{d}r + r^2 g_{S^1}$$

and we look for QLCs with σ assumed to be the flip on the basic 1-forms.

Proposition 2.10. *The metric g has a canonical Ricci flat $*$-preserving QLC and associated geometry*

$$\nabla\mathrm{d}t = \frac{1}{2r}\mathrm{d}r\otimes_s\mathrm{d}t,\quad \nabla\mathrm{d}r = -\frac{1}{2r}\mathrm{d}r\otimes\mathrm{d}r + \frac{r_H^2}{2r^3}\mathrm{d}t\otimes\mathrm{d}t + r_H g_{S^1},\quad \nabla f^\pm = -\frac{1}{r}\mathrm{d}r\otimes_s f^\pm,$$

$$\mathrm{R}_\nabla\mathrm{d}t = \frac{1}{r^2}\mathrm{d}t\wedge\mathrm{d}r\otimes\mathrm{d}r - \frac{r_H}{2r}\mathrm{d}t\wedge g_{S^1},\quad \mathrm{R}_\nabla\mathrm{d}r = -\frac{r_H^2}{r^4}\mathrm{d}r\wedge\mathrm{d}t\otimes\mathrm{d}t - \frac{r_H}{2r}\mathrm{d}r\wedge g_{S^1},$$

$$\mathrm{R}_\nabla f^\pm = -\frac{1}{2r^2}f^\pm\wedge\mathrm{d}r\otimes\mathrm{d}r + \frac{r_H^2}{2r^4}f^\pm\wedge\mathrm{d}t\otimes\mathrm{d}t + \frac{r_H}{r}f^\pm\wedge g_{S^1},$$

$$\square = -\frac{r}{r_H}\partial_t^2 + \frac{r_H}{r}\partial_r^2 + \frac{r_H}{r^2}\partial_r + \frac{1}{r^2}\square_{S^1},\quad \square_{S^1} = -\frac{4(1+(q-1)s\partial_q)}{(q+1)^2}(s\partial_q)^2,$$

where ∂_q is the standard q-derivative so that $\square_{S^1}$ on modes s^l has eigenvalue

$$\lambda_l = -\frac{4q^l[l]_q^2}{(q+1)^2},\quad [l]_q := \frac{1-q^l}{1-q}.$$

Proof. First, we can redo the discrete black hole model with $a(r,i) = ar^2$ for any constant factor a for the angular term $g_{\mathbb{Z}_n} = -ae^+\otimes_s e^-$ in the metric. This same factor enters in the connection in the $\nabla\mathrm{d}r$ as $g_{\mathbb{Z}_n}$ there. The same happens for R_∇ in the term where $e^+\otimes_s e^-$ entered. We then replace $g_{\mathbb{Z}_n}$ by g_{S^1} to get the connection as stated, noting that $f^\pm$ are a linear combination of $e^\pm$ so expressions linear in these have the same form. This version is constructed so as to be isomorphic to the discrete black hole when $q = e^{\frac{2\pi\imath}{n}}$ and $s^n = 1$ are imposed, but these properties do not enter into the computations for a QLC, so this also holds for generic q, and likewise for Ricci flatness and for being $*$-preserving when $|q| = 1$. One can also do a direct check of these features and see that ∇ is $*$-preserving also when q is real, as a consequence of g_{S^1} being real in the required sense.

For Ricci, the antisymmetric lift $i(f^+\wedge f^-) = \frac{1}{2}(f^+\otimes f^- - f^-\otimes f^+)$ of

$$f^+\wedge f^- = \left(\frac{q-1}{q+1}\right)(q-q^{-1})^2 e^+\wedge e^-$$

is equivalent to that of $e^+\wedge e^-$ when we use the correspondence (2.15). We also use the inverse metric which on the $f^\pm$ comes out as

$$(f^\pm, f^\pm) = -\frac{4q}{r^2(q+1)^2},\quad (f^\pm, f^\mp) = 2\frac{q^2+1}{r^2(q+1)^2}.$$

For the Laplacian, we use $\mathrm{d}s^l$ from proposition 2.2 and the inverse metric to compute $\Box s^l = (\ ,\ )\nabla\mathrm{d}s^l = -\frac{4q^{2+l}}{r^2(q+1)^2}[l]_q^2 s^l$, which we write as stated since $s\partial_q s^l = [l]_q s^l$ for the standard q-derivative $\partial_q f(s) = (f(qs)-f(s))/((q-1)s)$. The other values of $\Box$ on functions of r, t are unchanged from the discrete case. In the classical case with $s = e^{\imath l\theta}$, we have $s\frac{\partial}{\partial s} = -\imath\frac{\partial}{\partial\theta}$ as the limit of $s\partial_q$. $\Box$

This is a joint process $q \to 1$ *and* $f^+ = -f^-$, with the latter taking precedence so that $g_{S^1} \to -f^+ \otimes f^+ = \mathrm{d}\theta \otimes \mathrm{d}\theta$ as classically in our normalisation of g_{S^1}. In this way, one arrives as the classical 1+2-dimensional curved metric

$$g_{class} = -\frac{r_H}{r}\mathrm{d}t \otimes \mathrm{d}t + \frac{r}{r_H}\mathrm{d}r \otimes \mathrm{d}r + r^2\mathrm{d}\theta \otimes \mathrm{d}\theta,$$

which is not, however, Ricci flat. One finds in our conventions (which are $-1/2$ of the usual ones)

$$\mathrm{Ricci} = -\frac{1}{2}\left(\frac{r_H^2}{2r^4}\mathrm{d}t \otimes \mathrm{d}t - \frac{1}{2r^2}\mathrm{d}r \otimes \mathrm{d}r + \frac{r_H}{r}\mathrm{d}\theta \otimes \mathrm{d}\theta\right), \quad S = 0,$$

$$\Box = -\frac{r}{r_H}\partial_t^2 + \frac{r_H}{r}\partial_r^2 + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

The Laplacian agrees with the limit of the q-deformed geometry but Ricci does not. This is due to the 4D cotangent bundle in the quantum model, since the trace gives a different result from the trace in the quotient, where we impose $f^+ = -f^-$. Moreover, the dropped terms in the metric that are singular as $q \to 1$ contribute in the calculation of Ricci $= 0$ in the quantum model.

3.3. Discrete black hole model for $\beta(r) < 0$. Here we briefly analyse the case where $r_H < 0$ in our previous presentation of the discrete black hole. More precisely, we still define $r_H = 2GM > 0$ but replace r_H by $-r_H$ and we also replace t by r and r by t in all the formulae (2.161)-(2.169) in order to match the signature. Thus, the quantum metric and resulting quantum geometry are now

$$g = -\frac{t}{r_H}\mathrm{d}t \otimes \mathrm{d}t + \frac{r_H}{t}\mathrm{d}r \otimes \mathrm{d}r - t^2 e^+ \otimes_s e^-,$$

$$\nabla\mathrm{d}r = \frac{1}{2t}\mathrm{d}r \otimes_s \mathrm{d}t, \quad \nabla\mathrm{d}t = -\frac{1}{2t}\mathrm{d}t \otimes \mathrm{d}t + \frac{r_H^2}{2t^3}\mathrm{d}r \otimes \mathrm{d}r - r_H e^+ \otimes_s e^-,$$

$$\nabla e^\pm = -\frac{1}{t}\mathrm{d}r \otimes_s e^\pm, \quad \Box = -\frac{r_H}{t}\partial_t^2 + \frac{t}{r_H}\partial_r^2 - \frac{r_H}{t^2}\partial_t + \frac{2}{t^2}(\partial_+ + \partial_-)$$

with a curvature singularity now at $t = 0$. We next make a change of variable

$$t = (\frac{3\tau}{2})^{\frac{2}{3}} r_H^{\frac{1}{3}} = \eta(\tau)^2 r_H, \quad \eta(\tau) = \left(\frac{3\tau}{2r_H}\right)^{\frac{1}{3}}$$

in order to have a constant term in the 'time' coefficient of the metric, so that the quantum geometric structures become

$$
\begin{aligned}
g &= -\mathrm{d}\tau \otimes \mathrm{d}\tau + \eta^{-2}\mathrm{d}r \otimes \mathrm{d}r - \eta^4 r_H^2 e^+ \otimes_s e^-, \\
\nabla e^\pm &= -\frac{2}{3\tau}\mathrm{d}\tau \otimes_s e^\pm, \quad \nabla \mathrm{d}\tau = -\frac{1}{3\eta^2\tau}\mathrm{d}r \otimes \mathrm{d}r - \eta r_H e^+ \otimes_s e^-, \\
\nabla \mathrm{d}r &= \frac{1}{3\tau}\mathrm{d}r \otimes_s \mathrm{d}\tau, \quad \Box = -\partial_\tau^2 + \frac{1}{3\tau}\partial_\tau + \eta^2\partial_r^2 + \frac{2}{\eta^4 r_H^2}(\partial_+ + \partial_-).
\end{aligned}
$$

We now do the parallel analysis to Section 3.1. Using the above Laplacian for the Klein-Gordon equation, we first look for solutions of the form $\phi = e^{-\imath m\tau}\psi_l(\tau, r)$ where ψ_l is slowly varying in τ and with eigenvalue λ_l for the angular sector. Ignoring $\ddot{\psi}_l$, we have

$$
\imath\dot{\psi}_l = -\frac{\eta^2}{2m - \frac{\imath}{3\tau}}\left(\partial_r^2 + \frac{8\lambda_l}{9\tau^2}\right)\psi_l,
$$

where dot denotes ∂_τ. If we assume that we are very far from the $\tau = 0$ singularity in the sense

$$
\tau >> \frac{1}{m}
$$

(i.e. at macroscopic times much larger than the Compton wavelength in time units), we have

$$
\imath\dot{\psi}_l \approx -\frac{\eta^2}{2m}\left(\partial_r^2 + \frac{8\lambda_l}{9\tau^2}\right)\psi_l. \tag{2.170}
$$

This looks, as expected, a bit like quantum mechanics, not in the presence of a point source potential but rather with an overall time-dependent expansion factor and a time-dependent contribution of the angular momentum. Note that $e^{-\imath m\tau}$ does not itself obey the Klein-Gordon equation.

Next, we look for the 'comoving' behaviour, noting that solutions of the Klein-Gordon equation of mass m and $l = 0$ are in fact given by Hankel functions, of which we focus on the first type,

$$
\phi_m(\tau) = \tau^{\frac{2}{3}} H^{(1)}_{\frac{2}{3}}(m\tau).
$$

Here, the real and imaginary parts (Bessel J, K functions respectively) oscillate, $\phi_m(0)$ is a nonzero (imaginary) value, and $|\phi_m|^2$ gradually increases with time. This therefore plays the role of an exact plane wave. Relative to this, we look for solutions of the form

$$
\phi(\tau, r) = \phi_m(\tau)\psi_l(\tau, r)
$$

with ψ_l slowly varying in τ, leading to a Schroedinger-like equation

$$
\imath\dot{\psi}_l = -\frac{\eta^2 h(m\tau)}{2m}\left(\partial_r^2 + \frac{8\lambda_l}{9\tau^2}\right)\psi_l,
$$

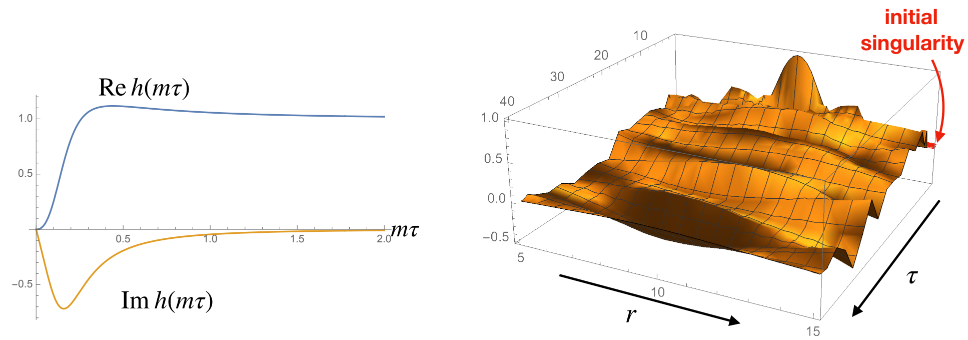

FIGURE 6. Function $h(m\tau)$ in definition of Schroedinger-like equation for discrete black hole metric and evolution of a Gaussian centred at $r = 10r_H$ at $\tau = 1/m$, for $r_H = m = 1$ and $l = 0$. The essentially zero initial values at $r = 0, 20r_H$ are held fixed. Figure as in [**3**]

where

$$h(s) = \imath \frac{H^{(1)}_{\frac{2}{3}}(s)}{H^{(1)}_{-\frac{1}{3}}(s) - \frac{1}{6s} H^{(1)}_{\frac{2}{3}}(s)} \approx 1$$

for large s, as shown on the left in Figure 6. Here, one can see that $h(m\tau)$ approaches 1 very rapidly as $\tau >> 1/m$. In other words, the behaviour near the $\tau = 0$ singularity is different but for larger τ the effective Schroedinger-like equation is now much more sharply approximated by (2.170) than before.

The numerical solution for the real part of this equation is shown on the right in Figure 6, where we used the exact function $h(m\tau)$ and set the initial Gaussian at $m\tau = 1$. The evolution becomes noticeably constant in r compared to regular quantum mechanics. Some of the noise in the picture comes from the numerical approximation.

CHAPTER 3

Cosmological Models over the Fuzzy Sphere

There is a general procedure for constructing a first-order calculus over an enveloping algebra. This construction can be found in [**14**, Chap. 1.6.1], where the main ingredients are theorem 1.41 and example 1.44. The quantum geometry of the fuzzy sphere is developed in [**31**], which we just include for completeness in the first section of this chapter. The rest of the results of the chapter are new and were published in [**3**].

1. Generalities of the fuzzy sphere and its classical limit

We start considering the angular momentum enveloping algebra $U(su_2)$ with basis $\{e_1, e_2, e_3\}$ and relations $[e_i, e_j] = \epsilon_{ijk} e_k$. Besides, we introduce the hermitian generators $x_i = 2\imath\lambda_p e_i$, where λ_p is a real dimensionless parameter, supposed to be of the order of the Planck scale relative to the actual sphere size.

A representation $\rho : U(su_2)^+ \to \Lambda^1$ is considered, where $U(su_2)^+$ is the enveloping algebra without constant terms and Λ^1 is the space expanded by the Pauli matrices σ_i plus the unity matrix. Then, a first-differential calculus can be constructed, where Λ^1 is the bimodule, the exterior derivative is $\mathrm{d}x_i = \rho(x_i) = \lambda_p \sigma_i$ and the bimodule structure is

$$x_i.\omega = x_i\omega, \quad \omega.x_i = x_i\omega + \omega\rho(x_i),$$

where $\omega \in \Lambda^1$. Nevertheless, we use a different calculus here, which uses this set up as a starting point. More details about this calculus can be found in [**14**, Chap 1.6.1]

The unit fuzzy sphere $A = \mathbb{C}_\lambda[S^2]$ in the sense of [**30, 14, 31**] just means the enveloping algebra $U(su_2)$ with an additional relation giving a fixed value of the quadratic Casimir. This is the standard coadjoint quantisation of the unit sphere with its Kirillov-Kostant bracket known since the 1970s, and in our conventions takes the form

$$[x_i, x_j] = 2\imath\lambda_p \epsilon_{ijk} x_k, \quad \sum_i x_i^2 = 1 - \lambda_p^2$$

The conventions are chosen so that the standard spin-j representation descends to a representation of the the fuzzy sphere if $\lambda_p = 1/(2j+1)$, but we are not restricted to these discrete values.

The more novel ingredient in [**14, 31**] is a rotationally invariant differential calculus in the sense of an exterior algebra (Ω, d) given by central basic 1-forms $s^i \in \Omega^1$ and exterior derivative

$$\mathrm{d}x_i = \epsilon_{ijk}x_j s^k, \quad \mathrm{d}s^i = -\frac{1}{2}\epsilon_{ijk}s^j \wedge s^k, \tag{3.1}$$

with associated partial derivatives defined by $\mathrm{d}f(x) = (\partial_i f)s^i$ in this basis (they act in the same way as orbital angular momentum). The s^i are preferable as they graded commute with everything, but they can be recovered in terms of the $\mathrm{d}x_i$ by [**31**]

$$s^i = \frac{1}{1-\lambda_p^2}(x^i\theta' + \epsilon_{ijk}\mathrm{d}x_j x_k); \quad \theta' = x_i s^i = \frac{x_i \mathrm{d}x_i}{2\imath\lambda_p}. \tag{3.2}$$

The basis $\{s^i\}$ is central, then the commutators are $x_i s^j = s^j x_i$. However, it is possible to obtain the commutators in the $\mathrm{d}x_i$ basis, which are

$$[\theta', x_i] = 2\imath\lambda_p \mathrm{d}x_i, \quad [x_i, \mathrm{d}x_j] = 2\imath\lambda_p(\delta_{ij}\theta' - \frac{x_j}{1-\lambda_p^2}(x_i\theta' + \epsilon_{imn}(\mathrm{d}x_m)x_n)). \tag{3.3}$$

These commutation relations are just a consequence of the equations 3.1, 3.2, and the centrality of the basis. Also, the centrality of the s^i elements can be proved from these commutation relations. There is also a $*$-operation with $x_i^* = x_i$ and $s^{i*} = s^i$. Then $*$ commutes with d and $\theta'^* = \theta'$.

A metric on the fuzzy sphere from the point of view of quantum Riemannian geometry means $g \in \Omega^1 \otimes_A \Omega^1$ subject to certain conditions and is shown in [**14, 31**] to be necessarily of the form

$$g = g_{ij}s^i \otimes s^j$$

for a real symmetric matrix g_{ij}. Here g, in order to have a bimodule inverse, needs to be central and this forces the g_{ij} to be constants. Quantum symmetry in the sense $\wedge(g) = 0$ requires the matrix to be symmetric and reality in the sense $\mathrm{flip}(*\otimes*)(g) = g$ then requires g_{ij} to be real-valued. We also need non-degeneracy in the sense of a bimodule map $(\ ,\ ) : \Omega^1 \otimes_A \Omega^1 \to A$ inverting g in the obvious way. Here 'bimodule map' means commuting with the product by elements of a from either side, i.e. fully tensorial from either side. In our case this is $(s^i, s^j) = g^{ij}$, the inverse matrix to g_{ij}. The rotationally invariant 'round metric' is $g_{ij} = \delta_{ij}$ or $g = s^i \otimes s^i$ (sum over i understood).

The new result in [**31**] was to find a quantum Levi-Civita connection $\nabla : \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ in the sense of torsion-free and metric compatible. This can be solved for the fuzzy sphere under the assumption that the coefficients are constant in the s^i basis, giving[**31**]

$$\nabla s^i = -\frac{1}{2}\Gamma^i{}_{jk}s^j \otimes s^k, \quad \Gamma^i{}_{jk} = g^{il}(2\epsilon_{lkm}g_{mj} + \mathrm{Tr}(g)\epsilon_{ljk}).$$

Moreover, as classically, we can just take the map i to be the antisymmetric lift, so

$$i(s^i \wedge s^j) = \frac{1}{2}(s^i \otimes s^j - s^j \otimes s^i).$$

The resulting Ricci curvature on the fuzzy sphere are in [**31**] but in the round metric case one has

$$\nabla s^i = -\frac{1}{2}\epsilon_{ijk}s^j \otimes s^k, \quad \mathrm{Ricci} = -\frac{1}{4}g, \quad S = -\frac{3}{4}.$$

The curvatures here are not the values you might have expected for a unit sphere even allowing for our conventions. Nor does the Einstein tensor (at least, if defined in the usual way) vanish as would be the case for a classical 2-manifold.

To understand this last point better, which is a small new result of this preliminary section, we look more carefully at the classical limit $\lambda_p \to 0$. By the Leibniz rule and the values above, we have for the round metric

$$\begin{aligned}\nabla(\theta') &= x_i \nabla s^i + \mathrm{d}x_i \otimes s^i = -\frac{1}{2}\epsilon_{ijk}x_i s^j \otimes s^k + \epsilon_{ijk}x_j s^k \otimes s^i = \frac{1}{2}\epsilon_{ijk}x_i s^j \otimes s^k \\ &= \frac{1}{2(1-\lambda_p^2)^2}\epsilon_{ijk}(x_j\theta' + \epsilon_{jmn}(\mathrm{d}x_m)x_n) \otimes (x_k\theta' + \epsilon_{kab}(\mathrm{d}x_a)x_b) \\ &= \frac{1}{2}\epsilon_{ijk}\mathrm{d}x^i \otimes (\mathrm{d}x^j)x^k + O(\lambda_p).\end{aligned}$$

This means that we cannot just set $\theta' = 0$ in the classical limit for the given quantum geometry. From the commutator relations (3.3), we see that the calculus is commutative and $x_i\mathrm{d}x_i = 0$ in the classical limit $\lambda_p \to 0$ as expected for the unit sphere, but θ' itself does not need to vanish, and we have seen that it cannot if we want to have a limit for ∇. Rather, we consider the classical limit as the classical sphere plus a single remnant θ' which graded-commutes with everything and (in the classical limit) does not arise from functions and differentials on the sphere. Indeed, this limit is not a strict differential calculus but a generalised one for this reason, but there is no such problem in the quantum case, where

$$\theta' = \frac{1}{2\imath\lambda_p}x_i\mathrm{d}x_i$$

shows its origin as 'normal' to the sphere as embedded in $\mathbb{R}^3$. We now note that the round metric has the limit

$$\begin{aligned}g = s^i \otimes s^i &= (x_i\theta' + \epsilon_{imn}(\mathrm{d}x_m)x_n) \otimes (x_i\theta' + \epsilon_{iab}(\mathrm{d}x_a)x_b) \\ &= (1-\lambda_p^2)\theta' \otimes \theta' + \epsilon_{imn}x_i\theta' \otimes_s (\mathrm{d}x_m)x_n + (\delta_{ma}\delta_{nb} - \delta_{mb}\delta_{na})(\mathrm{d}x_m)x_n \otimes (\mathrm{d}x_a)x_b \\ &= \theta' \otimes \theta' + \mathrm{d}x_i \otimes \mathrm{d}x_i + O(\lambda_p)\end{aligned}$$

since the calculus is commutative to $O(\lambda_p)$. Thus we see that the rotationally invariant 'round' metric actually has an extra direction required by the calculus. We can recover the completely classical S^2 by the limit $\lambda_p \to 0$ *and* projecting $\theta' = 0$, but traces taken for the Ricci curvature before we do this will remember the extra 'normal direction' and not map onto the classical values.

2. Expanding fuzzy sphere FLRW model

Here we work with the coordinate algebra $A = C^\infty(\mathbb{R})\otimes\mathbb{C}_\lambda[S^2]$ where the $\mathbb{R}$ has a classical time t variable with classical dt graded commuting with t, dt and with the generators x_i, s^i of the exterior algebra of the fuzzy sphere.

We first consider a general metric of the form

$$g = \beta \mathrm{d}t\otimes \mathrm{d}t + n_i(\mathrm{d}t\otimes s^i + s^i\otimes \mathrm{d}t) + g_{ij}s^i\otimes s^j \tag{3.4}$$

where g_{ij} is a symmetric 3×3 matrix of coefficients and β, n_i further coefficients, *a priori* all valued in A. The condition in definition 1.3 of chapter 1 forces the $n_i = 0$ and, the centrality of the metric makes the remaining coefficients to be in the center of $C_\lambda[S^2]$, which is trivial. Hence g_{ij}, β are functions only of the time t. The reality condition for quantum metrics forces them to be real-valued.

Next, a general QLC for the calculus has the form

$$\nabla s^i = -\frac{1}{2}\Gamma^i{}_{jk}s^j\otimes s^k + \gamma^i{}_j s^j\otimes_s \mathrm{d}t + \tau^i \mathrm{d}t\otimes \mathrm{d}t \tag{3.5}$$

$$\nabla \mathrm{d}t = \mu_{jk}s^j\otimes s^k + \eta_j s^j\otimes_s \mathrm{d}t + \Gamma \mathrm{d}t\otimes \mathrm{d}t \tag{3.6}$$

again with $\Gamma^i{}_{jk}, \gamma^i{}_j, \tau^i, \Gamma, \eta_j, \mu_{jk}\in A$. However, given that the spatial metric g_{ij} are functions only of t, it is natural to assume this also for the spatial Christoffel symbols Γ^i_{jk} just as is done for the fuzzy sphere alone in [**31**]. In this case, compatibility of ∇ with the relations of commutativity of dt, s^i with t, x_j and the natural assumption that the associated braiding σ has the classical 'flip' form when one of the arguments is dt, conditions needed for a bimodule connection, require that $\gamma^i_j, \tau^i, \Gamma, \eta_j, \mu_{jk}$ are also functions of time alone.

The non trivial conditions for ∇ to be torsion-free are

$$\frac{1}{2}(-\Gamma^i{}_{jk} + \epsilon^i{}_{jk})s^j\otimes s^k = 0, \quad \mu_{jk} = \mu_{kj}, \tag{3.7}$$

since d(dt) = 0. The conditions $\nabla g = 0$ for metric compatibility then produces

$$\begin{aligned}
&\mathrm{d}t\otimes s^i\otimes s^j : \dot g_{ij} + g_{il}\gamma^l{}_j + g_{lj}\gamma^l{}_i = 0,\\
&\mathrm{d}t\otimes \mathrm{d}t\otimes \mathrm{d}t : \dot\beta + 2\beta\Gamma = 0,\\
&s^l\otimes s^m\otimes s^j : -\frac{g_{ij}}{2}\Gamma^i{}_{lm} - \frac{g_{in}}{2}\Gamma^n{}_{pj}\sigma^{ip}{}_{lm} = 0,\\
&s^l\otimes \mathrm{d}t\otimes s^j : g_{ij}\gamma^i{}_l + \beta\mu_{lj} = 0,\\
&\mathrm{d}t\otimes \mathrm{d}t\otimes s^j : g_{ij}\tau^i + \beta\eta_j = 0,\\
&s^n\otimes s^p\otimes \mathrm{d}t : g_{ij}\gamma^j{}_l\sigma^{il}{}_{np} + \beta\mu_{np} = 0,\\
&\mathrm{d}t\otimes s^i\otimes \mathrm{d}t : g_{ij}\tau^j + \beta\eta_i = 0,\\
&s^m\otimes \mathrm{d}t\otimes \mathrm{d}t : 2\beta\eta_m = 0.
\end{aligned}$$

It is clear from the third of these and the first of (3.7) that Γ^i_{jk} is indeed the Christoffel symbol for the fuzzy sphere QLC as solved uniquely in the $*$-preserving case with constant coefficients in [**31**]. Also, the last equation implies that $\eta_m = 0$, and using this together with the fifth or seventh equation, we get $\tau^i = 0$. The second equation makes $\Gamma = -\dot{\beta}/(2\beta)$. Using $g_{ik}\gamma^k{}_j = \gamma_{ij}$ and the symmetry of g_{ij} in the first equation we get the value of $\gamma^i{}_j$, then this together with the 4th equation gives μ_{ij}, resulting in

$$\gamma^i{}_j = -\frac{1}{2}\dot{g}_{jk}g^{ik}, \quad \mu_{ij} = \frac{\dot{g}_{ij}}{2\beta}. \tag{3.8}$$

Note that μ_{ij} is proportional to the time derivative of the metric, which implies that it is also symmetric if the metric is, solving the second half of (3.7). Because $\gamma^i{}_j$ and μ_{ij} just depend on real functions, they are also real-valued functions. This leads to a reasonably canonical QLC.

