# Non-Perturbative Explorations of Chiral Rings in $4 \mathrm{~d} \mathcal{N}=2$ SCFTs 

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We study the conditions under which $4 \mathrm{~d} \mathcal{N}=2$ superconformal field theories (SCFTs) have multiplets housing operators that are chiral with respect to an $\mathcal{N}=1$ subalgebra. Our main focus is on the set of often-ignored and relatively poorly understood $\overline{\mathcal{B}}$ representations. These multiplets typically evade direct detection by the most popular non-perturbative 4 d $\mathcal{N}=2$ tools and correspondences. In spite of this fact, we demonstrate the ubiquity of $\overline{\mathcal{B}}$ multiplets and show they are associated with interesting phenomena. For example, we give a purely algebraic proof that they are present in all local unitary $\mathcal{N}>2$ SCFTs. We also show that $\overline{\mathcal{B}}$ multiplets exist in $\mathcal{N}=2$ theories with rank greater than one and a conformal manifold or a freely generated Coulomb branch. Using recent topological quantum field theory results, we argue that certain $\overline{\mathcal{B}}$ multiplets exist in broad classes of theories with the $\mathbb{Z}_{2}$-valued 't Hooft-Witten anomaly for $S p(N)$ global symmetry. Motivated by these statements, we then study the question of when $\overline{\mathcal{B}}$ multiplets exist in rank-one SCFTs with exactly $\mathcal{N}=2$ SUSY and vanishing 't Hooft-Witten anomaly. We conclude with various open questions.

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## 1. Introduction

Chiral operators play a fundamental role in $4 \mathrm{D} \mathcal{N}=2$ quantum field theories (QFTs) at all length scales. At short distances, the allowed $\mathcal{N}=2$-preserving relevant deformations of $\mathcal{N}=2$ superconformal field theories (SCFTs) are chiral [1]. Expectation values of chiral operators parameterize the $\mathcal{N}=2$ moduli spaces of these SCFTs and initiate renormalization group (RG) flows to vacua where low-energy vector multiplets and hypermultiplets live. These effective multiplets are also chiral 1

Since chiral operators are so ubiquitous, it is no surprise that many of the most important non-perturbative insights into $4 \mathrm{~d} \mathcal{N}=2$ QFTs are intimately connected with these operators. For example, Seiberg-Witten geometries [2] encode the exact infrared (IR) prepotential, which is a chiral object. Higgs branches enjoy various non-renormalization

[^1]theorems [3], and their associated chiral operators are closely related to 2d VOAs [4] and hidden infinite-dimensional symmetries.

Given their prominence and the powerful geometrical and algebraic constraints on their spectra, one may have the impression that chiral sectors of $\mathcal{N}=2$ theories are completely understood, simple to characterize, and probe phenomena that are well known. Each of these statements is far from the truth.

To better understand these points, it is helpful to first think about ultraviolet (UV) physics and understand which superconformal representations house chiral operators. As we will review in the next section, this question was answered in [5]. A basic but important point is that the operators directly connected with the relatively well-understood Coulomb branch physics of Seiberg-Witten theory and the physics of the Higgs branch sit in halfBPS multiplets 2 In the nomenclature of [6], these are the $\overline{\mathcal{E}}$ and $\hat{\mathcal{B}}$ multiplets (a more detailed discussion appears in the next section) $\sqrt[3]{ }$ In particular, vevs for the superconformal primaries (SCPs) of these multiplets parameterize the Coulomb and Higgs branches respectively.

However, $\mathcal{N}=2$ SCFTs contain less protected multiplets with chiral primaries, and these multiplets give rise to interesting physics. For example, $\overline{\mathcal{D}}$ multiplets are also intimately connected with 2 d VOAs [4] and probe various subtle properties of the topology and punctures of class $\mathcal{S}$ compactification surfaces (e.g., see [7]). Moreover, some $\overline{\mathcal{D}}$ multiplets contain the extra supercurrents of $\mathcal{N}>2 \mathrm{SCFTs}$ while others capture the physics of the Weinberg-Witten theorem.

Still, from a purely algebraic point of view, the above multiplets are not the most general representations containing chiral operators. Indeed, while these multiplets have primaries that depend on at most two quantum numbers, there are more general multiplets with primaries that depend on three quantum numbers. These are the $\overline{\mathcal{B}}$ multiplets and are the main focus of this paper.

Given the greater freedom in their quantum numbers, one might wonder if $\overline{\mathcal{B}}$ multiplets are ubiquitous in $4 \mathrm{~d} \mathcal{N}=2$ SCFTs. Unfortunately, the answer to this question is

[^2]obscured by the fact that these multiplets are under less control then the $\overline{\mathcal{E}}$, $\hat{\mathcal{B}}$, and $\overline{\mathcal{D}}$ representations. For example, $\overline{\mathcal{B}}$ multiplets are not captured by any of the special limits of the superconformal index. Moreover, Seiberg-Witten theory and the $4 \mathrm{~d} / 2 \mathrm{~d}$ correspondence of [4] do not directly detect these degrees of freedom. 4

One well-known instance where $\overline{\mathcal{B}}$ multiplets appear is whenever a UV theory has a "mixed" branch. This is a branch of moduli space where low-energy vector multiplets and hypermultiplets co-exist. In other words, mixed branches include a Coulomb branch and a Higgs branch component at common points in moduli space. Therefore, we expect a $\overline{\mathcal{B}}$ multiplet to appear in the following operator product

$$
\begin{equation*}
\overline{\mathcal{E}} \times \hat{\mathcal{B}} \ni \overline{\mathcal{B}}, \tag{1.1}
\end{equation*}
$$

where the $\overline{\mathcal{B}}$ primary is a composite built out of a Coulomb branch $\overline{\mathcal{E}}$ primary and a Higgs branch $\hat{\mathcal{B}}$ primary. Giving an expectation value to the $\overline{\mathcal{B}}$ primary gives a vev to both $\overline{\mathcal{E}}$ and $\hat{\mathcal{B}}$ primaries and initiates an RG flow to a mixed branch. However, in many theories, null relations set $\overline{\mathcal{B}}=0$ and lead to geometrically separate Coulomb and Higgs branches.

The main purpose of this paper is to explain a much broader array of phenomena that are captured by $\overline{\mathcal{B}}$ multiplets beyond the existence of a mixed branch. Indeed, we will argue that

- All local unitary $4 \mathrm{~d} \mathcal{N}>2$ SCFTs have $\overline{\mathcal{B}}$ multiplets. We give an algebraic proof of this fact that follows purely from locality and unitarity (see Section 3.1).
- All higher-rank $4 \mathrm{~d} \mathcal{N}=2$ SCFTs with conformal manifolds parameterized by gauge coupling $\sqrt{5}^{5}$ have $\overline{\mathcal{B}}$ multiplets that exist at all points on the conformal manifold ${ }^{6}$ (see Section (3.2).

[^3]- All higher-rank $4 \mathrm{~d} \mathcal{N}=2$ SCFTs with freely generated Coulomb branches have $\overline{\mathcal{B}}$ multiplets (see Section (3.2).
- Any $4 \mathrm{~d} \mathcal{N}=2$ SCFT with an $S p(N)$ symmetry having a $\mathbb{Z}_{2}$-valued 't Hooft anomaly [8] has a $\overline{\mathcal{B}}$ multiplet if its Coulomb branch has at least one point consisting of purely free fields (see Section (3.3).

Therefore, we will see that $\overline{\mathcal{B}}$ multiplets are indeed ubiquitous and that they are related to various interesting phenomena.

Given the above results, we are also motivated to study rank-one SCFTs with purely $\mathcal{N}=2$ SUSY, no $\mathbb{Z}_{2}$-valued 't Hooft anomaly, and no mixed branch. Indeed, the existence of $\overline{\mathcal{B}}$ multiplets in these theories is not implied by the above results. For example, in the case of the rank-one $\mathcal{N}=2$ theory studied in [5], such multiplets were shown to be absent. The main tool used in that paper was $\mathcal{N}=2$ superconformal representation theory coupled with the dynamics of $\mathcal{N}=1 \rightarrow \mathcal{N}=2$ SUSY enhancement along an RG flow to the IR. Using similar techniques, we will study various other rank-one theories amenable to such analysis. In all such isolated theories that we study (for simplicity, we stick to those of Argyres-Douglas type), we will see that $\overline{\mathcal{B}}$ multiplets are absent.

The plan of the paper is as follows. In the next section, we introduce various details of the superconformal analysis of chiral operators in $4 \mathrm{~d} \mathcal{N}=2 \mathrm{SCFTs}$. In section 3 we present the general results described in the bullet points above. We are then motivated to make some conjectures on the spectrum of $\overline{\mathcal{B}}$ multiplets in general theories. In the remainder of the paper, we study various rank-one SCFTs and conclude with a discussion of future directions.

## 2. $\overline{\mathcal{B}}$ multiplets and superconformal representation theory

In this section, we briefly discuss the superconformal representation theory and ring structure of $\mathcal{N}=2$ multiplets that contain an operator that is chiral with respect to an $\mathcal{N}=1 \subset \mathcal{N}=2$ subalgebra. We conclude by explaining where $\overline{\mathcal{B}}$ multiplets sit in this universe.

Recall that an $\mathcal{N}=2$ superconformal field theory has an $S U(2)_{R} \times U(1)_{R}$ symmetry with eight Poincaré and eight special supercharges transforming as doublets under the $R$ symmetry. Without loss of generality, we follow [5] and take our $\mathcal{N}=1$ subalgebra to be
generated by the following Poincaré supercharges

$$
\begin{equation*}
Q_{1 \alpha} \sim Q_{\alpha}^{2} \in\left(\frac{1}{2}, 0\right)_{-\frac{1}{2},-\frac{1}{2}}, \quad \bar{Q}_{\dot{\alpha}}^{1} \sim \bar{Q}_{2 \dot{\alpha}} \in\left(0, \frac{1}{2}\right)_{\frac{1}{2}, \frac{1}{2}} \tag{2.1}
\end{equation*}
$$

where the quantum numbers are $(j, \bar{j})_{R, r}$, with $(j, \bar{j})$ the left and right spin, $R$ the $S U(2)_{R}$ weight, and $r$ the $U(1)_{r}$ charge (note that, to get a superconformal subalgebra, we should also include the special supercharges $S_{2 \alpha} \sim S_{\alpha}^{1}$ and $\left.\bar{S}_{1 \dot{\alpha}} \sim \bar{S}_{\dot{\alpha}}^{2}\right)$.

With these conventions, the $\mathcal{N}=1$ chiral operators are those satisfying

$$
\begin{equation*}
\left[\bar{Q}_{\dot{\alpha}}^{1}, \mathcal{O}\right\}=0, \quad \mathcal{O} \neq\left\{\bar{Q}_{\dot{\alpha}}^{1}, \mathcal{O}^{\prime}\right] \tag{2.2}
\end{equation*}
$$

Such operators form a chiral ring (their OPEs are free from singularities), and the second condition in (2.2) is equivalent to demanding that $\mathcal{O}$ is non-trivial in this ring (here $\mathcal{O}^{\prime}$ is any well-defined local operator in the theory). Note that we have suppressed any $S U(2)_{R}$ and Lorentz quantum numbers of $\mathcal{O}$.

In the context of $\mathcal{N}=2$ SCFTs, operators satisfying (2.2) can sit in various representations. We find homes for our chiral operators in these multiplets by acting on chiral superconformal primaries with the (chiral part) of the subalgebra orthogonal to (2.1) [5]

$$
\begin{equation*}
Q_{2 \alpha} \sim Q_{\alpha}^{1} \in\left(\frac{1}{2}, 0\right)_{\frac{1}{2},-\frac{1}{2}}, \quad \bar{Q}_{\dot{\alpha}}^{2} \sim \bar{Q}_{1 \dot{\alpha}} \in\left(0, \frac{1}{2}\right)_{-\frac{1}{2}, \frac{1}{2}} . \tag{2.3}
\end{equation*}
$$

The analysis of [5] shows that operators satisfying (2.2) can only sit in the following positions in an $\mathcal{N}=2$ multiplet

$$
\begin{align*}
\mathcal{O}_{\alpha_{1} \cdots \alpha_{2 j}}^{11 \cdots 1} \in(j, 0)_{R, r} & \xrightarrow{Q_{\alpha}^{1}} \mathcal{O}_{\alpha_{1} \cdots \alpha_{2 j} \alpha}^{11 \cdots 11} \oplus \mathcal{O}_{\alpha_{1} \cdots \alpha_{2 j-1} \alpha}^{11 \cdots 11} \in\left(j+\frac{1}{2}, 0\right)_{R+\frac{1}{2}, r-\frac{1}{2}} \oplus\left(j-\frac{1}{2}, 0\right)_{R+\frac{1}{2}, r-\frac{1}{2}} \\
& \xrightarrow{\left(Q^{1}\right)^{2}} \mathcal{O}_{\alpha_{1} \cdots \alpha_{2 j}}^{11 \cdots 11} \in(j, 0)_{R+1, r-1}, \tag{2.4}
\end{align*}
$$

Here the leftmost operator is a chiral superconformal primary of an $\mathcal{N}=2$ multiplet (it has highest $S U(2)_{R}$ weight), and the remaining operators are successive $Q_{\alpha}^{1}$ descendants. Depending on the multiplet in question, some of the descendants may be null.

In the language of [6], solutions to (2.4) are exhausted by multiplets in the so-called "full chiral sector" (FCS) [5]

$$
\begin{equation*}
\mathrm{FCS}:=\overline{\mathcal{E}}_{r} \oplus \hat{\mathcal{B}}_{R} \oplus \overline{\mathcal{D}}_{R(j, 0)} \oplus \overline{\mathcal{B}}_{R, r(j, 0)} . \tag{2.5}
\end{equation*}
$$

Let us analyse these multiplets in turn:

- The $\overline{\mathcal{E}}_{r}$ primary is $U(1)_{r}$ charged but has $R=j=\bar{j}=0$ [9]. It is annihilated by all the $\bar{Q}_{\alpha}^{i}$. In this sense, it is maximally protected. According to the standard lore, it is also the most universal FCS multiplet: such multiplets exist in all known interacting $4 \mathrm{~d} \mathcal{N}=2$ SCFTs, and their vevs give coordinates on the Coulomb branch of these SCFTs. The superconformal primaries form the closed Coulomb branch chiral (sub)ring $]^{7}$ More generally, these multiplets house three $\mathcal{N}=1$ chiral operators from (2.4)

$$
\begin{equation*}
\mathcal{O} \in(0,0)_{0, r} \quad \xrightarrow{Q_{\alpha}^{1}} \mathcal{O}_{\alpha}^{1} \in\left(\frac{1}{2}, 0\right)_{1 / 2, r-1 / 2} \xrightarrow{\left(Q^{1}\right)^{2}} \mathcal{O}^{11} \in(0,0)_{1, r-1} \tag{2.6}
\end{equation*}
$$

The descendant states do not form part of the Coulomb branch chiral ring.

- The $\hat{\mathcal{B}}_{R}$ multiplets have $r=j=\bar{j}=0$ and $R>0$. All known examples of these multiplets parameterize Higgs branches of $\mathcal{N}=2$ SCFTs via the vevs of their primaries. They form a closed Higgs branch chiral (sub)ring. Therefore, while these multiplets are also common, they are more special than the $\overline{\mathcal{E}}$ type. Indeed, the SCFT in question has to have sufficient "matter" for a Higgs branch to exist. These multiplets form Virasoro primaries under the $4 \mathrm{~d} / 2 \mathrm{~d}$ map of [4] and hence are under good analytic control. $\hat{\mathcal{B}}_{R}$ multiplets house only one $\mathcal{N}=1$ chiral operator (the primary)

$$
\begin{equation*}
\mathcal{O} \in(0,0)_{R, 0} \quad \xrightarrow{Q_{\alpha}^{1}} 0 \tag{2.7}
\end{equation*}
$$

As a result, the primary is also anti-chiral with respect to the orthogonal algebra in (2.3). Like the $\overline{\mathcal{E}}_{r}$ multiplet, $\hat{\mathcal{B}}_{R}$ is therefore maximally protected under the SUSY algebra (although the primary is not annihilated by all of the same supercharges).

- The $\overline{\mathcal{D}}_{R(j, 0)}$ multiplet has $\bar{j}=0$ and $r=1+j$. In general, it has $R, j>0$, but it can also have $j=0$ and $R \geq 0$. This multiplet is ubiquitous in low energy effective theories: the $\overline{\mathcal{D}}_{0(0,0)}$ multiplet contains the chiral half of the free vector and is therefore present on the Coulomb branch of any theory. More generally, $\overline{\mathcal{D}}$ multiplets with $R>0$ appear if we also have decoupled hypermultiplets (or more complicated matter sectors with Higgs branches) appearing on the Coulomb branch 8 Indeed, consider

[^4]the following IR OPE
\[

$$
\begin{equation*}
\overline{\mathcal{D}}_{0,(0,0)} \times \hat{\mathcal{B}}_{R} \ni \overline{\mathcal{D}}_{R(0,0)} . \tag{2.8}
\end{equation*}
$$

\]

By studying the OPE of primaries (since there are no singularities, this is just the chiral ring product), we see that a $\overline{\mathcal{D}}_{R(0,0)}$ multiplet must appear on the righthand side.

