BPS Operators and Brane Geometries

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Abstract

In this thesis we explore the finite N spectrum of BPS operators in four-dimensional supersymmetric conformal field theories (CFT), which have dual AdS gravitational descriptions. In the first part we analyze the spectrum of chiral operators in the free limit of quiver gauge theories. We find explicit counting formulas at finite N for arbitrary quivers, construct an orthogonal basis in the free inner product, and derive the chiral ring structure constants. In order to deal with arbitrarily complicated quivers, we develop convenient diagrammatic techniques: the results are expressed by associating Young diagrams and Littlewood-Richardson coefficients to modifications of the original quiver. We develop the notion of a "quiver character", which is a generalization of the symmetric group character, obeying analogous orthogonality properties.

In the second part we analyze how the BPS spectrum changes at weak coupling, focusing on the $\mathcal{N} = 4$ supersymmetric Yang-Mills. We find a formal expression for the complete set of eighth-BPS operators at finite N, and use it to derive corrections to a near-BPS operator.

In the third part of this thesis we move on to the strong coupling regime, where the dual gravitational description applies. The BPS spectrum on the gravity side includes D3-branes wrapping arbitrary holomorphic surfaces, a generalization of the spherical giant gravitons. Quantizing this moduli space gives a Hilbert space, which, via duality and non-renormalization theorems, should map to the space of BPS operators derived in the weak coupling regime. We apply techniques from fuzzy geometry to study this correspondence between D3-brane geometries, quantum states, and BPS operators in field theory.

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Contents

Chapter 1

Introduction

Arguably the most important development in the high energy theoretical physics in the recent years has been the discovery of AdS/CFT correspondence [1, 2, 3]. It is a conjectured equivalence between two seemingly very different physical theories: one is a supersymmetric conformal field theory (CFT) in four dimensions, and the other is a string theory on a five-dimensional Anti de Sitter space (AdS) times a compact 5D manifold. String theory is a quantum theory of gravity, so the idea is that all the gravitational physics in the AdS space, from gravitational attraction to black holes, can be equivalently described by processes in a different theory on the four-dimensional boundary of AdS, which does not include gravity. This is reminiscent of a hologram, where a three dimensional picture can be recorded on a two dimension plate. In AdS/CFT it is the five-dimensional physics that can be captured by a four-dimensional physics on the boundary, thus the duality is also called "holographic correspondence".

The importance of the correspondence lies in the fact that it relates two major unsolved problems in theoretical physics. On one side there is quantum gravity, which is a long-standing problem in the quest to find a unified description of all fundamental forces in the universe. String theory has made tremendous progress in developing a consistent model of quantum gravity, however, it is far from solved, and many mysteries remain. On the other side we have a CFT, which is a four-dimensional gauge theory, in some aspects not too different from the quantum chromodynamics (QCD), a theory that explains strong force between quarks and gluons. Even though QCD is in principle quite well understood, it still poses a big theoretical challenge, because it is "strongly coupled", so the usual perturbative techniques of quantum field theory, such as Feynman diagrams, can not be applied. A lot of effort has been put into trying to develop a better theoretical framework for understanding strongly coupled field theories, but it still remains one of the biggest open problems. AdS/CFT correspondence connects the two, and opens up new possibilities for understanding both quantum gravity and strongly coupled field theories.

The research on AdS/CFT correspondence could be, very broadly, grouped into three categories. First, naturally, the duality needs to be tested, so one must find quantities that can be calculated on both sides and compared. This is challenging, because of the strong-weak nature of the duality, meaning that the weak coupling limit of CFT accessible to perturbative calculations is dual to the strongly coupled or "very quantum" gravity that we don't know much about. Conversely, semiclassical limit of gravity that is well understood is dual to the strongly coupled CFT, where good calculational techniques are missing. These issues, however, can be circumvented in various special cases, and numerous checks have been performed. The second research direction is to use the CFT results, assuming duality holds, to derive new results about string theory and quantum gravity, such as the microscopic counting of the black hole entropy. In a sense, AdS/CFT *defines* quantum gravity, at least on AdS spaces, in a non-perturbative and fully consistent way by equating it to the field theory. Finally, one can go the other way, and use the semiclassical gravity to learn something about the strongly coupled field theories, for example, about the quark-gluon plasma in QCD. In this work we mainly focus on the second direction, working in the weak coupling limit of the field theory to derive results which could tell us about non-perturbative quantum gravity.

In its original formulation [1, 2, 3] AdS/CFT relates Type IIB superstring theory on $AdS_5 \times S^5$ to the $\mathcal{N}=4$ super Yang-Mills theory (SYM) with $SU(N)$ gauge group. Many other dual pairs have been discovered since, but for the big part of this work we focus on the original $AdS_5 \times S^5$. The string theory has two adjustable parameters, string length squared α' and string coupling g_s. Parameters in SYM are the rank of the gauge group N and the gauge coupling g_{YM}, which does not run, making the theory conformal. If we use 't Hooft coupling $\lambda \equiv g_{\mathrm{YM}}^2 N$, the conjectured relationship is

$$\frac{R^4}{4\pi\alpha'^2} = \lambda, \quad g_s = \frac{\lambda}{N} \tag{1.1}$$

where R is the radius of AdS which is fixed, since only the ratio $\frac{R^2}{\alpha'}$ is measurable. This concretely realizes the idea by 't Hooft [4], that $SU(N)$ gauge theory at large N admits a new expansion in $1/N$, according to the genus of the double-line Feynman graph, which looks very much like the genus expansion of string theory, thus suggesting $g_s \sim 1/N$. In the limit $N \to \infty$, $g_s \to 0$ only the planar diagrams contribute. The effective coupling constant for SYM is then λ and to match with the supergravity approximation $\alpha' \ll R^2$ we have to take the very strong coupling limit $\lambda \gg 1$. One can also rewrite (1.1) as

$$l_s \sim \frac{R}{\lambda^{1/4}}, \quad l_P \sim \frac{R}{N^{1/4}} \tag{1.2}$$

where $l_s \sim \sqrt{\alpha'}$ is the string length, and $l_P \sim (g_s\alpha'^2)^{1/4}$ is the Planck length. This clearly shows that λ controls the "stringiness" and N controls the "quantumness" of the bulk theory.

Given the strong-weak nature of the duality any direct proof of the AdS/CFT correspondence is beyond reach, but ever since the discovery there has been a growing amount of evidence and checks. In order to compare the two quantum theories one, in principle, needs to check two ingredients: the spectrum and the correlators. The exact prescription for comparing CFT observables to those of string theory in AdS was given in [2, 3]. The spectrum of local operators in CFT should be matched with the spectrum of states in AdS, the scaling dimensions in CFT matching energies in AdS. More precisely, the representations of the global symmetry group $PSU(2,2|4)$ should match on both sides. Then the correlators of local operators in CFT should be identified with appropriate correlators in AdS, where the states are propagating from the boundary and interacting in the bulk.

Most of the checks have been performed in the planar limit $N \to \infty$, which suppresses string interactions. On the SYM side this suppresses the mixing of trace structures, thus one can focus on single-trace operators as the natural duals of single-particle states in AdS. The states that can be compared most easily are the BPS states. These are special short representations of $PSU(2,2|4)$, that are annihilated by some of the supercharges. As a consequence of that, the energy of the BPS states is algebraically related to the other charges and thus can not depend continuously on λ. This allows one to reliably calculate the BPS spectrum in SYM at $\lambda \ll 1$ and compare to the gravity regime at $\lambda \gg 1$. Indeed it was confirmed in [3] that the spectrum of half-BPS single trace operators matches the Kaluza-Klein modes of Type IIB supergravity. Furthermore, it was shown that the three-point function of half-BPS operators calculated in SYM at zero coupling is equal to the supergravity calculation [5]. This also suggested that the three-point function is λ-independent, and this was confirmed later by non-renormalization theorems [6].

In order to go beyond the supergravity approximation and see "stringy" effects, one needs to compare results at finite large λ, outside of the BPS sector. Even comparing the spectrum is extremely challenging because the operators acquire anomalous dimensions, which needs to be calculated in SYM at strong coupling. This problem was partly overcome with the BMN limit [7], which was to take an operator with a very large R-charge such as $\mathrm{tr}(Z^J)$ and add impurities making it "near-BPS". It turns out the anomalous dimension in SYM is then suppressed like $\frac{\lambda}{J^2}$ and the non-planar effects are of order $\frac{J^2}{N}$. This allows one to take the limit $1 \ll \lambda \ll J^2 \ll N$, which is perturbatively accessible at SYM, but at the same time has large λ. The AdS dual of this regime is a string with a large angular momentum on S^5. This can be seen as a string in the so-called pp-wave background, which can be quantized exactly, and the spectrum was matched with the SYM operator dimensions. The construction provided a concrete example, how the physics of impurities travelling on the closed spin-chain $\mathrm{tr}(Z^J)$, reconstructs the 1+1 dimensional world-sheet theory of the string.

Subsequently, the dilatation operator [8] was developed as an efficient method of computing anomalous dimensions in SYM, and it was realized that in the planar limit the complete problem of finding the spectrum is identical to solving the integrable $PSU(2,2|4)$ spin-chain [9]. With the discovery of integrability it was possible to use the powerful mathematical machinery, such as Bethe ansatz, to find the operator dimensions even to all-loop order in λ. On the other side, integrability was also found in the problem of calculating string energies on $AdS_5 \times S^5$ [10]. Many further developments followed, culminating in very precise tests of the spectrum, all the way from weak to strong λ, including both long and short strings, however, all in the *planar* limit. For a recent review on this vast subject see [11].

In a parallel development the effects of finite N in AdS/CFT were explored. Finite N is dual to finite non-zero string coupling g_s and Planck length l_P, thus it opens possibility to study truly non-perturbative quantum gravity effects, such as microscopic entropy of black holes, or the physics of D-branes. Since the integrability no longer applies, making the strong λ regime virtually inaccessible, the best prospect is to look at the BPS spectrum and compare λ-independent results. The BPS spectrum of the interacting string theory is much richer than just Kaluza-Klein modes of supergravity: it includes D3-branes wrapping various surfaces, supergravity geometries of varying topology asymptotic to AdS, and even supersymmetric black holes with macroscopic horizons. All these should, in principle, have dual BPS operators in the finite N SYM, that can be reliably calculated at weak coupling. However, the challenge in making the comparison is that the full BPS spectrum is not known on either side. In AdS we are dealing with non-perturbative objects, that are constructed as semiclassical solutions, but their exact quantization and Hilbert space is unknown. In SYM, even at weak coupling, in order to get finite N results one needs to deal with formidable combinatorics problems. These are, however, technical rather than conceptual, and can potentially be overcome by developing the right computational tools. Solving the BPS spectrum on the SYM side could then provide insights about the exact quantum gravity states, at least in the supersymmetric sector.

The half-BPS sector of the theory, annihilated by half of the supercharges, is the most supersymmetric and best understood. Each half-BPS multiplet has the highest weight state that in $\mathcal{N} = 4$ only involves one complex scalar Z. Thus in the planar limit the full half-BPS spectrum is just arbitrary products of $\mathrm{tr}(Z^J)$, which are dual to the multi-particle states of Kaluza-Klein gravitons with momentum on S^5. However, at finite N the spectrum is modified on both sides. On the AdS side it was found that the momentum of the graviton can not increase indefinitely, rather, at $J = O(N)$ the appropriate solution is a spherical D3-brane which rotates around S^5, dubbed "giant graviton" [12]. The radius of the brane increases with J and at $J = N$ it becomes equal to the S^5 radius, thus can not increase any further. This was a somewhat surprising explanation of the "stringy exclusion principle", the fact that at finite N some of the states disappear from the spectrum. Besides the giant graviton, another half-BPS solution was found where D3-brane is an S^3 expanding into AdS_5, called "dual giant graviton" [13, 14]. One can also build multi-giant states which are still half-BPS.

On the SYM side the change in the spectrum at finite N comes from the simple fact that $\mathrm{tr}(Z^{N+k})$ can be expanded as products of traces $\mathrm{tr}(Z^J)$ with $J \leq N$, so the basis is overcomplete. In fact, already at $J \sim \sqrt{N}$ the single traces are no longer orthogonal to multi-traces, because the planar expansion breaks down. The operators that were found suitable to describe giant gravitons were not single traces, but subdeterminants [15], with the maximal giant dual to $\det(Z)$, nicely capturing the cutoff. This was generalized by [16], who used symmetric group techniques to solve the two-point function exactly at finite N, and provided a complete orthogonal basis for the half-BPS sector. The basis states were found to be Schur polynomials $\chi_R(Z)$ labelled by $U(N)$ representations R, which are Young diagrams with n boxes given by the charge of the operator. The effect of finite N is the cutoff on the height of the diagram $l(R) \leq N$. Furthermore, the three-point functions were also calculated exactly in [16], and were found to be given by Littlewood-Richardson coefficients. The Schur polynomial basis beautifully showed how the different classical D3-brane solutions are different limits of the single Hilbert space: the giant graviton is R with a single tall column, the dual giant is a single long row, multi-giants are given by R with multiple rows or columns with different heights. But, crucially, it provides all the quantum states "in between", smoothly interpolating between gravitons, multi-giants and multi-dual-giants.

The Schur basis $\chi_R(Z)$ gives the half-BPS states at any energy scale, so one can explore the regime $n \sim N^2$. In AdS it is no longer well described by D3-branes, but instead these should be new geometries, asymptotic to AdS_5. The full classification of smooth half-BPS supergravity geometries was achieved by LLM [17], and it was found that their phase space can be matched to configurations of a fermion droplet on a 2D plane. The fermion droplet, in turn, was shown to be a semiclassical description of the half-BPS SYM sector [18]. This painted a complete picture of the half-BPS sector: each smooth LLM geometry is dual to a semiclassical fermion droplet configuration, which could be written as some coherent state of Schur polynomials, while individual basis states $\chi_R(Z)$ are dual to exact quantum states of appropriately quantized LLM geometries. Somewhat surprisingly, the quantization of *smooth* supergravity geometries was enough to recover the full Hilbert space, without including singular geometries or even new non-geometric backgrounds. The singular geometries were, instead, interpreted as statistical ensembles of the smooth ones [19]. This is an example where SYM provides previously unknown information about quantum states in gravity.

One important line of research building on the half-BPS sector results, was the analysis of D3-brane physics from the point of view of SYM. Since the defining feature of D3-branes is that their perturbative excitations are described by open strings, one would like to see the emergence of open strings from the operators dual to the giant gravitons. This was developed in [20, 21, 22, 23, 24, 25, 26] where it was shown that given an operator $\chi_R(Z)$ with long columns or rows, there is a natural way to "attach" open string operators. Their correlators can again be organized in $1/N$ genus expansion, now including worldsheets with boundaries, which was a further confirmation that Schur polynomial basis provides duals of the D3-branes. The new $1/N$ expansion allowed even to take a BMN-like limit for the open string [27] and some version of integrability was found, but it has not proved to be as powerful as in the closed string sector.

Another interesting class of perturbations of the half-BPS background can be constructed by taking two or more giant gravitons of different sizes, and considering their collective non-BPS excitations. There is no planar worldsheet expansion, but the symmetric group techniques for dealing with finite N in this regime were developed in [28, 29, 30, 31, 32], where the one-loop and two-loop excitation spectrum was calculated. Even though these results are at weak λ, because of the "near-BPS" nature of the states one might hope to compare them with the gravity regime. Furthermore, it was found that the energy spectrum is equal to a set of harmonic oscillators, suggesting that some version of integrability might apply in this non-planar sector. This offers an exciting prospect to explore the regime where both couplings N and λ are finite.

Finally, besides comparing the spectra and excitations, the SYM three point correlators were also recently checked against semiclassical AdS calculation involving giant gravitons [33, 34, 35]

The ultimate goal of this work is to extend these finite N half-BPS results to the less supersymmetric sectors of the theory, where the operator dimensions should still be protected. These are quarter-BPS, eighth-BPS and sixteenth-BPS sectors, preserving a respective fraction of supersymmetries. One would like to develop a similar story, where D3-brane configurations are mapped to $O(N)$ operators at weak coupling, and new geometries are mapped to $O(N^2)$ operators. However, there are significant new complications on both AdS and SYM side, and much less is known.

First, let us discuss the expectations from the gravity side. In the D3-brane regime, the generalization of giant gravitons to the eighth-BPS sector was found in [36], where it was shown that supersymmetric branes embedded in S^5 can be constructed from holomorphic surfaces in $\mathbb{C}^3$. Here one finds much richer configurations than in the half-BPS sector, including intersecting branes and deformed branes. This phase space of holomorphic surfaces has been geometrically quantized in [37, 38], providing a candidate eighth-BPS Hilbert space at finite N, that could be matched with SYM side. Even though the branes in S^5 do not span *all* classical eighth-BPS configurations – one also has, for example, generalization of dual giant gravitons expanding in AdS_5 [39] – the quantization of both gives the same Hilbert space, extending the similar duality in the half-BPS sector.

Going further, into the regime of new geometries, there are some constructions of the quarter-BPS and eighth-BPS geometries [40, 41, 42], but, unlike LLM, the classification is not complete.

Finally, in the sixteenth-BPS sector there is no classification of D3-brane configurations or geometries, but there is a known class of supersymmetric black holes with macroscopic horizons [43]. This makes it qualitatively different from all the sectors with more supersymmetry, and extremely interesting, with the possibility to perform precise microstate counting. However, this is also the most challenging sector.

On the SYM side the main complication is that the BPS spectrum changes from zero coupling to weak coupling. This is not the case for half-BPS, where the basis $\chi_R(Z)$ could be built at $\lambda = 0$ and it did not change at $\lambda > 0$. In the quarter- or eighth-BPS sectors the operators can be constructed by starting with the BPS operators in the *free* limit, spanned by holomorphic gauge invariant operators of two or three complex scalars Φ_a, and then finding the subspace annihilated by the one-loop dilatation operator. This is, in general, quite complicated at finite N.

If one is only interested in the spectrum – that is, counting of the operators – not the operators themselves, the problem is made easier by using the chiral ring. There is a one-to-one map between the eighth-BPS operators and the chiral ring elements, and the latter ones are, basically, spanned by *commuting* matrices Φ_a. The spectrum was calculated in [44], and was shown to be equal to that of N identical bosons in a 3D harmonic oscillator. It was also argued that the spectrum should not change from weak to strong λ, and thus could be compared to the gravity regime. Indeed, when the spectrum of quantized D3-branes was found in [38, 39], it matched the boson spectrum. Curiously, even though the D3-brane approximation should only work at $O(N)$ charge, the spectrum agreed with SYM exactly, all the way from graviton $O(1)$ to new geometry $O(N^2)$ regime.

The match with chiral ring counting, however, does not provide more detailed dynamical information, such as two-point or three-point functions. For this we need to go back to the harder problem of calculating BPS operators. One would like, in principle, to have a precise map between quantized D3-brane states and an orthogonal basis of operators. This would allow, for example, to study the excitations including open strings, but now propagating on more intricate world volumes, or stretched between intersecting branes. Furthermore, in the regime of $O(N^2)$ charges having a basis of operators might even help the classification of the possible eighth-BPS geometries.

The explicit construction of quarter-BPS operators was first studied in [45, 46], where a class of operators with small charges was constructed. In order to extend this to large charges, it is helpful to first have a complete basis at $\lambda = 0$ but finite N. A construction of the free orthogonal basis for quarter- and eighth-BPS was accomplished in [47, 48], using symmetric group techniques, quite analogous to the half-BPS construction of Schur polynomials. Since this basis transforms covariantly under the global $U(3)$, we refer to it as the covariant basis. A different free orthogonal basis was constructed in [49], which we call restricted Schur basis. Another orthogonal basis specific for the quarter-BPS was constructed in [50], based on Brauer algebra. In all these cases it is necessary to find the subspace of operators that remain BPS at small but non-zero λ. This can be accomplished by either finding the kernel of one-loop dilatation operator, or by finding orthogonal subspace to the descendants. Some progress was made in [51] with the covariant basis, and in [52] with the Brauer basis, and even a map to set of quarter-BPS geometries was suggested in [53]. However, the full problem of finding complete eighth-BPS basis and identifying dual gravity states is still unsolved.

There have been attempts to use the matrix model techniques to define a sensible two-point function directly on the N boson states arising from the chiral ring [54]. It is a tempting approach, as it gives rise to a nice semiclassical picture of emergent S^5. However, it is not clear that the two-point function constructed this way actually matches the correct one given by correlators of BPS operators.

Finally, in the sixteenth-BPS sector the results are very limited on the SYM side. The spectrum at $\lambda = 0$ was calculated in [44], but at weak coupling the chiral ring method is not available. Some progress was made in [55] for large N, using Q-cohomology similar to the chiral ring, but the finite N counting which could be compared, for example, to the black hole entropy, is still unknown. The large N results were also confirmed using the planar one-loop dilatation operator [56]. The construction of operators at finite N, however, would be much harder, because one would need to use the full non-planar dilatation operator.

In this thesis we continue the efforts to develop a map between quarter- or eighth-BPS states in $AdS_5 \times S^5$ and BPS operators in $\mathcal{N} = 4$ SYM. We are mainly interested in the regime where operators have charge $O(N)$ and on the gravity side the D3-brane probe approximation is valid. The thesis is organized into three main chapters: Chapter 3 discusses finite N orthogonal bases in *free* theories, Chapter 4 analyzes the problem of finding BPS states at *weak* coupling, and Chapter 5 makes the connection with the BPS states in the gravity, or *strong* coupling, limit. Additionally, in Chapter 2 we review some background material that we use in the main part. We finish with conclusions in Chapter 6 and supplemental material in appendices.

Chapter 3 is based on the paper [57]. Here we make a detour from $\mathcal{N} = 4$ SYM and develop general methods to find complete finite N bases for arbitrary four-dimensional quiver gauge theories in the zero superpotential limit. These results are an extension of previous group theoretic constructions [47, 49]. Here we develop convenient diagrammatic methods, using the quiver itself decorated with group theoretic data as a calculational tool. Infinite class of AdS/CFT pairs with $\mathcal{N} = 1$ or $\mathcal{N} = 2$ quiver gauge theories can be derived by placing D3-branes at orbifold Calabi-Yau singularities [58, 59, 60], or, more generally, at toric Calabi-Yau singularities [61, 62, 63, 64]. Our zero coupling results can be used as a starting point for weak coupling finite N calculations in these theories. On the other hand, the zero coupling limit itself should be dual to a tensionless string in $AdS_5 \times X^5$. Our counting formulae for various quivers give a rich set of data, which can be used to gain insight into the world-sheet theory.

Chapter 4 is partly based on the paper [65] but also contains new material. Here we go back to $\mathcal{N} = 4$ SYM, and address the problem of finding the complete BPS basis at weak coupling.

Chapter 5 is based on [66]. We analyze the Mikhailov's eighth-BPS D3-brane configurations [36] and their geometrically quantized Hilbert space [38]. We use techniques from fuzzy geometry to better understand the map between quantum states and classical configurations, and, finally, propose a map between the D3-brane states and the BPS operators found in Chapter 4.

Chapter 2

Background

2.1 $\mathcal{N}=4$ superconformal algebra

$\mathcal{N}=4$ SYM theory, being conformal, enjoys the group of global symmetries $PSU(2,2|4)$. Its generators are

$$\left(\begin{array}{cc|c} (J_1)^\alpha_\beta & P^{\alpha\dot\alpha} & Q^{\alpha i} \\ K_{\alpha\dot\alpha} & (J_2)^{\dot\alpha}_{\dot\beta} & \bar Q^{\dot\alpha}_i \\ \hline S_{\alpha i} & \bar S^i_{\dot\alpha} & R^i_j \end{array}\right), \quad H \tag{2.1}$$

The indices take values $\alpha = 1\dots2$, $\dot\alpha = 1\dots2$, $i = 1\dots4$. They transform as fundamentals of the bosonic subgroup $SU(2)\times SU(2)\times SU(4)$ spanned by the diagonal blocks $(J_1)^\alpha_\beta$, $(J_2)^\alpha_\beta$, R^i_j. Let us go over the generators:

- $(J_1)^\alpha_\beta$, $(J_2)^{\dot\alpha}_{\dot\beta}$ are the six Lorentz generators $M_{\mu\nu}$ rewritten as $SO(4) = SU(2)\times SU(2)$. Note these are traceless, $(J_1)^1_1 + (J_1)^2_2 = (J_2)^1_1 + (J_2)^2_2 = 0$.
- $P^{\alpha\dot\alpha}$ are the four momentum generators
- $K_{\alpha\dot\alpha}$ are the four special conformal generators
- H is the dilatation operator, generating scaling transformations. It is the remaining diagonal generator, which combines with J_1, J_2, P, K to make the total of 15 generators of $SU(2,2) \approx SO(4,2)$ conformal group.
- $Q^{\alpha i}, \bar Q^{\dot\alpha}_i$ are the 16 fermionic supersymmetry generators of $\mathcal{N}=4$
- $S_{\alpha i}, \bar S^i_{\dot\alpha}$ are the 16 extra fermionic superconformal generators
- R^i_j are the 15 generators of $SU(4)$ R-symmetry group

The key algebra commutation relations are[1]:

$$
\begin{aligned}
\{Q^{\alpha i},\bar{Q}^{\dot{\alpha}}_j\} &= P^{\alpha\dot{\alpha}}\delta^i_j \\
\{S_{\alpha i},\bar{S}^j_{\dot{\alpha}}\} &= K_{\alpha\dot{\alpha}}\delta^j_i \\
\{S_{\alpha i},Q^{\beta j}\} &= (J_1)^\beta_\alpha\delta^j_i + R^j_i\delta^\beta_\alpha + \frac{H}{2}\delta^j_i\delta^\beta_\alpha \\
\{\bar{S}^i_{\dot{\alpha}},\bar{Q}^{\dot{\beta}}_j\} &= (J_2)^{\dot{\beta}}_{\dot{\alpha}}\delta^i_j - R^i_j\delta^{\dot{\beta}}_{\dot{\alpha}} + \frac{H}{2}\delta^i_j\delta^{\dot{\beta}}_{\dot{\alpha}}
\end{aligned}
\tag{2.2}
$$

$$
[H,Q^{\alpha i}] = \frac{1}{2}Q^{\alpha i}, \quad [H,\bar{Q}^{\dot{\alpha}}_i] = \frac{1}{2}\bar{Q}^{\dot{\alpha}}_i, \quad [H,S_{\alpha i}] = -\frac{1}{2}S_{\alpha i}, \quad [H,\bar{S}^i_{\dot{\alpha}}] = -\frac{1}{2}\bar{S}^i_{\dot{\alpha}}
$$

The commuting Cartan subalgebra is spanned by the generators

$$
\begin{aligned}
&H \\
j_1 &= \frac{1}{2}(J_1)^2_2 - \frac{1}{2}(J_1)^1_1, \\
j_2 &= \frac{1}{2}(J_2)^2_2 - \frac{1}{2}(J_2)^1_1, \\
R_1 &= R^2_2 - R^1_1, \\
R_2 &= R^3_3 - R^2_2, \\
R_3 &= R^4_4 - R^3_3
\end{aligned}
\tag{2.3}
$$

Thus we can label the states by the corresponding eigenvalues

$$
|E; j_1, j_2; R_1, R_2, R_3\rangle \tag{2.4}
$$

j_1, j_2 label the states in the two $SU(2)$ representations respectively, while (R_1, R_2, R_3) labels a state in the $SU(4)$ representation.

In a CFT we consider the spectrum of local gauge invariant operators $\mathcal{O}(x)$, which transform under the (super-)conformal group. The action under transformations is given by commutators e.g. $[Q^{\alpha i}, \mathcal{O}(x)]$, and the eigenvalue of H is the *scaling dimension* of the operator. On the other hand, via the radial quantization, there is a one-to-one map between local operators $\mathcal{O}(x)$ and the states $|\mathcal{O}\rangle$ in the Hilbert space of the theory compactified on $\mathbb{R} \times S^3$. The point x around which we perform the radial quantization is chosen to be $x = 0$. The superconformal symmetry transformations induces the action on this Hilbert space

$$
Q^{\alpha i}|\mathcal{O}\rangle = |[Q^{\alpha i}, \mathcal{O}(0)]\rangle \tag{2.5}
$$

The dilatation operator H in particular is mapped to the time-translation operator, thus the spectrum of scaling dimensions of $\mathcal{O}(x)$ is the spectrum of energies on $\mathbb{R} \times S^3$ (it is discrete, because of the compact space). The inner product on this Hilbert space is defined by the Zamolodchikov metric using the CFT correlator

$$
\langle \mathcal{O}_1|\mathcal{O}_2\rangle = \lim_{x\to\infty} |x|^{2E} \left\langle \bar{\mathcal{O}}_1(x)\mathcal{O}_2(0)\right\rangle \tag{2.6}
$$

[1] for the full algebra see, for example, Appendix A of [44]

Essentially, $\langle\mathcal{O}|$ is obtained by acting by the inversion transformation $x^\mu = \frac{x^\mu}{x^2}$ on $|\mathcal{O}\rangle$. This inner product implies the following Hermitian conjugation

$$(Q^{\alpha i})^\dagger = S_{\alpha i}, \quad (\bar{Q}_i^{\dot{\alpha}})^\dagger = \bar{S}^i_{\dot{\alpha}} \tag{2.7}$$

The Hilbert space on S^3 is especially relevant for the AdS/CFT correspondence, since the boundary of AdS_5 is precisely $\mathbb{R} \times S^3$, and H is identified with the generator of global time translations in AdS_5. Thus the energy spectrum of states in the bulk (also discrete) *is* the spectrum of the scaling dimensions of local operators.

The operators $\mathcal{O}(x)$ (or equivalently states $|\mathcal{O}\rangle$ on S^3, which we will use interchangeably from now on) can be decomposed into irreducible representations (irreps) of $PSU(2,2|4)$. One can see from the algebra (2.2) that $Q^{\alpha i}, \bar{Q}_i^{\dot{\alpha}}$ increase the scaling dimension by $\frac{1}{2}$, while $S_{\alpha i}, \bar{S}^i_{\dot{\alpha}}$ decrease it. Since the spectrum of scaling dimensions in a unitary field theory is bounded from below, in every irrep there will be a set of lowest weight states, annihilated by all $S, \bar{S}$. Such states are called a *superconformal primaries.*

$$[S_{\alpha i}, \mathcal{O}_{sp}(x)] = [\bar{S}^i_{\dot{\alpha}}, \mathcal{O}_{sp}(x)] = 0 \tag{2.8}$$

The primaries will have equal E and appear in an irreducible representation of $SU(2) \times SU(2) \times SU(4)$. Thus we can label the irreducible representations of $PSU(2,2|4)$ by the representation $[E; j_1, j_2; R_1, R_2, R_3]$ of the primaries, here $[R_1, R_2, R_3]$ are the Dynkin labels of $SU(4)$. The representation will have a unique highest weight state with charges

$$|E; j_1, j_2; R_1, R_2, R_3\rangle_{\text{hw}} \tag{2.9}$$

which is annihilated by the raising operators $(J_1)_+, (J_2)_+, R^i_{i+1}$, as well as by $S_{\alpha i}, \bar{S}^i_{\dot{\alpha}}$.

2.2 Short multiplets

The anticommutators $\{S, Q\}$ in (2.2) and unitarity imply the following bounds on the representation labels of primary states [67]:

$$\begin{aligned} E &= \frac{3R_1 + 2R_2 + R_3}{2}, \quad j_1 = 0 \quad \text{or} \quad E \geq 2 + 2j_1 + \frac{3R_1 + 2R_2 + R_3}{2}, \quad \text{and} \\ E &= \frac{R_1 + 2R_2 + 3R_3}{2}, \quad j_2 = 0 \quad \text{or} \quad E \geq 2 + 2j_2 + \frac{R_1 + 2R_2 + 3R_3}{2} \end{aligned} \tag{2.10}$$

When the scaling dimension saturates the unitarity bound we get *short* representations of $PSU(2,2|4)$. In a generic (long) representation of $PSU(2,2|4)$ one can act on the superconformal primaries with all 16 raising operators $Q^{\alpha i}, \bar{Q}_i^{\dot{\alpha}}$. However, in the short representations, some of the Q's annihilate the primaries.

The short representations have been classified in [68], here we present a quick summary. They can be labelled by numbers $(t, \bar{t})$, specifying what fraction of the Q and $\bar{Q}$ supercharges annihilate the highest weight state. The possibilities for t and $\bar{t}$ are shown

t	Annihilated	Constraint
0	–	$E > 2 + 2j_1 + \frac{1}{2}(3R_1 + 2R_2 + R_3)$
$\frac{1}{8}$	$\tilde{Q}$	$E = 2 + 2j_1 + \frac{1}{2}(3R_1 + 2R_2 + R_3)$
$\frac{1}{4}$	$Q^{\alpha 1}$	$E = \frac{1}{2}(3R_1 + 2R_2 + R_3),\ j_1 = 0$
$\frac{1}{2}$	$Q^{\alpha 1}, Q^{\alpha 2}$	$E = \frac{1}{2}(2R_2 + R_3),\ R_1 = 0,\ j_1 = 0$

Table 2.1: Conditions for the highest weight state annihilation by $Q^{\alpha i}$

$\bar{t}$	Annihilated	Constraint
0	–	$E > 2 + 2j_2 + \frac{1}{2}(R_1 + 2R_2 + 3R_3)$
$\frac{1}{8}$	$\tilde{\bar{Q}}$	$E = 2 + 2j_2 + \frac{1}{2}(R_1 + 2R_2 + 3R_3)$
$\frac{1}{4}$	$\bar{Q}_4^{\dot\alpha}$	$E = \frac{1}{2}(R_1 + 2R_2 + 3R_3),\ j_2 = 0$
$\frac{1}{2}$	$\bar{Q}_3^{\dot\alpha}, \bar{Q}_4^{\dot\alpha}$	$E = \frac{1}{2}(R_1 + 2R_2),\ R_3 = 0,\ j_2 = 0$

Table 2.2: Conditions for the highest weight state annihilation by $\bar{Q}_i^{\dot\alpha}$

in Table 2.1 and Table 2.2 respectively. We denote by $\tilde{Q}$, $\tilde{\bar{Q}}$ the linear combinations

$$\tilde{Q} = Q^{11} - \frac{1}{2j_1 + 1}(J_1)_- Q^{21}, \quad \tilde{\bar{Q}} = \bar{Q}_4^2 - \frac{1}{2j_2 + 1}(J_2)_- \bar{Q}_4^1 \tag{2.11}$$

There is a variety of combinations $(t, \bar{t})$ that give rise to unitary short representations, shown in Table 2.3. The constraints on the energy and other charges for each case comes from combining the corresponding t and $\bar{t}$ conditions. For example, $(\frac{1}{2}, \frac{1}{2})$ imposes $j_1 = j_2 = R_1 = R_3 = 0$ which leads to the half-BPS representations

$$(\tfrac{1}{2}, \tfrac{1}{2}): \quad [E; 0, 0; 0, E, 0], \tag{2.12}$$

while $(\frac{1}{4}, \frac{1}{4})$ sets $E = \frac{1}{2}(R_1 + 2R_2 + 3R_3) = \frac{1}{2}(3R_1 + 2R_2 + R_3)$ which leads to the quarter-BPS representations

$$(\tfrac{1}{4}, \tfrac{1}{4}): \quad [p + 2q; 0, 0; q, p, q], \tag{2.13}$$

Because of these constraints which relate representation labels (integer or half-integer) to the scaling dimension, the BPS representations obey special non-renormalizability prop-

$(t, \bar{t})$	Annihilated	Name	Sector subgroup
$(\frac{1}{2}, \frac{1}{2})$	$Q^{\alpha 1}, Q^{\alpha 2}, \bar{Q}_3^{\dot\alpha}, \bar{Q}_4^{\dot\alpha}$	half-BPS	$U(1)$
$(\frac{1}{4}, \frac{1}{4})$	$Q^{\alpha 1}, \bar{Q}_4^{\dot\alpha}$	quarter-BPS	$SU(2) \times U(1)$
$(\frac{1}{4}, 0)$	$Q^{\alpha 1}$	eighth-BPS	$SU(2\|3)$
$(0, \frac{1}{4})$	$\bar{Q}_4^{\dot\alpha}$	eighth-BPS	$SU(2\|3)$
$(\frac{1}{8}, 0)$	$\tilde{Q}$	sixteenth-BPS	$PSU(1,2\|3) \times U(1)$
$(0, \frac{1}{8})$	$\tilde{\bar{Q}}$	sixteenth-BPS	$PSU(1,2\|3) \times U(1)$
$(\frac{1}{8}, \frac{1}{8})$	$\bar{Q}, \tilde{\bar{Q}}$	–	$PSU(1,1\|2) \times U(1) \times U(1)$
$(\frac{1}{4}, \frac{1}{8})$	$Q^{\alpha 1}, \tilde{\bar{Q}}$	–	$SU(1\|2) \times U(1)$
$(\frac{1}{8}, \frac{1}{4})$	$\bar{Q}_4^{\dot\alpha}, \tilde{Q}$	–	$SU(1\|2) \times U(1)$

Table 2.3: Short unitary representations of $PSU(2, 2|4)$

erties, since the scaling dimension can not vary continuously with coupling constant. It is, however, possible for two short multiplets to combine into a long multiplet and then acquire an anomalous scaling dimension. This commonly happens when the coupling changes from zero to non-zero.

For each shortening condition $(t, \bar{t})$ it is possible to define a reduced sector of the theory, which completely accounts for the corresponding multiplets [69]. Instead of considering the highest weight states annihilated by a set of Q, we take all states annihilated by Q and $S \sim Q^\dagger$. These do not have to be superconformal primaries, annihilated by all S. For example, for the eighth-BPS sector $(0, \frac{1}{4})$ that will be the main focus of this work, we take all states annihilated by $\bar{Q}_4^{\dot{\alpha}}, \bar{S}_{\dot{\alpha}}^4$

$$Q_4^{\dot{\alpha}} |\mathcal{O}_{SU(2|3)}\rangle = S_{\dot{\alpha}}^4 |\mathcal{O}_{SU(2|3)}\rangle = 0 \tag{2.14}$$

The states in this sector fall into representations of the residual symmetry $SU(2|3) \subset PSU(2,2|4)$ generated by the charges commuting with $\bar{Q}_4^{\dot{\alpha}}, \bar{S}_{\dot{\alpha}}^4$:

$$(J_1)^\alpha_\beta, R^i_j, Q^i_\alpha, S^\alpha_i, \quad i = 1 \dots 3 \tag{2.15}$$

The residual subgroup for each sector is listed in Table 2.3, and it is a convenient way to label them. Thus (2.14) is often referred to as $SU(2|3)$ sector, but we will also simply call it the eighth-BPS sector.

The relationship between sectors and short multiplets is that there is a one-to-one map between $SU(2|3)$ representations in the sector and the $(0, \frac{1}{4})$ (or more supersymmetric) multiplets, thus counting one gives the other. That is, acting on the $SU(2|3)$ sector with the full $PSU(2,2|4)$ will generate all states in the following short multiplets:

$$\bar{Q}_4^{\dot{\alpha}} = \bar{S}_{\dot{\alpha}}^4 = 0 \quad \rightarrow \quad (0, \tfrac{1}{4}), (\tfrac{1}{8}, \tfrac{1}{4}), (\tfrac{1}{4}, \tfrac{1}{4}), (\tfrac{1}{2}, \tfrac{1}{2}) \tag{2.16}$$

More supersymmetric multiplets come from the fact that requiring annihilation by $\bar{Q}_4^{\dot{\alpha}}, \bar{S}_{\dot{\alpha}}^4$ still allows them to be annihilated by additional supercharges.

What makes reduced sectors useful is that it is easier to impose $\bar{Q}_4^{\dot{\alpha}} = \bar{S}_{\dot{\alpha}}^4 = 0$ than to consider only highest weight states. Using the algebra we can write

$$\{\bar{S}_{\dot{\alpha}}^4, \bar{Q}_4^{\dot{\beta}}\} = (J_2)^{\dot{\beta}}_{\dot{\alpha}} - R^4_4 \delta^{\dot{\beta}}_{\dot{\alpha}} + \frac{H}{2} \delta^{\dot{\beta}}_{\dot{\alpha}} = 0 \tag{2.17}$$

thus all the states in the sector obey constraints

$$J_2 = 0, \quad E = 2R^4_4 - \frac{1}{2} R^i_i = \frac{1}{2}(R_1 + 2R_2 + 3R_3) \tag{2.18}$$

In fact, (2.18) is an equivalent definition of the $SU(2|3)$ sector, because $\{\bar{S}_{\dot{\alpha}}^4, \bar{Q}_4^{\dot{\beta}}\} = 0$ together with unitarity implies annihilation by both $\bar{S}_{\dot{\alpha}}^4, \bar{Q}_4^{\dot{\beta}}$. This form of the constraint is the most convenient to use for calculations

Note, that for purposes of comparing with AdS, the eighth-BPS sector defined by

$S^4_{\dot\alpha} = \bar{Q}^{\dot\beta}_4 = 0$ is also more convenient, than dealing with the highest weight states and multiplets. For example, these conditions can be translated directly into the requirement for configurations of D3-branes to preserve $\frac{1}{8}$ of the supercharges in AdS [36].

2.3 $\mathcal{N} = 4$ super Yang-Mills

Let us now look in detail at $\mathcal{N} = 4$ supersymmetric Yang-Mills. In this work we take the gauge group to be $U(N)$. The field content is a single $\mathcal{N} = 4$ vector multiplet, consisting of

$$\Phi_{ij}, \quad \Psi_{\alpha i}, \bar{\Psi}^i_{\dot\alpha}, \quad A_\mu \tag{2.19}$$

We have 6 real scalars (or 3 complex) in the antisymmetric tensor Φ_{ij} representation **6** of $SU(4)$, obeying reality condition

$$\Phi^*_{ij} = \Phi^{ij} = \frac{1}{2}\epsilon^{ijkl}\Phi_{kl}, \tag{2.20}$$

4 Weyl fermions $\Psi_{i\alpha}$ in the fundamental of $SU(4)$ with their conjugates $\bar{\Psi}^i_{\dot\alpha}$ in the anti-fundamental, and gauge fields A_μ. Thus in total we have (6+2) bosonic and 8 fermionic degrees of freedom on-shell. All the fields are adjoints of the gauge group, and so can be thought to be $N \times N$-matrix valued. In order to build gauge-invariant objects we use the covariant derivative

$$D_\mu \equiv \partial_\mu - igA_\mu \tag{2.21}$$

and the field strength tensor

$$F_{\mu\nu} = ig^{-1}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \tag{2.22}$$

where g is the gauge coupling constant.

The Lagrangian of the theory is uniquely fixed by the supersymmetry

$$\begin{aligned}\mathcal{L} = \mathrm{tr}\Big\{ &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\Phi_{ij}D^\mu\Phi^{ij} - \frac{1}{2}i\Psi^\alpha_i D_{\alpha\dot\alpha}\bar{\Psi}^{\dot\alpha i} \\ &-g\Psi^\alpha_i[\Psi_{\alpha j}, \Phi^{ij}] - g\bar{\Psi}^i_{\dot\alpha}[\bar{\Psi}^{\dot\alpha j}, \Phi_{ij}] + 2g^2[\Phi_{ij}, \Phi_{kl}][\Phi^{ij}, \Phi^{kl}]\Big\}\end{aligned} \tag{2.23}$$

It is convenient to bring all the fields into the $SU(2) \times SU(2) \times SU(4)$ covariant form, by replacing indices μ with $\alpha, \dot\beta$ via σ matrices

$$\begin{aligned} D_\mu &\to D_{\dot\alpha\beta} \\ F_{\mu\nu} &\to F_{\alpha\beta} \oplus \bar{F}_{\dot\alpha\dot\beta} \end{aligned} \tag{2.24}$$

$F_{\alpha\beta}, \bar{F}_{\dot\alpha\dot\beta}$ are the self-dual and anti-self-dual field strength, each a symmetric tensor of $SU(2)$ with 3 independent components, decomposing the 6 in the antisymmetric $F_{\mu\nu}$. The supersymmetry transformations are then:

$$\begin{aligned} [Q^i_\alpha, \Phi_{jk}] &= \delta^i_j \Psi_{\alpha k} - \delta^i_k \Psi_{\alpha j} \\ \{Q^i_\alpha, \Psi_{\beta j}\} &= 2i\delta^i_j F_{\alpha\beta} + ig\epsilon_{\alpha\beta}[\Phi_{jk}, \Phi^{ki}] \\ \{Q^i_\alpha, \bar{\Psi}^j_{\dot\beta}\} &= -2iD_{\alpha\dot\beta}\Phi^{ij} \\ [Q^i_\alpha, D_{\beta\dot\gamma}] &= -ig\epsilon_{\alpha\beta}\bar{\Psi}^i_{\dot\gamma} \end{aligned} \tag{2.25}$$

and the $\bar{Q}^{\dot\alpha}_i$ action is given by complex conjugation.

Let us note how the $PSU(2,2|4)$ representation theory applies here. For simplicity take the zero coupling limit $g = 0$, in which case the supersymmetry transformations close on the single field sector, schematically:

$$\begin{array}{lll} & \{Q, \Psi_{\alpha i}\} \sim F_{\alpha\beta} & \{Q, F_{\alpha\beta}\} \sim 0 \\ [Q, \Phi_{ij}] \sim \Psi_{\alpha i} & & \{\bar{Q}, F_{\alpha\beta}\} \sim \{Q, D_{a\dot\beta}\Phi_{ij}\} \sim D_{\alpha\dot\beta}\Psi_{\alpha i} \\ & \{\bar{Q}, \Psi_{\alpha i}\} \sim \{Q, \bar{\Psi}^i_{\dot\alpha}\} \sim D_{\alpha\dot\beta}\Phi_{ij} & \\ [\bar{Q}, \Phi_{ij}] \sim \bar{\Psi}^i_{\dot\alpha} & & \{Q, \bar{F}_{\dot\alpha\dot\beta}\} \sim \{\bar{Q}, D_{a\dot\beta}\Phi_{ij}\} \sim D_{\alpha\dot\beta}\bar{\Psi}^i_{\dot\alpha} \\ & \{\bar{Q}, \bar{\Psi}^i_{\dot\alpha}\} \sim \bar{F}_{\dot\alpha\dot\beta} & \{\bar{Q}, \bar{F}_{\dot\alpha\dot\beta}\} \sim 0 \end{array}$$

By repeatedly acting with $Q^i_\alpha, \bar{Q}_{\dot\alpha i}$ we can generate all possible fields of the form

$$\{\ D^k\Phi_{ij},\ D^k\Psi_{\alpha i},\ D^k\bar{\Psi}^i_{\dot\alpha},\ D^kF_{\alpha\beta},\ D^k\bar{F}_{\dot\alpha\dot\beta}\} \tag{2.26}$$

This is a single (infinite-dimensional) irreducible representation of $PSU(2,2|4)$, with superconformal primaries Φ_{ij}. It is called the *field strength multiplet*. The primaries Φ_{ij} are in the representation

$$[1;0,0;0,1,0] \tag{2.27}$$

thus according to (2.12) the multiplet is half-BPS. The highest weight state in the $[0,1,0]$ representation of $SU(4)$ (annihilated by all R^i_{i+1}) is

$$\Phi_{34} \equiv Z \tag{2.28}$$

One can also check explicitly that it is annihilated by half of supercharges

$$[Q^1_\alpha, \Phi_{34}] = [Q^2_\alpha, \Phi_{34}] = [\bar{Q}_{\dot\alpha 3}, \Phi_{34}] = [\bar{Q}_{\dot\alpha 4}, \Phi_{34}] = 0 \tag{2.29}$$

Strictly speaking, the multiplet (2.26) is not part of the spectrum of *gauge-invariant* local operators $\mathcal{O}(x)$, and can not be matched directly with the states in the bulk. Instead consider the operators

$$\mathrm{tr}(Z^n) \tag{2.30}$$

which are all highest weight states of $[n;0,0;0,n,0]$, and superconformal primaries of half-BPS multiplets. It can be shown that the states $\mathrm{tr}(Z^n)$ remain half-BPS as we vary the coupling, because they can not join with other representations to form a long multiplet [68]. Thus the scaling dimension remains protected, equal to the bare dimension n, which allows to compare against the strong coupling regime, where AdS description is valid. It turns out that the spectrum $\mathrm{tr}(Z^n)$ (plus descendants) indeed matches the single-particle states of the supergravity multiplet in $AdS_5 \times S^5$, which are half-BPS as well. This comparison was one of the first checks of the AdS/CFT correspondence.

In general, there is a one-to-one map between half-BPS multiplets and the highest weight states built purely from Z

$$\mathcal{O}_{1/2}(x) = \prod_{i=1}^{k} \mathrm{tr}(Z^{n_i}) \tag{2.31}$$

When working at finite N this basis is overcomplete, because there are non-trivial identifications between single- and multi-traces. A complete orthogonal basis for half-BPS was found in [16]: the states can be labelled by Young diagrams R of height $l(R) \leq N$ and the operators $\chi_R(Z)$ called Schur polynomials are appropriate linear combinations of traces. It has been confirmed that this completely captures the spectrum of half-BPS objects of Type IIB string theory in the bulk (not just supergravity!), including the giant graviton D3 branes, and smooth half-BPS geometries

2.4 Eighth-BPS chiral ring

In this work we will be focusing on the eighth-BPS sector of the $\mathcal{N} = 4$ SYM, annihilated by 2 supercharges $\bar{Q}_4^{\dot{\alpha}}$ and their Hermitian conjugates $\bar{S}^4_{\dot{\alpha}}$

$$\mathcal{H}_{\mathrm{BPS}} = \{\mathcal{O}(x) \mid [\bar{Q}_4^{\dot{\alpha}}, \mathcal{O}(x)] = [\bar{S}^4_{\dot{\alpha}}, \mathcal{O}(x)] = 0\} \tag{2.32}$$

As explained in Section 2.2, it contains part of the $(0, \frac{1}{4})$ or more supersymmetric multiplets (2.16). We could equivalently consider the sector annihilated by $Q^{\alpha 1}, S_{\alpha 1}$, leading to $(\frac{1}{4}, 0)$ multiplets and the anti-chiral ring.

The eighth-BPS sector can also be defined by the energy constraint (2.18)

$$E = \frac{1}{2}(R_1 + 2R_2 + 3R_3) \tag{2.33}$$

together with the requirement that states are singlets of J_2. It will be convenient to use the $SO(6)$ charges q_1, q_2, q_3 instead of the $SU(4)$ R_1, R_2, R_3, that are related as

$$\begin{aligned} q_1 &= \frac{1}{2}\left(+R^1_1 - R^2_2 - R^3_3 + R^4_4\right) = \frac{1}{2}(-R_1 + R_3) \\ q_2 &= \frac{1}{2}\left(-R^1_1 + R^2_2 - R^3_3 + R^4_4\right) = \frac{1}{2}(R_1 + R_3) \\ q_3 &= \frac{1}{2}\left(-R^1_1 - R^2_2 + R^3_3 + R^4_4\right) = \frac{1}{2}(R_1 + 2R_2 + R_3) \end{aligned} \tag{2.34}$$

The energy constraint (2.33) is then simply

$$E = q_1 + q_2 + q_3 \tag{2.35}$$

Consider again the free $g = 0$ limit of SYM. The part of the field strength multiplet (2.26) that obeys (2.33) is spanned by the fields

	j_1	E	$[R_1, R_2, R_3]$	$[q_1, q_2, q_3]$
Φ_{34}	0	1	$[0, 1, 0]$	$[0, 0, 1]$
Φ_{24}	0	1	$[1, -1, 1]$	$[0, 1, 0]$
Φ_{14}	0	1	$[-1, 0, 1]$	$[1, 0, 0]$
$\Psi_{\alpha 4}$	$\pm\frac{1}{2}$	$\frac{3}{2}$	$[0, 0, 1]$	$[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$

(2.36)

Let us define

$$\begin{aligned} \Phi_1 &\equiv X \equiv \Phi_{14} \\ \Phi_2 &\equiv Y \equiv \Phi_{24} \\ \Phi_3 &\equiv Z \equiv \Phi_{34} \\ \lambda_\alpha &\equiv \Psi_{\alpha 4} \end{aligned} \tag{2.37}$$

If we view the $\mathcal{N} = 4$ SYM from the $\mathcal{N} = 1$ perspective, where $Q^4_\alpha, \bar{Q}^{\dot{\alpha}}_4$ are the supercharges, then the complex scalars Φ_a are precisely the scalar components of the three chiral multiplets, while λ_α are the gauginos.

Let us denote the "letters" Φ_a, λ_α by W_A:

$$W_A \in \{\Phi_1, \Phi_2, \Phi_3, \lambda_+, \lambda_-\} \tag{2.38}$$

As already discussed in the previous section the fields W_A themselves are part of the half-BPS multiplet. But now consider products of traces built from arbitrary contractions of these letters[2]

$$\mathcal{O} = \mathrm{tr}(W_{A_1} W_{A_2} \ldots)\mathrm{tr}(W_{B_1} \ldots) \ldots \tag{2.39}$$

They are no longer half-BPS, but are contained in $\mathcal{H}_{\mathrm{BPS}}$, because the dimension and charges of $\mathcal{O}$ in the free theory is just the sum of the components, thus it will also obey $E = q_1 + q_2 + q_3$. Alternatively, one can see this from the fact that the generators $\bar{S}^4_{\dot{\alpha}}, \bar{Q}^{\dot{\beta}}_4$ just act on each field individually, and annihilate $\mathcal{O}$. Thus the full eighth-BPS sector at $g = 0$ is spanned by multi-trace operators (2.39) subject to finite N matrix relationships

$$\mathcal{H}^{\mathrm{free}}_{\mathrm{BPS}} = \{\mathcal{O} \mid \forall \mathcal{O} = \mathrm{tr}(W_{A_1} W_{A_2} \ldots)\mathrm{tr}(W_{B_1} \ldots) \ldots\} \tag{2.40}$$

Now consider the interacting theory $g \neq 0$. The complication arises because of the non-

[2]As before, all the field operators $\Phi_a(x)$, $\lambda_\alpha(x)$ are assumed to be at the same position x, which we will drop from now on. The operators $\mathcal{O}$ are in one-to-one correspondence with the states $|\mathcal{O}\rangle$ in the radial quantization around x.

linear terms in (2.25) proportional to g, which change how the operators are organized into representations. For example, the operator $\mathrm{tr}([Y,Z][Y,Z])$ is a superconformal primary at $g=0$ and part of the eighth-BPS sector, but at $g \neq 0$ it can appear on the right-hand side of Q^i_α action and, in fact, turns out to be a descendant of the Konishi primary operator $\mathrm{tr}(\Phi_{ij}\Phi^{ij})$. What happens is that some short representations join together to become long and acquire anomalous dimension, and thus the spectrum of eighth-BPS multiplets at $g \neq 0$ is different from that at $g=0$.

There are two approaches to finding the eighth-BPS sector in the interacting theory. First, note that solving $\bar{S}^4_{\dot\alpha} = 0$ explicitly is not possible in practice, because the action of $\bar{S}^4_{\dot\alpha}$ receives loop corrections at each order in g, and there is no straightforward method of deriving them. Instead, we can focus on solving the condition $E = q_1 + q_2 + q_3$. Scaling dimension E also receives loop corrections (anomalous dimension), but it is possible to use the dilatation operator techniques [8] to calculate them. Since the charges q_i are not renormalized, the operators satisfying the eighth-BPS sector condition are those, which are annihilated by the dilatation operator and have zero anomalous dimension $\Delta = 0$. It is furthermore conjectured, that it is enough for an operator to be annihilated by the *one-loop* dilatation operator Δ_2, in order to be annihilated at all-loops. Thus we can write

$$\mathcal{H}_{\mathrm{BPS}} = \mathrm{Ker}(\Delta) = \mathrm{Ker}(\Delta_2)|_{\mathcal{H}^{\mathrm{free}}_{\mathrm{BPS}}} \tag{2.41}$$

where Δ_2 is taken to act on the free $SU(2|3)$ sector. For example, the combination

$$\mathcal{O} = \mathrm{tr}(Z^2)\mathrm{tr}(Y^2) - \mathrm{tr}(ZY)\mathrm{tr}(ZY) + \frac{1}{N}\mathrm{tr}([Z,Y][Z,Y]) \tag{2.42}$$

is found to have zero anomalous dimension and thus is annihilated by $\bar{Q}^{\dot\alpha}_4, \bar{S}^4_{\dot\alpha}$. It is, in fact, in a quarter-BPS multiplet. We will come back to this in Chapter 4.

The other approach avoids dealing with the loop corrections altogether, and instead relies on the concept of the *chiral ring* [70]. The idea is to convert the problem of finding operators annihilated by $\bar{Q}^{\dot\alpha}_4, \bar{S}^4_{\dot\alpha}$ into the problem of $\bar{Q}^{\dot\alpha}_4$ cohomology, using the fact that $\bar{S}^4_{\dot\alpha} = (\bar{Q}^{\dot\alpha}_4)^\dagger$ in the Zamolodchikov metric. Consider $\mathrm{Ker}(\bar{Q}^{\dot\alpha}_4)$, the set of operators annihilated by $\bar{Q}^{\dot\alpha}_4$. Now in order for a state $|\mathcal{O}_{\mathrm{BPS}}\rangle$ in $\mathrm{Ker}(\bar{Q}^{\dot\alpha}_4)$ to be also annihilated by $\bar{S}^4_{\dot\alpha}$ we must have

$$\bar{S}^4_{\dot\alpha}|\mathcal{O}_{\mathrm{BPS}}\rangle = 0 \quad \leftrightarrow \quad \langle\mathcal{O}'|\bar{S}^4_{\dot\alpha}|\mathcal{O}_{\mathrm{BPS}}\rangle = 0 \quad \leftrightarrow \quad \langle\bar{Q}^{\dot\alpha}_4\mathcal{O}'|\mathcal{O}_{\mathrm{BPS}}\rangle = 0 \quad \forall\mathcal{O}' \tag{2.43}$$

That is, an operator $\mathcal{O}_{\mathrm{BPS}}$ is in eighth-BPS sector if and only if it is in $\mathrm{Ker}(\bar{Q}^{\dot\alpha}_4)$ and orthogonal to the image of $Q^{\dot\alpha}_4$

$$\mathcal{H}_{\mathrm{BPS}} = \mathrm{Ker}(\bar{Q}^{\dot\alpha}_4) \cap \mathrm{Im}(\bar{Q}^{\dot\alpha}_4)^\perp \tag{2.44}$$

There is one-to-one map between this space and the quotient space

$$\mathcal{C} = \mathrm{Ker}(\bar{Q}^{\dot\alpha}_4)/\mathrm{Im}(\bar{Q}^{\dot\alpha}_4), \tag{2.45}$$

that is, instead of considering subspace of $\text{Ker}(\bar{Q}_4^{\dot{\alpha}})$ orthogonal to $\text{Im}(\bar{Q}_4^{\dot{\alpha}})$, we consider $\text{Ker}(\bar{Q}_4^{\dot{\alpha}})$ *modulo* operators in $\text{Im}(\bar{Q}_4^{\dot{\alpha}})$. For example, within $\mathcal{C}$

$$\text{tr}([Z,Y][Z,Y]) \sim 0 \tag{2.46}$$

This is precisely how the chiral ring is defined in any $\mathcal{N}=1$ theory: operators annihilated by $\bar{Q}^{\dot{\alpha}}$, modulo $\bar{Q}^{\dot{\alpha}}$-descendants, or the $\bar{Q}^{\dot{\alpha}}$-cohomology. We will refer to $\mathcal{C}$ as the eighth-BPS cohomology or the eighth-BPS chiral ring.

Note that $\mathcal{H}_{BPS}$ and $\mathcal{C}$ are not the same space, in particular, $\mathcal{C}$ is not a Hilbert space in that it does not have a naturally defined inner product like $\mathcal{H}_{\text{BPS}}$. The elements of $\mathcal{C}$ are equivalence classes of operators, and each class contains exactly one state in $\mathcal{H}_{\text{BPS}}$. For example the operator 2.42 is simply identified with

$$\mathcal{O} \sim \text{tr}(Z^2)\text{tr}(Y^2) - \text{tr}(ZY)\text{tr}(ZY) \tag{2.47}$$

Even though we lose some information about the inner products, the main advantage of the chiral ring is that it can be found using the classical Q-action (2.25), without needing S or loop-corrections at all. This makes it particularly useful when studying partition functions, where all we need is counting and not correlators. Furthermore, $\mathcal{C}$ has a structure of a ring (hence the name), meaning that we can multiply two eighth-BPS operators (up to descendants) to get another eighth-BPS operator (up to descendants), which is not true for the operators in $\mathcal{H}_{\text{BPS}}$ themselves. This allows to use the power of algebraic geometry, see, for example, Appendix E.

