Quantum Groups and Noncommutative Complex Geometry

Réamonn Ó Buachalla

Submitted in Partial Fulfillment of the Requirements of the Degree of Doctor of Philosophy,
Queen Mary, University of London
2013

School of Mathematics

Supervisor: Prof. S. Majid

Dedication

I would like to sincerely thank my supervisor, Prof. Shahn Majid, for his guidance and support throughout the course of my PhD. I am grateful to him for the generous manner in which he shared ideas and insights, and also for the freedom he allowed me to pursue my own ideas. I would also like to thank Tomasz Breziński, Edwin Beggs, Stefan Kolb, Ulrich Krähmer, Adam Rennie, and Colin Reed for interesting and helpful discussions.

On a more personal level, I would also like to thank my parents Ina and John, for their support throughout my many years of education, and of course Eloise and Donaldo.

Abstract

Noncommutative Riemannian geometry is an area that has seen intense activity over the past 25 years. Despite this, noncommutative complex geometry is only now beginning to receive serious attention. The theory of quantum groups provides a large family of very interesting potential examples, namely the quantum flag manifolds. Thus far, only the irreducible quantum flag manifolds have been investigated as noncommutative complex spaces. In a series of papers, Heckenberger and Kolb showed that for each of these spaces, there exists a q-deformed Dolbeault double complex.

In this thesis a comprehensive framework for noncommutative complex geometry on quantum homogeneous spaces is introduced. The main ingredients used are covariant differential calculi and Takeuchi's categorical equivalence for faithfully flat quantum homogeneous spaces. A number of basic results are established, producing a simple set of necessary and sufficient conditions for noncommutative complex structures to exist. It is shown that when applied to the quantum projective spaces, this theory reproduces the q-Dolbeault double complexes of Heckenberger and Kolb. Furthermore, the framework is used to q-deform results from Borel–Bott–Weil theory, and to produce the beginnings of a theory of noncommutative Kähler geometry.

Contents

Chapter 1

Introduction

"La richesse kählérienne fait dire à certains que la géométrie kählérienne est plus importante que la géométrie riemannienne." Marcel Berger

Classical complex geometry is a subject of remarkable richness and beauty with deep connections to modern physics. Yet despite over twenty five years of noncommutative geometry, the development of noncommutative complex geometry is still in its infancy. What we do have is a large number of examples which demand consideration as noncommutative complex spaces. We cite, among others, noncommutative tori [22, 73], noncommutative projective algebraic varieties [69], fuzzy flag manifolds [54], and (most importantly from our point of view) examples arising from the theory of quantum groups [28, 51]. These objects are of central importance to areas such as the construction of spectral triples [23, 20, 67], noncommutative mirror symmetry [1, 66, 70], localisation for quantum groups [2, 3, 44], and the quantum Baum–Connes conjecture [78].

Thus far, there have been two attempts to formulate a general framework for noncommutative complex geometry. The first, due to Khalkhali, Landi, and van Suijlekom [34], was introduced to provide a context for their work on the noncommutative complex geometry of the Podleś sphere. This followed on from earlier work of Majid [51], Schwartz and Polishchuk [66], and Connes [12, 13]. Khalkhali and Moatadelro [35, 36] would go on to apply this framework to D'Andrea and Dąbrowski's work [20] on the higher order quantum projective spaces.

Subsequently, Beggs and Smith introduced a second more comprehensive approach to noncommutative complex geometry in [6]. Their motive was to provide a framework for quantising the intimate relationship between complex differential geometry and complex projective geometry. They foresee that the rich interaction between algebraic and analytic techniques occurring in the classical setting will carry over to the noncommutative world.

The more modest aim of this thesis is to begin the development of a theory of noncommutative complex geometry for quantum group homogeneous spaces. This will be done very much in the style of Majid's noncommutative Riemannian geometry [48, 51, 49]. The only significant difference being that here we will not need to assume that our quantum homogeneous spaces are Hopf–Galois extensions, while we will assume that they are faithfully flat. We first introduce the notion of a *covariant noncommutative complex structure* for a total differential calculus. Then, by calling on our assumption of faithful flatness, we use Takeuchi's categorical equivalence to establish a simple set of necessary and sufficient conditions for such noncommutative complex structures to exist. In subsequent work, it is intended to build upon these results and formulate noncommutative generalisations of Hodge theory and Kähler geometry for quantum homogeneous spaces [60]. Indeed, the first steps in this direction have already been taken in Chapter 7.

For this undertaking to be worthwhile, however, it will need to be applicable to a good many interesting examples. Recall that classically one of the most important classes of homogeneous complex manifolds is the family of generalised flag manifolds. As has been known for a long time, these spaces admit a direct q-deformation in terms of the Drinfeld–Jimbo quantum groups [42, 72, 76]. Somewhat more recently, q-deformations of the complex geometry of the flag manifolds have also begun to emerge. The first examples appeared in a series of works due to Heckenberger and Kolb [39, 26, 38]. In this series, they classified the covariant first-order differential calculi over the quantum Grassmannians, and in so doing identified a canonical first order calculus for the these spaces, along with a decomposition of this calculus into q-deformed analogues of the holomorphic and anti-holomorphic one-forms. Heckenberger and Kolb would go on to extend this work to include all the irreducible quantum flag manifolds [27]. For the special case of the Podleś sphere, Majid would independently reproduce this decomposition of one-forms using his framework for noncommutative Riemannian geometry. Moreover, he showed that the decomposition could be extended to a direct q-deformation of the Dolbeault double complex of $\mathbf{C}P^1$. A short time after, Heckenberger and Kolb [28] showed that such a q-Dolbeault double complex exists for all the irreducible flag manifolds. This group of results gives us one of the most important families of noncommutative complex structures that we have, and as such, provides an invaluable testing ground for any newly proposed theory of noncommutative complex geometry.

Heckenberger and Kolb undertook their work in the absence of a general framework for noncommutative complex geometry. While they produced an accomplished and comprehensive treatment of the q-deformed Dolbeault complexes, the fundamental processes at work were obscured by the complexity of the calculations. Moreover, their technical style of presentation is quite difficult to use as a basis for future work. Subsequent papers on the geometry of the quantum flag manifolds would follow a different approach [40, 20].

The general framework for noncommutative complex geometry that we introduce in this thesis is a refinement of Majid's approach to the complex geometry of the Podleś sphere. We show that, for the special case of the quantum projective spaces, the work of Heckenberger and Kolb can be understood in terms of our framework. This allows for a significant simplification of the required calculations, and helps identify some of the underlying general processes at work. It is foreseen that this work will prove easily extendable to all the irreducible quantum flag manifolds. Moreover, it is hoped to extend it even further to include all the quantum flag manifolds, and in so doing, produce new examples of noncommutative complex structures. As mentioned above, it is also hoped to use this new simplified presentation to identify noncommutative Hodge and Kähler structures hidden in the Dolbeault complexes of Heckenberger and Kolb.

One of the main motivations for studying noncommutative complex structures is that a number of important research programs in noncommutative geometry make central use of noncommutative generalisations of holomorphic vector bundles. We cite noncommutative mirror symmetry (as discussed in [1]), noncommutative Borel–Weil theory (as discussed in [51]), and the aforementioned efforts of Beggs and Smith to formulate a noncommutative version of the *géometrie-algebraique-*

géometrie-analytique principal. Now, the definition of a noncommutative complex structure over an algebra A naturally suggests a notion of holomorphic element of A. However, to define a holomorphic element of a projective module over A (the noncommutative analogue of a vector bundle) one needs to introduce a noncommutative holomorphic structure for these modules. This is essentially a special type of covariant derivative. A natural way to construct these covariant derivatives is using the theory of quantum principal bundles due to Breziński and Majid. We do this for negative index quantum line bundles of the quantum projective spaces, and then find an explicit description of the holomorphic subalgebras of these line bundles. In addition to the research projects just mentioned, this work is also foreseen to have applications to the theory of noncommutative holomorphic differential operators (as discussed in [4, 5]).

The thesis is organised as follows: In Chapter 2 we introduce some well-known material about quantum homogeneous spaces, Takeuchi's categorical equivalence, covariant differential calculi, the framing result of Majid, and the classification result of Hermisson. The presentation will differ somewhat from standard in the presentation of Majid and Hermisson's work.

In Chapter 3 we discuss the quantum special unitary group, its coquasi-triangular structure, and the quantum projective spaces. Moreover, we give an explicit presentation of the ideal corresponding to the Heckenberger–Kolb calculus for these spaces.

In Chapter 4 we introduce one of the fundamental results of the thesis. We show that we can restrict Takeuchi's equivalence to a monoidal equivalence between two subcategories of ${}^{G}_{M}\mathcal{M}_{M}$ and $\mathcal{M}^{H}_{M}$. Crucially, this allows us to take tensor products of framings. We build upon this work and show how to frame the maximal prolongation of a covariant first-order differential calculus. We then show how our method can be greatly simplified by making a suitable choice of calculus on the total space. The general theory is then applied to the Heckenberger–Kolb calculus for the quantum projective spaces, yielding a novel description of its maximal prolongation.

In Chapter 5 we introduce a new variation on the existing definitions of noncommutative complex structure, and provide a simple set of necessary and sufficient conditions for such structures to exist. The description of the maximal prolongation of the Heckenberger–Kolb calculus given in [28] is then presented as an example of a noncommutative complex structure.

In Chapter 6 we consider holomorphic connections and show how to construct such a connection for the quantum projective spaces using the theory of quantum principal bundles. We define the notion of a holomorphic structure, show that our connection induces a holomorphic structure for the negative index quantum line bundles of the quantum projective spaces, and calculate the holomorphic subalgebras of these bundles.

Finally, in Chapter 7 we take the first steps towards a noncommutative theory of Kähler geometry by q-deforming the the Kähler identities for the Podleś sphere.

Chapter 2

Preliminaries

In this section we recall Takeuchi's categorical equivalence for faithfully flat quantum homogeneous spaces, and some of the consequences of this result for the theory of covariant differential calculi. With the exception of the somewhat novel presentations of Majid's framing theorem and Hermisson's classification, all of the material found here is well-known.

2.1 Quantum Homogeneous Spaces and Takeuchi's Categorical Equivalence

Let G be a Hopf algebra with comultiplication Δ_G, counit ε_G, antipode S_G, unit 1_G, and multiplication m_G (where no confusion arises, we will drop explicit reference to G when denoting these operators). Throughout, we use Sweedler notation, as well as denoting $g^+ := g - \varepsilon(g)1$, for $g \in G$, and $V^+ = V \cap \ker(\varepsilon)$, for V a subspace of G. For a right G-comodule V with coaction Δ_R, we say that an element $v \in V$ is *coinvariant* if $\Delta_R(v) = v \otimes 1$. We denote the subspace of all coinvariant elements by V^G, and call it the *coinvariant subspace* of the coaction. More generally, a *covariant subspace* $W \subseteq V$ is defined to be a subspace that is also a sub-comodule of V. More generally, a *covariant subspace* $W \subseteq V$ is defined to be a subspace that is also a sub-comodule of V.

For H also a Hopf algebra, a *homogeneous* right H-coaction on G is a coaction of the form $(\mathrm{id} \otimes \pi) \circ \Delta$, where $\pi : G \to H$ is a Hopf algebra map. We call the coinvariant subspace $M := G^H$ of such a coaction a *quantum homogeneous space.* As is easy to see, M will always be a subalgebra of G. Moreover, it can be shown without difficulty that the coaction of G restricts to a right G-coaction on M, and that

$$\pi(m) = \varepsilon(m)1_H, \qquad\qquad (\text{for all } m \in M). \tag{2.1}$$

In this thesis we will *always* use the symbols G, H, π and M in this sense. We also note that G is itself a trivial example of a quantum homogeneous space, where $\pi = \varepsilon$.

Let us now introduce ${}^G_M\mathcal{M}_M$, the *category of associated bundles* of M, whose objects, the *associated bundles*, are the M-bimodules $\mathcal{E}$ endowed with a left G-coaction Δ_L, satisfying the compatibility condition

$$\Delta_L(mem') = m_{(1)}e_{(-1)}m'_{(1)} \otimes m_{(2)}e_{(0)}m'_{(2)}, \qquad (\text{for all } \ m, m' \in M, e \in \mathcal{E}),$$

and whose morphisms are both M-bimodule and left G-comodule maps. Moreover, let $\mathcal{M}^H_M$ denote the category whose objects V are the right M-modules endowed with a right H-coaction satisfying the compatibility condition

$$\Delta_R(vm) = v_{(0)}m_{(2)} \otimes S(\pi(m_{(1)}))v_{(1)}, \qquad (\text{for all } \ v \in V, m \in M), \tag{2.2}$$

and whose morphisms are both left M-module and right H-comodule maps. In what follows, for sake of clarity, we will denote the right M-action on an object in ${}^G_M\mathcal{M}_M$ by juxtaposition, while we will denote the right M-action on an object in $\mathcal{M}^H_M$ by $\triangleleft$.

For any object V in $\mathcal{M}^H_M$, we can associate to it a corresponding object in ${}^G_M\mathcal{M}_M$ as follows: Consider the coinvariant subspace $(G \otimes V)^H$, where $G \otimes V$ is endowed with the usual tensor product coaction. We can give $(G \otimes V)^H$ the structure of an object in ${}^G_M\mathcal{M}_M$ by defining right and left M-actions according to

$$m(\sum_i g^i \otimes v^i) = \sum_i mg^i \otimes v^i, \qquad (\sum_i g^i \otimes v^i)m = \sum_i g^i m_{(1)} \otimes (v^i \triangleleft m_{(2)}),$$

and defining a left G-coaction according to

$$\Delta_L(\sum_i g^i \otimes v^i) = \sum_i g^i_{(1)} \otimes g^i_{(2)} \otimes v^i.$$

A *framing*, for an object $\mathcal{E}$ in ${}^{G}_{M}\mathcal{M}_{M}$, is a pair (V, t) where V is an object in $\mathcal{M}^{H}_{M}$, and t is an isomorphism from $\mathcal{E}$ to $(G \otimes V)^{H}$. A natural question to ask is whether a framing exists for every object in ${}^{G}_{M}\mathcal{M}_{M}$, and when it does how many different choices of framing there are. In order to address this question, we will need to introduce some additional structures.

The right M-module structure of $\mathcal{E}$ clearly restricts to a right M-module structure on $\mathcal{E}/(M^{+}\mathcal{E})$. Moreover, it can be shown using (2.1) that the left G-module structure of $\mathcal{E}$ induces a right H-comodule structure on $\mathcal{E}/(M^{+}\mathcal{E})$ defined by

$$\Delta_R(\overline{e}) = \overline{e_{(0)}} \otimes S(\pi(e_{(-1)})), \qquad (e \in \mathcal{E}), \tag{2.3}$$

where $\overline{e}$ denotes the coset of e in $\mathcal{E}/(M^{+}\mathcal{E})$. To show that these two structures are compatible in the sense of (2.2) is routine. Thus, we have given $\mathcal{E}/(M^{+}\mathcal{E})$ the structure of an object in $\mathcal{M}^{H}_{M}$. Consider now the functors

$$\begin{aligned} \Phi_M &: {}^{G}_{M}\mathcal{M}_{M} \to \mathcal{M}^{H}_{M}, & \Phi_M(\mathcal{E}) &= \mathcal{E}/(M^{+}\mathcal{E}), \\ \Psi_M &: \mathcal{M}^{H}_{M} \to {}^{G}_{M}\mathcal{M}_{M}, & \Psi_M(V) &= (G \otimes V)^{H}. \end{aligned}$$

Where for $\mathcal{E}, \mathcal{F}$ two objects in ${}^{G}_{M}\mathcal{M}_{M}$, and $f : \mathcal{E} \to \mathcal{F}$ a morphism, we define $\Phi_M(f) : \Phi_M(\mathcal{E}) \to \Phi_M(\mathcal{F})$ to be the morphism to which f descends on $\Phi_M(\mathcal{E})$. While for $\varphi : V \to W$ a morphism in $\mathcal{M}^{H}_{M}$, we define $\Psi_M(\varphi) := 1 \otimes \varphi$. To show that both morphisms are well-defined is routine. Moreover, using some basic linear algebra arguments, it can also be shown that, for $\mathcal{E}, \mathcal{F}$ two objects in ${}^{G}_{M}\mathcal{M}_{M}$, and V, W two objects in $\mathcal{M}^{H}_{M}$, we have

$$\Phi(\mathcal{E} \oplus \mathcal{F}) = \Phi(\mathcal{E}) \oplus \Phi(\mathcal{F}), \qquad \Psi(V \oplus W) = \Psi(V) \oplus \Psi(W), \tag{2.4}$$

and if we further assume that $\mathcal{E} \subseteq \mathcal{F}$, and $V \subseteq W$, then

$$\Phi(\mathcal{E}/\mathcal{F}) = \Phi(\mathcal{E})/\Phi(\mathcal{F}), \qquad \Psi(V/W) = \Psi(V)/\Phi(W). \tag{2.5}$$

A natural question to ask is when this induces an equivalence of categories. This leads us to the notion of faithful flatness: We say that G is a *faithfully flat* module over M if the *tensor product functor* $G \otimes_M - : {}_{M}\mathcal{M} \to {}_{\mathbf{C}}\mathcal{M}$, from the category of left M-modules to the category of complex vector spaces, preserves and reflects exact sequences.

Theorem 2.1.1 (Takeuchi [75]) *Let $\pi : G \to H$ be a quantum homogeneous space for which G is a faithfully flat right module over $M = G^H$. An equivalence of categories between between ${}^G_M\mathcal{M}_M$ and $\mathcal{M}^H_M$ is determined by the functors Φ_M, and Ψ_M, and the natural isomorphisms*

$$\mathrm{frame}_M : \mathcal{E} \to \Psi_M \circ \Phi_M(\mathcal{E}), \qquad e \mapsto e_{(-1)} \otimes \overline{e_{(0)}}, \tag{2.6}$$

$$\mathrm{frame}^{\perp}_M : \Phi_M \circ \Psi_M(V) \to V, \qquad \sum_i \overline{g^i \otimes v^i} \mapsto \sum_i \varepsilon(g^i)v^i. \tag{2.7}$$

Thus we see that we have a framing $(\Phi(\mathcal{E}), \mathrm{frame}_M)$ for every object $\mathcal{E}$ in ${}^G_M\mathcal{M}_M$. Now for any other framing $s : \mathcal{E} \to (G \otimes V)^H$, for V some object in $\mathcal{M}^H_M$, we have the isomorphism

$$\sigma := \mathrm{frame}^{\perp}_M \circ \Phi_M(s) : \Phi_M(\mathcal{E}) \to V,$$

which gives us the re-expression $s = \Psi_M(\sigma) \circ \mathrm{frame}_M$. It follows that every framing of $\mathcal{E}$ is of the form $(V, \Psi_M(\sigma) \circ \mathrm{frame}_M)$, where V is some object in $\mathcal{M}^H_M$, and $\sigma : \Phi_M(\mathcal{E}) \to V$ is an isomorphism in $\mathcal{M}^H_M$.

This result allows us to introduce a quantum generalisation of the classical notion of vector bundle rank: For any object $\mathcal{E}$ in ${}^G_M\mathcal{M}_M$, we define the *rank* of $\mathcal{E}$ to be the vector space dimension of $\Phi_M(\mathcal{E})$. Moreover, we call an associated bundle of rank 1 a *quantum line bundle*.

We should note that the original presentation of this work by Takeuchi uses somewhat different conventions. Most noticeably, the notion of cotensor product $G \,\square_H\, \Phi_M(\mathcal{E})$ is used instead of coinvariant subspace $(G \otimes \Phi_M(\mathcal{E}))^H$. However, as is easily seen, the two notions are equivalent. Another important point to note is that the existence of the isomorphism from $\mathcal{E}$ to $\Psi_M \circ \Phi_M(\mathcal{E})$ does not depend on the assumption of faithful flatness, as the following lemma shows:

Lemma 2.1.2 *For a (not necessarily faithfully flat) quantum homogeneous space M, the map frame_M is an isomorphism, with inverse*

$$\mathrm{frame}^{-1}_M : \Psi_M \circ \Phi_M(\mathcal{E}) \to \mathcal{E}, \qquad \sum_i g^i \otimes \overline{e^i} \mapsto \sum_i g^i S(e_{(-1)})e_{(0)}. \tag{2.8}$$

Proof. Let us begin by showing that frame^{-1}_M is well-defined: For $\sum_i g^i \otimes m^i e^i$

an element in $(G \otimes \mathcal{E})^H$, with $m^i \in M^+$, for all i, we have

$$\begin{aligned}\sum_i f^i S((m^i e^i)_{(-1)})(m^i e^i)_{(0)} &= \sum_i f^i S(m^i_{(1)} e^i_{(-1)}) m^i_{(2)} e^i_{(0)} \\ &= \sum_i f^i S(e^i_{(-1)}) S(m^i_{(1)}) m^i_{(2)} e^i_{(0)} \\ &= \sum_i \varepsilon(m^i) f^i S(e^i_{(-1)}) e^i_{(0)} = 0,\end{aligned}$$

Thus, frame_M^{-1} descends to a well-defined map on $(G \otimes \Phi_M(\mathcal{E}))^H$. That frame_M^{-1} is indeed the inverse of frame_M follows from

$$\mathrm{frame}_M^{-1} \circ \mathrm{frame}_M(e) = \mathrm{frame}_M^{-1}(e_{(-1)} \otimes \overline{e_{(0)}}) = e_{-2} S(e_{(-1)}) e_{(0)} = \varepsilon(e_{(-1)}) e_{(0)} = e.$$

$\square$

Thus, we see that even in the absence of faithful flatness, a framing will exist for any $\mathcal{E} \in {}^G_M\mathcal{M}_M$. However, without faithful flatness we are not guaranteed the existence of an inverse for $\mathrm{frame}_M^{\perp}$.

2.2 Differential Calculi

In this section we recall the well known notions of a first order differential calculus, and a total differential calculus. These are very natural generalisations of the classical definitions of the Kähler forms, and the de Rham complex, of a variety respectively. They originate in the seminal work of Woronowicz [79].

2.2.1 Covariant First Order Differential Calculi

Let A be an algebra. (In what follows all algebras are assumed to be unital.) A *first-order differential calculus* over A is a pair (Ω^1, d), where Ω^1 is an A-A-bimodule and $\mathrm{d} : A \to \Omega^1$ is a linear map for which the *Leibniz rule* holds

$$\mathrm{d}(ab) = a(\mathrm{d}b) + (\mathrm{d}a)b, \qquad (a, b, \in A),$$

and for which $\Omega^1 = \mathrm{span}_{\mathbf{C}}\{a\mathrm{d}b \,|\, a, b \in A\}$. (Where no confusion arises we will drop explicit reference to d and denote a calculus by its bimodule Ω^1 alone.) We call an element of Ω^1 a *one-form.* The *universal first-order differential calculus* over A is the pair $(\Omega^1_u(A), \mathrm{d}_u)$, where $\Omega^1_u(A)$ is the kernel of the product map $m : A \otimes A \to A$ endowed with the obvious bimodule structure, and d_u is defined by

$$\mathrm{d}_u : A \to \Omega^1_u(A), \qquad\qquad a \mapsto 1 \otimes a - a \otimes 1.$$

It is not difficult to show (see [79] for details) that every calculus over A is of the form $(\Omega^1_u(A)/N, \mathrm{proj} \circ \mathrm{d}_u)$, where N is a A-sub-bimodule of $\Omega^1_u(A)$, and $\mathrm{proj} : \Omega^1_u(A) \to \Omega^1_u(A)/N$ is the canonical projection. Moreover, this association between calculi and sub-bimodules is bijective.

Now let $\Gamma \oplus \Gamma' = \Omega^1$ be a sub-bimodule of a first order differential calculus (Ω^1, d) over an algebra A. Define ∂ to be the composition of d with projection onto Γ, and define ∂' to be the composition of d with projection onto Γ. Since $\mathrm{d} = \partial + \partial'$, we must have that Γ and Γ' are spanned by elements of the form $a\partial b$ and $a\partial' b$ respectively, for $a, b \in A$. From the Leibniz rule for d we have that $\mathrm{d}(ab) = (\mathrm{d}a)b + a\mathrm{d}b$. This implies that

$$\partial(ab) + \overline{\partial}(ab) = (\partial a)b + (\overline{\partial} a)b + a\partial b + a(\overline{\partial} b).$$

But Γ and Γ' are both right submodules, so our direct sum decomposition says we must have

$$\partial(ab) = (\partial a)b + a\partial b, \qquad\qquad \overline{\partial}(ab) = (\overline{\partial} a)b + a\overline{\partial} b.$$

Hence, both (Γ, ∂) and (Γ', ∂') are first order differential calculi. We call such calculi *subcalculi* of Ω^1.

A differential calculus $\Omega^1(A)$ over a left G-comodule A is said to be *left-covariant* if there exists a (necessarily unique) left-coaction $\Delta_L : \Omega^1(A) \to G \otimes \Omega^1(A)$ such that

$$\Delta_L(a\mathrm{d}b) = \Delta(a)(\mathrm{id} \otimes \mathrm{d})\Delta(b), \qquad\qquad (a, b \in A).$$

Clearly this can happen if, and only if, the corresponding sub-bimodule $N \subseteq \Omega^1_u(A)$ is left-covariant, giving us a correspondence between left-covariant calculi and left-covariant sub-bimodules of $\Omega^1_u(A)$. Furthermore, for M a quantum homogeneous space, any left-covariant calculus has the structure of an object in ${}^G_M\mathcal{M}_M$. Thus, Takeuchi's theorem induces a correspondence between left-covariant calculi $\Omega^1(M)$

and sub-objects of $\Phi_M(\Omega^1_u(M))$ in $\mathcal{M}^H_M$. We define the *dimension* of the calculus to be its rank as an associated bundle.

A problem with this last classification is that our generator and relation presentation of $\Phi_M(\Omega^1_u(M))$ is not particularly easy to work with. However, the following very useful result tells us that there is an isomorphism between $\Phi_M(\Omega^1_u(M))$ and M^+, where we consider M^+ as an object in $\mathcal{M}^H_M$ according to the obvious right M-module structure, and the right H-comodule structure defined by $\Delta_{M,R}(m) = m_{(2)} \otimes S(\pi(m_{(1)}))$, for $m \in M^+$. (Note that the proof given here does not assume that G is a Hopf–Galois extension of M as is done in [48]. However, this more general result is implicit in the original proof.)

Theorem 2.2.1 *[Majid [48]] For a (not necessarily faithfully flat or Hopf–Galois) quantum homogeneous space M, we have an isomorphism*

$$\sigma : \Phi_M(\Omega^1_u(M)) \to M^+, \qquad \sum_i \overline{m^i \mathrm{d} n^i} \mapsto \sum_i \varepsilon(m^i)(n^i)^+, \tag{2.9}$$

and a corresponding framing $(M^+, s := \Psi(\sigma)\circ frame_M)$, which we call the canonical framing. *Explicitly s acts according to*

$$s : \Omega^1_u(M) \to (G \otimes M^+)^H, \qquad m\mathrm{d}n \mapsto mn_{(1)} \otimes (n_{(2)})^+. \tag{2.10}$$

Proof. We begin by showing that the map σ is well-defined as a right M-module map: Consider the right M-module map

$$\varepsilon \otimes \mathrm{id} : M \otimes M \to M, \qquad m \otimes n \mapsto \varepsilon(m)n.$$

It is clear from the definition of $\Phi_M(\Omega^1_u(M))$ that $\varepsilon \otimes \mathrm{id}$ descends to a well-defined mapping from $\Phi_M(\Omega^1_u(M))$ to M. Moreover, since

$$(\varepsilon \otimes \mathrm{id})(\overline{m\mathrm{d}n}) = (\varepsilon \otimes \mathrm{id})(\overline{m \otimes n - mn \otimes 1}) = \varepsilon(m)(n - \varepsilon(n)1) = \varepsilon(m)n^+,$$

it is clear that this restriction is exactly the map σ defined in (2.9). That σ is also a right H-comodule map is clear from

$$
\begin{aligned}
(\sigma \otimes \mathrm{id}) \circ \Delta_R(\overline{m\mathrm{d}n}) &= (\sigma \otimes \mathrm{id})(\overline{m_{(2)}\mathrm{d}n_{(2)}} \otimes S(\pi(m_{(1)}n_{(1)}))) \\
&= \varepsilon(m_{(2)})(n_{(2)})^+ \otimes S(\pi(m_{(1)}n_{(1)})) = (n_{(2)})^+ \otimes S(\pi(mn_{(1)})) \\
&= \varepsilon(m)(n_{(2)})^+ \otimes S(\pi(n_{(1)})) \\
&= \varepsilon(m)(n_{(2)} \otimes S(\pi(n_{(1)})) - 1 \otimes S(\pi(n))) \\
&= \varepsilon(m)(n_{(2)} \otimes S(\pi(n_{(1)})) - \varepsilon(n)1 \otimes 1_H) = \varepsilon(m)\Delta_{M,R}(n^+) \\
&= \Delta_{M,R} \circ \sigma(\overline{m\mathrm{d}n}),
\end{aligned}
$$

where we have used the relation $\pi(m) = \varepsilon(m)1_H$ from (2.1).

Now that we have shown that σ is a well defined morphism, we can move on to showing that it is, in fact, an isomorphism. To establish injectivity, we first note that the kernel of $\varepsilon \otimes \mathrm{id}$ is equal to $M^+ \otimes M$: Any element contained in the intersection of $M^+ \otimes M$ and $\Omega^1_u(M)$ will be of the form $\sum_i m^i \otimes n^i$, where each $m^i \in M^+$, and $\sum m^i n^i = 0$. Since

$$
\sum_i m^i \otimes n^i = \sum_i (m^i \otimes n^i - m^i n^i \otimes 1) = \sum_i m^i (1 \otimes n^i - n^i \otimes 1) = \sum_i m^i \mathrm{d}n^i,
$$

we must have that the kernel of $(\varepsilon \otimes \mathrm{id})|_{\Omega^1_u(M)}$ is contained in $M^+\Omega^1_u(M)$. Hence, we can conclude that σ is an injective map. Since the surjectivity of σ is clear, we can conclude that σ is an isomorphism.

Finally, we come to the framing in (2.10): It is clear from Lemma (2.1.2) that $\Psi_M(\sigma) \circ \mathrm{frame}_M$ is a framing for $\Omega^1_u(M)$. That the explicit action of s is as given above, follows from

$$
\begin{aligned}
s(m\mathrm{d}n) &= \Psi_M(\sigma) \circ \mathrm{frame}_M(m\mathrm{d}n) = \Psi_M(\sigma)(m_{(1)}n_{(1)} \otimes \overline{m_{(2)}\mathrm{d}n_{(2)}}) \\
&= m_{(1)}n_{(1)} \otimes \sigma(\overline{m_{(2)}\mathrm{d}n_{(2)}}) = m_{(1)}n_{(1)} \otimes \overline{\varepsilon(m_{(2)})(n_{(2)})^+} = mn_{(1)} \otimes \overline{(n_{(2)})^+}.
\end{aligned}
$$

□

Combining this result with the classification of covariant calculi on quantum homogeneous spaces discussed earlier, gives us the classification result of Hermisson:

Corollary 2.2.2 (Hermisson [29]) *For a faithfully flat quantum homogeneous space M, there is a bijective correspondence between left-covariant first-order differential calculi over M, and the sub-objects of M^+ in $\mathcal{M}^H_M$.*

Now for such a calculus $\Omega^1(M) \simeq \Omega^1_u(M)/N$, with its corresponding ideal $\sigma(\Phi_M(N))$, the canonical framing clearly descends to a framing

$$s : \Omega^1(M) \to (G \otimes V_M)^H, \qquad a\mathrm{d}b \mapsto ab_{(1)} \otimes \overline{(b_{(2)})^+},$$

where we have denoted $V_M := M^+/\sigma(\Phi_M(N))$. We will call (V_M, s) the *canonical framing* of the calculus. It is easy to see from (2.8) that an explicit presentation of the inverse of the canonical framing is given

$$s^{-1} : (G \otimes V_M)^H \to \Omega^1(M), \qquad \sum_i f^i \otimes v^i \mapsto \sum_i f^i S(v^i_{(1)})\mathrm{d}v^i_{(2)}. \tag{2.11}$$

If we drop the assumption of faithful flatness, then because of Lemma 2.1.2 we will still have a corresponding framing for every covariant calculus. However, we are not guaranteed an equivalence between calculi and ideals. This is essentially what is established in Majid's second framing theorem in [51]. For the special case of the trivial quantum homogeneous space G (where the faithful flatness condition is trivial), the results of Majid and Hermisson reduce to Woronowicz's celebrated classification of left-covariant calculi over a Hopf algebra G. For such a calculus $\Omega^1(G)$, we will denote its cotangent space by Λ^1_G, and call it the *space of left-invariant one forms.*

If (Ω^1, d) is a differential calculus over a $*$-algebra A such that the involution of A extends to an involutive conjugate-linear map $*$ on Ω^1, for which $(a\mathrm{d}b)^* = (\mathrm{d}b^*)a^*$, for all $a, b \in A$, then we say that (Ω^1, d) is a *first-order differential $*$-calculus.* It is easy to see that the universal calculus $\Omega^1_u(A)$ over any $*$-algebra A always has a unique $*$-calculus structure. Moreover, any non-universal calculus of the form $\Omega^1(A) = \Omega^1(A)/N$ is a $*$-calculus if, and only if, $N^* = N$.

Let us now assume that both G and H are Hopf $*$-algebras, and that π is a Hopf $*$-algebra map. It is easy to see that in this case M is a $*$-subalgebra of G. In general it is not known how to tell that a calculus $\Omega^1(M)$ over M is a $*$-calculus, directly from the corresponding sub-object of M^+. However, we can show without too much difficulty, that for the universal $*$-calculus $\Omega^1_u(G)$, the corresponding $*$-map on $G \otimes G^+$ acts according to

$$* : G \otimes G^+ \to G \otimes G^+, \qquad g \otimes \overline{v} \mapsto g^*_{(1)} \otimes \overline{S(v^*)g^*_{(2)}}. \tag{2.12}$$

Thus, for $\Omega^1(G)$ a non-universal calculus over G, with corresponding sub-object $I_G \subseteq G^+$, we have that $\Omega^1(G)$ is a $*$-calculus if, and only if,

$$\{S(v^*) \,|\, v \in I_G\} = I_G.$$

Now if $\Omega^1(G)$ restricts to $\Omega^1(M)$ on M, then since $(m\mathrm{d}n)^* = \mathrm{d}(n^*)m^* \in \Omega^1(M)$, for any $m, n \in M$, the $*$-structure on $\Omega^1(G)$ must induce a $*$-structure on $\Omega^1(M)$. This provides us with a crude method for establishing that $\Omega^1(M)$ has a $*$-structure.

Building upon the classification of left-covariant calculi, it can be shown that bicovariant calculi are in bijective correspondence with the Ad_R-covariant right ideals of H^+, where as usual the *right adjoint action* is defined by $\mathrm{Ad}_R(h) := h_{(2)} \otimes S(h_{(1)})h_{(3)}$, for $h \in H$.