The $\overline{\mathcal{D}}$ multiplet is less ubiquitous in interacting theories. This relative scarcity is because the $\overline{\mathcal{D}} U(1)_{r}$ charge is fixed in terms of the multiplet's spin to be at a unitarity bound. Moreover, recall that in flows to the Coulomb branch, $U(1)_{r}$ is broken but Lorentz symmetry is not. Therefore, it is common for $\overline{\mathcal{D}}$ multiplets to arise from UV multiplets with larger $U(1)_{r}$ charge like $\overline{\mathcal{E}}$ or $\overline{\mathcal{B}}$ multiplets.
However, $\overline{\mathcal{D}}$ multiplets always appear in local (interacting) theories with $\mathcal{N}>2$ SUSY since the $\overline{\mathcal{D}}_{1 / 2(0,0)}$ multiplet houses the extra SUSY currents. More generally, $\overline{\mathcal{D}}$ multiplets appear in certain interacting $\mathcal{N}=2$ theories as well. For example, they exist in class $\mathcal{S}$ theories whose corresponding Riemann surfaces have non-trivial $\pi_{1}$ (for more general examples, see [7]). These multiplets therefore seem to know interesting things about topology. They also give rise to Virasoro primaries under the $4 \mathrm{~d} / 2 \mathrm{~d}$ map of [4] and are therefore under stringent analytic control. For generic $R$ and $j$, they house the following $\mathcal{N}=1$ chiral primaries

$$
\begin{equation*}
\mathcal{O}_{\alpha_{1} \cdots \alpha_{2 j}}^{11 \cdots 1} \in(j, 0)_{R, j+1} \xrightarrow{Q_{\alpha}^{1}} \mathcal{O}_{\alpha_{1} \cdots \alpha_{2 j} \alpha}^{11 \cdots 11} \in\left(j+\frac{1}{2}, 0\right)_{R+\frac{1}{2}, j+\frac{1}{2}} \xrightarrow{\left(Q^{1}\right)^{2}} 0 \tag{2.9}
\end{equation*}
$$

- Finally, we consider the $\overline{\mathcal{B}}_{R, r(j, 0)}$ multiplets of interest. They are clearly the most general FCS multiplets in the sense that they have three independent quantum numbers $R>0, j$, and $r>1+j$ (only $\bar{j}=0$ ). Moreover, all states in (2.4) are present

$$
\begin{align*}
\mathcal{O}_{\alpha_{1} \cdots \alpha_{2 j}}^{11 \cdots 1} \in(j, 0)_{R, r} & \xrightarrow{Q_{\alpha}^{1}} \mathcal{O}_{\alpha_{1} \cdots \alpha_{2 j} \alpha}^{11 \cdots 11} \oplus \mathcal{O}_{\alpha_{1} \cdots \alpha_{2 j-1} \alpha}^{11 \cdots 11} \in\left(j+\frac{1}{2}, 0\right)_{R+\frac{1}{2}, r-\frac{1}{2}} \oplus\left(j-\frac{1}{2}, 0\right)_{R+\frac{1}{2}, r-\frac{1}{2}} \\
& \xrightarrow{\left(Q^{1}\right)^{2}} \mathcal{O}_{\alpha_{1} \cdots \alpha_{2 j}}^{11 \cdots 11} \in(j, 0)_{R+1, r-1} \tag{2.10}
\end{align*}
$$

On a mixed branch, these multiplets are as common as $\overline{\mathcal{D}}$ multiplets. For example, in the presence of free vectors, we have $\overline{\mathcal{E}}_{n}$ operators from the $n$-fold $\overline{\mathcal{D}}_{0(0,0)}^{\times n} \ni \overline{\mathcal{E}}_{n}$ OPE. We can then repeat the IR OPE in (2.8) but with $\overline{\mathcal{D}}_{0(0,0)} \rightarrow \overline{\mathcal{E}}_{n}$ and $\overline{\mathcal{D}}_{R(0,0)} \rightarrow \overline{\mathcal{B}}_{R, n(0,0)}$. We will show below that $\overline{\mathcal{B}}$ multiplets exist whenever a theory has a freely generated Coulomb branch of rank at least two (in this sense they are slightly less ubiquitous than the $\overline{\mathcal{D}}$ multiplets since they do not appear in the theory of a single free vector [5]).

In interacting theories, we expect such multiplets to be more common than $\overline{\mathcal{D}}$ multiplets. This is because $r>1+j$ is an inequality (as opposed to the equality in the $\overline{\mathcal{D}}$ case). For example, we expect

$$
\begin{equation*}
\overline{\mathcal{E}}_{r} \times \hat{\mathcal{B}}_{R} \ni \overline{\mathcal{B}}_{R, r(0,0)}, \tag{2.11}
\end{equation*}
$$

whenever the SCFT supports a mixed branch. The vevs of the corresponding $\overline{\mathcal{B}}_{R, r(0,0)}$ chiral primaries parameterize these mixed branches.

We can also find other chiral ring products giving rise to $\overline{\mathcal{B}}_{R, r(j, 0)}$ multiplets 9 For example, selection rules allow

$$
\begin{equation*}
\overline{\mathcal{D}}_{R(j, 0)} \times \overline{\mathcal{D}}_{R^{\prime}\left(j^{\prime}, 0\right)} \ni \overline{\mathcal{B}}_{R+R^{\prime}, j+j^{\prime}+2\left(j+j^{\prime}, 0\right)} . \tag{2.12}
\end{equation*}
$$

Unless all the $\overline{\mathcal{D}}$ primaries are minimally nilpotent in the chiral ring, they must give rise to corresponding $\overline{\mathcal{B}}$ multiplets. ${ }^{10}$ Combined with (2.11), this observation again suggests that $\overline{\mathcal{B}}$ multiplets should be more common than $\overline{\mathcal{D}}$ multiplets in interacting theories.

We can also imagine constructing $\overline{\mathcal{B}}$ multiplets via chiral ring products involving descendants of the multiplets discussed in previous bullets 11 For example, we can take

$$
\begin{array}{lll}
\overline{\mathcal{E}}_{r} \times \overline{\mathcal{E}}_{r^{\prime}} & \ni \mathcal{O} Q_{\alpha}^{1} \mathcal{O}^{\prime}+\kappa Q_{\alpha}^{1}(\mathcal{O}) \mathcal{O}^{\prime} \in \overline{\mathcal{B}}_{1 / 2, r+r^{\prime}-1 / 2(1 / 2,0)}, \\
\overline{\mathcal{E}}_{r} \times \overline{\mathcal{E}}_{r^{\prime}} & \ni & Q^{1 \alpha} \mathcal{O} Q_{\alpha}^{1} \mathcal{O}^{\prime}+\kappa_{1}\left(Q^{1}\right)^{2}(\mathcal{O}) \mathcal{O}^{\prime}+\kappa_{2} \mathcal{O}\left(Q^{1}\right)^{2}\left(\mathcal{O}^{\prime}\right) \in \overline{\mathcal{B}}_{1, r+r^{\prime}-1(0,0)} \tag{2.14}
\end{array}
$$

where $\kappa, \kappa_{1}, \kappa_{2} \in \mathbb{C}$ are required to make the above operators superconformal primaries 12

[^5]More generally, it is apriori possible that $\overline{\mathcal{B}}$ multiplets can appear as chiral ring generators 13

Finally, we note that, at the level of multiplication of superconformal primaries in the chiral ring, $\overline{\mathcal{B}}_{R, r(j, 0)}$ multiplets form (two-sided) ideals

$$
\begin{array}{rlll}
\overline{\mathcal{E}}_{r} & \times \overline{\mathcal{B}}_{R^{\prime}, r^{\prime}\left(j^{\prime}, 0\right)} & \ni & \overline{\mathcal{B}}_{R^{\prime}, r+r^{\prime}\left(j^{\prime}, 0\right)}, \quad \hat{\mathcal{B}}_{R} \times \overline{\mathcal{B}}_{R^{\prime}, r^{\prime}\left(j^{\prime}, 0\right)} \ni \overline{\mathcal{B}}_{R+R^{\prime}, r^{\prime}\left(j^{\prime}, 0\right)}, \\
\overline{\mathcal{D}}_{R(j, 0)} & \times \overline{\mathcal{B}}_{R^{\prime}, r^{\prime}\left(j^{\prime}, 0\right)} & \ni & \overline{\mathcal{B}}_{R+R^{\prime}, r^{\prime}+j+1\left(j+j^{\prime}, 0\right)}, \\
\overline{\mathcal{B}}_{R, r(j, 0)} & \times \overline{\mathcal{B}}_{R^{\prime}, r^{\prime}\left(j^{\prime}, 0\right)} & \ni & \overline{\mathcal{B}}_{R+R^{\prime}, r+r^{\prime}\left(j+j^{\prime}, 0\right)} \tag{2.15}
\end{array}
$$

Therefore, to summarize: in the absence of FCS chiral ring relations, we expect $\overline{\mathcal{B}}$ multiplets to be present whenever the theory is interacting (since we then expect $\overline{\mathcal{E}}$ multiplets). Moreover, we expect the corresponding chiral primaries to form ideals in the chiral ring and therefore to be crucial in understanding the full local operator algebras of interacting $4 \mathrm{~d} \mathcal{N}=2$ SCFTs.

However, $\mathcal{N}=2$ theories often have FCS chiral ring relations 14 (these relations will feature prominently in our rank-one discussion below), and so the above conclusion is too naive. Still, given how easy it is to generate such multiplets in the chiral ring, we expect $\overline{\mathcal{B}}$ multiplets to be present in broad classes of theories and to detect various physical phenomena. Indeed, we will arrive at a few general results on these multiplets in the next section.

## 3. General results

In this section, we discuss several abstract results on the presence of $\overline{\mathcal{B}}$ multiplets in broad classes of $4 \mathrm{~d} \mathcal{N}=2 \mathrm{SCFTs}$. These results are connected with various physical phenomena.

### 3.1. Local unitary SCFTs with $\mathcal{N} \geq 3$ SUSY

Any local unitary SCFT with $\mathcal{N} \geq 3$ supersymmetry has a $\overline{\mathcal{B}}$ multiplet. 15 Indeed, by locality, any such theory has an $\mathcal{N}=3$ stress tensor multiplet. As a result, from an

[^6]$\mathcal{N}=2$ perspective, the theory has a $U(1)$ flavor symmetry (descending from the $\mathcal{N}=3$ $R$ symmetry), which we will call $U(1)_{G}$. The Noether current for this symmetry sits in a corresponding $\hat{\mathcal{B}}_{1}^{0} \cong B_{1} \bar{B}_{1}[0 ; 0]^{(2 ; 0), 0}$ multiplet. Here we use the language of both [6] and [11]. 16 The reason we introduce new nomenclature is that we will make a claim regarding the presence of $\overline{\mathcal{B}}$ multiplets using $\mathcal{N}=3$ superconformal representation theory, and [6] only discusses $\mathcal{N}=2$ representations.
 morphic moment map, $M^{11}$. Together with the highest $S U(2)_{R}$-weight operator in the stress tensor multiplet, $\hat{\mathcal{C}}_{0(0,0)} \cong A_{2} \bar{A}_{\overline{2}}[0 ; 0]_{2}^{(0 ; 0)}$, and in the extra supercurrent multiplets, $\mathcal{D}_{\frac{1}{2}(0,0)} \oplus \overline{\mathcal{D}}_{\frac{1}{2}(0,0)} \cong A_{2} \bar{B}_{1}[0,0]^{(1 ; 2)} \oplus B_{1} \bar{A}_{2}[0 ; 0]^{(1 ;-2)}, M^{11}$ is related by the chiral algebra map of [4] to the generators of a $2 \mathrm{~d} \mathcal{N}=2$ super-Virasoro VOA [13]
\[

$$
\begin{equation*}
\chi\left(M^{11}\right)=J, \quad \chi\left(J_{+\dot{+}}^{11}\right)=T, \quad \chi\left(J_{+}^{11}\right)=G, \quad \chi\left(J_{\dot{+}}^{11}\right)=\bar{G} . \tag{3.1}
\end{equation*}
$$

\]

Here $J$ is a $U(1)$ affine current, $T$ is the 2 d EM tensor, and the remaining operators are the 2 d supercurrents (we refer the reader to [4, 13] for more detailed discussions of this correspondence).

We claim that in any local unitary $\mathcal{N} \geq 3$ theory, the 4 d OPE of the holomorphic moment maps contains the following dimension-four $S U(2)_{R}$ weight-two operator

$$
\begin{equation*}
M^{11} \times M^{11} \supset\left(M^{11}\right)^{2}, \tag{3.2}
\end{equation*}
$$

where $\left(M^{11}\right)^{2} \in \hat{\mathcal{B}}_{2}^{0} \cong B_{1} \bar{B}_{1}[0 ; 0]^{(4 ; 0), 0}$. We can argue for this statement by recalling the following selection rules (e.g., see [14]) ${ }^{17}$

$$
\begin{equation*}
\hat{\mathcal{B}}_{1} \times \hat{\mathcal{B}}_{1}=\hat{\mathcal{B}}_{1}+\hat{\mathcal{B}}_{2}+\sum_{\ell=0}^{\infty} \hat{\mathcal{C}}_{0\left(\frac{\ell}{2}, \frac{\ell}{2}\right)}+\sum_{\ell=0}^{\infty} \hat{\mathcal{C}}_{1\left(\frac{\ell}{2}, \frac{\ell}{2}\right)} \tag{3.3}
\end{equation*}
$$

and translating to the 2d VOA. Specializing to multiplets that can provide an $h=E-R=2$ operator in the OPE, it turns out that the righthand side of (3.3) reduces to $\hat{\mathcal{B}}_{1}+\hat{\mathcal{B}}_{2}+\hat{\mathcal{C}}_{0(0,0)}$. In the 2 d picture, the $\hat{\mathcal{B}}_{1}$ contribution to the OPE comes from $\chi\left(\partial M^{11}\right)=\partial J$.

To verify (3.2), we therefore need to check that there are no null relations involving $J J$, $\partial J$, and $T$ (otherwise, the 4 d normal-ordered product in (3.2) vanishes according to the

[^7]general prescription in [4]). We can use the bosonic part of the super-Virasoro algebra
\[

$$
\begin{align*}
{\left[L_{n}, L_{m}\right] } & =(n-m) L_{n+m}+\frac{c}{12} n\left(n^{2}-1\right) \delta_{n+m, 0} \\
{\left[J_{n}, J_{m}\right] } & =\frac{c}{12} n \delta_{n+m, 0} \\
{\left[L_{n}, J_{m}\right] } & =-m J_{n+m} \tag{3.4}
\end{align*}
$$
\]

and compute the matrix of inner products

$$
\begin{align*}
\left(\begin{array}{ccc}
\langle T \mid T\rangle & \langle T \mid J J\rangle & \langle T \mid \partial J\rangle \\
\langle J J \mid T\rangle & \langle J J \mid J J\rangle & \langle J J \mid \partial J\rangle \\
\langle\partial J \mid T\rangle & \langle\partial J \mid J J\rangle & \langle\partial J \mid \partial J\rangle
\end{array}\right) & =\left(\begin{array}{ccc}
\left\langle L_{2} L_{-2}\right\rangle & \left\langle L_{2} J_{-1} J_{-1}\right\rangle & \left\langle L_{2} J_{-2}\right\rangle \\
\left\langle J_{1} J_{1} L_{-2}\right\rangle & \left\langle J_{1} J_{1} J_{-1} J_{-1}\right\rangle & \left\langle J_{1} J_{1} J_{-2}\right\rangle \\
\left\langle J_{2} L_{-2}\right\rangle & \left\langle J_{2} J_{-1} J_{-1}\right\rangle & \left\langle J_{2} J_{-2}\right\rangle
\end{array}\right) \\
& =\left(\begin{array}{ccc}
c / 2 & c / 12 & 0 \\
c / 12 & c^{2} / 72 & 0 \\
0 & 0 & c / 6
\end{array}\right) . \tag{3.5}
\end{align*}
$$

The above matrix has determinant $c^{3}(c-1) / 864$ and is therefore non-invertible for $c=0,1$. Under the $4 \mathrm{~d} / 2 \mathrm{~d}$ map of [4], these central charges map to 4 d central charges $c_{4 d}=0,-1 / 12$ and correspond to non-unitary 4 d theories. Therefore, in a unitary 4 d theory with $\mathcal{N} \geq 3$ SUSY, we see that we necessarily have a $\hat{\mathcal{B}}_{2} \cong B_{1} \bar{B}_{1}[0 ; 0]^{(4 ; 0)}$ multiplet.

Where does the above multiplet sit in $\mathcal{N}=3$ representation theory? A moment's thought indicates it must sit inside the $\mathcal{N}=3$ stress-tensor multiplet self-OPE

$$
\begin{equation*}
B_{1} \bar{B}_{1}[0 ; 0]^{(1,1 ; 0)} \times B_{1} \bar{B}_{1}[0 ; 0]^{(1,1 ; 0)} \ni B_{1} \bar{B}_{1}[0 ; 0]^{(4 ; 0), 0} \cong \hat{\mathcal{B}}_{2} \tag{3.6}
\end{equation*}
$$

Note that the $B_{1} \bar{B}_{1}[0 ; 0]^{(1,1 ; 0)}$ multiplet transforms in the $\mathbf{8}$ of $S U(3)_{R}$. Now, recall that

$$
\begin{equation*}
8 \times 8=1+8+8+10+\overline{10}+27 . \tag{3.7}
\end{equation*}
$$

Clearly, $B_{1} \bar{B}_{1}[0 ; 0]^{(4 ; 0), 0}$ cannot sit in the first three $S U(3)_{R}$ representations above. Using branching rules, it is also easy to check that $\left(M^{11}\right)^{2}$ transforms as part of a 27 of $S U(3)_{R}$. In terms of $\mathcal{N}=2 S U(2)_{R}$ representations, we have scaling dimension four Lorentz-scalar primaries

$$
\begin{equation*}
27=1+2+2+3+3+3+4+4+5 . \tag{3.8}
\end{equation*}
$$

These operators must have $U(1)_{R}^{\mathcal{N}=3}$ charge zero and so the lefthand side is a representation of type $B_{1} \bar{B}_{1}[0 ; 0]^{(2,2 ; 0)}$. Therefore, on the righthand side the $U(1)_{R}^{\mathcal{N}=2}$ charges of the primaries satisfy

$$
\begin{equation*}
U(1)_{R}^{\mathcal{N}=2}=\frac{2}{3} U(1)_{S U(3)_{R}}=-2 U(1)_{G} \tag{3.9}
\end{equation*}
$$

In terms of our conventions relative to [11], $U(1)_{r}:=\frac{1}{2} U(1)_{R}^{\mathcal{N}}=2$.

Now we are ready to analyze the righthand side of (3.8). The first representation must be a long multiplet (since $R=r=0$ ). The next two representations have $R=1 / 2$ and $r=1$ and are also long multiplets (they cannot be $\hat{\mathcal{C}}$ multiplets since they have scalar primaries). The next three representations have $R=1$ and $r=2,0,-2$. This gives a $\overline{\mathcal{B}}_{1,2(0,0)} \oplus \hat{\mathcal{C}}_{1(0,0)} \oplus \mathcal{B}_{1,-2(0,0)}$ triple 18 The next two representations have $R=3 / 2$ and $r=1,-1$. These are $\overline{\mathcal{D}}_{3 / 2}(0,0) \oplus \mathcal{D}_{-3 / 2(0,0)}$. The final representation is $\hat{\mathcal{B}}_{2}$.

As a result, we have arrived at our promised statement: any local unitary $\mathcal{N} \geq 3$ SCFT has a $\overline{\mathcal{B}}$ multiplet. In $\mathcal{N}=2$ language, this multiplet can be understood as arising from the normal-ordered product of the extra supercurrent multiplet 19

$$
\begin{equation*}
\overline{\mathcal{D}}_{1 / 2(0,0)} \times \overline{\mathcal{D}}_{1 / 2(0,0)} \ni \overline{\mathcal{B}}_{1,2(0,0)} \tag{3.10}
\end{equation*}
$$

This is a particular example of the more general channel described in (2.12).
Note that this discussion does not require the existence of a moduli space of vacua (although all known examples of $\mathcal{N} \geq 3$ theories have such moduli spaces). Moreover, although the $4 \mathrm{~d} / 2 \mathrm{~d}$ VOA map of 4 ] doesn't directly detect $\overline{\mathcal{B}}$ multiplets, we see that we can combine that map with locality and $\mathcal{N}>2$ SUSY to deduce the existence of $\overline{\mathcal{B}}$ multiplets.

