The representation theory of Iwahori-Hecke algebras with unequal parameters

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Abstract

The Iwahori-Hecke algebras of finite Coxeter groups play an important role in many areas of mathematics. In this thesis we study the representation theory of the Iwahori-Hecke algebras of the Coxeter groups of type B_n and F_4, in the unequal parameter case. We denote these algebras H_Q and K_Q respectively. This follows on from work done by Dipper, James, Murphy and Norton. We are interested in the Iwahori-Hecke algebras of type B_n and F_4 since these are the only cases, apart from the dihedral groups, where the Coxeter generators lie in different conjugacy classes, and therefore the Iwahori-Hecke algebra can have unequal parameters. There are two parameters associated with these algebras, Q and q. Norton dealt with the case $Q = q = 0$, whilst Dipper, James and Murphy addressed the case $q \neq 0$ in type B_n. In this thesis we look at the case $Q \neq 0, q = 0$.

We begin by constructing the simple modules for H_Q, then compute the Ext quiver and find the blocks of H_Q. We continue by observing that there is a natural embedding of the algebra of type $n-1$ in the algebra of type n, and this gives rise to the notion of an induced module. We look at the structure of the induced module associated with a given simple H_Q-module. Here we are able to construct a composition series for the induced module and show that in a particular case the induced modules are self-dual.

Finally, we look at K_Q and find that the representation theory is related to representation theory of the Iwahori-Hecke algebra of type B_3. Using this relationship we are able to construct the simple modules for K_Q and begin the analysis of the Ext quiver.

For Irene, Bob, and Graham

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Chapter 1

Introduction

This thesis looks at the representation theory of Iwahori-Hecke algebras of Coxeter groups of type B_n and F_4. More precisely we will consider the unequal parameter case. We will see in Chapter 1 that an Iwahori-Hecke algebra can only have unequal parameters if the generators of the underlying Coxeter group lie in different conjugacy classes, and this only occurs in the dihedral groups and the type B_n and F_4 cases.

In Chapter 2 we define the Iwahori-Hecke algebra of a finite Coxeter group, denoted $H(G)$, as well as giving necessary background material used throughout this work. In the type B_n and F_4 cases there are two parameters, Q and q, associated to $H(G)$. It turns out that these parameters are crucial to the representation theory of $H(G)$. In the type B_n case Dipper, James and Murphy studied the case $q \neq 0$ in [4] and [5]. We close Chapter 2 with a description of their construction of the simple $H(G)$-modules, this is not only interesting but also necessary as it will allow us to classify the simple $H(G)$-modules in the F_4 case. We also include a brief description of the work of Norton, who classified the irreducible $H(G)$-modules when $Q = q = 0$ [11], and Fayers who classified the extensions between simple $H(G)$-modules when $Q = q = 0$ [6].

In Chapter 3 we study $H(G)$ when G is the Coxeter group of type B_n and $Q \neq 0, q = 0$, which we denote H_Q. The first part of this chapter is taken up with classifying the simple H_Q-modules, where we use a method similar to that used by Norton in [11]. We construct a filtration for H_Q and then refine this to a composition series using the properties of the underlying Coxeter group. We then turn our attention to extensions between simple H_Q-modules, and using a similar method to that used by Fayers in [6] we are eventually able to give the Ext quiver for H_Q. If H_Q is of type B_n and H'_Q is of type B_{n-1} then there is a natural way to embed H'_Q in H_Q. This leads us to consider induced modules, and we finish our study of H_Q by looking at modules induced from simple modules. Again using a method similar to Norton we construct a filtration for these induced modules which we then refine to a composition series. It is also possible to define the dual of an H_Q-module, and we show that for a one-dimensional simple H_Q-module the induced module is self-dual.

In Chapter 4 we turn our attention to the Iwahori-Hecke algebra of the Coxeter group of type F_4, which we denote K_Q. Interestingly the representation theory of K_Q is closely related to the representation theory of a subalgebra, R_Q, of K_Q. The algebra R_Q is of type B_3 and using the work of Dipper, James and Murphy we are able to construct the simple R_Q-modules, which in turn allows the construction of the simple K_Q-modules. We end Chapter 4 by looking at extensions between simple K_Q-modules.

Chapter 2

Background

In this chapter we will cover the necessary background material. We include details of the Iwahori-Hecke algebras and also pertinent results from representation theory. We conclude the chapter with an overview of previous work that is relevant to this thesis.

2.1 Iwahori-Hecke algebras

As the Iwahori-Hecke algebras are inextricably linked to Coxeter groups, we begin with some basic facts about these important objects.

2.1.1 Coxeter groups

Definition 1. [7, Section 5.1] A **Coxeter system** is defined to be a pair (G, S) consisting of a group G and a set of generators $S \subset G$, subject to **braid** relations of the form

$$(s_i s_j s_i \ldots)_{m(s_i,s_j)} = (s_j s_i s_j \ldots)_{m(s_i,s_j)},$$

and quadratic relations

$$s_i^2 = 1$$

where $m(s_i, s_j)$ is the order of $s_i s_j$ in G, and $m(s_i, s_i) = 1, m(s_i, s_j) = m(s_j, s_i) \geq 2$ for $s_i \neq s_j$ in S. In case no relation occurs for a pair s_i, s_j, we make the convention that

$m(s_i, s_j) = \infty$. Here $(aba\ldots)_m$ denotes an alternating product of m terms. We may refer to G itself as a **Coxeter group**.

Definition 2. [7, Section 5.2] Since the generators $s \in S$ have order 2 in G, each $w \neq 1$ in G can be written in the form $w = s_1 s_2 \cdots s_r$ for some s_i (not necessarily distinct) in S. If r is as small as possible, call it the **length** of w, written $l(w)$, and call any expression of w as a product of r elements of S a **reduced expression**. By convention, $l(1) = 0$.

Lemma 1. *[7, Section 1.7] Let $w = s_1 s_2 \cdots s_r$ (not necessarily reduced), where each $s_i \in S$. If $l(ws) < l(w)$ for some $s \in S$, then there exists an index i for which $ws = s_1 \cdots \hat{s}_i \cdots s_r$ (and thus $w = s_1 \cdots \hat{s}_i \cdots s_r s$, with a factor s exchanged for a factor s_i). In particular, w has a reduced expression ending in s if and only if $l(ws) < l(w)$.*

Lemma 2. *[7, Section 1.8] Let G be a Coxeter group. There is a unique longest element $w_0 \in G$.*

For example in the Coxeter group of type F_4 the longest element has length 24, and in type B_n the longest element has length n^2.

Theorem 3. *[10, Section 1.8]* ***(Matsumoto)*** *Suppose that $s_{i_1}, \ldots, s_{i_k}$ and $s_{j_1}, \ldots, s_{j_k}$ are elements of S such that $s_{i_1} \cdots s_{i_k}$ and $s_{j_1} \cdots s_{j_k}$ are two reduced expressions in G. Then $s_{i_1} \cdots s_{i_k} = s_{j_1} \cdots s_{j_k}$ if and only if $s_{i_1} \cdots s_{i_k}$ can be transformed into $s_{j_1} \cdots s_{j_k}$ using only the braid relations.*

Definition 4. [7, Section 1.10] For any subset $I \subset S$, define G_I to be the subgroup of G generated by all $s \in I$. All subgroups of G obtainable in this way are called **parabolic subgroups**.

Lemma 3. *[7, Section 1.10] Define $G^I = \{w \in G : l(sw) > l(w) \text{ for all } s \in I\}$. Given $w \in G$, there is a unique $u \in G^I$ and a unique $v \in G_I$ such that $w = uv$. Their lengths satisfy $l(w) = l(u) + l(v)$, and u is the unique element of smallest length in the coset wG_I.*

A similar result holds using right cosets.

Definition 5. The distinguished coset representatives G^I may be called **minimal coset representatives**.

If (G, S) is a Coxeter system, then as an abstract group G is determined up to isomorphism by the set of integers $m(s, s'), s, s' \in S$. This information can be encoded in a graph in the following way.

Definition 6. [7, Section 2.1] Let (G, S) be a Coxeter system. Let Γ be a graph with vertex set in bijection with S, join a pair of vertices corresponding to $s \neq s'$ by an edge whenever $m(s, s') \geq 3$, and when $m(s, s') > 3$ label such an edge with $m(s, s')$. This labelled graph is the **Coxeter graph** of G.

Definition 7. [7, Section 2.2] We say that a Coxeter system (G, S) is **irreducible** if the Coxeter graph of G is connected.

Throughout this work we will concentrate on the Iwahori-Hecke algebras of finite, irreducible Coxeter groups. This is because a reducible Coxeter group is a direct product of irreducible Coxeter groups [7, Section 2.2], and the Iwahori-Hecke algebra of a reducible Coxeter group is a tensor product of Iwahori-Hecke algebras of irreducible Coxeter groups. Finite, irreducible Coxeter groups have been classified, see [7, Chapter 2], and below we give the Coxeter graphs for type B_n, F_4 and $I_2(m)$.

$B_n (n \geq 2)$ $\quad s_0 \overset{4}{\text{———}} s_1 \text{———} s_2 \cdots\cdots s_{n-3} \text{———} s_{n-2} \text{———} s_{n-1}$

F_4 $\quad s_3 \text{———} s_0 \overset{4}{\text{———}} s_1 \text{———} s_2$

$I_2(m)$ $\quad s_0 \overset{m}{\text{———}} s_1$

It is only in these three cases that the Coxeter generators are not all in the same conjugacy class (for $I_2(m)$, m must be even for the generators to be in different conjugacy classes). In the next section we will see the definition of the Iwahori-Hecke algebra of a Coxeter group, and there we will see that it is only in these three cases that unequal parameters become possible.

2.1.2 Iwahori-Hecke algebras

We now come to the main objects of interest to us, we begin with a definition.

Definition 8. [8, Section 3.2] Let G be a Coxeter group with generators $s_0, s_1, \ldots, s_{n-1}$. Given a field $\mathbb{F}$ and $q_i \in \mathbb{F}$, subject to the condition that if s_i and s_j are in the same conjugacy class of G then we must have $q_i = q_j$, we define the **Iwahori-Hecke algebra** of G, denoted $H(G)$, to be the associative, unital algebra over $\mathbb{F}$ with generators $S_0, S_1, \ldots, S_{n-1}$ and relations

- $(S_i - 1)(S_i + q_i) = 0$,
- $(S_iS_jS_i \ldots)_{m(s_i,s_j)} = (S_jS_iS_j \ldots)_{m(s_i,s_j)}$,

for all $i \neq j$, here $(aba \ldots)_m$ denotes an alternating product of m terms.

Note that in the typr B_n case we will follow the convention in [6] so that the generators S_i that we are using here are the negatives of the generators T_i used in most of the literature. In the type F_4 case we use a different convention, given in Definition 12. With suitable adjustments Example 3 (see the end of Chapter 2) still works regardless of which convention for the generators is used. Note also that we follow the convention of writing S_{s_i} as S_i. Having seen this definition we observe how $H(G)$ can be thought of as a deformation of the group algebra of G. One feature of $H(G)$ that we will make use of throughout this work is the existence of the following basis.

Definition 9. [8, Section 3.2] For $w \in G$ we define $S_w \in H(G)$ by $S_w = S_{i_1}S_{i_2} \cdots S_{i_k}$ where $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression in G.

Note that by Matsumoto's theorem S_w is well defined.

Theorem 10. *[8, Section 3.3] Let G be a Coxeter group and let $H(G)$ be the Iwahori-Hecke algebra of G, then $\{S_w | w \in G\}$ is an $\mathbb{F}$-basis for $H(G)$.*

Iwahori-Hecke algebras have been studied extensively for certain values of the parameters, and we will outline the details of these results in Section 2.4. Next we introduce some notation.

Definition 11. Let G be the Coxeter group of type B_n, and let $H(G)$ be the Iwahori-Hecke algebra of G with $q_0 = Q, q_i = 0$ for $0 < i \leqslant n-1$. Then $H_Q := H(G)$.

Definition 12. Let G be the Coxeter group of type F_4, and let $H(G)$ be the Iwahori-Hecke algebra of G with $(S_0 - Q)(S_0 + 1) = (S_3 - Q)(S_3 + 1) = 0, S_1^2 = S_1, S_2^2 = S_2$. Then $K_Q := H(G)$.

Observe that H_Q has dimension $2^n n!$, and K_Q has dimension $2^7 3^2 = 1152$.

2.2 Some background on Rings, Modules and Representation theory

We now give some definitions and results which will be useful when studying the representation theory of $H(G)$. First we introduce the idea of a composition series.

2.2.1 Composition series

Definition 13. [2, Definition 2.5.1] Let R be a ring and M a nonzero left R-module. A **composition series** of length n for M is a chain of $n+1$ submodules $M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$ such that M_{i-1}/M_i is a simple module for $i = 1, 2, \ldots, n$. These simple modules are called the **composition factors** of the series.

Definition 14. [2, Section 2.5] If M has a composition series as defined above and $M = N_0 \supset N_1 \supset \cdots \supset N_p = 0$ is a second composition series, then the two series are said to be **equivalent** if $n = p$ and there is a permutation σ such that for $i = 0, 1, \ldots, n-1, M_i/M_{i+1} \cong N_{\sigma(i)}/N_{\sigma(i)+1}$.

The most important theorem regarding composition series is the Jordan-Hölder theorem.

Theorem 15. *[2, Theorem 2.5.2] If a module M has a composition series, then any other composition series for M is equivalent to it.*

Definition 16. [9, Chapter 8] Let R be a ring and M a nonzero left R-module. A **filtration** of length n for M is a chain of $n+1$ submodules $M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$.

If we restrict our attention to finite-dimensional modules for algebras then we have the following theorem.

Theorem 17. *Any filtration can be refined to give a composition series.*

Proof. Let R be a ring and let M be an R-module. Suppose $M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$ is a filtration for M. If M_i/M_{i+1} is not simple for some i then M_{i+1} is not maximal in M_i, that is to say there exists an R-module L such that $M_i \supset L \supset M_{i+1}$. Then $M = M_0 \supset M_1 \supset \cdots \supset M_i \supset L \supset M_{i+1} \supset \cdots \supset M_n = 0$ is a filtration for M. Since M has finite dimension we can repeat this process until we obtain a composition series for M. □

Composition series will be used in the following chapters to construct simple modules for H_Q. More precisely we need the following theorem:

Theorem 18. *[2, Proposition 2.1.8, Proposition 2.1.11] Let R be a ring. If M is a simple left R-module, then there is a maximal left ideal I of R such that $R/I \cong M$.*

It follows from this that any simple R-module M is isomorphic to a top composition factor in a composition series for R (considering R as a left R-module). Moreover, we know from the Jordan-Hölder theorem that given any composition series for R the composition factors of that series are the only possible candidates for this top factor. By definition the factors of a composition series are simple modules and so the factors of a composition series for R give us a complete list of simple modules for R up to isomorphism.

2.2.2 Dual modules

Dual modules will arise a number of times throughout this work, here we make clear what we mean by a dual module. We will restrict to the B_n case and make the following definition.

Definition 19. [6, Proposition 3.2] Let M be a left H_Q-module. There is an anti-automorphism of H_Q defined by $\phi : S_i \longmapsto S_i$ for all i. Now we can make the dual vector space M^* into a module with H_Q action $(h \cdot f)(m) = f(\phi(h)m)$ for $h \in H_Q, f \in M^*$ and $m \in M$.

2.3 Blocks

We now introduce a decomposition of a given algebra into a direct sum of subalgebras. A full exposition of this area can be found in [1] and [3], here we cover the main results and definitions that we will use later. Throughout this section all algebras are unital.

Theorem 20. *[1, Section 13.1] Let A be a finite dimensional algebra. Then A has a unique decomposition into a direct sum of subalgebras each of which is indecomposable as an algebra.*

Definition 21. [1, Section 13.1] If the unique decomposition of A is given by $A = A_1 + \cdots + A_r$ then the subalgebras $A_1, \ldots, A_r$ are called the **blocks** of A. If M is an A-module, $A_i M = M$ and $A_j M = 0$ for all $j \neq i$, then we say that M **lies in the block** A_i. Conversely, suppose that N is an A_i-module, we can make N into an A-module by demanding that each $A_j, j \neq i$ annihilates N, and it lies in A_i. Each A-module lying in A_i clearly arises in this way.

The following result shows how the study of A-modules reduces to the study of A_i-modules, and gives us some motivation for finding the blocks of A.

Proposition 1. *[1, Section 13.2] If M is an A-module then M has a unique direct decomposition $M = M_1 + \cdots + M_r$ where M_i lies in the block A_i.*

Now we need a useful way to determine the blocks of A. In order to do this we will first need the following definitions and proposition.

Definition 22. [9, Chapter 3] Let R be a ring and let M, M', M'' be left R-modules. Let $M' \xrightarrow{f} M \xrightarrow{g} M''$ be a sequence of homomorphisms. We will call this sequence **exact** if $\text{Im } f = \text{Ker } g$.

Proposition 2. *[9, Section 3.2] Let* $0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0$ *be an exact sequence of modules. The following conditions are equivalent:*

1. *There exists a homomorphism* $\varphi : M \to E$ *such that* $g \circ \varphi =$*id.*

2. *There exists a homomorphism* $\psi : E \to N$ *such that* $\psi \circ f =$*id.*

If these conditions are met, then we have isomorphisms $E = \mathit{Im}\ f \oplus \mathit{Ker}\ \psi$, $E = \mathit{Ker}\ g \oplus \mathit{Im}\ \varphi$, $E \cong N \oplus M$.

Definition 23. When the conditions of Proposition 2 are satisfied then we say that the exact sequence is **split**. Otherwise, we say that the exact sequence is **non-split**.

Definition 24. [9, Chapter XX] Let R be a ring and let M, N be left R-modules. An **extension** of M by N is an exact sequence $0 \to N \to E \to M \to 0$. If this exact sequence is split then we say that the extension is split, otherwise we say that the extension is non-split.

Definition 25. [3, Definition 2.6.1] Let R be a ring and let M, N be left R-modules. Let $0 \to N \to E \to M \to 0$ and $0 \to N \to E' \to M \to 0$ be extensions of M by N. These extensions are **isomorphic** if the following diagram commutes and ϕ and ψ are isomorphisms.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi} & & \downarrow & & \downarrow{\scriptstyle \psi} & & \\
0 & \longrightarrow & N & \longrightarrow & E' & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

If ϕ and ψ are the identity map then we say that the extensions are **equivalent**.

Now we are ready to give a useful condition for when two simple modules lie in the same block.

Proposition 3. *[1, Section 13.3] If* S *and* T *are simple* A*-modules then the following are equivalent:*

1. S and T lie in the same block.

2. There are simple A-modules $S = T_1, T_2, \ldots, T_n = T$ such that $T_i, T_{i+1}, 1 \leqslant i < n$, are equal or there is a non-split extension of one of them by the other.