Theorem 3.1. *Up to a reparametrisation of t, a quantum metric on the algebra $C^\infty(\mathbb{R})\otimes C_\lambda[S^2]$ has to have the form*

$$g = -\mathrm{d}t\otimes\mathrm{d}t + g_{ij}s^i\otimes s^j,$$

where g_{ij} is a time-dependent real 3×3 symmetric matrix. Moreover, this admits a canonical $*$-preserving QLC

$$\nabla\mathrm{d}t = -\frac{1}{2}\dot{g}_{ij}s^i\otimes s^j, \quad \nabla s^i = -\frac{1}{2}\Gamma^i{}_{jk}s^i\otimes s^j - \frac{1}{2}g^{ki}\dot{g}_{jk}s^j\otimes_s \mathrm{d}t,$$

where $\Gamma_{ijk} = 2\epsilon_{ikm}g_{mj} + \mathrm{Tr}(g)\epsilon_{ijk}$ as for the fuzzy sphere in [**31**]. *The associated Ricci scalar and Laplacian are*

$$\begin{aligned} 2S &= -g^{ij}\ddot{g}_{ij} - Tr(g) + \frac{1}{2}(Tr(g))^2 - \delta_{ij} - \frac{1}{4}\left(g^{ml}g^{ij}\dot{g}_{ml}\dot{g}_{ij} + g^{kl}g^{mn}\dot{g}_{nk}\dot{g}_{lm}\right),\\ \Box f &= \left(g^{ji}\partial_j\partial_i - \frac{1}{2}\left(g^{ij}\dot{g}_{ij}\right)\partial_t - \partial_t^2\right)f. \end{aligned}$$

Proof. The analysis for the metric was done above and we were forced by the requirement for the metric to be central (in order to be invertible) to $n_i = 0$ and $\beta(t)$, $g_{ij}(t)$ in (3.4). We add the $*$-reality of the metric in the form $\mathrm{flip}(*\otimes *)g = g$ to find β and g_{ij} real. Quantum symmetry also requires the latter to be symmetric. By a change of t variable, we can generically assume $\beta = -1$, but we do not need to do this.

Now substituting the obtained values so far in the analysis of the general form of the QLC (3.5), we have the connection

$$\nabla\mathrm{d}t = \frac{1}{2\beta}\dot{g}_{ij}s^i\otimes s^j - \frac{1}{2}\frac{\dot{\beta}}{\beta}\mathrm{d}t\otimes\mathrm{d}t; \quad \nabla s^i = -\frac{1}{2}\Gamma^i{}_{jk}s^j\otimes s^k - \frac{1}{2}g^{ki}\dot{g}_{jk}s^j\otimes_s \mathrm{d}t \tag{3.9}$$

for some unknown $\Gamma^i{}_{jk}(t)$, where we assumed that this does not depend on fuzzy sphere variables (which is reasonable given that the metric can not). The requirement of being

$*$-preserving yields

$$\dot g_{jk}(s^j\otimes s^k-\sigma(s^k\otimes s^j))=0,\quad \Gamma^i{}_{jk}s^j\otimes s^k-\overline{\Gamma}^i{}_{kj}\sigma(s^k\otimes s^j)=0. \tag{3.10}$$

with the second of these the same as for the fuzzy sphere in [**31**] at each fixed time. Here we used $\mathrm{d}t^*=\mathrm{d}t$. Thus all the equations for $\Gamma^i{}_{jk}$ are the same as in [**31**] and hence there is a unique solution for it in terms of $g_{ij}(t)$, as stated, under the assumption of no fuzzy sphere dependence. In this case, we know from [**31**] that σ =flip on $s^j\otimes s^k$ and hence the first of (3.10) is empty, as is the 6th of the metric compatibility equations in our previous analysis. The rest of the $*$-preserving conditions require $\Gamma,\eta_i,\gamma^i{}_j$ to be real-valued functions, which already holds as we have solved for them.

The curvature for the connection (3.9) is

$$\begin{aligned}
R_\nabla\mathrm{d}t=&\left(\frac{1}{2\beta}\ddot g_{ij}-\frac{\dot\beta}{4\beta^2}\dot g_{ij}-\frac{1}{4\beta}g^{ml}g_{il}g_{jm}\right)\mathrm{d}t\wedge s^i\otimes s^j\\
&+\frac{1}{4\beta}\left(-\dot g_{lk}\epsilon^l{}_{ij}+\dot g_{il}\Gamma^l{}_{jk}\right)s^i\wedge s^j\otimes s^k+\frac{1}{4\beta}g^{lm}\dot g_{jl}\dot g_{im}s^i\wedge s^j\otimes\mathrm{d}t\\
R_\nabla s^i=&\left(\frac14\Gamma^i{}_{jl}g^{ml}\dot g_{km}-\frac14\Gamma^l{}_{jk}g^{mi}\dot g_{lm}\right)\mathrm{d}t\wedge s^j\otimes s^k+\left(\frac14 g^{li}\dot g_{ml}\epsilon^m{}_{jk}-\frac14\Gamma^i{}_{jl}g^{ml}\dot g_{km}\right)s^j\wedge s^k\otimes\mathrm{d}t\\
&+\left(\frac14\Gamma^i{}_{ml}\epsilon^m{}_{jk}-\frac14\Gamma^i{}_{jm}\Gamma^m{}_{kl}+\frac{1}{4\beta}g^{mi}\dot g_{jm}\dot g_{kl}\right)s^j\wedge s^k\otimes s^l\\
&+\left(-\frac12(\dot g^{ki}\dot g_{jk}+g^{ki}\ddot g_{jk})+\frac{\dot\beta}{4\beta}g^{ki}\dot g_{jk}-\frac14 g^{ki}g^{ml}\dot g_{lk}\dot g_{jm}\right)\mathrm{d}t\wedge s^j\otimes\mathrm{d}t,
\end{aligned}$$

For this connection we have the Ricci tensor as follows

$$\begin{aligned}
2\mathrm{Ricci}=&\left(\frac{1}{2\beta}\left(\ddot g_{ij}-\frac{\dot\beta}{2\beta}\dot g_{ij}-g^{kl}\dot g_{il}\dot g_{jk}\right)+\frac12\Gamma^l{}_{mj}\epsilon^m{}_{li}-\frac14\Gamma^l{}_{lm}\Gamma^m{}_{ij}+\frac{1}{4\beta}g^{ml}\dot g_{ml}\dot g_{ij}\right.\\
&\left.+\ \frac14\Gamma^l{}_{im}\Gamma^m{}_{lj}-\frac{1}{4\beta}g^{ml}\dot g_{im}\dot g_{lj}\right)s^i\otimes s^j\\
&-\left(-\frac12\left(\dot g^{kl}\dot g_{lk}+g^{ij}\ddot g_{ij}\right)+\frac{\dot\beta}{4\beta}g^{ij}\dot g_{ij}-\frac14 g^{kl}g^{mn}\dot g_{nk}\dot g_{ml}\right)\mathrm{d}t\otimes\mathrm{d}t\\
&+\left(\frac12 g^{nl}\dot g_{mn}\epsilon^m{}_{il}-\frac14\Gamma^l{}_{lm}g^{nm}\dot g_{in}+\frac14\Gamma^l{}_{im}g^{nm}\dot g_{ln}\right)s^i\otimes\mathrm{d}t\\
&+\left(-\frac14\Gamma^l{}_{lm}g^{nm}\dot g_{in}+\frac14 g^{nl}\dot g_{mn}\Gamma^m{}_{li}\right)\mathrm{d}t\otimes s^i
\end{aligned}$$

Now taking $\beta = -1$, and the explicit value of $\Gamma^i{}_{jk}$, the Ricci tensor follows as

$$\begin{aligned}
2\mathrm{Ricci} = &\left(-\frac{\ddot{g}_{ij}}{2} + \frac{1}{2}g^{kl}\dot{g}_{il}\dot{g}_{jk} - \frac{1}{4}g^{ml}\dot{g}_{ml}\dot{g}_{ij} - g_{ij} - \delta_{ij} + \frac{1}{2}\mathrm{Tr}(g)g_{ij}\right) s^i \otimes s^j \\
&- \left(-\frac{1}{2}\left(\dot{g}^{kl}\dot{g}_{lk} + g^{ij}\ddot{g}_{ij}\right) - \frac{1}{4}g^{kl}g^{mn}\dot{g}_{nk}\dot{g}_{ml}\right) \mathrm{d}t \otimes \mathrm{d}t \\
&+ \left(\frac{1}{2}g^{nl}\dot{g}_{mn}\epsilon^m{}_{il} - \frac{1}{2}\epsilon^{kl}{}_m g^{nm}g_{kl}\dot{g}_{in} + \frac{1}{4}(2\epsilon^{kl}{}_m g_{ik} + \mathrm{Tr}(g)\epsilon^l{}_{im})g^{nm}\dot{g}_{ln}\right) s^i \otimes \mathrm{d}t \\
&+ \left(-\frac{1}{2}\epsilon^{kl}{}_m g_{kl}g^{nm}\dot{g}_{in} - \frac{1}{4}g^{nl}\dot{g}_{mn}(2\epsilon^{km}{}_i g_{kl} + \mathrm{Tr}(g)\epsilon^m{}_{li})\right) \mathrm{d}t \otimes s^i
\end{aligned}$$

Making the contraction with the inverse metric we recover the required Ricci scalar.

The Laplacian for a function $f = f(t, x^i)$ follows as

$$\square f = (\,,\,)\nabla(\mathrm{d}f) = (\,,\,)\nabla(\partial_i f s^i + \dot{f}\mathrm{d}t) = g^{ij}\partial_i\partial_j f - \partial_t^2 f - \frac{1}{2}g^{ij}\dot{g}_{ij}\partial_t f - \frac{1}{2}g^{jk}\Gamma^i{}_{jk}\partial_i f,$$

where the last term vanish when we take into account the explicit form of $\Gamma^i{}_{jk}$, recovering the required Laplacian. $\square$

The QLC here is unique under the reasonably assumption is in [**31**] that the $\Gamma^i{}_{jk}$ are constant on the fuzzy sphere, given that the g_{ij} have to be. The theorem applies somewhat generally but now we take the expanding round metric $g_{ij} = R^2(t)\delta_{ij}$ for the spatial part, so the metric, non-zero inverse metric entries, QLC, curvature and Laplacian are

$$g = -\mathrm{d}t \otimes \mathrm{d}t + R^2(t)s^i \otimes s^i, \quad (\mathrm{d}t, \mathrm{d}t) = -1, \quad (s^i, s^j) = \frac{\delta^{ij}}{R^2}, \tag{3.11}$$

$$\nabla \mathrm{d}t = -R\dot{R}s^i \otimes s^i; \quad \nabla s^i = -\frac{1}{2}\epsilon^i{}_{jk}s^j \otimes s^k - \frac{\dot{R}}{R}s^i \otimes_s \mathrm{d}t, \tag{3.12}$$

$$R_\nabla \mathrm{d}t = -R\ddot{R}\mathrm{d}t \wedge s^i \otimes s^i, \tag{3.13}$$

$$R_\nabla s^i = \left(\frac{1}{4}\epsilon^{pi}{}_n\epsilon_{pkm} - \dot{R}^2\delta^i{}_m\delta_{nk}\right) s^m \wedge s^n \otimes s^k + \frac{\ddot{R}}{R}\mathrm{d}t \wedge s^i \otimes \mathrm{d}t, \tag{3.14}$$

$$\mathrm{Ricci} = -(\dot{R}^2 + \frac{1}{2}R\ddot{R} + \frac{1}{4})s^i \otimes s^i + \frac{3}{2}\frac{\ddot{R}}{R}\mathrm{d}t \otimes \mathrm{d}t, \quad S = -3\left(\frac{\dot{R}^2}{R^2} + \frac{\ddot{R}}{R} + \frac{1}{4R^2}\right), \tag{3.15}$$

$$\square = \frac{1}{R^2}\sum_i \partial_i^2 - 3\frac{\dot{R}}{R}\partial_t - \partial_t^2. \tag{3.16}$$

Also of interest is the Einstein tensor and, in the absence of a general theory, we assume as before the 'naive definition' Eins = Ricci $- \frac{S}{2}g$, which works out as

$$\mathrm{Eins} = \left(\ddot{R} + \frac{1}{2}\dot{R}^2 + \frac{1}{8}\right) s^i \otimes s^i - \frac{3}{2}\left(\frac{1}{4R^2} + \frac{\dot{R}^2}{R^2}\right) \mathrm{d}t \otimes \mathrm{d}t \tag{3.17}$$

and is justified by checking that

$$\nabla \cdot \mathrm{Eins} = 0. \tag{3.18}$$

Here, if we have any tensor for the form $T = f\mathrm{d}t \otimes \mathrm{d}t + pR^2 s^i \otimes s^i$, then the divergence is

$$\nabla \cdot T = ((\ ,\ ) \otimes \mathrm{id})\nabla T = -\left(\dot f + 3(f+p)\frac{\dot R}{R}\right)\mathrm{d}t + \partial_i p s^i$$

and we use this now for the particular form of the Einstein tensor to establish (3.18). We also assume this form of T for the energy-momentum tensor of dust with pressure p and density f, in which case the continuity equation $\nabla \cdot T = 0$ for p a function only of t is

$$\dot f + 3(f+p)\frac{\dot R}{R} = 0$$

as usual, and Einstein's equation $\mathrm{Eins} + 4\pi G T = 0$ in our curvature conventions is

$$4\pi G f = \frac{3}{2}\left(\frac{\dot R^2}{R^2} + \frac{1}{4R^2}\right), \quad 4\pi G p = -\frac{\ddot R}{R} - \frac{1}{2}\frac{\dot R^2}{R^2} - \frac{1}{8R^2} = -\frac{\ddot R}{R} - \frac{4\pi G}{3} f$$

These are identical to the classical FLRW equations, see e.g.[**23**, Chap. 8], for a 4D closed universe with curvature constant $\kappa = 1/(4R_0^2)$ in the classical FLRW metric

$$-\mathrm{d}t \otimes \mathrm{d}t + R(t)^2\left(\frac{1}{r^2(1-\kappa r^2)}\mathrm{d}r \otimes \mathrm{d}r + g_{S^2}\right),$$

where g_{S^2} is the metric on a unit sphere, R_0 is a normalisation constant with dimension of length, and we have adapted $R(t)$ to include r in order to match our conventions.

3. Black hole with the fuzzy sphere

We assume a similar framework as in the previous section, but now with a 4D metric of a static form in polar coordinates. Thus, we add a radial variable r with differential $\mathrm{d}r$ and consider the Schwarzschild-like metric

$$g = -\beta(r)\mathrm{d}t \otimes \mathrm{d}t + H(r)\mathrm{d}r \otimes \mathrm{d}r + r^2 g_{ij} s^i \otimes s^j. \tag{3.19}$$

The algebra of functions here is $A = C^\infty(\mathbb{R} \times \mathbb{R}_{>0}) \otimes \mathbb{C}_\lambda[S^2]$ with classical variables and differentials $t, r, \mathrm{d}r, \mathrm{d}t$ for the $\mathbb{R}\times\mathbb{R}_{>0}$ part (so these graded commute among themselves and with the functions and forms on the fuzzy sphere). The coefficients g_{ij} define the metric on the fuzzy sphere, and centrality and reality of the metric dictates that these are constant real values. Thus, g_{ij} is a real symmetric invertible 3×3 matrix (it should also be positive definite for the expected signature) and $\beta(r), H(r)$ are real-valued functions.

We start with the general form of connection on the tensor product calculus,

$$\nabla s^i = -\frac{1}{2}\Gamma^i{}_{jk} s^j \otimes s^k + \alpha^i \mathrm{d}t \otimes \mathrm{d}t + \gamma^i \mathrm{d}r \otimes \mathrm{d}r + \Delta^i \mathrm{d}r \otimes_s \mathrm{d}t + \eta^i{}_j \mathrm{d}t \otimes_s s^j + \tau^i{}_j \mathrm{d}r \otimes_s s^j,$$

$$\nabla \mathrm{d}t = a_{ij} s^i \otimes s^j + b\mathrm{d}t \otimes \mathrm{d}t + c\mathrm{d}r \otimes \mathrm{d}r + d\mathrm{d}r \otimes_s \mathrm{d}t + e_j \mathrm{d}r \otimes_s s^j + f_j \mathrm{d}t \otimes_s s^j,$$

$$\nabla \mathrm{d}r = h_{ij} s^i \otimes s^j + \theta \mathrm{d}t \otimes \mathrm{d}t + R\mathrm{d}r \otimes \mathrm{d}r + \phi \mathrm{d}r \otimes_s \mathrm{d}t + \nu_j \mathrm{d}t \otimes_s s^j + \psi_j \mathrm{d}r \otimes_s s^j.$$

Assuming that $\sigma(\mathrm{d}t\otimes), \sigma(\otimes\mathrm{d}t), \sigma(\mathrm{d}r\otimes), \sigma(\otimes\mathrm{d}r)$ are the flip on the 1-forms $\mathrm{d}r, \mathrm{d}t, s^i$ and the natural restrictions needed for a bimodule connection, one finds that all the coefficients are functions of t and r alone (constant on the fuzzy sphere).

The torsion freeness conditions for $\nabla\mathrm{d}t$, $\nabla\mathrm{d}r$ and ∇s^i are

$$a_{ij} = a_{ji}, \quad h_{ij} = h_{ji}, \quad \Gamma^i{}_{jk} - \Gamma^i{}_{kj} + 2\epsilon^i{}_{jk} = 0, \tag{3.20}$$

respectively, and the conditions needed for the compatibility with the metric are

$$\begin{aligned}
\mathrm{d}r\otimes\mathrm{d}t\otimes\mathrm{d}t &: \partial_r\beta + 2\beta d = 0,\\
\mathrm{d}r^{\otimes 3} &: \partial_r H + 2HR = 0,\\
\mathrm{d}r\otimes s^l\otimes s^j &: 2rg_{lj} + r^2 g_{ij}\tau^i{}_l + r^2 g_{lm}\tau^m{}_j = 0,\\
s^m\otimes s^n\otimes\mathrm{d}t &: -\beta_{mn} + r^2 g_{ij}\eta^j{}_l\sigma^{il}{}_{mn} = 0,\\
\mathrm{d}t^{\otimes 3} &: -2\beta b = 0,\\
\mathrm{d}r\otimes\mathrm{d}r\otimes\mathrm{d}t/\mathrm{d}r\otimes\mathrm{d}t\otimes\mathrm{d}r &: -\beta c + H\phi = 0,\\
\mathrm{d}t\otimes\mathrm{d}r\otimes\mathrm{d}r &: 2H\phi = 0,\\
s^i\otimes\mathrm{d}t\otimes\mathrm{d}r/s^i\otimes\mathrm{d}r\otimes\mathrm{d}t &: -\beta e_i + H\nu_i = 0,\\
\mathrm{d}r\otimes s^j\otimes\mathrm{d}t/\mathrm{d}r\otimes\mathrm{d}t\otimes s^j &: -\beta e_j + r^2 g_{ij}\Delta^i = 0,\\
s^i\otimes\mathrm{d}t\otimes\mathrm{d}t &: -2\beta f_j = 0,\\
\mathrm{d}t\otimes\mathrm{d}t\otimes s^j/\mathrm{d}t\otimes s^j\otimes\mathrm{d}t &: -\beta f_j + r^2 g_{ij}\alpha^i = 0,
\end{aligned}$$

$$\begin{aligned}
s^i\otimes\mathrm{d}t\otimes s^j &: -\beta a_{ij} + r^2 g_{lj}\eta^l{}_i = 0,\\
\mathrm{d}t\otimes\mathrm{d}r\otimes\mathrm{d}t/\mathrm{d}t\otimes\mathrm{d}t\otimes\mathrm{d}r &: -\beta d + H\theta = 0,\\
\mathrm{d}r\otimes\mathrm{d}t\otimes\mathrm{d}t &: \partial_r\beta + 2\beta d = 0,\\
s^m\otimes s^n\otimes\mathrm{d}r &: Hh_{mn} + r^2 g_{ij}\tau^j{}_l\sigma^{il}{}_{mn} = 0,\\
\mathrm{d}t\otimes\mathrm{d}r\otimes s^i/\mathrm{d}t\otimes s^i\otimes\mathrm{d}r &: H\nu_i + r^2 g_{ij}\Delta^j = 0,\\
\mathrm{d}r\otimes\mathrm{d}r\otimes s^i/\mathrm{d}r\otimes s^i\otimes\mathrm{d}r &: H\psi_i + r^2 g_{ij}\gamma^j = 0,\\
s^i\otimes\mathrm{d}r\otimes\mathrm{d}r &: 2H\psi_i = 0,\\
s^i\otimes\mathrm{d}r\otimes s^j &: Hh_{ij} + r^2 g_{lj}\tau^l{}_i = 0,\\
s^p\otimes s^q\otimes s^m &: g_{lm}\Gamma^l{}_{pq} + g_{ij}\Gamma^j{}_{lm}\sigma^{il}{}_{pq} = 0,\\
\mathrm{d}t\otimes s^i\otimes s^j &: g_{lj}\eta^l{}_i + g_{il}\eta^l{}_j = 0.
\end{aligned}$$

We immediately note that $b = \phi = f_j = \psi_i = 0$ for the 5th, 7th, 10th, and 18th equations respectively. In this case, we have that $\alpha^i = \phi = \gamma^i = 0$ by the 11th, 6th and 17th equations respectively. Also, solving simultaneously the 8th, 9th, 16th equations, we obtain $\Delta^i = e_i = \nu_i = 0$. The value of d and R is deduced for the 1st and 2nd equations respectively, while θ comes from 13th and 1st equations, with result

$$d=-\frac{\partial_r\beta}{2\beta},\quad R=-\frac{\partial_r H}{2H},\quad \theta=-\frac{\partial_r\beta}{2H}. \tag{3.21}$$

The 3rd equations together with the symmetry of g_{ij} lead to $\tau^i{}_j=-\frac{1}{r}\delta^i{}_j$. Now, we can solve the 19th equation as

$$h_{ij}=\frac{r}{H}g_{ij}. \tag{3.22}$$

The 21st equation gives the condition $\eta_{ij}-\eta_{ji}=0$, where we used $\eta^k{}_j g_{ki}=\eta_{ij}$. But the 12th equation produces $a_{ij}=\frac{r^2}{\beta}\eta_{ij}$ so that a_{ij} is anti-symmetric, which together with the torsion freeness conditions imply that $a_{ij}=\eta^i{}_j=0$.

Theorem 3.2. *The static Schwarzschild-like metric with spatial part a fuzzy sphere,*

$$g=-\beta(r)\mathrm{d}t\otimes\mathrm{d}t+H(r)\mathrm{d}r\otimes\mathrm{d}r+r^2g_{ij}s^i\otimes s^j,$$

where g_{ij} is a real symmetric matrix with entries constant on the fuzzy sphere, has a canonical $*$-preserving QLC given by

$$\nabla\mathrm{d}t=-\frac{1}{2\beta}\partial_r\beta\mathrm{d}r\otimes_s\mathrm{d}t,\quad \nabla\mathrm{d}r=-\frac{1}{2H}\partial_rH\mathrm{d}r\otimes\mathrm{d}r+\frac{r}{H}g_{ij}s^i\otimes s^j-\frac{1}{2H}\partial_r\beta\mathrm{d}t\otimes\mathrm{d}t,$$
$$\nabla s^i=-\frac{1}{2}\Gamma^i{}_{jk}s^j\otimes s^k-\frac{1}{r}\mathrm{d}r\otimes_s s^i,$$

where $\Gamma_{ijk}=2\epsilon_{ikm}g_{mj}+\mathrm{Tr}(g)\epsilon_{ijk}$ is the fuzzy sphere QLC from [**31**]. *The corresponding Ricci scalar and Laplacian are*

$$\begin{aligned}S=&\frac{1}{2H\beta}\partial_r^2\beta-\frac{1}{4H\beta^2}(\partial_r\beta)^2-\frac{1}{4H^2\beta}\partial_r\beta\partial_rH+\frac{3}{2r^2}\\&+\frac{1}{4rH}(3+Tr(g))\left(\frac{\partial_r\beta}{\beta}-\frac{\partial_rH}{H}\right)+\frac{Tr(g)}{r^2H}(1-\frac{H}{2})+\frac{(Tr(g))^2}{4r^2},\end{aligned}$$
$$\square=-\frac{1}{\beta}\partial_t^2+\frac{1}{H}\partial_r^2+\left(\frac{3}{rH}-\frac{\partial_rH}{2H^2}+\frac{\partial_r\beta}{2H\beta}\right)\partial_r+\frac{g^{ij}}{r^2}\partial_i\partial_j.$$

Proof. Most of the analysis was done above. The torsion-freeness and metric compatibility conditions for Christoffel symbol Γ of the fuzzy sphere part are the same as in [**31**] as is the second half of the $*$-preserving conditions

$$h_{ij}s^i\otimes s^j-\overline{h}_{ji}\sigma(s^j\otimes s^i)=0,\quad \Gamma^i{}_{jk}s^j\otimes s^k-\overline{\Gamma}^i{}_{kj}\sigma(s^k\otimes s^j)=0$$

coming from $\nabla\mathrm{d}r$ and ∇s^i respectively, with $(s^i)^*=s^i,\mathrm{d}r^*=\mathrm{d}r$ and $\mathrm{d}t^*=\mathrm{d}t$. There is therefore a unique solution for Γ under the assumption that it consists of constants according to [**31**], and we use this solution. This has σ the flip on the s^i and hence Γ real. In this case, the other condition for $*$-preserving requires h_{ij} to be hermitian, which already holds because h_{ij} is real and symmetric for (3.22). The 4th and 15th metric compatibility equations also then hold. The connection stated is then obtained by substituting into the general form of the connection. This completes the analysis for the canonical QLC.

The curvature for this connection comes out as

$$\begin{aligned}
R_\nabla s^i &= \left(\frac{1}{4}\Gamma^i{}_{jk}\epsilon^j{}_{mn} - \frac{1}{4}\Gamma^i{}_{ml}\Gamma^l{}_{nk} + \frac{1}{H}g_{nk}\delta^i{}_m\right) s^m \wedge s^n \otimes s^k \\
&\quad - \frac{1}{2rH}\partial_r H s^i \wedge \mathrm{d}r \otimes \mathrm{d}r + \frac{1}{2r}\left(\epsilon^i{}_{jk} - \Gamma^i{}_{jk}\right) s^j \wedge s^k \otimes \mathrm{d}r - \frac{1}{2rH}\partial_r\beta s^i \wedge \mathrm{d}t \otimes \mathrm{d}t \\
R_\nabla \mathrm{d}t &= \left(\frac{\partial_r^2\beta}{2\beta} - \left(\frac{\partial_r\beta}{2\beta}\right)^2 - \frac{1}{4\beta H}\partial_r\beta\partial_r H\right)\mathrm{d}t \wedge \mathrm{d}r \otimes \mathrm{d}r + \frac{r}{2H\beta}\partial_r\beta g_{ij}\mathrm{d}t \wedge s^i \otimes s^j \\
R_\nabla \mathrm{d}r &= -\frac{r}{2H^2}g_{ij}\partial_r H \mathrm{d}r \wedge s^i \otimes s^j + \frac{r}{2H}(g_{ml}\Gamma^l{}_{nj} - g_{ij}\epsilon^i{}_{mn})s^m \wedge s^n \otimes s^j \\
&\quad + \left(\frac{1}{4H^2}\partial_r\beta\partial_r H - \frac{1}{2H}\partial_r^2\beta + \frac{1}{4\beta H}(\partial_r\beta)^2\right)\mathrm{d}r \wedge \mathrm{d}t \otimes \mathrm{d}t + \frac{g_{ij}}{H}s^i \wedge s^j \otimes \mathrm{d}r.
\end{aligned}$$

Taking the antisymmetric lift of products of the basic 1-forms and tracing gives the associated Ricci tensor

$$\begin{aligned}
4\mathrm{Ricci} &= \Big(\frac{\partial_r^2\beta}{\beta} - \frac{3}{rH}\partial_r H - \frac{1}{2\beta H}\partial_r\beta\partial_r H - \frac{1}{2}\left(\frac{\partial_r\beta}{\beta}\right)^2\Big)\mathrm{d}r \otimes \mathrm{d}r \\
&\quad + \frac{1}{H}\left(\frac{1}{2H}\partial_r\beta\partial_r H - \partial_r^2\beta + \frac{1}{2\beta}(\partial_r\beta)^2 - \frac{3}{r}\partial_r\beta\right)\mathrm{d}t \otimes \mathrm{d}t \\
&\quad + \Big(\frac{r}{H\beta}g_{ij}\partial_r\beta - \frac{r}{H^2}g_{ij}\partial_r H + 4\frac{g_{ij}}{H} - 2g_{ij} - 2\delta_{ij} + \mathrm{Tr}(g)g_{ij}\Big)s^i \otimes s^j.
\end{aligned}$$

This gives the Ricci scalar as stated. The Laplacian is also immediate from ∇ and the inverse metric. □

The QLC here is unique under the reasonable assumption as in [**31**] that the $\Gamma^i{}_{jk}$ are constant on the fuzzy sphere, given that the g_{ij} have to be. To do some physics we focus on the static rotationally invariant case where $g_{ij} = k\delta_{ij}$, for a positive constant k. In this case, it follows from the above that Ricci = 0 if and only if

$$H(r) = \frac{1}{\beta(r)}, \quad \beta(r) = \frac{1}{2}(\frac{1}{k} + 1) - \frac{3}{4}k + \frac{c_1}{r^2},$$

where c_1 is an arbitrary constants. The values

$$k = \frac{1}{3}(\sqrt{7} - 1), \quad c_1 = -r_H^2 \tag{3.23}$$

give the form of β for the Tangherlini black hole metric of mass M, see [**68, 69**], namely

$$\beta(r) = 1 - \frac{r_H^2}{r^2}, \quad r_H^2 = \frac{8}{3}G_5 M, \tag{3.24}$$

but note that the latter only makes sense in 5D spacetime due to an extra length dimension in the Newton constant G_5. We are thinking of our model as 4D so we will *not* take this value but just work with r_H as a free parameter. A different value of k can be absorbed in different normalisation of the t, r variables while r_H is more physical.

