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A spinorial double copy for $\mathcal{N} = 0$ supergravity

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Abstract: The Weyl double copy is a formula relating solutions of scalar, gauge and gravity theories, and is related to the BCJ double copy for scattering amplitudes. The latter relates Yang-Mills theory to $\mathcal{N} = 0$ supergravity, where an axion and dilaton are present in addition to the graviton. However, the traditional Weyl double copy applies only to pure gravity solutions, such that it remains to be seen whether or not it can be extended to the full spectrum of $\mathcal{N} = 0$ supergravity. We examine this question using recently developed twistor methods, showing that some sort of double copy formula for $\mathcal{N} = 0$ supergravity is indeed possible for certain solutions. However, it differs both from the traditional Weyl double copy form, and recent conjectures aimed at generalising the Weyl double copy to non-vacuum solutions.

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Contents

1 Introduction

Within the last couple of decades, a duality known as the *double copy* has generated a good deal of interest [1]. It was first observed in the 1980s as a relationship between tree-level scattering amplitudes for open and closed strings [2], known as the *KLT relations.* In taking the low energy limit, open and closed strings give rise to (non-abelian) gauge bosons and gravitons respectively, such that a relationship — the double copy [3, 4] — should hold between field theory scattering amplitudes. Importantly, the field theory double copy has been extended to loop level, going beyond its original stringy context, thus providing tantalising glimpses of deep common structures underlying our various theories of fundamental interactions. Further work has broadened the double copy beyond scattering amplitudes at fixed order in the coupling constant [5–8], and also to classical solutions. The latter may be exact [9–33], or perturbative [34–54], where the latter techniques are of high interest due to potential applications in gravitational wave physics. Non-perturbative aspects of the double copy have been explored in refs. [53, 55–66]. Recent comprehensive reviews of the double copy can be found in refs. [1, 67–70].

For classical solutions, two incarnations of the double copy have been particularly well-studied. The first, the Kerr-Schild double copy of ref. [9] (see also refs. [71–73]) uses the traditional tensorial formalism of field theory, and states that certain exact pure gravity solutions in position space can be written as simple products of kinematic information entering corresponding gauge theory solutions. This applies to a special class of solutions which happen to linearise their respective field theories, where a particular gauge is picked out (corresponding to the use of Kerr-Schild coordinates in gravity). A second exact classical double copy, relevant for the present paper, is the Weyl double copy of ref. [18]. This uses the spinorial formalism of field theory, and states that the gauge-independent Weyl spinor in General Relativity can be written as a certain combination of electromagnetic spinors, and a scalar field (see refs. [74, 75] for an earlier incarnation of this idea). This has been shown to work for abitrary vacuum Petrov type D solutions, as well as some type N solutions relevant for gravitational waves [21]. For solutions of linearised gauge / gravity theory, refs. [22, 24] found examples of other Petrov type, although argued that a generalisation of the Weyl double copy is needed for Petrov types I and III. Recently, the Weyl double copy has also been extended to describe non-vacuum solutions [28, 76].

It is an ongoing question to establish the general scope and validity of Weyl double copy-like formulae. We may also ask where they come from, and a number of works have recently shed light on this issue. Firstly, refs. [22, 24] formulated the double copy in twistor space. Certain "functions" in the latter are mapped to spacetime spinor fields by a formula known as the Penrose transform, and it was shown that a given product of twistor-space functions exactly reproduces the Weyl double copy in position space. A conceptual difficulty arises, however, in that the quantities that enter the Penrose transform can be subjected to certain equivalence transformations that leave the spacetime fields invariant. They are thus, strictly speaking, representatives of cohomology classes rather than functions, and as such cannot usually be meaningfully multipled together. This is not a problem for deriving the Weyl double copy in practice, but it does necessitate a prescription for picking out "special" representatives in twistor space, such that the correct position-space double copy is obtained. This was addressed further in refs. [26, 77], both of which gave suitable prescriptions, albeit with the irksome deficiency that neither choice obviously corresponds with the original twistor double copy of refs. [22, 24]. Finally, the situation was clarified recently in ref. [78], using ideas developed in refs. [79–82]. The latter references show that certain classical gauge / gravity solutions in position space[1] can be obtained as on-shell inverse Fourier transforms of momentum-space three-point amplitudes. Reference [80] then split such a transform into two stages, where the first maps amplitudes to twistor space, and the second corresponds to the Penrose transform from twistor to position space. The first stage turns out to correspond to a Laplace transform of the amplitude in energy, and this procedure will necessarily pick out a certain cohomology representative in twistor space. As shown in ref. [78], the representatives obtained for standard three-point amplitudes are precisely those used in the original twistor double copy of refs. [22, 24]. Thus, the BCJ double copy for 3-point amplitudes, the twistor double copy, and the position-space Weyl double copy amount to exactly the same thing, where overlap exists. Reference [78] also showed that the simple product-like form of the double copy in twistor space crucially relies on key properties of three-point amplitudes. In turn, exact position-space double

[1] As is made clear in refs. [80, 82] and below, these classical solutions are in (2,2) signature, rather than the conventional Minkowski spacetime with (1,3) signature. The latter can be obtained via analytic continuation of the coordinates.

copies are not expected to be generic. Thus, the twistor methods provide a way both to systematically derive Weyl double copy formulae, and to ascertain their scope.

An interesting analogue of the Weyl double copy was recently discovered in three spacetime dimensions. Dubbed the *Cotton double copy*, it relates classical solutions of topologically massive gauge and gravity theory, in the appropriate spinorial language [83, 84]. Unlike the 4d Weyl double copy, the Cotton double copy was found to only work for solutions of Petrov type N. Reference [85] confirmed this by applying similar twistor methods to the four-dimensional case of ref. [78], but now in a three-dimensional context. The appropriate twistor space in this case is called *minitwistor space*, and an appropriate generalisation of the Penrose transform must be used due to the presence of the topological mass. By applying similar methods to those developed in refs. [80–82], ref. [85] showed that this massive Penrose transform indeed arises naturally upon inverse Fourier transforming three-point amplitudes, entirely independently of its original presentation in the twistor literature [86]. The success of such methods in again deriving and constraining Weyl double-copy like formulae suggests that similar methods be used to look for possible Weyl double copies in cases that have not previously been considered.

With this motivation, we will here consider $\mathcal{N}=0$ supergravity, also known as NS-NS gravity. This theory first arose as the effective field theory emerging in the low energy limit of closed bosonic string theory (see e.g. refs. [87, 88] for textbook treatments). As such, it is the theory that is formally related to pure Yang-Mills theory by the double copy. Indeed, previous classical double copy formalisms have had to explain why solutions in pure gravity are obtained by the double copy, when one naturally expects additional degrees of freedom in the full $\mathcal{N}=0$ theory to be turned on [9, 46, 48, 49, 89]. The latter comprise a scalar — the *dilaton* — and a two-form field equivalent to a pseudo-scalar known as the *axion* in four spacetime dimensions. The situation appears to be that for classical solutions at linear order, one can choose whether or not one sources the dilaton and / or axion in the gravity theory. However, for higher-order classical and / or quantum corrections, one must introduce additional procedures to remove the non-gravitational degrees of freedom, should they be unwanted. This is indeed the most common situation for gravitational wave physics, but preserving the full spectrum offers the chance to ask conceptual questions about the double copy. In particular, exact position-space classical double copies for the full $\mathcal{N}=0$ theory are almost completely unexplored.[2] Furthermore, given that we now have techniques for systematically deriving Weyl double copy formulae, it surely makes sense to apply these to $\mathcal{N}=0$ supergravity, and then see what happens.

A number of recent studies have provided further inspiration. First, ref. [92] used similar methods to ref. [82] to write position-space solutions of $\mathcal{N}=0$ supergravity in terms of on-shell inverse Fourier transforms of momentum-space scattering amplitudes. They examined variants of the well-known Kerr and Taub-NUT solutions in pure gravity, but such that the dilaton and / or axion are also turned on. A direct consequence of this is that the solutions thus obtained are no longer vacuum solutions in gravity, and thus a

[2] Very interesting early work in this area set up a Kerr-Schild ansatz in the framework of double field theory [90, 91], although the solutions considered in this paper are distinct from this formalism.

greater variety of spinor fields in position space are needed, than in the traditional Weyl double copy of ref. [18]. The authors converted all spinor fields to the tensorial language, proving that the Riemann tensor can be written, in general, as a convolution of gauge-theoretic field strength tensors, and an inverse scalar field. An exact position-space double copy arises if this convolution is equivalent to a product, and ref. [92] showed explicitly how this occurs in the tensor language for pure gravity, but also that it is apparently broken when the dilaton and axion are present. Another source of inspiration is the recent work of refs. [28, 76], which has sought to extend the Weyl double copy in pure gravity to encompass non-vacuum solutions. The authors looked at solutions whose source currents / energy-momentum tensors can be written as a sum of terms which can each be meaningfully identified across different theories. Then, Weyl-double-copy-like formulae were proposed for the various spinor fields that enter the spinorial decomposition of the Riemann tensor. We will be able to compare our formulae in what follows with these conjectures.