In summary, we have the following result:
Statement 1: Any local unitary $4 \mathrm{~d} \mathcal{N} \geq 3$ SCFT has a $\overline{\mathcal{B}}_{1,2(0,0)}$ multiplet.

### 3.2. Higher-rank SCFTs

In the previous section, we saw that all local unitary $\mathcal{N} \geq 3$ SCFTs have $\overline{\mathcal{B}}$ multiplets. However, these theories are special by virtue of their enhanced symmetry. It is natural to then wonder if $\overline{\mathcal{B}}$ multiplets are always related to symmetry enhancement or other more special phenomena.

In this section, we will see the answer is no. In particular, we will demonstrate the ubiquity of $\overline{\mathcal{B}}$ multiplets. Indeed, we will see that, under relatively relaxed assumptions, higher-rank theories have such multiplets.

Let us begin with a rank $N \geq 2$ SCFT that is part of an $\mathcal{N}=2$ or $\mathcal{N}=4$ conformal manifold 20 All known exactly marginal deformations in $\mathcal{N}=2$ SCFTs involve gauge

[^8]couplings for some non-abelian group, $G$. Such theories always admit at least one weak gauge coupling limit in which the theory factorizes into a sector consisting of the vector multiplets, $V_{G}$, and one or more matter sectors, $\mathcal{T}_{i} 21$ If the theory has rank $N \geq 2$, this means that the we have at least two generators of the Coulomb branch chiral ring. One generator must be the quadratic Casimir, $\overline{\mathcal{E}}_{2} \in V_{G}$, and the other generator is either a higher Casimir of $V_{G}$ (e.g., if $\left.G=S U(3)\right)$ or else is a Coulomb branch generator, $\overline{\mathcal{E}}_{r} \in \mathcal{T}_{i} 22$ In general there is no invariant distinction between these two possibilities (e.g., as can be seen by looking at both sides of the duality in [16]).

Let us now set the $G$ gauge coupling to zero. Then, we see that

$$
\begin{equation*}
\mathcal{O}_{2, r, \alpha}:=\mathcal{O}_{2} Q_{\alpha}^{1}\left(\mathcal{O}_{r}\right)+\kappa Q_{\alpha}^{1}\left(\mathcal{O}_{2}\right) \mathcal{O}_{r} \tag{3.11}
\end{equation*}
$$

is a superconformal primary for a particular choice of $\kappa \in \mathbb{C}$. Here $\mathcal{O}_{2}$ is the $\overline{\mathcal{E}}_{2}$ superconformal primary (it is related by supersymmetry to the exactly marginal deformation), and $\mathcal{O}_{r}$ is some other Coulomb branch chiral ring generator. As a result, (3.11) is a primary of a $\overline{\mathcal{B}}_{1 / 2, r+3 / 2(1 / 2,0)}$ multiplet. Since $\overline{\mathcal{B}}_{1 / 2, r+3 / 2(1 / 2,0)}$ multiplets cannot recombine into long multiplets, this multiplet remains at all points on the conformal manifold (unlike generic operators of the form discussed in footnote (6). We therefore arrive at the following statement:

Statement 2: In any rank $N \geq 24 \mathrm{~d} \mathcal{N}=2$ SCFT with an exactly marginal gauge coupling, there is at least one $r$ such that $\overline{\mathcal{B}}_{1 / 2, r+3 / 2(1 / 2,0)}$ is in the spectrum.

Next let us consider rank $N \geq 2$ SCFTs that are not necessarily part of a conformal manifold. For simplicity, let us assume that these theories have an $N \geq 2$ dimensional Coulomb branch that is freely generated.

Then, in the free IR theory we flow to by turning on a vev for a chiral operator in the UV, we will encounter operators similar to those in (3.11). For example, we have

$$
\begin{equation*}
\mathcal{O}_{3, i, j \alpha}:=\phi_{i}\left(\phi_{i} Q_{\alpha}^{1} \phi_{j}-Q_{\alpha}^{1}\left(\phi_{i}\right) \phi_{j}\right) \in \overline{\mathcal{B}}_{1 / 2,5 / 2(1 / 2,0)}, \quad i \neq j \tag{3.12}
\end{equation*}
$$

where $i, j=1, \cdots, N$ denote the particular free vector component in the $U(1)^{N}$ free superMaxwell theory present at generic points on the Coulomb branch. For $N \geq 2$ such $\overline{\mathcal{B}}$ multiplets clearly exist, and this logic therefore explains the comment on higher-rank Coulomb branches below (2.10).

[^9]What can the multiplets in (3.12) descend from in the UV? Since superconformal recombination of $\overline{\mathcal{B}}_{1 / 2, r(1 / 2,0)}$ multiplets is forbidden, it is tempting to argue that the multiplets in (3.12) come from $\overline{\mathcal{B}}_{1 / 2, r(1 / 2,0)}$ multiplets in the UV theory. However, we should be careful: in the flow back up to the UV we break superconformal (and $\left.U(1)_{r}\right)$ symmetry. Therefore, we should understand whether these multiplets can sit inside larger non-conformal multiplets.

To that end, let $\left\{\lambda_{a}\right\}$ denote the (generally infinite) collection of irrelevant couplings in the deformed IR theory that flows back up to the UV theory in question. Since supersymmetry and $S U(2)_{R}$ symmetry are both preserved, these couplings should sit as superconformal primaries in background / spurion multiplets with quantum numbers

$$
\begin{equation*}
R\left(\lambda_{a}\right)=j\left(\lambda_{a}\right)=\bar{j}\left(\lambda_{a}\right)=0 \tag{3.13}
\end{equation*}
$$

If $\mathcal{O}_{3, i, j, \alpha}$ becomes part of a longer non-conformal representation but remains chiral and a (SUSY) primary after introducing the $\lambda_{a}$, then, in the UV theory, we expect a corresponding $\overline{\mathcal{B}}_{1 / 2, r(1 / 2,0)}$ multiplet when superconformal symmetry re-emerges. However, this scenario is incompatible with spontaneous superconformal symmetry breaking.

Instead, let us suppose that $\mathcal{O}_{3, i, j, \alpha}$ is not a SUSY primary or is no longer chiral after turning on the $\lambda_{a}$. To that end, first suppose $\mathcal{O}_{3, i, j, \alpha}$ is a SUSY primary satisfying

$$
\begin{equation*}
\bar{Q}_{\dot{\alpha}}^{1} \mathcal{O}_{3, i, j, \alpha}=\lambda_{a} \mathcal{O}_{3, i, j, \alpha \dot{\alpha}} \neq 0, \quad\left(\bar{Q}^{1}\right)^{2} \mathcal{O}_{3, i, j, \alpha}=0 \tag{3.14}
\end{equation*}
$$

In the first equation in (3.14) we could have considered a more general linear combination of operators multiplying different polynomials (series) in the $\lambda_{a}$. For simplicity, we have written a single such term with a single power of a coupling (but our arguments can be generalized straightforwardly to the most general case).

In the UV theory, the superconformal primary $\mathcal{O}_{3, i, j, \alpha}$ satisfies the shortening condition in (3.14). It is therefore in either a $\overline{\mathcal{B}}_{1 / 2, r(1 / 2,0)}$ or a $\overline{\mathcal{C}}_{1 / 2, r(1 / 2,0)}$ multiplet (note that a $\overline{\mathcal{D}}_{1 / 2,(1 / 2,0)}$ multiplet clearly contains too few degrees of freedom relative to the IR). We claim neither possibility is consistent.

To find the contradiction, let us first suppose that $\mathcal{O}_{3, i, j, \alpha \dot{\alpha}}$ is an IR superconformal primary. Then it is a primary of an $\operatorname{IR} \overline{\mathcal{C}}_{1, r^{\prime}(1 / 2,1 / 2)}$ or $\hat{\mathcal{C}}_{1(1 / 2,1 / 2)}$ multiplet. However, such multiplets cannot sit inside a UV $\overline{\mathcal{B}}_{1 / 2, r(1 / 2,0)}$ or $\overline{\mathcal{C}}_{1 / 2, r(1 / 2,0)}$ representation (they have more degrees of freedom). On the other hand, suppose that $\mathcal{O}_{3, i, j, \alpha \dot{\alpha}}$ is an IR superconformal descendant. Then it is a $\bar{Q}_{\dot{\alpha}}^{1}$ descendant, but this contradicts (3.14).

Next, suppose that we have

$$
\begin{equation*}
\bar{Q}_{\dot{\alpha}}^{1} \mathcal{O}_{3, i, j, \alpha}=\lambda_{a} \mathcal{O}_{3, i, j, \alpha \dot{\alpha}} \neq 0, \quad\left(\bar{Q}^{1}\right)^{2} \mathcal{O}_{3, i, j, \alpha} \neq 0 \tag{3.15}
\end{equation*}
$$

In this case, $\mathcal{O}_{3, i, j, \alpha \dot{\alpha}}$ is highest $S U(2)_{R}$ weight and satisfies

$$
\begin{equation*}
\left(\bar{Q}^{1}\right)^{2} \mathcal{O}_{3, i, j, \alpha \dot{\alpha}}=0 \tag{3.16}
\end{equation*}
$$

Let us now understand where $\mathcal{O}_{3, i, j, \alpha \dot{\alpha}}$ can sit in the IR superconformal representation theory. If $\mathcal{O}_{3, i, j, \alpha \dot{\alpha}}$ is a superconformal primary, the shortening condition in (3.16) is inconsistent unless $\mathcal{O}_{3, i, j, \alpha \dot{\alpha}}$ is in a $\overline{\mathcal{C}}_{1, r^{\prime}(1 / 2,1 / 2)}$ or $\hat{\mathcal{C}}_{1(1 / 2,1 / 2)}$ multiplet. However, (3.15) has at most the number of degrees of freedom of a long multiplet of type $\mathcal{A}_{1 / 2, r(1 / 2,0)}^{\Delta}$ and therefore has fewer degrees of freedom than a $\overline{\mathcal{C}}_{1, r^{\prime}(1 / 2,1 / 2)}$ multiplet. At the same time, it has sixteen more degrees of freedom than the direct sum $\hat{\mathcal{C}}_{1(1 / 2,1 / 2)} \oplus \overline{\mathcal{B}}_{1 / 2,5 / 2(1 / 2,0)}$; however, we also require in (3.15) that $\bar{Q}^{1 \dot{\alpha}} \mathcal{O}_{3, i, j, \alpha \dot{\alpha}} \neq 0$. This implies an additional multiplet with $R=3 / 2$, and $\mathcal{A}_{1 / 2, r(1 / 2,0)}^{\Delta}$ cannot accommodate the additional degrees of freedom.

Finally, if $\mathcal{O}_{3, i, j, \alpha \dot{\alpha}}$ is a superconformal descendant, then it must be a superconformal $\bar{Q}_{\dot{\alpha}}^{1}$ descendant, but this statement contradicts (3.15). Therefore, (3.15) is inconsistent.

Next let us consider the case that

$$
\begin{equation*}
\bar{Q}_{\dot{\alpha}}^{1} \mathcal{O}_{3, i, j, \alpha}=0 \tag{3.17}
\end{equation*}
$$

even after the irrelevant deformation. Then, to try to avoid a potential $\overline{\mathcal{B}}_{1 / 2, r(1 / 2,0)}$ multiplet in the UV, let us suppose $\mathcal{O}_{3, i, j, \alpha}$ is now a descendant

$$
\begin{equation*}
\bar{Q}^{1 \dot{\alpha}} \tilde{\mathcal{O}}_{3, i, j, \alpha \dot{\alpha}}=\lambda_{b} \mathcal{O}_{3, i, j, \alpha} \Rightarrow\left(\bar{Q}^{1}\right)^{2} \tilde{\mathcal{O}}_{3, i, j, \alpha \dot{\alpha}}=0 . \tag{3.18}
\end{equation*}
$$

Let us first imagine that $\tilde{\mathcal{O}}_{3, i, j, \alpha \dot{\alpha}}$ is a SUSY primary after the irrelevant deformation. Then, in the UV it satisfies the shortening condition in (3.18). UV superconformal invariance implies that $\tilde{\mathcal{O}}_{3, i, j, \alpha \dot{\alpha}}$ is a $\overline{\mathcal{C}}$ (or $\hat{\mathcal{C}}$ ) superconformal primary. However, (3.18) contradicts the spontaneous breaking of $U(1)_{r}$ in the flow to the IR.

Let us suppose instead that $\tilde{\mathcal{O}}_{3, i, j, \alpha \dot{\alpha}}$ is a SUSY descendant. Then, it is a $\bar{Q}_{\dot{\alpha}}^{1}$ descendant, and we require that

$$
\begin{equation*}
\left(\bar{Q}^{1}\right)^{2} \tilde{\mathcal{O}}_{\alpha}=\lambda_{b} \mathcal{O}_{3, i, j, \alpha} . \tag{3.19}
\end{equation*}
$$

Note that in the IR SCFT, $\tilde{\mathcal{O}}_{\alpha}$ cannot be a descendant since then it is a $\bar{Q}_{\dot{\alpha}}^{1}$ descendant, and (3.19) cannot hold. Suppose $\tilde{\mathcal{O}}_{\alpha}$ is an IR superconformal primary. Then, in the IR SCFT, it must be in a $\overline{\mathcal{D}}, \overline{\mathcal{B}}, \hat{\mathcal{C}}$, or $\overline{\mathcal{C}}$ multiplet with $R \geq 1 / 2$. From (3.19), we see that $\tilde{\mathcal{O}}_{\alpha}$ would be a member of such a multiplet with smaller scaling dimension than the operators in (3.12). More precisely, the irrelevant couplings have non-positive mass dimension and so in the IR SCFT $\Delta\left(\tilde{\mathcal{O}}_{\alpha}\right) \leq \Delta\left(\mathcal{O}_{3, i, j, \alpha}\right)-1=5 / 2$. Then, the only possibility is that $\tilde{\mathcal{O}}_{\alpha}$ is an IR $\overline{\mathcal{D}}_{1 / 2(1 / 2,0)}$ primary. However, there are always fewer such operators than $\overline{\mathcal{B}}$ operators of
the type described in (3.12) as long as the Coulomb branch is genuine (i.e., just consisting of $U(1)^{N}$ super-Maxwell theory at generic points). More generally, as long as the Coulomb branch is freely generated, we can repeat the analysis starting around (3.14) for $\overline{\mathcal{D}}_{1 / 2(1 / 2,0)}$ to arrive at

Statement 3: In any rank $N \geq 24 \mathrm{~d} \mathcal{N}=2$ SCFT with an $N$-dimensional freely generated Coulomb branch, there is at least one $r$ such that $\overline{\mathcal{B}}_{1 / 2, r(1 / 2,0)}$ is in the spectrum.

In particular, this result implies that $\overline{\mathcal{B}}_{1 / 2, r(1 / 2,0)}$ multiplets are ubiquitous: most known higher-rank $\mathcal{N}=2$ SCFTs have freely generated Coulomb branches.

## 3.3. $\overline{\mathcal{B}}$ multiplets and the Witten anomaly

In theories with an $S p(n)$ global flavor symmetry (here $S p(1) \cong S U(2)$ ), we may find a $\mathbb{Z}_{2}$-valued 't Hooft anomaly arising from large (background) gauge transformations associated with $\pi_{4}(S p(N)) \cong \mathbb{Z}_{2}$ [8]. We will argue that, under fairly lax assumptions, any theory possessing such an anomaly has a $\overline{\mathcal{B}}_{1, r(0,0)}$ multiplet (here $r$ is the $U(1)_{r}$ charge of a generator of the Coulomb branch chiral ring).

To understand this statement, we note that these anomalies are invariants of $S p(n)$ preserving RG flows. Since Coulomb branch operators are necessarily uncharged under flavor symmetries [17], RG flows onto the Coulomb branch triggered by turning on vevs for Coulomb branch chiral primaries preserve $S p(n)$ flavor symmetry.

Therefore, let us assume that the theory has a Coulomb branch, and let us study flows onto this space. To get a handle on the possibilities in the IR, note that the arguments in [18] show the $\mathbb{Z}_{2}$-valued anomaly cannot be saturated by a TQFT (see [19] for another application of this fact). Therefore, on the Coulomb branch, we require massless degrees of freedom that match the UV $S p(n)$ anomaly.

A simple example is the $S U(2) \mathcal{N}=4$ SYM theory. This theory has, from the $\mathcal{N}=2$ perspective, an $S p(1)$ flavor symmetry under which the components of the adjoint hypermultiplet, $\left(Q^{a}, \tilde{Q}^{a}\right)$ transform as doublets. ${ }^{23}$ Since $a=1,2,3$, we have an odd number of doublets and hence a $\mathbb{Z}_{2}$ anomaly. Now, consider the $S p(1)$-preserving RG flow gotten by turning on a vev for the vector multiplet scalars. This vev results in an IR theory which is just $U(1) \mathcal{N}=4 \mathrm{SYM}$. The abelian effective theory has a single doublet, $(q, \tilde{q})$, which realizes an $S p(1)$ symmetry and therefore also gives rise to a $\mathbb{Z}_{2}$ anomaly.

[^10]A more elaborate example involves the rank-one theory with $S p(5)$ symmetry discovered in [20]. There we can also turn on an expectation value for the Coulomb branch generator and flow to a Coulomb branch which has five hypermultiplets at generic points. These fields realize the $S p(5)$ symmetry and exhibit the $\mathbb{Z}_{2}$ anomaly as well (the hypermultiplets form a single 10 representation of $S p(5)$ ).

Now, let us suppose we flow onto the Coulomb branch of a theory exhibiting the $\mathbb{Z}_{2}$ 't Hooft anomaly by spontaneously breaking superconformal symmetry via a vev for a $\overline{\mathcal{E}}$ primary. Since no IR TQFT saturates the anomaly, we have a decoupled massless sector furnishing an $S p(n)$ holomorphic moment map, $\mu$ (we assume the flavor symmetry is locally realized). Therefore, in conjunction with the Coulomb branch operator, $\phi^{2}$, we can construct the normal-ordered product

$$
\begin{equation*}
\mathcal{O}_{W}:=\phi^{2} \mu \in \overline{\mathcal{B}}_{1,2(0,0)}, \tag{3.20}
\end{equation*}
$$

which is clearly a superconformal primary of the correct type (recall that $\mu$ has $R=1$ and $r=0)$. Note that this multiplet transforms in the adjoint of $S p(n)$. For technical reasons that will become apparent later, let us assume that the IR theory is completely free (i.e., it consists of free vectors and hypers).

Next, suppose we deform the theory and flow back up to the UV. What can (3.20) come from in the UV? We repeat the logic beginning around (3.14). In particular, to avoid a $\overline{\mathcal{B}}_{1, r(0,0)}$ multiplet in the UV, we need to have that (3.20) becomes a SUSY descendant or is no longer chiral after turning on some (generally infinite) irrelevant couplings, $\left\{\lambda_{a}\right\}$, and flowing back to the UV.