In order to make use of Proposition 3 we will need to introduce the functor Ext. A full explanation of the functor Ext can be found in [3], but for our purposes we are only interested in the space $\mathrm{Ext}_A^1(M, M')$, where M and M' are A-modules. We have the following definitions:

Definition 26. [3, Definition 2.3.1] A **chain complex** of modules consists of a collection, $\mathbf{C} = \{C_n | n \in \mathbb{Z}\}$, of modules indexed by the integers, together with maps $\delta_n : C_n \to C_{n-1}$ satisfying $\delta_n \circ \delta_{n+1} = 0$. A **cochain complex** of modules consists of a collection, $\mathbf{C} = \{C^n | n \in \mathbb{Z}\}$, of modules indexed by the integers, together with maps $\delta^n : C^n \to C^{n+1}$ satisfying $\delta^n \circ \delta^{n-1} = 0$.

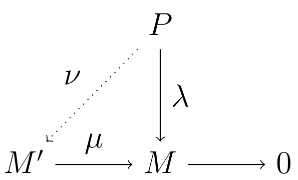
Definition 27. [3, Definition 2.3.2] The **homology** of a chain complex $\mathbf{C}$ is given by

$$H_n(\mathbf{C}) = H_n(\mathbf{C}, \delta_*) = \frac{\mathrm{Ker}(\delta_n : C_n \to C_{n-1})}{\mathrm{Im}(\delta_{n+1} : C_{n+1} \to C_n)}.$$

The **cohomology** of a cochain complex $\mathbf{C}$ is given by

$$H^n(\mathbf{C}) = H^n(\mathbf{C}, \delta^*) = \frac{\mathrm{Ker}(\delta^n : C^n \to C^{n+1})}{\mathrm{Im}(\delta^{n-1} : C^{n-1} \to C^n)}.$$

Definition 28. [3, Definition 1.5.1] A module P is **projective** if given modules M and M', a map $\lambda : P \to M$ and a surjective homomorphism $\mu : M' \to M$ there exists a map $\nu : P \to M'$ such that the following diagram commutes.



Definition 29. [3, Definition 2.4.1] Let R be a ring. A **projective resolution** of an R-module M is a long exact sequence

$$\cdots \to P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0$$

of modules with the P_n projective and with $P_0/\mathrm{Im}(\delta_1) \cong M$.

Definition 30. [3, Section 2.4] Let R be a ring. If M' is a left R-module and

$$\cdots \to P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0$$

is a projective resolution of a left R-module M, then we have a cochain complex

$$\mathrm{Hom}_R(P_0, M') \xrightarrow{\delta^0} \mathrm{Hom}_R(P_1, M') \xrightarrow{\delta^1} \mathrm{Hom}_R(P_2, M') \to \cdots$$

where δ^n is given by composition with δ_{n+1}. This complex is independent of choice of projective resolution, up to chain homotopy equivalence. Thus its cohomology groups are independent of this choice, and we define

$$\mathrm{Ext}^n_R(M, M') = H^n(\mathrm{Hom}_R(\mathbf{P}, M'), \delta^*).$$

It is possible to think of the space $\mathrm{Ext}^1_R(M, M')$ as the set of equivalence classes of extensions $0 \to M' \to M_0 \to M \to 0$, where the equivalence relation is isomorphism of short exact sequences, see [3, Section 2.4], and it is this interpretation of $\mathrm{Ext}^1_R(M, M')$ that we will use. More precisely we have the following lemma:

Lemma 4. *Let S and T be simple A-modules. Then*

- *if there are no non-split extensions of S by T then $\mathrm{Ext}^1_A(S, T) = 0$,*
- *if there is a non-split extension of S by T and all non-split extensions are isomorphic, then $\mathrm{Ext}^1_A(S, T)$ is one-dimensional.*

This leads to our next definition.

Definition 31. [3, Definition 4.1.6] Suppose A is a finite dimensional algebra over an algebraically closed field $\mathbb{F}$. Let $S_1, \ldots, S_r$ be the isomorphism classes of simple A-modules. The **Ext-quiver** is a directed graph (possibly with multiple arrows and loops) with vertices $v_1, \ldots, v_r$ corresponding to these simple modules, and the number of arrows from v_i to v_j is $\dim_{\mathbb{F}} \mathrm{Ext}^1_A(S_i, S_j)$.

Theorem 32. *Let L and K be simple A-modules, then $\text{Ext}^1_A(L, K)$ has the same dimension as $\text{Ext}^1_A(K^*, L^*)$.*

Proof. Let L and K be simple A-modules, and let E be an extension of K by L. Then E^* is an extension of L^* by K^*, and we observe that taking duals preserves equivalence of extensions. □

2.4 Previous work by Norton, James, Dipper, Murphy and Fayers

Now we give some details of the work done previously on Iwahori-Hecke algebras, beginning with Norton's work on the 0-Hecke algebras.

2.4.1 The 0-Hecke algebras

Let $H(G)$ be an Iwahori-Hecke algebra for an arbitrary finite Coxeter group G. If $q_i = 0$ for all i then we call $H(G)$ a 0-Hecke algebra. Norton studied the 0-Hecke algebras in [11] and was able to classify the simple modules and decompose the algebra into left ideals. Norton gives the following theorem:

Theorem 33. *[11, Section 3][6, Theorem 2.2] Let (G, S) be a Coxeter system and let $H(G)$ be the 0-Hecke algebra of G. Then $H(G)$ has 2^n distinct irreducible representations, where $n = |S|$. The irreducible $H(G)$-modules are one-dimensional, more explicitly, given a subset $J \subseteq \{0, 1, \ldots, n-1\}$, let M_J be the $H(G)$-module with basis $\{x\}$ and $H(G)$-action given by*

$$S_i x = \begin{cases} x & \text{if } i \in J \\ 0 & \text{if } i \notin J. \end{cases}$$

Then $\{M_J | J \subseteq \{0, 1, \ldots, n-1\}\}$ is a complete set of irreducible modules for $H(G)$.

It is worth noting that to obtain this theorem Norton uses a composition series for $H(G)$ which she constructs, in [11, 2], in the following way: first she lists the basis elements of $H(G)$ in order of increasing length. Then she renames these elements

$h_1, h_2, \ldots, h_{|G|}$ respectively. Then she constructs ideals of $H(G)$: $H_1, H_2, \ldots, H_{|G|}$, where H_i is the ideal of $H(G)$ generated by $\{h_m : m \geq i\}$. Then there is a composition series $H(G) = H_1 > H_2 \ldots > H_{|G|} > 0$.

We point this out since we will use the same approach to obtain our results for H_Q and K_Q. For the 0-Hecke algebras it turns out that the composition factors are one-dimensional. We will see later that when the parameters of the Iwahori-Hecke algebra are allowed more freedom, we end up with a slightly more complicated situation and find that irreducible representations of higher dimension occur.

In [6] Fayers was able to build on the work of Norton, and of particular relevance to this thesis is his result on extensions between simple $H(G)$-modules when $H(G)$ is a 0-Hecke algebra.

Theorem 34. *[6, Theorem 5.1] Suppose G is an arbitrary finite Coxeter group, and let $H(G)$ be the 0-Hecke algebra of G over an arbitrary field. Suppose $J, K \subseteq \{0, \ldots, n-1\}$ Then the dimension of $Ext^1_{H(G)}(M_J, M_K)$ is 1 if*

- *neither of J and K is contained in the other, and*
- *for any $j \in J \setminus K$ and $k \in K \setminus J$, we have $m(s_j, s_k) \geqslant 3$,*

and 0 otherwise.

Since in this case the simple $H(G)$-modules are one-dimensional the above result can be obtained by classifying the two-dimensional indecomposable $H(G)$-modules. Later, in Chapter 3, we will see that in type B_n when $Q \neq 0, q = 0$ the simple modules are also of low dimension, and we are able to use the same method in order to classify the extensions between simple modules.

2.4.2 Iwahori-Hecke algebras of type B_n

Now we will cover some of the background in the study of $H(G)$ when G is the Coxeter group of type B_n. This was begun by Dipper and James in [4]. They were able to construct a complete list of non-isomorphic simple $H(G)$-modules, however there were certain restrictions on the values of the parameters q and Q. Later in [5] Dipper, James and Murphy were able to remove these restrictions and give a set of $H(G)$-modules labelled by bipartitions of n, were the non-zero modules in this set give a complete set of non-isomorphic simple $H(G)$-modules. We note at this point that one important restriction does remain throughout the work of Dipper, James and Murphy, namely $q \neq 0$.

Here we will describe the method used in [5] to construct the set of simple $H(G)$-modules, and give an example for the B_3 case. We do this not only to allow the reader to become familiar with the algebra $H(G)$, but also because the Iwahori-Hecke algebra of type B_3 (under the restrictions made in [5]) will be important when we come to study the algebra K_Q.

Let G be the Coxeter group of type B_n with generators $s_0, s_1, \ldots, s_{n-1}$, and let $H(G)$ be the Iwahori-Hecke algebra of G over $\mathbb{F}$ with quadratic relations

1. $S_0^2 = Q + (Q-1)S_0$,
2. $S_i^2 = q + (q-1)S_i$ for $1 \leqslant i \leqslant n-1$,

subject to the condition that $q \neq 0$. Next we define some notation.

Definition 35. [4] Let $s_0, \ldots, s_{n-1}$ be a set of generators for a Coxeter group G of type B_n, for each pair $i, j > 0$, define $s_{i,j}$ inductively by $s_{i,i} = 1$ for all i and

$$s_{i,j} = \begin{cases} s_i s_{i+1,j} & \text{if } i < j, \\ s_{i,j+1} s_j & \text{if } i > j. \end{cases}$$

For example $s_{1,5} = s_1 s_2 s_3 s_4$ and $s_{6,2} = s_5 s_4 s_3 s_2$. We also let $S_{i,j} \in H(G)$ be given by $S_{i,j} = S_{s_{i,j}}$.

Definition 36. [5, Section 2.1] For $0 \leqslant a \leqslant n$ let the element $u_a^+ \in H(G)$ be given by

$$u_a^+ = \prod_{i=1}^{a} (q^{i-1} + S_{i,1} S_0 S_{1,i}).$$

Definition 37. [5, Section 2.4] Let $\mathfrak{S}_{\{a+1,\ldots,n\}}$ be the symmetric group on $\{a+1, \ldots, n\}$. For $1 \leqslant i \leqslant n-1$, we identify s_i with the transposition $(i, i+1) \in \mathfrak{S}_n$.

Definition 38. [5, Section 3] Let a be a non-negative integer. A **partition** of a is a non-increasing sequence of non-negative integers whose sum is a.

An a-**bipartition** of n is an ordered pair $(\lambda^{(1)}, \lambda^{(2)})$ of partitions, where $\lambda^{(1)}$ is a partition of a and $\lambda^{(2)}$ is a partition of $n-a$.

Definition 39. [5, Section 3] We define a partial order on the set of bipartitions of n as follows. Let $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ and $\mu = (\mu^{(1)}, \mu^{(2)})$ be bipartitions of n, with

$$\begin{aligned}
\lambda^{(1)} &= (\lambda_1^{(1)}, \lambda_2^{(1)}, \ldots), \\
\lambda^{(2)} &= (\lambda_1^{(2)}, \lambda_2^{(2)}, \ldots), \\
\mu^{(1)} &= (\mu_1^{(1)}, \mu_2^{(1)}, \ldots), \\
\mu^{(1)} &= (\mu_1^{(1)}, \mu_2^{(1)}, \ldots).
\end{aligned}$$

Choose the integer m large enough so that $\lambda_i^{(1)} = \mu_i^{(1)} = 0$ for all $i > m$. Then $\lambda \trianglerighteq \mu$ if

$$\sum_{i=1}^{j} \lambda_i^{(1)} \geqslant \sum_{i=1}^{j} \mu_i^{(1)} \text{ for all } j,$$

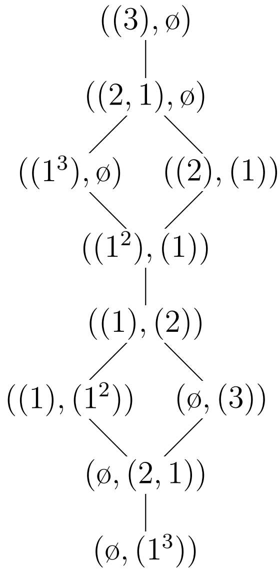
and

$$\sum_{i=1}^{m} \lambda_i^{(1)} + \sum_{i=1}^{j} \lambda_i^{(2)} \geqslant \sum_{i=1}^{m} \mu_i^{(1)} + \sum_{i=1}^{j} \mu_i^{(2)} \text{ for all } j.$$

If $\lambda \trianglerighteq \mu$ we say that λ dominates μ. If $\lambda \trianglerighteq \mu$ and $\lambda \neq \mu$ then we write $\lambda \triangleright \mu$.

We give an example of this partial order on the set of 3-bipartitions, here if μ is above λ then $\mu \triangleright \lambda$, and $\varnothing$ denotes the empty partition.

Example 1.



Definition 40. [5, Section 3] Let λ be a bipartition of n. We obtain a corresponding **diagram** $[\lambda]$ which consists of crosses in the plane, for example if $\lambda = ((3, 2), (2, 1))$ then

$$[\lambda] = \begin{array}{ccccccc} \times & \times & \times & & \times & \times \\ \times & \times & & & \times & \end{array}$$

We obtain a λ-**bitableau** from $[\lambda]$ by replacing each cross with one of the numbers $1, 2, \ldots, n$, allowing no repeats. A λ-bitableau $t = (t^{(1)}, t^{(2)})$ is **row-standard** if the entries increase from left to right in each row of $t^{(1)}$ and in each row of $t^{(2)}$. A row-standard λ-bitableau t is **standard** if the entries increase down each column.

Definition 41. [5, Section 3] Suppose that λ is an a-bipartition. Let $t^{\lambda} = (t^{\lambda(1)}, t^{\lambda(2)})$ be the standard λ-bitableau in which the numbers $1, 2, \ldots, a$ appear in order by rows in $t^{\lambda(1)}$ and the numbers $a + 1, \ldots, n$ appear in order by rows in $t^{(2)}$.

Definition 42. [5, Section 4] Let $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ be an a-bipartition of n. Let

$$x_{\lambda^{(1)}} = \sum \{S_w | w \in \mathfrak{S}_a \text{ and } w \text{ stabilizes the rows of } t^{(1)}\},$$

$$x_{\lambda^{(2)}} = \sum \{S_w | w \in \mathfrak{S}_{\{a+1,\ldots,n\}} \text{ and } w \text{ stabilizes the rows of } t^{(2)}\},$$

$$x_{\lambda} = u_a^{+} x_{\lambda^{(1)}} x_{\lambda^{(2)}}.$$

At this point we observe the fact that the three elements $u_a^+, x_{\lambda^{(1)}}$ and $x_{\lambda^{(2)}}$ commute, see [5, Section 3, Section 2]. To illustrate the above definition we give an example.

Example 2. Suppose $\lambda = ((3), \emptyset)$, then

$$\begin{aligned} u_3^+ &= \prod_{i=1}^{3}(q^{i-1} + S_{i,1}S_0S_{1,i}) \\ &= (1+S_0)(q+S_1S_0S_1)(q^2+S_2S_1S_0S_1S_2) \\ &= q^3 + q^3S_0 + q^2S_1S_0S_1 + q^2S_0S_1S_0S_1 + qS_2S_1S_0S_1S_2 + qS_0S_2S_1S_0S_1S_2 + \\ &\quad S_1S_0S_1S_2S_1S_0S_1S_2 + S_0S_1S_0S_1S_2S_1S_0S_1S_2, \end{aligned}$$

$x_{\lambda^{(1)}} = 1 + S_1 + S_2 + S_1S_2 + S_2S_1 + S_1S_2S_1$ and $x_{\lambda^{(2)}} = 1$.

Definition 43. [5, Section 4] Let * be the antiautomorphism of $H(G)$ defined by letting $S_w^* = S_{w^{-1}}$, and then extending to a linear map.

Definition 44. [5, Section 3] Let λ be a bipartition of n. Let t be a row-standard λ-bitableau, then we define $d(t)$ to be the element of $\mathfrak{S}_n$ which sends t^λ to t.

Definition 45. [5, Section 4] Suppose λ is a bipartition of n and ρ and σ are row-standard λ-bitableaux, then let

$$x_{\rho\sigma} = S_{d(\rho)}x_\lambda S_{d(\sigma)}^*$$

$$x_{\lambda\sigma} = S_{d(\sigma)}x_\lambda$$

The purpose of the previous definitions is to allow us to construct a set of $H(G)$-modules which in turn will give a complete list of simple $H(G)$-modules.

Definition 46. [5, Section 4] Let λ be a bipartition of n. Let $\overline{N}^\lambda$ be the $\mathbb{F}$-module spanned by

$$\{x_{rs} | r \text{ and } s \text{ are standard } \mu\text{-bitableaux for some bipartition } \mu \text{ of } n \text{ with } \mu \rhd \lambda\}$$

In fact this spanning set is an $\mathbb{F}$-basis for $\overline{N}^\lambda$, see [5, Corollary 4.15]. We are now in a position to define the left $H(G)$-module S^λ.

Definition 47. [5, Section 4] Let λ be a bipartition of n. Let

$$M^\lambda = H(G)x_\lambda,$$

$$\overline{M}^\lambda = M^\lambda \cap \overline{N}^\lambda,$$

$$S^\lambda = M^\lambda / \overline{M}^\lambda.$$

We call the module S^λ a Specht module for $H(G)$.

Theorem 48. *[5, Theorem 4.20] If λ is a bipartition of n then S^λ is $\mathbb{F}$-free, with basis*

$$\{x_{\lambda t} + \overline{M}^\lambda | t \text{ is a standard } \lambda\text{-bitableau}\}$$

Note that the Specht module S^λ has dimension equal to the number of standard λ-bitableaux.

Theorem 49. *[5, Theorem 4.22] Every simple $H(G)$-module is isomorphic to a composition factor of some S^λ. When $H(G)$ is semi-simple, each S^λ is absolutely irreducible, and as λ runs over bipartitions of n, S^λ runs over a complete set of simple $H(G)$-modules.*

We will now focus on an example. We consider $H(G)$ when G is the Coxeter group of type B_3. This example and the classification of the simple $H(G)$-modules that it contains is a deduction from the general results in [5].

