The measurement of labour content: an axiomatic approach*

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Abstract

This paper analyses the theoretical issues related to the measurement of the amount of labour used in the production of – or contained in – a bundle of goods for general technologies with heterogeneous labour. A novel axiomatic framework is used in order to formulate the key properties of the notion of labour content and analyse its theoretical foundations. The main measures of labour content used in various strands of the literature are then characterised. Quite surprisingly, a unique axiomatic structure can be identified which underlies measures of labour aggregates used in such diverse fields as neoclassical growth theory, input-output approaches, productivity analysis, and classical political economy.

JEL classification: D57 (Input-Output Analysis); D63 (Equity, Justice, Inequality, and Other Normative Criteria and Measurement); J24 (Human Capital; Skills; Occupational Choice; Labor Productivity); O33 (Technological Change: Choices and Consequences).

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1 Introduction

The measurement of the amount of labour used in the production of – or contained in – a bundle of goods plays a central role in many different fields and approaches in economics. The definition and measurement of labour aggregates (including human capital), for example, is crucial in debates on the determinants of growth and development (Barro and Sala-i-Martin [1] and Jones [16]) and in productivity analysis (Wolff and Howell [44], Jorgenson [17], Jorgenson et al [18], Flaschel et al. [11]).

In normative economics, the notion of labour content is fundamental in the theory of exploitation as the unequal exchange of labour (Roemer [35], Yoshihara [45, 46]; Veneziani [40]; Veneziani and Yoshihara [41, 42, 43]), but it also plays a pivotal – albeit often implicit – role in Kantian approaches to distributive justice (Roemer [36]).

Last but not least, labour content is a critical concept in classical approaches. It is central, for example, in structural macrodynamic models in the Ricardian tradition (Pasinetti [31, 32]); in classical analyses of the relation between technical change and profitability (Roemer [34]; Flaschel [10]; Flaschel et al. [11]; Flaschel et al. [12]; and Cogliano et al. [3]); and in classical price and value theory focusing on the notion of labour embodied (Kurz and Salvadori [20]; Flaschel [8, 9, 10]).[1]

Outside of simple technologies with a single type of homogeneous labour, however, the concept of labour content is elusive and controversial, and there exists no widely accepted approach to aggregate heterogeneous labour inputs. In productivity analysis, for example, different indices of quality-adjusted labour inputs have been used to study total factor productivity (Jorgenson [17]). In neoclassical growth theory, the controversy on the determinants of growth hinges upon different notions of labour input, or human capital (Jones [16]). In classical political economy, and in exploitation theory, many debates revolve around the appropriate extension of the notion of embodied labour to economies with complex technologies and heterogeneous labour inputs.

Two main approaches have been proposed to the measurement of labour content. In growth theory, for example, "If we do not consider variations in worker quality or in effort, then labor input is the sum of hours worked in a given period" (Barro and Sala-i-Martin [1], p.348). This can be called the *simple additive approach.* Alternatively, if quality and effort are taken into account, then "The overall input is the weighted sum over all categories, where the weights are the relative wage rates" (Barro and Sala-i-Martin [1], p.349). This can be called the *wage-additive approach.*

[1] A comprehensive discussion of theoretically sound measurement in classical political economy (and beyond) can be found in Kurz and Salvadori [22].

Interestingly, despite significant differences between the various strands of the literature, these two approaches are also the main ones in input-output theory, and in productivity analysis where the wage-additive approach is used to construct quality-adjusted indices of labour input. But also in classical political economy, and exploitation theory, where the wage-additive approach is often considered to reflect the classical economists' view on how to convert different types of labour into a single unit, whereby "the different kinds of labour are to be aggregated via the (gold) money wage rates" (Kurz and Salvadori [20], p.324).[2] According to Smith, for example,

> "It is often difficult to ascertain the proportion between two different quantities of labour. The time spent in two different sorts of work will not always alone determine this proportion. The different degrees of hardship endured, and of ingenuity exercised, must likewise be taken into account. There may be more labour in an hour's hard work, than in two hours easy business; or in an hour's application to a trade which it cost ten years labour to learn, than in a month's industry, at an ordinary and obvious employment. But it is not easy to find any accurate measure either of hardship or ingenuity. In exchanging, indeed, the different productions of different sorts of labour for one another, some allowance is commonly made for both. It is adjusted, however, not by any accurate measure, but by the higgling and bargaining of the market, according to that sort of rough equality which, though not exact, is sufficient for carrying on the business of common life" (Smith [38], ch. V, pp.34-35).

And one can similarly interpret Ricardo's arguments that "The estimation in which different quantities of labour are held, comes soon to be adjusted in the market with sufficient precision for all practical purposes, and depend much on the comparative skill of the labourer, and intensity of the labour performed" (Ricardo [33], ch. I, section II, p. 11).

Despite some debates on the concept of "abstract labour", the wage-additive measure is consistent also with Marx's ([24], pp.51-2) views on the conversion of complex labour into simple labour, although he refers to a social process, fixed by custom. See Morishima [27] and, especially, recent monetary approaches to classical value theory, such as the 'New Interpretation' (Duménil [6]; Foley [13]; Duménil et al. [7]) and the definition of 'actual labour values' by Flaschel [8, 10]. For an analysis of the reduction of

[2] Indeed, there are relevant similarities between the neoclassical notion of human capital and the treatment of skilled labour in classical political economy. See, e.g., Kurz and Salvadori [21].

complex labour to simple labour via wage differentials using input-output data, see Shaikh [37] and Tsoulfidis and Tsaliki [39].

More generally, virtually all of the measures of labour input, or labour content proposed in the literature belong to the class of *linear aggregators*: labour aggregates are defined as the weighted sum of heterogeneous labour inputs, where different approaches advocate different weights. In the simple additive approach, for example, the weights are assumed to be all equal to one; in the wage-additive approach, they coincide with the wages. In development accounting, however, other proxies of workers' skills – such as schooling duration – are sometimes used to measure efficiency units and convert different types of labour into a single measure (Jones [16]). In productivity analysis, job-based measures of labour skill requirements have also been used (Wolff and Howell [44]). In a classical perspective, Krause [19] has suggested that the weights be given by the *reduction vector*, which is defined as the Frobenius eigenvector of the matrix $\mathbf{H} =< h_{ij} >$, where h_{ij} is the amount of type-i labour required directly or indirectly to reproduce one unit of type-j labour (e.g. in the household and education sectors).[3]

This paper tackles the issue of the appropriate *measure of labour content* (henceforth, MLC) for general convex production technologies with heterogeneous labour inputs (described in section 2), by rigorously stating and explicitly discussing some foundational properties that a MLC should satisfy. The purpose is not to adjudicate between alternative approaches and provide *the* unique index of labour content appropriate for *all* strands of the literature mentioned above. Rather, we aim to highlight the common conceptual foundations of the main approaches *and* shed light on the implicit assumptions behind different measures. This is, in our view, a fundamental step in order to determine which measure is appropriate in which context.

One key, novel contribution of the paper is methodological: rather than proposing a MLC and comparing it with alternative measures, we adopt an axiomatic approach and discuss the appropriate way of measuring labour content starting from first principles. Although this approach is standard in theories of inequality and poverty measurement (Foster [14]), this paper provides the first application of axiomatic analysis to measures of labour content and quality-adjusted indices of labour inputs, and one of the first applications to classical political economy.[4]

By adopting the axiomatic method, we are able to characterise the class of linear aggregators used in the literature: the *generalised additive MLC* defines the labour content of a bundle of goods as the weighted sum of the

[3] A definition of weights independent of price information has been proposed also by Okishio [29, 30] and Fujimori [15]. For a discussion of additivity in the measurement of labour content in classical price and value theory, see Flaschel [10].

[4] Relevant exceptions include Flaschel et al. [11]; Yoshihara [45]; Veneziani and Yoshihara [41, 42, 43, 47].

amounts of different types of labour used in production. This characterisation allows one to precisely identify the common theoretical foundations of *all* of the main measures. Alternative approaches can then be conceptualised as special cases of the general additive class of MLCs advocating different restrictions to determine the weights.

To be specific, in section 3, a MLC is conceptualised as a *binary relation* defined over pairs of bundles of goods, associated production activities, and price vectors such that it is possible, and meaningful, to say that a certain bundle produced with a certain activity at some prices contains more or less labour than another one.

In section 4, we illustrate the basic properties of MLCs focusing on a special case: we study MLCs that are transitive and complete when comparing the labour content of produced goods *at given prices* – called, (p, w)-labour orderings. Three axioms are analysed which capture theoretically relevant properties of (p, w)-labour orderings. *Dominance* says that if the production of a bundle of goods requires a strictly higher amount of each type of labour, then its labour content is strictly higher. *Labour Trade-offs* rules out the possibility that the labour content of each and every bundle of produced goods is determined by looking at the amount of one type of labour input only. *Mixture Invariance* restricts the way in which measures of labour content vary when different production techniques are combined.

We prove that there is only one class of (p, w)-labour orderings that satisfies these three mild and intuitive properties (Proposition 1), namely the generalised additive MLC (formally defined in section 4). In other words, setting aside otherwise significant theoretical differences, the three axioms represent the core of all of the main approaches to labour measurement in the various strands of the literature cited above.

Section 5 develops the axiomatic analysis of MLCs in the general setting in which prices are allowed to vary. Two additional axioms are introduced. One states that although MLCs may depend on information about prices and wages, the latter should not be the *only* determinant of labour content. The other is a standard scale invariance property that requires the comparisons of the labour content to be invariant to certain perturbations, and changes in the units of measurement. Our main result, Theorem 1, shows that generalised versions of *Dominance*, *Labour Trade-offs*, and *Mixture Invariance*, together with these mild additional conditions, uniquely characterise the generalised additive MLC even when prices may vary.

In section 6, we explore the main refinements of the linear approach, and provide two additional characterisations. First, we show that the simple additive MLC is the only measure satisfying Dominance, Mixture Invariance, and a strengthening of Labour Trade-offs – called *Labour Equivalence* –

according to which no type of labour definitionally contributes more than others to the determination of labour content. Second, we introduce two mainly technical properties constraining the effect of changes in the price vector on MLCs – *Skill Substitutability* and *Independence* – and a new axiom, called *Consistency with Progressive Technical Change* which incorporates a classical intuition that capital-using labour-saving technical change should increase labour productivity and decrease labour content. We show that, within the generalised additive class, the wage-additive approach is the only one that satisfies these additional properties. This confirms the intuition that quality-adjusted measures of labour content capture the relation between technical change and labour productivity in market economies.