To that end, first suppose $\mathcal{O}_{W}$ is a SUSY primary satisfying ${ }^{24}$

$$
\begin{equation*}
\bar{Q}_{\dot{\alpha}}^{1} \mathcal{O}_{W}=\lambda_{a} \mathcal{O}_{W \dot{\alpha}} \neq 0, \quad\left(\bar{Q}^{1}\right)^{2} \mathcal{O}_{W}=0 \tag{3.21}
\end{equation*}
$$

At short distances, the superconformal primary, $\mathcal{O}_{W}$, satisfies the shortening condition in (3.21) and is therefore in either a $\overline{\mathcal{B}}_{1, r(0,0)}$ or a $\overline{\mathcal{C}}_{1, r(0,0)}$ multiplet (note that a $\overline{\mathcal{D}}_{1,(0,0)}$ multiplet clearly contains too few degrees of freedom relative to the IR). As in the related proof of statement 3 , neither possibility is consistent.

Indeed, let us first suppose that $\mathcal{O}_{W \dot{\alpha}}$ is an IR superconformal primary. Then it is a primary of an $\operatorname{IR} \overline{\mathcal{C}}_{3 / 2, r^{\prime}(0,1 / 2)}$ or $\hat{\mathcal{C}}_{3 / 2(0,1 / 2)}$ multiplet. However, such multiplets cannot sit inside a UV $\overline{\mathcal{B}}_{1, r(0,0)}$ or $\overline{\mathcal{C}}_{1, r(0,0)}$ representation (they have more degrees of freedom). On

[^11]the other hand, suppose that $\mathcal{O}_{W \dot{\alpha}}$ is an IR superconformal descendant. Then it is a $\bar{Q}_{\dot{\alpha}}^{1}$ descendant, but this contradicts (3.21).

Next, suppose that we have

$$
\begin{equation*}
\bar{Q}_{\dot{\alpha}}^{1} \mathcal{O}_{W}=\lambda_{a} \mathcal{O}_{W \dot{\alpha}} \neq 0, \quad\left(\bar{Q}^{1}\right)^{2} \mathcal{O}_{W} \neq 0 \tag{3.22}
\end{equation*}
$$

In this case, $\mathcal{O}_{W \dot{\alpha}}$ is highest $S U(2)_{R}$ weight and satisfies

$$
\begin{equation*}
\left(\bar{Q}^{1}\right)^{2} \mathcal{O}_{W \dot{\alpha}}=0 \tag{3.23}
\end{equation*}
$$

Let us now understand where $\mathcal{O}_{W \dot{\alpha}}$ can sit in the IR superconformal representation theory. If $\mathcal{O}_{W \dot{\alpha}}$ is a superconformal primary, the shortening condition in (3.23) is inconsistent unless $\mathcal{O}_{W \dot{\alpha}}$ is in a $\overline{\mathcal{C}}_{3 / 2, r^{\prime}(0,1 / 2)}$ or $\hat{\mathcal{C}}_{3 / 2(0,1 / 2)}$ multiplet. However, (3.22) has at most the number of degrees of freedom of a long multiplet of type $\mathcal{A}_{1, r(0,0)}^{\Delta}$ and therefore has fewer degrees of freedom than a $\overline{\mathcal{C}}_{3 / 2, r^{\prime}(0,1 / 2)}$ multiplet or a $\hat{\mathcal{C}}_{3 / 2(0,1 / 2)}$ multiplet.

Finally, if $\mathcal{O}_{W \dot{\alpha}}$ is a superconformal descendant, then it must be a superconformal $\bar{Q}_{\dot{\alpha}}^{1}$ descendant, but this statement contradicts (3.22). Therefore, (3.22) is inconsistent.

Next let us consider the case that

$$
\begin{equation*}
\bar{Q}_{\dot{\alpha}}^{1} \mathcal{O}_{W}=0 \tag{3.24}
\end{equation*}
$$

even after the irrelevant deformation. Then, to try to avoid a potential $\overline{\mathcal{B}}_{1, r(0,0)}$ multiplet in the UV, let us suppose $\mathcal{O}_{W}$ is now a descendant

$$
\begin{equation*}
\bar{Q}^{1 \dot{\alpha}} \tilde{\mathcal{O}}_{W \dot{\alpha}}=\lambda_{b} \mathcal{O}_{W} \Rightarrow\left(\bar{Q}^{1}\right)^{2} \tilde{\mathcal{O}}_{W \dot{\alpha}}=0 \tag{3.25}
\end{equation*}
$$

Let us first imagine that $\tilde{\mathcal{O}}_{W \dot{\alpha}}$ is a SUSY primary after the irrelevant deformation. Then, in the UV it satisfies the shortening condition in (3.25). UV superconformal invariance implies that $\tilde{\mathcal{O}}_{W \dot{\alpha}}$ is a $\overline{\mathcal{C}}$ (or $\hat{\mathcal{C}}$ ) superconformal primary. However, (3.25) contradicts the spontaneous breaking of $U(1)_{r}$ in the flow to the IR.

Let us suppose instead that $\tilde{\mathcal{O}}_{W \dot{\alpha}}$ is a SUSY descendant. Then, it is a $\bar{Q}_{\dot{\alpha}}^{1}$ descendant, and we require that

$$
\begin{equation*}
\left(\bar{Q}^{1}\right)^{2} \tilde{\mathcal{O}}_{W}=\lambda_{b} \mathcal{O}_{W} \tag{3.26}
\end{equation*}
$$

Note that in the IR SCFT, $\tilde{\mathcal{O}}_{W}$ cannot be a descendant since then it is a $\bar{Q}_{\dot{\alpha}}^{1}$ descendant, and (3.26) cannot hold. Suppose $\tilde{\mathcal{O}}_{W}$ is an IR superconformal primary. Then, in the IR SCFT, it must be in a $\overline{\mathcal{D}}, \overline{\mathcal{B}}, \hat{\mathcal{C}}$, or $\overline{\mathcal{C}}$ multiplet with $R \geq 0$. From (3.26), we see that $\tilde{\mathcal{O}}_{W}$ would be a member of such a multiplet with smaller scaling dimension than the operators in (3.20). More precisely, the irrelevant couplings have non-positive mass dimension and
so in the IR SCFT $\Delta\left(\tilde{\mathcal{O}}_{W}\right) \leq \Delta\left(\mathcal{O}_{W}\right)-1=3$. Then, the only possibilities are that $\tilde{\mathcal{O}}_{W}$ is an $\operatorname{IR} \overline{\mathcal{D}}_{1(0,0)}, \overline{\mathcal{B}}_{1 / 2,2(0,0)}, \hat{\mathcal{C}}_{1 / 2(0,0)}$, or a $\overline{\mathcal{C}}_{0,1(0,0)}$ primary. From (3.20), it is clear that this operator must transform in the adjoint of $S p(n)$.

Let us suppose that we can go to a point on the Coulomb branch where the theory is completely free (this includes free hypermultiplets). Then, we see that to get an $S p(n)$ adjoint primary, we require two hypermultiplet scalars. This logic leaves a single free vector scalar for us to adjoin to get an operator of dimension three. As a result, we can immediately rule out the $\overline{\mathcal{B}}, \hat{\mathcal{C}}$, and $\overline{\mathcal{C}}$ options 25

We are left with the $\overline{\mathcal{D}}$ option. For higher-rank Coulomb branches, we can always build more $\overline{\mathcal{B}}$ operators of type (3.20), but in the rank-one case we cannot. Instead, for rank one, we can repeat the logic around (3.21) for $\phi \mu \in \overline{\mathcal{D}}_{1(0,0)}$ rather than $\phi^{2} \mu \in \overline{\mathcal{B}}_{1,2(0,0)}$. Therefore, we arrive at

Statement 4: Consider a $4 \mathrm{~d} \mathcal{N}=2$ SCFT with an $S p(n)$ flavor symmetry having a non-vanishing $\mathbb{Z}_{2}$-valued anomaly. If the theory possesses points on the Coulomb branch consisting purely of free fields, then it has a $\overline{\mathcal{B}}_{1, r(0,0)}$ multiplet, where $r$ is the $U(1)_{r}$ charge of a Coulomb branch generator. This multiplet transforms in the adjoint of $S p(n){ }^{26}$

### 3.4. Free theories and a conjecture on general $4 d \mathcal{N}=2$ SCFTs

In this section, we would like to prove a few statements about the spectrum of $\overline{\mathcal{B}}$ multiplets in free $4 \mathrm{~d} \mathcal{N}=2$ SCFTs and then use these statements to conjecture a general constraint on $4 \mathrm{~d} \mathcal{N}=2$ SCFTs.

To that end, we begin with the following observation:
Fact 1: In a theory of $N$ free vectors and $M$ free hypermultiplets, all $\overline{\mathcal{B}}_{R, r(j, 0)}$ and $\overline{\mathcal{D}}_{R(j, 0)}$ multiplets satisfy $j \leq R$.

To understand this statement, note that a free vector and a free hypermultiplet have the chiral fields listed in table 1. Let us construct highest $S U(2)_{R}$ and Lorentz-weight primaries of $\overline{\mathcal{B}}$ and $\overline{\mathcal{D}}$ multiplets in a theory of $N$ free vectors and $M$ free hypers. The only sources of $j$ are the gauginos, $\lambda_{i, \alpha}^{1}$ (in particular, derivatives would lead to operators that are trivial in

[^12]|  | $R$ | $r$ | $j$ | $E$ | $\delta$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\phi_{i}$ | 0 | 1 | 0 | 1 | 0 |
| $\lambda_{i, \alpha}^{1}$ | $1 / 2$ | $1 / 2$ | $\pm 1 / 2$ | $3 / 2$ | 0 |
| $q_{a}$ | $1 / 2$ | 0 | 0 | 1 | 0 |
| $\tilde{q}_{a}$ | $1 / 2$ | 0 | 0 | 1 | 0 |

Table 1: List of chiral fields in a theory with $N$ abelian free vector multiplets ( $i=1, \cdots, N$ ) and $M$ free hypermultiplets $(a=1, \cdots, M)$.
the chiral ring and hence not highest-weight primaries of $\overline{\mathcal{B}}$ or $\overline{\mathcal{D}}$ multiplets). Since $j \leq R$ for the gaugino (and all other chiral fields), any word constructed out of letters in table 1 has $j \leq R$.

While the above argument provides a bound on spin versus $S U(2)_{R}$ quantum numbers, it is natural to ask if we can realize all of the above multiplets with $j \leq R$. Indeed, this is the case:

Fact 2: In a theory of $N=2 n$ free vectors and $M \geq 1$ free hypermultiplets, there exist values of $r$ such that we have at least one $\overline{\mathcal{B}}_{R, r(j, 0)}$ and one $\overline{\mathcal{D}}_{R(j, 0)}$ multiplet for all $R \leq(N-1) / 2$ and $j \leq R$.

To derive this set of facts, note that

$$
\begin{equation*}
\mathcal{O}_{u, v}=q^{u} \phi_{1}^{v} \tag{3.27}
\end{equation*}
$$

is a highest-weight primary of a $\overline{\mathcal{B}}_{u / 2, v(0,0)}$ multiplet if $v>1$ and a $\overline{\mathcal{D}}_{u / 2(0,0)}$ multiplet if $v=1$ (clearly the above component fields are superconformal primaries and therefore so too is the product).

Next, let us construct $\overline{\mathcal{D}}_{(2 m-1) / 2((2 m-1) / 2,0)}$ and $\overline{\mathcal{D}}_{m-1(m-1,0)}$ multiplets. To that end, note that

$$
\begin{align*}
\mathcal{O}_{(1,2 m),+{ }^{2 m-1}} & :=\sum_{i=1}^{2 m}(-1)^{i} \phi_{i} \prod_{j=1, j \neq i}^{2 m} \lambda_{j,+}^{1}, \\
\mathcal{O}_{(1,2 m-1),+2 m-2} & :=\sum_{i=1}^{2 m-1}(-1)^{i} \phi_{i} \prod_{j=1, j \neq i}^{2 m-1} \lambda_{j,+}^{1}, \tag{3.28}
\end{align*}
$$

are highest-weight primaries of a $\overline{\mathcal{D}}_{(2 m-1) / 2,((2 m-1) / 2,0)}$ multiplet and a $\overline{\mathcal{D}}_{m-1,(m-1,0)}$ multiplet respectively. Indeed, $r=1+j$ by construction, and the above operators are annihilated by
all $S$ and $\bar{S}$ supercharges. Now consider the operators

$$
\begin{equation*}
q^{u} \mathcal{O}_{(1,2 m),+^{2 m-1}} \in \overline{\mathcal{D}}_{(2 m-1+u) / 2((2 m-1) / 2,0)}, \quad q^{u} \mathcal{O}_{(1,2 m-1),+2 m-2} \in \overline{\mathcal{D}}_{m-1+u / 2(m-1,0)} \tag{3.29}
\end{equation*}
$$

This operator is a primary of a $\overline{\mathcal{D}}$ multiplet since $q$ is a superconformal primary with $r=j=0$. Allowing $u$ to be arbitrary, we have proven our claim for $\overline{\mathcal{D}}$ multiplets.

To arrive at the claim for $\overline{\mathcal{B}}$ multiplets, we can simply take the above operators and multiply by $\phi_{i}$

$$
\begin{align*}
\phi_{i} q^{u} \mathcal{O}_{(1,2 m),+^{2 m-1}} & \in \overline{\mathcal{B}}_{(2 m-1+u) / 2,(2 m+3) / 2((2 m-1) / 2,0)}, \\
\phi_{i} q^{u} \mathcal{O}_{(1,2 m-1),+^{2 m-2}} & \in \overline{\mathcal{B}}_{m-1+u / 2, m+1(m-1,0)} \tag{3.30}
\end{align*}
$$

Indeed, this statement follows from the fact that $\phi_{i}$ is a superconformal primary and has $r=1$ and $j=0$.

This logic also implies the following fact:
Fact 3: For any $j \leq R$, there exist values of $r$ and a local unitary $4 \mathrm{~d} \mathcal{N}=2 \mathrm{SCFT}, \mathcal{T}$, such that $\overline{\mathcal{B}}_{R, r(j, 0)}$ and $\overline{\mathcal{D}}_{R(j, 0)}$ is in the spectrum of $\mathcal{T}$.

If we consider the full set of free theories, it is reasonable to imagine that we see all possible superconformal representations up to deformations of the $U(1)_{r}$ charge. The heuristic reason for this belief is that, in a general theory, we expect interactions to lead to new null states. At the same time, interactions cannot change the $S U(2)_{R}$ and Lorentz-spin quantization (they can only lead to changes in the quantization of $\left.U(1)_{r}\right)$.

Moreover, the general arguments of [21] show that any local 4d $\mathcal{N}=2$ SCFT only has $\overline{\mathcal{D}}_{R(j, 0)}$ multiplets for $j \leq R$. Therefore, we are led to the following conjecture:

Conjecture: $\overline{\mathcal{B}}_{R, r(j, 0)}$ multiplets with $j>R$ are forbidden in general local unitary 4 d $\mathcal{N}=2$ SCFTs ${ }^{27}$

Note that a bound of the above form on $\overline{\mathcal{B}}$ implies the bound on $\overline{\mathcal{D}}$ found in 21]. Indeed, suppose this were not the case. Then, we would have a $\overline{\mathcal{D}}_{R(j, 0)}$ multiplet with $j>R$. Taking the product of the corresponding highest-weight primary with a free vector $\phi$ primary would give a $\overline{\mathcal{B}}$ multiplet with $j>R$. On the other hand, note that the $\overline{\mathcal{D}}$ bound does not imply the $\overline{\mathcal{B}}$ bound since not all $\overline{\mathcal{B}}$ operators need to come from a product of the form $\overline{\mathcal{D}} \times \overline{\mathcal{E}}$.

[^13]Note also that the above bound on $\overline{\mathcal{B}}$ implies a known bound on $\overline{\mathcal{E}}_{r(j, 0)}$ ruling out $j>0$ in these latter multiplets [9]. Indeed, suppose that there were a $j>0$ such that $\overline{\mathcal{E}}_{r(j, 0)}$ existed in some $\operatorname{SCFT}, \mathcal{T}$. Then, we could take $N$ decoupled copies of $\mathcal{T}$ to get $\overline{\mathcal{E}}_{r^{\prime}\left(j^{\prime}, 0\right)}$ with arbitrarily large $j^{\prime}>0$. Therefore, multiplying with a free hypermultiplet would give $\overline{\mathcal{B}}$ violating the conjecture by an arbitrarily large amount.

## 4. $\overline{\mathcal{B}}$ multiplets in Rank-one theories

We have shown that, from relatively minimal assumptions, theories with $\mathcal{N}>2$ SUSY, higher rank and a conformal manifold, a $\mathbb{Z}_{2}$-valued $S p(N)$ anomaly, or with freely generated higher-dimensional Coulomb branches must possess $\overline{\mathcal{B}}$ multiplets. Therefore, in this section, we specialize to rank-one theories that do not satisfy any of these properties (and also have no mixed branches) in order to understand whether any of these theories have $\overline{\mathcal{B}}$ multiplets.

We focus on a subset of such theories that can be described by certain non-conformal $\mathcal{N}=1$ Lagrangians with accidental IR enhancement to $\mathcal{N}=2$ (e.g., see [23]). In this context, a necessary condition for a Lagrangian to be useful in carrying out precision spectroscopy is for the IR superconformal $U(1)_{r}$ and $S U(2)_{R}$ Cartan to be visible in the UV and unbroken along the RG flow. This property is typically absent in $\mathcal{N}=2$ RG flows and hence explains the utility of constructions involving accidental SUSY enhancement.

### 4.1. General Strategy

As described in section 2, the spectrum of all multiplets in the FCS except the $\overline{\mathcal{B}}$ multiplets can easily be determined either from a Seiberg-Witten description (in the case of the $\overline{\mathcal{E}}$ multiplets) or from the associated 2 dVOA (in the case of the $\hat{\mathcal{B}}$ and $\overline{\mathcal{D}}$ multiplets). Therefore, to get a handle on the $\overline{\mathcal{B}}$ spectrum, our strategy will be similar to the one adopted in [5]: we will study rank-one $\mathcal{N}=2$ theories with weakly coupled UV descriptions in terms of an $\mathcal{N}=1$ gauge theory that is connected to our SCFT of interest by a sufficiently "smooth" RG flow (here we require that the superconformal IR symmetry is visible along the RG flow).

More precisely, we will use the fields in these Lagrangian descriptions to write down the full set of gauge-invariant local operators that are candidates to generate the $\mathcal{N}=1$ chiral ring (we assume the RG flow only acts to truncate this ring). This is the set of chiral operators that cannot be expressed as a product of two nontrivial chiral operators. 28
${ }^{28}$ The identity operator is defined as the trivial chiral operator.

We whittle down this list by demanding that these generators sit in certain IR $\mathcal{N}=2$ superconformal representations.

In fact, in all the rank-one theories we will consider, we never find a situation where an operator in this smaller list belongs to a $\overline{\mathcal{B}}$ multiplet. While we do not have a full understanding of when $\overline{\mathcal{B}}$ multiplets do and do not furnish generators of the chiral ring more generally, this observation plays an important role in further results we derive about the spectrum of chiral operators in the particular theories we study.