To recap, we have defined:

- Eighth-BPS multiplets: $PSU(2,2|4)$ multiplets, where the highest-weight superconformal primary is annihilated by $\bar{Q}_{\dot{\alpha}4}$
- Eighth-BPS sector (or $SU(2|3)$ sector): all states annihilated by $\bar{Q}_{\dot{\alpha}4}$ and $\bar{S}^{\dot{\alpha}4}$. This sector contains all the primaries of eighth-BPS or more supersymmetric multiplets, and their $SU(2|3)$ descendants, but not full multiplets. Equivalently the sector can be defined as:
  - States with $J_2 = 0$ and $E = q_1 + q_2 + q_3 = \frac{3}{2}R_3 + R_2 + \frac{1}{2}R_1$. This is easiest to implement in the free theory, while in the interacting theory need to use dilatation operator to find states with uncorrected E.
  - States annihilated by $\bar{Q}_{\dot{\alpha}4}$ and orthogonal (in Zamolodchikov metric) to $\bar{Q}_{\dot{\alpha}4}$-descendants
- Eighth-BPS chiral ring: the cohomology of $\bar{Q}_{\dot{\alpha}4}$. Elements of the chiral ring are equivalence classes of operators annihilated by $\bar{Q}_{\dot{\alpha}4}$, each containing exactly one element also annihilated by $\bar{S}^{\dot{\alpha}4}$. This is the easiest method in the interacting theory, avoiding loop corrections, but we don't get exact operators.

2.5 Chiral ring in $\mathcal{N} = 1$

In the previous section we arrived at the chiral ring from the analysis of eighth-BPS sector in $\mathcal{N} = 4$ SYM. Since in the Chapter 3 we will be dealing with more general $\mathcal{N} = 1$ SCFTs, let us here describe the chiral ring more generally from the $\mathcal{N} = 1$ perspective.

In a four-dimensional $\mathcal{N} = 1$ theory we have 4 supercharges $Q^\alpha, \bar{Q}_{\dot{\alpha}}$ and, if the theory is conformal, also 4 superconformal charges $S_\alpha, \bar{S}^{\dot{\alpha}}$. In the radial quantization with the Zamolodchikov inner product they are Hermitian conjugates of each other

$$S_\alpha = (Q^\alpha)^\dagger, \quad \bar{S}^{\dot{\alpha}} = (\bar{Q}_{\dot{\alpha}})^\dagger \tag{2.48}$$

The chiral ring is the cohomology of $\bar{Q}_{\dot{\alpha}}$ acting on local gauge invariant operators (or equivalently on states on S^3). The operators in the chiral ring have position-independent correlators

$$\frac{\partial}{\partial x_i} \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\dots\mathcal{O}_n(x_n)\rangle = 0 \tag{2.49}$$

so the VEVs $\langle \mathcal{O}_1\mathcal{O}_2\dots\mathcal{O}_n\rangle$ can be used to parametrize the vacuum moduli space. This is very useful for the non-perturbative studies of $\mathcal{N} = 1$ field theories [70, 71].

In a theory with gauge vector multiplet V, and any number of chiral matter multiplets Φ_a, the chiral operators (annihilated by $\bar{Q}_{\dot{\alpha}}$) are built from the complex scalars and gauginos

$$\Phi_a, \quad \lambda_\alpha \tag{2.50}$$

The image of $\bar{Q}_{\dot{\alpha}}$, which is identified with zero in the chiral ring, is spanned by

$$\frac{\partial W}{\partial \Phi_a} \propto \bar{Q}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}\bar{\Phi}_a, \quad [\lambda_\alpha, \Phi_a] \propto \bar{Q}^{\dot{\alpha}}(D_{\alpha\dot{\alpha}}\Phi_a), \quad \{\lambda_\alpha, \lambda_\beta\} \propto \bar{Q}^{\dot{\alpha}}(D_{\alpha\dot{\alpha}}\lambda_\beta) \tag{2.51}$$

where W is the superpotential. For example, in the $\mathcal{N} = 1$ description of $\mathcal{N} = 4$ there are 3 chiral adjoint fields Φ_a and the superpotential is

$$W = g\,\mathrm{tr}([\Phi_1, \Phi_2]\Phi_3) \tag{2.52}$$

Thus the image of $\bar{Q}_{\dot{\alpha}}$ at $g \neq 0$ is generated by

$$\frac{\partial W}{\partial \Phi_a} \propto \epsilon_{abc}[\Phi_b, \Phi_c], \quad [\lambda_\alpha, \Phi_a], \quad \{\lambda_\alpha, \lambda_\beta\} \tag{2.53}$$

so any (anti-)commutators are set to 0 in the chiral ring. This leads to the result that the chiral ring is spanned by gauge invariant operators built from (anti-)commuting matrices Φ_a, λ_α, which is isomorphic to the Hilbert space of N identical bosons, corresponding to matrix eigenvalues, in 3 bosonic and 2 fermionic dimensions.

In this work we focus only on the bosonic chiral ring, built from chiral scalars Φ_a but not gauginos[3]. All identifications are then given by the F-terms

$$F = \left\{ \frac{\partial W}{\partial \Phi_a} \right\} \tag{2.54}$$

The bosonic chiral ring can be identified with the coordinate ring on the vacuum moduli space. This has been studied extensively in the context of AdS/CFT correspondence. There is an infinite class of quiver SCFTs constructed from brane tilings [61, 62, 63, 64] where the chiral ring is found to be the coordinate ring of $\mathrm{Sym}^N(Y^6)$, where Y^6 is a toric Calabi-Yau threefold. This provides evidence that these theories, at the strongly coupled IR fixed point, describe the low energy theory of a stack of N D3-branes placed at a Y^6 singularity. The gravity dual of the SCFT can then be identified as the near horizon geometry $AdS_5 \times X_5$, where Y_6 is a cone over X_5.

As an example of $\mathcal{N} = 1$ SCFT let us take the conifold theory [72] with a gauge group $U(N) \times U(N)$, two chiral multiplets A_i in a bifundamental $(\bar{N}, N)$, and two B_i in a bifundamental $(N, \bar{N})$ representation (see quiver in Figure 3.9). The superpotential is

$$W = h\left(\mathrm{tr}(A_1 B_1 A_2 B_2) - \mathrm{tr}(A_1 B_2 A_2 B_1)\right) \tag{2.55}$$

At non-zero h the F-terms $\frac{\partial W}{\partial \Phi_a}$ are

$$\begin{aligned} F = \{B_1 A_2 B_2 - B_2 A_2 B_1, \quad & B_2 A_1 B_1 - B_1 A_1 B_2, \\ A_2 B_2 A_1 - A_1 B_2 A_2, \quad & A_1 B_1 A_2 - A_2 B_1 A_1\} \end{aligned} \tag{2.56}$$

In the interacting chiral ring they are identified with zero

$$F \sim 0 \tag{2.57}$$

The structure of (2.56) implies that within the chiral ring we can commute A's through B's and vice versa. The resulting mesonic chiral ring at large N is thus spanned by

$$S^{i_1 i_2 \ldots i_n} S^{j_1 j_2 \ldots j_n}\, \mathrm{tr}(A_{i_1} B_{j_1} A_{i_2} B_{j_2} \ldots A_{i_n} B_{j_n}) \tag{2.58}$$

where S is a symmetric tensor, and products of such symmetrized traces.

Note that to get a chiral ring at finite N we have to enforce both finite N constraints and the F-terms, which might not be independent. One can formulate this as follows. Let $V^{(\infty)}$ be the ring of chiral gauge invariant operators of the free theory at $N = \infty$, that is, treating operators as formal products of traces, without any finite N identifications. At finite N some operators in $V^{(\infty)}$ vanish – they form an ideal[4] $V_N \subset V^{(\infty)}$. The quotient $V^{(\infty)}/V_N$ is the free chiral ring at finite N. Now, let V_F be the space of all gauge invariant operators at $N = \infty$, which are identified with zero by F-terms. It is spanned by all

[3] There are some subtleties with fermionic chiral operators $\mathrm{tr}(\lambda_\alpha \lambda_\beta)$ related to Konishi anomaly that we will not get into, see [70].

[4] V_N is an ideal of $V^{(\infty)}$ because a product of vanishing operator and any other operator is also vanishing

operators containing an F-term anywhere within a trace. V_F is also an ideal of $V^{(\infty)}$. The $N = \infty$ interacting chiral ring is then the quotient $V^{(\infty)}/V_F$, which is spanned by products of symmetrized traces as in (2.58). Finally, the finite N interacting chiral ring is

$$V_{\text{int}}^{(N)} = V^{(\infty)}/(V_F \cup V_N) \tag{2.59}$$

that is, we identify operators in $V^{(\infty)}$ if they differ by V_F *or* V_N. This quotient can be implemented explicitly using computational algebraic geometry [73]. This is practical at small N but becomes computationally prohibitive at large N. We will solve the problem of finding V_N and $V^{(\infty)}/V_N$ in the next chapter, which gives the free chiral ring, however, taking the union $V_F \cup V_N$ remains a hard problem.

Finally, let us comment on the meaning of "free" limit of gauge theories, that will be the topic of Chapter 3. Consider the RG flow of the conifold theory, with the superpotential (2.55) (see [74] for a good review on the subject). Let us define a dimensionless coupling constant $\eta = h\mu$, with energy scale μ. The dimensionless couplings of the theory are then (g_1, g_2, η), where (g_1, g_2) are the gauge couplings of the two group factors. For simplicity let us restrict to the case where $g_1 = g_2 \equiv g$.

The theory is asymptotically free, so perturbatively g increases in the IR. The coupling η classically scales like μ and vanishes in the IR, corresponding to the fact that W is classically irrelevant. The full non-perturbative RG flow diagram, however, looks like in Figure 2.1. There is a line of fixed points in the (g, η) plane, which originates at the point $(g, \eta) = (g_*, 0)$ and extends up towards the strongly coupled regime. This means the theory in the IR has a marginal coupling, which controls the position on the fixed line. In the context of AdS/CFT correspondence this marginal coupling is related to α' in the bulk, and the supergravity regime corresponds to strong coupling, that is, being far up along the line. Note there is also a trivial free fixed point, disconnected from the

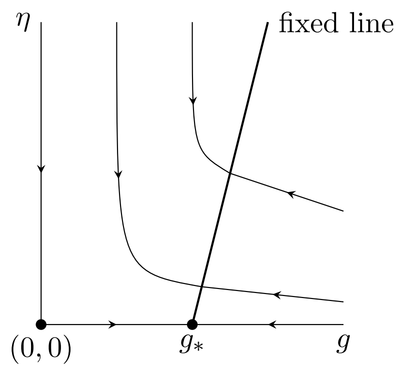


Figure 2.1: RG flow diagram of the conifold theory

line, at $(g, \eta) = (0, 0)$. Let us focus on the RG flow from this UV fixed point $(0, 0)$ to the IR fixed point $(g_*, 0)$. The theory in the IR is a strongly coupled CFT, but with zero superpotential. This fixed point is similar to the usual Seiberg fixed point in a $N_f = 2N_c$ SQCD, and is qualitatively different from the rest of the fixed line. With $W = 0$ the F-terms vanish, and the chiral ring is much larger compared to $\eta \neq 0$ theory.

Our main observation here, is that results in this work regarding the chiral ring in the “free” theory are valid not only in the UV free fixed point, but also in the IR interacting fixed point g_*, since the chiral ring is not changed along the flow.

Chapter 3

Free theory

The goal of this chapter is to find a complete basis for the chiral ring of various $\mathcal{N} = 1$ quiver gauge theories in the free limit but at finite N. For superconformal theories there is a correspondence between chiral ring and BPS operators, thus it also amounts to finding a finite N basis for BPS operators. The quiver gauge theories provide an infinite class of AdS/CFT examples, where the field theory on the world-volume of D3-branes at a Calabi-Yau singularity Y^6 is dual to a Type IIB superstring theory on $AdS_5 \times X^5$. These include orbifolds $\mathbb{C}^3/\mathbb{Z}_n$ [58, 59], conifold [72] or more general toric Calabi-Yaus [61, 62, 63, 64]

The zero coupling basis provides a good starting point for weak coupling calculations at finite N. These are explored in Chapter 4. However, the limit of zero coupling is also of intrinsic interest in the context of AdS/CFT. If duality holds in the full range of parameters λ, N, then the $\lambda = 0$ should be dual to tensionless, or critical tension, strings in AdS [75, 76]. The theory has a phase transition in this limit, the superconformal symmetry is enhanced to much larger higher spin symmetry, and the string theory should become topological. It is not known, however, how to handle the string theory in AdS exactly, so perhaps the free field theory calculations can provide some guidance. Similar directions have been explored in [77].

Even though we are not dealing with field theory interactions, just the combinatorics of finite N poses a significant challenge. In the half-BPS sector of $\mathcal{N} = 4$ SYM finding relationships between operators is not too hard: an operator $\mathrm{tr}(Z^{N+k})$ can always be expanded in terms of operators $\mathrm{tr}(Z^n)$ with $n \leq N$. In the eighth-BPS sector where we have three matrices and operators like $\mathrm{tr}(\Phi_1^{n_1}\Phi_2^{n_2}\Phi_3^{n_3})$, the relationships and cut-offs are much more intricate. The complete basis for this case was solved in [47] and [49]. Here we extend the construction to arbitrary quivers. For example, take the conifold theory, where the fields are bifundamentals of $U(N_1) \times U(N_2)$ and gauge invariant operators are of the form $\mathrm{tr}(A_iB_jA_kB_l\ldots)$ (see Section 3.5.1). The fields can be thought of as $N_1 \times N_2$ matrices, so how would one find all relationships, and write an independent basis? Similar question came up recently in the study of giant gravitons in ABJM [78]. Here we solve this problem in the most general case.

In Section 3.1 we start as a warm-up by not building the actual operators, but calculating the partition functions for counting the spectrum. We introduce the diagrammatic techniques and the "split-node" quiver, which concisely captures the counting at finite N. In the following sections we extend these objects to build the operators themselves.

In Section 3.2 we start with the $\mathcal{N}=4$ SYM, in which case the chiral ring corresponds to the eighth-BPS sector (see Section 2.4). Here we review the previous results on orthogonal bases. We also introduce the main symmetric group techniques, that we use in the following sections.

In Section 3.3 we generalize the restricted Schur basis of [49] to find a complete basis for an arbitrary quiver. Furthermore, we work out the multiplication of the basis operators, that is, the chiral ring structure constants. We find natural selection rules, where all Young diagrams in the operator labels combine according to the Littlewood-Richardson rule. This provides a new formula for the chiral ring structure constants even for $\mathcal{N}=4$, a slight extension of the previous results [79].

In Section 3.4 we find the generalization of the covariant basis [47] to an arbitrary quiver. We also calculate chiral ring structure constants in this basis, and find similar selection rules. This also gives an improved formula compared to previous $\mathcal{N}=4$ results [47].

In Section 3.5 we provide some concrete examples of quivers, including $\mathbb{C}^2/\mathbb{Z}_n \times \mathbb{C}$ orbifolds [58], $\mathbb{C}^3/\mathbb{Z}_3$ orbifold [59], and the conifold [72].

Both the restricted Schur and the covariant basis can, actually, be derived from a generic principle of "solving the invariance" of operators. We sketch it in the main text, but for a more detailed explanation see Appendix C.

Part of the motivation for the bases constructed in this chapter is that they are *orthogonal* in the free field metric (3.101). For $\mathcal{N}=4$ SYM this metric applies in the limit $g=0$, where both the gauge coupling and the superpotential vanishes. However, for other $\mathcal{N}=1$ SCFTs the RG flow diagram is more complicated, see Figure 2.1. There is a line of fixed points, and the interesting "free" limit from the AdS/CFT perspective is the point where superpotential vanishes, but the gauge coupling g_* is still strong. The free field metric is valid at the UV fixed point $g=0$, but as the theory flows to g_* in the IR, the Zamolodchikov metric gets modified, and our basis of operators will likely no longer be orthogonal. However, the chiral ring itself is not changed along the flow, so our basis will still be a *complete linearly independent finite N basis* for the chiral ring in the IR. From this perspective, the free two-point function could be seen as a particular inner product on the chiral ring states, which allows to solve the finite N constraints. Therefore, one of our key results, the chiral ring structure constants of the "free" operators (3.113) and (3.137), which depend only on the holomorphic information and not on the two-point function, are valid in the interacting fixed point g_*.

This chapter is based on the paper [57].

3.1 Partition functions

In this section we derive counting formulas for chiral gauge invariant operators in a general quiver gauge theory. We find that counting is neatly expressed in terms of the *split-node quiver*, which is a simple modification of the quiver diagram, with Young diagram labels on the edges, and Littlewood-Richardson multiplicities associated with the nodes. In the case of the covariant basis, we will also need Kronecker product multiplicities for the symmetric groups.

An $\mathcal{N} = 1$ supersymmetric quiver gauge theory is defined by a directed graph, called quiver, a gauge group factor associated to each quiver node, and a superpotential. In this chapter we consider a free theory, with vanishing superpotential. We take the gauge group to be $\prod_{a=1}^{G} U(N_a)$, where a runs over G nodes. Each arrow in the quiver between nodes a and b denotes a chiral multiplet transforming as $(N_a, \bar{N}_b)$. We denote the number of directed arrows from a to b by M_{ab}. The free theory has a global symmetry $\prod_{a,b} U(M_{ab})$. The full matter content is denoted by

$$\mathbf{\Phi} = \{\Phi_{ab;\alpha} : \ \alpha \in \{1, \dots, M_{ab}\} \} \tag{3.1}$$

An example that we will often use is the quiver for $\mathbb{C}^3/\mathbb{Z}_2$ theory, with a gauge group generalized to $U(N_1) \times U(N_2)$ shown in Figure 3.1. It is rich enough to demonstrate the different ingredients we will need to deal with the most general quiver.

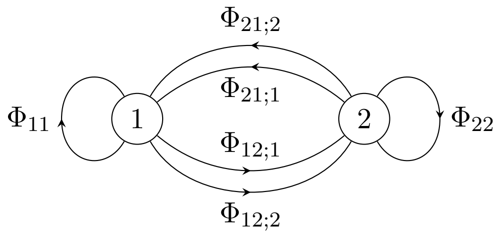


Figure 3.1: $\mathbb{C}^3/\mathbb{Z}_2$ quiver

Here we consider counting of chiral gauge invariant operators, such as, for the $\mathbb{C}^3/\mathbb{Z}_2$ example:

$$\mathrm{tr}(\Phi_{11}\Phi_{11}), \ \mathrm{tr}(\Phi_{12;1}\Phi_{21;2}), \ \mathrm{tr}(\Phi_{11}\Phi_{12;1}\Phi_{22}\Phi_{21;1}), \ \dots \tag{3.2}$$

graded by the number of times $\{n_{ab;\alpha}\}$ each field appears in the operator. The numbers $n_{ab;\alpha}$ determine the numbers of indices in the fundamental and anti-fundamental of each gauge group $U(N_a)$. These have to be equal by gauge invariance and they are denoted by n_a

$$n_a = \sum_b \sum_{\alpha=1}^{M_{ba}} n_{ba;\alpha} = \sum_b \sum_{\alpha=1}^{M_{ab}} n_{ab;\alpha} \tag{3.3}$$

In the limit $N_a \to \infty$ gauge invariant operators are in one-to-one correspondence with closed cycles in the quiver, but for finite N_a there are non-trivial identifications between operators. In Section 3.1.1 we use group integral formula to directly derive finite N_a

results, which is our main focus. Furthermore, in Section 3.1.2 we also show how in the $N_a \to \infty$ limit our results lead to particularly nice formulas for counting closed cycles in a directed graph.

3.1.1 The group integral formula

There is a group integral formula for the counting of gauge-invariant operators [80, 81, 82, 83]. It has been useful in the context of computation of indices recently. We will use the group integral formula to show that the finite N counting can be expressed in terms of Young diagrams R_a at the nodes with n_a boxes (i.e. $R_a \vdash n_a$), $r_{ab;\alpha} \vdash n_{ab;\alpha}$ at the edges and Littlewood-Richardson coefficients $\prod_a g(\cup_{b,\alpha} r_{ab;\alpha}; R_a) g(\cup_{b,\alpha} r_{ba;\alpha}; R_a)$ at the edges. The index α always appears on symbols carrying subscripts a, b which run over the pairs of gauge groups and range over $1 \le \alpha \le M_{ab}$. When $M_{ab} = 0$, all symbols carrying the corresponding α are dropped from the formulae.

The partition function for counting operators in any quiver is:

$$\mathcal{N}(\{t_{ab;\alpha}\}; \{N_a\}, \{M_{ab}\}) = \int \prod_a dU_a \quad e^{\sum_n \sum_{a,b,\alpha} \frac{(t_{ab;\alpha})^n}{n} \operatorname{tr} U_a^n \operatorname{tr}(U_b^\dagger)^n} \tag{3.4}$$

where $t_{ab;\alpha}$ are fugacities associated with $n_{ab;\alpha}$, and N_a is the size of U_a matrices. That is, if $\mathcal{N}(\{n_{ab;\alpha}\}; \{N_a\}, \{M_{ab}\})$ is the number of operators with charges $\{n_{ab;\alpha}\}$ then the partition function is

$$\mathcal{N}(\{t_{ab;\alpha}\}; \{N_a\}, \{M_{ab}\}) \equiv \sum_{\{n_{ab;\alpha}\}} \left(\prod_{a,b,\alpha} (t_{ab;\alpha})^{n_{ab;\alpha}} \right) \mathcal{N}(\{n_{ab;\alpha}\}; \{N_a\}, \{M_{ab}\}) \tag{3.5}$$

We will henceforth write $\int$ for $\int \prod_a dU_a$. Writing the exponential as a product and expanding in series

$$\begin{aligned}
&\mathcal{N}(\{t_{ab;\alpha}\}; \{N_a\}, \{M_{ab}\}) \\
&= \sum_{\{k^{(n)}_{ab;\alpha}\}=0}^{\infty} \int \prod_{a,b,\alpha,n} (t_{ab;\alpha})^{n k^{(n)}_{ab;\alpha}} \frac{(\operatorname{tr} U_a^n)^{k^{(n)}_{ab;\alpha}} (\operatorname{tr} U_b^{\dagger\, n})^{k^{(n)}_{ab;\alpha}}}{n^{k^{(n)}_{ab;\alpha}} k^{(n)}_{ab;\alpha}!} \\
&= \int \sum_{\{k^{(n)}_{ab;\alpha}\}=0}^{\infty} \prod_{a,b,\alpha} (t_{ab;\alpha})^{\sum_n n k^{(n)}_{ab;\alpha}} \prod_n \prod_{a,b,\alpha} \frac{(\operatorname{tr} U_a^n)^{k^{(n)}_{ab;\alpha}} (\operatorname{tr} U_b^{\dagger\, n})^{k^{(n)}_{ab;\alpha}}}{n^{k^{(n)}_{ab;\alpha}} k^{(n)}_{ab;\alpha}!} \\
&= \int \sum_{\{n_{ab;\alpha}\}=0}^{\infty} \prod_{a,b,\alpha} \frac{(t_{ab;\alpha})^{n_{ab;\alpha}}}{n_{ab;\alpha}!} \\
&\quad \times \sum_{\sigma_{ab;\alpha} \in S_{n_{ab;\alpha}}} \prod_a \sum_{R_a \vdash n_a} \chi_{R_a}(\cup_{b,\alpha} \sigma_{ab;\alpha}) \chi_{R_a}(U_a) \sum_{S_a \vdash n_a} \chi_{S_a}(\cup_{b,\alpha} \sigma_{ba;\alpha}) \chi_{S_a}(U_a^\dagger)
\end{aligned} \tag{3.6}$$

We have factored the powers $(t_{ab;\alpha})^{n_{ab;\alpha}}$, recognized that for fixed $n_{ab;\alpha}$, the sums over $k^{(n)}_{ab;\alpha}$ run over partitions of $n_{ab;\alpha}$, which correspond to conjugacy classes in $S_{n_{ab;\alpha}}$. We observe that

$$\prod_n \prod_{a,b,\alpha} (\mathrm{tr} U_a^n)^{k^{(n)}_{ab;\alpha}} = \sum_{\substack{R_a \vdash n_a \\ l(R_a) \leq N_a}} \chi_{R_a}(\cup_{a,b,\alpha} \sigma_{ab;\alpha}) \chi_{R_a}(U_a) \tag{3.7}$$

for $\sigma_{ab;\alpha}$ being a permutation in the conjugacy class of $n_{ab;\alpha}$ specified by $k^{(n)}_{ab;\alpha}$. Since the number of permutations in the specified conjugacy class is precisely

$$\frac{n_{ab;\alpha}!}{\prod_n n^{k^{(n)}_{ab;\alpha}} k^{(n)}_{ab;\alpha}!} \tag{3.8}$$

we have converted the sums over partitions to sums over permutations. We have also recognized that the traces can be expanded in terms of Schur Polynomials with coefficients given by the characters of these permutations. Note, crucially, the height of the Young diagram R_a is at most N_a, this fully captures the effect of finite N_a. Using the orthogonality of the Schur Polynomials under group integration

$$\int dU_a \; \chi_{R_a}(U_a) \chi_{S_a}(U_a^\dagger) = \delta_{R_a S_a} \tag{3.9}$$

we can expand characters in irreps R_a of S_{n_a} into characters of $\prod_{b,\alpha} r_{ab;\alpha}$ with expansion coefficients which are Littlewood-Richardson numbers

$$\chi_{R_a}(\cup_{b,\alpha} \sigma_{ab;\alpha}) = \sum_{r_{ab;\alpha} \vdash n_{ab;\alpha}} g(\cup_{b,\alpha} r_{ab;\alpha}; R_a) \prod_{b,\alpha} \chi_{r_{ab;\alpha}}(\sigma_{ab;\alpha}) \tag{3.10}$$

This leads to

$$\begin{aligned}
&\mathcal{N}(\{t_{ab;\alpha}\}; \{N_a\}, \{M_{ab}\}) \\
&= \sum_{\{n_{ab;\alpha}\}} \prod_{a,b,\alpha} \frac{(t_{ab;\alpha})^{n_{ab;\alpha}}}{n_{ab;\alpha}!} \sum_{\sigma_{ab;\alpha} \in S_{n_{ab;\alpha}}} \sum_{\substack{R_a \vdash n_a \\ l(R_a) \leq N_a}} \sum_{r_{ab;\alpha} \vdash n_{ab;\alpha}} \sum_{s_{ab;\alpha} \vdash n_{ab;\alpha}} \\
&\qquad \prod_a g(\cup_{b,\alpha} r_{ab;\alpha}; R_a) g(\cup_{b,\alpha} s_{ba;\alpha}; R_a) \prod_{a,b,\alpha} \chi_{r_{ab;\alpha}}(\sigma_{ab;\alpha}) \chi_{s_{ab;\alpha}}(\sigma_{ab;\alpha}) \\
&= \sum_{\{n_{ab;\alpha}\}} \prod_{a,b,\alpha} (t_{ab;\alpha})^{n_{ab;\alpha}} \sum_{\substack{R_a \vdash n_a \\ l(R_a) \leq N_a}} \sum_{r_{ab;\alpha} \vdash n_{ab;\alpha}} \prod_a g(\cup_{b,\alpha} r_{ab;\alpha}; R_a) g(\cup_{b,\alpha} r_{ba;\alpha}; R_a)
\end{aligned} \tag{3.11}$$

In the second line we used orthogonality of characters $\sum_\sigma \chi_r(\sigma)\chi_s(\sigma) = n! \delta_{rs}$. This form of the partition function, comparing with (3.5), gives explicit counting for each choice of charges $\{n_{ab;\alpha}\}$

$$\boxed{\mathcal{N}(\{n_{ab;\alpha}\}; \{N_a\}, \{M_{ab}\}) = \sum_{\substack{R_a \vdash n_a \\ l(R_a) \leq N_a}} \sum_{r_{ab;\alpha} \vdash n_{ab;\alpha}} \prod_a g(\cup_{b,\alpha} r_{ab;\alpha}; R_a) g(\cup_{b,\alpha} r_{ba;\alpha}; R_a)} \tag{3.12}$$

There is a simple diagrammatic description of this formula, deriving directly from the quiver itself:

Diagrammatic Rules for counting local operators in the quiver theory

- Choose integers $n_{ab;\alpha} \geq 0$ for all the edges of the quiver Q, subject to $n_a = \sum_b n_{ba}$.
- Replace each node with a pair of nodes, joined by a line labelled by a Young diagram R_a with n_a boxes. One of these two nodes, called the plus node, has all incoming lines and the other, called the minus node, has all outgoing lines. The resulting diagram is the *split-mode quiver*.
- To all the previously existing edges, attach Young diagrams $r_{ab;\alpha}$ with $n_{ab;\alpha}$ boxes.
- To each minus node attach a Littlewood-Richardson multiplicity $g(\bigcup_b \bigcup_{\alpha=1}^{M_{ab}} r_{ab;\alpha}; R_a)$ which couples all the incoming lines to R_a. To each plus node attach the LR multiplicity $g(\bigcup_b \bigcup_{\alpha=1}^{M_{ba}} r_{ba;\alpha}; R_a)$
- Take the product of LR-coefficients over all the nodes. This is the counting of free chiral operators with numbers $\{n_{ba;\alpha}\}$ of fields of type α transforming as $(N_a, \bar{N}_b)$.

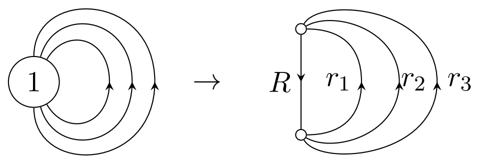


Figure 3.2: Split-node quiver for $\mathbb{C}^3$.

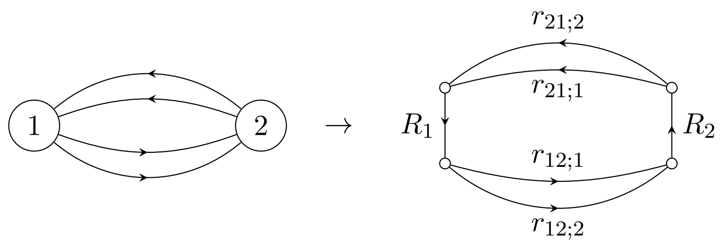


Figure 3.3: Split-node quiver for the conifold.

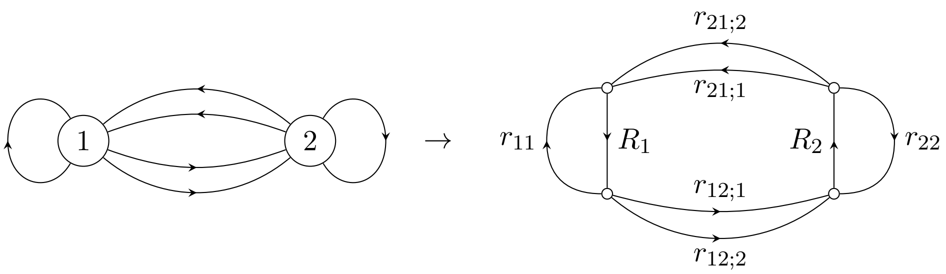


Figure 3.4: Split-node quiver for $\mathbb{C}^3/\mathbb{Z}_2$.

These steps are illustrated for $\mathbb{C}^3$ in Figure 3.2. We have suppressed the a, b indices labelling the nodes of the quiver, since there is only one node in this case.

$$\mathcal{N}_{\mathbb{C}^3}(n_1, n_2, n_3; N) = \sum_{\substack{R \vdash n \\ l(R) \leq N}} g(r_1, r_2, r_3; R)\, g(r_1, r_2, r_3; R) \tag{3.13}$$

This equation was given in [49, 84]. For $\mathcal{C}$, we read off the counting from (3.12) or by following the steps in Figure 3.3.

$$\mathcal{N}_{\mathcal{C}}(n_{12;1}, n_{12;2}, n_{21;1}, n_{21;2}; N_1, N_2) = \sum_{\substack{R_1 \vdash n_1 \\ l(R_1) \leq N_1}} \sum_{\substack{R_2 \vdash n_2 \\ l(R_2) \leq N_2}} \sum_{r_{12;1} \vdash n_{12;1}} \sum_{r_{12;2} \vdash n_{12;2}} \sum_{r_{21;1} \vdash n_{21;1}} \sum_{r_{21;2} \vdash n_{21;2}}$$
$$g(r_{12;1}, r_{12;2}; R_1)\, g(r_{12;1}, r_{12;2}; R_2)\, g(r_{21;1}; r_{21;2}, R_1)\, g(r_{21;1}, r_{21;2}; R_2) \tag{3.14}$$

This counting for the free conifold operators has not been given before. For $\mathbb{C}^3/\mathbb{Z}_2$, again following the steps above shown in Figure 3.4 or specializing (3.12), we get

$$\mathcal{N}_{\mathbb{C}^3/\mathbb{Z}_2}(n_{11}, n_{22}, n_{12;1}, n_{12;2}, n_{21;1}, n_{21;2}; N_1, N_2)$$
$$= \sum_{\substack{R_1 \vdash n_1 \\ l(R_1) \leq N_1}} \sum_{\substack{R_2 \vdash n_2 \\ l(R_2) \leq N_2}} \sum_{r_{11} \vdash n_{11}} \sum_{r_{22} \vdash n_{22}} \sum_{r_{12;1} \vdash n_{12;1}} \sum_{r_{12;2} \vdash n_{12;2}} \sum_{r_{21;1} \vdash n_{21;1}} \sum_{r_{21;2} \vdash n_{21;2}}$$
$$g(r_{11}, r_{12;1}, r_{12;2}, R_1)\, g(r_{22}, r_{12;1}, r_{12;2}, R_2)\, g(r_{11}, r_{21;1}, r_{21;2}, R_1)\, g(r_{22}, r_{21;1}, r_{21;2}, R_2) \tag{3.15}$$

There is another useful form of the counting formula where we do not specify $\{n_{ab;\alpha}\}$ but only $\{n_{ab}\}$

$$n_{ab} = \sum_{\alpha} n_{ab;\alpha} \tag{3.16}$$

that is, the total number of fields transforming under $U(M_{ab})$ global symmetry group. This will be related to the covariant basis, where we can count states according to representations of the global symmetry group $\prod_{ab} U(M_{ab})$. We group together representations $\cup_\alpha r_{ab;\alpha}$ corresponding to the same pair (a, b), and expand the multiplicities in (3.12) as

$$\begin{aligned} g(\cup_{b,\alpha} r_{ab;\alpha}; R_a) &= \sum_{\{s^-_{ab}\}} g(\cup_b s^-_{ab}; R_a) \prod_b g(\cup_\alpha r_{ab;\alpha}; s^-_{ab}) \\ g(\cup_{b,\alpha} r_{ba;\alpha}; R_a) &= \sum_{\{s^+_{ba}\}} g(\cup_b s^+_{ba}; R_a) \prod_b g(\cup_\alpha r_{ba;\alpha}; s^+_{ba}) \end{aligned} \tag{3.17}$$

$s^{\pm}_{ab}$ are intermediate representations in the reductions $R_a \to \{\cup_b s^-_{ab}\} \to \{\cup_{b,\alpha} r_{ab;\alpha}\}$ and $R_a \to \{\cup_b s^+_{ba}\} \to \{\cup_{b,\alpha} r_{ba;\alpha}\}$. Next, we apply (A.40) for fixed (a, b):

$$\sum_{\{r_{ab;\alpha}\}} g(\cup_\alpha r_{ab;\alpha}; s^+_{ab}) g(\cup_\alpha r_{ab;\alpha}; s^-_{ab}) = \sum_{\Lambda_{ab}} C(s^+_{ab}, s^-_{ab}, \Lambda_{ab}) g(\cup_\alpha [n_{ab;\alpha}]; \Lambda_{ab}) \tag{3.18}$$

where $\cup_\alpha [n_{ab;\alpha}]$ is the irrep of $\times_\alpha S_{n_{ab;\alpha}}$ consisting of the single row symmetric irreps $[n_{ab;\alpha}]$ for each factor. We find

$$\begin{aligned}\mathcal{N}(\{n_{ab;\alpha}\};\{N_a\},\{M_{ab}\}) = \sum_{\substack{R_a\vdash n_a\\ l(R_a)\leq N_a}} \sum_{s^+_{ab}\vdash n_{ab}} \sum_{s^-_{ab}\vdash n_{ab}} \sum_{\substack{\Lambda_{ab}\vdash n_{ab}\\ l(\Lambda_{ab})\leq M_{ab}}} \prod_a g(\cup_b s^-_{ab}; R_a) g(\cup_b s^+_{ba}; R_a)\\ \times \prod_{a,b} C(s^+_{ab}, s^-_{ab}, \Lambda_{ab}) g(\cup_\alpha [n_{ab;\alpha}]; \Lambda_{ab})\end{aligned} \tag{3.19}$$

The new labels Λ_{ab} are precisely the $U(M_{ab})$ representations. (3.19) can be understood by noting that the number of states in the irrep Λ_{ab}, a Young diagram of $U(M_{ab})$ with n_{ab} boxes, with specified charges $n_{ab;\alpha}$ under the diagonal $U(1)^{M_{ab}}$, is given by the Littlewood-Richardson multiplicity

$$g(\cup_\alpha [n_{ab;\alpha}]; \Lambda_{ab}) = \frac{1}{\prod_{a,b,\alpha} n_{ab;\alpha}!} \sum_{\sigma_{ab;\alpha}\in S_{n_{ab;\alpha}}} \chi_{\Lambda_{ab}}(\cup_\alpha \sigma_{ab;\alpha}) \tag{3.20}$$

Thus if we do not refine by $n_{ab;\alpha}$, but count all the states with fixed $\{n_{ab}\}$, we count the total number of states in the representation

$$\begin{aligned}\mathcal{N}(\{n_{ab}\};\{N_a\},\{M_{ab}\}) = \sum_{\substack{R_a\vdash n_a\\ l(R_a)\leq N_a}} \sum_{s^+_{ab}\vdash n_{ab}} \sum_{s^-_{ab}\vdash n_{ab}} \sum_{\substack{\Lambda_{ab}\vdash n_{ab}\\ l(\Lambda_{ab})\leq M_{ab}}} \prod_a g(\cup_b s^-_{ab}; R_a) g(\cup_b s^+_{ba}; R_a)\\ \times \prod_{a,b} C(s^+_{ab}, s^-_{ab}, \Lambda_{ab}) \mathrm{Dim}(\Lambda_{ab})\end{aligned} \tag{3.21}$$

where $\mathrm{Dim}(\Lambda_{ab})$ is the size of $U(M_{ab})$ irrep Λ_{ab}. We can also, instead of counting individual states, count how many times a particular global symmetry representation $\otimes_{ab}\Lambda_{ab}$ appears

$$\boxed{\mathcal{N}(\{\Lambda_{ab}\};\{N_a\},\{M_{ab}\}) = \sum_{\substack{R_a\vdash n_a\\ l(R_a)\leq N_a}} \sum_{\substack{s^+_{ab}\vdash n_{ab}\\ s^-_{ab}\vdash n_{ab}}} \prod_a g(\cup_b s^-_{ab}; R_a) g(\cup_b s^+_{ba}; R_a) \prod_{a,b} C(s^+_{ab}, s^-_{ab}, \Lambda_{ab})} \tag{3.22}$$

The following figures illustrate the structure of this formula to the case of $\mathbb{C}^3$, $\mathcal{C}$ and $\mathbb{C}^3/\mathbb{Z}_2$ quivers. The white nodes again represent LR multiplicities and the new black nodes represent Kronecker product multiplicities $C(s^+_{ab}, s^-_{ab}, \Lambda_{ab})$.

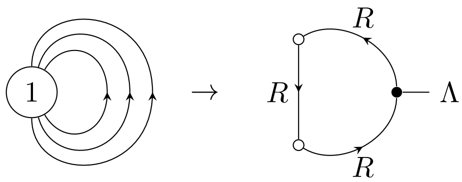


Figure 3.5: Covariant quiver for $\mathbb{C}^3$.

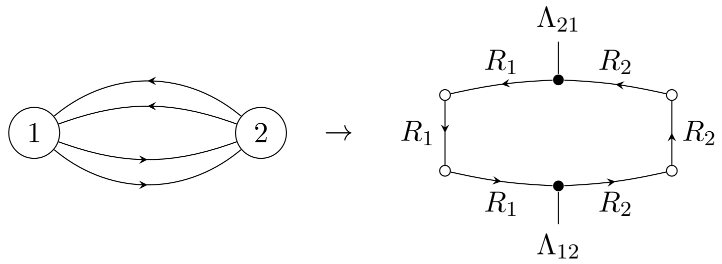


Figure 3.6: Covariant quiver for the conifold.

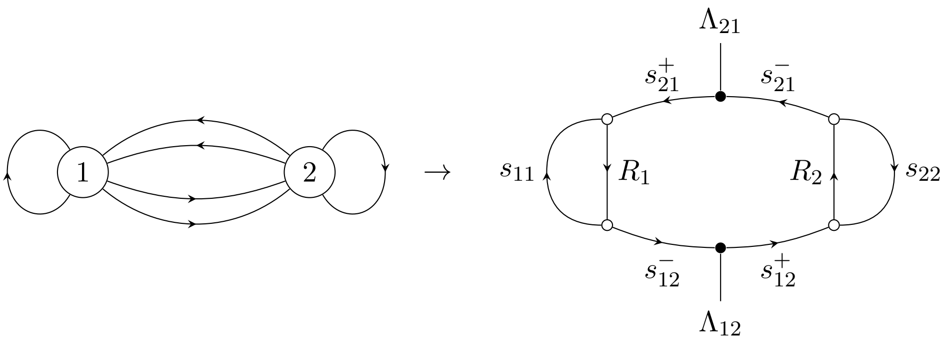


Figure 3.7: Covariant quiver for $\mathbb{C}^3/\mathbb{Z}_2$.

The corresponding formula for $\mathbb{C}^3$ according to Figure 3.5

$$\mathcal{N}_{\mathbb{C}^3}(\Lambda; N) = \sum_{\substack{R \vdash n \\ l(R) \leq N}} C(R, R, \Lambda) \tag{3.23}$$

It was first obtained in [85] and the matching construction of orthogonal operators given in [47]. Since there is only single incoming and outgoing arrow from the white branching nodes in Figure 3.5, there is no actual branching, and the labels on both sides are R. That is, compared to general formula (3.22) we have $s^+ = s^- = R$.

For conifold we have Figure 3.6

$$\mathcal{N}_{\mathcal{C}}(\Lambda_{12}, \Lambda_{21}; N) = \sum_{\substack{R_1 \vdash n \\ l(R_1) \leq N}} \sum_{\substack{R_2 \vdash n \\ l(R_2) \leq N}} C(R_1, R_2, \Lambda_{12}) C(R_2, R_1, \Lambda_{21}) \tag{3.24}$$

Again the white node multiplicities are trivial, setting $s^{\pm}_{ab}$ to R_a.

For $\mathbb{C}^3/\mathbb{Z}_2$ we find non-trivial branching multiplicities, following the diagram Figure 3.7:

$$\mathcal{N}_{\mathbb{C}^3/\mathbb{Z}_2}(\Lambda_{12}, \Lambda_{21}, n_{11}, n_{22}; N) = \sum_{\substack{R_1 \vdash n_1 \\ l(R_1) \leq N}} \sum_{\substack{R_2 \vdash n_2 \\ l(R_2) \leq N}} \sum_{s^-_{12} \vdash n_{12}} \sum_{s^+_{12} \vdash n_{12}} \sum_{s^-_{21} \vdash n_{12}} \sum_{s^+_{21} \vdash n_{12}} \sum_{s_{11} \vdash n_{11}} \sum_{s_{22} \vdash n_{22}}$$
$$g(s_{11}, s^-_{12}; R_1) g(s_{11}, s^+_{21}; R_1) g(s_{22}, s^-_{21}; R_2) g(s_{22}, s^+_{12}; R_2) C(s^-_{12}, s^+_{12}, \Lambda_{12}) C(s^-_{21}, s^+_{21}, \Lambda_{21}) \tag{3.25}$$

The only simplification compared to the generic formula (3.22) is that $s^+_{11} = s^-_{11} \equiv s_{11}$ and

$s_{22}^{+} = s_{22}^{-} \equiv s_{22}$, since the original quiver has $M_{11} = M_{22} = 1$, the corresponding global symmetry factor is abelian, and so $\Lambda_{11} = [n_{11}], \Lambda_{22} = [n_{22}]$ are trivial.

3.1.2 Infinite product generating functions

In this section we will use the covariant basis counting (3.21) to derive a simple infinite product formula valid when the numbers of fields are less than the ranks N_a. In this case counting gauge invariant operators is the same as counting closed loops in the quiver.

Counting the gauge invariant local operators for fixed ranks N_a, numbers M_{ab} of fields transforming in $(N_a, \bar{N}_b)$ in the theory, and numbers n_{ab} for the total number of fields of type $(N_a, \bar{N}_b)$ we have (3.21)

$$\mathcal{N}(\{n_{ab}\}; \{N_a\}, \{M_{ab}\}) = \sum_{\substack{R_a \vdash n_a \\ l(R_a) \leq N_a}} \sum_{s_{ab}^{+} \vdash n_{ab}} \sum_{s_{ab}^{-} \vdash n_{ab}} \sum_{\substack{\Lambda_{ab} \vdash n_{ab} \\ l(\Lambda_{ab}) \leq M_{ab}}} \prod_a g(\cup_b s_{ab}^{-}; R_a) g(\cup_b s_{ba}^{+}; R_a) \times \prod_{a,b} C(s_{ab}^{+}, s_{ab}^{-}, \Lambda_{ab}) \mathrm{Dim}(\Lambda_{ab}) \tag{3.26}$$

The finite N constraints are encoded in the requirement that the Young diagrams R_a have no more than N_a rows.

Let us convert it to a partition function with fugacities $\{t_{ab;\alpha}\}$ for numbers $\{n_{ab;\alpha}\}$. The contribution from a single irrep Λ_{ab} is

$$\chi_{\Lambda_{ab}}(\mathbb{T}_{ab}) \tag{3.27}$$

where $\mathbb{T}_{ab}$ is a square matrix of size M_{ab} with entries $t_{ab;\alpha}$ along the diagonal. Thus we can replace $\mathrm{Dim}(\Lambda_{ab})$ with $\chi_{\Lambda_{ab}}(\mathbb{T}_{ab})$ in (3.26) and sum over all representations without restriction on the number of boxes, to get the full partition function:

$$\mathcal{N}(\{t_{ab;\alpha}\}; \{N_a\}, \{M_{ab}\}) = \sum_{\substack{R_a \\ l(R_a) \leq N_a}} \sum_{s_{ab}^{+} \vdash n_{ab}} \sum_{s_{ab}^{-} \vdash n_{ab}} \sum_{\substack{\Lambda_{ab} \vdash n_{ab} \\ l(\Lambda_{ab}) \leq M_{ab}}} \prod_a g(\cup_b s_{ab}^{+}; R_a) g(\cup_b s_{ab}^{-}; R_a) \times \prod_{a,b} C(s_{ab}^{+}, s_{ab}^{-}, \Lambda_{ab}) \chi_{\Lambda_{ab}}(\mathbb{T}_{ab}) \tag{3.28}$$

Note this is the same partition function as in the derivation in the previous section (3.11), but now using the covariant basis we can conveniently package $(t_{ab;\alpha})^{n_{ab;\alpha}}$ into $\chi_{\Lambda_{ab}}(\mathbb{T}_{ab})$.

The counting formula (3.28) can be used to derive an elegant infinite product formula for large N_a. If we assume $n_a \leq N_a$ so sums over R_a are unconstrained, we can do the sums over $R_a, \Lambda_{ab}, s_{ab}^{\pm}$ to end up with a product of delta functions over the groups

$$\mathcal{N}(\{t_{ab;\alpha}\}; \{M_{ab}\}) = \sum_{\{\gamma_a\}} \sum_{\{\sigma_{ab}\}} \prod_a \delta_{S_{n_a}} \left((\otimes_b \sigma_{ba}) \, \gamma_a \, (\otimes_b \sigma_{ab}) \, \gamma_a^{-1} \right) \prod_{a,b} \mathrm{tr}_{n_{ab}}(\mathbb{T}_{ab} \sigma_{ab}) \tag{3.29}$$

where

$$\mathcal{N}(\{t_{ab;\alpha}\};\{M_{ab}\}) \equiv \mathcal{N}(\{t_{ab;\alpha}\};\{N_a=\infty\},\{M_{ab}\}) \tag{3.30}$$

The limit $N_a = \infty$ holds as long as $n_a \leq N_a$.

The sum is over permutations $\gamma_1, \gamma_2, \cdots \gamma_G$, one for each node (or group), with $\gamma_a \in S_{n_a}$; as well as a sum over permutations σ_{ab}, one for every pair (a, b) of nodes of the quiver which have a non-zero number M_{ab} of arrows from a to b. The σ_{ab} are permutations in $S_{n_{ab}}$. Note that $\otimes_b \sigma_{ba}$ is an outer product of permutations, e.g if there are 3 values of b for which n_{ba} is non-zero, say $b = 1, 2, 3$, then the product gives a permutation $\sigma_{11} \circ \sigma_{21} \circ \sigma_{31}$ which lives in the $S_{n_{1a}} \times S_{n_{2a}} \times S_{n_{3a}}$ subgroup of $S_{n_a} = S_{n_{1a}+n_{2a}+n_{3a}}$.

Consider cycles of length i. Let σ_{ab} have $p_{ab}^{(i)}$ cycles of this length. The delta functions associated with each node lead to the condition $\sum_b p_{ab}^{(i)} = \sum_b p_{ba}^{(i)}$. Given any γ_a, σ_{ab} which solve the delta function, we can generate the other solutions for the same σ_{ab}, by considering by multiplying γ_a on the right with permutations γ_a in the stabilizer of $(\otimes_b \sigma_{ab})$. This generates a multiplicity of

$$\prod_i \prod_a \left(\sum_b p_{ab}^{(i)}\right)! \, i^{\sum_b p_{ab}^{(i)}} \tag{3.31}$$

We can see that the sums over γ_a in (3.29) only depends on the conjugacy class of σ_{ab} in $S_{n_{ab}}$, since conjugating σ_{ab} by elements of $S_{n_{ab}}$ can be absorbed in $\gamma_a \in S_{n_a}$ the summations by exploiting the invariance of these sums under left or right multiplication by elements of the $S_{n_{ab}}$ subgroups of S_{n_a}. This means that the sums over σ_{ab} can be converted into sums over $p_{ab}^{(i)}$. There is a multiplicity

$$\prod_i \prod_{a,b} \frac{n_{ab}!}{i^{p_{ab}^{(i)}}(p_{ab}^{(i)})!} \tag{3.32}$$

Combining these facts we arrive at

$$\mathcal{N}(\{t_{ab;\alpha}\};\{M_{ab}\}) = \prod_{i=1}^{\infty}\left[\sum_{\{p_{ab}^{(i)}\}=0}^{\infty} \prod_a \delta\left(\sum_b p_{ba}^{(i)} - \sum_b p_{ab}^{(i)}\right)\left(\sum_b p_{ab}^{(i)}\right)! \prod_{a,b} \frac{\left(\sum_\alpha (t_{ab;\alpha})^i\right)^{p_{ab}^{(i)}}}{p_{ab}^{(i)}!}\right] \tag{3.33}$$

For each i we need to do a sum of the form

$$\mathcal{S}(\{t_{ab}\}) = \sum_{\{p_{ab}\}=0}^{\infty} \prod_a \delta\left(\sum_b p_{ba} - \sum_b p_{ab}\right)\left(\sum_b p_{ab}\right)! \prod_{a,b} \frac{(t_{ab})^{p_{ab}}}{p_{ab}!} \tag{3.34}$$

It is convenient to write the Kronecker delta as a contour integral, using

$$\delta(p) = \oint \frac{dz}{2\pi i z} z^p \tag{3.35}$$

which gives

$$\begin{aligned}\mathcal{S}(\{t_{ab}\}) &= \sum_{\{p_{ab}\}=0}^{\infty} \prod_a (\sum_b p_{ab})! \oint \frac{dz_a}{2\pi i z_a} z_a^{\sum_b p_{ba} - \sum_b p_{ab}} \prod_{a,b} \frac{(t_{ab})^{p_{ab}}}{p_{ab}!} \\ &= \oint \left(\prod_a \frac{dz_a}{2\pi i z_a} \right) \sum_{\{p_{ab}\}=0}^{\infty} \prod_a (\sum_b p_{ab})! \prod_{a,b} \frac{\left(z_a^{-1} z_b t_{ab}\right)^{p_{ab}}}{p_{ab}!} \\ &= \oint \left(\prod_a \frac{dz_a}{2\pi i z_a} \right) \prod_a \frac{1}{1 - \sum_b z_a^{-1} z_b t_{ab}}\end{aligned} \tag{3.36}$$

We can obtain the desired sum by calculating residues.

We find that the result can be expressed in an elegant and intuitive form. Let $\mathbb{V}$ be the set $\{1, 2, \cdots G\}$ of nodes in the quiver. We will let $V \subset \mathbb{V}$ be any subset, and define $\mathrm{Sym}(V)$ to be the group of all permutations of the elements in V. For each permutation σ we will define a monomial $T_\sigma(\{t_{ab}\})$ built from the set $\{t_{ab}\}$. Any permutation σ is a product of cycles $\sigma = \prod_j \sigma^{(j)}$. The monomial $T_\sigma(\{t_{ab}\})$ is a product over these cycles.

$$T_\sigma(\{t_{ab}\}) = \prod_j (-1)\ T_{\sigma^{(j)}}(\{t_{ab}\}) \tag{3.37}$$

For a cycle, such as $(a_1, a_2, \cdots a_k)$ with integers $a_1, \cdots a_k$ chosen from $\{1, \cdots, G\}$, the factor is

$$T_{(a_1, a_2 \cdots, a_k)}(\{t_{ab}\}) = t_{a_1 a_2} t_{a_2 a_3} \cdots t_{a_{k-1} a_k} t_{a_k a_1} \tag{3.38}$$

We find that

$$\boxed{\mathcal{S}(\{t_{ab}\}) = \frac{1}{(1 - \sum_{V \subset \mathbb{V}} \sum_{\sigma \in Sym(V)} T_\sigma(\{t_{ab}\})}} \tag{3.39}$$

The sign of each term is $(-1)^{C_\sigma}$ where C_σ is the number of cycles in the corresponding permutation. Each cycle $\sigma^{(i)}$ corresponds to an elementary closed loop in the quiver, elementary in the sense that it does not involve visiting any node more than once. The permutation σ corresponds to a product of disjoint elementary loops. For example, for a quiver with three nodes, this becomes

$$\begin{aligned}&\mathcal{S}(t_{11}, t_{22}, t_{33}, t_{12}, t_{13}, t_{23}) \\ &\quad = (1 - t_{11} - t_{22} - t_{33} + t_{11} t_{22} - t_{12} t_{21} + t_{22} t_{33} - t_{23} t_{32} + t_{11} t_{33} - t_{13} t_{31} \\ &\quad - t_{11} t_{22} t_{33} + t_{12} t_{21} t_{33} + t_{13} t_{31} t_{22} + t_{11} t_{23} t_{32} - t_{12} t_{23} t_{31} - t_{13} t_{32} t_{21})^{-1}\end{aligned} \tag{3.40}$$

The first three terms after 1 come from the 3 1-element subsets of $\mathbb{V} = \{1, 2, 3\}$. The next three pairs come from the 3 two-element subsets of $\mathbb{V}$. The first of each pair comes from the identity permutation of the subset, the second from the swap. The last line comes from permutations of $V = \mathbb{V}$.

The large N counting function can then be written as

$$\mathcal{N}(\{t_{ab;\alpha}\};\{M_{ab}\}) = \prod_{i=1}^{\infty} \mathcal{S}(\{t_{ab} \to \sum_{\alpha=1}^{M_{ab}} (t_{ab;\alpha})^i\}) \tag{3.41}$$

In this equation, we have the counting for a quiver with G nodes and any number of arrows for any specified pair of start and end points. When there are no arrows between a specified start and end point, we set the corresponding t_{ab} variable to zero.

Let us now explain how to specialize the above formula for some specific cases. Take the half-BPS sector of $\mathcal{N}=4$ SYM. This is described by one node and one arrow starting and ending at that node. The set $\mathbb{V}$ has one element $\{1\}$ and there is one t_{11} parameter. There are two subsets, $V=\emptyset$ or $V=\mathbb{V}$. In calculating $\mathcal{S}(t_{11})$, the monomial coming from the empty set is 1. The monomial from $V=\mathbb{V}$ is $-t_{11}$. So

$$\mathcal{N}_{\mathbb{C}}(t_{11}) = \prod_{i=1}^{\infty} \frac{1}{1-t_{11}^i} \tag{3.42}$$

For the one-node quiver with three lines starting and ending at the node, $\mathbb{V}=\{1\}$. The set of t-variables ("fugacities") is $\{t_{11;1}, t_{11;2}, t_{11;3}\}$.

$$\mathcal{S}_{\mathbb{C}^3}(t_{11}) = (1-t_{11})^{-1} \tag{3.43}$$

The counting function is

$$\mathcal{N}_{\mathbb{C}^3}(\{t_{11;\alpha}\}) = \prod_{i=1}^{\infty} \frac{1}{1-t_{11;1}^i - t_{11;2}^i - t_{11;3}^i} \tag{3.44}$$

This formula was written down in [69].

Beyond these examples, the analogous formulae have not been previously written down. For the conifold, we have $\mathbb{V}=\{1,2\}$. The $\mathcal{S}$ function is

$$\mathcal{S}_{\mathcal{C}}(t_{12}, t_{21}) = (1-t_{12}t_{21})^{-1} \tag{3.45}$$

The variables t_{11}, t_{22} are set to zero, since there are no arrows joining any node to itself. The 1 comes as usual from the empty set, the second term from the permutation (12) in $\mathrm{Sym}(V)$ for $V=\mathbb{V}$. All other terms are zero due to the vanishing of t_{11}, t_{22}. Since there is a multiplicity 2 for the arrows going from 1 to 2 and conversely from 2 to 1, we have variables $t_{12;1}, t_{12;2}, t_{21;1}, t_{21;2}$ and the counting function

$$\begin{aligned}\mathcal{N}_{\mathcal{C}}(\{t_{12;\alpha}, t_{21;\alpha}\}) &= \prod_{i=1}^{\infty} \frac{1}{1-(t_{12;1}^i + t_{12;2}^i)(t_{21;1}^i + t_{21;2}^i)} \\ &= \prod_{i=1}^{\infty} \frac{1}{1-t_{12;1}^i t_{21;1}^i - t_{12;2}^i t_{21;2}^i - t_{12;1}^i t_{21;2}^i - t_{12;2}^i t_{21;1}^i}\end{aligned} \tag{3.46}$$

For the example of $\mathbb{C}^3/Z_2$, the $\mathcal{S}$ function depends on $t_{11}, t_{22}, t_{12}, t_{21}$, The $\mathcal{N}$ function depends on $t_{11}, t_{22}, t_{12;1}, t_{12;1}, t_{21;1}, t_{21;2}$.

$$\mathcal{S}_{\mathbb{C}^3/Z_2}(t_{11}, t_{22}, t_{12}, t_{21}) = (1 - t_{11} - t_{22} - t_{12}t_{21} + t_{11}t_{22})^{-1} \tag{3.47}$$

Here $\mathbb{V} = \{1,2\}$. The monomials t_{11}, t_{22} come from choices $V = \{1\}$ and $V = \{2\}$. The term $t_{12}t_{21}$ comes from permutation (12) in Sym(V) for $V = \{1,2\}$. The term $t_{11}t_{22}$ comes from permutation (1)(2) in Sym(V) for $V = \{1,2\}$. The counting function is

$$\mathcal{N}_{\mathbb{C}^3/Z_2}(\{t_{11}, t_{22}, t_{12;\alpha}, t_{21;\alpha}\}) = \prod_{i=1}^{\infty} \frac{1}{1 - t_{11}^i - t_{22}^i - (t_{12,1}^i + t_{12,2}^i)(t_{21,1}^i + t_{21,2}^i) + t_{11}^i t_{22}^i} \tag{3.48}$$

3.2 $\mathcal{N} = 4$ SYM

Let us first review $\mathcal{N} = 4$ $U(N)$ SYM, for which the orthogonal basis of free chiral operators has been constructed before [47, 49]. We can view $\mathcal{N} = 4$ as a special case of $\mathcal{N} = 1$ quiver gauge theory with the quiver shown in Figure 3.8. Theory contains three

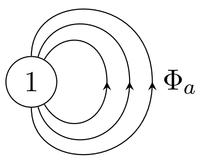


Figure 3.8: Quiver for $\mathbb{C}^3$, arrows correspond to three chiral multiplets Φ_1, Φ_2, Φ_3.

$\mathcal{N} = 1$ chiral multiplets Φ_a transforming in the adjoint of $U(N)$. There is a global $U(3)$ flavor symmetry. The chiral gauge invariant operators are built from the chiral adjoint scalars Φ_a, so we have single traces

$$\mathrm{tr}(\Phi_{a_1}\Phi_{a_2}\ldots\Phi_{a_n}) \tag{3.49}$$

and products of such traces. We will be interested in cases where N is finite and the operators involve more than N fields. In that case we need to take care of relationships between products of traces, arising from the fact that Φ_a are N-by-N matrices.

