Supplemental Material I: Macdonald index, arc space and the 4D theory

We begin by briefly reviewing the Macdonald index. This observable counts operators that obey the relations in (11) and (12) of the main text. These operators sit in $\hat{\mathcal{B}}_R$, $\mathcal{D}_{R,(0,\bar{j})}$, $\bar{\mathcal{D}}_{R,(j,0)}$, and $\hat{\mathcal{C}}_{R(j,\bar{j})}$ multiplets in the nomenclature of [1]. One can often find a closed-form expression for the Macdonald index. For example, in the $\mathcal{N}=2$ free massless hypermultiplet, one finds the following freely generated answer

$$\mathcal{I}_M^{\text{Free hyper}} = \frac{1}{(z\sqrt{t},q)_\infty(\frac{\sqrt{t}}{z},q)_\infty} \ , \tag{A.1}$$

where $(a,q)_n := \prod_{k=0}^{n-1}(1-aq^k)$, and z is a flavour fugacity.

Next, our goal is to explain the equality in (21) of the main text. For ease of reference, we reproduce it here

$$\text{HS}_{R_\infty^{\text{MAD}}}(q,T) = \mathcal{I}_M^{\text{MAD}}(q,T) \ . \tag{A.2}$$

Recall from the discussion below (14) of the main text that ∂_+ contributes q, and J contributes q^2T to the righthand side of this equation. The lefthand side is the Hilbert series that counts operators in R_∞^{MAD}. We will give a precise definition of this quantity below.

If (A.2) holds, then $\mathcal{I}_M^{\text{MAD}}$ can be obtained from words built out of J and the derivative ∂_+, subject to the condition $J^2=0$. This is because R_∞^{MAD} defined in (20) of the main text contains all such operators. Therefore, we would like to check whether

$$\begin{aligned}\mathcal{I}_M^{\text{MAD}}(q,T) &= \sum_{k=0}^{\infty}\frac{q^{k^2+k}}{(q)_k}T^k \\ &\stackrel{?}{=} \sum_{n,k=0}^{\infty}\dim V_{n,k}\ (q^2T)^k q^n \\ &= \sum_{n,k=0}^{\infty}\dim V_{n,k}\ q^{2k+n}T^k \ .\end{aligned} \tag{A.3}$$

Here $V_{n,k}$ is the set of operators built from k J's and n derivatives ∂_+ (subject to $J^2=0$), which, in general, take the form (we have suppressed complex coefficients in front of each term in the sum for simplicity)

$$\sum_{\substack{n_1,\cdots,n_k=0\\ \sum n_i=n}}^{\infty}\partial^{n_1}J\,\partial^{n_2}J\,\cdots\partial^{n_k}J \ , \tag{A.4}$$

and $\dim V_{n,k}$ is the dimension of each such linearly independent subspace.

As a result, to prove (A.2), we need to show that

$$\sum_{n,k=0}^{\infty}\dim V_{n,k}\ q^n p^k \stackrel{?}{=} \sum_{k=0}^{\infty}\frac{q^{k^2-k}}{(q)_k}p^k \ , \quad (p=q^2T) \ . \tag{A.5}$$

Interestingly, the RHS can be identified with a q-hypergeometric series [?]

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{q^{k^2-k}}{(q)_k}p^k &= {}_r\phi_{r+1}\left[\begin{matrix}a_1,\cdots,a_r\\ a_1,\cdots,a_r,0\end{matrix};q,p\right] \\ &= 1+p+pq+\left(p+p^2\right)q^2 \\ &\quad+\left(p+p^2\right)q^3+\left(p+2p^2\right)q^4+\cdots \ .\end{aligned} \tag{A.6}$$

for arbitrary r and a_i, where we have also explicitly written the perturbative expansion for the first few orders.

It is easy to compute $\dim V_{n,k}$ numerically to high order and verify this statement. Below, we will also prove it analytically.