As a consistency check of our method, we can make contact with known results on the non- $\overline{\mathcal{B}}$ part of the FCS when characterizing our generators. We can then use these operators to exhaustively attempt to construct $\overline{\mathcal{B}}$ operators through chiral ring fusion.

In particular, rank-one theories have a one-complex-dimensional Coulomb branch. We will study examples where the corresponding Coulomb branch chiral ring is freely generated. At the level of superconformal representation theory, this means that we have a single $\overline{\mathcal{E}}_{r}$ generator giving rise to an $\mathcal{N}=2$ chiral ring of primaries via the $n$-fold OPEs (for $n \in \mathbb{Z}_{>0}$ ), $\overline{\mathcal{E}}_{r}^{\times n} \ni \overline{\mathcal{E}}_{r n}$.

Most of the rank-one theories we study also have an $\mathcal{N}=2$ flavor symmetry. As we have seen in previous sections, the Noether currents sit in corresponding $\hat{\mathcal{B}}_{1}$ multiplets transforming in the adjoint of the flavor symmetry. These multiplets are associated with the Higgs branch. For all the theories we consider, the $\hat{\mathcal{B}}_{1}$ multiplets generate the Higgs branch chiral ring.

Moreover, all the examples we study here can be mapped to known 2d VOAs, and we can use these associated VOAs to conclude that there are no $\overline{\mathcal{D}}$ multiplets in our spectra. As a result, the $\overline{\mathcal{B}}$ generation channel in (2.12) is not available (recall that none of the theories we study here have $\mathcal{N}>2$ SUSY and so the result of section 3.1 does not apply).

Therefore, any $\overline{\mathcal{B}}$ multiplets in our theories of interest must be linear combinations of normal-ordered products of chiral operators sitting in $\overline{\mathcal{E}}_{r}$ and $/$ or $\hat{\mathcal{B}}_{1}$ multiplets. We construct all such products allowed by the OPE constraints and chiral ring relations in the theory, and we consider their linear combinations.

Two kinds of $\overline{\mathcal{B}}$ multiplets will be of particular interest in our analysis, so we single them out beforehand. These are obtained from the OPE in (2.11) and the second OPE in (2.14). In the rank-one theories we consider, these $\overline{\mathcal{B}}$ multiplets often vanish as a result of chiral ring relations arising from the dynamics of the $\mathcal{N}=1 \rightarrow \mathcal{N}=2$ RG flows.

We now proceed to our analysis of the $\mathcal{N}=1$ chiral spectra of individual rank-one SCFTs. For simplicity, we stick to isolated theories of Argyres-Douglas type and to the $S U(2)$ theory with four flavors.

| Fields | $S U(2)_{\text {gauge }}$ | $R$ | $r$ |
| :---: | :---: | :---: | :---: |
| $\phi$ | adj | 0 | $1 / 5$ |
| $\lambda_{\alpha}$ | $\operatorname{adj}$ | $1 / 2$ | $1 / 2$ |
| $q$ | 2 | $1 / 2$ | $2 / 5$ |
| $\tilde{q}$ | 2 | $1 / 2$ | $-1 / 5$ |
| $M$ | 1 | 0 | $6 / 5$ |
| $X$ | 1 | 1 | $3 / 5$ |

Table 2: UV fields in the $\mathcal{N}=1$ description of the $\left(A_{1}, A_{2}\right)$ theory [27,28]. Here $r$ and $R$ are the IR $U(1)_{r}$ and $S U(2)_{R}$ Cartan respectively.

### 4.2. The $\left(A_{1}, A_{2}\right) \cong \mathrm{MAD} \cong H_{0} \cong \mathfrak{a}_{0}$ theory

The chiral ring of this theory was analyzed in detail in [5] and so we merely summarize the story here. Recall that this is the original Argyres-Douglas theory 24] (referred to in [5, 15, 25] as the "Minimal" Argyres-Douglas (MAD) theory). It has a Coulomb branch chiral ring generator of dimension $6 / 5$ sitting as a primary in a $\overline{\mathcal{E}}_{6 / 5}$ multiplet. The theory has no Higgs branch, and, consistent with this fact, the associated 2d VOA is the LeeYang Virasoro vacuum module [26]. Arguments in 15 then imply that there are no $\hat{\mathcal{B}}$ or $\overline{\mathcal{D}}$ multiplets in this theory.

The $\mathcal{N}=1 S U(2)$ gauge theory Lagrangian with fields in Table $2{ }^{2} 9$ and superpotential [27, 28]

$$
\begin{equation*}
W=X \phi^{2}+M \phi q^{\prime} q^{\prime}+\phi q q \tag{4.1}
\end{equation*}
$$

was used in [5] to argue that there are no $\overline{\mathcal{B}}$ multiplets and

$$
\begin{equation*}
\operatorname{FCS}_{\left(A_{1}, A_{2}\right)}=\left\langle\mathcal{E}_{6 / 5}\right\rangle, \tag{4.2}
\end{equation*}
$$

where the chiral operators in $\overline{\mathcal{E}}_{6 / 5}$ are located as follows

$$
\begin{equation*}
M \in(0,0)_{0,6 / 5} \xrightarrow{Q_{\alpha}^{1}} \phi \lambda_{\alpha} \in\left(\frac{1}{2}, 0\right)_{1 / 2,7 / 10} \xrightarrow{\left(Q^{1}\right)^{2}} q q^{\prime} \in(0,0)_{1,1 / 5} \tag{4.3}
\end{equation*}
$$

The basic idea of the proof in [5] was to write down all possible chiral ring generators and argue that none can sit in $\overline{\mathcal{B}}$ representations (i.e., generators cannot sit at locations described in (2.10)). Then, dynamical constraints from the superpotential (4.1) rule out chiral ring products of operators in (4.3) giving rise to a $\overline{\mathcal{B}}$ multiplet (crucially, the channel described in (2.14) does not create $\overline{\mathcal{B}}$ chiral operators).

[^14]| Fields | $S U(2)_{\text {gauge }}$ | $R$ | $r$ |
| :---: | :---: | :---: | :---: |
| $\phi$ | adj | 0 | $1 / 3$ |
| $\lambda_{\alpha}$ | adj | $1 / 2$ | $1 / 2$ |
| $q$ | 2 | $1 / 2$ | $-1 / 6$ |
| $\tilde{q}$ | 2 | $1 / 2$ | $-1 / 6$ |
| $\alpha_{0}$ | 1 | 0 | $4 / 3$ |
| $\beta_{2}$ | 1 | 1 | $1 / 3$ |

Table 3: UV fields in the $\mathcal{N}=1$ description of the $\left(A_{1}, A_{3}\right)$ theory [27,32]. Here $r$ and $R$ are the IR $U(1)_{r}$ and $S U(2)_{R}$ Cartan.

### 4.3. The $\left(A_{1}, A_{3}\right) \cong H_{1} \cong \mathfrak{a}_{1}$ theory

This theory was not analyzed in [5]. It is the simplest Argyres-Douglas theory with flavor symmetry [29] $(S O(3)$ for the untwisted Hilbert space in this case [30]). The Coulomb branch chiral ring generator has dimension $4 / 3$ and sits as a primary in an $\overline{\mathcal{E}}_{4 / 3}$ multiplet. Unlike the previous case, the theory has a one-quaternionic-dimensional Higgs branch with the Higgs branch chiral ring generated by the holomorphic $S O(3)$ moment map $\mu^{a} \in \hat{\mathcal{B}}_{1}^{a}$ (here $a= \pm, 0$ indicates $S O(3)$ flavor weight) subject to the relation

$$
\begin{equation*}
\mu^{+} \mu^{-} \sim\left(\mu^{0}\right)^{2} \tag{4.4}
\end{equation*}
$$

This theory also has a known associated VOA [31]: the $\widehat{s u(2)}-4 / 3$ affine Kac-Moody (AKM) algebra. This fact allows us to immediately rule out $\overline{\mathcal{D}}$ multiplets. Indeed, $\widehat{s u(2)}-4 / 3$ is (strongly) generated by the affine current (related, by the map in [4, to $\mu^{a}$ which has $r=0$ ). This means that any operator in the 2 d VOA is built from normal-ordered products of (derivatives) of this current. Since the procedure in [4] used to construct the VOA respects the $U(1)_{r}$ symmetry, we conclude that all Schur operators in the $\left(A_{1}, A_{3}\right)$ theory are $U(1)_{r}$ neutral. Since $\overline{\mathcal{D}}$ Schur operators necessarily have $r \neq 0$, they cannot be present.

Therefore, the FCS can at most consist of $\overline{\mathcal{E}}, \hat{\mathcal{B}}$, and $\overline{\mathcal{B}}$ multiplets. To get a handle on these latter multiplets, we consider the $\mathcal{N}=1$ Lagrangian with fields given in Table 3 and superpotential [27,32]

$$
\begin{equation*}
W=\alpha_{0} q \tilde{q}+\beta_{2} \phi^{2} \tag{4.5}
\end{equation*}
$$

As in the $\left(A_{1}, A_{2}\right)$ case, the UV theory is an $S U(2) \mathcal{N}=2$ gauge theory. However, this time the superpotential preserves the $S U(2)$ flavor symmetry under which $(q, \tilde{q})$ transforms as a doublet. This fact is crucial in order to reproduce the IR symmetry discussion around (4.4).

| Operator | $R$ | $r$ | $j$ |
| :---: | :---: | :---: | :---: |
| $\alpha_{0}$ | 0 | $4 / 3$ | 0 |
| $\beta_{2}$ | 1 | $1 / 3$ | 0 |
| $\phi^{2}$ | 0 | $2 / 3$ | 0 |
| $\lambda^{2}$ | 1 | 1 | 0 |
| $q \tilde{q}$ | 1 | $-1 / 3$ | 0 |
| $\phi q q$ | 1 | 0 | 0 |
| $\phi q \tilde{q}$ | 1 | 0 | 0 |
| $\phi \tilde{q} \tilde{q}$ | 1 | 0 | 0 |
| $\phi \lambda_{\alpha}$ | $1 / 2$ | $5 / 6$ | $1 / 2$ |
| $q q \lambda_{\alpha}$ | $3 / 2$ | $1 / 6$ | $1 / 2$ |
| $q \tilde{q} \lambda_{\alpha}$ | $3 / 2$ | $1 / 6$ | $1 / 2$ |
| $\tilde{q} \tilde{q} \lambda_{\alpha}$ | $3 / 2$ | $1 / 6$ | $1 / 2$ |
| $\phi q q \lambda_{\alpha}$ | $3 / 2$ | $1 / 2$ | $1 / 2$ |
| $\phi q \tilde{q} \lambda_{\alpha}$ | $3 / 2$ | $1 / 2$ | $1 / 2$ |
| $\phi \tilde{q} \tilde{q} \lambda_{\alpha}$ | $3 / 2$ | $1 / 2$ | $1 / 2$ |
| $\phi \lambda_{\alpha} \lambda_{\beta}$ | 1 | $4 / 3$ | 1 |
| $q q \lambda_{\alpha} \lambda_{\beta}$ | 2 | $2 / 3$ | 1 |
| $q \tilde{q} \lambda_{\alpha} \lambda_{\beta}$ | 2 | $2 / 3$ | 1 |
| $\tilde{q} \tilde{q} \lambda_{\alpha} \lambda_{\beta}$ | 2 | $2 / 3$ | 1 |
| $\lambda_{\alpha} \lambda_{\beta} \lambda_{\gamma}$ | $3 / 2$ | $3 / 2$ | $3 / 2$ |

Table 4: List of $\mathcal{N}=1$ chiral generator candidates in the $\left(A_{1}, A_{3}\right)$ theory. Here $r$ and $R$ are the IR $U(1)_{r}$ and $S U(2)_{R}$ Cartan, and $j$ is the left spin.

Let us first study the potential chiral ring generators. To that end, we have listed the candidates in Table 4. In compiling this list, we have used the fact that $\delta^{a b} \sim \operatorname{Tr}\left(T^{a} T^{b}\right)$ and $\epsilon^{a b c} \sim \operatorname{Tr}\left(\left[T^{a}, T^{b}\right] T^{c}\right)$ to express any chiral ring generator in terms of at most three adjoints. Note that this does not imply that the chiral ring generators will obey classical relations in the quantum theory 30

We begin by identifying the known $\overline{\mathcal{E}}$ and $\hat{\mathcal{B}}$ generators of the FCS in terms of the operators in Table 4. To that end, note that unitarity bounds imply that the chiral operators in the $\hat{\mathcal{B}}_{1}$ multiplets and the primary and level-one descendants of $\overline{\mathcal{E}}_{4 / 3}$ cannot be composites built out of products of gauge-invariant operators (the level-two descendant of $\overline{\mathcal{E}}_{4 / 3}$ can at most be built out of a product of two gauge invariant operators). Therefore, we can immediately identify

$$
\begin{equation*}
\mu^{+}=\phi q q \in \hat{\mathcal{B}}_{1}^{+}, \quad \mu^{0}=\phi q \tilde{q} \in \hat{\mathcal{B}}_{1}^{0}, \quad \mu^{-}=\phi \tilde{q} \tilde{q} \in \hat{\mathcal{B}}_{1}^{-} . \tag{4.6}
\end{equation*}
$$

For the $\overline{\mathcal{E}}_{4 / 3}$ multiplet we have

$$
\begin{equation*}
\alpha_{0} \in(0,0)_{0,4 / 3} \xrightarrow{Q_{\alpha}^{1}} \phi \lambda_{\alpha} \in\left(\frac{1}{2}, 0\right)_{1 / 2,5 / 6} \xrightarrow{\left(Q^{1}\right)^{2}} \beta_{2} \in(0,0)_{1,1 / 3} . \tag{4.7}
\end{equation*}
$$

We have mapped $\beta_{2}$ to the level-two descendant of $\overline{\mathcal{E}}_{4 / 3}$ using the fact that there is no other candidate built from a single generator in Table 4 or a product of two such generators that has the correct quantum numbers (by construction, $\phi^{2}$ decouples from the IR chiral ring).

What about $\overline{\mathcal{B}}$ generators? Our analysis so far implies that any chiral operator, $\mathcal{O}$, that is not in an $\overline{\mathcal{E}}$ or $\hat{\mathcal{B}}$ multiplet must be in a $\overline{\mathcal{B}}$ multiplet. Therefore, $\mathcal{N}=2$ superconformal representation theory implies that $\mathcal{O}$ must have $r>j$ (and $r>1+j$ to be a superconformal primary in such a multiplet; see (2.10)).

We only have two fields in the list of candidate chiral generators that could potential sit as $\overline{\mathcal{B}}$ descendants, namely $\lambda^{2}$ and $\phi \lambda_{\alpha} \lambda_{\beta}$. Moreover, there are no candidates for $\overline{\mathcal{B}}$ primaries among the list in Table 4.

Now, if $\lambda^{2}$ is a level-one descendant, then the primary has $R=1 / 2, r=3 / 2, j=1 / 2$. But this is the superconformal primary of a $\overline{\mathcal{D}}_{1 / 2(1 / 2,0)}$ multiplet, which we know is absent. If $\lambda^{2}$ is a level 2 descendant, then the primary has $R=0, r=2, j=0$, which is the superconformal primary of a $\overline{\mathcal{E}}_{2}$ multiplet. We know that such a multiplet does not exist in the $\left(A_{1}, A_{3}\right)$ theory. Therefore, $\lambda^{2}$ must be trivial in the IR chiral ring.

Let us perform the same analysis for $\phi \lambda_{\alpha} \lambda_{\beta}$. If this is a level-two descendant, the primary has $R=0$ and $r=7 / 3$, which corresponds to an $\overline{\mathcal{E}}$ multiplet. However, no such

[^15]multiplet exists in the theory. If this is a level-one descendant, then the primary has $R=1 / 2, r=11 / 6, j=1 / 2$ (it cannot have $j=3 / 2$, since we would then require that $r>5 / 2)$. However, there is no product of generators in Table 4 that has these quantum numbers. Therefore, $\phi \lambda_{\alpha} \lambda_{\beta}$ must be trivial in the IR FCS.

As a result, we see that we have the following characterization of the generators of the FCS and the FCS itself

$$
\begin{equation*}
\operatorname{FCS}_{\left(A_{1}, A_{3}\right)}=\left\langle\overline{\mathcal{E}}_{4 / 3}, \hat{\mathcal{B}}_{1}^{a}\right\rangle / I \tag{4.8}
\end{equation*}
$$

where $I$ is the ideal generated by the constraint in (4.4). We would like to understand if these generators can give rise to a $\overline{\mathcal{B}}$ multiplet

$$
\begin{equation*}
\mathrm{FCS}_{\left(A_{1}, A_{3}\right)} \stackrel{?}{\ni} \overline{\mathcal{B}} . \tag{4.9}
\end{equation*}
$$

In particular, we can try to form $\overline{\mathcal{B}}$ multiplets by taking products of operators in (4.6) and (4.7). At the quadratic level, there are several possibilities, that we now proceed to study.

- $\alpha_{0}^{2}$ : This is the superconformal primary of the $\overline{\mathcal{E}}_{8 / 3}$ multiplet. In fact, since the Coulomb branch is freely generated, $\alpha_{0}^{k}$ will be the superconformal primary of the $\overline{\mathcal{E}}_{4 k / 3}$ multiplet for all $k \in \mathbb{N}$.
- $\alpha_{0} \phi \lambda_{\alpha}$ : This is the level one descendant of the $\overline{\mathcal{E}}_{8 / 3}$ multiplet. Similar to the case above, $\alpha_{0}^{k-1} \phi \lambda_{\alpha}$ will be the level-one descendant of the $\overline{\mathcal{E}}_{4 k / 3}$ multiplet for all $k \in \mathbb{N}$.
- $\beta_{2} \phi \lambda_{\alpha}$ : This is ruled out by the superpotential constraint,

$$
\begin{equation*}
\frac{\partial W}{\partial \phi^{a}} \cdot \lambda_{\alpha}^{a}=0 \tag{4.10}
\end{equation*}
$$

This constraint cannot receive quantum corrections because they would involve terms with $R=3 / 2$ and $r=7 / 6$ (recall that $\phi^{2}$ hits a unitarity bound and decouples).

- $\alpha_{0} \beta_{2}$ and $\left(\phi \lambda_{\alpha}\right)^{2}$ : One linear combination of these two operators will be the leveltwo descendant of the $\overline{\mathcal{E}}_{8 / 3}$ multiplet. Can we find a linear combination of these two operators that would be a superconformal primary? We see that the only candidate for the level-one descendant of such a multiplet is $\beta_{2} \phi \lambda_{\alpha}$. However, this has already been set to zero by (4.10). Therefore, no linear combination of these two operators can be a superconformal primary.
- $\beta_{2}^{2}(R=2, r=2 / 3, j=0)$ : Since $r<1+j$, this is not a superconformal primary. If it is a level-one descendant, then the primary has $(R=3 / 2, r=7 / 6, j=1 / 2)$.