Consider all possible multitrace operators with $U(1)^3 \subset U(3)$ charges $\boldsymbol{n} = (n_1, n_2, n_3)$ and bare dimension $n = n_1 + n_2 + n_3$. A natural way to label the operators is by using a permutation $\sigma \in S_n$:

$$\mathcal{O}(\boldsymbol{n}, \sigma) = \prod_{k=1}^{n_1} (\Phi_1)^{i_k}_{i_{\sigma(k)}} \prod_{k=n_1+1}^{n_1+n_2} (\Phi_2)^{i_k}_{i_{\sigma(k)}} \prod_{k=n_1+n_2+1}^{n_1+n_2+n_3} (\Phi_3)^{i_k}_{i_{\sigma(k)}} \tag{3.50}$$

That is, the operator involves a product of fields $(\Phi_1)^{n_1}(\Phi_2)^{n_2}(\Phi_3)^{n_3}$ and the permutation σ indicates that k'th upper index is contracted with $\sigma(k)$'th lower index. Each cycle in σ

corresponds to a single trace.

At this point let us introduce some convenient notation. $(\Phi_a)^i_j$ is a matrix, which can be thought of as linear operator acting on N-dimensional vector space V_N. Then the object:

$$\left(\Phi_1^{\otimes n_1} \otimes \Phi_2^{\otimes n_2} \otimes \Phi_3^{\otimes n_3}\right)^{i_1 \dots i_n}_{j_1 \dots j_n} \equiv \prod_{k=1}^{n_1} (\Phi_1)^{i_k}_{j_k} \prod_{k=n_1+1}^{n_1+n_2} (\Phi_2)^{i_k}_{j_k} \prod_{k=n_1+n_2+1}^{n_1+n_2+n_3} (\Phi_3)^{i_k}_{j_k} \tag{3.51}$$

is a linear operator acting on the N^n-dimensional vector space $V_N^{\otimes n}$. Permutations σ are also linear operators in $V_N^{\otimes n}$ which acts by permuting the V_N factors of the tensor product:

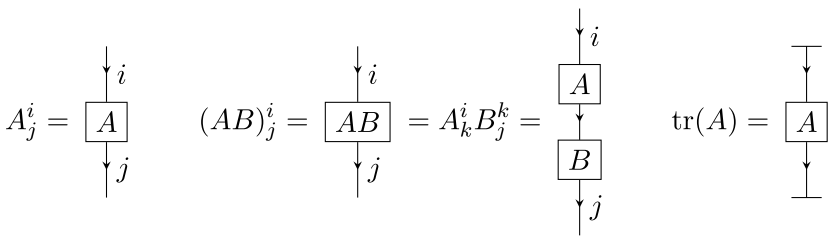
$$(\sigma)^{i_1 i_2 \dots i_n}_{j_1 j_2 \dots j_n} \equiv \delta^{i_1}_{j_{\sigma(1)}} \delta^{i_2}_{j_{\sigma(2)}} \dots \delta^{i_n}_{j_{\sigma(n)}} \tag{3.52}$$

Then (3.50) can be expressed as

$$\mathcal{O}(\boldsymbol{n}, \sigma) = \mathrm{tr}_{V_N^{\otimes n}} \left(\sigma \, \Phi_1^{\otimes n_1} \otimes \Phi_2^{\otimes n_2} \otimes \Phi_3^{\otimes n_3}\right) \tag{3.53}$$

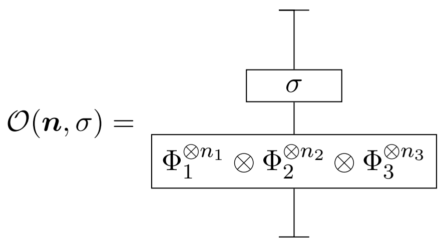
where the product of operators and the trace is over $V_N^{\otimes n}$, which means contracted indices of (3.51) and (3.51).

Let us also introduce diagrammatic notation for matrix multiplication and traces.



(3.54)

Incoming and outgoing arrows represent upper and lower indices respectively. Since in matrix multiplication conventionally lower index is contracted with upper, then in the diagram matrices are multiplied in the direction following arrows. When matrices are laid out vertically, the multiplication conventionally flows from top to bottom, and we can omit the arrows. The indices can, of course, belong to the vector space $V_N^{\otimes n}$, in which case lines represent the whole set $\{i_1 \dots i_n\}$ of contracted indices. Using this, we get a nice expression for the operator (3.53)



(3.55)

Note an operator is not labelled by a unique σ. $\mathcal{O}(\boldsymbol{n}, \sigma)$ does not change if we conjugate

σ by the subgroup:

$$\mathcal{O}(\boldsymbol{n}, \gamma\sigma\gamma^{-1}) = \mathcal{O}(\boldsymbol{n}, \sigma), \quad \gamma \in S_{n_1} \times S_{n_2} \times S_{n_3} \tag{3.56}$$

This can be seen from (3.50), where the conjugation can be brought from σ to act on $\Phi_1^{\otimes n_1} \otimes \Phi_2^{\otimes n_2} \otimes \Phi_3^{\otimes n_3}$, which is invariant. Furthermore, we still have the problem of finite N relationships.

One complete basis for the gauge invariant operators at finite N was constructed in [49], and is called "Restricted Schur" basis:

$$\mathcal{O}(\boldsymbol{L}) = \frac{1}{n_1!n_2!n_3!} \sum_{\sigma \in S_n} \chi_{R\to\boldsymbol{r}}^{\nu^-,\nu^+}(\sigma)\ \mathcal{O}(\boldsymbol{n}, \sigma) \tag{3.57}$$

The operators are uniquely specified by the set of group theoretic labels

$$\boldsymbol{L} = \{R, r_1, r_2, r_3, \nu^-, \nu^+\} \tag{3.58}$$

R, r_1, r_2, r_3 are Young diagrams

$$R \vdash n, \quad r_1 \vdash n_1, \quad r_2 \vdash n_2, \quad r_3 \vdash n_3 \tag{3.59}$$

R labels the representation of S_n and $\boldsymbol{r} = (r_1, r_2, r_3)$ labels the representation of the subgroup $S_{n_1} \times S_{n_2} \times S_{n_3} \subset S_n$ which appears in the decomposition of R in terms of subgroup irreps

$$R \to (r_1, r_2, r_3) \tag{3.60}$$

In case $\boldsymbol{r}$ appears in the decomposition more than once, the two numbers $\nu^\pm$ each label runs over the multiplicity given by Littlewood-Richardson coefficient $1 \leq \nu^\pm \leq g(r_1, r_2, r_3; R)$. For a summary of the facts about subgroup decomposition and branching coefficients see Appendix A.2. The finite N constraint appears simply as a cutoff on the number of rows in R:

$$l(R) \leq N \tag{3.61}$$

and there are no further relationships between the operators.

The key ingredient in (3.57) is the coefficient $\chi_{R\to\boldsymbol{r}}^{\nu^-,\nu^+}(\sigma)$ called "restricted character". It is a generalization of the usual character $\chi_R(\sigma) = \mathrm{tr}(D^R(\sigma))$ and defined as

$$\chi_{R\to\boldsymbol{r}}^{\nu^-,\nu^+}(\sigma) = \mathrm{tr}\left(P_{R\to\boldsymbol{r}}^{\nu^-,\nu^+} D^R(\sigma)\right) \tag{3.62}$$

$P^{\nu^-,\nu^+}_{R\to\boldsymbol{r}}$ is a projector-like operator [1]

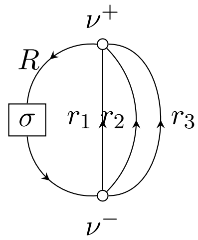
$$P^{\nu^-,\nu^+}_{R\to\boldsymbol{r}} = \sum_{l_1,l_2,l_3=1}^{d_r} |R;\boldsymbol{r},\nu^-,\boldsymbol{l}\rangle\langle R;\boldsymbol{r},\nu^+,\boldsymbol{l}| \tag{3.63}$$

or in terms of Branching coefficients (see (A.13))

$$(P^{\nu^-,\nu^+}_{R\to\boldsymbol{r}})_{ij} = \sum_{\boldsymbol{l}} B^{R\to\boldsymbol{r},\nu^-}_{i\to\boldsymbol{l}} B^{R\to\boldsymbol{r},\nu^+}_{j\to\boldsymbol{l}} \tag{3.64}$$

Using diagrammatic notation (A.15) we can represent the restricted character

$$\chi^{\nu^-,\nu^+}_{R\to\boldsymbol{r}}(\sigma) = \tag{3.65}$$



The edges now correspond to contracted indices in irreducible representations R, r_1, r_2, r_3, as labelled.

The basis (3.57) is not only complete, it is, in fact, orthogonal in the free field Zamolodchikov metric obtained from the two point function

$$\langle(\Phi_a)^i_j(\Phi^\dagger_b)^k_l\rangle = \delta_{ab}\delta^i_l\delta^k_j \tag{3.66}$$

Then

$$\langle O(R,\boldsymbol{r},\nu^-,\nu^+)O(\tilde{R},\tilde{\boldsymbol{r}},\tilde{\nu}^-,\tilde{\nu}^+)\rangle = \frac{h(R)f_N(R)}{h(r_1)h(r_2)h(r_3)}\delta_{R\tilde{R}}\delta_{r_1\tilde{r}_1}\delta_{r_2\tilde{r}_2}\delta_{r_3\tilde{r}_3}\delta_{\nu^+\tilde{\nu}^+}\delta_{\nu^-\tilde{\nu}^-} \tag{3.67}$$

$h(R)$ is the product of hooks of the Young diagram, and $f_N(R)$ is the weight of the diagram in $U(N)$. That is the only place that N dependence comes in, and it nicely captures the cutoff, because if the height of R exceeds N, then $f_N(R) = 0$, which means the operator is 0.

There is another complete orthogonal basis found in [47], where operators are organized into irreducible representation of the global symmetry $U(3)$. We will refer to it as "covariant basis", since operators transform covariantly with the global symmetry group. The operators are

$$\mathcal{O}(\boldsymbol{K}) = \frac{1}{n!}\sum_{\sigma\in S_n} B^{\Lambda\to[\boldsymbol{n}],\beta}_m S^{RR,\Lambda\tau}_{i\,j,\,m} D^R_{ij}(\sigma)\,\mathcal{O}(\boldsymbol{n},\sigma) \tag{3.68}$$

[1] If $\nu^- = \nu^+ \equiv \nu$, then $P^{\nu,\nu}_{R\to\boldsymbol{r}}$ is precisely the projector to $(\boldsymbol{r},\nu)$ in R. But the "off-diagonal" ones with $\nu^- \neq \nu^+$ are not strictly projectors, they are intertwining operators mapping between different copies of the same irrep $\boldsymbol{r}$ in R

The group theory labels in this case are

$$\boldsymbol{K} = \{R, \Lambda, \tau, \boldsymbol{n}, \beta\} \tag{3.69}$$

where $R, \Lambda \vdash n$ are Young diagrams with $n = n_1 + n_2 + n_3$ boxes. R is the same as before, with a cutoff of at most N rows, and Λ is an irrep of $U(3)$ with at most 3 rows. τ is the multiplicity label for the Kronecker product of S_n irreps

$$R \otimes R \to \Lambda \tag{3.70}$$

and $S^{RR,\Lambda\tau}_{i\,j,m}$ is the associated Clebsch-Gordan coefficient. For the review of the facts about Kronecker product and Clebsch-Gordan coefficients see Appendix A.3. $\boldsymbol{n} = (n_1, n_2, n_3)$ specifies how many fields of each flavor there are (note in $\boldsymbol{L}$ this information was contained in $\boldsymbol{r}$). $B^{\Lambda\to[\boldsymbol{n}],\beta}_m$ is the branching coefficient for the reduction from S_n irrep Λ to the trivial one-dimensional irrep $[n_1, n_2, n_3]$ of $S_{n_1} \times S_{n_2} \times S_{n_3}$, and β is the multiplicity label. In other words, β labels the invariants of Λ under $S_{n_1} \times S_{n_2} \times S_{n_3}$, and $B^{\Lambda\to[\boldsymbol{n}],\beta}_m$ are the invariant vectors. Note, compared with the usual branching coefficient notation $B^{\Lambda\to[\boldsymbol{n}],\beta}_{m\to i}$, we suppress the index i since $[\boldsymbol{n}]$ is one-dimensional.

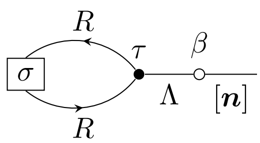
Again it will be useful to have a diagrammatic notation for the basis. Define

$$\chi(\boldsymbol{K}, \sigma) = B^{\Lambda\to[\boldsymbol{n}],\beta}_m S^{RR,\Lambda\tau}_{i\,j,m} D^R_{ij}(\sigma) \tag{3.71}$$

so that

$$\mathcal{O}(\boldsymbol{K}) = \frac{1}{n!} \sum_{\sigma\in S_n} \chi(\boldsymbol{K}, \sigma)\, \mathcal{O}(\boldsymbol{n}, \sigma) \tag{3.72}$$

The coefficient $\chi(\boldsymbol{K}, \sigma)$ can be expressed, using the diagrammatic notation (A.27) for the Clebsch-Gordan coefficient, as

$\chi(\boldsymbol{K}, \sigma) =$ 

(3.73)

The open line, which normally has an associated state label, corresponds to the unique $i = 1$ basis state of $[\boldsymbol{n}]$ in the branching $B^{\Lambda\to[\boldsymbol{n}],\beta}_{m\to i}$.

The two-point function between the operators is

$$\langle \mathcal{O}(\boldsymbol{K})\mathcal{O}(\tilde{\boldsymbol{K}})^\dagger \rangle = \frac{n_1!n_2!n_3!\mathrm{Dim}_N(R)}{d(R)^2} \delta_{R\tilde{R}} \delta_{\Lambda\tilde{\Lambda}} \delta_{\tau\tilde{\tau}} \delta_{\boldsymbol{n}\tilde{\boldsymbol{n}}} \delta_{\beta\tilde{\beta}} \tag{3.74}$$

3.3 Generalized restricted Schur basis

3.3.1 Complete basis

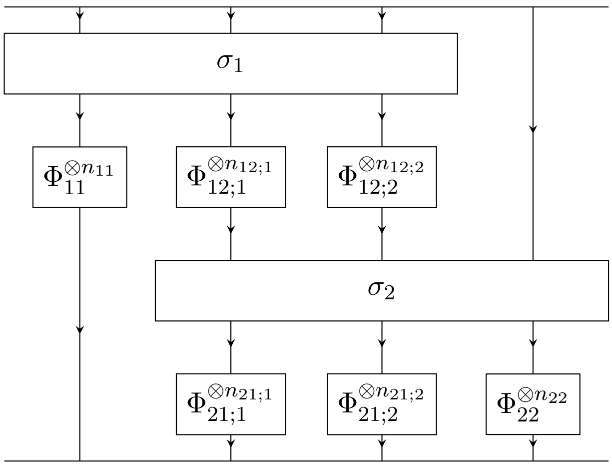
Let us assume we have a general quiver Q. We will often use $\mathbb{C}^3/\mathbb{Z}_2$ as an example, see Figure 3.1. In this section we derive the free orthogonal basis $\mathcal{O}_Q(\boldsymbol{L})$ for arbitrary quiver, analogous to the restricted Schur basis (3.57) in $\mathbb{C}^3$. We develop the alternative covariant basis $\mathcal{O}_Q(\boldsymbol{K})$ in the next Section 3.4.

In order to build a gauge-invariant operator[2] we contract the incoming and outgoing fields at each group node. In a more complicated quiver such as $\mathbb{C}^3/\mathbb{Z}_2$ there are different "paths" that an operator can take. We can build, for example:

$$\mathrm{tr}(\Phi_{11}\Phi_{11}),\ \mathrm{tr}(\Phi_{12;1}\Phi_{21;2}),\ \mathrm{tr}(\Phi_{11}\Phi_{12;1}\Phi_{22}\Phi_{21;1}),\ \dots \tag{3.75}$$

It is possible to capture all the different possibilities by fixing the number of times $n_{ab;\alpha}$ each field appears, and then contracting the indices corresponding to each group according to a permutation σ_a. This defines an operator which, in correspondence with (3.55), diagrammatically looks like:

$$\mathcal{O}_{\mathbb{C}^3/\mathbb{Z}_2}(\boldsymbol{n},\boldsymbol{\sigma}) = \tag{3.76}$$

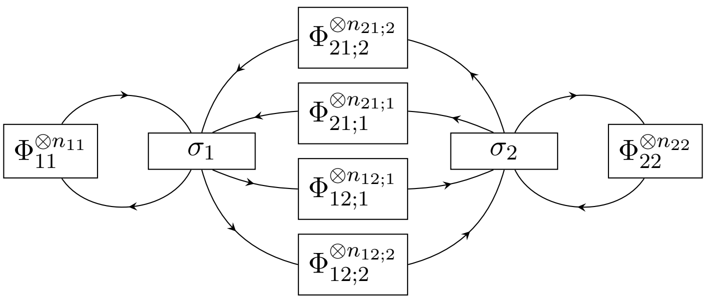


The lines represent indices in $V_N^{\otimes n_{ab;\alpha}}$. Note that if $n_{11} \neq n_{22}$, permutations σ_1, σ_2 are elements of symmetric groups of different size

$$\begin{aligned} \sigma_1 &\in S_{n_1}, \quad n_1 \equiv n_{11} + n_{12;1} + n_{12;2} \\ \sigma_2 &\in S_{n_2}, \quad n_2 \equiv n_{22} + n_{12;1} + n_{12;2} \end{aligned} \tag{3.77}$$

acting as operators in $V_{N_1}^{\otimes n_1}$ and $V_{N_2}^{\otimes n_2}$. If we rearrange the above diagram we get just the quiver itself with a permutation σ_a at each group node and an operator $(\Phi_{ab;\alpha})^{\otimes n_{ab;\alpha}}$ on each field line

$$\mathcal{O}_{\mathbb{C}^3/\mathbb{Z}_2}(\boldsymbol{n},\boldsymbol{\sigma}) = \tag{3.78}$$



[2]We restrict to the mesonic sector, or, in other words, $\prod_a U(N_a)$ gauge group, not $\prod_a SU(N_a)$.

It is clear that we can define $\mathcal{O}_Q(\boldsymbol{n},\boldsymbol{\sigma})$ in such a way for any quiver Q: it is a generalization of (3.53), but instead of contractions performed sequentially in a single trace, now the operators σ_a and $(\Phi_{ab;\alpha})^{\otimes n_{ab;\alpha}}$ are contracted along Q. With the diagrammatic representation of linear operators using boxes and lines, we are inserting the boxes for $(\Phi_{ab;\alpha})^{\otimes n_{ab;\alpha}}$ along the edge of the split-node quiver labelled α going from a to b, and we are inserting σ_a in the a'th line joining the a'th plus and minus nodes. Explicitly we can write:

$$\boxed{\mathcal{O}_Q(\boldsymbol{n},\boldsymbol{\sigma}) = \prod_{a,b}\prod_{\alpha=1}^{M_{ab}} \left(\Phi_{ab;\alpha}^{\otimes n_{ab;\alpha}}\right)^{\boldsymbol{I}_{ab;\alpha}}_{\boldsymbol{J}_{ab;\alpha}} \prod_a (\sigma_a)^{\bigcup_{b,\alpha}\boldsymbol{J}_{ba;\alpha}}_{\bigcup_{b,\alpha}\boldsymbol{I}_{ab;\alpha}}} \tag{3.79}$$

The indices a,b run over all group nodes, and it is understood that we skip the terms where $M_{ab}=0$. $\boldsymbol{I}_{ab;\alpha}$ and $\boldsymbol{J}_{ab;\alpha}$ are indices in the vector space $V_{N_a}^{\otimes n_{ab;\alpha}}$ and $\check{V}_{N_b}^{\otimes n_{ab;\alpha}}$, i.e $\boldsymbol{I}_{ab;\alpha}=\{i_1,\cdots,i_{n_{ab;\alpha}}\}$ and $\boldsymbol{J}_{ab;\alpha}=\{j_1,\cdots,j_{n_{ab;\alpha}}\}$ with the $i_1,i_2\cdots$ each living in V_{N_a} and $j_1,j_2,\cdots$ each in V_{N_b}. $(\Phi_{ab;\alpha})^{\otimes n_{ab;\alpha}}$ are linear maps $V_{N_a}^{\otimes n_{ab;\alpha}} \to V_{N_b}^{\otimes n_{ab;\alpha}}$, and σ_a are linear operators on $V_{N_a}^{\otimes n_a}$ where

$$n_a = \sum_{b,\alpha} n_{ab;\alpha} = \sum_{b,\alpha} n_{ba;\alpha} \tag{3.80}$$

The indices of σ_a are unions $\bigcup_{b,\alpha}\boldsymbol{J}_{ba;\alpha}$ and $\bigcup_{b,\alpha}\boldsymbol{I}_{ab;\alpha}$, meaning that upper indices of σ_a are contracted with lower indices of all fields $\Phi_{ba;\alpha}$ that enter node a, and lower indices of σ_a are contracted with upper indices of all fields $\Phi_{ab;\alpha}$ that leave node a.

As a basic example consider an operator in $\mathbb{C}^3/\mathbb{Z}_2$ with

$$\boldsymbol{n} = \{n_{11}, n_{22}, n_{12;1}, n_{12;2}, n_{21;1}, n_{21;2}\} = \{1,1,1,0,1,0\} \tag{3.81}$$

that is, build from fields $(\Phi_{11},\Phi_{22},\Phi_{12;1},\Phi_{21;1})$. We have

$$\mathcal{O}_{\mathbb{C}^3/\mathbb{Z}_2}(\boldsymbol{n},\sigma_1,\sigma_2) = (\Phi_{11})^{i_1}_{j_1}\,(\Phi_{22})^{i_2}_{j_2}\,(\Phi_{12;1})^{i_3}_{j_3}\,(\Phi_{21;1})^{i_4}_{j_4}\,(\sigma_1)^{j_1j_4}_{i_1i_3}(\sigma_2)^{j_2j_3}_{i_2i_4} \tag{3.82}$$

with $\sigma_1,\sigma_2\in S_2$. For different combinations of σ_a we get

$$\begin{aligned} \mathcal{O}(\mathbb{I},\mathbb{I}) &= \mathrm{tr}(\Phi_{11})\mathrm{tr}(\Phi_{22})\mathrm{tr}(\Phi_{12;1}\Phi_{21;1}) \\ \mathcal{O}((12),\mathbb{I}) &= \mathrm{tr}(\Phi_{11}\Phi_{12;1}\Phi_{21;1})\mathrm{tr}(\Phi_{22}) \\ \mathcal{O}(\mathbb{I},(12)) &= \mathrm{tr}(\Phi_{11})\mathrm{tr}(\Phi_{22}\Phi_{21;1}\Phi_{12;1}) \\ \mathcal{O}((12),(12)) &= \mathrm{tr}(\Phi_{11}\Phi_{12;1}\Phi_{22}\Phi_{21;1}) \end{aligned} \tag{3.83}$$

As in the previous section for the case of $\mathbb{C}^3$, the operators $\mathcal{O}_Q(\boldsymbol{n},\boldsymbol{\sigma})$ are not uniquely labelled by $\boldsymbol{\sigma}$, that is, the basis is overcomplete and different $\boldsymbol{\sigma}$ can correspond to the same operator. Specifically, we have an identification

$$\mathcal{O}_Q(\boldsymbol{n},\boldsymbol{\sigma}) = \mathcal{O}_Q(\boldsymbol{n},\mathrm{Adj}_{\boldsymbol{\gamma}}(\boldsymbol{\sigma})) \tag{3.84}$$

where

$$\boldsymbol{\gamma} = \{\gamma_{ab;\alpha}\} \in \bigotimes_{a,b,\alpha} S_{n_{ab;\alpha}} \tag{3.85}$$

and the adjoint action is defined as

$$\text{Adj}_{\boldsymbol{\gamma}}(\boldsymbol{\sigma}) = \left\{ (\otimes_{b,\alpha}\gamma_{ba;\alpha})\, \sigma_a \, (\otimes_{b,\alpha}\gamma^{-1}_{ab;\alpha}) \right\} \tag{3.86}$$

This is easily seen from the definition (3.79) and the fact that each $n_{ab;\alpha}$ block of identical fields is invariant under permutations

$$\left(\Phi_{ab;\alpha}^{\otimes n_{ab;\alpha}}\right) = \gamma^{-1}\left(\Phi_{ab;\alpha}^{\otimes n_{ab;\alpha}}\right)\gamma \tag{3.87}$$

These permutations can then be moved to act on $\boldsymbol{\sigma}$.

It is shown in [47, 48] that for $\mathbb{C}^3$ the complete orthogonal bases (3.57) and (3.68) can be derived by essentially "solving" the invariance (3.56). We will use the same method here to find generalized bases $\mathcal{O}_Q(\boldsymbol{L})$ and $\mathcal{O}_Q(\boldsymbol{K})$ for any quiver Q. As an illustration let us take the simplest example of half-BPS operators [16]. The idea is that the invariance

$$\mathcal{O}_{\mathbb{C}}(\sigma) = \frac{1}{n!}\sum_{\gamma\in S_n} \mathcal{O}_{\mathbb{C}}(\gamma^{-1}\sigma\gamma) \tag{3.88}$$

can be rewritten as

$$\mathcal{O}_{\mathbb{C}}(\sigma) = \sum_{\tau}\left(\frac{1}{n!}\sum_{\gamma}\delta(\gamma\sigma\gamma^{-1}\tau^{-1})\right)\mathcal{O}_{\mathbb{C}}(\tau) = \sum_{\tau}\left(\frac{1}{n!}\sum_{R\vdash n}\chi_R(\sigma)\chi_R(\tau)\right)\mathcal{O}_{\mathbb{C}}(\tau) \tag{3.89}$$

which looks like a projector to a lower-dimensional space labelled by Young diagram R. This motivates the Schur polynomial basis

$$\mathcal{O}_{\mathbb{C}}(R) = \frac{1}{n!}\sum_{\tau}\chi_R(\sigma)\mathcal{O}_{\mathbb{C}}(\sigma) \tag{3.90}$$

which indeed turns out to be complete and orthogonal. For $\mathbb{C}^3$ we have similarly (3.56) leading to

$$\mathcal{O}_{\mathbb{C}^3}(\boldsymbol{n},\sigma) \sim \sum_{\tau}\left(\sum_{R,\boldsymbol{r},\nu^-,\nu^+}\chi^{\nu^-,\nu^+}_{R\to\boldsymbol{r}}(\sigma)\chi^{\nu^-,\nu^+}_{R\to\boldsymbol{r}}(\tau)\right)\mathcal{O}_{\mathbb{C}^3}(\boldsymbol{n},\tau) \tag{3.91}$$

which suggests the basis (3.57). In order to generalize this to arbitrary quiver, we define "quiver characters" $\chi_Q(\boldsymbol{L},\boldsymbol{\sigma})$ obeying, schematically

$$\sum_{\boldsymbol{L}}\chi_Q(\boldsymbol{L},\boldsymbol{\sigma})\chi_Q(\boldsymbol{L},\boldsymbol{\tau}) \sim \sum_{\boldsymbol{\gamma}}\delta(\boldsymbol{\sigma},\text{Adj}_{\boldsymbol{\gamma}}(\boldsymbol{\tau})) \tag{3.92}$$

where $\boldsymbol{L}$ is a generalized set of group theory labels. With a help of quiver characters we can analogously express invariance (3.84) as

$$\mathcal{O}_Q(\boldsymbol{n},\boldsymbol{\sigma}) \sim \sum_{\boldsymbol{\tau}} \left(\sum_{\boldsymbol{L}} \chi_Q(\boldsymbol{L},\boldsymbol{\sigma})\chi_Q(\boldsymbol{L},\boldsymbol{\tau}) \right) \mathcal{O}_Q(\boldsymbol{n},\boldsymbol{\tau}) \tag{3.93}$$

leading to define a basis

$$\mathcal{O}_Q(\boldsymbol{L}) \sim \sum_{\boldsymbol{\sigma}} \chi_Q(\boldsymbol{L},\boldsymbol{\sigma})\mathcal{O}_Q(\boldsymbol{n},\boldsymbol{\sigma}) \tag{3.94}$$

The details of the derivation can be found in Appendix C, the result is that we can define restricted quiver characters as

$$\boxed{\begin{aligned} \chi_Q(\boldsymbol{L},\boldsymbol{\sigma}) &= \prod_a D^{R_a}_{i_a j_a}(\sigma_a) B^{R_a \to \bigcup_{b,\alpha} r_{ab;\alpha},\nu_a^-}_{j_a \to \bigcup_{b,\alpha} l_{ab;\alpha}} B^{R_a \to \bigcup_{b,\alpha} r_{ba;\alpha},\nu_a^+}_{i_a \to \bigcup_{b,\alpha} l_{ba;\alpha}} \\ \boldsymbol{L} &\equiv \{R_a, r_{ab;\alpha}, \nu_a^-, \nu_a^+\} \end{aligned}} \tag{3.95}$$

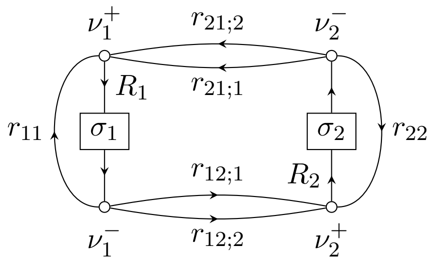
They obey the required invariance and orthogonality properties, listed in Appendix B.2, which are analogous to those of symmetric group characters. The complete basis of operators with a convenient normalization can then be defined as:

$$\boxed{\mathcal{O}_Q(\boldsymbol{L}) = \frac{1}{\prod n_a!}\sqrt{\frac{\prod d(R_a)}{\prod d(r_{ab;\alpha})}} \sum_{\boldsymbol{\sigma}} \chi_Q(\boldsymbol{L},\boldsymbol{\sigma})\mathcal{O}_Q(\boldsymbol{n},\boldsymbol{\sigma})} \tag{3.96}$$

The group theory labels $\boldsymbol{L}$ are:

- $r_{ab;\alpha}$: a Young diagram with $n_{ab;\alpha}$ boxes for each set of fields $\Phi_{ab;\alpha}$.
- R_a: a Young diagram for each group factor, labelling representation of S_{n_a}, where $n_a = \sum_{b,\alpha} n_{ba;\alpha} = \sum_{b,\alpha} n_{ab;\alpha}$ is the number of incoming and outgoing fields.
- ν_a^-: multiplicity index for outgoing field reduction $R_a \to \bigcup_{b,\alpha} r_{ab;\alpha}$.
- ν_a^+: multiplicity index for incoming field reduction $R_a \to \bigcup_{b,\alpha} r_{ba;\alpha}$.

The structure can most easily be seen with a diagram, which is the split-node quiver with permutations σ_a inserted

$\chi_{\mathbb{C}^3/\mathbb{Z}_2}(\boldsymbol{L},\boldsymbol{\sigma}) =$  (3.97)

Each group node carries a permutation in representation R_a (denoted by a box), which is then contracted via branching coefficients (denoted by white nodes) to representations $r_{ab;\alpha}$

associated with fields. There are multiplicities $\nu_a^\pm$ associated to each branching coefficient node. The lines denote contracted matrix indices $i_a, j_a, l_{ab;\alpha}$. Note that $\chi_Q(\boldsymbol{L},\boldsymbol{\sigma})$ reduces precisely to (3.65) for the $\mathbb{C}^3$ quiver! Also, for the trivial quiver $\mathbb{C}$ consisting of one node and one field Φ_{11}, corresponding to the half-BPS sector, we get $R_1 = r_{11}$, all the branching coefficients are unit matrices, and the quiver character *is* the usual symmetric group character.

Using the orthogonality properties of quiver characters we can write the inverse of the basis change (3.96):

$$\mathcal{O}_Q(\boldsymbol{n},\boldsymbol{\sigma}) = \sum_{\boldsymbol{L}} \sqrt{\frac{\prod d(R_a)}{\prod d(r_{ab;\alpha})}}\, \chi_Q(\boldsymbol{L},\boldsymbol{\sigma})\mathcal{O}_Q(\boldsymbol{L}) \tag{3.98}$$

3.3.2 Two-point function

We will show here that the general basis (3.96) is orthogonal in free field metric for any quiver Q

$$\boxed{\left\langle \mathcal{O}_Q(\boldsymbol{L})\mathcal{O}_Q(\tilde{\boldsymbol{L}})^\dagger \right\rangle = \delta_{\boldsymbol{L}\tilde{\boldsymbol{L}}} \frac{\prod n_{ab;\alpha}!}{\prod n_a!} \prod_a f_{N_a}(R_a)} \tag{3.99}$$

$f_{N_a}(R_a)$ is the product of weights of a $U(N_a)$ diagram R_a. We can see it is a straightforward generalization of the result (3.67) for $\mathbb{C}^3$, except with a different normalization, due to different normalization of the operators (3.96), compared to (3.57). It is important to note, that again N-dependence is in the factors $f_{N_a}(R_a)$ which vanish if the height of R_a exceeds N_a. So at finite N the Hilbert space consists of operators $\mathcal{O}_Q(\boldsymbol{L})$ where the height of all R_a does not exceed N_a

$$\mathcal{H} = \{\mathcal{O}_Q(\boldsymbol{L}) \mid \forall_a l(R_a) \leq N_a\} \tag{3.100}$$

The derivation of (3.99) is similar to that of (3.67) in [49], but now using analogous properties of quiver characters $\chi_Q(\boldsymbol{L},\boldsymbol{\sigma})$ from Appendix B.2. We have the free field metric

$$\left\langle (\Phi_{ab;\alpha})^i_j (\Phi^\dagger_{cd;\beta})^k_l \right\rangle = \delta_{ac}\delta_{bd}\delta_{\alpha\beta}\delta^i_l\delta^k_j \tag{3.101}$$

Then the two point function of $\mathcal{O}_Q(\boldsymbol{n},\boldsymbol{\sigma})$ operators is

$$\left\langle \mathcal{O}_Q(\boldsymbol{n},\boldsymbol{\sigma})\mathcal{O}_Q(\boldsymbol{n},\tilde{\boldsymbol{\sigma}})^\dagger \right\rangle = \sum_{\boldsymbol{\gamma}} \prod_a \mathrm{tr}_{V_{N_a}^{\otimes n_a}}(\mathrm{Adj}_{\boldsymbol{\gamma}}(\sigma_a)\tilde{\sigma}_a^{-1}) \tag{3.102}$$

The sum is over $\boldsymbol{\gamma} \equiv \{\gamma_{ab;\alpha} \in S_{n_{ab;\alpha}}\}$ – Wick contractions arising from each set of fields. For the derivation of (3.102) see Appendix G.2. Next, we apply (3.102) to the definition of $\mathcal{O}_Q(\boldsymbol{L})$ (3.96):

$$\left\langle \mathcal{O}_Q(\boldsymbol{L})\mathcal{O}_Q(\tilde{\boldsymbol{L}})^\dagger \right\rangle = c_{\boldsymbol{L}} c_{\tilde{\boldsymbol{L}}} \sum_{\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}},\boldsymbol{\gamma}} \chi_Q(\boldsymbol{L},\boldsymbol{\sigma})\chi_Q(\tilde{\boldsymbol{L}},\tilde{\boldsymbol{\sigma}}) \prod_a \mathrm{tr}_{V_{N_a}^{\otimes n_a}}(\mathrm{Adj}_{\boldsymbol{\gamma}}(\sigma_a)\tilde{\sigma}_a^{-1}) \tag{3.103}$$

where $c_{\boldsymbol{L}} = \frac{1}{\prod n_a!}\sqrt{\frac{\prod d(R_a)}{\prod d(r_{ab;\alpha})}}$ is the normalization coefficient appearing in front of the sum in (3.96). Note that $\chi_Q(\boldsymbol{L}, \boldsymbol{\sigma})$ is a real number, so we drop complex conjugation. Now redefining $\sigma_a \to \mathrm{Adj}_{\boldsymbol{\gamma}}(\sigma_a)$ and using invariance property (B.10) the dependence on $\boldsymbol{\gamma}$ drops out

$$\left\langle \mathcal{O}_Q(\boldsymbol{L})\mathcal{O}_Q(\tilde{\boldsymbol{L}})^\dagger \right\rangle = \left(c_{\boldsymbol{L}} c_{\tilde{\boldsymbol{L}}} \prod n_{ab;\alpha}! \right) \sum_{\boldsymbol{\sigma}, \tilde{\boldsymbol{\sigma}}} \chi_Q(\boldsymbol{L}, \boldsymbol{\sigma}) \chi_Q(\tilde{\boldsymbol{L}}, \tilde{\boldsymbol{\sigma}}) \prod_a \mathrm{tr}_{V_{N_a}^{\otimes n_a}} (\sigma_a \tilde{\sigma}_a^{-1}) \tag{3.104}$$

Next, applying (B.13)

$$\begin{aligned} \left\langle \mathcal{O}_Q(\boldsymbol{L})\mathcal{O}_Q(\tilde{\boldsymbol{L}})^\dagger \right\rangle &= \delta_{\boldsymbol{R}\tilde{\boldsymbol{R}}} \delta_{\boldsymbol{r}\tilde{\boldsymbol{r}}} \delta_{\boldsymbol{\nu}^- \tilde{\boldsymbol{\nu}}^-} \left({c_{\boldsymbol{L}}}^2 \prod n_{ab;\alpha}! \right) \\ &\times \sum_{\boldsymbol{\sigma}} \prod_a \frac{n_a!}{d(R_a)} \mathrm{tr} \left(D^{R_a}(\sigma_a) P^{\nu_a^+ \tilde{\nu}_a^+}_{R_a \to \bigcup_{b,\alpha} r_{ab;\alpha}} \right) \mathrm{tr}_{V_{N_a}^{\otimes n_a}} (\sigma_a) \end{aligned} \tag{3.105}$$

Finally (A.10) gives

$$\begin{aligned} \left\langle \mathcal{O}_Q(\boldsymbol{L})\mathcal{O}_Q(\tilde{\boldsymbol{L}})^\dagger \right\rangle &= \delta_{\boldsymbol{L}\tilde{\boldsymbol{L}}}\, {c_{\boldsymbol{L}}}^2 \, \frac{\prod n_{ab;\alpha}! \prod n_a! \prod d(r_{ab;\alpha})}{\prod d(R_a)} \prod_a f_{N_a}(R_a) \\ &= \delta_{\boldsymbol{L}\tilde{\boldsymbol{L}}} \frac{\prod n_{ab;\alpha}!}{\prod n_a!} \prod_a f_{N_a}(R_a) \end{aligned} \tag{3.106}$$

proving (3.99) .

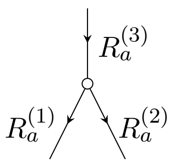
3.3.3 Chiral ring structure constants

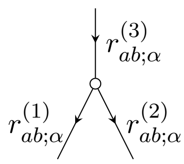
In this section we obtain general expressions for the chiral ring structure constants in the restricted Schur basis

$$\mathcal{O}_Q(\boldsymbol{L}^{(1)})\mathcal{O}_Q(\boldsymbol{L}^{(2)}) \equiv \sum_{\boldsymbol{L}^{(3)}} G(\boldsymbol{L}^{(1)}, \boldsymbol{L}^{(2)}; \boldsymbol{L}^{(3)}) \mathcal{O}_Q(\boldsymbol{L}^{(3)}) \tag{3.107}$$

The result can be expressed diagrammatically, and the main feature is that all Young diagram labels combine according to the Littlewood-Richardson rule. That is, the resulting diagram (3.113) involves the branching coefficients for $R_a^{(3)} \to (R_a^{(1)}, R_a^{(2)})$ and $r_{ab;\alpha}^{(3)} \to (r_{ab;\alpha}^{(1)}, r_{ab;\alpha}^{(2)})$

$$G(\boldsymbol{L}^{(1)}, \boldsymbol{L}^{(2)}; \boldsymbol{L}^{(3)}) \sim \prod_a \quad \text{[diagram 1]} \quad \prod_{a,b,\alpha} \quad \text{[diagram 2]} \tag{3.108}$$





and so the chiral ring structure constant vanishes unless the Littlewood-Richardson coefficients $g(R_a^{(1)}, R_a^{(2)}; R_a^{(3)})$ and $g(r_{ab;\alpha}^{(1)}, r_{ab;\alpha}^{(2)}; r_{ab;\alpha}^{(3)})$ are all non-zero.

Note that, if we consider correlators of n holomorphic operators and one anti-holomorphic, the coefficient would involve the appropriate Littlewood-Richardson coefficient for the branching $R_a^{(n+1)} \to (R_a^{(1)}, R_a^{(2)}, \ldots, R_a^{(n)})$ and so on for other labels. We leave it as an exercise for the reader to write out the explicit formulae for that case, following the analogous expressions we present for $n = 2$, i.e two holomorphic operators fusing into one.

Consider the product of operators (3.96)

$$
\begin{aligned}
&\mathcal{O}_Q(\boldsymbol{L}^{(1)})\mathcal{O}_Q(\boldsymbol{L}^{(2)}) \\
&= \frac{1}{\prod n_a^{(1)}!}\frac{1}{\prod n_a^{(2)}!}\sum_{\boldsymbol{\sigma}^{(1)}}\sum_{\boldsymbol{\sigma}^{(2)}}\hat{\chi}_Q(\boldsymbol{L}^{(1)},\boldsymbol{\sigma}^{(1)})\hat{\chi}_Q(\boldsymbol{L}^{(2)},\boldsymbol{\sigma}^{(2)})\mathcal{O}_Q(\boldsymbol{\sigma}^{(1)})\mathcal{O}_Q(\boldsymbol{\sigma}^{(2)}) \\
&= \frac{1}{\prod n_a^{(1)}!}\frac{1}{\prod n_a^{(2)}!}\sum_{\boldsymbol{\sigma}^{(1)}}\sum_{\boldsymbol{\sigma}^{(2)}}\hat{\chi}_Q(\boldsymbol{L}^{(1)},\boldsymbol{\sigma}^{(1)})\hat{\chi}_Q(\boldsymbol{L}^{(2)},\boldsymbol{\sigma}^{(2)})\mathcal{O}_Q(\boldsymbol{\sigma}^{(1)}\circ\boldsymbol{\sigma}^{(2)})
\end{aligned}
\tag{3.109}
$$

Here we use a conveniently normalized quiver character

$$
\hat{\chi}_Q(\boldsymbol{L},\boldsymbol{\sigma}) \equiv \sqrt{\frac{\prod d(R_a)}{\prod d(r_{ab;\alpha})}}\chi_Q(\boldsymbol{L},\boldsymbol{\sigma}) \tag{3.110}
$$

The outer product $\boldsymbol{\sigma}^{(1)}\circ\boldsymbol{\sigma}^{(2)}$ consists of pairs of permutations $\sigma_a^{(1)}\circ\sigma_a^{(2)}$ in $S_{n_a^{(1)}}\times S_{n_a^{(2)}}\subset S_{n_a^{(1)}+n_a^{(2)}}$. We can expand the permutation-basis operators as a sum of Fourier basis operators using (3.98) to get

$$
\begin{aligned}
&\mathcal{O}_Q(\boldsymbol{L}^{(1)})\mathcal{O}_Q(\boldsymbol{L}^{(2)}) \\
&= \frac{1}{\prod n_a^{(1)}!}\frac{1}{\prod n_a^{(2)}!}\sum_{\boldsymbol{\sigma}^{(1)}}\sum_{\boldsymbol{\sigma}^{(2)}}\sum_{\boldsymbol{L}^{(3)}}\hat{\chi}_Q(\boldsymbol{L}^{(1)},\boldsymbol{\sigma}^{(1)})\hat{\chi}_Q(\boldsymbol{L}^{(2)},\boldsymbol{\sigma}^{(2)})\hat{\chi}_Q(\boldsymbol{L}^{(3)},\boldsymbol{\sigma}^{(1)}\circ\boldsymbol{\sigma}^{(2)})\mathcal{O}_Q(\boldsymbol{L}^{(3)}) \\
&\equiv \sum_{\boldsymbol{L}^{(3)}} G(\boldsymbol{L}^{(1)},\boldsymbol{L}^{(1)};\boldsymbol{L}^{(3)})\mathcal{O}_Q(\boldsymbol{L}^{(3)})
\end{aligned}
\tag{3.111}
$$

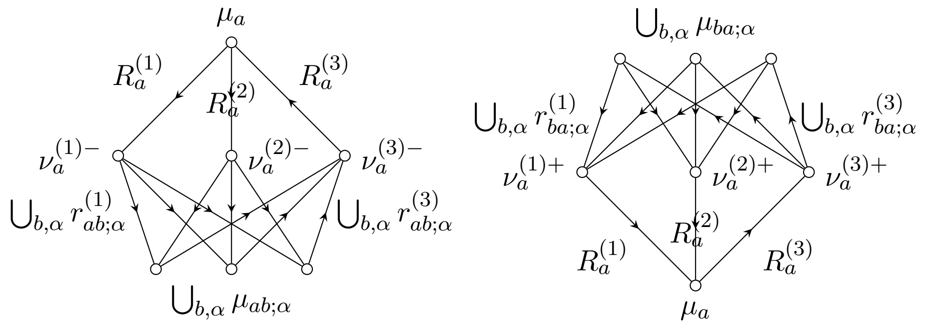
The sum $\boldsymbol{L}^{(3)}$ runs over labels with $\boldsymbol{n}^{(3)} = \boldsymbol{n}^{(1)} + \boldsymbol{n}^{(2)}$. This leads to the expression for the chiral ring structure constants

$$
G(\boldsymbol{L}^{(1)},\boldsymbol{L}^{(1)};\boldsymbol{L}^{(3)}) = \frac{1}{\prod n_a^{(1)}!}\frac{1}{\prod n_a^{(2)}!}\sum_{\boldsymbol{\sigma}^{(1)}}\sum_{\boldsymbol{\sigma}^{(2)}}\hat{\chi}_Q(\boldsymbol{L}^{(1)},\boldsymbol{\sigma}^{(1)})\hat{\chi}_Q(\boldsymbol{L}^{(2)},\boldsymbol{\sigma}^{(2)})\hat{\chi}_Q(\boldsymbol{L}^{(3)},\boldsymbol{\sigma}^{(1)}\circ\boldsymbol{\sigma}^{(2)}) \tag{3.112}
$$

which can be evaluating by doing the sums over $\boldsymbol{\sigma}^{(1)},\boldsymbol{\sigma}^{(2)}$.

Let us first describe the answer and then sketch the steps in the derivation. The final result is, diagrammatically:

$$G(\boldsymbol{L}^{(1)},\boldsymbol{L}^{(2)};\boldsymbol{L}^{(3)}) = f_{\boldsymbol{L}^{(1)}\boldsymbol{L}^{(2)}}^{\boldsymbol{L}^{(3)}} \sum_{\{\mu_a\}} \sum_{\{\mu_{ab;\alpha}\}} \prod_a \left(\quad \right) \tag{3.113}$$



with the constant prefactor

$$f_{\boldsymbol{L}^{(1)}\boldsymbol{L}^{(2)}}^{\boldsymbol{L}^{(3)}} = \sqrt{\frac{\prod_a d(R_a^{(1)})d(R_a^{(2)})d(R_a^{(3)})}{\prod_{a,b,\alpha} d(r_{ab;\alpha}^{(1)})d(r_{ab;\alpha}^{(2)})d(r_{ab;\alpha}^{(3)})} \frac{1}{\prod_a d(R_a^{(1)})d(R_a^{(2)})} \frac{1}{\prod_{a,b,\alpha} d(r_{ab;\alpha}^{(1)})d(r_{ab;\alpha}^{(2)})}} \tag{3.114}$$

For illustration purposes in (3.113) we draw three outgoing arrows $r_{ab;\alpha}$ from each branching node ν_a^- and three incoming arrows $r_{ba;\alpha}$ to each branching node ν_a^+. The precise structure depends on the quiver (on the other hand, the lines and nodes labelled by $^{(1),(2),(3)}$ are always three, associated with the three operators).

The explicit expression corresponding to (3.113) is

$$\begin{aligned} G(\boldsymbol{L}^{(1)},\boldsymbol{L}^{(2)};\boldsymbol{L}^{(3)}) = f_{\boldsymbol{L}^{(1)}\boldsymbol{L}^{(2)}}^{\boldsymbol{L}^{(3)}} \sum_{\{\mu_a\}} \sum_{\{\mu_{ab;\alpha}\}} \\ \prod_a \mathcal{F}\left(\cup_I R_a^{(I)}, \{\cup_{I,b,\alpha} r_{ab;\alpha}^{(I)}\}, \cup_I \nu_a^{(I)-}; \mu_a, \{\cup_{b,\alpha}\mu_{ab;\alpha}\}\right) \\ \times \mathcal{F}\left(\cup_I R_a^{(I)}, \{\cup_{I,b,\alpha} r_{ba;\alpha}^{(I)}\}, \cup_I \nu_a^{(I)+}; \mu_a, \{\cup_{b,\alpha}\mu_{ba;\alpha}\}\right) \end{aligned} \tag{3.115}$$

with the object $\mathcal{F}$ equal to the single connected piece in (3.113):

$$\begin{aligned} &\mathcal{F}\left(\cup_I R^{(I)}, \{\cup_{I,\alpha} r_\alpha^{(I)}\}, \cup_I \nu^{(I)}; \mu, \{\cup_\alpha \mu_\alpha\}\right) \\ &= B_{i^{(1)}\to\cup_\alpha l_\alpha^{(1)}}^{R^{(1)}\to\cup_\alpha r_\alpha^{(1)};\nu^{(1)+}} B_{i^{(2)}\to\cup_\alpha l_\alpha^{(2)}}^{R^{(2)}\to\cup_\alpha r_\alpha^{(2)};\nu^{(2)+}} B_{i^{(3)}\to\cup_\alpha l_\alpha^{(3)}}^{R^{(3)}\to\cup_\alpha r_\alpha^{(3)};\nu^{(3)+}} B_{i^{(3)}\to i^{(1)},i^{(2)}}^{R^{(3)}\to R^{(1)},R^{(2)};\mu} \prod_\alpha B_{l_\alpha^{(3)}\to l_\alpha^{(1)},l_\alpha^{(2)}}^{r_\alpha^{(3)}\to r_\alpha^{(1)},r_\alpha^{(2)};\mu_\alpha} \end{aligned} \tag{3.116}$$

The two pieces $\mathcal{F}(\cup_I R_a^{(I)}, \{\cup_{I,b,\alpha} r_{ab;\alpha}^{(I)}\}, \cup_I \nu_a^{(I)-}; \mu_a, \{\cup_{b,\alpha}\mu_{ab;\alpha}\})$ and $\mathcal{F}(\cup_I R_a^{(I)}, \{\cup_{I,b,\alpha} r_{ba;\alpha}^{(I)}\}, \cup_I \nu_a^{(I)+}; \mu_a, \{\cup_{b,\alpha}\mu_{ba;\alpha}\})$ originally appear with reversed arrows, but have the same expression (3.116) due to reality of branching coefficients.

The key feature of (3.113) is that sums μ_a are over multiplicity $g(R_a^{(1)}, R_a^{(2)}; R_a^{(3)})$ and $\mu_{ab;\alpha}$ are over $g(r_{ab;\alpha}^{(1)}, r_{ab;\alpha}^{(2)}; r_{ab;\alpha}^{(3)})$, and so the structure constant vanishes, unless all diagrams of $\boldsymbol{L}^{(3)}$ appear in the respective Littlewood-Richardson products

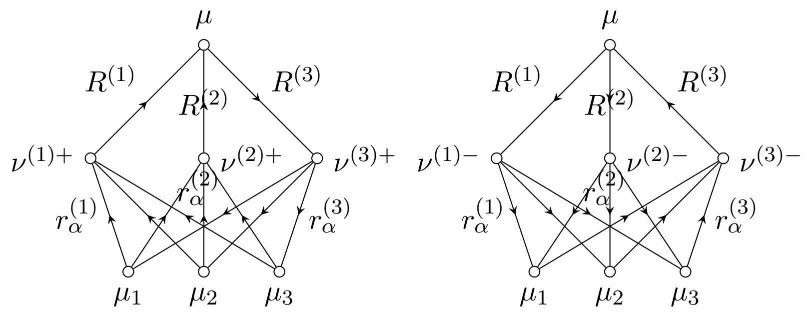
$$\boxed{\begin{array}{c} R_a^{(1)} \otimes R_a^{(2)} \to R_a^{(3)} \\ r_{ab;\alpha}^{(1)} \otimes r_{ab;\alpha}^{(2)} \to r_{ab;\alpha}^{(3)} \end{array}} \tag{3.117}$$

The branching coefficients in (3.113) are contracted in the natural way, given these selection rules. For each group node a there are two terms – one for ν^+ and one for ν^-. Within each term, the branching coefficients arising from each operator $B^{R_a^{(I)} \to \cup_{b,\alpha} r_{ab;\alpha}^{(I)}; \nu_a^{(I)\pm}}$ are coupled via extra branching coefficients: $B^{R_a^{(3)} \to R_a^{(1)}, R_a^{(2)}; \mu_a}$ for R_a's, and $B^{r_{ab;\alpha}^{(3)} \to r_{ab;\alpha}^{(1)}, r_{ab;\alpha}^{(2)}; \mu_{ab;\alpha}}$ for $r_{ab;\alpha}$'s.

Let us take as an example the structure constants of $\mathbb{C}^3$, which were discussed in the restricted Schur basis in [79]. The operators are defined via quiver characters (3.65), and for a single-node quiver (3.113) reduces to:

$$G_{\mathbb{C}^3}(\boldsymbol{L}^{(1)}, \boldsymbol{L}^{(2)}; \boldsymbol{L}^{(3)})$$

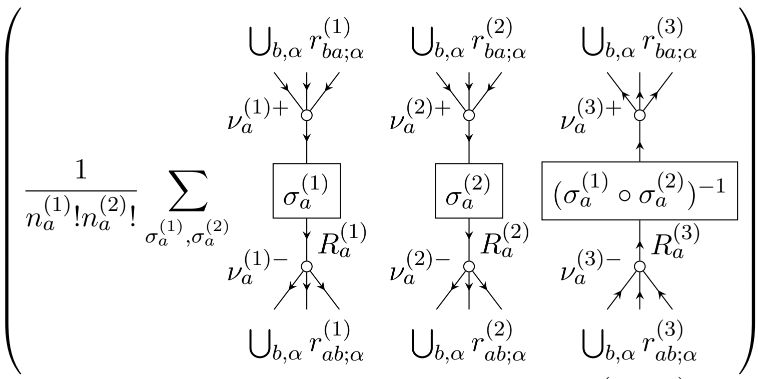
$$= f_{\boldsymbol{L}^{(1)}\boldsymbol{L}^{(2)}}^{\boldsymbol{L}^{(3)}} \sum_{\substack{\mu, \\ \mu_1,\mu_2,\mu_3}}$$



(3.118)

Let us now go over the steps in the derivation of (3.113). For clarity we mostly use diagrammatic notation. The starting point is the sum (3.112), which we write out as a trace of a product of group factors

$$G(\boldsymbol{L}^{(1)}, \boldsymbol{L}^{(2)}; \boldsymbol{L}^{(3)}) = \tilde{f}_{\boldsymbol{L}^{(1)}\boldsymbol{L}^{(2)}}^{\boldsymbol{L}^{(3)}} \mathbf{tr} \prod_a$$



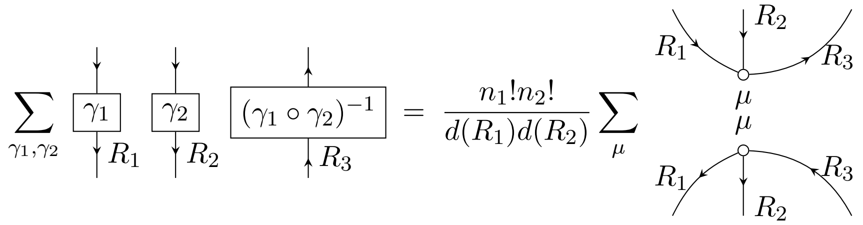
(3.119)

with a prefactor

$$\tilde{f}_{\boldsymbol{L}^{(1)}\boldsymbol{L}^{(2)}}^{\boldsymbol{L}^{(3)}} = \sqrt{\frac{\prod_a d(R_a^{(1)}) d(R_a^{(2)}) d(R_a^{(3)})}{\prod_{a,b,\alpha} d(r_{ab;\alpha}^{(1)}) d(r_{ab;\alpha}^{(2)}) d(r_{ab;\alpha}^{(3)})}} \tag{3.120}$$

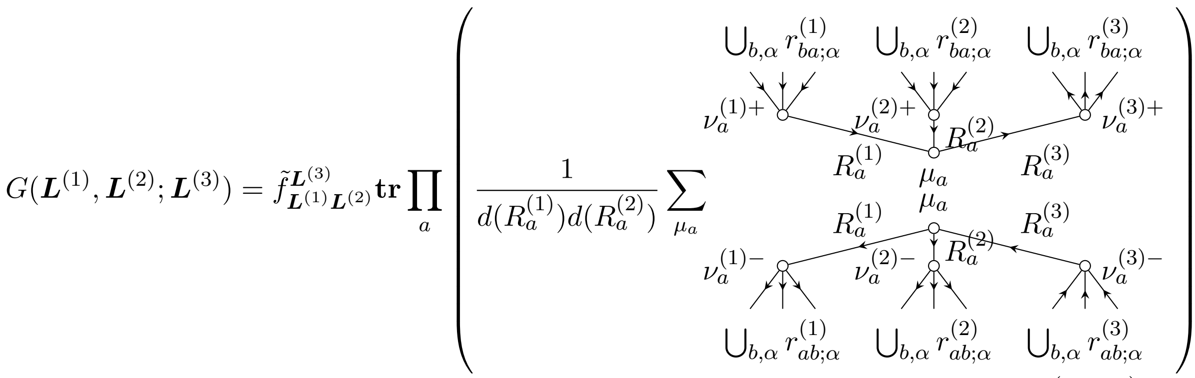
The trace $\mathbf{tr}$ refers to the contraction of the indices associated with the $\cup_{a,b} r_{ba;\alpha}^{(I)}$ at the top of the diagram to those of $\cup_{a,b} r^{(I)}_{ab;\alpha}$ at the bottom. The identification occurs across different group factors, to make up the quivers for the three quiver characters. The diagram, with free upper and lower external legs, corresponds to an expression with indices $\{\cup_{I,a,b,\alpha} i^{(I)}_{ba;\alpha}\}$ for the upper legs and $\{\cup_{I,a,b,\alpha} j^{(I)}_{ab;\alpha}\}$ for the lower legs, each set living in $\otimes_{I,a,b,\alpha} r^{(I)}_{ab;\alpha}$. The trace operation multiplies with $\prod_{I,a,b,\alpha} \delta_{i^{(I)}_{ab;\alpha}, j^{(I)}_{ab;\alpha}}$ and sums over the indices.