The aim of this paper is to apply the twistor methods of refs. [78, 80, 85] to the starting point of ref. [92]. That is, we will look at the position-space spinor fields generated by the inverse on-shell Fourier transforms of three-point amplitudes in $\mathcal{N} = 0$ supergravity, and split the transform into two stages, where the first takes us from momentum to twistor-space. This additional step will allow us to reveal exact relationships between spinor fields in position-space that appear to have been overlooked in ref. [92]. For the case of dilaton-axion solutions with no NUT charge or spin, they indeed take the form of local products of spinor fields in position space, albeit dressed by additional factors that mean that they are not of the traditional Weyl-double copy form. Furthermore, whilst one may write similar formulae for the case of non-vanishing spin / NUT charge, they do not have a straightforward double copy interpretation, thus providing additional insights into the results of ref. [92]. We also compare our results with the proposed non-vacuum Weyl double copy formulae of refs. [28, 76]. For the cases we look at, we do not arrive at the same results. This is not a problem, given that we are examining a different situation in a different theory. However, this comparison perhaps suggests that similar methods to those used in this paper might also prove fruitful in examining the non-vacuum pure-gravity case.

The structure of our paper is as follows. In section 2, we review relevant details of the spinorial formalism, as well as the arguments of refs. [78, 80, 82, 92] which will be needed in what follows. In section 3, we apply our twistor methods to the case of a $\mathcal{N} = 0$ supergravity solution with no spin or NUT charge, finding our first spinorial double copy formula in position space. In section 4, we extend the analysis to include the effects of non-zero NUT charge and spin. Finally, we discuss our results and conclude in section 5.

2 Review of key ideas

2.1 The spinorial formalism

Throughout this paper, we will use the spinorial formalism of field theory, in which all field equations can be written in terms of two-component Weyl spinors π_A, and conjugate spinors

$\omega_{\dot{A}}$. Indices may be raised and lowered using the two-dimensional Levi-Civita symbol:

$$\pi_A = \epsilon_{BA}\pi^B, \quad \pi^B = -\pi_A\epsilon^{AB}, \tag{2.1}$$

where

$$\epsilon_{AB}\epsilon^{CB} = \delta_A^C, \quad \epsilon_{01} = 1, \tag{2.2}$$

and similarly for the dotted equivalent $\epsilon^{\dot{A}\dot{B}}$. Any tensorial quantity may be converted into a (multi-index) spinor by contracting its spacetime indices[3] with the *Infeld-van-der-Waerden symbols*

$$\sigma^a_{A\dot{A}} = (\mathrm{I}, i\sigma_y, \sigma_z, \sigma_x), \tag{2.3}$$

expressed in terms of the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.4}$$

We have here matched conventions with refs. [82, 92], which work in (2,2) signature, for reasons that will be clarified below. As an example, the spinorial translation of a 4-vector is

$$V_{A\dot{A}} \equiv V_a\,\sigma^a_{A\dot{A}} = \begin{pmatrix} V_0 + V_2 & V_1 + V_3 \\ V_3 - V_1 & V_0 - V_2 \end{pmatrix}. \tag{2.5}$$

The determinant of this matrix is

$$|V_{A\dot{A}}| = \left(V_0^2 + V_1^2 - V_2^2 - V_3^2\right) = V^2, \tag{2.6}$$

which vanishes for null vectors, such that for the latter one may decompose eq. (2.5) into an outer product of a spinor and conjugate spinor:

$$V_{A\dot{A}} = \pi_A\tilde{\pi}_{\dot{A}}, \quad V^2 = 0. \tag{2.7}$$

For later use, we also note the formula

$$V \cdot W = \frac{1}{2}V_{A\dot{A}}W^{A\dot{A}}. \tag{2.8}$$

The widespread use of the spinorial formalism relies on the fact that it makes certain structures manifest, that are more difficult to see in the tensorial framework. This simplification relies on two key properties, both of which ultimately arise from the fact that each spinor index may assume one of only two values. The first property is that all multi-index spinor objects can be decomposed into sums of products of symmetric spinors, and Levi-Civita symbols. Relevant for this paper is the spinorial translation of the field strength tensor F_{ab} in (linearised) gauge theory:

$$F_{ab} \to F_{A\dot{A}B\dot{B}} = \phi_{AB}\epsilon_{\dot{A}\dot{B}} + \tilde{\phi}_{\dot{A}\dot{B}}\epsilon_{AB}, \tag{2.9}$$

[3]Throughout, we use lower-case Latin letters, capital Latin letters and Greek letters to correspond to Lorentz, spinor and twistor indices respectively.

JHEP05(2023)047

where ϕ_{AB} and $\tilde{\phi}_{\dot{A}\dot{B}}$ are symmetric in their indices, and represent the self-dual and anti-self-dual parts of the field respectively. In $\mathcal{N} = 0$ supergravity, ref. [92] introduced a generalised Riemann tensor $\mathfrak{R}_{abcd}$ for $\mathcal{N} = 0$ supergravity, whose components represent the combined graviton, dilaton and axion. Its spinorial translation is

$$\mathfrak{R}_{A\dot{A}B\dot{B}C\dot{C}D\dot{D}} = \mathbf{X}_{ABCD}\,\epsilon_{\dot{A}\dot{B}}\,\epsilon_{\dot{C}\dot{D}} + \tilde{\mathbf{X}}_{\dot{A}\dot{B}\dot{C}\dot{D}}\,\epsilon_{AB}\,\epsilon_{CD} \tag{2.10}$$

$$+ \mathbf{\Phi}_{AB\dot{C}\dot{D}}\,\epsilon_{\dot{A}\dot{B}}\,\epsilon_{CD} + \tilde{\mathbf{\Phi}}_{\dot{A}\dot{B}CD}\,\epsilon_{AB}\,\epsilon_{\dot{C}\dot{D}}\,, \tag{2.11}$$

which is directly analogous to the usual spinor decomposition of the Riemann tensor in General Relativity (see e.g. refs. [93, 94]). For vacuum solutions in pure gravity, the mixed-index spinors are absent, and the quantity $\mathbf{X}_{ABCD}$ becomes known as the *Weyl spinor*, which we will denote in that context by Φ_{ABCD}. As first presented in ref. [18], the Weyl spinors for certain solutions (those of Petrov type D) can be expressed as a symmetrised product of gauge theory field strength spinors, divided by a scalar field $S(x)$:

$$\Phi_{ABCD} = \frac{\Phi_{(AB}\Phi_{CD)}}{S}, \quad \tilde{\Phi}_{\dot{A}\dot{B}\dot{C}\dot{D}} = \frac{\tilde{\Phi}_{(\dot{A}\dot{B}}\tilde{\Phi}_{\dot{C}\dot{D})}}{\bar{S}}, \tag{2.12}$$

where $\bar{S}$ is the complex conjugate of S (in Lorentzian signature), and the two Weyl spinors represent the self-dual and anti-self-dual graviton degrees of freedom respectively. This is the *Weyl double copy*, and it is the potential generalisation of these formulae to $\mathcal{N} = 0$ supergravity that we are seeking in this paper.

2.2 Twistors

Twistor theory is a well-established framework combining elements of algebraic geometry, topology and complex analysis (see e.g. [95–97] for reviews), that has become increasingly prevalent in contemporary research on scattering amplitudes. One way of introducing twistors is as solutions of the *twistor equation*

$$\nabla^{(A}_{\dot{A}}\Omega^{B)} = 0, \tag{2.13}$$

where $\nabla_{A\dot{A}}$ is the spinorial translation of the spacetime covariant derivative, and Ω^B a spinor field. The general solution of this equation in Minkowski space is

$$\Omega^A = \omega^A - x^{A\dot{A}}\pi_{\dot{A}}, \tag{2.14}$$

such that each solution can be characterised by a four-component *twistor*, containing two spinors:

$$Z^\alpha = \left(\omega^A, \pi_{\dot{A}}\right). \tag{2.15}$$

Twistor space $\mathbb{T}$ then consists of the set of all such objects, and we may define a map from twistor space to spacetime by defining the "location" of a twistor in Minkowksi space to be the locus of points such that the spinor field Ω^A vanishes. This implies the *incidence relation*

$$\omega^A = x^{A\dot{A}}\pi_{\dot{A}}, \tag{2.16}$$

which is invariant under simultaneous rescalings of both sides:

$$\omega^A \to \lambda\omega^A, \quad \pi_{\dot{A}} \to \lambda\pi_{\dot{A}}, \quad \lambda \in \mathbb{C}. \tag{2.17}$$

Consequently, twistors obeying the incidence relation constitute points in *projective twistor space* $\mathbb{PT}$. The map between spacetime and twistor space is non-local: it may be shown that a point $x^{A\dot{A}}$ in spacetime maps to a Riemann sphere X in $\mathbb{PT}$. One way to appreciate this is to note that eq. (2.16) implies that, upon knowing $x^{A\dot{A}}$, a twistor is entirely fixed by the single 2-spinor $\pi_{\dot{A}}$. Given that the latter is only defined projectively, we may parametrise all possible twistors satisfying the incidence relation for a given spacetime point using the two independent parametrisations

$$\pi_{\dot{A}} = (1, \xi), \quad \text{or} \quad \pi_{\dot{A}} = (\eta, 1), \quad \xi, \eta \in \mathbb{C}. \tag{2.18}$$

The parameters ξ and η then indeed correspond to two conventional coordinate patches for a Riemann sphere, defined via stereographic projection from the north or south pole to a complex plane through the equator. On the overlap, one identifies $\xi \sim \eta^{-1}$.