Our results depend on the specific properties chosen: alternative axioms would yield different MLCs. We think that the axioms analysed in this paper have robust theoretical foundations and impose rather mild restrictions on MLCs. Indeed, they incorporate properties often explicitly or implicitly advocated in the literature. But, perhaps more importantly, we see this inherent indeterminacy of the axiomatic approach as a virtue, rather than a shortcoming. For the explicit statement of the properties that a MLC does, or should satisfy helps to clarify the theoretical foundations and properties of different measures. We return to this issue in the concluding section.

2 The basic framework

Consider general economies in which the production of commodities requires produced inputs and different types of labour. There are n produced goods, which may be consumed and/or used as inputs in different production activities. The set of types of labour inputs (potentially) used in production is $\mathcal{T} = \{1, ..., T\}$, with generic elements $\nu, \mu \in \mathcal{T}$.

A technology is described by a production set $P \subseteq \mathbb{R}^{2n+T}$ with elements – activities – of the form $a = (-a_l, -\underline{a}, \overline{a})$, where $a_l \equiv (a_{l\nu})_{\nu \in \mathcal{T}} \in \mathbb{R}^T_+$ is a profile of labour inputs measured in hours; $\underline{a} \in \mathbb{R}^n_+$ are the inputs of the produced goods; and $\overline{a} \in \mathbb{R}^n_+$ are the n outputs.[5]

This modelling of production is quite general and it allows for any type of heterogeneity in labour inputs. Simple production technologies with homogeneous labour are contained as special cases with $T = 1$. Different technologies requiring different types of heterogeneous labour can be represented by different production sets P. For instance, differences in *labour intensity* of each type of labour due to heterogeneous skills or human capital can be formalised as different production sets, since labour input vectors are

[5] For any integer $m > 0$, let $\mathbb{R}^m$ (*resp.*, $\mathbb{R}^m_+$, $\mathbb{R}^m_{++}$, $\mathbb{R}^m_-$) denote the (*resp.*, non-negative, strictly positive, non-positive) m-dimensional Euclidean space.

measured in hours.[6]

Let $\mathbf{0} = (0, ..., 0)$ denote the null vector. In what follows, some mild restrictions are imposed on the admissible class of production technologies.[7]

Assumption 0 (A0). P is a closed convex cone in $\mathbb{R}^{2n+T}$ and $\mathbf{0} \in P$.

Assumption 1 (A1). For all $a \in P$, if $\overline{a} \geq \mathbf{0}$ then $a_l \geq \mathbf{0}$.

Assumption 2 (A2). For all $c \in \mathbb{R}^n_+$, there is $a \in P$ such that $\overline{a} - \underline{a} \geqq c$.

Assumption 3 (A3). For all $a \in P$, and for all $(-a'_l, -\underline{a}', \overline{a}') \in \mathbb{R}^T_- \times \mathbb{R}^n_- \times \mathbb{R}^n_+$, if $(-a'_l, -\underline{a}', \overline{a}') \leqq a$ then $(-a'_l, -\underline{a}', \overline{a}') \in P$.

These assumptions are standard in all strands of the literature mentioned in the Introduction, including the canonical neoclassical growth model and input-output models. **A0** allows for general technologies with constant returns to scale. **A1** implies that *some* labour is indispensable to produce output. **A2** states that any non-negative commodity vector is producible as net output. **A3** is a standard *free disposal* condition.

The set of all production sets that satisfy **A0**-**A3** is denoted by $\mathcal{P}$. We shall analyse the issue of the appropriate measurement of labour content for all conceivable technologies in the set $\mathcal{P}$.

Let $p \in \mathbb{R}^n_+$ be the vector of prices of the n produced commodities and let $w \in \mathbb{R}^T_+$ be the vector of the wages of the T types of labour. At this stage, there is no reason to restrict (p, w) to be an equilibrium price vector, but in what follows, we shall focus on the economically relevant allocations with a strictly positive wage vector w.

3 Comparing labour content

The main purpose of our analysis is to identify some widely shared intuitions about the measurement of labour content, and then analyse what they imply in terms of the appropriate MLC. Consequently, we aim to identify a set of theoretically robust properties and formally weak restrictions that are widely (albeit possibly implicitly) endorsed in the literature.

As a starting point, we simply require that a *MLC be able to compare the labour content of produced goods.* This choice has two important implications. First, the existence of an appropriate definition of labour content

[6] Alternatively, one may define activity vectors by measuring each type of labour input in efficiency units, so that the amount of type-ν labour $a_{l\nu}$ would be the product of labour hours times the *intensity* of this type of labour. All of our results would continue to hold under this approach after appropriate changes in the axiomatic system. A focus on labour time is, however, in line with the literature.

[7] Vector inequalities: for all x, $y \in \mathbb{R}^m$, $x \geqq y$ if and only if $x_i \geqq y_i$ $(i = 1, \ldots, m)$; $x \geq y$ if and only if $x \geqq y$ and $x \neq y$; $x > y$ if and only if $x_i > y_i$ $(i = 1, \ldots, m)$.

for non-produced goods is set aside. This is an interesting theoretical question, for example, in environmental economics or in the economics of the household, but it is not the main focus of our analysis.[8]

Second, if a key property of a MLC is to allow one to make meaningful statements of the form: "the bundle of produced goods c contains more labour than the bundle c'," then it can be conceptualised as *a binary relation.*

It is a priori unclear what type of information – concerning, for example, technology, prices, market structures, and so on – is necessary in order to make such comparisons. We adopt the most general approach and allow the MLC to depend on *all* potentially relevant information. Formally, we consider *profiles* (c, a, p, w), where $c \in \mathbb{R}^n_+$ is a non-negative bundle of goods producible as net output by using activity $a \in \phi^P(c) \equiv \{a' \in P \mid \overline{a}' - \underline{a}' \geqq c\}$ for some $P \in \mathcal{P}$ at the price vector $(p, w) \in \mathbb{R}^{n+T}_+$.

Observe that very few restrictions are imposed on the variables in the admissible profiles. For example, they might be based on actual data, or they might be determined (possibly counterfactually) from optimal, equilibrium behaviour. Indeed, the only restriction imposed on two profiles $(c, a, p, w), (c', a', p', w')$ is that the vectors c and c' be productively feasible according to *some* technologies – a and a', respectively, – but a and a' are not even required to be *in the same production set.* In fact, it may be desirable in principle to compare the labour content of one (or more) vectors of net outputs, say, in nations with different technologies, or – in a dynamic perspective – as technology evolves over time.

Let the set of profiles (c, a, p, w) be denoted by $\mathcal{CP}$. Theoretically, there are no reasons to restrict our analysis, and it is a priori desirable to identify MLCs that can be applied to the largest possible set of conceivable scenarios. Hence, in what follows we shall focus on the universal domain $\mathcal{CP}$. Then:[9]

Definition 1 *A measure of labour content is an ordering $\succsim$ on $\mathcal{CP}$ such that for any (c, a, p, w), $(c', a', p', w') \in \mathcal{CP}$, vector c produced with a at (p, w)* contains at least as much labour as *vector c' produced with a' at (p', w') if and only if $(c, a, p, w) \succsim (c', a', p', w')$.*

Definition 1 provides a general framework to study labour measurement, whereby the specification of the desirable properties of a MLC can be seen

[8] Similarly, we focus on *human* labour and ignore the labour performed by working animals, consistent with most of the literature discussed in the Introduction. The issue of animal labour is quite interesting, and it has been raised by some – albeit not all – of the classical authors, but we leave it for further research. We thank an anonymous referee for raising this issue.

[9] Let $x \equiv (c, a, p, w)$. For any $x, x', x'' \in \mathcal{CP}$, $\succsim \subseteq \mathcal{CP} \times \mathcal{CP}$ is *reflexive* if and only if $x \succsim x$; *transitive* if and only if $x \succsim x'$ and $x' \succsim x''$ implies $x \succsim x''$; and *complete* if and only if $x \succsim x'$ or $x' \succsim x$. An *ordering* is a reflexive, transitive and complete binary relation.

as the identification of a set of restrictions on the binary relation $\succsim$ on $\mathcal{CP}$.[10]

For example, Definition 1 imposes no restriction on the role of prices in the measurement of labour content. A central question concerns whether prices should enter the definition of labour content and, if so, whether only equilibrium prices should matter. This is a rather controversial issue and various views have been proposed, depending also on the focus of the analysis. Definition 1 is compatible with different approaches: at this stage, we simply allow for the *possibility* that the measurement of labour content depends on (equilibrium or disequilibrium) prices.

In section 5, we identify a set of desirable properties for $\succsim$ and provide a characterisation of the class of generalised additive MLCs proposed in the literature. In order to illustrate the basic axioms, and the logic of the proof of Theorem 1, however, we first consider a special case.

4 Labour measurement: A special case

This section focuses on a subset of the set of possible MLCs by restricting attention to profiles with the *same price vector*. Formally:[11]

Definition 2 *For any (p,w), a MLC $\succsim$ on $\mathcal{CP}$ is a (p,w)-*labour ordering *if there exists an ordering $\succsim_{(p,w)}$ on $\mathbb{R}^T_+$ such that for any $(c,a,p,w),(c',a',p,w) \in \mathcal{CP}$, $(c,a,p,w) \succsim (c',a',p,w)$ if and only if $a_l \succsim_{(p,w)} a'_l$.*

The first property we impose on $\succsim_{(p,w)}$ is uncontroversial: it states that, given a price vector (p,w), if the production of a bundle of goods c requires a strictly higher amount of *every* type of labour than a bundle c', then it contains more labour. Formally:

Dominance (D): For any $(c,a,p,w),(c',a',p,w) \in \mathcal{CP}$, if $a_l > a'_l$, then $a_l \succ_{(p,w)} a'_l$.

It might be argued that it should be sufficient for the amount of *one* type of labour to be strictly greater in a_l than in a'_l to conclude that c contains more labour than c'. This seems reasonable, for example, in an input-output

[10] Let $x \equiv (c,a,p,w)$. For all $x,x' \in \mathcal{CP}$, the asymmetric part $\succ$ of $\succsim$ is defined by $x \succ x'$ if and only if $x \succsim x'$ and not $x' \succsim x$; and the symmetric part $\sim$ of $\succcurlyeq$ is defined by $x \sim x'$ if and only if $x \succsim x'$ and $x' \succsim x$. Here, $\succ$ and $\sim$ stand, respectively, for "contains strictly more labour than" and "contains the same amount of labour as".