To do so, it is useful to introduce the concept of an arc space. An arc space is a special kind of topological space that is intimately connected with the singularities of algebraic varieties. In the context of QFT, such spaces have appeared in various places (e.g., see [17, 19, 26]). In our case, the arc space encodes the operators in the Schur sector, and one can characterize the Schur spectrum from the associated arc space Hilbert series.

Here we follow [19] and start with the affine scheme

$$X = \text{Spec } R \ , \quad R = \mathbb{C}[x_1,\cdots,x_N]/\langle f_1,\cdots f_l\rangle \ , \tag{A.7}$$

where $f_i \in \mathbb{C}[x_1,\cdots,x_N]$ are polynomial relations.

From this structure, we have the jet scheme, X_m, which can be thought of as a generalization of the notion of a tangent space. It is given by

$$\begin{aligned}X_m &= \text{Spec } R_m \ , \\ R_m &= \mathbb{C}[x_1^{(i)},\cdots,x_N^{(i)}]/\langle f_1^{(i)},\cdots f_l^{(i)}\rangle \ , 0\le i\le m \ .\end{aligned} \tag{A.8}$$

In writing the above ideals, we introduced a derivation, D, such that $D(x_j^{(i)})=x_j^{(i+1)}$ if $0\le i<m$ and $D(x_j^{(i)})=0$ if $i=m$. This definition then specifies the action of D on all $\mathbb{C}[x_1^{(i)},\cdots,x_N^{(i)}]$. In particular, $f_j^{(i)} := D^i(f_j)$ is also a polynomial.

Given this discussion, we can consider the inverse limit and obtain the arc space

$$\begin{aligned}X_\infty &= \lim_{\leftarrow} X_m \simeq \text{Spec } R_\infty \ , \\ R_\infty &= \mathbb{C}[x_1^{(i)},\cdots,x_N^{(i)}]/\langle f_1^{(i)},\cdots f_l^{(i)}\rangle \ , \quad i\ge 0 \ .\end{aligned} \tag{A.9}$$

In [17, 19, 26], the above construction arises in the context of 2d VOAs, and $R=\mathcal{R}_{\mathcal{V}}$ is the associated Zhu's C_2 algebra. Roughly speaking, this is a commutative 2d algebra obtained by getting rid of all operators containing derivatives in the VOA $\mathcal{V}$

$$\mathcal{R}_{\mathcal{V}} = \mathcal{V}/\mathcal{C}_2(\mathcal{V}) \ , \quad \mathcal{C}_2 := \text{Span}\left\{a_{-h_a-1}b | a,b\in\mathcal{V}\right\} \ . \tag{A.10}$$

When the 4d SCFT has a Higgs branch, Zhu's C_2 algebra enables one to reconstruct this moduli space [26]. In the case of the MAD theory, there is no Higgs branch. However, Zhu's C_2 algebra still contains important information about this theory. Indeed, from (18) of the main text, it is easy to see that

$$\mathcal{R}_{\mathrm{Vir}_{c=-22/5}} = \mathbb{C}[x]/\langle x^2\rangle \ . \tag{A.11}$$

Constructing the arc space associated with (A.11) and showing that its operators are counted as in (A.3) is strong evidence for the fact that the arc space undoes the twisting of the MAD theory that led to the Lee-Yang VOA. At a physical level, the arc space therefore provides an inverse map from 2d to 4d for the case at hand (and also for the generalizations we discuss in section III of the Supplemental Material).