Example 3. Let G be the Coxeter group of type B_3, and let H be the Iwahori-Hecke algebra of G with $Q = 0$. Firstly, there are ten bipartitions of 3 and they are listed in Example 1. We will compute for $S^{(3),\emptyset}$:

$$\begin{aligned} u_3^+ &= q^3 + q^3S_0 + q^2S_1S_0S_1 + q^2S_0S_1S_0S_1 + qS_2S_1S_0S_1S_2 + qS_0S_2S_1S_0S_1S_2 + \\ &\quad S_1S_0S_1S_2S_1S_0S_1S_2 + S_0S_1S_0S_1S_2S_1S_0S_1S_2, \end{aligned}$$

$$\begin{aligned} x_{(3)} &= 1 + S_1 + S_2 + S_1S_2 + S_2S_1 + S_1S_2S_1, \\ x_\emptyset &= 1, \\ x_{(3),\emptyset} &= (1 + S_1 + S_2 + S_1S_2 + S_2S_1 + S_1S_2S_1)(q^3 + q^3S_0 + q^2S_1S_0S_1 + q^2S_0S_1S_0S_1 + \\ &\quad qS_2S_1S_0S_1S_2 + qS_0S_2S_1S_0S_1S_2 + S_1S_0S_1S_2S_1S_0S_1S_2 + S_0S_1S_0S_1S_2S_1S_0S_1S_2). \end{aligned}$$

There is only one standard $((3), \emptyset)$-bitableau, namely

$$t = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}\ , \ \emptyset.$$

Therefore $S^{(3),\emptyset}$ is one-dimensional. We also have $t^{(3),\emptyset} = t$, so $d(t) = 1$ and $x_{((3),\emptyset)t} = x_{(3),\emptyset}$. In this case there are no bipartitions μ of n such that $\mu \rhd ((3), \emptyset)$, so $S^{(3),\emptyset} = Hx_{(3),\emptyset}$. Now we give the action of H on $S^{(3),\emptyset}$. Observe that $S_i x_{(3)} = qx_{(3)}$ for $i = 1, 2$ and $S_0 u_3^+ = 0$ and recall that $x_{(3)}$ and u_3^+ commute. Hence S_0 acts on $S^{(3),\emptyset}$ as 0 and S_1 and S_2 act on $S^{(3),\emptyset}$ as q.

Now we compute for $S^{(2,1),\emptyset}$:

$$x_{(2,1)} = 1 + S_1$$

$$x_{(2,1),\emptyset} = u_3^+(1 + S_1)$$

There are two standard $((2, 1), \emptyset)$-bitableaux,

$$t_1 = \begin{array}{cc} 1 & 2 \\ 3 & \end{array}\ , \ \emptyset. \quad t_2 = \begin{array}{cc} 1 & 3 \\ 2 & \end{array}\ , \ \emptyset.$$

This tells us that $S^{(2,1),\emptyset}$ is two-dimensional, and we have $d(t_1) = 1, d(t_2) = s_2, x_{((2,1),\emptyset)t_1} = x_{(2,1),\emptyset}, x_{((2,1),\emptyset)t_2} = S_2 x_{(2,1),\emptyset}$. The only 3-bipartition that dominates $((2, 1), \emptyset)$ is $((3), \emptyset)$, and there is only one standard $((3), \emptyset)$-bitableau

$$t = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}\ , \ \emptyset.$$

So $\overline{N}^{(2,1),\emptyset}$ has a basis $\{x_{tt}\}$, where $x_{tt} = x_{(3),\emptyset} = u_3^+(1+S_1+S_2+S_1S_2+S_2S_1+S_1S_2S_1)$. Since $(1 + S_2 + S_1S_2)(1 + S_1) = (1 + S_1 + S_2 + S_1S_2 + S_2S_1 + S_1S_2S_1)$ we see that $\overline{M}^{(2,1),\emptyset} = \{bu_3^+(1 + S_1 + S_2 + S_1S_2 + S_2S_1 + S_1S_2S_1) | b \in \mathbb{F}\}$. So $S^{(2,1),\emptyset}$ has a basis $\{u_3^+(1+S_1) + \overline{M}^{(2,1),\emptyset}, u_3^+ S_2(1+S_1) + \overline{M}^{(2,1),\emptyset}\}$. Now we give the action of H on $S^{(2,1),\emptyset}$. S_0 acts on $S^{(2,1),\emptyset}$ as the zero matrix since $S_0 u_3^+ = 0$ and S_0 commutes with S_2. S_1 acts as

$$\begin{pmatrix} q & -1 \\ 0 & -1 \end{pmatrix},$$

since $S_1(1+S_1) = q(1+S_1)$ and $S_1S_2(1+S_1)u_3^+ = (S_1S_2 + S_1S_2S_1)u_3^+, -(1+S_1+S_2+S_2S_1)u_3^+ = -(1+S_1)u_3^+ - S_2(1+S_1)u_3^+$ and $(S_1S_2+S_1S_2S_1)u_3^+ + (1+S_1+S_2+S_2S_1)u_3^+ = (1+S_1+S_2+S_1S_2+S_2S_1+S_1S_2S_1)u_3^+$. Finally, S_2 acts as

$$\begin{pmatrix} 0 & q \\ 1 & q-1 \end{pmatrix}.$$

We give a full list of the Specht modules in this case below.

- one-dimensional: S_0 acts via $X_1 = (t)$, S_1 acts via $X_0 = (s)$, S_2 acts via $X_3 = (s)$ where $t \in \{0, 1\}$ and $s \in \{-1, q\}$.
  - $S^{(1^3),\emptyset}$: $t = 0, s = -1$,
  - $S^{(3),\emptyset}$: $t = 0, s = q$,
  - $S^{\emptyset,(1^3)}$: $t = 1, s = -1$,
  - $S^{\emptyset,(3)}$: $t = 1, s = q$.
- two-dimensional: S_0 acts via $X_1 = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$, S_1 acts via $X_0 = \begin{pmatrix} q & -1 \\ 0 & -1 \end{pmatrix}$, S_2 acts via $X_3 = \begin{pmatrix} 0 & q \\ 1 & q-1 \end{pmatrix}$ where $t \in \{0, 1\}$.
  - $S^{(2,1),\emptyset}$: $t = 0$,
  - $S^{\emptyset,(2,1)}$: $t = 1$.
- three-dimensional: S_0 acts via $X_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, S_1 acts via $X_0 = \begin{pmatrix} s & 0 & 0 \\ 0 & 0 & q \\ 0 & 1 & q-1 \end{pmatrix}$, S_2 acts via $X_3 = \begin{pmatrix} 0 & q & 0 \\ 1 & q-1 & 0 \\ 0 & 0 & s \end{pmatrix}$, or S_0 acts via $X_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, S_1 acts via $X_0 = \begin{pmatrix} 0 & q & 0 \\ 1 & q-1 & 0 \\ 0 & 0 & s \end{pmatrix}$, S_2 acts via $X_3 = \begin{pmatrix} s & 0 & 0 \\ 0 & 0 & q \\ 0 & 1 & q-1 \end{pmatrix}$, where $s \in \{-1, q\}$.

– $S^{(1),(1^2)}$: S_0 acts via $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $s = -1$,

– $S^{(1),(2)}$: S_0 acts via $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $s = q$,

– $S^{(1^2),(1)}$: S_0 acts via $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $s = -1$,

– $S^{(2),(1)}$: S_0 acts via $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $s = q$.

When we come to study the algebra K_Q we will need the Specht modules for the Iwahori-Hecke algebra of type B_3 with quadratic relations $S_0^2 = S_0, S_i^2 = Q + (Q-1)S_i$ for $i = 1, 2$, so we will replace q with Q and S_0 with $-S_0$.

Chapter 3

The Iwahori-Hecke algebra of type B_n

Here we will begin the study of the Iwahori-Hecke algebra of the Coxeter group of type B_n. We will first recall the definition and make clear some notation that will be used throughout this chapter. Suppose that G is the Coxeter group of type B with n generators $s_0, ..., s_{n-1}$ and the following relations:

$$
\begin{aligned}
(s_0 s_1)^4 &= 1, \\
(s_i s_j)^3 &= 1 \ \text{ if } j = i+1, 1 \leqslant i \leqslant n-2, \\
(s_i s_j)^2 &= 1 \ \text{ for } |i-j| \geqslant 2, \\
s_i^2 &= 1 \ \ 0 \leqslant i \leqslant n-1.
\end{aligned}
$$

Given a field $\mathbb{F}$ and elements Q, q of $\mathbb{F}$, we define the Iwahori-Hecke algebra of G, denoted $H_{Q,q}$, to be the associative algebra over $\mathbb{F}$ with generators $S_0, ..., S_{n-1}$ and relations

$$
\begin{aligned}
(S_0 - 1)(S_0 + Q) &= 0, \\
(S_i - 1)(S_i + q) &= 0, \quad i > 0, \\
S_0 S_1 S_0 S_1 &= S_1 S_0 S_1 S_0, \\
S_i S_j S_i &= S_j S_i S_j, \quad \text{if } j = i+1, 1 \leqslant i \leqslant n-2, \\
S_i S_j &= S_j S_i, \quad \text{if } j < i+1, 0 \leqslant i < j \leqslant n-1.
\end{aligned}
$$

We have seen in the previous chapter that this algebra has been well studied except for the case $Q \neq 0, q = 0$, it is this case that we will study now. Our first task is to understand the simple H_Q-modules. To do this we will construct a composition series for H_Q as a module for itself, and then analyse the composition factors of that series to give a classification of the simple H_Q-modules. In fact we will construct a filtration for H_Q as a module for itself and then refine that filtration to give the required composition series.

3.1 The algorithm LeftIdeal

To construct our filtration we will use an algorithm, and in order to state this we recall some information regarding the Coxeter group G. Firstly, G is a finite group of order $2^n n!$, and secondly we recall the exchange condition: if s is a generator for G and $w \in G$ then w has a reduced expression beginning in s if and only if $l(sw) < l(w)$, where l is the length function, see Lemma 1.

Whenever we have a reduced expression $s_{i_1} \cdots s_{i_r}$ for an element $w \in G$ we have a corresponding basis element $S_w = S_{i_1} \cdots S_{i_r}$. We want to construct the left ideal generated by S_w. To do this we need the following result:

Lemma 5. *[8, Section 3.2] Suppose $w \in G$. Then*

$$S_0 S_w = \begin{cases} S_{s_0 w} & \text{if } l(s_0 w) > l(w) \\ Q S_{s_0 w} + (1-Q) S_w & \text{if } l(s_0 w) < l(w) \end{cases}$$

and for $1 \leqslant i \leqslant n-1$

$$S_i S_w = \begin{cases} S_{s_i w} & \text{if } l(s_i w) > l(w) \\ S_w & \text{if } l(s_i w) < l(w). \end{cases}$$

Now we form the left ideal generated by a given S_w as follows: we begin by defining an algorithm which we call LeftIdeal.

Algorithm 1. LeftIdeal

Input: $w \in G$

let $A = \{w\}$

repeat

    let T be the empty set

    for $x \in A$ do

        for $1 \leqslant i \leqslant n-1$ do

            if $l(s_i x) > l(x)$ and $s_i x \notin A$ then

                let T become $T \cup \{s_i x\}$

            end if

        end for

        if $s_0 x \notin A$ then

            let T become $T \cup \{s_0 x\}$

        end if

    end for

    let A become $A \cup T$

until $T = \{\}$

Output: $A = \{w_1, ..., w_r\}$

Lemma 6. *The elements $S_{w_1}, ..., S_{w_r}$ span the left ideal, I, in H_Q generated by S_w.*

Proof. First note that the algorithm will terminate since either we reach a stage when T is empty before $A = G$ or we put every element of G in A and then T will be empty, and since G is finite this will happen after a finite number of steps.

To see that $B = \{S_{w_1}, ..., S_{w_r}\}$ is contained in I we use Lemma 5. Applying S_0 to any given S_x will result in either an $S_{s_0 x}$ or a linear combination containing $S_{s_0 x}$, so we should always put $s_0 x$ in our set A. For $1 \leqslant i \leqslant n-1$ we know that if $l(s_i x) < l(x)$ then S_i fixes S_x, but when $l(s_i x) > l(x)$ this gives the element $S_{s_i x}$, so we should put $s_i x$ in our set A precisely when $l(s_i x) > l(x)$. Now observe that an arbitrary element of I is a sum of terms of the form $aS_{i_1} S_{i_2} \cdots S_{i_p} S_w$ where $a \in \mathbb{F}$, and this is a linear combination of the elements of B by construction. $\square$

3.2 A filtration for H_Q

To construct a filtration for H_Q we will use an algorithm which we call Filtration. We take the set of elements of G, and run the algorithm LeftIdeal with $w = w_0$ (recall w_0 is the unique longest element in G). Call the left ideal that we construct W_0. Now throw out the set A from G and select an element of longest length from $G \setminus A$, then run the algorithm LeftIdeal with this new element to get another left ideal W_1. Repeat this process until we have exhausted G. We have a collection of left ideals $W_0, \dots, W_l$ with spanning sets $B_0, \dots, B_l$ respectively. We note for later use that we have a corresponding collection of sets of elements of G, $A_0, \dots, A_l$. Now the submodules spanned by $B_0, B_0 \cup B_1, \dots, B_0 \cup \cdots \cup B_l$ give the required filtration.

We want to show that the factors are two-dimensional. To do this we will show that at the ith stage of the algorithm Filtration we add exactly two elements to B_i that are not in B_j for $j \leqslant i$.

Theorem 50. *The factors of the filtration constructed by the algorithm Filtration are two-dimensional.*

Proof. We will use Lemma 3 to show that if w_i is the chosen longest element at the *ith* stage of the algorithm Filtration, then the only elements of B_i that are not in B_j for $j < i$ are S_{w_i} and $S_{s_0 w_i}$. We assume that $l(s_0 w_i) < l(w_i)$. It is clear that if $l(s_k w_i) > l(w_i)$ then $s_k w_i$ is in some $A_j, j < i$ by our choice of w_i. It will be sufficient to show that whenever $l(s_k s_0 w_i) > l(s_0 w_i)$ then $s_k s_0 w_i \in A_j$ for some $j < i$, for all $0 \leqslant k \leqslant n-1$.

For $2 \leqslant k \leqslant n-1$, fix k and let $H \leqslant G$ be the subgroup generated by s_0 and s_k. Then the coset Hw_i contains the elements $w_i, s_0 w_i, s_k w_i, s_k s_0 w_i$. Now if $l(s_0 w_i) < l(w_i)$ and $l(s_k w_i) < l(w_i)$ then by Lemma 3 $s_k s_0 w_i$ must be the unique element of shortest length in Hw_i, hence $l(s_k s_0 w_i) < l(s_0 w_i)$. If $l(s_0 w_i) < l(w_i)$ and $l(s_k w_i) > l(w_i)$ then $s_k w_i$ is in some $A_j, j < i$ and so $s_k s_0 w_i = s_0 s_k w_i$ is in some $A_j, j < i$.

Now we deal with the case of the generator s_1. We want to show that if $l(s_1 s_0 w_i) > l(s_0 w_i)$ then $s_1 s_0 w_i \in A_j$ for some $j < i$. The subgroup K generated by s_0, s_1 contains the elements $1, s_0, s_1, s_0 s_1, s_1 s_0, s_1 s_0 s_1, s_0 s_1 s_0, s_0 s_1 s_0 s_1$, and by Lemma 3 every coset of this subgroup will have the same structure as this with regard to length of elements. If we have $l(s_0w_i) < l(w_i)$ and $l(s_1s_0w_i) > l(s_0w_i)$, then this implies that s_0w_i is made longer by both s_0 and s_1 and this implies that s_0w_i is the unique element of shortest length in Kw_i. This in turn tells us that $l(s_0s_1s_0w_i) > l(w_i)$ and so $s_0s_1s_0w_i$ must appear in some A_j with $j < i$. So $s_1s_0w_i$ must appear in some A_j with $j < i$.

This completes the argument and we have shown that the factors in our filtration are indeed two-dimensional and the only elements in B_i that are not in B_j for $j < i$ are S_{w_i} and $S_{s_0w_i}$. □

We observe that the above argument gives us a basis for the filtration factors, namely $\{S_{w_i}, S_{s_0w_i}\}$.

3.3 Classification of simple modules for H_Q

We now turn our attention to the simple modules. We know from above that the factors of the filtration that we have constructed are two-dimensional. We know that every simple module is a quotient of the regular module, and hence by Jordan-Hölder any composition series for the regular module has all the simple modules among its composition factors. Since we have a filtration with two-dimensional factors we can refine this to a composition series and therefore any simple module is a composition factor of one of our two-dimensional factors. By considering the action of each of the Coxeter group generators on s_0w and w we will give the only possible actions for the Hecke algebra generators on the quotient module W_i/W_{i-1} consistent with the deformation relations $S_0^2 = (1-Q)S_0 + Q$ and $S_i^2 = S_i$, for $i > 0$. Note that $l(s_0w_i) < l(w_i)$.

Theorem 51. *The only possible actions for the Hecke algebra generator S_1 on the quotient module W_i/W_{i-1} with respect to the basis $\{S_{w_i}, S_{s_0w_i}\}$ are*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

For $2 \leqslant k$ the only possible actions for the Hecke algebra generator S_k on the quotient module W_i/W_{i-1} with respect to the basis $\{S_{w_i}, S_{s_0 w_i}\}$ are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

S_0 always acts on the quotient module W_i/W_{i-1} with respect to the basis $\{S_{w_i}, S_{s_0 w_i}\}$ as

$$\begin{pmatrix} 1-Q & 1 \\ Q & 0 \end{pmatrix}.$$

Furthermore whenever S_1 acts as

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

the quotient is a simple H_Q-module, otherwise our quotient splits into two one-dimensional simple H_Q-modules.

Proof. First note that for $k \geqslant 1$ and $S_w \in H_Q$, S_k can only either fix S_w or send it to something that appears earlier in the filtration (equivalent to sending it to zero in the quotient module), and similarly for $S_0 S_w$. So for S_1 it will be sufficient to show that if S_1 sends $S_0 S_w$ to zero then S_1 sends S_w to zero. This follows from the minimal coset representative argument used in Theorem 50, for if $l(s_1 s_0 w) > l(s_0 w)$ then $l(s_1 w) > l(w)$. For $S_k, k \geqslant 2$, we need to show that if S_k sends $S_0 S_w$ to zero then it sends S_w to zero, and if S_k fixes $S_0 S_w$ then it fixes S_w. This follows again from the minimal coset representative argument, since if $l(s_k s_0 w) > l(s_0 w)$ then $l(s_k w) > l(w)$, and if $l(s_k s_0 w) < l(s_0 w)$ then $l(s_k w) < l(w)$. It is clear that S_0 must act as stated.