Applying (A.19) and (A.5) we have



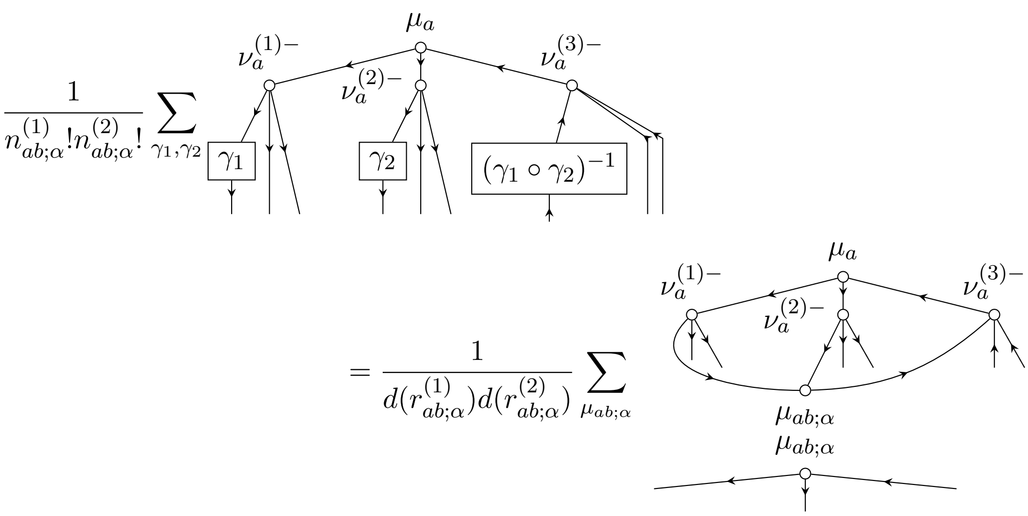
(3.121)

Using this to perform $\sigma^{(1)}_a, \sigma^{(2)}_a$ sums we get



(3.122)

At this point the diagram is still not factorized, because legs are contracted between different factors in $\prod_a$. Next, focus on the lower piece of the diagram, containing ν^- (equivalently we can pick the upper piece – they are symmetric). We can insert the following sum over γ_1, γ_2



(3.123)

On the left hand side γ_1 acts on one of the outgoing legs $r^{(1)}_{ab;\alpha}$ (for some choice of b, α), γ_2 acts on $r^{(2)}_{ab;\alpha}$, and $(\gamma_1 \circ \gamma_2)^{-1}$ acts on $r^{(3)}_{ab;\alpha}$. It is equal to the original ν^- factor in (3.123), because we can pull γ's through the branching coefficients and cancel. Next we can sum over all $\gamma_1 \circ \gamma_2 \in S_{n^{(1)}_{ab;\alpha}} \times S_{n^{(2)}_{ab;\alpha}}$, which allows us to apply (3.121) again, resulting in the right hand side. Performing this for each b, α we completely "cap off" the outgoing $r^{(I)}_{ab;\alpha}$ legs, contracting each $r^{(1)}_{ab;\alpha} \otimes r^{(2)}_{ab;\alpha} \to r^{(3)}_{ab;\alpha}$ respectively, and introducing $\{\mu_{ab;\alpha}\}$ sums. The leftover branching coefficient with $\mu_{ab;\alpha}$ (at the bottom of the right hand side) contracts the incoming legs $r^{(I)}_{ba;\alpha}$ of the respective ν^+ diagram in (3.123). Consequently, the diagram completely factorizes, and we get (3.113), with prefactor arising from

$$f^{\boldsymbol{L}^{(3)}}_{\boldsymbol{L}^{(1)}\boldsymbol{L}^{(2)}} = \frac{\tilde{f}^{\boldsymbol{L}^{(3)}}_{\boldsymbol{L}^{(1)}\boldsymbol{L}^{(2)}}}{\prod_a d(R^{(1)}_a) d(R^{(2)}_a) \prod_{a,b,\alpha} d(r^{(1)}_{ab;\alpha}) d(r^{(2)}_{ab;\alpha})} \tag{3.124}$$

The equations corresponding to the diagrammatic manipulations above are given in Appendix G.3.

3.4 Generalized covariant basis

3.4.1 Complete basis

We can define another complete, free orthogonal basis, which is a generalization of (3.68)

$$\boxed{\mathcal{O}_Q(\boldsymbol{K}) = \frac{\sqrt{\prod d(R_a)}}{\prod n_a!} \sum_{\boldsymbol{\sigma}} \chi_Q(\boldsymbol{K}, \boldsymbol{\sigma})\, \mathcal{O}_Q(\boldsymbol{n}, \boldsymbol{\sigma})} \tag{3.125}$$

We refer to it as the *covariant basis*, because the labels $\boldsymbol{K}$ include representations of the global symmetry group $\prod_{a,b} U(M_{ab})$. The basis arises from the possibility to "solve the invariance" as in (3.93) using *covariant* quiver characters:

$$\chi_Q(\boldsymbol{K}, \boldsymbol{\sigma}) = \left(\prod_a D^{R_a}_{i_a j_a}(\sigma_a) B^{R_a \to \cup_b s^-_{ab}, \nu^-_a}_{j_a \to \cup_b l^-_{ab}} B^{R_a \to \cup_b s^+_{ba}, \nu^+_a}_{i_a \to \cup_b l^+_{ba}} \right) \left(\prod_{a,b} B^{\Lambda_{ab} \to [\boldsymbol{n}_{ab}], \beta_{ab}}_{l_{ab}} S^{s^+_{ab}\, s^-_{ab}, \Lambda_{ab} \tau_{ab}}_{l^+_{ab}\, l^-_{ab}, l_{ab}} \right) \tag{3.126}$$

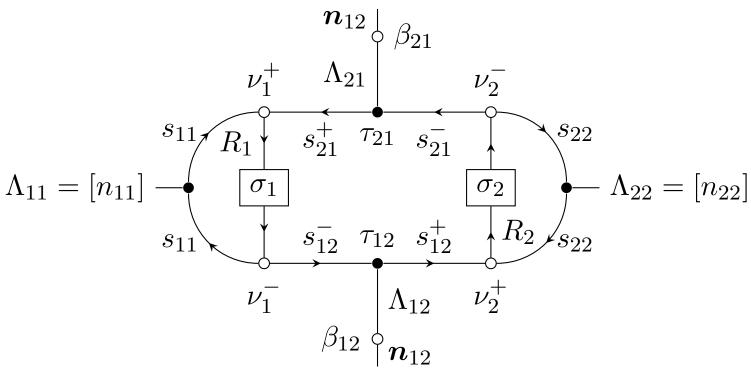
with a different set of labels

$$\boldsymbol{K} = \{R_a, s^+_{ab}, s^-_{ab}, \nu^+_a, \nu^-_a, \Lambda_{ab}, \tau_{ab}, n_{ab;\alpha}, \beta_{ab}\} \tag{3.127}$$

The covariant quiver characters $\chi_Q(\boldsymbol{K}, \boldsymbol{\sigma})$ also obey an analogous set of character orthogonality identities, listed in Appendix B.3. For the details of the derivation of the basis and how the two options $\chi_Q(\boldsymbol{L}, \boldsymbol{\sigma})$ and $\chi_Q(\boldsymbol{K}, \boldsymbol{\sigma})$ arise see Appendix C.

The covariant quiver characters are again most neatly expressed diagrammatically, as a modification of the original quiver. For $\mathbb{C}^3/\mathbb{Z}_2$ (3.126) becomes

$$\chi_{\mathbb{C}^3/\mathbb{Z}_2}(\boldsymbol{K},\boldsymbol{\sigma}) = \tag{3.128}$$



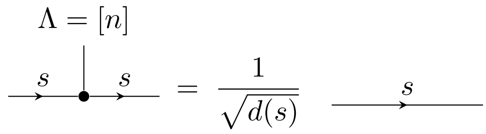
The labels involved are:

- $R_a \vdash n_a$ diagram associated to each group node factor is the same as before, with finite N cutoff $l(R_a) \leq N_a$.
- Each set of M_{ab} arrows between given pair of nodes is collapsed into one, and there is an associated diagram $\Lambda_{ab} \vdash n_{ab}$, where $n_{ab} = \sum_\alpha n_{ab;\alpha}$. It labels a representation of the global symmetry $U(M_{ab})$, and so $l(\Lambda_{ab}) \leq M_{ab}$. Since in $\mathbb{C}^3/\mathbb{Z}_2$ we have $M_{11} = M_{22} = 1$, the associated $\Lambda_{11}, \Lambda_{22}$ are fixed to be single-row diagrams, one-dimensional irreps.
- There are two additional diagrams $s^\pm_{ab} \vdash n_{ab}$ associated to each line. In case $M_{ab} = 1$ they are equal $s^+_{ab} = s^-_{ab}$ and the same as r_{ab} in the restricted basis.
- As in the restricted basis, we have branching at the white nodes $R_a \to \cup_b s^+_{ba}$ and $R_a \to \cup_b s^-_{ab}$ and the associated Littlewood-Richardson multiplicity labels $\nu^\pm_a$.
- There is a black node on each field line denoting Kronecker product $s^+_{ab} \otimes s^-_{ab} \to \Lambda_{ab}$ and the associated Clebsch-Gordan multiplicity label τ_{ab}.
- The extra labels β_{ab}, together with charges $\boldsymbol{n}_{ab} \equiv \{n_{ab;\alpha}\}$, identify a state in $U(M_{ab})$ irrep Λ_{ab}. That is equivalent to specifying a branching multiplicity label for $\Lambda_{ab} \to \cup_\alpha [n_{ab;\alpha}]$ reduction (see e.g. [86] for this fact).

Let us also note, that in the case of the trivial $\Lambda_{11}, \Lambda_{22}$ the corresponding Clebsch-Gordan coefficient still has to be included in (3.126)

$$S^{s^+ s^-,\ \Lambda=[n]}_{i\ j\ \ ,\ 1} = \delta_{s^+ s^-} \frac{\delta_{ij}}{\sqrt{d(s^+)}} \tag{3.129}$$

It forces $s^+ = s^-$, and is itself proportional to a delta function, but it includes the coefficient $\frac{1}{\sqrt{d(s)}}$. Diagrammatically



(3.130)

The key property of this basis is that the transformations under global symmetry group $\prod_{a,b} U(M_{ab})$ are made explicit

- $\{\Lambda_{ab}\}$ labels pick the representation of $\prod_{a,b} U(M_{ab})$
- $\{R_a, s^+_{ab}, s^-_{ab}, \nu^+_a, \nu^-_a, \tau_{ab}\}$ then distinguish different multiplets transforming under $\{\Lambda_{ab}\}$
- $\{\boldsymbol{n}_{ab}, \beta_{ab}\}$ label a state in $\{\Lambda_{ab}\}$.

The free two-point function in the covariant basis can be calculated in analogous way as in the previous section, now using the properties of covariant characters in Appendix B.3. With our normalization the result is exactly the same as (3.99):

$$\left\langle \mathcal{O}_Q(\boldsymbol{K}) \mathcal{O}_Q(\tilde{\boldsymbol{K}})^\dagger \right\rangle = \delta_{\boldsymbol{K}\tilde{\boldsymbol{K}}} \frac{\prod n_{ab;\alpha}!}{\prod n_a!} \prod_a f_{N_a}(R_a) \tag{3.131}$$

Finally, the inverse basis transformation is:

$$\mathcal{O}_Q(\boldsymbol{n}, \boldsymbol{\sigma}) = \sum_{\boldsymbol{K}} \sqrt{\prod d(R_a)}\, \chi_Q(\boldsymbol{K}, \boldsymbol{\sigma})\, \mathcal{O}_Q(\boldsymbol{K}) \tag{3.132}$$

3.4.2 Chiral ring structure constants

Here we calculate the chiral ring structure constants for the covariant basis (3.125) operators $\mathcal{O}_Q(\boldsymbol{K})$. As in the previous section, the product is

$$\mathcal{O}_Q(\boldsymbol{K}^{(1)}) \mathcal{O}_Q(\boldsymbol{K}^{(2)}) = \sum_{\boldsymbol{K}^{(3)}} G(\boldsymbol{K}^{(1)}, \boldsymbol{K}^{(2)}; \boldsymbol{K}^{(3)}) \mathcal{O}_Q(\boldsymbol{K}^{(3)}) \tag{3.133}$$

with the structure constants

$$\begin{aligned} &G(\boldsymbol{K}^{(1)}, \boldsymbol{K}^{(2)}; \boldsymbol{K}^{(3)}) \\ &\quad = \frac{1}{\prod n_a^{(1)}!} \frac{1}{\prod n_a^{(2)}!} \sum_{\boldsymbol{\sigma}^{(1)}} \sum_{\boldsymbol{\sigma}^{(2)}} \hat{\chi}_Q(\boldsymbol{K}^{(1)}, \boldsymbol{\sigma}^{(1)}) \hat{\chi}_Q(\boldsymbol{K}^{(2)}, \boldsymbol{\sigma}^{(2)}) \hat{\chi}_Q(\boldsymbol{K}^{(3)}, \boldsymbol{\sigma}^{(1)} \circ \boldsymbol{\sigma}^{(2)}) \end{aligned} \tag{3.134}$$

Here we use conveniently normalized covariant quiver characters

$$\hat{\chi}_Q(\boldsymbol{K}) \equiv \sqrt{\prod d(R_a)} \chi_Q(\boldsymbol{K}). \tag{3.135}$$

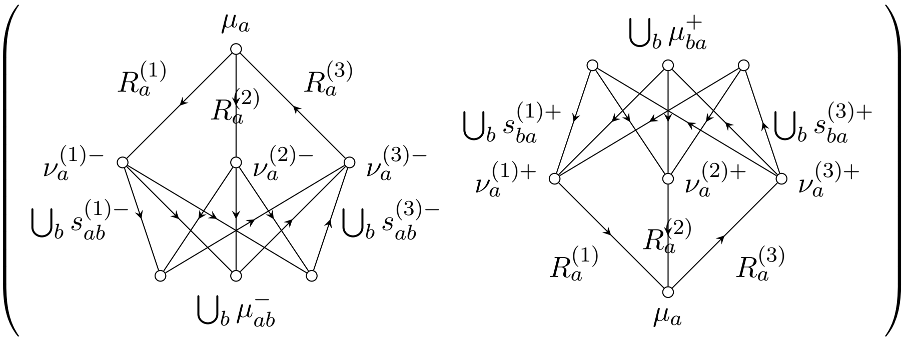
Let us first present the answer and some examples, and sketch the derivation afterwards. Recall from the definition (3.126) of the covariant quiver characters, that the labels are $\boldsymbol{K} = \{R_a, s^+_{ab}, s^-_{ab}, \nu^+_a, \nu^-_a, \Lambda_{ab}, \tau_{ab}, n_{ab;\alpha}, \beta_{ab}\}$, as displayed in (3.128). The result of the sum (3.134) is, like in the previous section, that *all* of the Young diagram labels multiply according to the Littlewood-Richardson rule

$$\boxed{\begin{array}{c} R_a^{(1)} \otimes R_a^{(2)} \to R_a^{(3)} \\ \Lambda_{ab}^{(1)} \otimes \Lambda_{ab}^{(2)} \to \Lambda_{ab}^{(3)} \\ s_{ab}^{(1)+} \otimes s_{ab}^{(2)+} \to s_{ab}^{(3)+} \\ s_{ab}^{(1)-} \otimes s_{ab}^{(2)-} \to s_{ab}^{(3)-} \end{array}} \tag{3.136}$$

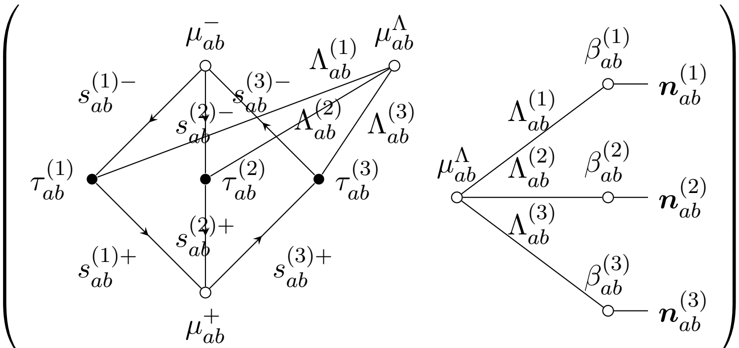
That is, $G(\boldsymbol{K}^{(1)}, \boldsymbol{K}^{(2)}; \boldsymbol{K}^{(3)})$ vanishes unless the labels from $\boldsymbol{K}^{(3)}$ are contained in the Littlewood-Richardson tensor product (also called outer product) of the Young diagrams. The non-vanishing coefficients are given, similarly as in (3.113), by connecting up all coupled legs via branching coefficients, and summing over the multiplicities for the new branchings. Specifically, we get:

$$G(\boldsymbol{K}^{(1)}, \boldsymbol{K}^{(2)}; \boldsymbol{K}^{(3)}) = f^{\boldsymbol{K}^{(3)}}_{\boldsymbol{K}^{(1)}\boldsymbol{K}^{(2)}} \sum_{\{\mu_a\}} \sum_{\{\mu^+_{ab}\}} \sum_{\{\mu^-_{ab}\}} \sum_{\{\mu^\Lambda_{ab}\}}$$

$$\prod_a \left(\quad \right)$$



$$\prod_{a,b} \left(\quad \right) \tag{3.137}$$



with

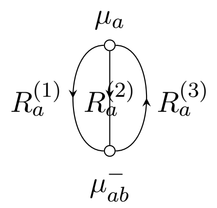
$$f^{\boldsymbol{K}^{(3)}}_{\boldsymbol{K}^{(1)}\boldsymbol{K}^{(2)}} = \frac{\sqrt{\prod_a d(R_a^{(1)}) d(R_a^{(2)}) d(R_a^{(3)})}}{\prod_a d(R_a^{(1)}) d(R_a^{(2)}) \prod_{a,b} d(s_{ab}^{(1)-}) d(s_{ab}^{(2)-}) d(s_{ab}^{(1)+}) d(s_{ab}^{(2)+}) d(\Lambda_{ab}^{(1)}) d(\Lambda_{ab}^{(2)})} \tag{3.138}$$

As for the restricted Schur basis, we get two factors of $\mathcal{F}$ defined in (3.116) for each group node, now $s^{\pm}_{ab}$ playing the role of $r_{ab;\alpha}$. In addition to that, for each edge in the quiver we get a factor coupling $\Lambda^{(1)}_{ab} \otimes \Lambda^{(2)}_{ab} \to \Lambda^{(3)}_{ab}$. Again for illustration we use three outgoing arrows from each branching node ν^-_a and three incoming arrows to each ν^+_a. The explicit expression is:

$$
\begin{aligned}
&G(\boldsymbol{K}^{(1)}, \boldsymbol{K}^{(2)}; \boldsymbol{K}^{(3)}) = f^{\boldsymbol{K}^{(3)}}_{\boldsymbol{K}^{(1)}\boldsymbol{K}^{(2)}} \sum_{\{\mu_a\}} \sum_{\{\mu^+_{ab}\}} \sum_{\{\mu^-_{ab}\}} \sum_{\{\mu^\Lambda_{ab}\}} \\
&\prod_a \mathcal{F}\left(\cup_I R^{(I)}_a, \{\cup_{I,b} s^{(I)-}_{ab}\}, \cup_I \nu^{(I)-}_a; \mu_a, \{\cup_b \mu^-_{ab}\}\right) \mathcal{F}\left(\cup_I R^{(I)}_a, \{\cup_{I,b} s^{(I)+}_{ba}\}, \cup_I \nu^{(I)+}_a; \mu_a, \{\cup_b \mu^+_{ba}\}\right) \\
&\prod_{a,b} \left(S^{s^{(1)+}_{ab} s^{(1)-}_{ab}, \Lambda^{(1)}_{ab} \tau^{(1)}_{ab}}_{l^{(1)+}_{ab} l^{(1)-}_{ab}, l^{(1)}_{ab}} S^{s^{(2)+}_{ab} s^{(2)-}_{ab}, \Lambda^{(2)}_{ab} \tau^{(2)}_{ab}}_{l^{(2)+}_{ab} l^{(2)-}_{ab}, l^{(2)}_{ab}} S^{s^{(3)+}_{ab} s^{(3)-}_{ab}, \Lambda^{(3)}_{ab} \tau^{(3)}_{ab}}_{l^{(3)+}_{ab} l^{(3)-}_{ab}, l^{(3)}_{ab}} \right. \\
&\qquad\qquad \left. \times B^{s^{(3)-}_{ab} \to s^{(1)-}_{ab}, s^{(2)-}_{ab}; \mu^-_{ab}}_{l^{(3)-}_{ab} \to l^{(1)-}_{ab}, l^{(2)-}_{ab}} B^{s^{(3)+}_{ab} \to s^{(1)+}_{ab}, s^{(2)+}_{ab}; \mu^+_{ab}}_{l^{(3)+}_{ab} \to l^{(1)+}_{ab}, l^{(2)+}_{ab}} B^{\Lambda^{(3)}_{ab} \to \Lambda^{(1)}_{ab}, \Lambda^{(2)}_{ab}; \mu^\Lambda_{ab}}_{l^{(3)}_{ab} \to l^{(1)}_{ab}, l^{(2)}_{ab}} \right) \\
&\times \left(B^{\Lambda^{(1)}_{ab} \to [\boldsymbol{n}^{(1)}_{ab}], \beta^{(1)}_{ab}}_{k^{(1)}_{ab}} B^{\Lambda^{(2)}_{ab} \to [\boldsymbol{n}^{(2)}_{ab}], \beta^{(2)}_{ab}}_{k^{(2)}_{ab}} B^{\Lambda^{(3)}_{ab} \to [\boldsymbol{n}^{(3)}_{ab}], \beta^{(3)}_{ab}}_{k^{(3)}_{ab}} B^{\Lambda^{(3)}_{ab} \to \Lambda^{(1)}_{ab}, \Lambda^{(2)}_{ab}; \mu^\Lambda_{ab}}_{k^{(3)}_{ab} \to k^{(1)}_{ab}, k^{(2)}_{ab}} \right)
\end{aligned}
\tag{3.139}
$$

In its most general form $G(\boldsymbol{K}^{(1)}, \boldsymbol{K}^{(2)}; \boldsymbol{K}^{(3)})$ looks more complicated than $G(\boldsymbol{L}^{(1)}, \boldsymbol{L}^{(2)}; \boldsymbol{L}^{(3)})$, because it has to deal with both $s^{\pm}_{ab}$ and Λ_{ab}. However, for linear quivers like $\mathbb{C}^3$ (3.73), conifold (3.161), dP_0 (3.177) it simplifies significantly, because $s^+_{ba} = R_a = s^-_{ab}$, so there are no $s^{\pm}_{ab}$ or $\nu^{\pm}_a$ labels at all. In that case the $\mathcal{F}$ factors reduce to

$$\mathcal{F}\left(R^{(I)}_a, R^{(I)}_a, \nu^{(I)-}_a = 1; \mu_a, \mu^-_{ab}\right) =$$

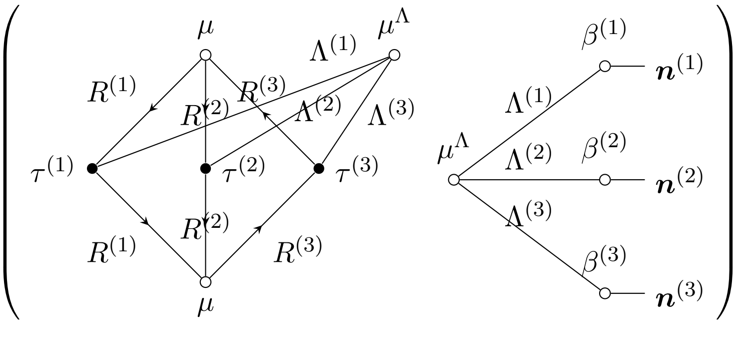


$$= \delta_{\mu_a \mu^-_{ab}}\, d(R^{(1)}_a)\, d(R^{(2)}_a) \tag{3.140}$$

using (A.17). Thus, for example, we can write the chiral ring structure constants for $\mathbb{C}^3$ as just the term for the single edge in the quiver

$$G_{\mathbb{C}^3}(\boldsymbol{K}^{(1)}, \boldsymbol{K}^{(2)}; \boldsymbol{K}^{(3)}) = \frac{\sqrt{d(R^{(1)})d(R^{(2)})d(R^{(3)})}}{d(R^{(1)})d(R^{(2)})d(\Lambda^{(1)})d(\Lambda^{(2)})}$$

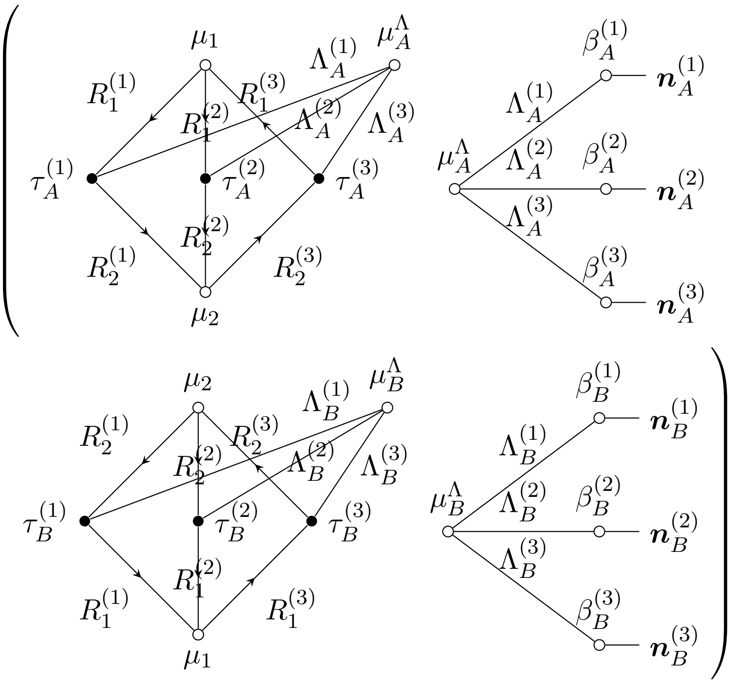
$$\times \sum_{\mu} \sum_{\mu^\Lambda} \left(\cdots \right) \tag{3.141}$$



A diagrammatic form of the fusion coefficient for the $\mathbb{C}^3$ case, manifestly exhibiting the $R^{(1)} \otimes R^{(2)} \to R^{(3)}$ LR-selection rule was given in [47]. For the conifold we have a product of two terms, one for each edge, using the labelling (3.162) $\boldsymbol{K} = \{R_1, R_2, \Lambda_A, \Lambda_B, \tau_A, \tau_B, \boldsymbol{n}, \beta_A, \beta_B\}$:

$$G_{\mathcal{C}}(\boldsymbol{K}^{(1)}, \boldsymbol{K}^{(2)}; \boldsymbol{K}^{(3)}) = \frac{\sqrt{d(R_1^{(1)})d(R_1^{(2)})d(R_1^{(3)})d(R_2^{(1)})d(R_2^{(2)})d(R_3^{(3)})}}{d(R_1^{(1)})d(R_1^{(2)})d(R_2^{(1)})d(R_2^{(2)})d(\Lambda_A^{(1)})d(\Lambda_A^{(2)})d(\Lambda_B^{(1)})d(\Lambda_B^{(2)})}$$

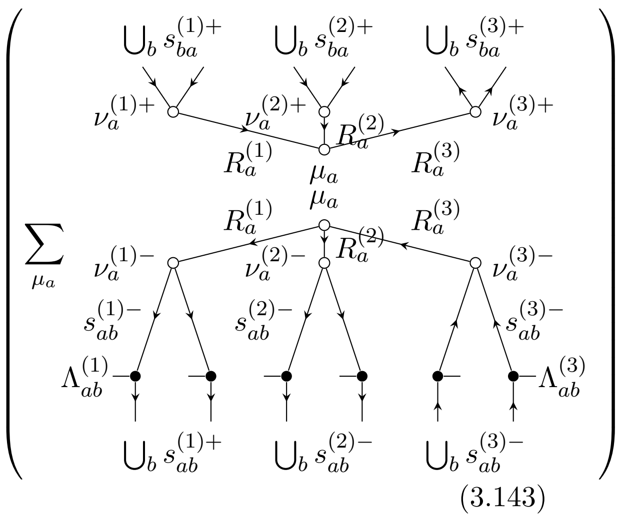
$$\times \sum_{\mu_1\mu_2} \sum_{\mu_A^\Lambda \mu_B^\Lambda}$$



(3.142)

The derivation of (3.137) parallels that of the last section, except in addition we have to deal with Clebsch-Gordan coefficient (black) nodes and Λ_{ab}. The sum over $\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)}$ in (3.134) is performed the same way as in (3.119) and we get analogously to (3.122):

$$G(\boldsymbol{K}^{(1)}, \boldsymbol{K}^{(2)}; \boldsymbol{K}^{(3)}) = \frac{\tilde{f}^{\boldsymbol{K}^{(3)}}_{\boldsymbol{K}^{(1)}\boldsymbol{K}^{(2)}}}{\prod d(R_a^{(1)})d(R_a^{(2)})} \mathbf{tr} \prod_a \sum_{\mu_a}$$



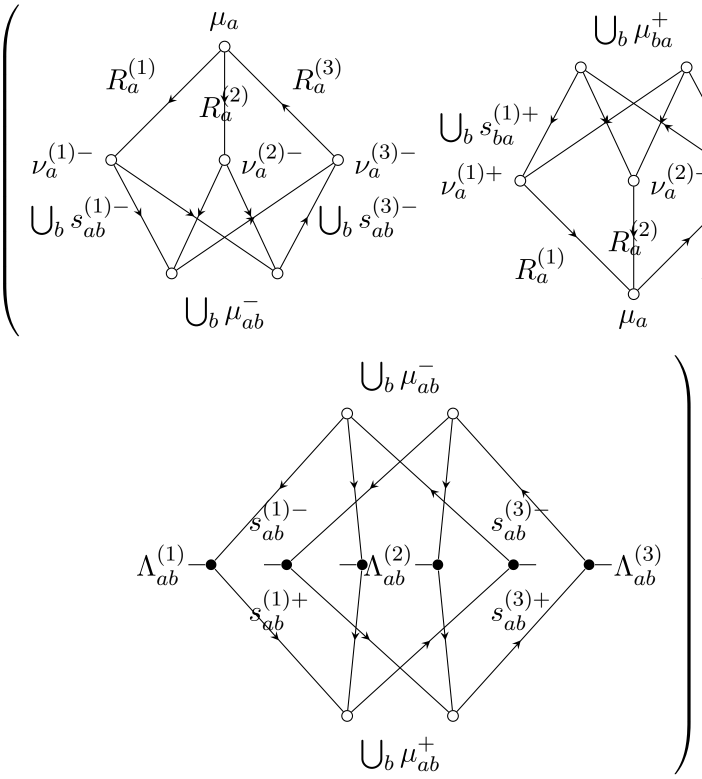
(3.143)

As before, the trace-operation identifies and sums the corresponding indices from $\cup_{a,b} s_{ba}^{(I)}$ at the top of the diagram to the indices from $\cup_{a,b} s_{ab}^{(I)}$. Now we have extra Clebsch-Gordan nodes between $s_{ab}^{(I)-}$ and $s_{ab}^{(I)+}$. Note the outgoing lines next to $\Lambda_{ab}^{(I)}$ are a shorthand for the whole collection of labels $(\tau_{ab}^{(I)}, \Lambda_{ab}^{(I)}, \beta_{ab}^{(I)}, \boldsymbol{n}_{ab}^{(I)})$ like in (3.128), including the β_{ab} white branching coefficient node.

In order to factorize this diagram we apply (3.123) *twice*: both on $s_{ab}^{(I)-}$ legs below $\nu_a^{(I)-}$ nodes, and on $s_{ab}^{(I)+}$ legs above $\nu_a^{(I)+}$. This introduces two sums over new branching coefficients μ_{ab}^+, μ_{ab}^- (compared to just one in the last section) and splits the diagram into *three* parts:

$$G(\boldsymbol{K}^{(1)}, \boldsymbol{K}^{(2)}; \boldsymbol{K}^{(3)}) = \frac{\tilde{f}^{\boldsymbol{K}^{(3)}}_{\boldsymbol{K}^{(1)}\boldsymbol{K}^{(2)}}}{\prod_a d(R_a^{(1)})d(R_a^{(2)}) \prod_{a,b} d(s_{ab}^{(1)-})d(s_{ab}^{(2)-})d(s_{ab}^{(1)+})d(s_{ab}^{(2)+})}$$

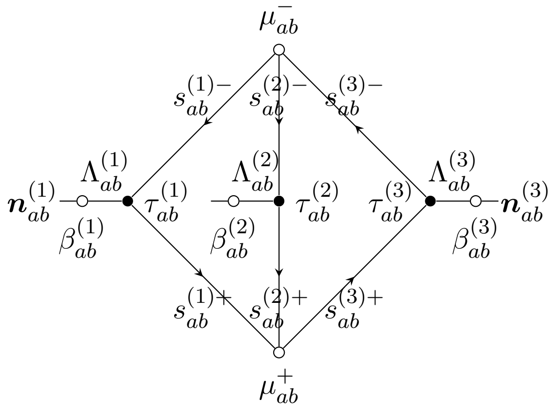
$$\times \sum_{\{\mu_a\}} \sum_{\{\mu_{ab}^+\}} \sum_{\{\mu_{ab}^-\}} \prod_a$$



(3.144)

The diagram involving $\Lambda_{ab}^{(I)}$ factorizes into a piece for each b, so we have (now including β_{ab} nodes):

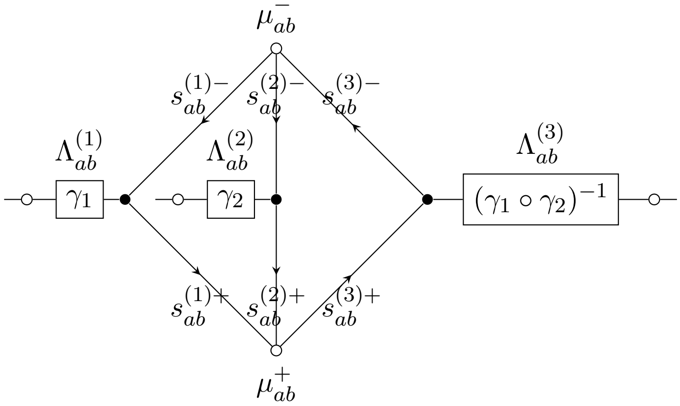
$$\prod_{a,b}$$



(3.145)

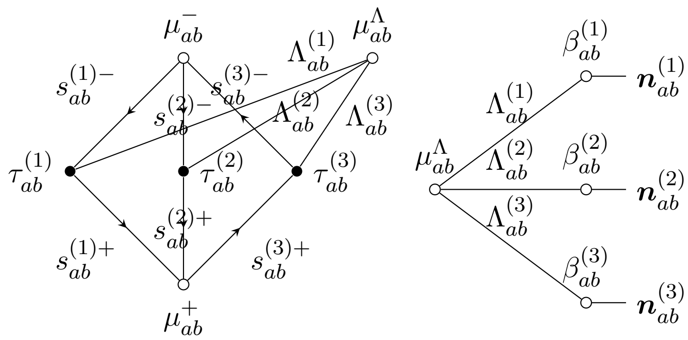
Finally, we couple $\Lambda^{(1)}_{ab} \otimes \Lambda^{(2)}_{ab} \to \Lambda^{(3)}_{ab}$ by inserting the following sum

$$\frac{1}{n^{(1)}_{ab}!n^{(2)}_{ab}!}\sum_{\gamma_1,\gamma_2}$$



(3.146)

$$= \frac{1}{d(\Lambda^{(1)}_{ab})d(\Lambda^{(2)}_{ab})}\sum_{\mu^\Lambda_{ab}}$$



The diagram on the left hand side with inserted γ_1, γ_2 is equal to (3.145), due to the property of Clebsch-Gordan coefficients (A.28), which allows to pull γ's through, and then cancel via μ^-_{ab} and μ^+_{ab} branching coefficients using (A.16). Then applying (3.121) again we get the right hand side. Plugging this in (3.144) gives the final answer (3.137).

3.5 Examples

Let us go over a few specific examples of quivers, to illustrate our general methods.

3.5.1 Conifold

The quiver for the Klebanov-Witten theory [72] describing D3-branes on a conifold singularity is shown in Figure 3.9. At non-zero coupling it is dual to $AdS_5 \times T^{1,1}$ where cone over $T^{1,1}$ is the conifold. The gauge group is $U(N_1) \times U(N_2)$ and we have bifundamental fields

$$A_1, A_2, B_1, B_2 \tag{3.147}$$

transforming in a global $U(2)\times U(2)$ flavor symmetry. Note according to the labelling in the previous section the fields correspond to $A_1 = \Phi_{12;1}, A_2 = \Phi_{12;2}, B_1 = \Phi_{21;1}, B_2 = \Phi_{21;2}$.

The gauge invariant mesonic operators are traces of alternating products $\mathrm{tr}(A_{i_1}B_{j_1}A_{i_2}B_{j_2}\ldots)$. According to the general prescription (3.79), a general gauge invariant operator can be

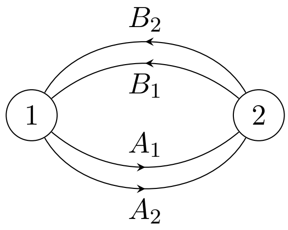


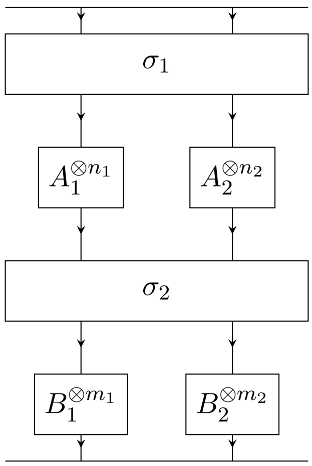
Figure 3.9: Quiver for the conifold theory.

specified by charges and two permutations

$$\mathcal{O}_{\mathcal{C}}(\boldsymbol{n},\{\sigma_1,\sigma_2\}) = \mathrm{tr}_{V_N^{\otimes n}}\left(\sigma_1(A_1^{\otimes n_1}\otimes A_2^{\otimes n_2})\sigma_2(B_1^{\otimes m_1}\otimes B_2^{\otimes m_2})\right) \tag{3.148}$$

or diagrammatically

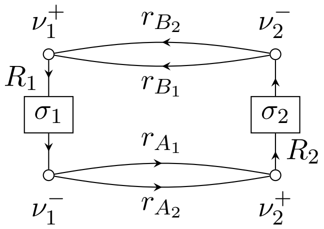
$$\mathcal{O}_{\mathcal{C}}(\boldsymbol{n},\{\sigma_1,\sigma_2\}) = \tag{3.149}$$



Here we denote $n = n_1 + n_2 = m_1 + m_2$ the total number of A's or B's, which has to be equal.

The counting is given by the split-node quiver, which was shown in Figure 3.3 and (3.14). Now the restricted quiver characters obtained by inserting (σ_1, σ_2) in the the same split-node quiver are

$$\chi_{\mathcal{C}}(\boldsymbol{L},\{\sigma_1,\sigma_2\}) = \tag{3.150}$$



leading to the restricted Schur basis operators (3.96):

$$\mathcal{O}_{\mathcal{C}}(\boldsymbol{L}) = \frac{1}{(n!)^2}\sqrt{\frac{d(R_1)d(R_2)}{d(r_{A_1})d(r_{A_2})d(r_{B_1})d(r_{B_2})}}\sum_{\sigma_1,\sigma_2}\chi_{\mathcal{C}}(\boldsymbol{L},\{\sigma_1,\sigma_2\})\,\mathcal{O}_{\mathcal{C}}(\boldsymbol{n},\{\sigma_1,\sigma_2\}) \tag{3.151}$$

The labels are

$$\boldsymbol{L} = \{R_1, R_2, r_{A_1}, r_{A_2}, r_{B_1}, r_{B_2}, \nu_1^{\pm}, \nu_2^{\pm}\} \tag{3.152}$$

where $R_1, R_2 \vdash n$ are Young diagrams associated with each of the group factors, limited to at most N_1, N_2 rows, $r_{A_1}, r_{A_2}, r_{B_1}, r_{B_2}$ are Young diagrams associated with each field type. They are constrained such that R_1, R_2 appear in the Littlewood-Richardson products

$$\begin{aligned} r_{A_1} \otimes r_{A_2} &\to R_1 \\ r_{A_1} \otimes r_{A_2} &\to R_2 \\ r_{B_1} \otimes r_{B_2} &\to R_1 \\ r_{B_1} \otimes r_{B_2} &\to R_2 \end{aligned} \tag{3.153}$$

and $\nu_1^{\pm}$, $\nu_2^{\pm}$ are the associated multiplicity labels, when R_1, R_2 appears more than once in the product.

In this case, as in (3.62) for $\mathbb{C}^3$, we can write the restricted quiver character $\chi_{\mathcal{C}}(\boldsymbol{L}, \boldsymbol{\sigma})$ as a sort of restricted trace. Define a projector

$$(P^{\nu^-,\nu^+}_{R\to\boldsymbol{r}\leftarrow S})_{ij} = \sum_l B^{R\to\boldsymbol{r}}_{i\to l} B^{S\to\boldsymbol{r}}_{j\to l} \tag{3.154}$$

which projects from two different representations R, S of S_n into the same irrep $\boldsymbol{r} = (r_1, r_2)$ of the subgroup $S_{n_1} \times S_{n_2}$. Then we can write the quiver character as

$$\chi_{\mathcal{C}}(\boldsymbol{L}, \{\sigma_1, \sigma_2\}) = \mathrm{tr}\left(D^{R_1}(\sigma_1) P^{\nu_1^-,\nu_2^+}_{R_1\to\boldsymbol{r}_A\leftarrow R_2} D^{R_2}(\sigma_2) P^{\nu_2^-,\nu_1^+}_{R_2\to\boldsymbol{r}_B\leftarrow R_2} \right) \tag{3.155}$$

The Restricted Schur basis operators are, explicitly:

$$\mathcal{O}_{\mathcal{C}}(\boldsymbol{L}) = c_{\boldsymbol{L}} \sum_{\sigma_1,\sigma_2} \mathrm{tr}\left(D^{R_1}(\sigma_1) P^{\nu_1^-,\nu_2^+}_{R_1\to\boldsymbol{r}_A\leftarrow R_2} D^{R_2}(\sigma_2) P^{\nu_2^-,\nu_1^+}_{R_2\to\boldsymbol{r}_B\leftarrow R_2} \right) \mathcal{O}_{\mathcal{C}}(\boldsymbol{n}, \{\sigma_1, \sigma_2\}) \tag{3.156}$$

Let us demonstrate the simplest example, with the charges $\boldsymbol{n} = \{1, 1, 1, 1\}$, that is, each field occurs once. The only choice for r diagrams is

$$r_{A_1} = r_{A_2} = r_{B_1} = r_{B_2} = \square \tag{3.157}$$

Littlewood-Richardson product is

$$\square \otimes \square \to \square\square \oplus \begin{smallmatrix}\square\\\square\end{smallmatrix} \tag{3.158}$$

each diagram appearing once, so there is no multiplicity. We can choose each R_1, R_2

independently to be either of the diagrams, giving 4 operators

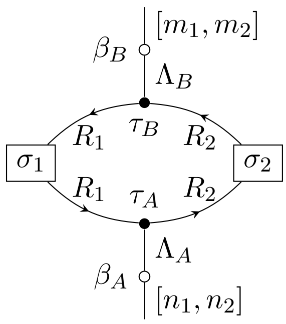
$$
\begin{aligned}
&\mathcal{O}(\yng(2),\yng(2))\\
&= \frac{1}{4}\left(\mathrm{tr}(A_1B_1)\mathrm{tr}(A_2B_2) + \mathrm{tr}(A_1B_2)\mathrm{tr}(A_2B_1) + \mathrm{tr}(A_1B_1A_2B_2) + \mathrm{tr}(A_1B_2A_2B_1)\right)\\
&\mathcal{O}(\yng(1,1),\yng(1,1))\\
&= \frac{1}{4}\left(\mathrm{tr}(A_1B_1)\mathrm{tr}(A_2B_2) + \mathrm{tr}(A_1B_2)\mathrm{tr}(A_2B_1) - \mathrm{tr}(A_1B_1A_2B_2) - \mathrm{tr}(A_1B_2A_2B_1)\right)\\
&\mathcal{O}(\yng(2),\yng(1,1))\\
&= \frac{1}{4}\left(\mathrm{tr}(A_1B_1)\mathrm{tr}(A_2B_2) - \mathrm{tr}(A_1B_2)\mathrm{tr}(A_2B_1) + \mathrm{tr}(A_1B_1A_2B_2) - \mathrm{tr}(A_1B_2A_2B_1)\right)\\
&\mathcal{O}(\yng(1,1),\yng(2))\\
&= \frac{1}{4}\left(\mathrm{tr}(A_1B_1)\mathrm{tr}(A_2B_2) - \mathrm{tr}(A_1B_2)\mathrm{tr}(A_2B_1) - \mathrm{tr}(A_1B_1A_2B_2) + \mathrm{tr}(A_1B_2A_2B_1)\right)
\end{aligned}
\tag{3.159}
$$

It can be checked that they are orthogonal in the free field metric. These operators are particularly easy to evaluate, since all the representations are one-dimensional, and so all branching coefficients are equal to 1. The only dependence comes from $D^{R_1}(\sigma_1)$, $D^{R_2}(\sigma_2)$. Note also the way this basis captures finite-N cutoff: if $N = 1$ the height of R_1, R_2 is limited to 1, so the only operator that survives is $\mathcal{O}(\yng(2),\yng(2))$. It is easy to see that the others are 0 if the fields are replaced by scalar values.

Covariant basis operators (3.125) for conifold are

$$
\mathcal{O}_{\mathcal{C}}(\boldsymbol{K}) = \frac{\sqrt{d(R_1)d(R_2)}}{(n!)^2} \sum_{\sigma_1,\sigma_2} \chi_{\mathcal{C}}(\boldsymbol{K},\{\sigma_1,\sigma_2\})\mathcal{O}_{\mathcal{C}}(\boldsymbol{n},\{\sigma_1,\sigma_2\}) \tag{3.160}
$$

$$
\chi_{\mathcal{C}}(\boldsymbol{K},\{\sigma_1,\sigma_2\}) = \tag{3.161}
$$



with the labels

$$
\boldsymbol{K} = \{R_1, R_2, \Lambda_A, \Lambda_B, \tau_A, \tau_B, \boldsymbol{n}, \beta_A, \beta_B\} \tag{3.162}
$$

The $R_1, R_2 \vdash n$ are Young diagrams associated to the group nodes like before. But now, instead of r_{A_i}, r_{B_i} we have global symmetry representation labels $\Lambda_A, \Lambda_B \vdash n$. They are constrained to appear in the irrep decomposition of the S_n Kronecker product

$$\begin{aligned} R_1 \otimes R_2 &\to \Lambda_A \\ R_1 \otimes R_2 &\to \Lambda_B \end{aligned} \tag{3.163}$$

If Λ_A, Λ_B appear multiple times in the decomposition, τ_A, τ_B is the multiplicity label. The remaining labels $\{n_A, n_B, \beta_A, \beta_B\}$ then label a specific state in the $U(M) \times U(M)$ irrep (Λ_A, Λ_B). Note, compared to the general case (3.127), we do not need additional labels $s_A^\pm, s_B^\pm, \nu_1^\pm, \nu_2^\pm$. This is because there is no "branching" in the quiver – all arrows outgoing from node 1 go to node 2 and vice-versa, which enforces $R_1 = s_A^- = s_B^+$ and $R_2 = s_A^+ = s_B^-$.

Let us again work out the example with $n = 2$, that is 2 A fields and 2 B fields. Like with Restricted Schur basis, we have 4 choices for R_1, R_2. In this simple case Λ_A, Λ_B are uniquely determined by the choice of R_1, R_2, since

$$\begin{aligned} \yng(2) \otimes \yng(2) &\to \yng(2) \\ \yng(2) \otimes \yng(1,1) &\to \yng(1,1) \\ \yng(1,1) \otimes \yng(1,1) &\to \yng(2) \end{aligned} \tag{3.164}$$

that is, only one irrep appears in the product, so $\Lambda_A = \Lambda_B = R_1 \otimes R_2$. For each choice of $R_1, R_2, \Lambda_A, \Lambda_B$ we list the highest-weight state in (Λ_A, Λ_B):

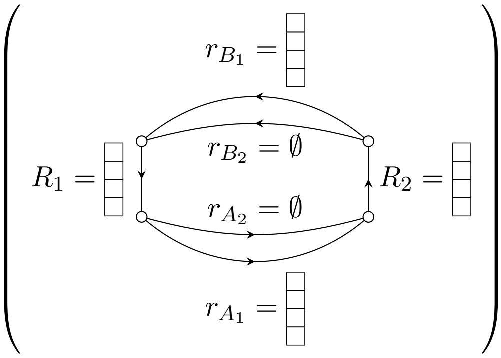
$$\begin{aligned}
&\mathcal{O}^{\rm hw}(R_1 = \yng(2), R_2 = \yng(2), \Lambda_A = \Lambda_B = \yng(2)) = \frac{1}{2}{\rm tr}(A_1B_1){\rm tr}(A_1B_1) + \frac{1}{2}{\rm tr}(A_1B_1A_1B_1) \\
&\mathcal{O}^{\rm hw}(R_1 = \yng(1,1), R_2 = \yng(1,1), \Lambda_A = \Lambda_B = \yng(2)) = \frac{1}{2}{\rm tr}(A_1B_1){\rm tr}(A_1B_1) - \frac{1}{2}{\rm tr}(A_1B_1A_1B_1) \\
&\mathcal{O}^{\rm hw}(R_1 = \yng(2), R_2 = \yng(1,1), \Lambda_A = \Lambda_B = \yng(1,1)) \\
&\quad = \frac{1}{4}\left({\rm tr}(A_1B_1){\rm tr}(A_2B_2) - {\rm tr}(A_1B_2){\rm tr}(A_2B_1) + {\rm tr}(A_1B_1A_2B_2) - {\rm tr}(A_1B_2A_2B_1)\right) \\
&\mathcal{O}^{\rm hw}(R_1 = \yng(1,1), R_2 = \yng(2), \Lambda_A = \Lambda_B = \yng(1,1)) \\
&\quad = \frac{1}{4}\left({\rm tr}(A_1B_1){\rm tr}(A_2B_2) - {\rm tr}(A_1B_2){\rm tr}(A_2B_1) - {\rm tr}(A_1B_1A_2B_2) + {\rm tr}(A_1B_2A_2B_1)\right)
\end{aligned} \tag{3.165}$$

3.5.2 Giant gravitons in the conifold

It is interesting to consider the operators in our basis, that have been identified with the giant gravitons in $T^{1,1}$ [87].

For two giants wrapping cycles a_1 and b_1, the dual operator is mesonic $\det(A_1B_1)$ and, in fact, can be expressed nicely in our basis

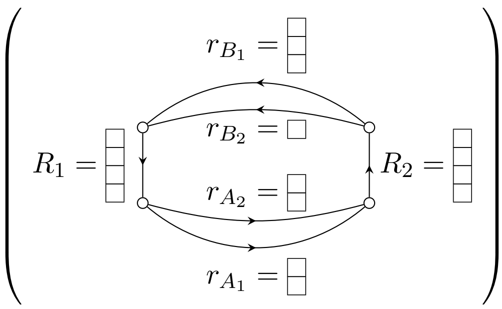
$$\mathcal{O}_{11} = \det(A_1B_1) = \mathcal{O}$$



(3.166)

Here the meaning of the diagram is just a convenient visualization of the labels $\boldsymbol{L}$, while the actual operator is $\mathcal{O}_{\mathcal{C}}(\boldsymbol{L})$ as defined in (3.156). The single-column Young diagrams are understood to have N boxes. Note because of the single-column diagrams there are no multiplicities at the white nodes, which makes them particularly nice examples. Next, if we fix $R_1 = R_2 = [1^N]$, but allow different numbers $(n_{A_1}, n_{A_2}, n_{B_1}, n_{B_2})$ of fields, subject to restriction $n_{A_1} + n_{A_2} = n_{B_1} + n_{B_2} = N$, we get one operator for each choice of charges

$$\mathcal{O}$$



(3.167)

This is because r_{A_i}, r_{B_i} diagrams are forced to be single-column by Littlewood-Richardson rule. In fact, that is precisely what is needed to fill out the $(N+1, N+1)$ representation of $SU(2) \times SU(2)$ of which $\det(A_1B_1)$ is the highest weight state. Using covariant basis we can label the whole representation by $R_1, R_2, \Lambda_A, \Lambda_B$ as in Figure 3.10.

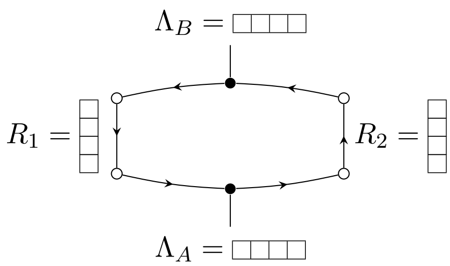


Figure 3.10: Representation containing giant gravitons expanding in $T^{1,1}$

In analogy with half-BPS states in $\mathbb{C}^3$, it is natural to suggest that if single-column operators $R_1 = R_2 = [1^N]$ correspond to giants expanding in the compact $T^{1,1}$, then single-row operators $R_1 = R_2 = [n]$ would be dual giants, expanding in AdS_5 and point-like in $T^{1,1}$. These states live in the $SU(2) \times SU(2)$ representation $(N+1, N+1)$ labelled as in Figure 3.11.

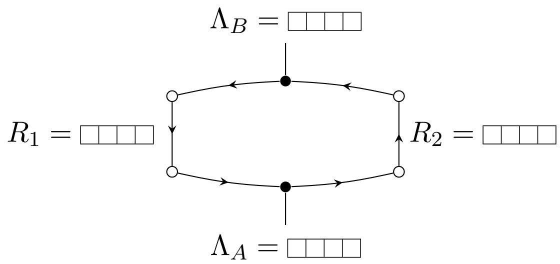


Figure 3.11: Representation containing *AdS* giants

It is important to note that, in principle, in order to match with the D3 brane states on the bulk side, we need to use the BPS operators of the *interacting* theory. The BPS operators are in one-to-one correspondence with the elements of the chiral ring, that is, in each equivalence class of operators modulo F-terms there is one BPS combination. The reason we can rely on the *free* chiral ring operators in these examples, is that the highest weight state involves only A_1, B_1 and no A_2, B_2. The F-terms only identify operators by symmetrizing A_1, A_2 and B_1, B_2, so for an operator like $\det(A_1 B_1)$ there are no F-term identifications. In other words, an operator only involving A_1, B_1 is the unique operator in its equivalence class in the interacting chiral ring, thus it must be BPS. Therefore, these operators have protected scaling dimension, proportional to the R-charge, and can be compared to states at strong coupling[3]. We can identify all such BPS operators: in order to have a highest weight state with only A_1, B_1, the $SU(2) \times SU(2)$ representation must be $(\Lambda_A, \Lambda_B) = ([n], [n])$, where Λ's are single-row. This is analogous to half-BPS operators in $\mathbb{C}^3$ having $\Lambda = [n]$. Then $R_1 = R_2$ can be anything, but are forced to be equal, in order to have $[n]$ in their product. Thus we have a class of operators in the chiral ring labelled by $R \vdash n$ for any n as in Figure 3.12.

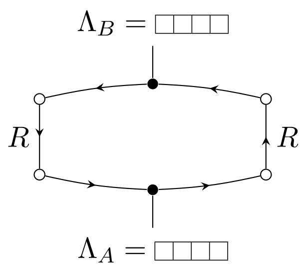


Figure 3.12: Protected representation

The highest weight operator in this representation can be expressed as

$$\mathcal{O}_{\mathcal{C}}(R) = \frac{1}{n!} \sum_{\sigma} \chi_R(\sigma) \mathrm{tr}_{V_N^{\otimes n}}(\sigma \, (A_1 B_1)^{\otimes n}) \tag{3.168}$$

[3] Note, however, they do not enjoy all the non-renormalization properties of the half-BPS operators in $\mathcal{N} = 4$ SYM, such as protected three-point functions [88].

3.5.3 $\mathbb{C}^3/\mathbb{Z}_2$

We have used the theory of $D3$ branes on a $\mathbb{C}^3/\mathbb{Z}_2$ singularity throughout, so here we just collect the references.

The quiver and the split-node quiver is displayed in Figure 3.4. The gauge symmetry is $U(N_1) \times U(N_2)$ and the global symmetry in the free limit is $U(2) \times U(2)$. The split-node quiver leads to counting (3.15). The restricted characters $\chi_{\mathbb{C}^3/\mathbb{Z}_2}(\boldsymbol{L}, \boldsymbol{\sigma})$ that give an explicit implementation of the counting are shown in (3.97). Combining with the operators $\mathcal{O}_{\mathbb{C}^3/\mathbb{Z}_2}(\boldsymbol{n}, \boldsymbol{\sigma})$ shown in (3.78) we get the basis $\mathcal{O}_{\mathbb{C}^3/\mathbb{Z}_2}(\boldsymbol{L})$ (3.96). The labels are

$$\boldsymbol{L} = \{R_1, R_2, r_{11}, r_{22}, r_{12;1}, r_{12;2}, r_{21;1}, r_{21;2}, \nu_1^{\pm}, \nu_2^{\pm}\} \tag{3.169}$$

The covariant basis $\mathcal{O}_{\mathbb{C}^3/\mathbb{Z}_2}(\boldsymbol{K})$ is built with covariant characters shown in (3.128).

3.5.4 dP_0

The theory of $D3$ branes on $\mathbb{C}^3/\mathbb{Z}^3$ singularity [59], also known as dP_0, has a quiver shown in Figure 3.13. The gauge group is $U(N_1) \times U(N_2) \times U(N_3)$, and we have a total of 9 bifundamental chiral multiplets

$$\{\Phi_{12;\alpha}, \Phi_{23;\alpha}, \Phi_{31;\alpha}\}, \quad \alpha \in \{1,2,3\} \tag{3.170}$$

There is a global flavor symmetry group $U(3) \times U(3) \times U(3)$. The counting of finite-N gauge invariant operators following (3.12) is given by the labelled split-node quiver, also in Figure 3.13:

$$\mathcal{N}_{dP_0}(\{n_{ab;\alpha}\}; N_1, N_2, N_3) = \sum_{\substack{R_1 \vdash n \\ l(R_1) \le N_1}} \sum_{\substack{R_2 \vdash n \\ l(R_1) \le N_1}} \sum_{\substack{R_3 \vdash n \\ l(R_1) \le N_1}} \sum_{\{r_{12;\alpha}\}} \sum_{\{r_{23;\alpha}\}} \sum_{\{r_{31;\alpha}\}}$$
$$g(\{r_{31;\alpha}\}; R_1)\, g(\{r_{12;\alpha}\}; R_1)\, g(\{r_{12;\alpha}\}; R_2)\, g(\{r_{23;\alpha}\}; R_2)\, g(\{r_{23;\alpha}\}; R_3)\, g(\{r_{31;\alpha}\}; R_3) \tag{3.171}$$

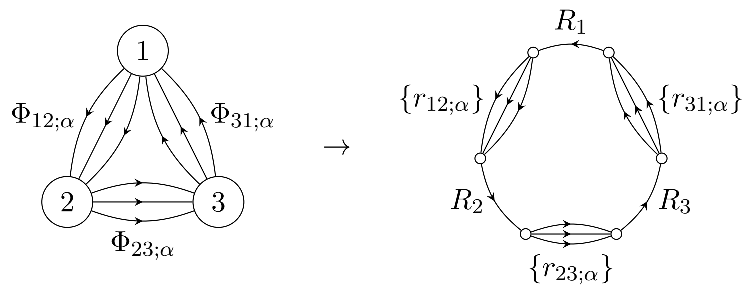


Figure 3.13: Quiver for dP_0 theory, and the split-node quiver for operator counting.

The gauge invariant mesonic operators are traces of products going around the quiver

$\mathrm{tr}(\Phi_{12;\alpha_1}\Phi_{23;\alpha_2}\Phi_{31;\alpha_3}\Phi_{12;\alpha_4}\ldots)$. According to the general prescription (3.79), a general gauge invariant operator can be specified by charges and three permutations

$$\mathcal{O}_{dP_0}(\boldsymbol{n},\{\sigma_1,\sigma_2,\sigma_3\}) = \mathrm{tr}_{V_N^{\otimes n}}\left(\sigma_1\,(\Phi_{12;\alpha})^{\otimes\{n_{12;\alpha}\}}\,\sigma_2\,(\Phi_{23;\alpha})^{\otimes\{n_{23;\alpha}\}}\,\sigma_3\,(\Phi_{31;\alpha})^{\otimes\{n_{31;\alpha}\}}\right) \tag{3.172}$$

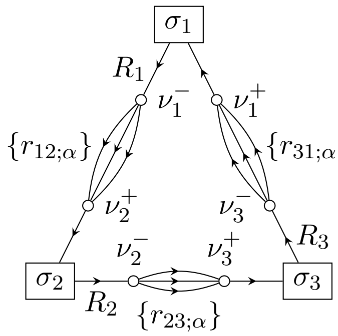
Here $n = \sum_\alpha n_{12;\alpha} = \sum_\alpha n_{23;\alpha} = \sum_\alpha n_{31;\alpha}$ is the total number of $\Phi_{12;\alpha}$'s or $\Phi_{23;\alpha}$'s or $\Phi_{31;\alpha}$'s . Since the dP_0 quiver is "linear", without any branchings like in $\mathbb{C}^3/\mathbb{Z}_2$, we can think of the operators $\mathcal{O}_{dP_0}(\boldsymbol{n},\boldsymbol{\sigma})$ as traces in $V_N^{\otimes n}$.