We say that H is *coquasi-triangular* if it is equipped with a convolution-invertible linear map $r : H \otimes H \to \mathbf{C}$ obeying

$$r(fg \otimes h) = r(f \otimes h_{(1)})r(g \otimes h_{(2)}), \quad r(f \otimes gh) = r(f_{(1)} \otimes h)r(f_{(2)} \otimes g), \tag{2.13}$$

and

$$g_{(1)}f_{(1)}r(f_{(2)} \otimes g_{(2)}) = r(f_{(1)} \otimes g_{(1)})f_{(2)}g_{(2)},$$

for all $f, g, h \in H$. For any coquasi-triangular Hopf algebra H, the *quantum Killing form* is the map

$$\mathcal{Q} : H \otimes H \to \mathbf{C} \qquad\qquad h \otimes g \mapsto r(g_{(1)} \otimes h_{(1)})r(h_{(2)} \otimes g_{(2)}).$$

If H has a set of generators $\{u^i_j \,|\, i, j = 1, \ldots, N\}$, for some $N \in \mathbf{N}$, then we can use $\mathcal{Q}$ to define a family of maps $\{Q_{kl} \,|\, k, l = 1, \ldots, N\}$ by setting

$$Q_{kl} : H \to \mathbf{C}, \qquad\qquad h \mapsto \mathcal{Q}(h \otimes u^k_l).$$

Using this family of maps, an N^2-dimensional representation Q can then be defined by

$$Q : H \to M_N(\mathbf{C}) \qquad\qquad h \mapsto [Q_{kl}(h)]_{kl}.$$

We call Q the *quantum Killing representation* of H. It can be shown [47] that $\ker(Q)^+$ is an Ad_R-covariant right ideal of H^+, and so, it corresponds to a bicovariant calculus. We call the corresponding calculus the *canonical bicovariant calculus* over H, and denote it by $\Omega^1_{\mathrm{bc},q}(H)$. When $H = \mathbf{C}_q[SU_2]$, it can be shown that one recovers Woronowicz's $4D_+$ calculus [79]. More generally, for $H = \mathbf{C}_q[SU_N]$, one recovers the bicovariant calculus introduced by Jurčo in [32].

2.2.2 Total Differential Calculi

We now come to noncommutative higher differential forms: For $(Y,+)$ a commutative semigroup, a *Y-graded algebra* is an algebra of the form $A = \bigoplus_{y\in Y} A^y$, where each A^y is a linear subspace of A, and $A^yA^z \subseteq A^{y+z}$, for all $y,z \in Y$. If $a \in A^y$, then we say that a is a *homogeneous element of degree y*. A *homogenous mapping of degree* d on A is a linear mapping $L : A \to A$ such that if $a \in A^y$, then $L(a) \in A^{y+d}$. We say that a subspace B of A is *homogeneous* if it admits a decomposition $B = \oplus_{y\in Y} B^y$, with $B^y \subseteq A^y$, for all $y \in Y$.

A triple $(A, \partial, \overline{\partial})$ is called a *double complex* if A is an N_0^2-graded algebra, ∂ is homogeneous mapping of degree $(1,0)$, $\overline{\partial}$ is homogeneous mapping of degree $(0,1)$, and

$$\partial^2 = \overline{\partial}^2 = 0, \qquad\qquad \partial \circ \overline{\partial} = -\overline{\partial} \circ \partial.$$

A *graded derivation* d on an $\mathbf{N}_0$-graded algebra A is a homogenous mapping of degree 1 that satisfies the *graded Liebniz rule*

$$\mathrm{d}(ab) = \mathrm{d}(a)b + (-1)^n a\mathrm{d}b, \qquad\qquad (\text{for all } a \in A^n,\ b \in A).$$

A pair (A, d) is called a *differential algebra* if A is an $\mathbf{N}_0$-graded algebra and d is a graded derivation on A such that $d^2 = 0$. The operator d is called the *differential* of the algebra.

Definition 2.2.3. A *total differential calculus* over an algebra A is a differential algebra $(\Omega(A), \mathrm{d})$, such that $\Omega^0 = A$, and

$$\Omega^k = \mathrm{span}_{\mathbf{C}}\{a_0\mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_k \,|\, a_0, \ldots, a_k \in A\}. \tag{2.14}$$

Following the classical example of the de Rham complex, we will always use $\wedge$ to denote the multiplication between total calculus elements, both of order greater than or equal to 1.

In commutative geometry the higher forms are constructed as exterior powers of the one-forms. In the noncommutative setting such a construction is not in general well-defined. However, there exists an alternative formulation of the higher forms which *is* well-defined for noncommutative algebras: For $(\Omega^1(A), \mathrm{d})$ a first-order differential calculus with corresponding sub-bimodule $N \subseteq \Omega^1_u(A)$, denote by $\Omega^\bullet(A)$

the quotient of the tensor algebra $\bigoplus_{k=0}^{\infty}(\Omega^1(A))^{\otimes_A k}$ by $\langle \mathrm{d}(N)\rangle$, where $\langle \mathrm{d}(N)\rangle$ is the subalgebra of the tensor algebra generated by $\mathrm{d}(N)$. As a little thought will confirm, the exterior derivative d has a unique extension to a map $\mathrm{d} : \Omega^\bullet(A) \to \Omega^\bullet(A)$, such that $(\Omega^\bullet(A), \mathrm{d})$ has the structure of a total differential calculus. We call this total differential calculus the *maximal prolongation* of $(\Omega^1(A), \mathrm{d})$. The maximal prolongation is easily seen to be unique, in the sense that any other calculus extending $(\Omega^1(A), \mathrm{d})$ can be obtained as a quotient of the maximal prolongation by an ideal of ker(d). It is clear that $\langle \mathrm{d}(N)\rangle$ is homogeneous with respect to the $\mathbf{N}_0$-grading of the tensor algebra. We will denote the corresponding decomposition by

$$\langle \mathrm{d}N\rangle = \bigoplus_{n\in\mathbf{N}_{\geq 2}} \langle \mathrm{d}N\rangle_k \,. \tag{2.15}$$

As is well known and easily seen, each $\langle \mathrm{d}N\rangle_k$ is an object in ${}^G_M\mathcal{M}_M$. This means that the natural comodule structure of the tensor algebra descends to a comodule structure on $\Omega^\bullet(A)$, giving it the structure of an object in ${}^G_M\mathcal{M}_M$. For the special case of the universal calculus, its maximal prolongation is just its tensor algebra. An important point to note is that the maximal prolongation of $\Omega^1(A)$ can also be constructed as the quotient of the tensor algebra of $\Omega^1_u(A)$ by the subalgebra $\langle N + \mathrm{d}N\rangle$, with the total derivative being obtained by restriction.

If $(\Omega^\bullet, \mathrm{d})$ is a differential calculus over a $*$-algebra A such that the involution of A extends to an involutive conjugate-linear map $*$ on $\Omega^\bullet$, for which $(\mathrm{d}\omega)^* = \mathrm{d}\omega^*$, for all $\omega \in \Omega$, and

$$(\omega_p \wedge \omega_q)^* = (-1)^{pq}\omega_q^* \wedge \omega_p^*, \qquad \text{(for all } \omega_p \in \Omega^p,\ \omega_q \in \Omega^q),$$

then we say that (Ω, d) is a *total $*$-differential calculus.* It is easy to see that if Ω^1 is a first order $*$-calculus, then its maximal prolongation is canonically a total $*$-calculus.

2.3 Quantum Principal Bundles

In this section we recall the general theory of quantum principal bundles. We will use quantum principal bundles in Chapter 6 to construct holomorphic structures for the line bundles over the quantum projective spaces. Reflecting standard presentation, we give the general form of the definition. In practice, however, we will only ever need to consider the homogeneous case.

Just as for the special case of quantum homogeneous spaces, the coinvariant subspace $M = P^H$ of a right H-comodule algebra P, is clearly a subalgebra of P. If the mapping

$$\mathrm{ver} = (m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta_R) : P \otimes_M P \to P \otimes H,$$

is an isomorphism, then we say that P is a *Hopf–Galois extension* of H. It is well-known, and not too difficult to show, that this condition is equivalent to exactness of the sequence

$$0 \longrightarrow P\Omega^1_u(M)P \xrightarrow{\iota} \Omega^1_u(P) \xrightarrow{ver} P \otimes H^+ \longrightarrow 0, \tag{2.16}$$

where $\Omega^1_u(M)$ is the restriction of $\Omega^1_u(P)$ to M, and ι is the inclusion map (see [50] for details). Now it is natural to look for a generalisation of this sequence to one using non-universal calculi. This brings us to one of the central structures used in this thesis:

Definition 2.3.1. A *quantum principal H-bundle* is a four-tuple (P, H, N, I_H), where H is a Hopf algebra; P a right H-comodule algebra such that P is a Hopf–Galois extension of $M = P^H$; N a sub-bimodule of $\Omega^1_u(P)$ determining a right-covariant calculus $\Omega^1(P)$; I_H an Ad_R-covariant right ideal of H^+ determining a bicovariant calculus $\Omega^1(H)$; for which holds the equality

$$\mathrm{ver}(N) = P \otimes I_H. \tag{2.17}$$

We usually omit explicit reference to the choice of calculi and refer to (P, H, N, I) as the quantum principal H-bundle $P \hookleftarrow M$. It is clear that every Hopf–Galois extension is a quantum principal bundle for the choice of the universal calculus on G. An immediate consequence of the definition is that for any quantum principal bundle (P, H, N, I_H), we have an exact sequence:

$$0 \longrightarrow P\Omega^1(M)P \xrightarrow{\iota} \Omega^1(P) \xrightarrow{\overline{ver}} P \otimes \Lambda^1_H \longrightarrow 0, \tag{2.18}$$

where $\Omega^1(M)$ is the restriction of $\Omega^1(P)$ to M, ι is the inclusion map, and $\overline{ver}$ the descent of ver to $\Omega^1(P)$ (which is well-defined since (2.17) holds).

Let us now restrict our attention to the special case of a quantum homogeneous space $\pi : G \to H$, where $\pi : G \to H$ is a surjective Hopf algebra map. If G is a Hopf–Galois extension of M, we say that M is a *Hopf–Galois quantum homogeneous space*. Let us now look at when non-universal choices of calculi give a Hopf–Galois quantum homogeneous space the structure of a quantum principal bundle: The map s can be used to let ver act on $G \otimes G^+$. As is easily seen,

$$\mathrm{ver} : G \otimes G^+ \mapsto G \otimes H, \qquad\qquad f \otimes g = f \otimes \pi(g). \tag{2.19}$$

Thus, for any left-covariant calculus on G with corresponding right ideal $I_G \subseteq G^+$, and left-covariant calculus on H with right ideal $I_H \subseteq H^+$, the requirement (2.17) is satisfied if, and only if, $I_H = \pi(I_G)$. Similarly, it is easy to show that $\Omega^1(G)$ is right-covariant if, and only if, $(\mathrm{id} \otimes \pi)(\mathrm{Ad}_R(I_G)) \subseteq I_G \otimes H$. In this case we have that

$$\mathrm{Ad}_R(\pi(I_G)) = (\pi \otimes \pi)\mathrm{Ad}_R(I_G) \subseteq (\pi \otimes \mathrm{id})(I_G \otimes H) = \pi(I_G) \otimes H,$$

and so, the calculus on H corresponding to I_H is bicovariant. We collect these observations in the following proposition:

Proposition 2.3.2 *[51] Let $\pi : G \to H$ be a Hopf–Galois quantum principal homogeneous space, and I_G a right ideal of G^+. If*

$$(\mathrm{id} \otimes \pi)\mathrm{Ad}_R(I_G) \subseteq I_G \otimes H, \tag{2.20}$$

then $(G, H, s(I_G), \pi(I_G))$ is a quantum principal bundle. We call such a quantum principal bundle a quantum principal homogeneous space*.*

A *connection* for a quantum principal H-bundle $P \hookleftarrow M$ is a left P-module projection $\Pi : \Omega^1(P) \to \Omega^1(P)$ such that $\ker(\Pi) = P\Omega^1(M)P$ and

$$\Delta_R \circ \Pi = (\Pi \otimes \mathrm{id}) \circ \Delta_R. \tag{2.21}$$

Connections are in bijective correspondence with linear maps $\omega : \Lambda^1_H \to \Omega^1(P)$ for which $\overline{\mathrm{ver}} \circ \omega = 1 \otimes \mathrm{id}$ and $\Delta_R \circ \omega = (\omega \otimes \mathrm{id}) \circ \overline{\mathrm{Ad}_{R,H}}$, where $\overline{\mathrm{Ad}_{R,H}}$ is the descent of $\mathrm{Ad}_{R,H}$ to the quotient Λ^1_H. We call such a map ω a *connection form*. Explicitly, the connection Π_ω corresponding to a connection form ω is given by

$$\Pi_\omega = m \circ (\mathrm{id} \otimes \omega) \circ \mathrm{ver}. \tag{2.22}$$

For a quantum principal homogeneous space $\pi : G \to H$ connection forms are in turn equivalent to linear maps $i : \Lambda^1_H \to \Lambda^1_G$ such that $\overline{\pi} \circ i = \mathrm{id}$ and

$$\overline{\mathrm{Ad}_{R,G}} \circ i = (i \otimes \mathrm{id}) \circ \mathrm{Ad}_{R,H}, \tag{2.23}$$

where $\overline{\pi}$ and $\overline{\mathrm{Ad}_{R,G}}$ are defined to be the unique mappings for which the following diagrams are commutative:

$$\begin{tikzcd} \ker(\varepsilon_G) \arrow[r, "\mathrm{Ad}_{R,G}"] \arrow[d, "\mathrm{proj}"'] & \ker(\varepsilon_G) \otimes G \arrow[d, "\mathrm{proj} \otimes \pi"] \\ \Lambda^1_G \arrow[r, "\overline{\mathrm{Ad}_{R,G}}"'] & \Lambda^1_G \otimes H, \end{tikzcd} \qquad \begin{tikzcd} \ker(\varepsilon_G) \arrow[r, "\pi"] \arrow[d, "\mathrm{proj}"'] & \ker(\varepsilon_H) \arrow[d, "\mathrm{proj}"] \\ \Lambda^1_G \arrow[r, "\overline{\pi}"'] & \Lambda^1_H. \end{tikzcd}$$

(Note that $\overline{\mathrm{Ad}_{R,G}}$ is well-defined because (2.20) is satisfied, while $\overline{\pi}$ is well-defined because $I_H = \pi(I_G)$.) We call such a map i a *bicovariant splitting map*. Explicitly, the connection form associated to i is $\omega = s \circ i$. For a more detailed presentation of connections, connection forms, and bicovariant splitting maps see [8, 9, 50, 49]. A connection Π is called *strong* if $(\mathrm{id} - \Pi)(\Omega^1(P)) \subseteq P\Omega^1(M)$. Strong connections are important because they allow us to construct covariant derivatives for all the associated bundles of a principal bundle. Recall that if $\mathcal{E}$ is a bimodule over an algebra A and $\Omega^1(A)$ is a differential calculus over A, then a *covariant derivative* for $\mathcal{E}$ is a linear mapping $\nabla : \mathcal{E} \to \mathcal{E} \otimes_A \Omega^1(A)$ such that

$$\nabla(sa) = \nabla(s)a + s \otimes \mathrm{d}a, \qquad (s \in \mathcal{E}, a \in A).$$

It was shown in [24], that for any associated line bundle $\mathcal{E}$ of a quantum principal bundle $P \hookleftarrow M$, a strong connection Π induces a covariant derivative ∇ on $\mathcal{E}$ defined by

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_M \Omega^1(M), \qquad e \mapsto (\mathrm{id} - \Pi)\mathrm{d}e, \tag{2.24}$$

where we identify $\mathcal{E} \otimes_M \Omega^1(M)$ with its canonical image in $\Omega^1(P)$.

Chapter 3

The Quantum Projective Spaces

In this chapter we introduce the quantum projective spaces and their Heckenberger–Kolb first order differential calculus. The description of $\mathbf{C}_q[\mathbf{C}P^{N-1}]$ given in the first half of the chapter is quite well-known. The material in the second half, however, is mainly novel. More explicitly, the calculus for SU_N is original, as is the explicit description of the ideal of the Heckenberger–Kolb calculus.

3.1 The Quantum Projective Spaces

In this the first half of the chapter we introduce the quantum projective spaces. We begin by recalling the well known construction of the quantum special unitary group. We then give the presentation of the quantum projective spaces originally introduced by Meyer in [53]. Finally, we discuss the Hopf–Galois property and faithful flatness for the quantum projective spaces.

3.1.1 The Quantum Special Unitary Group $\mathbf{C}_q[SU_N]$

We begin by fixing notation and recalling the various definitions and constructions needed to introduce the quantum unitary group and the quantum special unitary group. (Where proofs or basic details are omitted we refer the reader to [37, 62].) For $q \in (0,1]$ and $\nu := q - q^{-1}$, let $\mathbf{C}_q[M_N]$ be the quotient of the free noncommutative algebra $\mathbf{C}\left\langle u^i_j, \mid i,j=1,\ldots,N\right\rangle$ by the ideal generated by the elements

$$\begin{array}{lll} u^i_k u^j_k - q u^j_k u^i_k, & u^k_i u^k_j - q u^k_j u^k_i, & (1 \leq i < j \leq N, 1 \leq k \leq N); \\ u^i_l u^j_k - u^j_k u^i_l, & u^i_k u^j_l - u^j_l u^i_k - \nu u^i_l u^j_k, & (1 \leq i < j \leq N,\ 1 \leq k < l \leq N). \end{array}$$

These generators can be more compactly presented as

$$\sum_{w,x=1}^{N} R^{ac}_{wx} u^w_b u^x_d - \sum_{y,z=1}^{N} R^{yz}_{bd} u^a_y u^c_z, \qquad (1 \leq a,b,c,d \leq N), \tag{3.1}$$

where, for H the Heaviside step function with $H(0)=0$, we have denoted

$$R^{ik}_{jl} = q^{\delta_{ik}} \delta_{il}\delta_{kj} + \nu H(k-i)\delta_{ij}\delta_{kl}. \tag{3.2}$$

We can put a bialgebra structure on $\mathbf{C}_q[M_N]$ by introducing a coproduct Δ, and counit ε, uniquely defined by $\Delta(u^i_j) := \sum_{k=1}^N u^i_k \otimes u^k_j$, and $\varepsilon(u^i_j) := \delta_{ij}$. The *quantum determinant* of $\mathbf{C}_q[M_N]$ is the element

$$\det_N := \sum\nolimits_{\pi \in S_N} (-q)^{\ell(\pi)} u^1_{\pi(1)} u^2_{\pi(2)} \cdots u^N_{\pi(N)},$$

where summation is taken over all permutations π of the set of N elements, and $\ell(\pi)$ is the length of π. As is well-known, $\det_N$ is a central and grouplike element of the bialgebra. The centrality of $\det_N$ makes it easy to adjoin an inverse $\det_N^{-1}$. Both Δ and ε have extensions to this larger algebra, which are uniquely determined by $\Delta(\det_N^{-1}) = \det_N^{-1} \otimes \det_N^{-1}$, and $\varepsilon(\det_N^{-1}) = 1$. The result is a new bialgebra which we denote by $\mathbf{C}_q[GL_N]$. We can endow $\mathbf{C}_q[GL_N]$ with a Hopf algebra structure by defining

$$S(\det_N^{-1}) = \det_N, \qquad S(u^i_j) = (-q)^{i-j} \sum\nolimits_{\pi \in S_{N-1}} (-q)^{\ell(\pi)} u^{k_1}_{\pi(l_1)} u^{k_2}_{\pi(l_2)} \cdots u^{k_{N-1}}_{\pi(l_{N-1})} \det_N^{-1},$$

where $\{k_1,\ldots,k_{N-1}\} = \{1,\ldots,N\} \backslash \{j\}$, and $\{l_1,\ldots,l_{N-1}\} = \{1,\ldots,N\} \backslash \{i\}$ as ordered sets. Moreover, we can give $\mathbf{C}_q[GL_N]$ a Hopf $*$-algebra structure by setting $(\det_N^{-1})^* = \det_N$, and $(u^i_j)^* = S(u^j_i)$. We denote this Hopf $*$-algebra by $\mathbf{C}_q[U_N]$, and call it the *quantum unitary group of order* N. For $N=1$, we get the Hopf algebra $\mathbf{C}[U_1]$, where it is usual to denote $u^1_1 = t$, and $\det_N^{-1} = t^{-1}$. If we quotient $\mathbf{C}_q[U_N]$ by the ideal $\langle \det_N - 1 \rangle$, then the resulting algebra is again a Hopf $*$-algebra. We denote it by $\mathbf{C}_q[SU_N]$, and call it the *quantum special unitary group of order* N.

As is well-known [62], for each N^{th}-root $q^{\frac{1}{N}}$ of q, we have a map

$$r : \mathbf{C}_q[SU_N] \otimes \mathbf{C}_q[SU_N] \to \mathbf{C}, \qquad\qquad u^i_j \otimes u^k_l \mapsto q^{-\frac{1}{N}} R^{ki}_{jl},$$

which we call the *coquasi-triangular structure* map of $\mathbf{C}_q[SU_N]$, for $q^{\frac{1}{N}}$. We can use r to define a family of maps $\{Q_{kl} \mid k,l = 1,\ldots,N\}$ by setting

$$Q_{kl} : \mathbf{C}_q[SU_N] \to \mathbf{C}, \qquad\qquad f \mapsto \sum_{a=1}^{N} r(u^k_a \otimes f_{(1)}) r(f_{(2)} \otimes u^a_l). \tag{3.3}$$

Using this family of maps, an N^2-dimensional representation Q can be defined by

$$Q : \mathbf{C}_q[SU_N] \to M_N(\mathbf{C}) \qquad\qquad h \mapsto [Q_{kl}(h)]_{kl}.$$

We call Q the *quantum Killing representation* of $\mathbf{C}_q[SU_N]$.

For $N = 2$, we get the well-known Hopf algebra $\mathbf{C}_q[SU_2]$. Conforming to standard notation, we denote its generators u^1_1, u^1_2, u^2_1, and u^2_2 by a, b, c, and d respectively.

3.1.2 The Quantum Projective Spaces $\mathbf{C}_q[\mathbf{C}P^{N-1}]$ and the Quantum Line Bundles $\mathcal{E}_k$

We are now ready to introduce the quantum projective spaces. As mentioned earlier, they form a subfamily of the quantum flag manifolds, and they will serve as an invaluable testing ground for our general theory. We use a description, introduced in [53], that presents quantum $(N-1)$-projective space as the coinvariant subalgebra of a $\mathbf{C}_q[U_{N-1}]$-coaction on $\mathbf{C}_q[SU_N]$. This subalgebra is a q-deformation of the coordinate algebra of the complex manifold SU_N/U_{N-1}. (Recall that classically $\mathbf{C}P^{N-1}$ is isomorphic to SU_N/U_{N-1}.)

Definition 3.1.1. Let $\alpha_N : \mathbf{C}_q[SU_N] \to \mathbf{C}_q[U_{N-1}]$ be the surjective Hopf algebra map defined by setting $\alpha_N(u^1_1) = \det^{-1}_{N-1}$; $\alpha_N(u^1_i) = \alpha_N(u^i_1) = 0$, for $i = 2, \cdots, N$; and $\alpha_N(u^i_j) = u^{i-1}_{j-1}$, for $i,j = 2,\ldots,N$. *Quantum projective* $(N-1)$-*space* $\mathbf{C}_q[\mathbf{C}P^{N-1}]$ is defined to be the coinvariant subspace of the corresponding homogeneous coaction $\Delta_{SU_N,\alpha_N} = (\mathrm{id} \otimes \alpha_N) \circ \Delta$, that is,

$$\mathbf{C}_q[\mathbf{C}P^{N-1}] := \{f \in \mathbf{C}_q[SU_N] \mid \Delta_{SU_N,\alpha_N}(f) = f \otimes 1\}.$$

An important family of objects in ${}^{SU_N}_{\mathbf{C}P^{N-1}}\mathcal{M}_{\mathbf{C}P^{N-1}}$ is the family of *quantum line bundles* $\{\mathcal{E}_k \,|\, k \in \mathbf{Z}\}$: The module $\mathcal{E}_k$ is defined to be $\Psi_{\mathbf{C}P^{N-1}}(\mathbf{C})$, where $\mathbf{C}$ is considered as an object in $\mathcal{M}^{U_{N-1}}_{\mathbf{C}P^{N-1}}$ according to the unique $\mathbf{C}[U_1]$-coaction for which $1 \mapsto 1 \otimes \det^{-p}_{N-1}$, for $\lambda \in \mathbf{C}$. Clearly, we have that $\mathcal{E}_0 = \mathbf{C}P^{N-1}$. Moreover, identifying $\mathbf{C}_q[SU_N] \otimes \mathbf{C}$ and $\mathbf{C}_q[SU_N]$ allows us to consider $\mathcal{E}_k$ as a coinvariant subalgebra of $\mathbf{C}_q[SU_N]$. The corresponding coaction, which we denote by $\Delta^k_{SU_N,\alpha}$, is clearly a homogeneous coaction, whose Hopf algebra map we denote by α^k_N.

For practical purposes, it will later prove very useful to have a more concrete generator-and-relation description of the quantum projective spaces and their line bundles. We will find such a description using an alternative presentation of $\mathbf{C}_q[\mathbf{C}P^{N-1}]$ based upon the classical isomorphism between $\mathbf{C}_q[\mathbf{C}P^{N-1}]$ and S^{2N-1}/U_1, where S^{2N-1} is the $(2N-1)$-sphere. We begin by presenting a q-deformation of the coordinate algebra of S^{2N-1} which was first introduced in [74]. This deformation is based upon yet another classical isomorphism, this time the identification of S^{2N-1} and SU_N/SU_{N-1}.

Definition 3.1.2. For the surjective Hopf algebra map $\beta_N : \mathbf{C}_q[SU_N] \to \mathbf{C}_q[SU_{N-1}]$ defined by setting $\beta_N(u^1_1) = 1$; $\beta_N(u^1_i) = \beta_N(u^i_1) = 0$, for $i \neq 1$; and $\beta_N(u^i_j) = u^{i-1}_{j-1}$, for $i, j = 2, \ldots, N$, we have a homogeneous $\mathbf{C}_q[SU_{N-1}]$-coaction on $\mathbf{C}_q[SU_N]$ given by $\Delta_{SU_N,\beta} = (\mathrm{id} \otimes \beta_N) \circ \Delta$. The *quantum* $(2N-1)$*-sphere* $\mathbf{C}_q[S^{2N-1}]$ is the coinvariant subalgebra of $\Delta_{SU_N,\beta}$, that is,

$$\mathbf{C}_q[S^{2N-1}] = \{f \in \mathbf{C}_q[SU_N] \,|\, \Delta_{SU_N,\beta}(f) = f \otimes 1\}.$$

Now for $i = 1, \ldots, N$, we have

$$\Delta_{SU_N,\beta}(u^i_1) = (\mathrm{id} \otimes \beta_N)(\sum_{k=1}^{N} u^i_k \otimes u^k_1) = \sum_{k=1}^{N} u^i_k \otimes \beta_N(u^k_1) = u^i_1 \otimes 1,$$

and

$$\begin{aligned}\Delta_{SU_N,\beta}(S(u^1_i)) &= (\mathrm{id} \otimes \beta_N)(\sum_{k=1}^{N} S(u^k_i) \otimes S(u^1_k)) = \sum_{k=1}^{N} S(u^k_i) \otimes \beta_N(S(u^1_k)) \\ &= S(u^1_i) \otimes 1.\end{aligned}$$

Thus, u_1^i and $S(u_i^1)$ are contained in $\mathbf{C}_q[S^{2N-1}]$. Using representation theoretic methods, it was established in [74] that $\mathbf{C}_q[S^{2N-1}]$ is in fact generated as a algebra by the elements u_1^i and $S(u_i^1)$. It was also shown that a full set of relations is given by

$$u_1^i u_1^j = q u_1^j u_1^i \qquad (i < j); \qquad u_1^i S(u_j^1) = q S(u_j^1) u_1^i, \qquad (i \neq j);$$

$$u_1^i S(u_i^1) = S(u_i^1) u_1^i + q^{-1}\nu \sum_{k=i+1}^{N} q^{2(k-i)} u_1^k S(u_k^1); \qquad \sum_{i=1}^{N} S(u_i^1) u_1^i = 1. \tag{3.4}$$

(More easily accessible versions of the proof can be found in [37, 10].)

We now introduce a right $\mathbf{C}[U_1]$-coaction, γ_N^k, for $k \in \mathbf{Z}$, for the quantum $(2N-1)$-sphere and show that $\mathcal{E}_k$ arises as its coinvariant subalgebra. This alternative description of $\mathbf{C}_q[\mathbf{C}P^{N-1}]$ comes from [53].

Lemma 3.1.3 *Define a surjective Hopf algebra map* $\gamma_N^k : \mathbf{C}_q[SU_N] \to \mathbf{C}[U_1]$ *by setting* $\gamma_N^k(u_1^1) = t^{-k}$*;* $\gamma_N(u_l^l) = 1$*, for* $l = 2, \ldots, N-1$*;* $\gamma_N^k(u_N^N) = t^k$*; and* $\gamma_N^k(u_j^i) = 0$*, for* $i, j = 1, \ldots, N$*, and* $i \neq j$*. The map* $(\mathrm{id} \otimes \gamma_N^k) \circ \Delta$ *restricts to a* $\mathbf{C}[U_1]$*-coaction on* $\mathbf{C}_q[S^{2N-1}]$ *which we denote by* $\Delta^k_{S^{2N-1},\gamma}$*. Moreover,* $\mathcal{E}_k$ *is the coinvariant subalgebra of this coaction, that is,*

$$\mathcal{E}_k = \{ f \in \mathbf{C}_q[S^{2N-1}] \,|\, \Delta^k_{S^{2N-1},\gamma}(f) = f \otimes 1 \}.$$

Proof. That we have a $\mathbf{C}[U_1]$-coaction on $\mathbf{C}_q[S^{2N-1}]$ is clear from the fact that

$$\Delta^k_{S^{2N-1},\gamma}(u_1^i) = (\mathrm{id} \otimes \gamma_N^k) \sum_{k=1}^{N} u_k^i \otimes u_1^k = \sum_{k=1}^{N} u_k^i \otimes \gamma_N^k(u_1^k) = u_1^i \otimes t^{-k}, \tag{3.5}$$

and the fact that

$$\begin{aligned} \Delta_{S^{2N-1},\gamma}(S(u_i^1)) &= (\mathrm{id} \otimes \gamma_N^k) \sum_{k=1}^{N} S(u_i^k) \otimes S(u_k^1) = \sum_{k=1}^{N} S(u_i^k) \otimes \gamma_N^k(S(u_k^1)) \\ &= S(u_i^1) \otimes t^k. \end{aligned} \tag{3.6}$$

Let us now move onto showing that $\mathcal{E}_k$ is the coinvariant subalgebra of $\Delta_{S^{2N-1},\gamma_N^k}$. For the canonical projection $\delta_{N-1} : \mathbf{C}_q[U_{N-1}] \to \mathbf{C}_q[SU_{N-1}]$, we have that

$\delta_{N-1} \circ \alpha_N^k = \beta_N$, and so, the following diagram is commutative:

$$\begin{tikzcd}
\mathbf{C}_q[SU_N] \arrow[r, "\Delta^k_{SU_N,\alpha}"] \arrow[d, "\mathrm{id}"'] & \mathbf{C}_q[SU_N] \otimes \mathbf{C}_q[U_{N-1}] \arrow[d, "\mathrm{id}\otimes\delta_{N-1}"] \\
\mathbf{C}_q[SU_N] \arrow[r, "\Delta_{SU_N,\beta}"] & \mathbf{C}_q[SU_N] \otimes \mathbf{C}_q[SU_{N-1}].
\end{tikzcd}$$

It follows that $\mathbf{C}_q[\mathbf{C}P^{N-1}]$ is contained in $\mathbf{C}_q[S^{2N-1}]$. Now let us denote by $j : \mathbf{C}[U_1] \to \mathbf{C}_q[U_{N-1}]$ the canonical embedding of $\mathbf{C}[U_1]$ into $\mathbf{C}_q[U_{N-1}]$ uniquely defined by $j(t) = \det_N$ and $j(t^{-1}) = \det_N^{-1}$. Just as in (3.5) and (3.5), it is easy to show that $\Delta^k_{SU_N,\alpha}(u^i_1) = u^i_1 \otimes \det_N^{-k}$ and $\Delta^k_{SU_N,\alpha}(S(u^1_i)) = S(u^1_i) \otimes \det^k_N$, and so, we have another commutative diagram:

$$\begin{tikzcd}
\mathbf{C}_q[S^{2N-1}] \arrow[r, "\Delta^k_{S^{2N-1},\gamma}"] \arrow[d, "\iota"'] & \mathbf{C}_q[S^{2N-1}] \otimes \mathbf{C}[U_1] \arrow[d, "\iota\otimes j"] \\
\mathbf{C}_q[SU_N] \arrow[r, "\Delta^k_{SU_N,\alpha}"] & \mathbf{C}_q[SU_N] \otimes \mathbf{C}_q[U_{N-1}].
\end{tikzcd} \tag{3.7}$$

That $\mathcal{E}_k$ is the coinvariant subalgebra of $\Delta^k_{S^{2N-1},\gamma}$ follows easily from this. □

Corollary 3.1.4 *We have that $\mathbf{C}_q[\mathbf{C}P^{N-1}]$ is generated as an algebra by the elements $z_{ij} := u^i_1 S(u^1_j)$, for $i, j = 1, \ldots, N$. Moreover, for $k \in \mathbf{N}$, the algebras $\mathcal{E}_k$ is generated as right $\mathbf{C}_q[\mathbf{C}P^{N-1}]$-module by the finite set*

$$\mathcal{E}^0_k := \{(S(u^1_1))^{m_1} \cdots (S(u^1_N))^{m_N} \mid \sum_{i=1}^N m_i = k\},$$

while $\mathcal{E}_{-k}$ is generated as right $\mathbf{C}_q[\mathbf{C}P^{N-1}]$-module by the finite set

$$\mathcal{E}^0_{-k} := \{(u^1_1)^{m_1} \cdots (u^N_1)^{m_N} \mid \sum_{i=1}^N m_i = k\}.$$

Proof. Since $\Delta_{S^{2N-1},\gamma^k_N}$ is a $\mathbf{C}[U_1]$-coaction, it induces a $\mathbf{Z}$-grading on $\mathbf{C}_q[S^{2N-1}]$. This grading is uniquely determined by for $\deg(u^i_1) = -1$ and $\deg(S(u^1_i)) = 1$. The corollary now follows from the fact that $\mathbf{C}_q[S^{2N-1}]$ is generated by the elements u^i_1 and $S(u^1_i)$, and the relations (3.4). □

3.1.3 The Hopf–Galois Property and Faithful Flatness

Let us now present this quantum homogeneous space as a quantum principal homogeneous space. We begin by proving a general result:

Lemma 3.1.5 *For a Hopf algebra map $\pi : G \to H$, with corresponding quantum homogeneous space M, we have that G is a Hopf–Galois extension of M if $v(1 \otimes p) = 0$, for all $p \in \ker(\pi)$, where the map $v : G \otimes G \to G \otimes_M G$ is defined by setting $v(f \otimes g) = fS(g_{(1)}) \otimes g_{(2)}$, for $f, g \in G$.*

Proof. We will establish this result by introducing a map $\mathrm{ver}^{-1} : G \otimes H \to G \otimes_M G$ that acts as an inverse for ver whenever $v(1 \otimes p) = 0$, for all $p \in \ker(\pi)$. Let $i : H \to G$ be a linear mapping such that $\pi \circ i = \mathrm{id}$ (such a mapping can always be constructed) and set $\mathrm{ver}^{-1} = v \circ (\mathrm{id} \otimes i)$. We first show that $\mathrm{ver} \circ \mathrm{ver}^{-1} = \mathrm{id}$: For any $h \in H$,

$$\mathrm{ver} \circ \mathrm{ver}^{-1}(f \otimes h) = \mathrm{ver}(fS(i(h)_{(1)}) \otimes i(h)_{(2)}) = fS(i(h)_{(1)})i(h)_{(2)} \otimes \pi(i(h)_{(3)}) \tag{3.8}$$
$$= f\varepsilon(i(h)_{(1)}) \otimes \pi(i(h)_{(2)}) = f \otimes \pi(i(h)) = f \otimes h.$$

We now move on to showing that $\mathrm{ver}^{-1} \circ \mathrm{ver} = \mathrm{id}$: For any $x \in G$, the fact that $\pi \circ i = \mathrm{id}$, implies that $i(\pi(x)) = x + p_x$, for some $p_x \in \ker(\pi)$. This means that

$$\begin{aligned} \mathrm{ver}^{-1} \circ \mathrm{ver}(f \otimes g) &= \mathrm{ver}^{-1}(fg_{(1)} \otimes \pi(g_{(2)})) = v(fg_{(1)} \otimes i(\pi(g_{(2)}))) \\ &= v(fg_{(1)} \otimes g_{(2)}) + v(fg_{(1)} \otimes p_{g_{(2)}}) = fg_{(1)}S(g_{(2)}) \otimes g_{(3)} \\ &= f\varepsilon(g_{(1)}) \otimes g_{(2)} = f \otimes g. \end{aligned}$$

We note that ver^{-1} does not depend upon our choice for the map i. □

Using this lemma we now give a detailed proof of a result that was originally proposed in [53].