The candidate operators with $R=3 / 2$ and $j=1 / 2$ are $\alpha_{0}^{m} \beta_{2} \phi \lambda_{\alpha}=0$ (by (4.10)), $\alpha_{0}^{m}(\phi q q) \phi \lambda_{\alpha}, \alpha_{0}^{m}(\phi q \tilde{q}) \phi \lambda_{\alpha}, \alpha_{0}^{m}(\phi \tilde{q} \tilde{q}) \phi \lambda_{\alpha}$, but none of these can lead to $r=7 / 6$. If it is a level-two descendant, then the primary has $(R=1, r=5 / 3, j=0)$. The candidate operators with $R=1$ and $j=0$ are $\left(\phi \lambda_{\alpha}\right)^{2}$ and $\alpha_{0} \beta_{2}$, but since the levelone descendant of this potential $\overline{\mathcal{B}}$ multiplet vanishes, this is ruled out. Therefore, the operator $\beta_{2}$ must be nilpotent in the IR chiral ring (as implied by the discussion in (33]).

- $\alpha_{0} \phi q q, \alpha_{0} \phi q \tilde{q}$, and $\alpha_{0} \tilde{q} \tilde{q}$ : These are ruled out by the superpotential constraints,

$$
\begin{align*}
& \frac{\partial W}{\partial \tilde{q}^{j}}=\alpha_{0} q_{j}=0 \\
& \frac{\partial W}{\partial q^{j}}=\alpha_{0} \tilde{q}_{j}=0 \tag{4.11}
\end{align*}
$$

Therefore 31

$$
\begin{equation*}
\alpha_{0} q q=\alpha_{0} q \tilde{q}=\alpha_{0} \tilde{q} \tilde{q}=0 . \tag{4.12}
\end{equation*}
$$

These constraints cannot receive quantum corrections because they would involve operators with $R=1$ and $r=4 / 3$ (and having the same $S O(3)$ weight as the operators in question).

- $\left(\phi \lambda_{\alpha}\right)(\phi q q),\left(\phi \lambda_{\alpha}\right)(\phi q \tilde{q})$, and $\left(\phi \lambda_{\alpha}\right)(\phi \tilde{q} \tilde{q})(R=3 / 2, r=5 / 6, j=1 / 2)$ : Since $r<1+j$, this is not a superconformal primary. If it is a level-one descendant then the primary has $(R=1, r=4 / 3, j=0)$. The candidate operators with $R=1$ and $j=0$ are $\alpha_{0}^{m} \phi q q, \alpha_{0}^{m} \phi q \tilde{q}, \alpha_{0}^{m} \phi \tilde{q} \tilde{q}, \alpha_{0}^{m}\left(\phi \lambda_{\alpha}\right)^{2}$, and $\alpha_{0}^{m} \beta_{2}$. Among these operators only $\alpha_{0} \phi q \tilde{q}$ has compatible $r$ charge. However, it is removed by the superpotential constraint in (4.11). If it is a level-two descendant then the primary has $R=1 / 2, r=11 / 6$, $j=1 / 2$. The only candidate with $R=1 / 2$ is $\alpha_{0}^{m} \phi \lambda_{\alpha}$, but this does not have $r=11 / 6$. Therefore, this operator must be trivial in the IR chiral ring.
- $\beta_{2} \phi q q, \beta_{2} \phi q \tilde{q}$, and $\beta_{2} \phi \tilde{q} \tilde{q}$ : These operators vanish in the chiral ring as a result of the superpotential constraints,

$$
\begin{equation*}
\frac{\partial W}{\partial \phi^{a}} \cdot(q q)^{a}=\frac{\partial W}{\partial \phi^{a}} \cdot(q \tilde{q})^{a}=\frac{\partial W}{\partial \phi^{a}} \cdot(\tilde{q} \tilde{q})^{a}=0 . \tag{4.13}
\end{equation*}
$$

These constraints cannot receive quantum corrections because they would involve other operators with $R=2$ and $r=1 / 3$ (and having the same $S O(3)$ weight as the operators in question) since $\phi^{2}$ decouples.

[^16]| Field | Rep | $U(1)_{B}$ | $U(1)_{m}$ | $R$ | $r$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\phi$ | adj | 0 | 0 | 0 | $1 / 2$ |
| $\lambda_{\alpha}$ | adj | 0 | 0 | $1 / 2$ | $1 / 2$ |
| $q_{1}$ | $\square$ | 1 | 3 | $1 / 2$ | $1 / 4$ |
| $\tilde{q}_{1}$ | $\square$ | -1 | -3 | $1 / 2$ | $1 / 4$ |
| $q_{2}$ | $\square$ | 1 | -1 | $1 / 2$ | $-1 / 4$ |
| $\tilde{q}_{2}$ | $\square$ | -1 | 1 | $1 / 2$ | $-1 / 4$ |
| $M_{2}$ | 1 | 0 | 0 | 0 | $\frac{3}{2}$ |
| $\beta_{2}$ | 1 | 0 | 0 | 1 | 0 |

Table 5: UV fields in the $\mathcal{N}=1$ description of the $\left(A_{1}, D_{4}\right)$ theory [34, 35]. Here $r$ and $R$ are the IR $U(1)_{r}$ and $S U(2)_{R}$ Cartan. $U(1)_{B}$ and $U(1)_{m}$ are flavor symmetries corresponding to Cartans of the IR $S U(3)$ flavor symmetry.

We see that the only products that survive at the quadratic level are $\alpha_{0}^{2}, \alpha_{0} \phi \lambda_{\alpha},\left(\phi \lambda_{\alpha}\right)^{2}$, and $\alpha_{0} \beta_{2}$. Of these, the last two cannot be superconformal primaries (nor can any of their linear combinations be).

We therefore see that the most general product of operators from the generating set that we can write down and which is a superconformal primary has the form

$$
\begin{equation*}
\alpha_{0}^{m_{1}}, \quad(\phi q q)^{m_{2}}(\phi q \tilde{q})^{m_{3}}(\phi \tilde{q} \tilde{q})^{m_{4}}, \quad m_{1}, m_{2}, m_{4} \in \mathbb{N}, \quad m_{3}=0,1 . \tag{4.14}
\end{equation*}
$$

These are the Coulomb and Higgs branch operators respectively (recall the constraint in (4.4) that constraints $\left.m_{3}\right)$. Therefore, there are no $\overline{\mathcal{B}}$ multiplets in the $\left(A_{1}, A_{3}\right)$ theory.

### 4.4. The $\left(A_{1}, D_{4}\right) \cong H_{2} \cong \mathfrak{a}_{2}$ theory

This Argyres-Douglas theory has $S U(3)$ flavor symmetry and was originally discovered in [29]. Its Coulomb branch chiral ring generator has dimension $3 / 2$ and is a primary in a $\overline{\mathcal{E}}_{3 / 2}$ multiplet. This theory has a two-quaternionic-dimensional Higgs branch and a corresponding chiral ring generated by the holomorphic $S U(3)$ moment map transforming in the 8 (adjoint) representation, $\mu^{a} \in \hat{\mathcal{B}}_{1}^{a}$ (here we will take $a$ to be an adjoint index) subject to the Joseph ideal constraint.

As in the previous cases, this theory has a known associated 2d VOA [31]: the $\widehat{\operatorname{su}(3)}-3 / 2$ AKM algebra. Using the same logic we used in the case of the $\left(A_{1}, A_{3}\right)$ theory in previous subsection, we can again rule out $\overline{\mathcal{D}}$ multplets here too.

As a result, the FCS can again at most consist of $\overline{\mathcal{E}}, \hat{\mathcal{B}}$, and $\overline{\mathcal{B}}$ multiplets. To understand the spectrum of the $\overline{\mathcal{B}}$ multiplets we study an $\mathcal{N}=1$ Lagrangian theory with fields given
in Table 5 and superpotential [35]

$$
\begin{equation*}
W=M_{2} q_{2} \tilde{q}_{2}+\phi q_{1} \tilde{q}_{1}+\beta_{2} \phi^{2} \tag{4.15}
\end{equation*}
$$

Note that (as in the closely related $\left(A_{1}, A_{3}\right)$ case) the $\phi^{2}$ operator hits a unitarity bound and decouples in the IR. Moreover, only a $S U(2) \times U(1) \subset S U(3)$ flavor symmetry is manifest. Under this symmetry, $\left(q_{2}, \tilde{q}_{2}\right)$ transforms as a doublet (and $q_{1}, \tilde{q}_{1}$ are singlets).

We can construct a list of naive multiplets using exactly the same set of procedures as in the previous subsection. Although the number of fields involved here is larger, we have done this in Tables 6 and 7 .

It will again prove useful to identify the $\overline{\mathcal{E}}_{3 / 2}$ and $\hat{\mathcal{B}}_{1}$ chiral operators. Unitarity implies that the primaries of the $\hat{\mathcal{B}}_{1}$ multiplets and the primary and level-one descendant of the $\overline{\mathcal{E}}_{3 / 2}$ multiplet cannot be composites built out of products of gauge-invariant operators (the level-two descendant of $\overline{\mathcal{E}}_{3 / 2}$ can at most be built out of a product of two gauge invariant operators). Therefore, we can immediately identify the holomorphic moment maps

$$
\begin{equation*}
\mu^{a} \in\left\{\beta_{2}, \tilde{q}_{2} q_{1}, q_{1} q_{2}, \tilde{q}_{1} \tilde{q}_{2}, \tilde{q}_{1} q_{2}, \phi \tilde{q}_{2} \tilde{q}_{2}, \phi \tilde{q}_{2} q_{2}, \phi q_{2} q_{2}\right\} \ni \hat{\mathcal{B}}_{1}^{a} . \tag{4.16}
\end{equation*}
$$

For the $\overline{\mathcal{E}}_{3 / 2}$ multiplet we have

$$
\begin{equation*}
M_{2} \in(0,0)_{0,3 / 2} \quad \xrightarrow{Q_{\alpha}^{1}} \quad \phi \lambda_{\alpha} \in\left(\frac{1}{2}, 0\right)_{1 / 2,1} \xrightarrow{\left(Q^{1}\right)^{2}} \tilde{q}_{1} q_{1} \in(0,0)_{1,1 / 2} \tag{4.17}
\end{equation*}
$$

We have mapped $\tilde{q}_{1} q_{1}$ to the level-two descendant of $\overline{\mathcal{E}}_{3 / 2}$ using the fact that there is no other candidate built from a single generator in Table 4 or a product of two such generators that has the correct superconformal quantum numbers and is $S U(2) \times U(1) \subset S U(3)$ invariant (recall also that, by construction, $\phi^{2}$ decouples from the IR chiral ring).

In what follows, we will make use of the following superpotential constraints,

$$
\begin{align*}
\frac{\partial W}{\partial \beta_{2}} & =\operatorname{Tr}\left(\phi^{2}\right)=0, \\
\frac{\partial W}{\partial M_{2}} & =\operatorname{Tr}\left(\tilde{q}_{2} q_{2}\right)=0, \\
\frac{\partial W}{\partial \tilde{q}_{1}^{a}} & =\left(\phi q_{1}\right)_{a}=0, \\
\frac{\partial W}{\partial q_{1}^{a}} & ={ }_{a}\left(\tilde{q}_{1} \phi\right)=0 . \tag{4.18}
\end{align*}
$$

Let us now proceed to discuss potential $\overline{\mathcal{B}}$ chiral generators of the theory

- Any operators with $r=j$ in Tables 6 and 7 cannot sit in $\overline{\mathcal{B}}$ multiplets (see (2.10)). Since they are no $\overline{\mathcal{D}}$ multiplets, these operators (modulo those in (4.16)) are trivial

| Operators | $R$ | $r$ | $j$ | $U(1)_{B}$ | $U(1)_{m}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $M_{2}$ | 0 | $3 / 2$ | 0 | 0 | 0 |
| $\beta_{2}$ | 1 | 0 | 0 | 0 | 0 |
| $\phi^{2}$ | 0 | 1 | 0 | 0 | 0 |
| $\lambda^{2}$ | 1 | 1 | 0 | 0 | 0 |
| $\tilde{q}_{1} q_{1}$ | 1 | $1 / 2$ | 0 | 0 | 0 |
| $\tilde{q}_{2} q_{2}$ | 1 | $-1 / 2$ | 0 | 0 | 0 |
| $\tilde{q}_{2} q_{1}$ | 1 | 0 | 0 | 0 | 4 |
| $q_{1} q_{2}$ | 1 | 0 | 0 | 2 | 2 |
| $\tilde{q}_{1} \tilde{q}_{2}$ | 1 | 0 | 0 | -2 | -2 |
| $\tilde{q}_{1} q_{2}$ | 1 | 0 | 0 | 0 | -4 |
| $\phi q_{1} q_{1}$ | 1 | 1 | 0 | 2 | 6 |
| $\phi \tilde{q}_{1} q_{1}$ | 1 | 1 | 0 | 0 | 0 |
| $\phi \tilde{q}_{1} \tilde{q}_{1}$ | 1 | 1 | 0 | -2 | -6 |
| $\phi \tilde{q}_{2} \tilde{q}_{2}$ | 1 | 0 | 0 | -2 | 2 |
| $\phi \tilde{q}_{2} q_{2}$ | 1 | 0 | 0 | 0 | 0 |
| $\phi q_{2} q_{2}$ | 1 | 0 | 0 | 2 | -2 |
| $\phi \tilde{q}_{2} q_{1}$ | 1 | $1 / 2$ | 0 | 0 | 4 |
| $\phi q_{1} q_{2}$ | 1 | $1 / 2$ | 0 | 2 | 2 |
| $\phi \tilde{q}_{1} \tilde{q}_{2}$ | 1 | $1 / 2$ | 0 | -2 | -2 |
| $\phi \tilde{q}_{1} q_{2}$ | 1 | $1 / 2$ | 0 | 0 | -4 |
| $\phi \lambda_{\alpha}$ | $1 / 2$ | 1 | $1 / 2$ | 0 | 0 |
| $q_{1} q_{1} \lambda_{\alpha}$ | $3 / 2$ | 1 | $1 / 2$ | 2 | 6 |
| $\tilde{q}_{1} q_{1} \lambda_{\alpha}$ | $3 / 2$ | 1 | $1 / 2$ | 0 | 0 |
| $\tilde{q}_{1} \tilde{q}_{1} \lambda_{\alpha}$ | $3 / 2$ | 1 | $1 / 2$ | -2 | -6 |
| $q_{2} q_{2} \lambda_{\alpha}$ | $3 / 2$ | 0 | $1 / 2$ | 2 | -2 |
| $\tilde{q}_{2} q_{2} \lambda_{\alpha}$ | $3 / 2$ | 0 | $1 / 2$ | 0 | 0 |
| $\tilde{q}_{2} \tilde{q}_{2} \lambda_{\alpha}$ | $3 / 2$ | 0 | $1 / 2$ | -2 | 2 |
| $\tilde{q}_{2} q_{1} \lambda_{\alpha}$ | $3 / 2$ | $1 / 2$ | $1 / 2$ | 0 | 4 |
| $q_{1} q_{2} \lambda_{\alpha}$ | $3 / 2$ | $1 / 2$ | $1 / 2$ | 2 | 2 |
| $\tilde{q}_{1} \tilde{q}_{2} \lambda_{\alpha}$ | $3 / 2$ | $1 / 2$ | $1 / 2$ | -2 | -2 |
| $\tilde{q}_{1} q_{2} \lambda_{\alpha}$ | $3 / 2$ | $1 / 2$ | $1 / 2$ | 0 | -4 |
|  |  |  |  |  |  |
| ${ }^{2}+2$ |  |  |  |  |  |

Table 6: List of candidate chiral ring generators for the $\left(A_{1}, D_{4}\right)$ theory (continued in Table 7). Here $R$, $r$, and $j$ are the IR $S U(2)_{R}$ Cartan, $U(1)_{r}$ charge, and left spin. $U(1)_{B}$ and $U(1)_{m}$ are $\mathcal{N}=2$ flavor symmetries.

| Operators | $R$ | $r$ | $j$ | $U(1)_{B}$ | $U(1)_{m}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\phi q_{1} q_{1} \lambda_{\alpha}$ | $3 / 2$ | $3 / 2$ | $1 / 2$ | 2 | 6 |
| $\phi \tilde{q}_{1} q_{1} \lambda_{\alpha}$ | $3 / 2$ | $3 / 2$ | $1 / 2$ | 0 | 0 |
| $\phi \tilde{q}_{1} \tilde{q}_{1} \lambda_{\alpha}$ | $3 / 2$ | $3 / 2$ | $1 / 2$ | -2 | -6 |
| $\phi q_{2} q_{2} \lambda_{\alpha}$ | $3 / 2$ | $1 / 2$ | $1 / 2$ | 2 | -2 |
| $\phi \tilde{q}_{2} q_{2} \lambda_{\alpha}$ | $3 / 2$ | $1 / 2$ | $1 / 2$ | 0 | 0 |
| $\phi \tilde{q}_{2} \tilde{q}_{2} \lambda_{\alpha}$ | $3 / 2$ | $1 / 2$ | $1 / 2$ | -2 | 2 |
| $\phi \tilde{q}_{2} q_{1} \lambda_{\alpha}$ | $3 / 2$ | 1 | $1 / 2$ | 0 | 4 |
| $\phi q_{1} q_{2} \lambda_{\alpha}$ | $3 / 2$ | 1 | $1 / 2$ | 2 | 2 |
| $\phi \tilde{q}_{1} \tilde{q}_{2} \lambda_{\alpha}$ | $3 / 2$ | 1 | $1 / 2$ | -2 | -2 |
| $\phi \tilde{q}_{1} q_{2} \lambda_{\alpha}$ | $3 / 2$ | 1 | $1 / 2$ | 0 | -4 |
| $\phi \lambda_{\alpha} \lambda_{\beta}$ | 1 | $3 / 2$ | 1 | 0 | 0 |
| $q_{1} q_{1} \lambda_{\alpha} \lambda_{\beta}$ | 2 | $3 / 2$ | 1 | 2 | 6 |
| $\tilde{q}_{1} q_{1} \lambda_{\alpha} \lambda_{\beta}$ | 2 | $3 / 2$ | 1 | 0 | 0 |
| $\tilde{q}_{1} \tilde{q}_{1} \lambda_{\alpha} \lambda_{\beta}$ | 2 | $3 / 2$ | 1 | -2 | -6 |
| $q_{2} q_{2} \lambda_{\alpha} \lambda_{\beta}$ | 2 | $1 / 2$ | 1 | 2 | -2 |
| $\tilde{q}_{2} q_{2} \lambda_{\alpha} \lambda_{\beta}$ | 2 | $1 / 2$ | 1 | 0 | 0 |
| $\tilde{q}_{2} \tilde{q}_{2} \lambda_{\alpha} \lambda_{\beta}$ | 2 | $1 / 2$ | 1 | -2 | 2 |
| $\tilde{q}_{2} q_{1} \lambda_{\alpha} \lambda_{\beta}$ | 2 | 1 | 1 | 0 | 4 |
| $q_{1} q_{2} \lambda_{\alpha} \lambda_{\beta}$ | 2 | 1 | 1 | 2 | 2 |
| $\tilde{q}_{1} \tilde{q}_{2} \lambda_{\alpha} \lambda_{\beta}$ | 2 | 1 | 1 | -2 | -2 |
| $\tilde{q}_{1} q_{2} \lambda_{\alpha} \lambda_{\beta}$ | 2 | 1 | 1 | 0 | -4 |
| $\lambda_{\alpha} \lambda_{\beta} \lambda_{\gamma}$ | $3 / 2$ | $3 / 2$ | $3 / 2$ | 0 | 0 |