The quantum geometric structures in this 'fuzzy black hole' Ricci flat case are

$$g = -(1 - \frac{r_H^2}{r^2})\mathrm{d}t \otimes \mathrm{d}t + (1 - \frac{r_H^2}{r^2})^{-1}\mathrm{d}r \otimes \mathrm{d}r + r^2 k s^i \otimes s^i, \tag{3.25}$$

$$(\mathrm{d}t, \mathrm{d}t) = -\frac{r^2}{r^2 - r_H^2}, \quad (\mathrm{d}r, \mathrm{d}r) = 1 - \frac{r_H^2}{r^2}, \quad (s^i, s^j) = \frac{\delta^{ij}}{kr^2}, \tag{3.26}$$

$$\nabla \mathrm{d}t = -\frac{r_H^2}{r(r^2 - r_H^2)}\mathrm{d}r \otimes_s \mathrm{d}t, \tag{3.27}$$

$$\nabla \mathrm{d}r = \frac{r_H^2}{r(r^2 - r_H^2)}\mathrm{d}r \otimes \mathrm{d}r - \frac{r_H^2}{r^3}\left(1 - \frac{r_H^2}{r^2}\right)\mathrm{d}t \otimes \mathrm{d}t + rk\left(1 - \frac{r_H^2}{r^2}\right)s^i \otimes s^i, \tag{3.28}$$

$$\nabla s^i = -\frac{1}{2}\epsilon^i{}_{jk} s^j \otimes s^k - \frac{1}{r}\mathrm{d}r \otimes_s s^i, \tag{3.29}$$

$$\mathrm{R}_\nabla \mathrm{d}t = -\frac{3r_H^2}{r^2(r^2 - r_H^2)}\mathrm{d}t \wedge \mathrm{d}r \otimes \mathrm{d}r + \left(\frac{r_H}{r}\right)^2 k\mathrm{d}t \wedge s^i \otimes s^i, \tag{3.30}$$

$$\mathrm{R}_\nabla \mathrm{d}r = \left(\frac{r_H}{r}\right)^2 k\mathrm{d}r \wedge s^i \otimes s^i + 3r_H^2 \frac{r^2 - r_H^2}{r^6}\mathrm{d}r \wedge \mathrm{d}t \otimes \mathrm{d}t, \tag{3.31}$$

$$\begin{aligned} \mathrm{R}_\nabla s^i = &\left(-\frac{1}{4} + k\left(1 - \frac{r_H^2}{r^2}\right)\right)s^i \wedge s^j \otimes s^j + \left(\frac{r_H}{r}\right)^2 \frac{1}{r^2 - r_H^2} s^i \wedge \mathrm{d}r \otimes \mathrm{d}r \\ &+ \frac{r_H^2}{r^6}(r_H^2 - r^2)s^i \wedge \mathrm{d}t \otimes \mathrm{d}t, \end{aligned} \tag{3.32}$$

$$\Box = -\left(1 - \frac{r_H^2}{r^2}\right)^{-1} \partial_t^2 + \left(\frac{3}{r} - \frac{r_H^2}{r^3}\right)\partial_r + \left(1 - \frac{r_H^2}{r^2}\right)\partial_r^2 + \frac{1}{kr^2}\sum_i \partial_i^2. \tag{3.33}$$

For comparison, the classical Tangherilini 5D black hole metric has the form

$$g = -\left(1 - \frac{r_H^2}{r^2}\right)\mathrm{d}t \otimes \mathrm{d}t + \left(1 - \frac{r_H^2}{r^2}\right)^{-1} \mathrm{d}r \otimes \mathrm{d}r + r^2 g_{S^3}$$

with the Laplacian

$$\Box = -\left(1 - \frac{r_H^2}{r^2}\right)^{-1} \partial_t^2 + \left(\frac{3}{r} - \frac{r_H^2}{r^3}\right)\partial_r + \left(1 - \frac{r_H^2}{r^2}\right)\partial_r^2 + \frac{1}{r^2}\Box_{S^3}$$

where g_{S^3} denotes the metric element on a unit S^3. We see that this has just the same form as our metric and Laplacian except that our unit fuzzy sphere Laplacian $\sum_i \partial_i^2$ is replaced by the unit S^3 Laplacian

$$\Box_{S^3} = \frac{1}{\sin^2\psi}\partial_\psi(\sin^2\psi\partial_\psi) + \frac{1}{\sin^2\psi\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{\sin^2\psi\sin^2\theta}\partial_\phi^2$$

in standard angular coordinates.

We will also be interested in the spatial geometry of the fuzzy black hole as a time slice with respect to the t coordinate. This is easily achieved in our formalism.

Proposition 3.3. *Defining the spatial geometry of the fuzzy black hole as a slice of the 4D geometry by setting* $\mathrm{d}t = 0$*, gives*

$$g = \beta^{-1}\mathrm{d}r \otimes \mathrm{d}r + kr^2 s^i \otimes s^i,$$

$$\nabla \mathrm{d}r = \frac{r_H^2}{r^3\beta}\mathrm{d}r \otimes \mathrm{d}r + kr\beta s^i \otimes s^i, \quad \nabla s^i = -\frac{1}{2}\epsilon^i{}_{jk} s^j \otimes s^k - \frac{1}{r}\mathrm{d}r \otimes_s s^i,$$

$$R_\nabla \mathrm{d}r = k(\frac{r_H}{r})^2 \mathrm{d}r \wedge s^i \otimes s^i, \quad R_\nabla s^i = (k\beta - \frac{1}{4})s^i \wedge s^j \otimes s^j + \frac{r_H^2}{r^4\beta} s^i \wedge \mathrm{d}r \otimes \mathrm{d}r,$$

$$\mathrm{Ricci} = \frac{3r_H^2}{2r^4\beta}\mathrm{d}r \otimes \mathrm{d}r + \left(k(1 - \frac{r_H^2}{2r^2}) - \frac{1}{4}\right) s^i \otimes s^i, \quad S = \frac{3}{r^2}(1 - \frac{1}{4k})$$

using the antisymmetric lift as usual and $\beta = 1 - \frac{r_H^2}{r^2}$*. The spatial Einstein tensor* $\mathrm{Eins} = \mathrm{Ricci} - \frac{S}{2} g$ *comes out as*

$$\mathrm{Eins} = \frac{3}{2r^2\beta}(\frac{1}{4k} - \beta)\mathrm{d}r \otimes \mathrm{d}r + \frac{1}{2}(\frac{1}{4} - k(1 + \frac{r_H^2}{r^2}))s^i \otimes s^i$$

and is conserved in the sense $\nabla \cdot \mathrm{Eins} = 0$*.*

Proof. That setting $\mathrm{d}t = 0$ gives a QLC for the reduced metric and its curvature follows on general grounds but can be checked explicitly. The computation of Ricci is a trace of R_∇ as usual: we apply this to the second factors of g and then apply $(\mathrm{d}r, \mathrm{d}r) = \beta^{-1}$, $(s^i, s^j) = \frac{\delta_{ij}}{kr^2}$ (and other cases zero) to the first two tensor factors. The Ricci scalar S and Einstein tensor then follow. For its divergence, we first compute $\nabla\mathrm{Eins}$ by acting with ∇ on each tensor factor but keeping its left-most output to the far left,

$$\begin{aligned}\nabla\mathrm{Eins} = &\frac{1}{2}(\frac{1}{4} - k(1 + \frac{r_H^2}{r^2}))(-\frac{1}{r}s^i \otimes s^i \otimes \mathrm{d}r) + \mathrm{d}(\frac{3}{2r^2\beta}(\frac{1}{4k} - \beta)) \otimes \mathrm{d}r \otimes \mathrm{d}r \\ &+ \frac{3}{2r^2\beta}(\frac{1}{4k} - \beta)(\frac{2r_H^2}{r^3\beta}\mathrm{d}r \otimes \mathrm{d}r \otimes \mathrm{d}r + kr\beta s^i \otimes s^i \otimes \mathrm{d}r) + \cdots\end{aligned}$$

where $\cdots$ refers to terms that involve $\mathrm{d}r \otimes s^i$ or $s^i \otimes \mathrm{d}r$ in the first two tensor factors. The terms in $\nabla(s^i \otimes s^i)$ with s's in all tensor factors cancel. We then define $\nabla \cdot \mathrm{Eins}$ by applying $(\ ,\ )$ to the first two tensor factors to give

$$\nabla{\cdot}\mathrm{Eins} = \left(-\frac{3}{2r^3k}(\frac{1}{4} - k(1 + \frac{r_H^2}{r^2})) + \beta(\frac{1}{4} - k(1 + \frac{r_H^2}{r^2}))' + \frac{3}{2r^2}(\frac{1}{4k} - \beta)\frac{2r_H^2}{r^3\beta} + \frac{9}{2r^3}(\frac{1}{4k} - \beta)\right)\mathrm{d}r$$

from the displayed terms taken in order. We then check that the function in brackets vanishes. $\square$

3.1. Motion in the fuzzy black hole background. In terms of physical implications, since the radial form for the fuzzy black hole is the same as that of the Tangherilini solution, we can apply the usual logic that $g_{00} = -(1 + 2\Phi)$ to first approximation contains the gravitational potential Φ per unit mass governing geodesic motion for a mass m in the weak field limit, see [**23**, Chap. 4.1]. Therefore in our case, this should be

$$\Phi = -\frac{r_H^2}{2r^2} \tag{3.34}$$

but, because we are thinking of this as a 4D model, we do not set r_H to be the same as a Tangherilini 5D black hole. Rather, we think of r_H as the physical parameter and equate it for purpose of comparison with $r_H = 2GM$ so that the horizon occurs at the same r as for a Schwarzschild black hole of mass M. The weak field force law is no longer Newtonian gravity, having an inverse cubic form in r according to $\Phi = -2G^2M^2/r^2$. This is rather different from modified gravity schemes such as MOND for the modelling of dark matter [**42**] but could still be of interest.

To properly justify the above, we should study geodesics, which is possible but not easy on quantum spacetimes. Here, we instead reach the same conclusion from the point of view of quantum mechanics as the non-relativistic limit of the Klein-Gordon equation

$$\Box\phi = m^2\phi.$$

We already used this point of view in Section 3 of chapter 2. For the reference, we first do it for for a Schwarzschild black hole where we have the standard value $\beta(r) = 1 - \frac{r_H}{r}$, see [**23**, Chap. 5]. The Laplacian is

$$\Box_{Sch} = -\frac{1}{\beta}\frac{\partial^2}{\partial t^2} + \Delta_r + \frac{1}{r^2}\Delta_{S^2}; \quad \Box_r := \frac{1}{r^2}\frac{\partial}{\partial r}(\beta r^2\frac{\partial}{\partial r})$$

and $\Box_{S^2}$ is normalised to have eigenvalues $\lambda_l = -l(l+1)$ on the spherical harmonics of degree $l \in \mathbb{N} \cup \{0\}$ for the orbital angular momentum. We focus on waves of fixed l and look for solutions of the form

$$\phi = e^{-\imath mt}\psi_l(t,r)$$

with ψ_l slowly varying in t. Accordingly neglecting its double time derivative, the Klein-Gordon equation becomes the Schroedinger-like equation

$$\imath\dot{\psi}_l = -\frac{\beta}{2m}\left(\Box_r + \frac{\lambda_l}{r^2}\right)\psi_l + (\beta - 1)\frac{m}{2}\psi_l, \tag{3.35}$$

where $\beta(\Box_r + \frac{\lambda_l}{r^2})$ is a modification by β of the $\mathbb{R}^3$ Laplacian on ψ_l in polars, which we think of as the square of a modified momentum (the difference is anyhow suppressed at large r), and $(\beta - 1)m/2 = -GMm/r$ is the expected Newtonian potential for Schroedinger's equation in the presence of a point source of mass M.

Note that $e^{-\imath mt}$ is not itself a solution of the Klein-Gordon equation. Repeating the above but with reference to an actual solution would be analogous to finding the forces experienced by a particle in geodesic motion, where one only sees tidal forces. Continuing in the Schwarzschild case, we first solve (numerically) for spherical $l = 0$ solutions of the

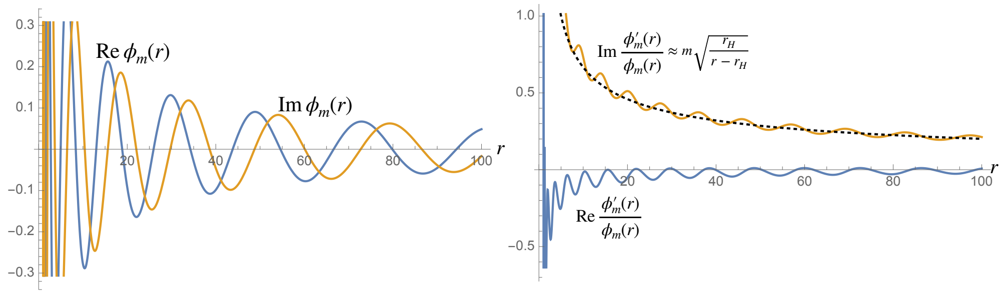

FIGURE 1. Radial solution $\phi_m(r)$ for the Klein-Gordon equation around a black hole shown for $m = 2/r_H, r_H = 1$ and asymptotic form of ϕ_m'/ϕ_m shown dashed. Figure as in [**3**]

form

$$\phi = e^{-\imath\omega t}\phi_\omega(r); \quad (\frac{\omega^2}{\beta} - m^2)\phi_\omega + \Delta_r\phi_\omega = 0 \tag{3.36}$$

with initial conditions specified at large r. We then look for a Schroedinger-like equation relative to ϕ_ω by solving the Klein-Gordon equation for solutions of the form

$$\phi = e^{-\imath\omega t}\phi_\omega(r)\psi_l(t,r) \tag{3.37}$$

with ψ_l of orbital angular momentum labelled by l and slowly varying in t. This time we obtain

$$\imath\dot{\psi}_l = -\frac{\beta}{2\omega}(\Box_r + \frac{\lambda_l}{r^2} + 2\beta\frac{\phi_\omega'}{\phi_\omega}\frac{\partial}{\partial r})\psi_l \tag{3.38}$$

with a new velocity-dependent correction but without the gravitational point source potential, as expected.

The natural choice for reference field here to focus on the case $\omega = m$. For large r, we can neglect β' relative to $2/r$ in $\Box_r$ and in this case one has an exact solution for ϕ_m in terms of Bessel I, K functions: we choose conditions which match to Bessel I, say, at large r. We assume $m > 1/r_H$ so that the Compton wavelength is less than r_H. Then ϕ_m'/ϕ_m is barely oscillatory for larger m and decays gradually as $r \to \infty$ according to

$$\frac{\phi_m'}{\phi_m} \approx \imath m\sqrt{\frac{r_H}{r - r_H}}, \quad r >> r_H. \tag{3.39}$$

The actual numerical solution as illustrated in Figure 1 is similar, although more oscillatory. We see that in this 'comoving frame' from a Klein-Gordon point of view, we do not experience the main force of gravity but we do see a novel radial velocity term in the effective Schroedinger-like equation approximated as

$$\imath\dot{\psi}_l \approx -\frac{\beta}{2m}(\Delta_r + \frac{\lambda_l}{r^2})\psi_l - \imath\beta^{\frac{3}{2}}\sqrt{\frac{2GM}{r}}\psi_l' \tag{3.40}$$

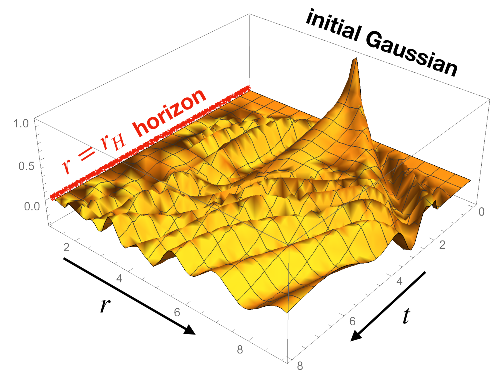

FIGURE 2. Schroedinger-like evolution relative to the ϕ_m in Figure 1. We see an initial Gaussian centred at $r = 5r_H$ evolving much as in quantum mechanics but decaying over time, with the essentially zero initial values at $r = 1.1r_H, 10r_H$ held fixed. Figure as in [**3**]

far from the horizon. Nearer the horizon, one needs to use the actual ϕ'_m/ϕ_m to avoid an instability coming in from the horizon. A numerical solution for ψ_l at $l = 0$ using the actual values is in Figure 2, showing an initial Gaussian centered at $r = 5r_H$ evolving much as in regular quantum mechanics but, unlike the latter, decaying over time. Some of the noise in the picture comes from the numerical approximation.

The above is for a regular black hole, but one can make a similar analysis for the different radial equations for our fuzzy black hole and thereby justify (3.34), provided we know something about the Laplacian operator in the fuzzy sphere, $\sum_i \partial_i^2$.

PROPOSITION 3.4. *$\frac{1}{2}\sum_i \partial_i^2$ on the fuzzy sphere has eigenvalues $\lambda_l = -l(l+1)$ as for the classical $\square_{S^2}$, with eigenspaces*

$$H_l = \{x^{i_1}x^{i_2}\cdots x^{i_l} f_{i_1\cdots i_l} \mid f \text{ totally symmetric and traceless}\}.$$

PROOF. Here, as vector spaces, $\mathbb{C}[x^1, \cdots, x^n] = \mathbb{C}[su_2^*] \cong U(su_2)$ by the Duflo map (as for any Lie algebra). This sends a commutative monomial in the x^i to an average of all orderings of its factors (it is an isomorphism because, although there are nontrivial commutation relations in the enveloping algebra, these are strong enough to reorder at the expense of lower degree.) This map is covariant for the coadjoint and adjoint actions, in our case, of su_2, and therefore descends to an isomorphism between polynomial functions on the classical sphere in cartesian coordinates on one side, and the fuzzy sphere $\mathbb{C}_\lambda[S^2]$ on the other side. Moreover, $\partial_k x_i = \epsilon_{ijk}x^j$ for our differential calculus on the latter acts as orbital angular momentum. Hence $\sum_i \partial_i^2$ acts as the quadratic Casimir and can be computed on the classical sphere, where it decomposes the polynomial functions into the spherical harmonics of each degree l. These then correspond to the H_l as stated. One can check this

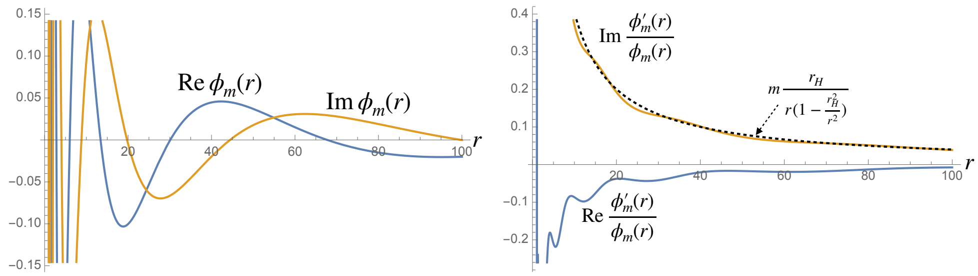

FIGURE 3. Radial solution $\phi_m(r)$ for the Klein-Gordon equation around a fuzzy black hole shown for $m = 4/r_H, r_H = 1$, and function $\phi_m'(r)/\phi_m(r)$. Figure as in [**3**]

directly on the fuzzy sphere on low degrees by hand, to fix the normalisation. For example, on degree $l = 1$, we have $\sum_k \partial_k^2 x^i = \partial_k \epsilon_{ijk} x^j = \epsilon_{jmk}\epsilon_{ijk} x^m = -2x^i$. □

Thus, we can solve the Laplacian and look at the non-relativistic limits by the same methods as we illustrated for the Schwarzschild black hole. The only difference is that the functions have values $\psi_l(t,r) \in \mathbb{C}_\lambda[S^2]$, but the differential equations themselves in t, r are purely classical according to

$$\Delta_{fuz} = -\frac{1}{\beta}\frac{\partial^2}{\partial t^2} + \Delta_r + \frac{2}{kr^2}\lambda_l; \quad \Delta_r := \frac{1}{r^3}\frac{\partial}{\partial r}(\beta r^3 \frac{\partial}{\partial r})$$

with $\beta = 1 - r_H^2/r^2$. Taking $e^{-\imath mt}$ as reference gives the same form as (3.35) but with $\frac{2\lambda_l}{kr^2}$ in a modified effective spatial Laplacian. Then $(\beta - 1)m/2 = -2G^2M^2m/r^2$ for the gravitational potential energy in agreement with (3.34).

Next, for the 'comoving' version, the $l = 0$ solutions of the Klein-Gordon equation are given by solving (3.36) as before and relative to this, slowly-varying ψ_l defined by (3.37) obey the Schroedinger-like equation (3.38) but now with $\frac{2\lambda_l}{kr^2}$ in place of $\frac{\lambda_l}{r^2}$. Focussing on the $\omega = m$ case, the main difference now is that ϕ_m decays more rapidly and in first approximation, if we leave out the β' term in Δ_r, is now solved by

$$\phi_m(r) \propto \frac{\left(r^2 - r_H^2\right)^{\frac{1}{2}\pm\frac{1}{2}\sqrt{1-m^2r_H^2}}}{r^2}.$$

We focus on the + case of the square root, which leads for $m >> 1/r_H$ to a fair approximation

$$\frac{\phi_m'}{\phi_m} \approx \imath m \frac{r_H}{r(1-\frac{r_H^2}{r^2})}, \quad r >> r_H$$

as illustrated in Figure 3. As a result, the long-range Schroedinger-like equation is

$$\imath\psi \approx -\frac{\beta}{2m}\left(\Delta_r + \frac{2\lambda_l}{kr^2}\right)\psi_l - \imath\beta\frac{2GM}{r}\psi_l'$$

if we use the Schwarzschild value of r_H, showing a coupling to the velocity term of the same size as the usual gravitational potential per unit mass. As before, near the horizon we need the actual ϕ'_m/ϕ_m values for stability of the solutions. An initial Gaussian breaks up and decays over time, looking much as before.

Finally, although we have used the Schwarzschild value of r_H for purposes of comparison, since the geometry is asymptotically flat, we could naively try to define an actual ADM mass (sometimes know as ADM energy, see [**23**, Chap. 6.4]), by copying its physical formulation in terms of the Einstein tensor of the spatial geometry[**4, 24, 41**], which in spherical polar amounts to the limit $r \to \infty$ of

$$\begin{aligned} M(r) &= \frac{2}{G(n-1)(n-2)\Omega_{n-1}} \int_{S_r^{n-1}} \mathrm{Eins}(r\partial_r, \sqrt{\beta}\partial_r)\mathrm{d}^{n-1}\Omega \\ &= \frac{2}{G(n-1)(n-2)} r^{n-1}\mathrm{Eins}(r\partial_r, \sqrt{\beta}\partial_r) \end{aligned}$$

for a spatial geometry of dimension n. Here there is a factor -2 compared to the usual definition because our Ricci and hence Einstein tensor conventions reduce in the classical case to $-\frac{1}{2}$ of the usual ones. Ω_{n-1} is the volume of a unit sphere of dimension $n-1$ and we integrate with measure $\mathrm{d}^{n-1}\Omega$ over the sphere S_r^{n-1} at radius r. The conformal Killing vector field in the general formula in [**41**] is just $r\partial_r$ in our case and the unit outward normal vector field is $\sqrt{\beta}\partial_r$ given the form of the spatial metric. As everything is rotationally invariant, the integration merely gives a factor $r^{n-1}\Omega_{n-1}$. For a usual Schwarzschild black hole of mass M, the Einstein tensor of the spatial geometry in our conventions can be extracted from [**14**, Cor. 9.9] to find

$$\mathrm{Eins}_{Sch}(r\partial_r, \sqrt{\beta}\partial_r) = \frac{1}{2r\sqrt{\beta}}(1-\beta), \quad M(r) = \frac{r_H}{2G\sqrt{\beta}} \to \frac{r_H}{2G}$$

as expected for the Schwarzschild $\beta(r) = 1 - \frac{r_H}{r}$. If we now use the fuzzy quantum black hole spatial geometry in Proposition 3.3, the radial sector is completely classical so it makes sense to read off $\mathrm{Eins}(\partial_r, \partial_r)$ as the coefficient of $\mathrm{d}r \otimes \mathrm{d}r$, resulting in our case in

$$\mathrm{Eins}_{fuz}(r\partial_r, \sqrt{\beta}\partial_r) = \frac{3}{2r\sqrt{\beta}}(\frac{1}{4k} - \beta),$$

which, since $\beta(r) = 1 - \frac{r_H^2}{r^2}$ and $k \neq \frac{1}{4}$, results in $M(r) \to \infty$. If we took $k = \frac{1}{4}$ then we would not have a Ricci flat metric in the spacetime quantum geometry and we would get $M(r) \to 0$, which is not reasonable either. These problems are a consequence of the dimension jump in the quantum model, evidently requiring a more sophisticated approach to ADM mass. Indeed, if we were to set $n = 4$ and $k = \frac{1}{4}$ then we would obtain $M(r) \to \frac{r_H^2}{2G}$, which is rather close to the value (3.24) for a classical 5D black hole.

CHAPTER 4

QRG of the Discrete Interval

The results of this chapter are published in [**10**]. Beside, we use the graded commutator $[\omega, \eta\} = \omega \wedge \eta - (-1)^m \eta \wedge \omega$ where $\eta, \omega \in \Omega, \eta \in \Omega^m$.

1. Exterior algebras on A_n and preprojective algebras

The preprojective algebra of a graph is a quotient of the path algebra of the graph viewed as bidirected (each edge is viewed as a pair of arrows, one in each direction). For the Dynkin graph of type A_n with nodes numbered in order $1, 2, \cdots, n$, we denote the edges

$$a_i = \omega_{i\to i+1}, \quad a'_i = \omega_{i+1\to i} = -a^*_i, \quad i = 1, \cdots, n-1.$$

In the maths literature, the notation a^*_i is used for what we denote a'_i; the two differ by a sign which just amounts to a different normalisation but is needed for our exterior algebras to become $*$-exterior algebras when working over $\mathbb{C}$. We also denote by δ_i the Kronecker δ-functions at the nodes. The path algebra then has the relations that all products of these generators are zero except

$$\delta_i^2 = \delta_i, \quad \delta_i a_i = a_i = a_i \delta_{i+1}, \quad \delta_{i+1} a'_i = a'_i = a'_i \delta_i, \quad a_i a_{i+1}, \quad a'_{i+1} a'_i, \quad a_i a'_i, \quad a'_i a_i$$

The dimension of the path algebra in degree 0 is n with basis δ_i. In degree 1 it is $2(n-1)$-dimensional with basis a_i, a'_i and in degree 2 it is $2(2n-3)$-dimensional with basis $a_i a_{i+1}, a'_{i+1} a'_i$ for $i = 1, \cdots, n-2$ and $a_i a'_i, a'_i a_i$ for $i = 1, \cdots, n-1$.