Now we find the eigenvalues and eigenvectors for these matrices and see which conditions give one-dimensional or two-dimensional simple modules for H_Q. First

$$\begin{pmatrix} 1-Q & 1 \\ Q & 0 \end{pmatrix}$$

has eigenvalues 1 and $-Q$ and eigenvectors

$$\begin{pmatrix} 1 \\ Q \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

respectively, and

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

has eigenvalues 1 and 0 and eigenvectors

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

respectively. So whenever S_1 acts as the matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

we have a candidate two-dimensional simple module for H_Q, and whenever S_1 acts as

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

we find that our two-dimensional factor decomposes into two candidate one-dimensional simple modules. □

Having found that every candidate simple H_Q-module is either one-dimensional or two-dimensional and having given the possible ways that S_i can act on a candidate simple H_Q-module for all i, we now introduce some notation for these candidate simple modules: Let M_{a,b_i} denote the candidate one-dimensional simple H_Q-module where S_0 acts as $a \in \{-Q, 1\}$ and for $i > 0$ S_i acts as $b_i \in \{1, 0\}$. Let N_{b_i} denote the candidate two-dimensional simple H_Q-module where for $i > 1$ S_i acts as

$$\begin{pmatrix} b_i & 0 \\ 0 & b_i \end{pmatrix}, b_i \in \{1, 0\}.$$

Theorem 52. *The candidate H_Q-modules M_{a,b_i} and N_{b_i} satisfy the defining relations of H_Q, and give all the simple H_Q-modules up to isomorphism. Hence there are 2^{n-2} two-dimensional simple H_Q-modules, and if $Q \neq -1$ we have 2^n one-dimensional simple H_Q-modules, if $Q = -1$ there are 2^{n-1} one-dimensional simple H_Q-modules.*

Proof. First we deal with the modules N_{b_i} and start by checking the defining relations for H_Q. It is clear that all the possible matrices for the actions of $S_1, \ldots, S_{n-1}$ are idempotent and that

$$\begin{pmatrix} 1-Q & 1 \\ Q & 0 \end{pmatrix}^2 = (1-Q)\begin{pmatrix} 1-Q & 1 \\ Q & 0 \end{pmatrix} + \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}.$$

Now we check the braid relations. If we have $j \neq i+1, 0 \leqslant i < j \leqslant n-1$ then at least one of S_i or S_j will act as the zero or identity matrix so the relation is satisfied since these matrices commute with all matrices. If we have $j = i+1, 1 \leqslant i \leqslant n-2$ then again one of S_i or S_j will act as the zero or identity matrix and since these matrices commute with all matrices the relation is satisfied. It remains to check the relation for S_0 and S_1. If S_1 acts as the zero or identity matrix then this holds trivially. If S_1 acts as the matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

then we find that both sides of the relation are zero. This shows that the H_Q-modules N_{b_i} satisfy the defining relations for H_Q. Now we look at the modules M_{a,b_i}. It is clear that if S_0 acts as $-Q$ or 1 then this satisfies the quadratic relation for S_0, and that 1 and 0 are idempotent. Now we check the braid relations. If we have $j \neq i+1, 0 \leqslant i < j \leqslant n-1$ then at least one of S_i or S_j will act as 0 or 1 so the relation is satisfied trivially. If we have $j = i+1, 1 \leqslant i \leqslant n-2$ then again one of S_i or S_j will act as 0 or 1 and again the relation is satisfied trivially. It remains to check the relation for S_0 and S_1, but S_1 can only act as 0 or 1 so this relation also holds trivially. We know from Theorem 51 that every simple H_Q-module must have one of these forms, therefore the H_Q-modules M_{a,b_i} and N_{b_i} give us a complete list of simple H_Q-modules up to isomorphism. □

We now observe that the simple H_Q-modules are self-dual. Recall that there is an anti-automorphism of H_Q defined by $\phi : S_i \longmapsto S_i$ for all i, and that if we have an H_Q-module M, we can make the dual vector space M^* into a module with H_Q action $(h \cdot f)(m) = f(\phi(h)m)$ for $h \in H_Q, f \in M^*$ and $m \in M$ (see definition 19). Also recall that if S_i acts via X_i on M with respect to the basis $\{e_1, ..., e_m\}$, then S_i acts via X_i^T on M^* with respect to the dual basis $\{e_1^*, ..., e_m^*\}$.

Theorem 53. *If M is a simple H_Q-module then M and M^* are isomorphic as H_Q-modules.*

Proof. Let M be a simple H_Q-module where S_i acts on M as the matrix X_i. If M is two-dimensional let $B = \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix}$, then $B^{-1}X_iB = X_i^T$. If M is one-dimensional then $X_i = X_i^T$. □

Now recall from Theorem 32 that if L and K are simple H_Q-modules then $\mathrm{Ext}^1_{H_Q}(L, K) = \mathrm{Ext}^1_{H_Q}(K^*, L^*)$. Since the simple H_Q-modules are self-dual we have the following theorem.

Theorem 54. *If L and K are simple H_Q modules then* $\mathrm{Ext}^1_{H_Q}(L, K) = \mathrm{Ext}^1_{H_Q}(K, L)$.

Proof. This follows from Theorems 32 and 53. □

Theorem 54 will be useful below.

3.4 Extensions of simple modules

We will now calculate the space $\mathrm{Ext}^1_{H_Q}(L, K)$ for simple H_Q-modules L and K.

First we give a lemma that will be used in the proofs of our theorems on extensions of simple modules.

Lemma 7. *Let I denote the identity matrix, 0 denote the zero matrix and X, Y denote matrices. If the matrices*

$$A = \begin{pmatrix} 0 & 0 \\ X & I \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & 0 \\ Y & I \end{pmatrix}$$

satisfy a braid relation $AB = BA$ or $ABA = BAB$, then $X = Y$.

If the matrices

$$A = \begin{pmatrix} I & 0 \\ X & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} I & 0 \\ Y & 0 \end{pmatrix}$$

satisfy a braid relation $AB = BA$ or $ABA = BAB$, then $X = Y$.

If the matrices

$$A = \begin{pmatrix} 0 & 0 \\ X & I \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} I & 0 \\ Y & 0 \end{pmatrix}$$

commute, then $X = -Y$*.*

Proof. These results follow immediately from calculation. □

Note that converse statements to those in Lemma 7 also hold, in the first two cases if $X = Y$ then $A = B$ and the braid relations are satisfied, in the third case if $X = -Y$ then $AB = 0 = BA$. Now we look at extensions of one-dimensional simple modules.

Theorem 55. *Let* $L = M_{l_0,l_i}, K = M_{k_0,k_i}$ *be one-dimensional simple* H_Q*-modules.*

When $Q \neq -1$ *the space* $\mathrm{Ext}^1_{H_Q}(L, K)$ *is one-dimensional if either*

(a) there is an $0 < i < n-1$ *such that* $k_i = l_{i+1} \neq l_i = k_{i+1}$ *and* $k_j = l_j$ *for all* $j \neq i, i+1$*, or*

(b) there is an $0 < i < n-2$ *such that* $k_i = l_{i+1} = k_{i+2} \neq l_i = k_{i+1} = l_{i+2}$ *and* $k_j = l_j$ *for all* $j \neq i, i+1, i+2$*.*

Otherwise, $\mathrm{Ext}^1_{H_Q}(L, K) = 0$*.*

Note that conditions b and c below are the same as conditions a and b above. When $Q = -1$ *the space* $\mathrm{Ext}^1_{H_Q}(L, K)$ *is one-dimensional if either*

(a) $k_i = l_i$ *for all* $i > 0$*, or*

(b) there is an $0 < i < n-1$ *such that* $k_i = l_{i+1} \neq l_i = k_{i+1}$ *and* $k_j = l_j$ *for all* $j \neq i, i+1$*, or*

(c) there is an $0 < i < n-2$ *such that* $k_i = l_{i+1} = k_{i+2} \neq l_i = k_{i+1} = l_{i+2}$ *and* $k_j = l_j$ *for all* $j \neq i, i+1, i+2$*.*

Otherwise, $\mathrm{Ext}^1_{H_Q}(L, K) = 0$*.*

Proof. Let M be a two-dimensional H_Q-module such that $0 < K < M$ and $M/K \cong L$. We can choose a basis $\{e_1, e_2\}$ for M such that $e_2 \in K$, let m_i be the matrix of the action of S_i on M with respect to this basis. Then with respect to this basis m_0 must be one of the following matrices:

$$\begin{pmatrix} -Q & 0 \\ x_0 & -Q \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ x_0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -Q & 0 \\ x_0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ x_0 & -Q \end{pmatrix},$$

and for $i > 0$ the m_i must be one of the matrices below:

$$\begin{pmatrix} 1 & 0 \\ x_i & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ x_i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ x_i & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ x_i & 0 \end{pmatrix}.$$

We now check the relations for H_Q to see if these do in fact give us extensions and if so whether or not these extensions are split. The first thing that the relations tell us is that when the diagonal entries in these matrices are equal and $Q \neq -1$ then $x_i = 0$ for $i \geqslant 0$, so in fact m_0 looks like one of

$$\begin{pmatrix} -Q & 0 \\ 0 & -Q \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -Q & 0 \\ x_0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ x_0 & -Q \end{pmatrix},$$

and for $i > 0$ (regardless of the value of Q) the m_i look like

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ x_i & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ x_i & 0 \end{pmatrix}.$$

First consider the case when $k_0 \neq l_0$ and $Q \neq -1$. Since we are classifying extensions up to isomorphism we are allowed to conjugate

$$\begin{pmatrix} -Q & 0 \\ x_0 & 1 \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} 1 & 0 \\ -x_0/(1+Q) & 1 \end{pmatrix}$$

to get

$$\begin{pmatrix} -Q & 0 \\ 0 & 1 \end{pmatrix}.$$

Conjugating

$$\begin{pmatrix} 0 & 0 \\ x_i & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ x_i & 0 \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} 1 & 0 \\ -x_0/(1+Q) & 1 \end{pmatrix}$$

still leaves us with something of the form

$$\begin{pmatrix} 0 & 0 \\ x & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ x & 0 \end{pmatrix}$$

respectively. So in this case we can simultaneously conjugate by $\begin{pmatrix} 1 & 0 \\ -x_0/(1+Q) & 1 \end{pmatrix}$ and we have $x_0 = 0$. Then by the braid relations we have $x_i = 0, i > 0$, so in this case the extension splits. Similarly when m_0 is $\begin{pmatrix} 1 & 0 \\ x_0 & -Q \end{pmatrix}$ we can simultaneously conjugate by $\begin{pmatrix} 1 & 0 \\ x_0/(1+Q) & 1 \end{pmatrix}$. Therefore whenever $k_0 \neq l_0$, we have a split extension. Now assume that $Q \neq -1$ and m_0 is

$$\begin{pmatrix} -Q & 0 \\ 0 & -Q \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now for $i > 0$, whenever

$$m_i = \begin{pmatrix} 0 & 0 \\ x_i & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ x_i & 0 \end{pmatrix}$$

we can simultaneously conjugate by

$$\begin{pmatrix} 1 & 0 \\ -x_i & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ x_i & 1 \end{pmatrix}$$

respectively and force $x_i = 0$. When we conjugate

$$\begin{pmatrix} 1 & 0 \\ x_j & 0 \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} 1 & 0 \\ -x_i & 1 \end{pmatrix}$$

or

$$\begin{pmatrix} 0 & 0 \\ x_j & 1 \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} 1 & 0 \\ x_i & 1 \end{pmatrix}$$

we get

$$\begin{pmatrix} 1 & 0 \\ x_i + x_j & 0 \end{pmatrix}.$$

By Lemma 7 we know that if $j \neq i \pm 1$ then $x_j = -x_i$. If $j = i \pm 1$ then we could have $x_i + x_j \neq 0$, in this case we observe that matrices which commute with $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are diagonal, whereas matrices that take $\begin{pmatrix} 1 & 0 \\ x & 0 \end{pmatrix}$ to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ by conjugation are of the form $\begin{pmatrix} a & 0 \\ ax & d \end{pmatrix}$. So if for $j = i+1$ we have $m_i = \begin{pmatrix} 1 & 0 \\ x_i & 0 \end{pmatrix}$ and $m_j = \begin{pmatrix} 0 & 0 \\ x_j & 1 \end{pmatrix}$ we see that we cannot force both x_i and x_j to be 0. Putting all this together we find that if there is no $i > 0$ such that $m_i = \begin{pmatrix} 0 & 0 \\ x_i & 1 \end{pmatrix}$ then the extension will split, if there is no $i > 0$ such that $m_i = \begin{pmatrix} 1 & 0 \\ x_i & 0 \end{pmatrix}$ then the extension splits, and if there is some i and j such that $j \neq i \pm 1$ with $m_i = \begin{pmatrix} 0 & 0 \\ x_i & 1 \end{pmatrix}$ and $m_j = \begin{pmatrix} 1 & 0 \\ x_i & 0 \end{pmatrix}$ then the extension will split. The only remaining cases are those that satisfy the statement of the theorem. Since we can conjugate $\begin{pmatrix} 1 & 0 \\ x & 0 \end{pmatrix}$ by $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ to get $\begin{pmatrix} 1 & 0 \\ y & 0 \end{pmatrix}$ we find that all non-split extensions are isomorphic. Finally we deal with the case $Q = -1$, where we have $m_0 = \begin{pmatrix} 1 & 0 \\ x_0 & 1 \end{pmatrix}$ but this time the quadratic relation does not force $x_0 = 0$. However if m_i is not scalar for some $i > 0$ then the braid relations force $x_0 = 0$ and we are back in one of the above mentioned cases. If for all $i > 0$ the m_i are scalar then we get non-split extensions which are isomorphic. □

3.5 Extensions involving two-dimensional and one-dimensional simple modules

Now we look at $\mathrm{Ext}^1_{H_Q}(L,K)$ when L is a two-dimensional simple H_Q-module and K is a one-dimensional simple H_Q-module.

Theorem 56. *Let $L = N_{l_i}$ and $K = M_{k_0,k_i}$. Then if $n > 3$ the space $\mathrm{Ext}^1_{H_Q}(L,K)$ is one-dimensional if and only if $l_2 = k_1$ and $k_2 \neq k_1$ and ($l_3 \neq k_1$ or $k_3 = k_1$) and $k_i = l_i$ for $i > 3$, and is 0 otherwise. If $n = 3$ the space $\mathrm{Ext}^1_{H_Q}(L,K)$ is one-dimensional if and only if $l_2 = k_1$ and $k_2 \neq k_1$, and is 0 otherwise. If $n = 2$ then the space $\mathrm{Ext}^1_{H_Q}(L,K)$ is zero-dimensional.*

Proof. Let M be a three-dimensional H_Q-module such that $0 < K < M$ and $M/K \cong L$. We can choose a basis $\{e_1, e_2, e_3\}$ for M such that $e_3 \in K$, and such that S_0, S_1 and S_i for $i > 1$ act on M/K as $\begin{pmatrix} 1-Q & 1 \\ Q & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $l_i I$ with respect to the basis $\{e_1 + K, e_2 + K\}$. Let S_i act on M as m_i, then the following matrices give the possible actions of the S_i on M with respect to this basis.

$$m_0 = \begin{pmatrix} 1-Q & 1 & 0 \\ Q & 0 & 0 \\ x_0 & x_0 & -Q \end{pmatrix} = A_{x_0,1} \quad \text{or} \quad m_0 = \begin{pmatrix} 1-Q & 1 & 0 \\ Q & 0 & 0 \\ x_0 & -x_0Q^{-1} & 1 \end{pmatrix} = A_{x_0,2}\ ,$$

$$m_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & y_1 & 0 \end{pmatrix} = B_{y_1,1} \quad \text{or} \quad m_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ x_1 & 0 & 1 \end{pmatrix} = B_{x_1,2}\ ,$$

and for $i > 1$

$$m_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_i & y_i & 0 \end{pmatrix} = C_{x_i,y_i,1} \quad \text{or} \quad m_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_i & y_i & 1 \end{pmatrix} = C_{x_i,y_i,2}\ ,$$

or m_i is the identity matrix or zero matrix.

We now consider cases. First the case when $m_0 = A_{x_0,1}, m_1 = B_{y_1,1}$. Here we can simultaneously conjugate by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_0 - y_1 Q & y_1 & 1 \end{pmatrix},$$

which gives $m_0 = A_{0,1}, m_1 = B_{0,1}$. Now we fix m_i for $i > 1$. If $m_i = C_{x_i,y_i,1}$ for $i > 1$ then the braid relations $S_0S_i = S_iS_0, S_1S_2S_1 = S_2S_1S_2, S_1S_i = S_iS_1, i > 2$, will force $x_i = 0 = y_i$ for $i > 1$. However if $m_i = C_{x_i,y_i,2}$ for $i > 1$ then the braid relations $S_0S_i = S_iS_0, S_1S_2S_1 = S_2S_1S_2, S_1S_i = S_iS_1, i > 2$, will force $x_i = 0 = y_i$ for $i > 2$, but tell us nothing regarding x_2 and y_2. Matrices that commute with $A_{0,1}$ and $B_{0,1}$ are of the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix},$$

and if we conjugate $C_{x_2,y_2,2}$ by a matrix of this form we cannot force $x_2 = 0 = y_2$. For the case $m_0 = A_{x_0,1}, m_1 = B_{x_1,2}$ we simultaneously conjugate by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_1 & Q^{-1}(x_1 - x_0) & 1 \end{pmatrix},$$

to get $m_0 = A_{0,1}, m_1 = B_{0,2}$. At this point fix m_i for $i > 1$. If $m_i = C_{x_i,y_i,1}$ then the braid relations $S_0S_i = S_iS_0, S_1S_2S_1 = S_2S_1S_2, S_1S_i = S_iS_1, i > 2$, will force $x_i = 0 = y_i$ for $i > 2$, but tell us nothing regarding x_2 and y_2. Then by the same argument as above we cannot force $x_2 = 0 = y_2$ by conjugation. However if $m_i = C_{x_i,y_i,2}$ then the braid relations $S_0S_i = S_iS_0, S_1S_2S_1 = S_2S_1S_2, S_1S_i = S_iS_1, i > 2$, will force $x_i = 0 = y_i$ for $i > 1$. Next the case when $m_0 = A_{x_0,2}, m_1 = B_{y_1,1}$ for some $i > 1$. Here we can simultaneously conjugate by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ y_1 - x_0 Q^{-1} & y_1 & 1 \end{pmatrix},$$

which gives $m_0 = A_{0,2}, m_1 = B_{0,1}$. Fix m_i for $i > 1$. Then if $m_i = C_{x_i,y_i,1}$ the braid relations $S_0S_i = S_iS_0, S_1S_2S_1 = S_2S_1S_2, S_1S_i = S_iS_1, i > 2$, will force $x_i = 0 = y_i$ for $i > 1$, however if $m_i = C_{x_i,y_i,2}$ then the braid relations $S_0S_i = S_iS_0, S_1S_2S_1 = S_2S_1S_2, S_1S_i = S_iS_1, i > 2$, will force $x_i = 0 = y_i$ for $i > 2$, but tell us nothing regarding x_2 and y_2. Matrices that commute with $= A_{0,2}$ and $B_{0,1}$ are of the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix},$$

and if we conjugate $C_{x_2,y_2,2}$ by a matrix of this form we cannot force $x_2 = 0 = y_2$. For the case $m_0 = A_{x_0,2}, m_1 = B_{x_1,2}$ we simultaneously conjugate by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_1 & -(x_1 + x_0Q^{-1}) & 1 \end{pmatrix},$$

to get $m_0 = A_{0,2}, m_1 = B_{0,2}$. Fix m_i for $i > 1$. If $m_i = C_{x_i,y_i,1}$ the braid relations $S_0S_i = S_iS_0, S_1S_2S_1 = S_2S_1S_2, S_1S_i = S_iS_1, i > 2$, will force $x_i = 0 = y_i$ for $i > 2$, but tell us nothing regarding x_2 and y_2. Then by the same argument as above we cannot force $x_2 = 0 = y_2$ by conjugation, but if $m_i = C_{x_i,y_i,2}$ then the braid relations $S_0S_i = S_iS_0, S_1S_2S_1 = S_2S_1S_2, S_1S_i = S_iS_1, i > 2$, will force $x_i = 0 = y_i$ for$i > 1$. Hence we can always force $x_0 = y_0 = x_1 = y_1 = 0$ by simultaneous conjugation, and the braid relations $S_0S_i = S_iS_0, S_1S_2S_1 = S_2S_1S_2, S_1S_i = S_iS_1, i > 2$, will force $x_i = 0 = y_i$ for $i > 2$. Then by also considering Lemma 7 we see that the only time when we cannot force $x_2 = 0 = y_2$ is when $m_1 = B_{y_1,1}, m_2 = C_{x_2,y_2,2}, m_3 = C_{x_3,y_3,1}$ and m_i is the identity matrix or zero matrix for $i > 3$, or when $m_1 = B_{x_1,2}, m_2 = C_{x_2,y_2,1}, m_3 = C_{x_3,y_3,2}$ and m_i is the identity matrix or zero matrix for $i > 3$. Furthermore y_2 is forced to be a scalar multiple of x_2 by the braid relations, and since we can conjugate by matrices of the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix},$$

we see that all non-split extensions are isomorphic, This proves the theorem. By Theorem 54 a similar theorem holds for the space $\mathrm{Ext}^1_{H_Q}(K, L)$. $\square$

3.6 Extensions of two-dimensional simple H_Q-modules by two-dimensional simple H_Q-modules

Finally we look at the space $\mathrm{Ext}^1_{H_Q}(L, K)$ when L and K are two-dimensional simple H_Q-modules.