[11] By Definition 2, the labour contained in two bundles c,c' produced with activities a,a' at prices (p,w) can be determined based only on the *direct* labour inputs used in production. It is worth stressing that this does not necessarily imply that other information about production techniques a,a', and in particular about *indirect* labour – that is, the labour contained in produced inputs used in the production process – is irrelevant. In fact, by **A0-A3**, focusing on the direct labour used to produce c as *net* output allows one to capture the *total* amount of labour contained in c, namely "the embodied labour – direct and indirect – in producing c from scratch" (Roemer [35], p.148).

analysis aimed at capturing labour multipliers. This view is not uncontroversial, though. Classical authors, for example, argued that some types of labour – for example, guard labour – are inherently unproductive and do not affect the labour content of produced goods. We need not adjudicate this issue here. Given that we aim to identify some *minimal* desirable properties of MLCs common to all approaches, it is theoretically appropriate to focus on the weaker, and less controversial, condition **D**.

The next property states that the MLC should allow for trade-offs between different types of labour used in production in at least a *minimal* subset of the set of conceivable profiles. To be precise, for a given price vector (p, w), for any pair of labour types ν and μ, there exist two production activities which only differ in the amount of labour of types ν and μ used and yield the same labour content, but one of them uses more of type-ν labour while the other uses more of type-μ labour.

Labour Trade-offs (LT): For all $\nu, \mu \in \mathcal{T}$, $\nu \neq \mu$, there exist $(c, a, p, w), (c', a', p, w) \in \mathcal{CP}$, such that $a_{l\nu} > a'_{l\nu}$, $a_{l\mu} < a'_{l\mu}$, and $a_{l\zeta} = a'_{l\zeta}$ for all $\zeta \neq \nu, \mu$, and $a_l \sim_{(p,w)} a'_l$.

Theoretically, **LT** rules out the possibility that the labour content of produced goods is determined by a single type of labour *for every conceivable profile*. **LT** does not preclude the possibility that some types of labour have a (possibly much) bigger weight in the determination of labour content than others in all profiles, or even that certain types of labour alone determine the labour content of most profiles. Yet, intuitively, if all types of labour are indeed used in at least some productive activities, then they should contribute to determine the labour content of at least *some* bundles of produced goods. Formally, **LT** is rather weak in that it only requires that, for any pair of labour types $\nu, \mu \in \mathcal{T}$, there exists *one* pair of activities in the set of all *conceivable* production techniques which yield the same amount of labour in producing *some* (possibly different) net output vectors.

The last axiom imposes a minimal requirement of consistency in labour measurement. It states that, for a given price vector (p, w), if two vectors of labour inputs dominate (in terms of corresponding labour content) another pair of vectors, then convex combinations of the former should dominate convex combinations of the latter.

Mixture Invariance (MI): Let $(c, a, p, w), (c', a', p, w), (\widetilde{c}, \widetilde{a}, p, w), (\widetilde{c}', \widetilde{a}', p, w) \in \mathcal{CP}$. Given $\tau \in (0, 1)$, let $a_l^\tau = \tau a_l + (1 - \tau)\, \widetilde{a}_l$ and $a_l'^\tau = \tau a'_l + (1 - \tau)\, \widetilde{a}'_l$. Then, $a_l^\tau \succ_{(p,w)} a_l'^\tau$, whenever $a_l \succ_{(p,w)} a'_l$ and $\widetilde{a}_l \succsim_{(p,w)} \widetilde{a}'_l$.

To see why **MI** is a desirable property, suppose that both a and $\widetilde{a}$ produce bundle c as net output, while a' and $\widetilde{a}'$ produce c'.[12] If **MI** were violated, then it would be possible to conclude that, overall, c' contains more labour than c when, say, a proportion $\tau \in (0,1)$ of the firms use a and a' to produce, respectively, c and c' (and a proportion $(1-\tau)$ use $\widetilde{a}$ and $\widetilde{a}'$ to produce, respectively, c and c'), even though for each individual activity a and a', c contains more labour than c', and the same holds for $\widetilde{a}$ and $\widetilde{a}'$. Or, consider firms 1 and 2 producing, respectively, c and c', and suppose that firm 1 (respectively, 2) uses technique a for a part $\tau \in (0,1)$ of the year and $\widetilde{a}$ for the rest of the year (respectively, a' and $\widetilde{a}'$). Then it would be possible to conclude that, overall, the labour contained in 1's net output is lower than that contained in 2's, despite the fact that in each part of the production period the opposite holds.

Observe that **MI** restricts the way in which a MLC ranks mixtures, starting from original profiles. However, it does *not* require that the amount of labour in a bundle should remain the same, or that the labour content of a mixture be equal to the convex combination of the labour contained in the original bundles. More generally, **MI** does not impose significant restrictions on the way in which the amount of labour contained in a bundle should vary.[13]

The three axioms capture widely shared views on the measurement of labour content and indeed all of the main approaches satisfy them. It is immediate to see, for example, that the MLCs used in standard productivity analysis, or in the growth literature, all satisfy **D**, **LT** and **MI**. Although it is less evident, the same holds for the standard definition of labour content in input-output theory. To see this, let the Leontief technology with a $n \times n$ non-negative and productive matrix, A, and a $1 \times n$ positive vector, L, of homogeneous labour requirements be represented by

$$P_{(A,L)} \equiv \left\{ a \in \mathbb{R}_{-} \times \mathbb{R}^{n}_{-} \times \mathbb{R}^{n}_{+} \mid \exists x \in \mathbb{R}^{n}_{+} : a \leqq (-Lx, -Ax, x) \right\},$$

and let $\mathcal{P}_{(\mathcal{A},\mathcal{L})} \subset \mathcal{P}$ be the set of all conceivable Leontief technologies.

For any $P_{(A,L)}$, the vector of labour multipliers is defined as $v = L(I - A)^{-1}$ and, for any $(c, a, p, w) \in \mathcal{CP}_{(\mathcal{A},\mathcal{L})}$ such that $a = (-Lx, -Ax, x)$ and $c = (I - A)x$, the labour content of c is defined as $vc = Lx$. To see that this MLC satisfies **D**, note that for any $(c, a, p, w), (c', a', p, w) \in \mathcal{CP}_{(\mathcal{A},\mathcal{L})}$, $Lx > L'x'$ immediately implies $a_l \succ_{(p,w)} a'_l$. To see that **MI** is met, consider (c, a, p, w), (c', a', p, w), $(\widetilde{c}, a, p, w)$, $(\widetilde{c}', a', p, w) \in \mathcal{CP}_{(\mathcal{A},\mathcal{L})}$ such that $Lx > L'x'$ and $\widetilde{L}\widetilde{x} \geqq \widetilde{L}'\widetilde{x}'$. Then, for any $\tau \in (0,1)$, $a^{\tau}_l = \tau Lx + (1-\tau)\widetilde{L}\widetilde{x} >$

[12] A similar, albeit less transparent, argument holds if $c \neq \widetilde{c}$ and $c' \neq \widetilde{c}'$.

[13] Note also that, by the definition of the universal set $\mathcal{P}$, for all $a_l, \widetilde{a}_l$, such that $(c, a; p, w), (\widetilde{c}, \widetilde{a}; p, w) \in \mathcal{CP}$ and for all $\tau \in (0,1)$, there exists a profile $(c^{\tau}, a^{\tau}; p, w) \in \mathcal{CP}$ such that $a^{\tau}_l = \tau a_l + (1-\tau)\widetilde{a}_l$.

$a_l'^{\tau} = \tau L'x' + (1-\tau)\widetilde{L}'\widetilde{x}'$, and so $a_l^{\tau} \succ_{(p,w)} a_l'^{\tau}$. Finally, because there is only one type of labour, **LT** is vacuously satisfied.

Generalised additive measures define the labour content of a bundle of produced goods as the weighted sum of the amount of time of different types of labour spent in its production. Formally:

Definition 3 *For any (p,w), a (p,w)-labour ordering $\succsim$ on $\mathcal{CP}$ is* generalised additive *if there is some strictly positive vector $\sigma_{(p,w)} \in \mathbb{R}^{\mathcal{T}}_{++}$ such that for all $(c,a,p,w), (c',a',p,w) \in \mathcal{CP}$, $a_l \succsim_{(p,w)} a_l'$ if and only if $\sigma_{(p,w)} a_l = \sum_{\nu \in \mathcal{T}} \sigma^{\nu}_{(p,w)} a_{l\nu} \geqq \sum_{\nu \in \mathcal{T}} \sigma^{\nu}_{(p,w)} a'_{l\nu} = \sigma_{(p,w)} a_l'$.*

Proposition 1 proves that the only measures that satisfy **D**, **LT** and **MI** are generalised additive.[14]

Proposition 1 *A (p,w)-labour ordering $\succsim$ on $\mathcal{CP}$ satisfies **Dominance**, **Labour Trade-offs**, and **Mixture Invariance** if and only if it is generalised additive.*

Although Proposition 1 does not characterise a unique ordering, it does identify a class of measures which share a common structure. This additive structure is often considered either as a fundamental property of a MLC, and thus implicitly postulated *as an axiom*, or as the consequence of marginal product pricing in perfectly competitive markets. Instead, additivity is here derived as a *result* starting from more foundational principles that are directly related to the properties of labour measurement, without any assumptions on market structure, equilibrium pricing, or the existence of differentiable production functions.

Although the main contribution of this paper is conceptual, it is worth noting that, from a purely formal viewpoint, Proposition 1 provides an independent characterisation of the so-called *weak weighted utilitarian* ordering which is analysed in social choice theory in the context of evaluating welfare profiles.[15] Axioms **D**, **LT** and **MI** are analogous to well-known Paretian, anonymity and independence properties in social choice theory. However, the similarity is purely at the formal level: the interpretation and justification are completely different, and some of the axioms are more defensible in the context of the measurement of labour content than in welfare economics. Diamond's [5] classic critique of utilitarianism, for example, is based on the

[14] All formal proofs can be found in the online Appendix A.

[15] Actually, standard results in social choice theory and in decision theory highlight the robustness of the main conclusions of this paper. For it is well-known that weak weighted utilitarianism, and weighted sum representations of individual preferences, can be characterised based on various different sets of axioms, focusing for example on *invariance conditions*. See, e.g., d'Aspremont and Gevers ([4], Theorem 4.2, p.509), Mitra and Ozbek ([25], Theorem 2, p.520). The axioms used in Proposition 1, however, are more intuitive and economically meaningful in the context of the measurement of labour content.

rejection of independence (or 'sure thing') principles analogous to **MI**. For 'mixing' welfare across different individuals may produce ethically relevant effects (Mariotti and Veneziani [23]). Clearly, this normative argument does not apply here.

