A MOD p JACQUET–LANGLANDS RELATION AND SERRE FILTRATION VIA THE GEOMETRY OF HILBERT MODULAR VARIETIES: SPLICING AND DICING

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Abstract. We consider the Hilbert modular varieties in characteristic p with Iwahori level at p and construct a geometric Jacquet–Langlands relation showing that the irreducible components are isomorphic to projective bundles over quaternionic Shimura varieties of level prime to p. We use this to establish a relation between mod p Hilbert and quaternionic modular forms that reflects the representation theory of GL_2 in characteristic p and generalizes a result of Serre for classical modular forms. Finally we study the fibres of the degeneracy map to level prime to p and prove a cohomological vanishing result that is used to associate Galois representations to mod p Hilbert modular forms.

1. Introduction

1.1. **Overview.** Suppose that $N \geq 4$ is an integer and p is a prime not dividing N, and let $X_1(N;p)$ denote the modular curve associated to the group $\Gamma_1(N) \cap \Gamma_0(p)$. According to a fundamental result of Deligne and Rapoport [DR73, V.1], the curve $X_1(N;p)$ has a semistable model over $\mathbf{Z}[1/N]$ whose reduction mod p is a union of two irreducible components, each isomorphic to the reduction of $X_1(N)$, the modular curve associated to $\Gamma_1(N)$. The model is defined by viewing $X_1(N;p)$ as parametrizing isogenies of degree p between elliptic curves (with a point of order N), the components in characteristic p are described by whether the isogeny or its dual has connected kernel, and they cross at the points corresponding to supersingular elliptic curves. In a similar vein, they describe (in [DR73, V.2]) a semistable model over $\mathbf{Z}[\zeta_p, 1/N]$ for the modular curve $X_1(Np)$ associated to $\Gamma_1(Np)$; its structure in characteristic p underpins an elegant relation, due to Serre (see [KM85, Thm. 12.8.8] and [Gro90, §8]), between mod p modular forms of weight 2 with respect to $\Gamma_1(Np)$ and those of weights ranging from 2 to $p+1$ with respect to $\Gamma_1(N)$.

The analogous situation becomes more complicated for Hilbert modular varieties, i.e., the Shimura varieties associated to $\mathrm{Res}_{F/\mathbf{Q}}\,\mathrm{GL}_2$ where F is a totally real number field. Recall that these varieties have dimension $d = [F : \mathbf{Q}]$, and can be viewed as (coarse) moduli spaces for certain d-dimensional abelian varieties with additional structure. We restrict our attention to the case where p is unramified in F, and consider the setting, analogous to the one above, of Hilbert modular varieties with Iwahori level at p. In [Pap95], Pappas defined models for these varieties over

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$\mathbf{Z}_{(p)}$, which he proved were flat local complete intersections of relative dimension d. The geometry of their reduction mod p has been studied by various authors, as will be discussed below. In brief, some collections of components can be identified with reductions of Hilbert modular varieties of level prime to p, as in the classical case, but there are also "intermediate" components with no such description. Our first main result is a geometric Jacquet–Langlands relation that describes them as products of projective bundles over quaternionic Shimura varieties of level prime to p.

In [Hel12] Helm proves results of a similar nature in the case of unitary Shimura varieties which imply the following: collections of components in a Hilbert modular variety with Iwahori level at p are related via *Frobenius factors* to a product of projective bundles over quaternionic Shimura varieties of level prime to p. While some of our geometric methods are inspired by Helm's work, we would like to point out an essential difference in the results: the quaternion algebras appearing in our work are *different*, leading to the existence of isomorphisms on the nose (as opposed to Frobenius factors). Apart from facilitating new applications, we consider the merit of these results to be the naturality of the relationships they establish between the mod p geometry of Shimura varieties associated to different reductive groups.

There is also an essential difference between our method and the one in [Hel12]: our construction of the above-mentioned isomorphisms is more direct in that it does not involve the degeneracy (or forgetful) map to the prime-to-p level. Indeed, the image of a point corresponding to an isogeny of abelian varieties is constructed by "splicing" the Dieudonné modules of the isogeny's source and target. As a result, we believe our method to be more amenable to generalization to Shimura varieties associated to higher rank groups.

Our second main set of results concerns a generalization of Serre's relation between mod p modular forms of weight 2 and level Np and those of weight $k \in [2, p+1]$ and level prime to p. More precisely, we obtain a filtration on the space of mod p Hilbert modular forms of parallel weight 2 and pro-p Iwahori level at p, and identify the graded pieces with spaces of quaternionic modular forms of level prime to p and weight (components) in $[2, p+1]$. We accomplish this by combining our geometric Jacquet-Langlands relation with an analysis of dualizing sheaves by a method we call "dicing". The motivation for the result, discussed further below, comes from the relation between algebraic and geometric Serre weights explored in [DS17]. In fact, this relation was a critical clue to our understanding of the more canonical choice of quaternion algebras.

Our final set of geometric results centers on the degeneracy map from the Hilbert modular variety of Iwahori level at p to level prime to p (i.e., the one intervening in Helm's approach as opposed to ours). In particular, we apply techniques from crystalline Dieudonné theory to determine the precise structure of its fibres (restricted to irreducible components). These results complement those given in the Key Lemma in [GK12], and are expected to have applications in the context of that work. We give a different application in this paper: combining this with our method of dicing dualizing sheaves, we prove a cohomological vanishing result which is a key ingredient in the construction in [DS17] of Galois representations associated to (non-paritious) Hilbert modular eigenforms in characteristic p.

1.2. **Geometric Jacquet–Langlands.** We first introduce the notation for our main objects of study: certain mod p Shimura varieties and automorphic bundles. Their precise definitions are given in §§2–4, along with various technical results that will be needed later in the paper.

We fix a totally real field F and a prime p unramified in F. We also fix embeddings $\overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_p$ and $\overline{\mathbf{Q}} \to \mathbf{C}$, and let Θ denote the set of embeddings $F \to \overline{\mathbf{Q}}$. We can thus identify Θ with the sets of homomorphisms:

$$\Theta_\infty := \{F \to \mathbf{R}\}, \quad \Theta_p := \{F \to \overline{\mathbf{Q}}_p\} \quad \text{and} \quad \overline{\Theta}_p := \{O_F/p \to \overline{\mathbf{F}}_p\}.$$

We let ϕ denote the Frobenius automorphism of $\overline{\mathbf{F}}_p$. Furthermore, we have a natural decomposition $\overline{\Theta}_p = \coprod_{v|p} \Theta_v$, where $\theta \in \Theta_v$ if and only if θ factors through $\mathcal{O}_F/v$, and each Θ_v is an orbit in $\overline{\Theta}_p$ under $\theta \mapsto \phi \circ \theta$.

Let $G = \mathrm{Res}_{F/\mathbf{Q}}(\mathrm{GL}_2)$, and let U be a sufficiently small open compact subgroup of $G(\mathbf{A_f}) = \mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}})$ containing $\mathrm{GL}_2(\mathcal{O}_{F,p})$. The Hilbert modular variety with complex points

$$\mathrm{GL}_2(F)\backslash((\mathbf{C} - \mathbf{R})^\Theta \times \mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}}))/U$$

has a canonical integral model over $\mathbf{Z}_{(p)}$, which we denote $Y_U(G)$, and write $\overline{Y} = Y_U(G) \times_{\mathbf{Z}_p} \overline{\mathbf{F}}_p$ for its geometric special fibre. Similarly for any non-empty $\Sigma \subset \Theta$ of even cardinality, consider the quaternion algebra $B = B_\Sigma$ over F ramified at precisely the set of infinite places corresponding to Σ, and let G_Σ denote the algebraic group over $\mathbf{Q}$ defined by $G_\Sigma(R) = (B \otimes_{\mathbf{Q}} R)^\times$. Choosing a maximal order $\mathcal{O}_B$ of B and an isomorphism $\widehat{\mathcal{O}}_B \cong M_2(\widehat{\mathcal{O}}_F)$, we can identify U with an open compact subgroup of $G_\Sigma(\mathbf{A_f}) \cong G(\mathbf{A_f})$ and consider the quaternionic Shimura variety with complex points

$$B^\times\backslash((\mathbf{C} - \mathbf{R})^{\Theta - \Sigma} \times (B \otimes \mathbf{A_f})^\times)/U.$$

We let $Y_U(G_\Sigma)$ denote its canonical integral model, defined over the localization at a prime over p in its reflex field; denote its geometric special fibre by $\overline{Y}_\Sigma$, and let $\overline{Y}_\varnothing = \overline{Y}$. Thus $\overline{Y}_\Sigma$ is a smooth variety over $\overline{\mathbf{F}}_p$ of dimension $|\Theta - \Sigma|$, which is proper if and only if $\Sigma \neq \varnothing$.

For each $\theta \in \Theta$, one can also define a rank two automorphic vector bundle $\mathcal{V}_\theta$ on $\overline{Y}_\Sigma$, together with a line bundle $\omega_\theta \subset \mathcal{V}_\theta$ whenever $\theta \notin \Sigma$. We let δ_θ denote the line bundle $\wedge^2_{\mathcal{O}_{\overline{Y}_\Sigma}} \mathcal{V}_\theta$ on $\overline{Y}_\Sigma$. For $(k, \ell) \in \mathbf{Z}^\Theta \times \mathbf{Z}^\Theta$ with $k_\theta \geq 2$ for all $\theta \in \Sigma$, we define the automorphic bundle of weight (k, ℓ) on $\overline{Y}_\Sigma$ as

$$\mathcal{A}_{k,\ell} = \left(\bigotimes_{\theta \notin \Sigma} \delta_\theta^{\ell_\theta} \omega_\theta^{k_\theta}\right) \bigotimes \left(\bigotimes_{\theta \in \Sigma} \delta_\theta^{\ell_\theta} \mathrm{Sym}^{k_\theta - 2} \mathcal{V}_\theta\right).$$

The space of mod p modular forms of weight (k, ℓ) and level U with respect to G_Σ is then defined as $H^0(\overline{Y}_\Sigma, \mathcal{A}_{k,\ell})$, under the assumption $F \neq \mathbf{Q}$. (For $F = \mathbf{Q}$, one has to extend the line bundle to the cusps in order to recover the usual notion.) The bundles are equipped with a natural action of $G_\Sigma(\mathbf{A}_\mathbf{f}^{(p)})$ defined compatibly with its action on the varieties $\overline{Y}_\Sigma$ for varying U, yielding a Hecke action on the spaces of forms.

Restrict attention now to the case $G = \mathrm{Res}_{F/\mathbf{Q}}(\mathrm{GL}_2)$ and consider the open compact subgroup

$$U_0(p) = \{g \in U \mid g_p \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \bmod p\mathcal{O}_{F,p}\}.$$

We let $Y_{U_0(p)}(G)$ denote the model over $\mathbf{Z}_{(p)}$ defined by Pappas in [Pap95] for the Hilbert modular variety of level $U_0(p)$, and let $\overline{Y}_0(p)$ denote its geometric special fibre. Pappas studied the local structure of $\overline{Y}_0(p)$ and proved that it is a flat local complete intersection of relative dimension $d = [F : \mathbf{Q}]$. The global geometry of $\overline{Y}_0(p)$ and its degeneracy map to $\overline{Y}$ were studied in [Sta97] in the case $d = 2$, and for general d in [GK12], through the introduction of a stratification. A similar stratification was considered in [Hel12] in the context of related unitary Shimura varieties, and a generalization to the context of Hilbert modular varieties with p ramified in F was studied in [ERX17].

The stratification on $\overline{Y}_0(p)$ is given by closed subvarieties indexed by pairs (I, J) of subsets of $\Theta = \overline{\Theta}_p$ satisfying $(\phi^{-1}I) \cup J = \Theta$. The d-dimensional strata have the form $\overline{Y}_0(p)_{I,J}$ where $I = \{\phi \circ \theta \mid \theta \notin J\}$, which we denote simply by $\overline{Y}_0(p)_J$. Thus $\overline{Y}_0(p) = \bigcup_{J\subset\Theta} \overline{Y}_0(p)_J$ where each $\overline{Y}_0(p)_J$ is a smooth d-dimensional variety over $\overline{\mathbf{F}}_p$, and each irreducible component of $\overline{Y}_0(p)$ lies in $\overline{Y}_0(p)_J$ for a unique $J \subset \Theta$. We can then state our geometric Jacquet–Langlands relation (proved in §5) as follows:

Theorem A. *For each $J \subset \overline{\Theta}_p$ and sufficiently small open compact subgroup U of $\mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}})$ containing $\mathrm{GL}_2(O_{F,p})$, there is a Hecke-equivariant isomorphism*

$$\overline{Y}_0(p)_J \xrightarrow{\sim} \prod_{\theta\in\Sigma} \mathbf{P}_{\overline{Y}_\Sigma}(\mathcal{V}_\theta),$$

where the product is a fibre product over $\overline{Y}_\Sigma$, and $\Sigma = \Sigma_J \subset \Theta_\infty$ corresponds under the identification $\Theta_\infty = \overline{\Theta}_p$ to $\{\theta \in J \mid \phi\circ\theta \notin J\} \cup \{\theta \notin J \mid \phi \circ \theta \in J\}$.

As we mentioned earlier, a similar result was proved in the context of related unitary Shimura varieties by Helm ([Hel12, Thm. 5.10]) though the morphism constructed by Helm is only proved to be bijective on points, i.e., a "Frobenius factor" in the terminology of [Hel12]. Using $'$ to denote the analogous unitary Shimura varieties, Helm's approach is to relate the $\overline{Y}'_0(p)_J$ to products of $\mathbf{P}^1$-bundles over lower-dimensional strata of $\overline{Y}'$, and to relate those in turn to products of $\mathbf{P}^1$-bundles over lower-dimensional Shimura varieties. We had initially hoped to use Helm's result for the application we had in mind, despite the presence of Frobenius factors in his construction. Indeed we were encouraged by the fact that results of Tian–Xiao [TX16] remove the Frobenius factor from the latter step. While the Frobenius factor is intrinsic to the former step (see Theorem D below), a more serious problem was that the set of ramified places for the quaternion algebra provided by the results in [Hel12] does not match the set Σ_J determining the vector bundles $\mathcal{V}_\theta$. This led us to the consideration of different quaternion algebras: in fact the ones that emerge naturally from our method of "splicing" described below, which is more direct and bypasses the projections to strata of $\overline{Y}'$.

To prove Theorem A, we first prove the analogous result in the context of related unitary Shimura varieties (as in [TX16]), so that the quaternionic Shimura varieties are replaced by ones which are moduli spaces for abelian varieties. Denoting the special fibres of the corresponding unitary Shimura varieties by $\overline{Y}'_0(p)$ and $\overline{Y}'_\Sigma$, the stratum $\overline{Y}'_0(p)_J$ parametrizes p-isogenies $A \to B$ such that the induced morphism on Dieudonné modules satisfies conditions determined by J. The idea is to define morphisms to (projective bundles over) $\overline{Y}'_\Sigma$ by splicing the Dieudonné modules of

A and B to obtain an abelian variety C corresponding to a point of $\overline{Y}'_\Sigma$. We then prove this yields an isomorphism analogous to the one we want, and explain how to transfer the result to the Hilbert/quaternionic setting to obtain Theorem A using results in §2. We remark that a key idea that enables us to obtain such clean results in comparison to [Hel12] and [TX16] lies in exploiting the possibility of allowing A and B to play symmetric roles.

1.3. **The Serre filtration.** We now describe in more detail the application of Theorem A we had in mind, and carried out in §6. Recall that we wish to generalize a result of Serre relating mod p modular forms of weight 2 and level $\Gamma_1(Np)$ to mod p modular forms of weights $k \in [2, p+1]$ and level $\Gamma_1(N)$ (see [KM85, Thm. 12.8.8]). More precisely, let $X_1(Np)$ denote the semistable model over $R = \mathbf{Z}_{(p)}[\mu_p]$ for the compact modular curve $X_1(Np)$ (as in [Gro90, §7]) and let $\mathcal{K}$ denote its dualizing sheaf. Then $H^0(X_1(Np), \mathcal{K})$ is a lattice over R in the space of weight two cusp forms with respect to $\Gamma_1(Np)$, and tensoring over R with $\overline{\mathbf{F}}_p$ yields $H^0(\overline{X}_1(Np), \overline{\mathcal{K}})$, where $\overline{X}_1(Np)$ is the special fibre of $X_1(Np)$ and its dualizing sheaf $\overline{\mathcal{K}}$ is its sheaf of regular (Rosenlicht) differentials. The space $H^0(\overline{X}_1(Np), \overline{\mathcal{K}})$ decomposes as a direct sum of eigenspaces with respect to the natural action of $(\mathbf{Z}/p\mathbf{Z})^\times$. Writing the characters $(\mathbf{Z}/p\mathbf{Z})^\times \to \mathbf{F}_p^\times$ as $\chi_m : a \mapsto a^m$ for $m = 1, 2, \ldots, p-1$, Serre's result, as refined by Gross (see Propositions 8.13 and 8.18 of [Gro90]), gives a Hecke-equivariant exact sequence

$$0 \to H^0(\overline{X}, \delta^m \omega^{p+1-m}(-C)) \to H^0(\overline{X}_1(Np), \overline{\mathcal{K}})^{\chi_m} \to H^0(\overline{X}, \omega^{m+2}(-C)) \to 0,$$

where $\overline{X}$ is the reduction of $X_1(N)$ and $\omega^k(-C)$ is the line bundle whose sections are (mod p) cusp forms of weight k with respect to $\Gamma_1(N)$, and δ is a trivial bundle whose presence has the effect of twisting the action of the Hecke operator T_q by q.

The above exact sequence can be viewed as a geometric counterpart to the one arising in the cohomology of the modular curve $\Gamma_1(N)\backslash\mathfrak{H}$ with coefficients in local systems associated to the right $\mathrm{GL}_2(\mathbf{F}_p)$-modules in the exact sequence:

$$0 \to \det{}^m \otimes \mathrm{Sym}^{p-1-m}\mathbf{F}_p^2 \to \mathrm{Ind}_P^{\mathrm{GL}_2(\mathbf{F}_p)}(1 \otimes \chi_m) \to \mathrm{Sym}^m \mathbf{F}_p^2 \to 0,$$

where P is the subgroup of upper-triangular matrices and $1 \otimes \chi_m$ is the character sending $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ to $\chi_m(a) = d^m$. From the point of view of Serre weight conjectures for Galois representations, and in particular the relation between algebraic and geometric notions of Serre weights as explored in [DS17], it is natural to seek a generalization of Serre's result to the context of Hilbert modular forms. The desired result should describe the space of mod p forms of parallel weight 2 and character χ with respect to pro-p-Iwahori level at p in terms of spaces of mod p forms of level prime to p and weights corresponding to the Jordan–Holder factors of the right representation $\mathrm{Ind}_P^{\mathrm{GL}_2(\mathcal{O}_F/p\mathcal{O}_p)}\chi$, where P is again the subgroup of upper-triangular matrices and $\chi : P \to \overline{\mathbf{F}}_p^\times$ is any character, which by twisting easily reduces to the case of characters the form $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \mapsto \chi(d)$ where χ is a character on $(\mathcal{O}_F/p\mathcal{O}_F)^\times$.

To that end, we maintain the notation from the discussion before Theorem A and now consider

$$U_1(p) = \{g \in U \mid g_p \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \bmod p\mathcal{O}_{F,p}\}.$$

Then Pappas provides us with a model for $Y_{U_1(p)}(G)$ which is finite and flat over $Y_{U_0(p)}(G)$, and hence Cohen–Macaulay over $\mathbf{Z}_{(p)}$. Let $\overline{Y}_1(p)$ denote its geometric special fibre, and let $\mathcal{K}$ (resp. $\overline{\mathcal{K}}$) denote the dualizing sheaf on $Y_{U_1(p)}(G)$ (resp. $\overline{Y}_1(p)$). Via the Kodaira–Spencer isomorphism $H^0(Y_{U_1(p)}(G),\mathcal{K})$ can be identified with a lattice over $\mathbf{Z}_{(p)}$ in the space of Hilbert modular forms of parallel weight 2 and level $U_1(p)$ over $\mathbf{Q}$, and we view $H^0(\overline{Y}_1(p),\overline{\mathcal{K}})$ as the space of mod p forms of parallel weight 2 and level $U_1(p)$. The natural action of the group $U_0(p)/U_1(p)\cong(\mathcal{O}_F/p\mathcal{O}_F)^\times$ on $H^0(\overline{Y}_1(p),\overline{\mathcal{K}})$ yields a decomposition

$$H^0(\overline{Y}_1(p),\overline{\mathcal{K}}) = \bigoplus_{\chi} H^0(\overline{Y}_1(p),\overline{\mathcal{K}})^\chi$$

into eigenspaces for the characters $\chi:(\mathcal{O}_F/p\mathcal{O}_F)^\times\to\overline{\mathbf{F}}_p^\times$.

Before stating the second main result, we recall two basic facts from the representation theory of $\mathrm{GL}_2(\mathcal{O}_F/p\mathcal{O}_F)$. Firstly, its irreducible (right) representations over $\overline{\mathbf{F}}_p$ are precisely those of the form:

$$V_{m,n} = \bigotimes_{\theta\in\overline{\Theta}_p} \det{}^{m_\theta}\otimes \mathrm{Sym}^{n_\theta} V_\theta$$

for $(m,n)\in\mathbf{Z}^\Theta\times\mathbf{Z}^\Theta$ with $0\le m_\theta,n_\theta\le p-1$ for all θ and $m_\theta<p-1$ for some θ in each Θ_v, and $V_\theta=\overline{\mathbf{F}}_p^2$ with $\mathrm{GL}_2(\mathcal{O}_F/p\mathcal{O}_F)$ acting via θ. Secondly (see [BP12, §2]), there is a decreasing filtration

$$0\subset \mathrm{Fil}^d V_\chi\subset \mathrm{Fil}^{d-1}V_\chi\subset\cdots\subset\mathrm{Fil}^1V_\chi\subset\mathrm{Fil}^0V_\chi=V_\chi$$

on the representation $V_\chi=\mathrm{Ind}_P^{\mathrm{GL}_2(\mathcal{O}_F/p\mathcal{O}_p)}\chi$ such that the graded part gr^jV_χ has the form $\bigoplus_{|J|=j}V_{\chi,J}$, where each $V_{\chi,J}$ is either irreducible or zero. We then prove the following theorem in §6.3.3 (see §6.3.4 for elaboration on the meaning of Hecke-equivariance):

Theorem B. *For each sufficiently small open compact subgroup U of $\mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}})$ containing $\mathrm{GL}_2(O_{F,p})$, there is a Hecke-equivariant spectral sequence*

$$E_1^{j,i}=\bigoplus_{|J|=j}H^{i+j}(\overline{Y}_{\Sigma_J},\mathcal{A}_{\chi,J})\Longrightarrow H^{i+j}(\overline{Y}_1(p),\overline{\mathcal{K}})^\chi,$$

where $\mathcal{A}_{\chi,J}=\mathcal{A}_{n+2,m}$ (resp. 0) if $V_{\chi,J}\cong V_{m,n}$ (resp. 0).

We thus obtain the following generalization of Serre's filtration, where the graded pieces take the same form as in the classical case, except that spaces of Hilbert modular forms are in general replaced by the quaternionic ones to which they correspond via Jacquet–Langlands.

Corollary C. *For each sufficiently small open compact subgroup U of $\mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}})$ containing $\mathrm{GL}_2(O_{F,p})$, there is a Hecke-equivariant decreasing filtration of length $d+1$ on $H^0(\overline{Y}_1(p),\overline{\mathcal{K}})^\chi$, together with a Hecke-equivariant inclusion:*

$$\mathrm{gr}^j\left(H^0(\overline{Y}_1(p),\overline{\mathcal{K}})^\chi\right)\hookrightarrow\bigoplus_{|J|=j}H^0(\overline{Y}_{\Sigma_J},\mathcal{A}_{\chi,J})$$

for $j=0,1,\dots,d$.

In order to prove Theorem B, we consider the direct image of $\mathcal{K}$ under the projection $\overline{Y}_1(p) \to \overline{Y}_0(p)$, decompose it into line bundles $\mathcal{K}_\chi$ on $\overline{Y}_0(p)$ under the action of $U_0(p)/U_1(p)$, and define a filtration on $\mathcal{K}_\chi$ by restricting to the strata. Using the fact that the $\overline{Y}_0(p)_J$ are obtained by successively bisecting $\overline{Y}_0(p)$ into local complete intersections, we obtain a description of the graded pieces of the filtration in terms of line bundles $\mathcal{K}_{\chi,J}$ on the $\overline{Y}_0(p)_J$. Our method of "dicing" the dualizing sheaf may be of independent interest, and is used again in the proof of Theorem E below. The proof of Theorem B is then completed by determining the line bundles to which the $\mathcal{K}_{\chi,J}$ correspond under the isomorphism of Theorem A.

1.4. **Degeneracy fibres.** Our final set of results, the subject of §7, concerns the degeneracy map $\overline{Y}_0(p) \to \overline{Y}$, or more precisely, its restriction to the stratum $\overline{Y}_0(p)_J$ (maintaining the above notation). This restriction is known (see [GK12]) to factor through a pointwise bijective morphism ξ_J from $\overline{Y}_0(p)_J$ to a product of $\mathbf{P}^1$-bundles over a lower-dimensional stratum in $\overline{Y}$; thus ξ_J is a Frobenius factor in the sense of [Hel12]. We make this more precise by showing that ξ_J is a factor of the Frobenius itself (rather than a power), and we go on to determine the precise structure of the fibres of $\overline{Y}_0(p)_J \to \overline{Y}$. In particular we prove the following (see Theorem 7.2.4 and the subsequent discussion for an even more precise version):

Theorem D. *If Z is a non-empty fibre of the morphism $\overline{Y}_0(p)_J \to \overline{Y}$ over a closed point of $\overline{Y}$, then Z is isomorphic to $(\mathbf{P}^1_{\overline{\mathbf{F}}_p})^r \times (\mathrm{Spec}(\overline{\mathbf{F}}_p[t]/(t^p)))^s$, where $r = |\Sigma_J|/2$ and $s = |J| - r$.*

Our approach to proving Theorem D relies heavily on crystalline Dieudonné theory. In particular, we use the full faithfulness of the Dieudonné crystal functor over smooth bases, due to Berthelot–Messing [BM90], in order to obtain the Frobenius factorization, which we then use to determine the local structure of the fibre Z. In order to determine the global structure of Z, we show that certain pointwise relations between the Dieudonné modules of A and B (where $A \to B$ is a universal isogeny) in fact arise from isomorphisms of crystals over Z.

We remark that the problem of describing the fibres of $\overline{Y}_0(p)_J \to \overline{Y}$ is also considered in [ERX17, §4.9], where a weaker result than Theorem D is used in their approach to constructing Hecke operators at primes dividing p. Related results, complementary to ones in this paper, are obtained in [GK12] where the degeneracy morphism is studied before restriction to the strata. Further motivation for such analysis of the degeneracy map is provided by its applications to p-adic analytic continuation of Hilbert modular forms via the dynamics of Hecke operators at primes over p, as in [Kas13] and [Sas19].

In this paper, we combine (the more precise version of) Theorem D with the method of dicing introduced in §6[1] in order to prove the following cohomological vanishing result.

Theorem E. *If $\pi : Y_{U_1(p)}(G) \to Y_U(G)$ is the natural projection (of models over $\mathbf{Z}_{(p)}$) and $\mathcal{K}$ is the dualizing sheaf on $Y_{U_1(p)}(G)$, then $R^i\pi_*\mathcal{K} = 0$ for all $i > 0$.*

We note that Theorem E (for $i = 1$) is a crucial ingredient in the proof of Theorem 6.1.1 of [DS17], which associates Galois representations to mod p Hilbert modular eigenforms of arbitrary weight.

[1]The results of §7 are, however, independent of those in §5.

1.5. **Questions.** We close the Introduction by listing several questions and directions for further research that are suggested by our work.

Question 1. Is there a more general framework for Theorems A and B where the group $G = \mathrm{Res}_{F/\mathbf{Q}}\,\mathrm{GL}_2$ is replaced by the one associated to a quaternion algebra over F (or an even more general reductive group), and the representation $\mathrm{Ind}\,\chi$ is replaced by any tamely ramified (or even more general) type? For a totally definite quaternion algebra, where the associated Shimura varieties are zero-dimensional, the analogues of the theorems are essentially tautologies, while the case of Shimura curves is related to the work of Newton–Yoshida [NY14].

Question 2. The flipping and twisting of weights that appear in the computation of the line bundles in the proof of Theorem B perfectly reflect the same phenomena in the computation of the Jordan–Hölder factors of the principal series types. Can one give a more conceptual explanation for this synchronized gymnastics?

Question 3. Note that Corollary C only produces an injection. The obstruction to proving that it is an isomorphism comes from terms of the form $H^1(\overline{Y}_{\Sigma_J}, \mathcal{A}_{\chi,J})$ in the spectral sequence, and one can construct examples where these do not vanish. If there is in fact a non-trivial cokernel, can one at least prove that the Hecke action on it is "Eisenstein"?

Question 4. A Hilbert modular eigenform f of parallel weight 2 and level $U_1(p)$, with coefficients in a finite extension $\mathcal{O}$ of $\mathbf{Z}_p$, determines a rank one submodule of the space of sections of the dualizing sheaf on $Y_{U_1(p)}(G)$ over $\mathcal{O}$, and hence a one-dimensional subspace of $H^0(\overline{Y}_1(p), \mathcal{K})^\chi$ for some χ. Motivated by the conjectures of [DS17], one can ask if its position in the filtration and inclusion of Corollary C is determined by the local Galois representations[2] $\rho_f|_{G_{F_v}}$ for $v|p$, or more precisely the invariants v_θ for $\theta \in \Theta_v$ associated to $\rho_f|_{G_{F_v}}$ as in the formulation of Breuil's Lattice Conjecture [Bre14, Conj. 1.2] (proved by Emerton–Gee–Savitt [EGS15]).

1.6. **Acknowledgments.** The authors would like to thank D. Helm for several useful conversations. We have already described how our results on geometric Jacquet–Langlands are inspired by those in his paper [Hel12], but we should also remark that this circle of ideas has deeper roots in Zink's work [Zin82], Serre's letters[3] [Ser96], Ribet's seminal paper [Rib90], Pappas's thesis [Pap92] and Ghitza's work [Ghi04]. It is a pleasure to acknowledge that some of the seeds for this paper were in fact planted by Pappas's description of the results in his thesis to one of the authors (F.D.) at Columbia in the 1990's; they were only germinated in the last few years by ideas diffusing from the geometric Serre weight conjectures in [DS17].

We also learned much from Y. Tian and L. Xiao's paper [TX16] and are grateful to the authors for responding to several questions about it.

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2. Shimura varieties

[2]Together with the U_v-eigenvalue if f is old at v.

[3]Written in 1987 and 1989.

2.1. **Hilbert modular varieties.** We begin with a slight variant, based on [DS17, §2], of the standard construction (e.g., in [Rap78, §1]) of integral canonical models for Hilbert modular varieties. The approach presented here to defining the moduli problem provides the most natural and convenient framework for our results.

2.1.1. *The Shimura datum.* We maintain the basic notation from the Introduction, so F is a totally real field of degree $d = [F : \mathbf{Q}]$ in which p is unramified, and we identify $\Theta = \{\theta : F \to \overline{\mathbf{Q}}\}$ with $\Theta_\infty = \{F \to \mathbf{R}\}$, $\Theta_p = \{F \to \overline{\mathbf{Q}}_p\}$ and $\overline{\Theta}_p = \{\mathcal{O}_F \to \overline{\mathbf{F}}_p\}$ via fixed embeddings $\overline{\mathbf{Q}} \to \mathbf{C}$ and $\overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_p$. We let $\mathbf{A}_F$ denote the adeles of F (omitting the subscript for $F = \mathbf{Q}$), $\mathbf{A}_{F,\mathbf{f}} = F \otimes \widehat{\mathbf{Z}}$ the finite adeles, and $\mathbf{A}_{F,\mathbf{f}}^{(p)} = F \otimes \widehat{\mathbf{Z}}^{(p)}$ the prime-to-p finite adeles.

Let $G = \mathrm{Res}_{F/\mathbf{Q}}\,\mathrm{GL}_2$ and $\mathbf{S} = \mathrm{Res}_{\mathbf{C}/\mathbf{R}}\mathbf{G}_m$, and consider the Shimura datum $(G, [h])$, where $[h]$ is the $G(\mathbf{R})$-conjugacy class of the homomorphism $h : \mathbf{S} \to G_{\mathbf{R}}$ defined on $\mathbf{R}$-points by the map $\mathbf{C}^\times \to \mathrm{GL}_2(\mathbf{R})^\Theta$ sending $x + iy \mapsto \left(\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix}\right)_{\theta\in\Theta}$. Let U be an open compact subgroup of $G(\mathbf{A}_\mathbf{f}) = \mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}})$ containing $\mathrm{GL}_2(\mathcal{O}_{F,p})$, which we assume is sufficiently small (as in [DS17]). Write $U = U^pU_p$, where $U^p \subset \mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}}^{(p)})$ and $U_p = \mathrm{GL}_2(\mathcal{O}_{F,p})$. The associated Shimura variety (of level U) is the Hilbert modular variety with complex points

$$\mathrm{GL}_2(F)\backslash((\mathbf{C} - \mathbf{R})^\Theta \times \mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}}))/U = \mathrm{GL}_2(F)_+\backslash(\mathfrak{H}^\Theta \times \mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}}))/U, \tag{1}$$

where $\mathfrak{H}$ is the complex upper-half plane and $\mathrm{GL}_2(F)_+$ denotes the subgroup of elements of $\mathrm{GL}_2(F)$ with totally positive determinant.

2.1.2. *The moduli problem.* Consider the functor $\widetilde{Y}_U(G)$ that associates to a $\mathbf{Z}_{(p)}$-scheme S the set of isomorphism classes of tuples $\underline{A} = (A, \iota, \lambda, \eta)$, where

- A is an abelian scheme over S of dimension d;
- ι is an embedding $\mathcal{O}_F \to \mathrm{End}_S(A)$ such that $\mathrm{Lie}(A/S)$ is, locally on S, free of rank one over $\mathcal{O}_F \otimes \mathcal{O}_S$;
- λ is a prime-to-p quasi-polarization[4] such that ι is invariant under the associated Rosati involution;
- η is a level U^p-structure on A, i.e., for a choice of geometric point $\overline{s}_i$ on each connected component S_i of S, the data of a $\pi_1(S_i, \overline{s}_i)$-invariant U^p-orbit of $\widehat{\mathcal{O}}_F^{(p)} = \mathcal{O}_F \otimes \widehat{\mathbf{Z}}^{(p)}$-linear isomorphisms

$$\eta_i : (\widehat{\mathcal{O}}_F^{(p)})^2 \to T^{(p)}(A_{\overline{s}_i}),$$

where $T^{(p)}$ denotes the product over $\ell \neq p$ of the ℓ-adic Tate modules, and $g \in U^p$ acts on η_i by pre-composing with right multiplication by g^{-1}.

[4]By a *prime-to-p quasi-polarization*, we mean a quasi-isogeny λ such that $n\lambda$ is a polarization of degree prime to p for some positive integer n prime to p.

Note that for any connected S and (A,ι,λ,η) as above, there is a unique $\epsilon \in (\mathbf{A}_{F,\mathbf{f}}^{(p)})^\times/(\det U^p)(\widehat{\mathbf{Z}}^{(p)})^\times$ such that the following diagram commutes

$$\begin{array}{ccc}
(\widehat{\mathcal{O}}_F^{(p)})^2 \times (\widehat{\mathcal{O}}_F^{(p)})^2 & \xrightarrow{\ \wedge^2\ } & \mathbf{A}_{F,\mathbf{f}}^{(p)} \\
\Big\downarrow{\scriptstyle (\eta,\eta)} & & \Big\downarrow{\scriptstyle \epsilon\otimes\zeta} \\
T^{(p)}(A_{\overline{s}}) \times T^{(p)}(A_{\overline{s}}) & & \mathbf{A}_{F,\mathbf{f}}^{(p)}(1) \\
\Big\downarrow{\scriptstyle (1,\lambda)} & & \Big\downarrow{\scriptstyle \mathrm{Tr}_{F/\mathbf{Q}}} \\
T^{(p)}(A_{\overline{s}}) \times (\mathbf{Q}\otimes T^{(p)}(A_{\overline{s}}^\vee)) & \xrightarrow{\mathrm{Weil}} & \mathbf{A}_{\mathbf{f}}^{(p)}(1)
\end{array}$$

for some compatible system of prime-to-p roots of unity, i.e., some isomorphism $\zeta : \widehat{\mathbf{Z}}^{(p)} \to \widehat{\mathbf{Z}}^{(p)}(1)$, where the top arrow is the standard $\mathcal{O}_F$-bilinear alternating pairing sending $((a,b),(c,d))$ to $ad-bc$. For each $\epsilon \in (\mathbf{A}_{F,\mathbf{f}}^{(p)})^\times/(\det U^p)(\widehat{\mathbf{Z}}^{(p)})^\times$, the corresponding subfunctor $\widetilde{Y}_U^\epsilon(G)$ is representable by smooth quasi-projective scheme over $\mathbf{Z}_{(p)}$ (in fact a PEL Shimura variety associated to the preimage of $\mathbf{G}_m$ under $\det : G \to \mathrm{Res}_{F/\mathbf{Q}}\mathbf{G}_m$). It follows that $\widetilde{Y}_U(G)$ is representable by an infinite disjoint union of smooth quasi-projective schemes over $\mathbf{Z}_{(p)}$, which we also denote by $\widetilde{Y}_U^\epsilon(G)$. Thus $\widetilde{Y}_U(G)$ is smooth over $\mathbf{Z}_{(p)}$; in particular it is locally of finite type. We remark that the complex points of $\widetilde{Y}_U(G)$ are in canonical (holomorphic) bijection with the double quotient

$$\mathrm{SL}_2(F)_+\backslash(\mathfrak{H}^\Theta \times \mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}}))/U.$$

2.1.3. *Descent and Hecke action.* We have an action of $\mathcal{O}_{F,(p),+}^\times$ on $\widetilde{Y}_U(G)$ defined by $\theta_\mu(A,\iota,\lambda,\eta) = (A,\iota,\mu\lambda,\eta)$ for $\mu \in \mathcal{O}_{F,(p),+}^\times$. Note that $(U \cap \mathcal{O}_F^\times)^2$ acts trivially, and [DS17, Lemma 2.4.1] shows that the resulting action by $\mathcal{O}_{F,(p),+}^\times/(U\cap\mathcal{O}_F^\times)^2$ is free (for sufficiently small U). Furthermore $\widetilde{Y}_U(G)$ is the union of the orbits of finitely many $\widetilde{Y}_U^\epsilon(G)$ and the stabilizer of each $\widetilde{Y}_U^\epsilon(G)$ is the finite group $(\mathcal{O}_{F,+}^\times \cap \det(U))/(U\cap\mathcal{O}_F^\times)^2$, so the quotient of $\widetilde{Y}_U(G)$ by this action is a smooth quasi-projective scheme over $\mathbf{Z}_{(p)}$, which we denote by $Y_U(G)$.

We now define an action[5] of $G(\mathbf{A}_{\mathbf{f}}^{(p)})$ on $\varprojlim_U Y_U(G)$. Suppose that U_1 and U_2 are open compact subgroups as above, and that $g \in \mathrm{GL}_2(\mathbf{A}_{\mathbf{f}}^{(p)})$ is such that $g^{-1}U_1g \subset U_2$. Let $\underline{A}_1 = (A_1,\iota_1,\lambda_1,\eta_1)$ denote the universal abelian variety over $\widetilde{Y}_{U_1}(G)$. Let A' denote the abelian variety over $S = \widetilde{Y}_{U_1}(G)$ which is (prime-to-p) isogenous to A and satisfies

$$T^{(p)}(A'_{\overline{s}_i}) = \eta_{1,i}((\widehat{\mathcal{O}}_F^{(p)})^2g^{-1})$$

for all i (indexing the connected components of S). Then A' inherits an $\mathcal{O}_F$-action ι' from the identification $\mathrm{End}_S(A_1)\otimes\mathbf{Q} = \mathrm{End}_S(A')\otimes\mathbf{Q}$ induced by the canonical quasi-isogeny $\pi \in \mathrm{Hom}_S(A_1,A')\otimes\mathbf{Z}_{(p)}$, which furthermore induces an isomorphism

[5]We shall, throughout the paper, only consider locally Noetherian schemes, so we simply mean an action on the projective system, without concern for whether the system is representable by a scheme.

$\mathrm{Lie}(A_1/S) \to \mathrm{Lie}(A'/S)$ compatible with the $\mathcal{O}_F$-actions. Note also that the quasi-polarization λ' on A' defined by $\lambda_1 = \pi^\vee \circ \lambda' \circ \pi$ induces the same Rosati involution as λ_1. Moreover $\eta' = \eta_1 \circ r_{g^{-1}}$ (where $r_{g^{-1}}$ denotes right multiplication by g^{-1}), defines a level U_2-structure on A'. We then define $\tilde{\rho}_g : \widetilde{Y}_{U_1}(G) \to \widetilde{Y}_{U_2}(G)$ by $\underline{A}_1 \mapsto \underline{A}' = (A', \iota', \lambda', \eta')$, i.e., if $\underline{A}_2$ is the universal abelian variety over $\widetilde{Y}_{U_2}(G)$, then $\tilde{\rho}_g^*(\underline{A}_2) = \underline{A}'$. Since $\tilde{\rho}_g$ commutes with the action of $\mathcal{O}_{F,(p),+}^\times$, it descends to a morphism $\rho_g : Y_{U_1}(G) \to Y_{U_2}(G)$. Furthermore the morphisms $\tilde{\rho}_g$ and ρ_g are finite and étale. Finally if h and U_3 (again sufficiently small) are such that $h^{-1}U_2 h \subset U_3$, then $\tilde{\rho}_g^* \tilde{\rho}_h^*(\underline{A}_3) \cong \tilde{\rho}_{gh}^*(\underline{A}_3)$, so that $\tilde{\rho}_h \circ \tilde{\rho}_g = \tilde{\rho}_{gh}$ and $\rho_h \circ \rho_g = \rho_{gh}$, giving the desired action of $G(\mathbf{A}_\mathbf{f}^{(p)})$ on $\varprojlim_U Y_U(G)$ (the limit being taken under the maps ρ_1 for $U_1 \subset U_2$).

Proceeding as in the case of the related PEL Shimura varieties (or alternatively deducing the analogous results from the PEL setting), one finds that $Y_U(G)$ defines the canonical model over $\mathbf{Q}$ for the Hilbert modular variety in (1) in the sense of [Del71] (see also [Mil05, §12]), and hence that the $Y_U(G)$ define a system of integral canonical models in the sense of [Kis10].

2.2. **Unitary Shimura varieties.** In this section, we recall the construction of integral canonical models for certain unitary Shimura varieties, partly following [TX16, §3], but treating the non-PEL setting as we did in §2.1.

2.2.1. *The Shimura data.* Let Σ be any subset of Θ_∞ of even cardinality, and let $B = B_\Sigma$ denote the quaternion algebra over F ramified at precisely the places in Σ; thus B is split at all finite places, and $B = \mathrm{GL}_2(F)$ if Σ is empty. We let G_Σ denote the algebraic group over $\mathbf{Q}$ defined by $G_\Sigma(R) = (B \otimes R)^\times$. For $\Sigma \neq \varnothing$, we choose isomorphisms $B \otimes_{F,\theta} \mathbf{R} \cong M_2(\mathbf{R})$ for each $\theta \notin \Sigma$ and define $h_\Sigma : \mathbf{S} \to G_{\Sigma,\mathbf{R}}$ by sending $x + iy \mapsto \left(\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix}\right)_{\theta \notin \Sigma}$. In §2.3 we will recall how canonical models for the Shimura varieties associated to the Shimura datum $(G_\Sigma, [h_\Sigma])$ can be described in terms of the ones for certain related unitary groups, which we now define.

Choose a quadratic CM extension E of F such that all primes v of F dividing p split in E, and let c denote the non-trivial element of $\mathrm{Gal}(E/F)$. We let $\Theta_E = \{\tau : E \to \overline{\mathbf{Q}}\}$, which we identify with $\Theta_{E,\infty} = \{E \to \mathbf{C}\}$, $\Theta_{E,p} = \{E \to \overline{\mathbf{Q}}_p\}$ and $\overline{\Theta}_{E,p} = \{\mathcal{O}_F \to \overline{\mathbf{F}}_p\}$. Choose a subset $\tilde{\Sigma}$ of Θ_E mapping bijectively to Σ under $\tau \mapsto \tau|_F$, and define $s_{\tilde{\theta}} \in \{0, 1, 2\}$ for $\theta \in \Theta_E$ by:

- $s_\tau = 1$ if $\tau|_F \notin \Sigma$;
- $s_\tau = 0$ and $s_{\tau^c} = 2$ if $\tau \in \tilde{\Sigma}$.

We let T_F denote the torus $\mathrm{Res}_{F/\mathbf{Q}} \mathbf{G}_m$, and similarly let $T_E = \mathrm{Res}_{E/\mathbf{Q}} \mathbf{G}_m$. We define G'_Σ to be the quotient $(G_\Sigma \times T_E)/Z$, where $Z = T_F$ is embedded in $G_\Sigma \times T_E$ via $x \mapsto (x, x^{-1})$. We define $i_{\tilde{\Sigma}} : \mathbf{S} \to T_{E,\mathbf{R}}$ as the homomorphism which on $\mathbf{R}$-points is the composite

$$\mathbf{C}^\times \to \prod_{\theta \in \Sigma} \mathbf{C}^\times \cong \prod_{\theta \in \Sigma} (E \otimes_{F,\theta} \mathbf{R})^\times \hookrightarrow \prod_{\theta \in \Theta} (E \otimes_{F,\theta} \mathbf{R})^\times,$$

where the first map is diagonal, the second is defined on the θ-component by the embedding τ^c for the $\tau \in \tilde{\Sigma}$ such that $\tau|_F = \theta$, and the third is the natural inclusion. We define $h'_{\tilde{\Sigma}} : \mathbf{S} \to G'_{\Sigma,\mathbf{R}}$ as the composite of $(h_\Sigma, i_{\tilde{\Sigma}})$ with the projection $(G_\Sigma \times T_E)_\mathbf{R} \to G'_{\Sigma,\mathbf{R}}$. We let L_Σ (resp. $L_{\tilde{\Sigma}}$) denote the reflex field of the Shimura datum $(G_\Sigma, [h_\Sigma])$ (resp. $(G'_\Sigma, [h'_{\tilde\Sigma}])$, i.e., the fixed field in $\overline{\mathbf{Q}}$ of the stabilizer in $\mathrm{Aut}\overline{\mathbf{Q}}$ of Σ (resp. $\tilde\Sigma$).

2.2.2. *The moduli problem.* Let $D = D_\Sigma = E \otimes_F B_\Sigma$, and let $u \mapsto \overline{u}$ denote the anti-involution on D defined by $c \otimes \iota$, where ι is the standard anti-involution. We can then identify G'_Σ with the algebraic group defined by

$$G'_\Sigma(R) = \{\, g \in D \otimes R \mid g\overline{g} \in (F \otimes R)^\times \,\}.$$

Let $\mathcal{O}_D$ be an order of D such that $\mathcal{O}_{D,p} = \mathcal{O}_E \otimes_{\mathcal{O}_F} \mathcal{O}_{B,p}$ for a maximal order $\mathcal{O}_{B,p}$ of B_p. We choose an element $\delta \in D^\times$ such that

- $\delta \in \mathcal{O}_{D,p}^\times$,
- $\overline{\delta} = -\delta$,
- the bilinear form on $D \otimes \mathbf{R}$ defined by
$$(v, w) \mapsto \mathrm{Tr}_{E/\mathbf{Q}}(\mathrm{tr}_{D/E}(v\overline{h'_{\tilde\Sigma}}(i)\overline{w}\delta))$$
is positive definite.

(Thus our δ is $\delta\sqrt{\mathfrak{d}}$ in the notation of [TX16, Lemma 3.8].)

We define an anti-involution $u \mapsto u^*$ on D by $u^* = \delta^{-1}\overline{u}\delta$, and a pairing $\psi_E : D \times D \to E$ by

$$\psi_E(v, w) = \mathrm{tr}_{D/E}(v\overline{w}\delta) = \mathrm{tr}_{D/E}(v\delta w^*),$$

so ψ_E satisfies $\psi_E(w, v) = -\psi_E(v, w)^c$ and $\psi_E(uv, w) = \psi_E(v, u^*w)$ for all $u, v, w \in D$. We also define $\psi_F : D \times D \to F$ by $\psi_F = \mathrm{Tr}_{E/F} \circ \psi_E$, so ψ_F is alternating and satisfies $\psi_F(uv, w) = \psi_F(v, u^*w)$; in particular ψ_F is F-bilinear.

Now suppose that U' is a sufficiently small open compact subgroup of $G'_\Sigma(\mathbf{A_f})$ containing the image U'_p of $\mathcal{O}_{B,p}^\times \times \mathcal{O}_{E,p}^\times$ (under the natural map $G_\Sigma \times T_E \to G'_\Sigma$); as usual write $U' = (U')^p U'_p$ where $(U')^p \subset G'(\mathbf{A}_\mathbf{f}^{(p)})$. We will define an integral canonical model for the Shimura variety $Y_{U'}(G'_\Sigma)$ as a quotient of a representable moduli problem in a manner similar to that for Hilbert modular varieties in §2.1.

Let $\mathcal{O}$ denote the localization of $\mathcal{O}_{L_{\tilde\Sigma}}$ at the prime over p determined by the embedding $\overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_p$, and consider the functor that associates to an $\mathcal{O}$-scheme S the set of isomorphism classes of tuples $\underline{A} = (A, \iota, \lambda, \eta, \epsilon)$, where

- A is an abelian scheme over S of dimension $4d$ (where $d = [F : \mathbf{Q}]$);
- ι is an embedding $\mathcal{O}_D \to \mathrm{End}_S(A)$ such that for all $\alpha \in \mathcal{O}_E$, the characteristic polynomial of $\iota(\alpha)$ on $\mathrm{Lie}(A/S)$ is
$$\prod_{\tau \in \Theta_E} (x - \tau(\alpha))^{2s_\tau} \quad \in \quad \mathcal{O}[x].$$
- λ is a prime-to-p quasi-polarization whose associated Rosati involution is compatible with $l \mapsto l^*$ on D;
- (η, ϵ) is a level U'-structure on (A, ι, λ), i.e., a $\pi_1(S_i, \overline{s}_i)$-invariant $(U')^p$-orbit of pairs (η_i, ϵ_i) for a geometric point $\overline{s}_i$ on each connected component S_i of S, where η_i is an $\widehat{\mathcal{O}}_D^{(p)} = \mathcal{O}_D \otimes \widehat{\mathbf{Z}}^{(p)}$-linear isomorphism $\widehat{\mathcal{O}}_D^{(p)} \to T^{(p)}(A_{\overline{s}_i})$ and ϵ_i is an $\mathbf{A}_{F,\mathbf{f}}^{(p)}$-linear isomorphism $\mathbf{A}_{F,\mathbf{f}}^{(p)} \to \mathbf{A}_{F,\mathbf{f}}^{(p)}(1)$

such that the following diagram commutes:

$$\begin{array}{ccc}
\widehat{\mathcal{O}}_D^{(p)} \times \widehat{\mathcal{O}}_D^{(p)} & \xrightarrow{\psi_F} & \mathbf{A}_{F,\mathbf{f}}^{(p)} \\
\downarrow{\scriptstyle (\eta_i,\eta_i)} & & \downarrow{\scriptstyle \epsilon_i} \\
T^{(p)}(A_{\overline{s}_i}) \times T^{(p)}(A_{\overline{s}_i}) & & \mathbf{A}_{F,\mathbf{f}}^{(p)}(1) \\
\downarrow{\scriptstyle (1,\lambda)} & & \downarrow{\scriptstyle \mathrm{Tr}_{F/\mathbf{Q}}} \\
T^{(p)}(A_{\overline{s}_i}) \times (\mathbf{Q} \otimes T^{(p)}(A_{\overline{s}_i}^\vee)) & \xrightarrow{\text{Weil}} & \mathbf{A}_{\mathbf{f}}^{(p)}(1),
\end{array}$$

where $g \in (U')^p$ on η_i is by precomposing with right multiplication by g^{-1}, and on ϵ_i by multiplication by $\nu(g)^{-1}$ where $\nu : G'_\Sigma \to T_F$ is defined by $g \mapsto g\overline{g}$.

As in §2.1.2 for U' sufficiently small, the functor is representable by an infinite disjoint union of smooth quasi-projective schemes over $\mathcal{O}$, but these are now Shimura varieties for the preimage of $\mathbf{G}_m$ under ν, indexed by the classes of ϵ mod $\nu((U')^p)(\widehat{\mathbf{Z}}^{(p)})^\times$. We denote the representing $\mathcal{O}$-scheme by $\widetilde{Y}_{U'}(G'_\Sigma)$

2.2.3. *Descent and Hecke action.* In order to define the descent data to obtain the Shimura variety from $\widetilde{Y}_{U'}(G'_\Sigma)$, we need the following rigidity lemma.

Lemma 2.2.1. *Suppose that k is an algebraically closed field, $\underline{A} = (A,\iota,\lambda,\eta,\epsilon)$ is a k-point of $\widetilde{Y}_{U'}(G')$, $\mu \in \mathcal{O}_{F,(p),+}^\times$, and $\alpha \in \mathrm{Aut}_k(A)$ induces an isomorphism from $(A,\iota,\mu\lambda,\eta,\mu\epsilon)$ to $\underline{A}$. Then $\alpha = \iota(\gamma)$ for some $\gamma \in U' \cap E^\times \subset \mathcal{O}_E^\times$ such that $\mu = \mathrm{Nm}_{E/F}(\gamma)$ (assuming U' is sufficiently small).*

Proof. This is presumably standard, but here is a sketch. We wish to prove that $\alpha = \iota(\gamma)$ for some $\gamma \in E$, from which the rest follows easily, so suppose this is not the case. Since α is compatible with the $\mathcal{O}_D$-action on A and $D \cong M_2(E)$, A is isogenous to $B \times B$ for some B with $E' = E(\alpha) \subset \mathrm{End}(B) \otimes \mathbf{Q}$ (identifying E with its image in $\mathrm{End}(A)$ and $\mathrm{End}(B)$). It then follows from the classification of endomorphism algebras of abelian varieties (since $\dim(B) = [E : \mathbf{Q}]$) that E' is a quadratic CM extension of E. Moreover since α is an automorphism, it is a unit in an order in E', so $\alpha \in \mathcal{O}_{E'}^\times$.

Next note that the commutativity of

$$\begin{array}{ccc}
A & \xrightarrow{\mu\lambda} & A^\vee \\
\downarrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle \alpha^\vee} \\
A & \xrightarrow{\lambda} & A^\vee
\end{array}$$

shows that the image of α under the λ-Rosati involution is $\mu\alpha^{-1} \in E'$, and hence that $\mathrm{Nm}_{E'/F'}(\alpha) = \alpha\overline{\alpha} = \mu$, where F' is the totally real subfield of E'. Since $\mathcal{O}_E^\times$ is a finite index subgroup of $\{\, \beta \in \mathcal{O}_{E'}^\times \mid \mathrm{Nm}_{E'/F'}(\beta) \in \mathcal{O}_F^\times \,\}$, it follows that $\alpha^r \in \mathcal{O}_E^\times$ for some $r > 0$. One can then proceed exactly as in the proof of [DS17, Lemma 2.4.1] for example (with E' and E replacing K and F). ◻

Now consider the action of $\mathcal{O}_{F,(p),+}^\times$ on $\widetilde{Y}_{U'}(G'_\Sigma)$ defined by $\theta_\mu(A,\iota,\lambda,\eta,\epsilon) = (A,\iota,\mu\lambda,\eta,\mu\epsilon)$ for $\mu \in \mathcal{O}_{F,(p),+}^\times$. Note that $\mathrm{Nm}_{E/F}(U' \cap E^\times)$ acts trivially, and the preceding lemma shows that the resulting action by $\mathcal{O}_{F,(p),+}^{\times}/\mathrm{Nm}_{E/F}(U' \cap E^{\times})$ is free. Arguing as in §2.1.3 we see that the quotient of $\widetilde{Y}_{U'}(G'_\Sigma)$ by this action is a smooth quasi-projective scheme over $\mathcal{O}$, which we denote by $Y_{U'}(G'_\Sigma)$.

We define an action of $G'_\Sigma(\mathbf{A}_\mathbf{f}^{(p)})$ on $\varprojlim_{U'} Y_{U'}(G'_\Sigma)$ as in §2.1.3. Suppose that $g \in G'_\Sigma(\mathbf{A}_\mathbf{f}^{(p)})$ and that U'_1 and U'_2 are sufficiently small open compact subgroups as above satisfying $g^{-1}U'_1 g \subset U'_2$. Let $\underline{A}_1 = (A_1, \iota_1, \lambda_1, \eta_1, \epsilon_1)$ denote the universal abelian variety over $\widetilde{Y}_{U'_1}(G'_\Sigma)$. Let A' denote the abelian variety over $\widetilde{Y}_{U'_1}(G'_\Sigma)$ which is (prime-to-p) isogenous to A and satisfies

$$T^{(p)}(A'_{\overline{s}_i}) = \eta_{1,i}(\widehat{\mathcal{O}}_D^{(p)} g^{-1})$$

for all i (indexing connected components). Then A' inherits an $\mathcal{O}_D$-action ι' from the identification $\mathrm{End}_S(A_1) \otimes \mathbf{Q} = \mathrm{End}_S(A') \otimes \mathbf{Q}$ induced by the canonical quasi-isogeny $\pi \in \mathrm{Hom}_S(A_1, A') \otimes \mathbf{Z}_{(p)}$, which furthermore induces an isomorphism $\mathrm{Lie}(A_1/S) \to \mathrm{Lie}(A'/S)$ compatible with the $\mathcal{O}_D$-actions. Moreover the quasi-polarization λ' on A' defined by $\lambda_1 = \pi^\vee \circ \lambda' \circ \pi$ induces the same Rosati involution as λ_1. Letting $\eta' = \eta_1 \circ r_{g^{-1}}$ (where $r_{g^{-1}}$ denotes right multiplication by g^{-1}) and $\epsilon' = \nu(g)^{-1}\epsilon_1$, we find that (η', ϵ') defines a level U'_2-structure on (A', ι', λ'). We then define $\tilde{\rho}_g : \widetilde{Y}_{U'_1}(G'_\Sigma) \to \widetilde{Y}_{U'_2}(G'_\Sigma)$ by $\underline{A}_1 \mapsto \underline{A}' = (A', \iota', \lambda', \eta', \epsilon')$, i.e., if $\underline{A}_2$ is the universal abelian variety over $\widetilde{Y}_{U'_2}(G'_\Sigma)$, then $\tilde{\rho}_g^*(\underline{A}_2) = \underline{A}'$. Since $\tilde{\rho}_g$ commutes with the action of $\mathcal{O}_{F,(p),+}^{\times}$, it descends to a morphism $\rho_g : Y_{U'_1}(G'_\Sigma) \to Y_{U'_2}(G'_\Sigma)$. Furthermore the morphisms $\tilde{\rho}_g$ and ρ_g are finite and étale. Finally if h and U'_3 (again sufficiently small) are such that $h^{-1}U'_2 h \subset U'_3$, then $\tilde{\rho}_g^*\tilde{\rho}_h^*(\underline{A}_3) \cong \tilde{\rho}_{gh}^*(\underline{A}_3)$, so that $\tilde{\rho}_h \circ \tilde{\rho}_g = \tilde{\rho}_{gh}$ and $\rho_h \circ \rho_g = \rho_{gh}$, giving the desired action of $G'_\Sigma(\mathbf{A}_\mathbf{f}^{(p)})$ on $\varprojlim_{U'} Y_{U'}(G'_\Sigma)$ (the limit being taken under the maps ρ_1 for $U'_1 \subset U'_2$).