Corollary 3.1.6 *The quantum homogeneous space $\alpha_N : \mathbf{C}_q[SU_N] \to \mathbf{C}_q[U_{N-1}]$ has a quantum principal bundle structure.*

Proof. Since $\langle u_1^i, u_i^1 \,|\, i \neq 1\rangle \subseteq \ker(\alpha_N)$, there exists a unique map proj' such that the following diagram is commutative:

$$\begin{tikzcd}
\mathbf{C}_q[SU_N] \arrow[rr, "\alpha_N"] \arrow[d, "\mathrm{proj}"'] & & \mathbf{C}_q[U_{N-1}] \\
\mathbf{C}_q[SU_N]/\langle u_1^i, u_i^1 \,|\, i \neq 1\rangle . \arrow[urr, "\mathrm{proj}'"'] & &
\end{tikzcd}$$

Now the mapping

$$(\mathrm{proj}')^{-1} : \mathbf{C}_q[U_{N-1}] \to \mathbf{C}_q[SU_N]/\left\langle u_1^i, u_i^1 \,|\, i \neq 1\right\rangle, \quad u_j^i \mapsto u_{j+1}^{i+1}, \quad \det_{N-1}^{-1} \mapsto u_1^1,$$

is well-defined because

$$(\mathrm{proj}')^{-1}(\det_{N-1}\det_{N-1}^{-1} - 1) = S(u_1^1)u_1^1 - 1 = \sum_{k=2}^{N} S(u_k^1)u_1^k = 0,$$

and

$$(\mathrm{proj}')^{-1}(\det_{N-1}^{-1}\det_{N-1} - 1) = 0.$$

Moreover, $(\mathrm{proj}')^{-1}$ is clearly inverse to proj'. This means that we must have

$$\left\langle u_1^i, u_i^1 \,|\, i \neq 1\right\rangle = \ker(\alpha_N).$$

Thus, we see that every $p \in \ker(\alpha_N)$ is of the form

$$p = \sum_{i=2}^{N} u_1^i f_i + \sum_{i=2}^{N} u_i^1 g_i, \qquad (f_i, g_i, \in \mathbf{C}_q[SU_N]). \tag{3.9}$$

Now, for any $f \in \mathbf{C}_q[SU_N]$, we have

$$\begin{aligned}
v(1 \otimes u_1^i f) &= \sum_{k=1}^{N} S(f_{(1)})S(u_k^i) \otimes u_1^k f_{(2)} = \sum_{k,l=1}^{N} S(f_{(1)})S(u_k^i) \otimes u_1^k S(u_l^1) u_1^l f_{(2)} \\
&= \sum_{k,l=1}^{N} S(f_{(1)})S(u_k^i)u_1^k S(u_l^1) \otimes u_1^l f_{(2)} = \sum_{k,l=1} S(f_{(1)})\varepsilon(u_1^i)S(u_l^1) \otimes u_1^l f_{(2)} \\
&= 0.
\end{aligned}$$

It can be shown in an exactly analogous manner that $v(1 \otimes u_i^1 g) = 0$, for any $g \in \mathbf{C}_q[SU_N]$. $\square$

We now come to the question of faithful flatness. In [55] Müller and Schneider undertook a general investigation of faithful flatness for quantum homogeneous spaces. They established the condition for a quite large class of homogeneous spaces, for which the quantum flag manifolds are a motivating family. We present their result here for the very special case of the quantum projective spaces.

Theorem 3.1.7 (Müller, Schneider) *The Hopf algebra $\mathbf{C}_q[SU_N]$ is a faithfully flat module over $\mathbf{C}_q[\mathbf{C}P^{N-1}]$.*

3.2 The Heckenberger–Kolb Calculus $\Omega^1_q(\mathbf{C}P^{N-1})$

In this section we will present the Heckenberger–Kolb calculus $\Omega^1_q[\mathbf{C}P^{N-1}]$ in terms of its classifying ideal. We will also consider a calculus on $\mathbf{C}_q[SU_N]$ that restricts to the $\Omega^1_q[\mathbf{C}P^{N-1}]$ on $\mathbf{C}_q[\mathbf{C}P^{N-1}]$. This calculus is not of interest in itself (it has highly non-classical dimension), instead it will serve as a very convenient calculating tool throughout the rest of the thesis.

3.2.1 A Distinguished Quotient of the Bicovariant Calculus for $\mathbf{C}_q[SU_N]$

As explained in Chapter 2, for every coquasi-triangular Hopf algebra H, there exists a canonical bicovariant differential calculus $\Omega^1_{\mathrm{bc},q}(H)$ over H, constructed using the quantum Killing representation. In this subsection we will recall what the calculus looks like for the case of $\mathbf{C}_q[SU_N]$; construct a certain quotient of it; and then explain why this quotient is important.

We begin by establishing some very useful formulae (given in terms of the coquasi-triangular structure specified in (3.2) for the action of Q on certain distinguished elements of $\mathbf{C}_q[SU_N]$.

Lemma 3.2.1 *For $\{Q_{kl} \,|\, k,l = 1,\ldots,N\}$ the family of maps defined in (3.3), we* *have the following formulae:*

$$Q_{kl}(u^i_j) = \sum_{a,z=1}^{N} q^{-\frac{2}{N}} R^{ik}_{za} R^{za}_{jl}, \tag{3.10}$$

$$Q_{kl}(S(u^g_h)) = \sum_{a,z=1}^{N} q^{2(a-h)+\frac{2}{N}} \overline{R}^{ak}_{zh} \overline{R}^{zg}_{al}, \tag{3.11}$$

$$Q_{kl}(u^i_j u^r_s) = \sum_{a,b,x,y,z=1}^{N} q^{-\frac{4}{N}} R^{rk}_{zb} R^{iz}_{ya} R^{ya}_{jx} R^{xb}_{sl}, \tag{3.12}$$

$$Q_{kl}(u^i_j S(u^g_h)) = \sum_{a,b,x,y,z=1}^{N} q^{2(b-h)} \overline{R}^{bk}_{zh} R^{iz}_{ya} R^{ya}_{jx} \overline{R}^{xg}_{bl}, \tag{3.13}$$

$$Q_{kl}(u^i_j S(u^g_h) u^r_s) = \sum_{a,b,c,v,w,x,y,z=1}^{N} q^{2(b-h)-\frac{2}{N}} R^{rk}_{zc} \overline{R}^{bz}_{yh} R^{iy}_{xa} R^{xa}_{jw} \overline{R}^{wg}_{bv} R^{vc}_{sl}. \tag{3.14}$$

Proof. The proof of the lemma consists of a series of routine calculations involving the definition of Q_{kl}, and the properties of a general coquasi-quasitriangular structure: For u^i_j, we have

$$Q_{kl}(u^i_j) = \sum_{z,a=1}^{N} r(u^k_z \otimes u^i_a) r(u^a_j \otimes u^z_l) = \sum_{z,a=1}^{N} q^{-\frac{2}{N}} R^{ik}_{za} R^{za}_{jl}.$$

For $u^i_j u^r_s$, we have

$$\begin{aligned} Q_{kl}(u^i_j u^r_s) &= \sum_{z,a,b}^{N} r(u^k_z \otimes u^i_a u^r_b) r(u^a_j u^b_s \otimes u^z_l) \\ &= \sum_{x,y,z,a,b}^{N} r(u^k_z \otimes u^r_b) r(u^z_y \otimes u^i_a) r(u^a_j \otimes u^y_x) r(u^b_s \otimes u^x_l) \\ &= \sum_{x,y,z,a,b}^{N} q^{-\frac{4}{N}} R^{rk}_{zb} R^{iz}_{ya} R^{ya}_{jx} R^{xb}_{sl}. \end{aligned}$$

We now move onto calculating $Q_{kl}(S(u^g_h))$:

$$
\begin{aligned}
Q_{kl}(S(u^g_h)) &= \sum_{a,z} r(u^k_z \otimes S(u^a_h))r(S(u^g_a) \otimes u^z_l) = \sum_{a,z} r(S(u^k_z) \otimes S^2(u^a_h))\overline{r}(u^g_a \otimes u^z_l) \\
&= \sum_{a,z} q^{2(a-h)} r(S(u^k_z) \otimes u^a_h)\overline{r}(u^g_a \otimes u^z_l) = \sum_{a,z} q^{2(a-h)}\overline{r}(u^k_z \otimes u^a_h)\overline{r}(u^g_a \otimes u^z_l) \\
&= \sum_{a,z=1}^{N} q^{2(a-h)+\frac{2}{N}} \overline{R}^{ak}_{zh}\overline{R}^{zg}_{al}.
\end{aligned}
$$

Next we take $Q_{kl}(u^i_j S(u^g_h))$:

$$
\begin{aligned}
Q_{kl}(u^i_j S(u^g_h)) &= \sum_{a,b,z=1}^{N} r(u^k_z \otimes (u^i_a S(u^b_h))r((u^a_j S(u^g_b)) \otimes u^z_l) \\
&= \sum_{a,b,x,y,z=1}^{N} r(u^k_z \otimes S(u^b_h))r(u^z_y \otimes u^i_a)r(u^a_j \otimes u^y_x)r(S(u^g_b) \otimes u^x_l) \\
&= \sum_{a,b,x,y,z=1}^{N} q^{2(b-h)}\overline{R}^{bk}_{zh} R^{iz}_{ya} R^{ya}_{jx}\overline{R}^{xg}_{bl},
\end{aligned}
$$

where we have used the fact that $r(f \otimes (gh)) = r(f_{(1)} \otimes h)r(f_{(2)} \otimes g)$.)
Finally, we establish the formula for $Q_{kl}(u^i_j S(u^g_h)u^r_s)$:

$$
\begin{aligned}
& Q_{kl}(u^i_j S(u^g_h)u^r_s) \\
&= \sum_{a,b,c,z=1}^{N} r(u^k_z \otimes u^i_a S(u^b_h)u^r_c)r(u^a_j S(u^g_b)u^c_s \otimes u^z_l) \\
&= \sum_{a,b,c,v,w,x,y,z=1}^{N} r(u^k_z \otimes u^r_c)r(u^z_y \otimes S(u^b_h))r(u^y_x \otimes u^i_a)r(u^a_j S(u^g_b)u^c_s \otimes u^z_l) \\
&= \sum_{a,b,c,v,w,x,y,z=1}^{N} q^{2(b-h)-\frac{1}{N}} R^{rk}_{zc}\overline{R}^{bz}_{yh} R^{iy}_{xa} r(u^a_j S(u^g_b)u^c_s \otimes u^z_l) \\
&= \sum_{a,b,c,v,w,x,y,z=1}^{N} q^{2(b-h)-\frac{1}{N}} R^{rk}_{zc}\overline{R}^{bz}_{yh} R^{iy}_{xa} r(u^a_j \otimes u^x_w)r(S(u^g_b) \otimes u^w_v)r(u^c_s \otimes u^v_l) \\
&= \sum_{a,b,c,v,w,x,y,z=1}^{N} q^{2(b-h)-\frac{2}{N}} R^{rk}_{zc}\overline{R}^{bz}_{yh} R^{iy}_{xa} R^{xa}_{jw}\overline{R}^{wg}_{bv} R^{vc}_{sl}.
\end{aligned}
$$

$\square$

With these formulae in hand we can now introduce a novel first-order differential calculus for $\mathbf{C}_q[SU_N]$.

Proposition 3.2.2 *The subspace* $I_{SU_N} = \ker(Q)^+ + D_1 + D_2$, *with*

$D_1 := \mathrm{span}_{\mathbf{C}}\{u^i_1 S(u^1_i) \,|\, i = 2, \ldots, N\}$, *and* $D_2 := \mathrm{span}_{\mathbf{C}}\{u^i_j \,|\, i, j = 2 \ldots, N; i \neq j\}$, *is a right ideal of* $\mathbf{C}_q[SU_N]^+$. *Moreover, for* $i = 1, \ldots, N-1$, *the elements*

$$e^-_i := \overline{u^{i+1}_1}, \qquad e^0 := \overline{u^1_1 - 1}, \qquad e^+_i := \overline{u^1_{i+1}}, \tag{3.15}$$

form a $(2N-1)$*-dimensional left-module basis of* $\Lambda^1_{SU_N} := \mathbf{C}_q[SU_N]^+ / I_{SU_N}$.

Proof. As discussed in Section 2.2, $\ker(Q)^+$ is a right ideal of $\mathbf{C}_q[SU_N]^+$, whose calculus is the standard bicovariant calculus for $\mathbf{C}_q[SU_N]$. We will begin by constructing a basis for $\Lambda^1_{bc,q,SU_N} := \mathbf{C}_q[SU_N]^+/\ker(Q)^+$: The map Q_{kl} acts on on u^i_j, for $i \neq j$, according to

$$Q_{kl}(u^i_j) = \sum_{a,z=1}^{N} q^{-\frac{2}{N}} R^{ik}_{za} R^{za}_{jl} = q^{-\frac{2}{N}} (R^{ik}_{ik} R^{ik}_{jl} + R^{ik}_{ki} R^{ki}_{jl}).$$

Since $i \neq j$, this gives a non-zero value if, and only if, $k = j$ and $l = i$, whereupon

$$Q_{ji}(u^i_j) = q^{-\frac{2}{N}} (R^{ij}_{ij} R^{ij}_{ji} + R^{ij}_{ji} R^{ji}_{ji}) = q^{-\frac{2}{N}} (\nu\theta(j-i) + \nu.\theta(i-j)) = q^{-\frac{2}{N}} \nu.$$

Thus, for E_{ij} the usual $(i,j)^{\text{th}}$-element of the canonical basis of $M_N(\mathbf{C})$, we have

$$Q(u^i_j) = q^{-\frac{2}{N}} \nu E_{ji} \tag{3.16}$$

For $i \geq 2$, the map Q_{kl} acts on $u^i_1 S(u^1_i)$ according to

$$\begin{aligned}
Q_{kl}(u^i_1 S(u^1_i)) &= \sum_{a,b,x,y,z=1}^{N} q^{2(b-i)} \overline{R}^{bk}_{zi} R^{iz}_{ya} R^{ya}_{1x} \overline{R}^{x1}_{bl} \\
&= q^{2(1-i)} \sum_{a,z=1}^{N} \overline{R}^{1k}_{zi} R^{iz}_{ya} R^{ya}_{1l} \overline{R}^{l1}_{1l} + q^{2(1-i)} \sum_{z=1}^{N} \overline{R}^{1k}_{zi} R^{iz}_{11} R^{11}_{11} \overline{R}^{11}_{11} \\
&= q^{2(1-i)} \sum_{z=1}^{N} \overline{R}^{1k}_{zi} R^{iz}_{1l} R^{1l}_{1l} \overline{R}^{l1}_{1l} + q^{2(1-i)} \sum_{z=1}^{N} \overline{R}^{1k}_{zi} R^{iz}_{l1} R^{l1}_{1l} \overline{R}^{l1}_{1l} + 0 \\
&= q^{2(1-i)} \overline{R}^{1k}_{1i} R^{i1}_{1i} R^{1i}_{1i} \overline{R}^{i1}_{1i} \delta_{li} + 0 \\
&= q^{2(1-i)} \overline{R}^{1i}_{1i} R^{i1}_{1i} R^{1i}_{1i} \overline{R}^{i1}_{1i} \delta_{li} \delta_{ki} \\
&= q^{2(1-i)} \nu^2 \delta_{li} \delta_{ki}.
\end{aligned}$$

Thus, this gives a non-zero answer if, and only if, $k = l = i$, and so,

$$Q(u^i_1 S(u^1_i)) = q^{2(1-i)}\nu^2 E_{ii}.$$

Now on u^1_1, the map Q_{11} operates as

$$Q_{11}(u^1_1) = q^{-\frac{2}{N}} \sum_{a,z=1}^{N} R^{11}_{za} R^{za}_{11} = q^{-\frac{2}{N}} R^{11}_{11} R^{11}_{11} = q^{2-\frac{2}{N}}.$$

Finally, we note that the general properties of a coquasi-triangular structure imply that $Q(1) = \mathbf{1_N}$, where $\mathbf{1_N}$ is the identity matrix of order N. Thus, $u^1_1 - 1$ has a non-zero image under Q_{11}.

All this tells us that the elements $Q(u^1_1 - 1)$; $Q(u^i_j)$, for $i \neq j$; and $Q(u^i_1 S(u^1_i))$, for $i \neq 1$, form a spanning set of $M_N(\mathbf{C})$, and hence a basis. It follows directly that the elements $\overline{u^1_1 - 1}$; $\overline{u^i_j}$, for $i \neq j$; and $\overline{u^i_1 S(u^1_i)}$, for $i \neq 1$, form an N^2-dimensional basis of Λ^1_{bc,q,SU_N}.

We can now move onto showing that I_{SU_N} is a submodule of $\mathbf{C}_q[SU_N]^+$. We first note that this is equivalent to the image of $D_1 + D_2$ in Λ^1_{bc,q,SU_N} being a right submodule. This is in turn equivalent the subspace

$$\{E_{ij} \,|\, i,j = 2,\dots,N\}.$$

being a submodule of $M_N(\mathbf{C})$. Thus, we see that I_{SU_N} is a submodule if, and only if,

$$Q_{1l}(u^i_j u^r_s) = Q_{k1}(u^i_j u^r_s) = Q_{1l}(u^i_1 S(u^1_i) u^r_s) = Q_{k1}(u^i_1 S(u^1_i) u^r_s) = 0,$$

for all $1 \leq k, l, r, s \leq N$. This is easily proved using (3.12) and (3.14). Let us begin with the action of Q_{1l} on $u^i_j u^r_s$:

$$Q_{1l}(u^i_j u^r_s) = q^{-\frac{4}{N}} \sum_{a,b,x,y,z=1}^{N} R^{r1}_{zb} R^{iz}_{ya} R^{ya}_{jx} R^{xb}_{sl} = q^{-\frac{4}{N}} \sum_{b,x=1}^{N} R^{r1}_{1r} R^{i1}_{1i} R^{1i}_{jx} R^{xb}_{sl} = 0$$

Next we take the action of Q_{k1} on $u^i_j u^r_s$:

$$Q_{k1}(u^i_j u^r_s) = q^{-\frac{4}{N}} \sum_{a,b,x,y,z=1}^{N} R^{rk}_{zb} R^{iz}_{ya} R^{ya}_{jx} R^{xb}_{s1} = q^{-\frac{4}{N}} \sum_{b,z=1}^{N} R^{rk}_{zb} R^{iz}_{1j} R^{1j}_{j1} R^{1s}_{s1} = 0.$$

For the action of Q_{1l} on $u^i_1 S(u^1_i)u^r_s$ we have

$$Q_{1l}(u^i_1 S(u^1_i)u^r_s) = q^{2(b-i)-1} \sum_{a,b,c,v,w,x,y,z=1}^{N} R^{r1}_{zc}\overline{R}^{bz}_{yi}R^{iy}_{xa}R^{xa}_{1w}\overline{R}^{w1}_{bv}R^{vc}_{sl}$$

$$= q^{2(b-i)-1} \sum_{c,v=1}^{N} R^{r1}_{1r}\overline{R}^{i1}_{1i}R^{i1}_{1i}R^{1i}_{1i}\overline{R}^{11}_{i1}R^{vc}_{sl} = 0.$$

Finally, we take the action of Q_{k1} on $u^i_1 S(u^1_i)u^r_s$

$$Q_{k1}(u^i_1 S(u^1_i)u^r_s) = q^{2(b-i)-1} \sum_{a,b,c,v,w,x,y,z=1}^{N} R^{rk}_{zc}R^{bz}_{yi}R^{iy}_{xa}R^{xa}_{1w}\overline{R}^{w1}_{bv}R^{vc}_{s1}$$

$$= q^{2(b-i)-1} \sum_{b,c,y=1}^{N} R^{rk}_{zc}R^{bz}_{yi}R^{iy}_{11}R^{11}_{11}\overline{R}^{11}_{11}R^{1s}_{s1} = 0.$$

It now follows directly that the set of elements given in (3.15) is a basis. □

We denote the calculus corresponding to I_{SU_N} by $\Omega^1(SU_N)$. We acknowledge that the dimension of this calculus is significantly less than the classical value, for $N > 2$. However, we are not interested in $\Omega^1_q(SU_N)$ as a quantum deformation in itself. Instead, we will view it as a useful mathematical tool to be exploited in our efforts to investigate the geometry of $\mathbf{C}_q[\mathbf{C}P^{N-1}]$. As we shall show below, the calculus that $\Omega^1_q(SU_N)$ restricts to on $\mathbf{C}_q[\mathbf{C}P^{N-1}]$ has classical dimension. By contrast, $\Omega^1_{\mathrm{bc},q}(SU_N)$ restricts to an (N^2-1)-dimensional calculus on $\mathbf{C}_q[\mathbf{C}P^{N-1}]$, a value much higher than the classical one. (The three-dimensional calculus induced by $\Omega^1_{\mathrm{bc},q}(SU_2)$ on $\mathbf{C}_q[\mathbf{C}P^1]$ was thoroughly investigated in [7].)

We finish our general discussion of the calculus $\Omega^1_q(SU_N)$ with the following technical lemma. It contains a number of explicit formulae which will prove invaluable in the chapters to come.

Lemma 3.2.3 *For the calculus $\Omega^1_q(SU_N)$ we have the following relations: For $i = 2,\ldots,N; j = 3,\ldots,N; i < j$, and $m = 1,\ldots,N$. The only non-zero actions of the generators on the basis elements $e^\pm_i$ are given by*

$$e^\pm_{i-1} \triangleleft u^m_m = q^{\delta_{m1}+\delta_{im}-\frac{2}{N}} e^\pm_{i-1}, \quad e^+_{i-1} \triangleleft u^j_i = q^{-\frac{2}{N}} \nu e^+_{j-1}, \quad e^-_{i-1} \triangleleft u^i_j = q^{-\frac{2}{N}} \nu e^-_{j-1}. \tag{3.17}$$

The only non-zero actions of the antipodes of the generators are given by

$$e^{+}_{i-1} \triangleleft S(u^j_i) = -q^{\frac{2}{N}}\nu e^{+}_{j-1}, \qquad e^{-}_{i-1} \triangleleft S(u^i_j) = -q^{\frac{2}{N}+2(i-j)}\nu e^{-}_{j-1}, \tag{3.18}$$

$$e^{\pm}_{i-1} \triangleleft S(u^m_m) = q^{\frac{2}{N}-\delta_{1j}-\delta_{ij}} e^{\pm}_{i-1}. \tag{3.19}$$

Moreover, we have the relations

$$\overline{S(u^i_1)} = -q^{\frac{4}{N}-1}e^{+}_{i-1}, \qquad \overline{S(u^1_i)} = -q^{\frac{4}{N}+1-2i}e^{-}_{i-1}; \tag{3.20}$$

and, that $\overline{u^m_m f} = q^{2\delta_{m1}-\frac{2}{N}}\overline{f}$, *and* $\overline{S(u^m_m)f} = q^{\frac{2}{N}-2\delta_{m1}}\overline{f}$, *for all* $f \in \mathbf{C}_q[SU_N]$.

Proof. We begin by introducing a variation on the operator Q_{kl}:

$$\widehat{Q}_{kl} := Q_{k1} + Q_{l1}(1-\delta_{1l}).$$

Clearly, we have that $\ker(\widehat{Q}) = I_{SU_N}$. We will now use $\widehat{Q}$, and (3.12), to calculate the action of the generator u^r_s on the basis element $\overline{u^i_1}$:

$$\begin{aligned}
\widehat{Q}_{kl}(u^i_1u^r_s) &= \sum_{a,b,x,y,z=1}^{N} q^{-\frac{4}{N}} R^{rk}_{zb}R^{iz}_{ya}R^{ya}_{1x}R^{xb}_{sl} \\
&= \sum_{b=1}^{N} q^{-\frac{4}{N}} R^{rk}_{1b}R^{i1}_{1i}R^{1i}_{1i}R^{ib}_{sl} + \sum_{b,x,y=1}^{N} q^{-\frac{4}{N}} R^{rk}_{zb}R^{iz}_{y1}R^{y1}_{1x}R^{xb}_{sl} \\
&= \sum_{b=1}^{N} q^{-\frac{4}{N}}\nu R^{rk}_{1b}R^{ib}_{sl} + 0 = q^{-\frac{4}{N}}\nu(R^{r1}_{1r}R^{ir}_{sl}\delta_{k1} + R^{1k}_{1k}R^{ik}_{sl}\delta_{1r}(1-\delta_{k1})) \\
&= q^{-\frac{4}{N}}\nu(R^{r1}_{1r}R^{ir}_{ir}\delta_{k1}\delta_{si}\delta_{lr} + R^{r1}_{1r}R^{ir}_{ri}\delta_{sr}\delta_{li} + 0) \\
&= q^{-\frac{4}{N}}\nu(\theta(r-i)\delta_{k1}\delta_{si}\delta_{lr} + q^{-\delta_{r1}-\delta_{ri}}\delta_{sr}\delta_{li}).
\end{aligned}$$

Thus, we see that $\widehat{Q}(u^i_1u^r_s)$ gives us a non-zero answer if and only if $s = i$ and $r > s$; or if $s = r$. For the first case we get that

$$\widehat{Q}(u^i_1u^r_i) = q^{-\frac{4}{N}}\nu^2E_{1r} = q^{-\frac{4}{N}}\nu^2(q^{-\frac{2}{N}}\nu)^{-1}\overline{u^r_1} = q^{-\frac{2}{N}}\nu\,\overline{u^r_1};$$

while for second case we get that

$$\widehat{Q}(u^i_1u^r_r) = q^{\delta_{r1}+\delta_{ri}-\frac{4}{N}}\nu E_{1i} = q^{\delta_{r1}+\delta_{ri}-\frac{4}{N}}\nu(q^{-\frac{2}{N}}\nu)^{-1}\overline{u^i_1} = q^{\delta_{r1}+\delta_{ri}-\frac{2}{N}}\overline{u^i_1};$$

where for both cases we have used (3.16). The formulae for the non-zero actions on e_i^+ now easily follow.

Using (3.12) once more, we now calculate the action of the generator u^r_s on the basis element $\overline{u^1_i}$:

$$\begin{aligned}
\widehat{Q}_{kl}(u^1_i u^r_s) &= \sum_{a,b,x,y,z=1}^{N} q^{-\frac{4}{N}} R^{rk}_{zb} R^{1z}_{ya} R^{ya}_{ix} R^{xb}_{sl} \\
&= \sum_{a,b,z=1}^{N} q^{-\frac{4}{N}} R^{rk}_{ib} R^{1i}_{1i} R^{1i}_{i1} R^{1b}_{sl} + \sum_{b,y,z=1}^{N} q^{-\frac{4}{N}} R^{rk}_{zb} R^{1z}_{y1} R^{y1}_{ix} R^{xb}_{sl} \\
&= \sum_{b=1}^{N} q^{-\frac{4}{N}} \nu R^{rk}_{ib} R^{1b}_{sl} + 0 = q^{-\frac{4}{N}} \nu (R^{ri}_{ir} R^{1r}_{sl} \delta_{ki} + R^{rk}_{ik} R^{1k}_{sl}) \\
&= q^{-\frac{4}{N}} \nu (R^{ri}_{ir} R^{1r}_{r1} \delta_{ki} \delta_{l1} \delta_{rs} + R^{is}_{is} R^{1s}_{s1} \delta_{ri} \delta_{l1} \delta_{ks}) \\
&= \nu q^{\delta_{ri} + \delta_{r1} - \frac{4}{N}} \delta_{ki} \delta_{l1} \delta_{rs} + q^{-\frac{4}{N}} \nu^2 \delta_{ri} \delta_{l1} \delta_{ks} \theta(s-i).
\end{aligned}$$

Thus, we see that $\widehat{Q}(u^1_i u^r_s)$ gives us a non-zero answer if, and only if, $r = s$; or $i = r$, and $s > i$. For the first case we get

$$\widehat{Q}(u^1_i u^r_r) = q^{\delta_{ri} + \delta_{r1} - \frac{4}{N}} \nu E_{1i} = q^{\delta_{ri} + \delta_{r1} - \frac{4}{N}} \nu (q^{-\frac{2}{N}} \nu)^{-1} \overline{u^i_1} = q^{\delta_{ri} + \delta_{r1} - \frac{2}{N}} \, \overline{u^i_1};$$

while for the second case we have

$$\widehat{Q}(u^1_i u^i_s) = q^{-\frac{4}{N}} \nu E_{s1} = q^{-\frac{4}{N}} \nu^2 (q^{-\frac{2}{N}} \nu)^{-1} \, \overline{u^1_i} = q^{-\frac{2}{N}} \nu \, \overline{u^1_i};$$

where for both cases we have again used (3.16). The formulae for the non-zero actions on e_i^- now easily follow.

Moving on, we use (3.13) to calculate the action of the element $S(u^r_s)$ on the basis element $\overline{u^i_1}$:

$$\begin{aligned}
\widehat{Q}_{kl}(u^i_1 S(u^g_h)) &= \sum_{a,b,x,y,z=1}^{N} q^{2(b-h)} \overline{R}^{bk}_{zh} R^{iz}_{ya} R^{ya}_{1x} \overline{R}^{xg}_{bl} \\
&= \sum_{b=1}^{N} q^{2(b-h)} \overline{R}^{bk}_{1h} R^{i1}_{1i} R^{1i}_{1i} \overline{R}^{ig}_{bl} + \sum_{b,x,y,z=1}^{N} q^{2(b-h)} \overline{R}^{bk}_{zh} R^{iz}_{y1} R^{y1}_{1x} \overline{R}^{xg}_{bl} \\
&= \nu (q^{2(1-h)} \overline{R}^{1h}_{1h} \overline{R}^{ig}_{1l} (1 - \delta_{h1}) \delta_{kh} + q^{2(h-h)} \overline{R}^{h1}_{1h} \overline{R}^{ig}_{hl} \delta_{k1}) + 0 \\
&= 0 + q^{-\delta_{1h}} \nu \overline{R}^{ig}_{hl} \delta_{k1} = \nu (\overline{R}^{ig}_{ig} \delta_{hi} \delta_{lg} + q^{-\delta_{1h}} \overline{R}^{ig}_{gi} \delta_{hg} \delta_{li}) \delta_{k1} \\
&= -\nu^2 \theta(g-i) \delta_{hi} \delta_{lg} \delta_{k1} + \nu q^{-\delta_{1h} - \delta_{ig}} \delta_{hg} \delta_{li} \delta_{k1}
\end{aligned}$$

Thus, we see that $\widehat{Q}(u^i_1 S(u^g_h))$ gives a non-zero answer if, and only if, we have $h = i$, and $g > i$; or $h = g$. For the first case we have

$$\widehat{Q}(u^i_1 S(u^g_i)) = \nu^2 E_{1g} = -\nu^2 (q^{-\frac{2}{N}}\nu)^{-1}\,\overline{u^g_1} = -q^{\frac{2}{N}}\nu\,\overline{u^g_1};$$

while for the second case we have

$$\widehat{Q}(u^i_1 S(u^g_g)) = \nu q^{-\delta_{1h}-\delta_{ig}} E_{1i} = \nu q^{-\delta_{1h}-\delta_{ig}}(q^{-\frac{2}{N}}\nu)^{-1}\,\overline{u^i_1} = q^{\frac{2}{N}-\delta_{1h}-\delta_{ig}}\,\overline{u^i_1};$$

where for both cases we have again used (3.16).

Using (3.13) again, we now calculate the action of the element $S(u^r_s)$ on the basis element $\overline{u^1_i}$:

$$\begin{aligned}
\widehat{Q}_{kl}(u^1_i S(u^g_h)) &= \sum_{a,b,x,y,z=1}^{N} q^{2(b-h)}\overline{R}^{bk}_{zh} R^{1z}_{ya} R^{ya}_{ix}\overline{R}^{xg}_{bl} \\
&= \sum_{b=1}^{N} q^{2(b-h)}\overline{R}^{bk}_{ih} R^{1i}_{1i} R^{1i}_{i1}\overline{R}^{1g}_{bl} \;+ \sum_{b,x,y,z=1}^{N} q^{2(b-h)}\overline{R}^{bk}_{zh} R^{1z}_{y1} R^{y1}_{ix}\overline{R}^{xg}_{bl} \\
&= \sum_{b=1}^{N} q^{2(b-h)}\nu\overline{R}^{bk}_{ih}\overline{R}^{1g}_{bl} \;+ 0 \\
&= q^{2(i-h)}\nu\overline{R}^{ih}_{ih}\overline{R}^{1i}_{i1}\delta_{gi}\delta_{kh}\delta_{l1}(1-\delta_{hi}) \;+ q^{2(h-h)}\nu\overline{R}^{hi}_{ih}\overline{R}^{1g}_{hl}\delta_{ki} \\
&= -q^{2(i-h)}\nu^2\delta_{gi}\delta_{kh}\delta_{l1}\theta(h-i) \;+ \nu\overline{R}^{hi}_{ih}\overline{R}^{1h}_{h1}\delta_{gh}\delta_{ki}\delta_{l1} \\
&= -q^{2(i-h)}\nu^2\delta_{gi}\delta_{kh}\delta_{l1}\theta(h-i) \;+ q^{-\delta_{h1}-\delta_{hi}}\nu\delta_{gh}\delta_{ki}\delta_{l1}.
\end{aligned}$$

Thus, we see that $\widehat{Q}(u^i_1 S(u^g_h))$ gives a non-zero answer if, and only if, we have $g = i$, and $h > i$; or $h = g$. For the first case we have

$$\widehat{Q}(u^1_i S(u^i_h)) = -q^{2(i-h)}\nu^2 E_{h1} = -q^{2(i-h)}\nu^2(q^{-\frac{2}{N}}\nu)^{-1}\,\overline{u^1_h} = -q^{\frac{2}{N}+2(i-h)}\nu\,\overline{u^1_h};$$

while for the second case we have

$$\widehat{Q}(u^1_i S(u^g_g)) = q^{-\delta_{1g}-\delta_{ig}}\nu E_{i1} = q^{-\delta_{1g}-\delta_{ig}}\nu(q^{-\frac{2}{N}}\nu)^{-1}\,\overline{u^1_i} = q^{\frac{2}{N}-\delta_{1g}-\delta_{ig}}\,\overline{u^1_i};$$

where for both cases we have, as usual, used (3.16).