Theorem 57. *Let $L = N_{l_i}$ and $K = N_{k_i}$. Then the space $\mathrm{Ext}^1_{H_Q}(L, K)$ is one-dimensional if either*

(a) there is an $1 < i < n-1$ such that $k_i = l_{i+1} \neq l_i = k_{i+1}$ and $k_j = l_j$ for all $j \neq i, i+1$, or

(b) there is an $1 < i < n-2$ such that $k_i = l_{i+1} = k_{i+2} \neq l_i = k_{i+1} = l_{1+2}$ and $k_j = l_j$ for all $j \neq i, i+1, i+2$.

Otherwise, $\mathrm{Ext}^1_{H_Q}(L, K) = 0$.

Proof. Let M be a four-dimensional H_Q-module such that $0 < K < M$ and $M/K \cong L$. Let S_i act on M as m_i. We can choose a basis $\{e_1, e_2, e_3, e_4\}$ for M such that $e_3, e_4 \in K$, and such that S_0, S_1 and S_i for $i > 1$ act on M/K as $\begin{pmatrix} 1-Q & 1 \\ Q & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $l_i I$ with respect to the basis $\{e_1 + K, e_2 + K\}$, and such that S_0, S_1 and S_i for $i > 1$ act on K as $\begin{pmatrix} 1-Q & 1 \\ Q & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $k_i I$ with respect to the basis $\{e_3, e_4\}$. Then by considering the quadratic and braid relations we can deduce that the following matrices give the possible actions of the S_i on M with respect to this basis.

$$m_0 = \begin{pmatrix} 1-Q & 1 & 0 & 0 \\ Q & 0 & 0 & 0 \\ x_0 & y_0 & 1-Q & 1 \\ -x_0(1-Q) - y_0 Q & -x_0 & Q & 0 \end{pmatrix} = A_{x_0, y_0},$$

$$m_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & y_1 & 0 & 0 \\ w_1 & 0 & 0 & 1 \end{pmatrix} = B_{y_1,w_1},$$

and for $i > 1$

$$m_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ x_i & y_i & 0 & 0 \\ w_i & z_i & 0 & 0 \end{pmatrix} = C_{i,1} \quad \text{or} \quad m_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ x_i & y_i & 1 & 0 \\ w_i & z_i & 0 & 1 \end{pmatrix} = C_{i,2},$$

or m_i is the identity or zero matrix.

Now we can conjugate A_{x_0,y_0} by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ y_0 & 0 & 1 & 0 \\ -x_0 & 0 & 0 & 1 \end{pmatrix}$$

and get $A_{0,0}$. Conjugating B_{y_1,w_1} by the above matrix gives us B_{y_1,w_1-x_0}. Then we conjugate B_{y_1,w_1-x_0} by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ y_1 Q & y_1 & 1 & 0 \\ x_0 - w_0 & 0 & 0 & 1 \end{pmatrix}$$

to get $B_{0,0}$, and note that the above matrix commutes with $A_{0,0}$. Now the braid relations $S_0S_i = S_iS_0, S_1S_2S_1 = S_2S_1S_2, S_1S_i = S_iS_1, i > 2$ tell us that for $C_{i,1}$ and $C_{i,2}$ we have $x_i = z_i$ and $y_i = 0 = w_i$. Then for a given i we can conjugate by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -x_i & 0 & 1 & 0 \\ 0 & -x_i & 0 & 1 \end{pmatrix}$$

and force $x_i = 0$. Then we apply Lemma 7 and see that if for $i > 1$ no S_i acts as $C_{i,1}$ then the extension will split, if no S_i acts as $C_{i,2}$ then the extension splits, and if we have for $i > 1$ some S_i acting as $C_{i,1}$ and some S_j acting as $C_{i,2}$ such that $j \neq i \pm 1$ then the extension splits. The only remaining cases are those that satisfy the statement of the theorem. Finally we can always simultaneously conjugate by a matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ x' - x & 0 & 1 & 0 \\ 0 & x' - x & 0 & 1 \end{pmatrix}$$

and see that all non-split extensions are isomorphic.

□

3.7 The blocks of H_Q

We now give the blocks of H_Q, an example of the Ext quiver for $n = 4$ is given after the following theorem.

Theorem 58. *When $Q \neq -1$ H_Q has five blocks. In particular the simple modules $M_{Q,0,\ldots,0}, M_{Q,1,\ldots,1}, M_{-1,0,\ldots,0}, M_{-1,1,\ldots,1}$ all lie in blocks by themselves, and all other simple modules lie in one block. When $Q = -1$ H_Q has three blocks, with $M_{-1,0,\ldots,0}$ lying in one block, $M_{-1,1,\ldots,1}$ lying in a second block, and all other simple modules lying in a third block.*

Proof. Assume that $Q \neq -1$. First we show that each of $M_{Q,0,\ldots,0}, M_{Q,1,\ldots,1}, M_{-1,0,\ldots,0}, M_{-1,1,\ldots,1}$ do not lie in the same block as any other one-dimensional simple H_Q-module, and that any two one-dimensional simple H_Q-modules K and L lie in the same block when $k_0 = l_0$ $(K, L \notin \{M_{Q,0,\ldots,0}, M_{Q,1,\ldots,1}, M_{-1,0,\ldots,0}, M_{-1,1,\ldots,1}\})$.

Recall that if $L = M_{l_0,l_i}, K = M_{k_0,k_i}$ then there is a non-split extension of K by L if either

(a) there is an $0 < i < n-1$ such that $k_i = l_{i+1} \neq l_i = k_{i+1}$ and $k_j = l_j$ for all $j \neq i, i+1$, or

(b) there is an $0 < i < n-2$ such that $k_i = l_{i+1} = k_{i+2} \neq l_i = k_{i+1} = l_{1+2}$ and $k_j = l_j$ for all $j \neq i, i+1, i+2$.

Otherwise, $\mathrm{Ext}^1_{H_Q}(L, K) = 0$.

Since we must have $k_i \neq k_{i+1}, l_i \neq l_{i+1}, k_0 = l_0$ for an extension to be non-split we see that each of $M_{Q,0,\ldots,0}, M_{Q,1,\ldots,1}, M_{-1,0,\ldots,0}, M_{-1,1,\ldots,1}$ must lie in blocks that contain no other one-dimensional simple H_Q-modules.

Now assume that $K = M_{k_0,k_i}$ with not all $k_i = 0$ and not all $k_i = 1$. First observe that there is a greatest m such that $k_m = 0$ (there must be one by the definition of K). Let

$$\mathscr{K}_m = \{M_{k_0,b_i} \mid b_m = 0 \text{ and } b_i = 1 \text{ for } i > m\}.$$

We will use reverse induction on m to show that there is a chain of simple H_Q-modules $K = R_1, R_2, \ldots, R_r = M_{k_0,b_1,\ldots,b_{n-2},0}$ such that for $1 \leq i < r$, $\mathrm{Ext}^1_{H_Q}(R_i, R_{i+1}) \neq 0$. The case when $m = n-1$ is trivial so assume $K \in \mathscr{K}_m, 1 \leq m < n-1$. We know that there is a non-split extension of K by L where $l_m = 1, l_{m+1} = 0, l_i = k_i$ for all other i. Now L is in $\mathscr{K}_{m+1}$ so by reverse induction there is a chain of simple H_Q-modules $L = S_1, S_2, \ldots, S_s = M_{k_0,b_1,\ldots,b_{n-2},0}$ such that for $1 \leq i < s$, $\mathrm{Ext}^1_{H_Q}(S_i, S_{i+1}) \neq 0$. Hence there is a chain of simple H_Q-modules $K = R_1, R_2, \ldots, R_r = M_{k_0,b_1,\ldots,b_{n-2},0}$ such that for $1 \leq i < r$, $\mathrm{Ext}^1_{H_Q}(R_i, R_{i+1}) \neq 0$. Therefore K must lie in the same block as some one-dimensional simple module $M_{k_0,b_1,\ldots,b_{n-2},0}$.

Next we will show that K lies in the same block as $M_{k_0,1,0,\ldots,0}$. First observe that for any given K there must be a greatest j such that $k_j = 1$. Let

$$\mathcal{K}_j = \{M_{k_0,b_i} \mid b_j = 1 \text{ and } b_i = 0 \text{ for } i > j\}.$$

We will use induction on j to show that if K is in any $\mathcal{K}_j$ for $1 \leq j < n-1$, then there is a chain of simple H_Q-modules $M_{k_0,1,0,\ldots,0} = T_1, T_2, \ldots, T_t = K$ such that $1 \leq i < t$, $\mathrm{Ext}^1_{H_Q}(T_i, T_{i+1}) \neq 0$. We may assume that $j \neq n-1$ since by the above argument we can always construct a chain of simple H_Q-modules $K = R_1, R_2, \dots, R_r = M_{k_0,b_1,\dots,b_{n-2},0}$ such that $1 \leq i < r$, $\mathrm{Ext}^1_{H_Q}(R_i, R_{i+1}) \neq 0$. The case $j = 1$ is trivial so assume that $1 < j < n-1$ and $K \in \mathcal{K}_j$. We consider three possibilities.

1. $k_j = 1, k_{j-1} = 0$. Here there is a non-split extension of K by K' where $k'_j = 0, k'_{j-1} = 1, k'_{j-2} = 0, k'_i = k_i$ for all other i (the condition $k'_{j-2} = 0$ only applies when $j > 2$).

2. $k_j = 1, k_{j-1} = 1, k_{j-2} = 0$ (this case does not apply to $j = 2$). Here there is a non-split extension of K by L where $l_j = 1, l_{j-1} = 0, l_{j-2} = 1, l_i = k_i$ for all other i. Then there is a non-split extension of L by K' (K' from case 1) .

3. $k_j = 1, k_{j-1} = 1, k_{j-2} = 1$ or $j = 2$. Here there is a non-split extension of K by M where $m_{j+1} = 1, m_j = 0, m_{j-1} = 1, m_{j-2} = 1, m_i = k_i$ for all other i (the condition $m_{j-2} = 0$ only applies when $j > 2$). Then there is a non-split extension of M by L, and there is a non-split extension of L by K' (L and K' from case 1 and 2).

This tells us that we can always construct a chain of simple H_Q-modules $K' = P_1, P_2, \dots, P_p = K$ such that for $1 \leq i < p$, $\mathrm{Ext}^1_{H_Q}(P_i, P_{i+1}) \neq 0$. Now $K' \in \mathcal{K}_{j-1}$, so by induction there is a chain of simple H_Q-modules $M_{k_0,1,0,\dots,0} = Q_1, Q_2, \dots, Q_q = K'$ such that for $1 \leq i < q$, $\mathrm{Ext}^1_{H_Q}(Q_i, Q_{i+1}) \neq 0$. Hence, there is a chain of simple H_Q-modules $M_{k_0,1,0,\dots,0} = T_1, T_2, \dots, T_t = K$ such that for $1 \leq i < t$, $\mathrm{Ext}^1_{H_Q}(T_i, T_{i+1}) \neq 0$. Therefore K always lies in the same block as the simple module $M_{k_0,1,0,\dots,0}$.

We have shown that all one-dimensional simple H_Q-modules, K, with $k_0 = Q$ lie in the same block provided $K \notin \{M_{Q,0,\dots,0}, M_{Q,1,\dots,1}\}$, and that all one-dimensional simple H_Q-modules, K, with $k_0 = -1$ lie in the same block provided $K \notin \{M_{-1,0,\dots,0}, M_{-1,1,\dots,1}\}$. Now we will show that all these modules together with the two-dimensional simple H_Q-modules in fact lie in one block. Suppose we have a two-dimensional simple H_Q-module $L = N_{l_i}$ that extends a one-dimensional simple H_Q-module $K = M_{k_0,k_i}$. Recall that if $n > 3$ the space $\mathrm{Ext}^1_{H_Q}(L, K)$ is one-dimensional if and only if $l_2 = k_1$ and $k_2 \neq k_1$ and ($l_3 \neq k_1$ or $k_3 = k_1$) and $k_i = l_i$ for $i > 3$, and is 0 otherwise. If $n = 3$ the space $\mathrm{Ext}^1_{H_Q}(L, K)$ is one-dimensional if and only if $l_2 = k_1$ and $k_2 \neq k_1$, and is 0 otherwise. If $n = 2$ then the space $\mathrm{Ext}^1_{H_Q}(L, K)$ is zero-dimensional. Since these conditions do not depend on k_0 we can find a K with $k_0 = Q$ such that there is a non-split extension of K by L, and a K with $k_0 = -1$ such that there is a non-split extension of K by L. Furthermore since we must have $k_1 \neq k_2$, we cannot have K being one of $M_{Q,0,\ldots,0}, M_{Q,1,\ldots,1}, M_{-1,0,\ldots,0}, M_{-1,1,\ldots,1}$. So all simple H_Q-modules except for $M_{Q,0,\ldots,0}, M_{Q,1,\ldots,1}, M_{-1,0,\ldots,0}, M_{-1,1,\ldots,1}$ lie in the same block and each of $M_{Q,0,\ldots,0}, M_{Q,1,\ldots,1}, M_{-1,0,\ldots,0}, M_{-1,1,\ldots,1}$ lie in a block by themselves.

If $Q = -1$ then recall that if $L = M_{l_0,l_i}, K = M_{k_0,k_i}$ then there is a non-split extension of K by L if either

(a) $k_i = l_i$ for all $i > 0$, or

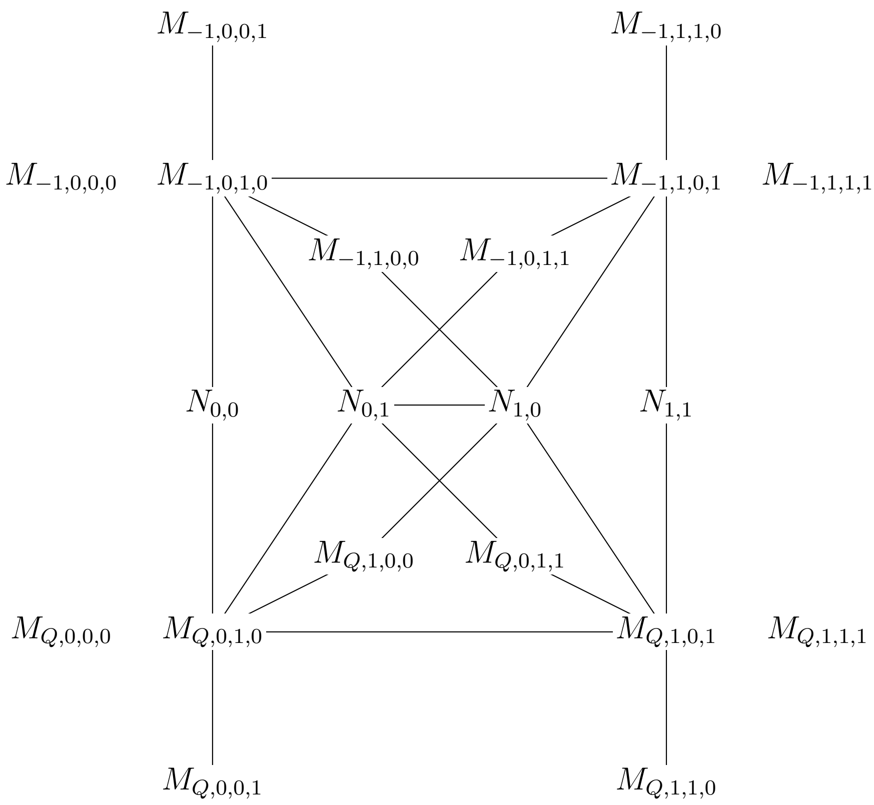
(b) there is an $0 < i < n - 1$ such that $k_i = l_{i+1} \neq l_i = k_{i+1}$ and $k_j = l_j$ for all $j \neq i, i + 1$, or

(c) there is an $0 < i < n - 2$ such that $k_i = l_{i+1} = k_{i+2} \neq l_i = k_{i+1} = l_{1+2}$ and $k_j = l_j$ for all $j \neq i, i + 1, i + 2$.

Otherwise, $\mathrm{Ext}^1_{H_Q}(L, K) = 0$. It follows from this and the above arguments that $M_{-1,0,\ldots,0}$ lies in a block by itself, $M_{-1,1,\ldots,1}$ lies in a second block by itself, and all other simple H_Q-modules lie in a third block. □

To illustrate the above theorem we give a diagram of the Ext quiver for $n = 4$ when $Q \neq -1$. In this case we have used a single line between two vertices to indicate an arrow going in each direction.

Example 4.