5 The foundations of labour measurement

Proposition 1 characterises MLCs in the special case where prices remain constant. Albeit insightful, this provides only limited insights on labour measurement when technical change takes place, across economies, or over time. In this section, we develop the general axiomatic analysis of MLCs which rank *all* profiles $(c,a,p,w),(c',a',p',w') \in \mathcal{CP}$.

As a first step, we reformulate the three core axioms presented in section 4 as restrictions on the MLC $\succsim \subseteq \mathcal{CP} \times \mathcal{CP}$.

Dominance (D): For any $(c,a,p,w),(c',a',p,w) \in \mathcal{CP}$, if $a_l > a'_l$ then $(c,a,p,w) \succ (c',a',p,w)$.

Labour Trade-offs (LT): For all $\nu,\mu \in \mathcal{T}$, $\nu \neq \mu$, and all $(p,w) \in \mathbb{R}_+^{n+T}$, there are $(c,a,p,w),(c',a',p,w) \in \mathcal{CP}$, such that $a_{l\nu} > a'_{l\nu}$, $a_{l\mu} < a'_{l\mu}$, and $a_{l\zeta} = a'_{l\zeta}$ for each $\zeta \neq \nu,\mu$, and $(c,a,p,w) \sim (c',a',p,w)$.

Mixture Invariance (MI): Let $(c,a,p,w),(c',a',p,w),(\widetilde{c},\widetilde{a},p,w),(\widetilde{c}',\widetilde{a}',p,w) \in \mathcal{CP}$. Given $\tau \in (0,1)$, let $a_l^\tau = \tau a_l + (1-\tau)\widetilde{a}_l$ and $a_l'^\tau = \tau a'_l + (1-\tau)\widetilde{a}'_l$. Then, $(c^\tau,a^\tau,p,w) \succ (c'^\tau,a'^\tau,p,w)$ holds, whenever $(c,a,p,w) \succ (c',a',p,w)$ and $(\widetilde{c},\widetilde{a},p,w) \succsim (\widetilde{c}',\widetilde{a}',p,w)$.

In order to generalise Proposition 1, we introduce two additional properties. The first states that different profiles should not be ordered lexicographically focusing only on goods' prices or wages: although we allow MLCs to depend on information about prices and wages, the latter should not be the *only* determinant of labour content. A bundle of goods c, produced as net output using activity a, at a price vector (p,w) should not contain strictly more (or less) labour than *all* other bundles c', produced as net output using *any* activity a', at a different price vector (p',w').

Minimal Equivalence (ME): For any $(p,w),(p',w') \in \mathbb{R}_+^{n+T}$, there exist two profiles $(c,a,p,w),(c',a',p',w') \in \mathcal{CP}$ with $a_{l\nu} = a_{l\mu} > 0$ and $a'_{l\nu} = a'_{l\mu} > 0$ for any $\nu,\mu \in \mathcal{T}$, such that $(c,a,p,w) \sim (c',a',p',w')$.

Formally, **ME** imposes quite a mild restriction on the MLC as it only requires the existence of *one* pair of profiles that are indifferent for any two different price vectors.[16] Theoretically, **ME** incorporates the intuition that the amount of time (of all types of labour) spent in producing a certain bundle should remain a key factor in determining the labour content of a bundle. Different price vectors may reflect different labour *intensities*, or *skills*, across profiles, but it should be possible to compensate such differences – at least in principle – by adjusting the amount of *time* (of all types of labour) spent in production.

The second property requires that the ranking of a pair of profiles be invariant to the scaling of the consumption bundle and the associated production activity. In other words, for any $k > 0$, if the labour content of a bundle of goods c, produced as net output of activity a at (p, w) is at least as much as the labour content of a bundle c', produced as net output of activity a' at (p', w') then the same is true for bundle kc, produced using ka at (p, w), when compared with kc' produced using ka' at (p', w'). Formally:

Scale Invariance (SINV): For any $(c, a, p, w), (c', a', p', w') \in \mathcal{CP}$ such that for any $\nu, \mu \in \mathcal{T}$, $a_{l\nu} = a_{l\mu}$ and $a'_{l\nu} = a'_{l\mu}$, and for any positive real number $k > 0$, $(c, a, p, w) \succsim (c', a', p', w')$ holds if and only if $(kc, ka, p, w) \succsim (kc', ka', p', w')$ holds.

Scale invariance properties are standard in the theory of inequality measurement, and in axiomatic social choice. They incorporate the intuition that the ranking of two objects should be invariant to certain changes in the measurement scale. Standard inequality measures, for example, typically satisfy such invariance properties with respect to all proportional changes. **SINV** is much weaker in that it only applies to a small subset of profiles (those with activities using the same amount of every labour input), and it seems particularly reasonable in the context of measuring labour content, especially given the convexity of production sets.[17] Indeed, if the scale of bundles of goods and their production activities in two profiles changes by the same proportion, then any technological condition, such as the composition of material and labour inputs and the difference of labour intensities or skills between these profiles, would not be altered and thus the relative ranking of labour content in these profiles should remain the same.

Finally, we extend the notion of generalised additive MLCs on $\mathcal{CP}$, according to which the labour content of a bundle of goods should be measured

[16] The condition that the activity vector of each profile should use the *same* amount of time of every type of labour is not particularly restrictive. If $(p, w) = (p', w')$, for example, then **ME** holds for any reflexive MLC.

[17] It is worth stressing, however, that **SINV** does not crucially hinge upon the convex cone assumption either theoretically or formally. Indeed, the axiom can be extended to hold also for a more general universal class of production sets $\mathcal{P}$.

as the weighted sum of the different types of labour used in its production, with the weights depending on the price vector.

Definition 4 *A MLC* $\succsim$ *on* $\mathcal{CP}$ *is* generalised additive *if, for all* (c,a,p,w), $(c',a',p',w') \in \mathcal{CP}$, *there exist some strictly positive vectors* $\sigma_{(p,w)}, \sigma_{(p',w')} \in \mathbb{R}^{T}_{++}$ *such that* $(c,a,p,w) \succsim (c',a',p',w')$ *if and only if* $\sigma_{(p,w)} a_l \geqq \sigma_{(p',w')} a'_l$ *.*

Our main result proves that the only MLCs satisfying **ME** and **SINV**, together with **D**, **LT**, **MI**, are the generalised additive ones.

Theorem 1 *A MLC* $\succsim$ *on* $\mathcal{CP}$ *satisfies* ***Dominance***, ***Labour Trade-offs***, ***Mixture Invariance***, ***Scale Invariance***, *and* ***Minimal Equivalence*** *if and only if it is generalised additive.*

In closing this section, it is worth noting that the generalised additive measure characterised by Theorem 1 is reminiscent of the social welfare function analysed by Negishi [28] in a pioneering contribution. The similarity holds at a broad formal level, though, as Negishi [28] focused on weighted sums of utilities, with the vector of weights depending on the efficient allocation, individuals' initial endowments, and prices. Yet, from a purely formal viewpoint, Theorem 1 may contribute to the analysis of the axiomatic structure of linear social welfare functions with variable weights.

6 Labour content: refinements

Theorem 1 highlights the theoretical foundations of, and the intuitions common to all of the main approaches. In this section, we explore further restrictions that allow us to characterise two of the most widely used measures – namely, the simple additive MLC and the wage-additive MLC – within the class identified by Theorem 1. Formally:

Definition 5 *A MLC* $\succsim$ *on* $\mathcal{CP}$ *is* additive *if, for all* $(c,a,p,w), (c',a',p',w') \in \mathcal{CP}$, $(c,a,p,w) \succsim (c',a',p',w')$ *if and only if* $\sum_{\nu \in \mathcal{T}} a_{l\nu} \geqq \sum_{\nu \in \mathcal{T}} a'_{l\nu}$.

Definition 6 *A MLC* $\succsim$ *on* $\mathcal{CP}$ *is* wage-additive *if, for all* $(c,a,p,w), (c',a',p',w') \in \mathcal{CP}$, $(c,a,p,w) \succsim (c',a',p',w')$ *if and only if* $wa_l \geqq w'a'_l$.

The key intuition behind the simple *additive* approach is that no type of labour always contributes more than others to the determination of labour content. This can be captured by the following strengthening of **LT**.

Labour Equivalence (LE): For all $(c,a,p,w), (c',a',p',w') \in \mathcal{CP}$ such that $a_{l\nu} = a'_{l\mu}$, $a_{l\mu} = a'_{l\nu}$, some $\nu, \mu \in \mathcal{T}$, and $a_{l\zeta} = a'_{l\zeta}$ for all $\zeta \neq \nu, \mu$, $(c,a,p,w) \sim (c',a',p',w')$.

Formally, **LE** is rather weak and a large number of conceivable, non-additive MLCs satisfy it: all types of labour contribute equally, for example, in multiplicative aggregators, such as the product of the different amounts of labour, or their geometric mean. Indeed, **LE** does not even imply that the amount of labour contained in a given bundle should always be obtained by aggregating *all* types of labour. For example, MLCs focusing either on the highest or on the lowest amount of labour spent in the production of a certain bundle (or on the difference between the two) satisfy **LE**.

The next result states that the combination of **D**, **MI**, and **LE**, implies that the labour content of a bundle of produced goods should be measured as the total amount of hours of labour of different types spent in its production.

Corollary 1 *A MLC* $\succsim$ *on* $\mathcal{CP}$ *satisfies* ***Dominance****,* ***Labour Equivalence****, and* ***Mixture Invariance*** *if and only if it is additive.*

A characterisation of the wage-additive MLC is less straightforward. Rather different arguments are used in various strands of the literature in order to justify the adoption of wage rates to aggregate heterogeneous labour. Here we propose three properties which aim to capture the intuitions common to *all* wage-additive approaches.

Two properties incorporate the intuition that changes in wages may be seen as reflecting changes in skills, or labour intensity. This is always true in the standard perfectly competitive framework, where wages are equal to the marginal productivity of different types of labour in equilibrium. The axioms discussed here are much less demanding, and therefore more general, in that they require wages to signal productive contributions only in a rather small subset of the set of conceivable cases.

The first axiom states that there exists a subset of profiles such that a uniform increase (resp., decrease) in wages can be interpreted as reflecting a generalised increase (resp., decrease) in labour productivity such that the amount of labour time necessary to produce a given bundle of goods, c, as net output decreases (resp., increases) proportionally and the labour content of c remains unchanged, even though the vector of produced inputs used in production and the output vector remain the same.