We now move onto finding a formula for $S(u^i_1)$ using (3.11):

$$\overline{S(u^i_1)} = \sum_{a,z=1}^{N} q^{2(a-1)+\frac{2}{N}}\overline{R}^{ak}_{z1}\overline{R}^{zi}_{al} = q^{2(1-1)+\frac{2}{N}}\overline{R}^{11}_{11}\overline{R}^{1i}_{1i}\delta_{li}\delta_{k1} = q^{\frac{2}{N}-1}\nu\delta_{li}\delta_{k1}.$$

Thus, we see that

$$\widehat{Q}(S(u^i_1)) = q^{\frac{2}{N}-1}\nu E_{1i} = q^{\frac{2}{N}-1}\nu(q^{-\frac{2}{N}}\nu)^{-1}\overline{u^i_1} = q^{\frac{4}{N}-1}\overline{u^i_1},$$

which gives us the first relation in (3.20). For $\widehat{Q}(S(u^1_i))$, we have

$$\widehat{Q}_{kl}(S(u^1_i)) = \sum_{a,z} q^{2(a-i)+\frac{2}{N}}\overline{R}^{ak}_{zi}\overline{R}^{z1}_{al} = q^{2(1-i)+\frac{2}{N}}\overline{R}^{1i}_{1i}\overline{R}^{11}_{11}\delta_{ki}\delta_{l1} = -q^{1-2i+\frac{2}{N}}\nu\delta_{ki}\delta_{l1}.$$

This gives us that

$$\widehat{Q}(S(u^1_i)) = q^{1-2i+\frac{2}{N}}\nu E_{il} = q^{1-2i+\frac{2}{N}}\nu(q^{-\frac{2}{N}}\nu)^{-1}\overline{u^1_i} = q^{\frac{4}{N}+1-2i}\overline{u^1_i},$$

from which follows the second relation in (3.20).

Finally, we come to $\overline{u^m_m f}$ and $\overline{S(u^m_m)f}$, for $f \in \mathbf{C}_q[SU_N]$, for $m = 1, \cdots, N$. Now as a little thought will confirm, we can canonically identify $\mathbf{C}_q[SU_N]/\ker(Q)$ and $\Lambda^1_{SU_N}$. This allows us to consider $\overline{1}$, $\overline{u^m_m}$, and $\overline{S(u^m_m)}$ as elements in $\Lambda^1_{SU_N}$. From (3.10) we now get

$$\widehat{Q}_{kl}(u^m_m) = \sum_{a,z=1}^{N} q^{-\frac{2}{N}}R^{mk}_{za}R^{za}_{ml} = q^{-\frac{2}{N}}R^{mk}_{km}R^{km}_{mk}\delta_{kl} = q^{-\frac{2}{N}}R^{m1}_{1m}R^{1m}_{m1}\delta_{k1}\delta_{l1} \tag{3.21}$$

$$= q^{2\delta_{m1}-\frac{2}{N}}\delta_{kl}\delta_{k1}, \tag{3.22}$$

which tells us that $\overline{u^m_m} = q^{2\delta_{m1}-\frac{2}{N}}\overline{1}$. From (3.11) we have

$$\widehat{Q}_{kl}(S(u^m_m)) = \sum_{a,z=1}^{N} q^{2(a-m)+\frac{2}{N}}\overline{R}^{ak}_{zm}\overline{R}^{zm}_{al} = q^{2(m-m)+\frac{2}{N}}\overline{R}^{mk}_{km}\overline{R}^{km}_{mk}\delta_{kl} = q^{\frac{2}{N}}\overline{R}^{m1}_{1m}\overline{R}^{1m}_{m1}\delta_{k1}\delta_{l1}$$

$$= q^{\frac{2}{N}-2\delta_{m1}}\delta_{k1}\delta_{l1},$$

which tells us that $\overline{S(u^m_m)} = q^{\frac{2}{N}-2\delta_{m1}}\overline{1}$. It now follows directly that $\overline{u^j_j f} = q^{2\delta_{j1}-\frac{2}{N}}\overline{f}$, and $\overline{S(u^j_j)f} = q^{\frac{2}{N}-2\delta_{j1}}\overline{f}$, for all $f \in \mathbf{C}_q[SU_N]$. $\square$

Example 3.2.4. Let us look now at the case of $N = 2$: The ideal I_{SU_2} corresponding to $\Omega^1_q(SU_2)$ is generated by the six elements

$$(a-q)(a-1), \quad bc, \quad b^2, \quad c^2, \quad (a-q)b, \quad (a-q)c,$$

or equivalently by the six elements

$$a + qd - (q+1), \quad bc, \quad b^2, \quad c^2, \quad (a-q)b, \quad (a-q)c.$$

A three-dimensional basis of $V_{\mathbf{C}P^1}$ is given by

$$e^+ := \overline{c}, \qquad\qquad e^0 := \overline{a-1}, \qquad\qquad e^+ := \overline{b}.$$

Explicitly, s^{-1} acts on these elements according to

$$s^{-1}(e_1^+) = a\mathrm{d}c - qc\mathrm{d}a, \quad s^{-1}(e^0) = d\mathrm{d}a - q^{-1}b\mathrm{d}c, \quad s^{-1}(e_1^-) = d\mathrm{d}b - q^{-1}b\mathrm{d}d.$$

While the exterior derivative acts according to

$$\mathrm{d}a = ae^0 + be_1^+, \quad \mathrm{d}b = ae_1^- - q^{-1}be^0, \quad \mathrm{d}c = ce^0 + de_1^+, \quad \mathrm{d}d = ce_1^- - q^{-1}de^0.$$

Finally, in matrix form, the right module relations are given by:

$$e^0 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} qa & q^{-1}b \\ qc & q^{-1}d \end{pmatrix} e^0 + (q-1) \begin{pmatrix} b & 0 \\ d & 0 \end{pmatrix} e^+ + (q-1) \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} e^-,$$

$$e^\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} e^\pm.$$

Since $\Omega^1_q(SU_2)$ is a three-dimensional calculus, it is natural to ask whether or not it is isomorphic to Woronowicz's well-known $3D$ calculus [79]. Recall that the ideal corresponding to the the $3D$ calculus is generated by the elements

$$a + q^{-2}d - (1+q^{-2}), \; bc, \; b^2, \; c^2, \; (a-1)b, \; (a-1)c. \tag{3.23}$$

Now Lemma 3.2.3 tells us that

$$\overline{(a-1)b} = \overline{b(qa-1)} = (q^2-1)\overline{b}, \qquad \overline{(a-1)c} = \overline{c(qa-1)} = (q^2-1)\overline{c},$$

and that

$$\overline{a + q^{-2}d - (1+q^{-2})} = (q + q^{-3} - (1+q^{-2}))\overline{1}.$$

As is easy to see, there is no value of $q \in \mathbf{C}$ for which these three elements are simultaneously zero, and so, the two calculi cannot be isomorphic. Alternatively, one can observe that since $(a-q)b - (a-1)b = (1-q)b$, any ideal containing both $(a-q)b$ and $(a-1)b$ will also contain b. Since this is not the case for either ideal, they cannot be equal. Moreover, a similar argument will show that $\Omega^1_q(SU_2)$ is not isomorphic to any of the other three-dimensional calculi presented in [71].

3.2.2 The Heckenberger–Kolb Calculus

We now introduce an ideal that will play a central role in this thesis:

$$I_{\mathbf{C}P^{N-1}} := \langle z_{ij},\, z_{i1}z_{j1},\, z_{1i}z_{1j} \mid i,j = 2,\ldots,N\rangle\,. \tag{3.24}$$

As will be shown, it is a left covariant subspace of $\mathbf{C}_q[\mathbf{C}P^{N-1}]^+$, and so, it has a corresponding calculus. We denote this calculus by $\Omega^1_q(\mathbf{C}P^{N-1})$ and call it the *Heckenberger–Kolb calculus.* The following proposition establishes the essential properties of the calculus.

Proposition 3.2.5 *The ideal $I_{\mathbf{C}P^{N-1}}$ is a left-covariant ideal of $\mathbf{C}_q[\mathbf{C}P^{N-1}]^+$. Moreover, the canonical map $\widehat{\iota}: V_{\mathbf{C}P^{N-1}} \to \Lambda^1_{SU_N}$ is an embedding, and $\widehat{\iota}(V_{\mathbf{C}P^{N-1}})$ has a basis given by*

$$\overline{z_{i1}} = q^{\frac{2}{N}-1}e^+_{i-1}, \qquad \overline{z_{1i}} = q^{\frac{2}{N}-2i+3}e^-_{i-1}, \qquad (i = 2,\ldots,N). \tag{3.25}$$

Proof. That $\Delta_{\mathbf{C}P^{N-1}}$ restricts to a $\mathbf{C}_q[U_{N-1}]$-coaction on $I_{\mathbf{C}P^{N-1}}$ is clear from the following calculations: For z_{ij} we have

$$\begin{aligned}
\Delta_{\mathbf{C}P^{N-1}}(z_{ij}) &= (\mathrm{id}\otimes S)\circ(\mathrm{id}\otimes\alpha_N)\Big(\sum_{a,b=1}^{N} u^a_1 S(u^1_b)\otimes u^i_a S(u^b_j)\Big)\\
&= (\mathrm{id}\otimes S)\Big(\sum_{a,b=2}^{N} u^a_1 S(u^1_b)\otimes \alpha_N(u^i_a S(u^b_j))\Big)\\
&= \sum_{a,b=2}^{N} z_{ab}\otimes S(\alpha_N(u^i_a S(u^b_j))) = I_{\mathbf{C}P^{N-1}}\otimes \mathbf{C}_q[U_{N-1}].
\end{aligned}$$

For $z_{i1}z_{j1}$ we have

$$\begin{aligned}
\Delta(z_{i1}z_{j1}) &= (\mathrm{id}\otimes S)\circ(\mathrm{id}\otimes\alpha_N)\Big(\sum_{a,b,c,d=1}^{N} u^a_1 S(u^1_b)u^c_1 S(u^1_d)\otimes u^i_a S(u^b_1)u^j_c S(u^d_1)\Big)\\
&= (\mathrm{id}\otimes S)\Big(\sum_{a,c=2}^{N} u^a_1 S(u^1_1)u^c_1 S(u^1_1)\otimes \alpha_N(u^i_a S(u^1_1)u^j_c S(u^1_1))\Big)\\
&= \sum_{a,c=2}^{N} z_{a1}z_{c1}\otimes S(\alpha_N(u^i_a u^j_c))\,\mathrm{det}^{-2}_{N-1} \in I_{\mathbf{C}P^{N-1}}\otimes \mathbf{C}_q[U_{N-1}].
\end{aligned}$$

Finally, for $z_{1i}z_{1j}$, we have

$$\begin{aligned}\Delta(z_{1i}z_{1j}) &= (\mathrm{id}\otimes S)\circ(\mathrm{id}\otimes\alpha_N)(\sum_{a,b,c,d=1}^{N} u^a_1 S(u^1_b)u^c_1 S(u^1_d)\otimes u^1_a S(u^b_i)u^1_c S(u^d_j))\\ &= (\mathrm{id}\otimes S)(\sum_{b,d=2}^{N} u^1_1 S(u^1_b)u^1_1 S(u^1_d)\otimes \alpha_N(u^1_1 S(u^b_i)u^1_1 S(u^d_j)))\\ &= \sum_{b,d=2}^{N} z_{1b}z_{1d}\otimes S(\alpha_N(S(u^b_i)S(u^d_j)))\,\mathrm{det}^2_{N-1}\in I_{\mathbf{C}P^{N-1}}\otimes \mathbf{C}_q[U_{N-1}].\end{aligned}$$

We will next establish that $\widehat{\iota}$ is well defined by showing that $I_{\mathbf{C}P^{N-1}}$ is contained in $\ker(\widehat{Q})^+$: For z_{ij}, we have

$$\widehat{Q}(z_{ij}) = \widehat{Q}(u^i_1 S(u^1_j)) = e^+_{i-1}\triangleleft S(u^1_j) = 0.$$

For $z_{i1}z_{j1}$, we have

$$\widehat{Q}(z_{i1}z_{j1}) = \widehat{Q}(u^i_1 S(u^1_1)u^j_1 S(u^1_l)) = e^+_{i-1}\triangleleft(S(u^1_1)u^j_1 S(u^1_1)) = q^{\frac{2}{N}-2}e^+_{i-1}\triangleleft(u^j_1 S(u^1_1)) = 0.$$

Finally, for $z_{1i}z_{1j}$, we have

$$\widehat{Q}(z_{1i}z_{1j}) = \widehat{Q}(u^1_1 S(u^1_i)u^1_1 S(u^1_j)) = q^{5-2i}e^-_{i-1}\triangleleft S(u^1_j) = 0.$$

Since $\ker(\widehat{Q})^+$ is a right ideal, it follows that $I_{\mathbf{C}P^{N-1}}$ is contained in $\ker(\widehat{Q})^+$.

Next we show that $\{\overline{z_{i1}},\overline{z_{1i}}\,|\,i=2,\ldots,N\}$ is a spanning set for $V_{\mathbf{C}P^{N-1}}$: Since

$$\sum_{i=1}^{N} q^{2(i-1)}z_{ii} = \sum_{i=1}^{N} q^{2(i-1)}u^i_1 S(u^1_i) = \sum_{i=1}^{N} S(u^1_i)u^i_1 = 1,$$

we must have that

$$0 = (\sum_{i=2}^{N}\overline{z_{ii}}) + \overline{z_{11}-1} = \overline{z_{11}-1}.$$

Thus, since $z_{kl}\in I_{\mathbf{C}P^{N-1}}$, for $(k,l)\neq(1,1)$, we need only consider monomials which have z_{i1}, or z_{1i}, as a first factor, for $i\neq 1$.

Recalling now the quantum sphere relations given in (3.4), we see that for any monomial $z_{i1}z_{kl}$, with $l \neq 1$, we have

$$\begin{aligned} z_{i1}z_{kl} &= u^i_1 S(u^1_1) u^k_1 S(u^1_l) = q^{-2} u^i_1 u^k_1 S(u^1_l) S(u^1_1) \\ &= q^{-2} u^i_1 S(u^1_l) u^k_1 S(u^1_1) - q^{-3}\delta_{kl}\nu \sum_{a=i+1}^{N} q^{2(a-i)} u^i_1 S(u^1_a) u^a_1 S(u^1_1) \\ &= q^{-2} z_{il} z_{k1} - q^{-3}\delta_{kl}\nu \sum_{a=i+1}^{N} z_{ia} z_{a1} \in I_{\mathbf{C}P^{N-1}}. \end{aligned}$$

Moreover, for the element $z_{1i}z_{kl}$, for $k \neq 1$, we have

$$\begin{aligned} z_{1i}z_{kl} &= u^1_1 S(u^1_i) u^k_1 S(u^1_l) = q^2 S(u^1_i) u^k_1 u^1_1 S(u^1_l) \\ &= q^2 u^k_1 S(u^1_i) u^1_1 S(u^1_l) - q^{-1}\delta_{ki}\nu \sum_{a=i+1}^{N} q^{2(a-i)} u^a_1 S(u^1_a) u^1_1 S(u^1_l) \\ &= q^2 z_{ki} z_{1l} - q^{-1}\delta_{ki}\nu \sum_{a=i+1}^{N} q^{2(a-i)} z_{aa} z_{1l} \in I_{\mathbf{C}P^{N-1}}. \end{aligned}$$

It follows directly that $\{\overline{z_{i1}}, \overline{z_{1i}} \,|\, i = 2, \ldots, N\}$ is a spanning set for $V_{\mathbf{C}P^{N-1}}$.

We can now finish by showing that $\{\overline{z_{i1}}, \overline{z_{1i}} \,|\, i = 2, \ldots, N\}$ is a basis for $\widehat{\iota}(V_M)$, and consequently that $\widehat{\iota}$ is an embedding: For $i = 2, \ldots, N$, we have

$$Q(z_{i1}) = Q(u^i_1 S(u^1_1)) = e^+_{i-1} \triangleleft S(u^1_1) = q^{\frac{2}{N}-1} e^+_{i-1},$$

and

$$Q(z_{1i}) = Q(u^1_1 S(u^1_i)) = q^{2-\frac{2}{N}} Q(S(u^1_i)) = q^{\frac{2}{N}-2i+3} e^-_{i-1}.$$

□

We will show how this calculus relates to the one-forms of Heckenberger and Kolb's q-deformed de Rham complex in Section 5.3, using the framework of noncommutative complex structures.

Chapter 4

Framing the Maximal Prolongation

A natural question to ask is whether or not one can extend the canonical framing of a left-covariant calculus $\Omega^1(M)$ to a framing for its maximal prolongation. In this chapter we will use Takeuchi's categorical equivalence to show that, for a distinguished class of calculi, this can indeed be done. We will do so in two parts, first we show how to frame tensor powers, and then we show how this framing restricts to the maximal prolongation.

4.1 Framings and Tensor Powers

In this section we will show how to frame tensor powers. This will require us to restrict our attention to a distinguished subcategory of ${}^G_M\mathcal{M}_M$, introduced in the subsection below. Following this we introduce the notion of a framing calculus which will serve as an invaluable tool for simplifying calculations throughout the rest of the thesis.

4.1.1 A Monoidal Equivalence of Categories

The category ${}^G_M\mathcal{M}_M$ has a natural monoidal structure $\otimes_M$, where for $\mathcal{E}, \mathcal{F}$ two objects in ${}^G_M\mathcal{M}_M$, we define $\mathcal{E} \otimes_M \mathcal{F}$ to be the usual bimodule tensor product endowed with the obvious left G-comodule structure

$$\Delta_L : \mathcal{E} \otimes_M \mathcal{F} \to G \otimes \mathcal{E} \otimes_M \mathcal{F}, \qquad e \otimes_M f \mapsto e_{(-1)} f_{(-1)} \otimes e_{(0)} \otimes_M f_{(0)}. \tag{4.1}$$

However, for $\mathcal{M}_M^H$ no such obvious monoidal structure exists. This leads us to consider a particular subcategory of $\mathcal{M}_M^H$ defined as follows: Let $\mathcal{M}_0^H$ be the strictly full monoidal subcategory of $\mathcal{M}_M^H$ whose objects V are those endowed with the trivial right action

$$v \triangleleft m = \varepsilon(m) v, \qquad\qquad (v \in V, m \in M).$$

This category *has* a natural monoidal structure $\otimes$, where for V, W two objects in $\mathcal{M}_0^H$, we define $V \otimes W$ to be the usual vector space tensor product, endowed with the trivial right M-action, and a right H-comodule structure given by

$$\Delta_R : V \otimes W \to V \otimes W \otimes H, \qquad v \otimes w \mapsto v_{(0)} \otimes w_{(0)} \otimes w_{(1)} v_{(1)}. \tag{4.2}$$

That these two structures are compatible in the sense of (2.2) follows easily from (2.1).

One should now of course ask what the corresponding subcategory of ${}^G_M\mathcal{M}_M$ is. As a candidate we propose the strictly full subcategory whose objects $\mathcal{E}$ are those which satisfy $\mathcal{E}M^+ \subseteq M^+\mathcal{E}$. As a moment's thought will confirm, for $\mathcal{E}, \mathcal{F}$ two objects in ${}^G_M\mathcal{M}_0$, their tensor product $\mathcal{E} \otimes_M \mathcal{F}$ is still an object in ${}^G_M\mathcal{M}_0$. Thus it is clear that ${}^G_M\mathcal{M}_0$ is a monoidal subcategory of ${}^G_M\mathcal{M}_M$. Moreover, as the following theorem demonstrates, it is monoidally equivalent to $\mathcal{M}_0^H$.

Theorem 4.1.1 *The functor Φ_M restricts to an equivalence of categories between ${}^G_M\mathcal{M}_0$ and $\mathcal{M}_0^H$. Moreover, for any two objects $\mathcal{E}, \mathcal{F} \in {}^G_M\mathcal{M}_0$, the natural transformation*

$$\mu_{\mathcal{E},\mathcal{F}} : \Phi_M(\mathcal{E} \otimes_M \mathcal{F}) \to \Phi_M(\mathcal{E}) \otimes \Phi_M(\mathcal{F}), \qquad \overline{v \otimes_M w} \mapsto \overline{v} \otimes \overline{w}, \tag{4.3}$$

gives an equivalence of monoidal categories between ${}^G_M\mathcal{M}_0$ and $\mathcal{M}_0^H$.

Proof. Let us first show that Φ restricts to an equivalence of categories between ${}^G_M\mathcal{M}_0$ and $\mathcal{M}_0^H$: If $\mathcal{E}$ is an object in ${}^G_M\mathcal{M}_0$, then for any $e \in \mathcal{E}$, and $m \in M^+$, we must have, from the definitions of ${}^G_M\mathcal{M}_0$ and $\Phi_M(\mathcal{E})$, that $\overline{e} \triangleleft m = 0$. Hence, for any $n \in M$, we have

$$\overline{e} \triangleleft n = \overline{e} \triangleleft (n^+ + \varepsilon(n)1) = \overline{e} \triangleleft n^+ + \overline{e} \triangleleft (\varepsilon(n)1) = \varepsilon(n)\overline{e}.$$

Thus, $\Phi_M(\mathcal{E})$ is well-defined as an object in $\mathcal{M}_0^H$. Conversely, if V is an object in $\mathcal{M}_0^H$, then for any element $\sum_i f^i \otimes v^i$ in $\Psi(V)$, the right action of M on $\Psi_M(V)$ must operate according to

$$(\sum_i f^i \otimes v^i)m = \sum_i f^i m_{(1)} \otimes (v^i \triangleleft m_{(2)}) = \sum_i f^i m_{(1)} \varepsilon(m_{(2)}) \otimes v^i = \sum_i f^i m \otimes v^i.$$

Now if $m \in M^+$, then $\sum_i f^i m \otimes v^i$ must be an element of $\ker(\mathrm{frame}_M^\perp)$. But since $\ker(\mathrm{frame}_M^\perp)$ is equal to $M^+\Psi_M(V)$, we must have that $(\sum_i f^i \otimes v^i)m$ is contained in $M^+\Psi_M(V)$. Hence $\Psi_M(V)$ is well-defined as an object in ${}^G_M\mathcal{M}_0$. That this gives an equivalence of categories now follows from the fact that $\Phi_M : {}^G_M\mathcal{M}_M \to \mathcal{M}_M^H$ is an equivalence of categories, and that ${}^G_M\mathcal{M}_0$, and $\mathcal{M}_0^H$, are both full subcategories of ${}^G_M\mathcal{M}_M$, and $\mathcal{M}_M^H$, respectively.

We now turn to showing that $\mu_{\mathcal{E},\mathcal{F}}$ is a natural isomorphism: It is trivial that $\mu_{\mathcal{E},\mathcal{F}}$ is well-defined as a right M-module map. To see that it is also a H-comodule map, note first that the right comodule structure on $\Phi_M(\mathcal{E} \otimes_M \mathcal{F})$ acts according to

$$\Delta_R : \overline{e \otimes_M f} \mapsto \overline{e_{(0)} \otimes_M f_{(0)}} \otimes S(e_{(-1)} f_{(-1)}), \qquad (e \in \mathcal{E}, f \in \mathcal{F}).$$

By (4.2), the right comodule structure on $\Phi_M(\mathcal{E}) \otimes \Phi_M(\mathcal{F})$ acts according to

$$\Delta_R : \overline{e} \otimes \overline{f} \mapsto \overline{e_{(0)}} \otimes \overline{f_{(0)}} \otimes S(f_{(-1)}) S(e_{(-1)}), \qquad (e \in \mathcal{E}, f \in \mathcal{F}).$$

Hence, $\mu_{\mathcal{E},\mathcal{F}}$ is indeed a morphism in $\mathcal{M}_0^H$. It remains to show that the inverse morphism, which would send $\overline{v} \otimes \overline{w}$ to $\overline{v \otimes w}$, is well-defined. But this follows directly from the fact that

$$(M^+\mathcal{E}) \otimes_M \mathcal{F} + \mathcal{E} \otimes_M (M^+\mathcal{F}) = M^+(\mathcal{E} \otimes_M \mathcal{F}).$$

□

This result allows us to identify $\Phi_M((\Omega^1(M))^{\otimes_M k})$ and $\Phi_M(\Omega^1(M))^{\otimes k}$, and gives us the following corollary.

Corollary 4.1.2 *Let $\Omega^1(M)$ be a left-covariant first-order differential calculus with canonical framing (V_M, σ). If $\Omega^1(M)$ is contained in the subcategory $\mathcal{M}_0^H$, then we have a framing $(V_M^{\otimes k}, \sigma^k)$, where*

$$\sigma^k : \Phi_M((\Omega^1(M))^{\otimes_M k}) \to V_M^{\otimes k}, \qquad \overline{\omega_1 \otimes \cdots \otimes \omega_k} \mapsto \sigma(\overline{\omega_1}) \otimes \cdots \otimes \sigma(\overline{\omega_k}).$$

4.1.2 Framing Calculi

For $\Omega^1(G)$ a left-covariant differential calculus over a Hopf algebra G, it can quite often happen that $\Omega^1(G)$ is not an object in ${}^G_G\mathcal{M}_0$, meaning we cannot frame its tensor powers using the above approach. An obvious example is the calculus $\Omega^1_q(SU_N)$ introduced in Chapter 3. For such calculi consider the framing $((\Lambda^1_G)^{\otimes k}, t^k)$, where

$$t^k := c^k \circ s^{\otimes k} : (\Omega^1(G))^{\otimes_G k} \to G \otimes (\Lambda^1_G)^{\otimes k}, \qquad\qquad (k \geq 2),$$

with $c^k : (G \otimes \Lambda^1_G)^{\otimes_G k} \to G \otimes (\Lambda^1_G)^{\otimes k}$ the obvious identification. We denote the corresponding isomorphism in $\mathcal{M}^G_G$ by

$$\tau^k : \Phi_G((\Omega^1(G))^{\otimes_G k}) \to (\Lambda^1_G)^{\otimes k}.$$

Explicitly, τ^k acts on $\overline{g^1 \mathrm{d}g^2 \otimes_G \mathrm{d}g^3 \otimes_G \cdots \otimes_G \mathrm{d}g^k}$ to give

$$\varepsilon(g^0)\overline{(g^1)^+ g^2_{(1)} \cdots g^k_{(1)}} \otimes \overline{(g^2_{(2)})^+ g^3_{(2)} \cdots g^k_{(2)}} \otimes \cdots \otimes \overline{(g^k_{(k-1)})^+}.$$

As we shall now show, for certain distinguished calculi on G, we can use τ^k to give a new framing for tensor powers of $\Omega^1(M)$:

Definition 4.1.3. For any first-order differential calculus $\Omega^1(M)$ over M, a *framing calculus* $\Omega^1(G)$ is a first-order differential calculus for G such that

1. $\Omega^1(G)$ *restricts* to $\Omega^1(M)$ on M, by which we mean
$$\Omega^1(M) = \{\sum_i m^i \mathrm{d}n^i \in \Omega^1(G) \mid m^i, n^i \in M, \text{ for all } i\};$$
2. $\Omega^1(M)G \subseteq G\Omega^1(M)$.

Now $\Omega^1(M)$ and $\Omega^1(G)$ live in two ostensibly different categories. For sake of clarity, we should spend a little time exploring the relationship between ${}^G_G\mathcal{M}_G$ and ${}^G_M\mathcal{M}_M$; as well as the relationship between $\mathcal{M}^G_G$ and $\mathcal{M}^H_M$. First we note that, since every G-G-bimodule is obviously an M-M-bimodule, we have the forgetful inclusion of ${}^G_G\mathcal{M}_G$ in ${}^G_M\mathcal{M}_M$, which remembers only the M-M-bimodule structure of the objects of ${}^G_G\mathcal{M}_G$. On the other side of Takeuchi's equivalence, it is easy to see that the only coaction on a right G-module that is compatible in the sense of

(2.2), is the trivial coaction. Thus, $\mathcal{M}_G^G$ must be equivalent to $\mathcal{M}_G$, giving us a forgetful inclusion of $\mathcal{M}_G^G$ in $\mathcal{M}_M^H$.

Let us denote by $i : \Omega^1(M) \to \Omega^1(G)$ the embedding of $\Omega^1(M)$ into $\Omega^1(G)$. With respect to the inclusion of ${}^G_G\mathcal{M}_G$ in ${}^G_M\mathcal{M}_M$, it is clear that i is a morphism in ${}^G_M\mathcal{M}_M$, as are its tensor powers $i^{\otimes k} : (\Omega^1(M))^{\otimes_M k} \to (\Omega^1(G))^{\otimes_G k}$, for $k \in \mathbf{N}$. An important question to ask is when $i^{\otimes k}$ is is an embedding, for $k \geq 2$. To address this question we will need to introduce two important commutative diagrams: First consider the maps

$$\iota^k := \mathrm{proj} \circ \Phi_M(i^{\otimes k}) : \Phi_M((\Omega^1(M))^{\otimes_M k}) \to \Phi_G((\Omega^1(G))^{\otimes_G k}), \qquad (k \geq 2)$$

where $\mathrm{proj} : \Phi_M((\Omega^1(G))^{\otimes_M k}) \to \Phi_G((\Omega^1(G))^{\otimes_G k})$ is the canonical projection. Since

$$i^{\otimes k} = \mathrm{frame}_G^{-1} \circ \Psi_M(\iota^k) \circ \mathrm{frame}_M,$$

it is clear that $i^{\otimes k}$ is an embedding if, and only if, ι^k is an embedding. We are now ready to introduce our first commutative diagram:

$$\begin{array}{ccc} \Phi_G(\Omega^1(G)) & \xrightarrow{\sigma} & \Lambda^1_G \\ {\scriptstyle\iota}\uparrow & & \uparrow{\scriptstyle\widehat{\iota}} \\ \Phi_M(\Omega^1(M)) & \xrightarrow[\sigma]{} & V_M, \end{array}$$

where $\widehat{\iota}$ is the descent of the embedding $M^+ \hookrightarrow G^+$. It is clear that $\widehat{\iota}$ is a morphism in $\mathcal{M}_M^H$, as are its tensors powers $\widehat{\iota}^{\,\otimes k} : V_M^{\otimes k} \hookrightarrow (\Lambda^1_G)^{\otimes k}$. For higher powers of k, we have the analogous diagram

$$\begin{array}{ccc} \Phi_G((\Omega^1(G))^{\otimes_G k}) & \xrightarrow{\tau^k} & (\Lambda^1_G)^{\otimes k} \\ {\scriptstyle\iota^k}\uparrow & & \uparrow{\scriptstyle\gamma^k} \\ \Phi_M((\Omega^1(M))^{\otimes_M k}) & \xrightarrow[\sigma^k]{} & V_M^{\otimes k}, \end{array} \tag{4.4}$$

where γ^k is the unique map for which the diagram is commutative. Explicitly, the action of γ^k is given by

$$\begin{aligned} \gamma^k(\overline{m^1} \otimes \cdots \otimes \overline{m^k}) &= \tau^k \circ \iota^k \circ (\sigma^k)^{-1}(\overline{m^1} \otimes \cdots \otimes \overline{m^k}) = \tau^k(\overline{\mathrm{d}m^1 \otimes_G \cdots \otimes_G \mathrm{d}m^k}), \\ &= \overline{m^1 m^2_{(1)} \cdots m^k_{(1)}} \otimes \cdots \otimes \overline{(m^{k-1}_{(k-2)})^+ m^k_{(k-2)}} \otimes \overline{(m^k_{(k-1)})^+}. \end{aligned}$$

Thus, unless $\Omega^1(G)$ is an object in ${}^G_G\mathcal{M}_0$, we have no guarantee that γ^k is equal to $\widehat{\iota}^{\,\otimes k}$. With these maps and diagrams in hand we are now ready to give a sufficient criteria for $i^{\otimes k}$ to be an embedding:

Lemma 4.1.4 *If $\Omega^1(M)$ is a finite dimensional calculus, then γ^k is an embedding, and hence ι^k and $i^{\otimes k}$ are embeddings.*

Proof. If the image of γ^k could be shown equal to $\widehat{\iota}^{\,\otimes k}(V_M^{\otimes k})$, then, since we are assuming $\Omega^1(M)$ to be finite dimensional, it would follow that γ^k was an isomorphism. As a first step towards establishing this, we note that $i(\Omega^1(M))$ is well-defined as an object in ${}^G_M\mathcal{M}_0$, and so, we can identify $\Phi_M(i(\Omega^1(M))^{\otimes_M k})$ and $\Phi_M(i(\Omega^1(M)))^{\otimes k}$, giving us the isomorphism

$$\sigma^k : \Phi_M(i(\Omega^1(M))^{\otimes_M k}) \to \widehat{\iota}^{\,\otimes k}(V_M^{\otimes k}).$$

Combining this fact with the commutative diagram in (4.4), gives us the new diagram

$$\begin{tikzcd}
\Phi_G(\Omega^1(G))^{\otimes_G k}) \arrow[r, "\tau^k"] & (\Lambda^1_G)^{\otimes k} \\
\Phi_M((i(\Omega^1(M))^{\otimes_M k}) \arrow[u, "\mathrm{proj}"] \arrow[r, "\sigma^k"'] & \widehat{\iota}^{\,\otimes k}(V_M^{\otimes k}), \arrow[u, "\gamma'^k"']
\end{tikzcd} \tag{4.5}$$

where proj is the canonical projection, and γ'^k is defined so as to make the diagram commutative. Now for an arbitrary element $\overline{m^1} \otimes \cdots \otimes \overline{m^k}$ in $\widehat{\iota}^{\,\otimes k}(V_M^{\otimes k})$, it follows from condition 2 of the framing calculus definition that

$$\overline{m^1 S(m^2_{(1)})} \otimes \overline{m^2_{(2)} S(m^3_{(2)})} \otimes \cdots \otimes \overline{m^{k-1}_{(2)} S(m^k_{(1)})} \otimes \overline{m^k_{(2)}} \in \widehat{\iota}^{\,\otimes k}(V_M^{\otimes k}).$$

Let us look at the image of this element under γ'^k, for the first few values of k: For $k = 2$, we have

$$\gamma'^2(\overline{m^1 S(m^2_{(1)})} \otimes \overline{m^2_{(2)}}) = \overline{m^1 S(m^2_{(1)}) m^2_{(2)}} \otimes \overline{(m^2_{(3)})^+} = \overline{m^1} \otimes \overline{m^2}.$$

For $k = 3$, we have

$$\begin{aligned}
& \gamma'^3(\overline{m^1 S(m^2_{(1)})} \otimes \overline{m^2_{(2)} S(m^3_{(1)}} \otimes \overline{m^3_{(2)}}) \\
= \ & \overline{m^1 S(m^2_{(1)}) m^2_{(2)} S(m^3_{(2)}) m^3_{(3)}} \otimes \overline{(m^2_{(3)} S(m^3_{(1)}))^+ m^3_{(4)}} \otimes \overline{(m^3_{(5)})^+}
\end{aligned}$$

$$
\begin{aligned}
&= \overline{m^1} \otimes \overline{(m^2 S(m^3_{(1)}))^+ m^3_{(2)}} \otimes \overline{(m^3_{(3)})^+} \\
&= \overline{m^1} \otimes \overline{m^2 S(m^3_{(1)}) m^3_{(2)}} \otimes \overline{(m^3_{(3)})^+} \\
&= \overline{m^1} \otimes \overline{m^2} \otimes \overline{(m^3)^+} = \overline{m^1} \otimes \overline{m^2} \otimes \overline{m^3}.
\end{aligned}
$$

Continuing in this manner for subsequent values of k, it becomes easy to see that in general

$$
\gamma'^k(\overline{m^1 S(m^2_{(1)})} \otimes \overline{m^2_{(2)} S(m^3_{(1)})} \otimes \cdots \otimes \overline{m^{k-1}_{(2)} m^k_{(1)} \otimes m^k_{(2)}}) = \overline{m^1} \otimes \overline{m^2} \cdots \otimes \overline{m^k}.
$$

Hence, $\widehat{\iota}^{\,\otimes k}(V_M^{\otimes k})$ is mapped surjectively onto itself by γ'^k, immediately implying that γ^k and i^k are embeddings. $\square$

As a direct consequence we get the following corollary:

Corollary 4.1.5 *The pair* $(V_M^{\otimes k}, \tau^k \circ \iota^k)$ *(or equivalently the pair* $(V_M^{\otimes k}, \gamma^k \circ \sigma^k)$*) is a framing for* $(\Omega^1(M))^{\otimes_M k}$*.*

We note that, if $\Omega^1(G)$ is an object in ${}^G_G\mathcal{M}_0$, then $\gamma^k = \widehat{\iota}^{\,\otimes k}$ and the two framings $(V_M^{\otimes k}, \tau^k \circ \iota^k)$ and (V_M, σ^k) are equal.