A key result of twistor theory known as the *Penrose transform* states that solutions of the linearised vacuum field equations for a spinor field of spin n can be obtained via the contour integral

$$\tilde{\Phi}_{\dot{A}_1\dot{A}_2\ldots\dot{A}_{2n}}(x) = \frac{1}{2\pi i}\oint_\Gamma \pi_{\dot{E}} d\pi^{\dot{E}}\, \pi_{\dot{A}_1}\pi_{\dot{A}_2}\ldots\pi_{\dot{A}_{2n}}\rho_x[f(Z^\alpha)], \tag{2.19}$$

where $f(Z^\alpha)$ is a holomorphic function of twistor variables, ρ_x denotes restriction to the Riemann sphere X corresponding to the spacetime point x^a, and the contour Γ on X is such that it separates any poles of the function $f(Z^\alpha)$. For this integral to make sense as being in *projective* twistor space, the function $f(Z^\alpha)$ must scale as

$$f(\lambda Z^\alpha) = \lambda^{-2n-2} f(Z^\alpha). \tag{2.20}$$

Furthermore, the quantities $f(Z^\alpha)$ entering the Penrose transform are not, strictly speaking, functions. One is free to redefine them according to the equivalence transformations:

$$f(Z^\alpha) \to f(Z^\alpha) + f_N(\alpha) + f_S(Z^\alpha), \tag{2.21}$$

where $f_N(Z^\alpha)$ ($f_S(Z^\alpha)$) has poles only in the northern (southern) hemisphere of the Riemann sphere X. Such contributions will vanish when performing the Penrose transform, due to having poles on only one side of the contour Γ. This infinite freedom to redefine $f(Z^\alpha)$ is stated more formally by referring to this quantity as a representative of a cohomology class [98], such that different representatives of the same cohomology class lead to the same spacetime field.

Note that the Penrose transform of eq. (2.19) gives only the anti-self-dual part of the field. One may obtain the self-dual part, given by an undotted spinor field, in different ways. One way is to consider the complex conjugate of the twistor equation of eq. (2.13)

$$\nabla_A^{(\dot{A}}\Lambda^{\dot{B})} = 0, \tag{2.22}$$

for which the general solution is:

$$\Lambda^{\dot{A}} = \mu^{\dot{A}} - x^{A\dot{A}}\lambda_A. \tag{2.23}$$

We may then combine the spinors appearing here to form a *dual twistor*

$$W_\alpha = \left(\lambda_A, \mu^{\dot{A}}\right), \tag{2.24}$$

satisfying the incidence relation

$$\mu^{\dot{A}} = x^{A\dot{A}}\lambda_A. \tag{2.25}$$

An inner product exists between (dual) twistors:

$$Z^\alpha W_\alpha = \omega^A \lambda_A + \mu^{\dot{A}}\pi_{\dot{A}}. \tag{2.26}$$

Furthermore, the analogue of the Penrose transform of eq. (2.19) in dual twistor space is

$$\tilde{\Phi}_{A_1 A_2 \ldots A_{2n}}(x) = \frac{1}{2\pi i}\oint_\Gamma \lambda_E d\lambda^E \, \lambda_{A_1}\lambda_{A_2}\ldots\lambda_{A_{2n}}\rho_x[f(W_\alpha)], \tag{2.27}$$

where again Γ is a contour on the Riemann sphere X in projective dual twistor space corresponding to the spacetime point x^a. To the uninitiated, the above concepts will be highly abstract, and we refer such a reader to detailed reviews for more details [95–97]. The relevance for the present study is that we will see the above Penrose transforms emerging naturally upon obtaining classical solutions in position space from momentum-space scattering amplitudes.

2.3 Spinor fields from amplitudes

Reference [81] introduced a systematic method for obtaining classical solutions from scattering amplitudes, which has become known as the *KMOC formalism*. This was subsequently used in refs. [82, 92] to construct linearised solutions in pure gravity and $\mathcal{N}=0$ supergravity. More specifically, the specific spinor fields entering eq. (2.11) can be obtained from on-shell inverse Fourier transforms of three-point amplitudes in (2,2) signature, where the reason for the latter is so that the relevant amplitudes are non-vanishing once all kinematic constraints are satisfied. Thus obtained, the fields can be analytically continued to conventional Lonrentzian (or indeed any other) signature. Let us define three-point amplitudes for a massive source particle emitting gravitons of a given helicity ($h^\pm$), dilaton (ϕ) or axion (B) radiation, with amplitudes $\mathcal{M}_X$, $X \in \{h, \phi, B\}$. Following ref. [92], we may define an alternative basis $\{\mathcal{M}_{\eta_1\eta_2}\}$ of amplitudes, where $\eta_i \in \{+1, -1\}$, and such that the physical amplitudes are given by

$$\mathcal{M}_{h^\pm} = \mathcal{M}_{\pm\pm}; \tag{2.28}$$

$$\begin{aligned}\mathcal{M}_\phi &= \frac{1}{2}\Big(\mathcal{M}_{+-} + \mathcal{M}_{-+}\Big);\\ \mathcal{M}_B &= \frac{1}{2}\Big(\mathcal{M}_{+-} - \mathcal{M}_{-+}\Big).\end{aligned} \tag{2.29}$$

JHEP05(2023)047

The motivation for this definition is the double copy for scattering amplitudes, which states that given degrees of freedom in $\mathcal{N}=0$ supergravity can be obtained as certain combinations of polarisation states from two separate gauge theories. In its simplest form, this holds for plane wave states, where it is known that the polarisation tensors for gravitons of given helicity can be obtained as outer products of photon polarisation vectors:

$$\epsilon^{\pm}_{ab} = \epsilon^{\pm}_{a}\epsilon^{\pm}_{b}. \tag{2.30}$$

Likewise, the two independent combinations

$$\frac{1}{2}\left(\epsilon^{+}_{a}\epsilon^{-}_{b} \pm \epsilon^{-}_{a}\epsilon^{+}_{b}\right) \tag{2.31}$$

turn out to describe the dilaton and axion respectively. The non-trivial statement of the double copy is that this product structure survives even when interactions are included, so that multi-point amplitudes involving gravitons, axions and dilatons can be obtained by combining gauge theory amplitudes in appropriate combinations. For three-point amplitudes involving radiation of a single $(h^{\pm},\phi,B)$ state this is very simple, and corresponds precisely to eq. (2.29), where the amplitudes $\mathcal{M}_{\eta_1\eta_2}$ are given by simple products of gauge theory three-point amplitudes with the appropriate helicity. Motivated by this, ref. [92] proposes a slightly more general relationship

$$\mathcal{M}_{\eta_L\eta_R} = -\frac{\kappa}{4\,Q^2}\, c_{\eta_L\eta_R}\, \mathcal{A}^{(L)}_{\eta_L}\, \mathcal{A}^{(R)}_{\eta_R}\,, \tag{2.32}$$

where $\kappa=\sqrt{32\pi G_N}$ is the gravitational coupling constant in terms of the Newton constant G_N, and Q the electromagnetic coupling of the source particle in the gauge theory. We also adopt the notation of ref. [92], such that the two gauge theories that are double-copied to make the gravity theory are referred to as the "left" (L) and "right" (R) theories. The additional constants $\{c_{\eta_L\eta_R}\}$ allow different normalisations for different helicity combinations, where the traditional BCJ double copy has these all equal to one.

Using the so-called *KMOC formalism* of ref. [81], ref. [92] showed that the classically observed values for the spinor fields appearing in eq. (2.11) at linearised level — derived as expectation values of quantum field operators — are given in terms of the above amplitudes via

$$\mathbf{X}_{ABCD} = -\frac{\kappa^2 c_{++}}{2Q^2}\ \mathrm{Re}\ i\int \mathrm{d}\Phi(k)\hat{\delta}(2p\cdot k)\,\mathcal{A}^{(L)}_{+}\mathcal{A}^{(R)}_{+}\ |k\rangle_A|k\rangle_B|k\rangle_C|k\rangle_D\, e^{-ik\cdot x}\ , \tag{2.33}$$

$$\tilde{\mathbf{X}}_{\dot{A}\dot{B}\dot{C}\dot{D}} = -\frac{\kappa^2 c_{--}}{2Q^2}\ \mathrm{Re}\ i\int \mathrm{d}\Phi(k)\hat{\delta}(2p\cdot k)\,\mathcal{A}^{(L)}_{-}\mathcal{A}^{(R)}_{-}\ [k|_{\dot{A}}[k|_{\dot{B}}[k|_{\dot{C}}[k|_{\dot{D}}\, e^{-ik\cdot x}\ , \tag{2.34}$$

$$\mathbf{\Phi}_{AB\dot{C}\dot{D}} = +\frac{\kappa^2 c_{+-}}{2\,Q^2}\ \mathrm{Re}\, i\int \mathrm{d}\Phi(k)\hat{\delta}(2p\cdot k)\,\mathcal{A}^{(L)}_{+}\mathcal{A}^{(R)}_{-}\ |k\rangle_A|k\rangle_B[k|_{\dot{C}}[k|_{\dot{D}}\, e^{-ik\cdot x}\,, \tag{2.35}$$

$$\tilde{\mathbf{\Phi}}_{\dot{A}\dot{B}CD} = +\frac{\kappa^2 c_{-+}}{2\,Q^2}\ \mathrm{Re}\, i\int \mathrm{d}\Phi(k)\hat{\delta}(2p\cdot k)\,\mathcal{A}^{(L)}_{-}\mathcal{A}^{(R)}_{+}\ [k|_{\dot{A}}[k|_{\dot{B}}|k\rangle_C|k\rangle_D\, e^{-ik\cdot x}\ . \tag{2.36}$$