Proceeding again as in (or deducing from) the PEL case, one finds that the $Y_{U'}(G'_\Sigma)$ define a system of canonical models over $\mathcal{O}$ for the Shimura varieties associated to $(G'_\Sigma, [h'_{\widetilde{\Sigma}}])$. In particular the complex manifold associated to $Y_{U'}(G'_\Sigma)$ is identified with

$$G'_\Sigma(\mathbf{Q})\backslash((\mathbf{C}-\mathbf{R})^{\Theta-\Sigma} \times G'_\Sigma(\mathbf{A}_\mathbf{f}))/U' = G'_\Sigma(\mathbf{Q})_+\backslash(\mathfrak{H}^{\Theta-\Sigma} \times G'_\Sigma(\mathbf{A}_\mathbf{f}))/U',$$

where $G'_\Sigma(\mathbf{Q})_+$ denotes the subgroup of $G'_\Sigma(\mathbf{Q})$ consisting of γ such that $\nu(\gamma)$ is totally positive. For clarity, we remark that

$$\widetilde{Y}_{U'}(G'_\Sigma)(\mathbf{C}) = G^1_\Sigma(\mathbf{Q})\backslash(\mathfrak{H}^{\Theta-\Sigma} \times G'_\Sigma(\mathbf{A}_\mathbf{f}))/U',$$

where $G^1_\Sigma = \ker(\nu : G'_\Sigma \to T_F)$.

2.3. **Quaternionic Shimura varieties.** We now recall and make explicit the way in which canonical models for the Shimura varieties associated to G_Σ can be described in terms of those for G'_Σ. In particular, for suitably chosen levels, a Shimura variety associated to $G_\Sigma \times T_E$ provides a common finite étale cover such that the projection maps induce isomorphisms on geometric connected components.

2.3.1. *Tori.* We first consider various tori that intervene in the relation.

Let T' denote the abelian quotient of G'_Σ, which we may identify with the quotient $(T_F \times T_E)/T_F$, where T_F is embedded in the product via $x \mapsto (x^2, x^{-1})$. Let $\nu' : G'_\Sigma \to T'$ denote the natural projection, so that ν' is induced by the map

$G \times T_E \to T_F \times T_E$ defined by $(g, y) \mapsto (\det(g), y)$ where det is the reduced norm. Note that the inclusion $T_F \to T'$ induced by $x \mapsto (x, 1)$ is split by the projection $(x, y) \mapsto x\mathrm{Nm}_{E/F}y$, so that

$$T' \cong T_F \times (T_E/T_F) \cong T_F \times T_E^1,$$

where T_E^1 is the kernel of $\mathrm{Nm}_{E/F} : T_E \to T_F$, the isomorphisms being defined by

$$(x, y)T_F \leftrightarrow (xyy^c, yT_F) \leftrightarrow (xyy^c, y/y^c).$$

We recall the description of canonical models over $\mathcal{O}$ for zero-dimensional Shimura varieties associated to the tori T_E, T_F and T'. More precisely, consider the Shimura data $(T_E, i_{\tilde{\Sigma}})$, (T_F, i_Σ) and $(T', i'_{\tilde{\Sigma}})$, where $i_{\tilde{\Sigma}}$ was defined in §2.2.1, $i_\Sigma = \det \circ h_\Sigma$ sends $z \in \mathbf{S}(\mathbf{R}) = \mathbf{C}^\times$ to $(z\overline{z})_{\theta\notin\Sigma} \in T_F(\mathbf{R}) = \prod_{\theta\in\Theta} \mathbf{R}^\times$, and $i'_{\tilde{\Sigma}}$ is the composite of $(i_\Sigma, i_{\tilde{\Sigma}})$ with the natural projection. We continue to suppress the Deligne homomorphism from the notation for the associated Shimura varieties, denoting them simply $Y_{V_E}(T_E)$ (for open compact $V_E \subset \mathbf{A}_{E,\mathbf{f}}^\times$), etc.

Assuming $\mathcal{O}_{E,p}^\times \subset V_E$ and writing $V_E = V_E^p\mathcal{O}_{E,p}^\times$ with $V_E^p \subset (\mathbf{A}_{E,\mathbf{f}}^{(p)})^\times$, the geometric points of $Y_{V_E}(T_E)$ are canonically identified with the finite set

$$C_{V_E} = (\mathbf{A}_{E,\mathbf{f}}^{(p)})^\times/\mathcal{O}_{E,(p)}^\times V_E^p = E^\times\backslash\mathbf{A}_{E,\mathbf{f}}^\times/V_E.$$

As a scheme over $\mathcal{O}$, it can be characterized by descent from the canonical isomorphism

$$Y_{V_E}(T_E) \times_{\mathcal{O}} \mathcal{O}_M \cong \coprod_{c\in C_{V_E}} \operatorname{Spec}\mathcal{O}_M,$$

where M is an abelian extension of the reflex field $L_{\tilde{\Sigma}}$ and the action of $\mathrm{Gal}(M/L_{\tilde{\Sigma}})$ is given by the Shimura reciprocity law for $i_{\tilde{\Sigma}}$ (see [TX16, 2.7]). In particular, since $\mathcal{O}_{E,p}^\times \subset V_E$, we can assume M is unramified at the primes of $L_{\tilde{\Sigma}}$ dividing p, so $Y_{V_E}(T_E)$ is étale over $\mathcal{O}$.

Similarly for $V_F = V_F^p\mathcal{O}_{F,p}^\times$ with $V_F^p \subset (\mathbf{A}_{F,\mathbf{f}}^{(p)})^\times$, the geometric points of $Y_{V_F}(T_F)$ are canonically identified with the finite set

$$C_{V_F} = (\mathbf{A}_{F,\mathbf{f}}^{(p)})^\times/\mathcal{O}_{F,(p),+}^\times V_F^p = F_+^\times\backslash\mathbf{A}_{F,\mathbf{f}}^\times/V_F$$

with Galois action determined by Shimura reciprocity for i_Σ. Note however that we will view $Y_{V_F}(T_F)$ as a finite étale scheme over $\mathcal{O}$ rather than a localization of the ring of integers of its reflex field L_Σ. Likewise for $V' \subset T'(\mathbf{A}_\mathbf{f}^{(p)})$ containing the image of $\mathcal{O}_{F,p}^\times \times \mathcal{O}_{E,p}^\times$, $Y_{V'}(T')$ is a finite étale $\mathcal{O}$-scheme whose geometric points are identified with $C_{V'} = T'(\mathbf{Q})_+\backslash T'(\mathbf{A}_\mathbf{f})/V'$, where $T'(\mathbf{Q})_+ = (F_+^\times \times E^\times)/F^\times$.

Suppose now that U is an open compact subgroup of $G_\Sigma(\mathbf{A}_{F,\mathbf{f}})$ containing $U_{B,p}^\times$. We say that V_E is *sufficiently small relative to* U if the following hold:

- $E^\times \cap V_E^1 = \{1\}$, where $V_E^1 := \{\, y/y^c \mid y \in V_E \,\} \subset \mathbf{A}_{E,\mathbf{f}}^\times$;
- $V_E \cap \mathbf{A}_{F,\mathbf{f}}^\times \subset U$;
- $\mathrm{Nm}_{E/F}(V_E) \subset \det(U)$.

Note that V_E can be chosen sufficiently small to satisfy the first condition (independently of U and of level prime to p) since $\mathrm{Nm}_{E/F} : \mathcal{O}_E^\times \to \mathcal{O}_F^\times$ has finite kernel. Indeed this finiteness ensures we can choose an open subgroup $U_E \subset \mathbf{A}_{E,\mathbf{f}}^\times$ such that if $\alpha \in E^\times \cap U_E$ and $\mathrm{Nm}_{E/F}(\alpha) = 1$, then $\alpha = 1$, and then choose V_E contained in the preimage of U_E under $y \mapsto y/y^c$. It follows that we can choose V_E (of level prime to p) satisfying all three conditions.

2.3.2. *Complex points.* We next describe some explicit relations among the complex points of the relevant Shimura varieties. Note that the commutative diagram

$$\begin{array}{ccc} G_\Sigma \times T_E & \longrightarrow & G'_\Sigma \\ \downarrow & & \downarrow \\ T_F \times T_E & \longrightarrow & T' \end{array}$$

is compatible with the Deligne homomorphisms $(h_\Sigma, i_{\tilde\Sigma})$, $h'_{\tilde\Sigma}$, $(i_\Sigma, i_{\tilde\Sigma})$ and $i'_{\tilde\Sigma}$, and therefore gives rise to commutative diagrams of Shimura varieties whenever the corresponding open compact subgroups satisfy the evident compatibilities. In particular for sufficiently small open compact subgroups $U \subset G_\Sigma(\mathbf{A_f})$, $V_E \subset \mathbf{A}_{E,\mathbf{f}}^\times$ and $U' \subset G'_\Sigma(\mathbf{A_f})$ such that U' contains the image of $U \times V_E$ under the natural projection, we get the commutative diagram

(2)

$$\begin{array}{ccc} (G_\Sigma(\mathbf{Q})_+\backslash(\mathfrak{H}^{\Theta-\Sigma} \times G_\Sigma(\mathbf{A_f}))/U) \times C_{V_E} & \longrightarrow & G'_\Sigma(\mathbf{Q})_+\backslash(\mathfrak{H}^{\Theta-\Sigma} \times G'_\Sigma(\mathbf{A_f}))/U' \\ \downarrow & & \downarrow \\ C_{\det(U)} \times C_{V_E} & \longrightarrow & C_{\nu'(U')} \end{array}$$

on complex points of the associated Shimura varieties, where $G_\Sigma(\mathbf{Q})_+$ is the subgroup of $G_\Sigma(\mathbf{Q})$ consisting of elements whose reduced norm is totally positive.

Lemma 2.3.1. *If V_E is sufficiently small relative to U and U' is the image of $U \times V_E$, then the preceding diagram is Cartesian.*

Proof. The lemma immediately reduces to the claim that the following diagram is Cartesian:

$$\begin{array}{ccc} G_\Sigma(\mathbf{Q})_+\backslash(\mathfrak{H}^{\Theta-\Sigma} \times G_\Sigma(\mathbf{A_f}))/U & \longrightarrow & G'_\Sigma(\mathbf{Q})_+\backslash(\mathfrak{H}^{\Theta-\Sigma} \times G'_\Sigma(\mathbf{A_f}))/U' \\ \downarrow & & \downarrow \\ C_{\det(U)} & \longrightarrow & C_{\nu'(U')} \end{array}$$

Recall that $C_{\det(U)} = F_+^\times\backslash\mathbf{A}_{F,\mathbf{f}}^\times/\det(U)$ and $C_{\nu'(U')} = T'(\mathbf{Q})_+\backslash T'(\mathbf{A_f})/\nu'(U')$ where $T'(\mathbf{Q})_+ = (F_+^\times \times E^\times)/F^\times$. Note in particular that the inclusion $F_+^\times \to T'(\mathbf{Q})_+$ is split by $(x,y) \mapsto x\mathrm{Nm}_{E/F}y$, as is the inclusion $\det(U) \to \nu'(U')$ under our assumptions that U' is generated by $U \times V_E$ and $\mathrm{Nm}_{E/F}(V_E) \subset \det(U)$. Therefore the bottom row of the diagram is injective.

Identifying $G_\Sigma(\mathbf{A_f})$ and $T_E(\mathbf{A_f}) = \mathbf{A}_{E,\mathbf{f}}^\times$ with their images in $G'_\Sigma(\mathbf{A_f})$, we remark that

$$G'_\Sigma(\mathbf{A_f}) = \mathbf{A}_{E,\mathbf{f}}^\times G_\Sigma(\mathbf{A_f}) \quad \text{and} \quad G'_\Sigma(\mathbf{Q})_+ = E^\times G_\Sigma(\mathbf{Q})_+.$$

Indeed viewing $G'_\Sigma(\mathbf{Q})_+ \subset D^\times$, note that if $\gamma \in G'_\Sigma(\mathbf{Q})_+$, then $\gamma^c\gamma^\iota \in F_+^\times$ and $\gamma\gamma^\iota \in E^\times$, so $\alpha := \gamma^{-1}\bar\gamma \in E^\times$. Moreover $\mathrm{Nm}_{E/F}(\alpha) = 1$, so $\alpha = \beta^{-1}\beta^c$ for some $\beta \in E^\times$, and we can write $\gamma = \beta\delta$ where $\delta = \beta^{-1}\gamma \in G_\Sigma(\mathbf{Q})_+$. Similarly we have $G'_\Sigma(\mathbf{Q}_q) = E_q^\times G_\Sigma(\mathbf{Q}_q)$ for all primes q and $\{g \in \mathcal{O}_{D,q}^\times \mid g\bar g \in \mathcal{O}_{F,q}^\times\} = \mathcal{O}_{E,q}^\times\mathcal{O}_{B,q}^\times$ for all but finitely many q, so that $G'_\Sigma(\mathbf{A_f}) = \mathbf{A}_{E,\mathbf{f}}^\times G_\Sigma(\mathbf{A_f})$.

We consider separately the cases where B is indefinite and B is totally definite. Recall that if B is indefinite, then the vertical arrows induce bijections on sets of connected components. So to prove the lemma, it suffices to prove the top row restricts to an isomorphism on each connected component. We must therefore show that if $g \in G_\Sigma(\mathbf{A_f})$, then the map

$$\Gamma_g \backslash \mathfrak{H}^{\Theta - \Sigma} \to \Gamma'_g \backslash \mathfrak{H}^{\Theta - \Sigma}$$

is an isomorphism, where $\Gamma_g = G_\Sigma(\mathbf{Q})_+ \cap gUg^{-1}$ and $\Gamma'_g := G'_\Sigma(\mathbf{Q})_+ \cap gU'g^{-1}$. We show that under our hypotheses, the inclusion $\Gamma_g \subset \Gamma'_g$ is in fact an equality. Suppose then that $\gamma \in \Gamma'_g$, and write $\gamma = \beta\delta$ for some $\beta \in E^\times$, $\delta \in G_\Sigma(\mathbf{Q})_+$. Then $\beta\delta = hy$ for some $h \in U$, $y \in V_E$, which implies that $\beta/\beta^c = y/y^c \in E^\times \cap V_E^1 = \{1\}$. We therefore have $\beta = \beta^c \in F^\times$ and $y = y^c \in \mathbf{A}_{F,\mathbf{f}}^\times \cap V_E \subset U$, so that $\gamma \in \Gamma_g$.

Suppose now that B is totally definite. Now $\Sigma = \Theta$, $G_\Sigma(\mathbf{Q})_+ = G_\Sigma(\mathbf{Q})$ and $G'_\Sigma(\mathbf{Q})_+ = G'_\Sigma(\mathbf{Q})$, making the top arrow

$$G_\Sigma(\mathbf{Q}) \backslash G_\Sigma(\mathbf{A_f})/U \to G'_\Sigma(\mathbf{Q}) \backslash G'_\Sigma(\mathbf{A_f})/U'.$$

We must show that this map is injective, and that its image consists precisely of the classes $G'_\Sigma(\mathbf{Q})gU'$ such that $g \in G'_\Sigma(\mathbf{A_f})$ and $T'(\mathbf{Q})_+\nu'(g)\nu'(U')$ is in the image of the bottom arrow.

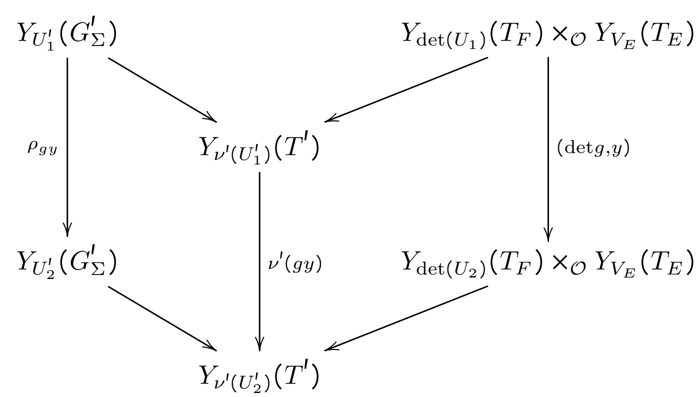
Suppose then that $h, h' \in G_\Sigma(\mathbf{A_f})$ and $h' \in G'_\Sigma(\mathbf{Q})hU'$. Since $G'_\Sigma(\mathbf{Q}) = E^\times G_\Sigma(\mathbf{Q})$ and $U' = UV_E$, we can write $h' = \beta\delta h g y$ for some $\beta \in E^\times$, $\delta \in G_\Sigma(\mathbf{Q})$, $g \in U$ and $y \in V_E$, so $\beta y = \beta^c y^c$. As in the indefinite case, we deduce that $\beta = \beta^c \in F^\times$ and $y = y^c \in U$, so that $h' \in G_\Sigma(\mathbf{Q})hU$, proving the injectivity.

Finally suppose that $g \in G'_\Sigma(\mathbf{A_f})$ is such that $T'(\mathbf{Q})_+\nu'(g)\nu'(U')$ is in the image of the bottom arrow. Since $G'_\Sigma(\mathbf{A_f}) = \mathbf{A}_{E,\mathbf{f}}^\times G_\Sigma(\mathbf{A_f})$, we can write $g = yh$ for some $y \in \mathbf{A}_{E,\mathbf{f}}^\times$ and $h \in G_\Sigma(\mathbf{A_f})$. The condition on $\nu'(g)$ implies $y \in E^\times \mathbf{A}_{F,\mathbf{f}}^\times V_E$, so we can write $y \in E^\times x V_E$ for some $x \in \mathbf{A}_{F,\mathbf{f}}^\times$, and conclude that $G'_\Sigma(\mathbf{Q})gU' = G'_\Sigma(\mathbf{Q})xhU'$ is in the image of the top arrow. □

2.3.3. *Construction of models.* Under the hypotheses of Lemma 2.3.1, we define

$$Y_{U \times V_E}(G_\Sigma \times T_E) = Y_{U'}(G'_\Sigma) \times_{Y_{\nu'(U')}(T')} (Y_{\det(U)}(T_F) \times_{\mathcal{O}} Y_{V_E}(T_E)).$$

Suppose now that $g \in G_\Sigma(\mathbf{A}_\mathbf{f}^{(p)})$ and that U_1 and U_2 are sufficiently small open compact subgroups (of level prime to p) in $G_\Sigma(\mathbf{A_f})$ satisfying $g^{-1}U_1 g \subset U_2$. For $y \in (\mathbf{A}_{E,\mathbf{f}}^{(p)})^\times$ and V_E (of level prime to p) sufficiently small relative to U_1 (hence also U_2), we have the commutative diagram



yielding a morphism $\rho_{(g,y)} : Y_{U_1 \times V_E}(G_\Sigma \times T_E) \to Y_{U_2 \times V_E}(G_\Sigma \times T_E)$.

The morphisms $\rho_{(g,y)}$ satisfy the usual compatibilities. In particular for $U = U_1 = U_2$, the automorphisms $\rho_{(1,y)}$ define a free action of the group $C_{V_E} = (\mathbf{A}^{(p)}_{E,\mathbf{f}})^\times/\mathcal{O}^\times_{E,(p)}V^p_E$ on $Y_{U\times V_E}(G_\Sigma \times T_E)$. We define $Y_U(G_\Sigma)$ as the quotient by this action. Note that this quotient is independent of the chosen V_E. Indeed if V_E and V'_E are of level prime to p and sufficiently small relative to U, then so is $V_E \cap V'_E$, so we can assume $V'_E \subset V_E$, in which case $Y_{U\times V_E}(G_\Sigma \times T_E)$ can be identified with the quotient of $Y_{U\times V'_E}(G_\Sigma \times T_E)$ by the action of $\ker(C_{V'_E} \to C_{V_E})$, compatibly with the action of C_{V_E}. Note also that $Y_{U\times V_E}(G_\Sigma \times T_E)$ is smooth and quasi-projective over $\mathcal{O}$ and hence so is $Y_U(G_\Sigma)$. Moreover the natural projections induce an isomorphism

$$Y_{U\times V_E}(G_\Sigma \times T_E) \xrightarrow{\sim} Y_U(G_\Sigma) \times_{\mathcal{O}} Y_{V_E}(T_E). \tag{3}$$

Returning to the general case of g, U_1, U_2 satisfying $g^{-1}U_1 g \subset U_2$, the morphism $\rho_{(g,1)}$ is compatible with the actions of C_{V_E} on the $Y_{U_i\times V_E}(G_\Sigma\times T_E)$, hence descends to a morphism

$$\rho_g : Y_{U_1}(G_\Sigma) \longrightarrow Y_{U_2}(G_\Sigma)$$

(independent of the choice of V_E). Furthermore if h and U_3 (again sufficiently small) are such that $h^{-1}U_2 h \subset U_3$, then $\rho_h \circ \rho_g = \rho_{gh}$, so we obtain an action of $G_\Sigma(\mathbf{A}^{(p)}_{\mathbf{f}})$ on $\varprojlim_U Y_U(G_\Sigma)$. Note that under the isomorphism of (3), the morphism $\rho_{(g,y)}$ corresponds to (ρ_g, y) on the product.

From the fact that the $Y_{U'}(G'_\Sigma)$ define canonical models over $L_{\widetilde{\Sigma}}$ for the Shimura varieties associated to $(G'_\Sigma, [h'_{\widetilde{\Sigma}}])$, it follows from their construction that so do the $Y_{U\times V_E}(G_\Sigma \times T_E)$ with respect to the Shimura data $(G_\Sigma \times T_E, [(h_\Sigma, i_{\widetilde{\Sigma}})])$, and hence that the $Y_U(G_\Sigma)$ define a system of integral canonical models with respect to $(G_\Sigma, [h_\Sigma])$. Note that we view $Y_U(G_\Sigma)$ as a scheme over $\mathcal{O}$, a localization of $L_{\widetilde{\Sigma}}$ rather than the reflex field L_Σ. We shall not consider its descent to L_Σ; on the contrary we need to extend scalars further in order to obtain our main results later. To that end it will be convenient to work over $W = W(\overline{\mathbf{F}}_p)$, and write S_W for the base-change to W of an $\mathcal{O}$-scheme S. The identity element of C_{V_E} then defines a section $\operatorname{Spec} W \to Y_{V_E}(T_E)_W \cong \coprod_{c\in C_{V_E}} \operatorname{Spec} W$. We thus obtain a morphism

$$Y_U(G_\Sigma)_W \to Y_U(G_\Sigma)_W \times_W Y_{V_E}(T_E)_W \to Y_{U\times V_E}(G_\Sigma \times T_E)_W \to Y_{U'}(G'_\Sigma)_W$$

for any U, V_E, U' as in Lemma 2.3.1. We have the following immediate consequence of the lemma and discussion above:

Lemma 2.3.2. *For U, V_E and U' as in Lemma 2.3.1, the diagram*

$$\begin{array}{ccc} Y_U(G_\Sigma)_W & \longrightarrow & Y_{U'}(G'_\Sigma)_W \\ \downarrow & & \downarrow \\ C_{\det(U)} & \longrightarrow & C_{\nu'(U')} \end{array}$$

is Cartesian (where $C_{\det(U)}$ and $C_{\nu'(U')}$ are viewed as schemes over W), and identifies $Y_U(G_\Sigma)_W$ with an open and closed subscheme of $Y_{U'}(G'_\Sigma)_W$. Moreover the inclusion is compatible with the Hecke action in the sense that the diagram

$$\begin{array}{ccc} Y_{U_1}(G_\Sigma)_W & \longrightarrow & Y_{U'_1}(G'_\Sigma)_W \\ {\scriptstyle\rho_g}\downarrow & & \downarrow{\scriptstyle\rho_g} \\ Y_{U_2}(G_\Sigma)_W & \longrightarrow & Y_{U'_2}(G'_\Sigma)_W \end{array}$$

commutes for any g, U_1, U_2 satisfying $g^{-1}U_1g \subset U_2$ (and sufficiently small V_E relative to U_1, with U_i' the image of $U_i \times V_E$ for $i = 1, 2$).

2.3.4. *Relation of Hilbert and unitary definitions.* Note that we have now given two definitions of the canonical model $Y_U(G)$ for $G = \mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_2$ over $\mathcal{O} = \mathbf{Z}_{(p)}$: the first as a Hilbert modular variety in §2.1.3, the second by taking $\Theta = \varnothing$ in the construction in §2.3.3. These are necessarily isomorphic by uniqueness of canonical models, but we will need to describe the isomorphism in terms of the moduli problems appearing in the two definitions.

For clarity we write $Y_U(G_\varnothing)$ for the model defined using $G'_\varnothing$, for which we choose $M_2(\mathcal{O}_E)$ for $\mathcal{O}_D$ and $\delta = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. We first define a morphism

$$\tilde{i} : \widetilde{Y}_U(G) \longrightarrow \widetilde{Y}_{U'}(G'_\varnothing) \tag{4}$$

for any sufficiently small $U \subset G(\mathbf{A_f})$ and $U' \subset G'_\varnothing(\mathbf{A_f})$ of level prime to p such that U' contains the image of U.

Let $(A, \iota, \lambda, \eta)$ be the universal object over $\widetilde{Y}_U(G)$. Define $A' = A \otimes_{\mathcal{O}_F} \mathcal{O}_E^2$ with $M_2(\mathcal{O}_E)$-action ι' defined by left-multiplication on $\mathcal{O}_E^2$. Then $(A')^\vee$ is canonically isomorphic to $A^\vee \otimes_{\mathcal{O}_F} \mathrm{Hom}_{\mathcal{O}_F}(\mathcal{O}_E^2, \mathcal{O}_F)$ (with $\alpha \in \mathcal{O}_F$ acting on $A^\vee$ as $\iota(\alpha)^\vee$). Define the quasi-polarization λ' on A' as the tensor product of λ with the $\mathcal{O}_F$-linear homomorphism $\mathcal{O}_E^2 \to \mathrm{Hom}_{\mathcal{O}_F}(\mathcal{O}_E^2, \mathcal{O}_F)$ induced by the pairing $(\alpha, \beta) \mapsto \mathrm{Tr}_{E/F}(\alpha\overline{\beta})$. We define the level structure on A' so that for each of the geometric points $\overline{s}_i$, η_i' is the unique isomorphism

$$M_2(\widehat{\mathcal{O}}_E^{(p)}) \longrightarrow T^{(p)}(A'_{\overline{s}_i}) = T^{(p)}(A_{\overline{s}_i}) \otimes_{\widehat{\mathcal{O}}_F^{(p)}} (\widehat{\mathcal{O}}_E^{(p)})^2$$

such that $\eta_i'\left((a,b)\begin{pmatrix} c \\ d \end{pmatrix}\right) = (\eta_i(a,b)) \otimes \begin{pmatrix} c \\ d \end{pmatrix}$ for all $a, b \in \widehat{\mathcal{O}}_F^{(p)}$, $c, d \in \widehat{\mathcal{O}}_E^{(p)}$. Finally we let $\epsilon' = \zeta \otimes \epsilon$, where ζ and ϵ are defined in the discussion following the definition of the functor $\widetilde{Y}_U(G)$. It is straightforward to check that $(A', \iota', \lambda', \eta', \epsilon')$ defines a tuple over $\widetilde{Y}_U(G)$ represented by $\widetilde{Y}_{U'}(G'_\varnothing)$, i.e., a morphism $\tilde{i} : \widetilde{Y}_U(G) \longrightarrow \widetilde{Y}_{U'}(G'_\varnothing)$.

The morphism $\tilde{i}$ is clearly compatible with action of $\mathcal{O}_{F,(p),+}^\times$, hence descends to a morphism $i : Y_U(G) \longrightarrow Y_{U'}(G'_\varnothing)$. Suppose now that V_E is an open compact subgroup of $\mathbf{A}_{E,\mathbf{f}}^\times$ contained in U' and of level prime to p. Since $\Sigma = \varnothing$, the Galois action on C_{V_E} defined by Shimura reciprocity is trivial, so we may identify $Y_{V_E}(T_E)$ with $\coprod_{C_{V_E}} \mathrm{Spec}(\mathbf{Z}_{(p)})$ as a scheme over $\mathbf{Z}_{(p)}$ and extend i to a morphism

$$i' : Y_U(G) \times Y_{V_E}(T_E) = \coprod_{C_{V_E}} Y_U(G) \longrightarrow Y_{U'}(G'_\varnothing)$$

by setting $i' = \rho_y \circ i$ on the component represented by $y \in (\mathbf{A}_{E,\mathbf{f}}^{(p)})^\times$.

It is straightforward to check that the morphism induced by i' on complex points is the same as the top line of (2). In particular the diagram

$$\begin{array}{ccc} Y_U(G) \times Y_{V_E}(T_E) & \longrightarrow & Y_{U'}(G'_\varnothing) \\ \downarrow & & \downarrow \\ Y_{\det U}(T_F) \times Y_{V_E}(T_E) & \longrightarrow & Y_{\nu'(U')} \end{array}$$

commutes, since it does so on complex points. Thus if V_E is sufficiently small relative to U, we obtain a morphism

$$Y_U(G) \times Y_{V_E}(T_E) \longrightarrow Y_{U\times V_E}(G_\varnothing \times T_E) \tag{5}$$

where the target is the fibre product defined after Lemma 2.3.1. Furthermore the morphisms of (5) for varying U and V_E are compatible with the action of $G(\mathbf{A}_{\mathbf{f}}^{(p)}) \times (\mathbf{A}_{E,\mathbf{f}}^{(p)})^\times$ in the usual sense, as can be deduced either from the definitions or the corresponding assertion on complex points. Since (5) is an isomorphism on complex points, it follows from uniqueness of canonical models that it is in fact an isomorphism. Combined with (3), we obtain isomorphisms

$$Y_U(G) \times Y_{V_E}(T_E) \longrightarrow Y_U(G_\varnothing) \times Y_{V_E}(T_E)$$

compatible with the action of $G(\mathbf{A}_{\mathbf{f}}^{(p)}) \times (\mathbf{A}_{E,\mathbf{f}}^{(p)})^\times$. Taking quotients by the action of C_{V_E} we obtain the desired isomorphism $Y_U(G) \to Y_U(G_\varnothing)$. Furthermore the isomorphisms are compatible with the Hecke action in the usual sense, and its composite with the inclusion of Lemma 2.3.2 is precisely the (base-change to W) of the morphism $i : Y_U(G) \to Y_{U'}(G'_\varnothing)$ defined above.

3. Automorphic vector bundles

3.1. **Construction of automorphic vector bundles.** In this section we will define automorphic vector bundles on the special fibres of the Shimura varieties $Y_U(G_\Sigma)$ (resp. $Y_{U'}(G'_\Sigma)$) for sufficiently small U (resp. U') of level prime to p.

3.1.1. *The Hilbert modular setting.* We begin with the case of $G = G_\varnothing = \mathrm{Res}_{F/\mathbf{Q}}\,\mathrm{GL}_2$, proceeding as in [DS17]. We assume $\mathbf{F} \subset \overline{\mathbf{F}}_p$ is sufficiently large to contain the image of $\overline{\theta} : \mathcal{O}_F \to \overline{\mathbf{F}}_p$ for all $\theta \in \Theta$ (for example take $\mathbf{F} = \overline{\mathbf{F}}_p$), and set $\overline{Y} = Y_U(G)_{\mathbf{F}}$ and $S = \widetilde{Y}_U(G)_{\mathbf{F}}$. We assume that U is sufficiently small that, in addition to the usual sense, we have $\alpha - 1 \in p\mathcal{O}_F$ for all $\alpha \in U \cap \mathcal{O}_F^\times$., i.e. *p-neat* in the terminology of [DS17, Def. 3.2.3]

Suppose that $\underline{A} = (A, \iota, \lambda, \eta) \in \widetilde{Y}_U(G)(S)$ and let $s : A \to S$ denote the structure morphism. Let $\tilde{\mathcal{V}} = \mathcal{H}^1_{\mathrm{dR}}(A/S)$ and consider the exact sequence

$$0 \to s_*\Omega^1_{A/S} \to \tilde{\mathcal{V}} \to R^1 s_*\mathcal{O}_A \to 0$$

of vector bundles on S, of rank d, $2d$ and d respectively. The action of $\mathcal{O}_F$ on A induces one on each of the vector bundles, making them sheaves of $\mathcal{O}_F \otimes \mathcal{O}_S$-modules. The assumption on ι implies that $s_*\Omega^1_{A/S}$ is locally (on S) free of rank one, the hypotheses on λ ensure the same is true of $R^1 s_*\mathcal{O}_A \cong \mathrm{Lie}(A^\vee/S)$, and it follows that $\mathcal{V}$ is locally free of rank two. Decomposing $\mathcal{O}_F \otimes \mathcal{O}_S \cong \bigoplus_{\theta\in\Theta} \mathcal{O}_S$, we obtain a corresponding decompositions $\tilde{\mathcal{V}} = \bigoplus_\theta \tilde{\mathcal{V}}_\theta$, $s_*\Omega^1_{A/S} = \bigoplus_\theta \tilde{\omega}_\theta$ and $R^1 s_*\mathcal{O}_A = \bigoplus_\theta \tilde{v}_\theta$, where each $\tilde{\omega}_\theta$ and $\tilde{v}_\theta$ is a line bundle on S, $\tilde{\mathcal{V}}_\theta$ is a vector bundle of rank two, and we have an exact sequence

$$0 \to \tilde{\omega}_\theta \to \tilde{\mathcal{V}}_\theta \to \tilde{v}_\theta \to 0.$$

We now define descent data on $\tilde{\mathcal{V}}$ with respect to the covering $S \to \overline{Y}$. Recall that the covering group is $\mathcal{O}_{F,(p),+}^\times/(U \cap \mathcal{O}_F^\times)^2$, and that for $\mu \in \mathcal{O}_{F,(p),+}^\times$, the action of μ on S is defined by $\theta_\mu(A, \iota, \lambda, \eta) = (A, \iota, \mu\lambda, \eta)$, so $\theta_\mu^* A$ is canonically $\mathcal{O}_F$-linearly isomorphic to A. We thus obtain a canonical $\mathcal{O}_F \otimes \mathcal{O}_S$-linear isomorphism

$\theta_\mu^*\tilde{\mathcal{V}} \to \tilde{\mathcal{V}}$, hence $\mathcal{O}_S$-linear isomorphisms $\theta_\mu^*\tilde{\mathcal{V}}_\theta \to \tilde{\mathcal{V}}_\theta$ for all θ, inducing isomorphisms $\theta_\mu^*\tilde{\omega}_\theta \to \tilde{\omega}_\theta$ and $\theta_\mu^*\tilde{v}_\theta \to \tilde{v}_\theta$. Furthermore if $\mu = \alpha^2$ for some $\alpha \in U \cap \mathcal{O}_F^\times$ (i.e., θ_μ is the identity map), then the canonical isomorphism $A \to \theta_\mu^* A = A$ is $\iota(\alpha^{-1})$, so the resulting automorphism of $\tilde{\mathcal{V}}$ is $\iota(\alpha^{-1})^*$, which is the identity since our hypotheses on U imply that $\alpha \equiv 1 \bmod p\mathcal{O}_F$. We thus obtain an action of the covering group on $\tilde{\mathcal{V}}$ over its action on S, inducing actions on the rank two vector bundles $\tilde{\mathcal{V}}_\theta$ as well as the line bundles $\tilde{\omega}_\theta$ and $\tilde{v}_\theta$. Since S is a disjoint union of finite étale schemes over $\overline{Y}$, the actions define effective descent data and hence vector bundles on $\overline{Y}$ fitting in an exact sequence

$$0 \to \omega_\theta \to \mathcal{V}_\theta \to v_\theta \to 0.$$

Suppose now that $U_1, U_2 \subset G(\mathbf{A_f})$ are sufficiently small (in the sense above) and of level prime to p, and $g \in G(\mathbf{A}_\mathbf{f}^{(p)})$ is such that $g^{-1}U_1 g \subset U_2$. Let $\overline{Y}_i = Y_{U_i}(G)_\mathbf{F}$, $S_i = \widetilde{Y}_U(G)_\mathbf{F}$, etc., for $i = 1, 2$. We thus have the morphism $\tilde{\rho}_g : S_1 \to S_2$ lying over $\rho_g : \overline{Y}_1 \to \overline{Y}_2$, and the canonical quasi-isogeny $\pi_g \in \mathrm{Hom}_{S_1}(A_1, \tilde{\rho}_g^* A_2) \otimes \mathbf{Z}_{(p)}$ induces an $\mathcal{O}_F \otimes \mathcal{O}_{S_1}$-linear isomorphism $\pi_g^* : \tilde{\rho}_g^*\tilde{\mathcal{V}}_2 \to \tilde{\mathcal{V}}_1$, and hence isomorphisms $\pi_g^* : \tilde{\rho}_g^*\tilde{\mathcal{V}}_{2,\theta} \to \tilde{\mathcal{V}}_{1,\theta}$ for each θ. It is straightforward to check that π_g^* is compatible with the descent data, hence defines an isomorphism $\rho_g^*\mathcal{V}_{2,\theta} \to \mathcal{V}_{1,\theta}$, and that these induce isomorphisms $\rho_g^*\omega_{2,\theta} \to \omega_{1,\theta}$ and $\rho_g^* v_{2,\theta} \to v_{1,\theta}$, all of which we also denote by π_g^*. Finally if $h \in G(\mathbf{A}_\mathbf{f}^{(p)})$ and U_3 is as above with $h^{-1}U_2 h \subset U_3$, then the relation $\pi_{gh} = \tilde{\rho}_g^*(\pi_h) \circ \pi_g$ ensures that $\pi_{gh}^* = \pi_g^* \circ \rho_g^*(\pi_h^*)$ for the sheaves on S_1, and hence on $\overline{Y}_1$.

3.1.2. *The unitary setting.* We now proceed similarly to define automorphic bundles on special fibres of Shimura varieties for G'_Σ. We assume $\mathbf{F}$ is sufficiently large as before, and now let $\overline{Y} = Y_{U'}(G'_\Sigma)_\mathbf{F}$ and $S = \widetilde{Y}_{U'}(G'_\Sigma)_\mathbf{F}$ where U' (of level prime to p) is sufficiently small that Lemma 2.2.1 holds and $\alpha - 1 \in p\mathcal{O}_F$ for all $\alpha \in U' \cap \mathcal{O}_E^\times$. Now let $s : A \to S$ denote the pull-back of the universal abelian variety over $\widetilde{Y}_{U'}(G'_\Sigma)$, let $\tilde{\mathcal{V}} = \mathcal{H}^1_{\mathrm{dR}}(A/S)$ and consider the exact sequence

$$0 \to s_*\Omega^1_{A/S} \to \tilde{\mathcal{V}} \to R^1 s_*\mathcal{O}_A \to 0$$

of vector bundles on S, of rank $4d$, $8d$ and $4d$ respectively.

The left action of $\mathcal{O}_D$ on A induces a right action on each of the bundles, making them sheaves of right $\mathcal{O}_D \otimes \mathcal{O}_S$-modules. Fix an isomorphism $\mathcal{O}_{B,p} \cong M_2(\mathcal{O}_{F,p})$. Recall that $\mathcal{O}_{D,p} = \mathcal{O}_E \otimes \mathcal{O}_{B,p}$, so we obtain an isomorphism $\mathcal{O}_{D,p} \cong M_2(\mathcal{O}_{E,p})$, and hence

$$\mathcal{O}_D \otimes \mathcal{O}_S \cong M_2(\mathcal{O}_E \otimes \mathcal{O}_S) = \bigoplus_{\tau \in \Theta_E} M_2(\mathcal{O}_S).$$

We thus obtain corresponding decompositions $\tilde{\mathcal{V}} = \bigoplus_\tau \tilde{\mathcal{V}}_\tau$, $s_*\Omega^1_{A/S} = \bigoplus_\tau \tilde{\omega}_\tau$ and $R^1 s_*\mathcal{O}_A = \bigoplus_\tau \tilde{v}_\tau$, with each factor inheriting the structure of a sheaf of right $M_2(\mathcal{O}_S)$-modules. The assumption on ι implies that $\tilde{\omega}_\tau$ is locally free of rank $2s_\tau$. Since $R^1 s_*\mathcal{O}_A \cong \mathrm{Lie}(A^\vee/S)$ as $\mathcal{O}_E \otimes \mathcal{O}_S$-modules with $\alpha \in \mathcal{O}_E$ acting as $\mathrm{Lie}(\iota(\alpha)^\vee)$, it follows from the hypotheses on λ that $\tilde{v}_\tau$ is locally free of rank $2s_{\tau^c}$, and so $\tilde{\mathcal{V}}_\tau$ is locally free of rank $2s_\tau + 2s_{\tau^c} = 4$. Let e_0 denote the idempotent $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in M_2(\mathcal{O}_S)$, and let $\tilde{\mathcal{V}}_\tau^0 = \tilde{\mathcal{V}}_\tau e_0$, a rank two vector bundle on S since $\tilde{\mathcal{V}}_\tau \cong \tilde{\mathcal{V}}_\tau^0 \oplus \tilde{\mathcal{V}}_\tau^0$. Similarly if $\tau \in \Theta_E$ is such that $\tau|_F \notin \Sigma$, then $s_\tau = s_{\tau^c} = 1$, so we have the line bundles $\tilde{\omega}^0_\tau = \tilde{\omega}_\tau e_0$, $\tilde{\upsilon}^0_\tau = \tilde{\upsilon}_\tau e_0$, and an exact sequence

$$0 \to \tilde{\omega}^0_\tau \to \tilde{\mathcal{V}}^0_\tau \to \tilde{\upsilon}^0_\tau \to 0.$$

We also define the line bundles $\tilde{\delta}_\tau = \wedge^2_{\mathcal{O}_S} \tilde{\mathcal{V}}^0_\tau$ for all $\tau \in \Theta_E$, canonically isomorphic to $\tilde{\omega}^0_\tau \otimes_{\mathcal{O}_S} \tilde{\upsilon}^0_\tau$ if $\tau|_F \notin \Sigma$.

We define descent data on $\tilde{\mathcal{V}}$ with respect to the covering $S \to \overline{Y}$ similarly to the case $G = G_\varnothing$. Under our assumptions on U', the covering group is again $\mathcal{O}^\times_{F,(p),+}/(U' \cap \mathcal{O}_F^\times)^2$. Now $\mu \in \mathcal{O}^\times_{F,(p),+}$ acts via $\theta_\mu(A,\iota,\lambda,\eta,\epsilon) = (A,\iota,\mu\lambda,\eta,\mu\epsilon)$. so $\theta_\mu^* A$ is canonically $\mathcal{O}_D$-linearly isomorphic to A over S, yielding $\mathcal{O}_S$-linear isomorphisms $\theta^*_\mu \tilde{\mathcal{V}}^0_\tau \to \tilde{\mathcal{V}}^0_\tau$ for all τ. Again our assumptions imply this isomorphism is the identity if $\mu \in (U' \cap \mathcal{O}_F)^2$, so we obtain vector bundles $\mathcal{V}^0_\tau$ on $\overline{Y}$ by descent, and we let $\delta_\tau = \wedge^2_{\mathcal{O}_{\overline{Y}}} \mathcal{V}^0_\tau$. Similarly if $\tau|_F \notin \Sigma$, then we obtain line bundles ω^0_τ, υ^0_τ on $\overline{Y}$, an exact sequence

$$0 \to \omega^0_\tau \to \mathcal{V}^0_\tau \to \upsilon^0_\tau \to 0,$$

and an isomorphism $\delta_\tau \cong \omega^0_\tau \otimes_{\mathcal{O}_{\overline{Y}}} \upsilon^0_\tau$.

Suppose now that U_1', $U_2' \subset G'_\Sigma(\mathbf{A_f})$ are sufficiently small (in the sense above) and of level prime to p, and $g \in G'_\Sigma(\mathbf{A}_\mathbf{f}^{(p)})$ is such that $g^{-1}U_1' g \subset U_2'$. Letting $\overline{Y}_i = Y_{U_i'}(G'_\Sigma)_\mathbf{F}$, $S_i = \widetilde{Y}_{U_i'}(G'_\Sigma)_\mathbf{F}$, etc as in the case of $G_\varnothing$, we have the morphism $\tilde{\rho}_g : S_1 \to S_2$ over $\rho_g : \overline{Y}_1 \to \overline{Y}_2$, and the quasi-isogeny π_g induces isomorphisms $\pi_g^* : \tilde{\rho}_g^* \tilde{\mathcal{V}}^0_{2,\tau} \to \tilde{\mathcal{V}}^0_{1,\tau}$ descending to isomorphisms $\rho_g^* \mathcal{V}^0_{2,\tau} \to \mathcal{V}^0_{1,\tau}$. Similarly we obtain isomorphisms $\rho_g^*\omega^0_{2,\tau} \to \omega^0_{1,\tau}$, $\rho_g^*\upsilon^0_{2,\tau} \to \upsilon^0_{1,\tau}$ if $\tau|_F \notin \Sigma$, and $\rho_g^*\delta_{2,\tau} \to \delta_{1,\tau}$ for all τ, all of which we denote π_g^*. Finally if $h \in G'_\Sigma(\mathbf{A}_\mathbf{f}^{(p)})$ and U_3' is as above with $h^{-1}U_2' h \subset U_3'$, then we have $\pi^*_{gh} = \pi^*_g \circ \rho^*_g(\pi^*_h)$.

3.1.3. *The quaternionic setting.* We now turn to the definition of the automorphic bundles in the case of G_Σ for arbitrary Σ (of even cardinality). Now we assume $\mathbf{F} = \overline{\mathbf{F}}_p$, and that $U \subset G_\Sigma(\mathbf{A}_\mathbf{f}^{(p)})$ is sufficiently small that we can choose V_E sufficiently small relative to U (see §2.3.1 so that $U' = UV_E \subset G'_\Sigma(\mathbf{A}_\mathbf{f}^{(p)})$ is sufficiently small in the sense of §3.1.2.

We decompose $\Theta = \overline{\Theta}_p = \coprod_{v|p} \Theta_v$, where $\theta \in \Theta_v$ if and only if θ factors through $\mathcal{O}_F/v$. For each prime v of F dividing p, we choose a prime $\tilde{v}$ of E dividing v, and let

$$\widetilde{\Theta} = \coprod_{v|p} \Theta_{E,\tilde{v}} \subset \overline{\Theta}_{E,p} = \Theta_E$$

(where $\tau \in \Theta_{E,\tilde{v}}$ if and only if τ factors through $\mathcal{O}_E/\tilde{v}$). Note that $\widetilde{\Theta}$ maps bijectively to Θ under $\tau \mapsto \tau|_F$; we do not assume $\tilde{\Sigma} \subset \widetilde{\Theta}$. For each $\theta \in \Theta$, we let $\tilde{\theta}$ denote the extension of θ in $\widetilde{\Theta}$.

Let $\overline{Y} = Y_U(G_\Sigma)_\mathbf{F}$, and consider the inclusion $i : \overline{Y} \to \overline{Y}' = Y_{U'}(G'_\Sigma)_\mathbf{F}$ obtained from the one in Lemma 2.3.2. For each $\theta \in \Theta$, we define the rank two vector bundle $\mathcal{V}_\theta$ on $\overline{Y}$ to be $i^*\mathcal{V}^0_{\tilde{\theta}}$. Similarly we define the line bundle $\delta_\theta = i^*\omega^0_{\tilde{\theta}} = \wedge^2_{\mathcal{O}_{\overline{Y}}} \mathcal{V}_\theta$, and if $\theta \notin \Sigma$, we have $\omega_\theta = i^*\omega^0_{\tilde{\theta}}$, $\upsilon_\theta = i^*\upsilon^0_{\tilde{\theta}}$ on $\overline{Y}$, the exact sequence

$$0 \to \omega_\theta \to \mathcal{V}_\theta \to \upsilon_\theta \to 0,$$

and the isomorphism $\delta_\theta \cong \omega_\theta \otimes_{\mathcal{O}_{\overline{Y}}} \upsilon_\theta$.

The vector bundles $\mathcal{V}_\theta$, ω_θ and υ_θ are independent of the choice of V_E. Moreover if $U_1, U_2 \subset G_\Sigma(\mathbf{A}_\mathbf{f}^{(p)})$ are as above and $g \in G_\Sigma(\mathbf{A}_\mathbf{f}^{(p)})$ is such that $g^{-1}U_1 g \subset U_2$, then we can choose V_E as above for U_1 and U_2, let $U_i' = U_i V_E$, $\overline{Y}_i = Y_{U_i}(G_\Sigma)_\mathbf{F}$, etc. for $i = 1, 2$, and define the isomorphism $\pi_g^* : \rho_g^* \mathcal{V}_{\theta,2} \to \mathcal{V}_{\theta,1}$ to be the pull-back via $i_1 : \overline{Y}_1 \to \overline{Y}_1'$ of the one so denoted on $\overline{Y}_1'$. We similarly define isomorphisms $\rho_g^* \omega_{\theta,2} \to \omega_{\theta,1}$ and $\rho_g^* \upsilon_{\theta,2} \to \upsilon_{\theta,1}$ for $\theta \notin \Sigma$, and $\rho_g^* \delta_{\theta,2} \to \delta_{\theta,1}$ for all $\theta \in \Theta$. As usual, it is straightforward to check that these are independent on the choice of V_E, and that $\pi_{gh}^* = \pi_g^* \circ \rho_g^*(\pi_h^*)$ for h and U_3 satisfying $h^{-1}U_2 h \subset U_3$.

Note that we have given two definitions of the bundles $\mathcal{V}_\theta$, δ_θ, ω_θ and υ_θ on $\overline{Y} = Y_U(G_\Sigma)_{\overline{\mathbf{F}}_p}$ for sufficiently small U in the case $\Sigma = \varnothing$, one by descent from $\tilde{\mathcal{V}} = \mathcal{H}^1_{\mathrm{dR}}(A/S)$ where A is the universal abelian variety on $S = \widetilde{Y}_U(G)_{\overline{\mathbf{F}}_p}$, the other by restriction and descent from $\tilde{\mathcal{V}}' = \mathcal{H}^1_{\mathrm{dR}}(A'/S')$ where A' is the universal abelian variety on $S' = \widetilde{Y}_U(G'_\varnothing)_{\overline{\mathbf{F}}_p}$. Writing $\tilde{i} : S \to S'$ for the base-change of the morphism in (4), we see from its construction that we have a canonical $\mathcal{O}_E \otimes \mathcal{O}_S$-linear isomorphism

$$\tilde{i}^*(\tilde{\mathcal{V}}' e_0) \cong \tilde{\mathcal{V}} \otimes_{\mathcal{O}_F} \mathcal{O}_E,$$

compatible with the actions of $\mathcal{O}_{F,(p),+}^\times$ on $\tilde{\mathcal{V}}$ and $\tilde{\mathcal{V}}' e_0$ over its actions on S and S'. It follows that the two vector bundles on $\overline{Y}$ denoted $\mathcal{V}_\theta$ are canonically isomorphic, hence also for those denoted δ_θ, and we see similarly the same holds for ω_θ and υ_θ. Furthermore, the isomorphisms are compatible with those denoted π_g^*.

Remark 3.1.1. Note that choices of the field E, set of places $\tilde{\Sigma}$ and isomorphisms $\mathcal{O}_{B,p} \cong M_2(\mathcal{O}_{F,p})$ are implicit in our definition of the automorphic bundles on $Y_U(G_\Sigma)_{\overline{\mathbf{F}}_p}$ (for $\Sigma \neq \varnothing$). We do not consider here the question of the bundles' dependence on these choices or descent to a suitably defined reflex field.

3.2. **Relations among vector bundles.** In this section we recall certain canonical relations among the automorphic bundles and cotangent bundles.

3.2.1. *The Kodaira–Spencer isomorphism.* We begin with the construction of the Kodaira–Spencer isomorphism, adapted to our setting.

Let $s : A \to S$ denote the universal abelian variety over $S = \widetilde{Y}_{U'}(G'_\Sigma)_\mathbf{F}$, where we assume $\mathbf{F}$ is sufficiently large and U' is sufficiently small and of level prime to p as in §3.1.2. The smooth morphisms $A \to S \to \operatorname{Spec} \mathbf{F}$ yield the exact sequence

$$0 \to s^*\Omega^1_{S/\mathbf{F}} \to \Omega^1_{A/\mathbf{F}} \to \Omega^1_{A/S} \to 0,$$

to which we apply $R^\bullet s_*$ to obtain the connecting morphism:

$$s_*\Omega^1_{A/S} \longrightarrow R^1 s_*(s^*\Omega^1_{S/\mathbf{F}}) \cong \Omega^1_{S/\mathbf{F}} \otimes_{\mathcal{O}_S} R^1 s_* \mathcal{O}_A. \tag{6}$$

Furthermore the morphism is right $\mathcal{O}_D \otimes \mathcal{O}_S$-linear (where $\mathcal{O}_D$ acts on the target via its action on $R^1 s_* \mathcal{O}_A$). We may therefore apply idempotents to obtain an $\mathcal{O}_S$-linear morphism

$$\tilde{\omega}^0_\tau \longrightarrow \Omega^1_{S/\mathbf{F}} \otimes_{\mathcal{O}_S} \tilde{\upsilon}^0_\tau,$$

and hence $\mathcal{H}om_{\mathcal{O}_S}(\tilde{\upsilon}^0_\tau, \tilde{\omega}^0_\tau) \to \Omega^1_{S/\mathbf{F}}$, for each τ such that $\tau|_F \notin \Sigma$. By a standard argument, the resulting morphism

$$\bigoplus_{\theta \in \Theta - \Sigma} \mathcal{H}om_{\mathcal{O}_S}(\tilde{\upsilon}^0_{\tilde{\theta}}, \tilde{\omega}^0_{\tilde{\theta}}) \longrightarrow \Omega^1_{S/\overline{\mathbf{F}}_p} \tag{7}$$

is an isomorphism. See in particular [Lan13, §2.1.7], where (6) is rewritten in such a way that the argument in the proof of [TX16, Cor. 3.17] shows that (7) is an isomorphism.

Furthermore the isomorphism is compatible with the action of $\mathcal{O}_{F,(p),+}^{\times}$, where the action is defined on $\mathcal{H}om_{\mathcal{O}_S}(\tilde{v}^0_{\tilde{\tau}}, \tilde{\omega}^0_{\tilde{\tau}})$ in §3.1.2, and on $\Omega^1_{S/\overline{\mathbf{F}}_p}$ via the canonical isomorphism $\theta_\mu^*\Omega^1_{S/\overline{\mathbf{F}}_p} \to \Omega^1_{S/\overline{\mathbf{F}}_p}$. This compatibility follows from the general fact that if

$$\begin{array}{ccc} A_1 & \xrightarrow{s_1} & S_1 \\ \pi\downarrow & & \downarrow\rho \\ A_2 & \xrightarrow{s_2} & S_2 \end{array}$$

is a commutative diagram of smooth morphisms of smooth schemes over $\mathbf{F}$ with π and ρ finite, then the resulting diagram

$$\begin{array}{ccc} \rho^* s_{2,*}\Omega^1_{A_2/S_2} & \longrightarrow & \rho^*\Omega^1_{S_2/\mathbf{F}} \otimes_{\mathcal{O}_{S_1}} \rho^* R^1 s_{2,*}\mathcal{O}_{A_2} \\ \downarrow & & \downarrow \\ s_{1,*}\Omega^1_{A_1/S_1} & \longrightarrow & \Omega^1_{S_1/\mathbf{F}} \otimes_{\mathcal{O}_{S_1}} R^1 s_{1,*}\mathcal{O}_{A,1} \end{array} \tag{8}$$

commutes, where the top arrow is $\rho^*(s_{2,*}\Omega^1_{A_2/S_2} \to \Omega^1_{S_2/\mathbf{F}} \otimes_{\mathcal{O}_{S_2}} R^1 s_{2,*}\mathcal{O}_{S_2})$ and the vertical arrows are obtained from the natural maps $\pi^*\Omega^1_{A_2/S_2} \to \Omega^1_{A_1/S_1}$ and $\rho^*\Omega^1_{S_2/\mathbf{F}} \to \Omega^1_{S_1/\mathbf{F}}$ by adjunction. Therefore (7) descends to an isomorphism

$$\bigoplus_{\theta\in\Theta-\Sigma} \mathcal{H}om_{\mathcal{O}_{\overline{Y}}}(v^0_{\tilde{\theta}}, \omega^0_{\tilde{\theta}}) \xrightarrow{\sim} \Omega^1_{\overline{Y}/\overline{\mathbf{F}}_p}$$

on $\overline{Y} = Y_{U'}(G'_\Sigma)_{\mathbf{F}}$, called the Kodaira–Spencer isomorphism. [6]

Suppose now that $U \subset G_\Sigma(\mathbf{A}_{\mathbf{f}}^{(p)})$ is sufficiently small and $U' = UV_E \subset G'_\Sigma(\mathbf{A}_{\mathbf{f}}^{(p)})$ is chosen as in §3.1.3. Let $\overline{Y} = Y_U(G_\Sigma)_{\mathbf{F}}$ where $\mathbf{F} = \overline{\mathbf{F}}_p$, and consider the inclusion $i : \overline{Y} \to \overline{Y}' = Y_{U'}(G'_\Sigma)_{\mathbf{F}}$. The Kodaira–Spencer isomorphism defined above on $\overline{Y}'$ then restricts via i to give the Kodaira–Spencer isomorphism

$$\bigoplus_{\theta\notin\Sigma} \mathcal{H}om_{\mathcal{O}_{\overline{Y}}}(v_\theta, \omega_\theta) \xrightarrow{\sim} \Omega^1_{\overline{Y}/\mathbf{F}}$$

on $\overline{Y}$. Since $\delta_\theta = \omega_\theta v_\theta$, this can also be written as

$$\bigoplus_{\theta\notin\Sigma} \delta_\theta^{-1}\omega_\theta^2 \xrightarrow{\sim} \Omega^1_{\overline{Y}/\mathbf{F}},$$

and taking determinants yields[7]

$$\bigotimes_{\theta\notin\Sigma} \delta_\theta^{-1}\omega_\theta^2 \xrightarrow{\sim} \mathcal{K}_{\overline{Y}/\mathbf{F}},$$

[6] We actually have such an isomorphism on the integral model over a sufficiently large $\mathcal{O}'$. Note that $(F^\times \cap K)^2$ acts trivially on $\mathcal{H}om_{\mathcal{O}_{S'}}(\tilde{v}^0_\tau, \tilde{\omega}^0_\tau)$ where $\tilde{v}^0_\tau$ and $\tilde{\omega}^0_\tau$ are defined as above, but on $S' = \widetilde{Y}_{U'}(G'_\Sigma)_{\mathcal{O}'}$, so it descends to a sheaf on $Y_{U'}(G'_\Sigma)_{\mathcal{O}'}$, even though $\tilde{v}^0_\tau$ and $\tilde{\omega}^0_\tau$ do not. Also note that [TX16] use the analogue of $\tilde{\omega}_{\tau^c}$ instead of $(\tilde{v}^0_\tau)^{-1}$; these are isomorphic, but the isomorphism depends on the polarization, so does not descend to the Shimura varieties associated to G'_Σ.