Table 7: Remaining candidate chiral ring generators for the $\left(A_{1}, D_{4}\right)$ theory (continued from Table 6). Here $R$, $r$, and $j$ are the IR $S U(2)_{R}$ Cartan, $U(1)_{r}$ charge, and left spin. $U(1)_{B}$ and $U(1)_{m}$ are $\mathcal{N}=2$ flavor symmetries.
in the IR chiral ring. This logic removes the following potential chiral ring generators

$$
\begin{align*}
& \tilde{q}_{2} q_{1} \lambda_{\alpha}, q_{1} q_{2} \lambda_{\alpha}, \tilde{q}_{1} \tilde{q}_{2} \lambda_{\alpha}, \tilde{q}_{1} q_{2} \lambda_{\alpha}, \phi q_{2} q_{2} \lambda_{\alpha}, \phi \tilde{q}_{2} q_{2} \lambda_{\alpha}, \phi \tilde{q}_{2} \tilde{q}_{2} \lambda_{\alpha}, \tilde{q}_{2} q_{1} \lambda_{\alpha} \lambda_{\beta}, \\
& q_{1} q_{2} \lambda_{\alpha} \lambda_{\beta}, \tilde{q}_{1} \tilde{q}_{2} \lambda_{\alpha} \lambda_{\beta}, \tilde{q}_{1} q_{2} \lambda_{\alpha} \lambda_{\beta}, \lambda_{\alpha} \lambda_{\beta} \lambda_{\gamma} . \tag{4.19}
\end{align*}
$$

- Similar logic rules out operators with $r<j$. This reasoning removes the following potential chiral ring generators

$$
\begin{equation*}
\tilde{q}_{2} q_{2}, q_{2} q_{2} \lambda_{\alpha}, \tilde{q}_{2} q_{2} \lambda_{\alpha}, \tilde{q}_{2} \tilde{q}_{2} \lambda_{\alpha}, q_{2} q_{2} \lambda_{\alpha} \lambda_{\beta}, q_{2} \tilde{q}_{2} \lambda_{\alpha} \lambda_{\beta}, \tilde{q}_{2} \tilde{q}_{2} \lambda_{\alpha} \lambda_{\beta} . \tag{4.20}
\end{equation*}
$$

- Other operators are removed from the IR chiral ring via superpotential constraints. Indeed, constraints in (4.18) remove

$$
\begin{equation*}
\phi q_{1} q_{1}, \phi \tilde{q}_{1} q_{1}, \phi \tilde{q}_{1} \tilde{q}_{1}, \phi \tilde{q}_{2} q_{1}, \phi q_{1} q_{2}, \phi \tilde{q}_{1} \tilde{q}_{2}, \phi \tilde{q}_{1} q_{2} . \tag{4.21}
\end{equation*}
$$

Note that there are no quantum corrections to these superpotenial removals since these are the unique operators with their superconformal and flavor quantum numbers ${ }^{32}$

- $\lambda^{2}$ has $(R, r(j, \bar{j}))=(1,1,(0,0))$. If it were a level-one descendant, the primary would have $(R, r(j, \bar{j}))=(1 / 2,3 / 2,(1 / 2,0))$. However, this would be a primary of a $\overline{\mathcal{D}}$ multiplet, which we know is absent 33 If it were a level-two descendant, the primary would have $(R, r(j, \bar{j}))=(0,2,(0,0))$, which would be an $\overline{\mathcal{E}}_{2}$ primary, which is again absent. Therefore, $\lambda^{2}$ must be trivial in the IR $\mathcal{N}=1$ chiral spectrum.
- $\phi q_{1} q_{1} \lambda_{\alpha}, \phi q_{1} \tilde{q}_{1} \lambda_{\alpha}$, and $\phi \tilde{q}_{1} \tilde{q}_{1} \lambda_{\alpha}$ have $(R, r(j, \bar{j}))=(3 / 2,3 / 2,(1 / 2,0))$. If they are levelone descendants, then the primary has $(R, r(j, \bar{j}))=(1,2,(1,0))$ (but then $r=1+j$, and we know there are no $\overline{\mathcal{D}}$ multiplets) or $(R, r(j, \bar{j}))=(1,2,(0,0))$ (but such a primary cannot be constructed out of the list of $S U(2)$-neutral generators). If it is a level-two descendant, then the primary has $(R, r(j, \bar{j}))=(1 / 2,5 / 2,(1 / 2,0))\left(M_{2} \phi \lambda_{\alpha}\right.$ has these quantum numbers, but it is a descendant in an $\overline{\mathcal{E}}_{3}$ multiplet).
- $\phi \lambda_{\alpha} \lambda_{\beta}$ has $(R, r(j, \bar{j}))=(1,3 / 2,(1,0))$. If it is a level-one descendant, then the primary has $(R, r(j, \bar{j}))=(1 / 2,2,(1 / 2,0))$ or $(R, r(j, \bar{j}))=(1 / 2,2,(3 / 2,0))$ (but neither can be constructed from the list of generators). If it is a level-two descendant, then the primary has $(R, r(j, \bar{j}))=(0,5 / 2,(1,0))$ (which cannot be constructed from the list of generators since $\phi^{2}$ decouples).

[^17]- $q_{1} q_{1} \lambda_{\alpha} \lambda_{\beta}, q_{1} \tilde{q}_{1} \lambda_{\alpha} \lambda_{\beta}$, and $\tilde{q}_{1} \tilde{q}_{1} \lambda_{\alpha} \lambda_{\beta}$ have $(R, r(j, \bar{j}))=(2,3 / 2,(1,0))$. If they are levelone descendants, then the primaries have $(R, r(j, \bar{j}))=(3 / 2,2,(1 / 2,0))$ (such operators cannot be constructed from the list of generators because of the vanishing of the operators (4.19) in the IR chiral ring and the vanishing of $\phi^{2}$ in the IR theory) or $(R, r(j, \bar{j}))=(3 / 2,2,(3 / 2,0))$ (these operators also cannot be constructed from the list of generators). If it is a level-two descendant, then the primary has $(R, r(j, \bar{j})=(1,5 / 2,(1,0))$ (such an operator cannot be constructed from the list of generators since $\phi^{2}$ decouples).
- $q_{1} q_{1} \lambda_{\alpha}, \tilde{q}_{1} q_{1} \lambda_{\alpha}$, and $\tilde{q}_{1} \tilde{q}_{1} \lambda_{\alpha}$ have $(R, r(j, \bar{j}))=(3 / 2,1,(1 / 2,0))$. These operators cannot be $\overline{\mathcal{B}}$ superconformal primaries since $r<1+j$. If they are level-one descendants, then the primary has $(R, r(j, \bar{j}))=(1,3 / 2,(0,0))$ (such operators can potentially be constructed by taking a products of $M_{2}$ and one of the eight $\hat{\mathcal{B}}_{1}$ primaries; we will say more about this multiplet later) or $(R, r(j, \bar{j}))=(1,3 / 2,(1,0))$ (this is ruled out, as $r<1+j$, and because such operators cannot be constructed out of the list of generators). If it is a level two descendant, then the primary has $(R, r(j, \bar{j}))=(1 / 2,2,(1 / 2,0))$ (which cannot be constructed from list of generators).
- $\phi \tilde{q}_{2} q_{1} \lambda_{\alpha}, \phi q_{1} q_{2} \lambda_{\alpha}, \phi \tilde{q}_{1} \tilde{q}_{2} \lambda_{\alpha}$, and $\phi \tilde{q}_{1} q_{2} \lambda_{\alpha}$ have $(R, r(j, \bar{j}))=(3 / 2,1,(1 / 2,0))$. These operators cannot be $\overline{\mathcal{B}}$ superconformal primaries since $r<1+j$. If they are levelone descendants, then the primaries have $(R, r(j, \bar{j}))=(1,3 / 2,(1,0))$ (which fails due to $r<1+j$ ) or $(R, r(j, \bar{j}))=(1,3 / 2,(0,0))$ (such primaries can potentially be constructed by multiplying $M_{2}$ with a $\hat{\mathcal{B}}_{1}$ primary; we will say more about these operators later). If these operators are level-two descendants, then the primary has $(R, r(j, \bar{j}))=(1 / 2,2,(1 / 2,0))$ (but such primaries cannot be constructed from list of generators since $\phi^{2}$ decouples).

Therefore, in order to construct a $\overline{\mathcal{B}}$ superconformal primary, we must build it out of products of chiral operators in the Coulomb branch multiplet $\overline{\mathcal{E}}_{3 / 2}$ (i.e., $\left\{M_{2}, \phi \lambda_{\alpha}, q_{1} \tilde{q}_{1}\right\}$ ) and the Higgs branch multiplets $\hat{\mathcal{B}}_{1}$ (i.e., $\left\{q_{1} q_{2}, q_{1} \tilde{q}_{2}, \tilde{q}_{1} q_{2}, \tilde{q}_{1} \tilde{q}_{2}, \phi q_{2} q_{2}, \phi q_{2} \tilde{q}_{2}, \phi \tilde{q}_{2} \tilde{q}_{2}, \beta_{2}\right\}$ ).

We can use the superpotential in (4.15) to constrain these products. To that end, we consider each quadratic product build out of the $\overline{\mathcal{E}}_{3 / 2}$ and $\hat{\mathcal{B}}_{1}$ chiral operators separately

- $M_{2}^{2}$ and $M_{2} \phi \lambda_{\alpha}$ : These are the primary and the level-one descendant of the $\overline{\mathcal{E}}_{3}$ multiplet.
- $M_{2} q_{1} \tilde{q}_{1}$ and $\left(\phi \lambda_{\alpha}\right)^{2}$ : One linear combination is the level-two descendant of $\overline{\mathcal{E}}_{3}$. The other independent linear combination, if nonzero, is a primary of a $\overline{\mathcal{B}}$ multiplet. There
is a unique candidate for the level-two descendant of this multiplet, $\left(q_{1} \tilde{q}_{1}\right)^{2}$. As we will show below, this operator vanishes in the IR chiral ring as a result of a superpotential constraint. Therefore, this $\overline{\mathcal{B}}$ multiplet cannot exist in the $\left(A_{1}, D_{4}\right)$ theory.
- $\left(q_{1} \tilde{q}_{1}\right)^{2}(R, r(j, \bar{j}))=(2,1,(0,0))$ : We have the following superpotential constraint

$$
\begin{equation*}
\frac{\partial W}{\partial \phi^{a}}\left(q_{1} \tilde{q}_{1}\right)^{a}=0 \tag{4.22}
\end{equation*}
$$

which leads to,

$$
\begin{equation*}
2 \beta_{2} \phi q_{1} \tilde{q}_{1}+\delta_{a b}\left(q_{1} \tilde{q}_{1}\right)^{a}\left(q_{1} \tilde{q}_{1}\right)^{b}=0 . \tag{4.23}
\end{equation*}
$$

We have already shown that $\phi q_{1} \tilde{q}_{1}$ is trivial in the IR $\mathcal{N}=1$ chiral spectrum. Therefore, we are left with,

$$
\begin{equation*}
\delta_{a b}\left(q_{1} \tilde{q}_{1}\right)^{a}\left(q_{1} \tilde{q}_{1}\right)^{b}=0 \tag{4.24}
\end{equation*}
$$

This constraint cannot receive quantum corrections since all other operators with the same superconformal quantum numbers have already been shown to vanish in the IR chiral ring.

- $M_{2} \hat{\mathcal{B}}_{1}$ has $(R, r(j, \bar{j}))=(1,3 / 2(0,0))$ : Here we consider the eight so-called "mixed branch" primaries consisting of the product of the primaries in the Coulomb and Higgs branch chiral ring generators. The following chiral ring relation causes them to vanish:

$$
\begin{equation*}
\frac{\partial W}{\partial q_{2}} \phi q_{2}=M_{2} \phi q_{2} \tilde{q}_{2}=0 \tag{4.25}
\end{equation*}
$$

By an $S U(3)$ flavor rotation, all other products of the form $M_{2} \hat{\mathcal{B}}_{1}$ also vanish in the IR chiral ring.

- $\phi \lambda_{\alpha} \hat{\mathcal{B}}_{1}$ has $((R, r(j, \bar{j}))=(3 / 2,1(1 / 2,0)))$ : These eight operators cannot form a $\overline{\mathcal{B}}$ primary since $r<1+j 34$
- $q_{1} \tilde{q}_{1} \hat{\mathcal{B}}_{1}$ has $((R, r(j, \bar{j}))=(2,1 / 2(0,0))$ : These operators cannot be $\overline{\mathcal{B}}$ primaries since $r<135$

[^18]| Field | R | r | j | $S U(2)_{\text {gauge }}$ | $S O(8)$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\phi$ | 0 | 1 | 0 | adj. | 1 |
| $\lambda_{\alpha}$ | $1 / 2$ | $1 / 2$ | $1 / 2$ | adj. | 1 |
| $Q^{a}$ | $1 / 2$ | 0 | 0 | 2 | 8 |

Table 8: List of chiral fields appearing in the Lagrangian of the $S U(2)$ theory with four fundamental flavors. Here $r, R$, and $j$ are the $U(1)_{r}, S U(2)_{R}$ Cartan, and left spin respectively. In the rightmost column we record the representation under the $S O(8)$ flavor symmetry (the matter fields transform in the vector representation). If we want to write our matter field charges in terms of $S U(4) \subset S O(8)$, we can define $q^{i}:=Q^{i}$ and $\tilde{q}^{i}:=Q^{i+4}$ where $i=1, \cdots, 4$.

Therefore, we see that we cannot construct a $\overline{\mathcal{B}}$ primary, and so there are no $\overline{\mathcal{B}}$ multiplets in the $\left(A_{1}, D_{4}\right)$ SCFT ${ }^{36}$ In particular, we see that

$$
\begin{equation*}
\operatorname{FCS}_{\left(A_{1}, D_{4}\right)}=\left\langle\overline{\mathcal{E}}_{3 / 2}, \hat{\mathcal{B}}_{1}\right\rangle / I, \quad \overline{\mathcal{B}} \notin \operatorname{FCS}_{\left(A_{1}, D_{4}\right)}, \tag{4.26}
\end{equation*}
$$

where $I$ is the Joseph ideal constraint.

## 4.5. $S U(2) S Q C D$ with $N_{f}=4$

Finally, we discuss the $\mathcal{N}=2$ Lagrangian theory of $S U(2)$ SQCD with four flavors. Unlike the previous cases, this theory is not isolated (it has an exactly marginal coupling).

In the interacting theory, we can again show that all chiral ring generators are in the Coulomb branch and Higgs branch chiral subrings ${ }^{37}$ In this case, this means the chiral ring generators live in $\overline{\mathcal{E}}_{2}$ or $\hat{\mathcal{B}}_{1}^{M}$ multiplets (with $M$ an $S O(8)$ adjoint index). Since the gauge group is $S U(2)$, the naive list of chiral generators is very similar to the lists in the Argyres-Douglas examples we treated before (which were based on $S U(2)$ gauge theory Lagrangians), so we will keep the discussion brief.

We use Table 8 to write down the list of UV chiral fields. The Lagrangian in this case is

$$
\begin{equation*}
W=\tau \phi\left(Q^{a} Q_{a}\right) \tag{4.27}
\end{equation*}
$$

then the primary has $((R, r(j, \bar{j}))=(1,3 / 2(0,0))$, but we have already eliminated this $\overline{\mathcal{B}}$ multiplet by the superpotential constraint.
${ }^{36}$ In fact, using the argument in footnotes 34 and 35, we directly see that all operators except those in the Coulomb branch and Higgs branch chiral rings vanish in the IR FCS.
${ }^{37}$ By the interacting theory, we mean the interacting theory at generic points on the conformal manifold.

Note that in terms of $S U(4) \subset S O(8)$, we can define $q^{i}:=Q^{i}$ and $\tilde{q}^{i}:=Q^{i+4}$ where $i=1, \cdots, 4$.

The naive chiral ring generators are the following:

- $\phi^{2}, \phi \lambda_{\alpha}, \lambda^{2}$ : These are, respectively, the primary, the level one, and the level-two descendants of the $\overline{\mathcal{E}}_{2}$ multiplet. In the interacting theory, $\lambda^{2}$ mixes with $\phi\left(Q^{a} Q_{a}\right)$ via (4.27).
- $M^{a b}=Q^{[a} Q^{b]}$ (there are 28 of these operators transforming in the adjoint of $S O(8)$ ). These are the $\hat{\mathcal{B}}_{1}^{A}$ Higgs branch generators housing the Noether currents of the flavor symmetry.
- The other possible operators with $j=0$ have the following form: $\phi Q Q(R=1$, $r=1$ ). If they are non-trivial in the chiral ring, these operators can only sit in $\overline{\mathcal{D}}$ multiplets. In the interacting theory, one linear combination with $\lambda^{2}$ becomes the level-two descendant of $\overline{\mathcal{E}}_{2}$.
- At $j=1 / 2$, the possible operators have either of two forms, (1) $Q Q \lambda$ ( $R=3 / 2$, $r=1 / 2$ ). If they are non-trivial in the chiral ring, these operators can only be descendants of $\overline{\mathcal{D}}_{1(0,0)}$ multiplets, or (2) $\phi Q Q \lambda(R=3 / 2, r=3 / 2)$. These operators can be primaries of $\overline{\mathcal{D}}_{\frac{3}{2},\left(\frac{1}{2}, 0\right)}$ multiplets or descendants of $\overline{\mathcal{B}}_{1,2(0,0)}, \overline{\mathcal{D}}_{1,(1,0)}$, or $\overline{\mathcal{B}}_{\frac{1}{2}, \frac{5}{2}\left(\frac{1}{2}, 0\right)}$ multiplets.
- At $j=1$, the possible generators are of the following forms, (1) $\phi \lambda \lambda(R=1, r=2)$. These can be a primary of a $\overline{\mathcal{D}}_{1(1,0)}$ multiplet or descendants of $\overline{\mathcal{B}}_{\frac{1}{2}, \frac{5}{2}\left(\frac{1}{2}, 0\right)}$ or $\overline{\mathcal{D}}_{\frac{1}{2}\left(\frac{3}{2}, 0\right)}$, or (2) $Q Q \lambda \lambda(R=2, r=1)$. These can only be descendants of $\overline{\mathcal{D}}_{\frac{3}{2},\left(\frac{1}{2}, 0\right)}$ or $\overline{\mathcal{B}}_{1,2(0,0)}$ multiplets.
- At $j=3 / 2$, the only possible generator is of the form $\lambda \lambda \lambda(R=r=3 / 2)$, which can only be the descendant of a $\overline{\mathcal{D}}_{1(1,0)}$ multiplet.