Restricted Schur basis operators (3.96) are:

$$\mathcal{O}_{dP_0}(\boldsymbol{L}) = c_{\boldsymbol{L}} \sum_{\sigma_1,\sigma_2,\sigma_3} \chi_{dP_0}(\boldsymbol{L},\{\sigma_1,\sigma_2,\sigma_3\})\,\mathcal{O}_{dP_0}(\boldsymbol{n},\{\sigma_1,\sigma_2,\sigma_3\}) \tag{3.173}$$

with the restricted quiver character as a further decorated split-node quiver:

$$\chi_{dP_0}(\boldsymbol{L},\{\sigma_1,\sigma_2,\sigma_3\}) = \tag{3.174}$$



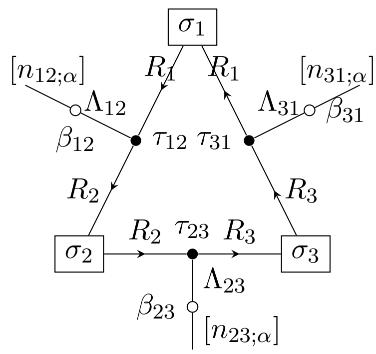
The labels are

$$\boldsymbol{L} = \{R_1, R_2, R_3, r_{12;\alpha}, r_{23;\alpha}, r_{31;\alpha}, \nu_1^{\pm}, \nu_2^{\pm}, \nu_3^{\pm}\} \tag{3.175}$$

Covariant basis operators (3.125) for dP_0 are

$$\mathcal{O}_{dP_0}(\boldsymbol{K}) = \frac{\sqrt{d(R_1)d(R_2)d(R_3)}}{(n!)^3} \sum_{\sigma_1,\sigma_2,\sigma_3} \chi_{dP_0}(\boldsymbol{K},\{\sigma_1,\sigma_2,\sigma_3\})\mathcal{O}_{dP_0}(\boldsymbol{n},\{\sigma_1,\sigma_2,\sigma_3\}) \tag{3.176}$$

$$\chi_{dP_0}(\boldsymbol{K},\{\sigma_1,\sigma_2,\sigma_3\}) = \tag{3.177}$$



with the labels

$$\boldsymbol{K} = \{R_1, R_2, R_3, \Lambda_{12}, \Lambda_{23}, \Lambda_{31}, \tau_{ab}, n_{ab;\alpha}, \beta_{ab}\} \tag{3.178}$$

That is, an operator $U(M)^3$ multiplet is defined by the global symmetry irrep $(\Lambda_{12}, \Lambda_{23}, \Lambda_{31})$, the diagrams $R_1, R_2, R_3 \vdash n$ for each gauge group factor and 3 multiplicity labels τ_{ab} for Clebsch-Gordan decompositions

$$\begin{aligned} R_1 \otimes R_2 &\to \Lambda_{12} \\ R_2 \otimes R_3 &\to \Lambda_{23} \\ R_3 \otimes R_1 &\to \Lambda_{31} \end{aligned} \tag{3.179}$$

3.5.5 $\mathbb{C}^2/\mathbb{Z}_n \times \mathbb{C}$

As a final example let us take the quiver of the $\mathbb{C}^2/\mathbb{Z}_n \times \mathbb{C}$ theory [58], Figure 3.14. In $\mathcal{N} = 1$ language it is a circular quiver with n nodes and fields $\Phi_{a,a+1}, \Phi_{a,a-1}, \Phi_{a,a}$.

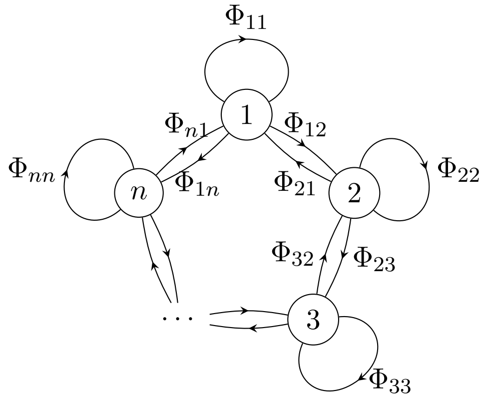


Figure 3.14: $\mathbb{C}^2/\mathbb{Z}_n \times \mathbb{C}$ quiver

The corresponding split-node quiver is shown in Figure 3.15. This leads to finite-N

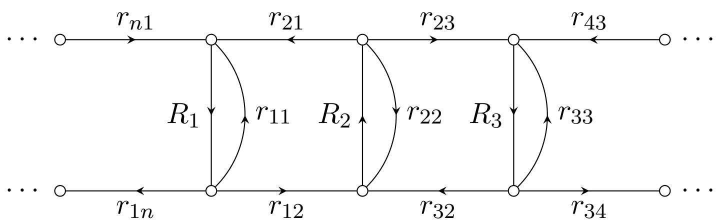


Figure 3.15: Split-node quiver for $\mathbb{C}^2/\mathbb{Z}_n \times \mathbb{C}$

counting of operators

$$\begin{aligned} \mathcal{N}_{\mathbb{C}^2/\mathbb{Z}_n\times\mathbb{C}}(\{n_{ab}\}, \{N_a\}) = &\sum_{\substack{\{R_a\vdash n_a\} \\ l(R_a)\leq N_a}} \sum_{\{r_{a,a+1}\}} \sum_{\{r_{a,a-1}\}} \sum_{\{r_{a,a}\}} \\ &\prod_a g(r_{a,a}, r_{a,a-1}, r_{a,a+1}; R_a)\; g(r_{a,a}, r_{a-1,a}, r_{a+1,a}; R_a) \end{aligned} \tag{3.180}$$

The restricted Schur basis $\mathcal{O}_Q(\boldsymbol{L})$ can be constructed by writing down quiver characters according to the split-node quiver, with the multiplicity labels $\nu_a^{\pm}$.

Chapter 4

Weak coupling

4.1 Chiral ring, wavefunctions and BPS operators

In this chapter we move on to the interacting $\mathcal{N} = 4$ SYM, working perturbatively in g at finite N. The goal is to find explicit operators in the eighth-BPS sector at one loop, that is, annihilated by the one loop dilatation operator Δ_2. It is conjectured [89] that such operators will also be annihilated by higher loop dilatation operators, and thus can be exactly identified with the eighth-BPS sector states at strong coupling and gravity. Furthermore, the two-point functions and extremal higher-point correlators for these operators should be exact at tree level (basically, because dilatation operator appears at higher loops), which would allow comparison with gravity correlators[1].

Let us start by elaborating on the connection between the chiral ring and the eighth-BPS sector described in Section 2.4. Let $\mathcal{H}$ be the Hilbert space of the *free* bosonic eighth-BPS sector spanned by all multi-trace operators built from the three scalars Φ_a

$$\mathcal{H} = \{\mathcal{O} \mid \forall \mathcal{O} = \mathrm{tr}(\Phi_{a_1}\Phi_{a_2}\ldots\Phi_{a_n})\mathrm{tr}(\Phi_{b_1}\ldots)\ldots\} \tag{4.1}$$

with the inner product given by the tree level contractions at finite N

$$\langle(\Phi_a)^i_j(\Phi^\dagger_b)^k_l\rangle = \delta_{ab}\delta^i_l\delta^k_j \tag{4.2}$$

The complete orthogonal basis for $\mathcal{H}$ is given by (3.57) or (3.68). At $g \neq 0$ these operators are annihilated by $\bar{Q}^{\dot\alpha}_4$, but not necessarily by $\bar{S}^4_{\dot\alpha}$. The eighth-BPS sector is the subspace $\mathcal{H}_{\mathrm{BPS}} \subset \mathcal{H}$ annihilated by $\bar{S}^4_{\dot\alpha}$ or equivalently, according to the conjecture above, the subspace annihilated by Δ_2

$$\mathcal{H}_{\mathrm{BPS}} = \mathrm{Ker}(\Delta_2) \tag{4.3}$$

where

$$\Delta_2 = -\frac{\lambda}{N}\mathrm{tr}([\Phi_a, \Phi_b][\check{\Phi}_a, \check{\Phi}_b]) \tag{4.4}$$

[1] In [89] this conjecture is explicitly stated for the quarter-BPS sector built only from scalars Z, Y, but it is believed to also extend to the eighth-BPS sector. If that were not the case, the results in this chapter would still apply to the quarter-BPS sector, and that is where we take all our examples from.

$\check{\Phi}_a$ acts as a formal derivative, generating Wick contractions on the fields to the right

$$(\check{\Phi}_a)^i_j (\Phi_b)^k_l \equiv \frac{\partial}{\partial(\Phi_a)^j_i} (\Phi_b)^k_l = \delta_{ab}\delta^i_l\delta^k_j \tag{4.5}$$

It is worth noting that originally in Section 2.4 the Hilbert space $\mathcal{H}_{\text{BPS}}$ is defined with the full Zamolodchikov inner product including all-loop corrections, while here we define $\mathcal{H}_{\text{BPS}}$ with the tree level inner product. However, according to the conjecture, there are no higher loop corrections, so it is the same.

There is another characterization of $\mathcal{H}_{\text{BPS}}$ that follows from (4.3). Δ_2 is a Hermitian operator on $\mathcal{H}$, so we can decompose the Hilbert space into orthogonal eigenstates of Δ_2 with real eigenvalues. $\mathcal{H}_{\text{BPS}}$ is spanned by all zero eigenvalue states, while the non-BPS operators with non-zero eigenvalues, must be $\bar{Q}^{\dot{\alpha}}_4$ descendants. The subspace of descendants $\mathcal{H}_D$ is spanned by all operators containing F-terms[2] (2.53)

$$\mathcal{H}_D = \{\mathcal{O}_D \mid \forall \mathcal{O}_D = \text{tr}([\Phi_{a_1}, \Phi_{a_2}]\Phi_{a_3}\ldots)\text{tr}(\Phi_{b_1}\ldots)\ldots\} \tag{4.6}$$

The two subspaces, having different eigenvalues of Δ_2, are orthogonal to each other, thus

$$\mathcal{H}_{\text{BPS}} = (\mathcal{H}_D)^{\perp} \tag{4.7}$$

This is analogous to the discussion around (2.44) but the difference here is that orthogonality is with respect to the *tree level* inner product. (4.7) was first used to find quarter-BPS operators in [46], however, the procedure of taking orthogonal complement of $\mathcal{H}_D$ turns out to be quite complicated even in tree level and possible in practice only for small number of fields.

Now, the eighth-BPS chiral ring is the quotient (2.45)

$$\mathcal{C} = \mathcal{H}/\mathcal{H}_D \tag{4.8}$$

As discussed in Section 2.4 there is a one to one map between $\mathcal{C}$ and $\mathcal{H}_{\text{BPS}}$, which is just the map between $\mathcal{H}/\mathcal{H}_D$ and $(\mathcal{H}_D)^{\perp}$. That is, each element of $\mathcal{C}$ is an equivalence class, a subspace of $\mathcal{H}$ differing only by $\mathcal{H}_D$, and it contains exactly one element in $\mathcal{H}_{\text{BPS}}$.

In this work we focus on the chiral ring as an intermediate step of finding the BPS operators. More precisely, we want to find the explicit map

$$\mathcal{C} \to \mathcal{H}_{\text{BPS}} \tag{4.9}$$

which would allow, given an element of the chiral ring, to find the corresponding BPS operator. This would be very useful for identifying gravity duals because, as we discuss in Chapter 5, in some cases there is a well developed map between gravity states and the chiral ring elements. If we could map those to the actual BPS states, that would allow a

[2] Note that because of finite N relations it might be possible to rewrite O_D without an F-term, but $\mathcal{H}_D$ is still well defined as anything that *can* be written with a commutator.

much more precise comparison, such as calculating correlators.

In order to develop the map $\mathcal{C} \to \mathcal{H}_{\text{BPS}}$ we first need a way to label elements of $\mathcal{C}$. A convenient way to do this is by using boson wavefunctions. This arises from the fact that $\mathcal{H}/\mathcal{H}_D$ is the coordinate ring of the symmetric product space $(\mathbb{C}^3)^N/S_N$ (see Appendix E). This ring is spanned by polynomials in $3N$ variables x_i, y_i, z_i where $i = 1 \dots N$, that are symmetric under simultaneous permutation $\sigma \in S_N$:

$$\mathcal{C} = \{\Psi(x_i, y_i, z_i) \mid \Psi(\sigma(x_i), \sigma(y_i), \sigma(z_i)) = \Psi(x_i, y_i, z_i),\ \forall \sigma \in S_N\} \tag{4.10}$$

To map from an operator $\mathcal{O} \in \mathcal{H}$ to its corresponding wavefunction $\Psi \in \mathcal{C}$ we simply take the matrices to be diagonal $Z = \text{diag}(z_i)$. Let us call this map P

$$P\ :\ \mathcal{H} \to \mathcal{C} \tag{4.11}$$

$$P(\mathcal{O}(X,Y,Z)) = \mathcal{O}(X = \text{diag}(x_i), Y = \text{diag}(y_i), Z = \text{diag}(z_i)) \tag{4.12}$$

For example

$$P\left(\text{tr}(Z^2Y^2)\right) = \sum_{i=1}^{N} z_i^2 y_i^2 \tag{4.13}$$

It can be shown that P maps operators to zero if and only if they are descendants

$$P(\mathcal{O}_D) = 0 \tag{4.14}$$

because $[\Phi_a, \Phi_b]$ vanishes for diagonal matrices[3]. The map also respects the addition and multiplication of operators

$$P(\mathcal{O}_1\mathcal{O}_2) = P(\mathcal{O}_1)P(\mathcal{O}_2), \quad P(\mathcal{O}_1 + \mathcal{O}_2) = P(\mathcal{O}_1) + P(\mathcal{O}_2) \tag{4.15}$$

thus if two operators differ by a descendent they are mapped to the same wavefunction.

Next we can ask how to reconstruct the unique BPS operator given a chiral ring element represented by a wavefunction $\Psi(x_i, y_i, z_i)$. This would be a particular inverse of the P map

$$P^{-1}:\ \mathcal{C} \to \mathcal{H}_{\text{BPS}} \tag{4.16}$$

where we map only to the BPS subspace of $\mathcal{H}$. First, we need *any* operator $\mathcal{O}$ such that $P(\mathcal{O}) = \Psi$. Then the BPS operator which maps to the same Ψ can only differ by a descendant:

$$\mathcal{O} = \mathcal{O}_{\text{BPS}} + \mathcal{O}_D \tag{4.17}$$

[3] The "only if" part can be proved by counting: there are as many symmetric polynomials as states in $\mathcal{H}/\mathcal{H}_D$, and the image of P covers all the polynomials, for example, using basis (4.31). Thus there can not be any additional non-descendent operators that have $P(\mathcal{O}) = 0$, because that would make $\mathcal{H}/\mathcal{H}_D$ larger than the space of polynomials.

Consider acting on $\mathcal{O}$ with the one-loop dilatation operator Δ_2:

$$\Delta_2\mathcal{O} = \Delta_2(\mathcal{O}_{\text{BPS}} + \mathcal{O}_D) = \Delta_2\mathcal{O}_D \tag{4.18}$$

Note that Δ_2 acts on $\mathcal{H}$ but always produces a descendant on the right-hand side. Consider an operator Δ_D equal to Δ_2 but only acting on the subspace $\mathcal{H}_D$

$$\Delta_D : \mathcal{H}_D \to \mathcal{H}_D \tag{4.19}$$

Unlike Δ_2, it has no zero eigenvalues, thus we can define the inverse

$$\Delta_D^{-1} : \mathcal{H}_D \to \mathcal{H}_D \tag{4.20}$$

By the definition

$$\Delta_D^{-1}\Delta_2\mathcal{O}_D = \mathcal{O}_D \tag{4.21}$$

so we can isolate and subtract the "non-BPS'ness" from $\mathcal{O}$

$$\boxed{\mathcal{O}_{\text{BPS}} = (1 - \Delta_D^{-1}\Delta_2)\mathcal{O}} \tag{4.22}$$

We can verify that the operator is BPS by acting with Δ_2:

$$\Delta_2\mathcal{O}_{\text{BPS}} = (\Delta_2 - \Delta_2\Delta_D^{-1}\Delta_2)\mathcal{O} = (\Delta_2 - \Delta_2)\mathcal{O} = 0 \tag{4.23}$$

Since in (4.22) we are manifestly subtracting an element of $\mathcal{H}_D$, $\mathcal{O}_{\text{BPS}}$ still corresponds to the same chiral ring element as $\mathcal{O}$. Thus we find

$$P^{-1} = 1 - \Delta_D^{-1}\Delta \tag{4.24}$$

in a sense that this will recover the BPS operator starting with any chiral ring representative. In general (4.22) is rather formal, since we have not found a systematic procedure to evaluate Δ_D^{-1}, even in the $1/N$ expansion. But we will see in Section 4.3 that in some cases the action of Δ_2 is simple enough so it can be inverted and Δ_D^{-1} can be calculated explicitly.

Finally, let us write a complete basis for $\mathcal{C}$, which in principle will define a complete basis for $\mathcal{H}_{\text{BPS}}$ via (4.22). The boson states can be labelled simply by specifying N excitation numbers

$$|\vec{m}_1, \vec{m}_2, \ldots, \vec{m}_N\rangle \tag{4.25}$$

where each $\vec{m}_i$ is in $\mathbb{Z}^3$, labelling a state of a 3D harmonic oscillator

$$\vec{m}_i = (m_i^{(x)}, m_i^{(y)}, m_i^{(z)}) \tag{4.26}$$

Because the bosons are identical, in order to get a unique labelling we impose some ordering

$$\vec{m}_1 \geq \vec{m}_2 \geq \ldots \geq \vec{m}_N \tag{4.27}$$

The wavefunction for a single boson is just

$$|\vec{m}\rangle = x^{m^{(x)}} y^{m^{(y)}} z^{m^{(z)}} \equiv \vec{z}^{\vec{m}} \tag{4.28}$$

and for N identical bosons

$$|\vec{m}_1, \vec{m}_2, \ldots, \vec{m}_N\rangle = \frac{1}{N!} \sum_{\sigma \in S_N} \prod_{i=1}^{N} \vec{z}_{\sigma(i)}^{\vec{m}_i} \tag{4.29}$$

Basically, it is a product of $\vec{z}_i^{\vec{m}_i}$ for each boson, but we have to sum over permutations. We also want a basis of representative operators, such that

$$P(\mathcal{O}(\vec{m}_1, \vec{m}_2, \ldots, \vec{m}_N)) = |\vec{m}_1, \vec{m}_2, \ldots, \vec{m}_N\rangle \tag{4.30}$$

Such basis was written in [37] and is given by the determinant-like operators

$$\mathcal{O}(\vec{m}_1, \vec{m}_2, \ldots, \vec{m}_N) = \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^\sigma (\Phi^{\vec{m}_1})^{i_1}_{i_{\sigma(1)}} (\Phi^{\vec{m}_2})^{i_1}_{i_{\sigma(2)}} \ldots (\Phi^{\vec{m}_N})^{i_1}_{i_{\sigma(N)}} \tag{4.31}$$

Here

$$\Phi^{\vec{m}} \equiv X^{m^{(x)}} Y^{m^{(y)}} Z^{m^{(z)}} \tag{4.32}$$

Thus putting everything together, the complete basis for the chiral ring $\mathcal{C}$ is

$$\mathcal{C} = \{\, |\vec{m}_1, \vec{m}_2, \ldots, \vec{m}_N\rangle \,\} \tag{4.33}$$

and the corresponding complete basis of the BPS operators is

$$\boxed{\mathcal{H}_{\text{BPS}} = \{\, (1 - \Delta_D^{-1}\Delta_2)\mathcal{O}(\vec{m}_1, \vec{m}_2, \ldots, \vec{m}_N) \,\}} \tag{4.34}$$

Here is a simple example showing the different states (up to normalization):

$$|(0,0,2),(0,2,0)\rangle = \sum_{i \neq j}^{N} y_i^2 z_j^2 \tag{4.35}$$

$$\mathcal{O}\left((0,0,2),(0,2,0)\right) = \text{tr}(Z^2)\text{tr}(Y^2) - \text{tr}(Z^2 Y^2) \tag{4.36}$$

$$\mathcal{O}_{\text{BPS}}\left((0,0,2),(0,2,0)\right) = \text{tr}(Z^2)\text{tr}(Y^2) - \text{tr}(Z^2 Y^2) - \frac{N-4}{6N}\text{tr}([Y,Z][Y,Z]) \tag{4.37}$$

Note that even though (4.34) gives a complete eighth-BPS basis, it does not give an *orthogonal* basis. One way to see this is to restrict to the half-BPS sector, where

$$\mathcal{O}(m_1, m_2, \ldots, m_N) = \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^\sigma (Z^{m_1})^{i_1}_{i_{\sigma(1)}} (Z^{m_2})^{i_1}_{i_{\sigma(2)}} \ldots (Z^{m_N})^{i_1}_{i_{\sigma(N)}} \tag{4.38}$$

It reproduces the single giant graviton states $\chi_{[1^n]}(Z) \propto \mathcal{O}(1,1,\ldots,1)$ but not the AdS giant $\chi_{[n]}(Z) \neq \mathcal{O}(n,0,0,\ldots) \propto tr(Z^n)$. One might hope that (4.34) at least gives a good approximation for orthogonal states dual to classical eighth-BPS giant configurations expanding in S^5 – precisely those that will be the topic of Chapter 5.

To go beyond the regime of giant gravitons one probably needs to diagonalize the basis (4.34), which is a very hard problem beyond the scope of this work.

4.2 Eighth-BPS by applying Ω^{-1}

In this section we develop a method to find exact eighth-BPS operators, that is especially suitable when the charges are small enough $n \leq N$, so there are no finite N relationships. It is based on the original paper [65]. We will see that in this case we can start with an operator built from *symmetrized* traces and then systematically apply $1/N$ corrections given by operator Ω^{-1}. This is a slightly different approach than (4.22), but has a virtue of having a systematic $1/N$ expansion. We will come back to (4.22) in the next section.

Let us start by reviewing the planar limit $N \to \infty$. Consider the basis of operators labelled by permutation $\sigma \in S_n$ as in (3.53), with a convenient normalization

$$\mathcal{O}(\sigma) = \frac{1}{N^{n/2}} \mathrm{tr}_{V_N^{\otimes n}} \left(\sigma \, \Phi_i^{\otimes n_i} \right) \tag{4.39}$$

Here $n = n_1 + n_2 + n_3$. The inner product at finite N is (3.102)

$$\begin{aligned} \langle \mathcal{O}(\sigma) | \mathcal{O}(\tilde{\sigma}) \rangle &= \frac{1}{N^n} \sum_{\gamma \in H} \mathrm{tr}_{V_N^{\otimes n}} (\gamma^{-1} \sigma^{-1} \gamma \tilde{\sigma}) \qquad (H = S_{n_1} \times S_{n_2} \times S_{n_3}) \\ &= \sum_{\gamma \in H} \sum_{\tau \in S_n} N^{C(\tau)-n} \delta(\gamma^{-1} \sigma^{-1} \gamma \tilde{\sigma} \tau) \\ &= \sum_{\gamma \in H} \delta(\gamma^{-1} \sigma^{-1} \gamma \tilde{\sigma} \, \Omega) \end{aligned} \tag{4.40}$$

The Ω factor

$$\Omega = \sum_{\sigma \in S_n} N^{C(\sigma)-n} \sigma \tag{4.41}$$

is a central element in the group algebra $\mathbb{C}(S_n)$, i.e. it commutes with any element of S_n. It has a $1/N$ expansion

$$\Omega = 1 + \frac{1}{N} \Sigma_{[2]} + \frac{1}{N^2} (\Sigma_{[3]} + \Sigma_{[2,2]}) + \frac{1}{N^3} (\Sigma_{[4]} + \Sigma_{[3,2]} + \Sigma_{[2,2,2]}) + \ldots \tag{4.42}$$

where Σ_T is the sum of all permutations in the conjugacy class T. If $n \ll \sqrt{N}$ then (4.40) is dominated by the term $\tau = 1$, in other words, we can take the planar approximation

$\Omega \approx 1$

$$\langle \mathcal{O}(\sigma)|\mathcal{O}(\tilde{\sigma})\rangle_\infty = \sum_{\gamma \in H} \delta(\gamma^{-1}\sigma^{-1}\gamma\tilde{\sigma}) \quad \propto \delta_{[\sigma],[\tilde{\sigma}]} \tag{4.43}$$

The subscript $\langle \dots | \dots \rangle_\infty$ denotes the planar inner product, and we can see it is non-zero only if σ is equal to $\tilde{\sigma}$ up to a conjugation by $\gamma \in H$. This is precisely when the operators are actually equal products of traces, thus this is the usual result that in the planar limit traces form an orthogonal basis.

It is useful to define a new Hilbert space $\mathcal{H}_\infty$ using the planar two-point function (4.43). It is the Hilbert space in the strict $N \to \infty$ limit, which is spanned by operators $\mathcal{O}(\sigma)$ defined up to conjugation

$$\mathcal{H}_\infty = \{\mathcal{O}(\sigma) \mid \mathcal{O}(\sigma) = \mathcal{O}(\gamma\sigma\gamma^{-1}),\ \gamma \in H\} \tag{4.44}$$

with no finite N identifications. It can be thought of as the space of conjugacy classes $[\sigma]$ (with respect to H), without any reference to N. There is the natural map we call $\mathcal{I}_N$

$$\mathcal{I}_N : \mathcal{H}_\infty \to \mathcal{H} \tag{4.45}$$

to the finite N Hilbert space $\mathcal{H}$, which is just taking $\mathcal{O}(\sigma)$ as an element of $\mathcal{H}$. It is important to note that $\mathcal{I}_N$ is not invertible, because an operator in $\mathcal{H}$ might be written using different $\mathcal{O}(\sigma)$ due to finite N relations. Ker($\mathcal{I}_N$) spans exactly the operators set to 0 at finite N.

Now Ω can be considered as a linear operator on $\mathcal{H}_\infty$

$$\Omega\, \mathcal{O}(\sigma) \equiv \mathcal{O}(\Omega\, \sigma) \equiv \sum_{\tau \in S_n} N^{C(\tau)-n} \mathcal{O}(\tau\sigma) \tag{4.46}$$

From the point of view of $\mathcal{H}_\infty$, N is just a parameter in Ω. Note this definition is consistent, because Ω is central so it is independent of which representative σ is chosen $\mathcal{O}(\Omega\gamma\sigma\gamma^{-1}) = \mathcal{O}(\gamma\Omega\sigma\gamma^{-1})$. Then the relationship between planar and finite N inner products (4.40) is

$$\langle \mathcal{O}(\sigma)|\mathcal{O}(\tilde{\sigma})\rangle = \langle \mathcal{O}(\sigma)|\Omega|\mathcal{O}(\tilde{\sigma})\rangle_\infty \tag{4.47}$$

Note also that Ω is Hermitian in $\mathcal{H}_\infty$, as can be seen from (4.40), it does not matter if it acts on σ or $\tilde{\sigma}$.

Next, we use the fact that $\mathcal{H}_{\text{BPS}} = (\mathcal{H}_D)^\perp$ (4.7). With the help of Ω we can write this as

$$\langle \mathcal{O}_D|\mathcal{O}_{\text{BPS}}\rangle = \langle \mathcal{O}_D|\Omega|\mathcal{O}_{\text{BPS}}\rangle_\infty = 0 \qquad \forall \mathcal{O}_D \in \mathcal{H}_D \tag{4.48}$$

What makes this particularly useful, is that it is very easy to find the orthogonal complement of $\mathcal{H}_D$ inside $\mathcal{H}_\infty$: it is spanned by symmetrized traces[4]

$$\langle \mathcal{O}_D | \mathcal{O}_S \rangle_\infty = 0, \quad \text{if } \mathcal{O}_D \in \mathcal{H}_D,\ \mathcal{O}_S \in \mathcal{H}_S \tag{4.49}$$

where

$$\mathcal{H}_S = \{\mathcal{O}_S \mid \forall \mathcal{O}_S = \mathrm{Str}(\Phi_{a_1}\Phi_{a_2}\ldots\Phi_{a_n})\mathrm{Str}(\Phi_{b_1}\ldots)\ldots\} \tag{4.50}$$

The orthogonality (4.49) can be shown by defining a symmetrization operator $\mathcal{P}$, which is a linear operator on $\mathcal{H}_\infty$ that acts by completely symmetrizing all traces. $\mathcal{P}$ is a Hermitian operator and

$$\mathcal{H}_D = \mathrm{Ker}(\mathcal{P}), \quad \mathcal{H}_S = \mathrm{Im}(\mathcal{P}) \tag{4.51}$$

so these two spaces must be orthogonal.

In general Ω is not invertible (because it maps some operators to 0), but it *is* invertible if $n \leq N$. That is, if we restrict to the subspace of $\mathcal{H}_\infty$ where the operator charges do not exceed N we can actually define Ω^{-1}. This object has appeared in the studies of 2D Yang-Mills, in the large N expansion (see [90]). It can be calculated explicitly for a fixed n or in the $1/N$ expansion when $n \ll N$, see Appendix A.5. Then we can get $\mathcal{H}_{\mathrm{BPS}}$ just by acting with Ω^{-1} on $\mathcal{H}_S$

$$\boxed{\mathcal{O}_{\mathrm{BPS}} = \Omega^{-1}\mathcal{O}_S, \qquad \text{if } n \leq N} \tag{4.52}$$

This follows trivially from (4.48) and (4.49)

$$\left\langle \mathcal{O}_D | \Omega^{-1}\mathcal{O}_S \right\rangle = \left\langle \mathcal{O}_D | \Omega\Omega^{-1}\mathcal{O}_S \right\rangle_\infty = \langle \mathcal{O}_D | \mathcal{O}_S \rangle_\infty = 0 \tag{4.53}$$

At $n \leq N$ there are no identifications between traces ($\mathcal{I}_N$ is invertible), which makes it easy to write a complete basis for $\mathcal{H}_{\mathrm{BPS}}$. Start with a complete basis for $\mathcal{H}_S$ which can be labelled by a set of three-dimensional integer vectors $(\vec{m}_1, \vec{m}_2, \ldots, \vec{m}_k)$

$$\mathcal{O}_S(\{\vec{m}_i\}) = \mathrm{Str}(\Phi_1^{m_1^{(1)}}\Phi_2^{m_1^{(2)}}\Phi_3^{m_1^{(3)}})\mathrm{Str}(\Phi_1^{m_2^{(1)}}\Phi_2^{m_2^{(2)}}\Phi_3^{m_2^{(3)}})\ldots\mathrm{Str}(\Phi_1^{m_k^{(1)}}\Phi_2^{m_k^{(2)}}\Phi_3^{m_k^{(3)}}) \tag{4.54}$$

where we impose some canonical ordering $\vec{m}_1 \geq \vec{m}_2 \geq \ldots \geq \vec{m}_k$. Then

$$\mathcal{O}_{\mathrm{BPS}}(\{\vec{m}_i\}) = \Omega^{-1}\mathcal{O}_S(\{\vec{m}_i\}) \tag{4.55}$$

is a basis for $\mathcal{H}_{\mathrm{BPS}}$.

Let us demonstrate the main result (4.52) with an example. Take

$$\mathcal{O}_S \equiv \mathcal{O}_S((0,0,2),(0,2,0)) = \mathrm{tr}(Z^2)\mathrm{tr}(Y^2) \tag{4.56}$$

[4] Strictly speaking here we can distinguish between $\mathcal{H}'_D \subset \mathcal{H}_\infty$, which is genuinely the space of all traces containing a commutator, while the original $\mathcal{H}_D \in \mathcal{H}$ is the image of $\mathcal{H}'_D$ under $\mathcal{I}_N$.

Using (A.46), the first $1/N$ correction is

$$\Omega^{-1}\mathcal{O}_S = \mathrm{tr}(Z^2)\mathrm{tr}(Y^2) - \frac{1}{N}\left(4\mathrm{tr}(Z^2Y^2) + \mathrm{tr}(Z)\mathrm{tr}(Z)\mathrm{tr}(Y^2) + \mathrm{tr}(Z^2)\mathrm{tr}(Y)\mathrm{tr}(Y)\right) + O(N^{-2}) \tag{4.57}$$

$\Sigma_{[2]}$ either joins the two traces or splits one of them. The full correction can be written by applying (A.49), but the answer is rather long, and not very illuminating. Note, however, it is very different from (4.22), which *only* has $1/N$ term in this case

$$(1 - \Delta_D^{-1}\Delta_2)\mathcal{O}_S = \mathrm{tr}(Z^2)(Y^2) + \frac{4}{6N}\mathrm{tr}([Y,Z][Y,Z]) \tag{4.58}$$

Now let us discuss the case $n > N$ from the perspective of Ω. Relationship (4.48) is still valid, and in principle it is possible to derive an analogous result to (4.52), but we need to take care inverting Ω. It is easiest to work in the basis labelled $U(N)$ representations R, such as the restricted Schur (3.57) or the covariant (3.68) basis. We will ignore the remaining labels and just refer to the operators as $\mathcal{O}(R)$. The basis for $\mathcal{H}_\infty$ is spanned by all R, and the projection to $\mathcal{H}$ simply cuts off the diagrams taller than N

$$\mathcal{I}_N\,\mathcal{O}(R) = \begin{cases} \mathcal{O}(R) & \text{if } l(R) \le N, \\ 0, & \text{if } l(R) > N \end{cases} \tag{4.59}$$

The action of Ω is diagonal in this basis

$$\Omega\,\mathcal{O}(R) = \frac{f(R)}{N^n}\mathcal{O}(R) \tag{4.60}$$

which also annihilates operators with $l(R) > N$. If $n \le N$ then $f(R)$ is always positive and the operator can be inverted

$$\Omega^{-1}\,\mathcal{O}(R) = \frac{N^n}{f(R)}\mathcal{O}(R) \tag{4.61}$$

But now note, that even if $n > N$ we can still act with Ω^{-1} on states with $l(R) \le N$, that is, on the $\mathrm{Im}(\mathcal{I}_N)$. Let us define such projected inversion

$$\Omega_I^{-1} = \mathcal{I}_N\,\Omega^{-1}\,\mathcal{I}_N \tag{4.62}$$

so that

$$\Omega_I^{-1}\,\mathcal{O}(R) = \begin{cases} \frac{N^n}{f(R)}\mathcal{O}(R) & \text{if } l(R) \le N, \\ 0, & \text{if } l(R) > N \end{cases} \tag{4.63}$$

Note

$$\Omega\Omega_I^{-1} = \mathcal{I}_N \tag{4.64}$$

Simply acting with Ω_I^{-1} on $\mathcal{O}_S$ will still not give a BPS operator, because following analogous steps as (4.53) we get $\langle\mathcal{O}_D|\mathcal{I}_N\,\mathcal{O}_S\rangle_\infty \neq 0$. The reason is that in general after $\mathcal{I}_N$ projection the operator is no longer symmetric.

What we need is to start with an operator which is symmetric *and* it is in $\text{Im}(\mathcal{I}_N)$. In other words, a symmetric operator, which can be written purely in terms of $l(R) \leq N$ diagrams

$$\mathcal{O}_{SI} = \sum_{\substack{R \\ l(R)\leq N}} c_R\, \mathcal{O}(R) \tag{4.65}$$

obeying

$$\mathcal{I}_N\, \mathcal{O}_{SI} = \mathcal{P}\, \mathcal{O}_{SI} = \mathcal{O}_{SI} \tag{4.66}$$

In that case it is indeed true that

$$\mathcal{O}_{\text{BPS}} = \Omega_I^{-1} \mathcal{O}_{SI} \tag{4.67}$$

is eighth-BPS, as can be shown by repeating the derivation (4.53)

$$\left\langle \mathcal{O}_D | \Omega_I^{-1} \mathcal{O}_{SI} \right\rangle = \left\langle \mathcal{O}_D | \Omega \Omega_I^{-1} \mathcal{O}_{SI} \right\rangle_\infty = \langle \mathcal{O}_D | \mathcal{I}_N \mathcal{O}_{SI} \rangle_\infty = \langle \mathcal{O}_D | \mathcal{O}_{SI} \rangle_\infty = 0 \tag{4.68}$$

Another way to formulate the result is as follows. We have $\text{Im}(\mathcal{P}) \in \mathcal{H}_\infty$, the space of symmetrized operators, and $\text{Im}(\mathcal{I}_N)$, the space of operators with $l(R) \leq N$. We define a projector $\mathcal{P}_I$ to the *intersection* of these two spaces

$$\text{Im}(\mathcal{P}_I) = \text{Im}(\mathcal{I}_N) \cap \text{Im}(\mathcal{P}) \tag{4.69}$$

$\text{Im}(\mathcal{P}_I)$ is spanned precisely by operators (4.65). Then the eighth-BPS operators are

$$\mathcal{H}_{\text{BPS}} = \Omega_I^{-1} \mathcal{P}_I\, \mathcal{H}_\infty \tag{4.70}$$

It is interesting to note that the counting of BPS operators at finite N is given by $|\text{Im}(\mathcal{I}_N) \cap \text{Im}(\mathcal{P})|$, so the symmetrization operator $\mathcal{P}$ is still relevant, while *a priori* one might expect it is only useful in $N \to \infty$ limit.

A drawback of this approach is that it is quite difficult to find the operators $\mathcal{O}_{SI}$ explicitly and unlike $\mathcal{H}_S$ we do not have a construction for the complete basis of $\text{Im}(\mathcal{P}_I)$. This limits our possibility of finding explicit operators $\mathcal{O}_{\text{BPS}}$ with $n > N$ to only small values of N, where we can construct the matrix elements of the operator $\mathcal{P}$ on $\mathcal{O}(R)$ basis, and calculate the intersection $\mathcal{P}_I$. Thus the results of this section are still more applicable to $n \leq N$, where can act with Ω^{-1} directly on symmetrized traces.

4.3 Eighth-BPS by subtracting descendants

In this section we go back to the idea of finding BPS operators based on (4.24). The main problem with (4.24) is that in general we do not have a way to systematically evaluate Δ_D^{-1}. However, if we start with a sufficiently "nice" basis, where there is small mixing between different Δ_2 eigenstates, then it might be possible to find an approximate inverse. Furthermore, if we start with an operator which is "near-BPS", then the corrections

$\Delta_D^{-1}\Delta_2$ will be small, and we can work order by order in $1/N$. We will show here an explicit example of this construction, in the sector where $n > N$.

As an example we take operators dual to a maximal giant graviton with attached open strings. These are single-column restricted Schurs with impurities [20, 22, 23, 24]

$$\begin{aligned}\chi_{R=[1^N],r=[1^{N-k}]}&(Z;W_1,\dots,W_k)\\ &= \frac{1}{(N-k)!}\sum_{\sigma\in S_N}(-1)^\sigma Z^{i_1}_{i_{\sigma(1)}}\dots Z^{i_{N-k}}_{i_{\sigma(N-k)}}(W_1)^{i_{N-k+1}}_{i_{\sigma(N-k+1)}}\dots(W_k)^{i_N}_{i_{\sigma(N)}}\end{aligned} \tag{4.71}$$

Since all the diagrams will involve a single column with impurities we will abbreviate

$$\chi(Z;W_1,\dots,W_k)\equiv\chi_{R=[1^N],r=[1^{N-k}]}(Z;W_1,\dots,W_k) \tag{4.72}$$

The attached "words" W_i can be any contraction of field letters, but unlike trace which is cyclic (closed string), it has the beginning and the end (open string). The bulk of the operator, carries charge Z^{N-k} and represents the brane. In case there are no strings attached it is just $\det(Z)$. It can be shown that correlators in this basis have a nice planar expansion, so that states with different strings are approximately orthogonal, and further $1/N$ corrections admit an interpretation as open string interactions, or interactions between the strings and the brane. See Appendix D for more details.

We focus on the operator

$$\mathcal{O}_{Y^L}=\frac{1}{\sqrt{N!N^L}}\chi(Z;Y^L) \tag{4.73}$$

which represents the ground state of the open string spin chain [27]. The operator is properly normalized

$$\langle\mathcal{O}_{Y^L}|\mathcal{O}_{Y^L}\rangle = 1+O(N^{-2}) \tag{4.74}$$

It is an accepted fact in the literature, that $\mathcal{O}_{Y^L}$ is "near-BPS" [20, 22, 23]. Let us examine this in detail. In general, the leading $O(\lambda)$ contribution to the one-loop dilatation operator Δ_2 comes from acting on the open string with a spin-chain Hamiltonian. The excited states would involve impurities $\chi(Z;Y^{l_1}ZY^{l_2}\dots)$, while for $\mathcal{O}_{Y^L}$ one-loop dimension is zero to this order. However, there are also subleading interactions between Y^L string and the brane. The expectation value for the one-loop dimension because of these effects is

$$\langle\mathcal{O}_{Y^L}|\Delta_2|\mathcal{O}_{Y^L}\rangle = 2\lambda\frac{L-1}{N} \tag{4.75}$$

This is taken as a sign that $\mathcal{O}_{Y^L}$ is "near-BPS", because anomalous dimension is suppressed if $N\gg\lambda L$.

From our perspective, however, we can refine the question of what "near-BPS" actually means. As discussed in Section 4.1, any operator can be uniquely expanded

$$\mathcal{O}_{Y^L}=c_{\text{BPS}}\mathcal{O}_{\text{BPS}}+c_D\mathcal{O}_D,\qquad |c_{\text{BPS}}|^2+|c_D|^2=1 \tag{4.76}$$

where $\mathcal{O}_{\mathrm{BPS}}$ is annihilated by Δ_2, and $\mathcal{O}_D$ is a mixture of non-zero-eigenvalue eigenstates of Δ_2 and vanishes in the chiral ring. All operators are assumed to be normalized. Our goal here is to find the BPS combination

$$\mathcal{O}_{Y^L} - c_D \mathcal{O}_D \tag{4.77}$$

and the relevant question is whether $|c_D|^2 \ll 1$ or not. If $\langle \mathcal{O}_{Y^L}|\Delta_2|\mathcal{O}_{Y^L}\rangle \ll 1$ that leaves, in principle, two possibilities

1. $\mathcal{O}_{Y^L}$ has a small mixture $|c_D| \ll 1$ with non-BPS states, that might have large eigenvalues $\langle \mathcal{O}_D|\Delta_2|\mathcal{O}_D\rangle \sim O(\lambda)$. In this case $\mathcal{O}_{Y^L}$ can indeed be considered as a good representative of the actual BPS state.
2. $\mathcal{O}_{Y^L}$ is actually dominated by the non-BPS states $|c_D| \approx 1$, that happen to have small (but non-zero) eigenvalues $\langle \mathcal{O}_D|\Delta_2|\mathcal{O}_D\rangle \ll 1$. In this case we should not treat $\mathcal{O}_{Y^L}$ as BPS.

These two possibilities were already pointed out in [22], but this question has not been conclusively resolved. Here we will find the dominant correction $c_D\mathcal{O}_D$ explicitly and will indeed show that $|c_D|^2 \sim O(N^{-1})$, that is, $\mathcal{O}_{Y^L}$ is a good approximation to a true (quarter-)BPS operator.

We will apply (4.24) to find the BPS component of $\mathcal{O}_{Y^L}$

$$\mathcal{O}^{\mathrm{BPS}}_{Y^L} = (1 - \Delta_D^{-1}\Delta_2)\mathcal{O}_{Y^L} \tag{4.78}$$

What makes it possible, is that we can use an appropriate planar approximation in the open string basis.

First, acting with Δ_2 gives

$$\begin{aligned}
\Delta_2 \mathcal{O}_{Y^L} &= \frac{\lambda}{N}\mathrm{tr}([Z,Y][\bar{Y},\bar{Z}])\,\frac{\chi(Z;Y^L)}{\sqrt{N!N^L}} \\
&= \frac{\lambda}{N\sqrt{N!N^L}}\,\mathrm{tr}([Z,Y][\underbrace{\bar{Y},\underbrace{\bar{Z}])\,\chi(Z;Z},Y^L})) \\
&= \frac{\lambda}{N\sqrt{N!N^L}}\,\sum_{i=1}^{L}\left(\chi(Z;Y^{i-1},Y^{L-i}[Z,Y]) - \chi(Z;Y^{i-1},[Z,Y]Y^{L-i})\right) \\
&\approx \frac{\lambda}{\sqrt{N}}\sum_{i=1}^{L-1}\frac{1}{\sqrt{N!N^{L+1}}}\left(\chi(Z;Y^{i-1},Y^{L-i}ZY) + \chi(Z;Y^{i-1},YZY^{L-i})\right)
\end{aligned} \tag{4.79}$$

The second line follows from (D.10). In the last line we have dropped the terms where Z appears at the end of the open string, such as

$$\chi(Z;Y^i,Y^{L-i}Z) \tag{4.80}$$

because their norm is suppressed by $\frac{1}{\sqrt{N}}$ compared to other operators. In general, operators with Z at the endpoints are not considered part of the open string basis (4.71), because they factorize and can be related to states containing closed string [22].

So the result is that

$$\Delta_2 \mathcal{O}_{Y^L} = \frac{\lambda}{\sqrt{N}} \mathcal{O}_\Delta \tag{4.81}$$

with

$$\mathcal{O}_\Delta = \frac{1}{\sqrt{N!N^{L+1}}} \sum_{i=1}^{L-1} \left(\chi(Z; Y^{i-1}, Y^{L-i} ZY) + \chi(Z; Y^{i-1}, YZY^{L-i}) \right) + O\left(\frac{1}{\sqrt{N}}\right) \tag{4.82}$$

Each operator in the sum is normalized

$$\left| \frac{1}{\sqrt{N!N^{L+1}}} \chi(Z; Y^{i-1}, Y^{L-i} ZY) \right|^2 \approx 1 \tag{4.83}$$

and represents a maximal giant with two attached strings, one in ground state Y^{i-1} and one with impurity YZY^{L-i}.

The next step is to find Δ_D^{-1} acting on $\mathcal{O}_\Delta$

$$\mathcal{O}_D = \Delta_D^{-1} \mathcal{O}_\Delta \tag{4.84}$$

In the sector of operators with an excited open string, the action of Δ_2 to the leading order $O(\lambda)$ is just the spin-chain Hamiltonian [27]

$$\Delta_2 \approx \lambda \sum_{l=1}^{L-1} (I_{l,l+1} - P_{l,l+1}) \tag{4.85}$$

where $I_{l,l+1}$ is the identity and $P_{l,l+1}$ is the exchange acting on adjacent sites $(l, l+1)$ on the string of length L. Note that the terms with Z in the first or last site should be set to zero. In particular, on a string with a single impurity

$$\begin{aligned} \Delta_2\, \chi(Z; YZY^j) &= 2\lambda\, \chi(Z; YZY^j) - \lambda\, \chi(Z; Y^2 ZY^{j-1}) \\ \Delta_2\, \chi(Z; Y^i ZY) &= 2\lambda\, \chi(Z; Y^i ZY) - \lambda\, \chi(Z; Y^{i-1} ZY^2) \\ \Delta_2\, \chi(Z; Y^i ZY^j) &= 2\lambda\, \chi(Z; Y^i ZY^j) - \lambda\, \chi(Z; Y^{i+1} ZY^{j-1}) - \lambda\, \chi(Z; Y^{i-1} ZY^{j+1}) \end{aligned} \tag{4.86}$$

In the operator $\mathcal{O}_\Delta$ we have, for each length of the open string, a sum of two terms where Z is in the first or last allowed site. It is easy to find the combination that gives the desired operator under Δ_2 action:

$$\frac{\Delta_2}{\lambda} \sum_{j=1}^{k-1} \chi(Z; Y^j ZY^{k-j}) = \chi(Z; YZY^{k-1}) + \chi(Z; Y^{k-1} ZY) \tag{4.87}$$

Thus we found the inverse Δ_D^{-1} in this particular sector, to the leading order in N:

$$\Delta_D^{-1}\left(\chi(Z;YZY^{k-1})+\chi(Z;Y^{k-1}ZY)\right)=\frac{1}{\lambda}\sum_{j=1}^{k-1}\chi(Z;Y^jZY^{k-j}) \tag{4.88}$$

We can apply this to each term in (4.82), since the extra string Y^i does not have an effect to leading order

$$\Delta_D^{-1}\,\mathcal{O}_\Delta=\frac{1}{\lambda}\frac{1}{\sqrt{N!N^{L+1}}}\sum_{i=1}^{L-1}\sum_{j=1}^{L-i}\chi(Z;Y^{i-1},Y^jZY^{L-i-j+1}) \tag{4.89}$$

Putting together (4.78) evaluates to

$$\boxed{\mathcal{O}_{Y^L}^{\text{BPS}}\approx\frac{\chi(Z;Y^L)}{\sqrt{N!N^L}}-\frac{1}{\sqrt{N}}\sum_{i=1}^{L-1}\sum_{j=1}^{L-i}\frac{\chi(Z;Y^{i-1},Y^jZY^{L-i-j+1})}{\sqrt{N!N^{L+1}}}} \tag{4.90}$$

This is the *leading order* correction to $\mathcal{O}_{Y^L}$. The important point is that it decreases the anomalous dimension of the operator parametrically: (4.90) has $\langle\Delta_2\rangle\sim O(\lambda N^{-2})$ while $\mathcal{O}_{Y^L}$ had $\langle\Delta_2\rangle\sim O(\lambda N^{-1})$. The terms in the sum in (4.90) are normalized to 1 and there are $O(L^2)$ of them, thus we find

$$|c_D|^2\sim O\left(\frac{L^2}{N}\right) \tag{4.91}$$

If $L\ll\sqrt{N}$ then the correction is indeed small $|c_D|\ll 1$. This shows that the anomalous dimension of $\mathcal{O}_{Y^L}$ is due to a small mixing with a non-BPS operator, rather than having a small non-zero eigenvalue itself.

Note that the approximation seems to break down when $L\sim O(\sqrt{N})$ which is when the BMN genus expansion parameter $g_2=\frac{L^2}{N}$ becomes order 1. This indicates that $\chi(Z;Y^L)$ is not a good approximation for a BPS operator in that limit.

Chapter 5

Strong coupling

In this chapter we move on to the strong coupling regime of $\mathcal{N} = 4$ SYM, described by string theory on $AdS_5 \times S^5$. We continue analysing the eighth-BPS sector, which is λ-independent, and thus can be compared against weak coupling results of the previous chapter. We focus on the finite N effects, giving us access to non-perturbative gravity states, such as D3-branes.

In the regime of energies where the D3-brane probe approximation is valid, the eighth-BPS phase space can be described in terms of Mikhailov's holomorphic surfaces [36]. If we consider a 4D algebraic curve in $\mathbb{C}^3$, given by the zero locus of a holomorphic polynomial, then its intersection with S^5 can be shown to give a D3-brane surface preserving $\frac{1}{8}$ of supersymmetries. This phase space includes various interesting configurations, such as intersecting branes and oscillating branes. The geometric quantization of the phase space was performed in [38], and the resulting spectrum was shown to agree with the finite N spectrum in SYM at weak coupling.

In this work we aim to make the correspondence between D3-brane configurations and the individual BPS operators more concrete. For that purpose we first need to better understand the map between classical D3-brane configurations and the quantum states in geometric quantization. We use techniques from fuzzy geometry to develop this map. In general, quantum states in the natural basis, being energy eigenstates, do not correspond to localized classical branes. However, we find a sector of intersecting *maximal* giants, that can be easily mapped to precise quantum states.

We further analyze the spectrum of BPS excitations around these classical configurations, and find a natural interpretation in terms of open and closed string states.

Given the rich set of quantum states on AdS side, we need a prescription to map those to SYM operators. There is a natural map from the Hilbert space of quantized branes to the SYM chiral ring. There is also a map from the chiral ring states to BPS operators, as described in Section 4.1. Combining these suggests a precise map from quantized brane states to BPS operators. We check that following this identification gives the expected BPS operators dual to eighth-BPS open strings attached to a half-BPS giant.

This chapter is based on the paper [66].

5.1 Quantization of eighth-BPS branes

In this section we will review how the phase space of eighth-BPS Mikhailov's solutions in $AdS_5 \times S^5$ is described by $\mathbb{CP}^\infty$. This phase space can be geometrically quantized to give a Hilbert space isomorphic to N bosons in a three-dimensional harmonic oscillator. The material in this section is largely based on [38] and we refer the reader there for the more complete treatment.

We first describe the moduli space of giant graviton solutions. The 3-brane action gives a symplectic form on this space, which gives it the structure of a phase space. We describe the symplectic form and then use the geometric quantization prescription [91] to build the Hilbert space and operators.

The starting point is the following construction by Mikhailov [36]. We consider D3-branes wrapping surfaces $\Sigma \subset S^5$ in $AdS_5 \times S^5$ which preserve 1/8 of supersymmetries (eighth-BPS). Mikhailov showed that all such surfaces Σ can be constructed by taking holomorphic functions in $\mathbb{C}^3$

$$P(x, y, z) = \sum_{n_1,n_2,n_3=0}^{\infty} c_{n_1,n_2,n_3}\, x^{n_1} y^{n_2} z^{n_3} \tag{5.1}$$

and intersecting the four-dimensional surface $P(x, y, z) = 0$ with the unit five-sphere $|x|^2 + |y|^2 + |z|^2 = 1$ embedded in $\mathbb{C}^3$. The intersection Σ is generically a three-dimensional surface in S^5 on which we wrap the D3-brane. More precisely, the shape of the D3-brane is a time-dependent solution given by polynomial

$$P(e^{it}x, e^{it}y, e^{it}z) = \sum_{n_1,n_2,n_3=0}^{\infty} c_{n_1,n_2,n_3}\, e^{i(n_1+n_2+n_3)t} x^{n_1} y^{n_2} z^{n_3} \tag{5.2}$$

That is, the time evolution keeps the shape of the D3-brane fixed, and it just rotates with a phase factor in all coordinates.

For the simplest example take the polynomial

$$P(x, y, z) = c\, z - 1. \tag{5.3}$$

Then the $P(x, y, z) = 0$ surface is $z = 1/c$ or time dependent $z(t) = e^{it}/c$ and intersection with S^5 is

$$|x|^2 + |y|^2 = 1 - \frac{1}{|c|^2}. \tag{5.4}$$

This defines a $S^3 \subset S^5$ with radius $r = \sqrt{1 - 1/|c|^2}$, which is the original half-BPS giant graviton of [12].

We now analyze the phase space[1] $\mathcal{M}$ of such eighth-BPS giants in S^5. Let us first consider the space $\mathcal{P}$ of holomorphic surfaces in $\mathbb{C}^3$ given by[2] $P(z) = 0$. The points in

[1] Note that surface $\Sigma \subset S^5$ defines a point in *phase space* rather than just configuration space, because it determines both position and velocity. This is a result of the BPS condition. See, for example, [92].

[2] We will often abbreviate $P(x, y, z)$ as $P(z)$, nevertheless, these are always polynomials of three complex

$\mathcal{P}$ are labelled by coefficients $\{c_{n_1,n_2,n_3}\}$. In fact, the coefficients are projective coordinates $\{c_{n_1,n_2,n_3}\} \sim \{\lambda c_{n_1,n_2,n_3}\}$, because multiplying them by a common factor λ keeps the surface $P(z) = 0$ unchanged. It is convenient to regularize the infinite-dimensional space $\mathcal{P}$ by considering a *finite*-dimensional subspace $\mathcal{P}_C \subset \mathcal{P}$ where only a subset $\{c_{n_1,n_2,n_3} \,|\, (n_1, n_2, n_3) \in C\}$ of coefficients are allowed to be non-zero. If n_C is the number of elements in C, then we get a space spanned by n_C complex projective coordinates, that is, topologically $\mathcal{P}_C = \mathbb{CP}^{n_C-1}$. For example, we could take $C = \{(0,0,0),\ (1,0,0),\ (0,1,0),\ (0,0,1)\}$ for which $\mathcal{P}_C$ is the space of linear polynomials (see (5.19) in the next section), topologically $\mathbb{CP}^3$. In the end the full $\mathcal{P}$ can be defined as a limit $\mathcal{P} = \lim_{d\to\infty} \mathcal{P}_{C_d}$, where C_d is a sequence which includes ever more monomials $C_d \subset C_{d+1}$. For example, C_d could be all coefficients that multiply monomials of degree up to d. The important aspect of this construction is that at every step we are dealing with a complex projective space $\mathbb{CP}^{n_C-1}$. The limiting case is $\mathcal{P} = \mathbb{CP}^\infty$.

Next, the intersection of each $P(z) = 0$ with S^5 is a surface $\Sigma(P) \subset S^5$ which defines the shape of a D3-brane and therefore labels a point in the phase space $\mathcal{M}$. That is, there is a map

$$\mathcal{P} \to \mathcal{M} \tag{5.5}$$

The regularized subspace $\mathcal{P}_C$ is mapped to $\mathcal{M}_C$, which is a finite-dimensional subspace of $\mathcal{M}$. One can argue that $\mathcal{M}_C$ is also $\mathbb{CP}^{n_C-1}$. One problem that has to be dealt with is that the map is many-to-one, that is, different polynomials $P(z) = 0$ can lead to the same intersection Σ. In fact, it was shown in [38] that two polynomials $P_1(z)$ and $P_2(z)$ have the same intersection with S^5 if and only if

$$P_1(z) = p(z)r_1(z), \quad P_2(z) = p(z)r_2(z) \tag{5.6}$$

where $r_1(z) = 0$ and $r_2(z) = 0$ do not intersect S^5. Therefore, in order to get the space $\mathcal{M}_C$ from $\mathcal{P}_C$, we need to identify

$$P(z) \sim P(z)r(z) \tag{5.7}$$

with any $r(z)$ that does not intersect S^5. Note that all polynomials $r(z)$ that do not intersect S^5 are themselves identified with a single polynomial $P(z) = 1$, which is the vacuum point ($\Sigma = \emptyset$) in the phase space. It was also shown in [38] that these identifications can be performed smoothly and the resulting space $\mathcal{M}_C$ is indeed still $\mathbb{CP}^{n_C-1}$. Let us denote the projective coordinates on $\mathcal{M}_C$ by $\{w_{n_1,n_2,n_3}\}$, with indices running over the same set C. The map $\mathcal{P}_C \to \mathcal{M}_C$ then takes the form of functions

$$w_{n_1,n_2,n_3} = w_{n_1,n_2,n_3}(c, \bar{c}) \tag{5.8}$$

They should be such that $w_{n_1,n_2,n_3}(c, \bar{c}) = w_{n_1,n_2,n_3}(c', \bar{c}')$ whenever points c_{n_1,n_2,n_3} and c'_{n_1,n_2,n_3} should be identified.

coordinates.

Let us now turn to the discussion of the symplectic form on the phase space $\mathcal{M}_C$, which is necessary for quantization. The starting point is the world-volume action on a single D3-brane with no world-volume field strength or fermions:

$$S = S_{\rm BI} + S_{\rm WZ} = \frac{1}{(2\pi)^3(\alpha')^2 g_s}\int_\Sigma \mathrm{d}^4\sigma\sqrt{-\tilde{g}} + \int_\Sigma A \tag{5.9}$$

Here $\tilde{g}$ is the induced metric, and A is the four-form background gauge field, such that field strength $F = \mathrm{d}A$ is proportional to S^5 volume form. The symplectic form can then be written as

$$\begin{aligned}\omega &= \int_\Sigma \mathrm{d}^3\sigma\, \delta\left(\frac{\delta S}{\delta \dot{x}^\mu}\right)\wedge \delta x^\mu \\ &= \frac{N}{2\pi^2}\int_\Sigma \mathrm{d}^3\sigma\,\delta\left(\sqrt{-g}g^{0\alpha}\frac{\partial x^\nu}{\partial\sigma^\alpha}G_{\mu\nu}\right)\wedge\delta x^\mu + \frac{2N}{\pi^2}\int_\Sigma \mathrm{d}^3\sigma \frac{\delta x^\lambda\wedge\delta x^\mu}{2}\left(\frac{\partial x^\nu}{\partial\sigma^1}\frac{\partial x^\rho}{\partial\sigma^2}\frac{\partial x^\sigma}{\partial\sigma^3}\right)\epsilon_{\lambda\mu\nu\rho\sigma}\end{aligned} \tag{5.10}$$

Now the metric $G_{\mu\nu}$ and the induced metric $g_{\alpha\beta} = G_{\mu\nu}\partial_\alpha x^\mu\partial_\beta x^\nu$ is taken on a unit radius S^5 (g is related to $\tilde{g}$ by rescaling). This symplectic form is defined on the phase space $\mathcal{M}_{\rm full}$ of *all* configurations of a D3-brane, supersymmetric or not. Space $\mathcal{M}_{\rm full}$ is, of course, much larger than the supersymmetric subspace $\mathcal{M}\subset\mathcal{M}_{\rm full}$. The "coordinates" on $\mathcal{M}_{\rm full}$ are fields $\{x^\mu(\sigma), \dot{x}^\mu(\sigma)\}$, whereas $\mathcal{M}$ is parametrized by "collective coordinates" $\{w_{n_1,n_2,n_3}\}$. In any case, we have a map $\mathcal{M}_C\to\mathcal{M}\to\mathcal{M}_{\rm full}$ and the pullback of (5.10) defines a symplectic form on $\mathcal{M}$ or $\mathcal{M}_C$. In fact, since we have a map $\mathcal{P}_C\to\mathcal{M}_C$ we can also take a pullback of ω on the space of holomorphic polynomials $\mathcal{P}_C$. This pullback will inevitably be degenerate and have singularities, but it can nevertheless be convenient for explicit calculations.