Skill Substitutability (SSUB): For any $(p, w), (p, w') \in \mathbb{R}_+^{n+T}$ such that $w' = \lambda_{(w,w')} w$ for some $\lambda_{(w,w')} > 0$, there exist $(c, a, p, w), (c, a', p, w') \in \mathcal{CP}$ such that for any $\nu, \mu \in \mathcal{T}$, $a_{l\nu} = a_{l\mu}$ and $a'_{l\nu} = a'_{l\mu}$, $a_l = \lambda_{(w,w')} a'_l$, and $(\underline{a}, \overline{a}) = (\underline{a}', \overline{a}')$, and that $(c, a, p, w) \sim (c, a', p, w')$.

Intuitively, the labour content of a bundle of goods c remains constant because a uniform increase in skills (reflected in the wages) compensates for a decrease in the amount of labour time spent in production of c. It is worth emphasising that the set of cases contemplated in **SSUB** is rather small. For the axiom applies only to a very small set of perturbations of a price vector (commodity prices must remain constant and wages must change by exactly the same factor) and, for any relevant pairs of price vectors, it only requires the existence of *one* pair of profiles with the required property.

Whereas **SSUB** focuses on changes in the wage *level*, the next property considers the effect of changes in *relative* wages. Consider any pair $(p, w), (p', w')$ such that relative wages are different but the aggregate wage level is the same, in the sense that w, w' both belong to the unit simplex. Then, there exist two profiles $(c, a, p, w), (c, a, p', w')$ such that the amount of labour contained in the bundle c produced with the given activity a – which uses the same amount of time of each type of labour – is constant.

Independence (IND): For any $(p, w), (p', w') \in \mathbb{R}_+^{n+T}$ such that $w \neq w'$ and $\sum w_\nu = \sum w'_\nu = 1$, there exist $(c, a, p, w), (c, a, p', w') \in \mathcal{CP}$ such that for any $\nu, \mu \in \mathcal{T}$, $a_{l\nu} = a_{l\mu}$ and $(c, a, p, w) \sim (c, a, p', w')$.

Intuitively, **IND** identifies a subset of the set of conceivable profiles $\mathcal{CP}$ whereby a constant aggregate wage level can be interpreted as reflecting a constant labour productivity, such that labour content is independent of changes in relative wages. Again, this subset is rather small as **IND** only applies to price vectors with wages belonging to the unit simplex and it only requires the existence of *one* pair of profiles with the desired property. Much like **SSUB**, it stipulates that wages reflect skills, and productive contributions more generally, in at least some cases while remaining silent in more general scenarios.

The last property focuses on the relation between technical change, productivity, and labour content. This is central in all of the strands of the literature mentioned in the Introduction, which emphasise the effect of profit-maximising behaviour and technological progress on labour productivity.

Our axiom is rooted in the classical tradition, and it provides a different perspective on the intuitions behind the wage-additive approach.[18] While the latter is often justified assuming marginal productivity pricing of labour in perfectly competitive markets, our axiom is independent of any assumptions on market structure and on differentiability of production functions and provides an alternative justification focusing on the kind of information that the MLC should capture.

The axiom generalises an insight originally proved by Roemer ([34]; see also Flaschel et al. [11]): any profitable (i.e., cost-reducing at current prices) technical change that is capital-using and labour-saving is *progressive*, – that

[18] For a thorough discussion of the link between labour content and labour productivity in the classical approach, see Flaschel et al. [11].

is, it decreases labour content (and increases labour productivity). In the Leontief models in which these results are derived, the definition of labour content is uncontroversial, and this insight is obtained as a result. However, the theoretical relevance of the link between technical change, productivity, and labour content in the literature is arguably such that its epistemological status is as a postulate: the appropriate MLC is one which preserves the link between profitable innovations, labour productivity, and labour content.

For all $c \in \mathbb{R}^n_+$, let $\phi(c) \equiv \left\{a' \in \mathbb{R}^{T+n}_- \times \mathbb{R}^n_+ \mid \exists P' \in \mathcal{P} : a' \in \phi^{P'}(c)\right\}$: $\phi(c)$ is the set of activities that belong to some production set $P' \in \mathcal{P}$ and that can produce c as net output. The next axiom captures the labour-reducing effect of profitable capital-using technical change.

Consistency with Progressive Technical Change (CPTC): For any $(p, w) \in \mathbb{R}^{n+T}_+$, there exist a profile $(c, a, p, w) \in \mathcal{CP}$ and a neighbourhood $\mathcal{N}(a) \subseteq \mathbb{R}^{T+n}_- \times \mathbb{R}^n_+$ of a such that for all $a' \in \mathcal{N}(a) \cap \phi(c)$, if $p\underline{a} + wa_l > p\underline{a}' + wa'_l$ and $\underline{a} \leq \underline{a}'$, then $(c, a, p, w) \succ (c, a', p, w)$.

CPTC captures the intuition that certain capital-using ($\underline{a} \leq \underline{a}'$) cost-reducing ($p\underline{a} + wa_l > p\underline{a}' + wa'_l$) innovations decrease the amount of labour necessary to produce a given bundle of commodities, c, thereby increasing labour productivity. It imposes a rather mild restriction on the MLC as it focuses on a small set of conceivable innovations. In fact, for any price vector (p, w), **CPTC** requires the existence of *one* profile $(c, a, p, w) \in \mathcal{CP}$ such that cost-reducing capital-using innovations increase productivity.[19] Further, **CPTC** focuses exclusively on (i) relatively small innovations – in a neighbourhood of a – that (ii) (weakly) increase *all* produced inputs used in a given process, and that (iii) change the technological conditions for the production of a *given* net output vector c.[20]

Two additional features of **CPTC** are worth noting. First, although no condition is explicitly imposed on labour inputs, the changes considered are, in a relevant sense, labour-saving. For, $p\underline{a} + wa_l > p\underline{a}' + wa'_l$ and $\underline{a} \leq \underline{a}'$ imply that the amount of at least one type of labour decreases, and for at least one profile, even if the amount of some labour input increases, this is more than outweighed by decreases in other types of labour. Second, it is

[19] **CPTC** focuses on innovations that are cost-reducing *at current prices*: the effect of technical change on prices and wages is ignored, since it is negligible at the timing of each capitalist's choice of the new technology. This is standard in the literature on progressive technical change (e.g., Morishima [27]; Roemer [34]; Flaschel et al. [11]).

[20] As a general definition of cost-reducing capital-using technical progress, one may argue that **CPTC** is too restrictive, and a larger set of innovations should be considered. This objection is not relevant here. Our results continue to hold if **CPTC** is strengthened to hold for a larger set of innovations. Besides, our aim is not to provide a general theory of technological change, and in the context of an axiomatic analysis of MLCs, focusing on a smaller set of innovations imposes milder restrictions. Similarly, **CPTC** is silent on the effect of innovations on fixed capital even though it can be adapted to include fixed capital, as in Roemer [34] and Flaschel et al [11].

immediate to show that the standard definition of labour content in Leontief models with homogeneous labour satisfies **CPTC** in $\mathcal{CP}_{(\mathcal{A},\mathcal{L})}$.[21]

Together with **D**, **LT**, **MI** and **CPTC**, if one endorses **SINV**, **SSUB** and **IND**, then one must conclude that the labour content of a bundle of produced goods should be measured as the weighted sum of the amount of time of different types of labour used in its production, with the weights given by the relevant wages, *even when the price vector changes.*

Theorem 2 *A MLC $\succsim$ on $\mathcal{CP}$ satisfies* ***Dominance, Labour Trade-offs, Mixture Invariance, Scale Invariance, Skill Substitutability, Consistency with Progressive Technical Change*** *and* ***Independence*** *if and only if it is wage-additive.*

Theorem 2 provides rigorous axiomatic foundations to the standard practice of measuring labour inputs based on wage costs in the input-output literature as well as in empirical studies on total factor productivity and growth. It is also consistent with the views of classical political economy on the so-called conversion of complex labour into simple labour using relative wages as the conversion factors.

7 Conclusion

This paper analyses the issue of the appropriate measurement of the amount of labour used in the production of - or contained in - a bundle of goods. Measures of labour content are formally conceptualised as binary relations comparing bundles of goods produced with certain activities at certain prices. An axiomatic approach is adopted in order to identify some foundational properties that every MLC should satisfy. Strikingly, it is shown that a small number of axioms incorporating some widely shared intuitions uniquely identify the class of linear MLCs, according to which the labour content of a bundle of goods is the weighted sum of the amount of time of different types of labour spent in its production. A linear aggregation of heterogeneous labour inputs is advocated in virtually *all* of the literature, and so our characterisation pins down the theoretical foundations and intuitions shared in such diverse approaches and fields as input-output theory, productivity analysis, neoclassical growth theory, and classical political economy. We also characterise the two main measures used in the literature, namely the simple additive MLC, according to which the labour content of a bundle

[21] To see this, given a price vector $(p,w) \in \mathbb{R}_+^{n+1}$, consider any $(c,a,p,w), (c,a',p,w) \in \mathcal{CP}_{(\mathcal{A},\mathcal{L})}$, such that $a = (-a_l, -Ax, x)$ and $a' = (-a_l', -A'x', x')$, where $a \in P_{(A,L)}$ and $a' \in P_{(A',L')}$. Suppose that labour intensity is identical at a and at a'. Then, without loss of generality, we can set $Lx = a_l$ and $L'x' = a_l'$. In this setting, if $pAx + wLx > pA'x' + wL'x'$ and $Ax \leq A'x'$, then $Lx > L'x'$ and so $a_l \succ a_l'$.

of produced goods corresponds to the total (unweighted) labour time spent in its production, and the wage-additive MLC, which uses relative wages in order to convert different types of labour into a single measure.

While, as we noted, our aim is not to identify *the* appropriate MLC, our results here can provide a rigorous framework to discuss this issue while clarifying the key conceptual differences between alternative measures. Thus, for example, the axiom of **Labour Equivalence** plays a key role in Corollary 1. Therefore one may argue that the simple additive MLC is particularly suitable in contexts in which it is appropriate to assume that no type of labour always contributes more than others to the determination of labour content. In the theory of exploitation as the unequal exchange of labour, this is implicitly assumed in the so-called "well-being view" (Yoshihara and Veneziani [47], p.404) according to which the concept of exploitation captures some inequalities in the distribution of material well-being and free hours that are normatively relevant. However, **Labour Equivalence** is also assumed in much of productivity analysis where the focus is on the productivity of labour time.[22] In contrast, if one is interested in the relation between technical change, productivity, and profitability, then Theorem 2 suggests that the wage additive measure – which satisfies **Consistency with Progressive Technical Change** – may be more appropriate. Similarly, in exploitation theory, the key intuition behind **Skill Substitutability** seems to underlie the so-called "contribution view" (Yoshihara and Veneziani [47], p.403) according to which exploitative relations are characterised by a mismatch between agents' contribution to the economy and their rewards. A similar intuition characterises Kantian approaches focusing on the so-called proportional solution (Roemer [36]).[23]

The axiomatic analysis developed in this paper is motivated by the idea that the theoretical strength of a MLC depends – to a large extent – on the foundational principles that underlie it. There are two important caveats about this, which also suggest directions for further research.