4.2 Framing the Maximal Prolongation

Let $\Omega^1(G)$ be a covariant first-order differential calculus over a Hopf algebra G, with corresponding submodule $N_G \subseteq \Omega^1_u(G)$, and ideal $I_G \subseteq G^+$. Since

$$
\mathrm{d}(N_G) = \{\mathrm{d}(S(v_{(1)})) \otimes_G \mathrm{d}v_{(2)} \,|\, v \in I_G\},
$$

we must have that

$$
\begin{aligned}
\tau^2(\Phi_G(\mathrm{d}(N_G))) &= \{\overline{S(v_{(1)})^+ v_{(2)}} \otimes \overline{(v_{(3)})^+} \,|\, v \in I_G\} \\
&= \{\overline{\varepsilon(v_{(1)})} \otimes \overline{(v_{(2)})^+} - \overline{v_{(1)}} \otimes \overline{(v_{(2)})^+} \,|\, v \in I_G\} \\
&= \{\overline{v_{(1)}} \otimes \overline{v_{(2)}} \,|\, v \in I_G\}.
\end{aligned}
$$

This result, usually referred to as the Maurer–Cartan formula [79, 37], allows one to give an explicit description of the higher forms for the maximal prolongation of $\Omega^1(G)$. In this section we will build on the earlier work of the chapter to construct an analogous result for calculi over quantum homogeneous spaces $M = G^H$, which are objects in ${}^G_M\mathcal{M}_0$. Throughout all calculi are assumed to be finite dimensional.

4.2.1 A Direct Approach

Let $\Omega^1(M)$ be a left-covariant first-order differential calculus over a quantum homogeneous space M, and let N_M be the corresponding sub-bimodule of the universal calculus over M. If we denote $I^k_M := \sigma^k(\Phi_M(\langle \mathrm{d}N_M\rangle_k))$, for $k \geq 2$, then it is clear from (2.5) that σ^k descends to an isomorphism

$$\sigma^k : \Phi_M(\Omega^k(M)) \to \sigma^k(\Phi_M((\Omega^1(M))^{\otimes_M k})/\sigma^k(\Phi_M(\langle \mathrm{d}N_M\rangle_k)) = V_M^{\otimes k}/I^k_M =: V^k_M.$$

In order for this isomorphism to be of use to us, we will need to find a convenient description of I^k_M. The following lemma brings us some way towards this goal.

Lemma 4.2.1 *For a left-covariant first-order differential calculus $\Omega^1(M)$, which is an object in ${}^G_M\mathcal{M}_0$, we have*

$$I^2_M = \{\sum_i \overline{m_i^+} \otimes \overline{n_i^+} \mid \sum_i m^i \mathrm{d}n^i \in N_M\}, \tag{4.6}$$

or equivalently that

$$I^2_M = \{\sum_i \overline{(f^i S(v^i_{(1)}))^+} \otimes \overline{(v_{(2)})^+} \mid \sum_i f^i \otimes v^i \in (G \otimes I_M)^H\}. \tag{4.7}$$

Moreover, for $k \geq 3$, we have $I^k_M = \bigoplus_{a+b=k-2} V_M^{\otimes a} \otimes I^2_M \otimes V_M^{\otimes b}$.

Proof. It follows immediately from the properties of the total derivative d, and the construction of the maximal prolongation, that

$$\Phi_M(\mathrm{d}(N_M)) = \{\sum_i \overline{\mathrm{d}m^i \otimes_M \mathrm{d}n^i} \mid \sum_i m^i \mathrm{d}n^i \in N_M\}. \tag{4.8}$$

Operating on (4.8) by σ^2 then gives us (4.6). One derives (4.7) from (2.11) in the same way.

For $k \geq 3$, the construction of the maximal prolongation tells us that

$$\langle \mathrm{d}(N_M)\rangle_k = \bigoplus_{a+b=k-2} (\Omega^1(M))^{\otimes_M a} \otimes_M \mathrm{d}(N_M) \otimes_M (\Omega^1(M))^{\otimes_M b}.$$

The fact that $\Omega^1(M)$ is an object in ${}^G_M\mathcal{M}_0$, and that $\mathrm{d}(N_M)$ is a sub-object of $\Omega^1(M)$ in ${}^G_M\mathcal{M}_M$, easily implies that $\Phi_M(\mathrm{d}(N_M))$ is an object in $\mathcal{M}^H_0$. This in turn tells us that $\mathrm{d}(N_M)$ is an object in ${}^G_M\mathcal{M}_0$. Thus, since Φ_M restricts to a monoidal functor on ${}^G_M\mathcal{M}_0$, we have

$$\Phi_M(\langle \mathrm{d}N_M\rangle_k) = \bigoplus_{a+b=k-2} (\Phi_M(\Omega^1(M)))^{\otimes a} \otimes \Phi_M(\mathrm{d}N_M) \otimes (\Phi_M(\Omega^1(M)))^{\otimes b}.$$

Operating on this by σ^k gives us the required expression for I^k_M. $\square$

4.2.2 Framing Calculi and the Maximal Prolongation

While Lemma 4.2.1 gives an explicit description of the ideal I^k_M, it requires a complete description of the generating relations of the calculus $\Omega^1(M)$ before one can begin calculating. This is more or less the approach followed in [28], and it leads to the type of heavily technical calculations that we are trying to avoid. Instead, in this section we will show that one can use a framing calculus to find a simple description of I^k_M in terms of any generating set of I_M.

Theorem 4.2.2 *Let $\Omega^1(G)$ be a framing calculus for $\Omega^1(M)$, with Λ^1_G its space of left-invariant one forms. We have the equality*

$$\widehat{\iota}^{\,\otimes 2}(I^2_M) = \mathrm{span}_{\mathbf{C}}\{\overline{S(z_{(1)})} \otimes \overline{(z_{(2)})^+} \,|\, z \in \mathrm{Gen}(I_M)\} \subseteq (\Lambda^1_G)^{\otimes 2},$$

where $\mathrm{Gen}(I_M)$ *is any subset of I_M that generates it as a right M-module.*

Proof. In the first part of the proof we establish the identity

$$\iota^2(\Phi(\mathrm{d}N_M)) = \{\overline{\mathrm{d}(S(z_{(1)})) \otimes_G \mathrm{d}(z_{(2)})} \,|\, z \in I_M\}.$$

We begin with the inclusion $\iota^2(\Phi(\mathrm{d}N_M)) \subseteq \{\overline{\mathrm{d}(S(z_{(1)})) \otimes_G \mathrm{d}(z_{(2)})} \,|\, z \in I_M\}$: It is clear from (2.11) that we have

$$i^{\otimes 2}(\mathrm{d}N_M) = \{\sum_i \mathrm{d}(g^i S(v^i_{(1)})) \otimes_G \mathrm{d}v^i_{(2)} \,|\, \sum g^i \otimes v^i \in (G \otimes I_M)^H\}.$$

For each i, since $S(v^i_{(1)})\mathrm{d}(v^i_{(2)}) = 0$ in $\Omega^1(G)$, it holds in $(\Omega^1(G))^{\otimes_G 2}$ that

$$\begin{aligned}\mathrm{d}(g^i S(v^i_{(1)})) \otimes_G \mathrm{d}v^i_{(2)} &= \mathrm{d}(g^i)S(v^i_{(1)}) \otimes_G \mathrm{d}v^i_{(2)} + g^i\mathrm{d}S(v^i_{(1)}) \otimes_G \mathrm{d}v^i_{(2)} \\ &= \mathrm{d}g^i \otimes_G S(v^i_{(1)})\mathrm{d}v^i_{(2)} + g^i\mathrm{d}S(v^i_{(1)}) \otimes_G \mathrm{d}v^i_{(2)} \\ &= g^i\mathrm{d}S(v^i_{(1)}) \otimes_G \mathrm{d}v^i_{(2)}.\end{aligned}$$

Thus, we have that

$$i^{\otimes 2}(\mathrm{d}N_M) = \{\sum_i g^i \mathrm{d}S(v^i_{(1)}) \otimes_G \mathrm{d}v^i_{(2)} \mid \sum_i g^i \otimes v^i \in (G \otimes I_M)^H\}.$$

This in turn implies that

$$\iota^2(\Phi_M(\mathrm{d}N_M)) = \{\sum_i \varepsilon(g^i)\overline{\mathrm{d}(S(v^i_{(1)})) \otimes_G \mathrm{d}v^i_{(2)}} \mid \sum_i g^i \otimes v^i \in (G \otimes I_M)^H\}. \quad (4.9)$$

From which it is clear that

$$\iota^2(\Phi_M(\mathrm{d}(N_M))) \subseteq \{\overline{\mathrm{d}(S(v_{(1)})) \otimes_G \mathrm{d}v_{(2)}} \mid v \in I_M\},$$

giving us the required inclusion.

We now turn to the opposite inclusion of $\{\overline{\mathrm{d}(S(v_{(1)})) \otimes_G \mathrm{d}(v_{(2)})} \mid v \in I_M\}$ in $\iota^2(\Phi_M(\mathrm{d}N_M))$: From Takeuchi's theorem we have that the image of $(G \otimes I_M)^H$ under $\mathrm{frame}^{\perp}_M$ is equal to I_M. In other words, for any $z \in I_M$, we have an element $\sum_i g^i \otimes v^i$ contained in $(G \otimes I_M)^H$ such that $z = \sum_i \varepsilon(g^i)v^i$. This gives us that

$$\mathrm{d}S(z_{(1)}) \otimes_G \mathrm{d}z_{(2)} = \sum_i \varepsilon(g^i)\mathrm{d}S(v^i_{(1)}) \otimes_G \mathrm{d}v^i_{(2)}.$$

Since (4.9) tells us that $\sum_i \varepsilon(g^i)\overline{\mathrm{d}S(v^i_{(1)}) \otimes \mathrm{d}v^i_{(2)}}$is an element of $\iota^2(\Phi_M(\mathrm{d}(N_M))$, we must have $\overline{\mathrm{d}S(z_{(1)}) \otimes \mathrm{d}z_{(2)}}$ contained in $\iota^2(\Phi(\mathrm{d}(N_M))$. This gives us the required opposite inclusion, and hence the required equality.

Let us now move onto the second part of the proof where we find the image of $\iota^2(\Phi_M(\mathrm{d}N_M))$ under τ^2:

$$\begin{aligned}
\tau^2 \circ \iota^2(\Phi_M(\mathrm{d}N_M))) &= \{\tau^2(\overline{\mathrm{d}S(z_{(1)})) \otimes_G \mathrm{d}(z_{(2)})}) \mid z \in I_M\} \\
&= \{\overline{(S(z_{(1)}))^+ z_{(2)}} \otimes \overline{(z_{(3)})^+}\} \mid z \in I_M\} \\
&= \{\overline{(\varepsilon(z_{(1)}) - z_{(1)})} \otimes \overline{(z_{(2)})^+} \mid z \in I_M\} \\
&= \{\overline{z_{(1)}} \otimes \overline{z_{(2)}} - \overline{1} \otimes \overline{z} - \overline{z} \otimes \overline{1}\} \mid z \in I_M\} \\
&= \{\overline{z_{(1)}} \otimes \overline{z_{(2)}} \mid z \in I_M\}
\end{aligned}$$

For any $m \in M^+$, the fact that V_M is an object in $\mathcal{M}^H_0$ means that

$$\begin{aligned}
\overline{(zm)_{(1)}} \otimes \overline{(zm)_{(2)}} &= \overline{z_{(1)}m_{(1)}} \otimes \overline{z_{(2)}m_{(2)}} = \overline{z_{(1)}m_{(1)}} \otimes \overline{z_{(2)}\varepsilon(m_{(2)})} \\
&= \overline{z_{(1)}m} \otimes \overline{z_{(2)}} = \varepsilon(m)\overline{z_{(1)}} \otimes \overline{z_{(2)}}.
\end{aligned}$$

Hence, for any generating subset $\mathrm{Gen}(I_M)$ of I_M, we have

$$\tau^2 \circ \iota^2(\Phi_M(\mathrm{d}N_M)) = \mathrm{span}_{\mathbf{C}}\{\overline{z_{(1)}} \otimes \overline{z_{(2)}} \mid z \in \mathrm{Gen}(I_M)\}.$$

We begin the final part of the proof by noting that, for any $z \in I_M$,

$$\overline{S(z_{(1)})^+} \otimes \overline{(z_{(2)})^+} = \overline{S(z_{(1)})} \otimes \overline{(z_{(2)})^+} - 1 \otimes \overline{z} = \overline{S(z_{(1)})} \otimes \overline{(z_{(2)})^+}$$

Thus, since $\gamma^2 = \gamma'^2 \circ \widehat{\iota}^{\,\otimes 2}$ (where we recall that γ'^2 is defined in the commutative diagram (4.5)), the theorem would follow if we could show that γ'^2 acted on $\mathrm{span}_{\mathbf{C}}\{\overline{(S(z_{(1)}))^+} \otimes \overline{(z_{(2)})^+} \mid z \in \mathrm{Gen}(I_M)\}$ to give $\mathrm{span}_{\mathbf{C}}\{\overline{z_{(1)}} \otimes \overline{z_{(2)}} \mid z \in \mathrm{Gen}(I_M)\}$. But this follows directly from the calculation

$$\begin{aligned}\gamma'^2(\overline{S(z_{(1)})^+} \otimes \overline{(z_{(2)})^+}) &= \tau^2(\overline{\mathrm{d}(S(z_{(1)})) \otimes_G \mathrm{d}(z_{(2)})}) = \overline{S(z_{(1)})^+ z_{(2)}} \otimes \overline{z^+_{(3)}} \\ &= \overline{\varepsilon(z_{(1)}) - z_{(1)}} \otimes \overline{z^+_{(2)}} = -\overline{z_{(1)}} \otimes \overline{z_{(2)}} + 1 \otimes \overline{z} \\ &= -\overline{z_{(1)}} \otimes \overline{z_{(2)}}.\end{aligned}$$

□

4.3 Framing the Maximal Prolongation of the Heckenberger–Kolb Calculus

In this section we will make two applications of the general theory developed earlier in this chapter. First we take the calculus $\Omega^1_q(SU_N)$ as a framing calculus for $\Omega^1_q(\mathbf{C}P^{N-1})$, and use it to explicitly describe the maximal prolongation of $\Omega^1_q(\mathbf{C}P^{N-1})$. Secondly, we take the famous three-dimensional Woronowicz calculus $\Gamma^1_q(SU_2)$ as a framing calculus for $\Omega^1_q(SU_2)$, and use it to describe the maximal prolongation of $\Omega^1_q(\mathbf{CP}^1)$. We see that these two descriptions for the maximal prolongation of $\Omega^1_q(SU_2)$ agree, as of course they should.

4.3.1 The Calculus $\Omega^1_q(SU_N)$ as a Framing Calculus for the Heckenberger–Kolb Calculus $\Omega^1_q(\mathbf{C}P^{N-1})$

From the relations given in (3.17), (3.18), and (3.19), it is clear that $\mathbf{C}_q[\mathbf{C}P^{N-1}]$ acts on $V_{\mathbf{C}P^{N-1}}$ according to

$$e^{\pm}_i \triangleleft z_{11} = e^{\pm}_i, \qquad e^{\pm}_i \triangleleft z_{ij} = 0, \qquad ((i,j) \neq (1,1)). \tag{4.10}$$

Thus, $\Omega^1(\mathbf{C}P^{N-1})$ is an object in ${}^G_M\mathcal{M}_0$, and we have a well-defined framing/Users/johnmccarthy/Lib CCC/NCCS-QHS/Ar Eagla/Ar Eagla na hEagla/Arxiv.v2.tex.pdf $(V^{\otimes k}_{\mathbf{C}P^{N-1}}, \sigma^k)$ for the k^{th}-tensor power over $\mathbf{C}_q[\mathbf{C}P^{N-1}]$ of $\Omega^1(\mathbf{C}P^{N-1})$. As one might expect, the calculus $\Omega^1_q(SU_N)$ introduced in Chapter 2 is a framing calculus for $\Omega^1_q(\mathbf{C}P^{N-1})$. To see this we first recall that $\Omega^1_q(SU_N)$ restricts to $\Omega^1_q(\mathbf{C}P^{N-1})$ on $\mathbf{C}_q[\mathbf{C}P^{N-1}]$. Moreover, the right actions given in Lemma 3.2.3 show that $V_{\mathbf{C}P^{N-1}}$ is a right submodule of $\Lambda^1_{SU_N}$. This means that we can use Theorem 4.2.2 to calculate the maximal prolongation of $\Omega^1_q(\mathbf{C}P^{N-1})$:

Theorem 4.3.1 *The subspace $I^2_{\mathbf{C}P^{N-1}}$ is spanned by the elements*

$$e^-_i \otimes e^+_j + q e^+_j \otimes e^-_i, \quad e^+_i \otimes e^-_i + q^{-2} e^-_i \otimes e^+_i - q^{2i-1}\nu \sum_{a=i+1}^{N-1} q^{-2a} e^-_a \otimes e^+_a, \tag{4.11}$$

$$e^-_i \otimes e^-_h + q e^-_h \otimes e^-_i, \qquad e^+_i \otimes e^+_h + q^{-1} e^+_h \otimes e^+_i, \qquad e^+_i \otimes e^+_i, \qquad e^-_i \otimes e^-_i, \tag{4.12}$$

for $h, i, j = 1, \ldots, N-1$, $i \neq j$, and $h < i$. Hence, $V^k_{\mathbf{C}P^{N-1}}$ is a $\binom{2(N-1)}{k}$-dimensional vector space, with a basis given by

$$\{e^+_{i_1} \wedge \cdots \wedge e^+_{i_m} \wedge e^-_{j_1} \wedge \cdots \wedge e^-_{j_m} \,|\, i_1 < \cdots < i_m;\, j_1 < \cdots < j_m\}.$$

***Proof*.** Beginning with the generators of the form z_{ij}, for $i, j \geq 2$, we see that

$$\begin{aligned}
\overline{S((z_{ij})_{(1)})} \otimes \overline{((z_{ij})_{(2)})^+} &= \sum_{a,b=1}^{N} \overline{S(u^i_a S(u^b_j))} \otimes \overline{(u^a_1 S(u^1_b))^+} \\
&= \sum_{a,b=1}^{N} \overline{S^2(u^b_j) S(u^i_a)} \otimes \overline{(u^a_1 S(u^1_b))^+} \\
&= \sum_{a,b=1}^{N} q^{2(b-j)} \overline{u^b_j S(u^i_a)} \otimes \overline{(u^a_1 S(u^1_b))^+}.
\end{aligned}$$

For the case $i \neq j$, Lemma 3.2.3 tells us that the summand $\overline{u^b_j S(u^i_a)} \otimes \overline{(u^a_1 S(u^1_b))^+}$ is non-zero only if $a = 1, b = j$ or $a = i, b = 1$. Thus, we must have

$$\begin{aligned}
\overline{S((z_{ij})_{(1)})} \otimes \overline{((z_{ij})_{(2)})^+} &= \overline{u^j_j S(u^i_1)} \otimes \overline{u^1_1 S(u^1_j)} + q^{2(1-j)} \overline{u^1_j S(u^i_i)} \otimes \overline{u^i_1 S(u^1_1)} \\
&= q^{2-\frac{4}{N}} \overline{S(u^i_1)} \otimes \overline{S(u^1_j)} + q^{\frac{4}{N}+1-2j} \overline{u^1_j} \otimes \overline{u^i_1} \\
&= q^{\frac{4}{N}+2-2j} \overline{u^i_1} \otimes \overline{u^1_j} + q^{\frac{4}{N}+1-2j} \overline{u^1_j} \otimes \overline{u^i_1} \\
&= (q^{\frac{4}{N}+2-2j})(e^+_{i-1} \otimes e^-_{j-1} + q^{-1} e^-_{j-1} \otimes e^+_{i-i}).
\end{aligned}$$

This gives us the first relation in (4.11).

For the case $i = j$, Lemma 3.2.3 tells us that the summand $\overline{u^b_i S(u^i_a)} \otimes \overline{(u^a_1 S(u^1_b))^+}$ is non-zero only if $a = 1, b = i$ or $a \geq i, b = 1$. Thus, we must have that $\overline{S((z_{ii})_{(1)})} \otimes \overline{((z_{ii})_{(2)})^+}$ is equal to

$$\overline{u^i_i S(u^i_1)} \otimes \overline{u^1_1 S(u^1_i)} + q^{2(1-i)} (\overline{u^1_i S(u^i_i)} \otimes \overline{u^i_1 S(u^1_1)} + \sum_{a=i+1}^{N} \overline{u^1_i S(u^i_a)} \otimes \overline{u^a_1 S(u^1_1)}).$$

This is easily seen to be equal to

$$q^{2-\frac{4}{N}} \overline{S(u^i_1)} \otimes \overline{S(u^1_i)} + q^{2(1-i)} (q^{\frac{4}{N}-2} \overline{u^1_i} \otimes \overline{u^i_1} - q^{\frac{4}{N}+2i-1} \nu \sum_{a=i+1}^{N} q^{-2a} \overline{u^1_a} \otimes \overline{u^a_1}),$$

which reduces to

$$q^{2+\frac{4}{N}-2i} \overline{u^i_1} \otimes \overline{u^1_i} + q^{\frac{4}{N}+2(1-i)} (q^{-2} \overline{u^1_i} \otimes \overline{u^i_1} - q^{2i-1} \nu \sum_{a=i+1}^{N} q^{-2a} \overline{u^1_a} \otimes \overline{u^a_1}),$$

giving us finally that

$$q^{\frac{4}{N}+2(1-i)} (e^+_{i-1} \otimes e^-_{i-1} + q^{-2} e^-_{i-1} \otimes e^+_{i-1} - q^{2i-1} \nu \sum_{a=i+1}^{N} q^{-2a} e^-_{a-1} \otimes e^+_{a-1}).$$

This gives us the second relation in (4.11).

We now come to the generators of the form $z_{i1} z_{j1}$, for $j \neq 1$, and calculate

$$\begin{aligned}
\overline{S((z_{i1} z_{j1})_{(1)})} \otimes \overline{((z_{i1} z_{j1})_{(2)})^+} &= \overline{S((u^i_1 S(u^1_1) u^j_1 S(u^1_1))_{(1)})} \otimes \overline{((u^i_1 S(u^1_1) u^j_1 S(u^1_1))_{(2)})^+} \\
&= \sum_{a,b,c,d=1}^{N} \overline{S(u^i_a S(u^b_1) u^j_c S(u^d_1))} \otimes \overline{(u^a_1 S(u^1_b) u^c_1 S(u^1_d))^+} \\
&= \sum_{a,b,c,d=1}^{N} q^{2(d+b-2)} \overline{u^d_1 S(u^j_c) u^b_1 S(u^i_a)} \otimes \overline{(u^a_1 S(u^1_b) u^c_1 S(u^1_d))^+}.
\end{aligned}$$

The module relations given in Lemma (3.2.3) imply that at most one element of $\{a, c\}$ is equal to 1, and that at least three elements of $\{a, b, c, d\}$ are equal to 1. Thus, we must have $b, d = 1$, and $(a, c) \neq (1, 1)$, giving us

$$\begin{aligned}
\overline{S((z_{i1}z_{j1})_{(1)})} \otimes \overline{((z_{i1}z_{j1})_{(2)})^+} &= \sum_{(a,c)\neq(1,1)} \overline{u^1_1 S(u^j_c) u^1_1 S(u^i_a)} \otimes \overline{u^a_1 S(u^1_1) u^c_1 S(u^1_1)} \\
&= q(\sum_{a=2}^{N} \overline{S(u^j_1)S(u^i_a)} \otimes \overline{u^a_1 u^1_1} + \sum_{c=2}^{N} \overline{S(u^j_c)S(u^i_1)} \otimes \overline{u^1_1 u^c_1}) \\
&= q^{2-\frac{2}{N}}(\sum_{a=2}^{N} \overline{S(u^j_1)S(u^i_a)} \otimes \overline{u^a_1} + q\sum_{c=2}^{N} \overline{S(u^j_c)S(u^i_1)} \otimes \overline{u^c_1}) \\
&= q^{2-\frac{2}{N}}(\sum_{a=2}^{N} \overline{S(u^j_1)S(u^i_a)} \otimes \overline{u^a_1} + q\,\overline{S(u^j_j)S(u^i_1)} \otimes \overline{u^j_1}) \\
&= -q^{\frac{2}{N}+1}(\sum_{a=2}^{N} \overline{u^j_1 S(u^i_a)} \otimes \overline{u^a_1} + q^{\frac{2}{N}+1}\,\overline{u^i_1} \otimes \overline{u^j_1}).
\end{aligned}$$

For the case of $i = j$, we have

$$\overline{(S(z_{i1}z_{i1})_{(1)})} \otimes \overline{((z_{i1}z_{ie^+_i \otimes e^+_i +1})_{(2)})^+} = -q^{\frac{2}{N}+1}(\overline{u^i_1 S(u^i_i)} \otimes \overline{u^i_1} + q^{\frac{2}{N}+1}\,\overline{u^i_1} \otimes \overline{u^i_1}),$$

which is just a linear multiple of $e^+_{i-1} \otimes e^+_{i-1}$, giving us the fourth element in (4.12). While for the case $i < j$, we have

$$\begin{aligned}
\overline{S((z_{i1}z_{j1})_{(1)})} \otimes \overline{((z_{i1}z_{j1})_{(2)})^+} &= -q^{\frac{2}{N}+1}(\overline{u^j_1 S(u^i_i)} \otimes \overline{u^i_1} + \overline{u^j_1 S(u^i_j)} \otimes \overline{u^j_1} + q^{\frac{2}{N}+1}\,\overline{u^i_1} \otimes \overline{u^j_1}) \\
&= -q^{\frac{4}{N}+1}(\overline{u^j_1} \otimes \overline{u^i_1} - \nu\overline{u^i_1} \otimes \overline{u^j_1} + q\,\overline{u^i_1} \otimes \overline{u^j_1}) \\
&= -q^{\frac{4}{N}+1}(\overline{u^j_1} \otimes \overline{u^i_1} + q^{-1}\overline{u^i_1} \otimes \overline{u^j_1}) \\
&= -q^{\frac{4}{N}+1}(e^+_{j-1} \otimes e^+_{i-1} + q^{-1}e^+_{i-1} \otimes e^+_{j-1}),
\end{aligned}$$

which gives us the second element in (4.12). For $i > j$, it is easy to see from the quantum sphere relations in (3.4), that $z_{i1}z_{j1} = q^{-1}z_{j1}z_{i1}$, which means that we just get us the second element in (4.12) again.

Finally, we come to the $z_{1i}z_{1j}$, for $j \neq 1$. We calculate that

$$\begin{aligned}
\overline{S((z_{1i}z_{1j})_{(1)})} \otimes \overline{((z_{1i}z_{1j})_{(2)})^+} &= \overline{S((u_1^1 S(u_i^1) u_1^1 S(u_j^1))_{(1)})} \otimes \overline{((u_1^1 S(u_i^1) u_1^1 S(u_j^1))_{(2)})^+} \\
&= \sum_{a,b,c,d=1}^{N} \overline{S(u_a^1 S(u_i^b) u_c^1 S(u_j^d))} \otimes \overline{(u_1^a S(u_b^1) u_1^c S(u_d^1))^+} \\
&= \sum_{a,b,c,d=1}^{N} q^{2(b-i+d-j)} \overline{u_j^d S(u_c^1) u_i^b S(u_a^1)} \otimes \overline{(u_1^a S(u_b^1) u_1^c S(u_d^1))^+}.
\end{aligned}$$

Now for $\overline{u_j^d S(u_c^1) u_i^b S(u_a^1)} \otimes \overline{(u_1^a S(u_b^1) u_1^c S(u_d^1))^+}$ to be non-zero, the relations in Lemma 3.2.3 imply that at least three elements of $\{a, b, c, d\}$ must be equal to 1, while at most one element of $\{b, d\}$ must be equal to one, giving us that

$$\begin{aligned}
\overline{S((z_{1i}z_{1j})_{(1)})} \otimes \overline{((z_{1i}z_{1j})_{(2)})^+} &= \sum_{a,b,c,d=1} q^{2(b-i+d-j)} \overline{u_j^d S(u_1^1) u_i^b S(u_1^1)} \otimes \overline{u_1^1 S(u_b^1) u_1^1 S(u_d^1)} \\
&= \sum_{a,b,c,d=1}^{N} q^{2(b-i+d-j)+1} \overline{u_j^d S(u_1^1) u_i^b} \otimes \overline{S(u_b^1) u_1^1 S(u_d^1)}.
\end{aligned}$$

For the case $i = j$, this tells us that $\overline{S((z_{1i}z_{1j})_{(1)})} \otimes \overline{((z_{1i}z_{1j})_{(2)})^+}$ is equal to

$$q^{2(1-i)+1} \overline{u_i^1 S(u_1^1) u_i^i} \otimes \overline{S(u_i^1) u_1^1 S(u_1^1)} + q^{2(1-i)+1} \overline{u_i^i S(u_1^1) u_i^1} \otimes \overline{S(u_1^1) u_1^1 S(u_i^1)},$$

which is just a linear multiple of $e_i^+ \otimes e_i^+$, giving us the last relation in (4.12). For the case of $i < j$, we have $\overline{S((z_{1i}z_{1j})_{(1)})} \otimes \overline{((z_{1i}z_{1j})_{(2)})^+}$ equal to

$$q^{3-2j}\, \overline{u_j^1 S(u_1^1) u_i^i} \otimes \overline{S(u_i^1) u_1^1 S(u_1^1)} + q^{3-2i}\, \overline{u_j^j S(u_1^1) u_i^1} \otimes \overline{S(u_1^1) u_1^1 S(u_j^1)}.$$

This easily reduces to

$$q^{\frac{4}{N}+3-2(i+j)}\, \overline{u_j^1} \otimes \overline{u_i^1} + q^{\frac{4}{N}+2-2(i+j)}\, \overline{u_i^1} \otimes \overline{u_j^1},$$

which is just a linear multiple of

$$q\, e_{j-1}^+ \otimes e_{i-1}^+ + e_{i-1}^+ \otimes e_{j-1}^+.$$

This gives the first relation in (4.12). For $i > j$, it is easy to see from the quantum sphere relations in (3.4), that $z_{1i}z_{1j} = qz_{1j}z_{1i}$, which means that we just get the first element in (4.12) again. $\square$

4.3.2 The Woronowicz Calculus $\Gamma^1_q(SU_2)$ as a Framing Calculus

In this section we specialise to the case of $\mathbf{C}_q[\mathbf{C}P^1]$, and use the three-dimensional Woronowicz calculus $\Omega^1_q(SU_2)$ on $\mathbf{C}_q[SU_2]$, as discussed in Chapter 3, as a framing calculus for the Heckenberger–Kolb calculus. We do this firstly to demonstrate that there can exist more than one framing calculus for any given quantum homogeneous space calculus, and secondly to highlight the fact that the description produced is independent of the choice of framing calculus.

Let us recall the ideal I_{SU_2} corresponding to the Woronowicz calculus given in (3.23). As is very well known (see [79, 37] for details), the cotangent space $V_{\mathbf{C}_q[\mathbf{C}P^1]} := \mathbf{C}_q[\mathbf{C}P^1]^+/I_{SU_2}$ has a basis given by

$$e^+ := \overline{c}, \qquad\qquad e^0 := \overline{a-1}, \qquad\qquad e^+ := \overline{b}.$$

Moreover, from the description of I_{SU_2} given in (3.23), it is easy to see that the non-zero actions of the generators of $\mathbf{C}_2[SU_2]$ on e^+ and e^- are given by

$$e^\pm \triangleleft a = q^{-1}e^\pm, \qquad\qquad e^\pm \triangleleft d = qe^\pm. \tag{4.13}$$

It is also clear that $I_{\mathbf{C}P^1} = \langle b^2, bc, c^2\rangle$, the ideal corresponding to the Heckenberger–Kolb calculus, is contained in I_{SU_2}, giving us a well-defined map $V_{\mathbf{C}P^1} \to \Lambda^1_{SU_2}$. With respect to this map, we have that $\overline{ab} = e^-$, and $\overline{cd} = qe^+$, showing that the map is in fact an inclusion. Since it is clear from (4.13) that $V_{\mathbf{C}P^1}$ is a right $\mathbf{C}_q[SU_2]$-submodule of $\Lambda^1_{SU_2}$, we have that $\mathbf{C}_q[SU_2]$ is a framing calculus for $\Omega^1_q(\mathbf{C}P^1)$. We can now use Theorem 4.2.2 to find a framing for the maximal prolongation of $\Omega^1_q(\mathbf{C}P^1)$:

Lemma 4.3.2 *It holds that*

$$I^2_{\mathbf{C}P^{N-1}} = \mathrm{span}_{\mathbf{C}}\{e^+\otimes e^+,\, e^-\otimes e^-,\, e^+\otimes e^- + q^{-2}e^-\otimes e^+\}, \tag{4.14}$$

and hence that $V^2_{\mathbf{C}P^1} = \mathbf{C}e^+\otimes e^-$, *while* $V^k_{\mathbf{C}P^1} = \{0\}$, *for all* $k \geq 3$.