To prove (A.3), we consider $N=1$ and $x_1 = J$ in (A.9). The derivation, D, can be regarded as the derivative acting on local operators. Then we can identify $x_1^{(i)} = \partial_+^i J$. The ideal is generated by $f_1^{(0)} = J^2$ and, more generally, all $f_1^{(i)} = \partial_+^i(J^2)$. As a result, the above arc space is exactly the space describing operators made out of J and derivatives in (A.4) subject to the constraint $J^2=0$. In other words, $R_\infty \to R_\infty^{\mathrm{MAD}}$, and we need to consider

$$X_\infty^{\mathrm{MAD}} = \mathrm{Spec}\ R_\infty^{\mathrm{MAD}}, \quad R_\infty^{\mathrm{MAD}} = \mathbb{C}[x^{(i)}]/\langle (x^2)^{(i)}\rangle, \quad i \geq 0\ , \tag{A.12}$$

or, as described around (20) of the main text using physical operators

$$R_\infty^{\mathrm{MAD}} = \mathbb{C}[J, \partial_+ J, \partial_+^2 J, \cdots]/\langle J^2, \partial_+(J^2), \cdots\rangle\ . \tag{A.13}$$

This ring is bi-graded, and we can assign weights $(i,1)$ to $x^{(i)} = \partial_+^i J$. These weights correspond to the $(E-3R, R)$ quantum numbers of operators in 4d.

To complete the proof, we need to first define the associated Hilbert series

$$\mathrm{HS}_{R_\infty^{\mathrm{MAD}}}(q,p) := \sum_{n,k=0}^{\infty} \dim(R_\infty^{\mathrm{MAD}})_{n,k}\ q^n p^k\ , \tag{A.14}$$

where $\dim(R_\infty^{\mathrm{MAD}})_{n,k}$ is the dimension of the subring with weight (n,k). As a result, we have $\dim(R_\infty^{\mathrm{MAD}})_{n,k} = \dim V_{n,k}$ for the number of such linearly independent operators.

The remaining goal is to compute the Hilbert series in (A.14). Fortunately, this has been done in [20]. Indeed, from (7.1) in that reference, we learn that

$$\begin{aligned}\mathrm{HS}_{R_\infty^{\mathrm{MAD}}}(q,p) &= \sum_{n,k=0}^{\infty} \dim(R_\infty)_{n,k}\ q^n p^k \\ &= \sum_{n,k=0}^{\infty} \dim V_{n,k}\ q^n p^k \\ &= \sum_{k=0}^{\infty} \frac{q^{k^2-k}}{(q)_k} p^k\ .\end{aligned} \tag{A.15}$$

This discussion thus proves the identity (A.5). Therefore, operators made out of J and derivatives in (A.4) subject to the constraint $J^2=0$ reproduce all the Schur operators and (upon the substitution $p \to q^2 T$) the associated Macdonald index as in (A.2).

Supplemental Material II: Leading ideals and a basis for the arc space

In this section, our goal is to show that the states in (22) of the main text form a basis for R_∞^{MAD}. For ease of reference, we reproduce these states here

$$\begin{aligned}&\partial_+^{n_1} J\, \partial_+^{n_2} J \cdots \partial_+^{n_k} J\ , \quad 0 \leq n_1 < n_2 < \cdots < n_k\ , \\ &n_{i+1} - n_i \geq 2\ , \ \sum_{i=1}^{k} n_i = n\ , \quad n \in \mathbb{Z}_{\geq 0}\ .\end{aligned} \tag{A.16}$$

For a given k and n, we wish to show that the above states form a basis for the space $V_{n,k}$.

To prove this statement, we first note that, following theorem 6.3 of [25] (see also proposition (5.2) of that reference), we have

$$\mathrm{HS}_{\mathbb{C}[x,\cdots]/I} = \mathrm{HS}_{\mathbb{C}[x,\cdots]/\mathrm{LT}(I)}\ , \tag{A.17}$$

where $\mathrm{LT}(I)$ is the so-called "leading ideal" of I. In the case of the MAD Schur ring, R_∞^{MAD}, I is given in (A.12). The corresponding leading ideal is then

$$\mathrm{LT}(\langle (x^2)^{(i)}\rangle) := \langle \left(x^{(i)}\right)^2\ ,\ x^{(i)}x^{(i+1)}\rangle\ . \tag{A.18}$$

In terms of the $\partial_+^i J$ operators, this statement implies that (A.16) forms a basis for the space $V_{n,k}$.