Proposition 4.1. *For a Dynkin graph of type A_n, $\Omega_{max} = \Omega_{med}$ is a quotient of the path algebra by the relations*

$$a_i a_{i+1} = 0, \quad a'_{i+1} a'_i = 0, \quad i = 1, \cdots, n-2 \tag{4.1}$$

and $\Omega_{min} = \Omega_{med'}$ is the further quotient by the relations

$$a_1 a'_1 = 0, \quad a'_{n-1} a_{n-1} = 0, \quad a_{i+1} a'_{i+1} + a'_i a_i = 0, \quad i = 1, \cdots, n-2. \tag{4.2}$$

The latter case is inner with $\mathrm{d} = [\theta, \}$*, where $\theta = \sum_i (a_i + a'_i)$, and Ω^2_{min} is $n-2$-dimensional while $\Omega^i_{min} = 0$ for $i \geq 3$.*

Proof. The dimensions up to Ω^2 are clear from the stated bases and quadratic nature of the relations. In degree 3 we consider all 3-step paths and their image in Ω^3_{min}. Since any 2-steps in the same direction vanish by (4.1), the only possible images in the quotient are

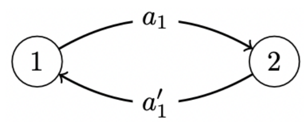

FIGURE 1. A_2 Graph

for zig-zag paths such as $a'_{i+1}a_{i+1}a'_{i+1} = -a'_{i+1}a'_i a_i = 0$ using (4.2) and then (4.1). Similarly for zig-zag the other way, $a_i a'_i a_i = -a_i a_{i+1} a'_{i+1} = 0$. □

The preprojective algebra Π_n has just the (4.2) relations and dimensions

$$n,\ 2(n-1),\ 3(n-2),\ ...,\ (n-1)2,\ n.$$

We see that Ω_{min} is a quotient of this by (4.1). Also, later, we will need a bimodule 'lifting' map $i : \Omega^2_{min} \to \Omega^1 \otimes_A \Omega^1$ such that following this by $\wedge$ is the identity. Given the description above, the natural choice is

$$i(a_i a'_i) = -i(a'_{i-1}a_{i-1}) = \frac{1}{2}(a_i \otimes a'_i - a'_{i-1} \otimes a_{i-1}), \quad i = 2, 3, \cdots, n-1 \tag{4.3}$$

where the product denotes wedge product. We take the same form of exterior algebra relations and lift map for the half-line $\mathbb{N}$, just without the upper bound on the indices i. Also note from the form of the path algebra that we can only have zero for the bimodule map α, i.e. bimodule connections ∇ are determined just from σ.

2. Explicit calculations for A_2, A_3, A_4, A_5

In this section, we give explicit geometries for small A_n. For $n \leq 4$, these are manageable by hand and we show the details of the calculation. For $n = 5$, we used Mathematica and Python (independently) and just list the final result.

2.1. A_2 geometry. The result is known from [**11**] by a different method, but here provides a warm up for the larger cases. We work over the directed graph $G(V, E)$ with vertices $V = \{1, 2\}$ and directed edges or 'arrows' $E = \{a_1, a'_1\}$ as in Figure 1. The products distinct from zero in $\Omega^1 \otimes_A \Omega^1$ are $a_1 \otimes a'_1, a'_1 \otimes a_1$. The exterior algebra Ω^2_{max} is 2-dimensional with basis $a_1 \wedge a'_1$ and $a'_1 \wedge a_1$. We work with Ω_{min} where these are set to zero.

Using the graded commutator for the exterior derivatives given that the calculus is inner with $\theta = a_1 + a'_1$,

$$\mathrm{d}a = [\theta, a_1\} = a'_1 \otimes a_1 + a_1 \otimes a'_1 = \mathrm{d}a'_1,$$

We necessarily take $\alpha = 0$ and the most general form of σ is

$$\sigma(a_1 \otimes a'_1) = \tau_1 a_1 \otimes a'_1, \quad \sigma(a'_1 \otimes a_1) = \tau'_1 a'_1 \otimes a_1$$

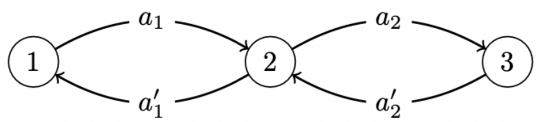

FIGURE 2. A_3 Graph

The general form of the metric is

$$g = f_1 a_1 \otimes a_1' + f_1' a_1' \otimes a_1$$

where f_1, f_1' are in the field, and real if we work over $\mathbb{C}$ and impose the reality condition for the metric.

The general form of the connection given that the calculus is inner is

$$\nabla a_1 = a_1' \otimes a_1 - \tau_1 a_1 \otimes a_1', \quad \nabla a_1' = a_1 \otimes a_1' - \tau_1' a_1' \otimes a_1$$

As we are working in Ω_{min}, there is no elements in Ω^2 for this case. Then there are no conditions for torsion freeness.

The metric compatibility conditions are

$$a_1 \otimes a_1' \otimes a_1 : f_1' - f_1 \tau_1 \tau_1' = 0$$
$$a_1' \otimes a_1 \otimes a_1' : f_1 - f_1' \tau_1 \tau_1' = 0$$

These conditions imply that there is a sign $\epsilon = \tau_1 \tau_1' = \pm 1$ with $f_1' = \epsilon f_1$. The $*$-preserving conditions

$$|\tau_1| = 1$$

with $\tau_1' = \epsilon \tau_1^{-1}$. We see that there is one sign and one overall normalisation in the metric

$$g = h_1 (a_1 \otimes a_1' + \epsilon a_1' \otimes a_1)$$

which allows a QLC with one parameter $\tau_1 = s$ in characteristic zero

$$\nabla a_1 = a_1' \otimes a_1 - s a_1 \otimes a_1', \quad \nabla a_1' = a_1 \otimes a_1' - \epsilon s^{-1} a_1' \otimes a_1.$$

and the further condition that h_1 is real and $|s| = 1$ for the reality property of the metric and for the connection to be $*$-preserving in the case over $\mathbb{C}$. All the connections are flat since $\Omega^2 = 0$.

2.2. A_3 **geometry.** We work over the directed graph $G(V, E)$ with vertices $V = \{1, 2, 3\}$ and directed edges $E = \{a_1, a_1', a_2, a_2'\}$ as in Figure 2. The products in $\Omega^1 \otimes_A \Omega^1$ different from zero are those where the head of the first arrow connects to the tail of the second arrow, giving the six non-zero elements $a_1 \otimes a_1', a_1 \otimes a_2, a_1' \otimes a_1, a_2 \otimes a_2', a_2' \otimes a_1', a_2' \otimes a_2$.

The exterior algebra Ω_{max} for the maximal prolongation has the relations

$$a_1 \wedge a_2 = a_2' \wedge a_1' = 0$$

and we work with the quotient Ω_{min} of this where we add the further relations

$$a_1 \wedge a_1' = a_2' \wedge a_2 = 0, \quad a_1' \wedge a_1 + a_2 \wedge a_2' = 0.$$

The dimensions of the vector spaces of Ω^i are therefore 3:4:1.

The exterior derivative is given by the graded commutator $\mathrm{d} = [\theta, \}$ with the inner element $\theta = a_1 + a_1' + a_2 + a_2'$ as

$$\mathrm{d}a_1 = a_1' \wedge a_1, \quad \mathrm{d}a_1' = a_1' \wedge a_1, \quad \mathrm{d}a_2 = -a_1' \wedge a_1, \quad \mathrm{d}a_2' = -a_1' \wedge a_1.$$

The metric, as it has to be central, has to have the form

$$g = f_1 a_1 \otimes a_1' + f_1' a_1' \otimes a_1 + f_2 a_2 \otimes a_2' + f_2' a_2' \otimes a_2,$$

where f_1, f_1', f_2, f_2' are in the field, and should be real if we work over $\mathbb{C}$ and impose the reality condition $\dagger \circ g = g$.

Given the calculus is inner, the torsion free connections have the form

$$\begin{aligned}
\nabla a_1 &= a_1' \otimes a_1 - \tau_1 a_1 \otimes a_1' - \sigma_1 a_1 \otimes a_2, \\
\nabla a_1' &= a_1 \otimes a_1' + a_2' \otimes a_1' - \tau_1' a_1' \otimes a_1 - (\tau_1' + 1) a_2 \otimes a_2', \\
\nabla a_2 &= a_1 \otimes a_2 + a_2' \otimes a_2 - \tau_2 a_2 \otimes a_2' - (\tau_2 + 1) a_1' \otimes a_1, \\
\nabla a_2' &= a_2 \otimes a_2' - \sigma_2' a_2' \otimes a_1' - \tau_2' a_2' \otimes a_2,
\end{aligned}$$

where the map $\alpha = 0$ and the braiding map is given by

$$\begin{aligned}
\sigma(a_1 \otimes a_1') &= \tau_1 a_1 \otimes a_1', \\
\sigma(a_1 \otimes a_2) &= \sigma_1 a_1 \otimes a_2 \\
\sigma(a_1' \otimes a_1) &= \tau_1' a_1' \otimes a_1 + (\tau_1' + 1) a_2 \otimes a_2', \\
\sigma(a_2 \otimes a_2') &= \tau_2 a_2 \otimes a_2' + (\tau_2 + 1) a_1' \otimes a_1, \\
\sigma(a_2' \otimes a_1') &= \sigma_2' a_2' \otimes a_1', \\
\sigma(a_2' \otimes a_2) &= \tau_2' a_2' \otimes a_2.
\end{aligned}$$

Metric compatibility (see equation 1.6 in Chapter 1) then produces

$$
\begin{aligned}
&a_1 \otimes a_1' \otimes a_1 : -f_1\tau_1\tau_1' + f_1' = 0,\\
&a_1 \otimes a_2 \otimes a_2' : -f_1\sigma_1(\tau_1' + 1) + f_2 = 0,\\
&a_1' \otimes a_1 \otimes a_1' : -f_1'\tau_1\tau_1' - f_2(\tau_2 + 1)\sigma_2' + f_1 = 0,\\
&a_2 \otimes a_2' \otimes a_2 : -f_1'\sigma_1(\tau_1' + 1) - f_2\tau_2\tau_2' + f_2' = 0,\\
&a_2' \otimes a_1' \otimes a_1 : -f_2'(\tau_2 + 1)\sigma_2' + f_1' = 0,\\
&a_2' \otimes a_2 \otimes a_2' : -f_2'\tau_2\tau_2' + f_2 = 0,\\
&a_2 \otimes a_2' \otimes a_1' : -f_1'\tau_1(\tau_1' + 1) - f_2\tau_2\sigma_2' = 0,\\
&a_1' \otimes a_1 \otimes a_2 : -f_1'\sigma_1\tau_1' - f_2(\tau_2 + 1)\tau_2' = 0.
\end{aligned}
$$

Under these conditions, we have two parameters and one sign in the metric as

$$g = h_1(\phi a_1 \otimes a_1' + \epsilon a_1' \otimes a_1) + h_2(\frac{1}{\phi}a_2 \otimes a_2' + \epsilon a_2' \otimes a_2), \quad \phi = \sqrt{2},$$

where the connection is

$$
\begin{aligned}
&\tau_1 = s, \quad \sigma_1 = \frac{h_2 s}{h_1\epsilon\phi(\epsilon\phi s + 1)}, \quad \tau_1' = \frac{1}{\epsilon\phi s},\\
&\tau_2 = -1 + \frac{1}{2 + \epsilon\phi s}, \quad \sigma_2' = \frac{h_1\epsilon\phi}{h_2}(s + \epsilon\phi), \quad \tau_2' = -\frac{1}{\epsilon\phi}\left(1 + \frac{1}{1 + \epsilon\phi s}\right)
\end{aligned}
$$

for a free parameter s. Notice that only the combination $\epsilon\phi$ enters. We do not consider $\phi = -\sqrt{2}$ in the metric as this would be equivalent to a redefinition of ϵ, h_1, h_2 by a change of sign. Finally, the *-preserving condition for the connection just requires

$$|s| = 1 \tag{4.4}$$

with no further constraints on h_i other than to be real.

2.3. A_4 **geometry.** We again work with Ω_{min} which now has vector space dimensions $4 : 6 : 2$ with $\Omega^i = 0$ for $i \geq 3$. Here, the path algebra is 10-dimensional in degree 2, Ω^2_{max} adds 4 relations and then we add further relations for Ω^2,

$$a_1 \wedge a_1' = a_3' \wedge a_3 = 0, \quad a_1' \wedge a_1 + a_2 \wedge a_2' = 0, \quad a_2' \wedge a_2 + a_3 \wedge a_3' = 0.$$

The metric, to be central, has to have the form

$$g = f_1 a_1 \otimes a_1' + f_1' a_1' \otimes a_1 + f_2 a_2 \otimes a_2' + f_2' a_2' \otimes a_2 + f_3 a_3 \otimes a_3' + f_3' a_3' \otimes a_3$$

where $f_1, f_1', f_2, f_2', f_3, f_3'$ are in the field, and real for the reality condition $\dagger \circ g = g$ when working over $\mathbb{C}$. We necessarily take $\alpha = 0$ and the torsion free connection and bimodule braiding map have to have the form

$$
\begin{aligned}
\nabla a_1 &= a'_1 \otimes a_1 - \tau_1 a_1 \otimes a'_1 - \sigma_1 a_1 \otimes a_2,\\
\nabla a'_1 &= a_1 \otimes a'_1 + a'_2 \otimes a'_1 - \tau'_1 a'_1 \otimes a_1 - (\tau'_1 + 1) a_2 \otimes a'_2,\\
\nabla a_2 &= a_1 \otimes a_2 + a'_2 \otimes a_2 - \tau_2 a_2 \otimes a'_2 - (\tau_2 + 1) a'_1 \otimes a_1 - \sigma_2 a_2 \otimes a_3,\\
\nabla a'_2 &= a_2 \otimes a'_2 + a'_3 \otimes a'_2 - \sigma'_2 a'_2 \otimes a'_1 - \tau'_2 a'_2 \otimes a_2 - (\tau'_2 + 1) a_3 \otimes a'_3,\\
\nabla a_3 &= a_2 \otimes a_3 + a'_3 \otimes a_3 - \tau_3 a_3 \otimes a'_3 - (\tau_3 + 1) a'_2 \otimes a_2,\\
\nabla a'_3 &= a_3 \otimes a'_3 - \sigma'_3 a'_3 \otimes a'_2 - \tau'_3 a'_3 \otimes a_3,
\end{aligned}
$$

$$
\begin{aligned}
\sigma(a_1 \otimes a'_1) &= \tau_1 a_1 \otimes a'_1,\\
\sigma(a_1 \otimes a_2) &= \sigma_1 a_1 \otimes a_2,\\
\sigma(a'_1 \otimes a_1) &= \tau'_1 a'_1 \otimes a_1 + (\tau'_1 + 1) a_2 \otimes a'_2,\\
\sigma(a_2 \otimes a'_2) &= \tau_2 a_2 \otimes a'_2 + (\tau_2 + 1) a'_1 \otimes a_1,\\
\sigma(a_2 \otimes a_3) &= \sigma_2 a_2 \otimes a_3,\\
\sigma(a'_2 \otimes a'_1) &= \sigma'_2 a'_2 \otimes a'_1,\\
\sigma(a'_2 \otimes a_2) &= \tau'_2 a'_2 \otimes a_2 + (\tau'_2 + 1) a_3 \otimes a'_3,\\
\sigma(a_3 \otimes a'_3) &= \tau_3 a_3 \otimes a'_3 + (\tau_3 + 1) a'_2 \otimes a_2,\\
\sigma(a'_3 \otimes a'_2) &= \sigma'_3 a'_3 \otimes a'_2,\\
\sigma(a'_3 \otimes a_3) &= \tau'_3 a'_3 \otimes a_3.
\end{aligned}
$$

The metric compatibility conditions are

$$a_1 \otimes a_1' \otimes a_1 : -f_1\tau_1\tau_1' + f_1' = 0, \tag{4.5}$$

$$a_1 \otimes a_2 \otimes a_2' : -f_1\sigma_1(\tau_1' + 1) + f_2 = 0, \tag{4.6}$$

$$a_1' \otimes a_1 \otimes a_1' : -f_1'\tau_1\tau_1' - f_2(\tau_2 + 1)\sigma_2' + f_1 = 0, \tag{4.7}$$

$$a_2 \otimes a_2' \otimes a_2 : -f_1'\sigma_1(\tau_1' + 1) - f_2\tau_2\tau_2' + f_2' = 0, \tag{4.8}$$

$$a_2 \otimes a_3 \otimes a_3' : -f_2\sigma_2(\tau_2' + 1) + f_3 = 0, \tag{4.9}$$

$$a_2' \otimes a_1' \otimes a_1 : -f_2'(\tau_2 + 1)\sigma_2' + f_1' = 0, \tag{4.10}$$

$$a_2' \otimes a_2 \otimes a_2' : -f_3(\tau_3 + 1)\sigma_3' - f_2'\tau_2\tau_2' + f_2 = 0, \tag{4.11}$$

$$a_3 \otimes a_3' \otimes a_3 : -f_3\tau_3\tau_3' - f_2'\sigma_2(\tau_2' + 1) + f_3' = 0, \tag{4.12}$$

$$a_3' \otimes a_2' \otimes a_2 : -f_3'(\tau_3 + 1)\sigma_3' + f_2' = 0, \tag{4.13}$$

$$a_3' \otimes a_3 \otimes a_3' : -f_3'\tau_3\tau_3' + f_3 = 0, \tag{4.14}$$

$$a_2 \otimes a_2' \otimes a_1' : -f_1'\tau_1(\tau_1' + 1) - f_2\tau_2\sigma_2' = 0, \tag{4.15}$$

$$a_1' \otimes a_1 \otimes a_2 : -f_1'\sigma_1\tau_1' - f_2(\tau_2 + 1)\tau_2' = 0, \tag{4.16}$$

$$a_3 \otimes a_3' \otimes a_2' : -f_3\tau_3\sigma_3' - f_2'\tau_2(\tau_2' + 1) = 0, \tag{4.17}$$

$$a_2' \otimes a_2 \otimes a_3 : -f_3(\tau_3 + 1)\tau_3' - f_2'\sigma_2\tau_2' = 0. \tag{4.18}$$

There are four metrics that allow one QLC each depending on a free parameter s. Here, the metric is found to be of the form

$$g = h_1(\phi a_1 \otimes a_1' + \epsilon a_1' \otimes a_1) + h_2(a_2 \otimes a_2' + \epsilon a_2' \otimes a_2) + h_3(\frac{1}{\phi} a_3 \otimes a_3' + \epsilon a_3' \otimes a_3),$$

where $\epsilon = \pm 1$ and

$$\phi = \frac{1 \pm \sqrt{5}}{2}.$$

We can chose either in what follows (so we have four metrics according to ϵ and the sign of the $\sqrt{5}$). For any choices of these we now solve for the connection and find for any value of s,

$$\tau_1 = s, \quad \tau_2 = -1 + \frac{1}{\phi + \epsilon s}, \quad \tau_3 = -1 + \frac{1}{\phi} \frac{\phi + \epsilon s}{(1 - \epsilon)(\phi + \epsilon s) + \epsilon},$$
$$\tau_1' = \epsilon \frac{-1 + \phi}{s}, \quad \tau_2' = \epsilon \frac{(\phi + \epsilon s)(1 - 1/\phi)}{-(\phi + \epsilon s) + 1}, \quad \tau_3' = \frac{\epsilon}{\phi \tau_3}, \quad \sigma_1 = \frac{h_2}{h_1 \phi} \frac{s}{\epsilon(-1 + \phi) + s},$$
$$\sigma_2 = \frac{h_2}{h_3 \phi} \frac{1}{\tau_2' + 1}, \quad \sigma_2' = \frac{h_1}{h_2}(\phi + \epsilon s), \quad \sigma_3' = \frac{h_2}{h_3} \phi \left(1 - \epsilon + \frac{\epsilon}{\phi + \epsilon s}\right).$$

Thus, for each metric we have a 1-parameter family of connections with parameter s. In the $*$-algebra case, reality of the metric demands h_i real and $*-$preserving for the connection is equivalent to

$$|s| = 1$$

with no further constraints on h_i. So, there is a still a 1-parameter moduli of connections for each of our four metrics, now with $|s| = 1$.

2.4. A_5 **geometry.** Now the path algebra has dimension 14 in degree 2 while Ω_{min} has vector space dimensions $5 : 8 : 3$ again with $\Omega^i = 0$ for $i \geq 3$. Proceeding similarly, the form of the braiding, connection, metric compatibility conditions and form of the metric are

$$
\begin{aligned}
\sigma(a_1 \otimes a_1') &= \tau_1 a_1 \otimes a_1', \\
\sigma(a_1 \otimes a_2) &= \sigma_1 a_1 \otimes a_2, \\
\sigma(a_1' \otimes a_1) &= \tau_1' a_1' \otimes a_1 + (\tau_1' + 1) a_2 \otimes a_2', \\
\sigma(a_2 \otimes a_2') &= \tau_2 a_2 \otimes a_2' + (\tau_2 + 1) a_1' \otimes a_1, \\
\sigma(a_2 \otimes a_3) &= \sigma_2 a_2 \otimes a_3, \\
\sigma(a_2' \otimes a_2) &= \tau_2' a_2' \otimes a_2 + (\tau_2' + 1) a_3 \otimes a_3', \\
\sigma(a_2' \otimes a_1') &= \sigma_2' a_2' \otimes a_1, \\
\sigma(a_3 \otimes a_3') &= \tau_3 a_3 \otimes a_3' + (\tau_3 + 1) a_2' \otimes a_2, \\
\sigma(a_3 \otimes a_4) &= \sigma_3 a_3 \otimes a_4, \\
\sigma(a_3' \otimes a_3) &= \tau_3' a_3' \otimes a_3 + (\tau_3' + 1) a_4 \otimes a_4', \\
\sigma(a_3' \otimes a_2') &= \sigma_3' a_3' \otimes a_2', \\
\sigma(a_4 \otimes a_4') &= \tau_4 a_4 \otimes a_4' + (\tau_4 + 1) a_3' \otimes a_3, \\
\sigma(a_4' \otimes a_3') &= \sigma_4' a_4' \otimes a_3', \\
\sigma(a_4' \otimes a_4) &= \tau_4' a_4' \otimes a_4;
\end{aligned}
$$

$$
\begin{aligned}
\nabla a_1 &= a_1' \otimes a_1 - \tau_1 a_1 \otimes a_1' - \sigma_1 a_1 \otimes a_2, \\
\nabla a_1' &= a_1 \otimes a_1' + a_2' \otimes a_1' - \tau_1' a_1' \otimes a_1 - (\tau_1' + 1) a_2 \otimes a_2', \\
\nabla a_2 &= a_2' \otimes a_2 + a_1 \otimes a_2 - \tau_2 a_2 \otimes a_2' - (\tau_2 + 1) a_1' \otimes a_1 - \sigma_2 a_2 \otimes a_3, \\
\nabla a_2' &= a_2 \otimes a_2' + a_3' \otimes a_2' - \tau_2' a_2' \otimes a_2 - (\tau_2' + 1) a_3 \otimes a_3' - \sigma_2' a_2' \otimes a_1, \\
\nabla a_3 &= a_3' \otimes a_3 + a_2 \otimes a_3 - \tau_3 a_3 \otimes a_3' - (\tau_3 + 1) a_2' \otimes a_2 - \sigma_3 a_3 \otimes a_4, \\
\nabla a_3' &= a_3 \otimes a_3' + a_4' \otimes a_3' - \tau_3' a_3' \otimes a_3 - (\tau_3' + 1) a_4 \otimes a_4' - \sigma_3' a_3' \otimes a_2', \\
\nabla a_4 &= a_4' \otimes a_4 + a_3 \otimes a_4 - \tau_4 a_4 \otimes a_4' - (\tau_4 + 1) a_3' \otimes a_3, \\
\nabla a_4' &= a_4 \otimes a_4' - \sigma_4' a_4' \otimes a_3' - \tau_4' a_4' \otimes a_4;
\end{aligned}
$$

$$a_1 \otimes a_1' \otimes a_1 : -f_1\tau_1\tau_1' + f_1' = 0,$$
$$a_1 \otimes a_2 \otimes a_2' : -f_1\sigma_1(\tau_1' + 1) + f_2 = 0,$$
$$a_1' \otimes a_1 \otimes a_1' : -f_1'\tau_1\tau_1' - f_2(\tau_2 + 1)\sigma_2' + f_1 = 0,$$
$$a_2 \otimes a_2' \otimes a_2 : -f_1'\sigma_1(\tau_1' + 1) - f_2\tau_2\tau_2' + f_2' = 0,$$
$$a_2 \otimes a_3 \otimes a_3' : -f_2\sigma_2(\tau_2' + 1) + f_3 = 0,$$
$$a_2' \otimes a_1' \otimes a_1 : -f_2'(\tau_2 + 1)\sigma_2' + f_1' = 0,$$
$$a_2' \otimes a_2 \otimes a_2' : -f_3(1 + \tau_3)\sigma_3' - f_2'\tau_2\tau_2' + f_2 = 0,$$
$$a_3 \otimes a_3' \otimes a_3 : -f_3\tau_3\tau_3' - f_2'\sigma_2(\tau_2' + 1) + f_3' = 0,$$
$$a_3 \otimes a_4 \otimes a_4' : -f_3\tau_3\sigma_3' + f_4 = 0,$$
$$a_3' \otimes a_2' \otimes a_2 : -f_3'(1 + \tau_3)\sigma_3' + f_2' = 0,$$
$$a_3' \otimes a_3 \otimes a_3' : -f_3'\tau_3\tau_3' - f_4(\tau_4 + 1)\sigma_4' + f_3 = 0,$$
$$a_4 \otimes a_4' \otimes a_4 : -f_3'\tau_3\sigma_3' - f_4\tau_4\tau_4' + f_4' = 0,$$
$$a_4' \otimes a_3' \otimes a_3 : -f_4'(\tau_4 + 1)\sigma_4' + f_3' = 0,$$
$$a_4' \otimes a_4 \otimes a_4' : -f_4'\tau_4\tau_4' + f_4 = 0,$$
$$a_2 \otimes a_2' \otimes a_1' : -f_1'\tau_1(\tau_1' + 1) - f_2\tau_2\sigma_2' = 0,$$
$$a_1' \otimes a_1 \otimes a_2 : -f_1'\sigma_1\tau_1' - f_2(\tau_2 + 1)\tau_2' = 0,$$
$$a_3 \otimes a_3' \otimes a_2' : -f_3\sigma_3'\tau_3 - f_2'\tau_2(\tau_2' + 1) = 0,$$
$$a_2' \otimes a_2 \otimes a_3 : -f_3(\tau_3 + 1)\tau_3' - f_2'\sigma_2\tau_2' = 0,$$
$$a_4 \otimes a_4' \otimes a_3' : -f_3'(\tau_3 + 1)\sigma_3' - f_4\tau_4\sigma_4' = 0,$$
$$a_3' \otimes a_3 \otimes a_4 : -f_3'\tau_3(\tau_3' + 1) - f_4(\tau_4 + 1)\tau_4' = 0;$$

$$g = h_1(\phi_1 a_1\otimes a_1' + \epsilon a_1'\otimes a_1) + h_2(\phi_2 a_2\otimes a_2' + \epsilon a_2'\otimes a_2) + h_3(\frac{1}{\phi_2} a_3\otimes a_3' + \epsilon a_3'\otimes a_3) + h_4(\frac{1}{\phi_1} a_4\otimes a_4' + \epsilon a_4'\otimes a_4),$$

where

$$\phi_1 = \sqrt{3}, \quad \phi_2 = \frac{2}{\sqrt{3}}$$

and ϵ is a sign. In the $*$-algebra case over $\mathbb{C}$ we need h_i real and $|s| = 1$ for the reality of the metric and for the connection to be $*$-preserving.