Here we have introduced the Lorentz-invariant phase space measure

$$d\Phi(k) = \frac{d^4k}{(2\pi)^4}\hat{\delta}(k^2)\Theta(k^0), \tag{2.37}$$

the normalised delta function

$$\hat{\delta}(x) = 2\pi\delta(x), \tag{2.38}$$

and the conventional bra-ket notation whereby the spinorial translation of the null radiation momentum k^a is written as

$$k_{A\dot{A}} = |k\rangle_A [k|_{\dot{A}}. \tag{2.39}$$

In words, eqs. (2.33)–(2.36) constitute the fact that classical spinor fields in $\mathcal{N}=0$ supergravity at linearised level can be expressed in terms of on-shell inverse Fourier transforms of three-point amplitudes, where the latter are obtained by double-copying gauge theory amplitudes according to eq. (2.32). Reference [92] used these formulae as a starting point for examining whether or not one may formulate exact position-space double copies for particular solutions, after converting all spinor fields into the tensorial language. We will consider the same family of solutions here, namely those constructed from gauge theory amplitudes

$$\mathcal{A}_\eta^{(L,R)} = -2Q(p\cdot\epsilon_\eta)e^{\eta(\theta_{L,R}+ik\cdot a_{L,R})}, \tag{2.40}$$

for some vectors $\{a_{R,L}\}$ and constant parameters $\theta_{L,R}$. Here

$$p^a = Mu^a \tag{2.41}$$

is the 4-momentum of the source, with mass M and 4-velocity u^a. We have also introduced polarisation vectors for the photon in each theory, explicit realisations of which are given by (see e.g. ref. [82])

$$\epsilon_-^a = (\epsilon_+^a)^* = -\frac{\langle k|\sigma^a|l]}{\sqrt{2}[kl]}, \tag{2.42}$$

where $|l]^{\dot{A}}$ is an arbitrary null reference spinor, corresponding to a gauge choice. The physical interpretation of the amplitudes in eq. (2.40) has been explored in detail in ref. [99]. Without the exponential factor (i.e. for $a^a=\theta=0$ in a given gauge theory), the amplitude describes photon emission from a spinless particle with electric charge, and thus gives rise to the Coulomb solution. The double copy of this in pure gravity is the Schwarzschild solution [9], and in $\mathcal{N}=0$ supergravity leads to the *JNW solution* [100], in which a static, spherically-symmetric black hole is dressed by a non-zero dilaton profile. Upon turning on the vector a^a in gauge theory or pure gravity, a spin is generated for the source particle, such that a^a can be identified with its Pauli-Lubanski spin pseudo-vector. In pure gravity the resulting field is that of the *Kerr solution* [101], and its gauge theory single copy is known as $\sqrt{\text{Kerr}}$. The exponential factor appearing in eqs. (2.40) then constitutes the *Newman-Janis shift*, a complex transformation that transforms the Schwarzschild into the Kerr black hole [102]. The additional constant factor in the exponent transforms to a different solution in pure gravity, namely the *NUT solution* of refs. [103, 104]. This is a generalisation of the Schwarzschild black hole that has a non-vanishing rotational character to the gravitational field at asymptotic infinity. Its single copy is a dyon, where the NUT charge in the gravity theory maps to the magnetic monopole charge in the gauge theory [10]. In $\mathcal{N}=0$ supergravity, double-copying the amplitudes of eq. (2.40) will thus lead to generalisations of the

JHEP05(2023)047

JNW black hole, in which both spin and NUT charge are present. This in turn entails the possibility that the axion may turn on in addition to the dilaton, as discussed in ref. [92].

In the remainder of this paper, we will adopt the starting point of eqs. (2.33)–(2.36) as in ref. [92], but take a different approach in looking for position-space double copies. That is, we will stick with the spinorial language, and also use the twistor methods of refs. [78, 80, 85], that have recently proved successful in deriving Weyl-double copy formulae in pure gravity. As a warm-up, we consider the simplest solution — corresponding to double-copying the Coulomb charge — in the following section.

3 A spinorial double copy for the JNW solution

3.1 From momentum to twistor space

In refs. [78, 85] (motivated by ref. [80]), the inverse Fourier transform appearing in eqs. (2.33)–(2.36) is split into two stages, such that one considers an intermediate twistor space between momentum and position space. This in turn provides additional insights allowing one to systematically derive position-space double copy formulae, and to ascertain when they apply. To illustrate this idea, let us focus on the gauge theory analogue of eqs. (2.33)–(2.36), namely the fact that the field strength spinors corresponding to the amplitudes of eq. (2.40) are given by

$$\langle \Phi_{AB} \rangle = \mathrm{Re}\, \frac{\sqrt{2}}{M} \int \mathrm{d}\Phi(k) \hat{\delta}(u \cdot k)\, \mathcal{A}_+ \, |k\rangle_A |k\rangle_B \, e^{-ik\cdot x} \,, \tag{3.1}$$

$$\langle \tilde{\Phi}_{\dot{A}\dot{B}} \rangle = \mathrm{Re}\, \frac{\sqrt{2}}{M} \int \mathrm{d}\Phi(k) \hat{\delta}(u \cdot k)\, \mathcal{A}_- \, [k_{\dot{A}}|\, [k_{\dot{B}}| \, e^{-ik\cdot x} \,. \tag{3.2}$$

In order to carry out the phase-space integral, one may make the change of variables

$$k_{A\dot{A}} = \omega \lambda_A \tilde{\lambda}_{\dot{A}} + \xi q_{A\dot{A}}, \tag{3.3}$$

where $q_{A\dot{A}}$ is the spinorial translation of an arbitrary constant null 4-vector, and the spinors λ_A and $\tilde{\lambda}_{\dot{A}}$ are defined only up to an overall scaling, which may be absorbed into ω. The various bra-ket symbols above are then given in these new variables by

$$|k\rangle_A = \omega^{1/2}\lambda_A, \quad \langle k|^A = \omega^{1/2}\lambda^A, \quad [k|_{\dot{A}} = \omega^{1/2}\tilde{\lambda}_{\dot{A}}, \quad |k]^{\dot{A}} = \omega^{1/2}\tilde{\lambda}^{\dot{A}}. \tag{3.4}$$

If we choose to parametrise the spinors via

$$\lambda_A = (1, z), \quad \tilde{\lambda}_{\dot{A}} = (1, \tilde{z}), \quad z, \tilde{z} \in \mathbb{C}, \tag{3.5}$$

then the change of variables is from the four components of k^a to the set $(\omega, \xi, z, \tilde{z})$. After evaluating the Jacobian, one finds

$$d\Phi(k) = \frac{dz d\tilde{z} d\omega d\xi \delta(\xi)\omega}{4(2\pi)^3}, \tag{3.6}$$

such that the field strength spinor with dotted indices becomes

$$\tilde{\Phi}_{\dot{A}\dot{B}} = \frac{2\sqrt{2}}{4M(2\pi)^2}\, \mathrm{Re} \int dz d\tilde{z} d\omega d\xi \, \delta(\xi)\delta(u\cdot k)\omega^2 \Theta(\omega) \mathcal{A}_- \, \tilde{\lambda}_{\dot{A}}\tilde{\lambda}_{\dot{B}} e^{-ik\cdot x}. \tag{3.7}$$

The ξ integral can be carried out immediately, and simply sets $\xi = 0$. This corresponds to the fact that ξ in eq. (3.3) parametrises how far k^a is from being null, and thus on-shell. However, eq. (3.1) manifestly contains an on-shell Fourier transform, and thus the vanishing of ξ enforces this on-shell condition. To go further in carrying out the integral, let us substitute the explicit form of the Coulomb amplitude, namely that obtained from eq. (2.40) by setting $\theta = a^a = 0$. From eq. (2.42) we find

$$\mathcal{A}_- = \frac{2MQ}{\sqrt{2}} \frac{u_{A\dot{A}}\lambda^A l^{\dot{A}}}{\tilde{\lambda}_{\dot{B}} l^{\dot{B}}}, \tag{3.8}$$

such that eq. (3.7) becomes

$$\begin{aligned} \tilde{\Phi}_{\dot{A}\dot{B}} = \frac{Q}{(2\pi)^2} \mathrm{Re} \int dz d\tilde{z} d\omega \delta \left(u_{A\dot{A}} \lambda^A \tilde{\lambda}^{\dot{A}} \right) \omega \Theta(\omega) \tilde{\lambda}_{\dot{A}} \tilde{\lambda}_{\dot{B}} e^{-\frac{i\omega}{2}\lambda_A \tilde{\lambda}_{\dot{A}} x^{A\dot{A}}} \\ \times \frac{u_{A\dot{A}}\lambda^A l^{\dot{A}}}{\tilde{\lambda}_{\dot{B}} l^{\dot{B}}}. \end{aligned} \tag{3.9}$$

The remaining delta function sets

$$\lambda_A \propto u_{A\dot{A}} \tilde{\lambda}^{\dot{A}} \tag{3.10}$$

which, as emphasised in ref. [78], can be turned into an equality by reparametrising

$$\lambda_A = \left(\frac{1}{\sqrt{z}}, \sqrt{z} \right), \quad \tilde{\lambda}_{\dot{A}} = \left(\frac{1}{\sqrt{-\tilde{z}}}, -\sqrt{-\tilde{z}} \right). \tag{3.11}$$