[7] We implicitly choose an ordering of factors when writing indexed tensor products; the resulting canonical isomorphisms with wedge products are only compatible up to sign with the canonical reordering isomorphisms on the tensor products, but we view the ordering as fixed.

where $\mathcal{K}_{\overline{Y}/\mathbf{F}}$ is the dualizing sheaf. Finally, using the commutativity of (8), it is straightforward to check that the above isomorphisms are independent of the choice of V_E and Hecke-equivariant in the usual sense, i.e., if U_1 and U_2 are sufficiently small, and $g \in G_\Sigma(\mathbf{A}_\mathbf{f}^{(p)})$ is such that $g^{-1}U_1 g \subset U_2$, then the diagram

$$\begin{array}{ccc} \bigotimes_{\theta\notin\Sigma} \rho_g^*\left(\delta_{\theta,2}^{-1}\omega_{\theta,2}^2\right) & \xrightarrow{\sim} & \rho_g^*\mathcal{K}_{\overline{Y}_2/\mathbf{F}} \\ \downarrow & & \downarrow \\ \bigotimes_{\theta\notin\Sigma} \delta_{\theta,1}^{-1}\omega_{\theta,1}^2 & \xrightarrow{\sim} & \mathcal{K}_{\overline{Y}_1/\mathbf{F}} \end{array}$$

commutes, where $\overline{Y}_i = Y_{U_i}(G_\Sigma)_\mathbf{F}$, etc., the left vertical arrow is the tensor product of the maps π_g^* defined in §3.1.3, and the right vertical arrow is the canonical map.

3.2.2. *Frobenius relations.* Let $s : A \to S$ denote the universal abelian variety over $S = \widetilde{Y}_{U'}(G'_\Sigma)_\mathbf{F}$ as above, and consider the relative Frobenius and Verschiebung morphisms:

$$\mathrm{Frob} : A \to A^{(p)} \quad \text{and} \quad \mathrm{Ver} : A^{(p)} \to A$$

over S, where $A^{(p)}$ denotes the pull-back of A along the absolute Frobenius on S. We then have the canonical exact sequences

$$0 \to \mathcal{H} \to \mathcal{H}^1_{\mathrm{dR}}(A/S) \to \mathcal{I} \to 0 \quad \text{and} \quad 0 \to \mathcal{I} \to \mathcal{H}^1_{\mathrm{dR}}(A^{(p)}/S) \to \mathcal{H} \to 0$$

of vector bundles on S, where

$$\mathcal{H} = \ker(\mathcal{H}^1_{\mathrm{dR}}(A/S) \xrightarrow{\mathrm{Ver}^*} \mathcal{H}^1_{\mathrm{dR}}(A^{(p)}/S) \text{ and } \mathcal{I} = \ker(\mathcal{H}^1_{\mathrm{dR}}(A^{(p)}/S) \xrightarrow{\mathrm{Frob}^*} \mathcal{H}^1_{\mathrm{dR}}(A/S)).$$

Moreover the above morphisms are $\mathcal{O}_D \otimes \mathcal{O}_S$-linear, so we can apply idempotents to obtain exact sequences

$$0 \to \mathcal{H}^0_\tau \to \mathcal{H}^1_{\mathrm{dR}}(A/S)^0_\tau \to \mathcal{I}^0_\tau \to 0 \quad \text{and} \quad 0 \to \mathcal{I}^0_\tau \to \mathcal{H}^1_{\mathrm{dR}}(A^{(p)}/S)^0_\tau \to \mathcal{H}^0_\tau \to 0$$

for all $\tau \in \Theta_E$.

Note that $\mathcal{H}^1_{\mathrm{dR}}(A^{(p)}/S)^0_\tau$ is canonically isomorphic to $(\mathcal{H}^1_{\mathrm{dR}}(A/S)^0_{\phi^{-1}\circ\tau})^{(p)}$ (where $\cdot^{(p)}$ again denotes pull-back by the absolute Frobenius on S), so that $\mathcal{I}^0_\tau$ is dual to $(\mathrm{Lie}(A/S)^0_{\phi^{-1}\circ\tau})^{(p)}$. It follows that $\mathcal{I}^0_\tau$ (resp. $\mathcal{H}^0_\tau$) has rank $s_{\phi^{-1}\circ\tau}$ (resp. $2 - s_{\phi^{-1}\tau}$). In particular if $\phi^{-1} \circ \tau|_F \in \Sigma$, then we obtain an isomorphism

$$\mathcal{H}^1_{\mathrm{dR}}(A/S)^0_\tau \cong \mathcal{H}^1_{\mathrm{dR}}(A^{(p)}/S)^0_\tau$$

induced by either Frob^* or Ver^* according to whether or not $\phi^{-1} \circ \tau \in \tilde{\Sigma}$. On the other hand if $\phi^{-1} \circ \tau|_F \notin \Sigma$, then we still have

$$\wedge^2_{\mathcal{O}_S}\mathcal{H}^1_{\mathrm{dR}}(A/S)^0_\tau \cong \mathcal{H}^0_\tau \otimes_{\mathcal{O}_S} \mathcal{I}^0_\tau \cong \wedge^2_{\mathcal{O}_S}\mathcal{H}^1_{\mathrm{dR}}(A^{(p)}/S)^0_\tau,$$

so we have

$$\tilde{\delta}_\tau = \wedge^2_{\mathcal{O}_S}\mathcal{H}^1_{\mathrm{dR}}(A/S)^0_\tau \cong \left(\wedge^2_{\mathcal{O}_S}\mathcal{H}^1_{\mathrm{dR}}(A/S)^0_{\phi^{-1}\circ\tau}\right)^{\otimes p} = \tilde{\delta}^{\otimes p}_{\phi^{-1}\circ\tau}$$

for all $\tau \in \Theta_E$. Furthermore the isomorphism is compatible with the action of $\mathcal{O}^\times_{F,(p),+}$, so it descends to an isomorphism $\delta_\tau \cong \delta^{\otimes p}_{\phi^{-1}\circ\tau}$ on $Y_{U'}(G'_\Sigma)_\mathbf{F}$, which in turn restricts to give an isomorphism $\delta_\theta \cong \delta^{\otimes p}_{\phi^{-1}\circ\theta}$ on $Y_U(G_\Sigma)_{\overline{\mathbf{F}}_p}$ for all $\theta \in \Theta$ and sufficiently small U (choosing suitable $U' = UV_E$). Finally, it is straightforward to check that the isomorphism is independent of the choice of V_E and Hecke equivariant in the sense that it is compatible with the maps π_g^* (for $g \in G_\Sigma(\mathbf{A}_\mathbf{f}^{(p)})$ and sufficiently small U_1 and U_2 such that $g^{-1}U_1 g \subset U_2$).

4. Iwahori level structures

4.1. **The Hilbert modular setting.** In this section we recall Pappas' definition of integral models of Hilbert modular varieties of level $U_0(p)$ and level $U_1(p)$, where U is a sufficiently small open compact subgroup of $G(\mathbf{A}_\mathbf{f}) = \mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}})$ of level prime to p, and

$$U_1(p) = \{g \in U \mid g_p \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \bmod p\mathcal{O}_{F,p}\} \subset U_0(p) = \{g \in U \mid g_p \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \bmod p\mathcal{O}_{F,p}\}.$$

Thus writing $U = U^p U_p$ with $U_p = \mathrm{GL}_2(\mathcal{O}_{F,p})$, we have $U_0(p) = U^p U_{0,p}$ and $U_1(p) = U^p U_{1,p}$ where $U_{0,p}$ is an Iwahori subgroup of U_p and $U_0(p)/U_1(p) \cong U_{0,p}/U_{1,p} \cong (\mathcal{O}_F/p)^\times$.

4.1.1. *Level* $U_0(p)$. For $U_0(p)$, we consider the functor which associates to a $\mathbf{Z}_{(p)}$-scheme S the set of isomorphism classes of triples $(\underline{A}_1, \underline{A}_2, f)$, where

- $\underline{A}_j = (A_j, \iota_j, \lambda_j, \eta_j)$ for $j = 1, 2$ represent S-points of $\widetilde{Y}_U(G)$, and
- $f : A_1 \to A_2$ is an isogeny of degree p^d such that $f \circ \iota_1(\alpha) = \iota_2(\alpha) \circ f$ for all $\alpha \in \mathcal{O}_F$, $p\lambda_1 = f^\vee \circ \lambda_2 \circ f$, $\eta_2 = f \circ \eta_1$ (as U^p-orbits for each $\overline{s}_i$), and $H = \ker f$ decomposes as $\prod_{v|p} H_v$ where $H_v \subset A_1[v]$ has rank $p^{[F_v:\mathbf{Q}_p]}$ for each prime v of $\mathcal{O}_F$ dividing p.

The results of [Pap95] (see in particular Theorem 2.2.2) show that this functor is representable by a flat local complete intersection over $\mathbf{Z}_{(p)}$ of constant relative dimension $d = [F : \mathbb{Q}]$. Moreover, letting $\widetilde{Y}_{U_0(p)}(G)$ denote the representing scheme, the forgetful morphism $\widetilde{Y}_{U_0(p)}(G) \to \widetilde{Y}_U(G)$ sending $(\underline{A}_1, \underline{A}_2, f)$ to $\underline{A}_1$ is projective.

By the proof of Lemma 2.1.2 of [GK12], the scheme $\widetilde{Y}_{U_0(p)}(G)$ also represents the functor sending a $\mathbf{Z}_{(p)}$-scheme S the set of isomorphism classes of pairs $(\underline{A}, H)$, where

- $\underline{A} = (A, \iota, \lambda, \eta)$ represents an S-point of $\widetilde{Y}_U(G)$, and
- H is a finite flat $(\mathcal{O}_F/p)$-submodule scheme of $A[p]$ over S which is *totally isotropic* in the sense that the λ-Weil pairing $A[p] \xrightarrow{\sim} A^\vee[p] = (A[p])^\vee$ induces an isomorphism
$$H \xrightarrow{\sim} \ker((A[p])^\vee \to H^\vee)$$

(where we use $\cdot^\vee$ to denote Cartier duals as well as dual abelian varieties). The natural isomorphism between the functors is defined by sending $(\underline{A}_1, \underline{A}_2, f)$ to $(\underline{A}_1, \ker f)$, and we use the two descriptions of $\widetilde{Y}_{U_0(p)}(G)$ interchangeably.

The group $\mathcal{O}_{F,(p),+}^\times$ acts on $\widetilde{Y}_{U_0(p)}(G)$ via its action on the quasi-polarization λ; as usual the action factors through a free action of $\mathcal{O}_{F,(p),+}^\times/(U \cap F^\times)^2$ (note that $U \cap F^\times = U_0(p) \cap F^\times$), and we denote the quotient scheme by $Y_{U_0(p)}(G)$. It follows that $Y_{U_0(p)}(G)$ is also a flat local complete intersection of relative dimension d over $\mathbf{Z}_{(p)}$ and the morphism $Y_{U_0(p)}(G) \to Y_U(G)$ is projective, so $Y_{U_0(p)}(G)$ is quasi-projective over $\mathbf{Z}_{(p)}$.

For open compact subgroups $U, V \subset G(\mathbf{A}_\mathbf{f})$ as above, and $g \in G(\mathbf{A}_\mathbf{f}^{(p)})$ such that $g^{-1}Ug \subset V$, the usual construction affords a finite étale morphism $\tilde{\rho}_g : \widetilde{Y}_{U_0(p)}(G) \to \widetilde{Y}_{V_0(p)}(G)$ descending to $\rho_g : Y_{U_0(p)}(G) \to Y_{V_0(p)}(G)$, and satisfying $\tilde{\rho}_h \circ \tilde{\rho}_g = \tilde{\rho}_{gh}$

and hence $\rho_h \circ \rho_g = \rho_{gh}$ for $h \in G(\mathbf{A}_{\mathbf{f}}^{(p)})$ and sufficiently small W containing $h^{-1}Vh$. Moreover the schemes $Y_{U_0(p)}(G)$ and morphisms ρ_g are integral models for the corresponding morphisms and varieties of canonical models over $\mathbf{Q}$ of level $U_0(p)$.

4.1.2. *Level $U_1(p)$ and Raynaud group schemes.* For $U_1(p)$, we consider the functor which associates to an $\mathbf{Z}_{(p)}$-scheme S the set of isomorphism classes of triples $(\underline{A}, H, P)$ where $\underline{A}$ and H are as above and

- $P \in H(S)$ is an $(\mathcal{O}_F/p)$-generator of H in the sense of Drinfeld–Katz–Mazur [KM85, 1.10].

The functor is represented by a $\mathbf{Z}_{(p)}$-scheme which we denote $\widetilde{Y}_{U_1(p)}(G)$, and the forgetful morphism $\widetilde{Y}_{U_1(p)}(G) \to \widetilde{Y}_{U_0(p)}(G)$ is finite flat ([Pap95, Thm. 2.3.3]).

Following [Pap95], the scheme $\widetilde{Y}_{U_1(p)}(G)$ can be described explicitly in terms of the universal H over $\widetilde{Y}_{U_0(p)}(G)$. We first recall the classification of certain finite flat group schemes considered by Raynaud in [Ray74]. Suppose that L is a number field containing the $(q-1)$-roots of unity, where $q-1$ is a divisible by the exponent of $(O_F/p)^\times$, let $\mathcal{O}$ be the localization of $\mathcal{O}_L$ at the prime over p determined by our choice of embeddings of $\overline{\mathbf{Q}}$, and let S be a scheme over $\mathcal{O}$. We say that a finite flat $(\mathcal{O}_F/p)$-module scheme H over S is *Raynaud* if condition $(\star\star)$ of [Ray74] is satisfied by the $(\mathcal{O}_F/v)$-vector space scheme $H[v]$ for each $v|p$. By [Ray74, Thm. 1.4.1], to give a Raynaud $(\mathcal{O}_F/p)$-module scheme H is equivalent to giving invertible $\mathcal{O}_S$-modules $\mathcal{L}_\theta$ for each $\theta \in \Theta$ and morphisms $s_\theta : \mathcal{L}_\theta^p \to \mathcal{L}_{\phi\circ\theta}$, $t_\theta : \mathcal{L}_{\phi\circ\theta} \to \mathcal{L}_\theta^p$ such that $s_\theta \circ t_\theta = w_v$ for all $v|p$ and $\theta \in \Theta_v$, where w_v is a certain fixed element of $p\mathcal{O}^\times$. The $(\mathcal{O}_F/p)$-module scheme corresponding to this data is given by

$$H = \mathbf{Spec}((\mathrm{Sym}_{\mathcal{O}_S}\mathcal{L})/\mathcal{I}),$$

where $\mathcal{L} = \bigoplus_{\theta\in\Theta}\mathcal{L}_\theta$, $\alpha \in \mathcal{O}_F$ acts on $\mathcal{L}_\theta$ as the Teichmuller lift of $\overline{\theta}(\alpha)$, and $\mathcal{I}$ is the sheaf of ideals generated by the $\mathcal{O}_S$-submodules $(s_\theta - 1)\mathcal{L}_\theta^p$ for $\theta \in \Theta$. The comultiplication $\mathcal{O}_H \to \mathcal{O}_H \otimes_{\mathcal{O}_S} \mathcal{O}_H$ defining the group law on H is given via duality by the scheme structure on the Cartier dual $H^\vee$, which is identified with

$$\mathbf{Spec}((\mathrm{Sym}_{\mathcal{O}_S}\mathcal{M})/\mathcal{J}),$$

where $\mathcal{M} = \mathcal{H}om_{\mathcal{O}_S}(\mathcal{L}, \mathcal{O}_S) = \bigoplus_{\theta\in\Theta_E}\mathcal{L}_\theta^{-1}$ and $\mathcal{J}$ is generated by $(t_\theta - 1)\mathcal{L}_\theta^{-p}$ for $\theta \in \Theta$. We write $\mathcal{R}_H$ for the direct image of $\mathcal{O}_H$ under the structure morphism $H \to S$, so that $\mathcal{R}_H$ is the sheaf of $\mathcal{O}_S$-algebras $(\mathrm{Sym}_{\mathcal{O}_S}\mathcal{L})/\mathcal{I}$, identified as $\mathcal{O}_S$-modules with

$$\bigoplus\left(\bigotimes_{\theta\in\Theta}\mathcal{L}_\theta^{m_\theta}\right),$$

where the direct sum is over $(m_\theta)_{\theta\in\Theta}$ with $0 \le m_\theta \le p-1$ for all θ, and similarly let

$$\mathcal{R}_{H^\vee} = (\mathrm{Sym}_{\mathcal{O}_S}\mathcal{M})/\mathcal{J} = \bigoplus\left(\bigotimes_{\theta\in\Theta}\mathcal{L}_\theta^{-m_\theta}\right)$$

denote the direct image of $\mathcal{O}_{H^\vee}$.

For later reference, we record a general result which we will use to filter Raynaud $(\mathcal{O}_F/p)$-module schemes in characteristic p.

Lemma 4.1.1. *Suppose that $p\mathcal{O}_S = 0$ and $I \subset \Theta$ has the property that $t_\theta = 0$ for all $\theta \in I$ and $s_\theta = 0$ for all $\theta \notin I$. Let $\mathcal{A}_I$ denote the sheaf of $\mathcal{R}_H$-ideals generated by the $\mathcal{L}_\theta$ for $\theta \notin I$, and let C_I denote the closed subscheme of H defined by $\mathcal{A}_I$. Then the scheme C_I is a finite flat $(\mathcal{O}_F/p)$-submodule scheme of H of rank $p^{|I|}$ over S, and $\mathrm{Lie}(C_I^\vee/S) = \bigoplus_{\theta\in I}\mathcal{L}_\theta$ as an $\mathcal{O}_F \otimes \mathcal{O}_S$-module.*

Proof. Since $s_\theta = 0$ for $\theta \notin I$, it follows that $\mathcal{A}_I$ is the direct sum of the invertible $\mathcal{O}_S$-submodules $\bigotimes_{\theta\in\Theta}\mathcal{L}_\theta^{m_\theta}$ of $\mathcal{R}_H$ for which $m_\theta > 0$ for some $\theta \notin I$. In particular $\mathcal{R}_H/\mathcal{A}_I$ is locally free over $\mathcal{O}_S$ of rank $p^{|I|}$, and is stable under the action of $\mathcal{O}_F$.

The assertion that C_I is a subgroup scheme amounts to the claim that the morphism $C_I \times_S C_I \to H$ defined by the group law on H factors through C_I:

$$\begin{array}{ccc} \mathcal{R}_H & \longrightarrow & \mathcal{R}_H \otimes_{\mathcal{O}_S} \mathcal{R}_H \\ \downarrow & & \downarrow \\ \mathcal{R}_H/\mathcal{A}_I & \dashrightarrow & (\mathcal{R}_H/\mathcal{A}_I) \otimes_{\mathcal{O}_S} (\mathcal{R}_H/\mathcal{A}_I). \end{array}$$

Applying $\mathcal{H}om_{\mathcal{O}_S}(\cdot, \mathcal{O}_S)$ to the diagram of morphisms of flat $\mathcal{O}_S$-modules renders the factorization equivalent to that of

$$\begin{array}{ccc} \mathcal{A}_I^\perp \otimes_{\mathcal{O}_S} \mathcal{A}_I^\perp & \dashrightarrow & \mathcal{A}_I^\perp \\ \downarrow & & \downarrow \\ \mathcal{R}_{H^\vee} \otimes_{\mathcal{O}_S} \mathcal{R}_{H^\vee} & \longrightarrow & \mathcal{R}_{H^\vee}, \end{array}$$

where the bottom arrow is multiplication and $\mathcal{A}_I^\perp$ is the kernel of the morphism

$$\mathcal{R}_{H^\vee} = \mathcal{H}om_{\mathcal{O}_S}(\mathcal{R}_H, \mathcal{O}_S) \to \mathcal{H}om_{\mathcal{O}_S}(\mathcal{A}_I, \mathcal{O}_S).$$

We must therefore prove that $\mathcal{A}_I^\perp$ is a subalgebra of $\mathcal{R}_{H^\vee}$, but

$$\mathcal{A}_I^\perp = \bigoplus\left(\bigotimes_{\theta\in I}\mathcal{L}_\theta^{-m_\theta}\right),$$

where the direct sum is over $(m_\theta)_{\theta\in I}$ with $0 \le m_\theta \le p-1$ for all $\theta \in I$, so this is immediate from the vanishing of the t_θ for $\theta \in I$. Furthermore since $C_I^\vee = \mathbf{Spec}(\mathcal{A}_I^\perp)$ and the augmentation ideal sheaf of $\mathcal{A}_I^\perp$ is generated by the $\mathcal{L}_\theta^{-1}$ for $\theta \in I$, the assertion about $\mathrm{Lie}(C_I^\vee/S)$ is also immediate from this vanishing. □

We now return to the discussion of integral models of Hilbert modular varieties of Iwahori level, and let $S = \widetilde{Y}_{U_0(p)}(G)_{\mathcal{O}}$ where $\mathcal{O}$ is as above. As in [Pap95, Lemma 4.2.2], the universal $(\mathcal{O}_F/p)$-module scheme H over S is Raynaud. Furthermore, the argument of [Pap95, 5.1] shows that $\widetilde{Y}_{U_1(p)}(G)_{\mathcal{O}}$ can be identified with the closed subscheme of H defined by the sheaf of ideals generated by the $(s_v - 1)\left(\bigotimes_{\theta\in\Theta_v}\mathcal{L}_\theta^{(p-1)}\right)$ for $v|p$, where $s_v = \bigotimes_{\theta\in\Theta_v} s_\theta$ is viewed as a morphism $\mathcal{L}_\theta^{(p-1)} \to \mathcal{O}_S$.

The action of the group $\mathcal{O}_{F,(p),+}^\times$ on $\widetilde{Y}_{U_1(p)}(G)$ via multiplication on λ now factors through a free action of $\mathcal{O}_{F,(p),+}^\times/(U_1(p)\cap F^\times)^2$, and we denote the quotient scheme by $Y_{U_1(p)}(G)$; thus $Y_{U_1(p)}(G)$ is finite flat over $Y_{U_0(p)}(G)$, so it is flat, Cohen–Macaulay and quasi-projective of relative dimension d over $\mathbf{Z}_{(p)}$.

Under the additional assumption that U is p-neat, i.e. $\alpha - 1 \in p\mathcal{O}_F$ for all $\alpha \in U \cap F^\times$ as in §3.1.1 (or equivalently $U_1(p) \cap F^\times = U \cap F^\times$), the canonical isomorphism $\theta_\mu^*\underline{A} \cong \underline{A}$ over $\widetilde{Y}_0(p)(G)_\mathcal{O}$ is the identity on H for all $\mu \in (U \cap F^\times)^2$, so H descends to a finite flat group scheme on $Y_{U_0(p)}(G)_\mathcal{O}$, which we also denote by H. It is also Raynaud in the above sense; in fact the line bundles $\mathcal{L}_\theta$ and sections s_θ, t_θ all descend to yield the same descriptions of H, $H^\vee$ and $Y_{U_1(p)}(G)_\mathcal{O}$ as spectra of finite flat $\mathcal{O}_S$-algebras over $S = Y_{U_0(p)}(G)_\mathcal{O}$. We remark also that under the same hypothesis on U, the natural free action of $(\mathcal{O}_F/p\mathcal{O}_F)^\times$ on $\widetilde{Y}_{U_1(p)}(G)$ defined by $c \cdot (\underline{A}, H, P) = (\underline{A}, H, c \cdot P)$ (for which $\widetilde{Y}_{U_0(p)}(G)$ is the quotient) descends to a free action of $(\mathcal{O}_F/p\mathcal{O}_F)^\times$ on $Y_{U_1(p)}(G)$ (with $Y_{U_0(p)}(G)$ as quotient).

As usual, for sufficiently small U, V of level prime to p and $g \in G(\mathbf{A}_\mathbf{f}^{(p)})$ such that $g^{-1}Ug \subset V$, we obtain a finite étale morphism $\tilde{\rho}_g : \widetilde{Y}_{U_1(p)}(G) \to \widetilde{Y}_{V_1(p)}(G)$, descending to $\rho_g : Y_{U_1(p)}(G) \to Y_{V_1(p)}(G)$ and satisfying $\tilde{\rho}_h \circ \tilde{\rho}_g = \tilde{\rho}_{gh}$ and $\rho_h \circ \rho_g = \rho_{gh}$ for suitable h and W. Furthermore the schemes $Y_{U_1(p)}(G)$ and morphisms ρ_g are integral models for the corresponding morphisms and varieties of canonical models over $\mathbf{Q}$ of level $U_1(p)$.

Finally, writing $(\underline{A}_U, H_U)$ (resp. $(\underline{A}_V, H_V)$) for the universal data over $\widetilde{Y}_{U_0(p)}(G)_\mathcal{O}$ (resp. $\widetilde{Y}_{V_0(p)}(G)_\mathcal{O}$) and $\mathcal{L}_{U,\theta}$ (resp. $\mathcal{L}_{V,\theta}$) for the associated Raynaud bundles, note that the canonical quasi-isogeny π_g from A_U to $\tilde{\rho}_g^*A_V$ induces an isomorphism $H_U \to \tilde{\rho}_g^*H_V$ and hence, by the functoriality of Raynaud's construction, isomorphisms $\pi_g^* : \tilde{\rho}_g^*\mathcal{L}_{V,\theta} \to \mathcal{L}_{U,\theta}$ satisfying the usual compatibility $\pi_{gh}^* = \pi_g^* \circ \tilde{\rho}_g^*(\pi_h^*)$. Moreover if V (and hence U) satisfies the additional assumption that $\alpha - 1 \in p\mathcal{O}_F$ for all $\alpha \in V \cap F^\times$, then the isomorphism $H_U \to \tilde{\rho}_g^*H_V$ descends to one over $Y_{U_0(p)}(G)_\mathcal{O}$, as do the isomorphisms π_g^* of Raynaud bundles, now satisfying $\pi_{gh}^* = \pi_g^* \circ \rho_g^*(\pi_h^*)$.

4.2. **The unitary setting.** We now define integral models for unitary Shimura varieties of Iwahori level proceeding similarly to the Hilbert case. We shall only need this for the analogue of $U_0(p)$, and only in the case of $\Sigma = \varnothing$.

4.2.1. *The moduli problem.* Let $G' = G'_\varnothing$, suppose as usual that U' is a sufficiently small open compact subgroup of $G'(\mathbf{A}_\mathbf{f})$ of level prime to p, and define

$$U'_0(p) = \{g \in U' \mid g_p \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \bmod p\mathcal{O}_{E,p}\}.$$

Here $G'(\mathbf{Q}_p)$ is viewed as the subgroup $\mathrm{GL}_2(F_p)E_p^\times$ of $\mathrm{GL}_2(E_p)$, so if $U' = (U')^pU'_p$, then $U'_0(p) = (U')^pU'_{0,p}$ where $U'_{0,p}$ is the image of $U_{0,p} \times \mathcal{O}_{E,p}^\times$ in $\mathrm{GL}_2(E_p)$.

Now consider the functor which associates to a $\mathbf{Z}_{(p)}$-scheme S the set of isomorphism classes of triples $(\underline{A}_1, \underline{A}_2, f)$, where

- $\underline{A}_j = (A_j, \iota_j, \lambda_j, \eta_j, \epsilon_j)$ for $j = 1, 2$ represent S-points of $\widetilde{Y}_{U'}(G')$, and
- $f : A_1 \to A_2$ is an isogeny of degree p^{4d} such that $f \circ \iota_1(\alpha) = \iota_2(\alpha) \circ f$ for all $\alpha \in \mathcal{O}_D$, $p\lambda_1 = f^\vee \circ \lambda_2 \circ f$, $\eta_2 = f \circ \eta_1$ (as $(U')^p$-orbits for each $\overline{s}_i$), $\epsilon_2 = p\epsilon_1$ and $H = \ker f$ decomposes as $\prod_{w|p} H_w$ where $H_w \subset A_1[w]$ has rank $p^{2[E_w:\mathbf{Q}_p]}$ for each prime w of $\mathcal{O}_E$ dividing p.

Again standard arguments show that the functor is representable by a scheme, which we denote by $\widetilde{Y}_{U'_0(p)}(G')$, and the forgetful morphism to $\widetilde{Y}_{U'}(G')$ sending $(\underline{A}_1, \underline{A}_2, f)$ to $\underline{A}_1$ is projective.

4.2.2. *Dieudonné theory over* $\overline{\mathbf{F}}_p$. We shall need to relate $\widetilde{Y}_{U_0(p)}(G)$ and $\widetilde{Y}_{U_0'(p)}(G')$, but first we recall some general facts from Dieudonné theory.[8] If A is an abelian variety over $\overline{\mathbf{F}}_p$ of dimension g, then the (contravariant) Dieudonné module $D = \mathbf{D}(A[p^\infty])$ is a free W-module of rank $2g$ equipped with ϕ-semilinear endomorphism Φ, induced by the absolute Frobenius on A, such that $pD \subset \Phi(D)$. Furthermore there is a canonical (i.e., functorial in A) isomorphism $D/pD = \mathbf{D}(A[p]) \cong H^1_{\mathrm{dR}}(A/\overline{\mathbf{F}}_p)$ identifying VD/pD with $H^0(A, \Omega^1_{A/\overline{\mathbf{F}}_p})$, where V is the ϕ^{-1}-semilinear endomorphism of D defined by $\Phi^{-1}p$. Letting $D^\vee = \mathbf{D}(A^\vee[p^\infty])$, we also have a canonical isomorphism $\mathrm{Hom}_W(D, W) \cong D^\vee$ under which Φ and V are semilinear dual, and the induced perfect pairing $D \times D^\vee \to W$ is anti-symmetric under the canonical isomorphism $A \cong (A^\vee)^\vee$.

Recall also that the functor $\mathbf{D}$ defines a contravariant equivalence of categories between finite (commutative) group schemes over $\overline{\mathbf{F}}_p$ killed by p and $\overline{\mathbf{F}}_p$-vector spaces equipped with a ϕ-semilinear endomorphism Φ and ϕ^{-1}-semilinear endomorphism V such that $\Phi V = V\Phi = 0$. For any such group scheme H, we have $\mathrm{rank}(H) = p^{\dim_{\overline{\mathbf{F}}_p}(\mathbf{D}(H))}$. Furthermore $\mathbf{D}$ is compatible with formation of Cartier duals in the obvious sense, and there are canonical isomorphisms

$$\mathbf{D}(H)/\Phi\mathbf{D}(H) \cong \mathrm{Hom}_{\overline{\mathbf{F}}_p}(\mathrm{Lie}(H)^{(p)}, \overline{\mathbf{F}}_p) \quad \text{and} \quad \ker(\mathbf{D}(H) \xrightarrow{V} \mathbf{D}(H)) \cong \mathrm{Lie}(H^\vee)^{(p)}.$$

Applying $\mathbf{D}$ to the canonical isomorphism $A^\vee[p] \cong (A[p])^\vee$ gives the reduction mod p of the isomorphism $D^\vee \cong \mathrm{Hom}_W(D, W)$ recalled above. In particular, if $\lambda : A \to A^\vee$ is a polarization of A, then the λ-Weil pairing $A[p] \to (A[p])^\vee$ corresponds under $\mathbf{D}$ to the morphism $\mathrm{Hom}_{\overline{\mathbf{F}}_p}(D/pD, \overline{\mathbf{F}}_p) \to D/pD$ obtained as the reduction mod p of the composite

$$\mathrm{Hom}_W(D, W) \xrightarrow{\sim} D^\vee \longrightarrow D.$$

Thus if $\lambda \in \mathrm{Hom}(A, A^\vee) \otimes \mathbf{Z}_{(p)}$ is a prime-to-p quasi-polarization, then the λ-Weil pairing on $A[p^\infty]$ induces an isomorphism $\mathrm{Hom}_W(D, W) \cong D$ corresponding to an alternating perfect pairing on D, and whose reduction mod p defines the isomorphism $\mathrm{Hom}_{\overline{\mathbf{F}}_p}(D/pD, \overline{\mathbf{F}}_p) \cong D/pD$.

4.2.3. *Relation between Hilbert and unitary settings.* Recall from §2.3.4 that for open compact $U \subset G(\mathbf{A_f})$ and $U' \subset G'(\mathbf{A_f})$ containing the image of U, we defined a morphism $\tilde{i} : \widetilde{Y}_U(G) \to \widetilde{Y}_{U'}(G')$ sending A to $A' = A \otimes_{\mathcal{O}_F} \mathcal{O}_E^2$. Furthermore under the hypotheses of Lemma 2.3.1, the morphism $\tilde{i}$ induces an isomorphism

$$\widetilde{Y}_U(G) \longrightarrow \widetilde{Y}_{U'}(G') \times_{C'} C$$

identifying $\widetilde{Y}_U(G)$ with a union of connected components of $\widetilde{Y}_{U'}(G')$, where C (resp. C') is the finite set $C_{\det(U)}$ (resp. $C_{\nu'(U')}$) viewed as a scheme over $\mathbf{Z}_{(p)}$. The construction of $\tilde{i}$ clearly extends to define a morphism $\widetilde{Y}_{U_0(p)}(G) \to \widetilde{Y}_{U_0'(p)}(G')$, which in turn induces a morphism

$$\widetilde{Y}_{U_0(p)}(G) \longrightarrow \widetilde{Y}_{U_0'(p)}(G') \times_{C'} C$$

[8] We follow the conventions of [Oda69], which differ from those of [Fon77] by a Frobenius twist. More precisely, our $\mathbf{D}(G)$ is the $D(G)$ of [BBM82, (4.2.8.1)], which is canonically isomorphic to $\mathbf{M}(G) = \underline{M}(G) \otimes_{W,\phi} W$ where $\mathbf{M}$ (resp. $\underline{M}$) is the functor defined in [Oda69] (resp. [Fon77]); see [BBM82, Thm. 4.2.14]. Here however we use the semilinear versions of Φ and V.

where the maps $\widetilde{Y}_{U_0(p)}(G) \to C$ and $\widetilde{Y}_{U_0'(p)}(G) \to C'$ are defining by composing with the forgetful morphisms.

Lemma 4.2.1. *Under the hypotheses of Lemma 2.3.1, the morphism*

$$\widetilde{Y}_{U_0(p)}(G) \longrightarrow \widetilde{Y}_{U_0'(p)}(G') \times_{C'} C$$

is an isomorphism.

Proof. First note that since $\widetilde{Y}_{U_0(p)}(G) \to \widetilde{Y}_U(G) \to \widetilde{Y}_{U'}(G')$ is projective, so is $\widetilde{Y}_{U_0(p)}(G) \to \widetilde{Y}_{U_0'(p)}(G)$. Furthermore since $\widetilde{Y}_U(G) \to \widetilde{Y}_{U'}(G)$ is injective on S-points for all S, it follows from the definitions of the corresponding functors that so is $\widetilde{Y}_{U_0(p)}(G) \to \widetilde{Y}_{U_0'(p)}(G)$. Being projective and quasi-finite, it must be finite, so in fact a closed immersion, and hence so is the morphism of the lemma. To prove it is an isomorphism, it therefore suffices to construct a section $S \to \widetilde{Y}_{U_0(p)}(G)$ for each connected component S of $\widetilde{Y}_{U_0'(p)}(G') \times_{C'} C$.

Let S be such a connected component, and let $(\underline{A}_1', \underline{A}_2', f')$ be the universal triple over S. By assumption, the S-point of $\widetilde{Y}_{U'}(G')$ defined by $\underline{A}_1'$ factors through $\tilde{i}$, meaning that $\underline{A}_1'$ is defined by $A_1 \otimes_{\mathcal{O}_F} \mathcal{O}_E^2$ (with its additional structure) for some S-point $\underline{A}_1$ of $\widetilde{Y}_U(G)$. Since $\widetilde{Y}_{U_0'(p)}(G')$ is locally of finite type over $\mathbf{Z}_{(p)}$, we may choose a geometric point $\overline{s} : \operatorname{Spec} k \to S$ such that either $k = \overline{\mathbf{F}}_p$ or k has characteristic zero.

Writing B_j for $e_0 A_j'$ where $e_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \mathcal{O}_D = M_2(\mathcal{O}_E)$ and identifying $A_j' = B_j \otimes_{\mathcal{O}_E} \mathcal{O}_E^2$ for $j = 1, 2$, the isogeny f' induces an isogeny $g : B_1 \to B_2$ such that $f' = g \otimes_{\mathcal{O}_E} \mathcal{O}_E^2$. Similarly since $e_0^* = e_0$, we have prime-to-p quasi-polarizations μ_j on the B_j such that $p\mu_1 = g^\vee \circ \mu_2 \circ g$. Moreover the quasi-polarization μ_1 on $B_1 = A_1 \otimes_{\mathcal{O}_F} \mathcal{O}_E$ is induced by λ_1 and the $\mathcal{O}_F$-linear pairing on $\mathcal{O}_E$ defined by $(\alpha, \beta) \mapsto \operatorname{tr}_{E/F}(\alpha\overline{\beta})$.

Let $H' = \ker(f_{\overline{s}}')$. We claim that $H' = H \otimes_{\mathcal{O}_F} \mathcal{O}_E^2$ for some totally isotropic subgroup $H \subset A_{1,\overline{s}}[p]$. First note that $H' = I \otimes_{\mathcal{O}_E} \mathcal{O}_E^2$ where $I = e_0 H' = \ker g_{\overline{s}}$ is a finite $(\mathcal{O}_E/p\mathcal{O}_E)$-submodule scheme of

$$B_{1,\overline{s}}[p] = (A_{1,\overline{s}} \otimes_{\mathcal{O}_F} \mathcal{O}_E)[p] = \prod_{w|p} (A_{1,\overline{s}} \otimes_{\mathcal{O}_F} \mathcal{O}_E)[w] = \prod_{w|p} B_{1,\overline{s}}[w],$$

the products being over primes w of $\mathcal{O}_E$ dividing p. Furthermore we may decompose $I = \prod_{w|p} I_w$ where each I_w is an $(\mathcal{O}_E/w)$-subspace scheme of $B_{1,\overline{s}}[w]$ of rank $p^{[E_w:\mathbf{Q}_p]}$. Note that for each prime w of $\mathcal{O}_F$, the isomorphism $\mathcal{O}_F/v \cong \mathcal{O}_E/w$ induces an isomorphism $A_{1,\overline{s}}[v] \cong B_{1,\overline{s}}[w]$, where v is the prime of $\mathcal{O}_F$ dividing w, and we define H_w to be the subgroup of $A_{1,\overline{s}}[v]$ corresponding to I_w. We will prove that H_w is isotropic and coincides with H_{w^c} for each w, from which the claim follows with $H = \prod_v H_{\tilde{v}}$.

Suppose first that $k = \overline{\mathbf{F}}_p$ and consider the Dieudonné modules $D = \mathbf{D}(A_{1,\overline{s}}[p^\infty])$ and $D_j = \mathbf{D}(B_{j,\overline{s}}[p^\infty])$ for $j = 1, 2$. Thus D is an $\mathcal{O}_F \otimes W$-module, hence decomposes as $\oplus_{\theta \in \Theta} D_\theta$, with Φ restricting to define ϕ-semilinear homomorphisms $D_\theta \to D_{\phi \circ \theta}$. Similarly D_1 and D_2 are $\mathcal{O}_E \otimes W$-module, hence decompose as $\oplus_{\tau \in \Theta_E} D_{j,\tau}$, with Φ restricting ϕ-semilinear homomorphisms $D_{j,\tau} \to D_{j,\phi\circ\tau}$. Furthermore each D_θ and $D_{j,\tau}$ is free of rank two over W, and we may identify D_1 with $D \otimes_{\mathcal{O}_F} \mathcal{O}_E$, giving canonical isomorphisms $D_{1,\tau} \cong D_\theta$ for each $\tau \in \Theta_E$ restricting to $\theta \in \Theta$. Note also that since $\mathrm{Lie}(B_j)_\tau$ is one-dimensional for each $\tau \in \Theta_E$, the successive quotients in the filtrations

$$pD_{j,\phi\circ\tau} \subset \Phi(D_{j,\tau}) \subset D_{j,\phi\circ\tau}$$

are all one-dimensional.

Now consider the inclusion $g^* : D_2 \to D_1$ induced by the isogeny g; it is $\mathcal{O}_E \otimes W$-linear, compatible with Φ, and has cokernel isomorphic to $\mathbf{D}(I)$. Note in particular that we have the inclusions

$$pD_{1,\tau} \subset g^*(D_{2,\tau}) \subset D_{1,\tau}$$

for each $\tau \in \Theta_E$, and since I_w has rank $p^{[E_w:\mathbf{Q}_p]}$, it follows that

$$\sum_{\tau\in\Theta_{E,w}} \dim_{\overline{\mathbf{F}}_p} D_{1,\tau}/g^*(D_{2,\tau}) = \dim_{\overline{\mathbf{F}}_p} \mathbf{D}(I_w) = [E_w : \mathbf{Q}_p].$$

Since $g^*(\Phi(D_{2,\tau})) = \Phi(g^*(D_{2,\tau}))$ for each τ, we have

$$\begin{aligned}
&\dim_{\overline{\mathbf{F}}_p}(D_{1,\phi\circ\tau}/\Phi(D_{1,\tau})) + \dim_{\overline{\mathbf{F}}_p}(D_{1,\tau}/g^*(D_{2,\tau}))\\
&\quad = \dim_{\overline{\mathbf{F}}_p}(D_{1,\phi\circ\tau}/\Phi(D_{1,\tau})) + \dim_{\overline{\mathbf{F}}_p}(\Phi D_{1,\tau}/\Phi(g^*(D_{2,\tau})))\\
&\quad = \dim_{\overline{\mathbf{F}}_p}(D_{1,\phi\circ\tau}/g^*(D_{2,\phi\circ\tau})) + \dim_{\overline{\mathbf{F}}_p}(g^* D_{2,\phi\circ\tau}/g^*(\Phi(D_{2,\tau})))\\
&\quad = \dim_{\overline{\mathbf{F}}_p}(D_{1,\phi\circ\tau}/g^*(D_{2,\phi\circ\tau})) + \dim_{\overline{\mathbf{F}}_p}(D_{2,\phi\circ\tau}/\Phi(D_{2,\tau})).
\end{aligned}$$

Since $\dim_{\overline{\mathbf{F}}_p}(D_{j,\phi\circ\tau}/\Phi(D_{j,\tau})) = 1$ for $j = 1, 2$, it follows that $\dim_{\overline{\mathbf{F}}_p} D_{1,\tau}/g^*(D_{2,\tau}) = 1$ for all τ. We have now shown that the $(\mathcal{O}_E/w)$-vector space scheme I_w has the property that $\mathbf{D}(I_w)$ is free of rank one over $(\mathcal{O}_E/w) \otimes \overline{\mathbf{F}}_p$, and hence $\mathbf{D}(H_w)$ is free of rank one over $(\mathcal{O}_F/v)\otimes\overline{\mathbf{F}}_p$. Since the pairing on $D/pD = \mathbf{D}(A_1[p])$ induced by the λ_1-Weil pairing is alternating and $(\mathcal{O}_F/p\mathcal{O}_F)$-bilinear, it follows that the composite

$$\mathbf{D}(H_w^\vee) = \mathrm{Hom}_{\overline{\mathbf{F}}_p}(\mathbf{D}(H_w), \overline{\mathbf{F}}_p) \hookrightarrow \mathrm{Hom}_{\overline{\mathbf{F}}_p}(\mathbf{D}(A_{1,\overline{s}}[w]), \overline{\mathbf{F}}_p) \stackrel{\sim}{\longrightarrow} \mathbf{D}(A_{1,\overline{s}}[w]) \twoheadrightarrow \mathbf{D}(H_w)$$

is trivial, and hence that H_w is isotropic.

We must now show that $H_w = H_{w^c}$, or equivalently that $I_{w^c} = (1\otimes c)I_w$. We will do this by proving that each of I_{w^c} and $(1\otimes c)I_w$ is the kernel of

$$B_{1,\overline{s}}[w^c] \to (B_{1,\overline{s}}[w])^\vee \to I_w^\vee,$$

where the first morphism induced by the μ_1-Weil pairing on B_1. Since the Rosati involution associated to μ_1 restricts to c on the image of $\mathcal{O}_E$ in $\mathrm{End}(B_{1,\overline{s}})$, the map $B_{1,\overline{s}}[w^c] \to (B_{1,\overline{s}}[w])^\vee$ is in fact an isomorphism, so comparing ranks, we see that it suffices to prove that each of I_{w^c} and $(1\otimes c)I_w$ is orthogonal to I_w. For $(1\otimes c)I_w$, this is immediate from the isotropy of H_w and the commutativity of the diagram

$$\begin{array}{ccc}
A_1[v] & \stackrel{\sim}{\longrightarrow} & A_1^\vee[v]\\
\wr\downarrow & & \wr\downarrow\\
B_1[w^c] & \stackrel{\sim}{\longrightarrow} & B_1^\vee[w]
\end{array}$$

implied by the relation between and λ_1 and μ_1. For I_{w^c}, this follows from the isotropy of I under the μ_1-Weil pairing, which is equivalent to the vanishing of the map $\mathbf{D}(I^\vee) \to \mathbf{D}(I)$ induced by μ_1. To see this vanishing, note that the image of $\mathbf{D}(I^\vee)$ in

$$\mathbf{D}((B_{1,\overline{s}}[p])^\vee) = \mathrm{Hom}_{\overline{\mathbf{F}}_p}(D_1/pD_1, \overline{\mathbf{F}}_p) = \mathrm{Hom}_W(D_1, W) \otimes_W \overline{\mathbf{F}}_p$$

is the image of $L = \{\psi \in \mathrm{Hom}_W(D_1, W) \mid \psi(g^*(D_2)) \subset pW\}$. If $\psi \in L$, then $\psi \circ g^* = p\xi$ for some $\xi \in \mathrm{Hom}_W(D_2, W)$, and the relation $p\mu_1 = g^\vee \circ \mu_2 \circ g$ implies that $\mu_1^*(\psi) = g^*\mu_2^*(\xi)$ is in the image of g^*, hence has trivial image in $\mathbf{D}(I)$.

This completes the proof of the claim in the case $k = \overline{\mathbf{F}}_p$. We omit the proof in the case k has characteristic zero, which is similar, but simpler, since one can argue using p-adic Tate modules instead of Dieudonné modules.

We have now shown that $\ker f'_{\overline{s}} = H \otimes_{\mathcal{O}_F} \mathcal{O}_E^2$ for some totally isotropic $H \subset A_{1,\overline{s}}[p]$. Letting $\pi : A_{1,\overline{s}} \to A_{2,\overline{s}} := A_{1,\overline{s}}/H$ denote the natural projection, we may endow $A_{2,\overline{s}}$ with additional structure making $(\underline{A}_{1,\overline{s}}, \underline{A}_{2,\overline{s}}, \pi)$ a k-point of $\widetilde{Y}_{U_0(p)}(G)$. Furthermore we obtain an isomorphism $A_{2,\overline{s}} \otimes_{\mathcal{O}_F} \mathcal{O}_E^2 \cong A'_{2,\overline{s}}$ whose compatibility with $f'_{\overline{s}}$ and $\pi \otimes 1$ implies that $\underline{A}'_{2,\overline{s}}$ corresponds to the image under $\tilde{i}$ of the k-point of $\widetilde{Y}_U(G)$ defined by $\underline{A}_{2,\overline{s}}$.

Since S is connected, it follows that the S-point of $\widetilde{Y}_{U'}(G')$ defined by $\underline{A}'_2$ factors through $\tilde{i}$, and hence that A'_2 is isomorphic to $A_2 \otimes_{\mathcal{O}_F} \mathcal{O}_E^2$ with its additional structure for some $\underline{A}_2$ corresponding to an S-point of $\widetilde{Y}_U(G)$. Moreover the fibre of $\underline{A}_2$ at $\overline{s}$ is isomorphic to the abelian variety with additional structure already denoted $\underline{A}_{2,\overline{s}}$, and $f'_{\overline{s}} = \pi \otimes 1$ for some $\mathcal{O}_F$-linear isogeny $\pi : A_{1,\overline{s}} \to A_{2,\overline{s}}$.

We will now show that $f' = f \otimes 1$ for some $\mathcal{O}_F$-linear isogeny $f : A_1 \to A_2$. To that end, we first observe that the natural maps

$$\mathrm{Hom}_{\mathcal{O}_F}(A_1, A_2) \otimes_{\mathcal{O}_F} \mathcal{O}_E \to \mathrm{Hom}_{\mathcal{O}_E}(A_1 \otimes_{\mathcal{O}_F} \mathcal{O}_E, A_2 \otimes_{\mathcal{O}_F} \mathcal{O}_E) \to \mathrm{Hom}_{\mathcal{O}_D}(A'_1, A'_2)$$

are isomorphisms (the first follows for example from the fact that it has torsion-free cokernel and becomes an isomorphism after tensoring with $\mathbf{Q}$,and the second is clear). Next note that the image of the map

$$\mathrm{Hom}_{\mathcal{O}_F}(A_1, A_2) \to \mathrm{Hom}_{\mathcal{O}_F}(A_1, A_2) \otimes_{\mathcal{O}_F} \mathcal{O}_E$$

defined by $f \mapsto f \otimes 1$ is precisely the set of elements fixed by the automorphism $1 \otimes c$. Moreover the same holds after replacing the A_j by their fibres $A_{j,\overline{s}}$. Since the natural map

$$\mathrm{Hom}_{\mathcal{O}_F}(A_1, A_2) \to \mathrm{Hom}_{\mathcal{O}_F}(A_{1,\overline{s}}, A_{2,\overline{s}})$$

is injective and $f'_{\overline{s}} = \pi \otimes 1$ for some $\pi \in \mathrm{Hom}_{\mathcal{O}_F}(A_{1,\overline{s}}, A_{2,\overline{s}})$, it follows that $f' = f \otimes 1$ for some $f \in \mathrm{Hom}_{\mathcal{O}_F}(A_1, A_2)$.

Finally, the compatibility of f' with the additional structure on $\underline{A}'_1$ and $\underline{A}'_2$ implies that of f with the additional structure on $\underline{A}_1$ and $\underline{A}_2$. Therefore $(\underline{A}_1, \underline{A}_2, f)$ defines an S-point of $\widetilde{Y}_{U_0(p)}(G)$ whose composite with $\tilde{i}$ is the inclusion of S in $\widetilde{Y}_{U'_0(p)}(G')$. □

4.2.4. *Descent, Hecke action, and Raynaud bundles.* We now return to the setting of arbitrary sufficiently small U' of level prime to p (not necessarily satisfying the hypotheses of Lemma 2.3.1). As in §4.1.1, we have a natural action of $\mathcal{O}_{F,(p),+}^\times$ on $\widetilde{Y}_{U'_0(p)}(G')$, now factoring through a free action of $\mathcal{O}_{F,(p),+}^\times/\mathrm{Nm}_{E/F}(U' \cap E^\times)$, and we denote the quotient scheme by $Y_{U'_0(p)}(G')$. Thus $Y_{U'_0(p)}(G') \to Y_{U'}(G')$ is projective, and $Y_{U'_0(p)}(G')$ is quasi-projective over $\mathbf{Z}_{(p)}$. As usual, for open compact subgroups U', $V' \subset G'(\mathbf{A_f})$ as above, and $g \in G'(\mathbf{A}_\mathbf{f}^{(p)})$ such that $g^{-1}U'g \subset V'$, we obtain a finite étale morphism $\tilde{\rho}_g : \widetilde{Y}_{U'_0(p)}(G') \to \widetilde{Y}_{V'_0(p)}(G')$ descending to $\rho_g : Y_{U'_0(p)}(G') \to Y_{V'_0(p)}(G')$. These satisfy $\tilde{\rho}_h \circ \tilde{\rho}_g = \tilde{\rho}_{gh}$ and hence $\rho_h \circ \rho_g = \rho_{gh}$

for $h \in G'(\mathbf{A}_\mathbf{f}^{(p)})$ and sufficiently small W' containing $h^{-1}V'h$, and the schemes $Y_{U_0'(p)}(G')$ and morphisms ρ_g are integral models for the corresponding morphisms and varieties of canonical models over $\mathbf{Q}$ of level $U_0'(p)$.

As in §4.1.2, let $\mathcal{O}$ be the localization of $\mathcal{O}_L$ at the distinguished prime over p for a sufficiently large L.

Proposition 4.2.2. *Let $S' = \widetilde{Y}_{U_0'(p)}(G')_\mathcal{O}$, let H' denote the kernel of the universal isogeny $A_1' \to A_2'$ on S', and let $I = e_0H'$. Then*

1. *the $\mathcal{O}_E/p\mathcal{O}_E$-module scheme I is Raynaud;*
2. *H' is totally isotropic with the respect to the λ_1'-Weil pairing on A_1';*
3. *for any $\overline{s} : \mathrm{Spec}(\overline{\mathbf{F}}_p) \to S'$, the Dieudonné module $\mathbf{D}(I_{\overline{s}})$ (resp. $D(H'_{\overline{s}})$) is free of rank one (resp. two) over $\mathcal{O}_E \otimes \overline{\mathbf{F}}_p$.*

Proof. (1) Suppose first that we are in the setting of Lemma 2.3.1, so that $U' = UV_E$ for sufficiently small U and V_E, and let H be the kernel of the universal isogeny on $\widetilde{Y}_{U_0(p)}(G)_\mathcal{O}$. Since $\tilde{i}^*I = H \otimes_{\mathcal{O}_F} \mathcal{O}_E$, it follows from Lemma 4.2.1 that the restriction of I to $S' \times_{C'} C$ is Raynaud. Furthermore, for any connected component S of S', we may choose $h \in (\mathbf{A}_{E,\mathbf{f}}^{(p)})^\times$ so that $\tilde{\rho}_h$ sends S to $S' \times_{C'} C$, and since the canonical quasi-isogeny π_h from $A_1' \to \tilde{\rho}_h^* A_1'$ induces an isomorphism $I \to \tilde{\rho}_h^* I$, it follows that I is Raynaud on S. Therefore I is Raynaud on S'. Returning to the setting of arbitrary U', note that we may choose sufficiently small U and V_E as above so that $V' = UV_E$ is contained in U'. Considering the natural projection $\tilde{\rho}_1 : \widetilde{Y}_{V_0'(p)}(G')_\mathcal{O} \to S'$, we see that since $\tilde{\rho}_1^* I$ is Raynaud, so is I.

Parts (2) and (3) may be similarly deduced from the corresponding properties of H and I on $S' \times_{C'} C$ in the setting of Lemma 4.2.1 (or indeed shown more directly using arguments in its proof). ◻

From the $\mathcal{O}_E/p\mathcal{O}_E$-module scheme I and embeddings $\tau \in \Theta_E$, we obtain as in §4.1.2 the Raynaud line bundles $\mathcal{L}_\tau$ on S', together with morphisms $s_\tau : \mathcal{L}_\tau^p \to \mathcal{L}_{\phi\circ\tau}$, $t_\tau : \mathcal{L}_{\phi\circ\tau} \to \mathcal{L}_\tau^p$ such that $s_\tau \circ t_\tau \in p\mathcal{O}^\times$. For sufficiently small U' and V', and $g \in G'(\mathbf{A}_\mathbf{f}^{(p)})$ such that $g^{-1}U'g \subset V'$, the canonical quasi-isogeny π_g from $A'_{1,U'}$ to $\tilde{\rho}_g^* A'_{1,V'}$ induces isomorphisms $I_{U'} \to \tilde{\rho}_g^* I_{V'}$ on $\widetilde{Y}_{U_0'(p)}(G')$, hence isomorphisms $\pi_g^* : \tilde{\rho}_g^*\mathcal{L}_{V',\theta} \to \mathcal{L}_{U',\theta}$ satisfying the usual compatibility $\pi_{gh}^* = \pi_g^* \circ \tilde{\rho}_g^*(\pi_h^*)$. Furthermore under the additional assumption that $\alpha - 1 \in p\mathcal{O}_E$ for all $\alpha \in U' \cap E^\times$, the $\mathcal{O}_E/p\mathcal{O}_E$-module scheme I, line bundles $\mathcal{L}_\tau$ and morphisms s_τ, t_τ all descend to objects, which we denote by the same symbol, on the quotient $Y_{U_0'(p)}(G')_\mathcal{O}$. If V' also satisfies this condition, then π_g^* descends to an isomorphism $\rho_g^*\mathcal{L}_{V',\theta} \to \mathcal{L}_{U',\theta}$, and these satisfy $\pi_{gh}^* = \pi_g^* \circ \rho_g^*(\pi_h^*)$.

Finally we note that for sufficiently small $U \subset G(\mathbf{A}_\mathbf{f})$ and $U' \subset G'(\mathbf{A}_\mathbf{f})$ such that U' contains the image of U, the identification $\tilde{i}^*I = H \otimes_{\mathcal{O}_F} \mathcal{O}_E$ allows us to identify $\tilde{i}^*\mathcal{L}_\tau$ with $\mathcal{L}_\theta$, where $\theta = \tau|_F$, under which $\tilde{i}^*s_\tau = s_\theta$ and $\tilde{i}^*t_\tau = t_\theta$. Furthermore the identifications are compatible with the descent data to $Y_{U_0'(p)}$ defined by the action of $\mathcal{O}_{F,(p)}^\times$, and with the morphisms π_g^* for suitable $g \in G(\mathbf{A}_\mathbf{f}^{(p)})$, $V \subset G(\mathbf{A}_\mathbf{f})$ and $V' \subset G'(\mathbf{A}_\mathbf{f})$.

4.3. **Stratifications.** We now define a stratification on the special fibre of $Y_{U_0(p)}(G)$, as in [GK12]. As usual, we first consider $\widetilde{Y}_{U_0(p)}(G)$ and then descend to $Y_{U_0(p)}(G)$.

4.3.1. *Closed strata.* Let U be a sufficiently small open compact subgroup of $G(\mathbf{A_f}) = \mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}})$ of level prime to p, and let $\mathbf{F}$ be a sufficiently large extension of $\mathbf{F}_p$ as in §3.1.1. Let $S = \widetilde{Y}_{U_0(p)}(G)_{\mathbf{F}}$, and let $(\underline{A}_1, \underline{A}_2, f)$ be the universal object over S. We thus have morphisms of line bundles $\mathrm{Lie}(f)_\theta : \mathrm{Lie}(A_1/S)_\theta \to \mathrm{Lie}(A_2/S)_\theta$ and $\mathrm{Lie}(f^\vee)_\theta : \mathrm{Lie}(A_2^\vee/S)_\theta \to \mathrm{Lie}(A_1^\vee/S)_\theta$.

For $I, J \subset \Theta$, we let $S_{\phi(I),J}$ denote the subscheme of S defined by the vanishing of the sections[9]

$$\{\, \mathrm{Lie}(f)_\theta \mid \theta \in I \,\} \cup \{\, \mathrm{Lie}(f^\vee)_\theta \mid \theta \in J \,\}.$$

We write simply S_J for $S_{\phi(I),J}$ if I is the complement of J in Θ.