Crucially, it was argued in [38], that not only $\mathcal{M}_C$ is topologically $\mathbb{CP}^{n_C-1}$, but also that the symplectic form ω is globally well defined, closed, and in the same cohomology class as $2\pi N\omega_{FS}$. This implies it is always possible to find such coordinates w_{n_1,n_2,n_3} that the pullback of (5.10) becomes *proportional to the Fubini-Study form*, with coefficient $2\pi N$:

$$\omega = 2\pi N\omega_{FS} = 2N\left[\frac{1}{1+|w|^2}\frac{\mathrm{d}\bar{w}_I\wedge\mathrm{d}w_I}{2i} - \frac{1}{(1+|w|^2)^2}\frac{w_I\bar{w}_J\,\mathrm{d}\bar{w}_I\wedge\mathrm{d}w_J}{2i}\right] \tag{5.11}$$

Here $|w|^2\equiv w_I\bar{w}_I$ and we use shorthand w_I for *inhomogeneous* coordinates on $\mathbb{CP}^{n_C-1}$. For example in the patch $w_{0,0,0}=1$ the index I runs over n_C-1 remaining (n_1,n_2,n_3) tuples in C.

Once we have the phase space manifold as $\mathbb{CP}^{n_C-1}$ with Fubini-Study form as the symplectic form, the geometric quantization is well known. The Hilbert space $\mathcal{H}_C$ is spanned by wavefunctions, which are holomorphic polynomials of the n_C projective coordinates

w_{n_1,n_2,n_3} of degree N

$$\mathcal{H}_C = \left\{ \prod_{(n_1,n_2,n_3)\in C} (w_{n_1,n_2,n_3})^{k_{n_1,n_2,n_3}} \;\Big|\; \sum k_{n_1,n_2,n_3} = N \right\} \tag{5.12}$$

or equivalently polynomials of $n_C - 1$ inhomogeneous coordinates w_{n_1,n_2,n_3} of degree up to N (if we take e.g. $w_{0,0,0} = 1$ in the patch). It is important to note how N enters the definition of Hilbert space purely through setting the scale of ω, which controls the effective Planck constant $1/N$ or the area in phase space that a single quantum state occupies. As we increase N, the area occupied by a state decreases, and we get more states in $\mathcal{H}_C$.

Finally, we need to discuss the conserved charges in the system. There is a natural $U(3)$ symmetry acting on the coordinates (x, y, z) which preserves the shape of Σ. The Cartan subgroup $U(1)^3$ rotating each coordinate by a phase will give three commuting charges L_i that we can use to label the states. The action $(x, y, z) \to (e^{i\alpha_1}x, e^{i\alpha_2}y, e^{i\alpha_3}z)$ induces transformation

$$c_{n_1,n_2,n_3} \to e^{in_1\alpha_1} e^{in_2\alpha_2} e^{in_3\alpha_3} c_{n_1,n_2,n_3} \tag{5.13}$$

on $\mathcal{P}_C$, as seen from (5.1). Now we also need to use the fact argued in [38] that the map $c_{n_1,n_2,n_3} \to w_{n_1,n_2,n_3}$ can be done in a $U(3)$ invariant way, so that the action on the final $\mathcal{M}_C \sim \mathbb{CP}^{n_C-1}$ phase space coordinates is also $w_{n_1,n_2,n_3} \to e^{in_1\alpha_1} e^{in_2\alpha_2} e^{in_3\alpha_3} w_{n_1,n_2,n_3}$. That means we have three vector fields on $\mathcal{M}_C$ generated by L_i

$$V_{L_i} = \sum_{n_1,n_2,n_3} i\, n_i w_{n_1,n_2,n_3} \partial_{n_1,n_2,n_3} - i\, n_i \bar{w}_{n_1,n_2,n_3} \bar{\partial}_{n_1,n_2,n_3} \tag{5.14}$$

We have used the abbreviation

$$\begin{aligned} \partial_{n_1,n_2,n_3} &\equiv \frac{\partial}{\partial w_{n_1,n_2,n_3}} \\ \bar{\partial}_{n_1,n_2,n_3} &\equiv \frac{\partial}{\partial \bar{w}_{n_1,n_2,n_3}} \end{aligned} \tag{5.15}$$

Upon geometric quantization these become operators on the Hilbert space

$$\hat{L}_i = \sum_{n_1,n_2,n_3} n_i w_{n_1,n_2,n_3} \partial_{n_1,n_2,n_3} \tag{5.16}$$

So that the charge of each excitation w_{n_1,n_2,n_3} is simply n_i under each of the respective $U(1)$, and the total charge of a state Ψ is the sum of all excitation charges. Note that the time evolution is given by an overall $U(1)$, generated by Hamiltonian

$$\hat{H} = \hat{L}_1 + \hat{L}_2 + \hat{L}_3 \tag{5.17}$$

This also reflects the BPS condition. Given the charge assignments we can write a partition function over the Hilbert space (5.12)

$$\mathcal{Z}_C(x_1,x_2,x_3) = \mathrm{Tr}_{\mathcal{H}_C}\left(x_1^{L_1}x_2^{L_2}x_3^{L_3}\right) = \left[\prod_{n_1,n_2,n_3\in C}\frac{1}{1-\nu x_1^{n_1}x_2^{n_2}x_3^{n_3}}\right]_{\nu^N} \tag{5.18}$$

The notation $[\ldots]_{\nu^N}$ denotes the coefficient of ν^N, which enforces the degree N of wavefunction. This matches the partition function over the chiral ring in $\mathcal{N}=4$ (see Section 4.1), and so reproduces the correct supersymmetric spectrum from quantizing giant gravitons.

A comment needs to be made on the validity of the D3 world-volume action (5.9). It certainly is a good description for large branes of energy $O(N)$, but not for small ones with high curvature. However, the spectrum of BPS gravitons at energies $O(1)$ is still correctly reproduced by $\mathcal{H}_C$ derived from ω, and that part of the spectrum comes precisely from very small D3-branes, where ω should not be valid. This may be a result of the fact that the full symplectic form, corrected for small branes, is still in the same cohomology class as ω and also $U(3)$ invariant.

5.2 Example: single half-BPS giant

In order to illustrate various concepts in the last section, let us go through an example of linear polynomials. It will also serve as a starting point for further calculations in this chapter. Take $C = \{(0,0,0),\,(1,0,0),\,(0,1,0),\,(0,0,1)\}$, then $\mathcal{P}_C \sim \mathbb{CP}^3$ is the space of hyperplanes

$$P(z) = c_{1,0,0}\,x + c_{0,1,0}\,y + c_{0,0,1}\,z + c_{0,0,0} = 0. \tag{5.19}$$

We abbreviate $c_0 = c_{0,0,0}$, $c_1 = c_{1,0,0}$, $c_2 = c_{0,1,0}$, $c_3 = c_{0,0,1}$. For inhomogeneous coordinates we set $c_0 = 1$.

Intersection with S^5 yields an S^3 of radius

$$r = \sqrt{1-\frac{1}{|c_1|^2+|c_2|^2+|c_3|^2}} \equiv \sqrt{1-\frac{1}{|c|^2}} \tag{5.20}$$

– the same as in (5.4), only $U(3)$-rotated. The energy and momenta of this solution are:

$$E = N\frac{|c|^2-1}{|c|^2}, \quad L_i = N|c_i|^2\frac{|c|^2-1}{|c|^4} \tag{5.21}$$

As typical, the map $\mathcal{P}_C \to \mathcal{M}_C$ is not one-to-one, the region $|c|^2 \le 1$ does not intersect S^5 and so maps to a single point: the vacuum. Good coordinates on $\mathcal{M}_C$ as $\mathbb{CP}^3$ can be constructed by rescaling:

$$w_i = \begin{cases}\sqrt{\frac{|c|^2-1}{|c|^2}}c_i & \text{if } |c|^2 \ge 1\\ 0 & \text{if } |c|^2 \le 1\end{cases} \tag{5.22}$$

As explained in detail in [38], this smoothly contracts “the hole” at $|c|^2 \le 1$ to a point $w_i = 0$.

The symplectic form, which in this case can be calculated explicitly using (5.10), takes the following form in c_i coordinates:

$$\omega = 2N\left[\left(\frac{1}{|c|^2}-\frac{1}{|c|^4}\right)\frac{\mathrm{d}\bar{c}_i\wedge \mathrm{d}c_i}{2i} - \left(\frac{1}{|c|^4}-\frac{2}{|c|^6}\right)\frac{c_i\bar{c}_j\,\mathrm{d}\bar{c}_i\wedge \mathrm{d}c_j}{2i}\right] \tag{5.23}$$

as long as $|c|^2 \geq 1$. In w_i coordinates this becomes

$$\omega = 2N\left[\frac{1}{1+|w|^2}\frac{\mathrm{d}\bar{w}_i\wedge \mathrm{d}w_i}{2i} - \frac{1}{(1+|w|^2)^2}\frac{w_i\bar{w}_j\,\mathrm{d}\bar{w}_i\wedge \mathrm{d}w_j}{2i}\right], \tag{5.24}$$

precisely $2\pi N$ times Fubini-Study form on $\mathbb{CP}^3$, with perfectly good behaviour at $|c|^2 = 1 \sim |w|^2 = 0$.

This $\mathbb{CP}^3$ can now be geometrically quantized to a Hilbert space $\mathcal{H}_C$ of wavefunctions

$$\Psi_{k_1,k_2,k_3} = (w_1)^{k_1}(w_2)^{k_2}(w_3)^{k_3}(w_0)^{N-k_1-k_2-k_3} \tag{5.25}$$

or in terms of only inhomogeneous coordinates (setting $w_0 = 1$)

$$\Psi_{k_1,k_2,k_3} = (w_1)^{k_1}(w_2)^{k_2}(w_3)^{k_3}, \quad \sum k_i \leq N \tag{5.26}$$

The momenta are[3]

$$\hat{L}_i\,\Psi_{k_1,k_2,k_3} = k_i\,\Psi_{k_1,k_2,k_3} \tag{5.27}$$

and total energy

$$\hat{E} = k_1 + k_2 + k_3 \tag{5.28}$$

Note the maximum energy of a state in this $\mathcal{H}_C$ is $E = N$, that of a maximal sphere giant, corresponding to $c_i \to \infty$ in (5.21).

Finally, let us emphasize one point which will be important later on: (5.24) is written in inhomogeneous coordinates where $w_1 = w_2 = w_3 = 0$ corresponds to the vacuum point with $E = 0$. But we can equally well take a different coordinate patch in $\mathbb{CP}^3$, for example where $w_3 = 1$ and (w_0, w_1, w_2) parametrize the point. The new inhomogeneous coordinates are expressed in terms of the old ones as

$$w_0' = \frac{1}{w_3}, \quad w_1' = \frac{w_1}{w_3}, \quad w_2' = \frac{w_2}{w_3} \tag{5.29}$$

The symplectic form (5.24) has the same form in terms of (w_0', w_1', w_2'). But now the point $w_0' = w_1' = w_2' = 0$ corresponds to $w_3 \to \infty$, $c_3 \to \infty$, which is the maximal giant arising from polynomial

$$P(z) = z = 0 \tag{5.30}$$

We can choose to write the wavefunctions in terms of these coordinates

$$\Psi'_{k_0',k_1',k_2'} = (w_0')^{k_0'}(w_1')^{k_1'}(w_2')^{k_2'} \tag{5.31}$$

[3] Remember (5.16) e.g. $L_1 = \sum n_1 w_{n_1,n_2,n_3}\partial_{n_1,n_2,n_3} = w_{1,0,0}\partial_{1,0,0} \equiv w_1\partial_1$

which is, of course, still the same Hilbert space as in (5.25), with a map $\Psi'_{k'_0,k'_1,k'_2} \to \Psi_{k_1=k'_1,k_2=k'_2,k_3=N-k'_0-k'_1-k'_2}$. One difference, though, is that now the vacuum $\Psi'_{0,0,0} = 1$ has $E = L_3 = N$, and the excitations can have negative charges:

$$\hat{L}_1 = w'_1\partial'_1, \quad \hat{L}_2 = w'_2\partial'_2, \quad \hat{L}_3 = -w'_0\partial'_0 - w'_1\partial'_1 - w'_2\partial'_2, \qquad \hat{H} = -w'_0\partial'_0 \tag{5.32}$$

Physically w'_1 and w'_2 quanta keep the giant energy the same, just rotate it in $U(3)$, while w'_0 takes the giant away from maximal by decreasing energy and L_3.

5.3 Fuzzy $\mathbb{CP}$

The Hilbert space $\mathcal{H}_C$ arising from geometric quantization of $\mathbb{CP}^{n_C-1}$ is closely related to *fuzzy* or non-commutative $\mathbb{CP}^{n_C-1}_N$. We will review this relation and use it to show how the holomorphic basis of the Hilbert space, is related to a discretization of the base in a description of $\mathbb{CP}^{n_C-1}_N$ as a toric fibration over a simplex in $\mathbb{R}^{n_C-1}$ [93, 94, 95, 96, 97, 98, 99]. The wavefunctions are localized at points on the toric base and spread out in the torus fibers. With the fuzzy $\mathbb{CP}$ technology in hand, we demonstrate this nice geometrical character of the states, using elementary calculations of expectation values of $SU(n_C)$ Lie algebra elements and their products, evaluated on states of $\mathcal{H}_C$. From this point of view, we find that the corners of the simplex, where the tori degenerate, correspond to distinguished states. We will return to these states in Section 5.4. We will show that they correspond to maximal giants, where the Mikhailov polynomials become monomials.

In the case of half-BPS giant gravitons, this will allow us to relate the Young diagram labels which arise in the construction of corresponding operators in the dual SYM theory, to the coordinates of points in the discretized toric base, which as we will explain is a simplex in $\mathbb{R}^{n_C-1}$. The Young diagram labels have a physical interpretation in terms of brane multiplicities for branes of different angular momenta.

This shows that fuzzy geometry can be a powerful tool in providing a precise connection between quantum states and localization, with its complementary non-locality due to quantum uncertainty, in the moduli space of solutions.

5.3.1 Fuzzy $\mathbb{CP}$ from operators on Hilbert space of giant states

The homogeneous coordinates for projective space $\mathbb{CP}^{n_C-1}$ are $W_0, W_1, \cdots, W_{n_C-1}$. A complete basis for rational functions is provided by

$$\frac{W_{I_1}W_{I_2}\cdots W_{I_n}\bar{W}_{J_1}\bar{W}_{J_2}\cdots\bar{W}_{J_n}}{|W|^{2n}} \tag{5.33}$$

where $|W|^2 = \sum_{i=0}^{n_C-1} W_I\bar{W}_I$. The denominator ensures that these functions are invariant under scaling by a complex number $W_I \to \lambda W_I$. These functions span the function space for $\mathbb{CP}^{n_C-1}$, which we will denote as $\mathrm{Fun}(\mathbb{CP}^{n_C-1})$. This decomposes into irreducible representations of $SU(n_C)$ as

$$\mathrm{Fun}(\mathbb{CP}^{n_C-1}) = \bigoplus_{n=0}^{\infty} V_{n,\bar{n}} \tag{5.34}$$

where $V_{n,\bar{n}}$ transforms as an irreducible representation corresponding to the Young diagram with n columns of length 1 and n columns of length of $n_C - 1$, which we denote as $[n, \bar{n}]$.

As we reviewed in Section 5.1, the geometric quantization of giant gravitons for $AdS_5 \times S^5$ with N units of flux, in a sector of polynomials of dimension n_C, leads to a quantization of the moduli space $\mathbb{CP}^{n_C-1}$ which produces a Hilbert space of holomorphic polynomials of degree N. This Hilbert space $\mathcal{H}_C$ consists of polynomials of degree N in the homogeneous coordinates $W_0, \cdots, W_{n_C-1}$. It can be viewed as the N-fold symmetric tensor product of the fundamental V_{n_C} of $SU(n_C)$

$$\mathcal{H}_C = \mathrm{Sym}^N(V_{n_C}) \tag{5.35}$$

This is also isomorphic to a Hilbert space of oscillators

$$(a^\dagger_{n_C-1})^{n_{n_C-1}} \cdots (a^\dagger_2)^{n_2}(a^\dagger_1)^{n_1}(a^\dagger_0)^{n_0}|0\rangle \tag{5.36}$$

with the constraint $n_0 + n_1 + n_2 + \cdots + n_{n_C-1} = N$. The dimension is

$$\mathrm{Dim}(\mathcal{H}_C) = \binom{N + n_C - 1}{N} \tag{5.37}$$

Given this Hilbert space, it is natural to consider the algebra of operators, i.e the endomorphism algebra $\mathrm{End}(\mathcal{H}_C)$. The decomposition into representations of $SU(n_C)$ is

$$\begin{aligned}\mathrm{End}(\mathcal{H}_C) &= \mathrm{Sym}^N(V_{n_C}) \otimes \mathrm{Sym}^N(\overline{V}_{n_C}) \\ &= \bigoplus_{n=0}^{N} V_{n,\bar{n}}\end{aligned} \tag{5.38}$$

where $V_{n,\bar{n}}$ transforms as the irreducible representation $[n, \bar{n}]$ described above. A basis for $\mathrm{End}(\mathcal{H}_C)$ is given by operators

$$W_{I_1} \cdots W_{I_n} \partial_{W_{J_1}} \cdots \partial_{W_{J_n}} \tag{5.39}$$

or in oscillator language

$$a^\dagger_{I_1} \cdots a^\dagger_{I_n} a_{J_1} \cdots a_{J_n} \tag{5.40}$$

The indices on the oscillators range from 0 to $n_C - 1$, and

$$[a_I, a^\dagger_J] = \delta_{IJ} \tag{5.41}$$

It is clear that the operators in (5.40) have a cut-off at $n = N$, since polynomials of degree N will be annihilated by more than N derivatives. The matrix algebra End($\mathcal{H}_C$) provides a finite dimensional approximation to Fun($\mathbb{CP}^{n_C-1}$) with an $SU(n_C)$ invariant cutoff at N, which is seen by comparing (5.38) with (5.34).

The algebra End($\mathcal{H}_{n_C}$) is generated by operators

$$E_{IJ} = W_I \partial_{W_J} \tag{5.42}$$

or in oscillator language

$$E_{IJ} = a_I^\dagger a_J \tag{5.43}$$

These form a basis for the algebra of $SU(n_C) \oplus U(1)$

$$[E_{IJ}, E_{KL}] = \delta_{JK} E_{IL} - \delta_{IL} E_{JK} \tag{5.44}$$

The traceless generators

$$\tilde{E}_{IJ} = E_{IJ} - \delta_{IJ} \frac{N}{n_C} \tag{5.45}$$

form the Lie algebra of $SU(n_C)$.

Using (5.41) along with the constraint

$$\sum_I a_I^\dagger a_I = N \quad \text{in } \mathcal{H}_{n_C} \tag{5.46}$$

we may also obtain the relations

$$\begin{aligned} E_{IJ}E_{JK} &= (N + n_C - 1)\hat{E}_{IK} \\ E_{IJ}E_{JI} &= N(N + n_C - 1) \\ \tilde{E}_{IJ}\tilde{E}_{JI} &= E_{IJ}E_{JI} - \frac{N^2}{n_C} \end{aligned} \tag{5.47}$$

We also have

$$E_{IJ}E_{KL} = a_I^\dagger a_J^\dagger a_K a_L + \delta_{JK} E_{IL} \tag{5.48}$$

The generators E_{IJ} correspond to the coordinate functions on $\mathbb{CP}^{n_C-1}$ that generate Fun($\mathbb{CP}^{n_C-1}$). In order to see the appearance of $\mathbb{CP}^{n_C-1}$ from the algebra, define rescaled generators

$$\begin{aligned} e_{IJ} &= \frac{1}{N} E_{IJ} \\ e_{IJ;KL} &= \frac{1}{N^2} a_I^\dagger a_J^\dagger a_K a_L \\ e_{I_1 \cdots I_n; J_1 \cdots J_n} &= \frac{1}{N^n} a_{I_1}^\dagger \cdots a_{I_n}^\dagger a_{J_1} \cdots a_{J_n} \end{aligned} \tag{5.49}$$

We have relations

$$\begin{aligned} [e_{IJ}, e_{KL}] &= \frac{1}{N} (\delta_{JK} e_{IL} - \delta_{IL} e_{JK}) \\ e_{IJ} e_{JI} &= 1 + \frac{(n_C - 1)}{N} \\ e_{IJ} e_{KL} &= e_{IJ;KL} + \frac{\delta_{JK}}{N} e_{IL} \end{aligned} \tag{5.50}$$

At large N these relationships simplify and we get a commutative algebra

$$\begin{aligned} [e_{IJ}, e_{KL}] &= 0 \\ e_{IJ} e_{JI} &= 1 \\ e_{IJ} e_{KL} &= e_{IJ;KL} = e_{IL} e_{JK} \end{aligned} \tag{5.51}$$

This defines $\mathbb{CP}^{n_C - 1}$ as an algebraic curve in the space spanned by coordinates e_{IJ}.

The homomorphism from these generators of End$(\mathcal{H}_C)$ to Fun$(\mathbb{CP}^{n_C-1})$ is given by

$$e_{I_1 \cdots I_n; J_1 \cdots J_n} \leftrightarrow \frac{1}{N^n} \frac{W_{I_1} \cdots W_{I_n} \bar{W}_{J_1} \cdots \bar{W}_{J_n}}{|W|^{2n}} \tag{5.52}$$

where $|W|^2 = \sum_{I=0}^{n_C-1} W_I \bar{W}_I$. The homomorphism property is easily established by verifying that the functions on the RHS of (5.52) obey relations (5.51). At finite N, the algebra End$(\mathcal{H}_C)$ is a fuzzy deformation of Fun$(\mathbb{CP}^{n_C-1})$. This can be made precise by using the map to define a star product on the classical algebra [93] [99] [94].

Toric geometry of $\mathbb{CP}$ from the Lie algebra embedding

The coordinates e_{IJ} give a description of $\mathbb{CP}^{n_C-1}$ as embedded in $\mathbb{R}^{n_C^2-1} \subset \mathbb{R}^{n_C^2}$ which is the Lie algebra of $SU(n_C) \subset U(n_C)$. This is an example of a general construction of co-adjoint orbits [93]. Another aspect of the geometry of $\mathbb{CP}^{n_C-1}$ will be of interest to us, namely the fact that it is a toric variety. Let us describe this in the cases of $\mathbb{CP}^2$, which generalizes to the general $n_C > 3$ case.

Given the homogeneous coordinates W_I, we can impose the equivalence $W_I \sim \lambda W_I$ by first setting

$$W_I \bar{W}_I = 1 \tag{5.53}$$

and then modding out by a phase $W_I \sim e^{i\theta} W_I$. This shows that $\mathbb{CP}^{n_C-1}$ is the base space of a fibration of S^{2n_C-1} with S^1 fiber.

Consider the case $n_C = 3$ where we have $\mathbb{CP}^2$. Lets us recall the toric description [100]. Keeping in mind that $|W_0|^2 = 1 - |W_1|^2 - |W_2|^2 \geq 0$, we can consider the quadrant parametrized by coordinates $|W_1|^2, |W_2|^2$. The allowed values of $|W_1|^2, |W_2|^2$ fall inside a triangle with vertices $(0,0), (0,1), (1,0)$. For each chosen point inside the triangle, there is, in the $\mathbb{CP}^2$, a T^2 of phases given by $(\theta_1 = \arg\ W_1, \theta_2 = \arg\ W_2)$. The cycle parametrized by θ_1 collapses on the vertical axis ($|W_1|^2 = 0$), the one parametrized by θ_2 on the horizontal axis ($|W_2|^2 = 0$), and the combination $\theta_1 + \theta_2$ collapses on the line $|W_0|^2 = 0$. See Figure 5.1.

This generalizes straightforwardly to $\mathbb{CP}^{n_C-1}$. The toric description has a base space which is a generalized tetrahedron or *simplex* in $\mathbb{R}^{n_C-1}$. There is a fiber T^{n_C-1} related to angles of W_I (modulo the overall $U(1)$).

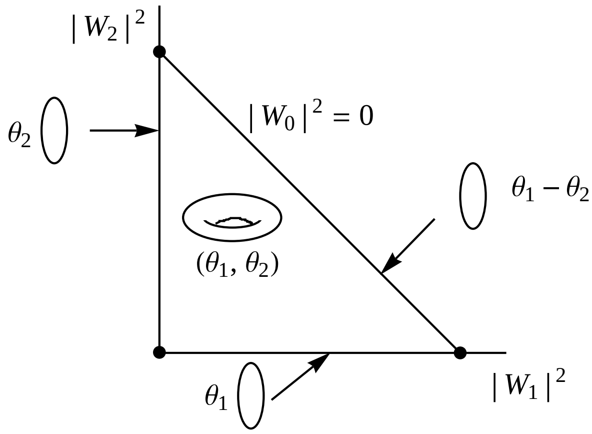


Figure 5.1: $\mathbb{CP}^2$ as a toric fibration. The base is the triangle (2-simplex) parametrized by $|W_1|^2, |W_2|^2$ and the fiber is the torus (θ_1, θ_2). The fiber degenerates to a circle on the edges of the triangle and to a point in the corners.

The identification

$$e_{IJ} = \frac{W_I \bar{W}_J}{|W|^2} \tag{5.54}$$

from (5.52) shows that the diagonal e_{II} are equal to the coordinates used to parametrize the toric base. The off-diagonal e_{IJ} are sensitive to the angles. Their magnitudes are completely determined once the diagonal generators are known since

$$e_{IJ}e_{JI} = e_{II}e_{JJ} \tag{5.55}$$

We can write

$$e_{IJ} = \sqrt{e_{II}e_{JJ}}\ \ e^{i(\theta_I - \theta_J)} \tag{5.56}$$

Hence, the off-diagonal elements of the Lie algebra are associated with the angular variables of the toric description and the diagonal ones with the base space.

Giants: points on toric base and delocalized on fiber

For a state $|\vec{n}\rangle \equiv |n_0, n_1, n_2, \cdots n_{n_C-1}\rangle$ described by the monomial $W_0^{n_0} W_1^{n_1} \cdots W_{n_C-1}^{n_{n_C-1}}$ we can calculate

$$\begin{aligned} \frac{\langle\vec{n}|e_{II}|\vec{n}\rangle}{\langle\vec{n}|\vec{n}\rangle} &= n_I \\ \langle\vec{n}|e_{II}^2|\vec{n}\rangle - \langle\vec{n}|e_{II}|\vec{n}\rangle^2 &= 0 \end{aligned} \tag{5.57}$$

This shows that the states $|\vec{n}\rangle$ have definite locations on the toric base. For the off-diagonal coordinates in the Lie algebra

$$\begin{aligned} X_{IJ} &= \frac{E_{IJ} + E_{JI}}{\sqrt{2}} \\ Y_{IJ} &= -i\frac{E_{IJ} - E_{JI}}{\sqrt{2}} \end{aligned} \tag{5.58}$$

we have

$$\begin{aligned} \langle\vec{n}|X_{IJ}|\vec{n}\rangle = 0 \quad &; \quad \langle\vec{n}|Y_{IJ}|\vec{n}\rangle = 0 \\ \frac{\langle\vec{n}|X_{IJ}^2|\vec{n}\rangle}{\langle\vec{n}|\vec{n}\rangle} &= \left(n_I n_J + \frac{n_I + n_J}{2}\right) \\ \frac{\langle\vec{n}|Y_{IJ}^2|\vec{n}\rangle}{\langle\vec{n}|\vec{n}\rangle} &= \left(n_I n_J + \frac{n_I + n_J}{2}\right) \end{aligned} \tag{5.59}$$

The variances of these off-diagonal coordinates are non-zero and change along the base of the toric fibration parametrized by $\langle E_{II}\rangle$.

A related way to describe where these states are localized and where they are spread, is to note that for any operator $\mathcal{O}$ in $\mathrm{End}(\mathcal{H}_C)$,

$$\langle\vec{n}|\mathcal{O}|\vec{n}\rangle = \mathrm{tr}(\mathcal{O}P_{\vec{n}}) \tag{5.60}$$

where the projector $P_{\vec{n}}$ is $|\vec{n}\rangle\langle\vec{n}|$. The trace is an $SU(n_C)$ invariant functional which becomes an integral $\int \mathrm{d}\Omega$ over $\mathbb{CP}^{n_C-1}$ in the large N limit. The explicit form of the measure can be derived, and will not be important. The projector $P_{\vec{n}}$ maps, under the correspondence (5.52) between $\mathrm{End}(\mathcal{H}_C)$ and $\mathrm{Fun}(\mathbb{CP})$ to

$$\frac{\prod_I (W_I \bar{W}_I)^{n_I}}{|W|^{2\sum_I n_I}} \tag{5.61}$$

This can be viewed as a density matrix associated with the state $|\vec{n}\rangle$. It is independent of the angular part of the W_I, which shows that these states are delocalized in the toric fiber.

To make the discussion more concrete let us take the set C to be the set of coefficients c_{n_1,n_2,n_3} with $n_1+n_2+n_3 \leq d$. We are now looking at polynomials of degree up to d. The index I in the above discussion runs over the triples (n_1, n_2, n_3) with $0 \leq n_1+n_2+n_3 \leq d$. The Hilbert space consists of polynomials in $w_{n_1,n_2,n_3} \equiv W_I$. The diagonal generators $W_I \partial_{W_I}$ of $U(n_C) \supset SU(n_C)$ parametrize points in the toric base. A state such as $w^N_{n_1,n_2,n_3}$ is an eigenstate for the corresponding diagonal generator with maximal eigenvalue N, and has vanishing eigenvalue for the other generators. This defines a corner point on the base simplex for the toric fibration. Consideration of (5.59) shows that these states have distinguished localization properties. Indeed if a single n_I is non-zero, and equal to N as required by the condition $\sum_I n_I = N$, then the uncertainty in the X_{IJ} coordinates, given by $\sqrt{\langle X^2_{IJ}\rangle - \langle X_{IJ}\rangle}$ is order $\sqrt{N}$. If a pair of n_I, n_J are non-zero, then the uncertainty in X_{IJ} is of order N. These states, maximally localized at specific points in the physical moduli space of giants parametrized by w_{n_1,n_2,n_3}, are natural candidates for the maximal (in the sense of being composites involving large giants described by $x = 0, y = 0, z = 0$) giants described by the polynomial $x^{n_1}y^{n_2}z^{n_3} = 0$. Making this precise requires taking into account the fact that there is a non-trivial relation between the coefficients c_{n_1,n_2,n_3} of the polynomials and the coordinates w_{n_1,n_2,n_3} on the moduli space. This will be done in Section 5.4.1 in the context of a discussion of quantum states near these corner points of the moduli space of giants in terms of physical (closed and open-string) excitations.

5.3.2 Examples

We now describe how the connection between the discretized simplex at the toric base and giant graviton states works in concrete sectors, starting with the fuzzy $\mathbb{CP}^1$ for a single half-BPS giant.

Fuzzy $\mathbb{CP}^1$ from single half-BPS giant

Let us start with the canonical example of a single giant in the half-BPS sector. The space of solutions can be parametrized by c in the polynomial $P(z) = cz - 1 = 0$. The coordinate c is mapped to w in $\mathbb{CP}^1 = S^2$ as reviewed in Section 5.1. We will see how fuzzy $\mathbb{CP}^1$ arises in this context.

The Hilbert space $\mathcal{H}$ is $\{w^{n_0}_{0,0,0} w^{n_1}_{0,0,1} \mid n_0+n_1 = N\}$. This has an action of $SU(2) \subset U(2)$. The Lie algebra generators are $E_{00}, E_{11}, E_{01}, E_{10}$ given by (5.42) with $W_0 = w_{0,0,0}, W_1 = w_{0,0,1}$. They generate the matrix algebra End($\mathcal{H}$) which approaches the algebra of functions on the sphere in the large N limit.

To relate to the usual description of fuzzy S^2 [101], we can define

$$\begin{aligned} E_{00} - E_{11} &= 2J_3 \\ E_{01} + E_{10} &= 2J_1 \\ -i(E_{10} + E_{01}) &= 2J_2 \end{aligned} \tag{5.62}$$

The J_i obey the usual $SU(2)$ relations

$$[J_i, J_j] = i\epsilon_{ijk} J_k \tag{5.63}$$

and the Casimir condition

$$J_1J_1 + J_2J_2 + J_3J_3 = \frac{N(N+2)}{4} \tag{5.64}$$

At $N \to \infty$ we rescale $x_i = \frac{2J_i}{N}$. Then $x_ix_i \approx 1$ and the commutators have a factor of $1/N$, so in the limit $N \to \infty$ the product becomes commutative.

We can write the expectation values

$$\begin{aligned} \frac{\langle n|J_3|n\rangle}{\langle n|n\rangle} &= n - \frac{N}{2} && \approx N\langle x_3\rangle \\ \frac{\langle n|J_{1,2}|n\rangle}{\langle n|n\rangle} &= 0 && \approx N\langle x_{1,2}\rangle \\ \frac{\langle n|(J_{1,2})^2|n\rangle}{\langle n|n\rangle} &= \frac{N}{4} + \frac{n(N-n)}{2} && \approx N^2\langle (x_{1,2})^2\rangle \end{aligned} \tag{5.65}$$

These expectation values give us the picture of the states on fuzzy S^2 as shown in Figure 5.2.

Note that generically the states in this basis are spread out with $\langle \Delta x_{1,2}^2\rangle \sim N^2$. However, there are two states which are maximally localized $\langle \Delta x_{1,2}^2\rangle \sim N$: those at the north and south pole. Since they have energies $E = 0$ and $E = N$, and are localized near classical points in phase space, they are interpreted as the states corresponding to the vacuum and the maximal giant.

In the context of geometric quantization, the map from $\mathbb{C}^\infty(S^2)$ to $\mathrm{End}(V_{N+1})$ is the way to go from classical functions on phase space to operators. This allows us to calculate expectation values of classical quantities on states in $\mathcal{H}_C$ by using just $SU(2)$ algebra and representations.

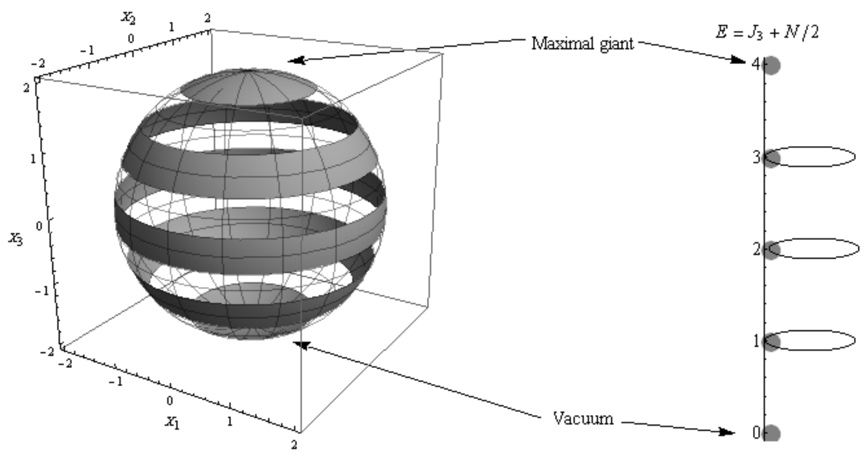


Figure 5.2: States on fuzzy $\mathbb{CP}^1_N$ with $N = 4$. As wavefunctions on S^2 they are localized in the regions shown. If $\mathbb{CP}^1$ is viewed as a circle T^1 fibered over the line segment base, then we get the picture on the right: states are localized on the base and spread out in the fiber. Importantly, the diagram on the base is just the $SU(2)$ weight diagram for V_5.

Fuzzy $\mathbb{CP}^n$ from n half-BPS giants

Let us now generalize the picture to the case of two half-BPS giants

$$P(z) = c_2 z^2 + c_1 z^1 - 1 = 0.$$

The phase space is $\mathbb{CP}^2$ and the Hilbert space $\mathcal{H}$ is $\{w_{0,0,0}^{n_0}, w_{0,0,1}^{n_1} w_{0,0,2}^{n_2} \,|\, n_0 + n_1 + n_2 = N\}$. It is isomorphic to the symmetric $[N, 0]$ representation of $SU(3)$. The space of operators $\mathrm{End}(\mathcal{H})$ approaches the algebra of functions on $\mathbb{CP}^2$ at large N. The action of $U(3) \supset SU(3)$ is given by (5.42) with the identification $W_0 = w_{0,0,0}, W_1 = w_{0,0,1}, W_2 = w_{0,0,2}$.

It is useful to consider the relations

$$\begin{aligned} &E_{00} + E_{11} + E_{22} = N \\ &E_{ij} E_{00} = E_{i0} E_{0j}, \qquad 1 \leq i, j \leq 2 \end{aligned} \tag{5.66}$$

the latter being valid in the large N limit.

For the construction of the fuzzy function algebra, the $SU(3)$ generators are again mapped to coordinates $E_{ij} \sim N e_{ij}$. The $\mathbb{CP}^2$ surface is embedded in this ambient 9-dimensional space. The following equations

$$e_{00} + e_{11} + e_{22} = 1, \quad e_{ij} e_{00} = e_{i0} e_{0j}. \tag{5.67}$$

allow us to formally express all the e_{ij} in terms of two complex e_{01}, e_{02}, as expected for 5 real constraints in 9 dimensional space. This leaves the 4 dimensional $\mathbb{CP}^2$.

The dimensionality can also be understood from the fact that the matrix algebra has dimension $\sim N^4$ at large N. This is a discrete geometry with coordinates such as E_{ii} ranging over N. The vector space dimension of the function space is N^4. A discrete classical space with extent of order N and N^4 elements has dimension 4. This gives a deduction of the dimensionality without explicitly solving the constraints. We expect that such arguments based on state counting, which is known from the BPS partition function, can be used to develop an understanding of giant gravitons for more general AdS/CFT duals. The space of quantum states in more general cases has been discussed from the point of view of dual giants, which are large in AdS and classically point-like in the compact directions [102]. A treatment of quantum states for the case of giants large in the compact directions is not yet available (for some efforts in this direction see [87]).

The $\mathbb{CP}^2$ can be represented as a T^2 fibered over a toric base $E_{11} + E_{22} \leq N$. The basis $|n_0, n_1, n_2\rangle$ gives $\langle E_{11}\rangle, \langle E_{22}\rangle$ localized at points on the base base, and spread out over the T^2 fiber see Figure 5.3.

The special states in this basis $|N, 0, 0\rangle, |0, N, 0\rangle, |0, 0, N\rangle$ again have a nice physical interpretation. They are localized near a point in phase space, which corresponds to vacuum, single maximal giant $z = 0$, and two maximal giants $z^2 = 0$ respectively. We will discuss this more generally in the next section.

Using energies and charges, the labels n_0, n_1, n_2 can be mapped to the Young diagram

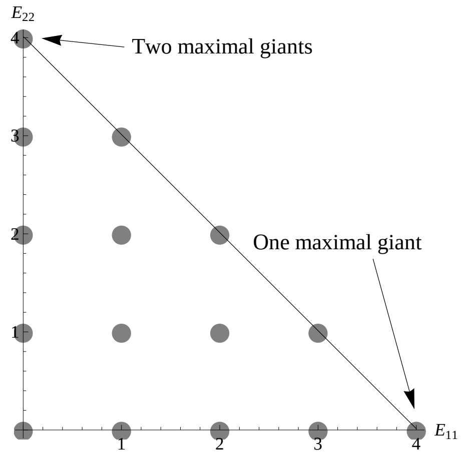


Figure 5.3: States on fuzzy $\mathbb{CP}^2_N$ with $N = 4$. As wavefunctions on $\mathbb{CP}^2$ they are localized on the toric base shown and spread out over T^2 fiber. The diagram is also just the $U(3)$ weight diagram for $[4, 0]$. The three corners of the base correspond to the points where T^2 shrinks to a point, and the states there are localized in all $\mathbb{CP}^2$ directions.

description of the sector of 2 half-BPS giants [16]. The Young diagram has n_1 rows of length 1 and n_2 rows of length 2. The label n_0 is determined as $N - n_1 - n_2$, and may be thought as the number of rows of length 0.

This generalizes to the case of n half-BPS giants. Polynomials are $c_0 + \sum_{i=1}^{n} c_i z_i = 0$. The moduli space can be mapped to $\mathbb{CP}^n$. The states in the Hilbert space map to points on the space described by $E_{11}, ..., E_{nn}$, by taking expectation values of these Lie algebra elements. These points provide a discretization of a simplex in $\mathbb{R}^n$. The corresponding Young diagrams have n_i rows of length i.

5.4 States to maximal giants

5.4.1 Maximal giant states

In this section we will use the picture established in Section 5.3 to analyze the physics of some concrete brane geometries. We will be able to identify configurations of maximal giants with specific states in the Hilbert space, and to classify the spectrum of excitations of these configurations.

First, let us make some general remarks about the correspondence between quantum states and points p in the phase space $\mathcal{M}_C$, where each point is associated to supersymmetric brane geometry. A quantum state $|\Psi_p\rangle \in \mathcal{H}_C$ which “corresponds” to p is a state whose wavefunction is maximally localized around p. Such states are called *coherent states* and can be built, for example, by translating a localized ground state in the phase space [91]. In our case we have $SU(n_C)$ acting transitively on $\mathcal{M}_C$, so we can write

$$|\Psi_p\rangle = e^{i\theta_{IJ}\hat{E}_{IJ}}|\Psi_0\rangle, \quad \text{if } p = e^{i\theta_{IJ}E_{IJ}}p_0, \quad e^{i\theta_{IJ}E_{IJ}} \in SU(n_C) \tag{5.68}$$

The states $|\Psi_p\rangle$ span the Hilbert space but are generically over-complete, since the Hilbert space $\mathcal{H}_C$ is finite dimensional.

On the other hand, if we take the basis such as (5.36), formed by eigenstates of the maximal set of commuting operators E_{II} in $SU(n_C)$, it is complete and orthogonal. Such states generically do not correspond to points on phase space. As we saw in Section 5.3 they are localized in the base simplex and spread out along the the torus fibers of a toric fibration structure of the phase space. However, there are special corner points (vertices of the simplex) in the base where the torus fiber degenerates. The energy eigenstates localized near these corners *are* in fact maximally localized and thus are also coherent states. Recall the wavefunction for such state is

$$\Psi^{\max}_{m_1,m_2,m_3} = (w_{m_1,m_2,m_3})^N \tag{5.69}$$

and it is localized near the point $w_{n_1,n_2,n_3} = \delta_{m_1 n_1}\delta_{m_2 n_2}\delta_{m_3 n_3}$ in w coordinates. Let us denote these points as $p^{\max}_{m_1,m_2,m_3}$. From the point of view of $\mathbb{CP}^{n_C-1}$, all these states look the same as the vacuum state $(w_{0,0,0})^N$, but in different inhomogeneous coordinate patches.

It turns out the corner points enjoy special $U(3)$ transformation properties, that make them particularly accessible to analysis. If we take a generic point in the phase space, its time evolution is generated by $U(1) \subset U(1)^3 \subset U(3)$ which acts as $w_{n_1,n_2,n_3} \to e^{i(n_1+n_2+n_3)\alpha} w_{n_1,n_2,n_3}$. This forms an orbit in the toric fiber. There are special regions in the phase space where this $U(1)$ degenerates to a point. They correspond to static solutions which map to the homogeneous Mikhailov's polynomials. Among these, there are points invariant under the whole $U(1)^3 \subset U(3)$ action $w_{n_1,n_2,n_3} \to e^{in_1\alpha_1}e^{in_2\alpha_2}e^{in_3\alpha_3}w_{n_1,n_2,n_3}$. If the points are labelled by homogeneous coordinates $\{w_{n_1,n_2,n_3}\}$, each coordinate transforms under a different phase under $U(1)^3$. Then the only points that can be $U(1)^3$ invariant are those where only single a coordinate w_{m_1,m_2,m_3} is non-zero, and the transformation acts with an overall phase. But these are precisely the discrete set of corner points $p^{\max}_{m_1,m_2,m_3}$. Thus we see that the corner points can be uniquely identified by their $U(3)$ transformation properties.

We now turn to the question of how to identify explicit brane geometries corresponding to the states $\Psi^{\max}_{m_1,m_2,m_3}$ at the corner points $p^{\max}_{m_1,m_2,m_3}$. In order to map a generic state on the moduli space $\mathcal{M}_C$ parametrized by w_{n_1,n_2,n_3} to an explicit brane geometry defined by a polynomial $P(z)$ we need to know the map between coordinates w_{n_1,n_2,n_3} and polynomial coefficients c_{n_1,n_2,n_3}. This is map is highly non-trivial and we do not know the precise mapping functions. However, the special $U(3)$ transformation properties of the corner points allows us to bypass this difficulty.

We know $p^{\max}_{m_1,m_2,m_3}$ is a fixed point of the $U(1)^3$ action. Since $U(3)$ transforms c_{n_1,n_2,n_3} and w_{n_1,n_2,n_3} coordinates in the same way, we can analogously argue that the corresponding point must have a single non-zero c_{n_1,n_2,n_3} coefficient, in order to be invariant under $c_{n_1,n_2,n_3} \to e^{in_1\alpha_1}e^{in_2\alpha_2}e^{in_3\alpha_3}c_{n_1,n_2,n_3}$. That is, the point in moduli space corresponds to a *monomial*

$$P(z) = x^{m_1} y^{m_2} z^{m_3} = 0\,. \tag{5.70}$$

The condition of being a fixed point does not uniquely pick the degrees (m_1, m_2, m_3) to match those in $p^{\max}_{m_1,m_2,m_3}$. We can complete this identification by checking the charges. The brane geometry specified by (5.70) consists of m_1, m_2, m_3 number of maximal giants wrapped on $x = 0, y = 0, z = 0$ surfaces respectively. It will have charges

$$L_i = N m_i, \quad E = N(m_1 + m_2 + m_3) \tag{5.71}$$

which does exactly match the charges of quantum state (5.69). So we conclude the correspondence

$$\boxed{P(z) = x^{m_1} y^{m_2} z^{m_3} = 0 \quad \leftrightarrow \quad \Psi^{\max}_{m_1,m_2,m_3} = (w_{m_1,m_2,m_3})^N} \tag{5.72}$$

Note that this is a very plausible correspondence: w_{n_1,n_2,n_3} in some sense "corresponds" to c_{n_1,n_2,n_3}, and this is the state at the maximal value of the diagonal E_{II} generator corresponding to w_{m_1,m_2,m_3}.

5.4.2 Spectrum of excitations

Having constructed the states corresponding to maximal giants, we now proceed to study physical properties of these configurations by looking at the spectrum of excitations.

On general grounds, we expect two kinds of low energy excitations in the background of D-branes: open and closed strings. The closed strings are bulk gravitons while the open strings correspond to world-volume excitations such as shape deformations of the branes. The eighth-BPS supersymmetric part of the excitation spectrum must already be included in our Hilbert space $\mathcal{H}_C$, and so we want to identify these states. From the point of view of classical phase space, the neighbourhood of the point $p^{\max}_{m_1,m_2,m_3}$ can be interpreted as small perturbations of the original configuration, as long as the difference in energy is $O(1)$. Therefore, the quantum states that live in this nearby region should be precisely the quantized supersymmetric excitations. We will return to this picture from the point of view of a local analysis of the symplectic form in Section 5.6.1.

It is easy to identify states in the full Hilbert space are "near" the configuration of maximal giants: they will be the ones that differ from the background $(w_{m_1,m_2,m_3})^N$ in $O(1)$ number of quanta. Applying the same notion to the trivial background $w^N_{0,0,0}$ gives the usual Kaluza-Klein spectrum of small graviton multiplets. We expect that the location of states nearby in this sense, computed with expectation values of $\langle X_{II} \rangle$ as in Section 5.3 will be nearby in terms of the Kähler metric on the moduli space. Expressing this more quantitatively would be interesting but is beyond the scope of this work. For example:

$$w_{i_1,i_2,i_3}(w_{m_1,m_2,m_3})^{N-1}, \quad w_{i_1,i_2,i_3} w_{j_1,j_2,j_3}(w_{m_1,m_2,m_3})^{N-2}, \quad \ldots \tag{5.73}$$

are nearby states in this sense. These nearby states form a nice Fock space structure[4] generated by operators:

$$A^\dagger_{k_1,k_2,k_3} \equiv w_{m_1+k_1,m_2+k_2,m_3+k_3}\partial_{m_1,m_2,m_3}, \quad k_i \geq -m_i,\ k_i \neq (0,0,0) \tag{5.74}$$

Around a fixed background the commutators between different modes vanish: $[A^\dagger_{k_1,k_2,k_3}, A^\dagger_{l_1,l_2,l_3}] = 0$. Individual $A^\dagger_{k_1,k_2,k_3}$ should then correspond to the spectrum of allowed single-particle (string) excitations, as long as $k_i \sim O(1)$. Note that the $U(1)^3$ charges of $A^\dagger_{k_1,k_2,k_3}$ are just (k_1,k_2,k_3), and some have negative contribution to energy. This is natural since we are not in the vacuum and there are directions in phase space which decrease the energy.

We make the following proposal for interpretation of the excitations:

- $A^\dagger_{k_1,k_2,k_3}$ with all $k_i \geq 0$ are the **closed string** excitations in the bulk. This is supported by the fact that the spectrum is the same as the graviton spectrum in the vacuum, which is what we expect. That is, there is a single mode for each choice of non-negative (k_1,k_2,k_3) except $(0,0,0)$, which reproduces the large N partition function $\mathcal{Z}(x_1,x_2,x_3) = \prod_{n_1,n_2,n_3} \frac{1}{1-x_1^{n_1}x_2^{n_2}x_3^{n_3}}$.
- $A^\dagger_{k_1,k_2,k_3}$ with at least one $k_i < 0$ are the **open string** excitations on the world-volume. This is a novel spectrum, not visible in the perturbations around the vacuum, but which can be extracted from full partition function of eighth-BPS sector by looking at states near a giant graviton background. The spectrum is dependent on m_i, and carries information about the geometry of the branes.

We will dedicate most of the remaining sections to give further support for our proposed open string spectrum, and to work out some physical implications. In this section we go through a list of examples of backgrounds and take a closer look at the spectra.

Single giant $z = 0$

As discussed in Section 5.2, the equation $z = 0$ describes a giant extended along the intersections of the x, y-planes with the S^5 in space-time. According to (5.72) the background is

$$\Psi^{\max}_{0,0,1} = (w_{0,0,1})^N\,. \tag{5.75}$$

The bulk gravitons are generated by $A^\dagger_{k_1,k_2,k_3} = w_{k_1,k_2,k_3+1}\partial_{0,0,1}$ with non-negative k_i. The world-volume excitations are generated by

$$A^\dagger_{k_1,k_2,-1} = w_{k_1,k_2,0}\,\partial_{0,0,1}, \quad k_1,k_2 \geq 0 \tag{5.76}$$

which is only a *two* parameter family. We interpret them as BPS wave modes on S^3 brane. Note that these excitations only carry momenta in L_1, L_2 but not in L_3 direction. In fact

[4] Strictly speaking, the commutators between the raising and lowering operators of different modes do not commute $[A^\dagger_{k_1,k_2,k_3}, A_{-l_1,-l_2,-l_3}] \neq \delta_{k_1,l_1}\delta_{k_2,l_2}\delta_{k_3,l_3}$, so it is not literally a Fock space algebra of independent oscillators. Nevertheless, for our purposes this does not have any effect, and we will take the liberty of referring to $A^\dagger_{k_1,k_2,k_3}$ as Fock space generators.

this makes sense. The world-volume is the intersection of $z = 0$ with the sphere, so it stretches in the x, y plane. Waves on the world-volume will have L_1, L_2 angular momenta but not L_3. We learn from the counting that there is in fact one BPS wave solution on the world-volume for each pair of charges (k_1, k_2).

There is a special excitation $A^\dagger_{0,0,-1}$ which decreases the energy and does not add momenta. It can be interpreted as the shrinking mode of the giant. The excitations $A^\dagger_{1,0,-1}$ and $A^\dagger_{0,1,-1}$ which do not change energy are just $U(3)$ generators for rotations in (x, z) and (y, z) directions.

Two giants $xy = 0$

This describes a composite of giants along $x = 0$ and $y = 0$. According to (5.72) the corresponding background quantum state is

$$\Psi^{\max}_{1,1,0} = (w_{1,1,0})^N \tag{5.77}$$

This is a configuration of two intersecting branes, as the equation $xy = 0$ has two branches

$$\begin{aligned} x = 0 \quad &: \quad |y|^2 + |z|^2 = 1 \\ y = 0 \quad &: \quad |z|^2 + |x|^2 = 1 \end{aligned} \tag{5.78}$$

There is also an interesting region where the branes intersect along S^1 at $|z|^2 = 1$. We can distinguish the bulk modes

$$A^\dagger_{k_1,k_2,k_3} = w_{k_1+1,k_2+1,k_3}\, \partial_{1,1,0} \tag{5.79}$$

and three types of world-volume excitations:

$$\begin{aligned} A^\dagger_{-1,k_2,k_3} &= w_{0,k_2+1,k_3}\, \partial_{1,1,0} \\ A^\dagger_{k_1,-1,k_3} &= w_{k_1+1,0,k_3}\, \partial_{1,1,0} \\ A^\dagger_{-1,-1,k_3} &= w_{0,0,k_3}\, \partial_{1,1,0} \end{aligned} \tag{5.80}$$

with $k_i \geq 0$. The first two types have analogous spectrum as (5.76) and can be interpreted as waves on $x = 0$ and $y = 0$ giants respectively. There is an additional one-parameter tower $A^\dagger_{-1,-1,k_3}$ which can be interpreted as modes living on the one-dimensional intersection of the $x = 0$ and $y = 0$ branes, or as strings stretching between the two branes. This intersection extends in the $\arg(z)$ direction and indeed this tower has z-charge.

Note that $A^\dagger_{-1,-1,k_3}$ is related to the classical deformation $xy = \epsilon z^{k_3}$. The deformation is indeed most significant near the intersection $x = y = 0$, where $x, y \sim \sqrt{\epsilon}$, consistent with the interpretation that that's where these open strings live. Since in that region $|z| \approx 1$, we can approximate $z = e^{i\psi}$ and so the intersection gets deformed as:

$$x = \frac{\epsilon\, e^{ik_3\psi}}{y} \tag{5.81}$$

that is, with a twist around S^1.

Again we should point out that two modes $A^\dagger_{-1,-1,0}$ and $A^\dagger_{-1,-1,1}$ in the one-parameter family have negative energy. They are related to the deformations $xy = \epsilon$ or $xy = \epsilon z$.

Three giants $xyz = 0$

This is a composite involving three giants $x = 0, y = 0, z = 0$. Following previous arguments, the background state

$$\Psi^{\max}_{1,1,1} = (w_{1,1,1})^N \tag{5.82}$$

The excitations can be summarized in the following table:

Operator	Interpretation
$A^\dagger_{k_1,k_2,k_3}$	Bulk gravitons
$A^\dagger_{-1,k_2,k_3}$	Waves on $x = 0$ giant
$A^\dagger_{k_1,-1,k_3}$	Waves on $y = 0$ giant
$A^\dagger_{k_1,k_2,-1}$	Waves on $z = 0$ giant
$A^\dagger_{k_1,-1,-1}$	Waves on $y = z = 0$ intersection
$A^\dagger_{-1,k_2,-1}$	Waves on $x = z = 0$ intersection
$A^\dagger_{-1,-1,k_3}$	Waves on $x = y = 0$ intersection
$A^\dagger_{-1,-1,-1}$	Composite deformation

(5.83)

Again in this classification $k_i \geq 0$. The first three families of world-volume excitations, each with two-parameter infinite sequences of modes, can be associated with each of the S^3 world-volume branches. The next three one-parameter families naturally correspond to each of the three S^1 intersections.

There is a single extra mode $A^\dagger_{-1,-1,-1}$ that does not fall into the categories discussed so far. It corresponds to the deformation $xyz = \epsilon$. Since $x = y = z = 0$ is not part of the brane world-volume, the largest effect of the deformation is again near the brane intersections. If we look at each pairwise intersection:

$$\begin{aligned} &\text{near } x = y = 0,\ z = e^{i\psi_3}: \qquad xy = \epsilon e^{-i\psi_3}, \\ &\text{near } y = z = 0,\ x = e^{i\psi_1}: \qquad yz = \epsilon e^{-i\psi_1}, \\ &\text{near } z = x = 0,\ y = e^{i\psi_2}: \qquad zx = \epsilon e^{-i\psi_2}. \end{aligned} \tag{5.84}$$

So all three intersections are simultaneously deformed with an "anti-holomorphic" twist, something that would locally come from $xy = \epsilon\bar{z}$, $yz = \epsilon\bar{x}$, $zx = \epsilon\bar{y}$. We are led to the conclusion that even though modes like $xy = \epsilon\bar{z}$ are individually not BPS, the particular *composite* described by $xyz = \epsilon$ is BPS. In other words, we interpret $A^\dagger_{-1,-1,-1}$ as a BPS state involving three open strings living on each S^1 intersection, which are individually not BPS.

Stack of branes $z^m = 0$

This can be interpreted as a limit of multiple branes located at $\prod_{i=1}^{m}(z-a_i)$ where the a_i tend to zero. Background is

$$\Psi^{\max}_{0,0,m} = (w_{0,0,m})^N \tag{5.85}$$

The bulk excitations are again $A^\dagger_{k_1,k_2,k_3} = w_{k_1,k_2,m+k_3}$ and we have world-volume excitations

$$A^\dagger_{k_1,k_2,-p} = w_{k_1,k_2,m-p}\,\partial_{0,0,m} \tag{5.86}$$

where $p = 1\cdots m$. It contains excitation spectrum of a single brane $A^\dagger_{k_1,k_2,-1}$, but there are in total m towers parametrized by p, with varying L_3 charge. This is related to the fact that there are m branes which we can excite, but the identification is not as straightforward, because the branes are coincident ant indistinguishable.

The fact that we do not have an infinite tower of states in the z-direction again makes sense, because we do not have world-volume extension in that direction. The fact that we do get an increasing number of excitations as m increases suggests that the non-abelian nature of the branes effectively blows up an internal circle in the world-volume of the brane in the z-plane, a circle which in ordinary geometry looks to be of vanishing size.

The excitation spectrum above can be viewed as a prediction, based on the eighth-BPS spectrum known from the chiral ring of the $\mathcal{N}=4$ $U(N)$ SYM. It should match calculations starting from the point of view of the non-abelian $U(m)$ gauge theory on coincident branes. The arguments for the spectrum of giant graviton physics developed so far have been based largely on the symplectic form derived from the abelian DBI. Unravelling excitations of specific geometries allows the possibility of using gauge-string duality to predict non-abelian physics of coincident branes. The use of dualities to predict non-abelian brane physics has been illuminating in the past [103, 104]

Three stacks $(xyz)^m = 0$

Finally, let us take a look at the configuration with m coincident branes wrapping each S^3. This will in fact display all features of a generic $x^{m_1}y^{m_2}z^{m_3} = 0$. The background state is

$$\Psi^{\max}_{m,m,m} = (w_{m,m,m})^N \tag{5.87}$$

and we can classify excitations similarly like before:

$$\begin{array}{c|l} \text{Operator} & \text{Interpretation} \\ \hline A^\dagger_{k_1,k_2,k_3} & \text{Bulk gravitons} \\ A^\dagger_{-p,k_2,k_3} & \text{Waves on } x=0 \text{ stack} \\ A^\dagger_{k_1,-p,k_3} & \text{Waves on } y=0 \text{ stack} \\ A^\dagger_{k_1,k_2,-p} & \text{Waves on } z=0 \text{ stack} \\ A^\dagger_{k_1,-p,-q} & \text{Waves on } y=z=0 \text{ intersection} \\ A^\dagger_{-p,k_2,-q} & \text{Waves on } x=z=0 \text{ intersection} \\ A^\dagger_{-p,-q,k_3} & \text{Waves on } x=y=0 \text{ intersection} \\ A^\dagger_{-p,-q,-r} & \text{Composite deformations} \end{array} \tag{5.88}$$

There are now extra parameters $0 < p, q, r \leq m$.

The structure is similar as for $xyz = 0$. First we get modes $A^\dagger_{-p,k_2,k_3}$, etc., living on each stack of branes.

Next, $A^\dagger_{k_1,-p,-q}$ are states living on the S^1 intersection. Now they are labelled by extra parameters (p, q) which can take m^2 values. This relates to the fact that between two stacks of m branes we have m^2 intersections. Just like for modes $A^\dagger_{-p,k_2,k_3}$, here the interpretation is obscured by the fact that the m^2 intersections are in fact identical, and (p, q) does not really label the intersection.

Finally, we have m^3 modes $A^\dagger_{-p,-q,-r}$, which are extensions of the composite BPS mode $A^\dagger_{-1,-1,-1}$ to the case of multiple branes.

5.5 Partition function

In this section we will show how the results of Section 5.4, which were interpreted in terms of the physics of branes, are reflected in the partition function. Since the partition function of BPS states is known from the dual $U(N)$ SYM field theory side, we can view the calculations in this section as recovering, from the dual field theory, without a priori information from branes, the factorization into bulk and world-volume states which is expected from AdS/CFT duality. Specifically we will extract the spectrum of BPS excitations around a single half-BPS sphere giant. The factorization of the spectrum into bulk graviton states and world-volume excitations (5.76) will be obtained here by considering a limit of the partition function which isolates the states discussed as being "near" the single giant $z = 0$ in Section 5.4.