3.8 Induced modules

Here we look at the structure of induced modules. First we change notation slightly and let $H_{Q,n}$ denote the Hecke algebra for the Coxeter group of type B_n. We observe that $H_{Q,n-1}$ is a subalgebra of $H_{Q,n}$ in a natural way and we also have the following result

Theorem 59. *$H_{Q,n}$ is free as an $H_{Q,n-1}$ module.*

Proof. Let G be the Coxeter group of type B_n and $K < G$ be the Coxeter group of type B_{n-1}. Let $H_{Q,n}$ be the Iwahori-Hecke algebra of G and $H_{Q,n-1}$ be the Iwahori-Hecke algebra of K. We know that $H_{Q,n}$ is free as an $\mathbb{F}$ module with basis $\{S_w | w \in G\}$. Now consider $H_{Q,n}$ as an $H_{Q,n-1}$-module. Recall Theorem 3, this tells us that for each $g \in G$ there is a unique minimal right coset representative u, and a unique $k \in K$ such that $g = ku$ and $l(g) = l(k) + l(u)$. Then by Theorem 3

$\{S_w | w$ is a minimal right coset representative of K in $G\}$

is a spanning set for $H_{Q,n}$, as an $H_{Q,n-1}$-module. Now we show linear independence, let $h_1S_{w_1}+h_2S_{w_2}+...+h_{|G:K|}S_{w_{|G:K|}}=0$, where $h_i=a_1^{(i)}S_{w'_1}+a_2^{(i)}S_{w'_2}+\cdots+a_{|K|}^{(i)}S_{w'_{|K|}}\in H_{Q,n-1}$, the w_r are minimal right coset representatives for K in G, and the w'_s are a complete list of the elements of K. Then by Theorem 3 we can rewrite $0=h_1S_{w_1}+h_2S_{w_2}+...+h_{|G:K|}S_{w_{|G:K|}}=a_1^{(1)}S_{w''_1}+a_2^{(1)}S_{w''_2}+\cdots+a_{|K|}^{|G:K|}S_{w''_{|G|}}$, where the w''_t are a complete list of the elements of G. Since $\{S_w|w\in G\}$ is an $\mathbb{F}$ basis for $H_{Q,n}$ we must have $a_j^{(i)}=0$ for all i,j, hence $h_i=0$ for all i. A similar result holds using minimal left coset representatives of K in G. □

This leads us to our definition of an induced module.

Definition 60. Let M be a simple $H_{Q,n-1}$ module, then $\mathrm{Ind}_{H_{Q,n-1}}^{H_{Q,n}}M=H_{Q,n}\otimes_{H_{Q,n-1}}M$

We will obtain a composition series for this induced module. Suppose G is the Coxeter group of type B_n and K is the Coxeter group of type B_{n-1} lying inside G. Let $\{w_1,\ldots,w_r\}$ be a set of minimal left coset representatives for K in G and let $\{e_1,\ldots,e_m\}$ be a basis for M. Then we know that $\{S_{w_i}e_j|1\leqslant i\leqslant r,1\leqslant j\leqslant m\}$ is a basis for $\mathrm{Ind}_{H_{Q,n-1}}^{H_{Q,n}}M$ since $\{S_{w_i}|1\leqslant i\leqslant r\}$ is a basis for $H_{Q,n}$ over $H_{Q,n-1}$. We now give a result due to Dipper and James, for this result we will need to recall Definition 35, where the elements $s_{i,j}$ are defined.

Lemma 8. *[4, Lemma 2.2] Let G be the Coxeter group of type B_n with generators $s_0,\ldots,s_{n-1}$, and let K be the Coxeter group of type B_{n-1} lying inside G. Then the set*

$$A=\{s_{i,n}|1\leq i\leq n\}\cup\{s_{i,1}s_0s_{1,n}|1\leq i\leq n\}$$

is a complete set of minimal left coset representatives for K in G.

Proof. To see that this is true first note that by Matsumoto's theorem if $u\in A$ then u is the unique reduced expression for that element of G, since there are no braid relations that allow us to write u in a different way. Now if u were not a minimal coset representative then there would be some $s\in\{s_0,\ldots,s_{n-2}\}$ such that $l(us)<l(u)$, which

is a contradiction since the exchange lemma tells us that this would imply that u had a reduced expression ending in s, and we know that it does not. Finally we observe that we have $2n$ coset representatives, which is the index of K in G. □

Recall the simple H_Q-modules M_{a,b_i} and N_{b_i} from the previous chapter. First we will compute a composition series for $\mathrm{Ind}_{H_{Q,n-1}}^{H_{Q,n}} N_{b_i}$. Suppose $b_2, \ldots, b_{n-1} \in \{0, 1\}$, and define two-dimensional $H_{Q,n}$-modules $C_1, \ldots, C_{2n}$ as follows: S_0 acts on C_t as $\begin{pmatrix} 1-Q & 1 \\ Q & 0 \end{pmatrix}$, S_1 acts on C_t as

$$\begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } t = n-1 \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{if } t = n+2 \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \text{otherwise,} \end{cases}$$

S_2 acts on C_t as

$$\begin{cases} \begin{pmatrix} b_2 & 0 \\ 0 & b_2 \end{pmatrix} & \text{if } t < n-2, t > n+3 \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } t = n-2, n, n+2 \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{if } t = n-1, n+1, n+3, \end{cases}$$

and for $k > 2$ S_k acts on C_t as

$$\begin{cases} \begin{pmatrix} b_k & 0 \\ 0 & b_k \end{pmatrix} & \text{if } t < n-k, t > n+k+1 \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } t = n-k, n+k \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{if } t = n-k+1, n+k+1 \\ \begin{pmatrix} b_{k-1} & 0 \\ 0 & b_{k-1} \end{pmatrix} & \text{if } n-k+1 < t < n+k. \end{cases}$$

Theorem 61. *Suppose $b_2, \dots, b_{n-2} \in \{0,1\}$. Then $Ind_{H_{Q,n-1}}^{H_{Q,n}} N_{b_i}$ has a filtration in which the factors are $C_1, \dots, C_{2n}$ from bottom to top. Hence the composition factors* *of* $Ind_{H_{Q,n-1}}^{H_{Q,n}} N_{b_i}$ *are*

$$N_{b_2\ldots,b_{n-2},1}$$
$$N_{b_2,\ldots,b_{n-3},1,0}$$
$$N_{b_2,\ldots,b_{n-4},1,0,b_{n-2}}$$
$$\vdots$$
$$N_{1,0,b_3,\ldots,b_{n-2}}$$
$$M_{Q,1,0,b_2,b_3,\ldots,b_{n-1}}$$
$$M_{-1,1,0,b_2,b_3,\ldots,b_{n-1}}$$
$$N_{0,b_2,\ldots,b_{n-2}}$$
$$N_{1,b_2,\ldots,b_{n-2}}$$
$$M_{Q,0,1,b_2,b_3,\ldots,b_{n-1}}$$
$$M_{-1,0,1,b_2,b_3,\ldots,b_{n-1}}$$
$$N_{0,1,b_3,\ldots,b_{n-2}}$$
$$N_{b_2,0,1,b_3,\ldots,b_{n-2}}$$
$$\vdots$$
$$N_{b_2,\ldots,b_{n-3},0,1}$$
$$N_{b_2,\ldots,b_{n-2},0}$$

Proof. We choose a basis $\{e_1, e_2\}$ for N_{b_i} such that the generators S_i act in the following way:

$$S_0 e_j = \begin{cases} (1-Q)e_1 + Qe_2 & \text{if } j = 1 \\ e_1 & \text{if } j = 2, \end{cases}$$

$$S_1 e_j = \begin{cases} 0 & \text{if } j = 1 \\ e_2 & \text{if } j = 2, \end{cases}$$

and for $i > 1$ $S_i e_j = b_i e_j$. In order to construct the required filtration we observe that the basis elements for $\mathrm{Ind}_{H_{Q,n-1}}^{H_{Q,n}} N_{b_i}$ can be arranged in pairs as follows:

$$P_i = \begin{cases} (S_{s_{n+1-i,1}s_0s_{1,n}}e_1, S_{s_{n+1-i,1}s_0s_{1,n}}e_2) & \text{if } 1 \le i \le n-1 \\ (S_{s_0s_{1,n}}e_2, S_{s_{1,n}}e_2) & \text{if } i = n \\ (S_{s_0s_{1,n}}e_1, S_{s_{1,n}}e_1) & \text{if } i = n+1 \\ (S_{s_{i-n,n}}e_1, S_{s_{i-n,n}}e_2) & \text{if } n+2 \le i \le 2n. \end{cases}$$

We construct a series of submodules of $\mathrm{Ind}_{H_{Q,n-1}}^{H_{Q,n}} N_{b_i}$ in the following way, let W_i be the submodule generated by the pairs P_j where $j \leqslant i$. Then

$$W_1 < W_2 < \cdots < W_{2n}$$

is a filtration for $\mathrm{Ind}_{H_{Q,n-1}}^{H_{Q,n}} N_{b_i}$.

Now we look at how each generator acts on the factors of this filtration. It is clear that S_0 acts as $\begin{pmatrix} 1-Q & 1 \\ Q & 0 \end{pmatrix}$ on all of the factors. Next consider the action of S_1. $S_1(S_2S_3 \cdots S_{n-1}e_j) = S_1S_2S_3 \cdots S_{n-1}e_j$, $S_1(S_1 \cdots S_{n-1}e_j) = S_1 \cdots S_{n-1}e_j$, $S_1(S_0S_1 \cdots S_{n-1}e_j) = S_1S_0S_1 \cdots S_{n-1}e_j$ and $S_1(S_1S_0S_1 \cdots S_{n-1}e_j) = S_1S_0S_1 \cdots S_{n-1}e_j$. For all other $w \in A$ $S_1we_j = wS_1e_j$. It follows that S_1 acts as stated in the theorem. Now we find the action of S_k for $k > 1$ by observing the following equalities:

- $S_k(S_i \cdots S_{n-1}e_j) = S_i \cdots S_{n-1}S_ke_j$ for $i > k+1$.
- $S_k(S_{k+1} \cdots S_{n-1}e_j) = S_kS_{k+1} \cdots S_{n-1}e_j$.
- $S_k(S_k \cdots S_{n-1}e_j) = S_k \cdots S_{n-1}e_j$.
- $S_k(S_i \cdots S_{n-1}e_j) = S_i \cdots S_{n-1}S_{k-1}e_j, i < k$.
- $S_k(S_i \cdots S_0S_1 \cdots S_{n-1}e_j) = S_i \cdots S_0S_1 \cdots S_{n-1}S_{k-1}e_j, i < k-1$.
- $S_k(S_{k-1} \cdots S_0S_1 \cdots S_{n-1}e_j) = S_kS_{k-1} \cdots S_0S_1 \cdots S_{n-1}e_j$.
- $S_k(S_k \cdots S_0S_1 \cdots S_{n-1}e_j) = S_k \cdots S_0S_1 \cdots S_{n-1}e_j$.

- $S_k(S_i \cdots S_0 S_1 \cdots S_{n-1} e_j) = S_i \cdots S_0 S_1 \cdots S_{n-1} S_k e_j, i > k.$

It follows that S_k acts as stated in the theorem. So we see that C_t is irreducible for $t \neq n-1, n+2$, and C_{n-1}, C_{n+2} have composition factors $M_{Q,0,1,b_2,b_3,\ldots,b_{n-1}}$, $M_{-1,0,1,b_2,b_3,\ldots,b_{n-1}}$, $M_{Q,1,0,b_2,b_3,\ldots,b_{n-1}}$, $M_{-1,1,0,b_2,b_3,\ldots,b_{n-1}}$. □

To illustrate the above theorem we give an example.

Example 5. Let $n = 3$ and let $\{e_1, e_2\}$ be a basis for N_{b_i}. Then

$$\{e_i, S_2e_i, S_1S_2e_i, S_0S_1S_2e_i, S_1S_0S_1S_2e_i, S_2S_1S_0S_1S_2e_i | i = 1, 2\}$$

is a basis for $\mathrm{Ind}_{H_{Q,n-1}}^{H_{Q,n}} N_{b_i}$. Therefore

$$\begin{aligned}
P_1 &= (S_2S_1S_0S_1S_2e_1, S_2S_1S_0S_1S_2e_2),\\
P_2 &= (S_1S_0S_1S_2e_1, S_1S_0S_1S_2e_2),\\
P_3 &= (S_0S_1S_2e_2, S_1S_2e_2),\\
P_4 &= (S_0S_1S_2e_1, S_1S_2e_1),\\
P_5 &= (S_2e_1, S_2e_2),\\
P_6 &= (e_1, e_2).
\end{aligned}$$

Now we give the action of the generators S_0, S_1 and S_2 on the factors of the filtration for

$\mathrm{Ind}_{H_{Q,n-1}}^{H_{Q,n}} N_{b_i}$ constructed as in the above theorem.

	S_0	S_1	S_2
C_1	$\begin{pmatrix} 1-Q & 1 \\ Q & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
C_2	$\begin{pmatrix} 1-Q & 1 \\ Q & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
C_3	$\begin{pmatrix} 1-Q & 1 \\ Q & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
C_4	$\begin{pmatrix} 1-Q & 1 \\ Q & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
C_5	$\begin{pmatrix} 1-Q & 1 \\ Q & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
C_6	$\begin{pmatrix} 1-Q & 1 \\ Q & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Now we will compute a composition series for $\mathrm{Ind}_{H_{Q,n-1}}^{H_{Q,n}} M_{a,b_i}$. Suppose $a \in \{Q, -1\}$ and $b_1, \ldots, b_{n-1} \in \{0, 1\}$, and define $H_{Q,n}$-modules $D_1, \ldots, D_{2n-1}$ as follows: D_t is two-dimensional if $t = n$ and one-dimensional otherwise, S_0 acts on D_t as

$$\begin{cases} \begin{pmatrix} 1-Q & 1 \\ Q & 0 \end{pmatrix} & \text{if } t = n \\ a & \text{otherwise ,} \end{cases}$$

S_1 acts on D_t as

$$\begin{cases} 1 & \text{if } t = n-1 \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } t = n \\ 0 & \text{if } t = n+1 \\ b_1 & \text{otherwise,} \end{cases}$$

and for $k > 1$ S_k acts on D_t as

$$\begin{cases} b_k & \text{if } t < n-k, t > n+k \\ 1 & \text{if } t = n-k, n+k-1 \\ 0 & \text{if } t = n-k+1, n+k \\ b_{k-1} & \text{if } n-k+1 < t < n+k \\ \begin{pmatrix} b_{k-1} & 0 \\ 0 & b_{k-1} \end{pmatrix} & \text{if } t = n. \end{cases}$$

Theorem 62. *Suppose $a \in \{Q, -1\}$ and $b_1, \ldots, b_{n-2} \in \{0, 1\}$. Then $Ind_{H_{Q,n-1}}^{H_{Q,n}} M_{a,b_i}$ has a filtration in which the factors are $D_1, \ldots, D_{2n-1}$ from bottom to top. Hence the composition factors of $Ind_{H_{Q,n-1}}^{H_{Q,n}} M_{a,b_i}$ are*

$$M_{a,b_1,b_2,b_3,\ldots,b_{n-2},1}$$
$$M_{a,b_1,b_2,b_3,\ldots,b_{n-3},1,0}$$
$$M_{a,b_1,b_2,b_3,\ldots,b_{n-4},1,0,b_{n-2}}$$
$$\vdots$$
$$M_{a,1,0,b_2,b_3,\ldots,b_{n-2}}$$
$$N_{b_1,b_2,b_3,\ldots,b_{n-2}}$$
$$M_{a,0,1,b_2,b_3,\ldots,b_{n-2}}$$
$$\vdots$$
$$M_{a,b_1,0,1,b_3,\ldots,b_{n-2}}$$
$$M_{a,b_1,b_2,b_3,\ldots,b_{n-3},0,1}$$
$$M_{a,b_1,b_2,b_3,\ldots,b_{n-2},0}$$

Proof. Choose a basis $\{e\}$ for M_{a,b_i}. We order our basis for $\mathrm{Ind}_{H_{Q,n-1}}^{H_{Q,n}} M_{a,b_i}$ in the following way:

$$J_i = \begin{cases} S_{s_{n+1-i,1}s_0 s_{1,n}}e & \text{if } 1 \le i < n \\ S_{s_{1,n}}e & \text{if } i = n \\ S_{s_0 s_{1,n}}e & \text{if } i = n+1 \\ S_{s_{i-n,n}}e & \text{if } n+1 < i \le 2n. \end{cases}$$

For $0 \leqslant i < n$ let W_i be the submodule generated by the elements J_j, where $j \le i$. For $n \leqslant i \leqslant 2n-1$ let W_i be the submodule generated by the elements J_j, where $j \le i+1$. Then

$$W_1 < \cdots < W_{2n-1}$$

is a filtration for $\mathrm{Ind}_{H_{Q,n-1}}^{H_{Q,n}} M_{a,b_i}$. The theorem follows from the equalities given in the proof of the previous theorem and by observing that all the factors of the filtration are irreducible. □

Again we illustrate the above theorem with an example.

Example 6. Let $n = 4$ and let $\{e\}$ be a basis for M_{a,b_i}. Then

$$\{e, S_3e, S_2S_3e, S_1S_2S_3e, S_0S_1S_2S_3e, S_1S_0S_1S_2S_3e, S_2S_1S_0S_1S_2S_3e, S_3S_2S_1S_0S_1S_2S_3e\}$$

is a basis for $\mathrm{Ind}_{H_{Q,n-1}}^{H_{Q,n}} M_{a,b_i}$. We give the action of the generators S_0, S_1, S_2 and S_3 on the factors of the filtration for $\mathrm{Ind}_{H_{Q,n-1}}^{H_{Q,n}} M_{a,b_i}$ constructed in the above theorem.

	S_0	S_1	S_2	S_3
D_1	a	b_1	b_2	1
D_2	a	b_1	1	0
D_3	a	1	0	b_2
D_4	$\begin{pmatrix} 1-Q & 1 \\ Q & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} b_1 & 0 \\ 0 & b_1 \end{pmatrix}$	$\begin{pmatrix} b_2 & 0 \\ 0 & b_2 \end{pmatrix}$
D_5	a	0	1	b_2
D_6	a	b_1	0	1
D_7	a	b_1	b_2	0

We will now continue the study of the induced modules by showing that $\mathrm{Ind}_{H_{Q,n-1}}^{H_{Q,n}} M_{a,b_i}$ is self-dual. It may be helpful when reading the proof of Theorem 63 to consider the matrices given in the example that follows the proof.

Theorem 63. *Let $M = M_{t_0,t_i}$, then the induced module $Ind_{H_{Q,n-1}}^{H_{Q,n}} M$ is self-dual.*

Proof. We choose a basis $\{e\}$ for M, this gives a basis for $\mathrm{Ind}_{H_{Q,n-1}}^{H_{Q,n}} M$, labelled by J_i, which we are familiar with from Theorem 62. We take the J_i and reverse the order that was given in Theorem 62, then the following matrices represent the action of each generator on $\mathrm{Ind}_{H_{Q,n-1}}^{H_{Q,n}} M$: S_0 acts as X_0 where

$$(X_0)_{p,q} = \begin{cases} 1-Q & \text{if } p=n, q=n \\ 1 & \text{if } p=n, q=n+1 \\ Q & \text{if } p=n+1, q=n \\ t_0 & \text{if } p=q \neq n, n+1 \\ 0 & \text{otherwise.} \end{cases}$$

Let S_1 act as X_1 where

$$(X_1)_{p,q} = \begin{cases} 1 & \text{if } p=n+1, q=n-1 \\ 1 & \text{if } p=n+2, q=n \\ 1 & \text{if } p=n+1, q=n+1 \\ 1 & \text{if } p=n+2, q=n+2 \\ t_1 & \text{if } p=q \neq n-1, n, n+1, n+2 \\ 0 & \text{otherwise.} \end{cases}$$

For $i > 1$ let S_i act as X_i where

$$(X_i)_{p,q} = \begin{cases} t_i & \text{if } p = q, 1 \leq p < n-i \\ 1 & \text{if } p = n-i+1, q = n-i \\ 1 & \text{if } p = n-i+1, q = n-i+1 \\ t_{i-1} & \text{if } p = q, n-i+1 < p < n+i \\ 1 & \text{if } p = n+i+1, q = n+i \\ 1 & \text{if } p = n+i+1, q = n+i+1 \\ t_i & \text{if } p = q, n+i+1 < p \leq 2n \\ 0 & \text{otherwise.} \end{cases}$$

Now recall that with respect to the dual basis S_i acts as X_i^T, so we construct a matrix A that simultaneously conjugates X_i^T to X_i for $0 \leqslant i \leqslant n-1$. We define the upper-triangular part of the matrix A as follows:

$$A_{p,q} = \begin{cases} 1-Q & \text{if } p = n, q = n \\ 1 & \text{if } p = 2n-q+1, q > n \\ 1 & \text{if } p = n+1, q = n+2 \\ 1 & \text{if } p = 2n-q+2, q > n+2 \\ \prod_{k=n+1-p}^{q-n-2} t_k & \text{if } 2n-q+2 < p < n, q > n+2 \\ \prod_{k=0}^{q-n-2} t_k & \text{if } p = n, q > n+1 \\ \prod_{k=1}^{q-n-2} t_k & \text{if } p = n+1, q > n+2 \\ \prod_{k=0}^{q-n-2} t_k & \text{if } n+1 < p \leq q, q > n+1 \\ 0 & \text{otherwise.} \end{cases}$$

Now let $A_{q,p} = A_{p,q}$ for $p < q$.