Note that if $I \subset I'$ and $J \subset J'$, then $S_{\phi(I'),J'}$ is a closed subscheme of $S_{\phi(I),J}$, and that for any I, J, I', J', we have

$$S_{\phi(I\cup I'),J\cup J'} = S_{\phi(I),J} \cap S_{\phi(I'),J'}.$$

Furthermore since $\mathrm{Lie}(\lambda_2)$ is an isomorphism and $\mathrm{Lie}(f^\vee \circ \lambda_2 \circ f) = 0$ on S, we have

$$S_{\phi(I),J} = S_{\phi(I\cup\{\theta\}),J} \cup S_{\phi(I),J\cup\{\theta\}}. \tag{9}$$

for $\theta \notin I \cup J$.

4.3.2. *Local structure.* For a closed point $Q \in S = \widetilde{Y}_{U_0(p)}(G)_{\mathbf{F}}$, define

$$I_Q = \{\theta \in \Theta : \mathrm{Lie}(f)_\theta(Q) = 0\} \quad \text{and} \quad J_Q = \{\theta \in \Theta : \mathrm{Lie}(f^\vee)_\theta(Q) = 0\}.$$

Note that $I_Q \cup J_Q = \Theta$. The following description of the completed local ring $\widehat{\mathcal{O}}_{S,Q}$ is an immediate consequence of the proof of Theorem 2.4.1 of [GK12]. (Note that although the moduli problems considered here and in [GK12] are slightly different, the same arguments apply since prime-to-p polarizations, quasi-polarizations and level structures deform uniquely.) In the statement, we view $\langle \mathrm{Lie}(f)_\theta \rangle$ and $\langle \mathrm{Lie}(f^\vee)_\theta \rangle$ via trivializations of the relevant line bundles in a neighborhood of Q.

Theorem 4.3.1. *Let Q be a closed point of $S = \widetilde{Y}_{U_0(p)}(G)_{\mathbf{F}}$. There is an isomorphism*

$$\widehat{\mathcal{O}}_{S,Q} \cong \widehat{\bigotimes_{\theta\in\Theta}} k_Q[[x_\theta, y_\theta]]/\langle c_\theta d_\theta \rangle,$$

where the completed tensor product is over the residue field k_Q at Q,

$$\begin{array}{rcccl} \langle c_\theta \rangle & = & \langle \mathrm{Lie}(f)_\theta \rangle & = & \begin{cases} \langle x_\theta \rangle, & \text{if } \theta \in I_Q, \\ \langle 1 \rangle, & \text{if } \theta \notin I_Q, \end{cases} \\ \text{and} \quad \langle d_\theta \rangle & = & \langle \mathrm{Lie}(f^\vee)_\theta \rangle & = & \begin{cases} \langle y_\theta \rangle, & \text{if } \theta \in J_Q, \\ \langle 1 \rangle, & \text{if } \theta \notin J_Q. \end{cases} \end{array}$$

Corollary 4.3.2. *The scheme $S_{\phi(I),J}$ over $\mathbf{F}$ is a reduced local complete intersection of constant dimension $d - |I \cap J|$, and is smooth if $I \cup J = \Theta$.*

[9]The reason for $\phi(I)$ $(= \{\, \theta \mid \phi^{-1} \circ \theta \in I \,\})$ instead of I in the notation is for consistency with the notation in [GK12].

Proof. If Q is a closed point of $S_{\phi(I),J}$, then $I \subset I_Q$ and $J \subset J_Q$. Theorem 4.3.1 and the definition of $S_{\phi(I),J}$ therefore imply that

$$\widehat{\mathcal{O}}_{S_{\phi(I),J},Q} \cong \widehat{\bigotimes_{\theta\in\Theta}} R_\theta, \quad \text{where} \quad R_\theta = \begin{cases} k_Q[[x_\theta, y_\theta]]/\langle c_\theta d_\theta \rangle, & \text{if } \theta \notin I \cup J, \\ k_Q[[x_\theta]], & \text{if } \theta \in J - I, \\ k_Q[[y_\theta]], & \text{if } \theta \in I - J, \\ k_Q, & \text{if } \theta \in I \cap J. \end{cases}$$

It follows that the completed local ring of $S_{\phi(I),J}$ at every closed point is a reduced complete intersection of dimension $d - |I \cap J|$, and is regular if $I \cup J = \Theta$. □

4.3.3. *Descent and Hecke action.* Note that the natural action of $\mathcal{O}^\times_{F,(p),+}$ on the line bundles

$$\mathcal{H}om_{\mathcal{O}_S}(\mathrm{Lie}(A_1/S)_\theta, \mathrm{Lie}(A_2/S)_\theta) \quad \text{and} \quad \mathcal{H}om_{\mathcal{O}_S}(\mathrm{Lie}(A_2^\vee/S)_\theta, \mathrm{Lie}(A_1^\vee/S)_\theta)$$

factors through the quotient $\mathcal{O}^\times_{F,(p),+}/(F^\times \cap U)^2$, so the line bundles descend to $\overline{Y}_0(p)$, as do the sections $\mathrm{Lie}(f)_\theta$ and $\mathrm{Lie}(f^\vee)_\theta$. We may thus similarly define closed subschemes $\overline{Y}_0(p)_{\phi(I),J}$ of $\overline{Y}_0(p) := Y_{U_0(p)}(G)_{\mathbf{F}}$ by the vanishing of $\mathrm{Lie}(f)_\theta$ for $\theta \in I$, and $\mathrm{Lie}(f^\vee)_\theta$ for $\theta \in J$, or equivalently as the quotient of $S_{\phi(I),J}$ by the action of $\mathcal{O}^\times_{F,(p),+}$. We again simply write $\overline{Y}_0(p)_J$ for $\overline{Y}_0(p)_{\phi(I),J}$ where $I = \Theta - J$. Analogous relations carry over with the $S_{\phi(I),J}$ replaced by $\overline{Y}_0(p)_{\phi(I),J}$, as does Corollary 4.3.2. In particular it follows that each irreducible component of $\overline{Y}_0(p)$, with its reduced induced structure, is smooth of dimension d and is contained in $\overline{Y}_0(p)_J$ for a unique $J \subset \Theta$, and that these irreducible subschemes are precisely the connected components of $\overline{Y}_0(p)_J$.

The stratification[10] is compatible with the Hecke action for varying U in the sense that if U and V are sufficiently small of level prime to p and $g \in \mathrm{GL}_2(\mathbf{A}^{(p)}_{F,\mathbf{f}})$ is such that $g^{-1}Ug \subset V$, then the morphisms denoted $\tilde{\rho}_g$ and ρ_g restrict to ones on the corresponding closed subschemes for each $I, J \subset \Theta$.

4.3.4. *The unitary setting.* We may similarly define closed subschemes of $S' := \widetilde{Y}_{U'_0(p)}(G')_{\mathbf{F}}$ for each $I, J \subset \Theta$, where $G' = G'_\varnothing$ and $U' \subset G'(\mathbf{A_f})$ are as in §4.2.1. Recall that we have fixed a subset $\widetilde{\Theta} \subset \Theta_E$ mapping bijectively to Θ under $\tau \mapsto \tau|_F$ and denoted the corresponding extension of each $\theta \in \Theta$ by $\tilde{\theta}$. We then define $S'_{\phi(I),J}$ by the vanishing of the sections

$$\{\, \mathrm{Lie}(f')^0_{\tilde{\theta}} \,|\, \theta \in I \,\} \cup \{\, \mathrm{Lie}(f'^\vee)^0_{\tilde{\theta}^c} \,|\, \theta \in J \,\},$$

where f' is the universal isogeny on S', and as usual ^{0} denotes the restriction to the direct summand obtained by applying the idempotent $e_0 = e_0^* \in \mathcal{O}_D$. We then obtain the same relations as above with S replaced by S', and again the sections descend to the quotient $\overline{Y}'_0(p) := Y_{U'_0(p)}(G')_{\mathbf{F}}$ to define closed subschemes $\overline{Y}'_0(p)_{\phi(I),J}$ satisfying analogous relations. Furthermore the morphisms $\tilde{\rho}_g$ and

[10]With respect to the obvious partial ordering on the set of pairs (I, J), the locally closed subschemes

$$\overline{Y}_0(p)_{\phi(I),J} - \bigcup_{(I',J')>(I,J)} \overline{Y}_0(p)_{\phi(I'),J'}$$

define a stratification of $\overline{Y}_0(p)$ in the usual sense, but we will make no direct use of this fact. Our interest is mainly in the subschemes $\overline{Y}_0(p)_J$, but we will use the term *stratification* as shorthand for the collections of closed subschemes indexed by (I, J).

ρ_g for $g \in G'(\mathbf{A}_{\mathbf{f}}^{(p)})$ induce ones on the corresponding subschemes $S'_{\phi(I),J}$ and $\overline{Y}'_0(p)_{\phi(I),J}$ for U and V satisfying the usual conditions. Again we write simply S'_J and $\overline{Y}'_0(p)_J$ when I is the complement of J.

For sufficiently small $U \subset G(\mathbf{A}_{\mathbf{f}})$ and $U' \subset G'(\mathbf{A}_{\mathbf{f}})$ such that U' contains the image of U, the morphism $\tilde{i} : S \to S'$ defined by applying $\otimes_{\mathcal{O}_F}\mathcal{O}_E^2$ yields an identification $\tilde{i}^*\mathrm{Lie}(A'_j/S')^0_{\tilde{\tau}} = \mathrm{Lie}(A_j/S)_\theta$ for $j = 1, 2$, under which $\mathrm{Lie}(f')^0_{\tilde{\theta}}$ pulls back to $\mathrm{Lie}(f)_\theta$. We have an analogous identification for the Lie algebras of the dual abelian varieties (taking into account the isomorphism $\mathcal{O}_E^2 \otimes \mathbf{F} \to \mathrm{Hom}_{\mathcal{O}_F}(\mathcal{O}_E^2, \mathcal{O}_F) \otimes \mathbf{F}$ induced by $(\alpha, \beta) \mapsto \mathrm{Tr}_{E/F}(\alpha\overline{\beta})$) under which $\mathrm{Lie}(f'^\vee)^0_{\tilde{\theta}^c}$ pulls back to $\mathrm{Lie}(f^\vee)_\theta$. It follows that $\tilde{i}$ respects the stratifications in the sense that $S_{\phi(I),J} = S \times_{S'} S'_{\phi(I),J}$, and hence that we may similarly identify

$$\overline{Y}_0(p)_{\phi(I),J} = \overline{Y}_0(p) \times_{\overline{Y}'_0(p)} \overline{Y}'_0(p)_{\phi(I),J}$$

under the morphism $i : \overline{Y}_0(p) \to \overline{Y}'_0(p)$. Furthermore under the hypotheses of Lemma 2.3.1, it follows from Lemma 4.2.1 that $\tilde{i}$ and i induce identifications $S_{\phi(I),J} = S'_{\phi(I),J} \times_{C'} C$ and $\overline{Y}_0(p)_{\phi(I),J} = \overline{Y}'_0(p)_{\phi(I),J} \times_{C'} C$.

Finally, we remark that the vanishing locus of $\mathrm{Lie}(f')^0_{\tilde{\theta}}$ is the same as that of $\mathrm{Lie}(f')^0_{\tilde{\theta}^c}$, and the same holds with f' replaced by $(f')^\vee$. To see this, note that by descent we can replace U' by a smaller open compact subgroup to reduce to the setting of Lemma 4.2.1, where it is immediate that the claim holds on $S = S' \times_{C'} C$. It follows that it holds on S' since for any connected component, we may choose $h \in (\mathbf{A}_{E,\mathbf{f}}^{(p)})^\times$ such that ρ_h maps the component to $S' \times_{C'} C$. Similarly the conclusions of Corollary 4.3.2 carry over with $S_{\phi(I),J}$ replaced by $S'_{\phi(I),J}$, and hence by $\overline{Y}'_0(p)_{\phi(I),J}$.

5. A Jacquet–Langlands relation

5.1. **Splicing.** The key new ingredient in obtaining our geometric relation will be the introduction of abelian varieties whose Dieudonné modules are described by "splicing" those of the source and target of the universal isogeny at each closed point of the Iwahori level moduli space.

5.1.1. *Preliminaries.* Fix a subset $J \subset \Theta$ and a sufficiently small level $U' \subset G'(\mathbf{A}_{\mathbf{f}})$ of level prime to p. Recall that the scheme S'_J is defined in §4.3.4 as the closed subscheme of $\widetilde{Y}_{U'_0(p)}(G')_{\mathbf{F}}$ defined by the vanishing of the sections

$$\{\, \mathrm{Lie}(f')^0_{\tilde{\theta}} \mid \theta \notin J\,\} \cup \{\, \mathrm{Lie}(f'^\vee)^0_{\tilde{\theta}^c} \mid \theta \in J\,\}$$

where $f' : A'_1 \to A'_2$ is the universal isogeny.

Since G will play no (direct) role in this section, we will write simply S for S'_J and $f : A_1 \to A_2$ for the universal isogeny on S, and let $H' = \ker(f)$ and $H = e_0 H'$. Recall from part (1) of Proposition 4.2.2 that H is a Raynaud $\mathcal{O}_E/p\mathcal{O}_E$-module scheme on S. This means that we have invertible $\mathcal{O}_S$-modules $\mathcal{L}_\tau$ for $\tau \in \Theta_E$ and morphisms $s_\tau : \mathcal{L}_\tau^p \to \mathcal{L}_{\phi\circ\tau}$, $t_\tau : \mathcal{L}_{\phi\circ\tau} \to \mathcal{L}_\tau^p$ such that $s_\tau \circ t_\tau = 0$, in terms of which we may write $H = \mathbf{Spec}\mathcal{R}_H$ where $\mathcal{R}_H$ is the sheaf of $\mathcal{O}_S$-algebras

$$(\mathrm{Sym}_{\mathcal{O}_S}\mathcal{L})/\mathcal{I} \cong \bigoplus\left(\bigotimes_{\tau\in\Theta_E}\mathcal{L}_\tau^{m_\tau}\right)$$

on S, $\mathcal{I}$ is the sheaf of ideals generated by the $\mathcal{O}_S$-submodules $(s_\tau - 1)\mathcal{L}_\tau^p$ for $\tau \in \Theta_E$, and the action of $\alpha \in \mathcal{O}_E$ on H is defined by multiplication by $\tau(\alpha)$ on $\mathcal{L}_\tau$ Similarly the Cartier dual $H^\vee$ is identified with

$$\mathbf{Spec}((\mathrm{Sym}_{\mathcal{O}_S}\mathcal{M})/\mathcal{J}),$$

where $\mathcal{M} = \mathcal{H}om_{\mathcal{O}_S}(\mathcal{L}, \mathcal{O}_S) = \bigoplus_{\tau\in\Theta_E}\mathcal{L}_\tau^{-1}$ and $\mathcal{J}$ is generated by $(t_\tau - 1)\mathcal{L}_\tau^{-p}$ for $\tau \in \Theta_E$.

Lemma 5.1.1. *Suppose that $\tau \in \Theta_E$. If $\tau|_F \notin J$, then $s_{\phi^{-1}\circ\tau} = 0$, and if $\tau|_F \in J$, then $t_{\phi^{-1}\circ\tau} = 0$.*

Proof. Suppose first that $\tau|_F \notin J$, so that $\tau = \tilde\theta$ or $\tilde\theta^c$ for some $\theta \notin J$, and hence $\mathrm{Lie}(f)^0_\tau = 0$ (see the discussion at the end of §4.3.4). Since Lie is left exact for morphisms of group schemes, we have that $\mathrm{Lie}(H/S)_\tau = \mathrm{Lie}(A_1/S)^0_\tau$ is invertible. In terms of the Raynaud data for H, the sheaf $\mathcal{A}$ of augmentation ideals is generated over $\mathcal{R}_H$ by $\bigoplus_\tau \mathcal{L}_\tau$, so that $\mathcal{A}/\mathcal{A}^2 = \bigoplus_\tau(\mathcal{L}_\tau/s_{\phi^{-1}\circ\tau}\mathcal{L}^p_{\phi^{-1}\circ\tau})$ as an $\mathcal{O}_E\otimes\mathcal{O}_S$-module, and therefore

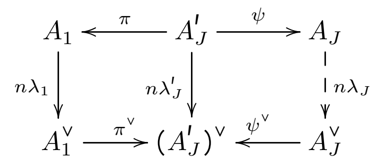
$$\mathrm{Lie}(H/S)_\tau = \mathcal{H}om_{\mathcal{O}_S}(\mathcal{L}_\tau/s_{\phi^{-1}\circ\tau}\mathcal{L}^p_{\phi^{-1}\circ\tau}, \mathcal{O}_S).$$

Since $\mathcal{L}_\tau$ is invertible, it follows that $s_{\phi^{-1}\circ\tau} = 0$.

Suppose now that $\tau|_F \in J$, so that $\mathrm{Lie}(f^\vee)^0_\tau = 0$. Since $H^\vee \cong \ker(f^\vee)^0$ as an $(\mathcal{O}_E/p\mathcal{O}_E)$-module scheme, we see as above that $\mathrm{Lie}(H^\vee/S)_\tau \cong \mathrm{Lie}(A_2^\vee/S)^0_\tau$ is invertible, and it similarly follows that $t_{\phi^{-1}\circ\tau} = 0$. □

5.1.2. *Universal splices.* In view of Lemma 5.1.1, we may apply Lemma 4.1.1 to the Raynaud $(\mathcal{O}_F/p)$-module scheme $\prod_v H[\tilde v]$ with $I = \phi^{-1}(J)$ to obtain a finite flat subgroup scheme C_J over S, which we view as a finite flat $\mathcal{O}_E/p\mathcal{O}_E$-submodule scheme of H. Thus C_J has rank $p^{|J|}$ over S, and $\mathrm{Lie}(C_J^\vee/S) = \bigoplus_{\theta\in J}\mathcal{L}_{\phi^{-1}\circ\tilde\theta}$ as an $\mathcal{O}_E\otimes\mathcal{O}_S$-module. Having defined $C_J \subset H = e_0H'$, we let C'_J denote the finite flat $\mathcal{O}_D$-submodule scheme $C_J\otimes_{\mathcal{O}_E}\mathcal{O}_E^2 \subset H'$. Note that in fact $C'_J \subset \prod_v H'[\tilde v] \subset A_1[\prod_v \tilde v]$ and hence $\lambda_1(C'_J) \subset A_1[\prod_v \tilde v^c]$ (where λ_1 is the quasi-polarization on A_1). Taking the dual of the natural projection $A_1^\vee \to A_1^\vee/\lambda_1(C'_J)$ yields an abelian scheme $A'_J := (A_1^\vee/\lambda_1(C'_J))^\vee$ with an action of $\mathcal{O}_D$, together with an $\mathcal{O}_D$-linear isogeny $\pi : A'_J \to A_1$ whose kernel is isomorphic to $(C'_J)^\vee$ (the Cartier dual of C'_J), compatibly with the $\mathcal{O}_D$-action under the anti-involution $\alpha \mapsto \alpha^* = (\alpha^c)^t$ of $\mathcal{O}_D$. Equivalently we may define $A'_J = A_1/(C'_J)^\perp$ with the isogeny π induced by multiplication by p on A_1. It is immediate from either description that π restricts an isomorphism $A'_J[\prod_v \tilde v] \xrightarrow{\sim} A_1[\prod_v \tilde v]$. We then define the *universal J-splice* to be the abelian scheme $A_J = A'_J/\pi^{-1}(C'_J)$. More generally for a triple $(\underline{A}_1, \underline{A}_2, f)$ corresponding to an S-point of S'_J for an arbitrary base S, we define the *J-splice* of $f : A_1 \to A_2$ to be the pull-back of the universal J-splice.

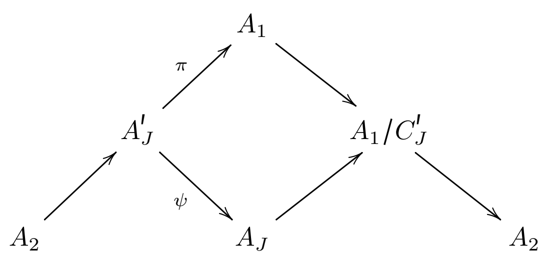
Note that the universal J-splice A_J inherits an $\mathcal{O}_D$-action from the $\mathcal{O}_D$-action on A'_J. Furthermore the prime-to-p quasi-polarization λ_1 induces one on A_J, which we denote λ_J. To see this, consider the commutative diagram:



where n is a positive integer prime to p such that $n\lambda_1$ is a polarization of degree prime to p, $\lambda'_J = \pi^\vee \circ \lambda_1 \circ \pi$ and ψ is the natural projection. By construction, $\pi^{-1}(C'_J)$ is contained in $\ker(n\lambda'_J)$, so $n\lambda'_J$ factors through an $\mathcal{O}_E$-antilinear isogeny $\xi : A_J \to (A'_J)^\vee$. Furthermore since $n\lambda'_J = \psi^\vee \circ \xi^\vee$, $\psi^\vee$ induces an isomorphism $A_J^\vee[\prod_v \tilde{v}^c] \to (A'_J)^\vee[\prod_v \tilde{v}^c]$ and $\xi^\vee$ is $\mathcal{O}_E$-antilinear, it follows that $\pi^{-1}(C'_J)$ is contained in the kernel of $\xi^\vee$, and so $\xi^\vee = \lambda \circ \psi$ for some polarization λ of A_J. Since π and ψ each have degree $p^{2|J|}$ and $n\lambda_1$ has degree prime to p, it follows that λ has degree prime to p, and we let $\lambda_J = n^{-1}\lambda$. Note that the associated Rosati involution is compatible with the anti-involution $*$ on D.

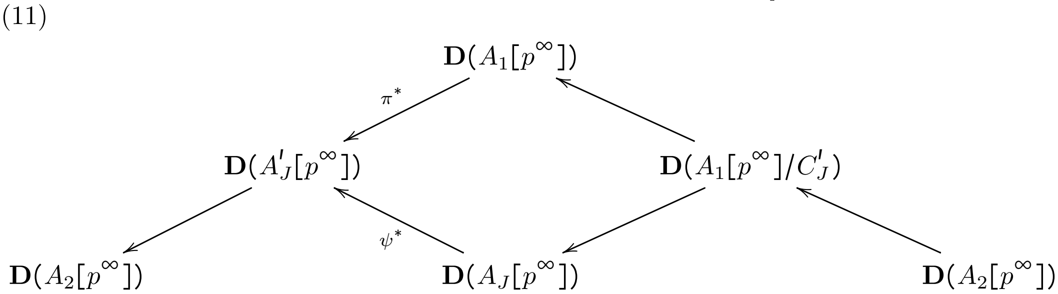
5.1.3. *Relation of Dieudonné modules.* We now explain the relation between the Dieudonné modules of A_1, A_2 and A_J at closed points of S'_J which motivates the construction (and name) of the J-splice. First note that since $H' = \ker(f)$ is totally isotropic with respect to λ_1 (by part (2) of Proposition 4.2.2) the projection $A_1 \to A'_J$ induced by multiplication by p factors through $f : A_1 \to A_2$, so we obtain a diagram of isogenies

(10)



such that the square commutes, the composite down the top right is $f : A_1 \to A_2$, and along the top left is the morphism $A_2 \to A_1$ whose composite with f is multiplication by p. For any $\bar{s} : \operatorname{Spec} \overline{\mathbf{F}}_p \to S'_J$, we have the morphisms of Dieudonné modules induced by the diagram (10) over $S = \operatorname{Spec}(\overline{\mathbf{F}}_p)$:

(11)



All the morphisms become isomorphisms after inverting p, so $(\pi^*)^{-1} \circ \psi^*$ identifies $\mathbf{D}(A_J[p^\infty])$ with an $\mathcal{O}_D$-stable W-lattice in $\mathbf{D}(A_1[p^\infty]) \otimes \mathbf{Q}$ whose components switch between those arising from A_1 and A_2 as dictated by the first two formulas in following proposition:

Proposition 5.1.2. *If $\theta \in \Theta$, then*

$$(\pi^*)^{-1}\psi^*(\mathbf{D}(A_J[p^\infty])_{\tilde\theta}) = \begin{cases} \mathbf{D}(A_1[p^\infty])_{\tilde\theta}, & \text{if } \theta \notin J, \\ f^*\mathbf{D}(A_2[p^\infty])_{\tilde\theta}, & \text{if } \theta \in J; \end{cases}$$

$$(\pi^*)^{-1}\psi^*(\mathbf{D}(A_J[p^\infty])_{\tilde\theta^c}) = \begin{cases} \mathbf{D}(A_1[p^\infty])_{\tilde\theta^c}, & \text{if } \theta \notin J, \\ p^{-1}f^*\mathbf{D}(A_2[p^\infty])_{\tilde\theta^c}, & \text{if } \theta \in J; \end{cases}$$

$$\dim_{\overline{\mathbf{F}}_p}(\mathrm{Lie}(A_J)_{\tilde\theta}) = \begin{cases} 4, & \text{if } \theta \in J \text{ and } \phi\circ\theta \notin J, \\ 0, & \text{if } \theta \notin J \text{ and } \phi\circ\theta \in J, \\ 2, & \text{otherwise;} \end{cases}$$

$$\text{and} \quad \dim_{\overline{\mathbf{F}}_p}(\mathrm{Lie}(A_J)_{\tilde\theta^c}) = 4 - \dim_{\overline{\mathbf{F}}_p}(\mathrm{Lie}(A_J)_{\tilde\theta}).$$

Proof. Comparing dimensions shows the natural inclusion $\mathrm{Lie}(C_J^\vee)^{(p)} \to \mathbf{D}(C_J)$ is an isomorphism, so $\dim_{\overline{\mathbf{F}}_p}\mathbf{D}(C_J)_\tau = 1$ if $\tau = \tilde\theta$ for some $\theta \in J$, and $\mathbf{D}(C_J)_\tau = 0$ otherwise; therefore $\dim_{\overline{\mathbf{F}}_p}\mathbf{D}(C'_J)_\tau = 2$ if $\tau = \tilde\theta$ for some $\theta \in J$, and $\mathbf{D}(C'_J)_\tau = 0$ otherwise. By part (3) of Proposition 4.2.2, we have $\dim_{\overline{\mathbf{F}}_p}\mathbf{D}(H')_\tau = 2$ for all τ, so if $\theta \in J$, then $\mathbf{D}(H'/C'_J)_{\tilde\theta} = 0$, and hence

$$\mathbf{D}(A_2[p^\infty])_{\tilde\theta} \longrightarrow \mathbf{D}(A_1[p^\infty]/C'_J)_{\tilde\theta}$$

is an isomorphism. On the other hand, if $\tau \notin \{\,\tilde\theta \,|\, \theta \in J\,\}$, then

$$\mathbf{D}(A_1[p^\infty]/C'_J)_\tau \longrightarrow \mathbf{D}(A_1[p^\infty])_\tau \quad \text{and} \quad \mathbf{D}(A_J[p^\infty])_\tau \longrightarrow \mathbf{D}(A'_J[p^\infty])_\tau$$

are isomorphisms. A similar analysis shows that if $\theta \in J$, then

$$\mathbf{D}(A'_J[p^\infty])_{\tilde\theta^c} \longrightarrow \mathbf{D}(A_2[p^\infty])_{\tilde\theta^c}$$

is an isomorphism, but if $\tau \notin \{\,\tilde\theta^c \,|\, \theta \in J\,\}$, then

$$\mathbf{D}(A_1[p^\infty])_\tau \longrightarrow \mathbf{D}(A'_J[p^\infty])_\tau \quad \text{and} \quad \mathbf{D}(A_1[p^\infty]/C'_J)_\tau \longrightarrow \mathbf{D}(A_J[p^\infty])_\tau$$

are isomorphisms. The first two formulas in the proposition then follow from composing isomorphisms arising in each case.

For $\tau \in \Theta_E$ and $A = A_1$, A_2 or A_J, we have

$$\begin{aligned} \dim_{\overline{\mathbf{F}}_p}(\mathrm{Lie}(A)_\tau) &= \dim_{\overline{\mathbf{F}}_p}(\mathrm{Lie}(A[p])_\tau) = \dim_{\overline{\mathbf{F}}_p}(\mathrm{Lie}(A[p])^{(p)})_{\phi\circ\tau} \\ &= \dim_{\overline{\mathbf{F}}_p}(\mathrm{Hom}_{\overline{\mathbf{F}}_p}(\mathrm{Lie}(A[p])^{(p)}, \overline{\mathbf{F}}_p)_{\phi\circ\tau}) = \dim_{\overline{\mathbf{F}}_p}(D_{\phi\circ\tau}/\Phi(D_\tau)) \end{aligned}$$

where $D = \mathbf{D}(A[p^\infty])$. This dimension is two for $A = A_1$, A_2, and the morphisms f^*, π^* and ψ^* commute with Φ. Thus if $\tau|_F = \theta$ with $\theta, \phi\circ\theta \notin J$, it follows from the first two formulas of the proposition that

$$\mathbf{D}(A_J[p^\infty])_{\phi\circ\tau}/\Phi\mathbf{D}(A_J[p^\infty])_\tau \cong \mathbf{D}(A_1[p^\infty])_{\phi\circ\tau}/\Phi\mathbf{D}(A_1[p^\infty])_\tau$$

has dimension two. Similarly using A_2 instead of A_1, we find the dimension is two if $\theta, \phi\circ\theta \in J$.

Suppose now that $\theta \in J$ and $\phi\circ\theta \notin J$. Applying $(\pi^*)^{-1}\psi^*$ to the inclusion $p\mathbf{D}(A_J[p^\infty])_{\phi\circ\tau} \subset \Phi(\mathbf{D}(A_J[p^\infty])_\tau)$ implies that

$$p\mathbf{D}(A_1[p^\infty])_{\phi\circ\tilde\theta} \subset \Phi(f^*\mathbf{D}(A_2[p^\infty])_{\tilde\theta}),$$

but both spaces have codimension two in $f^*\mathbf{D}(A_2[p^\infty])_{\phi\circ\tilde\theta}$, so equality holds, and it follows that

$$\mathbf{D}(A_J[p^\infty])_{\phi\circ\tilde\theta}/\Phi\mathbf{D}(A_J[p^\infty])_{\tilde\theta} \cong \mathbf{D}(A_1[p^\infty])_{\phi\circ\tilde\theta}/p\mathbf{D}(A_1[p^\infty])_{\phi\circ\tilde\theta}$$

has dimension four. On the other hand comparing codimensions arising from the inclusions

$$\Phi(p^{-1}f^*\mathbf{D}(A_2[p^\infty])_{\tilde\theta^c}) \subset \mathbf{D}(A_1[p^\infty])_{\phi\circ\tilde\theta^c} \subset p^{-1}f^*\mathbf{D}(A_2[p^\infty])_{\phi\circ\tilde\theta^c})$$

shows that $\Phi(p^{-1}f^*\mathbf{D}(A_2[p^\infty])_{\tilde\theta^c}) = \mathbf{D}(A_1[p^\infty])_{\phi\circ\tilde\theta^c}$, and hence that

$$\mathbf{D}(A_J[p^\infty])_{\phi\circ\tilde\theta^c}/\Phi\mathbf{D}(A_J[p^\infty])_{\tilde\theta^c} = 0.$$

(Alternatively, this can be deduced using duality from the formula in the case of $\tau = \tilde\theta$.)

We omit the proof in the case $\theta \notin J$ and $\phi\circ\theta \in J$, which is similar.

□

We have the following immediate consequence for the de Rham cohomology and Lie algebra sheaves of the universal J-splice over S'_J, and hence for the pull-back to any base S:

Corollary 5.1.3. *Let A_J denote the J-splice of an isogeny $f : A_1 \to A_2$ corresponding to an S-point of S'_J.*

(1) *If $\theta \in J$, then the isogenies $A_2 \to A_J$ and $A_J \to A_2$ (along the bottom of diagram (10)) induce isomorphisms:*

$$\mathcal{H}^1_{\mathrm{dR}}(A_J/S)_{\tilde\theta^c} \xrightarrow{\sim} \mathcal{H}^1_{\mathrm{dR}}(A_2/S)_{\tilde\theta^c} \quad \textit{and} \quad \mathcal{H}^1_{\mathrm{dR}}(A_J/S)_{\tilde\theta} \xleftarrow{\sim} \mathcal{H}^1_{\mathrm{dR}}(A_2/S)_{\tilde\theta}.$$

(2) *If $\tau \in \Theta_E$ and $\tau|_F \notin J$, then π and ψ induce isomorphisms*

$$\mathcal{H}^1_{\mathrm{dR}}(A_J/S)_\tau \xrightarrow{\sim} \mathcal{H}^1_{\mathrm{dR}}(A'_J/S)_\tau \xleftarrow{\sim} \mathcal{H}^1_{\mathrm{dR}}(A_1/S)_\tau.$$

(3) *If $\tau \in \Theta_E$, then $\mathrm{Lie}(A_J/S)_\tau$ is locally free of rank $2s_\tau$ over $\mathcal{O}_S$, where*

$$s_{\tilde\theta} = \begin{cases} 2, & \text{if } \theta \in J \text{ and } \phi\circ\theta \notin J,\\ 0, & \text{if } \theta \notin J \text{ and } \phi\circ\theta \in J,\\ 1, & \text{otherwise,}\end{cases}$$

and $s_{\tilde\theta^c} = 2 - s_{\tilde\theta}$.

5.2. **Unitary analogue of the theorem.** We continue to work with a fixed $J \subset \Theta$. In his section we will use the J-splice to construct an isomorphism analogous to the one in Theorem A, but in the context of the related unitary Shimura varieties.

5.2.1. *Preliminaries.* Let $\Sigma = \Sigma_J = \{\theta \in J \mid \phi\circ\theta \notin J\} \cup \{\theta \notin J \mid \phi\circ\theta \in J\}$, and let[11]

$$\tilde\Sigma = \{\tilde\theta^c \mid \theta \in J, \phi\circ\theta \notin J\} \cup \{\tilde\theta \mid \theta \notin J, \phi\circ\theta \in J\}.$$

Note that the cardinality of Σ is even.

Recall from §2.2 that B_Σ denotes the quaternion algebra over F ramified at precisely the places in Σ, G_Σ is the algebraic group over $\mathbf{Q}$ defined by $G_\Sigma(R) = (B_\Sigma \otimes R)^\times$, G'_Σ denotes $(G_\Sigma \times T_E)/T_F$, and $D_\Sigma = B_\Sigma \otimes_F E$. We continue to

[11]We apologize that in the exceptional situation where $\Sigma_J = \Theta$ (i.e., $\phi(J)$ is the complement of J, which can occur only when all primes over p have even degree), we have different meanings for $\tilde\Theta$ and $\tilde\Sigma$.

write B for $B_\varnothing = \mathrm{GL}_2(F)$ and similarly $G = G_\varnothing = \mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_2$, $G' = G'_\varnothing$ and $D = D_\varnothing = M_2(E)$.

We need to make some choices of algebraic data in order to define the Shimura variety to which we will relate $\overline{Y}'_0(p)_J$. We first choose an $\mathbf{A}_{F,\mathbf{f}}$-algebra isomorphism

$$\xi : B_\Sigma \otimes \widehat{\mathbf{Z}} \xrightarrow{\sim} B \otimes \widehat{\mathbf{Z}} = M_2(\mathbf{A}_{F,\mathbf{f}}),$$

and we let $\mathcal{O}_{B_\Sigma}$ denote the corresponding maximal order in B_Σ, i.e., $B_\Sigma \cap \xi(M_2(\widehat{\mathcal{O}}_F))$, and let $\mathcal{O}_{D_\Sigma} = \mathcal{O}_{B_\Sigma} \otimes_{\mathcal{O}_F} \mathcal{O}_E$. Thus $\mathcal{O}_{D_\Sigma}$ is a maximal order in D_Σ, and we choose an $\mathcal{O}_E$-algebra isomorphism

$$\vartheta : \mathcal{O}_{D_\Sigma} \xrightarrow{\sim} \mathcal{O}_D = M_2(\mathcal{O}_E).$$

Finally, by Lemma 5.4 of [TX16], we may choose an element $\delta_\Sigma \in D_\Sigma^\times$ satisfying the conditions in §2.2.2 with D replaced by D_Σ (for the above choice of $\widetilde{\Sigma}$) and such that the resulting anti-involution is compatible with $*$ under ϑ; more precisely:

- $\delta_\Sigma \in \mathcal{O}_{D_\Sigma,p}^\times$,
- $\overline{\delta}_\Sigma = -\delta_\Sigma$,
- the bilinear form on $D_\Sigma \otimes \mathbf{R}$ defined by
$$(v, w) \mapsto \mathrm{Tr}_{E/\mathbf{Q}}(\mathrm{tr}_{D_\Sigma/E}(v\overline{h}'_{\widetilde{\Sigma}}(i)\overline{w}\delta_\Sigma))$$
is positive definite,
- $\vartheta(\delta_\Sigma^{-1}\overline{(\alpha)}\delta_\Sigma) = \delta^{-1}\overline{\vartheta(\alpha)}\delta$ for all $\alpha \in D_\Sigma$.

Recall that we have chosen $\delta = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and that our notation differs slightly from that of [TX16], but the preceding assertion is nonetheless immediate from their Lemma 5.3. Having fixed such a choice of δ_Σ, we will abuse notation by writing $*$ for the corresponding anti-involution $\delta_\Sigma^{-1}\overline{\vartheta(\alpha)}\delta_\Sigma$ on D_Σ.

Note that the choices above give rise to two $\widehat{\mathcal{O}}_E$-algebra isomorphisms

$$\xi_E,\ \widehat{\vartheta} : \widehat{\mathcal{O}}_{D_\Sigma} \xrightarrow{\sim} \widehat{\mathcal{O}}_D,$$

so there exists $h \in \widehat{\mathcal{O}}_D^\times$ such that $h\xi_E(u)h^{-1} = \widehat{\vartheta}(u)$ for all $u \in \widehat{\mathcal{O}}_{D_\Sigma}$. Using the fact that $\xi_E(\overline{u}) = \overline{\xi_E(u)}$ for all u, it is straightforward to check that the element

$$\varepsilon := \overline{h}\delta h\xi_E(\delta_\Sigma^{-1}) \in \mathrm{GL}_2(\mathbf{A}_{E,\mathbf{f}})$$

is central and satisfies $\overline{\varepsilon} = \varepsilon$, so in fact $\varepsilon \in \mathbf{A}_{F,\mathbf{f}}^\times$; moreover $\varepsilon_p \in \mathcal{O}_{F,p}^\times$.

5.2.2. *Construction of the morphism.* For an open compact subgroup U' of $G'(\mathbf{A}_\mathbf{f})$, we let U'_Σ denote the corresponding open compact subgroup of $G'_\Sigma(\mathbf{A}_\mathbf{f})$ under the isomorphism $G'(\mathbf{A}_\mathbf{f}) \cong G'_\Sigma(\mathbf{A}_\mathbf{f})$ induced by ξ. We assume that U' is sufficiently small for both U' and U'_Σ to be sufficiently small in the usual sense, and that U' is of level prime to p, so that U'_Σ is necessarily so as well.

In §5.1.2, we defined the universal J-splice A_J on the closed subscheme S'_J of $\widetilde{Y}_{U'_0(p)}(G')_\mathbf{F}$. We will now endow it with the additional structure needed to define a morphism

$$\widetilde{\Psi}_J : S'_J \to \widetilde{Y}_{U'_\Sigma}(G'_\Sigma)_\mathbf{F}.$$

Firstly we define the action ι_J of $\mathcal{O}_{D_\Sigma}$ by composing $\vartheta : \mathcal{O}_{D_\Sigma} \to \mathcal{O}_D$ with the unique $\mathcal{O}_D$-action on A_J compatible with the quasi-isogeny $\psi \circ \pi^{-1} \in \mathrm{Hom}(A_1, A_J) \otimes \mathbf{Z}[1/p]$. Part (3) of Corollary 5.1.3 then implies that for $\alpha \in \mathcal{O}_{D_\Sigma}$, the action of $\iota_J(\alpha)$ on $\mathrm{Lie}(A_J/S'_J)$ has characteristic polynomial $\prod(x - \overline{\tau}(\alpha))^{2s_\tau} \in \mathbf{F}[x]$ as required. Next recall that we also defined the prime-to-p quasi-polarization λ_J on A_J to be compatible with $\psi\circ\pi^{-1}$, so the condition on the associated Rosati involution follows from the compatibility of ϑ with the anti-involutions denoted $*$. Finally we define η_J as the composite of the isomorphisms

$$\widehat{\mathcal{O}}_{D_\Sigma}^{(p)} \xrightarrow{\widehat{\xi}_E^{(p)}} \widehat{\mathcal{O}}_D^{(p)} \xrightarrow{h^{(p)}\cdot} \widehat{\mathcal{O}}_D^{(p)} \xrightarrow{\eta_{1,\overline{s}_i}} \widehat{T}^{(p)}(A_{1,\overline{s}_i}) \xrightarrow{\psi\circ\pi^{-1}} \widehat{T}^{(p)}(A_{J,\overline{s}_i})$$

for each $\overline{s}_i$, and we let $\epsilon_J = \varepsilon^{(p)}\epsilon_1$. The fact that (η_J,ϵ_J) is a level U'_Σ-structure on (A_J,ι_J,λ_J) then follows from the relations between ϑ, ξ, h and ε.

Recall that in §3.1.2 we defined the rank two vector bundle $\widetilde{\mathcal{V}}_\tau^0$ on $S=\widetilde{Y}_{U'_\Sigma}(G'_\Sigma)_{\mathbf{F}}$ for each $\tau\in\Theta_E$ as $\mathcal{H}^1_{\mathrm{dR}}(A/S)^0_\tau$, where A is the universal abelian scheme on S and $\cdot^0$ denotes the image under pull-back by the idempotent e_0, defined as the element of $\mathcal{O}_{D_\Sigma}\otimes\mathbf{F}_p$ corresponding to the usual $e_0=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\in M_2(\mathcal{O}_E/p\mathcal{O}_E)=\mathcal{O}_D\otimes\mathbf{F}_p$ under an isomorphism which we may take to be $\xi_E\otimes\mathbf{F}_p$.

We will now define an S-morphism $S'_J\to\mathbf{P}_S(\widetilde{\mathcal{V}}^0_{\tilde\theta})$ for each $\theta\in\Sigma$. To do so amounts to defining a surjective morphism from $\widetilde{\Psi}_J^*\widetilde{\mathcal{V}}^0_{\tilde\theta}$ to a line bundle on S'_J, but this sheaf is canonically identified $\mathcal{H}^1_{\mathrm{dR}}(A_J/S'_J)^0$. Recall from Part (1) of Corollary 5.1.3 that if $\theta\in J$, then the isogeny $A_J\to A_2$ induces an isomorphism $\mathcal{H}^1_{\mathrm{dR}}(A_2/S'_J)_{\tilde\theta}\to\mathcal{H}^1_{\mathrm{dR}}(A_J/S'_J)_{\tilde\theta}$ which is compatible with the actions of $\mathcal{O}_D\cong\mathcal{O}_{D_\Sigma}$ under the isomorphism ϑ. Precomposing with the morphism induced by $h_p^{-1}\in\mathrm{GL}_2(\mathcal{O}_E/p\mathcal{O}_E)$ yields an isomorphism compatible with the actions of $\mathcal{O}_D\otimes\mathbf{F}_p\cong\mathcal{O}_{D_\Sigma}\otimes\mathbf{F}_p$ under the isomorphism induced by ξ, and hence restricts to an isomorphism

$$\mathcal{H}^1_{\mathrm{dR}}(A_2/S'_J)^0_{\tilde\theta}\to\mathcal{H}^1_{\mathrm{dR}}(A_J/S'_J)^0_{\tilde\theta}=\widetilde{\Psi}_J^*\widetilde{\mathcal{V}}^0_{\tilde\theta}.$$

If $\theta\in J$ and $\phi\circ\theta\notin J$, then we define $S'_J\to\mathbf{P}_S(\widetilde{\mathcal{V}}^0_{\tilde\theta})$ corresponding to the surjective morphism in the exact sequence

$$0\to(s_*\Omega^1_{A_2/S'_J})^0_{\tilde\theta}\to\mathcal{H}^1_{\mathrm{dR}}(A_2/S'_J)^0_{\tilde\theta}\to(R_1s_*\mathcal{O}_{A_2})^0_{\tilde\theta}\to0$$

arising from the Hodge filtration on $\mathcal{H}^1_{\mathrm{dR}}(A_2/S'_J)$, where $s:A_2\to S'_J$ is the structure morphism. Similarly if $\theta\notin J$ and $\phi\circ\theta\in J$, then we define $S'_J\to\mathbf{P}_S(\widetilde{\mathcal{V}}^0_{\tilde\theta})$ using Part (2) of Corollary 5.1.3 and the Hodge filtration on $\mathcal{H}^1_{\mathrm{dR}}(A_1/S'_J)$. Taking the fibre product over S of the morphisms just defined for $\theta\in\Sigma$, we obtain the morphism

$$\tilde{\Xi}_J:S'_J\to\prod_{\theta\in\Sigma}\mathbf{P}_S(\widetilde{\mathcal{V}}^0_{\tilde\theta}).$$

5.2.3. *Proof of isomorphism.* We now prove the key result:

Theorem 5.2.1. *The morphism $\tilde{\Xi}_J$ is an isomorphism.*

Proof. Since the schemes are smooth of the same (constant) dimension over $\mathbf{F}$, it suffices to prove that the morphism is bijective on $\overline{\mathbf{F}}_p$-points and injective on tangent spaces at such points. To see this, note that the condition on tangent spaces implies that $\Omega^1_{S'_J/\prod_{\theta\in\Sigma}\mathbf{P}_S(\widetilde{\mathcal{V}}^0_{\tilde\theta})}=0$. Furthermore since the schemes are regular of the same dimension and the fibres at all closed points have dimension zero, it follows that $\tilde{\Xi}_J$ is flat, so in fact it is étale. By [Gro67, 18.2.8], we have that $\tilde{\Xi}_J$ is finite, so the condition at closed points implies that it is in fact an isomorphism.

To prove bijectivity on points, we will construct an inverse map. Let $\underline{A} = (A, \iota, \lambda, \eta, \epsilon)$ correspond to an $\overline{\mathbf{F}}_p$-point of $\widetilde{Y}_{U'_\Sigma}(G'_\Sigma)$ and for each $\theta \in \Sigma$, let L^0_θ be a line in $\mathbf{P}(H^1_{\mathrm{dR}}(A/\overline{\mathbf{F}}_p)^0_{\tilde\theta})$, so the datum $(\underline{A}, \{L^0_\theta\}_{\theta\in\Sigma})$ corresponds to an $\overline{\mathbf{F}}_p$-point of the target of $\tilde{\tilde{\Xi}}_J$. Let Δ denote the Dieudonné module $\mathbf{D}(A[p^\infty])$, with its right $M_2(\mathcal{O}_{E,p})$-action obtained from the left action of $\mathcal{O}_{D_\Sigma}$ on A and the isomorphism ξ_p. Decomposing $\Delta = \bigoplus_{\tau\in\Theta_E} \Delta_\tau$ where $\mathcal{O}_{E,p}$ acts on Δ_τ via τ, each Δ_τ is a right $M_2(W)$-module of rank 4 over W and the endomorphism Φ of Δ restricts to define injective maps $\Delta_\tau \to \Delta_{\phi\circ\tau}$ which are ϕ-semilinear with respect to the actions of $M_2(W)$. Considering the inclusions

$$p\Delta_{\phi\circ\tau} \subset \Phi(\Delta_\tau) \subset \Delta_{\phi\circ\tau},$$

the condition on $\mathrm{Lie}(A)$ means that the successive quotients are two-dimensional over $\overline{\mathbf{F}}_p$ unless $\theta = \tau|_F \in \Sigma$, in which case:

- $\Phi(\Delta_{\tilde\theta}) = \Delta_{\phi\circ\tilde\theta}$ and $\Phi(\Delta_{\tilde\theta^c}) = p\Delta_{\phi\circ\tilde\theta^c}$ if $\theta \notin J$, $\phi\circ\theta \in J$;
- $\Phi(\Delta_{\tilde\theta}) = p\Delta_{\phi\circ\tilde\theta}$ and $\Phi(\Delta_{\tilde\theta^c}) = \Delta_{\phi\circ\tilde\theta^c}$ if $\theta \in J$, $\phi\circ\theta \notin J$.

For each $\theta \in \Sigma$, we let $L_\theta = L^0_\theta \otimes_{\overline{\mathbf{F}}_p} \overline{\mathbf{F}}_p^2$, viewed as a right $M_2(\overline{\mathbf{F}}_p)$-submodule of $H^1_{\mathrm{dR}}(A/\overline{\mathbf{F}}_p)_{\tilde\theta}$, and let $\Lambda_{\tilde\theta}$ denote the preimage of L_θ under the natural projection

$$\Delta_{\tilde\theta} \longrightarrow \mathbf{D}(A[p])_{\tilde\theta} \cong H^1_{\mathrm{dR}}(A/\overline{\mathbf{F}}_p)_{\tilde\theta}.$$

Recall that the prime-to-p quasi-polarization λ gives rise to a perfect W-linear pairing on Δ, inducing an isomorphism

$$\Delta_{\tilde\theta} \longrightarrow \mathrm{Hom}_W(\Delta_{\tilde\theta^c}, W),$$

and we let $\Lambda_{\tilde\theta^c}$ denote the dual lattice to $\Lambda_{\tilde\theta}$ in $\Delta_{\tilde\theta^c} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ under the pairing. We thus have the $M_2(W)$-equivariant inclusions

$$p\Delta_{\tilde\theta} \subset \Lambda_{\tilde\theta} \subset \Delta_{\tilde\theta} \quad \text{and} \quad \Delta_{\tilde\theta^c} \subset \Lambda_{\tilde\theta^c} \subset p^{-1}\Delta_{\tilde\theta^c}$$

for each $\theta \in \Sigma$, where all the successive quotients are two-dimensional over $\overline{\mathbf{F}}_p$.

For each $\tau \in \Theta_E$, we define a W-lattice $\Delta_{1,\tau} \subset p^{-1}\Delta_\tau$ as follows:

- if $\theta \notin J$, then $\Delta_{1,\tilde\theta} = \Delta_{\tilde\theta}$ and $\Delta_{1,\tilde\theta^c} = \Delta_{\tilde\theta^c}$;
- if $\phi^{-1}\circ\theta \in J$ and $\theta \in J$, then $\Delta_{1,\tilde\theta} = p^{-1}\Phi(\Delta_{\phi^{-1}\circ\tilde\theta})$ and $\Delta_{1,\tilde\theta^c} = \Phi(\Delta_{\phi^{-1}\circ\tilde\theta^c})$;
- if $\phi^{-1}\circ\theta \notin J$ and $\theta \in J$, then $\Delta_{1,\tilde\theta} = p^{-1}\Phi(\Lambda_{\phi^{-1}\circ\tilde\theta})$ and $\Delta_{1,\tilde\theta^c} = \Phi(\Lambda_{\phi^{-1}\circ\tilde\theta^c})$;

and a W-lattice $\Delta_{2,\tau} \subset \Delta_\tau$ as follows:

- if $\theta \in J$, then $\Delta_{2,\tilde\theta} = \Delta_{\tilde\theta}$ and $\Delta_{2,\tilde\theta^c} = p\Delta_{\tilde\theta^c}$;
- if $\phi^{-1}\circ\theta \notin J$ and $\theta \notin J$, then $\Delta_{2,\tilde\theta} = \Phi(\Delta_{\phi^{-1}\circ\tilde\theta})$ and $\Delta_{2,\tilde\theta^c} = \Phi(\Delta_{\phi^{-1}\circ\tilde\theta^c})$;
- if $\phi^{-1}\circ\theta \in J$ and $\theta \notin J$, then $\Delta_{2,\tilde\theta} = p^{-1}\Phi(\Lambda_{\phi^{-1}\circ\tilde\theta})$ and $\Delta_{2,\tilde\theta^c} = p\Phi(\Lambda_{\phi^{-1}\circ\tilde\theta^c})$.

Setting $\Delta_j = \bigoplus_{\tau\in\Theta_E} \Delta_{j,\tau}$ for $j = 1, 2$, it is straightforward to check that

$$p\Delta_j \subset \Phi(\Delta_j) \subset \Delta_j.$$

It follows that Δ_j can be identified with $\mathbf{D}(A_j[p^\infty])$ for an abelian variety A_j isogenous to A. More precisely, since Δ_2 contains $p\Delta$ and is stable under the action of Φ and $V = p\Phi^{-1}$, the quotient Δ/Δ_2 of $\Delta/p\Delta = \mathbf{D}(A[p])$ corresponds to a finite flat subgroup scheme C of $A[p]$. Letting $A_2 = A/C$, the projection $g : A \to A_2$ induces an inclusion $\mathbf{D}(A_2[p^\infty]) \to \mathbf{D}(A[p^\infty]) = \Delta$ with image Δ_2. Similarly since $\Delta_2 \subset \Delta_1 \subset p^{-1}\Delta_2$, the quotient $\Delta_2/p\Delta_1$ corresponds to a finite flat subgroup scheme C' of $A_2[p]$. Letting $A_1 = A_2/C'$, the projection $A_2 \to A_1$

induces an inclusion $\mathbf{D}(A_1[p^\infty]) \to \mathbf{D}(A_2[p^\infty]) = \Delta_2$ with image $p\Delta_1$, and the isogeny $f : A_1 \to A_2$ induced by multiplication by p identifies $\mathbf{D}(A_1[p^\infty])$ with Δ_1.

Since the Δ_j are, by construction, stable under the action of $\mathcal{O}_{D_\Sigma} \otimes \mathbf{Z}_p$, the abelian varieties A_j inherit an action of $\mathcal{O}_{D_\Sigma}$ from the action on A. (Note for example that C is stable, so $\mathcal{O}_{D_\Sigma}$ acts on A_2, and that C' is stable under the resulting action, which therefore carries over to A_1.) We then define the actions ι_j of $\mathcal{O}_D$ on A_j by composing with the isomorphism $\vartheta^{-1} : \mathcal{O}_D \to \mathcal{O}_{D^\Sigma}$. The resulting action of $\mathcal{O}_{D,p} = M_2(\mathcal{O}_{E,p})$ on $\Delta_j = \mathbf{D}(A_j[p^\infty])$ is obtained by composing the conjugation $u \mapsto h_p^{-1} u h_p$ with the action inherited from the inclusion $\Delta_j \subset p^{-1}\Delta$. In particular, the inclusion is compatible with the action of $\mathcal{O}_{E,p}$ induced by ι_j, so this action yields the same decomposition $\Delta_j = \bigoplus_{\tau \in \Theta_E} \Delta_{j,\tau}$ as in the definition of Δ_j. It follows that $\dim_{\overline{\mathbf{F}}_p} \mathrm{Lie}(A_j)_\tau = \dim_{\overline{\mathbf{F}}_p}(\Delta_{j,\phi\circ\tau}/\Phi(\Delta_{j,\tau}))$, which it is straightforward to check is equal to two for all $\tau \in \Theta_E$.

Recall that the pairing on Δ induced by the quasi-polarization λ decomposes over the $\tau \in \Theta_E$ to define isomorphisms

$$\Delta_\tau \xrightarrow{\sim} \mathrm{Hom}_W(\Delta_{\tau^c}, W)$$

under which Φ is (ϕ-semi-linearly) adjoint to $V = p\Phi^{-1}$. It is straightforward to check from the definitions of the $\Delta_{j,\tau}$ that $\Delta_{1,\tau}$ is the dual lattice to Δ_{1,τ^c} and that $\Delta_{2,\tau}$ is the dual lattice to $p^{-1}\Delta_{2,\tau^c}$ for all $\tau \in \Theta_E$. It follows that the quasi-isogenies $\lambda_j \in \mathrm{Hom}(A_j, A_j^\vee) \otimes \mathbf{Q}$ defined by

$$\lambda_1 = (g^{-1} \circ f)^\vee \circ \lambda \circ (g^{-1} \circ f) \quad \text{and} \quad \lambda_2 = p(g^{-1})^\vee \circ \lambda \circ g^{-1}$$

induce isomorphisms $A_j[p^\infty] \cong A_j^\vee[p^\infty]$, and therefore define prime-to-p quasi-polarizations. The compatibility under ι_j between the anti-involution $*$ on D and the Rosati involution induced by λ_i then follows from the compatibility of $*$ on D_Σ with the λ-Rosati involution (under ι) and that of ϑ with the anti-involutions.

For $j = 1, 2$, we define η_j as the composite of the isomorphisms

$$\widehat{\mathcal{O}}_D^{(p)} \xrightarrow{(h^{(p)}\cdot)^{-1}} \widehat{\mathcal{O}}_D^{(p)} \xrightarrow{(\hat{\xi}_E^{(p)})^{-1}} \widehat{\mathcal{O}}_{D_\Sigma}^{(p)} \xrightarrow{\eta} \widehat{T}^{(p)}(A) \longrightarrow \widehat{T}^{(p)}(A_j),$$

where the last isomorphism is induced by $f^{-1} \circ g$ for $j = 1$ and by g for $j = 2$. Letting $\epsilon_1 = (\varepsilon^{(p)})^{-1}\epsilon$ and $\epsilon_2 = p\epsilon_1$, it is straightforward to check that (η_j, ϵ_j) defines a level U'-structure on A_j for $j = 1, 2$.

We have now shown that the data $\underline{A}_j = (A_j, \iota_j, \lambda_j, \eta_j, \epsilon_j)$ define $\overline{\mathbf{F}}_p$-points of $\widetilde{Y}_{U'}(G')$ for $j = 1, 2$. Furthermore the isogeny $f : A_1 \to A_2$ is compatible with the $\mathcal{O}_D$-action, and the equations $p\lambda_1 = f^\vee \circ \lambda_2 \circ f$, $\eta_2 = f \circ \eta_1$ and $\epsilon_2 = p\epsilon_1$ are immediate from the definitions. It is also straightforward to check from the definitions of the $\Delta_{j,\tau}$ that $\Delta_{1,\tau}/\Delta_{2,\tau}$ is two-dimensional over $\overline{\mathbf{F}}_p$ for all $\tau \in \Theta_E$, so $\mathbf{D}(\ker f) \cong \Delta_1/\Delta_2$ is free of rank two over $\mathcal{O}_E \otimes \overline{\mathbf{F}}_p$. It follows that the triple $(\underline{A}_1, \underline{A}_2, f)$ defines an $\overline{\mathbf{F}}_p$-point of $\widetilde{Y}_{U_0'(p)}(G')$.

Next we show that the ($\overline{\mathbf{F}}_p$-point defined by the) triple $(\underline{A}_1, \underline{A}_2, f)$ lies in the closed subscheme S_J', i.e., that $\mathrm{Lie}(f)_{\tilde\theta} = 0$ (or equivalently $\mathrm{Lie}(f)_{\tilde\theta^c} = 0$) if $\theta \notin J$, and $\mathrm{Lie}(f^\vee)_{\tilde\theta} = 0$ (or equivalently $\mathrm{Lie}(f^\vee)_{\tilde\theta^c} = 0$) if $\theta \in J$. Identifying $\mathbf{D}(A_j[p^\infty])$ with Δ_j for $j = 1, 2$, we see that the vanishing of $\mathrm{Lie}(f)_\tau$ is equivalent to the inclusion $V(\Delta_{2,\phi\circ\tau}) \subset p\Delta_{1,\tau}$, i.e., $\Delta_{2,\phi\circ\tau} \subset \Phi(\Delta_{1,\tau})$. In fact the equality

$\Delta_{2,\phi\circ\tau} = \Phi(\Delta_{1,\tau})$ if $\theta = \tau|_F \notin J$ is immediate from the definitions of the $\Delta_{i,\tau}$, using the equations $\Phi(\Delta_{\tilde\theta}) = \Delta_{\phi\circ\tilde\theta}$ and $\Phi(\Delta_{\tilde\theta^c}) = p\Delta_{\phi\circ\tilde\theta^c}$ if also $\phi\circ\theta \in J$. Similarly the vanishing of $\mathrm{Lie}(f^\vee)_\tau$ is equivalent to the inclusion $\Delta_{2,\tau} \subset V(\Delta_{1,\phi\circ\tau})$, i.e., $\Phi(\Delta_{2,\tau}) \subset p\Delta_{1,\phi\circ\tau}$, and if $\theta = \tau|_F \in J$, then in fact the equality $\Phi(\Delta_{2,\tau}) = p\Delta_{1,\phi\circ\tau}$ follows from the definitions, using that $\Phi(\Delta_{\tilde\theta}) = p\Delta_{\phi\circ\tilde\theta}$ and $\Phi(\Delta_{\tilde\theta^c}) = \Delta_{\phi\circ\tilde\theta^c}$ if also $\phi\circ\theta\notin J$.