One may then use the delta function to eliminate the z integral, yielding

$$\tilde{\Phi}_{\dot{A}\dot{B}} = \frac{Q}{(2\pi)^2} \mathrm{Re} \int d\tilde{z}\, \tilde{\lambda}_{\dot{A}} \tilde{\lambda}_{\dot{B}} \mathfrak{M}(\tilde{\lambda}_{\dot{A}}), \tag{3.12}$$

where

$$\begin{aligned} \mathfrak{M}(\tilde{\lambda}_{\dot{A}}) &= \int d\omega\, \omega\, \Theta(\omega) e^{-\frac{i\omega}{2} u_A{}^{\dot{B}} x^{A\dot{A}} \tilde{\lambda}_{\dot{A}} \tilde{\lambda}_{\dot{B}}} \\ &= -\frac{4}{[u_A{}^{\dot{B}} x^{A\dot{A}} \tilde{\lambda}_{\dot{A}} \tilde{\lambda}_{\dot{B}}]^2}. \end{aligned} \tag{3.13}$$

At this stage, we may reparametrise back to the original definition of $\tilde{z}$. From eq. (3.5), it then follows that

$$d\tilde{z} = \tilde{\lambda}_{\dot{E}} d\lambda^{\dot{E}} \tag{3.14}$$

is the projective measure on the Riemann sphere parametrised (in a particular coordinate patch) by $\tilde{z}$. Next, we may note that $\mathfrak{M}(\tilde{\lambda}_{\dot{A}})$ depends on $\tilde{\lambda}_{\dot{A}}$ through the specific combinations $\tilde{\lambda}_{\dot{A}}$ and

$$\omega^A = x^{A\dot{A}} \tilde{\lambda}_{\dot{A}},$$

where we can recognise the incidence relation of eq. (2.16). Thus, the quantity

$$Z^\alpha = (\tilde{\lambda}_{\dot{A}}, \omega^A) \tag{3.15}$$

is a bona fide point in projective twistor space $\mathbb{PT}$! We may also then write the functional dependence in eq. (3.13) as $\mathfrak{M}(\tilde{\lambda}_{\dot{A}}) \equiv \rho_x[\mathfrak{M}(Z^\alpha)]$, such that eq. (3.12) becomes

$$\tilde{\Phi}_{\dot{A}\dot{B}} = -\frac{Q}{(2\pi)^2}\,\mathrm{Re}\oint \tilde{\lambda}_{\dot{E}} d\tilde{\lambda}^{\dot{E}}\,\tilde{\lambda}_{\dot{A}}\tilde{\lambda}_{\dot{B}}\,\rho_x[\mathfrak{M}(Z^\alpha)]. \tag{3.16}$$

It is now straightforward to recognise the Penrose transform of eq. (2.19). Furthermore, the function of eq. (3.13) is homogeneous of degree -4 under rescalings of Z^α (and thus $\tilde{\lambda}_{\dot{A}}$), in agreement with eq. (2.20).

Let us summarise what has happened. We started by expressing spacetime (spinor) fields as inverse on-shell Fourier transforms of momentum-space scattering amplitudes. Next, we transformed to spinor variables, and found out that carrying out "half" of the Fourier transform takes our amplitude into twistor space. Indeed, eq. (3.13) is a variant of the so-called half Fourier transform used in the seminal work of ref. [105], which originated the modern use of twistor methods in scattering amplitude research. Here it takes the form of a Laplace transform in ω, which from eq. (3.3) can be interpreted as the energy of the radiation. Note that, in defining a precise form for $\mathfrak{M}(Z^\alpha)$, the half transform defines a particular cohomology representative in twistor space for a given classical spacetime field i.e. one that is "picked out" by the amplitude. Reference [78] used this observation to resolve cohomological ambiguities in the twistor double copy of refs. [22, 24].

3.2 Consistency relation between (anti-)self dual field strength spinors

In eq. (3.16), we have shown that the anti-self-dual field strength spinor can be obtained as an explicit Penrose transform of a cohomology representative derived from a momentum-space scattering amplitude. One may also perform a similar exercise for the self-dual spinor, starting from eq. (3.1), and such that the analogue of eq. (3.9) is found to be

$$\begin{aligned}\Phi_{AB} = \frac{Q}{(2\pi)^2}\,\mathrm{Re}\int dz d\tilde{z} d\omega\,\delta\left(u_{A\dot{A}}\lambda^A\tilde{\lambda}^{\dot{A}}\right)\omega\Theta(\omega)\lambda_A\lambda_B e^{-\frac{i\omega}{2}\lambda_A\tilde{\lambda}_{\dot{A}}x^{A\dot{A}}}\\ \times\frac{u_{A\dot{A}}l^A\tilde{\lambda}^{\dot{A}}}{l^A\lambda_A}.\end{aligned} \tag{3.17}$$

In this case, we can use the delta function to set

$$\tilde{\lambda}_{\dot{A}} = u_{A\dot{A}}\lambda^A, \tag{3.18}$$

namely the inverse relation of eq. (3.10). This eliminates the $\tilde{z}$ integral in eq. (3.17), such that one gets

$$\Phi_{AB} = \frac{Q}{(2\pi)^2}\,\mathrm{Re}\int dz \lambda_A\lambda_B\mathfrak{N}(\lambda_A), \tag{3.19}$$

where

$$\begin{aligned}\mathfrak{N}(\lambda_A) &= \int d\omega\omega\Theta(\omega)e^{\frac{i\omega}{2}u^B{}_{\dot{A}}x^{A\dot{A}}\lambda_A\lambda_B}\\ &= \frac{-4}{[u^B{}_{\dot{A}}x^{A\dot{A}}\lambda_A\lambda_B]^2}.\end{aligned} \tag{3.20}$$

Recognising that this depends only on λ_A and

$$\mu^{\dot{A}} = x^{A\dot{A}}\lambda_A,$$

we see that $\mathfrak{N}(\lambda_A) \equiv \rho_x[\mathfrak{N}(W_\alpha)]$ is defined on dual twistor space, where W_α is given as in eq. (2.24). Equation (3.19) then becomes

$$\Phi_{AB} = \frac{Q}{(2\pi)^2}\,\mathrm{Re}\oint \lambda_E d\lambda^E\,\lambda_A\lambda_B\rho_x[\mathfrak{N}(W_\alpha)], \tag{3.21}$$

and we recover the well-known result that anti-self-dual (self-dual) solutions are associated with Penrose transforms from twistor space (dual twistor space) respectively. However, a fact that was overlooked in the recent ref. [78] is that it is also possible to derive a consistency relation between the (anti-)self-dual spinor fields, for this particular solution. Returning to eq. (3.9), we may choose to eliminate $\tilde{\lambda}_{\dot{A}}$ using the delta function, rather than λ_A i.e. by using eq. (3.18) rather than eq. (3.10). We then find

$$\tilde{\Phi}_{\dot{A}\dot{B}} = u^A{}_{\dot{A}}u^B{}_{\dot{B}}\left[\frac{Q}{(2\pi)^2}\,\mathrm{Re}\int dz\,\lambda_A\lambda_B\rho_x[\mathfrak{N}(\lambda_A)]\right], \tag{3.22}$$

where $\mathfrak{N}(\lambda_A)$ is defined as in eq. (3.20). In other words, we have obtained the relationship

$$\tilde{\Phi}_{\dot{A}\dot{B}}(x) = u^A{}_{\dot{A}}u^B{}_{\dot{B}}\Phi_{AB}(x), \tag{3.23}$$

whose inverse — as may be derived by using eq. (3.10) in eq. (3.17) — is

$$\Phi_{AB} = u_A{}^{\dot{A}}u_B{}^{\dot{B}}\tilde{\Phi}_{\dot{A}\dot{B}}. \tag{3.24}$$

It is instructive to see how this relation actually works in practice, by finding the explicit forms of the spinor fields implied by the Penrose transforms of eqs. (3.16), (3.21). Let us first note that we may write the combination appearing in eq. (3.13), using the parametrisation of eq. (3.5), as

$$u_A{}^{\dot{B}}x^{A\dot{A}}\tilde{\lambda}_{\dot{A}}\tilde{\lambda}_{\dot{B}} = \tilde{N}^{-1}(x)(\tilde{z}-\tilde{z}_1)(\tilde{z}-\tilde{z}_2), \tag{3.25}$$

where an explicit calculation yields

$$\tilde{N}^{-1}(x) = x^{0\dot{1}}, \quad \tilde{z}_{1,2} = \frac{\left(x^{1\dot{1}} - x^{0\dot{0}}\right) \pm \sqrt{(x^{0\dot{0}})^2 + (x^{1\dot{1}})^2 - 2x^{0\dot{0}}x^{1\dot{1}} + 4x^{0\dot{1}}x^{1\dot{0}}}}{2x^{0\dot{1}}}. \tag{3.26}$$

From eqs. (3.12), (3.13), we then have

$$\tilde{\Phi}_{\dot{A}\dot{B}} = \frac{4Q}{4\pi^2}\tilde{N}^2(x)\,\mathrm{Re}\oint d\tilde{z}\,\frac{(1,\tilde{z})_{\dot{A}}(1,\tilde{z})_{\dot{B}}}{(\tilde{z}-\tilde{z}_1)^2(\tilde{z}-\tilde{z}_2)^2}, \tag{3.27}$$

and we may carry out the $\tilde{z}$ integral by enclosing the pole at $\tilde{z} = \tilde{z}_1$ (see e.g. ref. [24] for a similar calculation). The result is

$$\tilde{\Phi}_{\dot{A}\dot{B}} = -\frac{4Q}{2\pi}\frac{1}{[x^{0\dot{1}}]^2}\,\mathrm{Re}\,i\left[\frac{\tilde{\alpha}_{(\dot{A}}\tilde{\beta}_{\dot{B})}}{(\tilde{z}_1-\tilde{z}_2)^3}\right], \tag{3.28}$$

where the principal spinors are given by

$$\tilde{\alpha}_{\dot{A}} = (1, \tilde{z}_1), \quad \tilde{\beta}_{\dot{B}} = (1, \tilde{z}_2). \tag{3.29}$$