Recall the partition function is (5.18):

$$\mathcal{Z}_N(x_i) = \mathrm{Tr}_{\mathcal{H}}\left(x_1^{L_1} x_2^{L_2} x_3^{L_3}\right) = \sum_{n_1,n_2,n_3} x_1^{n_1} x_2^{n_2} x_3^{n_3} \mathcal{Z}_{N;n_1n_2n_3} \tag{5.89}$$

$$\mathcal{Z}(\nu; x_i) = \sum_{N=0}^{\infty} \nu^N \mathcal{Z}_N(x_i) = \prod_{n_1,n_2,n_3=0}^{\infty} \frac{1}{1 - \nu x_1^{n_1} x_2^{n_2} x_3^{n_3}}. \tag{5.90}$$

That is, $\mathcal{Z}_N(x_i)$ is the partition function counting operators at a fixed N, while $\mathcal{Z}(\nu; x_i)$ is the "grand canonical" partition function with chemical potential ν for N. Then $\mathcal{Z}_N(x_i)$ can be calculated as the coefficient of ν^N in the RHS of (5.90). It is equal to the $\mathcal{N}=4$ SYM chiral ring partition function, counting symmetric polynomials of the $3N$ values of the three diagonal complex scalar matrices Φ_i, after enforcing F-term constraints $[\Phi_i,\Phi_j]=0$, see Section 4.1.

In order to use $\mathcal{Z}_N(x_i)$ to extract the BPS spectrum around a half-BPS sphere giant[5] we must first find a way to isolate the state corresponding to the giant itself. The charges are obviously not enough, because once we fix $L_1=L_2=0,\ L_3=E$ we get *all* of the half-BPS states, and only one of them is a sphere giant (assuming $E\le N$). Recall the half-BPS states can be labelled by Young diagrams with E boxes and height $\le N$. The giant states that we want to focus on are those labelled by the single-column Young diagrams.

In order to find the single-column state we introduce an extra quantum number *size* S by which we label the eighth-BPS states. We can do this by using the N-dependence of the Hilbert spaces $\mathcal{H}_N$. It is natural to consider the sequence of subspaces

$$\mathcal{H}_1\subset\mathcal{H}_2\subset\ldots\subset\mathcal{H}_{N-1}\subset\mathcal{H}_N \tag{5.91}$$

For example, an operator like $\mathrm{tr}(Z^2)$ is considered "the same state" for any N. Then if we pick an operator $\mathcal{O}$ we can ask at what N it gets *excluded*. We label the operator to have size $S(\mathcal{O})$ if it gets excluded below S:

$$\mathcal{O}\in\mathcal{H}_N,\quad \text{iff}\ \ N\ge S \tag{5.92}$$

In the half-BPS sector the state R in the Schur basis gets excluded when N is below the height of the Young diagram $c_1(R)$, so $S=c_1(R)$. In the eighth-BPS sector if we represent states as (5.12), then S is just the number of excitations different from $w_{0,0,0}$. Or, in terms of N bosons, it is the number of bosons in excited states. It is reasonable that S has a physical interpretation in both gauge and the gravity side. It measures how close the state is to the "exclusion bound". For example a sphere giant has $S=E$, and maximal giants are those with $S=N$. A dual giant, on the other hand, has $S=1$.

The partition function for number of states refined by (S,L_1,L_2,L_3) is easy to get. If $Z_{S;n_1n_2n_3}$ is the number of such states then:

$$Z_{S;n_1,n_2,n_3}=\mathcal{Z}_{S;n_1,n_2,n_3}-\mathcal{Z}_{S-1;n_1,n_2,n_3} \tag{5.93}$$

and

$$\begin{aligned} Z(\nu;x_i)&\equiv\sum_{S;n_1,n_2,n_3}\nu^S x_1^{n_1}x_2^{n_2}x_3^{n_3}Z_{S;n_1,n_2,n_3}=(1-\nu)\mathcal{Z}(\nu;x_i)\\ &=\prod_{n_1+n_2+n_3>0}\frac{1}{1-\nu x_1^{n_1}x_2^{n_2}x_3^{n_3}} \end{aligned} \tag{5.94}$$

[5] Here we consider the giant to be of any size $E\le N$, not necessarily maximal

The only difference from (5.90) is that we do not have a term $1/(1-\nu)$, so we only count bosons in excited states.

Now we can uniquely identify the single sphere giant with energy E by specifying $(S, L_1, L_2, L_3) = (E, 0, 0, E)$. In terms of oscillators this is $(w_{0,0,1})^S$ as in (5.75), but not necessarily maximal. The excitations around this state should have charges differing by $O(1)$ from the background. Let us fix the size, and look at states with charges $(S, L_1, L_2, L_3) = (S, n_1, n_2, S + n_3)$ for small n_i. The number of such states is:

$$\widetilde{Z}_{S;n_1,n_2,n_3} \equiv Z_{S;n_1,n_2,n_3+S} \tag{5.95}$$

We can write the corresponding partition function

$$\widetilde{Z}(\nu; x_i) = \sum_{S;n_1,n_2,n_3} \nu^S x_1^{n_1} x_2^{n_2} x_3^{n_3} Z_{S;n_1,n_2,n_3+S} = Z\left(\frac{\nu}{x_3}; x_i\right) \tag{5.96}$$

where the RHS is known explicitly (5.94). Furthermore, we expect the counting $\widetilde{Z}_{S;n_1,n_2,n_3}$ to be independent of S if $n_i \ll S$. That is, the spectrum of excitations should not depend on the size of the giant. We can confirm this by taking $S \to \infty$ limit, which in terms of the partition function reads as

$$\widetilde{Z}(x_i) = \lim_{\nu \to 1} (1-\nu) \widetilde{Z}(\nu; x_i) = \prod_{\substack{n_1+n_2+n_3>0 \\ (n_1,n_2,n_3)\neq(0,0,1)}} \frac{1}{1 - x_1^{n_1} x_2^{n_2} x_3^{n_3-1}} \tag{5.97}$$

This produces finite counting for $O(1)$ charges. The partition function can be conveniently factored into pieces where $n_3 = 0$ and $n_3 > 0$:

$$\widetilde{Z}(x_i) = \left(\prod_{n_1+n_2>0} \frac{1}{1 - x_1^{n_1} x_2^{n_2} x_3^{-1}}\right)\left(\prod_{n_1+n_2+n_3>0} \frac{1}{1 - x_1^{n_1} x_2^{n_2} x_3^{n_3}}\right) \tag{5.98}$$

where in the second term we renamed $n_3 - 1 \to n_3$.

The spectrum (5.98) that we found is almost exactly (5.74). The first factor is generated by $A^\dagger_{n_1,n_2,-1}$ and interpreted as world-volume excitations. The second factor corresponds to $A^\dagger_{n_1,n_2,n_3}$ with non-negative n_i and generates the background graviton spectrum. We are missing here the negative energy mode $A^\dagger_{0,0,-1}$, but that's just because we fixed the size S to be constant in the derivation, while $(0, 0, -1)$ is precisely the mode that decreases size by 1.

We have now demonstrated how to take a limit of the partition function to achieve a factorization into closed and open strings. The same factorization was obtained in Section 5.4 by explicitly looking at the Fock space structure of the states. The quantum number S related to the exclusion of states with varying N, was the additional data beside R-charges we needed to accomplish this. For more general brane configurations discussed in Section 5.4 we would need additional quantum numbers such as the higher conserved charges which determine a Young diagram in the half-BPS case [16, 105]. These higher charges which exist in the oscillator Hilbert space have not yet been exhibited from the gauge theory point of view at weak coupling. The story at zero coupling in the eighth-BPS sector is developed in [105]. The strategy of extracting the expected open-closed factorization of states from the partition function should be specially instructive for unravelling the giant graviton physics in more general examples of AdS/CFT where the S^5 is replaced by a Sasaki-Einstein geometry. In these cases, the partition function is known from the dual quiver gauge theory but the matching of these states with giant gravitons extended in the Sasaki-Einstein space is a largely unexplored subject.

5.6 Local quantization of excitations

In this section we will see how to derive the spectrum (5.76) of the world-volume excitations on a sphere giant directly from the brane action. This provides a non-trivial check of the analysis in Section 5.4 without relying on the fact that the phase space is isomorphic to $\mathbb{CP}^{n_C-1}$ with Fubini-Study symplectic form.

We will mostly focus on the case of maximal giant $P(z) = z = 0$ as in (5.76), but the analysis here works for non-maximal sphere giants $P(z) = z - c_0 = 0$ too, see Appendix F. This in fact provides evidence that generic analysis in Section 5.4 should also work for non-maximal giants.

5.6.1 Structure of perturbations

We start in this section by revisiting the space of perturbations of a spherical giant.

The polynomial defining the unperturbed maximal giant is

$$P_0(z) = z = 0. \tag{5.99}$$

This is a point in the space of polynomials $\mathcal{P}$ and also in the phase space $\mathcal{M}$. In order to study perturbations, we need to identify the neighbourhood of P_0 in $\mathcal{M}$. Naively, one might guess that it is the image of the neighbourhood in $\mathcal{P}$, so a nearby point in $\mathcal{M}$ corresponds to

$$P_{\delta c}(z) = z + \sum_{n_1,n_2,n_3=0}^{\infty} \delta c_{n_1,n_2,n_3} x^{n_1} y^{n_2} z^{n_3} \,. \tag{5.100}$$

First, we note that any perturbation involving a non-zero power of z in fact does not deform the giant at all. This can be seen from the following factorization:

$$P_{\delta c}(z) = \left(z + \sum_{n_1,n_2} \delta c_{n_1,n_2,0}\, x^{n_1} y^{n_2} \right) \left(1 + \sum_{n_3>0,n_1,n_2} \delta c_{n_1,n_2,n_3}\, x^{n_1} y^{n_2} z^{n_3-1} \right) \tag{5.101}$$

where we drop $O(\delta c^2)$ terms. The second factor does not intersect S^5, so under the map

$\mathcal{P} \to \mathcal{M}$, $P_{\delta c}$ has to be identified with just the first factor:

$$P_{\delta b}(z) = z + \sum_{n_1,n_2} \delta b_{n_1,n_2}\, x^{n_1} y^{n_2} \tag{5.102}$$

This is the subspace of P_0 neighbourhood in $\mathcal{P}$ that corresponds to neighbourhood in the actual phase space $\mathcal{M}$.

There is another problem with the guess (5.100) in that it does not, in fact, explore the whole neighbourhood of P_0 in $\mathcal{M}$. Intuitively the reason is that we should be able to add an infinitesimally small disconnected surface by e.g. $P(z) = z(c\,z - 1)$ with $|c|^2 = 1 + \epsilon$, but this polynomial is not a small deformation of P_0 in $\mathcal{P}$. We can handle this case by recalling that there are many polynomials identified with the same point P_0 in $\mathcal{M}$, namely, any

$$\tilde{P}_0(z) = z\, Q(z) \tag{5.103}$$

where $Q(z) = 0$ does not intersect S^5. Any polynomials which are near $\tilde{P}_0(z)$ then also correspond to points in $\mathcal{M}$ near P_0. In particular, if we consider $Q_0(z)$ which just touches S^5 then a deformation

$$\tilde{P}(z) = z\ (Q_0(z) + \delta Q(z)) \tag{5.104}$$

corresponds to a new point in $\mathcal{M}$ near P_0, not included in (5.100). The physical interpretation of this class of deformations is clear from the factorized form of $\tilde{P}(z)$: with $\delta Q(z)$ we are adding infinitesimally small disconnected surfaces rather than deforming the shape of the original sphere giant.

The final conclusion of this section is then that the most general perturbation of a sphere giant $z = 0$ is given by

$$P(z) = \left(z + \sum_{n_1,n_2=0}^{\infty} \delta b_{n_1,n_2}\, x^{n_1} y^{n_2} \right) (Q_0(z) + \delta Q(z)) \tag{5.105}$$

such that $Q_0(z) = 0$ touches S^5 and $Q_0(z) + \delta Q(z)$ intersects it. The first factor involving $\delta b_{n_1,n_2}$ deforms the surface $z = 0$, while the second factor adds infinitesimally small disconnected surfaces. The action and the symplectic form for the two pieces is independent, because it involves an integral over each surfaces separately. That means we have a product structure to the phase space in the neighbourhood of P_0

$$\mathcal{M}_{P_0} = \mathcal{M}^{\rm wv}_{P_0} \times \mathcal{M}^{\rm bulk}_{P_0} \tag{5.106}$$

which has a natural interpretation as world-volume and bulk excitations.

If we perform the quantization locally, we will get a product of Hilbert spaces $\mathcal{H}_{P_0} = \mathcal{H}^{\rm wv}_{P_0} \times \mathcal{H}^{\rm bulk}_{P_0}$, as long as the excitation number is small so we stay in the neighbourhood. Furthermore, note that $\mathcal{M}^{\rm bulk}_{P_0}$ is exactly the same as the full phase space $\mathcal{M}$ around the vacuum point. That is, we might as well be considering quantization of $Q(z) = 0$ which barely intersects S^5, the existence of $z = 0$ brane does not have an effect. That means, we know what $\mathcal{H}_{P_0}^{\text{bulk}}$ is – it matches the low-energy spectrum of the full $\mathcal{H}$ and describes bulk gravitons, generated by Fock space of w_{n_1,n_2,n_3}. We identify $A^{\dagger}_{k_1,k_2,k_3}$ in (5.74) with non-negative k_i as the operators generating this "closed string" Fock space around a giant.

The remaining problem is then to get the world-volume spectrum $\mathcal{H}_{P_0}^{\text{wv}}$ arising from perturbations (5.102).

5.6.2 Quantization of world-volume excitations

We now turn to the analysis of the world-volume deformations of the maximal giant

$$P(z) = z + \sum_{m,n\geq 0} \delta b_{m,n}\, x^m y^n \tag{5.107}$$

We want to explicitly calculate the symplectic form on this slice of phase space (assuming $|\delta b|^2 \ll 1$) and subsequently quantize it. This process of quantizing the phase space "locally" around a solution is analogous to canonical quantization of first-order perturbations using quadratic effective action [106].

We are deforming an S^3 at $z = 0$:

$$|x|^2 + |y|^2 = 1 \tag{5.108}$$

Let us introduce some world-volume coordinates $(\sigma^1, \sigma^2, \sigma^3)$ on S^3, then $x(\sigma^i)$, $y(\sigma^i)$ are embedding functions. Small time-dependent perturbations around the spherical shape can be parametrized by the function $z(\sigma^i, t)$. Effectively these are the 2 real transverse coordinates to S^3 in S^5, which is a single complex scalar field on the world-volume. In principle for non-zero $z(\sigma^i, t)$ we need to modify $x(\sigma^i, t)$, $y(\sigma^i, t)$ such that $|x|^2+|y|^2+|z|^2 = 1$ still holds, however, for $|z| \ll 1$ this effect is second order in perturbation, and we can ignore it. In that case the full symplectic form (5.10) simplifies to (see Appendix F):

$$\omega = \frac{2N}{\pi^2}\int_{S^3} \mathrm{d}^3\sigma \left(\frac{\delta\bar{z}\wedge\delta z}{2i} - \frac{\delta\dot{\bar{z}}\wedge\delta z}{8} + \frac{\delta\bar{z}\wedge\delta\dot{z}}{8}\right) \tag{5.109}$$

where the integral $\mathrm{d}^3\sigma$ is over unit S^3 with its standard volume form.

If we put the time-dependence back in (5.107) according to (5.2) we get

$$z = \delta z = -\sum_{m,n\geq 0} \delta b_{m,n} e^{(m+n-1)it} x^m y^n\,. \tag{5.110}$$

Plugging this in (5.109) we find

$$\omega = \frac{2N}{2\pi^2}\int_{S^3} \mathrm{d}^3\sigma \sum_{m,n\geq 0} (m+n+1)|x|^{2m}|y|^{2n}\frac{\delta\bar{b}_{m,n}\wedge\delta b_{m,n}}{2i} \tag{5.111}$$

The integral is easy to do:

$$\int_{S^3} \mathrm{d}^3\sigma\, |x|^{2m}|y|^{2n} = 2\pi^2 \frac{m!\, n!}{(m+n+1)!} \tag{5.112}$$

Note that we never needed the explicit choice of the coordinate σ^i on the sphere. The final symplectic form evaluated at $P(z) = z$ is thus

$$\omega = 2N \sum_{m,n\geq 0} \frac{m!\, n!}{(m+n)!} \frac{\delta\bar{b}_{m,n} \wedge \delta b_{m,n}}{2i} \tag{5.113}$$

Symplectic form (5.113) is just that of a flat $\mathbb{C}^{n_C-1}$, and has a simple structure of decoupled harmonic oscillators $\delta b_{m,n}$. Quantization of these perturbations has a straightforward Fock space structure

$$\Psi = \prod_{m,n} (b_{m,n})^{k_{m,n}} \tag{5.114}$$

The $U(1)^3$ charges of the oscillators can be inferred from the transformation of $\delta b_{m,n}$ in (5.107) under $z^i \to e^{i\alpha_i} z^i$:

$$P(z) = z + \sum_{m,n\geq 0} \delta b_{m,n} x^m y^n \;\to\; e^{i\alpha_3} \left(z + \sum_{m,n\geq 0} \delta b_{m,n}\, e^{im\alpha_1 + in\alpha_2 - i\alpha_3} x^m y^n \right) \tag{5.115}$$

We have factored out an overall irrelevant phase, to keep z term unchanged. This means $b_{m,n}$ have charges $(L_1, L_2, L_3) = (m, n, -1)$. This does precisely match the spectrum of world-volume excitations $A^\dagger_{k_1,k_2,-1}$ proposed in (5.76).

One way to see this result, is as the derivation of the relationship between c_{n_1,n_2,n_3} coordinates on $\mathcal{P}$ and w_{n_1,n_2,n_3} on $\mathcal{M}$ in this particular region. If we expand the Fubini-Study form (5.11) in the inhomogeneous coordinate patch $w_{0,0,1} = 1$, then we know the symplectic form around $P(z) = z = 0$ must be

$$\omega = 2N \sum_{(n_1,n_2,n_3)\neq(0,0,1)} \frac{\mathrm{d}\bar{w}_{n_1,n_2,n_3} \wedge \mathrm{d}w_{n_1,n_2,n_3}}{2i} \tag{5.116}$$

for $|w|^2 \ll 1$. The $U(1)^3$ charges of w_{n_1,n_2,n_3} in this patch are $(n_1, n_2, n_3 - 1)$. Comparing with (5.113) we can thus identify the coordinates[6]:

$$w_{m,n,0} = \sqrt{\frac{m!\, n!}{(m+n)!}}\, \delta b_{m,n} \tag{5.117}$$

up to corrections of order $O(|\delta b|^2)$. The remaining coordinates w_{n_1,n_2,n_3} with $n_3 \geq 1$ must be associated with the directions in the phase space which add disconnected surfaces. Note, however, following the discussion in the previous section, we can not say that w_{n_1,n_2,n_3} is

[6] Perhaps it is clearer in terms of homogeneous coordinates: $\frac{w_{m,n,0}}{w_{0,0,1}} \approx \sqrt{\frac{m!\,n!}{(m+n)!}} \frac{c_{m,n,0}}{c_{0,0,1}}$, where $w_{0,0,1}, c_{0,0,1} \to \infty$

proportional to $\delta c_{n_1,n_2,n_3}$ in (5.100), although it does have the same charges.

Finally, let us say a word about the limits of approximation in this section. Given the symplectic form (5.113) in $\delta b_{m,n}$ coordinates, a single quantum state occupies an area in phase space

$$|\Delta b_{m,n}|^2 \sim \frac{1}{2N}\frac{(m+n)!}{m!\,n!} \tag{5.118}$$

If we require to stay in the region $\delta b_{m,n} \ll 1$, there is only a finite number of states available to fill, and so the number of excitations in state 5.114 should obey

$$k_{m,n} \ll 2N\frac{m!\,n!}{(m+n)!} \tag{5.119}$$

Note if both m, n are non-zero, the right-hand side could be much less than N. This limit is misleading, however. More precisely, the requirement for approximation (5.109) to be valid is that $\delta z, \delta\dot{z} \ll 1$ in (5.110). We can just as well require the whole integral over S^3 to be small, which, looking at (5.113) boils down to

$$\frac{m!\,n!}{(m+n)!}\,|\delta b_{m,n}|^2 \ll 1 \tag{5.120}$$

So the approximation can be valid even if $\delta b_{m,n} \gg 1$, given m, n are large. In fact, it is valid precisely where $w_{m,n,0} \ll 1$. This is just what we expect from the global picture, because at $w_{m,n,0} \sim O(1)$ the phase space starts looking like $\mathbb{CP}^{n_C-1}$ rather than just local $\mathbb{C}^{n_C-1}$. In $w_{m,n,0}$ coordinates (5.116) a single quantum state occupies area $|w_{m,n,0}|^2 \sim \frac{1}{N}$ so the true limit is

$$k_{m,n} \ll N \tag{5.121}$$

independent of the mode. This is consistent with the requirement $\sum k_{m,n} \leq N$, which we know from the global quantization.

The limit on the mode numbers m, n would be set not by the approximations in our derivation, but rather by the validity of DBI action itself. Since we are in the BPS sector, the string length does not play a role, but we can certainly worry if the waves on the brane have wavelengths of less than Planck length. Recall the Planck length is $N^{-1/4}$ in units of AdS radius, while the wavelengths for mode $\delta b_{m,n}$ are m^{-1} and n^{-1}. Requiring them to be longer than Planck length sets a limit

$$m, n \ll N^{1/4} \tag{5.122}$$

For states with higher quantum numbers the interpretation as waves on the brane will not hold.

5.7 From branes to BPS operators

The Hilbert space of geometrically quantized eighth-BPS branes is (5.12)

$$\mathcal{H}_{\text{D3}} = \left\{ \prod_{n_1,n_2,n_3=0}^{\infty} (w_{n_1,n_2,n_3})^{k_{n_1,n_2,n_3}} \;\Big|\; \sum k_{n_1,n_2,n_3} = N \right\} \tag{5.123}$$

It gives the same finite N spectrum of states, refined by the global charges, as the eighth-BPS chiral ring of SYM (4.33)

$$\mathcal{C} = \left\{ |\vec{m}_1, \vec{m}_2, \ldots, \vec{m}_N\rangle \;|\; \vec{m}_i = (m_i^{(x)}, m_i^{(y)}, m_i^{(z)}) \in \mathbb{Z}^3 \right\}, \quad \vec{m}_1 \geq \vec{m}_2 \geq \ldots \geq \vec{m}_N \tag{5.124}$$

Both can be interpreted as states of N identical particles in $\mathbb{C}^3$: $\vec{m}_i$ labels the state of each particle in some canonical order, while $k_{\vec{m}}$ is the occupation number of state $\vec{m}$. We can rewrite (5.123) equivalently

$$\mathcal{H}_{\text{D3}} = \left\{ \prod_{i=1}^{N} w_{\vec{m}_i} \;|\; \vec{m}_i \in \mathbb{Z}^3 \right\} \tag{5.125}$$

In principle, the equality of partition functions is not enough to establish precise correspondence between the states, because there are many states with any given global charges. However, the matching structures strongly suggest that we can simply identify the labels on both sides, so that the correspondence between quantized D3-brane states and the SYM chiral ring states is:

$$\prod_{i=1}^{N} w_{\vec{m}_i} \in \mathcal{H}_{\text{D3}} \quad \leftrightarrow \quad |\vec{m}_1, \vec{m}_2, \ldots, \vec{m}_N\rangle \in \mathcal{C} \tag{5.126}$$

Furthermore, there is a unique eight-BPS operator corresponding to each chiral ring element, thus via (4.34) we conjecture the dual operators of the D3-brane states to be

$$\boxed{\Psi = \prod_{i=1}^{N} w_{\vec{m}_i} \quad \leftrightarrow \quad \mathcal{O}_{\text{BPS}} = (1 - \Delta_D^{-1}\Delta_2)\mathcal{O}(\vec{m}_1, \vec{m}_2, \ldots, \vec{m}_N)} \tag{5.127}$$

with (4.31)

$$\mathcal{O}(\vec{m}_1, \vec{m}_2, \ldots, \vec{m}_N) = \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^\sigma (\Phi^{\vec{m}_1})^{i_1}_{i_{\sigma(1)}} (\Phi^{\vec{m}_2})^{i_1}_{i_{\sigma(2)}} \cdots (\Phi^{\vec{m}_N})^{i_1}_{i_{\sigma(N)}} \tag{5.128}$$

This is, basically, a more refined version of the proposal in [37].

Since we have identified a set of states in Section 5.4 that have an interpretation as localized semiclassical branes with excitations, we can compare them against known results in SYM. First, let us take a single half-BPS maximal giant (5.75). The identification implies

$$\Psi^{\max}_{0,0,1} = (w_{0,0,1})^N \quad \to \quad \mathcal{O}((0,0,1),(0,0,1),\ldots) = \det(Z) \tag{5.129}$$

which is the correct dual operator.

Now take the half-BPS giant with a single open string excitation $A^\dagger_{0,k,-1}$, as in (5.76). According to (5.127) we have

$$(w_{0,k,0})(w_{0,0,1})^{N-1} \;\to\; \mathcal{O} = (1-\Delta_D^{-1}\Delta_2)\,\chi(Z;Y^k) \tag{5.130}$$

The operator $\chi(Z;Y^k)$ follows from (5.128), using the notation defined in (4.72), up to normalization. As analysed in Section 4.3, this is the expected operator for the maximal giant with a BPS string attached, and we even know the leading order result (4.90) after applying $(1-\Delta_D^{-1}\Delta_2)$.

A single two-charge excitation $A^\dagger_{k_1,k_2,-1}$ gives

$$(w_{k_1,k_2,0})(w_{0,0,1})^{N-1} \;\to\; \mathcal{O} = (1-\Delta_D^{-1}\Delta_2)\,\chi(Z;\{X^{k_1}Y^{k_2}\}) \tag{5.131}$$

where $\{X^{k_1}Y^{k_2}\}$ is the symmetrized product. In fact, we could just as well take $\chi(Z;X^{k_1}Y^{k_2})$, since it only differs by permutations and thus corresponds to the same chiral ring element. Consequently, after applying $(1-\Delta_D^{-1}\Delta_2)$ the result will be proportional to the same BPS operator

$$(1-\Delta_D^{-1}\Delta_2)\,\chi(Z;\{X^{k_1}Y^{k_2}\}) \propto (1-\Delta_D^{-1}\Delta_2)\,\chi(Z;\;X^{k_1}Y^{k_2}) \tag{5.132}$$

However, the reason to take symmetrized product is that it will only be corrected by a small amount, because it is in the same $U(3)$ multiplet as $\chi(Z;\{Y^{k_1+k_2}\})$. The correction to $\chi(Z;X^{k_1}Y^{k_2})$, on the other hand, would be large.

We can extend the map to multiple excitations

$$(w_{k_1,k_2,0})(w_{l_1,l_2,0})(w_{0,0,1})^{N-2} \;\to\; \mathcal{O} = (1-\Delta_D^{-1}\Delta_2)\,\chi(Z;\{X^{k_1}Y^{k_2}\},\{X^{l_1}Y^{l_2}\}) \tag{5.133}$$

Since Δ_2 will still vanish on $\chi(Z;\{X^{k_1}Y^{k_2}\},\{X^{l_1}Y^{l_2}\})$ to leading order, the correction $\Delta_D^{-1}\Delta_2$ will similarly be small. In general, the action of open string creation operators $A^\dagger_{k_1,k_2,-1}$ translates to inserting words into $\chi(Z;W_1,W_2,\ldots)$ and the fact that we get approximately orthogonal states is ensured by the emerging open string planar expansion (see Appendix D).

Next, consider the generators $A^\dagger_{0,k,0}$ that we have identified with closed string states. The map to operators confirms this:

$$(w_{0,k,1}))(w_{0,0,1})^{N-1} \;\to\; \mathcal{O} = (1-\Delta_D^{-1}\Delta_2)\,\chi(Z;ZY^k) \sim \chi(Z)\mathrm{tr}(Y^k) \tag{5.134}$$

This is because $\chi(Z;W)$ factorizes when Z is found at the boundaries of the word W

$$\chi(Z;ZY^k) \sim \chi(Z)\mathrm{tr}(Y^k) \tag{5.135}$$

The resulting product of operators $\chi(Z)\mathrm{tr}(Y^k)$ is dual to a composition of the two states: the maximal giant $\chi(Z)$ and the closed string $\mathrm{tr}(Y^k)$. Note that an open string state such as $\chi(Z;YZY)$ would have the same charge, but it is non-BPS and thus does not appear in the spectrum of quantized BPS branes.

We conclude that the map (5.127) gives the expected results in the near-half-BPS sector, where we know the operators in SYM. The crucial problem is to go beyond half-BPS, and explore genuinely quarter- or eighth-BPS brane configurations. The map from maximal intersecting branes to states (5.72) suggests

$$P(z) = x^{n_1}y^{n_2}z^{n_3} \quad \leftrightarrow \quad \Psi = (w_{n_1,n_2,n_3})^N \quad \leftrightarrow \quad \mathcal{O} = (1-\Delta_D^{-1}\Delta_2)\mathrm{det}(X^{n_1}Y^{n_2}Z^{n_3}) \tag{5.136}$$

This is plausible, since the determinant is equal to

$$\mathrm{det}(X)^{n_1}\mathrm{det}(Y)^{n_2}\mathrm{det}(Z)^{n_3} \tag{5.137}$$

thus we get a product of operators dual to individual giants, and the product should correspond to composition of objects. The problem is that $\mathrm{det}(X^{n_1}Y^{n_2}Z^{n_3})$ is most likely not near-BPS, so $(1-\Delta_D^{-1}\Delta_2)$ will give a large correction, which we don't know how to calculate.

It is nevertheless interesting to consider, for example, the case of two intersecting giants (5.77)

$$xy = 0 \quad \leftrightarrow \quad \mathcal{O} = (1-\Delta_D^{-1}\Delta_2)\mathrm{det}(XY) \tag{5.138}$$

The various excitations (5.79), (5.80) would then correspond to

$$\begin{array}{lcl} A^\dagger_{k_1,k_2,k_3} & \to & P^{-1}\mathrm{det}(XY)\mathrm{Str}(X^{k_1}Y^{k_2}Z^{k_3}) \\ A^\dagger_{-1,k_2,k_3} & \to & P^{-1}\chi(XY;Y^{k_2+1}Z^{k_3}) \sim P^{-1}\mathrm{det}(Y)\chi(X;Y^{k_2}Z^{k_3}) \\ A^\dagger_{k_1,-1,k_3} & \to & P^{-1}\chi(XY;X^{k_1+1}Z^{k_3}) \sim P^{-1}\mathrm{det}(X)\chi(Y;X^{k_1}Z^{k_3}) \\ A^\dagger_{-1,-1,k_3} & \to & P^{-1}\chi(XY;Z^{k_3}) \end{array} \tag{5.139}$$

where we use the shorthand $P^{-1} = (1-\Delta_D^{-1}\Delta_2)$. The $\chi(XY;W)$ is still defined as in (4.72), substituting XY for Z. The factorization properties of determinants nicely capture the fact that $A^\dagger_{-1,k_2,k_3}$ is an excitation of $x=0$ giant, $A^\dagger_{k_1,-1,k_3}$ is an excitation of $y=0$ giant, while $A^\dagger_{-1,-1,k_3}$ lives on the intersection, and thus is non-factorizable. However, the potential problem is that if correction P^{-1} is large, it might completely change the states, and different excitations might not even be orthogonal. The calculation of the one-loop anomalous dimension of $\mathrm{det}(XY)$ was performed in [107], and found to be $O(\lambda)$, which is the same order as excited string states. This indicates that indeed $\mathrm{det}(XY)$ is likely to have large mixing with non-BPS states, and finding Δ_D^{-1} in this sector is a very hard problem.

Finally, let us make a note regarding orthogonality of the basis (5.127). We know that in general it is *not* orthogonal, because in the half-BPS sector it contains operators like $\mathrm{tr}(Z^n)$ (see also discussion near (4.38)). In general, in the half-BPS sector, the basis closely reproduces Schur polynomials $\chi_R(Z)$ in the regime of sphere giants (tall columns), but not in the regime of AdS giants (long rows). In fact, this nicely correlates with the fact that the states in $\mathcal{H}_{\mathrm{D3}}$ arise from quantizing D3-branes expanding in S^5. Thus in the regime of large giant gravitons the operators closely approximate classical states, and are appropriately orthogonal. On the other hand, states such as $\Psi = w_{0,0,n}$ correspond to holomorphic polynomials $cz^n - 1 = 0$ with many microscopic giants, the regime where DBI action should not be valid and the operators should not be expected to be dual to semiclassical D3-brane states. Thus one would expect the operator basis (5.127) to be orthogonal at least in the regime of states dual to large D3-brane configurations, such as (5.139), but not in the full Hilbert space.

Chapter 6

Conclusions

In this thesis we have explored the AdS/CFT in the non-planar finite N limit, focusing on the BPS sector.

One direction of research was the free (zero superpotential) limit of gauge theories. We found a complete finite N basis of BPS operators for arbitrary quiver gauge theories, and also computed the chiral ring structure constants in this basis. The key feature of our construction was the use of symmetric group and diagrammatic techniques: all results can be expressed nicely by modifying the defining quiver diagram itself, and interpreting the nodes and lines as group theoretic objects – representation labels, Littlewood-Richardson coefficients, branching coefficients.

Interestingly, our results are valid at the Seiberg-like fixed point of the SCFTs, where the superpotential vanishes, but the theory is at an interacting fixed point with strong gauge coupling. That is, we find a complete basis for the chiral ring and the structure constants at the IR fixed point. In the context of AdS/CFT, these fixed points should be dual to tensionless strings on various geometries. Unfortunately, the worldsheet theory can not be solved in this regime, so we can not perform any rigorous comparisons. On the other hand, our results from gauge theories give a wealth of information – such as counting formulae for various quivers – that could guide the string theory construction. In fact, there is a way to interpret the counting formulae in terms of coverings of Riemann surfaces, from which one may try to recover a worldsheet description. This approach was very useful in the study of 2D Yang-Mills [108], and some results in that direction are presented in our original paper [57].

The applicability of our results is not limited to four-dimensional SCFTs. Since everything is done in the free inner product and with no position dependence, one can interpret the calculations as being performed in a matrix model, where the two-point function is also $\langle (\Phi_a)^i_j (\Phi^\dagger_b)^k_l \rangle = \delta_{ab}\delta^i_l\delta^k_j$. As such, the orthogonal bases we have constructed can be applied whenever such matrix models appear. What our results then provide is a solution to the purely algebraic problem of finding the relationships between traces of multiple matrices.

The other direction of research was the non-zero coupling BPS spectrum, where we stick to the original $\mathcal{N} = 4$ SYM. The main goal was to find BPS operators dual to the eighth-BPS Mikhailov's brane configurations. We found a formal construction for the complete (but not orthogonal) BPS basis (4.34), and conjectured an identification with the quantized brane Hilbert space (5.127). One concrete example, where we could perform an explicit calculation, was the operator dual to a maximal half-BPS giant with an attached open string. The classical brane configuration is an oscillating S^3 maximal giant, given by the polynomial $z + \delta b\, y^L = 0$. When quantized, these modes correspond to BPS open strings, and the dual operator is commonly identified with $\chi(Z; Y^L)$. Using our prescription we have calculated the leading correction (4.90), which gives a much better approximation to the true BPS operator. By performing this calculation we also concluded that $\chi(Z; Y^L)$ *is* a good approximation of the BPS operator as long as $L \ll \sqrt{N}$, and the small anomalous dimension it has is due to mixing with the non-BPS component, which we have found explicitly.

Many open problems remain. One immediate question is how to interpret the corrections to $\chi(Z; Y^L)$ we found in (4.90). Each of the correction terms $\chi(Z; Y^k, Y^i Z Y^j)$ by itself can be identified with two strings attached to the giant, one of them in the ground state, the other one excited and non-BPS. It would be extremely interesting if this could be matched with the open string world-sheet theory in the plane wave limit, studied in [27, 109], but with non-zero string coupling. The suggestive interpretation is that when string interactions are turned on, the string can break into two, and the true BPS ground state of the corresponding Hamiltonian involves a mixture with these multi-string states. The corrections to the operator will also have an effect when calculating three-point functions between BPS states, perhaps those could be matched with the AdS calculation.

The main problem with the suggested map (5.127) is that the calculations of the correction P^{-1} are only tractable for the near-half-BPS solutions. The most interesting extension of this work would be to find the dual operator to a genuinely quarter-BPS configuration, such as two intersecting maximal giants $xy = 0$. According to our prescription, in order to do that, one has to find $P^{-1}\det(XY)$, and it is not likely that the correction will be small. Perhaps the resulting corrections can be interpreted as some "resolution" of the S^1 intersection of the branes.

One of the key results of Chapter 5 was the derivation of the BPS open string spectrum on various maximal giant configurations, including the strings living on the S^1 intersection (5.83). It would be very interesting if these results can be confirmed by world-sheet calculations. Furthermore, the operators dual to the BPS excitation spectrum of the $xy = 0$ solution are then given by (5.139). A crucial consistency check of the identification would be to see, if these operators are orthogonal to each other.

Solving these issues would open up a very exciting avenue to study microstates of near-extremal black holes analysed in [110, 111, 112]. From the AdS perspective, taking two stacks of intersecting branes $x^n y^m = 0$ creates an extremal black hole with the near-horizon geometry of AdS_3. The conjectured dual of this AdS_3 is the two-dimensional CFT arising from the open string excitations on the S^1 intersection. The analysis of the operators $\det(X^n Y^m)$, plausibly dual to these backgrounds, was initiated in [107]. It was found that they have a large anomalous dimension, thus unlikely to be BPS. According to our prescription, the BPS component can be recovered by finding $P^{-1}\det(X^n Y^m)$. Then the open string excitations on S^1 should be dual to operators such as $P^{-1}\chi(X^n Y^m; W)$.

Ultimately, we expect the BPS basis (4.34) to provide duals to semiclassical states only in the regime where Mikhailov's eighth-BPS branes is a good description. This is consistent with the identification (5.127). If we instead take states such as $\Psi = w_{0,0,n}$, containing many small branes, the dual operator is $\mathrm{tr}(Z^n)$, so we do not reproduce any semiclassical state, such as an AdS giant $\chi_{[n]}(Z)$. The guiding principle in the half-BPS sector was to look for orthogonal basis, and it would be very desirable to do that here. One can expect that having a complete orthogonal basis would interpolate between different solutions – sphere giants, AdS giants, new geometries – just like Schur polynomial basis. However, orthogonalizing (4.34) is an extremely challenging problem, given that we don't know how to compute Δ_D^{-1}.

There is one lesson that could be learned from Schur polynomials. The basis (4.34), when restricted to the half-BPS, in fact corresponds to N *bosons* on $\mathbb{C}$. The "correct" Schur polynomial basis instead corresponds to N *fermions* on $\mathbb{C}$. This can be shown to arise from Van der Monde determinant in the single matrix model [18]. So whereas (4.34) is a basis for N bosons on $\mathbb{C}^3$, perhaps we need some modification of it which is more fermion-like, and reduces to the fermion basis in the half-BPS sector. This approach has been attempted with the three-matrix model [54], but it used a conjectured form of the measure to account for integrated out off-diagonal terms. It should be possible, in principle, to derive the correct measure by using our map from chiral ring wavefunctions to operators.

Appendix A

Symmetric group formulae

A.1 General

$R \vdash n$ will denote a Young diagram with n boxes, associated with an irreducible representation (irrep) of S_n. A Young diagram R is also associated with an irrep of $U(N)$, when the length of the first column $l(R)$ obeys the constraint $l(R) \leq N$. $\mathrm{Dim}_N(R)$ denotes the dimension of $U(N)$ irrep R. $d(R)$ is the dimension of S_n irrep R.

$$\mathrm{Dim}_N(R) = \frac{f_N(R)}{h(R)}, \quad d(R) = \frac{n!}{h(R)} \tag{A.1}$$

$\mathrm{Dim}_N(R)$ is the dimension of $U(N)$ irrep R. $d(R)$ is the dimension of S_n irrep R. $f_N(R)$ is the (N-dependent) product of weights of boxes in the Young diagram. $h(R)$ is the product of hook lengths. Describing the boxes of a Young diagram with coordinates (i, j) running along rows and columns respectively, with r_i being the row lengths and c_j the column lengths

$$\begin{aligned} f_N(R) &= \prod_{i,j}(N - i + j) \\ h(R) &= \prod_{i,j}(r_i + c_j - i - j + 1) \end{aligned} \tag{A.2}$$

The Kronecker Delta over the symmetric group $\delta(\sigma)$, defined to be 1 if the argument is 1 and zero otherwise. It is also defined, by linearity, over formal sums of group elements with complex coefficients (the group algebra) by picking the coefficient of the identity permutation. It has an expansion in characters. There is a related character orthogonality relation, obtained by summing over irreps

$$\sum_{R \vdash n} \frac{d(R)}{n!} \chi_R(\sigma) = \delta(\sigma) \tag{A.3}$$

$$\sum_{R \vdash n} \chi_R(\sigma)\chi_R(\tau) = \sum_{\gamma \in S_n} \delta(\gamma\sigma\gamma^{-1}\tau^{-1}) \tag{A.4}$$

The characters are traces of matrix elements $\chi_R(\sigma) = \sum_i D^R_{ii}(\sigma)$. The matrix elements satisfy $D^R_{ij}(\sigma) = D^R_{ji}(\sigma^{-1})$). Orthogonality relations from summing over σ are

$$\sum_{\sigma\in S_n} D^R_{ij}(\sigma) D^S_{kl}(\sigma) = \frac{n!}{d(R)} \delta_{RS}\delta_{ik}\delta_{jl} \tag{A.5}$$

$$\sum_{\sigma\in S_n} \chi_R(\sigma)\chi_S(\sigma\tau) = \frac{n!}{d(R)} \delta_{RS}\chi_R(\tau) \tag{A.6}$$

$$\sum_{\sigma\in S_n} \chi_R(\sigma)\chi_S(\sigma) = n!\,\delta_{RS} \tag{A.7}$$

$$\sum_{\sigma\in S_n} D^R_{ij}(\sigma) N^{c(\sigma)} = \delta_{ij} f_N(R) \tag{A.8}$$

$$\sum_{\sigma\in S_n} \chi_R(\sigma) N^{c(\sigma)} = d(R) f_N(R) = n!\,\mathrm{Dim}_N(R) \tag{A.9}$$

$$\sum_{\sigma\in S_n} \mathrm{tr}\left(P^{\nu^-,\nu^+}_{R\to \boldsymbol{r}} D^R(\sigma)\right) N^{c(\sigma)} = \delta^{\nu^-\nu^+} d(\boldsymbol{r})\, f_N(R) \tag{A.10}$$

The last equation involves generalized projectors (intertwining operators) linking different copies (labelled by ν^+,ν^-) of the irrep $\boldsymbol{r}$ of a subgroup $H \subset S_n$. We will describe these subgroup reduction in more detail in the next subsection. For derivations of the above identities, the reader may consult e.g. [113].

A.2 Branching coefficients

Consider a subgroup $H \subset S_n$ of the form

$$H = S_{n_1} \times S_{n_2} \times \dots \tag{A.11}$$

An irrep R of S_n can be decomposed into irreps $\boldsymbol{r} = (r_1, r_2, \dots)$ of H

$$\begin{aligned} V_R^{(S_n)} &= \bigoplus_{\substack{r_1\vdash n_1 \\ r_2\vdash n_2}} V_{r_1}^{(S_{n_1})} \otimes V_{r_2}^{(S_{n_2})} \otimes V_R^{r_1 r_2} \\ |V_R^{r_1 r_2}| &= g(r_1, r_2; R) \end{aligned} \tag{A.12}$$

The states in R are spanned by the basis $|R; \boldsymbol{r}, \nu, \boldsymbol{l}\rangle$ where $\boldsymbol{r}, \nu$ labels the irrep of H (ν is the multiplicity label, if $\boldsymbol{r}$ appears multiple times in the decomposition), and $\boldsymbol{l} = (l_1, l_2, \dots)$ is a state in $\boldsymbol{r} = (r_1, r_2, \dots)$. Branching coefficients $B^{R\to \boldsymbol{r},\nu}_{i\to \boldsymbol{l}}$ are defined to be the components of the vector $|R; \boldsymbol{r}, \nu, \boldsymbol{l}\rangle$ in terms of any orthogonal basis for R.

$$B^{R\to \boldsymbol{r},\nu}_{i\to \boldsymbol{l}} = \langle R; i | R; \boldsymbol{r}, \nu, \boldsymbol{l}\rangle = \langle R; \boldsymbol{r}, \nu, \boldsymbol{l} | R; i\rangle \tag{A.13}$$

Since the representations of S_n can be chosen to be real, branching coefficients are real $(B^{R\to \boldsymbol{r},\nu}_{i\to \boldsymbol{l}})^* = B^{R\to \boldsymbol{r},\nu}_{i\to \boldsymbol{l}}$.

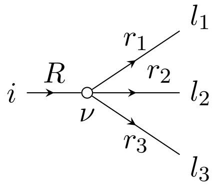
The multiplicities $g(r_1, r_2; R)$ are given by the Littlewood-Richardson rule, which instructs us to put together the boxes of r_2 alongside those of r_1, subject to some conditions (see e.g [86]). These are usually first encountered in physics in the context of irreps of $U(N)$ but the present description in terms of reduction $S_n \to H$ is related to that by Schur-Weyl duality. Some times we will informally write

$$r_1 \otimes r_2 = \bigoplus_R g(r_1, r_2; R)\, R \tag{A.14}$$

in place of the more accurate (A.12).

We use the following diagrammatic notation for the branching coefficients

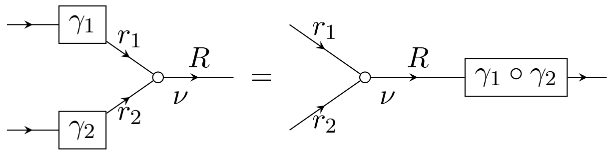
$$B^{R\to(r_1,r_2,r_3),\nu}_{i\to(l_1,l_2,l_3)} \equiv$$



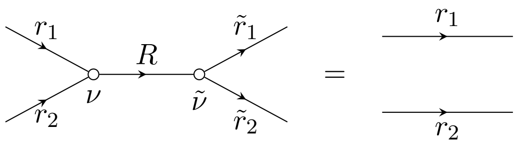
(A.15)

Because of reality, the diagram with arrows reversed is equal.

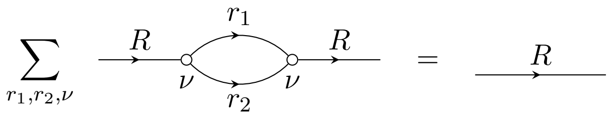
Here we list the properties of branching coefficients in the diagrammatic notation, followed by the corresponding equations. For illustration we take the subgroup $H = S_{n_1} \times S_{n_2}$, with the generalization to more factors being straightforward.



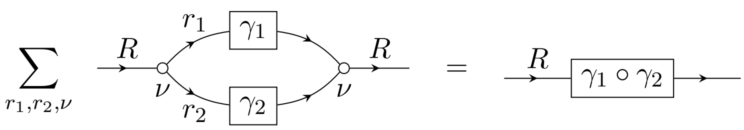
(A.16)



$\times\, \delta_{r_1\tilde{r}_1}\delta_{r_2\tilde{r}_2}\delta_{\nu\tilde{\nu}}$ (A.17)



(A.18)



(A.19)

The equations can be read off by assigning some state labels to each edge and branching coefficients for each white node. As usual with index notation, we need free indices matching on both sides of the equation for the open ends of lines, and repeated indices appearing on internal legs are assumed to be summed :

$$D^{r_1}_{i_1 j_1}(\gamma_1) D^{r_2}_{i_2 j_2}(\gamma_2) B^{R\to(r_1 r_2),\nu}_{k\to j_1,j_2} = B^{R\to(r_1 r_2),\nu}_{j\to i_1,i_2} D^R_{kj}(\gamma_1\circ\gamma_2) \tag{A.20}$$

$$B^{R\to(r_1,r_2);\nu}_{k\to i_1,i_2} B^{R\to(\tilde r_1,\tilde r_2);\tilde\nu}_{k\to j_1,j_2} = \delta_{i_1 j_1}\delta_{i_2 j_2}\delta_{\nu\tilde\nu}\delta_{r_1\tilde r_1}\delta_{r_2\tilde r_2} \tag{A.21}$$

$$\sum_{r_1,r_2,\nu} B^{R\to(r_1 r_2),\nu}_{i\to k_1,k_2} B^{R\to(r_1 r_2),\nu}_{j\to k_1,k_2} = \delta_{ij} \tag{A.22}$$

$$\sum_{r_1,r_2,\nu} B^{R\to(r_1 r_2),\nu}_{i\to j_1,j_2} D^{R_1}_{j_1 k_1}(\gamma_1) D^{R_2}_{j_2 k_2}(\gamma_2) B^{R\to(r_1 r_2),\nu}_{j\to k_1,k_2} = D^R_{ij}(\gamma_1\circ\gamma_2) \tag{A.23}$$

As an example of the generalization to $H = \times_b S_{n_b}$ with an arbitrary finite number of factors, the second equation above becomes :

$$B^{R\to\cup_b r_b;\nu}_{k\to\cup_b i_b} B^{R\to\cup_b \tilde r_b;\tilde\nu}_{k\to\cup_b j_b} = \delta_{\nu,\tilde\nu}\prod_b \delta_{r_b,\tilde r_b}\delta_{i_b j_b} \tag{A.24}$$

Another useful identity is

$$\chi_R(\gamma_1\circ\gamma_2) = \sum_{r_1,r_2} g(r_1,r_2;R)\chi_{r_1}(\gamma_1)\chi_{r_2}(\gamma_2) \tag{A.25}$$

which we get by taking the trace of (A.19).

A.3 Clebsch-Gordan coefficients

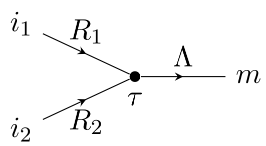
The standard tensor product of S_n irreps, where we take a tensor product of two irreps R, S of S_n and then decompose into irreps T of S_n with multiplicities $C(R,S,T)$, also plays a key role in this work.

$$\begin{aligned} V^{(S_n)}_R \otimes V^{(S_n)}_S &= \bigoplus_{T\vdash n} V_T \otimes V^T_{RS} \\ |V^T_{RS}| &= C(R,S,T) \end{aligned} \tag{A.26}$$

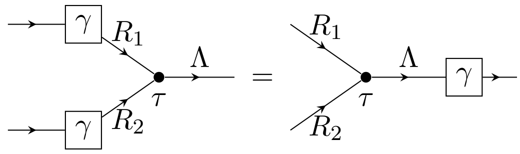
To distinguish the coupling of irreps $(r_1, r_2, \cdots)$ of $H = S_{n_1}\times S_{n_2}\cdots$ into irreps R of S_n (with $\sum_b n_b = n$) with the present decomposition relating three irreps of S_n, the former are sometimes called outer products of symmetric group irreps. while the latter are called Kronecker products. The Kronecker products are also called inner products sometimes but we will avoid that terminology, to avoid confusion with the scalar product of states within an irrep, which we will freely call inner product.

The diagrammatic notation for the Clebsch-Gordan coefficient will be a black node:

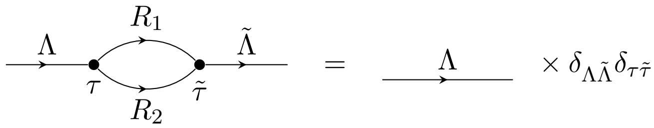
$$S^{R_1 R_2,\Lambda\,\tau}_{i_1\, i_2,\, m} = \tag{A.27}$$



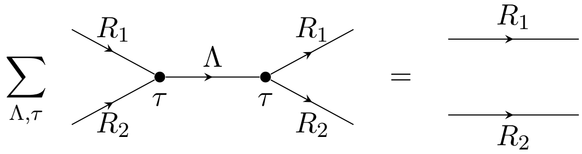
It obeys the following identities:



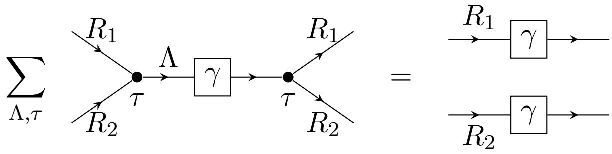
(A.28)



(A.29)



(A.30)



(A.31)

The corresponding equations are:

$$D^{R_1}_{i_1 j_1}(\gamma) D^{R_2}_{i_2 j_2}(\gamma) S^{R_1 R_2, \Lambda\,\tau}_{j_1\; j_2,\, m} = S^{R_1 R_2, \Lambda\,\tau}_{i_1\; i_2,\, l} D^{\Lambda}_{lm}(\gamma) \tag{A.32}$$

$$S^{R_1 R_2, \Lambda\,\tau}_{i_1\; i_2,\, l} S^{R_1 R_2, \tilde{\Lambda}\,\tilde{\tau}}_{i_1\; i_2,\, m} = \delta_{\Lambda\tilde{\Lambda}} \delta_{\tau\tilde{\tau}} \delta_{lm} \tag{A.33}$$

$$\sum_{\Lambda,\tau} S^{R_1 R_2, \Lambda\,\tau}_{i_1\; i_2,\, m} S^{R_1 R_2, \Lambda\,\tau}_{j_1\; j_2,\, m} = \delta_{i_1 j_1} \delta_{i_2 j_2} \tag{A.34}$$

$$\sum_{\Lambda,\tau} S^{R_1 R_2, \Lambda\,\tau}_{i_1\; i_2,\, l} D^{\Lambda}_{lm}(\gamma) S^{R_1 R_2, \Lambda\,\tau}_{j_1\; j_2,\, m} = D^{R_1}_{i_1 j_1}(\gamma) D^{R_2}_{i_2 j_2}(\gamma) \tag{A.35}$$

A.4 Multiplicities

Here we collect identities involving multiplicities $g(r_1, r_2; R)$ and $C(R_1, R_2, \Lambda)$.

Using (A.19) and (A.17) leads to:

$$\chi_R(\sigma_1 \circ \sigma_2) = \sum_{r_1 \vdash n_1} \sum_{r_2 \vdash n_2} g(r_1, r_2; R) \chi_{r_1}(\sigma_1) \chi_{r_2}(\sigma_2) \tag{A.36}$$

From this, Littlewood-Richardson multiplicity can be calculated as

$$g(r_1, r_2; R) = \frac{1}{n_1! n_2!} \sum_{\sigma_1 \in S_{n_1}} \sum_{\sigma_2 \in S_{n_2}} \chi_{r_1}(\sigma_1) \chi_{r_2}(\sigma_2) \chi_R(\sigma_1 \circ \sigma_2) \tag{A.37}$$

Analogously, for Clebsch-Gordan coefficients:

$$\chi_{R_1}(\sigma) \chi_{R_2}(\sigma) = \sum_{\Lambda \vdash n} C(R_1, R_2, \Lambda) \chi_\Lambda(\sigma) \tag{A.38}$$

and

$$C(R_1, R_2, \Lambda) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_{R_1}(\sigma)\chi_{R_2}(\sigma)\chi_\Lambda(\sigma) \tag{A.39}$$

Combining the above we find:

$$\begin{aligned} &\frac{1}{n_1!n_2!} \sum_{\sigma_1 \in n_1} \sum_{\sigma_2 \in n_2} \chi_{R_1}(\sigma_1 \circ \sigma_2)\chi_{R_2}(\sigma_1 \circ \sigma_2) \\ &= \sum_{r_1 \vdash n_1} \sum_{r_2 \vdash n_2} g(r_1, r_2; R_1) g(r_1, r_2; R_2) \\ &= \sum_{\Lambda \vdash n} C(R_1, R_2, \Lambda) g([n_1], [n_2]; \Lambda) \end{aligned} \tag{A.40}$$

where $[n_1]$ and $[n_2]$ are trivial representations for the corresponding groups, arising from $\frac{1}{n_1!n_2!} \sum_{\sigma_1,\sigma_2} \chi_\Lambda(\sigma_1 \circ \sigma_2)$.

A.5 Ω

Ω is a central element in the group algebra $\mathbb{C}(S_n)$, dependant on an integer parameter N:

$$\Omega = \sum_{\sigma \in S_n} N^{C(\sigma)-n} \sigma \tag{A.41}$$

where $C(\sigma)$ is the number of cycles in σ. We can also expand it as

$$\Omega = 1 + \frac{1}{N}\Sigma_{[2]} + \frac{1}{N^2}(\Sigma_{[3]} + \Sigma_{[2,2]}) + \frac{1}{N^3}(\Sigma_{[4]} + \Sigma_{[3,2]} + \Sigma_{[2,2,2]}) + \dots \tag{A.42}$$

where Σ_T is the sum of all permutations in the conjugacy class T. Ω is related to the dimension of $U(N)$

$$(\mathrm{Dim}_N(R))^m = \left(\frac{N^n d_R}{n!}\right)^m \frac{\chi_R(\Omega^m)}{d_R} \tag{A.43}$$

This can be inverted to find

$$\Omega^m = \frac{1}{n!} \sum_{\sigma \in S_n} \sum_{R \vdash n} d_R \left(\frac{n!\mathrm{Dim}_N(R)}{N^n d_R}\right)^m \chi_R(\sigma)\sigma \tag{A.44}$$

In particular, if $n \leq N$, the sum expression also makes sense for $m = -1$

$$\Omega^{-1} = \frac{N^n}{n!^2} \sum_{R \vdash n} \sum_{\sigma \in S_n} \frac{d_R^2}{\mathrm{Dim}_N(R)} \chi_R(\sigma)\sigma \tag{A.45}$$

since $\mathrm{Dim}_N(R)$ never vanishes. It can be expanded in $1/N$ to get a series valid at any $n \leq N$

$$\begin{aligned} \Omega^{-1} = 1 - \frac{1}{N}\Sigma_{[2]} + \frac{1}{N^2}\left(\frac{n(n-1)}{2} + 2\Sigma_{[3]} + \Sigma_{[2,2]}\right) \\ - \frac{1}{N^3}\left(\frac{n^2+3n-8}{2}\Sigma_{[2]} + 5\Sigma_{[4]} + 2\Sigma_{[3,2]} + \Sigma_{[2,2,2]}\right) + \dots \end{aligned} \tag{A.46}$$

Note that this series is infinite, unlike (A.42), it does not terminate at N^{-n+1}. For a given n it can be re-summed to get a closed expression. For example

$$\Omega^{-1}_{n=2} = \frac{N^2}{N^2-1}\left(1 - \frac{1}{N}\Sigma_{[2]}\right) \tag{A.47}$$

$$\Omega^{-1}_{n=3} = \frac{N^4}{(N^2-1)(N^2-4)}\left(1 - \frac{1}{N}\Sigma_{[2]} + \frac{1}{N^2}(-2 + 2\Sigma_{[3]})\right) \tag{A.48}$$

$$\begin{aligned}\Omega^{-1}_{n=4} = \frac{N^6}{(N^2-1)(N^2-4)(N^2-9)}\Bigg(1 - \frac{1}{N}\Sigma_{[2]} + \frac{1}{N^2}(-8 + 2\Sigma_{[3]} + \Sigma_{[2,2]}) \\ + \frac{1}{N^3}(4\Sigma_{[2]} - 5\Sigma_{[4]}) + \frac{1}{N^4}(6 - 3\Sigma_{[3]} + 6\Sigma_{[2,2]})\Bigg)\end{aligned} \tag{A.49}$$

Appendix B

Quiver characters

B.1 Symmetric group characters

The usual symmetric group characters $\chi_R(\sigma) \equiv \mathrm{tr}(D^R(\sigma))$ obey the following identities

$$\chi_R(\sigma) = \chi_R(\sigma^{-1}) \tag{B.1}$$
$$\chi_R(\sigma) = \chi_R(\gamma\sigma\gamma^{-1}) \tag{B.2}$$
$$\frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\chi_S(\sigma) = \delta_{RS} \tag{B.3}$$
$$\sum_{R\vdash n}\chi_R(\sigma)\chi_R(\tau) = \sum_{\gamma\in S_n}\delta(\sigma\gamma\tau\gamma^{-1}) \tag{B.4}$$

They could be summarized as: invariance under inversion (B.1); invariance under conjugation (B.2); orthogonality in representation labels (B.3); orthogonality in conjugacy classes (B.4). There is also a useful generalization of (B.3)

$$\frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\chi_S(\sigma\tau) = \delta_{RS}\frac{\chi_R(\tau)}{d(R)} \tag{B.5}$$

B.2 Restricted quiver characters

Restricted quiver character is defined as

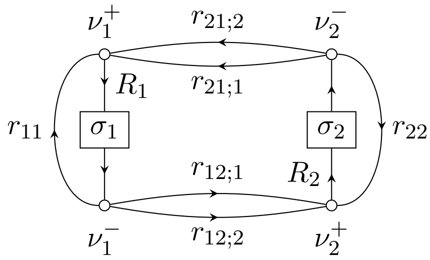
$$\chi_Q(\boldsymbol{L},\boldsymbol{\sigma}) = \prod_a D^{R_a}_{i_a j_a}(\sigma_a) B^{R_a\to\bigcup_{b,\alpha} r_{ab;\alpha},\nu_a^-}_{j_a\to\bigcup_{b,\alpha} l_{ab;\alpha}} B^{R_a\to\bigcup_{b,\alpha} r_{ba;\alpha},\nu_a^+}_{i_a\to\bigcup_{b,\alpha} l_{ba;\alpha}} \tag{B.6}$$

with

$$\boldsymbol{L} \equiv \{R_a, r_{ab;\alpha}, \nu_a^-, \nu_a^+\}, \qquad \boldsymbol{\sigma} \equiv \{\sigma_a\} \tag{B.7}$$

Diagrammatically, for the case $\mathbb{C}^3/Z_2$,

$$\chi_{\mathbb{C}^3/\mathbb{Z}_2}(\boldsymbol{L},\boldsymbol{\sigma}) = \tag{B.8}$$



Note that for the case of a trivial quiver with a single node and a single field, the quiver character is precisely the symmetric group character.