We can check the consistency of this discussion with (A.15) as follows. Define $B(n)$ to be the partition of n into arbitrary parts differing by at least two. Then we have

$$B(n) = \sum_{k=0}^{\infty} \dim V_{n-k,k}\ , \tag{A.19}$$

where the shift in n arises from the fact that we can have $n_1 = 0$ in (A.16) while $B(n)$ counts partitions with $n_1 > 0$. Finally, we can consider the corresponding partition function (e.g., see (5) of [27])

$$\sum_{n=0}^{\infty} B(n) q^n = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q)_k}\ , \tag{A.20}$$

which is consistent with (A.15) after setting $p \to q$ [?].

Supplemental Material III: Higher-rank theories

In this section, we consider the (A_1, A_{2r}) SCFTs for general $r \geq 1$ [28]. When $r = 1$, we are back to the case of the MAD theory (i.e., MAD $\cong (A_1, A_2)$).

These SCFTs are all quite similar to the (A_1, A_2) theory. Their Macdonald index is [12]

$$\mathcal{I}_M^{(A_1,A_{2r})}(q,T) = \sum_{n,k} \dim V_{n,k} q^{2k+n} T^k$$
$$= \sum_{N_1 \geq \cdots \geq N_r \geq 0}^{\infty} \frac{q^{N_1^2+\cdots N_r^2+N_1+\cdots+N_r}}{(q)_{N_1-N_2}\cdots(q)_{N_{r-1}-N_r}(q)_{N_r}} T^{N_1+\cdots+N_r} \ . \tag{A.21}$$

Explicitly for $r = 1, 2, 3$, we have

$$\begin{aligned}
\mathcal{I}^{(A_1,A_2)} &= 1 + Tq^2 + Tq^3 + Tq^4 + Tq^5 \\
&\quad + \left(T + T^2\right) q^6 + \cdots \ , \\
\mathcal{I}^{(A_1,A_4)} &= 1 + Tq^2 + Tq^3 + \left(T + T^2\right) q^4 + \left(T + T^2\right) q^5 \\
&\quad + \left(T + 2T^2\right) q^6 + \cdots \ , \\
\mathcal{I}^{(A_1,A_6)} &= 1 + Tq^2 + Tq^3 + \left(T + T^2\right) q^4 + \left(T + T^2\right) q^5 \\
&\quad + \left(T + 2T^2 + T^3\right) q^6 + \cdots \ .
\end{aligned} \tag{A.22}$$

Moreover, the associated chiral algebras are $(2, 2r + 3)$ Virasoro minimal models [22]

$$\chi((A_1, A_{2r})) = \mathrm{Vir}_{c=-\frac{2r(5+6r)}{3+2r}} \ . \tag{A.23}$$

Similarly to the case of the MAD theory (A.11), the corresponding Zhu's algebra is now given by

$$\mathcal{R}_{\mathrm{Vir}_{c=-\frac{2r(5+6r)}{3+2r}}} = \mathbb{C}[x]/\langle x^{r+1}\rangle \ . \tag{A.24}$$

Therefore, as described in the discussion section, since the corresponding hidden symmetry (Virasoro) is related to a conserved non-decoupling symmetry ($SU(2)_R$), it is natural to imagine that the bijection we saw between MAD Schur operators and free vector operators in (10) of the main text generalizes. Indeed, we will see this is the case.