The solutions are then as follows. The parts which looks similar to the A_4 case are

$$\tau_1 = s, \quad \sigma_1 = \frac{h_2\epsilon\phi_2}{h_1(\epsilon\phi_1 + \frac{1}{s})}, \quad \tau_1' = \frac{1}{\epsilon\phi_1 s}, \quad \tau_2 \quad = -1 + \frac{\epsilon\phi_2}{\epsilon\phi_1 + s}, \quad \sigma_2' = \frac{h_1}{h_2\epsilon\phi_2}(\epsilon\phi_1 + s)$$

and the remainder explicitly are

$$\tau_3 = \frac{-2\epsilon\phi_1 s + \epsilon\phi_1 + s - 2}{2\epsilon\phi_1(s-2) - 4s + 2}, \quad \tau_4 = \frac{\epsilon\phi_1(s-2) - 4s + 2}{\epsilon\phi_1(4-2s) + 6s - 3},$$

n	$\dim\Omega^i$	metrics with QLC	QLC	$*$-QLC
2	2:2:0:0	ϵ, h_1	s	$\|s\| = 1$
3	3:4:1:0	ϵ, h_1, h_2	s	$\|s\| = 1$
4	4:6:2:0	$\epsilon, \epsilon', h_1, h_2, h_3$	s	$\|s\| = 1$
5	5:8:3:0	$\epsilon, h_1, h_2, h_3, h_4$	s	$\|s\| = 1$
n	$n : 2(n-1) : n-2 : 0$	$\epsilon, \epsilon', h_1, ..., h_{n-1}$	s	$\|s\| = 1$

Table 1. Summary of vector space dimensions of the exterior algebra and the parameterisation of quantum Riemannian geometries found on A_n for $n \leq 5$.

$$\tau_2' = -\frac{\epsilon\phi_1 + s}{2(\epsilon\phi_1 s + 1)}, \quad \tau_3' = \frac{\epsilon\phi_1(s-2) - 2s + 1}{\epsilon\phi_1(s-2) - 6s + 3}, \quad \tau_4' = \frac{2(\epsilon\phi_1 - 1)s - \epsilon\phi_1 + 4}{\epsilon\phi_1(s-2) - 4s + 2},$$
$$\sigma_2 = \frac{3h_3(\epsilon\phi_1 s + 1)}{2h_2(\epsilon\phi_1(2s-1) - s + 2)}, \quad \sigma_3 = \frac{h_4(\epsilon\phi_1(2s-1) - s + 2)}{h_3(\epsilon\phi_1(4s-2) - 3s + 6)},$$
$$\sigma_3' = -\frac{2h_2(\epsilon\phi_1(s-2) - 2s + 1)}{3h_3(\epsilon\phi_1 + s)}, \quad \sigma_4' = \frac{h_3(2\epsilon\phi_1(s-2) - 6s + 3)}{h_4(\epsilon\phi_1(s-2) - 2s + 1)}.$$

3. Canonical metrics and QLC for A_n and the half-line $\mathbb{N}$

Here, we solve the system of equations for a quantum Riemannian geometry in general, building on our experience for small n.

3.1. Summary of computer results for $n \leq 8$. We summarise the results so far as the first entries in Table 1, where h_i are real variables for the reality conditions if we work over $\mathbb{C}$, ϵ a sign and ϵ' is a discrete parameter (not necessarily a binary choice) indicating a discrete moduli for certain numerical 'direction coefficients' $\{\phi_i\}$. The results found so far then fit the general format

$$g = \sum_{i=1}^{n-1} h_i(\phi_i a_i \otimes a_i' + \epsilon a_i' \otimes a_i); \quad \phi_{n-1} = \frac{1}{\phi_1}, \quad \phi_{n-2} = \frac{1}{\phi_2} \tag{4.19}$$

etc., as depicted in Figure 3. This means that only the first $\phi_1, \cdots, \phi_{\lfloor \frac{n}{2} \rfloor}$ have to be specified, the rest are inverse, and that in the even case $\phi_{\frac{n}{2}}^2 = 1$. We also note that

$$h_i \mapsto -h_i, \quad \phi_i \mapsto -\phi_i, \quad \epsilon \mapsto -\epsilon$$

is a symmetry of the metric in the odd case. Hence, without loss of generality, we may assume that $\phi_{\frac{n}{2}} = 1$ in the even case and in the odd case we do not need to list both a solution for $\{\phi_i\}$ and their negation. We then solved by computer for all remaining $n \leq 8$ and found that all solutions fit this general format with $\{\phi_i\}$ summarised in Table 2. Due to the above symmetry, for A_2, A_3, A_5 we do not list a discrete moduli of $\{\phi_i\}$ as this can be absorbed in a change of sign of the h_i and ϵ, while in the other cases we list them as separate rows. The solutions involve square roots (as we saw up to A_5) or are roots of higher-order polynomials. But with some work, we recognised all entries in a trigonometric form. For

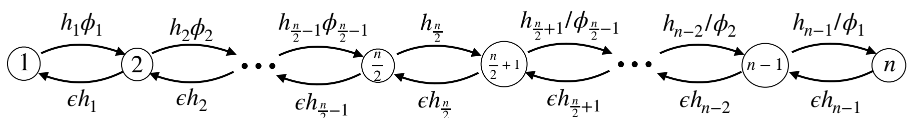

FIGURE 3. Metrics on A_n admitting a QLC have at the top square-lengths $h_i\phi_i$ travelling inwards to half way then with the inverse of the ϕ_i when travelling outward to the right. At the bottom, the square-lengths are ϵh_i with $\epsilon = 1$ the physical choice. We show the case of even n; the odd case is similar.

n	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7
2	1						
3	$2\cos\left(\frac{\pi}{4}\right)$	$\frac{1}{\phi_1}$					
4^+	$2\cos\left(\frac{\pi}{5}\right)$	1	$\frac{1}{\phi_1}$				
4^-	$-2\cos\left(\frac{2\pi}{5}\right)$	1					
5	$2\cos\left(\frac{\pi}{6}\right)$	$\sec\left(\frac{\pi}{6}\right)$	$\frac{1}{\phi_2}$	$\frac{1}{\phi_1}$			
6^+	$2\cos\left(\frac{\pi}{7}\right)$	$2\cos\left(\frac{2\pi}{7}\right)$	1	$\frac{1}{\phi_2}$	$\frac{1}{\phi_1}$		
6^0	$2\cos\left(\frac{3\pi}{7}\right)$	$-2\cos\left(\frac{\pi}{7}\right)$	1				
6^-	$-2\cos\left(\frac{2\pi}{7}\right)$	$-2\cos\left(\frac{3\pi}{7}\right)$	1				
7^+	$2\cos\left(\frac{\pi}{8}\right)$	$\sqrt{2}\sin\left(\frac{3\pi}{8}\right)$	$\csc\left(\frac{3\pi}{8}\right)$	$\frac{1}{\phi_3}$	$\frac{1}{\phi_2}$	$\frac{1}{\phi_1}$	
7^-	$2\cos\left(\frac{3\pi}{8}\right)$	$-\sqrt{2}\sin\left(\frac{\pi}{8}\right)$	$\csc\left(\frac{\pi}{8}\right)$				
8 (1)	$2\cos\left(\frac{\pi}{9}\right)$	$1+2\cos\left(\frac{4\pi}{9}\right)$	$\frac{2\sin\left(\frac{4\pi}{9}\right)}{\sqrt{3}}$	1	$\frac{1}{\phi_3}$	$\frac{1}{\phi_2}$	$\frac{1}{\phi_1}$
8 (2)	$-2\cos\left(\frac{4\pi}{9}\right)$	$1+2\cos\left(\frac{2\pi}{9}\right)$	$\frac{2\sin\left(\frac{2\pi}{9}\right)}{\sqrt{3}}$	1			
8 (3)	$-2\cos\left(\frac{4\pi}{9}\right)$	$1+2\cos\left(\frac{2\pi}{9}\right)$	$-\frac{2\sin\left(\frac{2\pi}{9}\right)}{\sqrt{3}}$	1			
8 (4)	$-2\cos\left(\frac{2\pi}{9}\right)$	$1-2\cos\left(\frac{\pi}{9}\right)$	$-\frac{2\sin\left(\frac{\pi}{9}\right)}{\sqrt{3}}$	1			

TABLE 2. Table of allowed direction coefficients ϕ_i for $n \leq 8$

example, in the 7^- row

$$\sqrt{2-\sqrt{2}} = 2\cos\left(\frac{3\pi}{8}\right), \quad \sqrt{1-\frac{1}{\sqrt{2}}} = \sqrt{2}\sin\left(\frac{\pi}{8}\right), \quad \sqrt{4+2\sqrt{2}} = \csc\left(\frac{\pi}{8}\right)$$

and so forth. From these 'experimental' results in Table 2, we make the following observations:

Remark 4.2. (1) In all cases in the table, we find

$$\phi_{i+1} = \phi_1 - \frac{1}{\phi_i}, \quad i = 1, 2, 3, 4, 5, 6, 7$$

except for 8(2) where this holds for $i = 1$ but not for $i = 2$, for example.

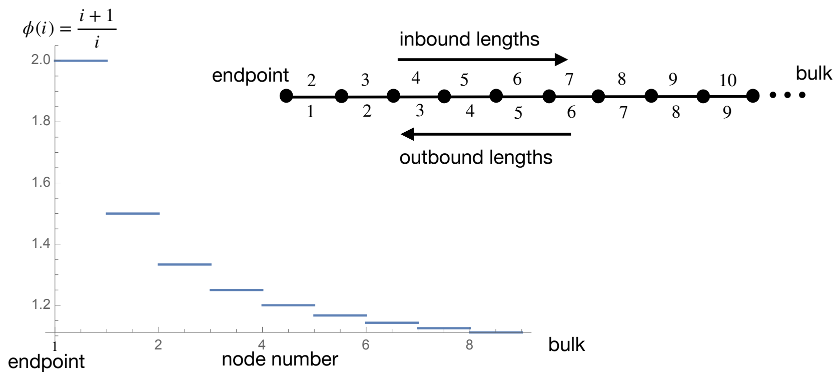

FIGURE 4. The direction coefficient $\phi(i) = \frac{i+1}{i}$ at node i on the half-line $\mathbb{N}$. Metrics that admit a QRG have an arbitrary real number at each edge but in the ratio shown for the inbound direction / outbound direction. Eg at the first link the inbound length is twice the outbound, at the second the ratio is 3:2, etc. The ratio tends rapidly to 1 as we enter the bulk showing that this is an effect due to the endpoint boundary.

(2) For each n in the table, there is a *unique* solution with $\phi_i > 0$, shown in the first row. These are reproduced by the single formula

$$\phi_i = \frac{\sin\left(\frac{(i+1)\pi}{n+1}\right)}{\sin\left(\frac{i\pi}{n+1}\right)} = \frac{(i+1)_q}{(i)_q}; \qquad (i)_q := \frac{q^i - q^{-i}}{q - q^{-1}}; \quad q = e^{\frac{\pi\imath}{n+1}} \tag{4.20}$$

in terms of symmetric q-integers. Here $i = 1, 2, \cdots, n-1$ and the values of ϕ_i obey

$$2 > \phi_1 > \phi_2 > \phi_3 > \cdots > \phi_{\lfloor \frac{n}{2} \rfloor} \geq 1$$

with equality in the even case, i.e. ϕ_i decreases from ϕ_1 at the endpoint towards 1 as we approach the midpoint (and equals 1 at the midpoint in the even case). After that, the $1/\phi_i$ follows the same pattern going back up to the other endpoint.

This, along with $\epsilon = 1$ and $h_i > 0$, is the *unique physical form of the metric* for each n in the sense of positive metric coefficients at each link. One might be able to give an interpretation of negative values as Lorentzian[**11**] but this does not seem reasonable in the present case where all the links are in a line. The metric coefficient or 'square-length' is $h_i\phi_i$ going from $i \to i+1$ and h_i going from $i+1 \to i$, and the 'direction coefficients' ϕ_i is the ratio of these. The above says that there is a longer 'square length' travelling into the bulk compared to travelling outward and that this ratio is most at the endpoints and tends to or is 1 in the middle.

(3) In the limit $n \to \infty$, the physical choice in (2) tends to $\phi(i) = \frac{i+1}{i}$ as in Figure 4. The values of ϕ_i for finite n approach these from below and we see that the finite n QRG is a q-deformation (for q a root of unity) of the $n \to \infty$ theory.

(4) For each n, the unique positive value of ϕ_1 are roots of a certain polynomial and ϕ_2 determined by (1) are roots of a similar polynomial of the same degree. The other allowed values of ϕ_1, ϕ_2 are then all joint solutions of (1) and these two polynomials, modulo the global symmetry mentioned above.

(5) For every allowed quantum metric for $n \le 8$, i.e. for every row in the table, there is a *unique form of QLC* up to the value of $\tau_1 = s$, which is a free parameter required to obey $|s| = 1$ for the connection to be $*$-preserving.

It is expected that the above patterns hold up for all n. In particular, it is clear that over $\mathbb{C}$ and with the required 'reality' structures, there should be a unique allowed form of quantum metric with positive square-lengths given by free parameters $h_1, \cdots, h_{n-1} > 0$ and 'direction coefficients' prescribed by (4.20).

3.2. General solution for the A_n and $\mathbb{N}$. We now solve for the quantum Riemannian geometry in general A_n motivated by our experience for $n \le 8$, which also serves as a check. We consider a general metric with coefficients

$$f_i = \phi_i h_i, \quad f'_i = \epsilon h_i, \quad h_i, \phi_i \neq 0, \quad \epsilon = \pm 1$$

for the metric weights as in (4.19) for increasing and decreasing arrows respectively.

Next, the general form of σ is forced by the bimodule map property and after including the torsion equation, but not yet solving for metric compatibility, to be of the form

$$\begin{aligned} \sigma(a_i \otimes a_{i+1}) &= \sigma_i a_i \otimes a_{i+1}, \quad i = 1, 2, \cdots, n-2, \\ \sigma(a'_i \otimes a'_{i-1}) &= \sigma'_i a'_i \otimes a'_{i-1}, \quad i = 2, 3, \cdots, n-1, \end{aligned} \tag{4.21}$$

$$\sigma(a_1 \otimes a'_1) = \tau_1 a_1 \otimes a'_1, \tag{4.22}$$

$$\begin{aligned} \sigma(a_i \otimes a'_i) &= \tau_i a_i \otimes a'_i + (1 + \tau_i) a'_{i-1} \otimes a_{i-1}, \quad i = 2, \cdots, n-1, \\ \sigma(a'_i \otimes a_i) &= \tau'_i a'_i \otimes a_i + (1 + \tau'_i) a_{i+1} \otimes a'_{i+1}, \quad i = 1, \cdots, n-2, \end{aligned} \tag{4.23}$$

$$\sigma(a'_{n-1} \otimes a_{n-1}) = \tau'_{n-1} a'_{n-1} \otimes a_{n-1}, \tag{4.24}$$

where the $4(n-2)+2$ parameters are organised into four families with $n-2$ values each for σ_i, σ'_i and $n-1$ values each for τ_i, τ'_i as shown. The pattern here is that 2-steps in the same line just have one constant as do the back-and-forth steps at the ends where one can only step one way, but when one can go back-and-forth both to the left and to the right, σ of one of these is a mixture of both possibilities.

We provide an inductive proof now that there is a QLC for all n, limiting attention to metrics of the form (4.19) with $\phi_1, \cdots, \phi_{n-1}$ initially unknown and $\epsilon = \pm 1$ arbitrary, and solving for the connection coefficients. In fact, it pays to consider the equations of 'A_∞' i.e. the natural numbers $\mathbb{N}$ as a discrete half-line and obtain any A_n of interest by truncation.

PROPOSITION 4.3. *For any quantum metric on the natural numbers $\mathbb{N}$ described by h_i, ϵ, there is a 1-parameter moduli of allowed direction coefficients ϕ_i and a 1-parameter family* *of QLCs as defined iteratively by*

$$\phi_{i+1} = \phi_1 - \frac{1}{\phi_i}, \quad \tau_{i+1} = -1 + \frac{\phi_{i+1}}{\phi_i + \epsilon\tau_i}$$

for arbitrary $\phi_1, \tau_1 \neq 0$. *The other connection coefficients are then given by*

$$\sigma_i = \frac{h_{i+1}\phi_{i+1}}{h_i\phi_i(1+\tau'_i)}, \quad \sigma'_i = \frac{h_{i-1}(\phi_{i-1}+\epsilon\tau_{i-1})}{h_i\phi_i}, \quad \tau'_i = \epsilon\frac{\phi_i - \phi_{i+1}}{\tau_i}.$$

Moreover, for h_i, ϕ_1 *real, the connection is* $*$*-preserving iff* $|\tau_1| = 1$.

Proof. Writing out the equations for metric compatibility, these break down into groups of increasing vertex. The first group is

$$-f_1\tau_1\tau'_1 + f'_1 = 0$$
$$-f'_1\tau_1\tau'_1 + f_1 - [f_2(\tau_2+1)\sigma'_2] = 0$$

(this is the same as we saw for A_2 except there we do not have the square bracket term because we do not have f_2, τ_2, σ'_2). The next group are 6 more equations for the 4 new variables $\sigma_1, \sigma'_2, \tau_2, \tau'_2$ and 2 new parameters f_2, f'_2

$$-f_1\sigma_1(\tau'_1+1) + f_2 = 0$$
$$-f'_1\sigma_1(\tau'_1+1) - f_2\tau_2\tau'_2 + f'_2 = 0$$
$$-f'_2(\tau_2+1)\sigma'_2 + f'_1 = 0$$
$$-f'_2\tau_2\tau'_2 + f_2 - [f_3(\tau_3+1)\sigma'_3] = 0$$
$$-f'_1\tau_1(\tau'_1+1) - f_2\tau_2\sigma'_2 = 0$$
$$-f'_1\sigma_1\tau'_1 - f_2(\tau_2+1)\tau'_2 = 0$$

(the equations so far are the same as we saw for A_3 except that there we do not have the square bracket term since there are no variables f_3, τ_3, σ'_3). Similarly we have 6 more equations for the 4 new variables $\sigma_2, \sigma'_3, \tau_3, \tau'_3$ and 2 new parameters f_3, f'_3

$$-f_2\sigma_2(\tau'_2+1) + f_3 = 0$$
$$-f'_2\sigma_2(\tau'_2+1) - f_3\tau_3\tau'_3 + f'_3 = 0$$
$$-f'_3(\tau_3+1)\sigma'_3 + f'_2 = 0$$
$$-f'_3\tau_3\tau'_3 + f_3 - [f_4(\tau_4+1)\sigma'_4] = 0$$
$$-f'_2\tau_2(\tau'_2+1) - f_3\tau_3\sigma'_3 = 0$$
$$-f'_2\sigma_2\tau'_2 - f_3(\tau_3+1)\tau'_3 = 0$$

(the equations so far are the same as we saw for A_4 except that there we do not have the square bracket term since no variables f_4, τ_4, σ'_4).

The general induction step here is to add a set of six equations for the 4 new variables $\sigma_{i-1}, \sigma'_i, \tau_i, \tau'_i$ and 2 new parameters f_i, f'_i,

$$\begin{aligned}
&-f_{i-1}\sigma_{i-1}(\tau'_{i-1}+1)+f_i=0\\
&-f'_{i-1}\sigma_{i-1}(\tau'_{i-1}+1)-f_i\tau_i\tau'_i+f'_i=0\\
&-f'_i(\tau_i+1)\sigma'_i+f'_{i-1}=0\\
&-f'_i\tau_i\tau'_i+f_i-[f_{i+1}(\tau_{i+1}+1)\sigma'_{i+1}]=0\\
&-f'_{i-1}\tau_{i-1}(\tau'_{i-1}+1)-f_i\tau_i\sigma'_i=0\\
&-f'_{i-1}\sigma_{i-1}\tau'_{i-1}-f_i(\tau_i+1)\tau'_i=0
\end{aligned}$$

(which to this point would be the equations for A_{i+1}, except for A_{i+1} itself we would not have the square bracket equations due to no $f_{i+1}, \tau_{i+1}, \sigma'_{i+1}$). This covers the half-line case, and we also noted for later how to extract the A_n solutions from the same analysis.

To solve the system, we rewrite the above ith step equations as

$$(\tau_i+1)\sigma'_i=\frac{f'_{i-1}}{f'_i},\quad \tau_i\tau'_i=\frac{f_i}{f'_i}-\frac{f_{i+1}}{f'_{i+1}},\quad \sigma_{i-1}(\tau'_{i-1}+1)=\frac{f_i}{f_{i-1}}$$

$$f_i\sigma'_i-f'_{i-1}\tau_{i-1}=f_{i-1},\quad f'_{i-1}\sigma_{i-1}-f_i\tau'_i=f'_i,\quad \frac{f_i}{f'_i}-\frac{f_{i+1}}{f'_{i+1}}=\frac{f'_i}{f_i}-\frac{f'_{i-1}}{f_{i-1}}$$

If we write $f_i=h_i\phi_i$ and $f'_i=\epsilon h_i$ for some unknown ϕ_i and $\epsilon=\pm1$ then the last equation is

$$\phi_{i+1}=\phi_i-\frac{1}{\phi_i}+\frac{1}{\phi_{i-1}}$$

which, starting off without the ϕ_{i-1} term, gives the iteration equation for ϕ_i as stated.

Also, from the first and 4th of these ith step equations, we get

$$\tau_{i+1}=-1+\frac{f'_i}{f'_{i+1}\sigma'_{i+1}}=-1+\frac{f'_if_{i+1}}{f'_{i+1}(f_i+f'_i\tau_i)}=-1+\frac{\phi_{i+1}}{\phi_i+\epsilon\tau_i}$$

as stated. Similarly, from the 3rd and the 5th, we get

$$\tau'_i=\frac{f'_{i-1}\sigma_{i-1}-f'_i}{f_i}=\frac{f'_{i-1}}{f_{i-1}(1+\tau'_{i-1})}=\frac{\epsilon}{\phi_{i-1}(1+\tau'_{i-1})}-\frac{\epsilon}{\phi_i}$$

as another (redundant) recursion relation. The 3rd, 4th and 2nd moreover give

$$\sigma_{i-1}=\frac{f_i}{f_{i-1}(1+\tau'_{i-1})},\quad \sigma'_i=\frac{f_{i-1}+f'_{i-1}\tau_{i-1}}{f_i},\quad \tau'_i=\epsilon\frac{\phi_i-\phi_{i+1}}{\tau_i}$$

which can be used to determine σ_{i-1}, σ'_i and obtain τ'_i from τ_i as stated (the two sequences are compatible with this relation).

The stated iterative equations have a unique solution given initial values of ϕ_1, τ_1. The τ'_1 is determined as

$$\tau'_1=\frac{1}{\tau_1}(\phi_1-\phi_1+\frac{1}{\phi_1})=\frac{1}{\tau_1\phi_1}.$$

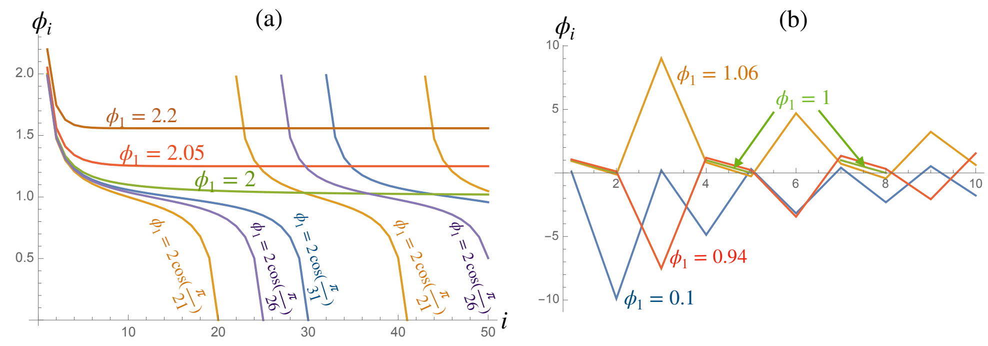

FIGURE 5. (a) $\phi_1 \geq 2$ leads to ϕ_i asymptotically constant while $\phi_1 < 2$ leads to ϕ_i oscillatory. Here $\phi_1 = 2\cos\left(\frac{\pi}{n+1}\right)$ is suitable for A_n and its ϕ_i blows up at i a multiple of $n+1$. (b) Smaller ϕ_1 including $\phi_1 = 2\cos\left(\frac{\pi}{3}\right) = 1$ and perturbations of it.

For the last part, for the connection to be $*$-preserving, we apply $\sigma^{-1} \circ \dagger = \dagger \circ \sigma$ to the relations (4.22) and (4.24) obtaining the conditions $|\tau_1| = |\tau'_{n-1}| = 1$. Applying the $*$-preserving conditions to (4.21), we require $\overline{\sigma_i} = 1/\sigma'_{i+1}$ which from their form in the proposition holds if

$$|\tau_i|^2 = 1 - \frac{\phi_i}{\phi_{i-1}}.$$

Here, we drop the last term for $i = 1$. This is solved with no conditions beyond $|\tau_1| = 1$, as we prove by induction: if the condition holds for $|\tau_i|^2$ then the recurrence relation for τ_{i+1} implies that $|\tau_{i+1}|^2 = 1 - \phi_{i+1}/\phi_i$ as expected. Similarly, the $*$-preserving conditions for (4.23) requires $|\Delta_i|^2 = 1$, where $\Delta_i = (1 + \tau_{i+1})(1 + \tau'_i) - \tau_{i+1}\tau'_i$. Using the form of τ'_i, this reduces to $|\tau_i|^2 = 1 - \phi_i/\phi_{i-1}$ again. $\square$

The iteration for ϕ_i here can be done in closed form. If $\phi_1 = x$, then

$$\phi_i = \frac{1}{2}\left(\sqrt{x^2-4}\left(\frac{1}{\frac{1}{2} - \frac{1}{2}\left(-\sqrt{x^2-4}-x\right)^{-i}\left(\sqrt{x^2-4}-x\right)^{i}} - 1\right) + x\right),$$

$$\phi_2 = x - \frac{1}{x}, \quad \phi_3 = \frac{x\left(x^2-2\right)}{x^2-1}, \quad \phi_4 = \frac{\left(x^4-3x^2+1\right)}{x\left(x^2-2\right)}, \quad \phi_5 = \frac{x\left(x^4-4x^2+3\right)}{x^4-3x^2+1},$$

etc., and we then use this solution to determine the recursion relations for τ_i. We see from ϕ_3 that demanding *edge-symmetry where* $\phi_i = 1$ *at all i, is not an option*. Qualitatively, we see from Figure 5 that there are two phases for the system:

(1) $\phi_1 \geq 2$ (Open phase): Here ϕ_i decays rapidly to an asymptote $\frac{1}{2}\left(\sqrt{\phi_1^2-4}+\phi_1\right)$. This case leads to solutions on the half-line graph $\mathbb{N}$.

(2) $0 < \phi_1 < 2$ (Finite phase): Here ϕ_i is oscillatory, could have zeros and singularities and be periodic for certain ϕ_1 (as illustrated). This case leads to solutions on A_n as a subgraph of $\mathbb{N}$.

The critical line started by $\phi_1 = 2$ between these two regions is particularly simple. It can be approached from either side but more naturally from above.

PROPOSITION 4.4. *For the half-line graph $\mathbb{N}$, the canonical choice of direction coefficients is given by $\phi_1 = 2$. If the metric h_i and initial τ_1 are rational then all coefficients of the quantum geometry are rational. In particular, for initial $\tau_1 = s = \pm 1$ and $\epsilon = 1$,*

$$\phi_i = \frac{i+1}{i}, \quad \tau_i = \frac{s(-1)^{i-1}}{i}; \quad \tau'_i = -\tau_{i+1}, \quad \sigma_i = \frac{h_{i+1}}{h_i}(1+\tau_{i+1}), \quad \sigma'_i = \frac{h_{i-1}}{h_i}\frac{1}{1+\tau_i}.$$

We refer to this as the canonical quantum Riemannian geometry of $\mathbb{N}$.