Substituting the result of eq. (3.26) and simplifying, one finds

$$\tilde{\Phi}_{\dot{A}\dot{B}} = \frac{4Q}{2\pi} \operatorname{Re} i \left[\frac{1}{\Lambda^{3/2}(x)} \begin{pmatrix} x^{0\dot{1}} & x^{1\dot{1}} - x^{0\dot{0}} \\ x^{1\dot{1}} - x^{0\dot{0}} & -x^{1\dot{0}} \end{pmatrix} \right], \tag{3.30}$$

where

$$\Lambda = (x^{0\dot{0}})^2 + (x^{1\dot{1}})^2 - 2x^{0\dot{0}}x^{1\dot{1}} + 4x^{0\dot{1}}x^{1\dot{0}}. \tag{3.31}$$

Likewise, one may carry out the Penrose transform in eq. (3.21), and the result is

$$\Phi_{AB} = -\frac{4Q}{2\pi} \operatorname{Re} i \left[\frac{1}{\Lambda^{3/2}(x)} \begin{pmatrix} x^{1\dot{0}} & x^{1\dot{1}} - x^{0\dot{0}} \\ x^{1\dot{1}} - x^{0\dot{0}} & -x^{0\dot{1}} \end{pmatrix} \right]. \tag{3.32}$$

To confirm the relation of eq. (3.23), we may note that the static 4-velocity $u^a = (1, \mathbf{0})$ implies

$$u^A{}_{\dot{A}} = \epsilon^{BA} u_{A\dot{A}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.33}$$

Then we have

$$u^A{}_{\dot{A}} u^B{}_{\dot{B}} \Phi_{AB} = -\frac{4Q}{2\pi} \operatorname{Re} i \left[\frac{1}{\Lambda^{3/2}(x)} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x^{1\dot{0}} & x^{1\dot{1}} - x^{0\dot{0}} \\ x^{1\dot{1}} - x^{0\dot{0}} & -x^{0\dot{1}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right],$$

which indeed agrees with eq. (3.30). Note that we have here chosen to express the field-strength spinor in terms of the components $x^{A\dot{A}}$, rather than substitute explicit spacetime coordinates. It is then straightforward to evaluate these formula in either (2,2) or (1,3) signature.

3.3 A double copy formula for the JNW solution

We now have all the ingredients we need to ascertain the existence — or otherwise — of a double copy formula for the JNW solution. Let us start with the spinor $\tilde{\mathbf{X}}_{\dot{A}\dot{B}\dot{C}\dot{D}}$, given in terms of an amplitude by eq. (2.33). Carrying out similar steps to that leading to eq. (3.16), we find

$$\tilde{\mathbf{X}}_{\dot{A}\dot{B}\dot{C}\dot{D}} = \frac{\kappa^2 c_{--} M}{(2\pi)^2} \operatorname{Re} \oint \tilde{\lambda}_{\dot{E}} d\tilde{\lambda}^{\dot{E}}\, \tilde{\lambda}_{\dot{A}} \tilde{\lambda}_{\dot{B}} \tilde{\lambda}_{\dot{C}} \tilde{\lambda}_{\dot{D}}\, \rho_x \left[\frac{4}{(\tilde{\lambda}_{\dot{A}} U_A{}^{\dot{B}} \tilde{\lambda}_{\dot{B}} x^{A\dot{A}})^3} \right], \tag{3.34}$$

where we have substituted the explicit form of the amplitudes of eq. (2.40), and again used the delta function $\delta(u \cdot k)$ to impose the condition of eq. (3.10). Carrying out the Penrose transform using the parametrisation of eq. (3.5) (see e.g. ref. [24] for a similar calculation), one finds

$$\tilde{\mathbf{X}}_{\dot{A}\dot{B}\dot{C}\dot{D}} = \frac{\kappa^2 c_{--} iM}{2\pi} \operatorname{Re} \left[\frac{\tilde{N}^3(x)}{(\tilde{z}_1 - \tilde{z}_2)^5} \tilde{\alpha}_{(\dot{A}} \tilde{\beta}_{\dot{B}} \tilde{\alpha}_{\dot{C}} \tilde{\beta}_{\dot{D})} \right], \tag{3.35}$$

JHEP05(2023)047

where $\tilde{N}(x)$, $\tilde{z}_{1,2}$ and the principal spinors $\tilde{\alpha}_{\dot{A}}$ and $\tilde{\beta}_{\dot{A}}$ have been defined in eqs. (3.26), (3.29). One may also define a scalar field

$$\tilde{S} \propto \frac{N(x)}{\tilde{z}_1 - \tilde{z}_2}, \tag{3.36}$$

which can be found to satisfy the massless Klein-Gordon equation. Then comparison of eq. (3.35) with eqs. (3.28), (3.36) implies the relationship

$$\tilde{\mathbf{X}}_{\dot{A}\dot{B}\dot{C}\dot{D}} = \frac{\tilde{\Phi}_{(\dot{A}\dot{B}}\tilde{\Phi}_{\dot{C}\dot{D})}}{\tilde{S}}, \tag{3.37}$$

where all constant factors have been absorbed into the scalar function $\tilde{S}(x)$. This is a precise analogue of the Weyl double copy formula of eq. (2.12), and indeed its derivation using twistor methods is exactly the same as in ref. [78]. Similar arguments may be used to verify the corresponding relationship

$$\mathbf{X}_{ABCD} = \frac{\Phi_{(AB}\Phi_{CD)}}{S}, \tag{3.38}$$

where the scalar field S is defined by

$$S(x) \propto \frac{N(x)}{z_1 - z_2}, \quad N^{-1}(x) = x^{1\dot{0}}. \tag{3.39}$$

Given the close analogues of these formulae with their pure gravity counterparts, it is perhaps not surprising that they occur. A more interesting question is whether or not a double copy formula emerges for the mixed-index spinors appearing in eq. (2.11), which have no counterpart for vacuum solutions in pure gravity. Let us first consider

$$\begin{aligned}\langle \mathbf{\Phi}_{AB\dot{C}\dot{D}}\rangle = \frac{\kappa^2 c_{+-} M^2}{4(2\pi)^2}\,\mathrm{Re}\, i \int dz d\tilde{z} d\omega \omega^3 \Theta(\omega)\delta(2p\cdot k)\, \lambda_A\lambda_B\tilde{\lambda}_{\dot{C}}\tilde{\lambda}_{\dot{D}}\, e^{-\frac{i\omega}{2}\lambda_A\tilde{\lambda}_{\dot{A}}x^{A\dot{A}}} \\ \times \left(\frac{u_{A\dot{A}}\lambda^A l^{\dot{A}}}{\tilde{\lambda}_{\dot{B}} l^{\dot{B}}}\right)\left(\frac{u_{D\dot{D}} l^D \tilde{\lambda}^{\dot{D}}}{l^C\lambda_C}\right),\end{aligned} \tag{3.40}$$

where we have substituted the amplitudes of eq. (2.40) into eq. (2.35). We may then use the delta function to implement either of the conditions of eq. (3.10) or eq. (3.18). Choosing the former, we get

$$\langle \mathbf{\Phi}_{AB\dot{C}\dot{D}}\rangle = \frac{\kappa^2 c_{+-} M}{4(2\pi)^2} u^C{}_{\dot{C}} u^D{}_{\dot{D}}\,\mathrm{Re}\, i \int dz d\omega \omega^2 \Theta(\omega)\, \lambda_A\lambda_B\lambda_C\lambda_D\, e^{\frac{i\omega}{2}\lambda_A u^B{}_{\dot{A}}\lambda_B x^{A\dot{A}}}, \tag{3.41}$$

where the integral appearing here is precisely that which generates $\mathbf{X}_{ABCD}$. Using eq. (3.38) we then immediately find

$$\mathbf{\Phi}_{AB\dot{C}\dot{D}} = u^C{}_{\dot{C}} u^D{}_{\dot{D}} \left(\frac{\Phi_{(AB}\Phi_{CD)}}{S}\right). \tag{3.42}$$

This is indeed a double copy formula for the mixed-index Riemann spinor. However, its form is different to formulae such as eq. (2.12) that have previously arisen in the Weyl double copy for vacuum pure gravity solutions. In particular, the numerator does not simply contain a product of electromagnetic field strength tensors, but instead involves an additional projector, that depends on the 4-velocity of the static source. Elucidating this structure crucially depended on transforming the momentum-space amplitude for the JNW solution to an intermediate twistor space, as it is this that allows us to ascertain the relation eq. (3.10), that is ultimately responsible for the additional prefactors in eq. (3.42). Indeed, eq. (3.42) bears a resemblance to the consistency relation obtained for the Coulomb solution in eq. (3.23), and this begs the question of whether eq. (3.42) is fully general, or whether such a form is highly specialised to the particular JNW solution we are considering here. We will address this point more fully in the following section, but first note that it is interesting to compare our results with the recent refs. [28, 76], which looked at generalising the Weyl double copy in pure gravity to non-vacuum solutions. Various formulae were proposed for the Riemann spinor, for different types of solution. They included combinations such as

$$\Phi_{AB\dot{C}\dot{D}} \propto \Phi_{AB}\tilde{\Phi}_{\dot{C}\dot{D}} \tag{3.43}$$

i.e. involving products of the two distinct gauge theory field strength spinors. Interestingly, the formula of eq. (3.42) does not have this form. To see this, note that the symmetrised brackets can be expanded to give

$$\boldsymbol{\Phi}_{AB\dot{C}\dot{D}} = u^C{}_{\dot{C}} u^D{}_{\dot{D}} \frac{1}{S} \left(\Phi_{AB}\Phi_{CD} + \Phi_{AC}\Phi_{BD} + \Phi_{AD}\Phi_{BC}\right). \tag{3.44}$$