Lemma 9. *A is invertible.*

Proof. The matrix A can be put into row echelon form simply by re-ordering the rows of A and swapping columns n and $n+1$. Then observe that all the rows are non-zero, so A has full rank. □

Since A is symmetric, if we can show that $X_i^T A$ is symmetric for $0 \leqslant i \leqslant n-1$ then we will have $X_i^T A = AX_i$, as required. Note that $t_i \in \{0,1\}$ for all $i > 0$. First the case when $i = 0$. If $1 \leqslant p < n$ and $1 \leqslant q < n$, or $n+1 < p \leqslant 2n$ and $1 \leqslant q < n$, or $1 \leqslant p < n$ and $n+1 < q \leqslant 2n$, or $n+1 < p \leqslant 2n$ and $n+1 < q \leqslant 2n$ then it is clear that $\left(X_0^T A\right)_{p,q} = t_0 A_{p,q} = t_0 A_{q,p} = \left(X_0^T A\right)_{q,p}$. If $p = n$ then $\left(X_0^T A\right)_{p,q} = (1-Q)A_{n,q} + QA_{n+1,q}$ and

$$\left(X_0^T A\right)_{q,p} = \begin{cases} 0 & q < n \\ (1-Q)A_{n,n} + QA_{n+1,n} & q = n \\ A_{n,n} & q = n+1 \\ t_0 A_{q,n} & q > n+1, \end{cases}$$

and by the definition of A we have $\left(X_0^T A\right)_{n,q} = \left(X_0^T A\right)_{q,n}$. If $p = n+1$ then $\left(X_0^T A\right)_{p,q} = A_{n,q}$ and

$$\left(X_0^T A\right)_{q,p} = \begin{cases} 0 & q < n \\ (1-Q) & q = n \\ 1 & q = n+1 \\ t_0 A_{q,n+1} & q > n+1, \end{cases}$$

and again by the definition of A we have $\left(X_0^T A\right)_{n+1,q} = \left(X_0^T A\right)_{q,n+1}$.

Now the case when $i = 1$. If $1 \leqslant p < n-1$ and $1 \leqslant q < n-1$, or $n+2 < p \leqslant 2n$ and $1 \leqslant q < n-1$, or $1 \leqslant p < n-1$ and $n+2 < q \leqslant 2n$, or $n+2 < p \leqslant 2n$ and $n+2 < q \leqslant 2n$ then it is clear that $\left(X_1^T A\right)_{p,q} = t_1 A_{p,q} = t_1 A_{q,p} = \left(X_1^T A\right)_{q,p}$. If $p = n-1$

then $\left(X_1^T A\right)_{p,q} = A_{n+1,q}$ and

$$\left(X_1^T A\right)_{q,p} = \begin{cases} 0 & q < n \\ 1 & q = n \\ 0 & q = n+1 \\ 1 & q = n+2 \\ t_i A_{q,n-1} & q > n+2, \end{cases}$$

and it follows from the definition of A that $\left(X_1^T A\right)_{n-1,q} = \left(X_1^T A\right)_{q,n-1}$. If $p = n$ then $\left(X_1^T A\right)_{p,q} = A_{n+2,q}$ and

$$\left(X_i^T A\right)_{q,p} = \begin{cases} 0 & q < n-1 \\ 1 & q = n-1 \\ t_0 & q = n \\ 1 & q = n+1 \\ t_0 & q = n+2 \\ t_1 A_{q,n} & q > n+2, \end{cases}$$

and it follows from the definition of A that $\left(X_1^T A\right)_{n-i+1,q} = \left(X_1^T A\right)_{q,n-i+1}$. If $p = n+1$ then $\left(X_1^T A\right)_{p,q} = A_{n+1,q}$ and

$$\left(X_1^T A\right)_{q,p} = \begin{cases} 0 & q < n \\ 1 & q = n \\ 0 & q = n+1 \\ 1 & q = n+2 \\ t_i A_{q,n+1} & q > n+2, \end{cases}$$

and it follows from the definition of A that $\left(X_1^T A\right)_{n+1,q} = \left(X_1^T A\right)_{q,n+1}$. If $p = n+2$ then

$\left(X_1^T A\right)_{p,q} = A_{n+2,q}$ and

$$\left(X_1^T A\right)_{q,p} = \begin{cases} 0 & q < n-1 \\ 1 & q = n-1 \\ t_0 & q = n \\ 1 & q = n+1 \\ t_0 & q = n+2 \\ t_1 A_{q,n+2} & q > n+2, \end{cases}$$

and it follows from the definition of A that $\left(X_1^T A\right)_{n+2,q} = \left(X_1^T A\right)_{q,n+2}$.

Now the case when $i > 1$. If $1 \leqslant p < n-i$ and $1 \leqslant q < n-i$, or $n+i+1 < p \leqslant 2n$ and $1 \leqslant q < n-i$, or $1 \leqslant p < n-i$ and $n+i+1 < q \leqslant 2n$, or $n+i+1 < p \leqslant 2n$ and $n+i+1 < q \leqslant 2n$ then it is clear that $\left(X_i^T A\right)_{p,q} = t_i A_{p,q} = t_i A_{q,p} = \left(X_i^T A\right)_{q,p}$. If $n-i+1 < p < n+i$ and $1 \leqslant q < n-i$ or $1 \leqslant p < n-i$ and $n-i+1 < q < n+i$ then it is clear that $\left(X_i^T A\right)_{p,q} = 0$. If $n-i+1 < p < n+i$ and $n-i+1 < q < n+i$ then $\left(X_i^T A\right)_{p,q} = t_{i-1} A_{p,q}$. If $n-i+1 < p < n+i$ and $n+i+1 < q \leqslant 2n$ then $\left(X_i^T A\right)_{p,q} = t_{i-1} A_{p,q} = t_i A_{q,p} = \left(X_i^T A\right)_{q,p}$ (note that $t_i \in \{0,1\}$ for all $i > 0$). If $p = n-i$ then $\left(X_i^T A\right)_{p,q} = A_{n-i+1,q}$ and

$$\left(X_i^T A\right)_{q,p} = \begin{cases} 0 & q < n+i \\ 1 & q = n+i \\ 1 & q = n+i+1 \\ t_i A_{q,n-i} & q > n+i+1, \end{cases}$$

and it follows from the definition of A that $\left(X_i^T A\right)_{n-i,q} = \left(X_i^T A\right)_{q,n-i}$. If $p = n-i+1$

then $\left(X_i^T A\right)_{p,q} = A_{n-i+1,q}$ and

$$\left(X_i^T A\right)_{q,p} = \begin{cases} 0 & q < n+i \\ 1 & q = n+i \\ A_{n+i+1,n-i+1} & q = n+i+1 \\ t_i A_{q,n-i+1} & q > n+i+1, \end{cases}$$

and it follows from the definition of A that $\left(X_i^T A\right)_{n-i+1,q} = \left(X_i^T A\right)_{q,n-i+1}$. If $p = n+i$ then $\left(X_i^T A\right)_{p,q} = A_{n+i+1,q}$ and

$$\left(X_i^T A\right)_{q,p} = \begin{cases} 0 & q < n-i \\ 1 & q = n-i \\ 1 & q = n-i+1 \\ t_{i-1} A_{q,n+i} & n-i+1 < q < n+i \\ A_{n+i+1,n+i} & q = n+i \\ A_{n+i+1,n+i} & q = n+i+1 \\ t_i A_{q,n+i} & q > n+i+1, \end{cases}$$

and it follows from the definition of A that $\left(X_i^T A\right)_{n+i,q} = \left(X_i^T A\right)_{q,n+i}$. If $p = n+i+1$ then $\left(X_i^T A\right)_{p,q} = A_{n+i+1,q}$ and

$$\left(X_i^T A\right)_{q,p} = \begin{cases} 0 & q < n-i \\ 1 & q = n-i \\ 1 & q = n-i+1 \\ t_{i-1} A_{q,n+i+1} & n-i+1 < q < n+i \\ A_{n+i+1,n+i+1} & q = n+i \\ A_{n+i+1,n+i+11} & q = n+i+1 \\ t_i A_{q,n+i+1} & q > n+i+1, \end{cases}$$

and it follows from the definition of A that $\left(X_i^T A\right)_{n+i+1,q} = \left(X_i^T A\right)_{q,n+i+1}$. Thus $X_i^T A$ is symmetric for $0 \leqslant i \leqslant n-1$. $\square$

We give the matrix A and the matrices that define the action of H_Q on $\mathrm{Ind}_{H_{Q,n-1}}^{H_{Q,n}} M_{t_0,t_i}$ when $n = 4$.

Example 7.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & t_2 \\ 0 & 0 & 0 & 1-Q & 1 & t_0 & t_1t_0 & t_2t_1t_0 \\ 0 & 0 & 0 & 1 & 0 & 1 & t_1 & t_2t_1 \\ 0 & 0 & 1 & t_0 & 1 & t_0 & t_1t_0 & t_2t_1t_0 \\ 0 & 1 & 1 & t_1t_0 & t_1 & t_1t_0 & t_1t_0 & t_2t_1t_0 \\ 1 & 1 & t_2 & t_2t_1t_0 & t_2t_1 & t_2t_1t_0 & t_2t_1t_0 & t_2t_1t_0 \end{pmatrix}$$

$$X_0 = \begin{pmatrix} t_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-Q & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & Q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_0 \end{pmatrix} \qquad X_1 = \begin{pmatrix} t_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_1 \end{pmatrix}$$

$$X_2 = \begin{pmatrix} t_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_2 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Chapter 4

The Iwahori-Hecke algebra of type F_4

We will look at the Iwahori-Hecke algebra of the Coxeter group of type F_4. We recall Definition 12. Suppose that G is the Coxeter group of type F_4. Given a field $\mathbb{F}$ and an element $Q \neq 0$ of $\mathbb{F}$, we define the Iwahori-Hecke algebra K_Q to be the associative algebra over $\mathbb{F}$ with generators S_0, S_1, S_2, S_3 and relations $(S_0 - Q)(S_0 + 1) = 0$,
$(S_3 - Q)(S_3 + 1) = 0$,
$S_1^2 = S_1$,
$S_2^2 = S_2$,
$S_0S_1S_0S_1 = S_1S_0S_1S_0$,
$S_0S_3S_0 = S_3S_0S_3$,
$S_1S_2S_1 = S_2S_1S_2$,
$S_0S_2 = S_2S_0, S_3S_1 = S_1S_3, S_3S_2 = S_2S_3$.

4.1 A filtration for K_Q

We will construct a filtration for the regular module K_Q, then by considering the factors we will classify the simple modules for K_Q. We construct the filtration for K_Q in the same way as we constructed the filtration for H_Q. Recall the algorithms LeftIdeal and

Filtration from the previous chapter. If we make a slight alteration to the algorithm LeftIdeal then we can run both of these algorithms when G is the Coxeter group of type F_4.

Algorithm 2. LeftIdeal2

Input: $w \in G$

let $A = \{w\}$

repeat

  let T be the empty set

  for $x \in A$ do

    if $l(s_1x) > l(x)$ and $s_1x \notin A$ then

      let T become $T \cup \{s_1x\}$

    end if

    if $l(s_2x) > l(x)$ and $s_2x \notin A$ then

      let T become $T \cup \{s_2x\}$

    end if

    if $s_0x \notin A$ then

      let T become $T \cup \{s_0x\}$

    end if

    if $s_3x \notin A$ then

      let T become $T \cup \{s_3x\}$

    end if

  end for

  let A become $A \cup T$

until $T = \{\}$

Output: $A = \{w_1, ..., w_r\}$

Now using LeftIdeal2 and Filtration we can construct a filtration for K_Q which we call F. We begin the analysis of this filtration with the following theorem.

Theorem 64. *The factors of F are six-dimensional and S_2 always acts as zero or the identity on these factors.*

Proof. We will use the techniques used in Theorem 50 to show that if w_i is the chosen longest element at the ith stage of the algorithm Filtration, then the only elements in B_i that are not in some B_j for $j < i$ are $S_{w_i}, S_{s_0 w_i}, S_{s_3 w_i}, S_{s_0 s_3 w_i}, S_{s_3 s_0 w_i}, S_{s_0 s_3 s_0 w_i}$. Because of our choice of w_i we can assume that $l(s_0 w_i) < l(w_i)$ and $l(s_3 w_i) < l(w_i)$. If $l(s_0 w_i) > l(w_i)$ then $s_0 w_i$ is in some B_j for $j < i$, but that implies that w_i is in some B_j for $j < i$, which contradicts our choice of w_i. Similarly for s_3. It will be sufficient to show that

1. if $l(s_k s_0 w_i) > l(s_0 w_i)$ then $s_k s_0 w_i \in A_j$ for $k = 1, 2$ and some $j < i$.

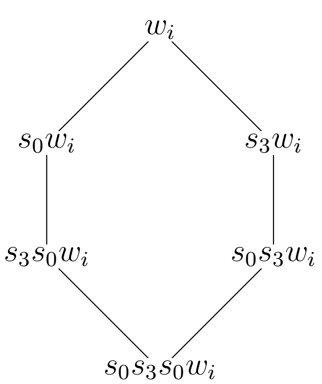
2. if $l(s_k s_3 w_i) > l(s_3 w_i)$ then $s_k s_3 w_i \in A_j$ for $k = 1, 2$ and some $j < i$.

3. if $l(s_k s_0 s_3 w_i) > l(s_0 s_3 w_i)$ then $s_k s_0 s_3 w_i \in A_j$ for $k = 1, 2$ and some $j < i$.

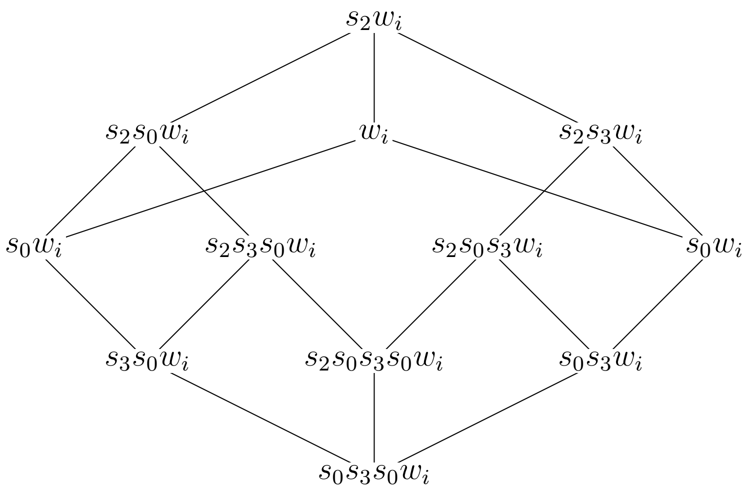
4. if $l(s_k s_3 s_0 w_i) > l(s_3 s_0 w_i)$ then $s_k s_3 s_0 w_i \in A_j$ for $k = 1, 2$ and some $j < i$.

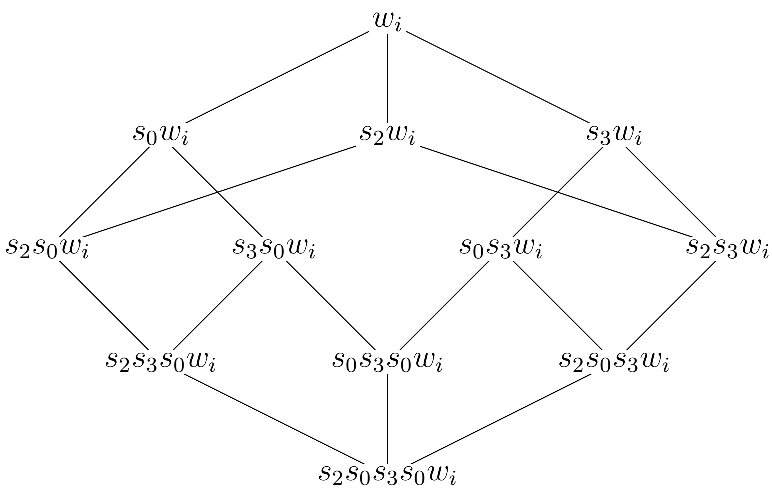
5. if $l(s_k s_0 s_3 s_0 w_i) > l(s_0 s_3 s_0 w_i)$ then $s_k s_0 s_3 s_0 w_i \in A_j$ for $k = 1, 2$ and some $j < i$.

It is clear that if $l(s_k w_i) > l(w_i)$ then $s_k w_i$ is in A_j for some $j < i$ by our choice of w_i. Recall Lemma 3, we will use this throughout the following arguments. First we deal with $k = 2$. Consider the subgroup $H < G$ generated by s_0, s_3, s_2 and the subgroup $K < G$ generated by s_0, s_3. Since we must have $l(s_0 w_i) < l(w_i)$ and $l(s_3 w_i) < l(w_i)$ it follows that every coset of the subgroup K will have the following structure with regard to the length of elements:



(here if x appears below y then y is longer than x). It follows from this that the only possible candidates for the unique shortest element of the coset Hw_i are $s_0s_3s_0w_i$ and $s_2s_0s_3s_0w_i$. When $s_0s_3s_0w_i$ is the unique shortest element we have $l(s_2w_i) > l(w_i)$, which in turn ensures that the required conditions are met. Furthermore in this case S_2 will act as zero on the factor W_i/W_{i-1}. When $s_2s_0s_3s_0w_i$ is the unique shortest element all the relevant inequalities are reversed and S_2 will act as the identity on the factor W_i/W_{i-1}. Below we give the length structure of the coset Hw_i when $s_0s_3s_0w_i$ is the unique shortest element, and when $s_2s_0s_3s_0w_i$ is the unique shortest element.