To summarize, we see that these operators may lie in the following multiplets: (1) $\overline{\mathcal{D}}$ multiplets. However, in the interacting theory, these multiplets are not present due to the same logic as in the $\left(A_{1}, A_{3}\right)$ and $\left(A_{1}, D_{4}\right)$ cases: the associated chiral algebra is of AKM type $\widehat{s o(8)}_{-2}$ in this case [4]). To see this statement more concretely, note that, when we turn on interactions, thirty six linear combinations of the $\phi Q Q$ and $\lambda^{2}$ operators discussed above pair up with thirty-six of the thirty-seven stress tensor multiplets in the free theory to become long multiplets (the remaining stress tensor multiplet is protected along the full conformal manifold; see [36] for further details of this argument). (2) As descendants in
a $\overline{\mathcal{B}}_{\frac{1}{2}, \frac{5}{2}\left(\frac{1}{2}, 0\right)}$ multiplet. However, we cannot construct a candidate for the superconformal primary of this multiplet in a rank-one theory. (3) As a descendant in a $\overline{\mathcal{B}}_{1,2(0,0)}$ multiplet (we can construct a candidate primary for this multiplet, but the multiplet itself is known to be absent in this theory since it does not have a mixed branch; more on this later). Therefore, we conclude that, in the interacting theory, the only non-trivial chiral generators lie in $\overline{\mathcal{E}}_{2}$ or $\hat{\mathcal{B}}_{1}$ multiplets.

Once again, we proceed to take normal-ordered quadratic products of the above Coulomb and Higgs branch generators. There are two possible sources of $\overline{\mathcal{B}}$ multiplets which we can form by taking products of the above generators. We now study each case individually.

The Coulomb branch multiplet, $\overline{\mathcal{E}}_{2}$, has the following chiral operators (for simplicity, we define $M_{2}$ at $\tau=0$ ),

$$
\begin{equation*}
M_{0}:=\phi^{2}, M_{1 \alpha}:=\phi \lambda_{\alpha}, M_{2}=\lambda^{2} \tag{4.28}
\end{equation*}
$$

Since the theory is rank one, there is no candidate for a superconformal primary of a $\overline{\mathcal{B}}_{1 / 2,3 / 2,(1 / 2,0)}$ multiplet.

We can take a linear combination of these operators

$$
\begin{equation*}
\mathcal{O}:=M_{0} M_{2}+\kappa M_{1}^{2}, \tag{4.29}
\end{equation*}
$$

where $\kappa \in \mathbb{C}$, and $\mathcal{O}$ is a superconformal primary of a $\overline{\mathcal{B}}_{1,3(0,0)}$ multiplet. It is trivial to check that this operator is present in the free theory. In the index, it is also easy to check that there is no $\overline{\mathcal{B}}_{1,3(0,0)}$ contribution at the leading order it can appear. The reason is that there are thirty-eight $\overline{\mathcal{C}}_{0,2(0,0)}$ multiplets in the free theory: thirty-seven from $\overline{\mathcal{E}}_{2} \times \hat{\mathcal{C}}_{0(0,0)}$ and one of the form $f_{A B C} \epsilon^{\alpha \beta} \epsilon_{I J} \lambda_{\alpha}^{I A} \lambda_{\beta}^{J B} \phi^{C}$, where we have contracted $S U(2)_{R}$ and Lorentz indices of the gauginos (note that the anti-symmetrization of gauge indices makes this latter operator a superconformal primary). On the other hand, there are thirtyseven $\overline{\mathcal{B}}_{1,3(0,0)}$ multiplets: thirty-six arising from $\overline{\mathcal{E}}_{2} \times \overline{\mathcal{D}}_{1(0,0)}$ and one linear combination as in (4.29). The corresponding index contributions cancel up to a net $\overline{\mathcal{C}}_{0,2(0,0)}$ contribution (note that three of the $\overline{\mathcal{C}}_{0,2(0,0)}$ multiplets are flavor singlets and so are two of the $\overline{\mathcal{B}}_{1,3(0,0)}$ contributions).

At leading order in the gauge coupling (i.e., at leading order in $\tau$ ), it is easy to see that two flavor-neutral $\overline{\mathcal{C}}_{0,2(0,0)}$ multiplets and the $\overline{\mathcal{B}}_{1,3(0,0)}$ multiplet in (4.29) remain as short multiplets (unlike other $\overline{\mathcal{C}}_{0,2(0,0)}$ and $\overline{\mathcal{B}}_{1,3(0,0)}$ pairs in the theory that become long multiplets at leading order when we turn on the superpotential in (4.27)). The $\overline{\mathcal{B}}_{1,3(0,0)}$ multiplet is not protected from recombination at higher orders in the coupling (but there is always a protected $\overline{\mathcal{C}}_{0,2(0,0)}$ multiplet).

Finally, we can form a potential primary of a $\overline{\mathcal{B}}_{1,2(0,0)}$ multiplet from the operator,

$$
\begin{equation*}
M_{0} \operatorname{Tr}(Q Q) \tag{4.30}
\end{equation*}
$$

which would correspond to a mixed branch. These operators vanish in the chiral ring at leading order in the coupling (they pair up with an adjoint-valued $\overline{\mathcal{C}}_{0,1(0,0)}$ multiplet to become long multiplet descendants). This statement is consistent with the fact that the theory does not have a mixed branch. In fact, any such candidate obtained from the $\overline{\mathcal{E}}_{2 m} \times \hat{\mathcal{B}}_{n}$ OPE for $m, n \in \mathbb{N}$ will not correspond to a $\overline{\mathcal{B}}_{n, 2 m(0,0)}$ multiplet for the same reason.

## 5. Conclusion

In this paper, we have explored various new roles that $\overline{\mathcal{B}}$ multiplets play in $4 \mathrm{~d} \mathcal{N}=2$ SCFTs. We have shown that these operators are ubiquitous and are connected to many interesting phenomena. Our work also raises several questions:

- Can our algebraic proof that the product of primaries in $\overline{\mathcal{D}}_{1 / 2(0,0)} \times \overline{\mathcal{D}}_{1 / 2(0,0)}$ is nonvanishing and produces a $\overline{\mathcal{B}}$ multiplet be generalized to show that any $\mathcal{N}>2$ theory has an infinite chiral ring generated by the primary of $\overline{\mathcal{D}}_{1 / 2(0,0)}$ ? 38 Since this multiplet houses the extra supercurrents in its higher components, it would be interesting to understand if such an infinite chiral ring exists as a consequence of physics related to Ward identities for extended SUSY. If so, this may be a step in an abstract CFTbased proof that all $\mathcal{N}>2$ theories have a mixed branch of moduli space.
- We saw that the existence of (adjoint valued) $\overline{\mathcal{B}}$ multiplets can be a diagnostic of the $\mathbb{Z}_{2}$-valued $S p(N)$ 't Hooft anomaly in [8]. What about more general 't Hooft anomalies? Can these operators help diagnose the existence of 2-groups and other more elaborate structures?
- We saw that $\overline{\mathcal{B}}$ multiplets form a natural ideal in the FCS ring. Can we use this fact to prove new results about chiral rings in $4 \mathrm{~d} \mathcal{N}=2$ SCFTs?
- The $\overline{\mathcal{B}}$ production channels we discussed do not lead to $\overline{\mathcal{B}}$ chiral ring generators. It would be interesting to understand the most general conditions under which $\overline{\mathcal{B}}$ multiplets can and cannot house chiral ring generators (see footnote 6). Can gauging discrete symmetries help?

[^19]- We conjectured a bound on the quantum numbers of $\overline{\mathcal{B}}$ multiplets. It would be interesting to prove (or disprove) this bound.
- We have seen that the particular isolated rank-one theories we studied do not have $\overline{\mathcal{B}}$ multiplets (of course, rank one theories that violate some of the criteria we imposed do have $\overline{\mathcal{B}}$ mulitplets; examples include $S U(2) \mathcal{N}=4$ SYM among others). It would be interesting to better understand why this is the case (and to extend our analysis to other cases with $\mathcal{N}=1$ Lagrangians). Our analysis relied on the existence of certain $\mathcal{N}=1$ Lagrangians that flow to $\mathcal{N}=2$ in the IR. Can we give an abstract argument, perhaps using properties of 't Hooft anomalies and $a$-maximization, that any rank-one IR SCFT of the general type we studied arising from an $\mathcal{N}=1$ Lagrangian has no $\overline{\mathcal{B}}$ multiplets? Can we extend our analysis to other rank-one theories even when there is no known Lagrangian? Can we give a more manifest and dynamical proof that the rank-one theories we studied do not have $\overline{\mathcal{B}}$ multiplets?

We hope to return to some of these questions soon.

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[^1]:    ${ }^{1}$ In this paper, unless otherwise specified, an operator is termed "chiral" if it is chiral with respect to some $\mathcal{N}=1 \subset \mathcal{N}=2$ subalgebra and is non-trivial in the corresponding ring (see section 2 for more details). A multiplet is considered chiral if it houses a chiral operator.

[^2]:    ${ }^{2}$ Even these sectors are not fully understood in general. For example, it is believed (without proof) that any interacting $4 \mathrm{~d} \mathcal{N}=2 \mathrm{SCFT}$ has a Coulomb branch. But even basic properties of the Coulomb branch, such as the most general conditions under which its corresponding chiral ring is freely generated, are not known.
    ${ }^{3}$ More precisely, $\hat{\mathcal{B}}$ multiplets are chiral with respect to half the supersymmetry and anti-chiral with respect to the other half. The remaining multiplets housing chiral operators (except for $\overline{\mathcal{E}}$ ) satisfy less restrictive shortening conditions. See the next section for further details.

[^3]:    ${ }^{4}$ Seiberg-Witten curves indirectly detect certain $\overline{\mathcal{B}}$ multiplets in the low energy description of the Coulomb branch.
    ${ }^{5}$ All known examples of $4 \mathrm{~d} \mathcal{N}=2$ conformal manifolds have a gauge coupling interpretation. Such families of SCFTs generally have "matter" sectors that are interacting isolated SCFTs (as opposed to only containing collections of free hypermultiplets whose symmetries are gauged).
    ${ }^{6}$ It is straightforward to construct $\overline{\mathcal{B}}$ multiplets that exist at special points on the conformal manifold (or, more generally, for special values of a gauge coupling). For example, at zero gauge coupling we can construct, for $S U(N)$ (and $N>2$ ), $\overline{\mathcal{B}}$ primaries of the schematic form $\operatorname{Tr} \phi^{2} \mathcal{O}$, where $\mathcal{O}$ is a $\hat{\mathcal{B}}_{R}$ primary transforming in the adjoint of $S U(N)$, and $\phi$ is the corresponding vector multiplet scalar. However, such operators are not protected from recombination and typically become part of long multiplets as we turn on the gauge coupling (here we have taken the generic case $R>1 / 2$; in the special case of a free matter sector with $R=1 / 2$, we do obtain a protected multiplet). Our interest is in $\overline{\mathcal{B}}$ multiplets that are robust against quantum corrections, are present everywhere on the conformal manifold, and do not require considering special matter sectors.

[^4]:    ${ }^{7}$ Such operators therefore give rise to coordinates in Seiberg-Witten geometries and their generalizations. This fact explains their ubiquity, although it is not completely clear to us if their ubiquity is also a consequence of the type of $4 \mathrm{~d} \mathcal{N}=2$ SCFTs we have been able to construct to date. Ideally, one would like to understand if such multiplets emerge from some more minimal set of algebraic criteria.
    ${ }^{8}$ Such a phenomenon indicates the presence of a mixed branch.

[^5]:    ${ }^{9}$ See 10 for $\overline{\mathcal{B}}$ production channels outside the chiral ring OPE (and footnote 6 for production channels that do not involve OPEs of bulk local operators). We will not discuss these channels in this paper.
    ${ }^{10}$ Indeed, in the next section, we will use algebraic techniques to show that the $\overline{\mathcal{D}}_{1 / 2(0,0)}$ multiplets housing extra $\mathcal{N}>2$ supercurrents are never minimally nilpotent.
    ${ }^{11}$ In some cases this is impossible. For example,

    $$
    \begin{align*}
    \overline{\mathcal{D}}_{R(j, 0)} \times \overline{\mathcal{D}}_{R^{\prime}\left(j^{\prime}, 0\right)} & \ni \mathcal{O}_{\alpha_{1} \cdots \alpha_{2 j}}^{1 \cdots 1} Q_{\alpha}^{1} \mathcal{O}_{\alpha_{1} \cdots \alpha_{2 j^{\prime}}}^{\prime 1 \cdots 1}+\kappa \mathcal{O}_{\alpha_{1} \cdots \alpha_{2 j}}^{\prime 1 \cdots 1} Q_{\alpha}^{1} \mathcal{O}_{\alpha_{1} \cdots \alpha_{2 j}}^{1 \cdots 1} \\
    & \in \overline{\mathcal{B}}_{R+R^{\prime}+1 / 2, j+j^{\prime}+3 / 2\left(j+j^{\prime}+1 / 2,0\right)}^{\cong} \overline{\mathcal{D}}_{R+R^{\prime}+1 / 2\left(j+j^{\prime}+1 / 2\right)}, \tag{2.13}
    \end{align*}
    $$

    where $\kappa \in \mathbb{C}$ is required to make the operator in question a superconformal primary. More generally, if we involve at most a single $\overline{\mathcal{D}}_{R(j, 0)}$ primary, we must also take spin contractions (this is because the descendant in (2.9) has $r=j$ ).
    ${ }^{12}$ Note that if $\mathcal{O}=\mathcal{O}^{\prime}$, then the $\overline{\mathcal{B}}_{1 / 2, r+r^{\prime}-1(1 / 2,0)}$ multiplet in (2.14) vanishes.

[^6]:    ${ }^{13}$ Although, outside of theories involving very special matter sectors, the only such examples we are aware of are of the unprotected form discussed in footnote 6, one may also consider the possibility of obtaining such generators from gauging a non-anomalous discrete symmetry.
    ${ }^{14}$ These relations need not involve only Coulomb branch primaries in general. Indeed, in the examples we discuss below, they do not.
    ${ }^{15}$ This statement is highly non-trivial for local (non-Lagrangian) $\mathcal{N}=3$ SCFTs. Note that it also applies to any potential (yet to be discovered) local non-Lagrangian $\mathcal{N}=4$ theories.

[^7]:    ${ }^{16}$ The additional superscript in $\hat{\mathcal{B}}_{1}^{0} \cong B_{1} \bar{B}_{1}[0 ; 0]^{(2 ; 0), 0}$ refers to the fact that this multiplet has zero $U(1)_{G}$ charge (we follow the conventions of [12]). We will only write this superscript explicitly in cases where the $U(1)_{G}$ charge is relevant to the argument.
    ${ }^{17}$ We only keep track of so-called "Schur" multiplets in these selection rules. The reason is that these multiplets house the operators subject to the correspondence in [4].

[^8]:    ${ }^{18}$ Note that the $\hat{\mathcal{C}}_{1(0,0)}$ multiplet is the universal multiplet described in [15] for an interacting theory with a flavor symmetry.
    ${ }^{19}$ Note that $\overline{\mathcal{D}}_{1 / 2(0,0)} \times \mathcal{D}_{1 / 2(0,0)} \ni \hat{\mathcal{C}}_{1(0,0)}$.
    ${ }^{20}$ Recall that there are no $\mathcal{N}=3$-preserving exactly marginal deformations.

[^9]:    ${ }^{21}$ The $\mathcal{T}_{i}$ need not be weakly coupled themselves. For example, consider the Minahan-Nemeschansky $E_{6}$ theory, $\mathrm{MN}_{E_{6}}$, appearing in the $S U(2)$ duality frame of [16].
    ${ }^{22}$ As an example of this latter phenomenon, consider the case of $\overline{\mathcal{E}}_{3} \in \mathrm{MN}_{E_{6}}$ in the example of [16].

[^10]:    ${ }^{23}$ The corresponding holomorphic moment maps are $Q^{a} Q_{a}, \tilde{Q}^{a} \tilde{Q}_{a}$, and $Q^{a} \tilde{Q}_{a}$.

[^11]:    ${ }^{24}$ As in the discussion around (3.14), we make the same simplifying assumptions on the appearance of the $\lambda_{a}$ in our multiplets described below. This is for the sake of simplicity of presentation.

[^12]:    ${ }^{25}$ For the latter case, note that to get $R=0$ we should anti-symmetrize the free hyper $S U(2)_{R}$ indices. This implies that the operator is not in the adjoint of the flavor symmetry.
    ${ }^{26}$ We can arrive at this result more simply if we are willing to invoke the lore that a mixed branch implies the existence of operators in the (2.11) channel. Indeed, the existence of the mixed branch follows from the discussion around (3.20).

[^13]:    ${ }^{27}$ In fact, the analysis of [21] can be used to directly prove our conjecture for the special case of $r<j+2$. We thank A. Manenti for pointing this fact out to us and for collaboration on upcoming work on this conjecture 22.

[^14]:    ${ }^{29}$ Our naming conventions differ slightly from [5]. In particular, $\tilde{q} \rightarrow q^{\prime}$.

[^15]:    ${ }^{30}$ Indeed, as in footnote 15 of [5], we can argue that generators in the quantum theory are built from traces involving two or three adjoints.

[^16]:    ${ }^{31}$ Alternatively, we could arrive at the same conclusion by using the first equation in (4.11) and invoking $S U(2)$ flavor covariance.

[^17]:    ${ }^{32}$ This logic relies on the fact that $\phi^{2}$ decouples due to unitarity bound violations or equivalently, for this purpose, $\phi^{2}$ is set to zero in the IR chiral ring by a superpotential constraint.
    ${ }^{33}$ Also, the primary cannot be constructed out of the list of chiral generators.

[^18]:    ${ }^{34}$ In fact, we can show these operators are trivial in the IR chiral ring. Indeed, if they form a levelone descendant, then the primary has $((R, r(j, \bar{j}))=(1,3 / 2(0,0))$, but this $\overline{\mathcal{B}}$ multiplet has already been removed by the superpotential constraint above. If they form a level-two descendant, then the primary has $(R, r(j, \bar{j}))=(1 / 2,2(1 / 2,0))$, and the only candidate operator is $M_{2} \phi \lambda_{\alpha}$. However, this is itself a level-one descendant.
    ${ }^{35}$ In fact, these operators are trivial in the IR chiral ring. Indeed, if they form a level-one descendant, then the primary has $(R, r(j, \bar{j}))=(3 / 2,1(1 / 2,0))$, but this also has $r<1+j$. If it is a level-two descendant,

[^19]:    ${ }^{38}$ Such a ring would contain an infinite number of $\overline{\mathcal{B}}$ multiplets.