PROOF. Here $\phi_1 = 2$ is best approached for the analytic solution from above but one can see directly that the ϕ_i stated has this initial value and solves the required recursion relation. For τ_i, τ'_i, (which are independent of the metric) the recursion has an analytic solution using Pochhammer functions. For example

$$\tau_1 = s, \quad \tau_2 = -\frac{16s+8}{2(8s+16)}, \quad \tau_3 = \frac{120s+24}{18(4s+20)}, \quad \tau_4 = -\frac{2016s+864}{288(12s+28)}$$

etc. This simplifies greatly when $|s| = 1$, namely if $s = e^{\imath\theta}$, then

$$\tau_i = -\frac{\left(6i+(-1)^i+3\right)e^{\imath\theta} + 2i + 3(-1)^i + 1}{i\left(((-1)^i(2i+1)+3)\,e^{\imath\theta} + 3(-1)^i(2i+1)+1\right)}.$$

We show the result when $s = \pm 1$ and also for this case the resulting $\tau'_i, \sigma_i, \sigma'_i$. □

We are not claiming this as unique but it it represents by far the simplest solution. Moreover, although we typically work over $\mathbb{C}$ in mathematical physics, it is striking that this canonical quantum geometry on $\mathbb{N}$ works over the rational numbers $\mathbb{Q}$. Finally, we turn to the finite case as promised.

COROLLARY 4.5. *For the finite interval graph A_n with metric defined by h_i, ϵ, ϕ_i, there is a quantum Riemannian geometry of the form in Proposition 4.3 provided ϕ_1 is such that the stated iteration leads to $\phi_n = 0$ and all preceding $\phi_i \neq 0$. In this case*

$$\phi_{n-i} = \frac{1}{\phi_i}.$$

The physical choice where $\phi_i > 0$ for $i = 1, \cdots, n-1$ is provided by $\phi_1 = 2\cos\left(\frac{\pi}{n+1}\right)$ and results in (4.20). E.g., if $\tau_1 = s = \pm 1$ and $\epsilon = 1$ then

$$\tau_i = \frac{s(-1)^{i-1}}{(i)_q}, \quad q = e^{\frac{\imath\pi}{n+1}}$$

n	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6	τ_7	τ_8
2	1	-1						
3	1	$-\frac{1}{\sqrt{2}}$	1					
4	1	$\frac{1}{2}(1-\sqrt{5})$	$-\tau_2$	-1				
5	1	$-\frac{1}{\sqrt{3}}$	$\frac{1}{2}$	τ_2	1			
6	1	$-\frac{1}{2\cos(\frac{\pi}{7})}$	$(-1)^{3/7}-(-1)^{4/7}$	$-\tau_3$	$-\tau_2$	-1		
7	1	$-\sqrt{1-\frac{1}{\sqrt{2}}}$	$\sqrt{2}-1$	$-\frac{1}{\sqrt{2}}\sqrt{1-\frac{1}{\sqrt{2}}}$	τ_3	τ_2	1	
8	1	$-\frac{1}{2\cos(\frac{\pi}{9})}$	$\frac{1}{1+2\cos(\frac{2\pi}{9})}$	$-\frac{1}{4\cos(\frac{\pi}{9})\cos(\frac{2\pi}{9})}$	$-\tau_4$	$-\tau_3$	$-\tau_2$	-1

TABLE 3. Table of connection coefficients τ_i for A_n for $n \leq 8$, $s = 1$ and the canonical ϕ_1. We adjoined τ_n according to the symmetry.

as a q-deformation of the solution on $\mathbb{N}$ in Proposition 4.4 with $\tau_i', \sigma_i, \sigma_i'$ given in terms of this by the same formulae as there. We refer to this as the canonical quantum Riemannian geometry of A_n

PROOF. We solve the iterative system as before but there is no f_n etc for our truncated graph, so we need

$$f_{n-1} = f_{n-1}'\tau_{n-1}\tau_{n-1}'$$

in order that the relevant equation in the last group of 6 holds without that square bracketed term. This is $\tau_{n-1}\tau_{n-1}' = \phi_{n-1}$, which comparing with the general formula $\tau_{n-1}\tau_{n-1}'$ needs $\phi_n = 0$. As the preceding ϕ_i should all be nonzero, this is the first time this should happen. Next, it follows from the inductive formula for ϕ_i in Proposition 4.3 that $\phi_1 = \frac{1}{\phi_{n-1}}$. Moreover, assuming $\phi_i = 1/\phi_{n-i}$ as induction hypothesis and using the recursive relation for ϕ_{i+1} and ϕ_{n-i} gives

$$\phi_{i+1} = \phi_1 - \frac{1}{\phi_i} = \phi_1 - \phi_{n-i} = \frac{1}{\phi_{n-i-1}}$$

as required, proving the stated assertion. For $\epsilon = 1$ as here, one can check that (4.20) obeys this has has the required positivity provided we start with $\phi_1 = 2\cos\left(\frac{\pi}{n+1}\right)$.

To find τ_i for this choice of ϕ_1, we iterated the recursion relation in Proposition 4.3 to fill out a table of τ_i values for small n values, see Table 3. Note that standard recursion methods e.g. on Mathematica do not yield a general answer in closed form. We then 'recognised' the general formula as stated. Once found, it is easy enough to check that τ_i obeys the recursion relation in Proposition 4.3 for $\phi_i = (i+1)_q/(i)_q$ and compute the other values. □

The canonical choice of ϕ_1 is illustrated in Figure 5. It is also worth noting that for $s = \pm 1$, the τ_i in this case enjoy symmetries similar to those of ϕ_i, namely for odd n:

$$\tau_1 = \pm 1, \quad \tau_{n-1} = \tau_2, \quad \tau_{n-2} = \tau_3, \quad \tau_{\frac{n-1}{2}} = \tau_{\frac{n+3}{2}}, \quad \tau_{\frac{n+1}{2}} = (-1)^{\frac{n-1}{2}}\sin\left(\frac{\pi}{n+1}\right)$$

and for even n:

$$\tau_1 = \pm 1, \quad \tau_{n-1} = -\tau_2, \quad \tau_{n-2} = -\tau_3, \quad \tau_{\frac{n}{2}+1} = -\tau_{\frac{n}{2}}.$$

Similarly iterating for small n but general $\tau_1 = s$ and computing the associated $\tau_i, \tau'_i, \sigma_i, \sigma'_i$ recovers for $A_2, \cdots, A_5$ the explicit values reported in Section 2 for $\epsilon = 1$.

More generally, without requiring positivity, one can start with $\phi_1 = 2\cos\left(\frac{j\pi}{n+1}\right)$ for $j = 1, 2, \cdots, n-1$, but some of these differ only by a sign so the solution they generate can be absorbed in ϵ, and others can be excluded as some of the ϕ_i they generate vanish. Also, for specific n there can be further 'irregular' solutions not generated by our method. For example, if we set $n = 8$ then all the solutions for ϕ_1 such that $\phi_8 = 0$ are given by

$$\phi_1 : \quad \pm 2\cos\left(\frac{\pi}{9}\right),\ \pm 2\cos\left(\frac{2\pi}{9}\right),\ \pm 2\cos\left(\frac{3\pi}{9}\right),\ \pm 2\cos\left(\frac{4\pi}{9}\right)$$

of which the 3rd solution is just ± 1 and can be excluded as not all the $\phi_2, \cdots, \phi_7$ are nonzero. The other three feature in Table 2 (we only listed one of the signs since the other can be absorbed in the choice of ϵ). This explains the 'regular' part of the table but we also see from row 8(2) that for specific n, we do not generate all the solutions by our method. Indeed, row 8(2) has the same initial ϕ_1 as row 8(3) while our method only gives one set of ϕ_i for an initial ϕ_1.

4. Laplacian and elements of QFT on A_3

In the remainder of the paper, we work with only the canonical QRGs on $\mathbb{N}$ and A_n in Proposition 4.4 and Corollary 4.5, with h_i real, $s = \pm 1$ and $\epsilon = 1$ there. Given this, we will now repurpose $\epsilon > 0$ as a lattice spacing for the continuum limit of the geometry on $\mathbb{N}$. In this section, we compute and study the Laplacian $\Box\psi = (\ ,\ )\nabla \mathrm{d}\psi$ for a field ψ on A_n and $\mathbb{N}$, and in Section 5 the Ricci curvature. In both cases, we need the inverse metric, which now looks like

$$(a_i, a'_i) = \frac{\delta_i}{h_i}, \quad (a'_i, a_i) = \frac{\delta_{i+1}}{h_i \phi_i}$$

for $i = 1, \cdots, n-1$ and is otherwise zero on basis elements.

Lemma 4.6. *The Laplacian operator* $\square f$ *of a function* $f = \sum_i f(i)\delta_i$ *on the vertices of the* A_n *graph with* $n \geq 3$ *has the form*

$$\begin{aligned}\square f &= \sum_{i=1}^{n-1}(f(i+1)-f(i))(\,,\,)(\nabla a_i - \nabla a_i')\\ &= \left((f(1)-f(2))\frac{\tau_1+1}{h_1}\right)\delta_1\\ &+ \sum_{i=2}^{n-1}\Big((f(i)-f(i-1))(\tau_{i-1}'+1)+(f(i)-f(i+1))(\tau_i+1)\Big)\left(\frac{1}{h_i}+\frac{1}{h_{i-1}\phi_{i-1}}\right)\delta_i\\ &+ (f(n)-f(n-1))\left(\frac{\tau_{n-1}'+1}{h_{n-1}\phi_{n-1}}\right)\delta_n\end{aligned}$$

Proof. Because the calculus is inner, we have $\mathrm{d}f = [\theta, f] = \sum_1^{n-1}(f(i+1)-f(i))(a_i - a_i')$ where $\theta = \sum_1^{n-1} a_i + a_i'$. Using this in the definition of the Laplacian $\square f = (\,,\,)\nabla\mathrm{d}f$ for an arbitrary function f we gives the first expression for the Laplacian.

Next, the general form for the connection corresponding to a_1, a_{n-1} and a_i (valid only for $1 < i < n-1$) using again that the calculus is inner, is

$$\begin{aligned}\nabla a_1 &= a_1' \otimes a_1 - \tau_1 a_1 \otimes a_1' - \sigma_1 a_1 \otimes a_2,\\ \nabla a_1' &= a_1 \otimes a_1' + a_2' \otimes a_1' - \tau_1' a_1' \otimes a_1 - (\tau_1'+1)a_2 \otimes a_2',\\ \nabla a_{n-1} &= a_{n-2} \otimes a_{n-1} + a_{n-1}' \otimes a_{n-1} - \tau_{n-1}a_{n-1} \otimes a_{n-1}' - (\tau_{n-1}+1)a_{n-2}' \otimes a_{n-2},\\ \nabla a_{n-1}' &= a_{n-1} \otimes a_{n-1}' - \sigma_{n-1}' a_{n-1}' \otimes a_{n-2}' - \tau_{n-1}' a_{n-1}' \otimes a_{n-1},\\ \nabla a_i &= a_i' \otimes a_i + a_{i-1} \otimes a_i - \tau_i a_i \otimes a_i' - (\tau_i+1)a_{i-1}' \otimes a_{i-1} - \sigma_i a_i \otimes a_{i+1},\\ \nabla a_i' &= a_i \otimes a_i' + a_{i+1}' \otimes a_i' - \tau_i' a_i' \otimes a_i - (\tau_i'+1)a_{i+1} \otimes a_{i+1}' - \sigma_i' a_i' \otimes a_{i-1}'.\end{aligned}$$

Arranging the terms and applying the inverse metric, we recover the explicit form stated. □

We also use this for the case of $\mathbb{N}$ but without the final values.

4.1. $\mathbb{N}$ and its continuum limit. The Laplacian for the connection of the Proposition 4.4 with general values of the metric h_i comes out as

$$\begin{aligned}(\square f)(1) &= (f(1)-f(2))\frac{1+s}{h_1}\\ (\square f)(i) &= \left(-(\Delta_{\mathbb{Z}}f)(i) + \frac{(-1)^i s}{i}(f(i+1)-f(i-1))\right)\left(\frac{1}{h_i}+\frac{1}{h_{i-1}(1+\frac{1}{i-1})}\right); \quad i > 1\end{aligned}$$

in terms of the usual discrete Laplacian $(\Delta_{\mathbb{Z}}f)(i) = f(i+1)+f(i-1)-2f(i)$. For the sake of discussion, we now take $s = 1$ in order to avoid $(\square f)(1) = 0$ for all f. An alternative, which amounts to ignoring the $(-1)^i$ term, would be to average the Laplacian between $s = \pm 1$.

For reference,

$$(\Box_h f)(i) = -(f(i+1) + f(i-1) - 2f(i))(\frac{1}{h_{i-1}} + \frac{1}{h_i})$$

was the Laplacian for an infinite line graph $\mathbb{Z}$ with metric weights h_i as found in [**2**]. Compared to this, we see two effects of the truncation to $\mathbb{N}$, both going as $1/i$ so that they are not visible far from the boundary at $i = 1$:

(1) The metric-dependent factor $\frac{1}{h_{i-1}} + \frac{1}{h_i}$ for $\mathbb{Z}$ decreases slightly as $i \to 1$;
(2) There is a derivative correction but with an alternating sign $(-1)^i$.

We now look at both of these in the context of a lattice approximation of $(0, \infty) \subset \mathbb{R}$ sampled at $x = \epsilon i$, where $i \in \mathbb{N}$ and $\epsilon > 0$ now denotes the lattice spacing. We consider a function f as either $f(x)$ or $f(i)$ via this correspondence. We implement the lattice spacing by a constant value $h_i = \epsilon^2$ in the inbound (increasing i direction), but one also has similar results for the more symmetrical $h_i = \epsilon^2 \sqrt{\frac{i}{i+1}}$. Then the metric-dependent factor is

$$\frac{1}{h_i} + \frac{1}{h_{i-1}(1 + \frac{1}{i-1})} = \frac{1}{\epsilon^2}\beta^{-1}, \quad \beta^{-1}(i) = 1 + \frac{1}{1 + \frac{1}{i-1}} = 2 - \frac{1}{i} \tag{4.25}$$

(1) We first ignore the term with $(-1)^i$ as this clearly has no continuum limit and we will argue that its effects are minimal. In this case, we have as $\epsilon \to 0$,

$$\beta(x) = \frac{1}{2} + \frac{\epsilon}{4x} + O(\epsilon^2), \quad \frac{1}{\epsilon^2}\Delta_{\mathbb{Z}} f = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + O(\epsilon^2)$$

To interpret what happens at order ϵ we consider solving the time-independent Schroedinger equation $\Box f = 4mEf$ for a mass m and energy E in our normalisation of $\Box$. We set $\hbar = 1$ for present purposes. Then to $O(\epsilon^2)$, the equation we are solving is

$$\left(-\frac{1}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{E\epsilon}{2x}\right) f = Ef.$$

This does not have an immediate parallel with quantum mechanics as the 'potential' term is energy-dependent but we can see that energy E shifts by an amount which is $E\epsilon$ times a 'potential' $-\frac{1}{2x}$, with eigenfunctions to $O(\epsilon^2)$ obtained by solving this. One could therefore think of this as like a $\frac{1}{x^2}$ force driving solutions towards the boundary as $x = 0$. We illustrate this in Figure 6. This shows the solution to

$$\Box f = 4mEf, \quad f(1) = 1 - \alpha, \quad f(2) = 1 - 2\alpha, \quad \alpha = \frac{2mE\epsilon^2}{1 + 2mE\epsilon^2}$$

where the initial conditions are such that the linearly extrapolated value at the origin is $f(0) = 1$ and $4mEf(1) = (\Box f)(1) = 2(f(1) - f(2))/\epsilon^2$ as required. The effect of the $(-1)^i$ is to produce ripples in the solution which are less pronounced at larger x and which get faster and smaller as $\epsilon \to 0$ (since there are more steps in the range $(0, x)$ for any finite x).

(2) For a theoretical picture of the term with the $(-1)^i$ factor, we discuss two ways to think about this, at least intuitively. One is to live with the lack of continuity and just keep the $(-1)^i$ factor which in the limit of $\epsilon \to 0$ stands for an infinitely-rapidly alternating

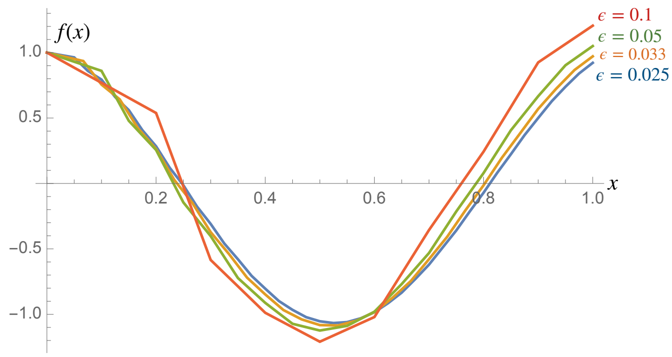

FIGURE 6. Numerical solutions of $\Box f = 4mEf$ at $mE = 15$ and different values of $h_i = \epsilon^2$, converging to a smooth solution as $\epsilon \to 0$.

function of $x = i\epsilon$, but which makes sense for any finite $\epsilon > 0$. In this case, the other parts of the expression have a limit and we obtain

$$\Box f = -2\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + (-1)^i \frac{4}{x}\frac{\mathrm{d}f}{\mathrm{d}x} + O(\epsilon)$$

in so far as this makes sense. The other approach is to sample our functions only at even i and replace $(-1)^i \frac{1}{i}$ by its average value at i and $i+1$, i.e. by $\frac{1}{2}(\frac{1}{i} - \frac{1}{i+1}) = \frac{1}{2i(i+1)}$, which tends to $\frac{\epsilon^2}{2x^2}$ plus higher order in ϵ. In this case, one can say, again intuitively, that

$$\Box f = (-\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \frac{\epsilon}{x^2}\frac{\mathrm{d}f}{\mathrm{d}x})\beta^{-1}(x) + O(\epsilon^2) = -(\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + 4\beta'\frac{\mathrm{d}f}{\mathrm{d}x})\beta^{-1} + O(\epsilon^2)$$

where we recognise $\beta'(x) = -\frac{\epsilon}{4x^2}$. This leads to a further term $\frac{\epsilon}{2mx^2}f'$ added to the effective 'Hamiltonian' in our previous analysis. This contribution no longer has the flavour of a potential energy but rather of a coupling to an effective background gauge potential. Note that the expression here is not quite $\beta^{-1}(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{\beta'}{2\beta}\frac{\mathrm{d}}{\mathrm{d}x})$, the classical Laplacian for metric $g = \beta \mathrm{d}x \otimes \mathrm{d}x$ and connection $\nabla \mathrm{d}x = -\frac{\beta'}{2\beta}\mathrm{d}x \otimes \mathrm{d}x$ in our notations. (Namely, it differs by a factor -4 in the β' coefficient given that $\beta \approx 1/2$.) We also recall that the limit of the quantum geometry is more precisely a 2-dimensional noncommutative differential calculus rather than a classical calculus.

The overall picture is that the direction-dependent quantum metric on $\mathbb{N}$ cannot be avoided as we approach the $i = 1$ boundary and cannot be mapped to an effective continuum limit, but we can begin to get a feel for its physical significance as something like an effective force pushing solutions towards the boundary and possibly a further velocity dependent force. This analysis was for constant metrics h_i or similar. We will mention another natural family of metrics in Section 5.

4.2. QFT on a finite lattice interval A_n. The Laplacian from Lemma 4.6 for the A_n geometry given for the Corollary 4.5 is

$$\square f(1) = (f(1) - f(2))\frac{s+1}{h_1}, \quad \square f(n) = (f(n) - f(n-1))\frac{1+(-1)^n s}{h_{n-1}}(2)_q,$$
$$\square f(i) = \left(-\Delta_{\mathbb{Z}} f(i) + \frac{(-1)^i s}{(i)_q}(f(i+1) - f(i-1))\right)\left(1 + \frac{h_i(i-1)_q}{h_{i-1}(i)_q}\right)\frac{1}{h_i}$$

for $i = 2, \cdots, n-1$. We used that $(n)_q = 1$ and $(n-1)_q = (2)_q$. It is convenient to write the Laplacian in the form

$$\square f = (Lf)\beta^{-1}; \quad \beta^{-1}(1) = \frac{1}{h_1}, \quad \beta^{-1}(n) = \frac{(2)_q}{h_{n-1}}, \quad \beta^{-1}(i) = \left(1 + \frac{h_i(i-1)_q}{h_{i-1}(i)_q}\right)\frac{1}{h_i},$$

for $i = 2, \cdots, n-1$. Then for the free field QFT partition functions etc., we are interested in

$$Z = \int \prod_{i=1}^{n} \mathrm{d}\psi(i)\mathrm{d}\bar{\psi}(i) e^{\frac{\imath}{\alpha}\sum_{i=1}^{n}\mu_i\bar{\psi}(i)((L\psi)(i)\beta^{-1}(i) - m^2\psi(i))}$$

with α a real coupling constant and $\mu_i > 0$ a measure of 'integration' (now a sum) on A_n. This action is quadratic and hence can be evaluated as a determinant. This also applies in the real scalar field case where we have $\prod_i \mathrm{d}\psi(i)$.

As an example, we let $n = 3$ and $s = 1$. We have $(2)_q = \sqrt{2}$ and $(3)_q = 1$ and weights μ_i, we have an action for a complex (or real) 3-vector $\psi = (\psi_1, \psi_2, \psi_3)$. In these terms, the action is quadratic,

$$\begin{aligned} S[\psi] &= \bar{\psi}_1\mu_1\left(\frac{2(\psi_1 - \psi_2)}{h_1} - m^2\psi_1\right) + \bar{\psi}_3\mu_3\left(-m^2\psi_3\right) \\ &\quad + \bar{\psi}_2\mu_2\left(\left(\frac{1}{h_2} + \frac{1}{\sqrt{2}h_1}\right)\left(\left(-\frac{1}{\sqrt{2}} - 1\right)\psi_1 + \left(\frac{1}{\sqrt{2}} - 1\right)\psi_3 + 2\psi_2\right) - m^2\psi_2\right) \\ &= \bar{\psi}.B.\psi \end{aligned}$$

for a matrix

$$B = \begin{pmatrix} \mu_1\left(\frac{2}{h_1} - m^2\right) & -\frac{\mu_1 2}{h_1} & 0 \\ -\mu_2\left(1 + \frac{1}{\sqrt{2}}\right)\left(\frac{1}{h_2} + \frac{1}{\sqrt{2}h_1}\right) & \mu_2\left(2\left(\frac{1}{h_2} + \frac{1}{\sqrt{2}h_1}\right) - m^2\right) & \mu_2\left(\frac{1}{\sqrt{2}} - 1\right)\left(\frac{1}{h_2} + \frac{1}{\sqrt{2}h_1}\right) \\ 0 & 0 & \mu_3\left(-m^2\right) \end{pmatrix}$$

now including the mass term. Hence, the partition function Z is again given as usual for a Gaussian via the determinant

$$\begin{aligned} \det(B) = \frac{\mu_1\mu_2\mu_3 m^2}{h_1^2 h_2^2}\Big(&h_1^2 h_2 m^2\left(-h_2 m^2 + 2\right) + \left(\sqrt{2} + 2\right)h_1 h_2^2 m^2 \\ &+ \left(\sqrt{2} - 2\right)h_1 h_2 - h_2^2\left(\sqrt{2} - 1\right)\Big). \end{aligned}$$

For the constant case $h_1 = h_2$, this simplifies to

$$\det(B) = -\frac{\mu_1\mu_2\mu_3 m^2}{h_1^2}\left(h_1 m^2\left(h_1 m^2 - \sqrt{2} - 4\right) + 1\right).$$

One can similarly compute correlation functions.

5. Curvatures and elements of quantum gravity on $n = 3$

In this section, we compute the curvatures in terms of the h_i real parameters and $s = \pm 1$ and $\epsilon = 1$, i.e. for the canonical QRGs on $\mathbb{N}$ in Proposition 4.4 and on A_n by Corollary 4.5 with $\phi_1 = 2\cos\left(\frac{\pi}{n+1}\right)$. We then use the lifting map (4.3) to define the Ricci tensor as in [**14**] and the Ricci scalar by $S = (\ ,\ )(\mathrm{Ricci})$ where $(\ ,\ )$ is the inverse metric. Curvature tensors will depend on the h_i only through the ratio

$$\rho_i = \frac{h_{i+1}}{h_i},$$

which we use throughout the section. We also repurpose ϵ as the lattice spacing in the case of $\mathbb{N}$.

In general, the Riemann curvature for our form of connection reduces to

$$\begin{aligned}
R_\nabla a_1 &= 0,\\
R_\nabla a_1' &= (\tau_1'(\sigma_2' - \tau_1) + \sigma_2')a_1' \wedge a_1 \otimes a_1' + (\tau_1'(\tau_2' - \sigma_1) + \tau_2')a_1' \wedge a_1 \otimes a_2,\\
R_\nabla a_i &= (\tau_i(\sigma_{i-1} - \tau_i') + \sigma_{i-1} - \sigma_i(\tau_{i+1} + 1))a_i \wedge a_i' \otimes a_i + (\tau_i(\tau_{i-1} - \sigma_i')\\
&\quad + \tau_{i-1})a_i \wedge a_i' \otimes a_{i-1}',\\
R_\nabla a_i' &= (\tau_i'(\sigma_{i+1}' - \tau_i) + \sigma_{i+1}' - \sigma_i'(\tau_{i-1}' + 1))a_i' \wedge a_i \otimes a_i' + (\tau_i'(\tau_{i+1}' - \sigma_i)\\
&\quad + \tau_{i+1}')a_i' \wedge a_i \otimes a_{i+1},\\
R_\nabla a_{n-1} &= (\tau_{n-1}(\sigma_{n-2} - \tau_{n-1}') + \sigma_{n-2})a_{n-1} \wedge a_{n-1}' \otimes a_{n-1}\\
&\quad + (\tau_{n-1}(\tau_{n-2} - \sigma_{n-1}') + \tau_{n-2})a_{n-1} \wedge a_{n-1}' \otimes a_{n-2}',\\
R_\nabla a_{n-1}' &= 0
\end{aligned}$$

for $i = 2, \cdots, n-2$ on A_n and the same without the final cases on $\mathbb{N}$.

5.1. Curvatures for $\mathbb{N}$. The results for the canonical solution in Proposition 4.4 with $s = \pm 1$ are as follows. For the Riemann curvature, we find

$$\begin{aligned}
\mathrm{R}_\nabla a_1 &= 0,\\
\mathrm{R}_\nabla a'_1 &= \left(\rho_1\frac{(1-2s)}{4} - \frac{(1+2s)}{6}\right)a'_1\wedge a_1\otimes a_2 - \left(\frac{(s+2)}{\rho_1(s-2)}+\frac{1}{2}\right)a'_1\wedge a_1\otimes a'_1,\\
\mathrm{R}_\nabla a_i &= \left(\rho_{i-1}\frac{\left(i-(-1)^i s\right)^2}{i^2} - \frac{\rho_i\left(i+1+(-1)^i s\right)^2}{(i+1)^2} - \frac{1}{i(i+1)}\right)a_i\wedge a'_i\otimes a_i\\
&\quad + (-1)^i s\left(\frac{\left(i-(-1)^i s\right)}{(i-1)i} + \frac{1}{\rho_{i-1}\left(i-(-1)^i s\right)}\right)a_i\wedge a'_i\otimes a'_{i-1},\\
\mathrm{R}_\nabla a'_i &= \left(-\frac{\left(i+(-1)^i s\right)}{\rho_{i-1}\left(i-(-1)^i s\right)} + \frac{\left(i+1-(-1)^i s\right)}{\rho_i\left(i+1+(-1)^i s\right)} - \frac{1}{i(i+1)}\right)a'_i\wedge a_i\otimes a'_i\\
&\quad + \frac{(-1)^i s}{(i+1)}\left(\frac{\left(i+1-(-1)^i s\right)}{(i+2)} + \rho_i\frac{\left(i+1+(-1)^i s\right)}{(i+1)}\right)a'_i\wedge a_i\otimes a_{i+1}
\end{aligned}$$

for $i \geq 2$. The Ricci tensor for the lift (4.3) is then

$$\begin{aligned}
\mathrm{Ricci} = &\left(\rho_1\frac{(s-2)s}{4} - \frac{s(s+2)}{6}\right)a_1\otimes a_2 - \left(\frac{(s+2)}{\rho_1(s-2)}+\frac{1}{2}\right)a_1\otimes a'_1\\
&+ \frac{1}{2}\sum_{i\geq 2}\Big\{\phi_i\left(-\frac{\left(i+(-1)^i s\right)}{\rho_{i-1}\left(i-(-1)^i s\right)} + \frac{\left(i+1-(-1)^i s\right)}{\rho_i\left(i+1+(-1)^i s\right)} - \frac{1}{i(i+1)}\right)a_i\otimes a'_i\\
&\quad + \frac{(-1)^i s\phi_i}{(i+1)}\left(\frac{\left(i+1-(-1)^i s\right)}{(i+2)} + \rho_i\frac{\left(i+1+(-1)^i s\right)}{(i+1)}\right)a_i\otimes a_{i+1}\\
&\quad + \frac{1}{\phi_i}\left(\rho_{i-1}\frac{\left(i-(-1)^i s\right)^2}{i^2} - \frac{\rho_i\left(i+1+(-1)^i s\right)^2}{(i+1)^2} - \frac{1}{i(i+1)}\right)a'_i\otimes a_i\\
&\quad + \frac{(-1)^i s}{\phi_i}\left(\frac{\left(i-(-1)^i s\right)}{(i-1)i} + \frac{1}{\rho_{i-1}\left(i-(-1)^i s\right)}\right)a'_i\otimes a'_{i-1}\Big\}.
\end{aligned}$$

There are no Ricci flat solutions but note that the only coefficients that do not decay $O(\frac{1}{i})$ for generic ρ_i are the coefficients of $a_i\otimes a'_i$ and $a'_i\otimes a_i$, which asymptote to $\frac{1}{\rho_i} - \frac{1}{\rho_{i-1}}$ and $\rho_{i-1} - \rho_i$ respectively.