Proof. Take the generating set $\{b^2, bc, c^2\}$ for $I_{\mathbf{C}P^1}$. For b^2 we have that

$$\begin{aligned}\overline{S((b^2)_{(1)})}\otimes\overline{((b^2)_{(2)})^+} &= \overline{S(a^2)}\otimes\overline{b^2} + (1+q^{-2})\overline{S(ab)}\otimes\overline{bd} + \overline{S(b^2)}\otimes\overline{(d^2)^+}\\ &= (1+q^{-2})(-q^{-1}\overline{bd}\otimes\overline{bd}) = -(1+q^2)qe^-\otimes e^-.\end{aligned}$$

For c^2, we have that

$$\begin{aligned}\overline{S((c^2)_{(1)})} \otimes \overline{((c^2)_{(2)})^+} &= \overline{S(c^2)} \otimes \overline{(a^2)^+} + (1+q^{-2})\overline{S(cd)} \otimes \overline{ac} + \overline{S(d^2)} \otimes \overline{c^2} \\ &= (1+q^2)(-q\,\overline{ac} \otimes \overline{ac}) = -q^{-5}(1+q^2)e^+ \otimes e^+.\end{aligned}$$

Finally, for bc, we have that

$$\begin{aligned}\overline{S((bc)_{(1)})} \otimes \overline{(bc)_{(2)}} &= \overline{S(ac)} \otimes \overline{ba} + \overline{S(ad)} \otimes \overline{bc} + \overline{S(bc)} \otimes \overline{(da)^+} + \overline{S(bd)} \otimes \overline{dc} \\ &= -q\,\overline{cd} \otimes \overline{ba} - q^{-1}\overline{ab} \otimes \overline{dc} = -q\,\overline{c} \otimes \overline{b} - q^{-1}\overline{b} \otimes \overline{c} \\ &= -q(e^- \otimes e^+ + q^2 e^+ \otimes e^-).\end{aligned}$$

This gives the three elements in (4.14), along with the implied descriptions of the the higher forms. $\square$

Chapter 5

Covariant Complex Structures

In this chapter, which can be considered the central chapter of the thesis, we introduce complex structures and covariant complex structures. While such objects have been considered elsewhere [6, 34], this is the first presentation of a simple set of sufficient conditions for such structures to exist.

5.1 Almost Complex Structures

We begin this section by introducing our definition of an almost complex structure over a general algebra. We then specialise to the case where this algebra is a quantum homogeneous space, and give a simple set of necessary and sufficient conditions for such an almost complex structure to exist. Finally, we apply this general theory to the Heckenberger–Kolb calculus for the quantum projective spaces.

5.1.1 Almost Complex Structures for a Not Neccessarily Covariant Calculus

Let us first introduce the *wedge map* $\wedge$ for a total differential calculus $\Omega^\bullet(A)$, by defining

$$\wedge : \Omega^k(A) \otimes_A \Omega^l(A) \to \Omega^{k+l}(A), \qquad\qquad \omega \otimes_A \omega' \mapsto \omega \wedge \omega'.$$

Next, we introduce the central definition of the thesis:

Definition 5.1.1. An *almost complex structure* for a total $*$-differential calculus $\Omega^\bullet(A)$ over a $*$-algebra A, is an $\mathbf{N}_0^2$-algebra grading $\bigoplus_{(p,q)\in\mathbf{N}_0^2}\Omega^{(p,q)}$ for $\Omega^\bullet(A)$ such that, for all $(p,q)\in\mathbf{N}_0^2$:

1. $\Omega^k(A)=\bigoplus_{p+q=k}\Omega^{(p,q)}$;
2. the wedge map restricts to isomorphisms
$$\wedge:\Omega^{(p,0)}\otimes_A\Omega^{(0,q)}\to\Omega^{(p,q)}, \qquad \wedge:\Omega^{(0,q)}\otimes_A\Omega^{(p,0)}\to\Omega^{(p,q)}; \tag{5.1}$$
3. $*(\Omega^{(p,q)})=\Omega^{(q,p)}$.

We call an element of $\Omega^{(p,q)}$ a (p,q)*-form*.

Classically every decomposition of the cotangent bundle into two sub-bimodules extends to an almost complex structure. As the following proposition shows, things are more complicated in the noncommutative setting. The proof requires us to consider the unique $\mathbf{N}_0^2$-grading of the tensor algebra $\bigoplus_{k=0}^\infty(\Omega^1(A))^{\otimes_A k}$ of $\Omega^1(A)$ extending a bimodule decomposition $\Omega^1(A)=\Omega^{(1,0)}\oplus\Omega^{(0,1)}$. Explicitly, the decomposition $\Omega^{\otimes(\bullet,\bullet)}:=\bigoplus_{(p,q)\in\mathbf{N}_0^2}\Omega^{\otimes(p,q)}$ is defined by

$$\Omega^{\otimes(p,q)}:=\{w\in\Omega^{p+q}(A)\,|\,\pi(\omega)\in(\Omega^{(1,0)})^{\otimes_A p}\otimes_A(\Omega^{(0,1)})^{\otimes_A q},\ \text{for some } \pi\in S_{p+q}\},$$

where S_{p+q} is the permutation group on $p+q$ objects, acting $\mathbf{C}$-linearly on $\Omega^{p+q}(A)$ in the obvious way.

Theorem 5.1.2 *For $\Omega^1(A)$ a first-order differential calculus over an algebra A, and $\Omega^1(A)=\Omega^{(1,0)}\oplus\Omega^{(0,1)}$ a decomposition of $\Omega^1(A)$ into sub-bimodules, we have that:*

1. *the decomposition has at most one extension, satisfying condition (1), to an $\mathbf{N}_0^2$-grading of the maximal prolongation of $\Omega^1(A)$;*
2. *such an extension exists if, and only if, $\mathrm{d}(N)$ is homogeneous with respect to the decomposition*
$$(\Omega^1(A))^{\otimes_A 2}=\Omega^{\otimes(2,0)}\oplus\Omega^{\otimes(1,1)}\oplus\Omega^{(0,2)}. \tag{5.2}$$

3. *When this decomposition exists, the maps in condition 2 of the almost complex structure definition are isomorphisms if, and only if,* $\wedge$ *restricts to isomorphisms*

$$\wedge : \Omega^{(1,0)} \otimes_A \Omega^{(0,1)} \to \Omega^{(1,1)}, \qquad \wedge : \Omega^{(0,1)} \otimes_A \Omega^{(1,0)} \to \Omega^{(1,1)}; \tag{5.3}$$

4. *moreover, condition 3 holds if, and only if,* $*(\Omega^{(1,0)}) = \Omega^{(0,1)}$*, or equivalently if, and only if,* $*(\Omega^{(0,1)}) = \Omega^{(1,0)}$.

Proof. We begin by giving a sufficient condition for an $\mathbf{N}_0^2$-grading, extending the decomposition of $\Omega^1(A)$, to exist: For some $\omega \in \mathrm{d}(N)$, we denote the decomposition of ω with respect to (5.2) by $\omega := \omega_1 + \omega_2 + \omega_3$. By definition $\mathrm{d}(N)$ is homogeneous with respect to (5.2) if, for each ω, we have $\omega_1, \omega_2, \omega_3 \in \mathrm{d}(N)$. In this case, for any homogeneous elements ν, ν' in the tensor algebra of $\Omega^1(A)$, the decomposition of the element $\nu \otimes_A \omega \otimes_A \nu'$, with respect to $\Omega^{\otimes(\bullet,\bullet)}$, is given by

$$\nu \otimes_A \omega \otimes_A \nu' = \nu \otimes_A \omega_1 \otimes_A \nu' + \nu \otimes_A \omega_2 \otimes_A \nu' + \nu \otimes_A \omega_3 \otimes_A \nu'.$$

It is clear that $\nu \otimes_A \omega_i \otimes_A \nu' \in \langle \mathrm{d}(N) \rangle$, for $i = 1, 2, 3$. Now since every element of $\langle \mathrm{d}(N) \rangle$ is a sum of elements of the from $\nu \otimes \omega \otimes \nu'$, we see that homogeneity of $\mathrm{d}(N)$ with respect to (5.2), implies homogeneity of $\langle \mathrm{d}(N) \rangle$ with respect to $\Omega^{\otimes(\bullet,\bullet)}$. In this case, $\Omega^{\otimes(\bullet,\bullet)}$ clearly descends to a grading $\Omega^{(\bullet,\bullet)}$ on the maximal prolongation. Finally, we note that if $\mathrm{d}(N)$ is not homogeneous with respect to to (5.2), then clearly $\Omega^{\otimes(\bullet,\bullet)}$ cannot descend to a grading on the maximal prolongation.

We will now show that this grading is the only possible $\mathbf{N}_0^2$-grading on the maximal prolongation extending the decomposition of $\Omega^1(A)$: For another such distinct grading $\Gamma^{(\bullet,\bullet)}$ to exist, there would have to be an element $\omega \in \Omega^{\otimes(p,q)}$, for some $(p, q) \in \mathbf{N}_0^2$, such that the image of ω in $\Omega^\bullet(A)$ was not contained in $\Gamma^{(p,q)}$. Now it is clear from the definition of $\Omega^{\otimes(p,q)}$ that every element of $\Omega^{\otimes(p,q)}$ is of the form

$$\omega := \sum_{i=1} \omega_1^i \otimes \cdots \otimes \omega_{p+q}^i, \tag{5.4}$$

where each $\omega_1^i \otimes \cdots \otimes \omega_{p+q}^i$ has exactly p of its factors contained in $\Omega^{(1,0)}$, and q of its factors contained in $\Omega^{(0,1)}$. However, the general properties of a graded algebra imply that the image of such an element in $\Omega^\bullet(A)$ must be contained in $\Gamma^{(p,q)}$. Thus, we can conclude that there exists no other grading on the maximal prolongation extending the decomposition $\Omega^1(A)$. This gives us the first and second parts of the theorem.

Now we come to showing that when this $\mathbf{N}_0^2$-grading exists, condition 2 of the definition of an almost complex structure holds if, and only if, the maps in (5.3) are isomorphisms. Let us begin by establishing that surjectivity of the first map in (5.1) follows from surjectivity of the first map in (5.3): Let

$$\omega := \sum_i \omega_1^i \wedge \cdots \wedge \omega_{p+q}^i, \tag{5.5}$$

be a general element of $\Omega^{(p,q)}$, where, just as in (5.4), each $\omega_1^i \wedge \cdots \wedge \omega_{p+q}^i$ has exactly p of its factors contained in $\Omega^{(1,0)}$, and q of its factors contained in $\Omega^{(0,1)}$. If for each of these summands, there exists no pair of adjacent factors $\omega_k^i \wedge \omega_{k+1}^i$, for some $1 \leq k < p+q$, such that $\omega_k \in \Omega^{(0,1)}$, and $\omega_{k+1} \in \Omega^{(1,0)}$, then it is clear that ω is contained in the image of $\Omega^{(p,0)} \otimes_A \Omega^{(0,q)}$ under $\wedge$. If such an adjacent pair does exist, then since we are assuming the first map in (5.3) to be surjective, there must exist an element $\sum_j \nu_j \otimes_A \nu_j'$ in $\Omega^{(1,0)} \otimes_A \Omega^{(0,1)}$, such that

$$\sum_j \nu_j \wedge \nu_j' = \omega_i \wedge \omega_{i+1}.$$

If upon inserting this relation into ω we obtain a presentation of ω whose summands contain no other such pairs of adjacent factors, then it is clear that ω is contained in the image of $\Omega^{(p,0)} \otimes_A \Omega^{(0,q)}$ under $\wedge$. If such adjacent pairs do exist, then it is easy to see that by successive applications of this process, one will eventually arrive at a presentation of ω containing none. Thus, it is clear that ω is contained in the image of $\Omega^{(p,0)} \otimes_A \Omega^{(0,q)}$ under $\wedge$, which is to say that surjectivity of the first map in (5.1) follows from surjectivity of the first map in (5.3). That surjectivity of the second map in (5.1) follows from surjectivity of the second map in (5.3) is established in an exactly analogous manner.

We now move on to establishing injectivity. As a little thought will confirm, the first map of (5.1) would be seen to be injective if it could be shown that, for all $(p,q) \in \mathbf{N}_0^2$,

$$\langle \mathrm{d}(N) \rangle \cap (\Omega^{\otimes(p,0)} \otimes_A \Omega^{\otimes(0,q)}) = \langle \mathrm{d}N \rangle_{(p,0)} \otimes_A \Omega^{\otimes(0,q)} + \Omega^{\otimes(p,0)} \otimes_A \langle \mathrm{d}N \rangle_{(0,q)}, \tag{5.6}$$

where $\langle \mathrm{d}N \rangle_{(p,0)}$, and $\langle \mathrm{d}N \rangle_{(0,q)}$, are the $\otimes(p,0)$, and $\otimes(0,q)$, homogeneous components of $\langle \mathrm{d}N \rangle$ respectively. To see that this is so, consider the general element $\sum_i \nu_i \otimes \omega_i \otimes \nu'_i$ of $\langle \mathrm{d}(N) \rangle$, with each ν_i, ν'_i contained in the tensor algebra of $\Omega^1(A)$, and each $w_i \in \mathrm{d}(N)$. Since the first mapping in (5.3) is an isomorphism, it must hold that

$$\mathrm{d}(N) \cap (\Omega^{(1,0)} \otimes_A \Omega^{(0,1)}) = \{0\}.$$

This implies that $\sum_i \nu_i \otimes \omega_i \otimes \nu'_i$ is contained in $\Omega^{\otimes(p,0)} \otimes_A \Omega^{\otimes(0,q)}$ only if $\omega^i \in \Omega^{\otimes(2,0)}$, or $\omega^i \in \Omega^{\otimes(0,2)}$. It now follows that (5.6) holds, and hence that the first map of condition 3 is injective. That the second map of (5.1) is injective is established analogously. Thus, we have established the third part of the theorem.

We now come to the fourth and final part of the theorem. Note first that since the $*$-map is involutive, assuming $*(\Omega^{(1,0)}) = \Omega^{(0,1)}$ is clearly equivalent to assuming $*(\Omega^{(0,1)}) = \Omega^{(1,0)}$. Next we note that, for a general element ω in $\Omega^{(p,q)}$ as given in (5.5), the properties of graded $*$-algebra imply that

$$\omega^* := \sum_{i=1} (\omega_1^i \wedge \cdots \wedge \omega_{p+q}^i)^* = \sum_{i=1} (-1)^{\frac{(p+q)(p+q-1)}{2})} (\omega_{p+q}^i)^* \wedge \cdots \wedge (\omega_1^i)^*. \qquad (5.7)$$

Our two equivalent assumptions, and the properties of a graded algebra, now imply that ω^* must be contained in $\Omega^{(q,p)}$, giving us that $*(\Omega^{(p,q)}) \subseteq \Omega^{(q,p)}$. The opposite inclusion is established analogously, giving us the desired equality. $\square$

An interesting question to ask here is whether one can find a first-order differential calculus Ω^1 with a decomposition $\Omega^1 = \Omega^+ \oplus \Omega^-$ that does not extend to an $\mathbf{N}_0^2$-grading of the maximal prolongation of Ω^1. At present there is no obvious candidate for such a calculus.

Another interesting question to consider is that of almost complex structures on total calculi other than maximal prolongations. Recall that every total calculus extending $(\Omega^1(A), \mathrm{d})$ can be obtained as a quotient of the maximal prolongation by an ideal $I \subseteq \ker(\mathrm{d})$. It is not difficult to see that a decomposition of $\Omega^1(A) = \Omega^{(1,0)} \oplus \Omega^{(0,1)}$ is extendable to an almost complex structure on such a total calculus if, and only if, it is extendable to an almost complex structure on the maximal prolongation, and for which I is homogeneous with respect to the associated $\mathbf{N}_0^2$-grading. This gives us a classification of all almost complex structures over an algebra A. However, since at present we have no interesting examples of such structures, we will not pursue this observation here.

5.1.2 Covariant Almost Complex Structures

We say that an almost complex structure $\Omega^{(\bullet,\bullet)}$ for a quantum homogeneous space $M = G^H$ is *left-covariant* if we have

$$\Delta_L(\Omega^{(p,q)}) \subseteq G \otimes \Omega^{(p,q)}, \qquad\qquad \text{(for all } (p,q) \in \mathbf{N}^2).$$

As a little thought will confirm, an almost complex structure will be covariant if, and only if,

$$\Delta_L(\Omega^{(1,0)}) \subseteq G \otimes \Omega^{(1,0)}, \qquad\qquad \Delta_L(\Omega^{(0,1)}) \subseteq G \otimes \Omega^{(0,1)}.$$

For covariant almost complex structures we will of course have each $\Omega^{(p,q)}$ contained as an object in ${}^G_M\mathcal{M}_M$. For the special case that $\Omega^1(M)$ is an object in ${}^G_M\mathcal{M}_0$, we denote

$$V_M^{\otimes(p,q)} := \sigma^{p+q}(\Phi_M(\Omega^{\otimes(p,q)})).$$

Clearly, it follows from the definition of an almost complex structure that we have $V_M^k = \bigoplus_{p+q=k} V_M^{(p,q)}$. Another important fact is that since $\wedge$ is clearly a morphism in ${}^G_M\mathcal{M}_0$, we have a corresponding morphism $\Phi_M(\wedge)$ in $\mathcal{M}_0^H$. Moreover, since we have given Φ_M the structure of a monoidal functor, we can consider $\Phi_M(\wedge)$ as a morphism

$$\Phi_M(\wedge) : (\Phi_M(\Omega^1(M)))^{\otimes 2} \to \Phi_M(\Omega^2(M)).$$

We use this to define a new morphism

$$\wedge_\sigma := \sigma^2 \circ \Phi_M(\wedge) \circ (\sigma^2)^{-1} : V_M^{\otimes 2} \to V_M^2.$$

The following corollary shows that for covariant complex structures, we have a convenient reformulation of Theorem 5.1.2.

Corollary 5.1.3 *For a left-covariant first-order differential calculus $\Omega^1(M)$, with canonical framing (V_M, s), we have that:*

1. *Decompositions of $\Omega^1(M) = \Omega^{(1,0)} \oplus \Omega^{(0,1)}$ into left-covariant bimodules correspond to decompositions of $V_M = V_M^{(1,0)} \oplus V_M^{(0,1)}$ into right-covariant right comodules.*

2. *Such a decomposition extends to an $\mathbf{N}_0^2$-grading of the maximal prolongation of $\Omega^1(M)$ if, and only if, I_M^2 is homogeneous with respect to the decomposition*
$$V_M^{\otimes 2} = V_M^{\otimes(2,0)} \oplus V_M^{\otimes(1,1)} \oplus V_M^{\otimes(0,2)}. \tag{5.8}$$

3. *If $\Omega^1(M)$ is contained as an object in the category ${}^G_M\mathcal{M}_0$, then condition 3 of the almost complex structure definition is satisfied if, and only if, we have isomorphisms*
$$\wedge_\sigma : V_M^{(1,0)} \otimes V_M^{(0,1)} \to V_M^{(1,1)}, \qquad \wedge_\sigma : V_M^{(0,1)} \otimes V_M^{(1,0)} \to V_M^{(1,1)}. \tag{5.9}$$

Proof. Since Φ obeys (2.4), every covariant bimodule decompositions of $\Omega^1(M)$ induces a covariant right module decomposition of $\Phi(\Omega^1(M)$. Conversely, since Ψ obeys (2.4), every covariant right module decomposition of $\Phi(\Omega^1(M))$ induces a covariant bimodule decomposition of $\Omega^1(M)$. This gives an equivalence between decompositions of $\Omega^1(M)$ and decompositions of $\Phi(\Omega^1(M))$. The first part of the proof now follows from the fact that s is an isomorphism in ${}^G_M\mathcal{M}_M$, and σ is an isomorphism in $\mathcal{M}_M^H$.

Turning now to the second part of the proof, we see that, using an analogous argument to the one above, one can establish an equivalence between decompositions of $(\Omega^1(M))^{\otimes_M 2}$ and decompositions of $V_M^{\otimes 2}$. Moreover, the decomposition of $(\Omega^1(M))$ given in (5.2), corresponds to the decomposition of $V_M^{\otimes 2}$ given in (5.8). Properties (2.4) and (2.5) of the functors Ψ and Φ now imply that $\mathrm{d}(N)$ is homogeneous with respect to (5.2) if, and only if, I_M^2 is homogeneous with respect to (5.2). Part 2 of the corollary now follows from part 2 of Theorem 5.1.2.

For the last part of the proof, we note that since σ^2 is an isomorphism, the functorial properties of Φ_M imply that the maps in (5.3) are isomorphisms if, and only if, the maps in (5.9) are isomorphisms. Part 3 of the corollary now follows from part 3 of Theorem 5.1.2. □

Finally, we come to finding an easily verifiable reformulation of the $*$-condition. As for first order differential $*$-calculi, the fact that the $*$-map is not a bimodule map means that Takeuchi's equivalence will be of no use here. However, just as for first order differential $*$-calculi, there exists a convenient direct reformulation.

Proposition 5.1.4 *Let $\Omega^1(M)$ be a first order differential $*$-calculus in ${}^G_M\mathcal{M}_M$, and let $\Omega^{(\bullet,\bullet)}$ be an $\mathbf{N}_0^2$-grading for its maximal prolongation $\Omega^\bullet(M)$ satisfying the first condition of an almost complex structure. If $\Omega^1(G)$ is a framing calculus for $\Omega^1(M)$, with respect to which*

$$\Omega^{(1,0)}G \subseteq G\Omega^{(1,0)}, \qquad \Omega^{(0,1)}G \subseteq G\Omega^{(0,1)}, \tag{5.10}$$

then we have $*(\Omega^{(p,q)}) = \Omega^{(q,p)}$ if, and only if,

$$\{\overline{S(m)^*} \,|\, \overline{m} \in V^{(1,0)}\} = V^{(0,1)}. \tag{5.11}$$

Proof. From (2.12), we see that if (5.10) and (5.11) holds, then

$$(G \otimes V^{(1,0)})^* = G \otimes V^{(0,1)}.$$

From this it easily follows that $*(\Omega^{(1,0)}) = \Omega^{(0,1)}$. Part 4 of Theorem 5.1.2 now implies that $*(\Omega^{(p,q)}) = \Omega^{(q,p)}$.

Conversely, let us assume that there exists a $\overline{v} \in V^{(1,0)}$ such that $\overline{S(v)^*} \notin V^{(0,1)}$. With respect to the choice of framing calculus, we have

$$(s^{-1}(1 \otimes \overline{v}))^* = s^{-1}((1 \otimes \overline{v})^*) = s^{-1}(1 \otimes \overline{S(v)^*}) \notin s^{-1}(G \otimes V^{(0,1)}) = G\Omega^{(0,1)}. \tag{5.12}$$

However, we must also have $s^{-1}(1 \otimes v) = \sum_i a_i\omega_i$, for some $a_i \in G, \omega_i \in \Omega^{(1,0)}$. If we had an almost complex structure, then ω_i^* would be contained in $\Omega^{(0,1)}$, for all i, giving us that

$$\sum_i (a_i\omega_i)^* = \sum_i (\omega_i)^* a_i^* \in \Omega^{(1,0)}G \subseteq G\Omega^{(0,1)}.$$

Since this contradicts (5.12), we are forced to conclude that, for some ω_i, we have $\omega_i^* \notin \Omega^{(0,1)}$, and consequently, that we do not have an almost complex structure. □

5.2 Integrability and Complex Structures

In this section we will show how the classical notion of integrability transfers directly to the noncommutative setting. Mirroring the classical picture, we demonstrate how integrability of an almost complex structure implies the existence of a quantum Dolbeault double complex. Moreover, with respect to a choice of framing calculus, we give a simple set of sufficient criteria for a complex structure to be integrable.

5.2.1 Integrability for a General Almost Complex Structure

In this subsection we discuss integrability for complex structures without the assumption of covariance. We begin with two lemmas whose proofs carry over directly from the classical case. (It should be noted that these results have already appeared in [6], where one can find a more comprehensive treatment of integrability in the noncommutative setting.)

Lemma 5.2.1 *If $\bigoplus_{(p,q)\in\mathbf{N}^2}\Omega^{(p,q)}$ is an almost-complex structure for a total calculus $\Omega^\bullet(A)$ over an algebra A, then the following two conditions are equivalent:*

1. $\mathrm{d}(\Omega^{(1,0)}) \subseteq \Omega^{(2,0)} \oplus \Omega^{(1,1)}$,
2. $\mathrm{d}(\Omega^{(0,1)}) \subseteq \Omega^{(1,1)} \oplus \Omega^{(0,2)}$.

Proof. For any $\omega \in \Omega^{(0,1)}$, the properties of an almost complex structure imply that $\omega^* \in \Omega^{(1,0)}$. Thus if we assume 1, it must hold that $d\omega^* \in \Omega^{(2,0)} \oplus \Omega^{(1,1)}$. This in turn implies that $\mathrm{d}\omega = (\mathrm{d}\omega^*)^* \in \Omega^{(1,1)} \oplus \Omega^{(0,2)}$, showing us that 2 holds. The proof in other other direction is entirely analogous. □

If these conditions hold for an almost-complex structure, then we say that it is *integrable*. We will usually call an integrable almost-complex structure a *complex structure*. (To see how the formulation of integrability that we have generalised is equivalent to the more standard formulation, see [30]).

With a view to exploring some of the consequences of integrability, we now introduce two new operators: For $\bigoplus_{(p,q)\in\mathbf{N}^2}\Omega^{(p,q)}$ an almost complex structure, we define ∂, and $\overline{\partial}$, to be the unique order $(1,0)$, and $(0,1)$ respectively, homogeneous operators for which

$$\partial|_{\Omega^{(p,q)}} = \mathrm{proj}_{\Omega^{(p+1,q)}} \circ \mathrm{d}, \qquad \overline{\partial}|_{\Omega^{(p,q)}} = \mathrm{proj}_{\Omega^{(p,q+1)}} \circ \mathrm{d},$$

where $\mathrm{proj}_{\Omega^{(p+1,q)}}$, and $\mathrm{proj}_{\Omega^{(p,q+1)}}$, are the projections onto $\Omega^{(p+1,q)}$, and $\Omega^{(p,q+1)}$ respectively.

Lemma 5.2.2 *If an almost complex structure $\bigoplus_{(p,q)\in \mathbf{N}_0^2} \Omega^{(p,q)}$ is integrable, then*

1. $\mathrm{d} = \partial + \overline{\partial}$;
2. $(\bigoplus_{(p,q)\in\mathbf{N}^2} \Omega^{(p,q)}, \partial, \overline{\partial})$ *is a double complex;*
3. $\partial(a^*) = (\overline{\partial}a)^*$, *and* $\overline{\partial}(a^*) = (\partial a)^*$, *for all* $a \in A$;
4. *both* ∂ *and* $\overline{\partial}$ *satisfy the graded Liebniz rule.*

Proof. We begin by proving that $\mathrm{d} = \partial + \overline{\partial}$: Since $\Omega^{(p,q)}$ is spanned by products of p elements of $\Omega^{(1,0)}$, and q elements of $\Omega^{(0,1)}$, it follows from the Liebniz rule and the assumption of integrability that

$$\mathrm{d}\omega \in \Omega^{(p+1,q)} \oplus \Omega^{(p,q+1)}, \qquad (\text{for all } \omega \in \Omega^{(p,q)}).$$

Thus, we must have that $d = \overline{\partial} + \partial$.

Let us now move on to the second part of the proof: Since $\mathrm{d}^2 = 0$, we have

$$0 = \mathrm{d}^2 = (\partial + \overline{\partial}) \circ (\partial + \overline{\partial}) = \partial^2 + (\overline{\partial} \circ \partial + \partial \circ \overline{\partial}) + \overline{\partial}^2.$$

For any $\omega \in \Omega^k(M)$, it is easy to see that any non-zero images of ω under ∂^2, $\overline{\partial}\partial + \partial\overline{\partial}$, and $\overline{\partial}^2$, would lie in complementary subspaces of $\Omega^{k+2}(M)$. Thus, it must hold that

$$\partial^2 = 0, \qquad \partial \circ \overline{\partial} = -\overline{\partial} \circ \partial, \qquad \overline{\partial}^2 = 0,$$

showing that we have a double complex.

For the third part of the proof, we first note that since $\mathrm{d}(a^*) = (\mathrm{d}a)^*$, we have

$$\partial(a^*) + \overline{\partial}(a^*) = (\partial a)^* + (\overline{\partial} a)^*.$$

Now $\partial(a^*)$ and $(\overline{\partial} a)^*$ both lie in $\Omega^{(1,0)}$, while $\overline{\partial}(a^*)$ and $(\partial a)^*$ both lie in $\Omega^{(0,1)}$. Since these are again complementary subspaces of $\Omega^1(M)$, we must have $\partial(a^*) = (\overline{\partial} a)^*$, and $\overline{\partial}(a^*) = (\partial a)^*$.

The fourth part of the lemma is an analogously consequence of the Liebniz rule of d. $\square$

Thus we see that integrability in the noncommutative setting has many of the same properties as classical integrability. Inspired by the classical case we call the double complex $(\bigoplus_{(p,q)\in \mathbf{N}^2} \Omega^{(p,q)}, \partial, \overline{\partial})$ the *quantum Dolbeault double complex* of an integrable complex structure.

5.2.2 Integrability for a Covariant Complex Structure

Directly verifying that an almost complex structure is integrable can lead to quite involved calculations. So we would like to use the assumption of covariance to find a simple set of sufficient criteria (analogous to our method for verifying the existence of an almost-complex structure given in the previous section). This will require us to make a choice of linear complement $V_M^\perp$ to $\widehat{\iota}(V_M)$ in Λ^1_G. With respect to this choice of complement, we will write

$$(V_M^{\otimes 2})^\perp := (\widehat{\iota}(V_M) \otimes V_M^\perp) \oplus (V_M^\perp \otimes \widehat{\iota}(V_M)) \oplus (V_M^\perp)^{\otimes 2},$$

for the corresponding linear complement to $\widehat{\iota}(V_M)^{\otimes 2}$ in $(\Lambda^1_G)^{\otimes 2}$. Moreover, we will say that a subset $\{m^j\}_j \subseteq M^+$ *descends* to a spanning set of $V^{(1,0)}$, if we have $\mathrm{span}_{\mathbf{C}}\{\overline{m^j}\}_j = V^{(1,0)}$. We state the result in terms of the holomorphic cotangent space, however, as is clear from the proof, an exactly analogous result holds for the anti-holomorphic cotangent space $V^{(0,1)}$.

Proposition 5.2.3 *Let $\Omega^{(\bullet,\bullet)}$ be a covariant almost-complex structure over $M = G^H$ a quantum homogeneous space, $V_M^\perp$ a choice of linear complement to $\widehat{\iota}(V_M)$ in Λ^1_G, and $\{m^j\}_j$ a subset of M^+ that descends to a spanning set of $V^{(1,0)}$. It holds that $\Omega^{(\bullet,\bullet)}$ is integrable if, for all $m^j \in \{m^j\}_j$, and $v \in \Lambda^1_G$, we have that*

$$(v \triangleleft S(m^j_{(1)})) \otimes \overline{(m^j_{(2)})^+} \in \widehat{\iota}(V_M^{\otimes(2,0)}) \oplus \widehat{\iota}(V_M^{\otimes(1,1)}) \oplus (V_M^{\otimes 2})^\perp. \tag{5.13}$$

Proof. It is clear that $\mathrm{d}(\Omega^{(1,0)})$ is contained in $\Omega^{(2,0)} \oplus \Omega^{(1,1)}$ if we have

$$\Phi_M(\mathrm{d}(\Omega^{(1,0)})) \subseteq \Phi_M(\Omega^{(2,0)}) \oplus \Phi_M(\Omega^{(1,1)}). \tag{5.14}$$

We will establish the proposition by demonstrating that this happens when (5.13) holds: From (2.11) it is clear that

$$\Omega^{(1,0)} = s^{-1}((G \otimes V^{(1,0)})^H) = \{\sum_j f^j S(m^j_{(1)})\mathrm{d}m^j_{(2)} \mid \sum_j f^j \otimes \overline{m^j} \in (G \otimes V^{(1,0)})^H\}.$$

This in turn implies that

$$\Phi_M(\mathrm{d}(\Omega^{(1,0)})) = \{\sum_j \overline{\mathrm{d}(f^j S(m^j_{(1)})) \otimes \mathrm{d}(m^j_{(2)})} \mid \sum_j f^j \otimes \overline{m^j} \in (G \otimes V^{(1,0)})^H\},$$

giving us the equality

$$\sigma^2(\Phi_M(\mathrm{d}(\Omega^{(1,0)})) = \{\sum_j \overline{(f^j S(m^j_{(1)}))^+} \otimes \overline{(m^j_{(2)})^+} \mid \sum_j f^j \otimes \overline{m^j} \in (G \otimes V^{(1,0)})^H\}.$$

Now as a little thought will confirm, this means that (5.14) holds if, for each such $\sum_j (f^j S(m^j_{(1)}))^+ \otimes (m^j_{(2)})^+$, we have that its image in $(\Lambda^1_G)^{\otimes 2}$ under $\widehat{\iota}^{\otimes 2}$ satisfies

$$\sum_j \overline{(f^j S(m^j_{(1)}))^+} \otimes \overline{(m^j_{(2)})^+} \ \in \ \widehat{\iota}(V_M^{\otimes(2,0)}) \oplus \widehat{\iota}(V_M^{\otimes(1,1)}).$$

As a little more thought will confirm, this will hold if, for each j, we have

$$\overline{(f^j S(m^j_{(1)}))^+} \otimes \overline{(m^j_{(2)})^+} \ \in \ \widehat{\iota}^{\otimes 2}(V_M^{\otimes(2,0)}) \oplus \widehat{\iota}^{\otimes 2}(V_M^{\otimes(1,1)}) \oplus (V_M^{\otimes 2})^{\perp}.$$

That (5.14) is implied by the requirements of the proposition now follows from the identity

$$\overline{(f^j S(m^j_{(1)}))^+} \otimes \overline{(m^j_{(2)})^+} = \overline{(f^j S(m^j_{(1)}))} \otimes \overline{(m^j_{(2)})^+}.$$

□

5.2.3 Integrability and an Alternative Construction of the Maximal Prolongation

For an almost complex structure $\Omega^{(\bullet,\bullet)}$, the pairs $(\Omega^{(1,0)}, \partial)$ and $(\Omega^{(0,1)}, \overline{\partial})$ are each first order differential calculi. Thus, one can consider their maximal prolongations.

Let us denote the k-forms of the maximal prolongation of $\Omega^{(1,0)}$ by $(\Omega^{(1,0)})^k$, and the k-forms of the maximal prolongation of $\Omega^{(1,0)}$ by $(\Omega^{(0,1)})^k$. It is natural to ask when we have

$$(\Omega^{(1,0)})^k = \Omega^{(k,0)}, \qquad\qquad (\Omega^{(0,1)})^k = \Omega^{(0,k)}. \tag{5.15}$$

The following result tells us that this condition is in fact equivalent to integrability.