Since the rank of the (A_1, A_{2r}) theory is r, the natural generalization of (4) of the main text is

$$\hat{\mathcal{C}}_{0(0,0)} \ni J := J^{11}_{++} \longrightarrow \Lambda_r := \sum_{i=1}^{r} \lambda^1_{i,+}\bar{\lambda}^1_{i,\dot{+}} \in \hat{\mathcal{C}}^{(\mathrm{Free})^{\times r}}_{0(0,0)} \ , \tag{A.25}$$

and the natural generalization of our proposal in (10) of the main text is

$$\begin{aligned}
\mathcal{S}_{(A_1,A_{2r})} &\ni \partial_+^{i_1} J \cdots \partial_+^{i_n} J \\
&\longrightarrow \partial_+^{i_1}\Lambda_r \cdots \partial_+^{i_n}\Lambda_r \\
&\quad \in \tilde{\mathcal{S}}_{(\mathrm{Free\ Vector})^{\times r}} \subset \mathcal{S}_{(\mathrm{Free\ Vector})^{\times r}} \ .
\end{aligned} \tag{A.26}$$

It is straightforward to numerically check that (A.26) correctly reproduces the states computed by the Macdonald index. Since these are bosonic (as follows from the identification (A.23)), this constitutes strong evidence of the proposal. Here we will give some analytic evidence as well.

Let us now argue for the map in (A.26). Our first step is to understand the (A_1, A_{2r}) Schur ring. As in the case of (18) of the main text, the (A_1, A_{2r}) theory has a null vector involving T_{2d} and its derivatives (except now at $h = 2(r + 1)$; this is the origin of (A.24)). The 4d interpretation of this null relation generalizes (19) of the main text

$$J^r(z)J(0) \supset J^{r+1}(0) = 0 \ . \tag{A.27}$$

This can also be understood from (A.24). In the flow to the IR, this null relation follows from (A.25) and Fermi statistics.

As in the MAD case, it is natural to conjecture that the 4d Schur ring is generated by $\partial_+^i J$ subject to the constraint $J^{r+1} = 0$. More precisely, the analog of (A.12) is just

$$\begin{aligned}
X_\infty^{(A_1,A_{2r})} &= \mathrm{Spec}\ R_\infty^{(A_1,A_{2r})} \ , \\
R_\infty^{(A_1,A_{2r})} &= \mathbb{C}[x^{(i)}]/\langle (x^{r+1})^{(i)}\rangle \ , \quad i \geq 0 \ .
\end{aligned} \tag{A.28}$$

The idea is to compare the Hilbert series associated with (A.28) with the Macdonald index. However, unlike the MAD case, the higher r Hilbert series are not known in refined form. Instead, the unrefined case corresponding to $p = q$ is given by (see theorem 5.6 of [25])

$$\begin{aligned}
\mathrm{HS}(q, p = q) &= \sum_{n,k} \dim V_{n-k,k} q^n = \sum_{n,k} \dim V_{n,k} q^{n+k} \\
&= \prod_{i>0,\ i\neq 0,r+1,r+2 \mod (2r+3)} \frac{1}{1-q^i} \ , \\
&= \begin{cases} 1 + q + q^2 + q^3 + 2q^4 + \cdots \ , & r = 1 \\ 1 + q + 2q^2 + 2q^3 + 3q^4 + \cdots \ , & r = 2 \\ 1 + q + 2q^2 + 3q^3 + 4q^4 + \cdots \ , & r = 3 \\ \quad \vdots & \end{cases}
\end{aligned} \tag{A.29}$$

Therefore, we will need to compare this quantity with a somewhat unorthodox fugacity slice of the Macdonald index (A.21) gotten by setting $T = 1/q$ (note that this is *not* the Schur index, where we would instead set $T = 1$):

$$\mathcal{I}_M^{(A_1,A_{2r})}(q, T = 1/q) = \sum_{n,k} \dim V_{n,k} q^{n+k}$$
$$= \sum_{N_1 \geq \cdots \geq N_r \geq 0}^{\infty} \frac{q^{N_1^2+\cdots N_r^2}}{(q)_{N_1-N_2}\cdots(q)_{N_{r-1}-N_r}(q)_{N_r}} \ . \tag{A.30}$$

Intriguingly, these quantities can be written in terms of r-fold q-hypergeometric series [15]. It would be interesting to understand if the refined index for $r > 1$ can be expressed in terms of a generalization of (A.6) (and to understand the connection between these various types of hypergeometric functions).