We now check that $(\underline{A}, \{L^0_\theta\}_{\theta\in\Sigma}) \mapsto (\underline{A}_1, \underline{A}_2, f)$ defines an inverse of $\tilde\Xi_J$ on $\overline{\mathbf{F}}_p$-points. Suppose first that $(\underline{A}, \{L^0_\theta\}_{\theta\in\Sigma})$ and $(\underline{A}_1, \underline{A}_2, f)$ are as above, and let A_J denote the J-splice of f. Letting π and ψ be as in the construction of A_J, note that Proposition 5.1.2 and the formulas

- $\Delta_{1,\tilde\theta} = \Delta_{\tilde\theta}$ and $\Delta_{1,\tilde\theta^c} = \Delta_{\tilde\theta^c}$ if $\theta\notin J$,
- $\Delta_{2,\tilde\theta} = \Delta_{\tilde\theta}$ and $\Delta_{2,\tilde\theta^c} = p\Delta_{\tilde\theta^c}$ if $\theta\in J$,

imply that the lattice $(\pi^*)^{-1}\psi^*(\mathbf{D}(A_J[p^\infty]) \subset \mathbf{D}(A_1[p^\infty])\otimes\mathbf{Q} = \Delta_1\otimes\mathbf{Q}$ coincides with Δ. It follows that there is an isomorphism $\sigma : A_J \xrightarrow{\sim} A$ compatible with the quasi-isogenies $\psi\circ\pi^{-1}$ and $g^{-1}\circ f$ from A_1. Unravelling the definitions of the auxiliary data for $\underline{A}_J$ in terms of that for $\underline{A}_1$, and in turn in terms of that for $\underline{A}$, one finds that these are compatible with σ. Furthermore if $\theta\notin J$ and $\phi\circ\theta\in J$, then $\Lambda_{\tilde\theta}\subset\Delta_{\tilde\theta}$ coincides with $V(\Delta_{1,\phi^{-1}\circ\tilde\theta})$, so that $L_\theta \subset \Delta_{\tilde\theta}/p\Delta_{\tilde\theta} = H^1_{\mathrm{dR}}(A/\overline{\mathbf{F}}_p)_{\tilde\theta}$ corresponds to $H^0(A_1, \Omega^1_{A_1/\overline{\mathbf{F}}_p})$ under the isomorphism $H^1_{\mathrm{dR}}(A/\overline{\mathbf{F}}_p)_{\tilde\theta} \cong H^1_{\mathrm{dR}}(A_1/\overline{\mathbf{F}}_p)_{\tilde\theta}$ induced by the quasi-isogeny $g^{-1}\circ f$. It follows that $\sigma^*L_\theta \subset H^1_{\mathrm{dR}}(A_J/\overline{\mathbf{F}}_p)$ corresponds to $H^0(A_1, \Omega^1_{A_1/\overline{\mathbf{F}}_p})$ under the isomorphism $H^1_{\mathrm{dR}}(A_J/\overline{\mathbf{F}}_p)_{\tilde\theta} \cong H^1_{\mathrm{dR}}(A_1/\overline{\mathbf{F}}_p)_{\tilde\theta}$ of Corollary 5.1.3(2), and hence that $\sigma^*L^0_\theta$ defines the same line in $H^1_{\mathrm{dR}}(A_J/\overline{\mathbf{F}}_p)^0_{\tilde\theta}$ as $\tilde\Xi_J(\underline{A}_1, \underline{A}_2, f)$. If $\theta\in J$ and $\phi\circ\theta\notin J$, then one similarly finds that the same conclusion holds, now using that $\Lambda_{\tilde\theta} = V(\Delta_{2,\phi^{-1}\circ\tilde\theta})$ and that the isogeny $g\circ\sigma$ induces the isomorphism $H^1_{\mathrm{dR}}(A_J/\overline{\mathbf{F}}_p)_{\tilde\theta} \cong H^1_{\mathrm{dR}}(A_2/\overline{\mathbf{F}}_p)_{\tilde\theta}$ of Corollary 5.1.3(1). We now shown that $\tilde\Xi_J(\underline{A}_1, \underline{A}_2, f)$ defines the same point as the data $(\underline{A}, \{L^0_\theta\}_{\theta\in\Sigma})$.

Suppose now that $(\underline{A}_1, \underline{A}_2, f) \in S'_J(\overline{\mathbf{F}}_p)$, and let $(\underline{A}, \{L^0_\theta\}_{\theta\in\Sigma}) = \tilde\Xi_J(\underline{A}_1, \underline{A}_2, f)$, so in particular $A = A_J$ is the J-splice of f and the L^0_θ are lines in $H^1_{\mathrm{dR}}(A/\overline{\mathbf{F}}_p)^0_{\tilde\theta}$. We must show that the triple $(\underline{A}'_1, \underline{A}'_2, f')$ arising from the above construction is isomorphic to $(\underline{A}_1, \underline{A}_2, f)$.

With π and ψ as in diagram (10), we prove that $\Delta_1 = (\psi^*)^{-1}\pi^*\mathbf{D}(A_1)$ as lattices in $\Delta\otimes\mathbf{Q} = \mathbf{D}(A)\otimes\mathbf{Q}$ by showing that

$$\Delta_{1,\tau} = (\psi^*)^{-1}\pi^*\mathbf{D}(A_1)_\tau$$

for all $\tau\in\Theta_E$. For τ such that $\theta = \tau|_F \notin J$, this is immediate from the definition of $\Delta_{1,\tau}$ and Proposition 5.1.2. Suppose now that θ is such that $\phi^{-1}\circ\theta, \theta\in J$. The vanishing of $(\mathrm{Lie} f^\vee)_\tau$ means that $\Phi(f^*\mathbf{D}(A_2)_{\phi^{-1}\circ\tau}) = p\mathbf{D}(\mathbf{A}_1)_\tau$, so Proposition 5.1.2 implies that

$$(\psi^*)^{-1}\pi^*\mathbf{D}(A_1)_\tau = p^{-1}\Phi((\psi^*)^{-1}\pi^*f^*\mathbf{D}(A_2)_{\phi^{-1}\circ\tau}) = p^\delta\Phi(\mathbf{D}(A)_{\phi^{-1}\circ\tau}) = \Delta_{1,\tau},$$

where $\delta = 0$ (resp. -1) if $\tau = \tilde\theta$ (resp. $\tilde\theta^c$). Finally if $\phi^{-1}\circ\theta\notin J$ and $\theta\in J$, then the line $L^0_{\phi^{-1}\circ\theta}$ given by the construction of $\tilde\Xi_J$ is $((\psi^*)^{-1}\pi^*H^0(A_1, \Omega_{A_1/\overline{\mathbf{F}}_p}))^0$, so that $L_{\phi^{-1}\circ\theta} = (\psi^*)^{-1}\pi^*H^0(A_1, \Omega_{A_1/\overline{\mathbf{F}}_p})$. It follows that $\Lambda_{\phi^{-1}\circ\tilde\theta} = (\psi^*)^{-1}\pi^*V(\mathbf{D}(A_1)_{\tilde\theta})$, and the compatibility of $\pi\circ\psi^{-1}$ with the quasi-polarizations λ_1 and $\lambda = \lambda_J$ inducing the perfect pairings on $\mathbf{D}(A_1)$ and Δ then implies that its dual lattice $\Lambda_{\phi^{-1}\circ\tilde{\theta}^c}$ is $\Phi^{-1}((\psi^*)^{-1}\pi^*\mathbf{D}(A_1)_{\tilde{\theta}^c})$. We then conclude the desired equation from the definition of $\Delta_{1,\tau}$ for $\tau = \tilde{\theta}$ and $\tau = \tilde{\theta}^c$. It follows that there is an isomorphism $\sigma_1 : A_1 \to A_1'$ compatible with the quasi-isogenies $\psi \circ \pi^{-1}$ and $g^{-1} \circ f'$, where $g : A \to A_2'$ is the quotient map in the construction of $(\underline{A}_1', \underline{A}_2', f')$. Similarly one checks that $\Delta_2 = (\psi^*)^{-1}\pi^* f^*\mathbf{D}(A_2)$ and deduces that there is an isomorphism $\sigma_2 : A_2 \to A_2'$ compatible with g and the isogeny $A \to A_2$ of diagram (10). It follows that $\sigma_2 \circ f = f' \circ \sigma_1$, and it is straightforward to check that the σ_i respect the auxiliary data for $\underline{A}_i$ for $i = 1, 2$, and hence define an isomorphism between $(\underline{A}_1, \underline{A}_2, f)$ and $(\underline{A}_1', \underline{A}_2', f')$.

Recall that crystalline deformation theory, in particular the Grothendieck–Messing Theorem, gives an equivalence of categories between abelian varieties over $T = \overline{\mathbf{F}}_p[\epsilon]/\epsilon^2$ and pairs $(A, \tilde{L})$ where A is an abelian variety over $\overline{\mathbf{F}}_p$ and $\tilde{L}$ is a free T-submodule of $H^1_{\mathrm{dR}}(A/\overline{\mathbf{F}}_p) \otimes_{\overline{\mathbf{F}}_p} T$ such that $\tilde{L} \otimes_T \overline{\mathbf{F}}_p = H^0(A, \Omega^1_{A/\overline{\mathbf{F}}_p})$. The equivalence is defined by the functor sending $\tilde{A}$ to $(A, \tilde{L})$, where $A = \tilde{A} \otimes_T \overline{\mathbf{F}}_p$ and $\tilde{L} = H^0(\tilde{A}, \Omega^1_{\tilde{A}/T})$ is identified with a submodule of $H^1_{\mathrm{dR}}(A/\overline{\mathbf{F}}_p) \otimes_{\overline{\mathbf{F}}_p} T$ via its canonical isomorphisms with $H^1_{\mathrm{crys}}(A/T) \cong H^1_{\mathrm{dR}}(\tilde{A}/T)$; in particular the functor $\cdot \otimes_T \overline{\mathbf{F}}_p$ is faithful.

Suppose that $(\underline{A}_1, \underline{A}_2, f)$ is an element of $S_J'(\overline{\mathbf{F}}_p)$ with image $(\underline{A}, \{L_\theta^0\}_{\theta\in\Sigma})$ under $\tilde{\Xi}_J$, and that $(\underline{\tilde{A}}_1, \underline{\tilde{A}}_2, \tilde{f})$ and $(\underline{\tilde{A}}_1', \underline{\tilde{A}}_2', \tilde{f}')$ are two elements of $S_J'(T)$ lifting $(\underline{A}_1, \underline{A}_2, f)$ and having the same image $(\underline{\tilde{A}}, \{\tilde{L}_\theta^0\}_{\theta\in\Sigma})$ under $\tilde{\Xi}_J$. Letting $\tilde{L}_j = H^0(\tilde{A}_j, \Omega^1_{\tilde{A}_j/T})$ and $\tilde{L}_j' = H^0(\tilde{A}_j', \Omega^1_{\tilde{A}_j'/T})$, it suffices to prove that $\tilde{L}_j = \tilde{L}_j'$ as submodules of $H^1_{\mathrm{dR}}(A_j/\overline{\mathbf{F}}_p) \otimes_{\overline{\mathbf{F}}_p} T$ for $j = 1, 2$. Since $\tilde{L}_j$ is stable under the action of $\mathcal{O}_D \otimes T \cong \prod_{\tau\in\Theta_E} M_2(T)$ induced by ι_j, we may decompose $\tilde{L}_j = \oplus_{\tau\in\Theta_E}\tilde{L}_{j,\tau}$, and similarly $\tilde{L}_j' = \oplus_{\tau\in\Theta_E}\tilde{L}_{j,\tau}'$. Furthermore since $\tilde{L}_j$ is totally isotropic with respect to the perfect T-valued pairing on $H^1_{\mathrm{dR}}(A_j/\overline{\mathbf{F}}_p) \otimes_{\overline{\mathbf{F}}_p} T$ induced by λ_j, we have $\tilde{L}_{j,\tau^c} = \tilde{L}_{j,\tau}^\perp$, and similarly $\tilde{L}_{j,\tau^c}' = (\tilde{L}_{j,\tau}')^\perp$, so it suffices to prove that $\tilde{L}_{j,\tilde{\theta}} = \tilde{L}_{j,\tilde{\theta}}'$ for all $\theta \in \Theta$ and $j = 1, 2$.

If $\theta \notin J$, then note that $H^0(\tilde{A}_1, \Omega^1_{\tilde{A}_1/T})_{\tilde{\theta}}$ and $H^0(\tilde{A}_1', \Omega^1_{\tilde{A}_1'/T})_{\tilde{\theta}}$ have the same image in $H^1_{\mathrm{dR}}(\tilde{A}/T)_{\tilde{\theta}}$ under the isomorphism of part (2) of Corollary 5.1.3(2), this being $H^0(\tilde{A}, \Omega^1_{\tilde{A}/T})_{\tilde{\theta}}$ if $\theta \notin \Sigma$ and $\tilde{L}_\theta^0 \otimes_T T^2$ if $\theta \in \Sigma$, and it follows that $\tilde{L}_{1,\tilde{\theta}} = \tilde{L}_{1,\tilde{\theta}}'$ in this case. On the other hand since $\mathrm{Lie}(\tilde{f})_{\tilde{\theta}} = 0$ if $\theta \notin J$, we have that $\tilde{L}_{2,\tilde{\theta}}$ is the kernel of $H^1_{\mathrm{dR}}(\tilde{A}_2/T)_{\tilde{\theta}} \to H^1_{\mathrm{dR}}(\tilde{A}_1/T)_{\tilde{\theta}}$, and this is identified with $H^0(A_2, \Omega_{A_2/\overline{\mathbf{F}}_p})_{\tilde{\theta}} \otimes_{\overline{\mathbf{F}}_p} T$. Similarly the fact that $\mathrm{Lie}(\tilde{f}')_{\tilde{\theta}} = 0$ implies that $\tilde{L}_{2,\tilde{\theta}}' = H^0(A_2, \Omega_{A_2/\overline{\mathbf{F}}_p})_{\tilde{\theta}} \otimes_{\overline{\mathbf{F}}_p} T$ and we conclude that $\tilde{L}_{2,\tilde{\theta}} = \tilde{L}_{2,\tilde{\theta}}'$.

If $\theta \in J$, then we similarly find that $\tilde{L}_{2,\tilde{\theta}} = \tilde{L}_{2,\tilde{\theta}}'$ coincide since they have the same images under the isomorphism of Corollary 5.1.3(1). Finally the vanishing of $\mathrm{Lie}(\tilde{f}^\vee)_{\tilde{\theta}}$ implies that $\tilde{L}_{2,\tilde{\theta}}$ is the image of $H^1_{\mathrm{dR}}(\tilde{A}_2/T)_{\tilde{\theta}} \to H^1_{\mathrm{dR}}(\tilde{A}_1/T)_{\tilde{\theta}}$, and this is $H^0(A_1, \Omega_{A_1/\overline{\mathbf{F}}_p})_{\tilde{\theta}} \otimes_{\overline{\mathbf{F}}_p} T$, and the same holds for $\tilde{L}_{2,\tilde{\theta}}'$.

We thus have in all cases that $\tilde{L}_{j,\tilde{\theta}} = \tilde{L}'_{j,\tilde{\theta}}$, from which it follows that $(\underline{\tilde{A}}_1, \underline{\tilde{A}}_2, \tilde{f})$ and $(\underline{\tilde{A}}'_1, \underline{\tilde{A}}'_2, \tilde{f}')$ define the same point of $S'_J(T)$. This completes the proof of injectivity on tangent spaces, and hence of the theorem.

□

5.2.4. *Descent and Hecke equivariance.* Recall from §4.2.4 and §2.2.3 that the reductions mod p of the Shimura varieties associated to G' (of level $U'_0(p)$) and G'_Σ (of level U'_Σ) are obtained as quotients of the schemes considered above by the action of $\mathcal{O}^\times_{F,(p),+}$ on the quasi-polarizations λ (and multipliers ϵ) appearing in the moduli problems. Furthermore for sufficiently small U'_Σ, the vector bundles $\tilde{\mathcal{V}}^0_\tau$ on $\widetilde{Y}_{U'_\Sigma}(G'_\Sigma)_{\mathbf{F}}$ descend to the vector bundles denoted $\mathcal{V}^0_\tau$ on the quotient $Y_{U'_\Sigma}(G'_\Sigma)_{\mathbf{F}}$, which we denote as $\overline{Y}'_\Sigma$. The evident compatibility of the isomorphism $\tilde{\Xi}_J$ with the action on $\mathcal{O}^\times_{F,(p),+}$ therefore implies that it descends to an isomorphism[12]

$$\Xi'_J : \overline{Y}_0(p)'_J \longrightarrow \prod_{\theta\in\Sigma} \mathbf{P}_{\overline{Y}'_\Sigma}(\mathcal{V}^0_{\tilde{\theta}}),$$

where the product is a fibre product over $\overline{Y}'_\Sigma$.

Furthermore the isomorphisms Ξ'_J for varying U' are compatible with the Hecke action in a sense we make precise as follows. Recall that for sufficiently small open compact subgroups U'_1 and U'_2 of $G'(\mathbf{A_f})$ of level prime to p and $g \in G'(\mathbf{A}^{(p)}_{\mathbf{f}})$ such that $g^{-1}U'_1 g \subset U'_2$, we defined the finite étale morphism $\tilde{\rho}_g : \widetilde{Y}_{U'_{1,0}(p)}(G')_{\mathbf{F}} \to \widetilde{Y}_{U'_{2,0}(p)}(G')_{\mathbf{F}}$, restricting to such a morphism of the closed subschemes $S'_{1,J} \to S'_{2,J}$. Letting $g_\Sigma = \xi_E^{-1}(g)$, we have $g_\Sigma^{-1}U'_{1,\Sigma}g_\Sigma \subset U'_{2,\Sigma}$, hence also morphisms

$$\tilde{\rho}_{g_\Sigma} : \widetilde{Y}_{U_{1,\Sigma}}(G'_\Sigma)_{\mathbf{F}} \to \widetilde{Y}_{U_{2,\Sigma}}(G'_\Sigma)_{\mathbf{F}} \quad \text{and} \quad \pi^*_{g_\Sigma} : \tilde{\rho}^*_{g_\Sigma}\tilde{\mathcal{V}}^0_{2,\tau} \stackrel{\sim}{\longrightarrow} \tilde{\mathcal{V}}^0_{1,\tau},$$

where $\tilde{\mathcal{V}}^0_{i,\tau}$ denotes the automorphic bundle $\tilde{\mathcal{V}}^0_\tau$ on $S_i := \widetilde{Y}_{U_{i,\Sigma}}(G'_\Sigma)_{\mathbf{F}}$ for $i = 1, 2$. These in turn induce a morphism

$$\prod_{\theta\in\Sigma} \mathbf{P}_{S_1}(\tilde{\mathcal{V}}^0_{1,\tilde{\theta}}) \to \prod_{\theta\in\Sigma} \mathbf{P}_{S_2}(\tilde{\mathcal{V}}^0_{2,\tilde{\theta}})$$

which we also denote by $\tilde{\rho}_{g_\Sigma}$, and it is straightforward to check that the resulting diagram

$$\begin{array}{ccc} S'_{1,J} & \xrightarrow{\tilde{\Xi}_{1,J}} & \prod_{\theta\in\Sigma} \mathbf{P}_{S_1}(\tilde{\mathcal{V}}^0_{1,\tilde{\theta}}) \\ \downarrow{\scriptstyle \tilde{\rho}_g} & & \downarrow{\scriptstyle \tilde{\rho}_{g_\Sigma}} \\ S'_{2,J} & \xrightarrow{\tilde{\Xi}_{2,J}} & \prod_{\theta\in\Sigma} \mathbf{P}_{S_2}(\tilde{\mathcal{V}}^0_{2,\tilde{\theta}}) \end{array}$$

[12]We remark that since $(U'_\Sigma \cap F)^2$ acts via scalars on $\tilde{\mathcal{V}}^0_\tau$, the additional condition imposed on U'_Σ to ensure the vector bundles descend to the quotient by $\mathcal{O}_{F,(p),+}/(U'_\Sigma \cap F)^2$ is not actually needed in order to descend the associated projective bundles.

commutes. Taking quotients by the action of $\mathcal{O}_{F,(p),+}$ then yields a commutative diagram

$$\begin{array}{ccc} \overline{Y}_{1,0}(p)'_J & \xrightarrow{\Xi'_{1,J}} & \prod_{\theta\in\Sigma} \mathbf{P}_{\overline{Y}'_{1,\Sigma}}(\mathcal{V}^0_{1,\tilde{\theta}}) \\ \downarrow{\scriptstyle \rho_g} & & \downarrow{\scriptstyle \rho_{g_\Sigma}} \\ \overline{Y}_{2,0}(p)'_J & \xrightarrow{\Xi'_{2,J}} & \prod_{\theta\in\Sigma} \mathbf{P}_{\overline{Y}'_{2,\Sigma}}(\mathcal{V}^0_{2,\tilde{\theta}}) \end{array} \tag{12}$$

We have now proved the analogue of Theorem A with G replaced by G':

Theorem 5.2.2. *For each sufficiently small open compact subgroup U' of $G'(\mathbf{A}_\mathbf{f})$ of level prime to p, there is an isomorphism*

$$\Xi'_J : \overline{Y}_0(p)'_J \longrightarrow \prod_{\theta\in\Sigma} \mathbf{P}_{\overline{Y}'_\Sigma}(\mathcal{V}^0_{\tilde{\theta}}).$$

The morphisms Ξ'_J are compatible with the Hecke action in the sense that if $g \in G'(\mathbf{A}^{(p)}_\mathbf{f})$ is such that $g^{-1}U'_1 g \subset U'_2$, then the diagram (12) commutes.

5.3. **Proof of Theorem A.** We now explain how to deduce Theorem A from Theorem 5.2.2. The task at hand is similar to that of proving that Theorem 5.2 of [TX16] follows from their Theorem 5.8 and Corollary 5.9, but we were not able to supply a proof of this using the ingredients provided in [TX16][13], so instead we give an argument based on our Lemmas 2.3.2 and 4.2.1 (similar to Construction 2.12 of [TX19]),

5.3.1. *Compatibility on components.* Note that Lemmas 2.3.2 and 4.2.1 (and the discussion at the end of §4.3.4) describe both varieties in Theorem A (over $\mathbf{F} = \overline{\mathbf{F}}_p$) as fibre products of those of Theorem 5.2.2 with C over C', where $C = C_{\det(U)} = C_{\det(U_\Sigma)}$ indexes the set of components of $\overline{Y} = Y_U(G)_{\overline{\mathbf{F}}_p}$ (and $\overline{Y}_\Sigma = Y_{U_\Sigma}(G_\Sigma)_{\overline{\mathbf{F}}_p}$ if $\Sigma \neq \varnothing$) and $C' = C_{\nu'(U')} = C_{\nu'(U'_\Sigma)}$ indexes the set of components of $\overline{Y}' = Y'_{U'}(G')_{\overline{\mathbf{F}}_p}$ (and $\overline{Y}'_\Sigma = Y'_{U'_\Sigma}(G'_\Sigma)_{\overline{\mathbf{F}}_p}$ if $\Sigma \neq \varnothing$). Thus if the isomorphism of Theorem 5.2.2 were compatible with the natural projections to C', we could obtain Theorem A by pulling back along the inclusion $C \to C'$. Unfortunately we do not know this compatibility, but it will suffice for our purposes to prove that the diagram

$$\begin{array}{ccccccc} \overline{Y}_0(p)'_J & \longrightarrow & \overline{Y}_0(p)' & \longrightarrow & \overline{Y}' & \longrightarrow & C' \\ \downarrow{\scriptstyle \Xi'_J} & & & & & & \uparrow \\ \prod_{\theta\in\Sigma} \mathbf{P}_{\overline{Y}'_\Sigma}(\mathcal{V}^0_{\tilde{\theta}}) & & \longrightarrow & & \overline{Y}'_\Sigma & \longrightarrow & C' \end{array} \tag{13}$$

commutes for some automorphism of C' defined by multiplication by an element $t \in T'(\mathbf{A}^{(p)}_\mathbf{f})$ independent of U'. Let π denote the composite along the top row of

[13] The proof of the implication in [TX16] seems to be non-trivial, but is omitted from the paper. In any case, our situation is different in that we also need to transfer the vector bundles whose projectivizations are involved.

the diagram, let Ψ_J denote the composite $\overline{Y}_0(p)'_J \to \overline{Y}'_\Sigma$, and let π_Σ denote the projection $\overline{Y}'_\Sigma \to C'$.

Suppose first that $\Sigma \neq \Theta$. In this case π_Σ has connected fibres, and since Ψ_J is closed and has connected fibres, it follows that $\pi_\Sigma \circ \Psi_J$ has connected fibres. Thus if $y_1, y_2 \in \overline{Y}_0(p)'_J(\overline{\mathbf{F}}_p)$ have the same image under $\pi_\Sigma \circ \Psi_J$, then $\pi(y_1) = \pi(y_2)$. Since $\pi_\Sigma \circ \Psi_J$ is surjective, it follows that there is a unique endomorphism $\delta = \delta_{U'}$ of C' such that $\pi = \delta \circ \pi_\Sigma \circ \Psi_J$. Furthermore if $U'_1, U'_2 \subset G'(\mathbf{A_f})$ and $g \in G'(\mathbf{A}_\mathbf{f}^{(p)})$ are such that $g^{-1}U'_1 g \subset U'_2$, then the compatibility of Ψ_J, π and π_Σ with the Hecke action (via ξ_E and ν') implies that the diagram

$$\begin{array}{ccc} C_1 & \xrightarrow{\delta_1} & C_1 \\ {\scriptstyle \nu'(g)}\downarrow & & \downarrow{\scriptstyle \nu'(g)} \\ C_2 & \xrightarrow{\delta_2} & C_2 \end{array}$$

commutes, where $C_j = C_{\nu'(U'_j)}$ and $\delta_j = \delta_{U'_j}$ for $j = 1, 2$. We may therefore take the limit over sufficiently small $(U')^p \subset G'(\mathbf{A}_\mathbf{f}^{(p)})$ to obtain an endomorphism δ of

$$\varprojlim C_{\nu'(U')} = \varprojlim T'(\mathbf{Q})^{(p)}_+ \backslash T'(\mathbf{A}_\mathbf{f}^{(p)})/\nu'(U')^p = \overline{T'(\mathbf{Q})^{(p)}_+} \Big\backslash T'(\mathbf{A}_\mathbf{f}^{(p)}),$$

compatible with the action of $T'(\mathbf{A}_\mathbf{f}^{(p)}) = \nu'(G'(\mathbf{A}_\mathbf{f}^{(p)}))$ (where $T'(\mathbf{Q})^{(p)}$ denotes the subgroup of p-integral elements of $T'(\mathbf{Q})$ and $\bar{\cdot}$ denotes its closure). It follows that δ is defined by multiplication by t for some $t \in T'(\mathbf{A}_\mathbf{f}^{(p)})$ independent of U'.

Suppose on the other hand that $\Sigma = \Theta$, so that B_Σ is a totally definite quaternion algebra, $\overline{Y}'_\Sigma$ is the finite set

$$G'_\Sigma(\mathbf{Q})\backslash G'_\Sigma(\mathbf{A_f})/U'_\Sigma = G'_\Sigma(\mathbf{Q})^{(p)}\backslash G'_\Sigma(\mathbf{A}_\mathbf{f}^{(p)})/(U'_\Sigma)^p$$

viewed as a scheme over $\overline{\mathbf{F}}_p$ (where $G'_\Sigma(\mathbf{Q})^{(p)}$, and the morphism

$$\pi_\Sigma : \overline{Y}'_\Sigma \longrightarrow C' = T'(\mathbf{Q})^{(p)}_+\backslash T'(\mathbf{A}_\mathbf{f}^{(p)})/(\nu'(U'_\Sigma)^p)$$

is induced by $\nu' : G'_\Sigma \to T'$. The same argument as above still yields a unique morphism $\delta = \delta_{U'} : \overline{Y}'_\Sigma \longrightarrow C'$ such that $\pi = \delta \circ \Psi_J$. Furthermore if $U'_1, U'_2 \subset G'(\mathbf{A_f})$ and $g \in G'(\mathbf{A}_\mathbf{f}^{(p)})$ are such that $g^{-1}U'_1 g \subset U'_2$, then the diagram

$$\begin{array}{ccc} \overline{Y}'_{1,\Sigma} & \xrightarrow{\delta_1} & C_1 \\ {\scriptstyle \rho_{g\Sigma}}\downarrow & & \downarrow{\scriptstyle \nu'(g)} \\ \overline{Y}'_{2,\Sigma} & \xrightarrow{\delta_2} & C_2 \end{array}$$

commutes (where now $\overline{Y}'_{j,\Sigma} = G'_\Sigma(\mathbf{Q})\backslash G'_\Sigma(\mathbf{A_f})/U'_{j,\Sigma}$ for $j = 1, 2$). Taking the limit over sufficiently small $(U')^p \subset G'(\mathbf{A}_\mathbf{f}^{(p)})$ now yields a map

$$\delta : \overline{G'_\Sigma(\mathbf{Q})^{(p)}} \Big\backslash G'_\Sigma(\mathbf{A}_\mathbf{f}^{(p)}) \longrightarrow \overline{T'(\mathbf{Q})^{(p)}_+} \Big\backslash T'(\mathbf{A}_\mathbf{f}^{(p)})$$

such that $\delta(yg) = \nu'(g)\delta(y)$ for all $g \in G'_\Sigma(\mathbf{A}_\mathbf{f}^{(p)})$, and it follows just the same that $\delta(y) = t\pi_\Sigma(y)$ for some $t \in T'(\mathbf{A}_\mathbf{f}^{(p)})$ independent of U'.

5.3.2. *Construction of the isomorphism.* We have now shown that the diagram (13) commutes, where the right vertical arrow is multiplication by t^{-1} for some $t \in T'(\mathbf{A}_{\mathbf{f}}^{(p)})$ independent of U. (Note from the proof that such a t is only unique up to multiplication by an element of $\overline{T'(\mathbf{Q})_+^{(p)}}$.) Recall that $T' = (T_F \times T_E)/T_F$ where T_F is embedded in the product via $x \mapsto (x^2, x^{-1})$, so we may choose $u \in T_E(\mathbf{A}_{\mathbf{f}}^{p)})$ such that $(1,u)t^{-1}$ is in the image of the embedding $T_F(\mathbf{A}_{\mathbf{f}}^{(p)}) \to T'(\mathbf{A}_{\mathbf{f}}^{(p)})$ defined by $v \mapsto (v, 1)$. We then have a commutative diagram

$$\begin{array}{ccccc} \overline{Y}_0(p)'_J & \xrightarrow{\rho_u} & \overline{Y}_0(p)'_J & \xrightarrow{\Xi'_J} & \prod_{\theta\in\Sigma} \mathbf{P}_{\overline{Y}'_\Sigma}(\mathcal{V}^0_{\tilde\theta}) \\ \downarrow & & \downarrow & & \downarrow \\ C' & \xrightarrow[\cdot(1,u)]{} & C' & \xrightarrow[\cdot t^{-1}]{} & C' \end{array} \qquad (14)$$

where the downward arrows are the natural maps defined by the rows of (13).

Suppose now that U is an open compact subgroup of $G(\mathbf{A}_{\mathbf{f}})$ of level prime to p, and let $U_\Sigma = \xi^{-1}(U) \subset G_\Sigma(\mathbf{A}_{\mathbf{f}})$. Suppose that V_E is an open compact subgroup of $\mathbf{A}_{E,\mathbf{f}}^\times$ of level prime to p, sufficiently small relative to U (and hence U_Σ) in the sense of §2.3.1, and let U' denote the image of $U \times V_E$ in $G'(\mathbf{A}_{\mathbf{f}})$; note that $U'_\Sigma = \xi^{-1}(U')$ is also the image of $U_\Sigma \times V_E$ in $G'_\Sigma(\mathbf{A}_{\mathbf{f}})$. We assume that U is sufficiently small that so are U_Σ, U' and U'_Σ.

By Lemma 2.3.2, the natural maps give rise to a Cartesian diagram

$$\begin{array}{ccc} \overline{Y}_\Sigma & \xrightarrow{i} & \overline{Y}'_\Sigma \\ \downarrow & & \downarrow \\ C_{\det(U)} & \longrightarrow & C_{\nu'(U')}, \end{array}$$

where $\overline{Y}_\Sigma = Y_{U_\Sigma}(G_\Sigma)_{\overline{\mathbf{F}}_p}$ and where $\overline{Y}'_\Sigma = Y_{U'_\Sigma}(G'_\Sigma)_{\overline{\mathbf{F}}_p}$. Furthermore the automorphic vector bundle $\mathcal{V}_\theta$ is defined as $i^*\mathcal{V}^0_{\tilde\theta}$, so we obtain from this a Cartesian diagram

$$\begin{array}{ccc} \prod_{\theta\in\Sigma} \mathbf{P}_{\overline{Y}_\Sigma}(\mathcal{V}_\theta) & \longrightarrow & \prod_{\theta\in\Sigma} \mathbf{P}_{\overline{Y}'_\Sigma}(\mathcal{V}^0_{\tilde\theta}) \\ \downarrow & & \downarrow \\ C_{\det(U)} & \longrightarrow & C_{\nu'(U')}. \end{array}$$

On the other hand, by the discussion at the end of §4.3.4, Lemma 4.2.1 gives rise to a Cartesian diagram

$$\begin{array}{ccc} \overline{Y}_0(p)_J & \longrightarrow & \overline{Y}_0(p)'_J \\ \downarrow & & \downarrow \\ C_{\det(U)} & \longrightarrow & C_{\nu'(U')}. \end{array}$$

Note that the bottom row of (14) restricts to multiplication by v on $C_{\det(U)}$ for some $v \in T_F(\mathbf{A}_{\mathbf{f}}^{(p)})$ independent of U. It follows that the fibre product of the isomorphisms

$$\Xi'_J \circ \rho_u : \overline{Y}_0(p)'_J \to \prod_{\theta\in\Sigma} \mathbf{P}_{\overline{Y}'_\Sigma}(\mathcal{V}^0_{\tilde\theta}) \quad \text{and} \quad \cdot v : C_{\det(U)} \to C_{\det(U)}$$

over $\cdot v : C_{\nu'(U')} \to C_{\nu'(U')}$ defines an isomorphism

$$\Xi_J : \overline{Y}_0(p)_J \to \prod_{\theta\in\Sigma} \mathbf{P}_{\overline{Y}_\Sigma}(\mathcal{V}_\theta).$$

Furthermore if U_1 and U_2 are sufficiently small open compact subgroups of $g \in G(\mathbf{A}_\mathbf{f}^{(p)})$ is such that $g^{-1}U_1 g \subset U_2$, then letting $g_\Sigma = \xi^{-1}(g) \in G_\Sigma(\mathbf{A}_\mathbf{f}^{(p)})$ and assuming $V_{E,1} \subset V_{E,2}$, the morphisms

$$\overline{Y}_{1,0}(p)_J \xrightarrow{\rho_g} \overline{Y}_{2,0}(p)_J \quad \text{and} \quad \mathbf{P}_{\overline{Y}_{1,\Sigma}}(\mathcal{V}_{1,\theta}) \xrightarrow{\rho_{g_\Sigma}} \mathbf{P}_{\overline{Y}_{2,\Sigma}}(\mathcal{V}_{2,\theta})$$

are the restrictions of the ones obtained from the images of g in $G'(\mathbf{A}_\mathbf{f}^{(p)})$ and $g_\Sigma \in G'_\Sigma(\mathbf{A}_\mathbf{f}^{(p)})$. Since u is central in $G'(\mathbf{A}_\mathbf{f}^{(p)})$, ρ_u commutes with ρ_g, and the commutativity of (12) implies that of

$$\begin{array}{ccc} \overline{Y}_{1,0}(p)_J & \xrightarrow{\Xi_{1,J}} & \prod_{\theta\in\Sigma} \mathbf{P}_{\overline{Y}_{1,\Sigma}}(\mathcal{V}_{1,\theta}) \\ \downarrow{\scriptstyle \rho_g} & & \downarrow{\scriptstyle \rho_{g_\Sigma}} \\ \overline{Y}_{2,0}(p)_J & \xrightarrow{\Xi_{2,J}} & \prod_{\theta\in\Sigma} \mathbf{P}_{\overline{Y}_{2,\Sigma}}(\mathcal{V}_{2,\theta}). \end{array} \tag{15}$$

In particular taking $U_1 = U_2$ and $g = 1$, we see that Ξ_J is independent of the choice of V_E. (Note however that several choices, namely ϑ, ξ and u, were made in the construction of Ξ_J, and the choice of $\widetilde{\Theta}$ is even implicit in the definition of $\mathcal{V}_\theta$, but these were all independent of U.)

This completes the proof of Theorem A:

Theorem 5.3.1. *For each sufficiently small open compact subgroup U of $G(\mathbf{A}_\mathbf{f})$ of level prime to p, there is an isomorphism*

$$\Xi_J : \overline{Y}_0(p)_J \longrightarrow \prod_{\theta\in\Sigma} \mathbf{P}_{\overline{Y}_\Sigma}(\mathcal{V}_\theta).$$

The morphisms Ξ_J are compatible with the Hecke action in the sense that if $g \in G(\mathbf{A}_\mathbf{f}^{(p)})$ is such that $g^{-1}U_1 g \subset U_2$, then the diagram (15) commutes.

Remark 5.3.2. It is natural to expect that, like $\overline{Y}_0(p)_J$, the product $\prod_{\theta\in\Sigma} \mathbf{P}_{\overline{Y}_\Sigma}(\mathcal{V}_\theta)$ admits canonical descent data to $\mathbf{F}_J$ (and even to $\mathbf{F}_\Sigma$), where $\mathbf{F}_J$ denotes the fixed field of the stabilizer of J in $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$. However one easily sees that if $\Sigma \neq \varnothing$, then Ξ_J is not compatible with the natural Galois action on the sets of components, and hence does not descend to a morphism over $\mathbf{F}_J$. One can then consider whether there is a natural description of the obstruction, for example viewed as a class in $H^1(\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_J), \mathrm{Aut}_{\overline{\mathbf{F}}_p}(\overline{Y}_0(p)_J))$.

5.4. **Comparison of vector bundles.** In this section, we will relate the Raynaud bundles on $\overline{Y}_0(p)_J$ to the tautological line bundles on $\prod_{\theta\in\Sigma} \mathbf{P}_{\overline{Y}_\Sigma}(\mathcal{V}_\theta)$ and automorphic vector bundles on $\overline{Y}_\Sigma$.

5.4.1. *The unitary setting.* We first prove an analogous result in the context of Theorem 5.2.2. As in the construction of the morphisms $\widetilde{\Psi}_J$ and $\widetilde{\Xi}_J$ in §5.2.2, we let A_J denote the J-splice of the universal isogeny $f : A_1 \to A_2$ over the closed subscheme S'_J of $\widetilde{Y}_{U'_0(p)}(G')_\mathbf{F}$. We write H for the Raynaud group scheme $e_0 \ker(f)$

on S'_J, and let $\pi : H \to S'_J$, $s_J : A_J \to S'_J$ and $s_j : A_j \to S'_J$ for $j = 1, 2$ denote the structure morphisms.

Recall that for $\tau \in \Theta_E$, the rank two automorphic bundle $\widetilde{\mathcal{V}}^0_\tau$ on $\widetilde{Y}_{U'_\Sigma}(G'_\Sigma)_{\mathbf{F}}$ is defined in §3.1.2 in terms of the de Rham cohomology of the universal abelian scheme; its determinant bundle is denoted $\widetilde{\delta}_\tau$, and if $\tau|_F \notin \Sigma$, then the Hodge filtration on $\widetilde{\mathcal{V}}^0_\tau$ is given by the line bundles $\widetilde{\omega}^0_{\tilde\theta}$ and $\widetilde{\upsilon}^0_{\tilde\theta}$.

The Raynaud bundles $\mathcal{L}_\tau$ on S'_J are defined by the identification

$$H = \mathbf{Spec}\left(\left(\mathrm{Sym}_{\mathcal{O}_{S'_J}}\left(\oplus_{\tau\in\Theta_E}\mathcal{L}_\tau\right)\right)/\langle(s_\tau - 1)\mathcal{L}^p_\tau\rangle_{\tau\in\Theta_E}\right).$$

Recall also from Lemma 5.1.1 and its proof that if $\theta \notin J$, then $s_{\phi^{-1}\circ\tilde\theta} = 0$ and $\mathrm{Lie}(H/S'_J)_{\tilde\theta} = \mathrm{Lie}(A_1/S'_J)^0_{\tilde\theta}$, so we have canonical isomorphisms

$$\mathcal{L}_{\tilde\theta} = (\pi_*\Omega^1_{H/S'_J})_{\tilde\theta} = (s_{1,*}\Omega^1_{A_1/S'_J})^0_{\tilde\theta}.$$

Similarly if $\theta \in J$, then $t_{\phi^{-1}\circ\tilde\theta} = 0$ and $\mathrm{Lie}(H^\vee/S'_J)_{\tilde\theta} = \mathrm{Lie}(A_2^\vee/S'_J)^0_{\tilde\theta}$, giving canonical isomorphisms

$$\mathcal{L}_{\tilde\theta} = \mathrm{Lie}(H^\vee/S'_J)_{\tilde\theta} = (R^1 s_{2,*}\mathcal{O}_{A_2})^0_{\tilde\theta}.$$

Next recall from the construction of the morphisms

$$\begin{array}{ccc} S'_J & \xrightarrow{\tilde\Xi_J} & \prod_{\theta\in\Sigma}\mathbf{P}_S(\widetilde{\mathcal{V}}^0_{\tilde\theta}) \\ & \searrow^{\widetilde{\Psi}_J} & \downarrow \\ & & S = \widetilde{Y}_{U'_\Sigma}(G'_\Sigma)_{\mathbf{F}}, \end{array}$$

that Corollary 5.1.3 yields an isomorphism

$$\mathcal{H}^1_{\mathrm{dR}}(A_j/S'_J)^0_{\tilde\theta} \longrightarrow \mathcal{H}^1_{\mathrm{dR}}(A_J/S'_J)^0_{\tilde\theta} = \widetilde{\Psi}^*_J\widetilde{\mathcal{V}}^0_{\tilde\theta}, \tag{16}$$

where $j = 2$ or 1 according to whether or not $\theta \in J$. Furthermore, if $\theta \notin \Sigma$, then the isomorphism is compatible with the Hodge filtration and so induces isomorphisms

$$(s_{j,*}\Omega^1_{A_j/S'_J})^0_{\tilde\theta} \cong \widetilde{\Psi}^*_J\widetilde{\omega}^0_{\tilde\theta} \quad \text{and} \quad (R^1 s_{j,*}\Omega^1_{A_j/S'_J})^0_{\tilde\theta} \cong \widetilde{\Psi}^*_J\widetilde{\upsilon}^0_{\tilde\theta} \cong \widetilde{\Psi}^*_J\widetilde{\delta}_{\tilde\theta} \otimes_{\mathcal{O}_{S'_J}} \widetilde{\Psi}^*_J(\widetilde{\omega}^0_{\tilde\theta})^{-1}.$$

On the other hand if $\theta \in \Sigma$, then the Hodge filtration on $\mathcal{H}^1_{\mathrm{dR}}(A_j/S'_J)^0_{\tilde\theta}$ defines the morphism $S'_J \to \mathbf{P}_S(\widetilde{\mathcal{V}}^0_{\tilde\theta})$ and hence induces isomorphisms

$$(s_{j,*}\Omega^1_{A_j/S'_J})^0_{\tilde\theta} \cong \widetilde{\Psi}^*_J\widetilde{\delta}_{\tilde\theta} \otimes_{\mathcal{O}_{S'_J}} \tilde\Xi^*_J\mathcal{O}(-1)_\theta \quad \text{and} \quad (R^1 s_{j,*}\Omega^1_{A_j/S'_J})^0_{\tilde\theta} \cong \tilde\Xi^*_J\mathcal{O}(1)_\theta.$$

We thus obtain an isomorphism

$$\mathcal{L}_{\tilde\theta} \xrightarrow{\sim} \tilde\Xi^*_J\mathcal{M}_\theta, \quad \mathcal{M}_\theta := \begin{cases} \widetilde{\omega}_{\tilde\theta}, & \text{if } \theta \notin J,\ \theta \notin \Sigma; \\ \widetilde{\delta}^0_{\tilde\theta}(-1)_\theta, & \text{if } \theta \notin J,\ \theta \in \Sigma; \\ \widetilde{\delta}^0_{\tilde\theta}(\widetilde{\omega}_{\tilde\theta})^{-1}, & \text{if } \theta \in J,\ \theta \notin \Sigma; \\ \mathcal{O}(1)_\theta, & \text{if } \theta \in J,\ \theta \in \Sigma, \end{cases} \tag{17}$$

where we write $\widetilde{\delta}_{\tilde\theta}$ and $\widetilde{\omega}^0_{\tilde\theta}$ for their pull-back to $\prod_{\theta\in\Sigma}\mathbf{P}_S(\widetilde{\mathcal{V}}^0_{\tilde\theta})$, $\mathcal{O}(1)_\theta$ for the pull-back of the twisting sheaf on the θ-component of the product, and $\mathcal{F}(n)_\theta$ for the twist of $\mathcal{F}$ by $\mathcal{O}(n)_\theta = \mathcal{O}(1)^n_\theta$.

The isomorphism of (17) is Hecke-equivariant in the following sense. Suppose as usual that $U_1', U_2' \subset G'(\mathbf{A_f})$ are as above and $g \in G'(\mathbf{A}_\mathbf{f}^{(p)})$ is such that $g^{-1}U_1'g \subset U_2'$, It is then straightforward to check that the resulting diagram

$$
\begin{array}{ccc}
\tilde{\rho}_g^*\mathcal{L}_{2,\tilde{\theta}} & \xrightarrow{\tilde{\rho}_g^*(\sigma_2)} & \tilde{\rho}_g^*\tilde{\Xi}_{2,J}^*\mathcal{M}_{2,\theta} = \tilde{\Xi}_{1,J}^*\tilde{\rho}_{g_\Sigma}^*\mathcal{M}_{2,\theta} \\
\downarrow{\scriptstyle \pi_g^*} & & \downarrow{\scriptstyle \tilde{\Xi}_{1,J}^*(\pi_{g_\Sigma}^*)} \\
\mathcal{L}_{1,\tilde{\theta}} & \xrightarrow[\sigma_1]{} & \tilde{\Xi}_{1,J}^*\mathcal{M}_{1,\theta},
\end{array}
\tag{18}
$$

where $\sigma_i : \mathcal{L}_{i,\tilde{\theta}} \to \tilde{\Xi}_{i,J}^*\mathcal{M}_{\theta,i}$ is the isomorphism of (17) with $U' = U_i'$ for $i = 1, 2$, $\pi_g^* : \tilde{\rho}_g^*\mathcal{L}_{2,\tilde{\theta}} \xrightarrow{\sim} \mathcal{L}_{1,\tilde{\theta}}$ is defined at the end of §4.2.4, and $\pi_{g_\Sigma}^* : \tilde{\rho}_{g_\Sigma}^*\mathcal{M}_{2,\tilde{\theta}} \xrightarrow{\sim} \mathcal{M}_{1,\tilde{\theta}}$ is induced by the isomorphisms $\tilde{\rho}_{g_\Sigma}^*\tilde{\mathcal{V}}_{2,\tilde{\theta}}^0 \to \tilde{\mathcal{V}}_{1,\tilde{\theta}}^0$, with determinant $\tilde{\rho}_{g_\Sigma}^*\tilde{\delta}_{2,\tilde{\theta}} \to \tilde{\delta}_{1,\tilde{\theta}}$ and restriction $\tilde{\rho}_{g_\Sigma}^*\tilde{\omega}_{2,\tilde{\theta}}^0 \to \tilde{\omega}_{1,\tilde{\theta}}^0$ if $\theta \notin \Sigma$, all denoted $\pi_{g_\Sigma}^*$ in §3.1.2.

The isomorphism of (17) is also compatible with the descent data associated to the action of $\mathcal{O}_{F,(p),+}$ (for sufficiently small U'), and hence descends to an isomorphism $\sigma : \mathcal{L}_{\tilde{\theta}} \to (\Xi_J')^*\mathcal{M}_\theta$ on $\overline{Y}_0(p)_J'$ in the context of Theorem 5.2.2, where the line bundle $\mathcal{M}_\theta$ on $\prod_{\theta\in\Sigma}\mathbf{P}_{\overline{Y}_\Sigma'}(\mathcal{V}_{\tilde{\theta}}^0)$ is defined by the same formula, but using the automorphic bundles $\omega_{\tilde{\theta}}^0$ and $\delta_{\tilde{\theta}}$ on $\overline{Y}_\Sigma'$. Furthermore the resulting isomorphism is Hecke equivariant in the sense that the diagram analogous to (18) commutes.

5.4.2. *The Hilbert setting.* To obtain the desired relation on $\overline{Y}_0(p)_J$, recall that the isomorphism Ξ_J of Theorem 5.3.1 is defined as defined as the restriction of the composite $\Xi_J' \circ \rho_u$ for suitably chosen $u \in (\mathbf{A}_{E,\mathbf{f}}^{(p)})^\times$ (independent of U) and U'. We thus obtain an isomorphism

$$
\mathcal{L}_\theta \xrightarrow{\sim} \Xi_J^*\mathcal{M}_\theta, \quad \mathcal{M}_\theta := \begin{cases} \omega_\theta, & \text{if } \theta \notin J,\ \theta \notin \Sigma; \\ \delta_\theta(-1)_\theta, & \text{if } \theta \notin J,\ \theta \in \Sigma; \\ \delta_\theta\omega_\theta^{-1}, & \text{if } \theta \in J,\ \theta \notin \Sigma; \\ \mathcal{O}(1)_\theta, & \text{if } \theta \in J,\ \theta \in \Sigma \end{cases}
\tag{19}
$$

as the restriction to $\overline{Y}_0(p)_J$ of the composite of the isomorphisms

$$
\mathcal{L}_\theta \xrightarrow{(\pi_u^*)^{-1}} \rho_u^*\mathcal{L}_\theta \xrightarrow{\rho_u^*\sigma} \rho_u^*(\Xi_J')^*\mathcal{M}_\theta.
$$

Furthermore if U_1 and U_2 are sufficiently small open compact subgroups of $G(\mathbf{A_f})$ and $g \in G(\mathbf{A}_\mathbf{f}^{(p)})$ is such that $g^{-1}U_1g \subset U_2$, then it is straightforward to check that the resulting diagram

$$
\begin{array}{ccc}
\rho_g^*\mathcal{L}_{2,\theta} & \xrightarrow{\rho_g^*(\sigma_2)} & \rho_g^*\Xi_{2,J}^*\mathcal{M}_{2,\theta} = \Xi_{1,J}^*\rho_{g_\Sigma}^*\mathcal{M}_{2,\theta} \\
\downarrow{\scriptstyle \pi_g^*} & & \downarrow{\scriptstyle \Xi_{1,J}^*(\pi_{g_\Sigma}^*)} \\
\mathcal{L}_{1,\theta} & \xrightarrow[\sigma_1]{} & \Xi_{1,J}^*\mathcal{M}_{1,\theta}
\end{array}
\tag{20}
$$

commutes.

5.4.3. *Relation of determinant bundles.* We also relate the bundles denoted δ_θ on the varieties $\overline{Y}$ and $\overline{Y}_\Sigma$. Again we first make the comparison in the context of the unitary Shimura varieties $\overline{Y}'$ and $\overline{Y}'_\Sigma$. Note that (16) yields an isomorphism

$$\wedge^2_{\mathcal{O}_{S'_J}} \mathcal{H}^1_{\mathrm{dR}}(A_j/S'_J)^0_{\tilde\theta} \longrightarrow \widetilde{\Psi}^*_J \tilde\delta_{\tilde\theta}$$

where $j = 2$ or 1 according to whether or not $\theta \in J$, and that the first line bundle is simply the pull-back of $\tilde\delta_{\tilde\theta}$ on $\widetilde{Y}_{U'}(G')_{\mathbf{F}}$ under the forgetful morphism $\tilde\pi_j$ sending $(\underline{A}_1, \underline{A}_2, f)$ to $\underline{A}_j$. Furthermore the isomorphism descends to an isomorphism on $\overline{Y}_0(p)'_J$, restricting to an isomorphism

$$\pi_j^*\delta_\theta \xrightarrow{\sim} \Psi_J^*\delta_\theta$$

on $\overline{Y}_0(p)_J$, where the first δ_θ is on $\overline{Y}$, the second is on $\overline{Y}_\Sigma$, and π_j is the morphism $\overline{Y}_0(p)_J$ obtained from $\tilde\pi_j$.

On the other hand, letting g denote the transpose of f, so $f \circ g = p$, we have the exact sequences

$$\begin{array}{rcccccccc} & 0 & \to & \ker(f^*) & \to & \mathcal{H}^1_{\mathrm{dR}}(A_2/S'_J) & \to & \ker(g^*) & \to & 0 \\ \text{and} & 0 & \to & \ker(g^*) & \to & \mathcal{H}^1_{\mathrm{dR}}(A_1/S'_J) & \to & \ker(f^*) & \to & 0, \end{array}$$

of right $M_2(\mathcal{O}_E) \otimes \mathcal{O}_{S'_J}$-modules, to which we may apply idempotents to obtain the exact sequences

$$\begin{array}{rcccccccc} & 0 & \to & \ker(f^*)^0_{\tilde\theta} & \to & \mathcal{H}^1_{\mathrm{dR}}(A_2/S'_J)^0_{\tilde\theta} & \to & \ker(g^*)^0_{\tilde\theta} & \to & 0 \\ \text{and} & 0 & \to & \ker(g^*)^0_{\tilde\theta} & \to & \mathcal{H}^1_{\mathrm{dR}}(A_1/S'_J)^0_{\tilde\theta} & \to & \ker(f^*)^0_{\tilde\theta} & \to & 0. \end{array}$$

Since $\ker(f^*)^0_{\tilde\theta}$ and $\ker(g^*)^0_{\tilde\theta}$ are invertible $\mathcal{O}_{S'_J}$-modules, this yields an isomorphism

$$\tilde\pi_2^*\tilde\delta_{\tilde\theta} \quad\cong\quad \ker(f^*)^0_{\tilde\theta} \otimes_{\mathcal{O}_{S'_J}} \ker(g^*)^0_{\tilde\theta} \quad\cong\quad \tilde\pi_1^*\tilde\delta_{\tilde\theta},$$

which in turn descends to $\overline{Y}_0(p)'_J$ and restricts to an isomorphism

$$\pi_2^*\delta_\theta \cong \pi_1^*\delta_\theta$$

on $\overline{Y}_0(p)_J$. Again abusing notation and writing δ_θ for its pull-back to $\prod_{\theta\in\Sigma} \mathbf{P}_{\overline{Y}_\Sigma}(\mathcal{V}_\theta)$, we have now constructed isomorphisms

$$\pi_j^*\delta_\theta \xrightarrow{\sim} \Xi_J^*\delta_\theta$$

for all $\theta \in \Theta$, $J \subset \Theta$ and $j = 1, 2$. Furthermore it is straightforward to check that the isomorphisms are Hecke-equivariant in the same sense as (20).

Finally, we note that just as in [DS17, §3.4], the line bundle $\bigotimes_{\theta\in\Theta} \delta_\theta$ on $\widetilde{Y}_U(G)_{\mathcal{O}}$ is equipped with a canonical trivialization which descends to $Y_U(G)_{\mathcal{O}}$ and transforms by $||\det g||^{-1}$ under π_g^* (for $g \in G(\mathbf{A}_{\mathbf{f}}^{(p)})$ and varying U). It follows that the same holds for $\bigotimes_{\theta\in\Theta} \delta_\theta$ on $\overline{Y}$, and this in turn pulls back to such trivializations of $\pi_j^*\left(\bigotimes_{\theta\in\Theta} \delta_\theta\right)$ for $j = 1$ and 2, which one can check are compatible with the isomorphism $\pi_2^*\left(\bigotimes_{\theta\in\Theta} \delta_\theta\right) \cong \pi_1^*\left(\bigotimes_{\theta\in\Theta} \delta_\theta\right)$ constructed above. Applying $\Xi_{J,*}$, we obtain a trivialization of $\Psi_J^*\left(\bigotimes_{\theta\in\Theta} \delta_\theta\right)$ which transforms by $||\det g_\Sigma||^{-1}$ under $\pi^*_{g_\Sigma}$ (for $g \in G_\Sigma(\mathbf{A}_{\mathbf{f}}^{(p)})$ varying U_Σ). Since the natural map $\mathcal{O}_{\overline{Y}_\Sigma} \to \Psi_{J,*}\mathcal{O}_X$ is an isomorphism for $X = \prod_{\theta\in\Sigma} \mathbf{P}_{\overline{Y}_\Sigma}(\mathcal{V}_\theta)$, this in fact yields a trivialization of $\bigotimes_{\theta\in\Theta} \delta_\theta$ on $\overline{Y}_\Sigma$ which transforms by $||\det g_\Sigma||^{-1}$ under $\pi^*_{g_\Sigma}$.

6. The Serre filtration

6.1. **Dualizing sheaves.** Our aim now is to analyze the dualizing sheaf on the special fibre of $Y_{U_1(p)}(G)$.

6.1.1. *Generalities.* We begin with some general properties of dualizing sheaves, all of which are recorded in [StaX, Chapter 46].

First recall that if $f : Y \to S$ is a Cohen–Macaulay morphism of constant relative dimension n, then f admits a relative dualizing sheaf, which we denote $\mathcal{K}_{Y/S}$. We will write this as $\mathcal{K}_{Y/R}$ if $S = \operatorname{Spec} R$, and simply as $\mathcal{K}_Y$ if R is a field evident from the context. Then $\mathcal{K}_{Y/S}$ is a coherent sheaf on Y, flat over S. If moreover f is Gorenstein (i.e., all fibres are Gorenstein), then $\mathcal{K}_{Y/S}$ is invertible. If moreover f is smooth, then $\mathcal{K}_{Y/S}$ is identified with the canonical sheaf $\Omega^n_{Y/S} = \wedge^n_{\mathcal{O}_Y}\Omega^1_{Y/S}$. We recall also that formation of the dualizing sheaf commutes with base-change; i.e., if $f : Y \to S$ is Cohen-Macaulay of (constant relative) dimension n, and $S' \to S$ is any morphism, then letting Y' denote $Y \times_S S'$ and $\pi : Y' \to Y$ the projection, we have a canonical isomorphism $\pi^*\mathcal{K}_{Y/S} \cong \mathcal{K}_{Y'/S'}$.