Lemma 5.2.4 *For an almost complex structure $\Omega^{(\bullet,\bullet)}$, the equalities in (5.15) are equivalent to each other, and to integrability.*

Proof. Let $\{\omega_i^-\}_i$, be a subset of $\Omega^1_u(M)$, such that $\mathrm{span}_{\mathbf{C}}\{\omega_i^-\} = \Omega^{(0,1)}$, where by abuse of notation we have used the same symbol for ω_i^-, as for its coset in $\Omega^1(M)$. If N_M is the sub-bimodule of $\Omega^1_u(M)$ corresponding to $\Omega^1(M)$, then it is clear that the sub-bimodule of $\Omega^1_u(M)$ corresponding to $(\Omega^{(1,0)}, \partial)$ is given by

$$N_M^+ := N_M \,+\, \mathrm{span}_{\mathbf{C}}\{\omega_i^-\}_i.$$

Now from the definition of the maximal prolongation, we have that

$$(\Omega^{(1,0)})^k = (\Omega^{(1,0)})^{\otimes_M k} / \left\langle \partial N_M^+ \right\rangle_k,$$

while Theorem 5.1.2 tells us that

$$\Omega^{(k,0)} = \Omega^{\otimes (k,0)} / \left\langle \mathrm{d}N_M \right\rangle_{(k,0)} = (\Omega^{(1,0)})^{\otimes_M k} / \left\langle \mathrm{d}N_M \right\rangle_{(k,0)}.$$

It is easy to see that

$$\left\langle \partial N_M^+ \right\rangle_k = \bigoplus_{a+b=k-1} (\Omega^{(1,0)})^{\otimes_M a} \otimes_M (\partial N_M^+) \otimes_M (\Omega^{(1,0)})^{\otimes_M b}$$

and

$$\left\langle \mathrm{d}N_M \right\rangle_{(0,k)} = \bigoplus_{a+b=k-1} (\Omega^{(1,0)})^{\otimes_M a} \otimes_M \left\langle \mathrm{d}N_M \right\rangle_{(1,0)} \otimes_M (\Omega^{(1,0)})^{\otimes_M b}.$$

Thus, the first equality in (5.15) is equivalent to $\partial N_M^+ = \left\langle \mathrm{d}N_M \right\rangle_{(1,0)}$. As a little careful thought will confirm, we have

$$\left\langle \mathrm{d}N_M \right\rangle_{(1,0)} = \partial N_M,$$

Thus, the first equality in (5.15) amounts to having $\partial\omega_i^- = 0$, for all i. But this holds if, and only if, our almost complex structure is integrable.

That the second equality in (5.15) is equivalent to integrability is proved in exactly the same way.

□

5.3 The Heckenberger–Kolb Calculus

In this section we apply the machinery developed in this chapter to the Heckenberger–Kolb calculus. We show that the total calculus $\Omega^\bullet(\mathbf{C}P^{N-1})$ has an almost complex, and that this almost complex structure is integrable. We also demonstrate that the holomorphic top form, and the anti-holomorphic top form, are isomorphic to the line bundles $\mathcal{E}_N$, and $\mathcal{E}_{-N}$, respectively. Finally, we show explicitly how our construction of the de Rham complex relates to that of Heckenberger and Kolb.

5.3.1 An Almost Complex Structure

Consider the canonical linear decomposition $V_{\mathbf{C}P^{N-1}} = V^{(1,0)} \oplus V^{(0,1)}$, where

$$V^{(1,0)} := \mathrm{span}_{\mathbf{C}}\{e_i^+ \,|\, i = 2, \ldots, N\}, \qquad V^{(0,1)} := \mathrm{span}_{\mathbf{C}}\{e_i^- \,|\, i = 2, \ldots, N\}.$$

The relations in (4.10) tell us that this is a decomposition into right submodules. In fact, as the following lemma shows, it induces an almost complex structure on $\Omega_q^\bullet(\mathbf{C}P^{N-1})$.

Lemma 5.3.1 *The decomposition $V_{\mathbf{C}P^{N-1}} = V^{(1,0)} \oplus V^{(0,1)}$ is a decomposition into right-covariant right submodules, and the corresponding decomposition of $\Omega_q^1(\mathbf{C}P^{N-1})$ extends to an almost complex structure for $\Omega_q^\bullet(\mathbf{C}P^{N-1})$.*

Proof. That the decomposition is right-covariant is clear from the following calculations: For z_{i1} we have

$$\Delta_{\mathbf{C}P^{N-1}}(z_{i1}) = (\mathrm{id} \otimes S) \circ (\mathrm{id} \otimes \alpha_N)(\sum_{a,b=1}^{N} u_1^a S(u_b^1) \otimes u_a^i S(u_1^b))$$

$$= (\mathrm{id} \otimes S)(\sum_{a=2}^{N} u^a_1 S(u^1_1) \otimes \alpha_N(u^i_a S(u^1_1))$$
$$= \sum_{a,b=2}^{N} z_{a1} \otimes S(\alpha_N(u^i_a)) \det{}_N,$$

while for z_{1i} we have

$$\Delta_{\mathbf{C}P^{N-1}}(z_{1i}) = (\mathrm{id} \otimes S) \circ (\mathrm{id} \otimes \alpha_N)(\sum_{a,b=1}^{N} u^a_1 S(u^1_b) \otimes u^1_a S(u^b_i))$$
$$= (\mathrm{id} \otimes S)(\sum_{b=2}^{N} u^1_1 S(u^b_1) \otimes \alpha_N(u^1_1 S(u^b_1))$$
$$= \sum_{a,b=2}^{N} z_{a1} \otimes S(\alpha_N(S(u^b_1))) \det{}_N^{-1}.$$

That $I^2_{\mathbf{C}P^{N-1}}$ is homogeneous with respect to the decomposition from part 2 of Corollary 5.1.3, follows directly from Proposition 4.3.1, as does the fact the maps in (5.9) are isomorphisms. That the first two conditions of an almost complex structure are satisfied now follows directly from Proposition 4.3.1.

We now come to the third condition of an almost complex structure. That (5.10) holds follows directly from the module relations given in (3.17). Moreover, (5.11) follows from the fact that for $i = 2, \ldots, N$, we have

$$\overline{S(u^i_1)^*} = \overline{S^{-1}((u^i_1)^*)} = \overline{S^{-1} \circ S(u^1_i)} = \overline{u^1_i} = e^-_{i-1} \in V^{(0,1)},$$

where we have used the standard Hopf $*$-algebra identity $* \circ S = S^{-1} \circ *$. Proposition 5.1.4 now directly implies that the third, and final, condition of an almost complex structure holds. □

Classically, we have that $\Omega^{(N-1,0)}$ is isomorphic to $\mathcal{E}_{-N}$, and that $\Omega^{(0,N-1)}$ is isomorphic to $\mathcal{E}_{-N}$. As a consequence, it also holds that $\mathbf{C}P^{N-1}$ is orientable, which is to say that $\Omega^{(N-1,N-1)}$ is isomorphic to $\mathbf{C}_q[\mathbf{C}P^{N-1}]$. These are very important properties and one would naturally hope that they generalise to the quantum setting. The following proposition tells us that this is indeed the case.

Proposition 5.3.2 *It holds that:* $\Omega^{(N-1,0)} \simeq \mathcal{E}_{-N}$, $\Omega^{(0,N-1)} \simeq \mathcal{E}_N$, *and, as a direct consequence, that*

$$\Omega^{(N-1,N-1)}(\mathbf{C}P^{N-1}) \simeq \mathbf{C}_q[\mathbf{C}P^{N-1}].$$

Proof. We begin with the anti-holomorphic forms, and the action of Δ_M^{N-1} on $V^{(N-1,0)} \simeq \mathbf{C}e_1^- \wedge \cdots e_{N-1}^-$:

$$\begin{aligned}\Delta_M^{N-1}(e_1^- \wedge \cdots \wedge e_{N-1}^-) &= \sum_{l=1}^{N-1}\sum_{k_l=1}^{N-1} e_{k_1}^- \wedge \cdots \wedge e_{k_{N-1}}^- \otimes S^2(u_1^{k_1}) \cdots S^2(u_{N-1}^{k_{N-1}}) \det{}^{N-1} \\ &= \sum_{l=1}^{N-1}\sum_{k_l=1}^{N-1} e_{k_1}^- \wedge \cdots \wedge e_{k_{N-1}}^- \otimes S^2(u_1^{k_1} \cdots u_{N-1}^{k_{N-1}}) \det{}^{N-1}.\end{aligned}$$

Since any summand with a repeated basis element in the first tensor factor will be zero, we must have

$$\Delta_M^{N-1}(e_1^- \wedge \cdots e_{N-1}^-) = \sum_{\pi \in S_{N-1}} e_{\pi(1)}^- \wedge \cdots \wedge e_{\pi(N-1)}^- \otimes S^2(u_1^{\pi(1)} \cdots u_{N-1}^{\pi(N-1)}) \det{}^{N-1}.$$

Now as a little thought will confirm $e_{\pi(1)}^- \wedge \cdots \wedge e_{\pi(N-1)}^- = (-q)^{\mathrm{sgn}(\pi)} e_1^- \wedge \cdots \wedge e_{N-1}^-$. Hence, since

$$\sum_{\pi \in S_{N-1}} (-q)^{\mathrm{sgn}(\pi)} u_1^{\pi(1)} \cdots u_{N-1}^{\pi(N-1)} = \det,$$

we must have

$$\begin{aligned}\Delta_M^{N-1}(e_1^- \wedge \cdots e_{N-1}^-) &= e_1^- \wedge \cdots \wedge e_{N-1}^- \otimes S^2(\det) \det{}^{N-1} \\ &= e_1^- \wedge \cdots \wedge e_{N-1}^- \otimes \det{}^N.\end{aligned}$$

Thus, $\Omega^{(N-1,0)}(\mathbf{C}P^{N-1}) \simeq \mathcal{E}_N$ as one would have hoped.

Let us now turn to the anti-holomorphic forms. We begin with the action of Δ_M^{N-1} on $V^{(0,N-1)} \simeq \mathbf{C}e_1^+ \wedge \cdots e_{N-1}^+$:

$$\begin{aligned}\Delta_M^{N-1}(e_1^+ \wedge \cdots \wedge e_{N-1}^+) &= \sum_{l=1}^{N-1}\sum_{k_l=1}^{N-1} e_{k_1}^+ \wedge \cdots \wedge e_{k_{1-N}}^+ \otimes S(u_{k_1}^1) \cdots S(u_{k_{N-1}}^{N-1}) \det{}^{1-N} \\ &= \sum_{l=1}^{N-1}\sum_{k_l=1}^{N-1} e_{k_1}^+ \wedge \cdots \wedge e_{k_{N-1}}^+ \otimes S(u_{k_{N-1}}^{N-1} \cdots u_{k_1}^1) \det{}^{1-N}.\end{aligned}$$

Now since any summand with a repeated basis element in the first tensor factor will be zero, we must have

$$\Delta_M^{N-1}(e_1^+ \wedge \cdots e_{N-1}^+) = \sum_{\pi \in S_{N-1}} e_{\pi(1)}^+ \wedge \cdots \wedge e_{\pi(1-N)}^+ \otimes S(u_{\pi(N-1)}^{N-1} \cdots u_{\pi(1)}^1) \det^{N-1}.$$

As a little thought will confirm $e_{\pi(1)}^+ \wedge \cdots \wedge e_{\pi(N-1)}^+ = (-q)^{-\mathrm{sgn}(\pi)} e_1^+ \wedge \cdots \wedge e_{N-1}^+$, for any $\pi \in S_{N-1}$. Thus, since it is clear that

$$\sum_{\pi \in S_{N-1}} (-q)^{-\mathrm{sgn}(\pi)} u_{\pi(N-1)}^{N-1} \cdots u_{\pi(1)}^1 = \det,$$

we must have

$$\begin{aligned}\Delta_M^{N-1}(e_1^+ \wedge \cdots e_{N-1}^+) &= e_1^+ \wedge \cdots \wedge e_{N-1}^+ \otimes S(\det) \det^{1-N} \\ &= e_1^+ \wedge \cdots \wedge e_{N-1}^+ \otimes \det^{-N}.\end{aligned}$$

Thus, as we would have hoped, it holds that $\Omega^{(0,N-1)}(\mathbf{C}P^{N-1}) \simeq \mathcal{E}_{-N}$. An exactly analogous proof can be used to establish that $\Omega^{(N-1,0)}(\mathbf{C}P^{N-1})$ is isomorphic to $\mathcal{E}_N$.

Finally, we note that the fact that $\Omega^{(N-1,N-1)}(\mathbf{C}P^{N-1})$ is isomorphic to $\mathbf{C}_q[\mathbf{C}P^{N-1}]$ is a direct consequence of these two results. $\square$

5.3.2 Integrability of the Almost-Complex Structure

Finally, we come to establishing integrability for this almost complex structure using Proposition 5.2.3.

Proposition 5.3.3 *The almost-complex structure $\Omega_q^{(\bullet,\bullet)}(\mathbf{C}P^{N-1})$ is integrable.*

Proof. We will establish the proposition by showing that (5.13) holds for the calculus, where as a choice of complement to $\widehat{\iota}^{\otimes 2}(V_{\mathbf{C}P^{N-1}}^{\otimes 2})$ in $\Lambda_{SU_N}^1$, we take $V_{\mathbf{C}P^{N-1}}^{\perp} = \mathbf{C}e^0$.

For $\overline{z_{i1}} = \overline{u^i_1 S(u^1_1)} \in V^{(1,0)}$, with $i = 2, \ldots, N$, we have, for $k = 1, \ldots, N-1$, that

$$\begin{aligned}(e^+_k \triangleleft S((z_{i1})_{(1)})) \otimes \overline{((z_{i1})_{(2)})^+} &= \sum_{a,b=1}^{N} (e^+_k \triangleleft S(u^i_a S(u^b_1))) \otimes \overline{(u^a_1 S(u^1_b))^+} \\ &= \sum_{a,b=1}^{N} q^{2(b-1)} (e^+_k \triangleleft (u^b_1 S(u^i_a))) \otimes \overline{(u^a_1 S(u^1_b))^+} \\ &= \sum_{a=1}^{N} (e^+_k \triangleleft (u^1_1 S(u^i_a))) \otimes \overline{(u^a_1 S(u^1_1))^+} \\ &= \sum_{a=2}^{N} (e^+_k \triangleleft S(u^i_a)) \otimes \overline{u^a_1} \\ &= (e^+_k \triangleleft S(u^i_i)) \otimes \overline{u^i_1} + (e^+_k \triangleleft S(u^i_{k+1})) \otimes \overline{u^{k+1}_1}.\end{aligned}$$

From the relations given in Section 3.3, it is clear that $(e^+_k \triangleleft S(u^i_i)) \otimes \overline{u^i_1}$ is equal to a linear multiple of $e^+_k \otimes e^+_{i-1}$, while $(e^+_k \triangleleft S(u^i_{k+1})) \otimes \overline{u^{k+1}_1}$ is equal to a linear multiple of $e^+_{i-1} \otimes e^+_k$. Thus, we have that $(e^+_k \triangleleft S((z_{i1})_{(1)}) \otimes \overline{((z_{i1})_{(2)})^+}$ is contained in $\widehat{\iota}(V^{(2,0)})$.

Moreover, for e^-_k, with $k = 1, \ldots, N-1$, we have

$$\begin{aligned}(e^-_k \triangleleft S((z_{i1})_{(1)})) \otimes \overline{((z_{i1})_{(2)})^+} &= \sum_{a,b=1}^{N} (e^-_k \triangleleft S(u^i_a S(u^b_1))) \otimes \overline{(u^a_1 S(u^1_b))^+} \\ &= \sum_{a,b=1}^{N} q^{2(b-1)} (e^-_k \triangleleft (u^b_1 S(u^i_a))) \otimes \overline{(u^a_1 S(u^1_b))^+} \\ &= \sum_{a=1}^{N} (e^-_k \triangleleft (u^1_1 S(u^i_a))) \otimes \overline{(u^a_1 S(u^1_1))^+} \\ &= \sum_{a=2}^{N} (e^-_k \triangleleft S(u^i_a)) \otimes \overline{u^a_1} \\ &= (e^-_k \triangleleft S(u^i_i)) \otimes \overline{u^i_1} + \delta_{ki} \sum_{a=i+1}^{N} (e^+_k \triangleleft S(u^k_a)) \otimes \overline{u^a_1}.\end{aligned}$$

From the relations given in Lemma 3.2.3, it is clear that $(e^-_k \triangleleft S(u^i_i)) \otimes \overline{u^i_1}$ is equal to a linear multiple of $e^-_k \otimes e^+_{i-1}$, while $(e^-_k \triangleleft S(u^k_a)) \otimes \overline{u^a_1}$ is equal to a linear multiple of $e^-_{a-1} \otimes e^+_{a-1}$. Thus, we have that $(e^+_k \triangleleft S((z_{i1})_{(1)}) \otimes \overline{((z_{i1})_{(2)})^+}$ is contained in $\widehat{\iota}(V^{(1,1)})$.

Finally, we come to e^0. To simplify our calculations we identify $\mathbf{C}_q[SU_N]/\ker(Q)$ and $\Lambda^1_{SU_N}$, just as in Lemma 3.2.3. With respect to this identification, (3.21) implies that $e^0 = \overline{u^1_1 - 1} = (q^{2-\frac{2}{N}} - 1)\overline{1}$. Denoting for sake of presentation $\mu := (q^{2-\frac{2}{N}} - 1)$, we get that

$$\begin{aligned} e^0 \triangleleft S((z_{i1})_{(1)})) \otimes \overline{((z_{i1})_{(2)})^+} &= \mu\, \overline{S((z_{i1})_{(1)})} \otimes \overline{((z_{i1})_{(2)})^+} \\ &= \mu\, (\overline{(S((z_{i1})_{(1)})^+} \otimes \overline{((z_{i1})_{(2)})^+} - \overline{1} \otimes \overline{z_{i1}}) \\ &= \mu\, \overline{(S((z_{i1})_{(1)})^+} \otimes \overline{((z_{i1})_{(2)})^+} - e^0 \otimes \overline{z_{i1}}, \end{aligned}$$

which, by our earlier calculations, is contained in $\widehat{\iota}(V^{\otimes(2,0)}) \oplus \widehat{\iota}(V^{\otimes(1,1)}) \oplus \mathbf{C}e^0$.

Hence, the requirements of (5.13) are satisfied, and our almost-complex structure is in fact a complex structure. □

5.3.3 Relationship with the Heckenberger–Kolb Construction

We will finish this chapter by explicitly demonstrating how the q-deformed de Rham complex we have constructed for the quantum projective spaces relates to the q-deformed de Rham complex constructed by Heckenberger and Kolb in [27, 28]. We begin by recalling the celebrated classification result, for the special case of the quantum projective spaces. Just before, however, we will need to recall a simple definition: A left-covariant first-order calculus over an algebra A is called *irreducible* if it does not possess any non-trivial quotients by a left-covariant A-bimodule. We now state the result:

Theorem 5.3.4 *[27] There exist exactly two non-isomorphic finite-dimensional irreducible left-covariant first-order differential calculi over quantum projective $(N-1)$-space. Each has dimension $N-1$.*

Since both $\Omega^{(1,0)}_q$ and $\Omega^{(0,1)}_q$ have dimension $N-1$, they must both be irreducible (since otherwise there would exist an irreducible left-covariant calculus of dimension strictly less than $N-1$ in contradiction of the theorem). Moreover, it is easy to see that $\Omega^{(1,0)}_q$ and $\Omega^{(0,1)}_q$ correspond to different ideals of $\mathbf{C}_q[\mathbf{C}P^{N-1}]^+$, and consequently are non-isomorphic. This gives us the following corollary:

Corollary 5.3.5 *The two calculi identified in Theorem 5.3.4 are $\Omega^{(1,0)}_q$ and $\Omega^{(0,1)}_q$.*

Heckenberger and Kolb constructed a total differential calculus extending the direct sum calculus $\Omega_q^{(1,0)} \oplus \Omega_q^{(0,1)}$ as follows: They took the maximal prolongations of $\Omega_q^{(1,0)}$, and $\Omega_q^{(0,1)}$, and defined

$$\Omega^k(\mathbf{C}P^{N-1})_q := \bigoplus_{a+b=k} (\Omega_q^{(1,0)})^a \otimes_{\mathbf{C}_q[\mathbf{C}P^{N-1}]} (\Omega_q^{(0,1)})^b,$$

where $(\Omega_q^{(1,0)})^a$ is the space of a-forms of $\Omega_q^{(1,0)}$, and $(\Omega_q^{(0,1)})^b$ is the space of b-forms of $\Omega_q^{(0,1)}$. They then showed that the partial derivatives ∂ and $\overline{\partial}$ could be extended to operators on the direct sum $\oplus_{k=1}^{2(N-1)}$ giving it the structure of a double complex. That Heckenberger and Kolb's construction of $\Omega^k(\mathbf{C}P^{N-1})_q$ is isomorphic to ours follows from Lemma 5.2.4 and the integrability of our calculus. That the two constructions of the exterior derivative agree follows from the fact that there exists only one exterior derivative on the maximal prolongation of a first-order differential calculus.

Chapter 6

Holomorphic Structures

As discussed in the introduction, one of the primary motivations for studying noncommutative complex structures is the need for a general framework in which to understand noncommutative holomorphic vector bundles. In this chapter we follow [6, 36] and formulate a definition of holomorphic vector bundle based upon the classical Koszul–Malgrange characterization of holomorphic structures. We then specialise to the quantum homogeneous space case, and formulate noncommutative versions of the basic facts underlying geometric representation theory. This general picture is then realised in detail for the specific case of the negative charge quantum line bundles over the quantum projective spaces.

6.1 Holomorphic Structures and Corepresentations

In this section we will discuss three topics: First we present a general framework for noncommutative holomorphic structures; then we consider a very tractable type of holomorphic structure which generalises the classical notion of a globally generated holomorphic vector bundle; finally we specialise to the quantum homogeneous space setting and establish direct links with the corepresentation theory of quantum groups.

6.1.1 Noncommutative Holomorphic Sructures

For a complex structure $\Omega^{(\bullet,\bullet)}$ over an algebra A, we define a *holomorphic element* of A to be an element of the subalgebra

$$A^{(1,0)} := \{\overline{\partial}(a) = 0 \,|\, a \in A\}.$$

This of course directly generalises the classical notion of a holomorphic function.

Generalising the classical notion of a holomorphic section of a vector bundle will prove a little more involved: An *anti-holomorphic covariant derivative* for a right A-module $\mathcal{E}$, is a linear map $\overline{\nabla} : \mathcal{E} \to \mathcal{E} \otimes_A \Omega^{(0,1)}$, such that

$$\overline{\nabla}(ea) = \overline{\nabla}(e)a + e \otimes_A \overline{\partial}(a), \qquad\qquad (a \in A, e \in \mathcal{E}).$$

Now anti-holomorphic covariant derivatives can easily be constructed from ordinary covariant derivatives: Given a covariant derivative $\nabla : \mathcal{E} \to \mathcal{E} \otimes_A \Omega^1(A)$, we have a canonical decomposition $\nabla = \nabla^{(1,0)} + \nabla^{(0,1)}$, where for $\Pi^{(1,0)}$, and $\Pi^{(0,1)}$, the projections onto $\Omega^{(1,0)}$, and $\Omega^{(0,1)}$, respectively, we have denoted

$$\nabla^{(1,0)} := (\mathrm{id} \otimes \Pi^{(1,0)}) \circ \nabla, \qquad\qquad \nabla^{(0,1)} := (\mathrm{id} \otimes \Pi^{(0,1)}) \circ \nabla.$$

Since $\Pi^{(0,1)}$ is a right A-module map, we must have

$$\begin{aligned}\nabla^{(0,1)}(ea) &= (\mathrm{id} \otimes \Pi^{(0,1)}) \circ \nabla(ea) = (\mathrm{id} \otimes \Pi^{(0,1)})(\nabla(e)a + e \otimes_A \mathrm{d}a)\\ &= \nabla^{(0,1)}(e)a + e \otimes_A (\Pi^{(0,1)}(\mathrm{d}a)) = \nabla^{(0,1)}(e)a + e \otimes_A \overline{\partial}a.\end{aligned}$$

Hence, $\nabla^{(0,1)}$ is an anti-holomorphic covariant derivative.
The following important lemma is a direct generalisation of the classical case.

Lemma 6.1.1 *Let $\overline{\nabla}$ be an anti-holomorphic covariant derivative for a right A-module $\mathcal{E}$. For $k \in \mathbf{N}_0$, the map*

$$\overline{\nabla} : \mathcal{E} \otimes_A \Omega^{(0,k)} \to \mathcal{E} \otimes_A \Omega^{(0,k+1)}, \qquad e \otimes_A \omega \mapsto \overline{\nabla}(e) \wedge \omega + e \otimes_A \overline{\partial}\omega,$$

is a well-defined extension of $\overline{\nabla}$. Moreover, the operator

$$\overline{\nabla}^2 : \mathcal{E} \to \mathcal{E} \otimes_A \Omega^{(0,2)},$$

which we call the curvature *of $\overline{\nabla}$, is a right A-module map.*

Proof. First we show that the map is well-defined: For $a \in A$, we have

$$
\begin{aligned}
\overline{\nabla}(ea \otimes \omega) &= \overline{\nabla}(ea) \wedge \omega + (ea) \otimes \overline{\partial}\omega \\
&= \overline{\nabla}(e)a \wedge \omega + e \otimes \overline{\partial}a \wedge \omega + (ea) \otimes \overline{\partial}\omega \\
&= \overline{\nabla}(e) \wedge (a\omega) + e \otimes \overline{\partial}a \wedge \omega + e \otimes a\overline{\partial}\omega \\
&= \overline{\nabla}(e) \wedge (a\omega) + e \otimes \overline{\partial}(a\omega) \\
&= \overline{\nabla}(e \otimes a\omega).
\end{aligned}
$$

Thus, we see that this extension of ∇ is indeed well-defined.

To show that $\overline{\nabla}^2$ is a left A-module map, we first need to note that, for $t := \sum_i e^i \otimes_A \nu^i \in \mathcal{E} \otimes_A \Omega^{(0,k)}$, and $\omega \in \Omega^l$, we have

$$
\begin{aligned}
\overline{\nabla}(t \wedge \omega) &= \sum_i \overline{\nabla}(e^i \otimes_A \nu^i \wedge \omega) = \sum_i \overline{\nabla}(e^i) \otimes_A \nu^i \wedge \omega + \sum_i e^i \otimes_A \overline{\partial}(\nu^i \wedge \omega) \\
&= \sum_i \overline{\nabla}(e^i) \otimes_A \nu^i \wedge \omega + \sum_i e^i \otimes_A \overline{\partial}\nu^i \wedge \omega + (-1)^l \sum_i e^i \otimes_A \nu^i \wedge \overline{\partial}\omega \\
&= \sum_i \overline{\nabla}(e^i \otimes_A \nu^i) \wedge \omega + (-1)^l \sum_i e^i \otimes_A \nu^i \wedge \overline{\partial}\omega \\
&= \overline{\nabla}(t) \wedge \omega + (-1)^l t \wedge \overline{\partial}\omega.
\end{aligned}
$$

With this result in hand, we can now see that

$$
\begin{aligned}
\overline{\nabla}^2(ea) &= \overline{\nabla}(\overline{\nabla}(e)a + e \otimes_A \overline{\partial}a) = \overline{\nabla}^2(e)a - \overline{\nabla}(e) \wedge \overline{\partial}a + \overline{\nabla}(e) \wedge \overline{\partial}a + e \otimes_A \overline{\partial}^2 a \\
&= \overline{\nabla}^2(e)a.
\end{aligned}
$$

Thus, we see that $\overline{\nabla}^2(e)$ is indeed a left A-module map. □

If the curvature of a holomorphic covariant derivative $\overline{\nabla}$ is equal to the zero map, then we say that $\overline{\nabla}$ is *flat*. For any complex structure $\Omega^{(\bullet,\bullet)}$ over an algebra A, a *holomorphic vector bundle* over A is a pair $(\mathcal{E}, \overline{\nabla})$, where $\mathcal{E}$ is a right A-module, and $\overline{\nabla}$ is a flat holomorphic covariant derivative for $\mathcal{E}$. We call an element $e \in \mathcal{E}$ *holomorphic* if $\overline{\nabla}(e) = 0$, and denote the space of holomorphic elements by $\mathcal{E}^{(1,0)}$.

The Koszul–Malgrange theorem tells us that this is a direct generalisation of the classical definition of the space of holomorphic sections of a holomorphic vector bundle. (See [41, 6] for a more detailed presentation of this correspondence.)

6.1.2 Globally Generated Holomorphic Vector Bundles

In general, proving that a covariant derivative is flat can lead to quite tedious calculations. There does, however, exist a special class of covariant derivatives for which the situation is much simpler: For an anti-holomorphic covariant derivative $\overline{\nabla} : \mathcal{E} \to \mathcal{E} \otimes \Omega^{(0,1)}$, we call a subspace $\mathcal{E}^0 \subseteq \mathcal{E}$ a *holomorphic generating set* if $\mathcal{E}^0 \subseteq \ker(\overline{\nabla})$, and $\mathcal{E}^0$ generates $\mathcal{E}$ as a right A-module. The importance of such subspace is demonstrated by the following lemma:

Lemma 6.1.2 *Let $\overline{\nabla} : \mathcal{E} \to \mathcal{E} \otimes_A \Omega^{(0,1)}$ be an anti-holomorphic covariant derivative for a right A-module $\mathcal{E}$. If there exists a holomorphic generating set for $\overline{\nabla}$, then it is flat, and moreover,*

$$\mathrm{span}_{\mathbf{C}}\{ea \,|\, e \in \mathcal{E}^0,\, a \in ker(\overline{\partial})\} \subseteq \mathcal{E}^{(1,0)}. \tag{6.1}$$

Proof. Since $\mathcal{E}^0$ is a generating set of $\mathcal{E}$ as a right A-module, we have that every element of $\mathcal{E}$ is of the form $\sum_i e^i a^i$, for $e^i \in \mathcal{E}^0$, and $a \in A$. Now from Lemma 6.1.1, we have

$$\overline{\nabla}^2(\sum_i e^i a^i) = \sum_i \overline{\nabla}^2(e^i)a^i = 0, \tag{6.2}$$

giving us that $\overline{\nabla}^2 = 0$.

Let now consider the identity

$$\overline{\nabla}(\sum_i e^i a^i) = \sum_i \overline{\nabla}(e^i)a^i + \sum_i e^i \otimes_A \overline{\partial} a^i = \sum_i e^i \otimes_A \overline{\partial} a^i.$$

Clearly, $\overline{\nabla}(\sum_i e^i a^i)$ is zero if $a^i \in \ker(\overline{\partial})$, for each i. □

We will call a holomorphic vector bundle for which there exists a holomorphic generating set a *globally generated* holomorphic vector bundle. This definition generalises the classical notion of a globally generated holomorphic vector bundle (see [57] for details).

6.1.3 Covariant Holomorphic Structures and Corepresentations

Let $M = G^H$ be a quantum homogeneous space endowed with a covariant complex structure $\Omega^{(\bullet,\bullet)}$. Since $(\Omega^{(0,1)}, \partial)$ is a left-covariant first-order differential calculus, it must hold that

$$\Delta_L \circ \overline{\partial} = (\mathrm{id} \otimes \overline{\partial}) \circ \Delta_L.$$

This implies that, for a holomorphic element $m \in M^{(1,0)}$, we have

$$(\mathrm{id} \otimes \overline{\partial}) \circ \Delta_L(m) = \Delta_L \circ \overline{\partial}(m) = 0. \tag{6.3}$$

Hence, $\Delta_L(M^{(1,0)}) \subseteq G \otimes M^{(1,0)}$, or in other words, $M^{(1,0)}$ is a right G-comodule.

Let us now try to generalise this fact for objects $\mathcal{E}$ in ${}^G_M\mathcal{M}_M$. Denoting the left G-coaction of $\mathcal{E}$ by Δ_L, we say that an anti-holomorphic covariant derivative $\overline{\nabla} : \mathcal{E} \to \mathcal{E} \otimes_M \Omega^{(0,1)}$ is *homogeneous* if it holds that

$$(\mathrm{id} \otimes \overline{\nabla}) \circ \Delta_L = \Delta_L^{\otimes 2} \circ \overline{\nabla}. \tag{6.4}$$

The following result generalises the G-comodule structure of $M^{(1,0)}$:

Lemma 6.1.3 *If an anti-holomorphic covariant derivative $\overline{\nabla} : \mathcal{E} \to \mathcal{E} \otimes_M \Omega^{(0,1)}$ is homogeneous, then its space $\mathcal{E}^{(1,0)}$ of holomorphic elements is a right G-comodule. Moreover, if $\mathcal{E}$ has a holomorphic generating set, then this is an if, and only if, statement.*

Proof. For $e \in \mathcal{E}^{(1,0)}$, we have

$$(\mathrm{id} \otimes \overline{\nabla}) \circ \Delta_L(e) = \Delta_L^{\otimes 2} \circ \overline{\nabla}(e) = 0.$$

Hence, $\Delta_L(\mathcal{E}^{(0,1)}) \subseteq G \otimes \mathcal{E}^{(1,0)}$, and $\mathcal{E}^{(1,0)}$ is a right G-comodule.

Let us now assume the existence of a holomorphic generating set $\mathcal{E}^0$. For $a \in A$, and $e \in \mathcal{E}^0$, we have

$$\Delta_L^{\otimes 2} \circ \overline{\nabla}(ea) = \Delta_L^{\otimes 2}(e \otimes_A \overline{\partial} a) = e_{(-1)}a_{(1)} \otimes e_{(0)} \otimes_A \overline{\partial} a_{(2)},$$

and

$$\begin{aligned}(\mathrm{id} \otimes \overline{\nabla}) \circ \Delta_L(ea) &= e_{(-1)}a_{(1)} \otimes e_{(0)}a_{(2)} = e_{(-1)}a_{(1)} \otimes \overline{\nabla}(e_{(0)}a_{(2)}) \\ &= e_{(-1)}a_{(1)} \otimes \overline{\nabla}(e_{(0)})a_{(2)} + e_{(-1)}a_{(1)} \otimes e_{(0)} \otimes_A \overline{\partial} a_{(2)}.\end{aligned}$$

Now if $\mathcal{E}^{(1,0)}$ is a left G-comodule, then $\overline{\nabla}(e_{(0)}) = 0$, giving us that

$$\Delta_L^{\otimes 2} \circ \overline{\nabla}(ea) = (\mathrm{id} \otimes \overline{\nabla}) \circ \Delta_L(ea).$$

Hence, $\overline{\nabla}$ is homogeneous. $\square$

The theory of quantum principal bundles provides us with an important method for constructing homogeneous anti-holomorphic covariant derivatives for line bundles: We define a covariant derivative $\nabla : \mathcal{E} \to \mathcal{E} \otimes_A \Omega^1(M)$ to be *homogeneous* if it satisfies the obvious analogue of (6.4):

$$\Delta_L^{\otimes 2} \circ \nabla = (\mathrm{id} \otimes \nabla) \circ \Delta_L.$$

Now if the complex structure on M is covariant, then it is easy to see that the holomorphic part of ∇ will be a homogeneous anti-holomorphic covariant derivative. Moreover, if $M = G^H$ is a quantum principal homogeneous space, then it is easy to see that any covariant derivative for $\mathcal{E}$ induced by a strong connection $\Pi : \Omega^1(G) \to \Omega^1(G)$ will be homogeneous.

We finish this section with a short lemma linking the definitions of a strong connection and a framing calculus.

Lemma 6.1.4 *For a quantum principal homogeneous space $M = G^H$ such that $\Omega^1(G)$ is a framing calculus for $\Omega^1(M)$, all connections are strong.*

Proof. From the general definition, any connection $\Pi : \Omega^1(G) \to \Omega^1(G)$ satisfies

$$(\mathrm{id} - \Pi)(\Omega^1(M)) = G\Omega^1(M)G.$$

But if $\Omega^1(G)$ is a framing calculus for $\Omega^1(M)$, then by definition we also have that $\Omega^1(M)G \subseteq G\Omega^1(M)$, implying that

$$(\mathrm{id} - \Pi)(\Omega^1(M)) \subseteq G\Omega^1(M).$$

Hence, Π is a strong connection. $\square$

6.2 A Holomorphic Structure for the Quantum Projective Space Line Bundles

In this section we construct holomorphic structures for the quantum line bundles of $\mathbf{C}_q[\mathbf{C}P^{N-1}]$ which are indexed by the negative integers. We begin by showing that the differential structure we have been using for $\mathbf{C}_q[SU_N]$ gives the quantum homogeneous space $\alpha_N : \mathbf{C}_q[SU_N] \to \mathbf{C}_q[U_{N-1}]$ the structure of a quantum principal bundle. We use the general theory of quantum principal bundles to construct a connection for the bundle, and then show that this connection induces holomorphic covariant derivatives on the line bundles $\mathcal{E}_{-k}$, for $k \in \mathbf{N}$. Finally, we show the corresponding spaces of holomorphic sections contain the standard $\binom{N+k-1}{N-1}$-dimensional corepresentations of $\mathbf{C}_q[SU_N]$.