First, although additive measures possess many desirable features from both the theoretical and the empirical viewpoint, alternative MLCs can certainly be proposed that capture different intuitions, and have different properties. From this perspective, an axiomatic analysis aims precisely at making the relevant assumptions and intuitions explicit and open to scrutiny.

Second, it is certainly desirable for a MLC to have sound theoretical

[22] Observe that the literature mentioned here focuses on the simple additive MLC for analytical reasons and does not necessarily restrict to situations with a uniform wage rate, as differential skills (and effort levels) will be reflected in different wages in a competitive setting (see Botwinick [2]).

[23] Conversely, if one believes that in the theory of value and income distribution only the wage-additive approach makes sense, then Theorem 2 allows one to precisely articulate the underpinning intuitions.

foundations. Yet one may argue that its relevance ultimately rests on the insights that can be gained from it, and the fruitfulness of the MLCs considered in this paper can only be judged when they are applied to economically relevant problems. From this viewpoint, this paper should be seen as a first, and preliminary step into a wider research programme.

A Appendix (to be made available online): Proofs

First of all, we prove two results that are of some interest in their own right. Lemma 1 derives some convexity properties of a (p,w)-*labour ordering* $\succsim$.

Lemma 1 *Let the ordering $\succsim_{(p,w)}$ on $\mathbb{R}^T_+$ satisfy* ***Mixture Invariance****. Consider any set $\left\{a_l^1,...,a_l^K\right\}$, $K>1$, such that $\left(c^k,a^k,p,w\right)\in\mathcal{CP}$, for all $k=1,...,K$ and $a_l^i\sim_{(p,w)}a_l^j$, for all $i,j\in\{1,...,K\}$. Then, for all $\{\tau_1,...,\tau_K\}$ such that $\tau_i\in[0,1]$ all $i\in\{1,...,K\}$ and $\sum_{i=1}^K\tau_i=1$, $\sum_{i=1}^K\tau_i a_l^i\sim_{(p,w)}a_l^j$, for all $j\in\{1,...,K\}$.*

Proof. 1. First of all, note that by the definition of the universal set $\mathcal{P}$, for all $\left\{a_l^1,...,a_l^K\right\}$, such that $\left(c^k,a^k,p,w\right)\in\mathcal{CP}$, for all $k=1,...,K$, and for all $\{\tau_1,...,\tau_K\}$ such that $\tau_i\in[0,1]$ all $i\in\{1,...,K\}$ and $\sum_{i=1}^K\tau_i=1$, there exists a profile $(c^\tau,a^\tau,p,w)\in\mathcal{CP}$ such that $a_l^\tau=\sum_{i=1}^K\tau_i a_l^i$.

2. Note that if $\tau_i=1$, some $i\in\{1,...,K\}$, then the result holds by assumption. Therefore in what follows we focus on the case where $\tau_i\in[0,1)$, all $i\in\{1,...,K\}$.

3. We proceed by induction on K.

($K=2$) Consider any pair $\left(c^1,a^1,p,w\right),\left(c^2,a^2,p,w\right)\in\mathcal{CP}$ such that $a_l^1\sim_{(p,w)}a_l^2$. Suppose, by way of contradiction, that there exists some $\tau\in(0,1)$, such that $\tau a_l^1+(1-\tau)a_l^2\nsim_{(p,w)}a_l^i$, for some $i\in\{1,2\}$. Let $a_l^\tau\equiv\tau a_l^1+(1-\tau)a_l^2$. By completeness, suppose that $a_l^\tau\succ_{(p,w)}a_l^i$, for some $i\in\{1,2\}$, without loss of generality. By transitivity, $a_l^\tau\succ_{(p,w)}a_l^i$, for all $i\in\{1,2\}$. But then **MI** implies $a_l^\tau\succ_{(p,w)}ta_l^1+(1-t)a_l^2$ for all $t\in(0,1)$. Setting $t=\tau$ yields the desired contradiction.

(Inductive step) Suppose that the result holds for all $K-1\geqq 2$. Consider $\left\{a_l^1,...,a_l^K\right\},K>1$, such that $\left(c^k,a^k,p,w\right)\in\mathcal{CP}$, for all $k=1,...,K$, and $a_l^i\sim_{(p,w)}a_l^j$, for all $i,j\in\{1,...,K\}$. Take any $\{\tau_1,...,\tau_K\}$ such that $\tau_i\in[0,1)$ all $i\in\{1,...,K\}$ and $\sum_{i=1}^K\tau_i=1$. We need to prove that $\sum_{i=1}^K\tau_i a_l^i\sim_{(p,w)}a_l^j$, for all $j\in\{1,...,K\}$.

If $\tau_i=0$, some $i\in\{1,...,K\}$, then the result follows from the induction hypothesis and transitivity. So suppose that $\tau_i\in(0,1)$, all $i\in\{1,...,K\}$. Note that for any $k\in\{1,...,K\}$, $\sum_{i=1}^K\tau_i a_l^i=\sum_{j\neq k}\tau_j\left(\sum_{i\neq k}\frac{\tau_i}{\sum_{j\neq k}\tau_j}a_l^i\right)+\tau_k a_l^k$ and by construction $\frac{\tau_i}{\sum_{j\neq k}\tau_j}\in(0,1)$, all $i\in\{1,...,K\}\setminus\{k\}$, and

$\sum_{i \neq k} \frac{\tau_i}{\sum_{j \neq k} \tau_j} = 1$. Therefore by the induction hypothesis and transitivity, $\sum_{i \neq k} \frac{\tau_i}{\sum_{j \neq k} \tau_j} a_l^i \sim_{(p,w)} a_l^h$ for all $h \in \{1, ..., K\}$. Then the result follows by noting that $\sum_{j \neq k} \tau_j = 1 - \tau_k \in (0,1)$ and by invoking the the induction hypothesis and transitivity again. ■

Remark: The restriction $K > 1$ in Lemma 1 is without loss of generality, as the result trivially holds in the case $K = 1$.

The next Lemma proves that any two vectors with the same amount of labour content actually identify a direction in the T-dimensional space along which all vectors have the same labour content.

Lemma 2 *Let the ordering $\succsim_{(p,w)}$ on $\mathbb{R}^T_+$ satisfy **Mixture Invariance**. Suppose $(c, a, p, w), (c', a', p, w) \in \mathcal{CP}$ and $a_l \sim_{(p,w)} a'_l$. If $(c'', a'', p, w) \in \mathcal{CP}$ and there exists $t \in (0,1)$ such that $a_l = ta''_l + (1-t)a'_l$, then $a''_l \sim_{(p,w)} a_l \sim_{(p,w)} a'_l$.*

Proof. 1. Suppose that $(c, a, p, w), (c', a', p, w) \in \mathcal{CP}$ and $a_l \sim_{(p,w)} a'_l$. Suppose, by way of contradiction, that $(c'', a'', p, w) \in \mathcal{CP}$ and there exists $t \in (0,1)$ such that $a_l = ta''_l + (1-t)a'_l$, but $a''_l \nsim_{(p,w)} a'_l$. By completeness, suppose $a''_l \succ_{(p,w)} a'_l$, without loss of generality.

2. By **MI**, and noting that by the reflexivity of $\succsim_{(p,w)}$, $a''_l \sim_{(p,w)} a''_l$ and $a'_l \sim_{(p,w)} a'_l$, it follows that $a''_l \succ_{(p,w)} \tau a''_l + (1-\tau)a'_l \succ_{(p,w)} a'_l$ holds for all $\tau \in (0,1)$. The desired contradiction follows setting $\tau = t$. ■

We can now prove Proposition 1.[24]

Proof of Proposition 1. (*Necessity*) It is immediate that if a (p,w)-labour ordering $\succsim$ on $\mathcal{CP}$ is generalised additive, it satisfies the axioms.

(*Sufficiency*) Consider a (p,w)-labour ordering $\succsim$ on $\mathcal{CP}$ that satisfies **D**, **LT**, and **MI**. In order to show that $\succsim$ is generalised additive, we first show that any (p,w)-labour ordering $\succsim$ on $\mathcal{CP}$ that satisfies **D**, **LT**, and **MI** has an additive feature: that is, there is some $\sigma_{(p,w)} \in \mathbb{R}^T$, $\sigma_{(p,w)} > \mathbf{0}$, such that for all $(c, a, p, w), (c', a', p, w) \in \mathcal{CP}$, $a_l \succsim_{(p,w)} a'_l$ if and only if $\sigma_{(p,w)} a_l \geqq \sigma_{(p,w)} a'_l$.

Step 1. We prove that for any $(c, a, p, w), (c', a', p, w) \in \mathcal{CP}$, $a_l \succsim_{(p,w)} a'_l$ implies $a_l + y \succsim_{(p,w)} a'_l + y$, for all $y \in \mathbb{R}^T$ such that $a_l + y, a'_l + y \in \mathbb{R}^T_+$. To see this, suppose, by way of contradiction, that there exist $(c, a, p, w), (c', a', p, w) \in \mathcal{CP}$, and $y \in \mathbb{R}^T$ such that $a_l \succsim_{(p,w)} a'_l$ and $a_l + y, a'_l + y \in \mathbb{R}^T_+$, but $a_l + y \succsim_{(p,w)} a'_l + y$ dose not hold. By completeness, this implies $a'_l + y \succ_{(p,w)} a_l + y$. Then, by **MI**, for all $\tau \in (0,1)$,

[24] The properties in Proposition 1, and in the other characterisation results below, are independent.

$\tau a_l + (1-\tau)(a'_l + y) \succ_{(p,w)} \tau a'_l + (1-\tau)(a_l + y)$. For $\tau = \frac{1}{2}$, the latter expression becomes

$$\frac{1}{2}a_l + \frac{1}{2}(a'_l + y) \succ_{(p,w)} \frac{1}{2}a'_l + \frac{1}{2}(a_l + y)$$

which violates reflexivity.