Contracting with (,), the Ricci scalar is then

$$\mathrm{S}(1)=-\frac{1}{2h_1}\left(1+\frac{2(s+2)}{\rho_1(s-2)}\right),$$
$$\mathrm{S}(2)=-\frac{1}{8h_2}\left(1-\frac{6(s+2)}{\rho_1(s-2)}+\frac{6(s-3)}{\rho_2(s+3)}\right),$$
$$\mathrm{S}(i)=-\frac{1}{2h_i}\Bigg(\frac{1}{i^2}+\rho_{i-1}\frac{(i-1)}{i^3}-\rho_{i-1}\rho_{i-2}\frac{\left(i-1+(-1)^is\right)^2}{i^2}+\rho_{i-1}^2\frac{(i-1)^2\left(i-(-1)^is\right)^2}{i^4}$$
$$+\frac{(i+1)\left(i+(-1)^is\right)}{\rho_{i-1}i\,(i-(-1)^is)}-\frac{(i+1)\left(i+1-(-1)^is\right)}{\rho_i i\,(i+1+(-1)^is)}\Bigg),$$

with

$$S(i)=-\frac{1}{2h_i}\left(\rho_{i-1}(\rho_{i-1}-\rho_{i-2})+\frac{1}{\rho_{i-1}}-\frac{1}{\rho_i}\right)+O(\frac{1}{i})$$

for generic h_i. In the constant h_i case, however, we have to look to the next order and then one finds

$$S(i)=-\frac{(1+4(-1)^is)}{h_1i^2}+O(\frac{1}{i^3})$$

which has a non-continuum alternating term suppressed for large i, in line with such a term term in the Laplacian in Section 4.

Alternatively, we can land exactly on $S=0$, in fact on any prescribed function for the curvature, provided we use an oscillatory h_i which will then not have a classical limit itself. We explore this option next.

Proposition 4.7. *On $\mathbb{N}$, there is a unique metric $\{h_i\}$ up to normalisation such that $S=0$, given by*

$$s=1:\quad h_i^{flat}=2h_1\frac{(2\lfloor\frac{i}{2}\rfloor+1)^2}{(i+1)}=2h_1\begin{cases}i+1 & i\ even\\ \frac{i^2}{(i+1)} & i\ odd\end{cases},$$

$$s=-1:\quad h_i^{flat}=2h_1\frac{\lceil\frac{i}{2}\rceil^2}{(i+1)}=\frac{h_1}{2}\begin{cases}\frac{i^2}{(i+1)} & i\ even\\ i+1 & i\ odd\end{cases},$$

for any inital value h_1.

Proof. From the form of $S(i)$, it is clear that we can solve iteratively to find h_i for any initial h_1. Doing this for $s=\pm1$ gives the solutions shown. □

For the rest of the section, we focus on $s=1$ but there is a similar story for $s=-1$. First note that setting $s=1$ and $h_1=\epsilon^3$ for a small number $\epsilon>0$ and repeating the analysis in Section 4, the metric-dependent factor in the Laplacian becomes

$$\frac{1}{h_i}+\frac{1}{h_{i-1}(1+\frac{1}{i-1})}=\frac{1}{\epsilon^3}\beta^{-1}(i);\quad \beta^{-1}(i)=\frac{1}{i}+O(\frac{1}{i^2})$$

in place of (4.25). We can take the continuum limit with the leading order $\beta^{-1}(i) = \frac{1}{i} = \frac{\epsilon}{x}$ and do the parallel analysis to (1) in Section 4. Ignoring the $(-1)^i$ differential term as we did before, gives that $\Box f = 4mEf$ becomes the Airy equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + 4mExf = 0 \tag{4.26}$$

with a real decaying cosine-wave-like solution if $4mE > 0$ and, say, $f(0) = 1, f'(0) = 0$. In addition, we can expect ripples in the discrete solution visible for small i due to the even values of $\beta^{-1}(i)$ and due to the $(-1)^i$ differential term as discussed in Section 4. Meanwhile, the QFT action depends on the measure μ_i and if we take the obvious choice $\mu_i = h_i$ then this cancels the $1/x$ in the continuum limit and we obtain a multiple of the free field action (again ignoring the suppressed $(-1)^i$ term in the Laplacian), which is perhaps reasonable as the curvature is zero.

Next, we consider metrics near to the above flat one in a conformal sense,

$$h_i = h_i^{flat} g_i; \quad \rho_i = \rho_i^{flat}\eta_i; \quad \eta_i = \frac{g_{i+1}}{g_i}$$

with h^{flat} from Proposition 4.7 for $s = 1$. Then a calculation with $h_1 = \epsilon^3$ and $i = x/\epsilon$ as above and working to leading order in ϵ, gives

$$S(x) = -\frac{1}{2h_i^{flat} g_i}\left(\eta_{i-1}(\eta_{i-1} - \eta_{i-2}) + \frac{1}{\eta_{i-1}} - \frac{1}{\eta_i}\right) = -\frac{1}{4\epsilon x g}(\frac{1}{\eta^2} + \eta)\frac{\mathrm{d}\eta}{\mathrm{d}x},$$

where

$$\eta_i = 1 + \frac{g_{i+1} - g_i}{g_i} = 1 + \epsilon g^{-1}\frac{\mathrm{d}g}{\mathrm{d}x} + O(\epsilon^2).$$

Putting this in, we have to leading oder in ϵ,

$$S(x) = -\frac{g^{-1}}{4x}\frac{\mathrm{d}}{\mathrm{d}x}(g^{-1}\frac{\mathrm{d}g}{\mathrm{d}x}) = -\frac{e^{-\psi}}{4x}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}$$

if we write $g(x) = e^{\psi(x)}$ for a real scalar field ψ.

We briefly consider the Einstein-Hilbert action for such metric fluctuations near the scalar-flat metric, expressed in $\psi(x)$. We need to fix the measure μ_i in

$$S[h] = \sum_i \mu_i S(i)$$

and based on experience in [**38**] for $\mathbb{Z}$, we take $\mu_i = h_i = h_i^{flat} g_i$. The theory behind how to choose this measure is not clearly understood, but we expect some power of the metric. Classically, one would have $\sqrt{\det(g)}$ for the measure but in [**38**] it gave more reasonable answers not to take a square root, related to Ω^1 there being 2-dimensional. Our Ω^1 is not exactly a free module but is more like this far from the boundary. In this case,

$$S[\psi] = \sum_{i=1}^{\infty} h_i^{flat} g_i S(i) \to -\frac{\epsilon}{2}\int_0^\infty \mathrm{d}x \frac{\mathrm{d}}{\mathrm{d}x}(\frac{\mathrm{d}\psi}{\mathrm{d}x}) = \frac{\epsilon}{2}(\frac{\mathrm{d}\psi}{\mathrm{d}x})(0^+)$$

to leading order as $\epsilon \to 0$, given that $h_i^{flat} = 2\epsilon^2 x$ to leading order and assuming our fields decay at ∞. The ϵ can be absorbed in μ or in a coupling constant in front of the action. The action here is topological and appears to amount to a trivial theory on the boundary at $x = 0^+$ (approaching from the bulk), but could be more interesting before we take the continuum limit and if we look more closely at the boundary.

5.2. Curvatures for A_n. We proceed from the general expression for the curvature and put in the connection in Corollary 4.5. However, the $\tau_i, \tau'_i, \sigma_i, \sigma'_i$ are exactly a q-deformation of the formulae for $\mathbb{N}$ in the sense that all integers $i-1, i, i+1, i+2$ are replaced by $(i-1)_q, (i)_q, (i+1)_q, (i+2)_q$ respectively. With that change, the formulae for R_∇ for are exactly as before except that now $R_\nabla a'_{n-1} = 0$ and $R_\nabla a_{n-1}$ drops the term with ρ_{n-1}. One can also simplify with $(n)_q = 1$ and $(n-1)_q = (2)_q$. Likewise $R_\nabla a_1 = 0$ and $R_\nabla a'_1$ drops the term with $1/\rho_0$. Thus,

$$R_\nabla a_1 = 0, \quad R_\nabla a'_1 = -\left(\frac{s+(2)_q}{\rho_1(s-(2)_q)} + \frac{1}{(2)_q}\right) a'_1 \wedge a_1 \otimes a'_1,$$

$$R_\nabla a_{n-1} = \left(\frac{\rho_{n-2}\left((2)_q + (-1)^n s\right)^2}{(2)_q^2} - \frac{1}{(2)_q}\right) a_{n-1} \wedge a'_{n-1} \otimes a_{n-1}, \quad R_\nabla a'_{n-1} = 0.$$

For the Ricci tensor the sum over i is q-deformed and truncated, but for the $i = 1$ term we drop the $1/\rho_0$ in the coefficient of $a_1 \otimes a'_1$ and the $a'_1 \otimes$ terms altogether, and for the $i = n-1$ terms we drop the ρ_{n-1} in the coefficient of $a'_{n-1} \otimes a_{n-1}$ and the $a_{n-1}\otimes$ terms altogether. Thus,

$$\begin{aligned}
\mathrm{Ricci} = {} & \frac{s}{2}\left(\rho_1 \frac{s-(2)_q}{(2)_q} - \frac{(2)_q + s}{(3)_q}\right) a_1 \otimes a_2 - \frac{1}{2}\left(\frac{(2)_q((2)_q + s)}{\rho_1(s-(2)_q)} + 1\right) a_1 \otimes a'_1 \\
& + \frac{1}{2}\sum_{i=2}^{n-2}(q-\text{def previous}) + \frac{1}{2}\left(\rho_{n-2}\frac{((2)_q + (-1)^n s)^2}{(2)_q} - 1\right) a'_{n-1} \otimes a_{n-1} \\
& - \frac{(-1)^n s}{2}\left(\frac{(2)_q + (-1)^n s}{(3)_q} + \frac{(2)_q}{\rho_{n-2}((2)_q + (-1)^n s)}\right) a'_{n-1} \otimes a'_{n-2}
\end{aligned}$$

There are no Ricci flat solutions. The Ricci scalar is then

$$\begin{aligned}
\mathrm{S}(1) &= -\frac{1}{2h_1}\left(1 + \frac{(2)_q(s+(2)_q)}{\rho_1(s-(2)_q)}\right),\\
\mathrm{S}(2) &= -\frac{(3)_q}{2h_2(2)_q}\left(\frac{1}{(2)_q(3)_q} - \frac{s+(2)_q}{\rho_1(s-(2)_q)} + \frac{s-(3)_q}{\rho_2(s+(3)_q)}\right),\\
\mathrm{S}(i) &= -\frac{(i+1)_q}{2h_i(i)_q}\left(\frac{1}{(i)_q(i+1)_q} + \frac{((i)_q+(-1)^i s)}{\rho_{i-1}((i)_q-(-1)^i s)} - \frac{((i+1)-(-1)^i s)}{\rho_i((i+1)_q+(-1)^i s)}\right)\\
&\quad - \frac{(i-1)_q^2}{2h_i(i)_q^2}\rho_{i-1}\left(\frac{1}{(i-1)_q(i)_q} - \rho_{i-2}\frac{((i-1)_q+(-1)^i s)^2}{(i-1)_q^2} + \rho_{i-1}\frac{((i)_q-(-1)^i s)^2}{(i)_q^2}\right),\\
\mathrm{S}(n-1) &= -\frac{(3)_q^2}{2h_{n-2}(2)_q^2}\left(\frac{1}{(2)_q(3)_q} + \rho_{n-2}\frac{((2)_q+(-1)^n s)^2}{(2)_q} - \rho_{n-3}\frac{((3)_q-(-1)^n s)^2}{(3)_q}\right),\\
\mathrm{S}(n) &= -\frac{1}{2h_{n-1}}\left((2)_q - \rho_{n-2}((2)_q+(-1)^n s)^2\right)
\end{aligned}$$

for $n \geq 3$. For $n = 3$ we have $S(2) = S(n-1) = 0$. These formulae show how the geometry of A_n q-deforms that of $\mathbb{N}$. As with Proposition 4.7, there is again a unique metric h^{flat} up to overall scale such that $S = 0$.

We conclude with a small example for $n = 3$, $s = 1$. Then $(2)_q = \sqrt{2}$, $(3)_q = 1$ and we obtain

$$S = \left(\frac{(3+2\sqrt{2})}{\sqrt{2}h_2} - \frac{1}{2h_1}\right)\{1, 0, -(3-2\sqrt{2})\}$$

at the three points. This vanishes at $h_2 = (4+3\sqrt{2})h_1$. Next, if we write $\mu_1 = h_1, \mu_3 = h_2$ then get for the Einstein-Hilbert action

$$S[\rho] := \sum_i \mu_i S(i) = \frac{(3+2\sqrt{2})}{\sqrt{2}\rho} + \frac{(3-2\sqrt{2})}{2}\rho - \frac{1}{\sqrt{2}} - \frac{1}{2}; \quad \rho = \frac{h_2}{h_1}$$

but note that we can get any coefficients for the two powers of ρ by scaling μ_i. Sticking with the obvious values, if we ignore the constant then

$$Z = \int \mathrm{d}h_1 \mathrm{d}h_2 e^{-\frac{1}{G}(\frac{c}{\rho}+\rho)} = \int_0^\infty h_1 \mathrm{d}h_1 \int_0^\infty \mathrm{d}\rho e^{-\frac{1}{G}(\frac{c}{\rho}+\rho)}; \quad c = 24 + 17\sqrt{2}$$

for a real positive coupling constant G. The first integral is an infinite volume which we ignore, while the ρ integrals converge for the calculation of expectation values,

$$\int_0^\infty \mathrm{d}\rho e^{-\frac{1}{G}(\frac{c}{\rho}+\rho)}\rho^m = 2c^{\frac{m+1}{2}} K_{m+1}\left(\frac{2\sqrt{c}}{G}\right)$$

as BesselK functions. These diverge as $G \to 0$ and as $G \to \infty$, but the expectation behave like

$$\langle \rho^m \rangle \to \begin{cases} c^{\frac{m}{2}} & G \to 0 \\ \infty & G \to \infty \end{cases},$$

while, for example, the relative uncertainty increases from 0 to a limit

$$\frac{\Delta\rho}{\langle\rho\rangle} := \frac{\sqrt{\langle\rho^2\rangle - \langle\rho\rangle^2}}{\langle\rho\rangle} \to 1; \quad \frac{\langle\rho^2\rangle}{\langle\rho\rangle^2} \to 2$$

as $G \to \infty$ (the 'strong gravity' limit). This looks quite reasonable for a theory of quantum gravity on 3 points in the sense that it follows the same pattern as other models[**11, 2, 31**].

APPENDIX A

Non $*$-preserving solutions for $\mathbb{Z}_n$

In this appendix, we list the solutions over $\mathbb{C}$ which do not obey the unitarity or 'reality' condition, hence are not included in chapter 2 Section 1.1. Because we use symbolic algebra these results are also valid at least for any field of characteristic zero. These could be of interest in different contexts over other fields, for example, to obtain 'digital' quantum geometries over $\mathbb{F}_2$ in the setting of [**56**] (in this case there could be other solutions also, as the field then has non-zero characteristic).

For $n \geq 3$ **odd**, there are two further independent solutions:

(i)
$$\sigma(e^+ \otimes e^+) = -\rho e^+ \otimes e^+, \quad \sigma(e^- \otimes e^+) = -e^+ \otimes e^- - 2e^- \otimes e^+,$$
$$\sigma(e^+ \otimes e^-) = e^- \otimes e^+, \quad \sigma(e^- \otimes e^-) = R_-^2(\rho^{-1}) e^- \otimes e^-,$$

giving the geometric structures

$$\begin{aligned}
\nabla e^+ &= (1+\rho)e^+ \otimes e^+, \quad \nabla e^- = (1 - R_-^2(\rho^{-1}))e^- \otimes e^- + 2(e^+ \otimes e^- + e^- \otimes e^+)\\
R_\nabla e^+ &= -\partial_-(\rho) e^+ \wedge e^- \otimes e^+,\\
R_\nabla e^- &= -\partial_-(R_-(\rho^{-1}))e^+ \wedge e^- \otimes e^- - 2(1 - R_-(\rho))e^+ \wedge e^- \otimes e^+,\\
\mathrm{Ricci} &= \frac{1}{2}\left(-\partial_-(R_-(\rho))e^- \otimes e^+ + 2(1-\rho)e^+ \otimes e^+ + \partial_-(\rho^{-1})e^+ \otimes e^-\right),\\
S &= \frac{1}{2}\left(\frac{\partial_-(\rho^{-1})}{a} - \frac{\partial_-(R_-(\rho))}{R_- a}\right),\\
\Delta f &= \frac{1}{a}(R_- f - R_+(f))(R_-(\rho) + 1).
\end{aligned}$$

For $n = 3$, we may freely add a map α given by $\alpha(e^-) = \lambda R_+(a) e^+ \otimes e^+$ to ∇e^- for a free parameter λ, and $\alpha(e^+) = 0$, so no change to ∇e^+. This agrees with the triangle analysis in [**14**, Ex. 8.19] aside from a different definition of ρ.

(ii)
$$\sigma(e^+ \otimes e^+) = \rho e^+ \otimes e^+, \quad \sigma(e^+ \otimes e^-) = -2e^+ \otimes e^- - e^- \otimes e^+,$$
$$\sigma(e^- \otimes e^+) = e^+ \otimes e^-, \quad \sigma(e^- \otimes e^-) = -R_-^2(\rho^{-1}) e^- \otimes e^-,$$

giving the geometric structures

$$
\begin{aligned}
\nabla e^+ &= (1-\rho)e^+\otimes e^+ + 2(e^+\otimes e^- + e^-\otimes e^+),\quad \nabla e^- = (1+R_-^2(\rho^-))e^-\otimes e^-,\\
R_\nabla e^+ &= -\partial_-\rho e^+\wedge e^-\otimes e^+ + 2(1-R_-(\rho^{-1}))e^+\wedge e^-\otimes e^-,\\
R_\nabla e^- &= -\partial_-(R_-(\rho^{-1}))e^+\wedge e^-\otimes e^-,\\
\mathrm{Ricci} &= \frac{1}{2}\left(-\partial_-(R_-(\rho))e^-\otimes e^+ + 2(1-R_-^2(\rho^{-1}))e^-\otimes e^- + \partial_-(\rho^{-1})e^+\otimes e^-\right),\\
S &= \frac{1}{2}\left(\frac{\partial_-(\rho^{-1})}{a} - \frac{\partial_-(R_-(\rho))}{R_-a}\right),\\
\Delta f &= \frac{1}{a}(R_+(f)-R_-(f))(R_-(\rho)+1).
\end{aligned}
$$

For $n=3$, we may freely add a map α given by $\alpha(e^+) = \lambda R_+(a)e^-\otimes e^-$ to ∇e^+ for a free parameter λ, and $\alpha(e^-)=0$, so no change to ∇e^-. This again agrees with the triangle analysis in [**14**] aside from a different definition of ρ.

For $n\geq 4$ **even**, there are two further independent solutions each with a free nonzero parameter q, from which we define a function

$$
Q = q^{(-1)^i} = \begin{pmatrix} q\\ q^{-1}\\ \vdots \end{pmatrix}.
$$

Then

(i)
$$
\sigma(e^+\otimes e^+) = \rho e^+\otimes e^+,\quad \sigma(e^+\otimes e^-) = (Q-1)e^+\otimes e^- + Qe^-\otimes e^+,
$$
$$
\sigma(e^-\otimes e^+) = e^+\otimes e^-,\quad \sigma(e^-\otimes e^-) = R_-^2(\rho^{-1})Qe^-\otimes e^-,
$$

giving the geometric structures

$$
\begin{aligned}
\nabla e^+ &= (1-\rho)e^+\otimes e^+ + (1-Q)(e^-\otimes e^+ + e^+\otimes e^-),\quad \nabla e^- = (1-R_-^2(\rho^{-1})Q)e^-\otimes e^-,\\
R_\nabla e^+ &= \partial_-(\rho R_+(Q))e^+\wedge e^-\otimes e^+ + (R_+(Q-1)R_-(\rho^{-1}) - (Q-1))e^+\wedge e^-\otimes e^-,\\
R_\nabla e^- &= \partial_-(R_-(\rho^{-1})R_+(Q))e^+\wedge e^-\otimes e^+,\\
\mathrm{Ricci} &= \frac{1}{2}\left(\partial_-(R_-(\rho)Q)e^-\otimes e^+ + \partial_+(R_+(Q)R_-(\rho^{-1}))e^+\otimes e^- + ((Q-1)R_-^2(\rho^{-1}) - R_-(Q-1))e^-\otimes e^-\right),\\
S &= \frac{1}{2a}\left(\partial_+(R_+(Q)R_-(\rho^{-1})) - R_-(\rho)\partial_-(R_-(\rho)Q)\right),\\
\Delta f &= -\left(\frac{1}{R_-(a)} + \frac{1}{a}\right)(\partial_- f + Q\partial_+ f).
\end{aligned}
$$

(ii)
$$
\sigma(e^+\otimes e^+) = \rho Qe^+\otimes e^+,\quad \sigma(e^-\otimes e^-) = R_-^2(\rho^{-1})e^-\otimes e^-,
$$
$$
\sigma(e^+\otimes e^-) = e^-\otimes e^+,\quad \sigma(e^-\otimes e^+) = Qe^+\otimes e^- + (Q-1)e^-\otimes e^+,
$$

giving the geometric structures

$$\nabla e^+ = (1-\rho Q)e^+\otimes e^+,\quad \nabla e^- = (1-R_-^2(\rho^{-1}))e^-\otimes e^- + (1-Q)(e^+\otimes e^- + e^-\otimes e^+),$$

$$R_\nabla e^+ = \partial_-(\rho Q)e^+\wedge e^-\otimes e^-,$$

$$R_\nabla e^- = (-R_+(Q-1)R_-(\rho)+Q-1)e^+\wedge e^-\otimes e^+ + \partial_-(QR_-(\rho^{-1}))e^+\wedge e^-\otimes e^-,$$

$$\mathrm{Ricci} = \frac{1}{2}\Big(\partial_-(R_-(\rho Q))e^-\otimes e^- - (\partial_-(R_+(Q)\rho^{-1})e^+\otimes e^- + (\rho(Q-1)-R_+(Q-1))e^+\otimes e^+\Big),$$

$$S = -\frac{1}{2a}\partial_-(R_+(Q)\rho^{-1}),$$

$$\Delta f = -\left(\frac{1}{R_-(a)}+\frac{1}{a}\right)(Q\partial_- f + \partial_+ f).$$

For $n=4$, we have a further more general form for the generalised braiding

$$\sigma(e^+\otimes e^+) = \sigma_0 e^+\otimes e^+ + \sigma_6 e^-\otimes e^-,\quad \sigma(e^+\otimes e^-) = \sigma_1 e^+\otimes e^- + \sigma_2 e^-\otimes e^+,$$
$$\sigma(e^-\otimes e^+) = \sigma_3 e^+\otimes e^- + \sigma_4 e^-\otimes e^+,\quad \sigma(e^-\otimes e^-) = \sigma_5 e^-\otimes e^- + \sigma_7 e^+\otimes e^+$$

for which the conditions for zero torsion are the same as before but metric compatibility now has a more complicated form due to the two extra parameters σ_6,σ_7. The QLCs turn out to fall into 10 families of which 3 are the ones with $\sigma_6=\sigma_7=0$ already covered above. *In addition* we have

(i) a 4-parameter solution with a free nonzero function $\gamma=(\gamma_0,\gamma_1,\gamma_2,\gamma_3)$ and

$$\sigma(e^+\otimes e^+) = \gamma e^-\otimes e^-,\quad \sigma(e^+\otimes e^-) = -e^+\otimes e^-,$$
$$\sigma(e^-\otimes e^+) = -e^-\otimes e^+,\quad \sigma(e^-\otimes e^-) = R_-(\gamma^{-1})R_+(\rho')e^+\otimes e^+,$$
$$\nabla e^+ = e^+\otimes e^+ + e^-\otimes e^+ + e^+\otimes e^- - \gamma e^-\otimes e^-,$$
$$\nabla e^- = e^-\otimes e^- + e^+\otimes e^- + e^-\otimes e^+ - R_-(\gamma^{-1})R_+(\rho')e^+\otimes e^+,$$

where

$$\rho' = \frac{1}{\rho R_+\rho}.$$

This is ∗-preserving if and only if γ has the 2-parameter form such that $R_+^2(\gamma)=\bar\gamma^{-1}$ as in the main text.