The equality of (A.29) and (A.30) follows from the Andrews-Gordon identity. This is strong evidence that the 4d Schur ring is as described in (A.28) (i.e., that it is generated by $\partial_+^i J$ subject to $J^{r+1}=0$).

We can then further conjecture the refined Hilbert series by setting $T\to p/q^2$ in the Macdonald index:

$$\mathrm{HS}(q,p)=\sum_{N_1\geq\cdots\geq N_r\geq 0}^{\infty}\frac{q^{N_1^2+\cdots N_r^2-N_1-\cdots-N_r}p^{N_1+\cdots+N_r}}{(q)_{N_1-N_2}\cdots(q)_{N_{r-1}-N_r}(q)_{N_r}}\ . \tag{A.31}$$

At low orders, we have verified this conjecture explicitly by enumerating the corresponding elements of the arc space (A.28).

To make contact with the IR description and the proposal in (A.26), we again use leading ideals to generate a convenient UV basis for the arc space. To that end, we have (see proposition 5.2 of [25])

$$\begin{aligned}\mathrm{LT}(\langle(x^{r+1})^{(i)}\rangle)\ &=\ \langle\left(x^{(j)}\right)^s\left(x^{(j+1)}\right)^{r+1-s}\rangle\ ,\\ &\quad j\geq 0\ ,\quad s=0\ ,\ 1\ ,\ \cdots\ ,\ r\ .\end{aligned} \tag{A.32}$$

This discussion shows that a basis of operators is given by

$$\begin{aligned}&(\partial^{n_1}J)^{Q_1}\,(\partial^{n_2}J)^{Q_2}\ \cdots(\partial^{n_s}J)^{Q_s}\ ,\\ &\sum_{i=1}^{s}Q_in_i=n\ ,\qquad \sum_{i=1}^{s}Q_i=k\ ,\end{aligned} \tag{A.33}$$

where

$$\begin{aligned}&0\leq n_1<n_2<\cdots n_s\ ,\quad Q_i\leq r\ ,\\ &Q_i+Q_{i+1}\leq r\quad \text{if}\quad n_{i+1}=n_i+1\ .\end{aligned} \tag{A.34}$$

As a result, in (A.33), n_i can repeat at most r times. Moreover, n_i and n_i+1 (if $n_i+1=n_{i+1}$) together can repeat at most r times. From Gordon's Partition Theorem, the generating function of this partition is exactly given by (A.29).

To show that the map in (A.25) does not affect the counting of the basis of states (and that therefore (A.28) holds), we define

$$P(i,m):=\partial^{\lfloor\frac{m}{2}\rfloor}\lambda^1_{i,+}\partial^{\lfloor\frac{m+1}{2}\rfloor}\bar\lambda^1_{i,\dot+}\in\partial^mJ\ . \tag{A.35}$$

Then we have the following one-to-one correspondence

$$\begin{aligned}\prod_{i=1}^{s}(\partial^{n_i}J)^{Q_i}\leftrightarrow&\prod_{i=1}^{s}\Big[P(1,n_i)P(2,n_i)\cdots P(Q_i,n_i)\Big]^{A_i}\\ &\times\Big[P(r,n_i)P(r-1,n_i)\cdots P(r-Q_i+1,n_i)\Big]^{1-A_i}\ ,\end{aligned} \tag{A.36}$$

where $A_i=i\mod 2$, and $A_{i+1}=1-A_i$. Note that the fermionic expression on the righthand side is never zero thanks to the condition (A.34). Therefore, as in the MAD case, we have shown that we can also reproduce the Macdonald index from r gauginos.

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[34] For more details, see https://en.wikipedia.org/wiki/Basic_hypergeometric_series.
[35] Note this partition function is the one appearing in the Rogers–Ramanujan identity.