Step 2. By **LT,** for all $\nu, \mu \in \mathcal{T}$, there are $(c, a, p, w), (c', a', p, w) \in \mathcal{CP}$ such that $a_{l\nu} > a'_{l\nu}$, $a_{l\mu} < a'_{l\mu}$, and $a_{l\zeta} = a'_{l\zeta}$, $\zeta \neq \nu, \mu$, and $a_l \sim_{(p,w)} a'_l$. Take $\nu = 1$: by **LT** for all $\mu \in \mathcal{T}\backslash\{1\}$, there exist $(c^\mu, a^\mu, p, w), (c'^\mu, a'^\mu, p, w) \in \mathcal{CP}$ such that $a^\mu_{l1} > a'^\mu_{l1}$, $a^\mu_{l\mu} < a'^\mu_{l\mu}$, and $a^\mu_{l\zeta} = a'^\mu_{l\zeta}$, $\zeta \neq 1, \mu$, and $a^\mu_l \sim_{(p,w)} a'^\mu_l$. Let the set of all $2(T-1)$ vectors $\{a^\mu_l, a'^\mu_l\}_{\mu\in\mathcal{T}\backslash\{1\}}$ be denoted as I^1. Construct $\sigma_{(p,w)} = \left(\sigma^1_{(p,w)}, ..., \sigma^T_{(p,w)}\right)$ as follows: for all $\mu \in \mathcal{T}\backslash\{1\}$, $\frac{\sigma^1_{(p,w)}}{\sigma^\mu_{(p,w)}} = \frac{a'^\mu_{l\mu} - a^\mu_{l\mu}}{a^\mu_{l1} - a'^\mu_{l1}}$ and $\sum_{\nu\in\mathcal{T}} \sigma^\nu_{(p,w)} = 1$. By construction $\sigma_{(p,w)} > \mathbf{0}$ and, for all $\mu \in \mathcal{T}\backslash\{1\}$, $\sum_{\nu\in\mathcal{T}} \sigma^\nu_{(p,w)} a^\mu_{l\nu} = \sum_{\nu\in\mathcal{T}} \sigma^\nu_{(p,w)} a'^\mu_{l\nu}$. We show that, starting from I^1, one iso-labour surface can be constructed such that for all $(c, a, p, w), (c', a', p, w) \in \mathcal{CP}$ with $\sum_{\nu\in\mathcal{T}} \sigma^\nu_{(p,w)} a_{l\nu} = \sum_{\nu\in\mathcal{T}} \sigma^\nu_{(p,w)} a'_{l\nu} = k$, we have $a_l \sim_{(p,w)} a'_l$.

Step 3. Consider $a^2_l, a'^2_l \in I^1$: by construction $(c^2, a^2, p, w), (c'^2, a'^2, p, w) \in \mathcal{CP}$ are such that $a^2_{l1} > a'^2_{l1}$, $a^2_{l2} < a'^2_{l2}$, and $a^2_{l\zeta} = a'^2_{l\zeta}$, $\zeta \neq 1, 2$, and $a^2_l \sim_{(p,w)} a'^2_l$. Choose $y^2 \in \mathbb{R}^T_+$ such that $a^{\max}_l \equiv a^2_l + y^2 \geqq a^\mu_l$, for all $a^\mu_l \in I^1$. [If $a^2_l \geqq a^\mu_l$ for all $\mu \in \mathcal{T}\backslash\{1\}$, then $y^2 = \mathbf{0}$ can be chosen.] By Step 1, $a^2_l \sim_{(p,w)} a'^2_l$ implies $a^{\max}_l \equiv a^2_l + y^2 \sim_{(p,w)} a'^2_l + y^2$.

Similarly, consider any $a^\mu_l, a'^\mu_l \in I^1$, $\mu \in \mathcal{T}\backslash\{1,2\}$. By construction $(c^\mu, a^\mu, p, w), (c'^\mu, a'^\mu, p, w) \in \mathcal{CP}$ are such that $a^\mu_{l1} > a'^\mu_{l1}$, $a^\mu_{l\mu} < a'^\mu_{l\mu}$, and $a^\mu_{l\zeta} = a'^\mu_{l\zeta}$, $\zeta \neq 1, \mu$, and $a^\mu_l \sim_{(p,w)} a'^\mu_l$. For all $\mu \in \mathcal{T}\backslash\{1,2\}$, define $y^\mu \in \mathbb{R}^T_+$ such that for any $a^\mu_l, a'^\mu_l \in I^1$: $a^\mu_l + y^\mu = a^{\max}_l$. By Step 1, $a^\mu_l \sim_{(p,w)} a'^\mu_l$ implies $a^{\max}_l = a^\mu_l + y^\mu \sim_{(p,w)} a'^\mu_l + y^\mu$, for all $\mu \in \mathcal{T}\backslash\{1,2\}$.

Therefore, we obtain a set of T vectors $\left\{a^{\max}_l, \left(a'^\mu_l + y^\mu\right)_{\mu\in\mathcal{T}\backslash\{1\}}\right\} \subset \mathbb{R}^T_+$ such that $a^{\max}_l \sim_{(p,w)} a'^\mu_l + y^\mu$, for all $\mu \in \mathcal{T}\backslash\{1\}$, and by transitivity, $a'^\eta_l + y^\eta \sim_{(p,w)} a'^\mu_l + y^\mu$, for all $\mu, \eta \in \mathcal{T}\backslash\{1\}$. Moreover, by the construction of $\sigma_{(p,w)}$ in Step 2, and noting that $a^{\max}_{l\nu} \geq \mathbf{0}$, $\sum_{\nu\in\mathcal{T}} \sigma^\nu_{(p,w)} \left(a'^\mu_{l\nu} + y^\mu\right) = \sum_{\nu\in\mathcal{T}} \sigma^\nu_{(p,w)} a^{\max}_{l\nu} = k > 0$, for all $\mu \in \mathcal{T}\backslash\{1\}$. Finally, noting that the addition of y^μ to each pair of vectors preserves the original inequalities, the T vectors are easily shown to be affinely independent.

Step 4. Let $\Delta\left(a^{\max}_l, \left(a'^\mu_l + y^\mu\right)_{\mu\in\mathcal{T}\backslash\{1\}}\right)$ be the closed $T-1$ simplex defined by $\left\{a^{\max}_l, \left(a'^\mu_l + y^\mu\right)_{\mu\in\mathcal{T}\backslash\{1\}}\right\} \subset \mathbb{R}^T_+$. Next, let $\Delta\left(e^1, ..., e^T\right)$ be the closed $T-1$ simplex defined by $\left\{e^1, ..., e^T\right\} \subset \mathbb{R}^T_+$, where for all $\nu \in \mathcal{T}$, $e^\nu \equiv \left(0, ..., \frac{k}{\sigma^\nu_{(p,w)}}, ..., 0\right)$. By construction, $\Delta\left(a^{\max}_l, \left(a'^\mu_l + y^\mu\right)_{\mu\in\mathcal{T}\backslash\{1\}}\right) \subseteq \Delta\left(e^1, ..., e^T\right) = \left\{a_l \in \mathbb{R}^T_+ : \sum_{\nu\in\mathcal{T}} \sigma^\nu_{(p,w)} a_{l\nu} = k\right\}$.

Step 5. For all $(c, a, p, w) \in \mathcal{CP}$ such that $a_l \in \Delta\left(a_l^{\max}, \left(a_l^{\prime\mu} + y^\mu\right)_{\mu\in\mathcal{T}\setminus\{1\}}\right)$, Lemma 1 implies $a_l \sim_{(p,w)} a_l^{\max}$. For all $(c, a, p, w) \in \mathcal{CP}$ such that $a_l \in \Delta\left(e^1, ..., e^T\right) \setminus \Delta\left(a_l^{\max}, \left(a_l^{\prime\mu} + y^\mu\right)_{\mu\in\mathcal{T}\setminus\{1\}}\right)$, there exist $(\widetilde{c}, \widetilde{a}, p, w), (\widetilde{c}', \widetilde{a}', p, w) \in \mathcal{CP}$ and $t \in (0, 1)$ such that $\widetilde{a}_l, \widetilde{a}_l' \in \Delta\left(a_l^{\max}, \left(a_l^{\prime\mu} + y^\mu\right)_{\mu\in\mathcal{T}\setminus\{1\}}\right)$ and $\widetilde{a}_l = ta_l + (1-t)\widetilde{a}_l'$. Then, noting that by the previous argument (together with transitivity) $\widetilde{a}_l \sim_{(p,w)} \widetilde{a}_l'$, by Lemma 2 it follows that $a_l \sim_{(p,w)} \widetilde{a}_l \sim_{(p,w)} \widetilde{a}_l'$.

Therefore by transitivity, we conclude that for all $(c, a, p, w), (c', a', p, w) \in \mathcal{CP}$ such that $\sum_{\nu\in\mathcal{T}} \sigma_{(p,w)}^\nu a_{l\nu} = \sum_{\nu\in\mathcal{T}} \sigma_{(p,w)}^\nu a_{l\nu}' = k$, we have $a_l \sim_{(p,w)} a_l'$.

Step 6. Next, we show that for all $(c, a, p, w), (c', a', p, w) \in \mathcal{CP}$ such that $\sum_{\nu\in\mathcal{T}} \sigma_{(p,w)}^\nu a_{l\nu} = \sum_{\nu\in\mathcal{T}} \sigma_{(p,w)}^\nu a_{l\nu}' = k' \neq k$, we have $a_l \sim_{(p,w)} a_l'$. Suppose first that $k' > k$. By Step 3, consider any $\left\{\left(c^i, a^i, p, w\right)\right\}_{i=1,...,T} \subset \mathcal{CP}$ such that $\sum_{\nu\in\mathcal{T}} \sigma_{(p,w)}^\nu a_{l\nu}^i = k$ for all $i = 1, ..., T$, and $\left\{a_l^i\right\}_{i=1,...,T} \subset \mathbb{R}_+^T$ is a set of T affinely independent vectors. By Step 5, we have $a_l^i \sim_{(p,w)} a_l^j$, for all $i, j \in \{1, ..., T\}$. Let $y = (k' - k, k' - k, ..., k' - k) > \mathbf{0}$. Then $\left\{a_l^i + y\right\}_{i=1,...,T} \subset \mathbb{R}_+^T$ is a set of T affinely independent vectors such that $\sum_{\nu\in\mathcal{T}} \sigma_{(p,w)}^\nu \left(a_{l\nu}^i + y_\nu\right) = k'$, for all $i = 1, ..., T$, and by Step 1, $a_l^i + y \sim_{(p,w)} a_l^j + y$, for all $i, j \in \{1, ..., T\}$. Therefore the argument in Steps 4 and 5 can be applied to conclude that for all $(c, a, p, w), (c', a', p, w) \in \mathcal{CP}$ such that $\sum_{\nu\in\mathcal{T}} \sigma_{(p,w)}^\nu a_{l\nu} = \sum_{\nu\in\mathcal{T}} \sigma_{(p,w)}^\nu a_{l\nu}' = k$, we have $a_l \sim_{(p,w)} a_l'$.