In the first term, the consistency relation of eq. (3.23) may be used to express the mixed-index Riemann spinor in terms of a product of gauge theory field-strengths. However, this is not true for the second and third terms, which will be non-zero in general. To see this, one may set

$$\Phi_{AB} \propto \alpha_{(A}\beta_{B)}, \quad I_{ABCD} = u^C{}_{\dot{C}} u^D{}_{\dot{D}} \Big[\Phi_{AC}\Phi_{BD} + \Phi_{AD}\Phi_{BC}\Big],$$

and perform an explicit calculation to obtain

$$I_{0011} \propto \alpha_0^2 \beta_0^2. \tag{3.45}$$

If I_{ABCD} is to vanish, then we must have $\alpha_0 = 0$ or $\beta_0 = 0$, where we may choose the former without loss of generality. We then find

$$I_{0000} \propto \alpha_1^2 \beta_0^2. \tag{3.46}$$

We cannot now choose $\alpha_1 = 0$ without the entire spinor field $\boldsymbol{\Phi}_{AB\dot{C}\dot{D}}$ vanishing. Thus, we must choose $\beta_0 = 0$, which further implies

$$I_{1111} = \alpha_2^2 \beta_2^2. \tag{3.47}$$

We are now forced to choose either $\alpha_2 = 0$ or $\beta_2 = 0$, such that the only way that I_{ABCD} can vanish is if $\boldsymbol{\Phi}_{AB\dot{C}\dot{D}=0}$. Thus, the second and third terms in eq. (3.44) are indeed non-zero in general. As a consequence, the double copy formula that the twistor methods arrive at is distinctly different to other double-copy ansätze appearing in the literature. This is in itself not surprising, given the findings of ref. [92], namely that a pure double copy of gauge theory field strengths in position space was not possible for the JNW solution in the tensorial language.

Finally, we note that the counterpart of eq. (3.42) for the other mixed-index Riemann spinor is derived to be

$$\tilde{\mathbf{\Phi}}_{\dot{A}\dot{B}CD} = u_C{}^{\dot{C}} u_D{}^{\dot{D}} \left(\frac{\tilde{\Phi}_{(\dot{A}\dot{B}}\tilde{\Phi}_{\dot{C}\dot{D})}}{\tilde{S}} \right), \tag{3.48}$$

which is simply related to eq. (3.42), such that similar observations to the above apply.

In this section, we have examined the JNW solution in $\mathcal{N}=0$ supergravity, finding that it is indeed possible to construct spinorial double copy formulae for all spinors entering the decomposition of eq. (2.11). Whilst two of these formulae are straightforward counterparts of the Weyl double copy for vacuum solutions in pure gravity, the mixed-index double copy formulae are different to anything encountered before. In order to probe how general such formulae are, we must move away from the spinless and magnetic-chargeless case of eq. (2.40). This is the subject of the following section.

4 The case of non-zero spin and NUT charge

We can generalise the Coulomb solution to include a non-zero magnetic monopole charge and spin by using the full amplitudes of eq. (2.40), in which the parameters $\alpha \equiv (a, \theta)$ are turned on in each theory. In order to examine the implications of this for generalising the spinorial double copy, it is instructive to first review the results of ref. [78], in pure gravity.

4.1 The Kerr-Taub-NUT solution in pure gravity

In section 3, we have expressed gravity amplitudes leading to the JNW solution in terms of gauge theory amplitudes according to eq. (2.32). However, for amplitudes in pure gravity corresponding to non-zero spin and / or NUT charge, it is conventional to write these in the form (see e.g. refs. [78, 92])

$$\mathcal{M}_\pm \sim \frac{\mathcal{A}_\pm \mathcal{A}_\pm}{\mathcal{A}_\pm^{\text{scal.}}}. \tag{4.1}$$

Here $\mathcal{A}_\pm^{\text{scal.}}$ is a three-point amplitude for emission of a massless scalar from a spinning and / or "magnetically" charged particle. The classical solution corresponding to this amplitude is the so-called zeroth copy of the $\sqrt{\text{Kerr}}$ or Taub-NUT solution, that comprises a solution of linearised biadjoint scalar field theory. The zeroth copy field is indeed different for the (anti-)self-dual cases, hence the $\pm$ label on the scalar amplitude. To understand further the reason why the scalar amplitude is needed in eq. (4.1), it is sufficient to consider the Kerr solution with spin a^a, for which the relevant 3-point amplitude for the anti-self-part is given by

$$\mathcal{M}_+ = e^{ik\cdot a}\mathcal{M}_+^{(0)}, \tag{4.2}$$

where $\mathcal{M}^{(0)}$ denotes the spinless (Schwarzschild) amplitude. The photon amplitude for the corresponding $\sqrt{\text{Kerr}}$ solution is

$$\mathcal{A}_+ = e^{ik\cdot a}\mathcal{A}_+^{(0)}, \tag{4.3}$$

where $\mathcal{A}_+^{(0)}$ denotes the Coulomb amplitude. Thus, if we were simply to square eq. (4.3) according to eq. (2.32), we would instead generate the gravitational amplitude

$$e^{2ika}\mathcal{M}_+^{(0)},$$

which has twice the spin of the usual Kerr solution. In order to formulate the double copy between the conventional $\sqrt{\text{Kerr}}$ and Kerr solutions, one must then introduce the scalar amplitude

$$\mathcal{A}_+^{\text{scal.}} = e^{ik\cdot a}\mathcal{A}^{(0),\text{scal.}}, \tag{4.4}$$

where $\mathcal{A}^{\text{scal.}}$ is the amplitude for emission of a scalar from a spinless particle (and indeed is just a constant). The combination of eq. (4.1) then performs the relevant double copy. Reference [92] has argued that this leads to an ambiguity in the double copy, namely that one may choose *different* scalar functions, such that the spin of the Kerr solution is apportioned in different ways in the two gauge theory amplitudes. It remains, true, however, that only one such choice matches the original Kerr-Schild and Weyl double copies for the Kerr solution [9, 18]. Indeed, the requirement of having a local double copy in position space itself fixes the relevant scalar amplitude, as follows from the arguments of ref. [78]. There, it was observed that eqs. (2.34), (3.2), and their counterpart for a massless scalar field:

$$\phi = \text{Re}\, i\int d\Phi(k)\hat{\delta}(2p\cdot k)\mathcal{A}_+^{\text{scal.}}e^{-ik\cdot x}, \tag{4.5}$$

generate cohomology representatives in twistor space for each theory that are related in certain circumstances by the simple product-like relationship

$$\mathfrak{M}_{\text{grav.}} = \frac{\mathfrak{M}_{\text{EM}}\mathfrak{M}_{\text{EM}}}{\mathfrak{M}_{\text{scal.}}}. \tag{4.6}$$

This is precisely the twistor double copy of refs. [22, 24], which it was shown leads to the position-space Weyl double copy of ref. [18] as a consequence of the Penrose transform. Furthermore, the fact that each twistor quantity $\mathfrak{M}$ arises from a precise integral transform of a scattering amplitude provides a rule for picking out a particular cohomology representative, thus resolving the puzzle for how one is allowed to "multiply" together functions in twistor space.

As ref. [78] makes clear, the fact that the double copy has a local product structure in twistor space is not generic, but relies on the special properties of certain three-point scattering amplitudes. We see, for example, in eq. (3.13) that the integral that takes one from momentum space to twistor space is a Laplace transform in the energy ω. Given that the double copy of three-point amplitudes is indeed a product in momentum-space, it can only be true that a local product is obtained in twistor space provided the integrands of the ω integral for each amplitude consist of functions whose convolution is equivalent to a product of similar functions. This is true for pure power-like functions of ω, which indeed correspond to the three-point amplitudes for spinless particles (i.e. those leading to the Coulomb and Schwarzschild solutions). This is not the only possibility: one is also free to shift the conjugate variable in the Laplace transform:

$$e^{\omega U} \to e^{\omega(U+V)}, \tag{4.7}$$

given that this operation commutes with the convolution. This is in fact what eqs. (4.2), (4.3), (4.4) do, as the 4-momentum k^a is linear in ω. Thus, the Newman-Janis shift acting on a 3-point amplitude is such that it preserves the local double copy in twistor space. Crucially, however, this will only work if the shifts acting on the gravity, gauge and scalar amplitudes *are the same*, and the physics of this is that the corresponding spacetime fields must be related by the conventional single and zeroth copies. Once a local product in twistor space is obtained, a local spacetime spinorial double copy will emerge automatically from the Penrose transform, as shown in ref. [78].