Now we deal with $k = 1$. We can use exactly the same method as we used in Theorem 50 to show that conditions 1 and 2 hold. For conditions $3, 4$ and 5 we need to consider the subgroup $L < G$ generated by s_0, s_3, s_1 and the coset Lw_i (L will be a copy of the Coxeter group of type B_3). If $l(s_1s_0s_3s_0w_i) > l(s_0s_3s_0w_i)$ or $l(s_1s_0s_3w_i) > l(s_0s_3w_i)$ then this implies that $s_0s_3s_0w_i$ is the unique shortest element of L, and this in turn tells us that $l(s_0s_3s_0s_1s_0s_3s_0w_i) > l(w_i)$ and $l(s_3s_0s_1s_0s_3w_i) > l(w_i)$. Therefore $s_1s_0s_3s_0$ and $s_1s_0s_3$ are in A_j for some $j < i$. Lastly, if $l(s_1s_3s_0w_i) > l(s_3s_0w_i)$ then there are two cases depending on $l(s_1s_0)$. If $l(s_1s_0w_i) > l(s_1s_3s_0w_i)$ then by looking at the subgroup $P < G$ generated by s_0, s_1 and the coset Pw_i we see that $s_0s_1s_0w_i \in A_j, j < i$, which implies that $s_1s_3s_0w_i \in A_j, j < i$. If $l(s_1s_0w_i) < l(s_1s_3s_0w_i)$ then we know that $s_1s_3s_0w_i$ is made shorter by s_1 and s_3, however $s_1s_3s_0w_i$ is not the longest element of Lw_i so $l(s_0s_1s_3s_0w_i) > l(s_1s_3s_0)$. We assume that $l(s_1s_0s_3s_0w_i) < l(s_0s_3s_0w_i)$ (otherwise the required condition holds trivially) and this tells us that $l(s_0s_1s_0s_1s_3s_0w_i) < l(s_1s_3s_0w_i)$. It follows that $l(s_1s_0s_1s_3s_0w_i) < l(s_0s_1s_3s_0w_i)$ and since $s_0s_1s_3s_0w_i$ is not the longest element of Lw_i we must have $l(s_3s_0s_1s_3s_0w_i) > l(s_0s_1s_3s_0w_i)$. Then this gives us $l(s_3s_0s_1s_3s_0w_i) > l(w_i)$ so $s_3s_0s_1s_3s_0w_i \in A_j, j < i$, which implies that $s_1s_3s_0w_i \in A_j, j < i$. □

4.2 Classification of the simple K_Q-modules

Since S_2 always acts as zero or the identity on the factors of F we can consider the subalgebra generated by S_0, S_1, S_3, this is a Hecke algebra of type B_3 which we denote R_Q. This leads to the following proposition.

Proposition 4. *Every simple R_Q-module yields two simple K_Q-modules. The K_Q-modules obtained in this way give a complete list of simple K_Q-modules.*

Proof. Suppose M is a simple R_Q-module. If we want to turn M into a simple K_Q-module then we let S_0, S_1 and S_3 act on M as defined by the R_Q-action, but we must say how S_2 acts on M. We Know that S_2 always acts as zero or the identity on the factors of F, by Theorem 64, so S_2 must act as zero or the identity on a simple K_Q-module. We check that this action satisfies the K_Q-relations, which it does, so we have a K_Q-action on M. Since S_2 acts on M as zero or the identity it is clear that M is simple as a K_Q-module. Conversely if N is a simple K_Q-module then we know that S_2 must act on N as zero or the identity by Theorem 64, hence N is simple as an R_Q-module. Therefore in order to determine the actions of S_0, S_1, S_3 on N it is sufficient to consider the simple R_Q-modules. $\square$

We know from [5] that we can construct Specht modules for R_Q and by Theorem 49 every simple module for R_Q is a quotient of a Specht module. There are ten Specht modules, see Example 3, and we give the actions of the generators below.

- one-dimensional: S_1 acts via $X_1 = (t)$, S_0 acts via $X_0 = (s)$, S_3 acts via $X_3 = (s)$ where $t \in \{0, 1\}$ and $s \in \{-1, Q\}$.
  - $S^{(1^3),\emptyset}$: $t = 0, s = -1$,
  - $S^{(3),\emptyset}$: $t = 0, s = Q$,
  - $S^{\emptyset,(1^3)}$: $t = 1, s = -1$,
  - $S^{\emptyset,(3)}$: $t = 1, s = Q$.
- two-dimensional: S_1 acts via $X_1 = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$, S_0 acts via $X_0 = \begin{pmatrix} Q & -1 \\ 0 & -1 \end{pmatrix}$, S_3 acts via $X_3 = \begin{pmatrix} 0 & Q \\ 1 & Q-1 \end{pmatrix}$ where $t \in \{0, 1\}$.
  - $S^{(2,1),\emptyset}$: $t = 0$,
  - $S^{\emptyset,(2,1)}$: $t = 1$.
- three-dimensional: S_1 acts via $X_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, S_0 acts via $X_0 = \begin{pmatrix} s & 0 & 0 \\ 0 & 0 & Q \\ 0 & 1 & Q-1 \end{pmatrix}$, S_3 acts via $X_3 = \begin{pmatrix} 0 & Q & 0 \\ 1 & Q-1 & 0 \\ 0 & 0 & s \end{pmatrix}$, or S_1 acts via $X_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, S_0 acts via $X_0 = \begin{pmatrix} 0 & Q & 0 \\ 1 & Q-1 & 0 \\ 0 & 0 & s \end{pmatrix}$, S_3 acts via $X_3 = \begin{pmatrix} s & 0 & 0 \\ 0 & 0 & Q \\ 0 & 1 & Q-1 \end{pmatrix}$, where $s \in \{-1, Q\}$.

– $S^{(1),(1^2)}$: S_1 acts via $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $s = -1$,

– $S^{(1),(2)}$: S_1 acts via $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $s = Q$,

– $S^{(1^2),(1)}$: S_1 acts via $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $s = -1$,

– $S^{(2),(1)}$: S_1 acts via $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $s = Q$.

Lemma 10. *The Specht modules given above are non-isomorphic as R_Q-modules.*

Proof. Let M and N be R_Q-modules. Observe that if S_i acts as X_i on M and as Y_i on N, then for M and N to be isomorphic there must be an invertible matrix T, such that $TX_iT^{-1} = Y_i$ for all i. We start with the one-dimensional Specht modules: $S^{(1^3),\emptyset}$ is not isomorphic to $S^{(3),\emptyset}$ since $tr(X_0^{(1^3),\emptyset}) \neq tr(X_0^{(3),\emptyset})$. $S^{(1^3),\emptyset}$ is not isomorphic to $S^{\emptyset,(1^3)}$ or $S^{\emptyset,(3)}$ since $tr(X_1^{(1^3),\emptyset}) \neq tr(X_1^{\emptyset,(1^3)})$ and $tr(X_1^{(1^3),\emptyset}) \neq tr(X_1^{\emptyset,(3)})$. $S^{(3),\emptyset}$ is not isomorphic to $S^{\emptyset,(1^3)}$ or $S^{\emptyset,(3)}$ since $tr(X_1^{(3),\emptyset}) \neq tr(X_1^{\emptyset,(1^3)})$ and $tr(X_1^{(3),\emptyset}) \neq tr(X_1^{\emptyset,(3)})$. $S^{\emptyset,(1^3)}$ is not isomorphic to $S^{\emptyset,(3)}$ since $tr(X_0^{\emptyset,(1^3)}) \neq tr(X_0^{\emptyset,(3)})$. Next we consider the two-dimensional Specht modules: $S^{(2,1),\emptyset}$ is not isomorphic to $S^{\emptyset,(2,1)}$ since $tr(X_1^{(2,1),\emptyset}) \neq tr(X_1^{\emptyset,(2,1)})$. Finally we look at the three-dimensional Specht modules: $S^{(1),(1^2)}$ is not isomorphic to $S^{(1),(2)}$ since $tr(X_0^{(1),(1^2)}) \neq tr(X_0^{(1),(2)})$. $S^{(1),(1^2)}$ is not isomorphic to $S^{(1^2),(1)}$ or $S^{(2),(1)}$ since $tr(X_1^{(1),(1^2)}) \neq tr(X_1^{(1^2),(1)})$ and $tr(X_1^{(1),(1^2)}) \neq tr(X_1^{(2),(1)})$. $S^{(1),(2)}$ is not isomorphic to $S^{(1^2),(1)}$ or $S^{(2),(1)}$ since $tr(X_1^{(1),(2)}) \neq tr(X_1^{(1^2),(1)})$ and $tr(X_1^{(1),(2)}) \neq tr(X_1^{(2),(1)})$. $S^{(1^2),(1)}$ is not isomorphic to $S^{(2),(1)}$ since $tr(X_0^{(1^2),(1)}) \neq tr(X_0^{(2),(1)})$. □

Before we can classify the simple K_Q-modules we give the following definition from [4]: Let e be the least positive integer such that $1 + Q + Q^2 + \cdots + Q^{e-1} = 0$. If no such integer exists then let $e = \infty$. Note that the case $e = 2$ corresponds to $Q = -1$ in type B_n, and the $e \geqslant 3$ case corresponds to $Q \neq -1$. We now classify the simple K_Q-modules depending on e.

Theorem 65. *When $e > 3$ there are eight one-dimensional simple K_Q-modules, four two-dimensional simple K_Q-modules and eight three-dimensional simple K_Q-modules. When $e = 3$ there are eight one-dimensional simple K_Q-modules and eight three-dimensional simple K_Q-modules. When $e = 2$ there are four one-dimensional simple K_Q-modules, four two-dimensional simple K_Q-modules and four three-dimensional simple K_Q-modules.*

Proof. By [5, Theorem 4.22] we know that every simple R_Q-module occurs as a quotient of a Specht module. Therefore it will be sufficient to find the simple quotients of the Specht modules listed above. For $e > 3$ we find that there are no simultaneous eigenvectors for the matrices that define the three-dimensional Specht modules, we also find that there are no simultaneous eigenvectors for the transposed matrices. Therefore the three-dimensional modules have no one-dimensional or two-dimensional submodules, so they are irreducible. The same is true for the two-dimensional Specht modules and so we have 20 simple K_Q-modules.

For $e = 3$ the one-dimensional and three-dimensional Specht modules are still irreducible and give us 8 simple K_Q-modules. However since $Q^2 + Q + 1 = 0$ we now have a simultaneous eigenvector in the two-dimensional case, so the two-dimensional Specht module is reducible and gives us a copy of the one-dimensional Specht module where $S_0 = (-1), S_3 = (-1)$.

For $e = 2$ the Specht modules are all irreducible, however since $Q + 1 = 0$ we have only two of each dimension giving 12 simple K_Q-modules. □

4.3 Extensions involving one-dimensional simple K_Q-modules

For the next theorem it will be useful to label the one-dimensional simple K_Q-modules: let S_{l_0,l_1,l_2} be the one-dimensional simple K_Q-module on which S_i acts as l_i for each i, with $l_3 = l_0$.

Theorem 66. *Suppose $l_0, p_0 \in \{Q, -1\}$ and $l_1, l_2, p_1, p_2 \in \{0, 1\}$.*

1. *If $e > 3$, then*

$$\dim \operatorname{Ext}^1_{K_Q}(S_{l_0,l_1,l_2}, S_{p_0,p_1,p_2}) = \begin{cases} 1 & \text{if } l_0 = p_0, l_1 \neq p_1, l_2 \neq l_1, p_2 \neq l_2, \\ 0 & \text{otherwise.} \end{cases}$$

2. *If $e = 3$, then*

$$\dim \operatorname{Ext}^1_{K_Q}(S_{l_0,l_1,l_2}, S_{p_0,p_1,p_2}) = \begin{cases} 1 & \text{if } l_0 = p_0, l_1 \neq p_1, l_2 \neq l_1, p_2 \neq l_2, \\ 1 & \text{if } l_0 \neq p_0, l_1 = p_1, l_2 = p_2, \\ 0 & \text{otherwise.} \end{cases}$$

3. *If $e = 2$, then*

$$\dim \operatorname{Ext}^1_{K_Q}(S_{l_0,l_1,l_2}, S_{p_0,p_1,p_2}) = \begin{cases} 1 & \text{if } l_1 = p_1, l_2 = p_2, \\ 1 & \text{if } l_1 \neq p_1, l_2 \neq l_1, p_2 \neq l_2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $P = S_{p_0,p_1,p_2}$, $L = S_{l_0,l_1,l_2}$, and let M be a 2-dimensional K_Q-module such that $0 < P < M$ and $M/P \cong L$. We can choose a basis $\{e_1, e_2\}$ for M such that $e_2 \in P$. Let S_i act on M as m_i, then with respect to this basis m_0 and m_3 can be one of the following

$$\begin{pmatrix} Q & 0 \\ x_i & Q \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ x_i & -1 \end{pmatrix}, \quad \begin{pmatrix} Q & 0 \\ x_i & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ x_i & Q \end{pmatrix}.$$

Note that m_3 must have the same diagonal entries as m_0. Furthermore m_1 and m_2 can be one of the following

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ x_i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ x_i & 1 \end{pmatrix}.$$

First the case when $e > 3$. If $l_0 \neq p_0$ then the braid relation $S_0S_3S_0 = S_3S_0S_3$ tells us that $Q^2x_0+Qx_0+x_0 = Q^2x_3+Qx_3+x_3$ and since $Q^2+Q+1 \neq 0$ we deduce that $x_0 = x_3$. We can conjugate m_0 so that $x_0 = 0$, then the braid relations $S_0S_2 = S_2S_0$ and $S_3S_1 = S_1S_3$ force $x_1 = 0$ and $x_2 = 0$, so the extension splits in this case, so $Ext^1(L,P) = 0$. If $l_0 = p_0$ then both m_0 and m_3 are scalar matrices. Consider the subalgebra generated by S_0, S_1, S_2, this is a Hecke algebra of type B_3, call this subalgebra H. Now since m_3 is scalar we see that $Ext^1_{K_Q}(L,P) = Ext^1_H(L,P)$ and we have already calculated $Ext^1_H(L,P)$ when studying the B_n case.

For $e = 3$ we have $Q^2 + Q + 1 = 0$ so it is possible to have $x_0 \neq x_3$ when $l_0 \neq p_0$ but we can still conjugate m_3 so that $x_3 = 0$. Assume at least one of m_1 or m_2 is non-scalar, say m_j is non-scalar where j is 1 or 2, then the braid relation $S_3S_j = S_jS_3$ forces $x_j = 0$. Then if $j = 1$ the braid relation $S_0S_1S_0S_1 = S_1S_0S_1S_0$ forces $x_0 = 0$, and if $j = 2$ the braid relation $S_0S_2 = S_2S_0$ forces $x_0 = 0$. However, if both m_1 and m_2 are scalar then the braid relations can be satisfied with one of x_0, x_3 being non-zero, say $x_0 \neq 0$, and we can't make it zero by conjugating, so we have a non-split extension. We can simultaneously conjugate by

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$$

so x_0 becomes λx_0, therefore $\mathrm{Ext}^1_{K_Q}(L,P)$ will be 1-dimensional. When $l_0 = p_0$ we have exactly the same conditions as for the case $e > 3$.

For $e = 2$ m_0 and m_3 look like

$$\begin{pmatrix} -1 & 0 \\ x_i & -1 \end{pmatrix}$$

and the braid relation $S_0S_3S_0 = S_3S_0S_3$ forces $x_0 = x_3$, however we cannot conjugate to get $x_3 = 0$. If exactly one of m_1 or m_2 is non scalar then by the braid relations

$S_3S_1 = S_1S_3$ or $S_3S_2 = S_2S_3$ we can force $x_3 = x_1 = x_2 = 0$. If $l_1 = p_1, l_2 = p_2$ then the braid relations allow $x_3 \neq 0$ and the extension is non-split. If $l_1 \neq p_1, l_2 \neq l_1, p_2 \neq l_2$ then by the braid relation $S_3S_1 = S_1S_3$ we have $x_3 = 0$, but we cannot simultaneously conjugate m_1 and m_2 to get both x_1 and x_2 equal to zero. □

4.4 Extensions involving one and two dimensional simple K_Q-modules

For the next theorem we will use the same notation for the one-dimensional simple K_Q-modules as in the previous theorem and introduce the following notation for the two-dimensional simple K_Q-modules: let M_{l_1,l_2} be the 2-dimensional simple K_Q-module on which S_i acts as l_iI for $i = 1, 2$, where I is the 2×2 identity matrix, and S_0 acts as

$$\begin{pmatrix} Q & -1 \\ 0 & -1 \end{pmatrix}$$

and S_3 acts as

$$\begin{pmatrix} 0 & Q \\ 1 & Q-1 \end{pmatrix}$$

Theorem 67. *Suppose $l_0 \in \{Q, -1\}$ and $l_1, l_2, p_1, p_2 \in \{0, 1\}$. If $e > 3$, then*

$$\dim \operatorname{Ext}^1_{K_Q}(M_{p_1,p_2}, S_{l_0,l_1,l_2}) = 0.$$

Proof. Let M be a 3-dimensional K_Q-module such that $0 < S_{l_0,l_1,l_2} < M$ and $M/S_{l_0,l_1,l_2} \cong M_{p_1,p_2}$. we can choose a basis $\{e_1, e_2, e_3\}$ for M such that $e_3 \in P$. Let S_i act on M as m_i then with respect to this basis and after considering the quadratic relations m_0 can be one of the following

$$\begin{pmatrix} Q & -1 & 0 \\ 0 & -1 & 0 \\ 0 & y_0 & Q \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} Q & -1 & 0 \\ 0 & -1 & 0 \\ (Q+1)y_0 & y_0 & -1 \end{pmatrix},$$

m_3 can be one of the following

$$\begin{pmatrix} 0 & Q & 0 \\ 1 & Q-1 & 0 \\ -y_3 & y_3 & Q \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & Q & 0 \\ 1 & Q-1 & 0 \\ y_3 & Qy_3 & -1 \end{pmatrix},$$

and m_1 and m_2 can be one of the following

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_i & y_i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_i & y_i & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Firstly we can simultaneously conjugate by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ y_0 & y_3 & 1 \end{pmatrix} \quad \text{if } l_0 = -1, \text{ or} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -y_0 & -y_3 & 1 \end{pmatrix}$$

if $l_0 = Q$. This allows us to set $y_0 = y_3 = 0$ and the conjugation does not change the form of m_1 or m_2. If m_1 and m_2 are both scalar then the extension clearly splits. If m_1 has the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_i & y_i & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_i & y_i & 1 \end{pmatrix},$$

then the braid relations $S_0S_1S_0S_1 = S_1S_0S_1S_0$ and $S_1S_3 = S_3S_1$ force $x_1 = y_1 = 0$. If m_2 has the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_i & y_i & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_i & y_i & 1 \end{pmatrix},$$

then the braid relations $S_0S_2 = S_2S_0$ and $S_2S_3 = S_3S_2$ force $x_2 = y_2 = 0$. So in all cases the extension will split. □

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