If $h : X \to Y$ and $f : Y \to S$ are both Cohen-Macaulay of constant relative dimension and either f or h is Gorenstein, then $\mathcal{K}_{X/S}$ is canonically isomorphic to $\mathcal{K}_{X/Y} \otimes_{\mathcal{O}_X} h^*\mathcal{K}_{Y/S}$. In particular if h is étale, then $\mathcal{K}_{X/S} = h^*\mathcal{K}_{Y/S}$.

Suppose now that $h : X \to Y$ is finite and that $f : Y \to S$ and $f \circ h : X \to S$ are Cohen–Macaulay of the same relative dimension. Then $h_*\mathcal{K}_{X/S}$ is canonically isomorphic to $\mathcal{H}om_{\mathcal{O}_Y}(h_*\mathcal{O}_X, \mathcal{K}_{Y/S})$. This can be seen as an immediate consequence of the Duality Theorem, or indeed as a special case of the construction of dualizing sheaves. We will apply this in §6.2.2 in the case where h is a closed immersion and both f and $f \circ h$ are local complete intersections. However we will first apply it in §6.1.4 in a setting where h is finite flat and f is Cohen–Macaulay; note that in this case $f \circ h$ is automatically Cohen–Macaulay of the same relative dimension as f.

6.1.2. *The Kodaira–Spencer isomorphism.* Recall from §3.2.1 that the Kodaira–Spencer isomorphism on Shimura varieties associated to G_Σ relates their dualizing sheaves to certain automorphic bundles. We defined this isomorphism (and indeed the integral models and automorphic bundles) via the Shimura varieties associated to the unitary group G'_Σ, but for $G = \operatorname{Res}_{F/\mathbf{Q}} \operatorname{GL}_2$ (i.e., $\Sigma = \varnothing$), we can establish the relation directly from the moduli problem considered in §2.1.2. We briefly recall this now, but working integrally instead of modulo p as we did in §3.2.1.

Suppose that $L \subset \overline{\mathbf{Q}}$ contains the Galois closure of F, and let $\mathcal{O}$ be the localization of $\mathcal{O}_L$ at the prime over p determined by our fixed embedding $\overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_p$, and let A be the universal abelian variety over $S = \widetilde{Y}_U(G)_{\mathcal{O}}$ where U is a sufficiently small open compact subgroup of $\operatorname{GL}_2(\mathbf{A}_{F,\mathbf{f}})$ of level prime to p. Proceeding exactly as in §3.1.1 we can define line bundles $\tilde{\omega}_\theta$, $\tilde{v}_\theta$ and $\tilde{\delta}_\theta$ on S by the Hodge filtration and determinant on the θ-component of $\mathcal{H}^1_{\mathrm{dR}}(A/S)$ for each $\theta \in \Theta$. For $k, \ell \in \mathbf{Z}^\Theta$, we define

$$\widetilde{\mathcal{A}}_{k,l} = \bigotimes_{\theta\in\Theta} \tilde{\delta}_\theta^{\ell_\theta}\tilde{\omega}_\theta^{k_\theta}.$$

If (k, ℓ) is paritious in the sense of [DS17, Def. 3.2.1], i.e., $k_\theta + 2\ell_\theta$ is independent of θ, then $\widetilde{\mathcal{A}}_{k,\ell}$ descends to a line bundle $\mathcal{A}_{k,\ell}$ on $Y = Y_U(G)_{\mathcal{O}}$; in particular the line bundles $\tilde{v}_\theta^{-1}\tilde{\omega}_\theta = \tilde{\delta}_\theta^{-1}\tilde{\omega}_\theta^2$ descend to $Y = Y_U(G)_{\mathcal{O}}$. The same construction as in

§3.2.1 then produces isomorphisms

$$\bigoplus_{\theta\in\Theta}\tilde{\delta}_\theta^{-1}\tilde{\omega}_\theta^2 \xrightarrow{\sim} \Omega^1_{S/\mathcal{O}} \quad\text{and}\quad \bigotimes_{\theta\in\Theta}\tilde{\delta}_\theta^{-1}\tilde{\omega}_\theta^2 \xrightarrow{\sim} \mathcal{K}_{S/\mathcal{O}}$$

which descend to isomorphisms on Y.[14] We thus obtain the Kodaira–Spencer isomorphism $\mathcal{A}_{2,-1} \cong \mathcal{K}_{Y/\mathcal{O}}$. Using the commutativity of the diagram

$$\begin{array}{ccccccccc} 0 & \to & (s')^*\Omega^1_{S/\mathcal{O}} & \to & \Omega^1_{A'/\mathcal{O}} & \to & \Omega^1_{A'/S} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \Delta_* s^*\Omega^1_{S/\mathcal{O}} & \to & \Delta_*\Omega^1_{A/\mathcal{O}} & \to & \Delta_*\Omega^1_{A/S} & \to & 0 \end{array}$$

where $A' = A\otimes_{\mathcal{O}_F}\mathcal{O}_E^2$, $s : A \to S$, $s' : A' \to S$ are the structure morphisms, $\Delta : A \to A'$ is the diagonal embedding and the vertical arrows are adjoint to the canonical morphisms, one sees that the base-change to $\mathbf{F} = \overline{\mathbf{F}}_p$ of the Kodaira–Spencer isomorphism just defined coincides with the one already defined on $\overline{Y} = Y_U(G)_{\mathbf{F}}$ in §3.2.1. On the other hand, working over L instead of $\mathcal{O}$, one can dispense with the hypothesis that the open compact subgroup has level prime to p. For any sufficiently small open compact subgroup U, the analogous constructions yield automorphic bundles $\mathcal{A}_{k,\ell}$ on $Y_U(G)_L$ for paritious (k,ℓ) and the Kodaira–Spencer isomorphism $\mathcal{A}_{2,-1}\cong\mathcal{K}_{Y_U(G)_L}$.

6.1.3. *Hecke action.* Suppose now that U_1 and U_2 are sufficiently small open compact subgroups of $\mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}})$ of level prime to p and that $g\in\mathrm{GL}_2(\mathbf{A}^{(p)}_{F,\mathbf{f}})$ is such that $g^{-1}U_1g\subset U_2$. Letting $Y_j = Y_{U_j}(G)_{\mathcal{O}}$ for $j=1,2$, we have a morphism $\rho_g : Y_1\to Y_2$ and a canonical isomorphism $\rho_g^*\mathcal{K}_{Y_1/\mathcal{O}}\to\mathcal{K}_{Y_2/\mathcal{O}}$, satisfying the usual compatibility with $\rho_h^*\mathcal{K}_{Y_2/\mathcal{O}}\to\mathcal{K}_{Y_3/\mathcal{O}}$ if $h^{-1}U_2h\subset U_3$. We thus obtain a natural action of $\mathrm{GL}_2(\mathbf{A}^{(p)}_{F,\mathbf{f}})$ on

$$\varinjlim H^0(Y_U(G)_{\mathcal{O}},\mathcal{K}_{Y_U(G)_{\mathcal{O}}/\mathcal{O}}),$$

where the direct limit is over open compact subgroups $U\subset\mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}})$ of level prime to p.

For any paritious (k,ℓ), we similarly define an action of $\mathrm{GL}_2(\mathbf{A}^{(p)}_{F,\mathbf{f}})$ on the space of Hilbert modular forms (over $\mathcal{O}$) of weight (k,ℓ) and level prime to p, i.e., the direct limit over such U of the $\mathcal{O}$-modules

$$M_{k,\ell}(U,\mathcal{O}) := H^0(Y_U(G)_{\mathcal{O}},\mathcal{A}_{k,\ell}),$$

incorporating a twist by the character $||\det||$ as in [DS17, §4.2] for consistency with standard conventions. As recalled in §5.4.3, the line bundle $\widetilde{\mathcal{A}}_{0,1} = \otimes_\theta\tilde{\delta}_\theta$ on $\widetilde{Y}_U(G)_{\mathcal{O}}$ is equipped with a canonical trivialization which descends to $Y_U(G)_{\mathcal{O}}$ and transforms by $||\det g||^{-1}$ under π_g^*. We thus obtain an isomorphism $\mathcal{A}_{2,0}\cong\mathcal{A}_{2,-1}$ whose composite with the Kodaira–Spencer isomorphism $\mathcal{A}_{2,-1}\cong\mathcal{K}_{Y/\mathcal{O}}$ is compatible with the Hecke action in the sense that the resulting diagram

$$\begin{array}{ccc} M_{2,0}(U_2,\mathcal{O}) & \xrightarrow{\sim} & H^0(Y_2,\mathcal{K}_{Y_2/\mathcal{O}}) \\ \downarrow & & \downarrow \\ M_{2,0}(U_1,\mathcal{O}) & \xrightarrow{\sim} & H^0(Y_1,\mathcal{K}_{Y_1/\mathcal{O}}) \end{array}$$

[14] In fact identifying $\oplus\tilde{v}_\theta^{-1}\tilde{\omega}_\theta$ with $\mathcal{H}om_{\mathcal{O}_S\otimes\mathcal{O}_F}(R^1s_*\mathcal{O}_A, s_*\Omega^1_{A/S})$, the vector bundles and isomorphisms descend to $Y_U(G)$ over $\mathbf{Z}_{(p)}$.

commutes, and hence the isomorphism

$$\varinjlim_U M_{2,0}(U,\mathcal{O}) \cong \varinjlim_U H^0(Y_U(G)_{\mathcal{O}}, \mathcal{K}_{Y_U(G)_{\mathcal{O}}/\mathcal{O}})$$

is $\mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}}^{(p)})$-equivariant.

Similarly for any $k,\ell \in \mathbf{Z}^{\Theta}$ (not necessarily paritious) and $\mathbf{F}$ containing the residue field of $\mathcal{O}$, we have an action of $\mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}}^{(p)})$ on the space

$$\varinjlim_U M_{k,\ell}(U,\mathbf{F}) = \varinjlim_U H^0(Y_U(G)_{\mathbf{F}}, \mathcal{A}_{k,\ell})$$

of Hilbert modular forms over $\mathbf{F}$ of weight (k,ℓ) and level prime to p. Note that for any paritious (k,ℓ), the natural map

$$M_{k,\ell}(U,\mathcal{O}) \otimes_{\mathcal{O}} \mathbf{F} \to M_{k,\ell}(U,\mathbf{F})$$

is Hecke-equivariant and injective, but not obviously surjective, the cokernel being isomorphic to $\mathrm{Tor}_1^{\mathcal{O}}(H^1(Y_U(G)_{\mathcal{O}}, \mathcal{A}_{k,\ell}), \mathbf{F})$. The Kodaira–Spencer isomorphism in this context gives an isomorphism

$$\varinjlim_U M_{2,0}(U,\mathbf{F}) \cong \varinjlim_U H^0(Y_U(G)_{\mathbf{F}}, \mathcal{K}_{Y_U(G)_{\mathbf{F}}}).$$

compatible with the one over $\mathcal{O}$ and the action of $\mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}}^{(p)})$.

Similarly for paritious (k,ℓ) and fields K containing L, we have an action of $\mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}})$ on the space

$$\varinjlim_U M_{k,\ell}(U,K) = \varinjlim_U H^0(Y_U(G)_K, \mathcal{A}_{k,\ell})$$

of Hilbert modular forms over K of weight (k,ℓ), where the limit is now over all sufficiently small open compact $U \subset \mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}})$, and $M_{k,\ell}(U,\mathbf{C})$ is identified with the usual space of holomorphic Hilbert modular forms of weight (k,ℓ) and level U. The Kodaira–Spencer isomorphism in this context gives a $\mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}})$-equivariant isomorphism

$$\varinjlim_U M_{2,0}(U,K) \cong \varinjlim_U H^0(Y_U(G)_K, \mathcal{K}_{Y_U(G)_K}).$$

If U has level prime to p, then the natural map

$$M_{k,\ell}(U,\mathcal{O}) \otimes_{\mathcal{O}} K \to M_{k,\ell}(U,K)$$

is a Hecke-equivariant isomorphism, compatible with the Kodaira–Spencer isomorphism for $(k,\ell) = (2,0)$.

6.1.4. *Forms of parallel weight* 2*, level* $U_1(p)$*.* Our main object of study for the rest of the paper will be the dualizing sheaf on the special fibre of the Pappas model $Y_{U_1(p)}(G)$, defined in §4.1.2. Recall that $U_1(p) = U^pU_{1,p}$ where U^p is a sufficiently small open compact subgroup of $\mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}}^{(p)})$ and $U_{1,p} \subset U_{0,p} \subset \mathrm{GL}_2(\mathcal{O}_{F,p})$ with $U_0(p)/U_1(p) \cong (\mathcal{O}_F/p)^{\times}$. We will write simply $Y_i(p)$ for $Y_{U_i(p)}(G)_{\mathcal{O}}$ for $i = 1,2$. Since $Y_0(p)$ is a flat local complete intersection over $\mathcal{O}$ and $Y_1(p)$ is finite flat over $Y_0(p)$, it follows that $Y_1(p)$ is Cohen–Macaulay over $\mathcal{O}$. Therefore $Y_0(p)$ and $Y_1(p)$ admit dualizing sheaves $\mathcal{K}_{Y_0(p)/\mathcal{O}}$ and $\mathcal{K}_{Y_1(p)/\mathcal{O}}$, and $\mathcal{K}_{Y_0(p)/\mathcal{O}}$ is invertible.

If U and U' are sufficiently small open compact subgroups of $\mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}})$ of level prime to p and that $g \in \mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}}^{(p)})$ is such that $g^{-1}Ug \subset U'$, then we have the finite étale morphisms $\rho_g : Y_i(p) \to Y_i'(p) = Y_{U_i(p)}(G)_{\mathcal{O}}$ and canonical isomorphisms $\rho_g^*\mathcal{K}_{Y_i'(p)/\mathcal{O}} \to \mathcal{K}_{Y_i(p)/\mathcal{O}}$ for $i = 0, 1$ satisfying the usual compatibilities. We thus obtain a natural action of $\mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}}^{(p)})$ on

$$\varinjlim H^0(Y_i(p), \mathcal{K}_{Y_i(p)/\mathcal{O}}),$$

where the direct limit is over open compact subgroups $U \subset \mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}})$ of level prime to p. Since

$$H^0(Y_i(p), \mathcal{K}_{Y_i(p)/\mathcal{O}}) \otimes_{\mathcal{O}} L = H^0(Y_{U_i(p)}(G)_L, \mathcal{K}_{Y_{U_i(p)}(G)_L}) \cong M_{2,0}(U_i(p), L),$$

we may identify $H^0(Y_i(p), \mathcal{K}_{Y_i(p)/\mathcal{O}})$ with an $\mathcal{O}$-lattice in $M_{2,0}(U_i(p), L)$ which we denote $M_{2,0}(U_i(p), \mathcal{O})$; the identification is Hecke-equivariant in the sense that it is compatible with the $\mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}}^{(p)})$-action on direct limits. Similarly we may consider the dualizing sheaf $\mathcal{K}_{\overline{Y}_i(p)}$ on $\overline{Y}_i(p) := Y_{U_i(p)}(G)_{\mathbf{F}}$ for $\mathbf{F}$ containing the residue field of $\mathcal{O}$. Letting

$$M_{2,0}(U_i(p), \mathbf{F}) = H^0(\overline{Y}_i(p), \mathcal{K}_{\overline{Y}_i(p)}),$$

we obtain a natural action of $\mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}}^{(p)})$ on $\varinjlim M_{2,0}(U_i(p), \mathbf{F})$ and Hecke-equivariant injective homomorphisms

$$M_{2,0}(U_i(p), \mathcal{O}) \otimes_{\mathcal{O}} \mathbf{F} \to M_{2,0}(U_i(p), \mathbf{F}).$$

Recall also that we have a natural action of $(\mathcal{O}_F/p)^\times \cong U_0(p)/U_1(p)$ on $Y_1(p)$, hence on $M_{2,0}(U_1(p), \mathcal{O}) = H^0(Y_1(p), \mathcal{K}_{Y_1(p)/\mathcal{O}})$. Furthermore the action is compatible with the Hecke action (in the usual sense) and with the natural action of $(\mathcal{O}_F/p)^\times$ on $M_{2,0}(U_1(p), L)$. If L is sufficiently large as in §4.1.2 (i.e., L contains the $(q-1)$-roots of unity where $q-1$ is divisible by the exponent of $(\mathcal{O}_F/p)^\times$), then we may decompose

$$M_{2,0}(U_1(p), \mathcal{O}) = \bigoplus_{\chi} M_{2,0}(U_1(p), \mathcal{O})^\chi$$

as the direct sum of χ-eigenspaces over the characters $\chi : (\mathcal{O}_F/p)^\times \to \mathcal{O}^\times$. Thus $M_{2,0}(U_1(p), \mathcal{O})^\chi$ is an $\mathcal{O}$-lattice in the space $M_{2,0}(U_1(p), L)^\chi$ of Hilbert modular forms of parallel weight 2, level $U_1(p)$ and character χ, and $\mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}}^{(p)})$ acts on $\varinjlim_U M_{2,0}(U_1(p), \mathcal{O})^\chi$ where the direct limit is over sufficiently small open compact U of level prime to p. Similarly we may decompose

$$M_{2,0}(U_1(p), \mathbf{F}) = \bigoplus_{\chi} M_{2,0}(U_1(p), \mathbf{F})^\chi,$$

and we obtain an action of $\mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}}^{(p)})$ on $\varinjlim M_{2,0}(U_1(p), \mathbf{F})^\chi$ compatible with the injective maps

$$M_{2,0}(U_1(p), \mathcal{O})^\chi \otimes_{\mathcal{O}} \mathbf{F} \to M_{2,0}(U_1(p), \mathbf{F})^\chi.$$

Since the morphism $h : Y_1(p) \to Y_0(p)$ is finite, we have

$$h_*\mathcal{K}_{Y_1(p)/\mathcal{O}} = \mathcal{H}om_{\mathcal{O}_{Y_0(p)}}(h_*\mathcal{O}_{Y_1(p)}, \mathcal{K}_{Y_0(p)/\mathcal{O}})$$

and $R^j h_*\mathcal{K}_{Y_1(p)/\mathcal{O}} = 0$ for $j > 0$. Recall that the explicit description of $Y_1(p)$ in terms of a Raynaud $(\mathcal{O}_F/p)$-module scheme on $Y_0(p)$ shows that

$$h_*\mathcal{O}_{Y_1(p)} = (\mathrm{Sym}_{\mathcal{O}_{Y_0(p)}}\mathcal{L})/\mathcal{I}'$$

where $\mathcal{L}$ is the direct of the Raynaud line bundles $\mathcal{L}_\theta$ on $Y_0(p)$ and $\mathcal{I}'$ is the ideal generated by the $\mathcal{O}_S$-submodules $(s_\theta - 1)\mathcal{L}_\theta^p$ for $\theta \in \Theta$ and $(s_v - 1) \otimes_{\theta\in\Theta_v} \mathcal{L}_\theta^{p-1}$ for $v|p$. We therefore have

$$h_*\mathcal{O}_{Y_1(p)} = \bigoplus \left(\bigotimes_{\theta\in\Theta} \mathcal{L}_\theta^{m_\theta} \right)$$

where the direct sum is over $m \in \mathbf{Z}^\Theta$ such $0 \le m_\theta \le p-1$ for all θ and $m_\theta < p-1$ for some θ in each Θ_v. Since $(\mathcal{O}_F/p)^\times$ acts on $\mathcal{L}_\theta$ via the Teichmuller lift of $\overline{\theta}$ and each character χ of $(\mathcal{O}_F/p)^\times$ is the Teichmuller lift of $\prod_{\theta\in\Theta} \overline{\theta}^{m_{\chi,\theta}}$ for a unique such $m = m_\chi$, we may rewrite this decomposition as

$$h_*\mathcal{O}_{Y_1(p)} = \bigoplus_\chi \mathcal{L}_\chi$$

where $(\mathcal{O}_F/p)^\times$ acts on the line bundle $\mathcal{L}_\chi := \mathcal{L}_\theta^{m_{\chi,\theta}}$ via χ. We therefore have

$$h_*\mathcal{K}_{Y_1(p)/\mathcal{O}} = \bigoplus_\chi \mathcal{L}_{\chi^{-1}}^{-1} \mathcal{K}_{Y_0(p)/\mathcal{O}}$$

so that $H^i(Y_1(p), \mathcal{K}_{Y_1(p)/\mathcal{O}})^\chi = H^i(Y_0(p), \mathcal{L}_{\chi^{-1}}^{-1}\mathcal{K}_{Y_0(p)/\mathcal{O}})$ for $i \ge 0$, and in particular

$$M_{2,0}(U_1(p), \mathcal{O})^\chi = H^0(Y_0(p), \mathcal{L}_{\chi^{-1}}^{-1}\mathcal{K}_{Y_0(p)/\mathcal{O}}).$$

Furthermore these identifications are compatible with the natural Hecke action arising from the identifications $\rho_g^*\mathcal{K}_{Y_j'(p)/\mathcal{O}} = \mathcal{K}_{Y_j(p)/\mathcal{O}}$ for $j = 0, 1$ and the isomorphism $\pi_g^* : \rho_g^*\mathcal{L}_\theta' \cong \mathcal{L}_\theta$ defined at the end of §4.1.2 (where $g \in \mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}}^{(p)})$, $g^{-1}U'g \subset U$, $Y_j'(p) = Y_{U_j'(p)}(G)_\mathcal{O}$ and $\mathcal{L}_\theta'$ is the Raynaud bundle on $Y_0'(p)$).

Similarly if we let $\overline{Y}_j(p) = Y_{U_j(p)}(G)_\mathbf{F}$ for $j = 0, 1$ and $\overline{h} : \overline{Y}_1(p) \to \overline{Y}_0(p)$, then we obtain

$$\overline{h}_*\mathcal{K}_{\overline{Y}_1(p)} = \bigoplus_\chi \mathcal{L}_{\chi^{-1}}^{-1}\mathcal{K}_{\overline{Y}_0(p)} \tag{21}$$

where $\mathcal{L}_\chi = \mathcal{L}_\theta^{m_{\chi,\theta}}$ as above, but now χ is viewed as a character $(\mathcal{O}_F/p\mathcal{O}_F)^\times \to \mathbf{F}^\times$ and $\mathcal{L}_\theta$ is the Raynaud bundle on $\overline{Y}_0(p)$. It follows that $H^i(\overline{Y}_1(p), \mathcal{K}_{\overline{Y}_1(p)})^\chi = H^i(\overline{Y}_0(p), \mathcal{L}_{\chi^{-1}}^{-1}\mathcal{K}_{\overline{Y}_0(p)})$ and in particular

$$M_{2,0}(U_1(p), \mathbf{F})^\chi = H^0(\overline{Y}_0(p), \mathcal{L}_{\chi^{-1}}^{-1}\mathcal{K}_{\overline{Y}_0(p)}),$$

compatibly with the Hecke action and the corresponding identifications over $\mathcal{O}$. Our analysis of the space $M_{2,0}(U_1(p), \mathbf{F})$, and more generally the cohomology of $\mathcal{K}_{\overline{Y}_1(p)}$, thus reduces to the study of the line bundles $\mathcal{L}_{\chi^{-1}}^{-1}\mathcal{K}_{\overline{Y}_0(p)}$ on $\overline{Y}_0(p)$.

6.2. **Dicing.** In this section, we describe the dualizing sheaf $\mathcal{K}_{\overline{Y}_0(p)}$ on $\overline{Y}_0(p)$ in terms of those on the subschemes $\overline{Y}_0(p)_J$. Recall that $\overline{Y}_0(p)_J$ is defined as the vanishing locus of

$$\{\, \mathrm{Lie}(f)_\theta \,|\, \theta \notin J \,\} \cup \{\, \mathrm{Lie}(f^\vee)_\theta \,|\, \theta \in J \,\}$$

where $f : A_1 \to A_2$ is the universal isogeny on $\widetilde{Y}_{U_0(p)}(G)_\mathbf{F}$; the schemes $\overline{Y}_0(p)_J$ are smooth over $\mathbf{F}$, and every irreducible component of $\overline{Y}_0(p)$ is a connected component of $\overline{Y}_0(p)$ for a unique J. We will define a filtration on $\mathcal{K}_{\overline{Y}_0(p)}$ whose associated graded is a direct sum of sheaves supported on the subschemes $\overline{Y}_0(p)_J$. We refer to the process of analyzing the sheaf by dividing the scheme into smaller ones of the same dimension as *dicing*.

6.2.1. *Dicing the structure sheaf.* We first define a filtration on $\mathcal{O}_{\overline{Y}_0(p)}$ by the sheaves of ideals

$$\mathcal{I}_j := \langle \prod_{\theta\in J} \mathrm{Lie}(f)_\theta \rangle_{|J|=j}$$

for $j \geq 0$, where $\mathrm{Lie}(f)_\theta$ is viewed locally as a section of $\mathcal{O}_{\overline{Y}_0(p)}$ via any choice of trivialization of the descent of

$$\mathcal{H}om_{\mathcal{O}_S}(\mathrm{Lie}(A_1/S)_\theta, \mathrm{Lie}(A_2/S)_\theta)$$

to $\overline{Y}_0(p)$. Note that

$$\mathcal{O}_{\overline{Y}_0(p)} = \mathcal{I}_0 \supset \mathcal{I}_1 \supset \cdots \supset \mathcal{I}_d \supset \mathcal{I}_{d+1} = 0.$$

For each J, let i_J denote the closed immersion $\overline{Y}_0(p)_J \to \overline{Y}_0(p)$. We let $\mathcal{P}_J$ denote the ideal sheaf on $\overline{Y}_0(p)$ defining $\overline{Y}_0(p)_J$, and let $\mathcal{I}_J$ denote the ideal sheaf on $\overline{Y}_0(p)_J$ generated by $\prod_{\theta\in J}\mathrm{Lie}(f)_\theta$. We see from Theorem 4.3.1 that the ideal generated by $\prod_{\theta\in J}\mathrm{Lie}(f)_\theta$ is non-zero in the regular local ring obtained by completing $\overline{Y}_0(p)_J$ at each closed point, and it follows that $\mathcal{I}_J$ is an invertible sheaf. Similarly we let $\mathcal{J}_J$ denote the invertible sheaf of ideals on $\overline{Y}_0(p)_J$ generated by $\prod_{\theta\notin J}\mathrm{Lie}(f^\vee)_\theta$.

Note that if $|J| = |J'|$ and $J' \neq J$, then the section $\prod_{\theta\in J'}\mathrm{Lie}(f)_\theta$ vanishes on $\overline{Y}_0(p)_J$, so we have a natural map $\mathcal{I}_j \to \bigoplus_{|J|=j} i_{J,*}\mathcal{I}_J$. Furthermore if $|J'| = j+1$, then $\prod_{\theta\in J'}\mathrm{Lie}(f)_\theta$ vanishes, so $\mathcal{I}_{j+1}$ is contained in the kernel.

Lemma 6.2.1. *The natural map* $\mathrm{gr}^j\mathcal{O}_{\overline{Y}_0(p)} \to \bigoplus_{|J|=j} i_{J,*}\mathcal{I}_J$ *is an isomorphism.*

Proof. It suffices to prove the morphism is an isomorphism on completions of stalks at each closed point Q of $\overline{Y}_0(p)$. We do this using the description

$$\widehat{\mathcal{O}}_{S,Q} \cong \widehat{\bigotimes_{\theta\in\Theta}} k_Q[[x_\theta,y_\theta]]/\langle c_\theta d_\theta\rangle$$

provided by Theorem 4.3.1. Using the notation of that theorem and writing c_J for $\prod_{\theta\in J}c_\theta$ and $\bar{\cdot}$ for images in the quotient $k_Q[[x_\theta,y_\theta]]/\langle c_\theta d_\theta\rangle$, the above morphism corresponds to the natural map

$$\begin{array}{ccc} I_j/I_{j+1} & \to & \bigoplus_{|J|=j}(\langle \bar{c}_J\rangle + P_J)/P_J \\ \left(\sum_{|J|=j} r_J\bar{c}_J\right) + I_{j+1} & \mapsto & (r_J\bar{c}_J + P_J)_J\,, \end{array} \tag{22}$$

where $I_m = \langle \bar{c}_J\rangle_{|J|=m}$ and $P_J = \langle \bar{c}_\theta\rangle_{\theta\notin J} + \langle \bar{d}_\theta\rangle_{\theta\in J}$. Note that $Q \in \overline{Y}_0(p)_J$ if and only if $J \subset J_Q$ and $I_Q \cup J = \Theta$, in which case P_J corresponds to a minimal prime of $\widehat{\mathcal{O}}_{S,Q}$; otherwise P_J corresponds to $\widehat{\mathcal{O}}_{S,Q}$.

Since (22) is obviously surjective, we just need to prove that if $r_J\bar{c}_J \in P_J$, then $r_J\bar{c}_J \in I_{j+1}$. If $\theta \notin I_Q$ for some $\theta\notin J$, then $\bar{c}_\theta$ is a unit, so $r_J\bar{c}_J \in \langle \bar{c}_{J\cup\{\theta\}}\rangle \subset I_{j+1}$, and if $\theta\notin J_Q$ for some $\theta \in J$, then $\bar{c}_\theta = 0$, so $r_J\bar{c}_J = 0 \in I_{j+1}$. We may therefore assume that P_J corresponds to a prime ideal of $\widehat{\mathcal{O}}_{\overline{Y}_0(p),Q}$. Note that since $\bar{c}_\theta \notin P_J$ for each $\theta\in J$, we have $\bar{c}_J \notin P_J$, and therefore $r_J \in P_J$. Furthermore $\bar{c}_J\bar{d}_\theta = 0$ for $\theta \in J$, so

$$r_J\bar{c}_J \in \bar{c}_J P_J = \bar{c}_J\langle \bar{c}_\theta\rangle_{\theta\notin J} \subset I_{j+1}.$$

This completes the proof that (22) is an isomorphism. □

Note that the proof that (22) is an isomorphism shows that $I_j = \bigcap_{|J|<j} P_J$, which translates to the statement that $\mathcal{I}_j = \bigcap_{|J|<j} \mathcal{P}_J$, hence defines $\bigcup_{|J|<j} \overline{Y}_0(p)_J$ with its reduced induced subscheme structure. Note in particular that $\langle \overline{c}_\Theta \rangle = I_d = \bigcap_{J \neq \Theta} P_J$; furthermore the same argument shows that if $J \subset \Theta$, then

$$\langle \overline{c}_J \overline{d}_{\Theta - J} \rangle = \bigcap_{J' \neq J} P_{J'}. \tag{23}$$

Since the ring is reduced and all its minimal primes are of the form P_J, it follows that $\langle \overline{c}_J \overline{d}_{\Theta - J} \rangle$ is the annihilator of P_J and that $\langle \overline{c}_J \overline{d}_{\Theta - J} \rangle \cap P_J = \{0\}$.

6.2.2. *Dicing the dualizing sheaf.* Recall from §6.1.1 that since $i_J : \overline{Y}_0(p)_J \to \overline{Y}_0(p)$ is a finite morphism of Cohen–Macaulay schemes of the same dimension, we have a canonical isomorphism

$$i_{J,*}\mathcal{K}_{\overline{Y}_0(p)_J} \cong \mathcal{H}om_{\mathcal{O}_{\overline{Y}_0(p)}}(i_{J,*}\mathcal{O}_{\overline{Y}_0(p)_J}, \mathcal{K}_{\overline{Y}_0(p)}).$$

Moreover since $\overline{Y}_0(p)_J$ and $\overline{Y}_0(p)$ are local complete intersections, and hence Gorenstein, their dualizing sheaves are line bundles; in particular we have

$$\mathcal{H}om_{\mathcal{O}_{\overline{Y}_0(p)}}(i_{J,*}\mathcal{O}_{\overline{Y}_0(p)_J}, \mathcal{K}_{\overline{Y}_0(p)}) = \mathcal{H}om_{\mathcal{O}_{\overline{Y}_0(p)}}(i_{J,*}\mathcal{O}_{\overline{Y}_0(p)_J}, \mathcal{O}_{\overline{Y}_0(p)}) \otimes_{\mathcal{O}_{\overline{Y}_0(p)}} \mathcal{K}_{\overline{Y}_0(p)}.$$

The surjective morphism $\mathcal{O}_{\overline{Y}_0(p)} \to \mathcal{O}_{\overline{Y}_0(p)}/\mathcal{P}_J = i_{J,*}\mathcal{O}_{\overline{Y}_0(p)_J}$ induces an injective morphism

$$\mathcal{H}om_{\mathcal{O}_{\overline{Y}_0(p)}}(i_{J,*}\mathcal{O}_{\overline{Y}_0(p)_J}, \mathcal{O}_{\overline{Y}_0(p)}) \to \mathcal{O}_{\overline{Y}_0(p)}$$

whose image is the sheaf of ideals $\mathcal{A}nn_{\mathcal{O}_{\overline{Y}_0(p)}}\mathcal{P}_J$. It follows from (23) and the subsequent discussion that $\mathcal{A}nn_{\mathcal{O}_{\overline{Y}_0(p)}}\mathcal{P}_J$ is generated by $\prod_{\theta \in J} \mathrm{Lie}(f)_\theta \prod_{\theta \notin J} \mathrm{Lie}(f^\vee)_\theta$, so that

$$i_J^* \mathcal{A}nn_{\mathcal{O}_{\overline{Y}_0(p)}}\mathcal{P}_J = \mathcal{I}_J \mathcal{J}_J;$$

moreover since $\mathcal{P}_J \cap \mathcal{A}nn_{\mathcal{O}_{\overline{Y}_0(p)}}\mathcal{P}_J = 0$, the natural map

$$\mathcal{A}nn_{\mathcal{O}_{\overline{Y}_0(p)}}\mathcal{P}_J \to i_{J,*}i_J^*\mathcal{A}nn_{\mathcal{O}_{\overline{Y}_0(p)}} = i_{J,*}(\mathcal{I}_J\mathcal{J}_J)$$

is an isomorphism. Tensoring with $\mathcal{K}_{\overline{Y}_0(p)}$ therefore yields a canonical isomorphism

$$i_{J,*}\mathcal{K}_{\overline{Y}_0(p)_J} \xrightarrow{\sim} i_{J,*}(\mathcal{I}_J\mathcal{J}_J) \otimes_{\mathcal{O}_{\overline{Y}_0(p)}} \mathcal{K}_{\overline{Y}_0(p)}. \tag{24}$$

Now consider the *dicing filtration* on $\mathcal{K}_{\overline{Y}_0(p)}$ defined by

$$\mathrm{Fil}^j \mathcal{K}_{\overline{Y}_0(p)} = \mathcal{I}_j \mathcal{K}_{\overline{Y}_0(p)} = \mathcal{I}_j \otimes_{\mathcal{O}_{\overline{Y}_0(p)}} \mathcal{K}_{\overline{Y}_0(p)},$$

where the latter equality follows from the invertibility of $\mathcal{K}_{\overline{Y}_0(p)}$. Lemma 6.2.1 thus yields a canonical isomorphism

$$\mathrm{gr}^j \mathcal{K}_{\overline{Y}_0(p)} \xrightarrow{\sim} \bigoplus_{|J|=j} i_{J,*}\mathcal{I}_J \otimes_{\mathcal{O}_{\overline{Y}_0(p)}} \mathcal{K}_{\overline{Y}_0(p)}.$$

We may then apply (24) to compute the summands as

$$i_{J,*}\mathcal{J}_J^{-1} \otimes_{\mathcal{O}_{\overline{Y}_0(p)}} i_{J,*}(\mathcal{I}_J\mathcal{J}_J) \otimes_{\mathcal{O}_{\overline{Y}_0(p)}} \mathcal{K}_{\overline{Y}_0(p)} = i_{J,*}(\mathcal{J}_J^{-1}\mathcal{K}_{\overline{Y}_0(p)_J}).$$

We therefore have a canonical isomorphism

$$\mathrm{gr}^j \mathcal{K}_{\overline{Y}_0(p)} \cong \bigoplus_{|J|=j} i_{J,*}(\mathcal{J}_J^{-1}\mathcal{K}_{\overline{Y}_0(p)_J}). \tag{25}$$

It is straightforward to check that the dicing filtration is compatible with the Hecke action in the usual sense, as is the isomorphism (25). Combining this isomorphism with the conclusion of §6.1.4, we see that the analysis of the cohomology of $\mathcal{K}_{\overline{Y}_1(p)}$ reduces to that of the line bundles $\mathcal{J}_J^{-1}\mathcal{K}_{\overline{Y}_0(p)_J} i_J^* \mathcal{L}_{\chi^{-1}}^{-1}$ on the smooth varieties $\overline{Y}_0(p)_J$.

More precisely, we define a filtration on each component of $\overline{h}_*\mathcal{K}_{\overline{Y}_1(p)}$ under the decomposition (21) by setting

$$\mathrm{Fil}^j(\mathcal{K}_{\overline{Y}_0(p)}\mathcal{L}_{\chi^{-1}}^{-1}) = \mathcal{I}_j \mathcal{K}_{\overline{Y}_0(p)}\mathcal{L}_{\chi^{-1}}^{-1}.$$

The filtration induces one on $H^i(\overline{Y}_1(p), \mathcal{K}_{\overline{Y}_1(p)})^\chi = H^i(\overline{Y}_0(p), \mathcal{K}_{\overline{Y}_0(p)}\mathcal{L}_{\chi^{-1}}^{-1})$ defined by $\mathrm{Fil}^j(H^i(\overline{Y}_0(p), \mathcal{K}_{\overline{Y}_0(p)}\mathcal{L}_{\chi^{-1}}^{-1}) =$

$$\mathrm{im}\left(H^i(\overline{Y}_0(p), \mathrm{Fil}^j(\mathcal{K}_{\overline{Y}_0(p)}\mathcal{L}_{\chi^{-1}}^{-1})) \to H^i(\overline{Y}_0(p), \mathcal{K}_{\overline{Y}_0(p)}\mathcal{L}_{\chi^{-1}}^{-1})\right),$$

for which the graded pieces $E_\infty^{j,i} = E_{d+1}^{j,i} = \mathrm{gr}^j\left(H^{i+j}(\overline{Y}_0(p), \mathcal{K}_{\overline{Y}_0(p)}\mathcal{L}_{\chi^{-1}}^{-1})\right)$ are computed by the spectral sequence

$$E_1^{j,i} = H^{i+j}(\overline{Y}_0(p), \mathrm{gr}^j(\mathcal{K}_{\overline{Y}_0(p)}\mathcal{L}_{\chi^{-1}}^{-1})) \Longrightarrow H^{i+j}(\overline{Y}_0(p), \mathcal{K}_{\overline{Y}_0(p)}\mathcal{L}_{\chi^{-1}}^{-1}).$$

Note that (25) gives a canonical isomorphism

$$\mathrm{gr}^j(\mathcal{K}_{\overline{Y}_0(p)}\mathcal{L}_{\chi^{-1}}^{-1}) \cong \bigoplus_{|J|=j}\left(i_{J,*}(\mathcal{J}_J^{-1}\mathcal{K}_{\overline{Y}_0(p)_J}) \otimes_{\mathcal{O}_{\overline{Y}_0(p)}} \mathcal{L}_{\chi^{-1}}^{-1}\right),$$

from which it follows that

$$E_1^{j,i} \cong \bigoplus_{|J|=j} H^{i+j}(\overline{Y}_0(p)_J,\ \mathcal{J}_J^{-1}\mathcal{K}_{\overline{Y}_0(p)_J} i_J^*\mathcal{L}_{\chi^{-1}}^{-1}).$$

6.2.3. *Hecke equivariance.* The above spectral sequence and isomorphisms are compatible with the Hecke action in the following sense. Suppose as usual that U and U' are sufficiently small open compact subgroups of $G(\mathbf{A_f})$ of level prime to p, and $g \in G(\mathbf{A}_\mathbf{f}^{(p)})$ is such that $g^{-1}Ug \subset U'$. Using $'$ to denote the relevant objects defined with U replaced by U', we have the corresponding spectral sequence

$$E_1'^{j,i} = H^{i+j}(\overline{Y}_0'(p), \mathrm{gr}^j(\mathcal{K}_{\overline{Y}_0'(p)}\mathcal{L}'^{-1}_{\chi^{-1}})) \Longrightarrow H^{i+j}(\overline{Y}_0'(p), \mathcal{K}_{\overline{Y}_0'(p)}\mathcal{L}'^{-1}_{\chi^{-1}})$$

and isomorphism

$$E_1'^{j,i} \cong \bigoplus_{|J|=j} H^{i+j}(\overline{Y}_0'(p)_J,\ \mathcal{J}_J'^{-1}\mathcal{K}_{\overline{Y}_0'(p)_J} i_J'^*\mathcal{L}'^{-1}_{\chi^{-1}}).$$

The canonical isomorphism $\rho_g^*\mathcal{K}_{\overline{Y}_1'(p)} \cong \mathcal{K}_{\overline{Y}_1(p)}$ is compatible with the decompositions (21) and corresponding isomorphisms of summands (with $\rho_g^*\mathcal{L}'^{-1}_{\chi^{-1}} \cong \mathcal{L}_{\chi^{-1}}^{-1}$ defined by π_g^*). As $\rho_g^*\mathcal{I}_j' = \mathcal{I}_j$, the isomorphisms of summands preserve the filtrations and hence induce morphisms $E_r'^{j,i} \to E_r^{j,i}$ compatible with the differentials $d_r^{j,i} : E_r^{j,i} \to E_r^{j+r,i-r+1}$, $d_r'^{j,i} : E_r'^{j,i} \to E_r'^{j+r,i-r+1}$ and identifications $E_{r+1}^{j,i} = \ker(d_r^{j,i})/\mathrm{im}(d_r^{j-r,i+r-1})$, $E_{r+1}'^{j,i} = \ker(d_r'^{j,i})/\mathrm{im}(d_r'^{j-r,i+r-1})$. Furthermore the descriptions of $\mathrm{gr}^j\mathcal{K}_{\overline{Y}_0(p)}$ and $\mathrm{gr}^j\mathcal{K}_{\overline{Y}_0'(p)}$ are compatible with ρ_g^* in the obvious sense, so the resulting diagram

$$\begin{array}{ccc} E_1^{\prime j,i} & \xrightarrow{\sim} & \bigoplus_{|J|=j} H^{i+j}(\overline{Y}'_0(p)_J,\ \mathcal{J}_J^{\prime -1}\mathcal{K}_{\overline{Y}'_0(p)_J} i_J^{\prime *}\mathcal{L}'^{-1}_{\chi^{-1}}) \\ \downarrow & & \downarrow \\ E_1^{j,i} & \xrightarrow{\sim} & \bigoplus_{|J|=j} H^{i+j}(\overline{Y}_0(p)_J,\ \mathcal{J}_J^{-1}\mathcal{K}_{\overline{Y}_0(p)_J} i_J^{*}\mathcal{L}^{-1}_{\chi^{-1}}) \end{array}$$

commutes, where the arrow on the right is induced by the canonical isomorphisms $\rho_g^*\mathcal{J}'_J \cong \mathcal{J}_J$, $\rho_g^*\mathcal{K}_{\overline{Y}'_0(p)_J} \cong \mathcal{K}_{\overline{Y}_0(p)_J}$ and $\pi_g^* : \rho_g^*\mathcal{L}'_{\chi^{-1}} \xrightarrow{\sim} \mathcal{L}_{\chi^{-1}}$. To sum up so far, we have constructed a Hecke-equivariant spectral sequence with

$$E_1^{j,i} \cong \bigoplus_{|J|=j} H^{i+j}(\overline{Y}_0(p)_J,\ \mathcal{J}_J^{-1}\mathcal{K}_{\overline{Y}_0(p)_J} i_J^*\mathcal{L}^{-1}_{\chi^{-1}}) \Longrightarrow H^{i+j}(\overline{Y}_1(p), \mathcal{K}_{\overline{Y}_1(p)})^\chi. \tag{26}$$

6.3. **Proof of Theorem B.** We now proceed to prove Theorem B, relating the cohomology of $\mathcal{K}_{\overline{Y}_1(p)}$ to that of automorphic sheaves on the quaternionic Shimura varieties $\overline{Y}_\Sigma$. We do this by determining the line bundles corresponding to the $\mathcal{J}_J^{-1}\mathcal{K}_{\overline{Y}_0(p)_J} i_J^*\mathcal{L}_\chi^{-1}$ under the isomorphism Ξ_J of Theorem 5.3.1, and in turn their (higher) direct images on the $\overline{Y}_\Sigma$. (Recall that $\mathcal{J}_J$ and i_J were defined in §6.2.1, and $\mathcal{L}_\chi$ in §6.1.4.) We now assume $\mathbf{F} = \overline{\mathbf{F}}_p$ to ensure that Ξ_J is defined.

6.3.1. *The factors $\mathcal{J}$, $\mathcal{K}$ and $\mathcal{L}$.* We first write $\chi = \prod_{\theta\in\overline{\Theta}_p} \theta^{m_\theta}$ for a unique $m = m_\theta \in \mathbf{Z}^\Theta$ such that $0 \le m_\theta \le p-1$ for all θ and $m_\theta > 0$ for some θ in each Θ_v. (Recall that we identify $\overline{\Theta}_p$ with $\Theta = \coprod_{v|p}\Theta_v$ via our fixed choices of embeddings, and note the slight difference with the choice made in §6.1.4 in that we take $m_\theta = p-1$, instead of 0, for all $\theta \in \Theta_v$ if χ is trivial on $(\mathcal{O}_F/v)^\times$.) We then have $\chi^{-1} = \prod_{\theta\in\overline{\Theta}_p} \theta^{p-1-m_\theta}$, so that

$$\mathcal{L}^{-1}_{\chi^{-1}} = \bigotimes_{\theta\in\Theta} \mathcal{L}_\theta^{m_\theta - p+1}. \tag{27}$$

Next we write the factor $\mathcal{J}_J^{-1}$ in terms of the Raynaud bundles on $\overline{Y}_0(p)_J$. Recall that $\mathcal{J}_J$ is the ideal sheaf on $\overline{Y}_0(p)_J$ defined by $\prod_{\theta\notin J}\mathrm{Lie}(f^\vee)_\theta$. Though we make no direct use of the following description, we remark that $\mathcal{J}_J^{-1}$ can be identified with the line bundle $\mathcal{O}_{\overline{Y}_0(p)_J}(D)$ where

$$D = \sum_{\theta\notin\Theta}\left(\overline{Y}_0(p)_J \cap \overline{Y}_0(p)_{J\cup\{\theta\}}\right) = \sum_{\theta\notin\Theta} \overline{Y}_0(p)_{\Theta-\phi(J),J\cup\{\theta\}},$$

viewed as a Weil divisor on $\overline{Y}_0(p)_J$. Recall that $\mathrm{Lie}(f^\vee)_\theta$ is a section of the line bundle defined by

$$\mathcal{H}om_{\mathcal{O}_S}(\mathrm{Lie}(A_2^\vee/S)_\theta, \mathrm{Lie}(A_1^\vee/S)_\theta)$$

with its canonical descent data, where $f : A_1 \to A_2$ is the universal isogeny on S of $\widetilde{Y}_{U_0(p)}(G)_{\mathbf{F}}$. Letting $s_j : A_j \to S$ denote the structure morphism for $j = 1,2$, we have canonical isomorphisms

$$\mathrm{Lie}(A_j^\vee/S)_\theta \cong \tilde{v}_{j,\theta} \cong \tilde{\delta}_{j,\theta}\tilde{\omega}_{j,\theta}^{-1},$$

where $\tilde{v}_{j,\theta} = R^1 s_{j,*}\mathcal{O}_{A_j}$, $\tilde{\omega}_{j,\theta}^{-1} = s_{j,*}(\Omega^1_{A_j/S})_\theta$ and $\tilde{\delta}_{j,\theta} = \wedge^2_{\mathcal{O}_S}\mathcal{H}^1_{\mathrm{dR}}(A_j/S)_\theta$. As in §5.4.3, the exact sequences

$$\begin{array}{lccccccccc} & 0 & \to & \ker(f^*) & \to & \mathcal{H}^1_{\mathrm{dR}}(A_2/S) & \to & \ker(g^*) & \to & 0 \\ \text{and} & 0 & \to & \ker(g^*) & \to & \mathcal{H}^1_{\mathrm{dR}}(A_1/S) & \to & \ker(f^*) & \to & 0 \end{array}$$

(where $g : A_2 \to A_1$ denotes the transpose of f, i.e., $f \circ g = p$) yield an isomorphism $\tilde{\delta}_{1,\theta} \cong \tilde{\delta}_{2,\theta}$. We thus obtain an isomorphism

$$\begin{aligned} \mathcal{H}om_{\mathcal{O}_S}(\tilde{v}_{2,\theta}, \tilde{v}_{1,\theta}) &\cong \mathcal{H}om_{\mathcal{O}_S}(\tilde{\delta}_{2,\theta}\tilde{\omega}_{2,\theta}^{-1}, \tilde{\delta}_{1,\theta}\tilde{\omega}_{1,\theta}^{-1}) \\ &\cong \mathcal{H}om_{\mathcal{O}_S}(\tilde{\delta}_{1,\theta}\tilde{\omega}_{2,\theta}^{-1}, \tilde{\delta}_{1,\theta}\tilde{\omega}_{1,\theta}^{-1}) \cong \mathcal{H}om_{\mathcal{O}_S}(\tilde{\omega}_{1,\theta}, \tilde{\omega}_{2,\theta}) \end{aligned}$$

under which $\mathrm{Lie}(f^\vee)$ corresponds to g^*.

Suppose now that $\theta \notin J$, let $\tilde{i}_J$ denote the closed immersion $S_J \to S$, and consider the restriction of g^* to S_J. Recall from the proof of Lemma 5.1.1 that the inclusion $H = \ker(f) \subset A_1$ induces an isomorphism $\mathrm{Lie}(H/S_J)_\theta \cong \mathrm{Lie}(A_1/S_J)_\theta$ and hence

$$\tilde{i}_J^*\tilde{\omega}_{1,\theta} \cong \tilde{i}_J^*\mathcal{L}_\theta.$$

Furthermore if also $\phi^{-1} \circ \theta \notin J$, then we similarly have $\tilde{i}_J^*\tilde{\omega}_{1,\phi^{-1}\circ\theta} \cong \tilde{i}_J^*\mathcal{L}_{\phi^{-1}\circ\theta}$, and since $\mathrm{Ver}^*(f^*\mathcal{H}^1_{\mathrm{dR}}(A_2/S_J)) = f^*(\tilde{i}_J^*\tilde{\omega}^p_{2,\phi^{-1}\circ\theta}) = 0$ in this case, we have the exact sequence

$$0 \longrightarrow f^*\mathcal{H}^1_{\mathrm{dR}}(A_2/S_J) \longrightarrow \mathcal{H}^1_{\mathrm{dR}}(A_1/S_J) \xrightarrow{\mathrm{Ver}^*} \tilde{i}_J^*\tilde{\omega}^p_{1,\phi^{-1}\circ\theta} \longrightarrow 0,$$

which combined with the exact sequence

$$0 \longrightarrow f^*\mathcal{H}^1_{\mathrm{dR}}(A_2/S_J) \longrightarrow \mathcal{H}^1_{\mathrm{dR}}(A_1/S_J) \xrightarrow{g^*} \tilde{i}_J^*\tilde{\omega}_{2,\theta} \longrightarrow 0$$

yields an isomorphism

$$\tilde{i}_J^*\tilde{\omega}_{2,\theta} \cong \tilde{i}_J^*\tilde{\omega}^p_{1,\phi^{-1}\circ\theta} \cong \tilde{i}_J^*\mathcal{L}^p_{\phi^{-1}\circ\theta}.$$

On the other hand if $\phi^{-1} \circ \theta \in J$, then the proof of Lemma 5.1.1 shows that the inclusion $H^\vee \subset A_2^\vee$ induces an isomorphism $\mathrm{Lie}(H^\vee/S_J)_\theta \cong \mathrm{Lie}(A_2^\vee/S_J)_\theta$, and hence $\tilde{i}_J^*\mathcal{L}_{\phi^{-1}\circ\theta} \cong \tilde{i}_J^*\tilde{v}_{2,\phi^{-1}\circ\theta}$. In this case however, the fact that $\phi^{-1} \circ \theta \in J$ and $\theta \notin J$ implies that

$$\Phi(\mathbf{D}(A_{2,\overline{s}})_{\phi^{-1}\circ\theta}) = g^*(\mathbf{D}(A_{1,\overline{s}}))_\theta = V(\mathbf{D}(A_{2,\overline{s}})_{\phi\circ\theta})$$

for all $\overline{s} \in S_J(\overline{\mathbf{F}}_p)$. It follows that $\mathrm{Frob}^* : \mathcal{H}^1_{\mathrm{dR}}(A_2/S_J)^{(p)}_{\phi^{-1}\circ\theta} \to \mathcal{H}^1_{\mathrm{dR}}(A_2/S_J)_\theta$ induces an isomorphism $\tilde{i}_J^*\tilde{v}^p_{2,\phi^{-1}\circ\theta} \to \tilde{i}_J^*\tilde{\omega}_{2,\theta}$, so that now

$$\tilde{i}_J^*\mathcal{L}^p_{\phi^{-1}\circ\theta} \cong \tilde{i}_J^*\tilde{v}^p_{2,\phi^{-1}\circ\theta} \cong \tilde{i}_J^*\tilde{\omega}_{2,\theta}.$$

Therefore in both cases we obtain an isomorphism

$$\tilde{i}_J^*(\tilde{\omega}_{1,\theta}^{-1}\tilde{\omega}_{2,\theta}) \cong \tilde{i}_J^*(\mathcal{L}_\theta^{-1}\mathcal{L}^p_{\phi^{-1}\circ\theta}).$$

We now show that the section corresponding to g^* under this isomorphism is simply the one induced on the Raynaud line bundles by $\mathrm{Ver} : H^{(p)} \to H$; i.e., the diagram

$$\begin{array}{ccc} \tilde{i}_J^*\tilde{\omega}_{1,\theta} & \xrightarrow{\sim} & \mathcal{L}_\theta \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle \mathrm{Ver}^*} \\ \tilde{i}_J^*\tilde{\omega}_{2,\theta} & \cong & \mathcal{L}^p_{\phi^{-1}\circ\theta} \end{array}$$

commutes, where the top arrow is induced by the inclusion $H \subset A_1$ and bottom arrow is the isomorphism defined above. It suffices to check the commutativity on fibres at all $\overline{s} \in S_J(\overline{\mathbf{F}}_p)$, which we do by writing the resulting diagram in terms of the Dieudonné modules of $H_{\overline{s}}$, $A_{1,\overline{s}}[p]$ and $A_{2,\overline{s}}[p]$. More precisely, under the canonical descriptions of cotangent and tangent spaces in terms of Dieudonné modules, the diagram on fibres becomes

$$\begin{array}{ccccc} \mathbf{D}(A_{1,\overline{s}}^{(p^{-1})}[p])_\theta/\Phi(\mathbf{D}(A_{1,\overline{s}}^{(p^{-1})}[p])_{\phi^{-1}\circ\theta}) & & \xrightarrow{\sim} & & \mathbf{D}(H_{\overline{s}}^{(p^{-1})})_\theta\ , \\ {\scriptstyle g^*}\downarrow & & & & \downarrow{\scriptstyle \mathrm{Ver}^*} \\ \mathbf{D}(A_{2,\overline{s}}^{(p^{-1})}[p])_\theta/\Phi(\mathbf{D}(A_{2,\overline{s}}^{(p^{-1})}[p])_{\phi^{-1}\circ\theta}) & \cong & \Delta & \cong & \mathbf{D}(H_{\overline{s}})_\theta \end{array}$$

where the top arrow is given by the inclusion $H_{\overline{s}} \subset A_{1,\overline{s}}[p]$ and the bottom isomorphisms are the ones induced by those defined above, where

- $\Delta = \mathbf{D}(A_{1,\overline{s}}[p])_\theta/\Phi(\mathbf{D}(A_{1,\overline{s}}[p])_{\phi^{-1}\circ\theta})$ if $\phi^{-1}\circ\theta \notin J$, and
- $\Delta = \ker(\Phi) = \ker(V)$ on $\mathbf{D}(A_{2,\overline{s}}[p])_\theta$ if $\phi^{-1}\circ\theta \in J$.

In the first case, the desired compatibility is immediate from the fact that the isomorphism $\Delta \to \mathbf{D}(H_{\overline{s}})_\theta$ is again defined by the inclusion $H_{\overline{s}} \subset A_{1,\overline{s}}[p]$ and the isomorphism $\mathbf{D}(A_{2,\overline{s}}^{(p^{-1})}[p])_\theta/\Phi(\mathbf{D}(A_{2,\overline{s}}^{(p^{-1})}[p])_{\phi^{-1}\circ\theta}) \cong \Delta$ is the one arising from the commutative diagram

$$\begin{array}{ccccc} & & H^0(A_{1,\overline{s}}, \Omega^1_{A_{1,\overline{s}}/\overline{\mathbf{F}}_p})_\theta & & \\ & {\scriptstyle g^*}\swarrow & & \searrow{\scriptstyle \mathrm{Ver}^*} & \\ H^0(A_{2,\overline{s}}, \Omega^1_{A_{2,\overline{s}}/\overline{\mathbf{F}}_p})_\theta & & \cong & & H^0(A_{1,\overline{s}}^{(p)}, \Omega^1_{A_{1,\overline{s}}^{(p)}/\overline{\mathbf{F}}_p})_\theta \end{array}$$

and the canonical isomorphisms $\mathbf{D}(A[p])/\Phi(\mathbf{D}(A[p])) \cong H^0(A^{(p)}, \Omega^1_{A^{(p)}/\overline{\mathbf{F}}_p})$ induced by Ver^* on $\mathbf{D}(A[p]) \cong H^1_{\mathrm{dR}}(A/\overline{\mathbf{F}}_p)$ for $A = A_{1,\overline{s}}^{(p^{-1})}$, $A_{2,\overline{s}}^{(p^{-1})}$ and $A_{1,\overline{s}}$. In the second case, one finds that $\mathbf{D}(H_{\overline{s}})_\theta \xrightarrow{\sim} \Delta$ is induced by $g : A_{2,\overline{s}}[p] \to H_{\overline{s}} \subset A_{1,\overline{s}}[p]$, whereas $\Delta \xrightarrow{\sim} \mathbf{D}(A_{2,\overline{s}}^{(p^{-1})}[p])_\theta/\Phi(\mathbf{D}(A_{2,\overline{s}}^{(p^{-1})}[p])_{\phi^{-1}\circ\theta})$ is the composite of

- the inverse of the canonical isomorphism
$$H^1(A_{2,\overline{s}}^{(p)}, \mathcal{O}_{A_{2,\overline{s}}^{(p)}})_\theta \cong \Delta \subset \mathbf{D}(A_{2,\overline{s}}[p])_\theta \cong H^1_{\mathrm{dR}}(A_{2,\overline{s}}/\overline{\mathbf{F}}_p)_\theta$$
induced by $\mathrm{Frob}^* : H^1_{\mathrm{dR}}(A_{2,\overline{s}}^{(p)}/\overline{\mathbf{F}}_p)_\theta \to H^1_{\mathrm{dR}}(A_{2,\overline{s}}/\overline{\mathbf{F}}_p)_\theta$,
- the isomorphism $H^1(A_{2,\overline{s}}^{(p)}, \mathcal{O}_{A_{2,\overline{s}}^{(p)}})_\theta \to H^0(A_{2,\overline{s}}, \Omega^1_{A_{2,\overline{s}}/\overline{\mathbf{F}}_p})_\theta$ induced by Frob^* (in view of the fact that this is indeed the image),

- and the inverse of the canonical isomorphism

$$\mathbf{D}(A_{2,\overline{s}}^{(p^{-1})}[p])_\theta/\Phi(\mathbf{D}(A_{2,\overline{s}}^{(p^{-1})}[p])_{\phi^{-1}\circ\theta}) \cong H^0(A_{2,\overline{s}},\Omega^1_{A_{2,\overline{s}}/\overline{\mathbf{F}}_p})_\theta$$

induced by Ver^* on $\mathbf{D}(A_{2,\overline{s}}^{(p^{-1})}[p]) \cong H^1_{\mathrm{dR}}(A_{2,\overline{s}}^{(p^{-1})}/\overline{\mathbf{F}}_p)$.