4.2 Generalisation to $\mathcal{N} = 0$ supergravity

Returning to the full spectrum of $\mathcal{N} = 0$ supergravity, we now wish to see whether the double copy formulas of eqs. (3.37), (3.38), (3.42), (3.48) generalise to the presence of non-zero spin and / or NUT charge. Given that the arguments are similar in both cases, we will here restrict ourselves to a non-zero spin vector a^a for each source particle. Then the KMOC formulae of eqs. (2.33)–(2.36) become

$$\mathbf{X}_{ABCD} = -\frac{\kappa^2 c_{++}}{2Q^2}\,\mathrm{Re}\, i\int \mathrm{d}\Phi(k)\hat{\delta}(2p\cdot k)e^{-ik\cdot(a_L+a_R)}\mathcal{A}_+^{(0)}\mathcal{A}_+^{(0)}\,|k\rangle_A|k\rangle_B|k\rangle_C|k\rangle_D e^{-ik\cdot x}\ , \quad (4.8)$$

$$\tilde{\mathbf{X}}_{\dot{A}\dot{B}\dot{C}\dot{D}} = -\frac{\kappa^2 c_{--}}{2Q^2}\,\mathrm{Re}\, i\int \mathrm{d}\Phi(k)\hat{\delta}(2p\cdot k)e^{ik\cdot(a_L+a_R)}\mathcal{A}_-^{(0)}\mathcal{A}_-^{(0)}\,[k|_{\dot{A}}[k|_{\dot{B}}[k|_{\dot{C}}[k|_{\dot{D}} e^{-ik\cdot x}\ , \quad (4.9)$$

$$\mathbf{\Phi}_{AB\dot{C}\dot{D}} = +\frac{\kappa^2 c_{+-}}{2Q^2}\,\mathrm{Re}\, i\int \mathrm{d}\Phi(k)\hat{\delta}(2p\cdot k)e^{-ik\cdot(a_L-a_R)}\mathcal{A}_+^{(0)}\mathcal{A}_-^{(0)}\,|k\rangle_A|k\rangle_B[k|_{\dot{C}}[k|_{\dot{D}} e^{-ik\cdot x}, \quad (4.10)$$

$$\tilde{\mathbf{\Phi}}_{\dot{A}\dot{B}CD} = +\frac{\kappa^2 c_{-+}}{2Q^2}\,\mathrm{Re}\, i\int \mathrm{d}\Phi(k)\hat{\delta}(2p\cdot k)e^{ik\cdot(a_L-a_R)}\mathcal{A}_-^{(0)}\mathcal{A}_+^{(0)}\,[k|_{\dot{A}}[k|_{\dot{B}}|k\rangle_C|k\rangle_D e^{-ik\cdot x}\ , \quad (4.11)$$

where $a^a_{L,R}$ are the spin vectors of the two gauge theory solutions, and $\mathcal{A}_\pm^{(0)}$ the photon amplitudes in the spinless case. Following arguments exactly analogous to the previous section, we can write each product of amplitudes (and spin factor) as a combination of spinless scalar and gauge theory amplitudes, each shifted by a common exponential factor:

$$e^{\alpha_{\eta_L\eta_R}}\mathcal{A}_{\eta_L}\mathcal{A}_{\eta_R} \to \frac{[e^{\alpha_{\eta_L\eta_R}}\mathcal{A}^{(0)}_{\eta_L}][e^{\alpha_{\eta_L\eta_R}}\mathcal{A}^{(0)}_{\eta_R}]}{[e^{\alpha_{\eta_L\eta_R}}\mathcal{A}^{(0),\mathrm{scal.}}]}, \quad (4.12)$$

where we have defined

$$\alpha_{\eta_L\eta_R} = -ik\cdot(\eta_L a_L + \eta_R a_R). \quad (4.13)$$

It is now straightforward to apply the arguments of section 3, and the result is that eqs. (3.37), (3.38), (3.42), (3.48) are replaced by

$$\tilde{\mathbf{X}}_{\dot{A}\dot{B}\dot{C}\dot{D}}[a_R+a_L] = \frac{\Phi_{(\dot{A}\dot{B}}[a_R+a_L]\Phi_{\dot{C}\dot{D})}[a_R+a_L]}{\tilde{S}[a_R+a_L]}, \quad (4.14)$$

$$\mathbf{X}_{ABCD}[a_R+a_L] = \frac{\Phi_{(AB}[a_R+a_L]\Phi_{\dot{C}\dot{D})}[a_R+a_L]}{S[a_R+a_L]}, \quad (4.15)$$

and

$$\mathbf{\Phi}_{AB\dot{C}\dot{D}}[a_R - a_L] = u^C{}_{\dot{C}} u^D{}_{\dot{D}} \left(\frac{\Phi_{(AB}[a_R - a_L]\Phi_{CD)}[a_R - a_L]}{S[a_R - a_L]} \right), \tag{4.16}$$

$$\mathbf{\Phi}_{\dot{A}\dot{B}CD}[a_R - a_L] = u_C{}^{\dot{C}} u_D{}^{\dot{D}} \left(\frac{\tilde{\Phi}_{(\dot{A}\dot{B}}[a_R - a_L]\tilde{\Phi}_{\dot{C}\dot{D})}[a_R - a_L]}{\tilde{S}[a_R - a_L]} \right). \tag{4.17}$$

Here all electromagnetic spinors and scalar fields correspond to the $\sqrt{\text{Kerr}}$ and zeroth copy Kerr solutions respectively, but where the spin is taken to be a particular combination of a_R and a_L, as indicated by the arguments of each field. As for the JNW solution considered in the previous section, the double copy formula for the mixed-index spinors involves a prefactor that depends upon the 4-velocity of the source particle. Interestingly, this is the same factor that appears in the spinless case, a fact which is ultimately due to the static nature of the solution, in that the prefactor arises from the delta function $\delta(2p \cdot k)$ in eqs. (4.8)–(4.11). However, the spin arguments in eqs. (4.14)–(4.17) are such that these formulae do not admit a strict double-copy interpretation. In order to satisfy the requirements of a local position-space double copy, as outlined in the previous section, the spin parameters a^a of each electromagnetic spinor and scalar field must be the same. This in turn means that the combinations of spin vectors appearing in each individual electromagnetic spinor depend on *both* a_R and a_L i.e. the spin vectors from both the gauge theories appearing in the double copy. There is, of course, a special case in which we may indeed obtain a double-copy interpretation, namely $a_R = a_L$. This matches what happens for the Kerr solution in pure gravity, but is such that the axion will automatically vanish. Furthermore, the dilaton will look like a dilaton that is generated by a non-spinning particle, and it is not clear if such a solution can be made to satisfy the Einstein-dilaton equations of motion beyond linearised order (see e.g. ref. [106] for related work).

It is perhaps worth stressing that eqs. (4.14)–(4.17), even if they lack a strict double copy interpretation, nevertheless constitute exact position-space relations for solutions of $\mathcal{N} = 0$ supergravity at linearised order. They may therefore be useful for something, however restricted in scope.

5 Discussion

In this paper, we have addressed the question of whether position-space double copy formulae exist for $\mathcal{N} = 0$ supergravity, that are analogous to the well-known Weyl double copy [18]. To this end, we have used recently developed methods that express classical solutions in terms of on-shell inverse Fourier transforms of scattering amplitudes [81], together with arguments that split this transform into two stages [80]. The first stage takes amplitudes into twistor space, such that the twistor double copy of refs. [22, 24] is obtained. The second stage then consists of the well-known Penrose transform from twistor to position space, and allows one to discern a position-space double copy formula, if it exists. This chain of arguments has been previously used to derive the original Weyl double copy [78], and also the so-called Cotton double copy for topologically massive solutions in three spacetime dimensions [83–85]. Thus, it is natural to try to apply it to the case of $\mathcal{N} = 0$ supergravity, which after all is the known "full" double copy of Yang-Mills theory.

Whether or not a position-space double copy exists for $\mathcal{N} = 0$ supergravity has been recently considered in ref. [92], which indeed inspired the present study. The authors in that case found that no double copy exists if the dilaton and / or axion are turned on, where the tensorial formalism was used. Our arguments in this paper show that this is not quite true, and that one can indeed write double copy formulae for all spinor fields appearing in the generalised Riemann tensor of eq. (2.11), at least for the JNW solution sourced by a spinless particle with no NUT charge. For those spinors with a single type of index, the formulae precisely mirror those for the case of pure gravity. For the mixed-index spinors, however, there are additional factors involving the 4-velocity of the source particle. That these were not considered in ref. [92] may be due to its focus on tensorial formulae for the position-space double copy, given that the translation from spinors to tensors can obscure simple properties in the former language. Furthermore, the twistor methods considered here proved crucial in deriving the presence of the additional prefactor, which we note is also absent in recent conjectures for how to double-copy spinors for non-vacuum gravity solutions [28, 76].

When non-zero spin and / or NUT charge are present, it is yet again possible to write formal position-space double copy formulae for solutions of $\mathcal{N} = 0$ supergravity, which again involve similar prefactors to the JNW case. However, the interpretation of these formulae is not natural, given that they must involve products of electromagnetic spinors, each of which involves the spin / NUT parameters of the full gravity solution. Whether or not such formulae are useful is a matter of debate, but it is in any case interesting that the twistor methods, as in refs. [78, 85], are again able to ascertain both the presence of spinorial double copy formulae, but also their scope and applicability. It would be interesting to investigate whether similar methods could shed light on the generalised (non-vacuum) double copies explored in refs. [28, 76], or indeed to other theories and / or types of solution.

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JHEP05(2023)047

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JHEP05(2023)047

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JHEP05(2023)047