A similar argument holds for the case $k' < k$, restricting attention to the profiles $\left(c^i, a^i, p, w\right) \in \mathcal{CP}$ such that $\sum_{\nu\in\mathcal{T}} \sigma_{(p,w)}^\nu a_{l\nu}^i = k$ and such that if $y = (k' - k, k' - k, ..., k' - k)$ then $a_l^i + y \in \mathbb{R}_+^T$.

Step 7. The previous arguments prove that if $(c, a, p, w), (c', a', p, w) \in \mathcal{CP}$ are such that $\sum_{\nu\in\mathcal{T}} \sigma_{(p,w)}^\nu a_{l\nu} = \sum_{\nu\in\mathcal{T}} \sigma_{(p,w)}^\nu a_{l\nu}'$, then $a_l \sim_{(p,w)} a_l'$. Then, by **D** and transitivity, it follows that for all $(c, a, p, w), (c', a', p, w) \in \mathcal{CP}$ such that $\sum_{\nu\in\mathcal{T}} \sigma_{(p,w)}^\nu a_{l\nu} > \sum_{\nu\in\mathcal{T}} \sigma_{(p,w)}^\nu a_{l\nu}'$, it must be $a_l \succ_{(p,w)} a_l'$. ■

Proof of Theorem 1. (*Necessity*) It is immediate that if a labour ordering $\succsim$ on $\mathcal{CP}$ is generalised additive, it satisfies the axioms.

(*Sufficiency*) By Proposition 1, for each $(p, w) \in \mathbb{R}_+^{n+T}$, and for any $(c, a, p, w), (c', a', p, w) \in \mathcal{CP}$, there exists $\sigma_{(p,w)} \in \mathbb{R}_{++}^T$ such that $(c, a, p, w) \succsim (c', a', p, w)$ if and only if $\sigma_{(p,w)} \cdot a_l \geqq \sigma_{(p,w)} \cdot a_l'$. Note that $\sum_{\nu\in\mathcal{T}} \sigma_{(p,w)}^\nu = 1$ holds by the construction in the proof of Proposition 1.

By axiom **ME**, for any $(p, w), (p', w') \in \mathbb{R}_+^{n+T}$, there exist $(c, a, p, w), (c', a', p', w') \in \mathcal{CP}$ such that for any $\nu, \mu \in \mathcal{T}$, $a_{l\nu} = a_{l\mu} > 0$ and $a_{l\nu}' = a_{l\mu}' > 0$, and that $(c, a, p, w) \sim (c', a', p', w')$. Without loss of generality, let $\sigma_{(p,w)} \cdot a_l \neq \sigma_{(p',w')} \cdot a_l'$. Then, there exists $\lambda > 0$ such that $\sigma_{(p,w)} \cdot a_l = \lambda\sigma_{(p',w')} \cdot a_l'$. Let $\widetilde{\sigma}_{(p',w')} \equiv \lambda\sigma_{(p',w')}$, so that $\sigma_{(p,w)} \cdot a_l = \widetilde{\sigma}_{(p',w')} \cdot a_l'$. Then, by **SINV** and the transitivity of $\succsim$, it follows that for any $(c'', a'', p, w), (c^*, a^*, p', w') \in \mathcal{CP}$, $(c'', a'', p, w) \succsim (c^*, a^*, p', w')$ if and only if $\sigma_{(p,w)} \cdot a_l'' \geqq \widetilde{\sigma}_{(p',w')} \cdot a_l^*$.

Consider any $(p,w),(p',w'),(p'',w'')\in\mathbb{R}_{+}^{n+T}$. Let $\lambda_{(p,w;p',w')}>0$ be such that $\sigma_{(p,w)}\cdot a_l=\lambda_{(p,w;p',w')}\sigma_{(p',w')}\cdot a_l'$ for $(c,a,p,w),(c',a',p',w')$ with $(c,a,p,w)\sim(c',a',p',w')$; let $\lambda_{(p',w';p'',w'')}>0$ be such that $\sigma_{(p',w')}\cdot a_l'=\lambda_{(p',w';p'',w'')}\sigma_{(p'',w'')}\cdot a_l''$ for $(c',a',p',w'),(c'',a'',p'',w'')$ with $(c',a',p',w')\sim(c'',a'',p'',w'')$; and let $\lambda_{(p,w;p'',w'')}>0$ be such that $\sigma_{(p,w)}\cdot a_l=\lambda_{(p,w;p'',w'')}\sigma_{(p'',w'')}\cdot a_l''$ for $(c'',a'',p'',w''),(c,a,p,w)$. The proof is concluded by showing that $\lambda_{(p,w;p'',w'')}=\lambda_{(p,w;p',w')}\lambda_{(p',w';p'',w'')}$ holds.

Suppose, by way of contradiction, that $\lambda_{(p,w;p'',w'')}\neq\lambda_{(p,w;p',w')}\lambda_{(p',w';p'',w'')}$. By $\sigma_{(p,w)}\cdot a_l=\lambda_{(p,w;p',w')}\sigma_{(p',w')}\cdot a_l'$ and $\sigma_{(p',w')}\cdot a_l'=\lambda_{(p',w';p'',w'')}\sigma_{(p'',w'')}\cdot a_l''$, it follows that $(c,a,p,w)\sim(c',a',p',w')\sim(c'',a'',p'',w'')$, and $\sigma_{(p,w)}\cdot a_l=\lambda_{(p,w;p',w')}\sigma_{(p',w')}\cdot a_l'=\lambda_{(p,w;p',w')}\lambda_{(p',w';p'',w'')}\sigma_{(p'',w'')}\cdot a_l''$ holds. By the transitivity of $\succsim$, $(c,a,p,w)\sim(c'',a'',p'',w'')$ holds. Then, $\sigma_{(p,w)}\cdot a_l=\lambda_{(p,w;p'',w'')}\sigma_{(p'',w'')}\cdot a_l''$. However, $\sigma_{(p,w)}\cdot a_l=\lambda_{(p,w;p',w')}\lambda_{(p',w';p'',w'')}\sigma_{(p'',w'')}\cdot a_l''$ and $\lambda_{(p,w;p'',w'')}\neq\lambda_{(p,w;p',w')}\lambda_{(p',w';p'',w'')}$, which is a contradiction. Therefore, $\lambda_{(p,w;p'',w'')}=\lambda_{(p,w;p',w')}\lambda_{(p',w';p'',w'')}$ holds. ■

Proof of Corollary 1. Straightforward and therefore omitted. ■

Proof of Theorem 2. (*Necessity*) It is immediate that if a labour ordering $\succsim$ on $\mathcal{CP}$ is wage-additive, it satisfies the axioms.

(*Sufficiency*) Take any pair of profiles $(c,a,p,w),(c',a',p',w')\in\mathcal{CP}$. Note that by the universality of $\mathcal{P}$, it is possible that $(c',a',p,w),(c,a,p',w')\in\mathcal{CP}$. Note that it follows from **CPTC** that $(c,a,p,w)\succsim(c',a',p,w)$ if and only if $w\cdot a_l\geqq w\cdot a_l'$. Likewise, $(c,a,p',w')\succsim(c',a',p',w')$ if and only if $w'\cdot a_l\geqq w'\cdot a_l'$.

Let $w^*>\mathbf{0}$ be such that $\sum w_\nu^*=1$ and $w^*=\lambda_{(w,w^*)}w$ for some $\lambda_{(w,w^*)}>0$. Also, let $w'^*>\mathbf{0}$ be such that $\sum w_\nu'^*=1$ and $w'^*=\lambda_{(w',w'^*)}w'$ for some $\lambda_{(w',w'^*)}>0$. Then, by **IND**, there exist $(c^*,a^*,p,w^*),(c^*,a^*,p',w'^*)\in\mathcal{CP}$ such that for any $\nu,\mu\in\mathcal{T}$, $a_{l\nu}^*=a_{l\mu}^*$, and that $(c^*,a^*,p,w^*)\sim(c^*,a^*,p',w'^*)$ for $w^*a_l^*=w'^*a_l^*$. Moreover, by **SSUB**, there exist $(c^{**},a^{**},p,w^*),(c^{***},a^{***},p,w)\in\mathcal{CP}$ such that for any $\nu,\mu\in\mathcal{T}$, $a_{l\nu}^{**}=a_{l\mu}^{**}$ and $a_{l\nu}^{***}=a_{l\mu}^{***}$; $\lambda_{(w,w^*)}a_l^{**}=a_l^{***}$; and $(\underline{a}^{**},\overline{a}^{**})=(\underline{a}^{***},\overline{a}^{***})$, and that $(c^{**},a^{**},p,w^*)\sim(c^{***},a^{***},p,w)$. By the same argument applying **SSUB**, there exist $(c'^{**},a'^{**},p',w'^*),(c'^{***},a'^{***},p',w')\in\mathcal{CP}$ such that $(c'^{**},a'^{**},p',w'^*)\sim(c'^{***},a'^{***},p',w')$.

Note that there exists $k>0$ such that $a_l^*=ka_l^{**}$. Then, $(c^*,a^*,p,w^*)\sim(kc^{**},ka^{**},p,w^*)$ by $w^*a_l^*=w^*ka_l^{**}$ and **CPTC**. Then, $(kc^{**},ka^{**},p,w^*)\sim(kc^{***},ka^{***},p,w)$ by **SINV**. Thus, $(c^*,a^*,p,w^*)\sim(kc^{***},ka^{***},p,w)$ by the transitivity of $\succsim$. Likeiwse, there exists $k'>0$ such that $a_l^*=k'a_l'^{**}$. Then, $(c^*,a^*,p',w'^*)\sim(k'c'^{**},k'a'^{**},p',w'^*)$ by $w'^*a_l^*=w'^*k'a_l'^{**}$ and **CPTC**. Then, $(k'c'^{**},k'a'^{**},p',w'^*)\sim(k'c'^{***},k'a'^{***},p',w')$ by **SINV**. Thus, $(c^*,a^*,p',w'^*)\sim(k'c'^{***},k'a'^{***},p',w')$ by the transitivity of $\succsim$.

In conclusion, by the transitivity of $\succsim$, $(kc^{***}, ka^{***}, p, w) \sim (k'c'^{***}, k'a'^{***}, p', w')$ holds for $wka_l^{***} = w'k'a_l'^{***}$. Then, by **D**, **SINV**, and the transitivity of $\succsim$, we obtain for $(c, a, p, w), (c', a', p', w') \in \mathcal{CP}$, $(c, a, p, w) \succsim (c', a', p', w')$ holds if and only if $w \cdot a_l \geqq w' \cdot a_l'$ holds. ■

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