6.2.1 A Quantum Principal Bundle Structure

We carefully show that the calculus $\Omega^1_q(SU_N)$ induces the structure of a quantum principal bundle on the Hopf–Galois extension $\mathbf{C}_q[SU_N] \hookleftarrow \mathbf{C}_q[\mathbf{C}P^{N-1}]$. (The proof used here is a more concise version of the original proof in [58].)

Proposition 6.2.1 *It holds that $(\mathbf{C}_q[SU_N], \mathbf{C}_q[U_{N-1}], I_{SU_N}, \alpha_N(I_{SU_N}))$ is a quantum principal homogeneous space.*

Proof. We have already proved that $\alpha_N : \mathbf{C}_q[SU_N] \to \mathbf{C}_q[U_{N-1}]$ is a Hopf–Galois principal homogeneous space. Thus, Proposition 2.3.2 tells us that all we need to show is that (2.20) holds for I_{SU_N}. Recall that $I_{SU_N} = \ker(Q)^+ + D_1 + D_2$, where $D_1 = \mathrm{span}_{\mathbf{C}}\{u^i_1 S(u^1_i) \,|\, i = 2, \ldots, N\}$, and $D_2 = \mathrm{span}_{\mathbf{C}}\{u^i_j \,|\, i \neq j; i, j = 2, \ldots, N\}$. Now since $\ker(Q)^+$ is an Ad_R-stable ideal, it is clear that

$$(\mathrm{id} \otimes \alpha_N)\mathrm{Ad}_R(\ker(Q)^+) \subseteq \ker(Q)^+ \otimes \mathbf{C}_q[U_{N-1}].$$

For D_1, we begin by noting that

$$\begin{aligned}(\mathrm{id} \otimes \alpha_N)\mathrm{Ad}_R(u^i_1 S(u^1_i)) &= \sum_{a,b,c,d=1}^{N} u^a_b S(u^c_d) \otimes \alpha_N(S(u^i_a S(u^d_i)) u^b_1 S(u^1_c)) \\ &= \sum_{a,d=2}^{N} u^a_1 S(u^1_d) \otimes S(u^{i-1}_{a-1} S(u^{d-1}_{i-1}))\, \mathrm{det}^{-1}_{N-1} \mathrm{det}_{N-1} \\ &= \sum_{a,d=2}^{N} u^a_1 S(u^1_d) \otimes S(u^{i-1}_{a-1} S(u^{d-1}_{i-1})).\end{aligned}$$

For $a = d$, we have $u^a_1 S(u^1_a) \in D_1$ by definition. For $a \neq d$, it follows directly from the relations in Lemma 3.2.3 that $u^a_1 S(u^1_d) \in I_{SU_N}$. Hence, we have that

$(\mathrm{id} \otimes \alpha_N)\mathrm{Ad}_R(D_1)$ is contained in $I_{SU_N} \otimes \mathbf{C}_q[U_{N-1}]$. Turning now to $u^i_j \in D_2$, we see that

$$(\mathrm{id} \otimes \alpha_N)\mathrm{Ad}_R(u^i_j) = (\mathrm{id} \otimes \alpha_N)(\sum_{k,l=1}^{N} u^k_l \otimes S(u^i_k)u^l_j) = \sum_{k,l=2}^{N} u^k_l \otimes S(u^{i-1}_{k-1})u^{l-1}_{j-1}.$$

For $k \neq l$, we have $u^k_l \in D_2$ by definition. It remains for us to show that $\sum_{k=2}^{N} u^k_k \otimes S(u^{i-1}_{k-1})u^{k-1}_{j-1}$ is contained in $I_{SU_N} \otimes \mathbf{C}_q[U_{N-1}]$. We will do this by showing that its image in $\Lambda^1_{SU_N} \otimes \mathbf{C}_q[U_{N-1}]$ is zero: Denoting $\lambda := \frac{q^{-\frac{2}{N}}}{(q^{2-\frac{2}{N}}-1)}$, we have

$$\begin{aligned}\sum_{k=2}^{N} \overline{u^k_k} \otimes S(u^{i-1}_{k-1})u^{k-1}_{j-1} &= \sum_{k=2}^{N} \lambda e^0 \otimes S(u^{i-1}_{k-1})u^{k-1}_{j-1} = \lambda e^0 \otimes (\sum_{k=1}^{N-1} S(u^{i-1}_k)u^k_{j-1}) \\ &= \lambda e^0 \otimes \varepsilon(u^{i-1}_{j-1}) = 0.\end{aligned}$$

(Note that the summation $\sum_{k=1}^{N-1} S(u^{i-1}_k)u^k_{j-1}$ takes place in $\mathbf{C}_q[U_{N-1}]$.) Thus, we have shown that (2.20) holds for I_{SU_N}. □

6.2.2 An Anti-Holomorphic Covariant Derivative

In this section we will use the general theory presented in Chapter 2 to construct a connection for the bundle $\alpha_N : \mathbf{C}_q[SU_N] \to \mathbf{C}_q[\mathbf{C}P^{N-1}]$.

Lemma 6.2.2 *A bicovariant splitting map is given by*

$$i : \Lambda^1_{U_{N-1}} \to \Lambda^1_{SU_N}, \qquad \overline{\det^{-1}_{N-1} - 1} \mapsto \overline{u^1_1 - 1},$$

Moreover, the corresponding connection $\Pi : \Omega^1_q(SU_N) \to \Omega^1_q(SU_N)$ *is strong and satisfies*

$$\Pi(e^0) = e^0, \qquad \Pi(e^+_i) = \Pi(e^-_i) = 0, \qquad (i = 1, \ldots, N-1).$$

Proof. That i satisfies $\alpha_N \circ i = \mathrm{id}$ is obvious, while (2.23) follows from

$$\begin{aligned}\overline{\mathrm{Ad}_{R,SU_N}} \circ i(\overline{\det^{-1}_N - 1}) = \overline{\mathrm{Ad}_{R,SU_N}}(e^0) &= \sum_{k,l=1}^{N} (\overline{u^k_l} \otimes \alpha_N(S(u^1_k u^l_1)) - 1 \otimes 1 \\ &= \overline{u^1_1} \otimes 1 - 1 \otimes 1 = e^0 \otimes 1 = i(\overline{\det^{-1}_N - 1}) \otimes 1 \\ &= (i \otimes \mathrm{id}) \circ \overline{\mathrm{Ad}_{R,U_{N-1}}}(\overline{\det^{-1}_N - 1}).\end{aligned}$$

Denoting the connection form corresponding to i by ω, we have

$$\Pi_\omega(e^0) = m \circ (\mathrm{id} \otimes \omega) \circ \mathrm{ver}(e^0) = \omega \circ \alpha_N(\overline{u_1^1 - 1}) = \omega(\overline{\det_{N-1}^{-1} - 1}) = \overline{u_1^1 - 1} = e^0.$$

Similarly, $\alpha_N(u_1^{i+1}) = \alpha_N(u_{i+1}^1) = 0$ implies that $\Pi_\omega(e_i^+) = \Pi_\omega(e_i^-) = 0$.
The fact that Π is strong follows from Lemma 6.1.4 and the fact that $\Omega_q^1(SU_N)$ is a framing calculus for $\Omega_q^1(\mathbf{C}P^{N-1})$. $\square$

For any object $\mathcal{E}$ in ${}^G_M\mathcal{M}_M$, we denote the corresponding covariant derivative by $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathbf{C}P^{N-1}} \Omega_q^1(\mathbf{C}P^{N-1})$, and the corresponding anti-holomorphic covariant derivative by $\nabla^{(0,1)} : \mathcal{E} \to \mathcal{E} \otimes_{\mathbf{C}P^{N-1}} \Omega_q^1(\mathbf{C}P^{N-1})$.

6.2.3 A Holomorphic Structure for the Modules $\mathcal{E}_{-k}$

A natural question to ask is for which objects $\mathcal{E}$ in ${}^G_M\mathcal{M}_M$, is the pair $(\mathcal{E}, \nabla^{(0,1)})$ a holomorphic vector bundle. As first step towards answering this, let us look at the quantum line bundles $\mathcal{E}_{-k}$, for all $k \in \mathbf{N}$.

Proposition 6.2.3 *It holds that*

$$\{(u_1^1)^{m_1} \cdots (u_1^N)^{m_N} \mid \sum_{i=1}^N m_i = k\} \subseteq \mathcal{E}_{-k}^{(1,0)}, \tag{6.5}$$

is a holomorphic generating set for $\nabla^{(0,1)}$, and so, $(\mathcal{E}_{-k}, \nabla^{(0,1)})$ is a globally generated holomorphic vector bundle.

Proof. Since we established in Corollary 3.1.4 that $\mathcal{E}_{-k}^{(1,0)}$ is a generating set for $\mathcal{E}_{-k}$, all we need to do is show that the elements of $\mathcal{E}_{-k}^0$ are contained in the kernel of $\nabla^{(0,1)}$. To this end, we note that

$$\mathrm{d}u_1^i = s^{-1}(\sum_{a=1}^N u_a^i \otimes \overline{(u_1^a)^+}) = \sum_{a=1}^{N-1} u_{a+1}^i e_a^+ \;\in\; \mathcal{E}_k \otimes_{\mathbf{C}P^{N-1}} (\Omega_q^{(1,0)} \oplus \mathbf{C}e^0)$$

Now from Lemma 3.2.3 it is easy to see that $\Omega_q^{(1,0)} \oplus \mathbf{C}e^0$ is closed under right multiplication by elements of the form u_1^i. This fact, when combined with the Liebniz rule, implies that

$$\mathrm{d}e \;\in\; \mathcal{E}_k \otimes_{\mathbf{C}P^{N-1}} (\Omega_q^{(0,1)} \oplus \mathbf{C}e^0), \qquad \text{(for all } e \in \mathcal{E}_{-k}^0).$$

It now follows that $\Pi^{(0,1)} \circ \Pi \circ \mathrm{d}(e) = 0$, and so, $\mathcal{E}^0_{-k} \subseteq \mathcal{E}^{(1,0)}_{-k}$, implying that $\mathcal{E}^0_{-k}$ is a holomorphic generating set for $\nabla^{(0,1)}$. Lemma 6.1.2 now tells us that $\nabla^{(0,1)}$ is flat. $\square$

Thus we see that $\mathcal{E}^{(1,0)}_{-k}$ contains the standard $\binom{N+k-1}{N-1}$-dimensional corepresentation of $\mathbf{C}_q[SU_N]$. This generalises the classical Borel-Weil theorem, and extends Majid's result for $\mathbf{C}_q[\mathbf{C}P^1]$ [51]. The extension of this work to include the quantum Grassmannians will be considered elsewhere, as will the explicit relationship of these results to the work of Khalkhali, Landi, Moatadelro, and van Suijlekom [34, 35, 36].

Finally, we finish with a very natural conjecture:

Conjecture 6.2.4 *It holds that*

$$\mathcal{E}^{(1,0)}_k = \{0\}, \qquad \textit{and} \qquad \mathcal{E}^{(1,0)}_{-k}, = \{(u^1_1)^{m_1} \cdots (u^N_1)^{m_N} \mid \sum_{i=1}^N m_i = k\}, \quad (k \in \mathbf{N}).$$

Chapter 7

The Noncommutative Kähler Geometry of $\mathbf{C}_q[\mathbf{C}P^1]$

In this chapter we take the first steps towards a general theory of noncommutative Kähler geometry by establishing a noncommutative generalisation of the Kähler identities for $\mathbf{C}_q[\mathbf{C}P^1]$. While we use a minimum of formalism here, a fuller treatment incorporating a noncommutative metric and a noncommutative fundamental Kähler form will appear in [60].

7.1 Hodge, Lefschetz, and Laplace Operators

We will now introduce a direct generalisation of the classical Hodge $*$-map for the Fubini–Study metric on $\mathbf{C}P^1$: First we define

$$*: \Omega^1_q(\mathbf{C}P^{N-1}) \to \Omega^1_q(\mathbf{C}P^{N-1}), \qquad *: fe^- \mapsto -ife^-, \quad *: fe^+ \mapsto ife^+.$$

We see that $*$ squares to give $-\mathrm{id}$ on $\Omega^1_q(\mathbf{C}P^{N-1})$, as one would want. Next we choose $e^+ \wedge e^-$ as our top form, and define

$$*: \Omega^2_q(\mathbf{C}P^{N-1}) \to \mathbf{C}_q[\mathbf{C}P^{N-1}], \qquad fe^+ \wedge e^- \mapsto f.$$

We then complete the picture by defining the inverse map

$$*: \mathbf{C}_q[\mathbf{C}P^{N-1}] \to \Omega^2_q(\mathbf{C}P^{N-1}), \qquad f \mapsto fe^+ \wedge e^-.$$

We note that $*$ squares to the the identity on $\Omega_q^2(\mathbf{C}P^{N-1})$ and $\mathbf{C}_q[\mathbf{C}P^{N-1}]$, as one would want.

Directly generalising the classical picture again, we define the *Lefschetz operator*

$$L : \mathbf{C}_q[\mathbf{C}P^1] \to \Omega_q^2(\mathbf{C}P^1), \qquad\qquad f \mapsto -fe^+ \wedge e^- = - * (f),$$

and the *dual-Lefschetz operator*

$$\Lambda : \Omega_q^2(\mathbf{C}P^1) \to \mathbf{C}_q[\mathbf{C}P^1], \qquad\qquad \omega \mapsto * \circ L \circ *(\omega) = - * (f).$$

We extend L and Λ to operators on the total calculus by defining them to be zero on all other forms.

We next define the *codifferentials* by

$$\mathrm{d}^* = - * \mathrm{d} *, \qquad\qquad \partial^* = - * \overline{\partial} *, \qquad\qquad \overline{\partial}^* = - * \partial * .$$

Finally, we define the *Laplace operators* Δ, Δ_∂, and $\Delta_{\overline{\partial}}$, by

$$\Delta_\partial := (\partial + \partial^*)^2 = \partial \circ \partial^* + \partial^* \circ \partial, \qquad \Delta_{\overline{\partial}} := (\overline{\partial} + \overline{\partial}^*)^2 = \overline{\partial} \circ \overline{\partial}^* + \overline{\partial}^* \circ \overline{\partial},$$

$$\Delta := (\mathrm{d} + \mathrm{d}^*)^2 = \mathrm{d} \circ \mathrm{d}^* + \mathrm{d}^* \circ \mathrm{d}.$$

7.2 The Kähler Identities

In this section we will show the Hodge, Lefschetz, and dual Lefschetz operators defined above satisfy a generalisation of the Kähler identities for the projective line. We will then use this result to show that the three Laplacians $\Delta_{\mathrm{d}}, \Delta_\partial$, and $\Delta_{\overline{\partial}}$, are simple scalar multiples of each other.

Proposition 7.2.1 *Using the notation of the previous section, we have the following relations:*

$$[L, \partial^*] = i\overline{\partial}, \qquad [L, \overline{\partial}^*] = -i\partial, \qquad [L, \partial] = 0, \qquad [L, \overline{\partial}] = 0,$$
$$[\Lambda, \partial] = i\overline{\partial}^*, \qquad [\Lambda, \overline{\partial}] = -i\partial^*, \qquad [\Lambda, \partial^*] = 0, \qquad [\Lambda, \overline{\partial}^*] = 0.$$

Proof. The relations

$$[L, \partial] = [L, \overline{\partial}] = [\Lambda, \partial^*] = [\Lambda, \overline{\partial}^*] = 0$$

are direct consequences of the definition of L and Λ. The remaining relations are easily verified by direct calculation. We begin with the relation $[L, \partial^*] = i\overline{\partial}$: Note that $[L, \partial^*]$, and $i\overline{\partial}$, have non-zero actions only on $\mathbf{C}_q[\mathbf{C}P^1]$ and $\Omega_q^{(1,0)}$. For $f \in \mathbf{C}_q[\mathbf{C}P^1]$, we have

$$[L, \partial^*]f = (-L \circ *\overline{\partial} * + * \overline{\partial} * \circ L)f = *\overline{\partial} * \circ L(f) = - * (\overline{\partial} f) = i\overline{\partial} f.$$

While for $f\partial g \in \Omega_q^{(1,0)}$, we have

$$[L, \partial^*](f\partial g) = (-L \circ *\overline{\partial} * + * \overline{\partial} * \circ L)f\partial g = -L \circ *\overline{\partial} * (f\partial g) = i\overline{\partial}(f\partial g),$$

which establishes the relation.

Next we turn to $[L, \overline{\partial}^*]$ and $-i\partial$: Note that both operators have non-zero actions only on $\mathbf{C}_q[\mathbf{C}P^{N-1}]$ and $\Omega_q^{(0,1)}$. For $f \in \mathbf{C}_q[\mathbf{C}P^{N-1}]$, we have

$$[L, \overline{\partial}^*](f) = (L \circ \overline{\partial}^* - \overline{\partial}^* \circ L)(f) = -\overline{\partial}^* \circ L(f) = - * (\partial f) = -i\partial f.$$

For $f\overline{\partial} g \in \Omega^{(0,1)}$, we have

$$\begin{aligned} [L, \overline{\partial}^*](f\overline{\partial} g) &= (L \circ \overline{\partial}^* - \overline{\partial}^* \circ L)(f\overline{\partial} g) = L \circ \overline{\partial}^*(f\overline{\partial} g) \\ &= \partial(*(f\overline{\partial} g)) = -i\partial(f\overline{\partial} g), \end{aligned}$$

which establishes the identity.

Now $[\Lambda, \partial]$ and $i\overline{\partial}^*$, both have non-zero actions only on $\Omega_q^{(0,1)}$ and $\Omega_q^2(\mathbf{C}P^{N-1})$. For $f\overline{\partial} g \in \Omega_q^{(0,1)}$, we have

$$\begin{aligned} [\Lambda, \partial](f\overline{\partial} g) =& (\Lambda \circ \partial - \partial \circ \Lambda)(f\overline{\partial} g) = \Lambda \circ \partial(f\overline{\partial} g) = - * \partial(f\overline{\partial} g) \\ &= - * \partial * (if\overline{\partial} g) = i\overline{\partial}^*(f\overline{\partial} g). \end{aligned}$$

For $fe^+ \otimes e^- \in \Omega_q^2(\mathbf{C}P^1)$, we have

$$\begin{aligned} [\Lambda, \partial](fe^+ \otimes e^-) &= (\Lambda \circ \partial - \partial \circ \Lambda)(fe^+ \otimes e^-) = -\partial \circ \Lambda(fe^+ \otimes e^-) \\ &= \partial * (fe^+ \otimes e^-) = - * \partial * (ife^+ \otimes e^-) = i\overline{\partial}^*(fe^+ \otimes e^-), \end{aligned}$$

giving us the required equality.

Finally, we come to $[\Lambda, \overline{\partial}]$ and $-i\partial^*$. Note that both operators have non-zero actions only on $\Omega_q^{(1,0)}$ and $\Omega_q^2(\mathbf{C}P^{N-1})$. For $f\partial g \in \Omega^{(1,0)}$, we have

$$\begin{aligned}[\Lambda, \overline{\partial}](f\partial g) =& (\Lambda \circ \overline{\partial} - \overline{\partial} \circ \Lambda)(f\partial g) = \Lambda \circ \overline{\partial}(f\partial g) = - * \overline{\partial}(f\partial g) \\ &= *\overline{\partial} * (if\partial g) = -i\partial^*(f\partial g).\end{aligned}$$

For $fe^+ \otimes e^- \in \Omega_q^2(\mathbf{C}P^1)$, we have

$$\begin{aligned}[\Lambda, \overline{\partial}](fe^+ \otimes e^-) &= (\Lambda \circ \overline{\partial} - \overline{\partial} \circ \Lambda)(fe^+ \otimes e^-) = -\overline{\partial} \circ \Lambda(fe^+ \otimes e^-) \\ &= \overline{\partial} * (fe^+ \otimes e^-) = *\overline{\partial} * (ife^+ \otimes e^-) = -i\partial^*(fe^+ \otimes e^-),\end{aligned}$$

establishing the last required identity. □

We can now follow the standard classical proof [30] and use the Kähler identities to establish equality of the three Laplacians.

Corollary 7.2.2 *The Laplace operators are related by* $\Delta = 2\Delta_\partial = 2\Delta_{\overline{\partial}}$.

Proof. First note that

$$\begin{aligned}-i(\overline{\partial}\partial^* + \partial^*\overline{\partial}) &= \overline{\partial}[\Lambda, \overline{\partial}] + [\Lambda, \overline{\partial}]\overline{\partial} = \overline{\partial}\Lambda\overline{\partial} - \overline{\partial}^2\Lambda - \Lambda\overline{\partial}^2 - \overline{\partial}\Lambda\overline{\partial} \\ &= \overline{\partial}\Lambda\overline{\partial} - \overline{\partial}\Lambda\overline{\partial} = 0,\end{aligned}$$

and similarly

$$\partial\overline{\partial}^* + \overline{\partial}^*\partial = \partial[\Lambda, \partial] + [\Lambda, \partial]\partial = 0.$$

This gives that

$$\begin{aligned}\Delta = \mathrm{d} \circ \mathrm{d}^* + \mathrm{d}^* \circ \mathrm{d} &= (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) + (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}) \\ &= (\partial\partial^* + \partial^*\partial) + (\overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}) + (\overline{\partial}\partial^* + \partial^*\overline{\partial}) + (\partial\overline{\partial}^* + \overline{\partial}^*\partial) \\ &= (\partial\partial^* + \partial^*\partial) + (\overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}) + 0 + 0 \\ &= \Delta_\partial + \Delta_{\overline{\partial}}.\end{aligned}$$

It remains to show that $\Delta_\partial = \Delta_{\overline{\partial}}$. But this is an easy consequence of the calculation:

$$\begin{aligned}-i\Delta_\partial &= -i(\partial\partial^* + \partial^*\partial) = \partial[\Lambda, \overline{\partial}] + [\Lambda, \overline{\partial}]\partial = \partial\Lambda\overline{\partial} - \partial\overline{\partial}\Lambda + \Lambda\overline{\partial}\partial - \overline{\partial}\Lambda\partial \\ &= \partial\Lambda\overline{\partial} + \overline{\partial}\partial\Lambda - \overline{\partial}\Lambda\partial = [\partial, \Lambda]\overline{\partial} + \overline{\partial}[\partial, \Lambda] = -i\overline{\partial}^*\overline{\partial} - i\overline{\partial}\overline{\partial}^* = -i\Delta_{\overline{\partial}}.\end{aligned}$$

□

One could now attempt to extend this result to the higher order quantum projective spaces by direct calculation. However, such an approach seems likely to be overly laborious. A more promising idea is to build upon our framework of noncommutative complex geometry, and develop a general theory of noncommutative Kähler geometry for quantum homogeneous spaces. Such an approach is at present being developed [61].

Bibliography

[1] D. Auroux, L. Katzarkov, D. Orlov, Mirror symmetry for weighted projective planes and their noncommutative deformations, *Ann. of Math.*, **167**, 867-943, (2008)

[2] E. Backelin, K. Kremnizer, Quantum flag varieties, equivariant quantum D-modules, and localization of quantum groups, *Adv. Math.*, **203**, 408-429, (2006)

[3] E. Backelin, K. Kremnizer, Localization for quantum groups at a root of unity, *J. Amer. Math. Soc.*, **21**, 1001-1018, (2008)

[4] E. Beggs, T. Brzeziński, Noncommutative differential operators, Sobolev spaces and the centre of a category, preprint, arXiv:1108.5047

[5] E. Beggs, Differential and holomorphic differential operators on noncommutative algebras, preprint, arXiv:1209.3900

[6] E. Beggs, S. P. Smith, Noncommutative complex differential geometry, arXiv:1209.3595

[7] S. Brain, G. Landi, The $3D$ spin geometry of the quantum two-sphere, Rev. Math. Phys., **22**, 963-993, (2010)

[8] T. Brzeziński, S. Majid, Quantum group gauge theory on quantum spaces, Comm. Math. Phys., **157**, 591-638, (1993)

[9] T. Brzeziński, S. Majid, Quantum differentials and the q-Monopole Revisited, *Acta Appl. Math.*, **54**, 185-232, (1998)

[10] V. Chari and A. Pressley, *A Guide to Quantum Groups*, Cambridge University Press, Cambridge, 1994

[11] P. S. Chakraborty, A. Pal, Equivariant spectral triples on the quantum SU(2) group, *K-Theory*, **28**, 107-126, (2003)

[12] A. Connes, J. Cuntz, Quasi homomorphismes, cohomologie cyclique et positivité, *Comm. Math. Phys.*, **114**, 515–526, (1988).

[13] A. Connes, *Noncommutative Geometry*, Academic Press, San Diego, 1994

[14] F. D'Andrea, L. Dabrowski, G. Landi, The Noncommutative geometry of the quantum projective plane, *Rev. Math. Phys.*, **20**, 979–1006, (2008)

[15] A. Connes, G. Landi, *Noncommutative manifolds, the instanton algebra and isospectral deformations*, Comm. Math. Phys., **221** (2001), 141–159.

[16] L. Dąbrowski, G. Landi, A. Sitarz, W. van Suijlekom, J. C. Várilly, *The Dirac operator on* $\mathrm{SU}_q(2)$, Comm. Math. Phys., **259** (2005), 729–759.

[17] L. Dąbrowski, A. Sitarz, Dirac operator on the standard Podleś quantum sphere, *Noncommutative geometry and quantum groups (Warsaw, 2001)*, Banach Center Publ., **61**, 49–58, Polish Acad. Sci., Warsaw, (2003).

[18] F. D'Andrea, G. Landi, Anti-selfdual connections on the quantum projective plane: monopoles, *Comm. Math. Phys.*, **297**, 841-893, (2010)

[19] F. D'Andrea, G. Landi, Geometry of Quantum Projective Spaces, preprint, arXiv:1203.0621

[20] F. D'Andrea, L. Dąbrowski, Dirac operators on quantum projective spaces, *Comm. Math. Phys.*, **295**, 731-790, (2010)

[21] H. Figueroa, J, C. Varilly, J. M. Gracia–Bondia, *Elements of Noncommutative Geometry*, Birkhäuser, Boston, 2001

[22] M. Dieng, A. Schwarz, Differential and complex geometry of two-dimensional noncommutative tori, *Lett. Math. Phys.*, **61**, (2002)

[23] J. Fröchlich, O. Grandjean, A. Recknagel, Supersymmetric quantum theory and (non-commutative) differential geometry, *Comm. Math. Phys.*, **193**, 527–594, (1998)

[24] P. Hajac, S. Majid, Projective module description of the q-monopole, *Comm. Math. Phys.*, **206**, 246-264, (1999)

[25] I. Heckenberger, Hodge and Laplace–Beltrami operators for bicovariant differential calculi on quantum groups, Compositio Math. **1**23, 329-354, (2000)

[26] I. Heckenberger and S. Kolb, Differential Calculus on Quantum Complex Grassmann Manifolds II: Classification, unpublished, arXiv:math/0112155

[27] I. Heckenberger and S. Kolb, The locally finite part of the dual coalgebra of quantised irreducible flag manifolds, *Proc. Lon. Math. Soc.* (3) **89**, 457–484, (2004)

[28] I. Heckenberger and S. Kolb, De Rham complex for quantized irreducible flag manifolds, *J. Algebra*, **305**, 704-741, (2006)

[29] U. Hermisson, Derivations with quantum group action, *Comm. Algebra*, **30**, (2002)

[30] D. Huybrechts, Complex geometry - an introduction, Springer, Universitext, 2004

[31] V. Lakshmibai, N. Reshetikhin, Quantum deformations of flag and Schubert schemes. *C.R. Acad. Sci.* Paris 313, Série I, **121**. (1991).

[32] B. Jurčo, Differential calculus on quantized simple Lie groups, *Lett. Math. Phys.*, **22**, 177–186, (1991)

[33] C. Kassel, *Quantum Groups*, Springer–Verlag, New York–Heidelberg-Berlin, 1995

[34] M. Khalkhali, G. Landi, W. van Suijlekom, Holomorphic structures on the quantum projective line, *Int. Math. Res. Not. IMRN,* (2010)

[35] M. Khalkhali, A. Moatadelro, The homogeneous coordinate ring of the quantum projective plane, *J. Geom. Phys.*, **61**, (2011)

[36] M. Khalkhali, A. Moatadelro, Noncommutative complex geometry of the quantum projective space, *J. Geom. Phys.,* **61**, 2436–2452, (2011)

[37] A. Klimyk, K. Schmüdgen, *Quantum Groups and their Representations*, Springer Verlag, Heidelberg–New York, 1997

[38] S. Kolb, Differential Calculus on Quantum Complex Grassmann Manifolds, PhD thesis (Dr. rer. nat.), Universität Leipzig, (2002).

[39] S. Kolb, Differential Calculus on Quantum Complex Grassmann Manifolds I: Construction, unpublished, arXiv:math/0112154

[40] U. Krähmer, Dirac operators on quantum flag manifolds, *Lett. Math. Phys.*, **67**, 49–59, (2004)

[41] J.-L. Koszul, B. Malgrange, Sur certaines structures fibrées complexes, *Arch. Math. (Basel)*, **9**, 102–109, (1958)

[42] V. Lakshmibai, N. Reshetikhin, Quantum deformations of flag and Schubert schemes, *C. R. Acad. Sci. Paris*, **313**, 121–126, (1991)

[43] V. Lakshmibai, N. Reshetikhin, Quantum flag and Schubert schemes, *Cont. Math.* **134**, 145–181, (1992)

[44] V. A. Lunts, A. L. Rosenberg, Localization for quantum groups, *Selecta Mathematica*, **5**, 123-159, (1999)

[45] G. Landi, C. Reina, A. Zampini, Gauged Laplacians on quantum Hopf bundles, *Comm. Math. Phys.*, **287**, 179 - 209, (2009)

[46] G. Landi, A. Zampini, Calculi, Hodge operators and Laplacians on a quantum Hopf fibration, Preprint (2010)

[47] S. Majid, Classification of bicovariant differential calculi, *J. Geom. Phys.*, **25**, 119–140, (1998)

[48] S. Majid, Quantum and braided group Riemannian geometry, *J. Geom. Phys.*, **30**, 113–146, (1999)

[49] S. Majid, A Quantum Groups Primer, London Mathematical Society Lecture Note Series, *Cambridge University Press*, (2002)

[50] S. Majid, *Foundations of Quantum Group Theory*, Cambridge University Press, Cambridge, 2000

[51] S. Majid, Noncommutative Riemannian and spin geometry of the standard q-sphere, *Comm. Math. Phys.*, **256**, 255–285, (2005)

[52] S. Majid, q-Fuzzy spheres and quantum differentials on $B_q[SU_2]$ and $U_q(su_2)$, *Lett. Math. Phys.*, **98**, 167–191, (2011)

[53] U. Meyer, Projective quantum spaces, *Lett. Math. Phys.*, **35**, 91–97, (1995)

[54] S. Murray, C. Sämann, Quantization of flag manifolds and their supersymmetric extensions, *Adv. Theor. Math. Phys.*, **12**, 641–710, (2008)

[55] E. F. Müller, H.-J. Schneider, Quantum homogeneous spaces with faithfully flat module structures, *Israel Journal of Mathematics*, **111**, 157–190, (1999)

[56] S. Neshveyev, L. Tuset, The Dirac operator on compact quantum groups, *J. Reine Angew. Math.*, **641**, 1–20, (2010)

[57] C. Okonek, M. Schneider, H. Spindler, Vector Bundles on Complex Projective Spaces, Birkhäuser Verlag, 1980

[58] R. Ó Buachalla, Quantum bundle description of quantum projective spaces, Comm. Math. Phys., **316**, 345–373, (2013)

[59] R. Ó Buachalla, Noncommutative complex structures on quantum homogeneous spaces, preprint, arXiv:1108.2374

[60] R. Ó Buachalla, The noncommutative Kähler geometry of the standard q-sphere, (in preparation)

[61] R. Ó Buachalla, Noncommutative Kähler structures on quantum homogeneous spaces, (in preparation)

[62] B. Parshall, J. Wang, Quantum linear groups, *Memoirs AMS*, **439**, (1991)

[63] P. Podleś, Quantum spheres, *Lett. Math. Phys.*, **14**, 193–202, (1987)

[64] G. B. Podkolzin, L.I. Vainerman, Quantum Stiefel manifold and double cosets of quantum unitary group, *Pac J. Math.*, **188**, 179–199, (1999)

[65] P. Podleś, Differential calculus on quantum spheres, *Lett. Math. Phys.*, **18**, 107–119, (1989)

[66] A. Polishchuk, A. Schwarz, Categories of holomorphic vector bundles on noncommutative two-tori, *Comm. Math. Phys.*, **236**, 135-159, (2003)

[67] A. Rennie, R. Senior, The resolvent cocycle in twisted cyclic cohomology and a local index formula for the Podleś sphere, preprint, arXiv:1111.5862

[68] Y. S. Soĭbel'man, The algebra of functions on a compact quantum group, and its representations, *Leningrad Math. J.*, **2**, 161–178, (1991)

[69] J. T. Stafford, M. Van den Bergh, Noncommutative curves and noncommutative surfaces, *Bull. Amer. Math. Soc.,* **38**, 171–216, (2001)

[70] Y. Soĭbel'man, Quantum tori, mirror symmetry and deformation theory, *Lett. Math. Phys.*, **56**, 99–125, (2001)

[71] K. Schmüdgen, A. Schüler, Left-covariant differential calculi on $SL_q(2)$ and $SL_q(3)$, *J. Geom. Phys.,* **20**, 87–115, (1996)

[72] Y. S. Soĭbel'man, On quantum flag manifolds, *Func. Ana. Appl.* **25**, 225–227, (1992)

[73] A. Schwarz. Theta functions on noncommutative tori, *Lett. Math. Phys.*, **58**, 81–90, (2001)

[74] Y. S. Soĭbel'man, L. Vaksman, The algebra of functions on the quantum group $SU(n+1)$ and odd-dimensional quantum spheres, *Leningrad Math. J.* **2**, 1023–1042, (1991)

[75] M. Takeuchi, Relative Hopf modules - equivalences and freeness conditions, *J. Algebra*, **60**, 452-471, (1979)

[76] E. Taft, J. Towber, Quantum deformation of flag schemes and Grassmann schemes I. A q-deformation of the shape-algebra for $GL(n)$, *J. Algebra*, **142**, 1–36, (1991)

[77] G. B. Podkolzin, L.I. Vainerman, Quantum Stiefel manifold and double cosets of quantum unitary group, *Pac. J. Math.*, **188**, 179–199, (1999)

[78] C. Voigt, The Baum–Connes conjecture for free orthogonal quantum groups, *Adv. Math.*, **227**, 1873 – 1913, (2011)

[79] S. L. Woronowicz, Differential calculus on compact matrix pseudogroups (quantum groups), *Comm. Math. Phys.*, **122**, 125–170, (1989)