It follows that this composite is simply the inverse of an isomorphism induced by $\mathrm{Ver} : A_{2,\overline{s}}[p] \to A_{2,\overline{s}}^{(p^{-1})}[p]$, and the desired compatibility is an immediate consequence.

The isomorphisms

$$\mathcal{H}om_{\mathcal{O}_{S_J}}(\mathrm{Lie}(A_2^\vee/S_J)_\theta, \mathrm{Lie}(A_1^\vee/S_J)_\theta) \cong \tilde{i}_J^*(\tilde{\omega}_{1,\theta}^{-1}\tilde{\omega}_{2,\theta}) \cong \tilde{i}_J^*(\mathcal{L}_\theta^{-1}\mathcal{L}^p_{\phi^{-1}\circ\theta})$$

are compatible with the canonical descent data relative to the cover $S_J \to \overline{Y}_0(p)_J$, and hence descend to isomorphisms of line bundles on $\overline{Y}_0(p)_J$. Furthermore the section defined by the morphism Ver^* (or equivalently $\mathrm{Lie}(f^\vee)_\theta$ or g^*) descends to a section over $\overline{Y}_0(p)_J$, and taking the product over $\theta \notin J$ yields, by the very definition of $\mathcal{J}_J$, an isomorphism

$$\mathcal{J}_J^{-1} \xrightarrow{\sim} \bigotimes_{\theta\notin J} i_J^*(\mathcal{L}_\theta^{-1}\mathcal{L}^p_{\phi^{-1}\circ\theta}). \tag{28}$$

The isomorphism is compatible with the Hecke action in the usual sense: if U and U' are sufficiently small open compact subgroups of $G(\mathbf{A_f})$ of level prime to p and $g \in G(\mathbf{A_f}^{(p)})$ is such that $g^{-1}Ug \subset U'$, then the resulting diagram

$$\begin{array}{ccc} \rho_g^*\mathcal{J}_J'^{-1} & \longrightarrow & \rho_g^* i_J'^*(\mathcal{L}_\theta'^{-1}\mathcal{L}'^p_{\phi^{-1}\circ\theta}) \\ \downarrow & & \downarrow \pi_g^* \\ \mathcal{J}_J^{-1} & \longrightarrow & i_J^*(\mathcal{L}_\theta^{-1}\mathcal{L}^p_{\phi^{-1}\circ\theta}) \end{array}$$

with the obvious notation commutes.

Recall that the line bundles corresponding to the Raynaud bundles under the isomorphism

$$\Xi_J : \overline{Y}_0(p)_J \to \prod_{\theta\in\Sigma} \mathbf{P}_{\overline{Y}_\Sigma}(\mathcal{V}_\theta)$$

of Theorem 5.3.1 were determined in §5.4.2, so now we have the ingredients in place to compute the line bundles corresponding to the factors $\mathcal{J}_J^{-1}$ and $i_J^*\mathcal{L}^{-1}_{\chi^{-1}}$. On the other hand, $\Xi_{J,*}\mathcal{K}_{\overline{Y}_0(p)_J}$ is canonically isomorphic to the dualizing sheaf on $\prod_{\theta\in\Sigma}\mathbf{P}_{\overline{Y}_\Sigma}(\mathcal{V}_\theta)$, which we now compute.

To that end, note that if $X = \mathbf{P}_S(\mathcal{V})$ where $\mathcal{V}$ is a rank two vector bundle on a scheme S, then the dualizing sheaf $\mathcal{K}_{X/S}$ relative to the projection $\psi : X \to S$ is canonically isomorphic to

$$\Omega^1_{X/S} \cong \psi^*(\wedge_{\mathcal{O}_S}\mathcal{V})(-2). \tag{29}$$

It follows that if $X = \prod_{\theta\in\Sigma}\mathbf{P}_{\overline{Y}_\Sigma}(\mathcal{V}_\theta)$ and $\Psi_J : X \to \overline{Y}_\Sigma$ denotes the projection, then $\mathcal{K}_X$ is canonically isomorphic to

$$\Psi_J^*\mathcal{K}_{\overline{Y}_\Sigma} \bigotimes \left(\bigotimes_{\theta\in\Sigma}\Psi_J^*(\delta_\theta)(-2)_\theta\right),$$

where $\delta_\theta = \wedge^2_{\mathcal{O}_{\overline{Y}_\Sigma}}\mathcal{V}_\theta$, all tensor products are over $\mathcal{O}_X$, and as usual, $(n)_\theta$ denotes the twist by $\mathcal{O}(n)_\theta = \mathcal{O}(1)_\theta^n$ in the θ-component. Combining this with the Kodaira–Spencer isomorphism from §3.2.1, we conclude that $\Xi_{J,*}\mathcal{K}_{\overline{Y}_0(p)_J}$ is isomorphic to

$$\mathcal{K}_X \cong \left(\bigotimes_{\theta\notin\Sigma}\delta_\theta^{-1}\omega_\theta^2\right)\bigotimes\left(\bigotimes_{\theta\in\Sigma}\delta_\theta(-2)_\theta\right), \tag{30}$$

where we have again written δ_θ and ω_θ for their pull-back via Ψ_J. Furthermore the compatibility of (29) with isomorphisms of vector bundles and that of the Kodaira–Spencer isomorphism with the Hecke action ensures that (30) is compatible with the Hecke action in the usual sense.

6.3.2. *Jordan-Hölder factors and automorphic bundles.* Before completing our description of the bundles $\Xi_{J,*}(\mathcal{J}_J^{-1}\mathcal{K}_{\overline{Y}_0(p)_J}i_J^*\mathcal{L}_\chi^{-1})$, we recall the explicit formula of Bardoe and Sin [BS00] (as presented in [BP12]) for the Jordan-Hölder factors of the right representation $\mathrm{Ind}_P^{\mathrm{GL}_2(\mathcal{O}_F/p\mathcal{O}_F)}(1\otimes\chi)$, where P is the subgroup of upper-triangular matrices and $1\otimes\chi$ denotes the character $\left(\begin{smallmatrix} a & b\\ 0 & d\end{smallmatrix}\right)\mapsto\chi(d)$.

Theorem 6.3.1. *There exists a decreasing filtration*

$$V_\chi = \mathrm{Fil}^0V_\chi \supset \mathrm{Fil}^1V_\chi \supset\cdots\supset \mathrm{Fil}^{d-1}V_\chi\supset\mathrm{Fil}^dV_\chi\supset\mathrm{Fil}^{d+1}V_\chi = 0$$

on $V_\chi = \mathrm{Ind}_P^{\mathrm{GL}_2(\mathcal{O}_F/p\mathcal{O}_F)}(1\otimes\chi)$ *such that* $\mathrm{gr}^jV_\chi\cong\oplus_{|J|=j}V_{\chi,J}$ *with*

$$V_{\chi,J} = \bigotimes_{\theta\in\Theta}(\theta\circ\det)^{\ell_{J,\theta}}\mathrm{Sym}^{n_{J,\theta}}(V_{\mathrm{st}}\otimes_{\mathcal{O}_F,\theta}\mathbf{F}),$$

where V_{st} *denotes the standard representation of* $\mathrm{GL}_2(\mathcal{O}_F/p\mathcal{O}_F)$ *on* $(\mathcal{O}_F/p\mathcal{O}_F)^2$ *and*

$$(\ell_{J,\theta},n_{J,\theta}) = \begin{cases}(0,m_\theta), & \text{if } \theta\notin J,\ \phi\circ\theta\notin J;\\ (0,m_\theta-1), & \text{if } \theta\in J,\ \phi\circ\theta\notin J;\\ (m_\theta+1,p-2-m_\theta), & \text{if } \theta\notin J,\ \phi\circ\theta\in J;\\ (m_\theta,p-1-m_\theta), & \text{if } \theta\in J,\ \phi\circ\theta\in J.\end{cases}$$

Note that the integers $n_{J,\theta}$ in the statement of the theorem satisfy $-1\le n_{J,\theta}\le p-1$; thus each non-zero $V_{\chi,J}$ is irreducible. (We adopt the usual convention that $\mathrm{Sym}^{-1}\mathbf{F}^2 = 0$.) Theorem 6.3.1 is immediate from Lemma 2.3 and Theorem 2.4 of [BP12], which treats similarly defined left representations of each $\mathrm{GL}_2(\mathcal{O}_F/v)$. To obtain Theorem 6.3.1, simply write $\chi = \prod_{v|p}\chi_v$ where χ_v is a character of $(\mathcal{O}_F/v)^\times$, so that $V_\chi^\iota\cong\bigotimes_{v|p}\mathrm{Ind}_{P_v}^{\mathrm{GL}_2(\mathcal{O}_F/v)}(\chi_v\otimes 1)$ as a left representation of $\mathrm{GL}_2(\mathcal{O}_F/p\mathcal{O}_F) = \prod_{v|p}\mathrm{GL}_2(\mathcal{O}_F/v)$, where P_v denotes the subgroup of upper-triangular matrices of $\mathrm{GL}_2(\mathcal{O}_F/v)$, and $\cdot^\iota$ is the equivalence of categories between right and left representations of $\mathrm{GL}_2(\mathcal{O}_F/p\mathcal{O}_F)$ associated to its standard anti-involution. Furthermore the filtration in the theorem is in fact the socle, as well as co-socle, filtration on V_χ provided $m_\theta < p-1$ for some θ in each Θ_v. (More generally, the socle and co-socle filtrations of V_χ coincide and have length $1+|\Theta_\chi|$ where Θ_χ is the union of the Θ_v for which χ_v is non-trivial, and the constituents of the j^{th} graded piece of the co-socle filtration are the non-zero $V_{\chi,J}$ such that $|J\cap\Theta_\chi| = j$.)

We let $\mathcal{A}_{\chi,J}$ denote the automorphic vector bundle

$$\left(\bigotimes_{\theta\notin\Sigma}\delta_\theta^{\ell_{J,\theta}}\omega_\theta^{n_{J,\theta}+2}\right)\left(\bigotimes_{\theta\in\Sigma}\delta_\theta^{\ell_{J,\theta}+1}\mathrm{Sym}^{n_{J,\theta}}_{\mathcal{O}_{\overline{Y}_\Sigma}}\mathcal{V}_\theta\right)\tag{31}$$

on $\overline{Y}_\Sigma$, where $\ell_J,\ n_J \in \mathbf{Z}^\Theta$ are as in the statement of Theorem 6.3.1, the tensor products are over $\mathcal{O}_{\overline{Y}_\Sigma}$, and we suppress the central tensor product symbols here and below. We view $H^0(\overline{Y}_\Sigma,\mathcal{A}_{\chi,J})$ as the space of automorphic forms for G_Σ over $\mathbf{F}$ of weight (n_J+2,ℓ_J) and level U_Σ; indeed for paritious (n_J,ℓ_J), the same definition with $\mathbf{F}$ replaced by $\mathbf{C}$ yields the usual space of automorphic forms for G_Σ of weight (n_J+2,ℓ_J) and level U_Σ.

Recall that if U_Σ and U'_Σ are sufficiently small open compact subgroups of $G_\Sigma(\mathbf{A_f})$ of level prime to p, and $g_\Sigma \in G_\Sigma(\mathbf{A}_\mathbf{f}^{(p)})$ is such that $g_\Sigma^{-1}U_\Sigma g_\Sigma \subset U'_\Sigma$, then we have morphisms

$$\rho_{g_\Sigma}:\overline{Y}_\Sigma\to\overline{Y}'_\Sigma \quad\text{and}\quad \pi^*_{g_\Sigma}:\rho^*_{g_\Sigma}\mathcal{A}'_{\chi,J}\tilde{\to}\mathcal{A}_{\chi,J}$$

satisfying the usual compatibilities, where $\mathcal{A}'_{\chi,J}$ is the automorphic vector bundle defined on $\overline{Y}'_\Sigma := Y_{U'_\Sigma}(G_\Sigma)_\mathbf{F}$ by (31). Letting g_Σ act as the composite

$$H^i(\overline{Y}'_\Sigma,\mathcal{A}'_{\chi,J})\xrightarrow{\rho^*_{g_\Sigma}}H^i(\overline{Y}_\Sigma,\rho^*_{g_\Sigma}\mathcal{A}'_{\chi,J})\xrightarrow{||\det(g_\Sigma)||\pi^*_{g_\Sigma}}H^i(\overline{Y}_\Sigma,\mathcal{A}_{\chi,J})$$

we obtain a left action of $G_\Sigma(\mathbf{A}_\mathbf{f}^{(p)})$ on $\varinjlim H^i(\overline{Y}_\Sigma,\mathcal{A}_{\chi,J})$, where the direct limit is over open compact $U_\Sigma \subset G_\Sigma(\mathbf{A_f})$ of level prime to p.

6.3.3. *Completion of the proof.* We now return to the task of computing the bundles $\Xi_{J,*}(\mathcal{J}_J^{-1}\mathcal{K}_{\overline{Y}_0(p)_J}i_J^*\mathcal{L}_\chi^{-1})$, with a view to relating them to the bundles $\mathcal{A}_{\chi,J}$. First note that the isomorphisms (27) and (28) immediately give

$$\begin{aligned}\mathcal{J}_J^{-1}i_J^*\mathcal{L}_{\chi^{-1}}^{-1} &\cong \left(\bigotimes_{\theta\in\Theta}i_J^*\mathcal{L}_\theta^{m_\theta-p+1}\right)\left(\bigotimes_{\theta\notin J}i_J^*\mathcal{L}_\theta^{-1}\right)\left(\bigotimes_{\phi\circ\theta\notin J}i_J^*\mathcal{L}_\theta^{p}\right)\\ &= \left(\bigotimes_{\theta\notin J,\phi\circ\theta\notin J}i_J^*\mathcal{L}_\theta^{m_\theta}\right)\left(\bigotimes_{\theta\notin J,\phi\circ\theta\in J}i_J^*\mathcal{L}_\theta^{m_\theta-p}\right)\left(\bigotimes_{\theta\in J,\phi\circ\theta\in J}i_J^*\mathcal{L}_\theta^{m_\theta+1}\right)\left(\bigotimes_{\theta\in J,\phi\circ\theta\in J}i_J^*\mathcal{L}_\theta^{m_\theta-p+1}\right).\end{aligned}$$

Applying (19) then gives an isomorphism

$$\begin{aligned}\Xi_{J,*}(\mathcal{J}_J^{-1}i_J^*\mathcal{L}_\chi^{-1}) \cong &\left(\bigotimes_{\theta\notin J,\phi\circ\theta\notin J}\omega_\theta^{m_\theta}\right)\left(\bigotimes_{\theta\notin J,\phi\circ\theta\in J}\delta_\theta^{m_\theta-p}(p-m_\theta)_\theta\right)\\ &\cdot\left(\bigotimes_{\theta\in J,\phi\circ\theta\in J}\mathcal{O}(m_\theta+1)_\theta\right)\left(\bigotimes_{\theta\in J,\phi\circ\theta\in J}\delta_\theta^{m_\theta-p+1}\omega_\theta^{p-1-m_\theta}\right).\end{aligned}$$

Combining this with (30) and the isomorphism $\delta_\theta \cong \delta^p_{\phi^{-1}\circ\theta}$ of §3.2.2 therefore yields an isomorphism

$$\begin{aligned}
\Xi_{J,*}(\mathcal{J}_J^{-1}\mathcal{K}_{\overline{Y}_0(p)_J} i_J^*\mathcal{L}_\chi^{-1}) &\cong \left(\bigotimes_{\theta\notin J,\phi\circ\theta\notin J} \delta_\theta^{-1}\omega_\theta^{m_\theta+2}\right)\left(\bigotimes_{\theta\notin J,\phi\circ\theta\in J} \delta_\theta^{m_\theta-p+1}(p-2-m_\theta)_\theta\right)\\
&\quad\cdot\left(\bigotimes_{\theta\in J,\phi\circ\theta\in J} \delta_\theta(m_\theta-1)_\theta\right)\left(\bigotimes_{\theta\in J,\phi\circ\theta\in J} \delta_\theta^{m_\theta-p}\omega_\theta^{p+1-m_\theta}\right) \qquad (32)\\
&\cong \left(\bigotimes_{\theta\notin\Sigma} \delta_\theta^{\ell_{J,\theta}-1}\omega_\theta^{n_{J,\theta}+2}\right)\left(\bigotimes_{\theta\in\Sigma} \delta_\theta^{\ell_{J,\theta}}(n_{J,\theta})_\theta\right)
\end{aligned}$$

of line bundles on $\prod_{\theta\in\Sigma}\mathbf{P}_{\overline{Y}_\Sigma}(\mathcal{V}_\theta)$, where $\ell_J,\ n_J \in \mathbf{Z}^\Theta$ are as in the statement of Theorem 6.3.1. Furthermore the isomorphism is Hecke-equivariant in the sense that the diagram analogous to (20) commutes. More precisely, suppose that U and U' are sufficiently small open compact subgroups of $G(\mathbf{A_f})$ of level prime to p and $g \in G(\mathbf{A}_\mathbf{f}^{(p)})$ is such that $g^{-1}Ug \subset U'$. Let

$$\mathcal{F} = \mathcal{J}_J^{-1}\mathcal{K}_{\overline{Y}_0(p)_J} i_J^*\mathcal{L}_\chi^{-1}, \quad \mathcal{G} = \left(\bigotimes_{\theta\notin\Sigma} \delta_\theta^{\ell_{J,\theta}-1}\omega_\theta^{n_{J,\theta}+2}\right)\left(\bigotimes_{\theta\in\Sigma} \delta_\theta^{\ell_{J,\theta}}(n_{J,\theta})_\theta\right)$$

and similarly define $\mathcal{F}'$ and $\mathcal{G}'$ with U replaced by U'. The isomorphisms $\sigma : \Xi_{J,*}\mathcal{F} \tilde{\to} \mathcal{G}$ and $\sigma' : \Xi'_{J,*}\mathcal{F}' \tilde{\to} \mathcal{G}'$ of (32) are then compatible with the isomorphisms $\pi_g^* : \rho_g^*\mathcal{F}' \tilde{\to} \mathcal{F}$ and $\pi_{g_\Sigma}^* : \rho_{g_\Sigma}^*\mathcal{G}' \tilde{\to} \mathcal{G}$, meaning that the diagram

$$\begin{array}{ccc}
\Xi_{J,*}\rho_g^*\mathcal{F}' = \rho_{g_\Sigma}^*\Xi'_{J,*}\mathcal{F}' & \xrightarrow{\rho_{g_\Sigma}^*(\sigma')} & \rho_{g_\Sigma}^*\mathcal{G}' \\
\downarrow{\scriptstyle \Xi_{J,*}(\pi_g^*)} & & \downarrow{\scriptstyle \pi_{g_\Sigma}^*} \\
\Xi_{J,*}\mathcal{F} & \xrightarrow[\sigma]{} & \mathcal{G}
\end{array}$$

commutes.

Recall that for a rank two vector bundle $\mathcal{V}$ on a scheme S, if $\psi : X = \mathbf{P}_S(\mathcal{V}) \to S$ denotes the natural projection and $n \geq -1$, then $R^i\psi_*\mathcal{O}(n) = 0$ for all $i \geq 1$ and $\psi_*\mathcal{O}(n)$ is canonically isomorphic to $\mathrm{Sym}^n_{\mathcal{O}_S}\mathcal{V}$ (where $\mathrm{Sym}^{-1}_{\mathcal{O}_S}(V) = 0$ as usual). Since $n_{J,\theta} \geq -1$ for all θ, it follows from (32) that

$$R^i(\Psi_{J,*}\Xi_{J,*})(\mathcal{J}_J^{-1}\mathcal{K}_{\overline{Y}_0(p)_J} i_J^*\mathcal{L}_\chi^{-1}) = R^i\Psi_{J,*}(\Xi_{J,*}(\mathcal{J}_J^{-1}\mathcal{K}_{\overline{Y}_0(p)_J} i_J^*\mathcal{L}_\chi^{-1})) = 0$$

for all $i \geq 1$, and that

$$\Psi_{J,*}\Xi_{J,*}(\mathcal{J}_J^{-1}\mathcal{K}_{\overline{Y}_0(p)_J} i_J^*\mathcal{L}_\chi^{-1}) \cong \left(\bigotimes_{\theta\notin\Sigma} \delta_\theta^{\ell_{J,\theta}-1}\omega_\theta^{n_{J,\theta}+2}\right)\left(\bigotimes_{\theta\in\Sigma} \delta_\theta^{\ell_{J,\theta}}\mathrm{Sym}^{n_{J,\theta}}_{\mathcal{O}_{\overline{Y}_\Sigma}}(\mathcal{V}_\theta)\right).$$

Note that the resulting line bundle on $\overline{Y}_\Sigma$ is $\left(\bigotimes_{\theta\in\Theta}\delta_\theta^{-1}\right)\mathcal{A}_{\chi,J}$. Multiplying by the trivialization of $\bigotimes_{\theta\in\Theta}\delta_\theta$ defined in §5.4.3, we obtain an isomorphism

$$\varsigma : \Psi_{J,*}\Xi_{J,*}(\mathcal{J}_J^{-1}\mathcal{K}_{\overline{Y}_0(p)_J} i_J^*\mathcal{L}_\chi^{-1}) \xrightarrow{\sim} \mathcal{A}_{\chi,J}$$

whose compatibility with the Hecke action is expressed by the commutativity of the resulting diagram

$$\begin{array}{ccc}
\Psi_{J,*}\Xi_{J,*}\rho_g^*\mathcal{F}' = \rho_{g_\Sigma}^*\Psi'_{J,*}\Xi'_{J,*}\mathcal{F}' & \xrightarrow{\rho^*_{g_\Sigma}(\varsigma')} & \rho^*_{g_\Sigma}\mathcal{A}'_{\chi,J} \\
\Big\downarrow {\scriptstyle \Psi_{J,*}\Xi_{J,*}(\pi_g^*)} & & \Big\downarrow {\scriptstyle ||\det g_\Sigma||\pi^*_{g_\Sigma}} \\
\Psi_{J,*}\Xi_{J,*}\mathcal{F} & \xrightarrow[\varsigma]{} & \mathcal{A}_{\chi,J}.
\end{array}$$

In view of the vanishing of $R^i(\Psi_{J,*}\Xi_{J,*})(\mathcal{J}_J^{-1}\mathcal{K}_{\overline{Y}_0(p)_J}i_J^*\mathcal{L}_\chi^{-1})$ for all $i > 0$, we conclude that ς induces a Hecke-equivariant isomorphism

$$H^i(\overline{Y}_0(p)_J,\, \mathcal{J}_J^{-1}\mathcal{K}_{\overline{Y}_0(p)_J}i_J^*\mathcal{L}_\chi^{-1}) \xrightarrow{\sim} H^i(\overline{Y}_\Sigma, \mathcal{A}_{\chi,J})$$

for all $i \geq 0$. Combining this with (26) completes the proof of Theorem B.

6.3.4. *Hecke equivariance.* For clarity and convenience, we review the sense in which the resulting spectral sequence

$$E_1^{j,i} = \bigoplus_{|J|=j} H^{i+j}(\overline{Y}_\Sigma, \mathcal{A}_{\chi,J}) \Longrightarrow H^{i+j}(\overline{Y}_1(p), \mathcal{K}_{\overline{Y}_1(p)})^\chi \tag{33}$$

is Hecke-equivariant. First recall that for each $\Sigma = \Sigma_J$ appearing in the direct sum we have fixed an isomorphism

$$G(\mathbf{A_f}) = \mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}}) \cong G_\Sigma(\mathbf{A_F}),$$

and for open compact subgroups U (resp. elements g) of $G(\mathbf{A_f})$, we write U_Σ (resp. g_Σ) for its image in $G_\Sigma(\mathbf{A_f})$. Thus if U and U' are sufficiently small open compact subgroups of $G(\mathbf{A_f})$ of level prime to p and $g \in G(\mathbf{A}_\mathbf{f}^{(p)})$ is such that $g^{-1}Ug \subset U'$, we have $\mathbf{F}$-linear maps we denote

$$\begin{array}{rll}
[g] = [g]_{U,U'} & : H^{i+j}(\overline{Y}'_1(p), \mathcal{K}_{\overline{Y}'_1(p)})^\chi & \to \ H^{i+j}(\overline{Y}_1(p), \mathcal{K}_{\overline{Y}_1(p)})^\chi \\
\text{and} \quad [g_\Sigma] = [g_\Sigma]_{U_\Sigma,U'_\Sigma} & : H^{i+j}(\overline{Y}'_\Sigma, \mathcal{A}'_{\chi,J}) & \to \ H^{i+j}(\overline{Y}_\Sigma, \mathcal{A}_{\chi,J}),
\end{array}$$

where as usual $\cdot'$ denotes the object defined with U replaced by U', and if g' and U'' are as above with $g'^{-1}U'g' \subset U''$, then $[g]\circ[g'] = [gg']$ and $[g_\Sigma]\circ[g'_\Sigma] = [g_\Sigma g'_\Sigma]$.

The existence of the spectral sequence in (33) means that

- there are $\mathbf{F}$-linear differentials $d_r^{j,i} : E_r^{j,i} \to E_r^{j+r,i-r+1}$ for all $r \geq 1$, $j \geq 0$, $i \geq -j$, where $E_{r+1}^{j,i}$ is defined inductively for $r \geq 1$ by
$$E_{r+1}^{j,i} = \ker(d_r^{j,i})/\mathrm{im}(d_r^{j-r,i+r-1});$$
- there is a decreasing filtration of length $d+1$ on $H^{i+j}(\overline{Y}_1(p), \mathcal{K}_{\overline{Y}_1(p)})^\chi$ and $\mathbf{F}$-linear isomorphisms
$$\alpha^{j,i} : E_\infty^{j,i} = E_{d+1}^{j,i} \xrightarrow{\sim} \mathrm{gr}^j\left(H^{i+j}(\overline{Y}_1(p), \mathcal{K}_{\overline{Y}_1(p)})^\chi\right).$$

The Hecke-equivariance of the spectral sequence means that for g, U, U' as above (again using $\cdot'$ to denote the corresponding objects with U replaced by U'), we have

$$[g]\left(\mathrm{Fil}^j\left(H^{i+j}(\overline{Y}'_1(p), \mathcal{K}_{\overline{Y}'_1(p)})^\chi\right)\right) \subset \mathrm{Fil}^j\left(H^{i+j}(\overline{Y}_1(p), \mathcal{K}_{\overline{Y}_1(p)})^\chi\right),$$

and for all r, j, i as above there are $\mathbf{F}$-linear $g_r^{j,i} : E_r'^{j,i} \to E_r^{j,i}$ such that

- $g_1^{j,i} = ([g_{\Sigma_J}])_{|J|=j} : \bigoplus_{|J|=j} H^{i+j}(\overline{Y}'_\Sigma, \mathcal{A}'_{\chi,J}) \to \bigoplus_{|J|=j} H^{i+j}(\overline{Y}_\Sigma, \mathcal{A}_{\chi,J})$,
- the diagram

$$\begin{array}{ccc} E_r'^{j,i} & \xrightarrow{d_r'^{j,i}} & E_r'^{j+r,i-r+1} \\ \downarrow g_r^{j,i} & & \downarrow g_r^{j+r,i-r+1} \\ E_r^{j,i} & \xrightarrow{d_r^{j,i}} & E_r^{j+r,i-r+1} \end{array}$$

  commutes,
- $g_{r+1}^{j,i}$ is induced by $g_r^{j,i}$,
- and the diagram

$$\begin{array}{ccc} E_\infty'^{j,i} = E_{d+1}'^{j,i} & \xrightarrow{\alpha'^{j,i}} & \mathrm{gr}^j\left(H^{i+j}(\overline{Y}'_1(p), \mathcal{K}_{\overline{Y}'_1(p)})^\chi\right) \\ \downarrow g_{d+1}^{j,i} & & \downarrow \mathrm{gr}^j([g]) \\ E_\infty^{j,i} = E_{d+1}^{j,i} & \xrightarrow{\alpha^{j,i}} & \mathrm{gr}^j\left(H^{i+j}(\overline{Y}_1(p), \mathcal{K}_{\overline{Y}_1(p)})^\chi\right) \end{array}$$

  commutes.

We remark that we have not given an intrinsic description of the differentials $d_1^{j,i}$ in terms of the spaces $H^{i+j}(\overline{Y}_\Sigma, \mathcal{A}_{\chi,j})$; rather they are instead defined via the isomorphisms induced by ς for varying J.

Taking the direct limit over sufficiently small U of level prime to p of the spectral sequences in Theorem B and defining the action of $\mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}}^{(p)})$ on each $\varinjlim H^{i+j}(\overline{Y}_\Sigma, \mathcal{A}_{\chi,j})$ via the isomorphism with $G_\Sigma(\mathbf{A}_\mathbf{f})$ gives the following:

Corollary 6.3.2. *There is a spectral sequence of smooth* $\mathbf{F}$*-representations of* $\mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}}^{(p)})$

$$E_1^{j,i} = \bigoplus_{|J|=j} \varinjlim \left(H^{i+j}(\overline{Y}_{\Sigma_J}, \mathcal{A}_{\chi,J})\right) \Longrightarrow \varinjlim \left(H^{i+j}(\overline{Y}_1(p), \mathcal{K}_{\overline{Y}_1(p)})^\chi\right).$$

6.3.5. *The Serre filtration.* Specializing the preceding corollary to the case of $i+j = 0$ immediately gives:

Corollary 6.3.3. *There is a filtration on* $\varinjlim H^0(\overline{Y}_1(p), \mathcal{K}_{\overline{Y}_1(p)})^\chi$ *of length* $d+1$ *by* $\mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}}^{(p)})$*-subrepresentations such that for each* $j = 0, 1, \ldots, d+1$*, there is a* $\mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}}^{(p)})$*-equivariant injection*

$$\mathrm{gr}^j\left(\varinjlim H^0(\overline{Y}_1(p), \mathcal{K}_{\overline{Y}_1(p)})\right)^\chi \longrightarrow \bigoplus_{|J|=j} \varinjlim \left(H^0(\overline{Y}_{\Sigma_J}, \mathcal{A}_{\chi,J})\right).$$

We immediately obtain Corollary C by specializing Theorem B (or alternatively by taking U^p-invariants in the preceding corollary). With the notation from the discussion above, the Hecke-equivariance in the statement of Corollary C means that

$$[g]\mathrm{Fil}^j\left(H^0(\overline{Y}'_1(p), \mathcal{K}_{\overline{Y}'_1(p)})^\chi\right) \subset \mathrm{Fil}^j\left(H^0(\overline{Y}_1(p), \mathcal{K}_{\overline{Y}_1(p)})^\chi\right)$$

and that the diagram

$$\begin{array}{ccc} \mathrm{gr}^j\left(H^0(\overline{Y}'_1(p), \mathcal{K}_{\overline{Y}'_1(p)})^\chi\right) & \longrightarrow & \bigoplus_{|J|=j} H^0(\overline{Y}'_{\Sigma_J}, \mathcal{A}'_{\chi,J}) \\ \Big\downarrow {\scriptstyle \mathrm{gr}^j([g])} & & \Big\downarrow {\scriptstyle ([g_{\Sigma_J}])_{|J|=j}} \\ \mathrm{gr}^j\left(H^0(\overline{Y}_1(p), \mathcal{K}_{\overline{Y}_1(p)})^\chi\right) & \longrightarrow & \bigoplus_{|J|=j} H^0(\overline{Y}_{\Sigma_J}, \mathcal{A}_{\chi,J}) \end{array}$$

commutes.

Remark 6.3.4. The representations in Corollary 6.3.3 are furthermore admissible, provided $F \neq \mathbf{Q}$. For $F = \mathbf{Q}$, working instead with the compactified modular curves yields admissible representations, and taking $U_1(N)$-invariants recovers the exact sequence described in the Introduction. We note also that in the case $F = \mathbf{Q}$, the spectral sequences of Theorems B and Corollary 6.3.2 (and slightly less obviously their analogues for the compact curves) degenerate at E_1.

Remark 6.3.5. The obstruction to the injection in Corollary C being an isomorphism is measured by the image of the differentials $d_r^{j,-j}$, and these take values in subquotients of direct sums of spaces of the form $H^1(\overline{Y}_{\Sigma_J}, \mathcal{A}_{\chi,J})$ for $|J| = j+r > j$. We note that these spaces need not vanish. For example, suppose $d = 3$ and $p\mathcal{O}_F = v_1v_2$ with $|\mathcal{O}_F/v_i| = p^i$, and write $\overline{\Theta}_p = \Theta_{v_1} \coprod \Theta_{v_2}$ where $\Theta_{v_1} = \{\theta_1\}$ and $\Theta_{v_2} = \{\theta_2, \theta_3\}$. Note that for $J = \{\theta_1, \theta_2\}$, we have $\Sigma = \Theta_{v_2}$, so $\overline{Y}_\Sigma$ is a curve. If $\chi = \theta_2^{p-2}\theta_3$, then we have

$$m_{\theta_1} = p-1, \quad m_{\theta_2} = p-2, \quad m_{\theta_3} = 1,$$

so the formula for (ℓ_J, n_J) and the Kodaira–Spencer isomorphism give

$$\mathcal{A}_{\chi,J} = \omega_{\theta_1}^2 \delta_{\theta_2} \delta_{\theta_3}^{p-1} \cong \delta_{\theta_1}\delta_{\theta_2}\delta_{\theta_3}^{p-1}\mathcal{K}_{\overline{Y}_\Sigma}.$$

For sufficiently small U, the line bundles δ_{θ_i} on $\overline{Y}$ are (non-canonically) trivializable (see [DS17, §4.5]), and arguing as we did in §5.4.3 for $\otimes_{\theta\in\Theta}\delta_\theta$, we see that the δ_{θ_i} on $\overline{Y}_\Sigma$ are (non-canonically) trivializable. Furthermore for sufficiently small U, the curve $\overline{Y}_\Sigma$ has components of positive genus, so $H^1(\overline{Y}_{\Sigma_J}, \mathcal{A}_{\chi,J}) \neq 0$. Note that this does not preclude the vanishing of the differentials $d_r^{j,-j}$. We remark also that even if the differentials do not vanish, so the injections are not isomorphisms, it may be the case that their cokernels are in some sense "Eisenstein."

7. Degeneracy fibres

7.1. **Frobenius factorization.** In the next two sections we study the fibres of the natural degeneracy map $\overline{\psi} : \overline{Y}_0(p) \to \overline{Y}$ over $\mathbf{F} = \overline{\mathbf{F}}_p$, where $\overline{Y}_0(p) = Y_{U_0(p)}(G)_{\mathbf{F}}$, $\overline{Y} = Y_U(G)_{\mathbf{F}}$ and $\overline{\psi}$ is induced by the forgetful morphism $\widetilde{Y}_{U_0(p)}(G) \to \widetilde{Y}_U(G)$. In particular, we will improve on results of [GK12, §2.6] and [ERX17, §4.9][15] (see also [Hel12, §5] in the unitary setting) with a view to proving Theorem E.

[15] There is however a serious gap in the argument in [ERX17, §4.9], specifically in the claim about rearranging choices of local parameters in the paragraph after (4.9.3).

7.1.1. *Preliminaries.* First recall that $\overline{Y}$ is equipped with a stratification defined by the vanishing of the partial Hasse invariants $h_\theta \in H^0(\overline{Y}, \overline{\omega}_\theta^{-1}\overline{\omega}_{\phi^{-1}\circ\theta}^p)$. Recall (see for example [DS17, §5.1]) that these are defined by descent from sections $\widetilde{h}_\theta$ on $T := \widetilde{Y}_U(G)_{\mathbf{F}}$ induced by the Verschiebung morphism $A^{(p)} \to A$, where A is the universal abelian scheme over T, and that for each $J \subset \Theta$, the closed subscheme T_J of T (resp. $\overline{Y}_J$ of $\overline{Y}$) defined by the vanishing of the $\widetilde{h}_\theta$ (resp. h_θ) for $\theta \in J$ is smooth of dimension $d - |J|$. By [GK12, Thm. 2.6.4], the restriction of the forgetful morphism in $S = \widetilde{Y}_{U_0(p)}(G)_{\mathbf{F}} \to T$ to S_J factors through $T_{J'}$ where $J' := \{\theta \in J \,|\, \phi^{-1} \circ \theta \notin J\}$. Furthermore letting $J'' := \{\theta \notin J \,|\, \phi^{-1} \circ \theta \in J\}$, the morphisms

$$\mathcal{H}^1_{\mathrm{dR}}(A_2/S)_\theta \to \mathcal{H}^1_{\mathrm{dR}}(A_1/S)_\theta$$

induced by the universal isogeny $f : A_1 \to A_2$ on S_J give rise to a morphism

$$(34) \qquad \tilde{\xi}_J : S_J \to P_J := \prod_{\theta \in J''} \mathbf{P}_{T_{J'}}(\mathcal{H}^1_{\mathrm{dR}}(A/T_{J'})_\theta)$$

which is bijective on geometric closed points (the injectivity is clear and the surjectivity follows from the calculation in [GK12, Lemma 2.6.6]). Since $S \to T$ is projective, so is $\tilde{\xi}_J$, and since S_J and P_J are locally of finite type of the same dimension over $\mathbf{F}$, it follows that $\tilde{\xi}_J$ is finite. Since the two schemes are smooth over $\mathbf{F}$, it follows by "miracle flatness" that $\tilde{\xi}_J$ is flat, and hence $S_J \to T_{J'}$ is Cohen–Macaulay.

Recall that $\mathcal{H}^1_{\mathrm{dR}}(A/T_{J'})_\theta$ is the (restriction to $T_{J'}$) of the vector bundle $\tilde{\mathcal{V}}_\theta$ defined in §3.1.1, and that these descend to the automorphic bundles on $\overline{Y}_{J'}$ denoted $\mathcal{V}_\theta$. Taking quotients in (34) by the action of $\mathcal{O}^\times_{F,(p),+}$, we see that the restriction of $\overline{\psi}$ to $\overline{Y}_0(p)_J$ similarly factors as a composite

$$\overline{Y}_0(p)_J \xrightarrow{\xi_J} \prod_{\theta \in J''} \mathbf{P}_{T_{J'}}(\mathcal{V}_\theta) \to \overline{Y}_{J'} \to \overline{Y}$$

where ξ_J is finite flat (and bijective on geometric closed points) and the next two maps are the canonical projection and closed immersion, so that $\overline{Y}_0(p)_J \to \overline{Y}_{J'}$ is Cohen–Macaulay.

7.1.2. *Relative Dieudonné theory.* The study of the fibres of $\overline{Y}_0(p)_J \to \overline{Y}$ at closed points reduces to understanding the fibres of $S_J \to T_{J'}$, whose sets of closed points are seen from the above discussion to be in bijection with products of projective lines; however we must deal with the fact that these fibres are not in general reduced. In order to proceed, we will first analyze the morphism $\tilde{\xi} : S_J \to P_J$ more carefully, for which we will make use of Dieudonné theory for finite flat group schemes over more general bases than perfect fields, as developed in [BBM82]. More precisely, if H is a finite flat group scheme over a scheme S of characteristic p, then [BBM82, Déf. 3.1.5] associates to H a Dieudonné crystal $\mathbf{D}(H)$ over S, i.e., a crystal of $\mathcal{O}_{S/\mathbf{Z}_p}$-modules equipped with morphisms[16] $\Phi : \mathbf{D}(H)^{(p)} \to \mathbf{D}(H)$ and $V : \mathbf{D}(H) \to \mathbf{D}(H)^{(p)}$ such that $\Phi \circ V$ and $V \circ \Phi$ are multiplication by p. The functor $\mathbf{D}$ is contravariant in H and the morphisms Φ and V are induced by $\mathrm{Frob}_H : H \to H^{(p)}$

[16] Recall we use Φ instead of F since F denotes our fixed totally real field; here however we use the linearized versions of Φ and V.

and $\mathrm{Ver}_H : H^{(p)} \to H$; furthermore $\mathbf{D}$ is compatible with base change $S \to S'$ in the obvious sense, and reduces to the usual Dieudonné theory when $S = \operatorname{Spec} k$ for a perfect field k (see[BBM82, Thm. 4.2.14]; note that we have already used $\mathbf{D}(H)$ to denote $\varprojlim_n \Gamma(\mathrm{Spec}(W_n), \mathbf{D}(H))$ when $k = \overline{\mathbf{F}}_p$, and this coincides with $\Gamma(S, \mathbf{D}(H))$ when H is killed by p). We shall in fact only consider group schemes H killed by p, on which we may view $\mathbf{D}$ as an exact functor to the category of crystals of locally free $\mathcal{O}_{S/\mathbf{F}_p}$-modules equipped with Φ and V such that $\Phi \circ V$ and $V \circ \Phi$ are trivial (Prop. 4.3.1 and Lemma 4.3.5 of [BBM82]).

Recall also from [BBM82, 4.3.4] that if $\mathcal{F}$ is a quasi-coherent sheaf of $\mathcal{O}_S$-modules, then pull-back via Frobenius endomorphisms on divided power thickenings over $\mathbf{F}_p$ yields a crystal of $\mathcal{O}_{S/\mathbf{F}_p}$-modules, which we denote $\Pi^*\mathcal{F}$ (rather than $\Phi^*\mathcal{F}$). The proof of [BBM82, Prop. 4.3.6] yields a natural morphism

$$\Pi^*\mathrm{Lie}(H^\vee) \to \mathbf{D}(H)$$

which is an isomorphism if $\mathrm{Ver}_H = 0$. On the other hand [BBM82, Prop. 4.3.10] yields a natural morphism

$$\mathbf{D}(H) \to \Pi^*\omega_H$$

where ω_H denotes the sheaf of invariant differentials, and this is an isomorphism if $\mathrm{Frob}_H = 0$. Furthermore recall that ω_H is canonically isomorphic to $e^*\Omega^1_{H/S}$ where $e : S \to H$ is the zero section, and we have a natural map $\mathrm{Lie}(H^\vee) \to \omega_H$, defined for example by identifying $\mathrm{Lie}(H^\vee)$ with $\mathcal{H}om_S(H, \mathbf{G}_a)$ and sending a section η to the invariant differential $\eta^*(dX)$. We remark also that in the case $S = \operatorname{Spec} k$ for a perfect field k, the two propositions in [BBM82] yield the isomorphisms

$$\mathrm{Lie}(H^\vee)^{(p)} = \ker(V)_S \qquad \text{and} \qquad \omega_H^{(p)} = \mathrm{coker}(\Phi)_S$$

which appeared in our previous discussion in the case $k = \overline{\mathbf{F}}_p$.

Lemma 7.1.1. *Suppose that k is a field and H is a finite flat group scheme over $S = \operatorname{Spec} k$ killed by p, then the composite*

$$\Pi^*\mathrm{Lie}(H^\vee) \to \mathbf{D}(H) \to \Pi^*\omega_H$$

is $\Pi^*\delta_H$, where δ_H is the natural map $\mathrm{Lie}(H^\vee) \to \omega_H$.

Proof. First note that the desired compatibility amounts to the claim that the composite $\mathrm{Lie}(H^\vee)^{(p)} \to \mathbf{D}(H)_S \to \omega_H^{(p)}$ is $\delta_H^{(p)}$, and we may extend scalars so as to assume k is perfect. Letting $H_0 = \ker(\mathrm{Frob}_H)$, the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Lie}(H^\vee)^{(p)} & \hookrightarrow & \mathbf{D}(H)_S & \twoheadrightarrow & \omega_H^{(p)} \\
\downarrow & & \downarrow & & \downarrow \wr \\
\mathrm{Lie}(H_0^\vee)^{(p)} & \hookrightarrow & \mathbf{D}(H_0)_S & \xrightarrow{\sim} & \omega_{H_0}^{(p)}
\end{array}$$

shows that the compatibility for H_0 implies it for H, so we may assume $\mathrm{Frob}_H = 0$. Similarly letting $H_1 = \mathrm{coker}(\mathrm{Ver}_H)$, the diagram

$$\begin{array}{ccccc} \mathrm{Lie}(H_1^\vee)^{(p)} & \xrightarrow{\sim} & \mathbf{D}(H_1)_S & \xrightarrow{\sim} & \omega_{H_1}^{(p)} \\ \downarrow \wr & & \downarrow & & \downarrow \\ \mathrm{Lie}(H^\vee)^{(p)} & \hookrightarrow & \mathbf{D}(H)_S & \xrightarrow{\sim} & \omega_H^{(p)} \end{array}$$

shows we may also assume $\mathrm{Ver}_H = 0$. Since Frob_H and Ver_H are both trivial, we have $H \cong \alpha_p^m$ for some $m \geq 0$, so we may further assume $H = \alpha_p$. Finally in the case $H = \alpha_p = \mathrm{Spec}\, k[Y]/(Y^p)$ with $\mu(Y) = Y \otimes 1 + 1 \otimes Y$, we can compute the isomorphisms $\mathrm{Lie}(H^\vee)^{(p)} \cong \mathbf{D}(H)_S$ and $\mathbf{D}(H)_S \cong \omega_H^{(p)}$ of Prop. 4.3.6 and Prop. 4.3.10 of [BBM82] using their descriptions in [BBM82, 4.3.7, 4.3.12]. More precisely, take the basis element of $\mathbf{D}(H)_S \cong \mathrm{Ext}^{\natural}_S(H, \mathbf{G}_a)$ defined by the extension E associated in [BBM82, 4.3.12(i)] to the pair (f, ω) where

$$f = -\sum_{i=1}^{p-1} \binom{p}{i} Y^i \otimes Y^{p-i} \quad \text{and} \quad \omega = Y^{p-1} dY.$$

According to [BBM82, 4.3.7], the image of the class in $\mathrm{Lie}(H^\vee)^{(p)}$ is defined by the homomorphism $H^{(p)} \to \mathbf{G}_a = \mathrm{Spec}\, k[X]$ through which $\mathrm{Ver}_E : E^{(p)} \to E$ factors, which one can check directly from the definition in [DG70, Exp. VII, 4.3] is given by $X \mapsto Y$. On the other hand according to [BBM82, 4.3.12(ii)], its image in $\omega_{H^{(p)}}$ is given by $C(\omega) = dY$. ◻

Remark 7.1.2. The preceding lemma would also follow easily from knowing that the relations between (co)tangent spaces and Dieudonné modules provided by Propositions 4.3.6 and 4.3.10 of [BBM82] were compatible (via their Theorem 4.2.14) with the ones given by Propositions III.3.2 and III.4.3 of [Fon77]. However we were not able to verify the commutativity of the resulting diagram.

We now determine the Dieudonné crystal of a Raynaud $(\mathcal{O}_F/p)$-module scheme H over a suitable base S in characteristic p. Let q be a power of p such that $q-1$ is divisible by the exponent of $(\mathcal{O}_F/p)^\times$.

Proposition 7.1.3. *Suppose that S is a smooth scheme over a field containing $\mathbf{F}_q$ and H is the Raynaud $(\mathcal{O}_F/p)$-module scheme over S associated to the data $(\mathcal{L}_\theta, s_\theta, t_\theta)_{\theta\in\Theta}$ (see §4.1.2). Then $\mathbf{D}(H)$ is canonically isomorphic to $\Pi^*\mathcal{L}$ with $\Phi = \Pi^*(s)$ and $V = \Pi^*(t)$, where $\mathcal{L} = \bigoplus_{\theta\in\Theta}\mathcal{L}_\theta$, $s = \bigoplus_{\theta\in\Theta} s_\theta$ and $t = \bigoplus_{\theta\in\Theta} t_\theta$.*

Proof. First note that the action of $\mathcal{O}_F$ on H induces an action of $\mathcal{O}_F \otimes \mathbf{F}_q$ on the crystal, and hence a decomposition $\mathbf{D}(H) = \bigoplus_{\theta\in\Theta}\mathbf{D}(H)_\theta$ under which Φ and V restrict to morphisms

$$\Phi : \mathbf{D}(H)_\theta^{(p)} \to \mathbf{D}(H)_{\phi\circ\theta} \qquad \text{and} \qquad V : \mathbf{D}(H)_{\phi\circ\theta} \to \mathbf{D}(H)_\theta^{(p)}.$$

To prove the proposition, we may assume S is connected and hence integral, so the equation $s_\theta t_\theta = 0$ implies that either $s_\theta = 0$ or $t_\theta = 0$, and we may therefore choose a set I such that $s_\theta = 0$ (resp. $t_\theta = 0$) for all $\theta \notin I$ (resp. $\theta \in I$). Applying Lemma 4.1.1 to H and $H^\vee$ now yields an exact sequence of finite flat $(\mathcal{O}_F/p)$-module schemes

$$0 \to C \to H \to C' \to 0$$

such that $\mathrm{Ver}_C = 0$, $\mathrm{Frob}_{C'} = 0$, and the natural maps $\mathrm{Lie}(H^\vee) \to \mathcal{L} \to \omega_H$ induce isomorphisms

$$\mathrm{Lie}(C^\vee) \cong \bigoplus_{\theta\in I} \mathcal{L}_\theta \quad \text{and} \quad \bigoplus_{\theta\notin I} \mathcal{L}_\theta \cong \omega_{C'}.$$

The $(\mathcal{O}_F/p)$-equivariant exact sequence

$$0 \to \mathbf{D}(C') \to \mathbf{D}(H) \to \mathbf{D}(C) \to 0$$

and isomorphisms $\mathbf{D}(C) \cong \Pi^*\mathrm{Lie}(C^\vee)$, $\mathbf{D}(C') \cong \Pi^*\omega_{C'}$ thus yield an $(\mathcal{O}_F/p)$-equivariant isomorphism

$$\alpha : \mathbf{D}(H) \xrightarrow{\sim} \bigoplus_{\theta\in I} \Pi^*\mathcal{L}_\theta$$

of crystals of locally free $\mathcal{O}_{S/\mathbf{F}_p}$-modules, which we write as $\bigoplus \alpha_\theta$.

We must prove that α is compatible with Φ and V. Since S is smooth and the constructions are compatible with base-change, we can apply [BM90, Cor. 1.3.4] to replace S by its generic point and hence assume $S = \mathrm{Spec}\, k$ for a field k. We just give the argument for Φ; the proof for V is similar[17].

First note that since $\mathrm{Frob}_{C'} = 0$, it follows that Φ annihilates $\mathbf{D}(C')^{(p)}$ and Frob_H factors through $C^{(p)}$. In particular if $\phi^{-1}\circ\theta \notin I$, then Φ is trivial on

$$\mathbf{D}(H)_\theta^{(p)} \cong \mathbf{D}(C')_\theta^{(p)} \cong \Pi^*\mathcal{L}^p_{\phi^{-1}\circ\theta},$$

as is $\Pi^* s_{\phi^{-1}\circ\theta}$. On the other hand if $\phi^{-1}\circ\theta \in I$, then $t_{\phi^{-1}\circ\theta} = 0$, so that $\omega_{H^\vee,\theta} = \mathcal{L}_\theta^{-1}$ and $\mathrm{Lie}(H^\vee)_\theta = \mathcal{L}_\theta$. Furthermore the right-most square of the diagram

$$(35)\qquad \begin{array}{ccccccc}
\mathbf{D}(H)_\theta^{(p)} & \xrightarrow{\sim} & \mathbf{D}(C)_\theta^{(p)} & \xleftarrow{\sim} & \Pi^*\mathrm{Lie}(C^\vee)^{(p)}_{\phi^{-1}\circ\theta} & \xrightarrow{\sim} & \Pi^*\mathcal{L}^p_{\phi^{-1}\circ\theta} \\
& \searrow^{\Phi} & \downarrow & & \downarrow & & \downarrow{\scriptstyle \Pi^* s_{\phi^{-1}\circ\theta}} \\
& & \mathbf{D}(H)_{\phi\circ\theta} & \longleftarrow & \Pi^*\mathrm{Lie}(H^\vee)_\theta & \xrightarrow{\sim} & \Pi^*\mathcal{L}_\theta
\end{array}$$

commutes since $\mathrm{Frob}_H : H \to C^{(p)} \to H^{(p)}$ is defined by the morphism of $\mathcal{O}_S$-algebras induced by the s_θ, and the rest of the diagram commutes by functoriality. Since the top row is $\alpha_\theta^{(p)}$, it just remains to show that the bottom row of (35) is $\alpha^{-1}_{\phi\circ\theta}$. If $\theta \in I$, this is clear from the definitions. On the other hand if $\theta \notin I$ and $\phi^{-1}\circ\theta \in I$, then we find that $\mathrm{Lie}(C'^\vee)_\theta \cong \mathrm{Lie}(H^\vee)_\theta = \mathcal{L}_\theta$, and by functoriality and Lemma 7.1.1 the squares in the diagram

$$\begin{array}{ccccc}
\mathbf{D}(H)_{\phi\circ\theta} & \xleftarrow{\sim} & \mathbf{D}(C')_{\phi\circ\theta} & \xrightarrow{\sim} & \Pi^*\omega_{C',\theta} \\
\uparrow & & \uparrow & & \uparrow\wr \\
\Pi^*\mathrm{Lie}(H^\vee)_\theta & \xleftarrow{\sim} & \Pi^*\mathrm{Lie}(C'^\vee)_\theta & \xrightarrow{\sim} & \Pi^*\mathcal{L}_\theta
\end{array}$$

commute. The composite along the top and right is $\alpha_{\phi\circ\theta}$, and along the bottom and left is the bottom row of (35).

[17] As will be clear from the argument, the compatibility of Φ (resp. V) on $\mathbf{D}(H)_\theta^{(p)}$ (resp. $\mathbf{D}(H)_{\phi\circ\theta}$) is straightforward if $\phi^{-1}\circ\theta \notin I$ or $\theta \in I$ (resp. $\phi^{-1}\circ\theta \in I$ or $\theta \notin I$); the appeal to faithfulness under base-change is only needed to address the remaining case.

Finally we note that the isomorphism α is independent of choice of I. To prove this we can again apply [BM90, Cor. 1.3.4] to reduce to the case of $S = \operatorname{Spec} k$ for a field k, and then the above argument shows that if $\phi^{-1}\circ\theta \in I$, then $\alpha_{\phi\circ\theta}^{-1}$ is the composite $\Pi^*\mathcal{L}_\theta \xrightarrow{\sim} \Pi^*\mathrm{Lie}(H^\vee)_\theta \to \mathbf{D}(H)_{\phi\circ\theta}$, and similarly if $\phi^{-1}\circ\theta \notin I$, then $\alpha_{\phi\circ\theta}$ is the composite $\mathbf{D}(H)_{\phi\circ\theta} \to \Pi^*\omega_{H,\theta} \xrightarrow{\sim} \Pi^*\mathcal{L}_\theta$. If $s_{\phi^{-1}\circ\theta}$ and $t_{\phi^{-1}\circ\theta}$ are both trivial, then Lemma 7.1.1 implies that the composite of these maps is the identity, so $\alpha_{\phi\circ\theta}$ is independent of whether or not $\phi^{-1}\circ\theta \in I$. ◻

Remark 7.1.4. It is natural to expect that the proposition in fact holds over an arbitrary $\mathbf{F}_q$-scheme S, and indeed our proof only required that locally $\mathcal{O}_S$ be integral and have a p-basis in the sense of [BM90, 1.1.1], but the result stated above will suffice for our purpose.

7.1.3. *Construction of an isogeny over P_J.* We first construct Raynaud data on the variety P_J (defined in §7.1.1) to which we will apply Proposition 7.1.3. Let $u : A \to P_J$ denote the pull-back of the universal abelian scheme over $T_{J'}$ and consider the $\mathcal{O}_F \otimes \mathcal{O}_S$-linear morphisms

$$\mathrm{Frob}^* : \mathcal{H}^1_{\mathrm{dR}}(A^{(p)}/P_J) \to \mathcal{H}^1_{\mathrm{dR}}(A/P_J) \quad \text{and} \quad \mathrm{Ver}^* : \mathcal{H}^1_{\mathrm{dR}}(A/P_J) \to \mathcal{H}^1_{\mathrm{dR}}(A^{(p)}/P_J).$$

For each $\theta \in \Theta$ we define line bundles $\mathcal{M}_\theta$ and $\mathcal{L}_\theta$ fitting in an exact sequence

$$0 \to \mathcal{M}_\theta \to \mathcal{H}^1_{\mathrm{dR}}(A/P_J) \to \mathcal{L}_\theta \to 0 \tag{36}$$

as follows:

- if $\theta \in J$, then $\mathcal{M}_\theta = (u_*\Omega^1_{A/P_J})_\theta$, $\mathcal{L}_\theta = (R^1u_*\mathcal{O}_A)_\theta$ and (36) is the Hodge filtration;
- if $\phi^{-1}\circ\theta \notin J$, then $\mathcal{M}_\theta = \mathrm{Frob}^*(\mathcal{H}^1_{\mathrm{dR}}(A^{(p)}/P_J)_\theta) \cong (R^1u_*\mathcal{O}_A)^p_{\phi^{-1}\circ\theta}$ and
$$\mathcal{L}_\theta = \mathcal{H}^1_{\mathrm{dR}}(A/P_J)_\theta/\mathcal{M}_\theta \cong \mathrm{Ver}^*(\mathcal{H}^1_{\mathrm{dR}}(A/P_J)_\theta) = (u_*\Omega^1_{A/P_J})^p_{\phi^{-1}\circ\theta};$$
- if $\theta \in J''$, then (36) is the tautological filtration defining the projection $P_J \to \mathbf{P}_{T_{J'}}(\mathcal{H}^1_{\mathrm{dR}}(A/P_J)_\theta)$, so $\mathcal{L}_\theta = \mathcal{O}(1)_\theta$ and $\mathcal{M}_\theta \cong \wedge^2(\mathcal{H}^1_{\mathrm{dR}}(A/P_J))_\theta(-1)_\theta$.