(ii) a 3-parameter solution with parameter β and functions

$$\gamma=(p,q,p,q),\quad \delta=\frac{pq-1}{R_+(\gamma)-1}=(pq-1)(\frac{1}{q-1},\frac{1}{p-1},\frac{1}{q-1},\frac{1}{p-1}),$$

$$\sigma(e^+\otimes e^+) = \rho(1-\delta)e^+\otimes e^+ + \beta(\gamma-1)\rho' e^-\otimes e^-,\quad \sigma(e^+\otimes e^-) = (\gamma-1)e^+\otimes e^- + \gamma e^-\otimes e^+,$$
$$\sigma(e^-\otimes e^+) = (1-\delta)e^+\otimes e^- - \delta e^-\otimes e^+,\quad \sigma(e^-\otimes e^-) = -\frac{\delta}{\beta R_+^2\rho'}e^+\otimes e^+ + \frac{\gamma}{R_+^2\rho}e^-\otimes e^-,$$

where

$$\rho' = (\frac{\rho_0}{\rho_2},\rho_0\rho_1,1,\rho_0\rho_3),$$

giving the QLC

$$\nabla e^+ = (1-\rho(1-\delta))e^+\otimes e^+ + (1-\gamma)(e^-\otimes e^+ + e^+\otimes e^-) + \beta\rho'(1-\gamma)e^-\otimes e^-,$$
$$\nabla e^- = (1-\frac{\gamma}{R_+^2\rho})e^-\otimes e^- + \delta(e^+\otimes e^- + e^-\otimes e^+) + \frac{\delta}{\beta R_+^2\rho'}e^+\otimes e^+.$$

(iii) a 3-parameter solution with parameters β and functions

$$\gamma = (p,0,q,0),\quad \delta = (1,\frac{q}{p},1,\frac{p}{q}),$$

,

$$\sigma(e^+\otimes e^+) = R_-\left(\frac{\gamma}{\gamma-1}\right)\rho e^+\otimes e^+ + \frac{\beta\delta\rho'}{1-R_-(\gamma)}e^-\otimes e^-,$$
$$\sigma(e^+\otimes e^-) = (\gamma-1)e^+\otimes e^- + \gamma e^-\otimes e^+,$$
$$\sigma(e^-\otimes e^+) = R_+\left(\frac{\gamma}{\gamma-1}\right)e^+\otimes e^- + \frac{1}{R_+(\gamma-1)}e^-\otimes e^+,$$
$$\sigma(e^-\otimes e^-) = \frac{R_-(\delta)}{\beta R_+^2(\rho')}(1-\gamma)e^+\otimes e^+ + R_+^2(\frac{\gamma}{\rho})e^-\otimes e^-,$$

where

$$\rho' = (\frac{\rho_0}{\rho_2},\rho_0\rho_1,1,\rho_0\rho_3),$$

giving the QLC

$$\nabla e^+ = (1+R_-(\frac{\gamma}{1-\gamma})\rho)e^+\otimes e^+ + (1-\gamma)(e^-\otimes e^+ + e^+\otimes e^-) - \frac{\beta\delta\rho'}{1-R_-(\gamma)}e^-\otimes e^-,$$
$$\nabla e^- = \left(1-R_+^2\left(\frac{\gamma}{\rho}\right)\right)e^-\otimes e^- + \frac{1}{1-R_+(\gamma)}(e^+\otimes e^- + e^-\otimes e^+) - \frac{R_-(\delta)}{\beta R_+^2\rho'}(1-\gamma)e^+\otimes e^+.$$

(iv) a 3-parameter solution with parameters β and the functions

$$\gamma = (0,p,0,q),\quad \delta = (\frac{p}{q},1,\frac{q}{p},1),$$

,

$$\sigma(e^+\otimes e^+) = \rho R_-(\frac{\gamma}{\gamma-1})e^+\otimes e^+ + \frac{\beta\delta\rho'}{1-R_-(\gamma)}e^-\otimes e^-,$$
$$\sigma(e^+\otimes e^-) = (\gamma-1)e^+\otimes e^- + \gamma e^-\otimes e^+,$$
$$\sigma(e^-\otimes e^+) = R_+(\frac{\gamma}{\gamma-1})e^+\otimes e^- + \frac{1}{R_+(\gamma-1)}e^-\otimes e^+,$$
$$\sigma(e^-\otimes e^-) = \frac{R_-(\delta)}{\beta R_+^2(\rho')}(1-\gamma)e^+\otimes e^+ + R_+^2(\frac{\gamma}{\rho})e^-\otimes e^-,$$

where

$$\rho' = (\frac{\rho_0}{\rho_2},\rho_0\rho_1,1,\rho_0\rho_3),$$

giving the QLC

$$\nabla e^+ = (1 + R_-(\frac{\gamma}{1-\gamma})\rho)e^+ \otimes e^+ + (1-\gamma)(e^- \otimes e^+ + e^+ \otimes e^-) - \frac{\beta\delta\rho'}{1 - R_-(\gamma)} e^- \otimes e^-,$$
$$\nabla e^- = (1 - R_+^2(\frac{\gamma}{\rho}))e^- \otimes e^- + \frac{1}{1 - R_+(\gamma)}(e^- \otimes e^+ + e^- \otimes e^+) - \frac{R_-(\delta)}{\beta R_+^2 \rho'}(1-\gamma)e^+ \otimes e^+.$$

(v) a 2-parameter solution with parameter β and $Q = (q, q^{-1}, q, q^{-1})$ as usual,

$$\sigma(e^+ \otimes e^+) = \rho e^+ \otimes e^+, \quad \sigma(e^+ \otimes e^-) = (Q-1)e^+ \otimes e^- + Qe^- \otimes e^+,$$
$$\sigma(e^- \otimes e^+) = e^+ \otimes e^-, \quad \sigma(e^- \otimes e^-) = \beta\rho' e^+ \otimes e^+ + R_+^2(\rho^{-1})Qe^- \otimes e^-,$$

where

$$\rho' = (1, -\frac{\rho_1\rho_2}{q}, \frac{\rho_2}{\rho_0}, -\frac{\rho_2\rho_3}{q}),$$

giving the QLC

$$\nabla e^+ = (1-\rho)e^+ \otimes e^+ + (1-Q)(e^+ \otimes e^- + e^- \otimes e^+),$$
$$\nabla e^- = (1 - R_+^2(\rho^{-1})Q)e^- \otimes e^- - \beta\rho' e^+ \otimes e^+.$$

(vi) a 2-parameter solution with parameter β and $Q = (q, q^{-1}, q, q^{-1})$ as usual,

$$\sigma(e^+ \otimes e^-) = e^- \otimes e^+, \quad \sigma(e^- \otimes e^+) = Qe^+ \otimes e^- + (Q-1)e^- \otimes e^+,$$
$$\sigma(e^+ \otimes e^+) = \rho Q e^+ \otimes e^+, \quad \sigma(e^- \otimes e^-) = \beta\rho' e^+ \otimes e^+ + R_+^2(\rho^{-1})e^- \otimes e^-,$$

where

$$\rho' = (1, -\frac{\rho_1\rho_2}{q}, \frac{\rho_2}{\rho_0}, -\frac{\rho_2\rho_3}{q}),$$

giving the QLC

$$\nabla e^+ = (1 - \rho Q)e^+ \otimes e^+,$$
$$\nabla e^- = (1 - R_+^2(\rho^{-1}))e^- \otimes e^- + (1-Q)(e^+ \otimes e^- + e^- \otimes e^+) - \beta\rho' e^+ \otimes e^+.$$

(vii) a 2-parameter solution with parameter β and $Q = (q, q^{-1}, q, q^{-1})$ as usual,

$$\sigma(e^+ \otimes e^+) = -\rho'\rho Q e^+ \otimes e^+ + \beta\rho'' e^- \otimes e^-, \quad \sigma(e^+ \otimes e^-) = e^- \otimes e^+,$$
$$\sigma(e^- \otimes e^+) = -\rho' Q e^+ \otimes e^- - (\rho' Q + 1)e^- \otimes e^+, \quad \sigma(e^- \otimes e^-) = R_+^2(\rho^{-1})e^- \otimes e^-,$$

where

$$\rho' = (\rho_1\rho_0, \rho_0^{-1}\rho_1^{-1}, \rho_1\rho_0, \rho_0^{-1}\rho_1^{-1}), \quad \rho'' = (\frac{\rho_0}{\rho_2}q, 1, q, \frac{\rho_3}{\rho_1}),$$

giving the QLC

$$\nabla e^+ = (1 + \rho'\rho Q)e^+ \otimes e^+ - \beta\rho'' e^- \otimes e^-,$$
$$\nabla e^- = (1 - R_+^2(\rho^{-1}))e^- \otimes e^- + (1 + \rho' Q)(e^+ \otimes e^- + e^- \otimes e^+).$$

Note that $\mathbb{Z}_4$ here is a different group from $\mathbb{Z}_2 \times \mathbb{Z}_2$ treated in [**11**][**14**, Ex. 8.20], even though in both cases the graph is a square. This means that, although Ω^1 and the metric can be made to match up and hence the metric compatibility part of the QLC condition is the same, Ω^2 and hence the condition for torsion freeness are different. The $\mathbb{Z}_2$ case is also treated in [**11**].

APPENDIX B

Generalities of Quantum field theory and Quantum Gravity

This appendix has the purpose of presenting some generalities that help to read some parts of the thesis. In the first section, we cover the basic structures of the path integral approach, which can be consulted in a more general way in any QFT reference. We recommend the next one [**59**]. The second section gives a sort of algorithm for constructing a quantum gravity model in our approach. It clarifies the process that we followed in the models. The material of the second section was reported in [**9**] .

1. Quantum Field theory

In general, the starting point for QFT is to choose a time-space setting with a pseudo-Riemannian manifold $(\mathcal{M}, g)$ with dimension d. Over this manifold, we define the 'fields' $\phi : \mathcal{M} \to \Bbbk$, where $\Bbbk$ is some field like $\mathbb{R}, \mathbb{C}$, which is called target space. The next structure is the configuration space C which is defined over $\mathcal{M}$ as all possible 'states' of a field ϕ, i.e. each point in C represents a value of the target space assigned to each component of ϕ on $\mathcal{M}$. Usually this is an infinite dimensional space.

The next ingredient is a functional $S : C \to \mathbb{R}$ called action. Then for each configuration of ϕ, a real number is assigned. Usually, the action of a field ϕ is denoted as $S[\phi]$. We are interested in the critical set Crit(S) $= \{\phi_0 \in C \mid \delta S[\phi_0] = 0\}$, where δ is the exterior derivative that obeys $\delta^2 = 0$. Thus

$$\delta = \int_{\mathcal{M}} \delta\phi(x) \frac{\delta}{\delta\phi(x)}$$

where $\delta\phi(x)$ is a 1-form in C and δ acts as

$$\frac{\delta}{\delta\phi(x)} \int_{\mathcal{M}} \mathrm{d}^d y\ \phi(y)^2 = 2\phi(x), \qquad \frac{\delta}{\delta\phi(x)}\phi(y) = \delta^d(x-y),$$

where the last term is a Dirac-delta in d dimensions.

For a scalar field, the set Crit(S) coincides with the space of solution of the Klein-Gordon equation ($-\Box\phi + m^2\phi = 0$) defined in the same time-space setting as the QFT.

When setting up our QFT, we often assume that the action is local, meaning that it can be written as

$$S\phi = \int_{\mathcal{M}} \mathrm{d}^d x\ \sqrt{|g|}\ \mathcal{L}(\phi(x), \partial\phi(x), \dots),$$

where the Lagrangian density $\mathcal{L}$ depends on the value of ϕ and finitely many derivatives at just a single point in $\mathcal{M}$. Also, in order to have a local action, each term of the Lagrangian density must have the form

$$\int_{\mathcal{M}} \mathrm{d}^d x\, \lambda(x)\, \partial^{q_1}\phi(x), \partial^{q_2}\phi(x), \ldots, \partial^{q_n}\phi(x),$$

of a monomial of degree n in the fields with all fields and derivatives evaluated at the same $x \in \mathcal{M}$. We usually restrict the function $\lambda : x \to \mathbb{R}$ to be constant, known as coupling constant. Physically, each term is interpreted as an interaction, either between several different fields or between a field and itself.

The path integral is defined as

$$\int_{C} [\mathcal{D}\phi] \exp\left\{-\frac{1}{G} S\,[\phi]\right\},$$

where G is a real constant with dimensions as action related to the quantization. In the usual path integral formulation $G = \hbar$. This integral is defined over the infinite space C with a measure $[\mathcal{D}\phi]e^{-\frac{1}{G}S}$, that weights the contribution of each field configuration $\phi \in C$ by $e^{-S/G}$. Much of the work using the path integral is how to get finite quantities, even when the integral is defined over infinite space.

The partition function is one of the most important object to compute, it has the form

$$\mathcal{Z}_{(\mathcal{M},g)}(\lambda, \ldots) = \int_{C} [\mathcal{D}\phi] \exp\left\{-\frac{S\,[\phi]}{G}\right\}.$$

Note that $\mathcal{Z}$ does not depend on the fields itself. These are just dummy variables that we have integrated out in computing the partition function. However, depends in all the settings that we made before as the time-space $(\mathcal{M}, g)$ and the coupling constants.

After the partition function, the most important objects are the correlation functions. These are path integrals with further insertions, of the general form

$$\int_{C} [\mathcal{D}\phi] \exp\left\{-\frac{S\,[\phi]}{G}\right\} \prod_{i=1}^{n} O_i[\phi]$$

where the insertions O_i are functions on C. We usually normalise the correlation functions by the partition function as follows

$$\left\langle \prod_{i=1}^{n} O_i[\phi] \right\rangle = \frac{1}{\mathcal{Z}} \int_{C} [\mathcal{D}\phi] \exp\left\{-\frac{S\,[\phi]}{G}\right\} \prod_{i=1}^{n} O_i[\phi]$$

The idea of this normalisation is both to ensure that $\langle 1 \rangle = 1$ and to separate out the effect of inserting the operator O_i into the path integral from effects that are there in the basic partition function already. Mathematically, normalised correlation functions compute various moments of the probability distribution. These correlation functions correspond to dynamical processes in the target space.

Note that although the operator insertions depend on the values of the fields, the correlation functions themselves do not. Rather, our correlators are functions

$$F_{(\mathcal{M},g)}(x_1,\dots,x_n;\lambda,\dots) = \Big\langle \prod_{i=1}^{n} O_i[x_i] \Big\rangle$$

that depend on all the same data as the partition function $\mathcal{Z}$ together with some restrictions we want to impose on the correlation.

Correlation functions and the partition function are very close because the operators that can appear in the action. For example, assuming that the term λO is in the action, we have that

$$-\frac{G}{\mathcal{Z}}\frac{\partial}{\partial\lambda}\mathcal{Z} = \langle O\rangle.$$

Then, knowing all the correlation functions of the operators in the action is equivalent to knowing $\mathcal{Z}$ as a function of the coupling constants in the action.

The operators that appear in the action are integrated over all of $\mathcal{M}$. It is convenient to extend the idea above so as to obtain correlators of local operators $O_i(x)$ that depend on the value of (and perhaps finitely many derivatives) just at one point $x \in \mathcal{M}$. To do this, we include source terms such as

$$\mathcal{S}[\phi] \to \mathcal{S}[\phi] + \int_{\mathcal{M}} \mathrm{d}^d x\; J_i(x)\, O_i(x)$$

in the action. The source $J_i(x)$ is, like the field ,a function on $\mathcal{M}$.

Really, this is just another case of the choices we made in picking our action, allowing the coupling 'constant' $\lambda \to \lambda(x)$ to still vary over $\mathcal{M}$, but the name 'source' and use of the letter J(x) is conventional. We do not integrate over J in performing the path integral, so the partition function itself becomes a functional $\mathcal{Z} \to \mathcal{Z}_{(\mathcal{M},g)}[J_i]$ depending on the choice of functions J_i in addition to the other data. Varying this partition function w.r.t. the value of the source at some point $x \in \mathcal{M}$,we obtain formally

$$-\frac{\delta}{\delta\, J_i(x)}\mathcal{Z}[J_i] = \int \mathcal{D}\phi\, e^{\frac{1}{\beta}S[\phi]+\frac{1}{\beta}\int\sum_{i=1}^{n} J_i(t)\phi_i(t)}$$

and thus

$$\langle O_1(x_1)O_2(x_2)\dots O_n(x_n)\rangle = \frac{(-G)^n}{\mathcal{Z}}\frac{\delta^n\mathcal{Z}[J]}{\delta J(x_1))\delta J(x_))\dots J(x_n)}\Big|_{J=0}$$

Relations such as these show the close connection between correlation functions and the partition function. We see that correlators probe the response of the partition function to a change in the background structures we chose in setting up the theory.

2. Quantum Gravity with Path Integral

The general idea is to quantize the gravitational field over a time-space $(\mathcal{M}, g)$. There is no general theory or background to cover quantum gravity, each approach has different philosophies and approaches. However, historically the action used is the Hilbert-Einstein action

$$\int_{\mathcal{M}} \mathrm{d}x \sqrt{|g|}\, S,$$

where S is the Ricci scalar of the geometry of the time-space and the rest is a covariant measure, with $|g|$ as the absolute value of the determinant of metric. Next, It is used the Hilbert-Einstein action on the path integral for constructing a partition function and calculating the correlation functions. We adapt this idea to our formalism, finding some difficulties as choose a measure for the action.

After the development of the models in this thesis, we identify some steps that can be followed to construct quantum gravity on a quantum spacetime as follows.

(1) Choose a unital $*$-algebra A.

(2) Make A into a $*$-differential algebra at least to order Ω^2 (we can do without 3-forms or higher for the pure gravity sector.)

(3) Choose a class of quantum metrics $g \in \Omega^1 \otimes_A \Omega^1$ to further quantise, nondegenerate in the sense of having an inverse $(\ ,\ )$ and preferably quantum symmetric or subject to some other similar condition (such as edge-symmetric in the graph case). Describe this moduli explicitly.

(4) Solve for the moduli of QLC's $\nabla : \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ with associated 'generalised braiding' $\sigma : \Omega^1 \otimes_A \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ for each quantum metric in the class is step 3. Among your solutions, try to identify a canonical choice that works across the whole moduli of metrics.

   – It may be that there is more than one but one is natural (e.g. in having a classical limit).

   – It may be that there is a moduli of QLCs but no preferred one. In that case the quantum gravity theory has to be a functional integral over the joint moduli of metric-QLC pairs, not just over metrics.

   – Or it there may be that a QLC does not exist for the metrics in your class. In that case go back to step 3.

(5) Compute the Riemann curvature for the moduli of QLCs in step 4.

(6) Choose a lifting map $i : \Omega^2 \to \Omega^1 \otimes_A \Omega^1$ compatible with Ω^2 and compute Ricci with respect to it using the curvatures from step 5. Usually, there will be an obvious choice of i.

   – If not, apply some criterion such as that you want Ricci to have the same quantum symmetry and $*$-properties as the metric.

   – Or parameterise the possible i as a parameter to your quantum gravity theory.

(7) Compute the Ricci scalar $R = (\ ,\ )\mathrm{Ricci}$ from Ricci in step 6.

(8) Choose an integration map $\int : A \to \mathbb{C}$ preferably obeying at least the positivity of

$$\overline{\int a} = \int a^*, \quad \int a^* a \geq 0. \tag{B.1}$$

Similarly to i, there will often be an obvious choice or obvious ansatz which you can search among according to what works well.

(9) Choose your measure of functional integration on the moduli of quantum metrics as a classical manifold. Again, there will usually be an obvious choice or an obvious ansatz suggested by the classical geometry of the moduli space.

(10) At this point, a candidate for quantum gravity in a functional integral formulation has been constructed. Explore a bit to see if it looks sensible:

– Compute some expectation values, cutting off any UV or IR divergences in the metric field strengths with parameters but remembering that only the ratio of integrals enter into the expectation values.

– If these expectation values still diverge then look at the relative theory of expectation values relative to field expectation values.

– If the theory does not look very physical then go back and revisit your choices in reverse order (particularly your choice of $\int$ and your choice of i).

– Also look at the relative theory where only fluctuations relative to a mean or background metric are quantised (this tends to have more structure than the fully integrated theory).

Bibliography

[1] E. Alesci, S. Bahrami, D. Pranzetti, Quantum gravity predictions for black hole interior geometry, Phys. Lett. B 797 (2019) 134908 (7pp)

[2] J. N. Argota-Quiroz and S. Majid, *Quantum gravity on polygons and* $\mathbb{R}\times\mathbb{Z}_n$ *FLRW model*, Class. Quantum Grav. (2020) 245001 (43pp)

[3] J. N. Argota-Quiroz and S. Majid, *Fuzzy and discrete black hole models*, Class. Quantum Grav. 145020, (2021) 38pp.

[4] A. Ashtekar and R.O. Hansen, A unified treatment of null and spatial infinity in general relativity. I. Universal structure, asymptotic symmetries, and conserved quantities at spatial infinity, J. Math. Phys. 19 (1978) 1542–1566

[5] A. Bassi, A. Grossardt and H. Ulbricht, *Gravitational Decoherence*, Class. Quantum Grav. 34 (2017) 193002

[6] E.J. Beggs, Noncommutative geodesics and the KSGNS construction, J. Geom. Phys. 158 (2020) 103851

[7] S. Majid, Reconstruction and quantization of Riemannian structures, J. Math. Phys. 61 (2020) 022501 (32pp)

[8] E.J. Beggs and S. Majid, *Gravity induced by quantum spacetime*, Class. Quantum Grav. 31 (2014) 035020 (39pp)

[9] J. N. Argota-Quiroz and S. Majid , *Quantum gravity on finite spacetimes and dynamical mass*, PoS **CORFU2021** (2022), 210 doi:10.22323/1.406.0210

[10] J. N. Argota-Quiroz and S. Majid, *Quantum Riemannian geometry of the discrete interval and q-deformation*, in press J. Math. Phys. (Editor's Choice award).

[11] S. Majid *Quantum gravity on a square graph* Class. Quantum Grav. 245009 (2019) 36pp

[12] S. Majid *Quantum Riemannian geometry and particle creation on the integer line* Class. Quantum Grav. 135011 (2019) 36pp

[13] S. Majid, Noncommutative Riemannian geometry of graphs, J. Geom. Phys. 69 (2013) 74–93

[14] E.J. Beggs and S. Majid, *Quantum Riemannian Geometry*, Grundlehren der mathematischen Wissenschaften, Vol. 355, Springer (2020) 809pp.

[15] S. Majid, Noncommutative Riemannian geometry of graphs, J. Geom. Phys. 69 (2013) 74–93

[16] E.J. Beggs and S. Majid, Quantization by cochain twists and nonassociative differentials, J. Math. Phys., 51 (2010) 053522, 32pp

[17] E.J. Beggs and S. Majid, Quantum geodesics in quantum mechanics, arXiv:1912.13376 (math-ph)

[18] E. Beggs and S. Majid, Quantum geodesics and curvature, arXiv: 2201.08244 (math.QA)

[19] A. Ghobadi, Isotopy quotients of Hopf algebroids and the fundamental groupoid of digraphs, arXiv:2108.07525

[20] U. Carow-Watamura and S. Watamura *Noncommutative geometry and gauge theory on fuzzy sphere* Commun. Math. Phys. 212, (2000) 395–413

[21] N.D. Birrell and P.C.W. Davies, *Quantum Fields in Curved Space*, Cambridge University Press (1984)

[22] J. Mourad, *Linear connections in noncommutative geometry*, Class. Quant. Grav. 12 (1995) 965-974

[23] S.M. Carroll, *Spacetime and Geometry*, Cambridge University Press, 2019.

[24] P. Chruściel, A remark on the positive-energy theorem, Class. Quantum Grav. 3 (1986), 115–121

[25] A. Connes, *Noncommutative Geometry*, Academic Press (1994)

[26] S. Doplicher, K. Fredenhagen and J. E. Roberts, The quantum structure of spacetime at the Planck scale and quantum fields, Commun. Math. Phys. 172 (1995) 187–220

[27] M. Dubois-Violette and P.W. Michor, Connections on central bimodules in noncommutative differential geometry, J. Geom. Phys. 20 (1996) 218–232

[28] L. Freidel and S. Majid, Noncommutative harmonic analysis, sampling theory and the Duflo map in 2+1 quantum gravity, Class. Quantum Grav. 25 (2008) 045006 (37pp)

[29] C. Fritz and S. Majid, Noncommutative spherically symmetric spacetimes at semiclassical order, Class. Quantum Grav, 34 (2017) 135013 (50pp)

[30] G. 't Hooft, Quantization of point particles in 2+1 dimensional gravity and space- time discreteness, Class. Quantum Grav. 13 (1996) 1023

[31] E. Lira-Torres and S. Majid, Quantum gravity and Riemannian geometry on the fuzzy sphere, arXiv:2004.14363 [math.QA]

[32] J. Lukierski, H. Ruegg, A. Nowicki and V.N. Tolstoi, q-Deformation of Poincare algebra, Phys. Lett. B 264 (1991) 331

[33] S. Majid, Hopf algebras for physics at the Planck scale, Class. Quantum Grav. 5 (1988) 1587–1607

[34] S. Majid, Noncommutative model with spontaneous time generation and Planckian bound, J. Math. Phys. 46 (2005) 103520, 18pp

[35] S. Majid, Newtonian gravity on quantum spacetime, Euro Phys. J. Web of Conferences, 70 (2014) 00082 (10pp)

[36] S. Majid, Almost commutative Riemannian geometry: wave operators, Commun. Math. Phys. 310 (2012) 569–609

[37] S. Majid, q-Fuzzy spheres and quantum differentials on $B_q[SU_2]$ and $U_q(su_2)$, Lett. Math. Phys. 98 (2011) 167–191

[38] S. Majid, Quantum Riemannian geometry and particle creation on the integer line, Class. Quantum Grav. 36 (2019) 135011 (22pp)

[39] S. Majid and H. Ruegg, Bicrossproduct structure of the κ-Poincaré group and non-commutative geometry, Phys. Lett. B. 334 (1994) 348–354

[40] S. Majid and W.-Q. Tao, Cosmological constant from quantum spacetime, Phys. Rev. D 91 (2015) 124028 (12pp)

[41] P. Miao and L-F. Tam, Evaluation of the ADM mass and center of mass via the Ricci tensor, Proc. AMS 144 (2016) 753–761

[42] M. Milgrom, A modification of the Newtonian dynamics as a possible alternative to the hidden mass hypothesis, Astrophysical J. 270 (1983) 365–370

[43] V. Mukhanov and S. Winitzki, *Introduction to Quantum Effects in Gravity*, Cambridge University Press (2007)

[44] L. Parker and D. Toms, *Quantum Field Theory in Curved Spacetime: Quantized Fields and Gravity*, Cambridge University Press (2009)

[45] H.S. Snyder, Quantized space-time, Phys. Rev. D 67 (1947) 38–41

[46] F. R. Tangherlini, Schwarzschild field in n dimensions and the dimensionality of space problem. Nuovo Cimento 27 (1963) 636–651

[47] S. Majid, Quantum geometry of Boolean algebras and de Morgan duality, in press J. Noncomm. Geom. (2021) 37pp

[48] L. Parker, On the magnetic moment of a charged particle in a changing magnetic field, Nuovo Cimento 40B (1965) 99

[49] L. Parker, Quantized fields and particle creation in expanding universes. 1, Phys. Rev. **183** (1969) 1057

[50] L. Parker, Particle creation and particle number in an expanding universe, J. Phys. A **45** (2012) 374023

[51] L. Parker and J. Navarro-Salas, Fifty years of cosmological particle creation, arXiv:1702.07132 (physics.hist-ph)

[52] L. Parker and D. Toms, *Quantum Field Theory in Curved Spacetime: Quantized Fields and Gravity*, Cambridge University Press (2009)

[53] V. Mukhanov and S. Winitzki, *Introduction to Quantum Effects in Gravity*, Cambridge University Press (2007)

[54] N.D. Birrell and P.C.W. Davies, *Quantum Fields in Curved Space*, Cambridge University Press (1984)

[55] Sage Developers, *The Sage Mathematics Software System (Version 8.4)*, 2020, https://www.sagemath.org.

[56] S. Majid and A. Pachol, Digital finite quantum Riemannian geometries, J. Phys. A 53 (2020) 115202 (40pp)

[57] M. Bronstein, *Republication of: Quantum theory of weak gravitational fields*, General Relativity and Gravitation, vol. 44, no. 1, pp. 267–283, 2012.

[58] N. Bohr and L. Rosenfeld, *Zur Frage der Messbarkeit der elektromagnetischen Feldgrössern*, Det Kongelige Danske Videnskabernes Selskabs, Mathematiks-fysike Meddeleser., vol. 12, no. 65, 1933.

[59] Sanjeev S. Seahra, *Path Integrals in Quantum Field Theory*, `https://sanjeev.seahra.ca/wp-content/uploads/2022/01/path_integrals.pdf`, 2000.

[60] J. Ambjorn, J. Jurkiewicz and R. Loll, Dynamically triangulating Lorentzian quantum gravity, Nucl. Phys. B610 (2001) 347–382

[61] F. Dowker, Introduction to causal sets and their phenomenology, Gen. Rel. Grav, 45 (2013)1651–1667

[62] M. Hale, Path integral quantisation of finite noncommutative geometries, J. Geom. Phys. 44 (2002) 115–128

[63] L. Parker, Quantized fields and particle creation in expanding universes. 1, Phys. Rev. **183** (1969) 1057

[64] L. Parker, Particle creation and particle number in an expanding universe, J. Phys. A **45** (2012) 374023

[65] L. Parker and J. Navarro-Salas, Fifty years of cosmological particle creation, arXiv:1702.07132 (physics.hist-ph)

[66] L. Parker and D. Toms, *Quantum Field Theory in Curved Spacetime: Quantized Fields and Gravity*, Cambridge University Press (2009)

[67] V. Mukhanov and S. Winitzki, *Introduction to Quantum Effects in Gravity*, Cambridge University Press (2007)

[68] J. Matyjasek and P. Sadurski *Inside the Schwarzschild-Tangherlini black holes*, Phys. Rev. D 92, 044023 – Published 14 August 2015

[69] S. Tomizawa and H. Ishihara *Exact Solutions of Higher Dimensional Black Holes*, Progress of Theoretical Physics Supplement No.189,2011