Note the first two conditions are not exclusive, but if $\theta \in J$ and $\phi^{-1}\circ\theta \notin J$, then $\theta \in J'$, in which case $\mathrm{Frob}^*(\mathcal{H}^1_{\mathrm{dR}}(A^{(p)}/P_J)_\theta) = (u_*\Omega^1_{A/P_J})_\theta$ by the definition of $T_{J'}$. Furthermore note that $\mathrm{Frob}^*(\mathcal{M}^p_{\phi^{-1}\circ\theta}) \subset \mathcal{M}_\theta$ for all θ (since $\mathrm{Frob}^*(\mathcal{M}^p_{\phi^{-1}\circ\theta}) = 0$ if $\phi^{-1}\circ\theta \in J$), and similarly $\mathrm{Ver}^*(\mathcal{M}_\theta) \subset \mathcal{M}^p_{\phi^{-1}\circ\theta}$ for all θ (since $\mathrm{Ver}^*(\mathcal{H}^1_{\mathrm{dR}}(A/P_J)_\theta) = \mathcal{M}^p_{\phi^{-1}\circ\theta}$ if $\phi^{-1}\circ\theta \in J$, and $\mathrm{Ver}^*(\mathcal{M}_\theta) = 0$ if $\phi^{-1}\circ\theta \notin J$). We can therefore define $s_\theta : \mathcal{M}^p_\theta \to \mathcal{M}_{\phi\circ\theta}$ and $t_\theta : \mathcal{M}_{\phi\circ\theta} \to \mathcal{M}^p_\theta$ as the restrictions of Frob^* and Ver^* and so obtain Raynaud data $(\mathcal{M}_\theta, s_\theta, t_\theta)$ on P_J, and we let C denote the corresponding $(\mathcal{O}_F/p)$-module scheme. Similarly the morphisms induced by Frob^* on $\mathcal{L}^p_\theta$ and Ver^* on $\mathcal{L}_{\phi\circ\theta}$ give rise to Raynaud data on P_J, and we let H denote the corresponding $(\mathcal{O}_F/p)$-module scheme.

Letting $\mathcal{M} = \bigoplus \mathcal{M}_\theta$ and $\mathcal{L} = \bigoplus \mathcal{L}_\theta$, we obtain from (36) an exact sequence of Dieudonné crystals

$$0 \to \Pi^*\mathcal{M} \to \Pi^*\mathcal{H}^1_{\mathrm{dR}}(A/P_J) \to \Pi^*\mathcal{L} \to 0,$$

where Φ (resp. V) is defined on the terms by $\Pi^*\mathrm{Frob}^*$ (resp. $\Pi^*\mathrm{Ver}^*$), and according to Proposition 7.1.3, we have canonical isomorphisms $\Pi^*\mathcal{M} \cong \mathbf{D}(C)$ and $\Pi^*\mathcal{L} \cong$

$\mathbf{D}(H)$. On the other hand we also have the canonical isomorphisms

$$\Pi^*\mathcal{H}^1_{\mathrm{dR}}(A/P_J) \cong \Pi^*\mathbf{D}(A[p])_{P_J} \cong \mathbf{D}(A[p])^{(p)} \cong \mathbf{D}(A^{(p)}[p])$$

of Dieudonné crystals provided by [BBM82, (3.3.7.3),(4.3.7.1)], allowing us to interpret the above exact sequence as

$$0 \to \mathbf{D}(C) \to \mathbf{D}(A^{(p)}[p]) \to \mathbf{D}(H) \to 0.$$

Since P_J is smooth over $\mathbf{F}$, the functor $\mathbf{D}$ is fully faithful by [BM90, Thm. 4.1.1], so we obtain an exact sequence

$$0 \to H \to A^{(p)}[p] \to C \to 0$$

of finite flat group schemes on P_J. Since $\mathbf{D}(H)_{P_J,\theta}$ is locally free of rank one for each $\theta \in \Theta$, we see as in the proof of Lemma 4.2.1 that H is totally isotropic with respect to the λ-Weil pairing on $A^{(p)}$, and hence the pair $(\underline{A}^{(p)}, H)$ defines a morphism $P_J \to S = \widetilde{Y}_{U_0(p)}(G)_{\mathbf{F}}$. Letting $(\underline{A}_1, \underline{A}_2, f)$ denote the universal triple on S_J, note that the pull-back of $\underline{A}^{(p)}$ to S_J via $\tilde{\xi}_J$ is $\underline{A}_1^{(p)}$. Furthermore, the definition of $\mathcal{M}$ ensures that $\tilde{\xi}_J^*\mathcal{M}_\theta$ is the image of $\mathcal{H}^1_{\mathrm{dR}}(A_2/S_J)_\theta$ for each θ. (If $\theta \in J$, this is immediate from the vanishing of $\mathrm{Lie}(f^\vee)_\theta$, if $\theta \in J''$, this is part of the definition of $\tilde{\xi}_J$, and if $\phi^{-1} \circ \theta \notin J$, note that

$$\mathrm{Ver}^*_{A_1}(f^*\mathcal{H}^1_{\mathrm{dR}}(A_2/S_J)_\theta) = f^{(p),*}(\mathrm{Ver}^*_{A_2}\mathcal{H}^1_{\mathrm{dR}}(A_2/S_J)_\theta) = 0$$

since $\mathrm{Ver}^*_{A_2}\mathcal{H}^1_{\mathrm{dR}}(A_2/S_J)_\theta = (u_{2,*}\Omega^1_{A_2/S_J})^p_{\phi^{-1}\circ\theta}$.) It follows that the composite

$$\Pi^*\mathbf{D}(A_2[p])_{S_J} \to \Pi^*\mathbf{D}(A_1[p])_{S_J} \to \Pi^*(\tilde{\xi}_J^*\mathcal{L})$$

is trivial, but this is the same as

$$\mathbf{D}(A_2^{(p)}[p])_{S_J} \to \mathbf{D}(A_1^{(p)}[p])_{S_J} \to \mathbf{D}(\tilde{\xi}_J^* H).$$

Therefore H is contained in $\ker(f^{(p)})$, and comparing ranks, we see that $H = \ker(f^{(p)})$. It follows that the composite $S_J \xrightarrow{\tilde{\xi}_J} P_J \to S$ is the Frobenius morphism on S_J, or more precisely the composite

$$S_J \to S_J^{(p)} \cong S_{\phi(J)} \subset S$$

where $S_J \to S_J^{(p)}$ is the Frobenius morphism relative to $\operatorname{Spec}\mathbf{F}$ and $S_J^{(p)} \cong S_{\phi(J)}$ is the canonical isomorphism induced by the Frobenius automorphism of $\mathbf{F}$. We record this as a lemma:

Lemma 7.1.5. *There exists a morphism $P_J \to S_J^{(p)}$ whose composite with $\xi_J : S_J \to P_J$ is the Frobenius morphism $S_J \to S_J^{(p)}$.*

7.1.4. *Local structure.* We now study the local structure of the degeneracy fibres by computing the effect of the morphisms $S_J \to P_J \to T_{J'}$ on tangent spaces at closed points. As in the proof of Theorem 5.2.1, we apply the equivalence of categories between abelian schemes over $T = \mathbf{F}[\epsilon]/(\epsilon^2)$ and pairs $(A, \tilde{L})$ where A is an abelian variety over $\mathbf{F}$ and $\tilde{L}$ is a free T-submodule of $H^1_{\mathrm{dR}}(A/\mathbf{F}) \otimes_{\mathbf{F}} T$ such that $\tilde{L} \otimes_T \mathbf{F} = H^0(A, \Omega^1_{A/\mathbf{F}})$.

Let $(\underline{A}_1, \underline{A}_2, f)$ be (the data corresponding to) an element of $z \in S_J(\mathbf{F})$, and let y (resp. x) denote its image in $P_J(\mathbf{F})$ (resp. $T_{J'}(\mathbf{F})$). Thus x corresponds to $\underline{A}_1$ and y corresponds to $(\underline{A}_1, (L_\theta)_{\theta\in J''})$ where L_θ is the image of $H^1_{\mathrm{dR}}(A_2/\mathbf{F})_\theta \to$

$H^1_{\rm dR}(A_1/\mathbf{F})_\theta$. Note that if $(\underline{\tilde{A}}_1, \underline{\tilde{A}}_2, \tilde{f})$ is a lift of z to $S_J(T)$, then $H^0(\tilde{A}_2, \Omega^1_{\tilde{A}_2/T})_\theta$ is the kernel of

$$\tilde{f}^*_\theta : H^1_{\rm dR}(\tilde{A}_2/T)_\theta \to H^1_{\rm dR}(\tilde{A}_1/T)_\theta$$

for $\theta \notin J$, and this corresponds to $H^0(A_2, \Omega^1_{A_2/\mathbf{F}})_\theta \otimes_{\mathbf{F}} T$ under the isomorphisms with $H^1_{\rm crys}(A_2/T)_\theta$. Similarly if $\theta \in J$, then $H^0(\tilde{A}_1, \Omega^1_{\tilde{A}_1/T})_\theta$ is the image of $\tilde{f}^*_\theta$, and this corresponds to $H^0(A_1, \Omega^1_{A_1/\mathbf{F}})_\theta \otimes_{\mathbf{F}} T$. It follows from a standard argument that the tangent space of S_J at z is in canonical bijection with the set of tuples

$$\left((\tilde{L}_{1,\theta})_{\theta\notin J},\, (\tilde{L}_{2,\theta})_{\theta\in J}\right)$$

where each $\tilde{L}_{j,\theta}$ is a free T-submodule of $H^1_{\rm dR}(A_j/\mathbf{F}) \otimes_{\mathbf{F}} T$ lifting $H^0(A_j, \Omega^1_{A_j/\mathbf{F}})$. Similarly if $\underline{\tilde{A}}_1$ is a lift of x to $T_{J'}(T)$, then we find that $H^0(\tilde{A}_1, \Omega^1_{\tilde{A}_1/T})_\theta$ corresponds to $H^0(A_1, \Omega^1_{A_1/\mathbf{F}})_\theta \otimes_{\mathbf{F}} T$ if $\theta \in J'$, so the tangent space of $T_{J'}$ at x is identified with

$$(\tilde{L}_{1,\theta})_{\theta\notin J'}$$

where each $\tilde{L}_{1,\theta}$ is a lift of $H^0(A_i, \Omega^1_{A_1/\mathbf{F}})$ to $H^1_{\rm dR}(A_1/\mathbf{F}) \otimes_{\mathbf{F}} T$. Finally it follows from the definition of P_J that its tangent space at y is identified with

$$\left((\tilde{L}_{1,\theta})_{\theta\notin J'},\, (\tilde{L}_{\theta})_{\theta\in J''}\right)$$

where each $\tilde{L}_{1,\theta}$ (resp. $\tilde{L}_\theta$) is a lift of $H^0(A_1, \Omega^1_{A_1/\mathbf{F}})$ (resp. L_θ) to $H^1_{\rm dR}(A_1/\mathbf{F}) \otimes_{\mathbf{F}} T$. Furthermore the map on tangent spaces induced by $P_J \to T_{J'}$ corresponds to the natural projection under the above descriptions. As for the map on tangent spaces induced by ξ_J, suppose that $\left((\tilde{L}_{1,\theta})_{\theta\notin J},\, (\tilde{L}_{2,\theta})_{\theta\in J}\right)$ corresponds to $(\underline{\tilde{A}}_1, \underline{\tilde{A}}_2, \tilde{f})$ and let $\left((\tilde{L}_{1,\theta})_{\theta\notin J'},\, (\tilde{L}_{\theta})_{\theta\in J''}\right)$ correspond to its image in $P_J(T)$. Note that if $\theta \in J$ but $\theta \notin J'$ (i.e., $\phi^{-1} \circ \theta$ is also in J), then $\tilde{L}_{1,\theta}$, being the image of $\tilde{f}^*_\theta$, is $H^0(A_1, \Omega^1_{A_1/F})_\theta \otimes_{\mathbf{F}} T$, and similarly if $\theta \in J''$, then $\tilde{L}_\theta = L_\theta \otimes_{\mathbf{F}} T$.

Recall that letting $\operatorname{Spec}(R_1)$ denote the first infinitesimal neighborhood of x in $T_{J'}$ and choosing trivializations $R^2 \cong H^1_{\rm dR}(A_1^{\rm univ}/R)_\theta$ in a Zariski neighborhood $\operatorname{Spec} R$ compatible with the canonical isomorphism

$$H^1_{\rm dR}(A_1^{\rm univ}/R_1)_\theta \cong H^1_{\rm crys}(A_1/R_1)_\theta \cong H^1_{\rm dR}(A_1/\mathbf{F})_\theta \otimes_{\mathbf{F}} R_1$$

yields a regular system of parameters $\{\, t_\theta \,|\, \theta \notin J' \,\}$ at x, so that

$$\mathbf{F}[[t_\theta]]_{\theta\notin J'} \xrightarrow{\ \sim\ } \widehat{\mathcal{O}}_{T_{J'},x}.$$

Furthermore the resulting parameters are compatible with the above description of the tangent space in the sense that the span of each t_σ in the cotangent space $\mathfrak{m}_x/\mathfrak{m}_x^2$ is orthogonal to the set of $(\tilde{L}_{1,\theta})_{\theta\notin J'}$ such that $\tilde{L}_{1,\sigma} = L_{1,\sigma} \otimes_{\mathbf{F}} T$. We similarly obtain isomorphisms

$$\begin{array}{rcl} \mathbf{F}[[t_\theta]]_{\theta\notin J'} \,\widehat{\otimes}\, \mathbf{F}[[u_\theta]]_{\theta\in J''} & \xrightarrow{\ \sim\ } & \widehat{\mathcal{O}}_{P_J,y} \\ \text{and}\quad \mathbf{F}[[t_\theta]]_{\theta\notin J} \,\widehat{\otimes}\, \mathbf{F}[[v_\theta]]_{\theta\in J} & \xrightarrow{\ \sim\ } & \widehat{\mathcal{O}}_{S_J,z} \end{array}$$

for which the parameters are compatible with the above descriptions of the tangent spaces of P_J at y and S_J at z. The homomorphisms of completed local rings induced by the morphisms $S_J \to P_J \to T_{J'}$ then take the form

$$\mathbf{F}[[t_\theta]]_{\theta\notin J'} \hookrightarrow \mathbf{F}[[t_\theta]]_{\theta\notin J'} \,\widehat{\otimes}\, \mathbf{F}[[u_\theta]]_{\theta\in J''} \hookrightarrow \mathbf{F}[[t_\theta]]_{\theta\notin J} \,\widehat{\otimes}\, \mathbf{F}[[v_\theta]]_{\theta\in J},$$

where the first arrow is the natural inclusion and the second sends t_θ to t_θ for $\theta \notin J$ and the remaining parameters to elements of $\mathfrak{m}_z^2$. Furthermore Lemma 7.1.5 implies that the image of the second map contains v_θ^p for all $\theta \in J$. Taking the quotient of each ring by the ideal generated by the t_θ for $\theta \notin J$, the resulting homomorphism

$$\mathbf{F}[[t_\theta]]_{\theta\in J-J'} \widehat{\otimes} \mathbf{F}[[u_\theta]]_{\theta\in J''} \to \mathbf{F}[[v_\theta]]_{\theta\in J}$$

is trivial on the cotangent space at the closed point and has image containing v_θ^p for all $\theta \in J$. It therefore follows from [KN82, Cor. 2] that the image is precisely $\mathbf{F}[[v_\theta^p]]_{\theta\in J}$, so we may replace the parameters v_θ by parameters w_θ in $\widehat{\mathcal{O}}_{S_J,z}$ such that

$$\begin{array}{rcll} t_\theta & \equiv & w^p_{\phi^{-1}\circ\theta} \bmod I & \text{if } \theta \in J,\ \phi^{-1}\circ\theta \in J \\ \text{and} \quad u_\theta & \equiv & w^p_{\phi^{-1}\circ\theta} \bmod I & \text{if } \theta \notin J,\ \phi^{-1}\circ\theta \in J, \end{array}$$

where I is the ideal generated by the t_θ for $\theta \notin J$. We conclude that the quotient of $\widehat{\mathcal{O}}_{S_J,z}$ by the ideal generated by the t_θ for $\theta \notin J'$ is isomorphic to

$$\mathbf{F}[[w_\theta]]_{\theta\in J}/\langle w_\theta^p\rangle_{\theta,\phi\circ\theta\in J}.$$

Lemma 7.1.6. *Let Z be the fibre of $S_J \to T_{J'}$ at a closed point x of $T_{J'}$, and let Z_{red} be its reduced subscheme. If z is a closed point of Z, then*

$$\widehat{\mathcal{O}}_{Z,z} \cong \mathbf{F}[[w_\theta]]_{\theta\in J}/\langle w_\theta^p\rangle_{\theta,\phi\circ\theta\in J} \quad \text{and} \quad \widehat{\mathcal{O}}_{Z_{\mathrm{red}},z} \cong \mathbf{F}[[w_\theta]]_{\theta\in J,\phi\circ\theta\notin J};$$

in particular Z_{red} is smooth of dimension $|J'| = |J''|$ over $\mathbf{F}$.

Proof. The assertion concerning $\widehat{\mathcal{O}}_{Z,z}$ is immediate from the above discussion; the assertions for Z_{red} follow since Z is locally of finite type over $\mathbf{F}$, and in particular Nagata, so that $\widehat{\mathcal{O}}_{Z_{\mathrm{red}},z} = \widehat{\mathcal{O}}_{Z,z}^{\mathrm{red}}$. □

7.2. **Crystallization.** We maintain the notation of the preceding section.

7.2.1. *A crystallization lemma.* Note that at each closed point of S_J and $\theta \in J$, the map

$$H^1_{\mathrm{dR}}(A_2/\mathbf{F})_\theta \to H^1_{\mathrm{dR}}(A_1/\mathbf{F})_\theta$$

induced by the fibre $A_1 \to A_2$ of the universal isogeny has image $H^0(A_1, \Omega^1_{A_1/\mathbf{F}})_\theta$, which coincides with the image of the map induced by $\mathrm{Ver} : A_1 \to A_1^{(p^{-1})}$. It follows that the maps

$$\mathbf{D}(A_2[p^\infty])_\theta \to \mathbf{D}(A_1[p^\infty])_\theta \quad \text{and} \quad \mathbf{D}(A_1^{(p^{-1})}[p^\infty])_\theta \to \mathbf{D}(A_1[p^\infty])_\theta$$

have the same image, and hence we obtain an isomorphism

$$\mathbf{D}(A_2[p^\infty])_\theta \cong \mathbf{D}(A_1^{(p^{-1})}[p^\infty])_\theta,$$

which in turn induces an isomorphism

$$H^1_{\mathrm{dR}}(A_2/\mathbf{F})_\theta \to H^1_{\mathrm{dR}}(A_1^{(p^{-1})}/\mathbf{F})_\theta.$$

We wish to prove this in fact arises from an isomorphism of sheaves on each fibre of $S_J \to T_{J'}$, for which we appeal to the following crystallization lemma:

Lemma 7.2.1. *Suppose that X is a smooth scheme over $\mathbf{F}$, A, B_1 and B_2 are abelian schemes over X with O_F-action, and $\theta \in \Theta$. Let $\alpha_i : A \to B_i$ for $i = 1, 2$ be O_F-linear isogenies such that*

- $\ker(\alpha_i) \cap A[p^\infty] \subset A[p]$ *for $i = 1, 2$, and*

- $\alpha_{1,x}^*\mathbf{D}(B_{1,x}[p^\infty])_\theta = \alpha_{2,x}^*\mathbf{D}(B_{2,x}[p^\infty])_\theta$ *for all* $x \in X(\mathbf{F})$

Then there is a unique isomorphism $\mathcal{H}^1_{\mathrm{dR}}(B_1/X)_\theta \cong \mathcal{H}^1_{\mathrm{dR}}(B_2/X)_\theta$ *of coherent sheaves on* X *whose fibres are compatible with the isomorphisms*

$$\mathbf{D}(B_{1,x}[p])_\theta \cong \mathbf{D}(B_{2,x}[p])_\theta$$

induced by $(\alpha_{2,x}^*)^{-1}\alpha_{1,x}^*$ *for all* $x \in X(\mathbf{F})$. *Furthermore letting* $j : \mathrm{Spec}(R_1) = X_1 \to X$ *denote the first infinitesimal neighborhood of* x, *the isomorphism is also compatible with the isomorphisms*

$$H^1_{\mathrm{dR}}(B_{i,x}/\mathbf{F}) \otimes_{\mathbf{F}} R_1 \cong H^1_{\mathrm{dR}}(j^*B_i/R_1)$$

induced by their canonical isomorphisms with $H^1_{\mathrm{cris}}(B_{i,x}/R_1)$ *for* $i = 1,2$.

Proof. First note that $\mathcal{H}^1_{\mathrm{dR}}(A/X)_\theta$, being a direct summand of $\mathcal{H}^1_{\mathrm{dR}}(A/X)$, is locally free, and similarly for $\mathcal{H}^1_{\mathrm{dR}}(B_i/X)_\tau$ for $i = 1,2$. Furthermore the rank of the cokernel of $\alpha_{i,x}^* : H^1_{\mathrm{dR}}(B_{i,x}/\mathbf{F}) \to H^1_{\mathrm{dR}}(A_x/\mathbf{F})$ is the p-part of the degree of α_i, hence is locally constant, and the rank of the cokernel of each summand $H^1_{\mathrm{dR}}(B_{i,x}/\mathbf{F})_{\theta'} \to H^1_{\mathrm{dR}}(A_x/\mathbf{F})_{\theta'}$ is upper semi-continuous, so it follows that these ranks are also locally constant, and hence that the cokernel of $\alpha_i^* : \mathcal{H}^1_{\mathrm{dR}}(B_i/X)_\theta \to \mathcal{H}^1_{\mathrm{dR}}(A/X)_\theta$ is locally free (since X is reduced and its closed points are dense). The hypothesis that $\alpha_{1,x}^*\mathbf{D}(B_{1,x})_\theta = \alpha_{2,x}^*\mathbf{D}(B_{2,x})_\theta$ implies that the fibre at x of the composite

$$\mathcal{H}^1_{\mathrm{dR}}(B_1/X)_\theta \xrightarrow{\alpha_1^*} \mathcal{H}^1_{\mathrm{dR}}(A/X)_\theta \longrightarrow \mathrm{coker}(\alpha_2^*)$$

is trivial. Since $\mathcal{O}_X(U)$ has trivial (Jacobson) radical for all open $U \subset X$, morphisms to $\mathcal{O}_X$, hence to any locally free coherent sheaf on X, are determined by their fibres, so in fact the above composite is trivial. Therefore $\mathrm{im}(\alpha_2^*) \subset \mathrm{im}(\alpha_1^*)$ and similarly the opposite inclusion, and hence equality, holds.

Again using the fact that morphisms to locally free coherent sheaves on X are determined by fibres, we see that the desired isomorphism, if it exists, is unique. Furthermore we may work locally on X and hence assume that there is a smooth scheme $\widetilde{X}$ over $W = W(\mathbf{F})$ such that $\widetilde{X} \times_W \mathbf{F} = X$ (see for example [StaX, Lemma 10.135.19]).

Letting $s : A \to X$ and $t_i : B_i \to X$ (for $i = 1,2$) denote the structure morphisms, consider the crystals of locally free $\mathcal{O}_{X/\mathbf{Z}_p}$-modules $\mathcal{C} := R^1 s_{\mathrm{cris},*}\mathcal{O}_{A/\mathbf{Z}_p}$ and $\mathcal{E}_i := R^1_{t_i,\mathrm{cris},*}\mathcal{O}_{B_i/\mathbf{Z}_p}$ (see [BBM82, Cor. 2.5.5]). Since the functor sending an abelian scheme $u : C \to X$ to $R^1 u_{\mathrm{cris},*}\mathcal{O}_{C/\mathbf{Z}_p}$ is additive, the crystals $\mathcal{C}$ and $\mathcal{E}_i$ inherit an action of $\mathcal{O}_F \otimes W$, and hence decompose as direct sums of crystals of locally free $\mathcal{O}_{X/\mathbf{Z}_p}$-modules indexed by Θ. Since the isogenies α_i are $\mathcal{O}_F$-linear, they induce morphisms of crystals $\mathcal{E}_{i,\theta} \to \mathcal{C}_\theta$ for $i = 1,2$.

We claim that there is an isomorphism $\mathcal{E}_{1,\theta} \xrightarrow{\sim} \mathcal{E}_{2,\theta}$ such that the diagram

(37)
$$\begin{array}{ccc} \mathcal{E}_{1,\theta} & \longrightarrow & \mathcal{E}_{2,\theta} \\ & \searrow \quad \swarrow & \\ & \mathcal{C}_\theta & \end{array}$$

commutes. To prove this we use the equivalence of categories provided by [BO78, Cor. 6.8] (with $S = W$ and $S_0 = \mathbf{F}$). If $\mathcal{F}$ denotes the quasi-coherent sheaf (with integrable quasi-nilpotent connection) on $\widetilde{X}$ corresponding to $\mathcal{C}_\theta$, then $\mathcal{F}$ is locally free and its restriction to X is canonically isomorphic to $(\mathcal{C}_\theta)_X \cong \mathcal{H}^1_{\mathrm{dR}}(A/X)_\theta$ ([BBM82, (3.3.7.3)]). Similarly (for $i = 1, 2$) the underlying quasi-coherent sheaf $\mathcal{G}_i$ corresponding to $\mathcal{E}_{i,\theta}$ is a locally free sheaf on $\widetilde{X}$ whose restriction to X is identified with $\mathcal{H}^1_{\mathrm{dR}}(B_i/X)_\theta$. The claim is thus equivalent to the existence of an isomorphism $\mathcal{G}_1 \xrightarrow{\sim} \mathcal{G}_2$ which is compatible with connections and makes the diagram

$$\begin{array}{ccc} \mathcal{G}_1 & \longrightarrow & \mathcal{G}_2 \\ & {\scriptstyle \tilde{\alpha}_1^*}\searrow \quad \swarrow{\scriptstyle \tilde{\alpha}_2^*} & \\ & \mathcal{C}_\theta & \end{array}$$

commute, where each $\tilde{\alpha}_i^*$ is the morphism corresponding to $\mathcal{E}_{i,\theta} \to \mathcal{C}_\theta$. Since the $\tilde{\alpha}_i^*$ are compatible with connections, it suffices to prove that they are injective and have the same image.

Now let $\beta_i : B_i \to A$ be an isogeny such that $\alpha_i \circ \beta_i = d$ with $p||d$, and let $\tilde{\beta}_i^*$ be the morphism $\mathcal{F} \to \mathcal{G}_i$ corresponding to the one induced on crystals. Then the composite $\tilde{\beta}_i^* \circ \tilde{\alpha}_i^*$ is also multiplication by d, since it is so for the corresponding morphism of crystals by additivity. The composite is therefore injective, and therefore so is $\tilde{\alpha}_i^*$. On the other hand considering the composite $\beta_i \circ \alpha_i = d$, we see that the image of $\tilde{\alpha}_i^*$ contains $p\mathcal{F}$. Therefore to conclude $\tilde{\alpha}_1^*$ and $\tilde{\alpha}_2^*$ have the same image, it suffices to prove their reductions modulo p, or equivalently their restrictions to X, have the same image, but this is precisely the fact that $\mathrm{im}(\alpha_1^*) = \mathrm{im}(\alpha_2^*)$.

The isomorphism of crystals now yields an isomorphism

$$\mathcal{H}^1_{\mathrm{dR}}(B_1/X)_\theta \cong (\mathcal{E}_{1,\theta})_X \xrightarrow{\sim} (\mathcal{E}_{2,\theta})_X \cong \mathcal{H}^1_{\mathrm{dR}}(B_1/X)_\theta$$

of coherent sheaves on X. The desired compatibility with homomorphisms of Dieudonné modules follows from the commutativity of (37) and the fact that $\alpha_{i,x}^* : \mathbf{D}(B_{i,x}[p^\infty])_\theta \to \mathbf{D}(A_x[p^\infty])_\theta$ is realized as the inverse limit of the morphisms

$$\mathcal{E}_{i,\theta}(\{x\}, \mathrm{Spec}(W_n)) \longrightarrow \mathcal{C}_\theta(\{x\}, \mathrm{Spec}(W_n))$$

under the canonical isomorphisms provided by [BBM82, (3.3.7.2), Thm. 4.2.14]. Finally, since the isomorphisms $H^1_{\mathrm{dR}}(B_{i,x}/\mathbf{F}) \otimes_{\mathbf{F}} R_1 \cong H^1_{\mathrm{dR}}(j^*B_i/R_1)$ are realized as the composite of the crystalline transition maps

$$\mathcal{E}_{i,\theta}(\{x\}, \{x\}) \otimes_{\mathbf{F}} R_1 \xrightarrow{\sim} \mathcal{E}_{i,\theta}(\{x\}, X_1) \xleftarrow{\sim} \mathcal{E}_{i,\theta}(X_1, X_1)$$

under [BBM82, (3.3.7.3)], the second desired compatibility follows from the fact that $\mathcal{E}_{1,\theta} \to \mathcal{E}_{2,\theta}$ is an isomorphism of crystals. □

7.2.2. *The reduced degeneracy fibre.* We now return to the setup of Lemma 7.1.6, so Z is the fibre of $S_J \to T_{J'}$ at a closed point x of $T_{J'}$, Z_{red} is its reduced subscheme, and z is a closed point of Z. Let $\underline{A}_0$ be the abelian scheme over $\mathbf{F}$ corresponding to x and consider

$$A = A_0 \times_{\mathbf{F}} Z_{\mathrm{red}} = A_1 \times_{S_J} Z_{\mathrm{red}} \quad \text{and} \quad B = A_2 \times_{S_J} Z_{\mathrm{red}},$$

where as usual $f : A_1 \to A_2$ is the universal isogeny on S_J. We have already observed that if $\theta \in J$, then

$$f^* : H^1_{\mathrm{dR}}(A_{2,z}/\mathbf{F})_\theta \to H^1_{\mathrm{dR}}(A_0/\mathbf{F})_\theta \quad \text{and} \quad \mathrm{Ver}^* : H^1_{\mathrm{dR}}(A_0^{(p^{-1})}/\mathbf{F})_\theta \to H^1_{\mathrm{dR}}(A_0/\mathbf{F})_\theta$$

have the same image, so we may apply Lemma 7.2.1 with $B_1 = B$, α_1 as the restriction of f, $B_2 = A_0^{(p^{-1})} \times_{\mathbf{F}} Z_{\mathrm{red}}$ and α_2 as the pull-back of Ver to obtain an isomorphism

$$\mathcal{H}^1_{\mathrm{dR}}(B/Z_{\mathrm{red}})_\theta \xrightarrow{\sim} H^1_{\mathrm{dR}}(A_0^{(p^{-1})}/\mathbf{F})_\theta \otimes_{\mathbf{F}} \mathcal{O}_{Z_{\mathrm{red}}} \tag{38}$$

whose fibre at each closed point z is compatible with the isomorphism in the top row of the commutative diagram

$$\begin{array}{ccc} \mathbf{D}(B_z[p^\infty])_\theta & \xrightarrow{\sim} & \mathbf{D}(A_0^{(p^{-1})}[p^\infty])_\theta \\ & {\scriptstyle f^*}\searrow \quad \swarrow{\scriptstyle \mathrm{Ver}^*} & \\ & \mathbf{D}(A_0[p^\infty])_\theta. & \end{array} \tag{39}$$

Furthermore letting $j : \operatorname{Spec} R_1 \to Z_{\mathrm{red}}$ denote the first infinitesimal neighborhood of z, the diagram

$$\begin{array}{ccc} H^1_{\mathrm{dR}}(B_z/\mathbf{F})_\theta \otimes_{\mathbf{F}} R_1 & \longrightarrow & H^1_{\mathrm{dR}}(j^*B/R_1)_\theta \\ & \searrow \quad \swarrow & \\ & H^1_{\mathrm{dR}}(A_0^{(p^{-1})}/\mathbf{F})_\theta \otimes_{\mathbf{F}} R_1 & \end{array} \tag{40}$$

also commutes, where the top arrow is given by the canonical isomorphisms with $H^1_{\mathrm{cris}}(B_z/R_1)$ and the diagonal arrows by the restrictions of (38) to z and $\operatorname{Spec} R_1$. Moreover note that if also $\phi \circ \theta \in J$, then the commutativity of (39) implies that of

$$\begin{array}{ccccc} \mathbf{D}(B_z[p^\infty])_\theta^{(p)} & \xrightarrow{\sim} & \mathbf{D}(B_z^{(p)}[p^\infty])_{\phi\circ\theta} & \xrightarrow{\mathrm{Frob}^*} & \mathbf{D}(B_z[p^\infty])_{\phi\circ\theta} \\ \downarrow \wr & & & & \downarrow \wr \\ D(A_0^{(p^{-1})}[p^\infty])_\theta^{(p)} & \xrightarrow{\sim} & \mathbf{D}(A_0[p^\infty])_{\phi\circ\theta} & \xrightarrow{\mathrm{Frob}^*} & \mathbf{D}(A_0^{(p^{-1})}[p^\infty])_{\phi\circ\theta}, \end{array}$$

and hence that (38) restricts to

$$(s_*\Omega^1_{B/Z_{\mathrm{red}}})_\theta \xrightarrow{\sim} H^0(A_0^{(p^{-1})}, \Omega^1_{A_0^{(p^{-1})}/\mathbf{F}})_\theta \otimes \mathcal{O}_{Z_{\mathrm{red}}} \tag{41}$$

since it does so on fibres (where $s : B \to Z_{\mathrm{red}}$ is the structure morphism). On the other hand if $\phi \circ \theta \notin J$, i.e. $\phi \circ \theta \in J''$, then the Hodge filtration on $H^1_{\mathrm{dR}}(B/Z_{\mathrm{red}})_\theta$ defines a morphism $Z_{\mathrm{red}} \to \mathbf{P}(H^1_{\mathrm{dR}}(A_0^{(p^{-1})}/\mathbf{F})_\theta)$ under which $\mathcal{O}(1)$ pulls back via (38) to $(R^1 s_* \mathcal{O}_B)_\theta$.

Theorem 7.2.2. *The morphism*

$$\psi = \psi_{J,x} : Z_{\mathrm{red}} \to \prod_{\phi\circ\theta \in J''} \mathbf{P}(H^1_{\mathrm{dR}}(A_0^{(p^{-1})}/\mathbf{F})_\theta)$$

is an isomorphism.

Proof. As in the proof of Theorem 5.2.1, it suffices to prove that the morphism is bijective on $\overline{\mathbf{F}}_p$-points and injective on tangent spaces at such points.

For the bijectivity on points, recall that the morphism $\tilde{\xi}_J$ of (34) is bijective on points, and therefore so is its restriction to fibres over x, and hence so is

$$\tilde{\xi}_{J,x} : Z_{\mathrm{red}} \to \prod_{\theta \in J''} \mathbf{P}(H^1_{\mathrm{dR}}(A_0/\mathbf{F})_\theta).$$

If the point $z \in Z_{\mathrm{red}}(\mathbf{F})$ corresponds to the isogeny $f : A_0 \to B_z$, then the θ-component of $\tilde{\xi}_{J,x}(z)$ is the image of the W-submodule

$$f^*\mathbf{D}(B_z[p^\infty])_\theta \subset \mathbf{D}(A_0[p^\infty])_\theta$$

in $\mathbf{D}(A_0[p^\infty])_\theta/p\mathbf{D}(A_0[p^\infty])_\theta = \mathbf{D}(A_0[p])_\theta \cong H^1_{\mathrm{dR}}(A_0/\mathbf{F})_\theta$. On the other hand, since $H^0(B_z, \Omega^1_{B_z/\mathbf{F}})_{\phi^{-1}\circ\theta}$ is the reduction mod p of

$$\mathrm{Ver}^*(\mathbf{D}(B_z^{(p^{-1})}[p^\infty])_{\phi^{-1}\circ\theta} \subset \mathbf{D}(B_z[p^\infty])_{\phi^{-1}\circ\theta},$$

it follows from the commutativity of (39) and the compatibility $f \circ \mathrm{Ver} = \mathrm{Ver} \circ f^{(p^{-1})}$ that the $\phi^{-1} \circ \theta$-component of $\psi(z)$ is the image of the W-submodule

$$\left(f^{(p^{-1})}\right)^* \mathbf{D}(B_z^{(p^{-1})}[p^\infty])_{\phi^{-1}\circ\theta} \subset \mathbf{D}(A_0^{(p^{-1})}[p^\infty])_{\phi^{-1}\circ\theta}$$

in $H^1_{\mathrm{dR}}(A_0^{(p^{-1})}/\mathbf{F})_{\phi^{-1}\circ\theta} = H^1_{\mathrm{dR}}(A/\mathbf{F})_\theta^{(p^{-1})}$. Thus if $\tilde{\xi}_{J,x}(z) = (L_\theta)_{\theta\in J''}$, then $\psi(z) = (L_\theta^{(p^{-1})})_{\theta \in J''}$, so $\psi \circ \tilde{\xi}_{J,x}^{-1}$ is bijective on $\mathbf{F}$-points.

To prove the injectivity on tangent spaces, let $T = \mathbf{F}[\epsilon]/(\epsilon^2)$ and suppose $(\underline{A}_{0,T}, \underline{\widetilde{B}}_z, \tilde{f})$ and $(\underline{A}_{0,T}, \underline{\widetilde{B}}'_z, \tilde{f}')$ are two lifts of $z \in Z_{\mathrm{red}}(\mathbf{F})$ to $Z_{\mathrm{red}}(T)$ with the same image in $\prod_{\theta\in J''} \mathbf{P}(H^1_{\mathrm{dR}}(A_0^{(p^{-1})}/\mathbf{F})_\theta)(T)$. This means that $L_\theta := H^0(\widetilde{B}_z, \Omega^1_{\widetilde{B}_z/T})_\theta$ and $L'_\theta := H^0(\widetilde{B}'_z, \Omega^1_{\widetilde{B}'_z/T})_\theta$ have the same image in $H^1_{\mathrm{dR}}(A_0^{(p^{-1})}/\mathbf{F})_\theta \otimes_{\mathbf{F}} T$ under (38) for all $\theta \in J$ such that $\phi \circ \theta \notin J$. On the other hand if $\theta, \phi \circ \theta \in J$, then the same holds by (41). In either case, the commutativity of (40), tensored over R_1 with T, implies that L_θ and L'_θ have the same image in $H^1_{\mathrm{dR}}(B_z/\mathbf{F})_\theta \otimes_{\mathbf{F}} T$. On the other hand if $\theta \notin J$, then the same assertion follows from the local analysis preceding Lemma 7.1.6. Therefore the Grothendieck–Messing Theorem implies that the identity on B_z lifts to an isomorphism $\widetilde{B}_z \cong \widetilde{B}'_z$. The compatibility of the isomorphism with the auxiliary data follows from the faithfulness of $\cdot \otimes_T \mathbf{F}$ by a standard argument, so we conclude that $(\underline{A}_{0,T}, \underline{\widetilde{B}}_z, \tilde{f})$ and $(\underline{A}_{0,T}, \underline{\widetilde{B}}'_z, \tilde{f}')$ define the same element of the tangent space. □

7.2.3. *Thickening.* In order to extend this to obtain a complete description of the fibre Z, we recall the general version of the Grothendieck–Messing Theorem (see [MM74]) in characteristic p states that if $i : X_0 \hookrightarrow X$ is a nilpotent thickening of $\mathbf{F}_p$-schemes whose defining ideal sheaf is equipped with a divided power structure, then the category of abelian schemes over X is equivalent (under the obvious functor) to the category of pairs $(C, \mathcal{L})$ where $u : C \to X_0$ is an abelian scheme and

$$\mathcal{M} \subset (R^1 u_{\mathrm{cris},*}\mathcal{O}_{C/\mathbf{F}_p})_{(X_0,X)}$$

is a locally free $\mathcal{O}_X$-submodule such that $i^*\mathcal{M}$ corresponds to the image of $u_*\Omega^1_{C/X_0}$ under the canonical isomorphism with $\mathcal{H}^1_{\mathrm{dR}}(C/X_0)$.

We apply this with $X = X_0 \times_{\mathbf{F}} X_1$, where $X_0 = Z_{\mathrm{red}}$ and X_1 is defined as the fibre product

$$\begin{array}{ccc} X_1 & \longrightarrow & \prod_{\phi,\phi\circ\theta\in J} \mathbf{P}(H^1_{\mathrm{dR}}(A_0^{(p^{-1})}/\mathbf{F})_\theta) \\ \downarrow & & \downarrow \\ \operatorname{Spec}\mathbf{F} & \longrightarrow & \prod_{\phi,\phi\circ\theta\in J} \mathbf{P}(H^1_{\mathrm{dR}}(A_0/\mathbf{F})_{\phi\circ\theta}), \end{array}$$

where the bottom arrow is defined by $H^0(A_0, \Omega^1_{A_0/\mathbf{F}})_{\phi\circ\theta}$ and the right vertical arrow is the (relative Frobenius) morphism sending $(M_\theta)_\theta$ to $(M_\theta^p)_\theta$, where we view

$$L_\theta^p \subset H^1_{\mathrm{dR}}(A_0^{(p^{-1})}/\mathbf{F})_\theta^{(p)} \cong H^1_{\mathrm{dR}}(A_0/\mathbf{F})_{\phi\circ\theta}.$$

Note that $X_1 \cong \prod \operatorname{Spec}\mathbf{F}[w_\theta]/(w_\theta^p)$, so the closed immersion $\operatorname{Spec}\mathbf{F} \hookrightarrow X_1$ is a nilpotent thickening with divided powers, and hence so is the resulting closed immersion $i : X_0 \hookrightarrow X$.

Letting $\pi : X \to X_0$ denote the projection, we have a canonical isomorphism

$$(R^1 s_{\mathrm{cris},*}\mathcal{O}_{B/\mathbf{F}_p})_{(X_0,X)} \cong \pi^*\mathcal{H}^1_{\mathrm{dR}}(B/X_0), \tag{42}$$

and we let $\mathcal{M} = \bigoplus_{\theta\in\Theta} \mathcal{M}_\theta$ where

- $\mathcal{M}_\theta$ corresponds under (38) to the pull-back of the tautological bundle on $\mathbf{P}(H^1_{\mathrm{dR}}(A_0^{(p^{-1})}/\mathbf{F})_\theta)$ if $\theta, \phi\circ\theta \in J$;
- $\mathcal{M}_\theta$ corresponds under (42) to $\pi^*(s_*\Omega^1_{B/X_0})_\theta$ otherwise.

We then see (using (41) if $\theta, \phi\circ\theta \in J$) that $i^*\mathcal{M}_\theta = (s_*\Omega^1_{B/X_0})_\theta$ for all θ, so that the data $(B, \mathcal{M})$ corresponds to an abelian scheme $\widetilde{B}$ over X lifting B. Furthermore writing f for the restriction to Z_{red} of the universal isogeny, and using that

$$f^*(\mathcal{H}^1_{\mathrm{dR}}(B/X_0)_\theta) = H^0(A_0, \Omega^1_{A_0/\mathbf{F}})_\theta \otimes_{\mathbf{F}} \mathcal{O}_{X_0}$$

if $\theta, \phi\circ\theta \in J$, we see that the image of $\mathcal{M}$ under is contained in $H^0(A_0, \Omega^1_{A_0/\mathbf{F}})_\theta \otimes_{\mathbf{F}} \mathcal{O}_X$, from which it follows that f lifts (uniquely) to an isogeny $\tilde{f} : A_0 \times_{\mathbf{F}} X \to \widetilde{B}$ such that $\mathrm{Lie}(\tilde{f})_\theta = 0$ if $\theta \notin J$ and $\mathrm{Lie}(\tilde{f}^\vee)_\theta = 0$ if $\theta \in J$. Equipping $\widetilde{B}$ with the (unique) auxiliary data lifting that on B, we thus obtain a triple $(\underline{A}_0 \times_{\mathbf{F}} X, \underline{\widetilde{B}}, \tilde{f})$ corresponding to a morphism $\tilde{\psi} : X \to Z$ extending the inclusion $Z_{\mathrm{red}} \hookrightarrow Z$.

Lemma 7.2.3. *The morphism $\tilde{\psi} : X \to Z$ is an isomorphism.*

Proof. Since the morphism is finite and bijective on closed points, it suffices to prove it induces isomorphisms on the completed local rings at closed points.

Letting y be a closed point of X and $z = \tilde{\psi}(y)$, we first note that $\tilde{\psi}$ is injective on tangent spaces; indeed the injectivity of the composite $T_y(X) \to T_z(Z) \subset T_z(S_J)$ is immediate from the construction of $\tilde{\psi}$ and the description of $T_z(S_J)$ preceding Lemma 7.1.6. Therefore the homomorphism $\tilde{\psi}^* : \widehat{\mathcal{O}}_{Z,z} \to \widehat{\mathcal{O}}_{X,y}$ is surjective.

Now we let $\mathfrak{a}$ and $\mathfrak{b}$ denote the respective nilradicals and show by induction that $\tilde{\psi}^*$ induces an isomorphism $\widehat{\mathcal{O}}_{Z,z}/\mathfrak{a}^n \to \widehat{\mathcal{O}}_{X,y}/\mathfrak{b}^n$ for all $n \geq 1$. For $n = 1$, this follows from the identification of each quotient with the regular local ring $\widehat{\mathcal{O}}_{Z_{\mathrm{red}},z}$.

For the induction step, apply the snake lemma to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{a}^n/\mathfrak{a}^{n+1} & \longrightarrow & \widehat{\mathcal{O}}_{Z,z}/\mathfrak{a}^{n+1} & \longrightarrow & \widehat{\mathcal{O}}_{Z,z}/\mathfrak{a}^n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow\wr & & \\ 0 & \longrightarrow & \mathfrak{b}^n/\mathfrak{b}^{n+1} & \longrightarrow & \widehat{\mathcal{O}}_{X,y}/\mathfrak{b}^{n+1} & \longrightarrow & \widehat{\mathcal{O}}_{X,y}/\mathfrak{b}^n & \longrightarrow & 0. \end{array}$$

The definition of X and the explicit description of $\widehat{\mathcal{O}}_{Z,z}$ in Lemma 7.1.6 imply that $\mathfrak{a}^n/\mathfrak{a}^{n+1}$ and $\mathfrak{b}^n/\mathfrak{b}^{n+1}$ are finite free $\widehat{\mathcal{O}}_{Z_{\mathrm{red}},z}$-modules of the same rank, so the surjectivity of the left vertical arrow implies its injectivity, and hence that of the middle arrow. Taking n sufficiently large (namely $p^{|J|-|J'|}$) yields the desired isomorphism, and hence the lemma. □

Combining Theorem 7.2.2 and Lemma 7.2.3 yields a canonical isomorphism

$$Z \xrightarrow{\sim} \prod_{\theta\in J,\phi\circ\theta\notin J} \mathbf{P}(H^1_{\mathrm{dR}}(A_0^{(p^{-1})}/\mathbf{F})_\theta) \times \prod_{\theta,\phi\circ\theta\in J} T_\theta, \tag{43}$$

where T_θ is a fibre of the Frobenius morphism on $\mathbf{P}(H^1_{\mathrm{dR}}(A_0^{(p^{-1})}/\mathbf{F})_\theta)$. defined by Frobenius on the relevant factors. In particular, the degeneracy fibre Z is isomorphic to a product of $|J'|$ copies of $\mathbf{P}^1$ and $|J|-|J'|$ copies of $\operatorname{Spec}\mathbf{F}[x]/(x^p)$.

Note also that we may view (42) as defining an isomorphism

$$\mathcal{H}^1_{\mathrm{dR}}(\widetilde{B}/Z) \cong (R^1 s_{\mathrm{cris},*}\mathcal{O}_{B/\mathbf{F}_p})_{(Z_{\mathrm{red}},Z)} \cong \pi^*\mathcal{H}^1_{\mathrm{dR}}(B/Z_{\mathrm{red}})$$

(where we have written $s:\widetilde{B}\to Z$ for the restriction of the universal A_2 and $\pi: Z\to Z_{\mathrm{red}}$ for $\pi\circ\psi^{-1}$), which combined with (38) yields an isomorphism[18]

$$\mathcal{H}^1_{\mathrm{dR}}(\widetilde{B}/Z)_\theta \cong H^1_{\mathrm{dR}}(A_0^{(p^{-1})}/\mathbf{F})_\theta \otimes_{\mathbf{F}} \mathcal{O}_Z \tag{44}$$

for $\theta\in J$. Unravelling the definitions, one finds that (43) is the morphism whose θ-component is defined by the Hodge filtration on $\mathcal{H}^1_{\mathrm{dR}}(\widetilde{B}/Z)_\theta$ under (44). We record this as follows:

Theorem 7.2.4. *Let $x\in T_{J'}(\mathbf{F})$ correspond to $\underline{A}_0$, let Z be the fibre over x of the projection $S_J\to T_{J'}$, and let $s:\widetilde{B}\to Z$ denote the pull-back of $A_2\to S_J$. Then there is a canonical closed immersion*

$$j: Z \longrightarrow \prod_{\theta\in J}\mathbf{P}(H^1_{\mathrm{dR}}(A_0^{(p^{-1})}/\mathbf{F})_\theta)$$

under which $j^*\mathcal{O}(1)_\theta$ is identified with $(R^1 s_*\mathcal{O}_{\widetilde{B}})_\theta$ for all $\theta\in J$, and Z is identified with the fibre over $(H^0(A_0,\Omega^1_{A_0/F})_{\phi\circ\theta})_\theta$ of the morphism

$$\begin{array}{ccc} \prod_{\theta\in J}\mathbf{P}(H^1_{\mathrm{dR}}(A_0^{(p^{-1})}/\mathbf{F})_\theta) & \longrightarrow & \prod_{\theta,\phi\circ\theta\in J}\mathbf{P}(H^1_{\mathrm{dR}}(A_0/\mathbf{F})_{\phi\circ\theta}) \\ (M_\theta)_\theta & \longmapsto & (M_\theta^p)_\theta. \end{array}$$

[18] This isomorphism can in fact be seen more directly as arising from the isomorphism of crystals obtained in the proof of Lemma 7.2.1.

Finally we observe that the construction of the immersion j and the identifications in the theorem are independent of the quasi-polarization in the data defining the closed point x of $T_{J'}$, and hence is invariant under the action of $\mathcal{O}_{F,(p),+}^{\times}$. It follows that if $x \in \overline{Y}_{J'}(\mathbf{F})$, then the fibre over x of the degeneracy map $\pi_1 : \overline{Y}_0(p)_J \to \overline{Y}$ induced by $(\underline{A}_1, \underline{A}_2, f) \mapsto \underline{A}_2$ is canonically isomorphic to a fibre of the morphism

$$\begin{array}{ccc} \prod_{\theta \in J} \mathbf{P}(V_{\phi\circ\theta}^{(p^{-1})}) & \longrightarrow & \prod_{\theta, \phi\circ\theta \in J} \mathbf{P}(V_{\phi\circ\theta}) \\ (M_\theta)_\theta & \longmapsto & (M_\theta^p)_\theta, \end{array}$$

where $V_{\phi\circ\theta}$ is the fibre at x of the automorphic bundle $\mathcal{V}_{\phi\circ\theta}$ on $\overline{Y}$. Furthermore the isomorphism identifies the pull-back of $\mathcal{O}(1)_\theta$ for each $\theta \in J$ with the pull-back of the automorphic bundle $\upsilon_\theta = \omega_\theta^{-1}\delta_\theta$ on $\overline{Y}$ under the degeneracy map $\pi_2 : \overline{Y}_0(p)_J \to \overline{Y}$ induced by $(\underline{A}_1, \underline{A}_2, f) \mapsto \underline{A}_2$.

7.3. **Cohomological vanishing.** In this section we prove Theorem E. First note that in the statement, we may replace $\mathbf{Z}_{(p)}$ by the faithfully flat extension $\mathcal{O}$, so the task is to prove that $R^i\pi_*\mathcal{K}_{Y_1(p)/\mathcal{O}} = 0$ for all $i > 0$ and sufficiently small open compact subgroups $U \subset G(\mathbf{A_f})$ of level prime to p, where π is the natural projection from $Y_1(p) := Y_{U_1(p)}(G)_{\mathcal{O}}$ to $Y := Y_U(G)_{\mathcal{O}}$ and $G = \mathrm{Res}_{F/\mathbf{Q}}(\mathrm{GL}_2)$. We proceed in a series of steps to reduce the proof to showing the vanishing of the cohomology of certain line bundles on the fibres of π which we have already described.

7.3.1. *Reduction steps.* We first reduce the problem to the setting of the special fibres of $Y_1(p)$ and Y. Note that since π is projective, $R^i\pi_*\mathcal{K}_{Y_1(p)/\mathcal{O}}$ is a coherent sheaf on Y. Since $\pi \otimes_{\mathcal{O}} L$ is finite, it follows that $R^i\pi_*\mathcal{K}_{Y_1(p)/\mathcal{O}}$ is supported on $\overline{Y} = Y_{\mathbf{F}}$. Letting $j_0 : \overline{Y} \to Y$ and $j_1 : \overline{Y}_1(p) \to Y_1(p)$ denote the immersions of special fibres and $\overline{\pi}$ the restriction of π, the natural map

$$j_0^* R^i\pi_*\mathcal{K}_{Y_1(p)/\mathcal{O}} \longrightarrow R^i\overline{\pi}_*(j_1^*\mathcal{K}_{Y_1(p)/\mathcal{O}}) = R^i\overline{\pi}_*\mathcal{K}_{\overline{Y}_1(p)}$$

is injective, so it suffices to prove that $R^i\overline{\pi}_*\mathcal{K}_{\overline{Y}_1(p)} = 0$ for all $i > 0$. Furthermore we may extend scalars and assume $\mathbf{F} = \overline{\mathbf{F}}_p$.

Next we reduce to the setting of the irreducible components of $\overline{Y}_0(p)$. Recall that $\overline{\pi} = \overline{\psi} \circ \overline{h}$ where $\overline{h} : \overline{Y}_1(p) \to \overline{Y}_0(p)$ is finite flat and $\overline{\psi} : \overline{Y}_0(p) \to \overline{Y}$ is the natural degeneracy map induced by $(\underline{A}_1, \underline{A}_2, f) \mapsto f$, so we have

$$R^i\overline{\pi}_*\mathcal{K}_{\overline{Y}_1(p)} = R^i\overline{\psi}_*(\overline{h}_*\mathcal{K}_{\overline{Y}_1(p)}).$$

Furthermore in §6.2.2, we defined a filtration on $\overline{h}_*\mathcal{K}_{\overline{Y}_1(p)}$ whose graded pieces are direct sums of direct images of line bundles on the strata $\overline{Y}_0(p)_J$. Recall also from §7.1.1 that the restriction of ψ to $\overline{Y}_0(p)_J$ factors through a projective Cohen–Macaulay morphism $\psi_J : \overline{Y}_0(p)_J \to \overline{Y}_{J'}$, where the stratum $\overline{Y}_{J'}$ is the smooth closed subscheme of $\overline{Y}$ defined by the vanishing of the partial Hasse invariants h_θ for $\theta \in J' := \{\theta \in J \mid \phi^{-1} \circ \theta \notin J\}$. Thus from the description of $\mathrm{gr}(\overline{h}_*\mathcal{K}_{\overline{Y}_1(p)})$ in §6.2.2, we see that it suffices to prove that

$$R^i\psi_{J,*}\left(\mathcal{J}_J^{-1}\mathcal{K}_{\overline{Y}_0(p)_J} i_J^*\mathcal{L}_{\chi^{-1}}^{-1}\right) = 0$$

for all characters $\chi : (\mathcal{O}_F/p\mathcal{O}_F)^\times \to \mathbf{F}^\times$, subsets $J \subset \Theta$ and integers $i > 0$.

Now we reduce to considering the fibres of the morphism from $\overline{Y}_0(p)_J$ to $\overline{Y}$ for each J. We let $s : \operatorname{Spec}\mathbf{F} \to \overline{Y}_{J'}$ be a closed point, let Z be the fibre of ψ_J at s, and let $j : Z \to \overline{Y}_0(p)_J$ be the inclusion. Since ψ_J is flat and projective, it suffices to prove that

$$H^i(Z, j^*(\mathcal{J}_J^{-1}\mathcal{K}_{\overline{Y}_0(p)_J} i_J^*\mathcal{L}_{\chi^{-1}}^{-1})) = 0$$

for all χ, J, s as above and $i > 0$. Furthermore since ψ_J is Cohen–Macaulay and $\overline{Y}_{J'}$ is smooth, we have

$$\mathcal{K}_{\overline{Y}_0(p)_J} = \mathcal{K}_{\overline{Y}_0(p)_J/\overline{Y}_{J'}} \otimes_{\mathcal{O}_{\overline{Y}_0(p)_J}} \mathcal{K}_{\overline{Y}_{J'}},$$

so that $j^*\mathcal{K}_{\overline{Y}_0(p)_J}$ is isomorphic to $\mathcal{K}_Z$. It therefore suffices to prove that

$$H^i(Z, \mathcal{K}_Z \otimes_{\mathcal{O}_Z} j^*(\mathcal{J}_J^{-1} i_J^*\mathcal{L}_{\chi^{-1}}^{-1})) = 0$$

for all χ, J, s as above and $i > 0$.

7.3.2. *Completion of the proof.* Recall that the fibre Z was computed in Theorem 7.2.4, where it is shown to be isomorphic to a product of projective lines and copies of $\operatorname{Spec}(\mathbf{F}[x]/(x^p))$. More precisely, we have that Z is isomorphic to

$$\prod_{\theta\in J\cap\Sigma} \mathbf{P}(V_{\phi\circ\theta}^{(p^{-1})})_\theta \times \prod_{\theta\in J-\Sigma} T_\theta \tag{45}$$

where T_θ is a finite subscheme of $\mathbf{P}(V_{\phi\circ\theta}^{(p^{-1})})_\theta$. Furthermore, this isomorphism identifies the pull-back of each $\mathcal{O}(1)_\theta$ with the restriction to Z of the line bundle obtained by descent from $(R^1 s_{2,*}\mathcal{O}_{A_2})_\theta$, where $f : A_1 \to A_2$ is the universal isogeny on $\widetilde{Y}_0(p)_{J,\mathbf{F}}$. Recall also that for $\theta \in J$, we have a canonical identification of $(R^1 s_{2,*}\mathcal{O}_{A_2})_\theta$ with the Raynaud line bundle $\mathcal{L}_\theta$ on $\widetilde{Y}_0(p)_{J,\mathbf{F}}$, and this identification descends to $\overline{Y}_0(p)_J$. In particular, if $\theta \in J \cap \Sigma$, then the restriction of $\mathcal{L}_\theta$ to Z corresponds under the above isomorphism to $\mathcal{O}(1)_\theta$. Similarly if $\theta \in J - \Sigma$, then the restriction of $\mathcal{L}_\theta$ corresponds to the restriction of $\mathcal{O}(1)_\theta$, which is (non-canonically) isomorphic to $\mathcal{O}_Z$. On the other other hand, if $\theta \notin J$, then we obtain a trivialization of $\mathcal{L}_\theta$ on Z from its identification with $s_{1,*}\Omega^1_{A_1/\mathbf{F}}$ on $\widetilde{Y}_0(p)_{J,\mathbf{F}}$. Combining this with the formulas (27) and (28), we conclude that

$$j^*(\mathcal{J}_J^{-1} i_J^*\mathcal{L}_{\chi^{-1}}^{-1}) \cong \bigotimes_{\theta\in J\cap\Sigma} \mathcal{O}(n_\theta)_\theta$$

where $n_\theta = m_\theta + 1 > 0$ for all $\theta \in J \cap \Sigma$. Since the dualizing sheaf of (45) is isomorphic to $\bigotimes_{\theta\in J\cap\Sigma} \mathcal{O}(-2)_\theta$, it follows that

$$H^i(Z, \mathcal{K}_Z \otimes_{\mathcal{O}_Z} j^*(\mathcal{J}_J^{-1} i_J^*\mathcal{L}_{\chi^{-1}}^{-1})) \cong H^i\left(\prod_{\theta\in J\cap\Sigma} \mathbf{P}^1_{\mathbf{F}}, \bigotimes_{\theta\in J\cap\Sigma} \mathcal{O}(n_\theta - 2)_\theta\right)^{p^{|J-\Sigma|}} = 0$$

for all $i > 0$, and this completes the proof of Theorem E.

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