They obey analogous identities to (B.1), (B.2), (B.3), (B.4):

$$\chi_Q(\boldsymbol{L},\boldsymbol{\sigma}) = \chi_Q(\boldsymbol{L},\boldsymbol{\sigma}^{-1}) \tag{B.9}$$

$$\chi_Q(\boldsymbol{L},\boldsymbol{\sigma}) = \chi_Q(\boldsymbol{L},\mathrm{Adj}_{\boldsymbol{\gamma}}(\boldsymbol{\sigma})) \tag{B.10}$$

$$\frac{1}{\prod_a n_a!}\sum_{\boldsymbol{\sigma}}\frac{\prod_a d(R_a)}{\prod_{a,b,\alpha} d(r_{ab;\alpha})}\chi_Q(\boldsymbol{L},\boldsymbol{\sigma})\chi_Q(\tilde{\boldsymbol{L}},\boldsymbol{\sigma}) = \delta_{\boldsymbol{R}\tilde{\boldsymbol{R}}}\delta_{\boldsymbol{r}\tilde{\boldsymbol{r}}}\delta_{\boldsymbol{\nu}^+\tilde{\boldsymbol{\nu}}^+}\delta_{\boldsymbol{\nu}^-\tilde{\boldsymbol{\nu}}^-} \tag{B.11}$$

$$\sum_{\boldsymbol{L}}\frac{\prod_a d(R_a)}{\prod_{a,b,\alpha} d(r_{ab;\alpha})}\chi_Q(\boldsymbol{L},\boldsymbol{\sigma})\chi_Q(\boldsymbol{L},\boldsymbol{\tau}) = \frac{\prod_a n_a!}{\prod_{a,b,\alpha} n_{ab;\alpha}!}\sum_{\boldsymbol{\gamma}}\prod_a \delta(\mathrm{Adj}_{\boldsymbol{\gamma}}(\sigma_a)\tau_a^{-1}) \tag{B.12}$$

For the proofs see Appendix G.1.

The generalization of (B.5) is

$$\sum_{\boldsymbol{\sigma}}\chi_Q(\boldsymbol{L},\boldsymbol{\tau\sigma})\chi_Q(\tilde{\boldsymbol{L}},\boldsymbol{\sigma}) = \delta_{\boldsymbol{R}\tilde{\boldsymbol{R}}}\delta_{\boldsymbol{r}\tilde{\boldsymbol{r}}}\delta_{\boldsymbol{\nu}^-\tilde{\boldsymbol{\nu}}^-}\prod_a \frac{n_a!}{d(R_a)}\mathrm{tr}\left(D^{R_a}(\tau_a)P^{\nu_a^+\tilde{\nu}_a^+}_{R_a\to\bigcup_{b,\alpha} r_{ba;\alpha}}\right) \tag{B.13}$$

where

$$(P^{\nu_a^+\tilde{\nu}_a^+}_{R_a\to\bigcup_{b,\alpha} r_{ba;\alpha}})_{i_a\tilde{i}_a} \equiv B^{R_a\to\bigcup_{b,\alpha} r_{ba;\alpha},\nu_a^+}_{i_a\to\bigcup_{b,\alpha} l_{ba;\alpha}}B^{R_a\to\bigcup_{b,\alpha} r_{ba;\alpha},\tilde{\nu}_a^+}_{\tilde{i}_a\to\bigcup_{b,\alpha} l_{ba;\alpha}} \tag{B.14}$$

B.3 Covariant quiver characters

The covariant quiver characters are defined as

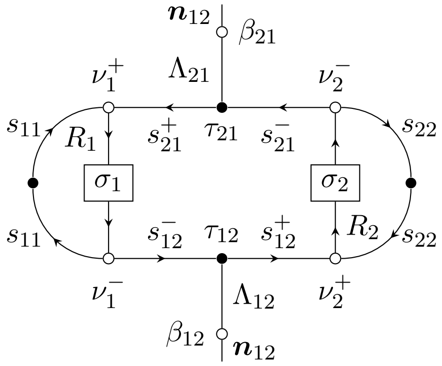
$$\chi_Q(\boldsymbol{K},\boldsymbol{\sigma}) = \left(\prod_a D^{R_a}_{i_a j_a}(\sigma_a)B^{R_a\to\bigcup_b s^-_{ab},\nu_a^-}_{j_a\to\bigcup_b l^-_{ab}}B^{R_a\to\bigcup_b s^+_{ba},\nu_a^+}_{i_a\to\bigcup_b l^+_{ba}}\right)\left(\prod_{a,b} B^{\Lambda_{ab}\to[\boldsymbol{n}_{ab}],\beta_{ab}}_{l_{ab}} S^{\;s^+_{ab}\;s^-_{ab},\;\Lambda_{ab}\tau_{ab}}_{\;l^+_{ab}\;\tilde{l}^-_{ab},\;l_{ab}}\right) \tag{B.15}$$

with

$$\boldsymbol{K} = \{R_a, s^+_{ab}, s^-_{ab}, \nu_a^+, \nu_a^-, \Lambda_{ab}, \tau_{ab}, n_{ab;\alpha}, \beta_{ab}\}, \qquad \boldsymbol{\sigma} = \{\sigma_a\} \tag{B.16}$$

Diagrammatically, for the $\mathbb{C}^3/\mathbb{Z}_2$ case,

$$\chi_{\mathbb{C}^3/\mathbb{Z}_2}(\boldsymbol{K},\boldsymbol{\sigma}) = \tag{B.17}$$



They obey identities:

$$\chi_Q(\boldsymbol{K},\boldsymbol{\sigma}) = \chi_Q(\boldsymbol{K},\boldsymbol{\sigma}^{-1}) \tag{B.18}$$

$$\chi_Q(\boldsymbol{K},\boldsymbol{\sigma}) = \chi_Q(\boldsymbol{K},\mathrm{Adj}_{\boldsymbol{\gamma}}(\boldsymbol{\sigma})) \tag{B.19}$$

$$\frac{1}{\prod_a n_a!}\sum_{\boldsymbol{\sigma}}\left(\prod_a d(R_a)\right)\chi_Q(\boldsymbol{K},\boldsymbol{\sigma})\chi_Q(\tilde{\boldsymbol{K}},\boldsymbol{\sigma}) = \delta_{\boldsymbol{K}\tilde{\boldsymbol{K}}} \tag{B.20}$$

$$\sum_{\boldsymbol{K}}\left(\prod_a d(R_a)\right)\chi_Q(\boldsymbol{K},\boldsymbol{\sigma})\chi_Q(\boldsymbol{K},\boldsymbol{\tau}) = \frac{\prod_a n_a!}{\prod_{a,b,\alpha} n_{ab;\alpha}!}\sum_{\boldsymbol{\gamma}}\prod_a \delta(\mathrm{Adj}_{\boldsymbol{\gamma}}(\sigma_a)\tau_a^{-1}) \tag{B.21}$$

And

$$\sum_{\boldsymbol{\sigma}}\chi_Q(\boldsymbol{K},\boldsymbol{\tau\sigma})\chi_Q(\tilde{\boldsymbol{K}},\boldsymbol{\sigma}) = \delta_{\boldsymbol{R}\tilde{\boldsymbol{R}}}\delta_{\boldsymbol{s}^-\tilde{\boldsymbol{s}}^-}\delta_{\boldsymbol{\nu}^-\tilde{\boldsymbol{\nu}}^-} \times\prod_a\left(\frac{n_a!}{d(R_a)}D^{R_a}_{i_a\tilde{i}_a}(\tau_a)B^{R_a\to\bigcup_b s_{ba}^+,\nu_a^+}_{i_a\to\bigcup_b l_{ba}}B^{R_a\to\bigcup_b \tilde{s}_{ba}^+,\tilde{\nu}_a^+}_{\tilde{i}_a\to\bigcup_b \tilde{l}_{ba}}\prod_b S^{s_{ba}^+s_{ba}^-;\Lambda_{ba}\tau_{ba}\beta_{ba}\boldsymbol{n}_{ba}}_{l_{ba}k_{ba}}S^{\tilde{s}_{ba}^+s_{ba}^-;\tilde{\Lambda}_{ba}\tilde{\tau}_{ba}\tilde{\beta}_{ba}\boldsymbol{n}_{ba}}_{\tilde{l}_{ba}k_{ba}}\right) \tag{B.22}$$

$$\sum_{\boldsymbol{\sigma},\boldsymbol{\tau}}\chi_Q(\boldsymbol{K},\boldsymbol{\tau\sigma})\chi_Q(\tilde{\boldsymbol{K}},\boldsymbol{\sigma})\prod_a N^{c(\tau_a)} = \delta_{\boldsymbol{K}\tilde{\boldsymbol{K}}}\prod_a\frac{n_a!}{d(R_a)}f_N(R_a) \tag{B.23}$$

Appendix C

General basis from invariance

Here we want to show how solving the invariance (3.84)

$$\mathcal{O}_Q(\boldsymbol{n}, \boldsymbol{\sigma}) = \mathcal{O}_Q(\boldsymbol{n}, \mathrm{Adj}_{\boldsymbol{\gamma}}(\boldsymbol{\sigma})) \tag{C.1}$$

naturally leads to the complete bases (3.96)

$$\mathcal{O}_Q(\boldsymbol{L}) = \frac{1}{\prod n_a!}\sqrt{\frac{\prod d(R_a)}{\prod d(r_{ab;\alpha})}} \sum_{\boldsymbol{\sigma}} \chi_Q(\boldsymbol{L}, \boldsymbol{\sigma}) \mathcal{O}_Q(\boldsymbol{n}, \boldsymbol{\sigma}) \tag{C.2}$$

and (3.125)

$$\mathcal{O}_Q(\boldsymbol{K}) = \frac{\sqrt{\prod d(R_a)}}{\prod n_a!} \sum_{\boldsymbol{\sigma}} \chi_Q(\boldsymbol{K}, \boldsymbol{\sigma}) \mathcal{O}_Q(\boldsymbol{n}, \boldsymbol{\sigma}) \tag{C.3}$$

C.1 Review of $\mathbb{C}$

First, let us start with the familiar example of half-BPS operators. Those are described by a trivial quiver $\mathbb{C}$, with single node and single field Φ_{11}. The operators obey invariance

$$\mathcal{O}_{\mathbb{C}}(\sigma) = \mathcal{O}_{\mathbb{C}}(\gamma\sigma\gamma^{-1}), \quad \gamma \in S_n \tag{C.4}$$

We want to express this as a projection to the invariant subspace

$$\mathcal{O}_{\mathbb{C}}(\sigma) = \frac{1}{n!}\sum_{\gamma\in S_n} \mathcal{O}_{\mathbb{C}}(\gamma\sigma\gamma^{-1}) = \sum_{\rho\in S_n}\left(\frac{1}{n!}\sum_{\gamma\in S_n}\delta(\gamma\sigma\gamma^{-1}\rho^{-1})\right)\mathcal{O}_{\mathbb{C}}(\rho) \tag{C.5}$$

Now

$$P(\sigma, \rho) = \frac{1}{n!}\sum_{\gamma\in S_n}\delta(\gamma\sigma\gamma^{-1}\rho^{-1}) \tag{C.6}$$

is a projector, and we want to find the explicit basis that it projects to. That amounts to being able to write $P(\sigma, \rho) = \sum_L \Psi_L(\sigma)\Psi_L^*(\rho)$ for some labels L and wavefunctions

$\Psi_L(\sigma)$. In order to do that, we rewrite $\delta(\sigma)$ using (A.3)

$$\begin{aligned} P(\sigma,\rho) &= \sum_{R\vdash n} \frac{d(R)}{(n!)^2} \sum_{\gamma} \chi_R(\gamma\sigma\gamma^{-1}\rho^{-1}) \\ &= \sum_{R\vdash n} \frac{d(R)}{(n!)^2} \sum_{\gamma} D^R_{ij}(\gamma) D^R_{jk}(\sigma) D^R_{kl}(\gamma^{-1}) D^R_{li}(\rho^{-1}) \end{aligned} \tag{C.7}$$

This allows us to perform γ sum using (A.5), resulting in

$$P(\sigma,\rho) = \frac{1}{n!} \sum_{R\vdash n} \chi_R(\sigma)\chi_R(\rho) \tag{C.8}$$

which is the desired explicit form for the projector. It leads to the complete basis (up to normalization, chosen for convenience) – Schur polynomial basis

$$\mathcal{O}_{\mathbb{C}}(R) = \frac{1}{n!} \sum_{\sigma} \chi_R(\sigma) \mathcal{O}_{\mathbb{C}}(\sigma) \tag{C.9}$$

C.2 Review of $\mathbb{C}^3$

Now let us see how the same procedure is applied to $\mathbb{C}^3$. The operators are invariant under (3.56)

$$\mathcal{O}_{\mathbb{C}^3}(\boldsymbol{n}, \gamma\sigma\gamma^{-1}) = \mathcal{O}_{\mathbb{C}^3}(\boldsymbol{n}, \sigma), \quad \gamma \in S_{n_1} \times S_{n_2} \times S_{n_3} \equiv H \subset S_n \tag{C.10}$$

This leads to a projection

$$\mathcal{O}_{\mathbb{C}^3}(\boldsymbol{n}, \sigma) = \sum_{\rho\in S_n} P(\sigma,\rho) \mathcal{O}_{\mathbb{C}^3}(\boldsymbol{n}, \rho) \tag{C.11}$$

with

$$P(\sigma,\rho) = \frac{1}{|H|} \sum_{\gamma\in H} \delta(\gamma\sigma\gamma^{-1}\rho^{-1}) \tag{C.12}$$

Again introducing sum over R we get the same as (C.7)

$$P(\sigma,\rho) = \sum_{R\vdash n} \frac{d(R)}{|H|\, n!} \sum_{\gamma\in H} D^R_{ij}(\gamma) D^R_{jk}(\sigma) D^R_{km}(\gamma^{-1}) D^R_{mi}(\rho^{-1}) \tag{C.13}$$

with a key difference that now the sum

$$\sum_{\gamma\in S_{n_1}\times S_{n_2}\times S_{n_3}} D^R_{ij}(\gamma) D^R_{km}(\gamma^{-1}) \tag{C.14}$$

is only over the subgroup of S_n.

There are two ways to evaluate (C.14), eventually leading to the two different bases (3.57) and (3.68). For the first one, we introduce explicit representations for the subgroup

$S_{n_1} \times S_{n_2} \times S_{n_3}$ by inserting a delta function as a sum over projectors (3.64)

$$\delta_{ij} = \sum_{r_1,r_2,r_3,\nu} (P^{\nu,\nu}_{R\to r_1,r_2,r_3})_{ij} = \sum_{r_1,r_2,r_3,\nu} B^{R\to \boldsymbol{r},\nu}_{i\to \boldsymbol{l}} B^{R\to \boldsymbol{r},\nu}_{j\to \boldsymbol{l}} \tag{C.15}$$

When $\gamma \in S_{n_1} \times S_{n_2} \times S_{n_3}$, $D^R(\gamma)$ can be moved through the branching coefficients, which allows us to express

$$D^R_{ij}(\gamma_1 \circ \gamma_2 \circ \gamma_3) = \sum_{r_1,r_2,r_3,\nu} B^{R\to \boldsymbol{r},\nu}_{i\to \boldsymbol{l}} D^{r_1}_{l_1\tilde{l}_1}(\gamma_1) D^{r_2}_{l_2\tilde{l}_2}(\gamma_2) D^{r_3}_{l_3\tilde{l}_3}(\gamma_3) B^{R\to \boldsymbol{r},\nu}_{j\to \tilde{\boldsymbol{l}}} \tag{C.16}$$

Applying this to both terms in (C.14) we get

$$\begin{aligned}\sum_{\gamma\in H} D^R_{ij}(\gamma) D^R_{km}(\gamma^{-1}) = \sum_{\boldsymbol{r}^+,\nu^+} \sum_{\boldsymbol{r}^-,\nu^-} \sum_{\gamma_1,\gamma_2,\gamma_3} & B^{R\to \boldsymbol{r}^+,\nu^+}_{i\to \boldsymbol{l}^+} D^{r_1^+}_{l_1^+\tilde{l}_1^+}(\gamma_1) D^{r_2^+}_{l_2^+\tilde{l}_2^+}(\gamma_2) D^{r_3^+}_{l_3^+\tilde{l}_3^+}(\gamma_3) B^{R\to \boldsymbol{r}^+,\nu^+}_{j\to \tilde{\boldsymbol{l}}^+} \\ & \times B^{R\to \boldsymbol{r}^-,\nu^-}_{k\to \boldsymbol{l}^-} D^{r_1^-}_{l_1^-\tilde{l}_1^-}(\gamma_1^{-1}) D^{r_2^-}_{l_2^-\tilde{l}_2^-}(\gamma_2^{-1}) D^{r_3^-}_{l_3^-\tilde{l}_3^-}(\gamma_3^{-1}) B^{R\to \boldsymbol{r}^-,\nu^-}_{m\to \tilde{\boldsymbol{l}}^-}\end{aligned} \tag{C.17}$$

Now the $\gamma_1, \gamma_2, \gamma_3$ sums give $(\delta^{r_1^+ r_1^-} \delta_{l_1^+\tilde{l}_1^-} \delta_{l_1^-\tilde{l}_1^+})$ etc, which reconnect the branching coefficients. The final answer for (C.14) is thus

$$\begin{aligned}\sum_{\gamma\in H} D^R_{ij}(\gamma) D^R_{km}(\gamma^{-1}) &= \sum_{\boldsymbol{r},\nu^+,\nu^-} \frac{n_1! n_2! n_3!}{d(r_1)d(r_2)d(r_3)} B^{R\to \boldsymbol{r},\nu^-}_{m\to \boldsymbol{l}} B^{R\to \boldsymbol{r},\nu^+}_{i\to \boldsymbol{l}} B^{R\to \boldsymbol{r},\nu^-}_{k\to \tilde{\boldsymbol{l}}} B^{R\to \boldsymbol{r},\nu^+}_{j\to \tilde{\boldsymbol{l}}} \\ &= \sum_{\boldsymbol{r},\nu^+,\nu^-} \frac{n_1! n_2! n_3!}{d(r_1)d(r_2)d(r_3)} (P^{\nu^-,\nu^+}_{R\to \boldsymbol{r}})_{mi} (P^{\nu^-,\nu^+}_{R\to \boldsymbol{r}})_{kj}\end{aligned} \tag{C.18}$$

The projector (C.13) is thus

$$P(\sigma,\rho) = \frac{1}{n!} \sum_{R,\boldsymbol{r},\nu^+,\nu^-} \frac{d(R)}{d(r_1)d(r_2)d(r_3)} \operatorname{tr}(P^{\nu^-,\nu^+}_{R\to \boldsymbol{r}} D^R(\sigma)) \operatorname{tr}(P^{\nu^-,\nu^+}_{R\to \boldsymbol{r}} D^R(\rho)) \tag{C.19}$$

This is again of the form $\sum_L \Psi_L(\sigma)\Psi^*_L(\rho)$, with labels $\boldsymbol{L} = \{R, r_1, r_2, r_3, \nu^+, \nu^-\}$, thus we conclude that the complete basis is (3.57)

$$\mathcal{O}_{\mathbb{C}^3}(\boldsymbol{L}) \sim \sum_\sigma \operatorname{tr}(P^{\nu^-,\nu^+}_{R\to \boldsymbol{r}} D^R(\sigma))\, \mathcal{O}_{\mathbb{C}^3}(\boldsymbol{n},\sigma) \tag{C.20}$$

up to a normalization coefficient.

An alternative way to evaluate the sum (C.14) is to observe that $D^R_{ij}(\gamma) D^R_{mk}(\gamma)$ is a representation matrix of γ in the tensor product $R \otimes R$ rep. We can decompose this into irreps using S_n Clebsch-Gordan coefficients

$$D^R_{ij}(\gamma) D^R_{mk}(\gamma) = \sum_{\Lambda,\tau} D^\Lambda_{l\tilde{l}}(\gamma) S^{RR,\Lambda\tau}_{i\,m,\,l} S^{RR,\Lambda\tau}_{j\,k,\,\tilde{l}} \tag{C.21}$$

Now the γ only appears in a single $D(\gamma)$, and the sum over $\gamma \in S_{n_1} \times S_{n_2} \times S_{n_3}$ is simply a projection to invariants under the subgroup

$$\sum_{\gamma \in S_{n_1} \times S_{n_2} \times S_{n_3}} D^{\Lambda}_{l\tilde{l}}(\gamma) = n_1! n_2! n_3! \sum_{\beta=1}^{g([\boldsymbol{n}];\Lambda)} B_l^{\Lambda \to [\boldsymbol{n}],\beta} B_{\tilde{l}}^{\Lambda \to [\boldsymbol{n}],\beta} \tag{C.22}$$

The branching coefficients have the same meaning as before: $[\boldsymbol{n}]$ denotes three single-row Young diagrams of length n_1, n_2, n_3, which is the trivial one-dimensional representation of $S_{n_1} \times S_{n_2} \times S_{n_3}$. Since it is one-dimensional, we suppress the index for it. β is the multiplicity for how many times $[\boldsymbol{n}]$ appears in Λ. Branching coefficient $B_l^{\Lambda \to [\boldsymbol{n}],\beta}$ itself is a vector in Λ, which is invariant under $S_{n_1} \times S_{n_2} \times S_{n_3}$, labelled by β. Note the number of invariants is $g([n_1],[n_2],[n_3];\Lambda)$, that is, how many ways there are to combine three single-row diagrams into Λ using Littlewood-Richardson rule. It vanishes if Λ has more than three rows, so we have a constraint

$$l(\Lambda) \leq 3 \tag{C.23}$$

Λ is a representation of the global symmetry $U(3)$. The full sum (C.14) is thus

$$\sum_{\gamma \in H} D^R_{ij}(\gamma) D^R_{km}(\gamma^{-1}) = n_1! n_2! n_3! \sum_{\Lambda,\tau,\beta} \left(B_l^{\Lambda \to [\boldsymbol{n}],\beta} S^{RR,\Lambda\tau}_{i\,m,\,l} \right) \left(B_{\tilde{l}}^{\Lambda \to [\boldsymbol{n}],\beta} S^{RR,\Lambda\tau}_{j\,k,\,\tilde{l}} \right) \tag{C.24}$$

and the projector (C.13) evaluates to

$$P(\sigma,\rho) = \sum_{R,\Lambda,\tau,\beta} \frac{d(R)}{n!} \left(B_l^{\Lambda \to [\boldsymbol{n}],\beta} S^{RR,\Lambda\tau}_{i\,m,\,l} D^R_{im}(\rho) \right) \left(B_{\tilde{l}}^{\Lambda \to [\boldsymbol{n}],\beta} S^{RR,\Lambda\tau}_{j\,k,\,\tilde{l}} D^R_{jk}(\sigma) \right) \tag{C.25}$$

This leads to the basis (3.68)

$$\mathcal{O}(\boldsymbol{K}) \sim \sum_{\sigma \in S_n} B_m^{\Lambda \to [\boldsymbol{n}],\beta} S^{RR,\Lambda\tau}_{i\,j,\,m} D^R_{ij}(\sigma)\, \mathcal{O}(\boldsymbol{n},\sigma) \tag{C.26}$$

up to normalization.

C.3 General quiver

Now let us extend this derivation for a general quiver. We need to solve the invariance (3.84)

$$\mathcal{O}_Q(\boldsymbol{n},\boldsymbol{\sigma}) = \mathcal{O}_Q(\boldsymbol{n}, \mathrm{Adj}_{\boldsymbol{\gamma}}(\boldsymbol{\sigma})) \tag{C.27}$$

that is, to evaluate the projector

$$\begin{aligned} P(\boldsymbol{\sigma},\boldsymbol{\rho}) &= \frac{1}{|H|}\sum_{\boldsymbol{\gamma}\in H}\delta(\mathrm{Adj}_{\boldsymbol{\gamma}}(\boldsymbol{\sigma})\boldsymbol{\rho}^{-1}) \\ &= \frac{1}{|H|}\sum_{\boldsymbol{\gamma}\in H}\prod_a \delta(\mathrm{Adj}_{\boldsymbol{\gamma}}(\sigma_a)\rho_a^{-1}) \end{aligned} \tag{C.28}$$

in analogy with (C.12). The invariance group is

$$H = \bigotimes_{a,b,\alpha} S_{n_{ab;\alpha}}, \qquad |H| = \prod_{a,b,\alpha} n_{ab;\alpha}! \tag{C.29}$$

Note beforehand, that $\chi_Q(\boldsymbol{L},\boldsymbol{\sigma})$ obeys exactly the required identity (B.12), which allows to write (C.28) like $\sum_{\boldsymbol{L}}\chi_Q(\boldsymbol{L},\boldsymbol{\sigma})\chi_Q(\boldsymbol{L},\boldsymbol{\rho})$, leading to the $\mathcal{O}_Q(\boldsymbol{L})$ basis. The same is true of $\chi_Q(\boldsymbol{K},\boldsymbol{\sigma})$, which obeys (B.21), leading to $\mathcal{O}_Q(\boldsymbol{K})$ basis. The purpose here, however, is to constructively *derive* $\chi_Q(\boldsymbol{L},\boldsymbol{\sigma})$ and $\chi_Q(\boldsymbol{K},\boldsymbol{\sigma})$ as the basis diagonalizing $P(\boldsymbol{\sigma},\boldsymbol{\rho})$.

Like before, we expand the deltas in terms of characters

$$\begin{aligned} P(\boldsymbol{\sigma},\boldsymbol{\rho}) &= \frac{1}{|H|}\sum_{\boldsymbol{R}}\sum_{\boldsymbol{\gamma}\in H}\prod_a \frac{d(R_a)}{n_a!}\chi_{R_a}(\mathrm{Adj}_{\boldsymbol{\gamma}}(\sigma_a)\rho_a^{-1}) \\ &= \frac{1}{|H|}\sum_{\boldsymbol{R}}\sum_{\boldsymbol{\gamma}\in H}\prod_a \frac{d(R_a)}{n_a!} D^{R_a}_{i_a j_a}(\otimes_{b,\alpha}\gamma_{ba;\alpha})\, D^{R_a}_{j_a k_a}(\sigma_a)\, D^{R_a}_{k_a m_a}(\otimes_{b,\alpha}\gamma^{-1}_{ab;\alpha})\, D^{R_a}_{m_a i_a}(\rho_a^{-1}) \end{aligned} \tag{C.30}$$

The question is, again, how to perform the $\gamma_{ab;\alpha}$ sum

$$\sum_{\boldsymbol{\gamma}\in H}\prod_a D^{R_a}_{i_a j_a}(\otimes_{b,\alpha}\gamma_{ba;\alpha}) D^{R_a}_{k_a m_a}(\otimes_{b,\alpha}\gamma^{-1}_{ab;\alpha}) \tag{C.31}$$

Note each $\gamma_{ab;\alpha}$ and $\gamma^{-1}_{ab;\alpha}$ occurs exactly once.

One way, in analogy to the restricted Schur basis, is to insert the branching coefficients around γ's:

$$D^{R_a}_{i_a j_a}(\otimes_{b,\alpha}\gamma_{ba;\alpha}) = \sum_{\bigcup_{b,\alpha} r_{ba;\alpha}}\sum_{\nu} B^{R_a\to\bigcup_{b,\alpha} r_{ba;\alpha},\nu_a}_{i_a\to\bigcup_{b,\alpha} l_{ba;\alpha}} B^{R_a\to\bigcup_{b,\alpha} r_{ba;\alpha},\nu_a}_{j_a\to\bigcup_{b,\alpha} \tilde{l}_{ba;\alpha}} \prod_{b,\alpha} D^{r_{ba;\alpha}}_{l_{ba;\alpha}\tilde{l}_{ba;\alpha}}(\gamma_{ba;\alpha}) \tag{C.32}$$

Replacing all $D(\gamma)$ and $D(\gamma^{-1})$ we get analogous expansion to (C.17), which allows us to perform $\gamma_{ab;\alpha}$ sums. They generate delta functions which contract the branching coefficients in analogy to (C.17) as follows:

$$\sum_{\gamma \in H} \prod_a D^{R_a}_{i_a j_a}(\otimes_{b,\alpha}\gamma_{ba;\alpha}) D^{R_a}_{k_a m_a}(\otimes_{b,\alpha}\gamma^{-1}_{ab;\alpha})$$
$$= \sum_{\{r_{ab;\alpha}\}} \sum_{\{\nu_a^+\}} \sum_{\{\nu_a^-\}} \frac{\prod n_{ab;\alpha}!}{\prod d(r_{ab;\alpha})} \prod_a \left(B^{R_a \to \bigcup_{b,\alpha} r_{ab;\alpha}, \nu_a^-}_{m_a \to \bigcup_{b,\alpha} l_{ab;\alpha}} B^{R_a \to \bigcup_{b,\alpha} r_{ba;\alpha}, \nu_a^+}_{i_a \to \bigcup_{b,\alpha} l_{ba;\alpha}} \right) \times \left(B^{R_a \to \bigcup_{b,\alpha} r_{ab;\alpha}, \nu_a^-}_{k_a \to \bigcup_{b,\alpha} \tilde{l}_{ab;\alpha}} B^{R_a \to \bigcup_{b,\alpha} r_{ba;\alpha}, \nu_a^+}_{j_a \to \bigcup_{b,\alpha} \tilde{l}_{ba;\alpha}} \right) \tag{C.33}$$

This leads to

$$P(\boldsymbol{\sigma}, \boldsymbol{\rho}) = \frac{1}{\prod n_a!} \sum_{\boldsymbol{R}, \boldsymbol{r}, \boldsymbol{\nu}^+, \boldsymbol{\nu}^-} \frac{\prod d(R_a)}{\prod d(r_{ab;\alpha})} \chi_Q(\boldsymbol{L}, \boldsymbol{\sigma}) \chi_Q(\boldsymbol{L}, \boldsymbol{\rho}) \tag{C.34}$$

with $\chi_Q(\boldsymbol{L}, \boldsymbol{\sigma})$ defined as in (3.95) and thus the basis

$$\mathcal{O}_Q(\boldsymbol{L}) = \frac{1}{\prod n_a!} \sqrt{\frac{\prod d(R_a)}{\prod d(r_{ab;\alpha})}} \sum_{\boldsymbol{\sigma}} \chi_Q(\boldsymbol{L}, \boldsymbol{\sigma}) \mathcal{O}_Q(\boldsymbol{\sigma}). \tag{C.35}$$

An alternative way of evaluating (C.31) is to use Clebsch-Gordan coefficients, leading to the covariant basis. In order to apply (C.24) we need a term $D(\gamma)D(\gamma^{-1})$ with γ in some subgroup of S_n. In general, however, (C.31) does not have that form, because $D(\otimes_{b,\alpha}\gamma_{ba;\alpha})$ contains permutations corresponding to fields incoming to a, and $D(\otimes_{b,\alpha}\gamma^{-1}_{ab;\alpha})$ contains outgoing. Therefore, first we have to insert branching coefficients to separate fields between different quiver nodes

$$\sum_{\gamma \in H} \prod_a D^{R_a}_{i_a j_a}(\otimes_{b,\alpha}\gamma_{ba;\alpha}) D^{R_a}_{k_a m_a}(\otimes_{b,\alpha}\gamma^{-1}_{ab;\alpha})$$
$$= \sum_{\gamma \in H} \prod_a \left(\sum_{\bigcup_b s^+_{ba}} \sum_{\nu_a^+} B^{R_a \to \bigcup_b s^+_{ba}, \nu_a^+}_{i_a \to \bigcup_b l^+_{ba}} B^{R_a \to \bigcup_b s^+_{ba}, \nu_a^+}_{j_a \to \bigcup_b \tilde{l}^+_{ba}} \prod_b D^{s^+_{ba}}_{l^+_{ba} \tilde{l}^+_{ba}}(\otimes_\alpha \gamma_{ba;\alpha}) \right) \times \left(\sum_{\bigcup_b s^-_{ab}} \sum_{\nu_a^-} B^{R_a \to \bigcup_b s^-_{ab}, \nu_a^-}_{k_a \to \bigcup_b l^-_{ab}} B^{R_a \to \bigcup_b s^-_{ab}, \nu_a^-}_{m_a \to \bigcup_b \tilde{l}^-_{ab}} \prod_b D^{s^-_{ab}}_{l^-_{ab} \tilde{l}^-_{ab}}(\otimes_\alpha \gamma^{-1}_{ab;\alpha}) \right) \tag{C.36}$$

Now for each ordered pair of quiver nodes (a, b), where we have M_{ab} fields labelled by α, we can apply (C.24)

$$\sum_{\bigcup_\alpha \gamma_{ab;\alpha}} D^{s^+_{ab}}_{l^+_{ab} \tilde{l}^+_{ab}}(\otimes_\alpha \gamma_{ab;\alpha}) D^{s^-_{ab}}_{\tilde{l}^-_{ab} l^-_{ab}}(\otimes_\alpha \gamma_{ab;\alpha})$$
$$= (\prod_\alpha n_{ab;\alpha}!) \sum_{\Lambda_{ab}, \tau_{ab}, \beta_{ab}} \left(B^{\Lambda_{ab} \to [\boldsymbol{n}_{ab}], \beta_{ab}}_{l_{ab}} S^{s^+_{ab}\, s^-_{ab}, \Lambda \tau_{ab}}_{l^+_{ab}\, l^-_{ab}, l_{ab}} \right) \left(B^{\Lambda_{ab} \to [\boldsymbol{n}_{ab}], \beta_{ab}}_{\tilde{l}_{ab}} S^{s^+_{ab}\, s^-_{ab}, \Lambda \tau_{ab}}_{\tilde{l}^+_{ab}\, l^-_{ab}, \tilde{l}_{ab}} \right) \tag{C.37}$$

Note that the effect on (C.36) is to reconnect l^+_{ab} with $\tilde{l}^-_{ab}$ via the Clebsch-Gordan coefficient

$S^{s^+_{ab}\, s^-_{ab}, \Lambda \tau_{ab}}_{l^+_{ab}\, \tilde{l}^-_{ab}, l_{ab}}$, and the same for $\tilde{l}^+_{ab}$ with l^-_{ab}. This produces the right structure where the branching coefficients factor into two quivers. The end result, putting everything back into (C.30) is

$$P(\boldsymbol{\sigma}, \boldsymbol{\rho}) = \frac{1}{\prod n_a!} \sum_{\boldsymbol{K}} \left(\prod_a d(R_a) \right) \chi_Q(\boldsymbol{K}, \boldsymbol{\sigma}) \chi_Q(\boldsymbol{K}, \boldsymbol{\rho}) \tag{C.38}$$

where the label $\boldsymbol{K}$ includes

$$\boldsymbol{K} = \{R_a, s^+_{ab}, s^-_{ab}, \nu^+_a, \nu^-_a, \Lambda_{ab}, \tau_{ab}, \beta_{ab}, n_{ab;\alpha}\} \tag{C.39}$$

and $\chi_Q(\boldsymbol{K}, \boldsymbol{\sigma})$ is as in (3.126). The basis is then

$$\mathcal{O}_Q(\boldsymbol{K}) = \frac{\sqrt{\prod d(R_a)}}{\prod n_a!} \sum_{\boldsymbol{\sigma}} \chi_Q(\boldsymbol{K}, \boldsymbol{\sigma}) \mathcal{O}_Q(\boldsymbol{n}, \boldsymbol{\sigma}) \tag{C.40}$$

Appendix D

Planar approximation for open strings

D.1 Basis for open string operators

Here we describe a basis of operators dual to branes with open strings attached. For now we consider the simplest possible case of *single maximal giant*. They are, basically, *restricted Schur* operators analyzed in many papers [20, 22, 23, 24]. Here we will just establish convenient notation and normalizations, which will make calculations easy. The main object is a single-column restricted Schur with distinct impurities:

$$\begin{aligned}&\chi_{R=[1^N],r=[1^{N-k}]}(Z;W_1,\ldots,W_k)\\&\quad=\frac{1}{(N-k)!}\sum_{\sigma\in S_N}(-1)^\sigma Z^{i_1}_{i_{\sigma(1)}}\ldots Z^{i_{N-k}}_{i_{\sigma(N-k)}}(W_1)^{i_{N-k+1}}_{i_{\sigma(N-k+1)}}\ldots(W_k)^{i_N}_{i_{\sigma(N)}}\end{aligned} \tag{D.1}$$

Since all the diagrams for now will involve a single column with impurities we will abbreviate

$$\chi(Z;W_1,\ldots,W_k)\equiv\chi_{R=[1^N],r=[1^{N-k}]}(Z;W_1,\ldots,W_k) \tag{D.2}$$

Note the non-maximal giants can be written in this notation by setting $W_i=\mathbb{I}$ – identity operator

$$\chi(Z;\underbrace{\mathbb{I},\ldots,\mathbb{I}}_{k})=k!\,\chi_{[1^{N-k}]}(Z) \tag{D.3}$$

Right-hand side is the usual half-BPS Schur polynomial, but it comes with $k!$ factor.

D.2 Rules for correlators

The strategy in calculating correlators of open string operators is:

1. Contract Z operators that build up the giant exactly

2. Contract remaining $O(1)$ operators in planar limit

The following formula allows us to do exact contraction of maximal giants[1]

$$\begin{aligned}&\big\langle \chi(Z;W_1,\ldots,W_k)\chi(\bar{Z};\bar{V}_1,\ldots,\bar{V}_k)\big\rangle\\ &\quad= (N-k)!\sum_{\sigma,\tau\in S_k}(-1)^\sigma(-1)^\tau\left\langle (W_1)^{i_1}_{j_{\sigma(1)}}\ldots(W_k)^{i_k}_{j_{\sigma(k)}}(\bar{V}_1)^{j_1}_{i_{\tau(1)}}\ldots(\bar{V}_k)^{j_k}_{i_{\tau(k)}}\right\rangle\end{aligned} \tag{D.4}$$

The contractions construct all possible gluings of open strings W_i and $\bar{V}_j$ into multi-trace combinations, subject to restriction that W_i is followed by $\bar{V}_j$ within trace. Let us denote the gluing operation by

$$\big[W_1,\ldots,W_k,\bar{V}_1,\ldots,\bar{V}_k\big] \equiv \sum_{\sigma,\tau\in S_k}(-1)^\sigma(-1)^\tau\left\langle (W_1)^{i_1}_{j_{\sigma(1)}}\ldots(W_k)^{i_k}_{j_{\sigma(k)}}(\bar{V}_1)^{j_1}_{i_{\tau(1)}}\ldots(\bar{V}_k)^{j_k}_{i_{\tau(k)}}\right\rangle \tag{D.5}$$

Then (D.4) can be written as

$$\big\langle \chi(Z;W_1,\ldots,W_k)\chi(\bar{Z};\bar{V}_1,\ldots,\bar{V}_k)\big\rangle = (N-k)!\big\langle\big[W_1,\ldots,W_k,\bar{V}_1,\ldots,\bar{V}_k\big]\big\rangle \tag{D.6}$$

Explicitly we have for $k=1$:

$$\big[W,\bar{V}\big] = \mathrm{tr}(W\bar{V}) \tag{D.7}$$

for $k=2$:

$$\big[W_1,W_2,\bar{V}_1,\bar{V}_2\big] = \mathrm{tr}(W_1\bar{V}_1)\mathrm{tr}(W_2\bar{V}_2)+\mathrm{tr}(W_1\bar{V}_2)\mathrm{tr}(W_1\bar{V}_2)-\mathrm{tr}(W_1\bar{V}_1W_2\bar{V}_2)-\mathrm{tr}(W_1\bar{V}_2W_2\bar{V}_1) \tag{D.8}$$

and for $k=3$, schematically:

$$\begin{aligned}&\big[W_1,W_2,W_3,\bar{V}_1,\bar{V}_2,\bar{V}_3\big] =\\ &\quad \mathrm{tr}(W_1\bar{V}_1)\mathrm{tr}(W_2\bar{V}_2)\mathrm{tr}(W_3\bar{V}_3) + (5\text{ permutations of }\bar{V}_i)\\ &\quad -\mathrm{tr}(W_1\bar{V}_1)\big(\mathrm{tr}(W_2\bar{V}_2W_3\bar{V}_3)+\mathrm{tr}(W_2\bar{V}_3W_3\bar{V}_2)\big) - (8\text{ other choices of }\mathrm{tr}(W_i\bar{V}_j))\\ &\quad +\mathrm{tr}(W_1\bar{V}_1W_2\bar{V}_2W_3\bar{V}_3) + (5\text{ permutations of }\bar{V}_i)\\ &\quad +\mathrm{tr}(W_1\bar{V}_1W_3\bar{V}_2W_2\bar{V}_3) + (5\text{ permutations of }\bar{V}_i)\end{aligned} \tag{D.9}$$

In planar limit the traces are associated with boundary of the string worldsheet, and so we can already see the emergence of open string worldsheet, with states W_i, $\bar{V}_j$ inserted at the boundary in all possible combinations. This happens as a result of contracting fields Z corresponding to the brane, so we can interpret this as demonstration, how the presence of D-brane allows for open strings to propagate. We will make the open string interpretation more clear in the next section by using diagrams.

So far we have assumed Z in the giant are only contracted between themselves. We will also allow Z to appear in open strings or closed strings, and we need to handle contractions

[1] In what follows we use $\bar{Z}$ to mean $Z^\dagger$ in order to reserve the superscript slot for more useful things

between those and the giant. The rule can be written simply as:

$$\chi(\underbrace{Z; W_1, \ldots) \ldots \bar{Z}} \ldots = \chi(Z; \underbrace{Z, W_1, \ldots) \ldots \bar{Z}} \ldots \tag{D.10}$$

where $\bar{Z}$ has to be part of closed or open string (i.e. not part of another giant). This equation states that the Z that is being contracted can be considered as an additional attached open string. What makes this work is the coefficient $\frac{1}{(N-k)!}$ in the definition (D.1): since the contraction can be made with any of $(N-k)$ operators in the giant, we pick up a coefficient $(N-k)$ when we pull Z out, but that combines into correct normalization $\frac{1}{(N-k-1)!}$ for a giant with $(k+1)$ strings attached.

Using (D.10) we can correct (D.6) into the following expansion, which captures the interaction between open strings and the brane:

$$\begin{aligned}\big\langle \chi(Z; W_1, \ldots, W_k)\chi(\bar{Z}; \bar{V}_1, \ldots, \bar{V}_k)\big\rangle = &\\ (N-k)!\,\big\langle \big[W_1, \ldots, W_k, \bar{V}_1, \ldots, \bar{V}_k\big]\big\rangle +&\\ (N-k-1)!\,\big\langle \big[W_1, \ldots, W_k, \bar{V}_1, \ldots, \bar{V}_k, : Z, \bar{Z} :\big]\big\rangle +&\\ (N-k-2)!\,\big\langle \big[W_1, \ldots, W_k, \bar{V}_1, \ldots, \bar{V}_k, : Z, Z, \bar{Z}, \bar{Z} :\big]\big\rangle + \ldots&\end{aligned} \tag{D.11}$$

The normal ordering around $Z, \bar{Z}$ indicates that in the remaining evaluation of correlator they should not be contracted between themselves. Here is a concrete example applying this formula:

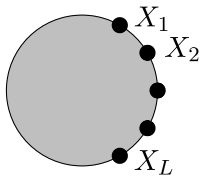
$$\begin{aligned}\big\langle \chi(Z; YZY)\,\chi(\bar{Z}; \bar{Y}\bar{Z}\bar{Y})\big\rangle &= (N-1)!\,\big\langle \big[YZY, \bar{Y}\bar{Z}\bar{Y}\big]\big\rangle + (N-2)!\,\big\langle \big[YZY, \bar{Y}\bar{Z}\bar{Y}, : Z, \bar{Z} :\big]\big\rangle \\ &= (N-1)!\,\big\langle \mathrm{tr}(YZY\bar{Y}\bar{Z}\bar{Y})\big\rangle \\ &+ (N-2)!\left\langle \mathrm{tr}(Y\underbrace{ZY\bar{Y}\underbrace{\bar{Z}\bar{Y})\mathrm{tr}(Z}\bar{Z}}) + \mathrm{tr}(Y\underbrace{ZY\bar{Z}})\mathrm{tr}(\underbrace{Z\bar{Y}\bar{Z}}\bar{Y})\right\rangle \\ &- (N-2)!\left\langle \mathrm{tr}(Y\underbrace{ZY\bar{Y}\underbrace{\bar{Z}\bar{Y}Z}\bar{Z}}) + \mathrm{tr}(Y\underbrace{ZY\bar{Z}}\underbrace{Z\bar{Y}\bar{Z}}\bar{Y})\right\rangle\end{aligned} \tag{D.12}$$

The explicit Wick contractions are indicated to emphasize that we are not allowed to contract $Z, \bar{Z}$ both arising from the giant. Note that the expansion (D.11) here is terminated after a single $: Z, \bar{Z} :$ insertion, because there is only a single pair of $Z, \bar{Z}$ in the open strings to be contracted with.

D.3 Diagrammatic notation

In this section we develop diagrammatic notation for calculating correlators involving maximal giants with open strings. The key simplification in using diagrams is that it will let us see the planar expansion clearly.

First, recall closed string planar diagrams. Define a single trace operator vertex:



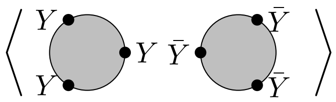
$$\equiv \frac{1}{\sqrt{N^L}}\mathrm{tr}(X_1 X_2 \ldots X_L) \tag{D.13}$$

which includes $N^{-L/2}$ factor where L is the number of fields. A contraction of such diagrams according to genus expansion is equal to

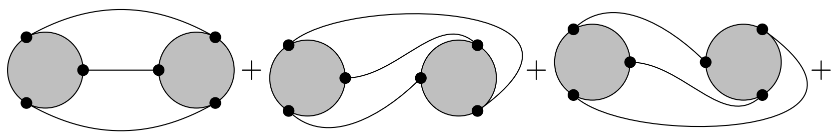
$$N^{2C-2G-V} \tag{D.14}$$

where $C =$ number of connected components, $G =$ total genus of the surface where planar diagram is drawn, and $V =$ number of single trace vertices, also understood as number of boundaries of the surface. For example:

$$\frac{1}{N^3}\left\langle \mathrm{tr}(Y^3)\mathrm{tr}(\bar{Y}^3)\right\rangle \equiv$$



$$=$$



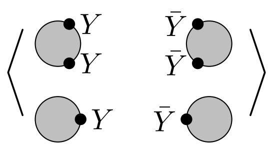
$$+\, O\left(\frac{1}{N^2}\right)$$

$$= 3 + O\left(\frac{1}{N^2}\right) \tag{D.15}$$

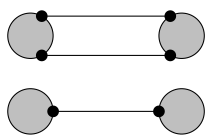
Each diagram contributes 1 because $C = 1$, $G = 0$, $V = 2$. There are three more contractions which are non-planar, and so have $G = 1$ and contribute $\frac{3}{N^2}$.

We can check that multi-string states are still correctly normalized

$$\left\langle \frac{\mathrm{tr}(Y^2)}{N}\frac{\mathrm{tr}(Y)}{\sqrt{N}}\frac{\mathrm{tr}(\bar{Y}^2)}{N}\frac{\mathrm{tr}(\bar{Y})}{\sqrt{N}}\right\rangle \equiv$$

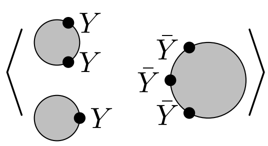


$$= 2 \times$$

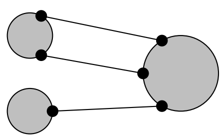


$$= 2 + O\left(\frac{1}{N^2}\right) \tag{D.16}$$

Now we have $C = 2$, $V = 4$, result is still $O(1)$. The $\frac{1}{N^2}$ correction comes from connected diagram, which has $C = 1$. This represents four-point string coupling. We can check three-point function as well:



$$= 6 \times$$

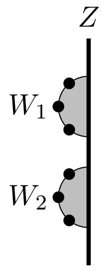


$$= \frac{6}{N} \tag{D.17}$$

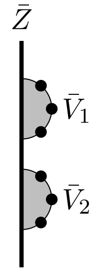
Here we have $C = 1, V = 3$, and coefficient 6 comes from number of ways to connect the points, all of which give the same diagram.

Let us now turn to giants with open strings. We propose the following notation:

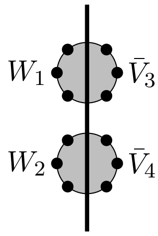
$$\frac{1}{\sqrt{N!N^{L_1}N^{L_2}}}\chi(Z;W_1,W_2) \equiv$$



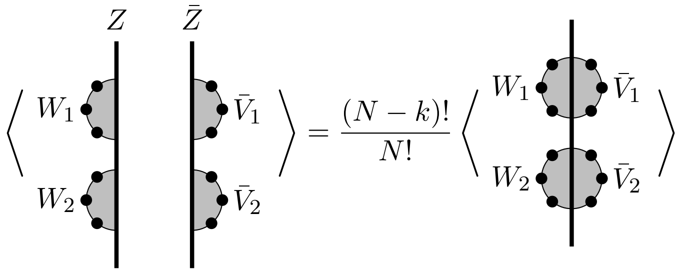
$$, \quad \frac{1}{\sqrt{N!N^{L_1}N^{L_2}}}\chi(\bar{Z};\bar{V}_1,\bar{V}_2) \equiv \quad \text{(D.18)}$$



$$\frac{1}{\sqrt{N^{L_1}N^{L_2}N^{L_3}N^{L_4}}}\left[W_1,W_2,\bar{V}_3,\bar{V}_4\right] \equiv \quad \text{(D.19)}$$

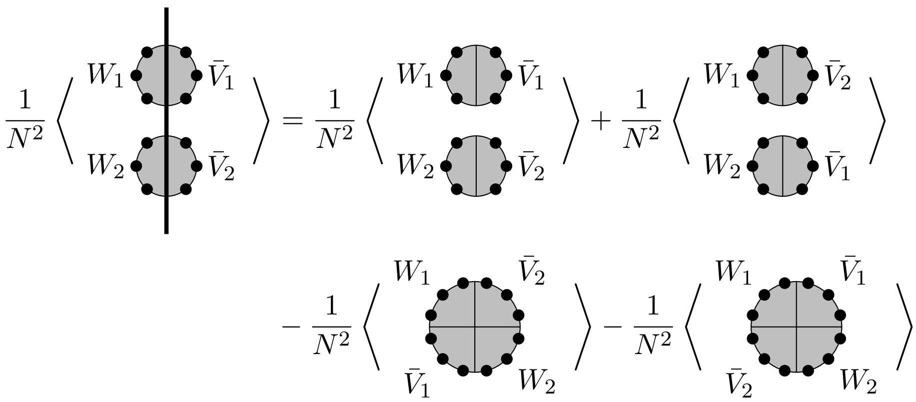


The relation (D.6) then translates into:



(D.20)

Here $k = 2$ is the number of attached open strings, and to leading order $\frac{(N-k)!}{N!} \approx \frac{1}{N^k}$. Next we apply (D.8):

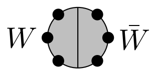


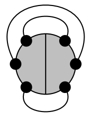
(D.21)

The operators on the right-hand-side are now simple products of traces following (D.13). We only draw internal lines to indicate where the boundaries of original open string words were.

If incoming and outgoing open string states are the same, we get correct normalization:

$$\left\langle \frac{\chi(Z;W)}{\sqrt{N!N^L}} \frac{\chi(\bar{Z};\bar{W})}{\sqrt{N!N^L}} \right\rangle = \frac{1}{N} \left\langle \quad \right\rangle = \frac{1}{N} \quad + \ldots$$

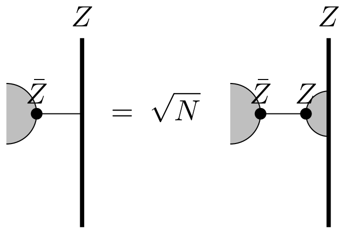




$$= 1 + O\left(\frac{1}{N^2}\right) \tag{D.22}$$

Note the open string propagation diagram contributes a factor N (from $C = 1, V = 1$) which cancels $\frac{1}{N}$ coming from (D.20) for each attached string.

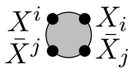
The interaction of strings with the brane are captured by contracting $\bar{Z}$ with the brane as in (D.10). Diagrammatically this can be represented as



(D.23)

The correlators involving the dilatation operator can be calculated by inserting

$$\Delta_2 = \frac{\lambda}{N} : \mathrm{tr}[X_i, X^i][\bar{X}^j, \bar{X}_j] := \lambda N : \quad : \tag{D.24}$$



where $[X_i, X^i] \equiv ZY - YZ$, and contracting it according to the rules.

Appendix E

Chiral ring from geometric invariant theory

We would like to *prove* that the chiral ring of $\mathbb{C}^3$ theory corresponds to $\mathrm{Sym}^N(\mathbb{C}^3)$.

The main point, which was proven in [114], is the following: the chiral ring

$$R = (\mathbb{C}[X_a]/F)^{G^c} \tag{E.1}$$

defines a variety, which is isomorphic to

$$\mathcal{M} = \mathcal{F}//G^c. \tag{E.2}$$

Here are the ingredients of this claim. $\mathbb{C}[X_a]$ is the polynomial ring of matrix elements of X_1, X_2, X_3 (not just gauge invariant). F is the ideal of $\mathbb{C}[X_a]$ generated by F-terms:

$$F = \langle\langle\, [X_a, X_b]_{ij}\,\rangle\rangle. \tag{E.3}$$

$\mathcal{F}$ is the variety defined by the ideal F, that is, $\mathcal{F} = I(F)$ is the space where all elements of the ideal F vanish. That is just the space of commuting matrices

$$\mathcal{F} = \{X_a \,|\, [X_a, X_b] = 0\} \tag{E.4}$$

G^c denotes the complexified gauge group which in our case is $G^c = GL(N, \mathbb{C})$. Then $R = (\mathbb{C}[X_a]/F)^{G^c}$ denotes the G^c-invariant polynomials in the quotient ring $\mathbb{C}[X_a]/F$, which is the definition of the chiral ring.

Now, $\mathcal{F}//G^c$ denotes the "geometric invariant theory quotient". According to [114] that means that elements of $\mathcal{F}//G^c$ are the "extended orbits" of G^c acting on $\mathcal{F}$. Extended orbit of X_a includes not only points gX_ag^{-1} but also limits of the group action

$$\lim_{n\to\infty} g_n X_a g_n^{-1} \tag{E.5}$$

for any sequence $g_n \in G^c$. In practice that means the following. It is a fact in linear algebra that commuting matrices can be *simultaneously triangularized* by conjugating with some g. So every orbit of G^c in $\mathcal{F}$ includes a configuration where all X_a are upper-triangular. The importance of *extended* orbit is that includes a configuration where X_a are *diagonal*. This can be achieved by acting on the upper-triangular configuration with

$$g = \begin{pmatrix} 1 & & & \\ & \lambda & & \\ & & \lambda^2 & \\ & & & \dots \end{pmatrix} \tag{E.6}$$

which takes

$$(gX_ag^{-1})_{ij} = (X_a)_{ij}\,\lambda^{i-j} \tag{E.7}$$

Since all elements in upper triangle have $i < j$, taking $\lambda \to \infty$ limit sets them to zero, while keeping the diagonal elements unchanged. This shows that every extended orbit of G^c in $\mathcal{F}$ contains a point where X_a are diagonal. The diagonal elements, being the eigenvalues, can not be changed by conjugation, only permuted, which shows that

$$\mathcal{M} = \mathcal{F}//G^c = \mathrm{Sym}^N(\mathbb{C}^3) \tag{E.8}$$

Since according to the theorem, $(\mathbb{C}[X_a]/F)^{G^c}$ is the ring associated with $\mathcal{F}//G^c$, then it must be the ring of $\mathrm{Sym}^N(\mathbb{C}^3)$.

Appendix F

Symplectic form for perturbations of sphere giant

In this appendix we derive the symplectic form for arbitrary perturbations of a non-maximal half-BPS giant. Our derivation is along the lines of that found in Appendix F in [38], and we use some results from there. The unperturbed solution is defined by the polynomial:

$$P(z) = z - c_0 \tag{F.1}$$

The surface in S^5 is:

$$\begin{aligned} |x|^2 + |y|^2 &= 1 - |c_0|^2 \\ z &= e^{it} c_0 \end{aligned} \tag{F.2}$$

where we have also put the time-dependence back in. We pick world-volume coordinates $(\sigma^1, \sigma^2, \sigma^3)$ to be some coordinates on a unit S^3 embedded in $\mathbb{C}^2$, so that we have functions $x_0(\sigma^i)$ and $y_0(\sigma^i)$ satisfying

$$|x_0(\sigma^i)|^2 + |y_0(\sigma^i)|^2 = 1. \tag{F.3}$$

The unperturbed surface in terms of the world-volume coordinates is

$$\begin{aligned} x(\sigma^i, t) &= \sqrt{1 - |c_0|^2}\, x_0(\sigma^i) \\ y(\sigma^i, t) &= \sqrt{1 - |c_0|^2}\, y_0(\sigma^i) \\ z(\sigma^i, t) &= e^{it} c_0 \end{aligned} \tag{F.4}$$

Small perturbations around the spherical shape are parametrized by a complex function

$$\delta z(\sigma^i, t) = z(\sigma^i, t) - e^{it} c_0 \tag{F.5}$$

Effectively these are the 2 real transverse coordinates to S^3 in S^5.

The general expression for symplectic form is (5.10):

$$\begin{aligned}\omega &= \omega_{\mathrm{BI}} + \omega_{\mathrm{WZ}} \\ \omega_{\mathrm{BI}} &= \frac{N}{2\pi^2}\int_\Sigma \mathrm{d}^3\sigma\, \delta\left(\sqrt{-g}g^{0\alpha}\frac{\partial x^\nu}{\partial\sigma^\alpha}G_{\mu\nu}\right)\wedge\delta x^\mu \\ \omega_{\mathrm{WZ}} &= \frac{2N}{\pi^2}\int_\Sigma \mathrm{d}^3\sigma\frac{\delta x^\lambda\wedge\delta x^\mu}{2}\left(\frac{\partial x^\nu}{\partial\sigma^1}\frac{\partial x^\rho}{\partial\sigma^2}\frac{\partial x^\sigma}{\partial\sigma^3}\right)\epsilon_{\lambda\mu\nu\rho\sigma}\end{aligned} \tag{F.6}$$

$G_{\mu\nu}$ is the metric on unit $S^5\times\mathbb{R}$ and $g_{\alpha\beta}$ is the induced metric on the world-volume $\Sigma\times\mathbb{R}$. Note

$$\omega_{\mathrm{BI}} = \frac{N}{2\pi^2}\int_\Sigma \mathrm{d}^3\sigma\,\delta p_\mu\wedge\delta x^\mu \tag{F.7}$$

with the definition of conjugate momentum

$$p_\mu = \sqrt{-g}g^{0\alpha}\frac{\partial x^\nu}{\partial\sigma^\alpha}G_{\mu\nu}\,. \tag{F.8}$$

We will see now that these expressions can be simplified significantly for the case at hand. First, the only perturbation of the surface δx^μ can be taken to be δz. In principle the surface has to be deformed in δx and δy away from (F.4), but those are higher order in δz and can be dropped. That results in:

$$\begin{aligned}\omega_{\mathrm{BI}} &= \frac{N}{2\pi^2}\int_\Sigma \mathrm{d}^3\sigma\;(\delta p_z\wedge\delta z + \delta\bar{p}_z\wedge\delta\bar{z}) \\ p_z &= \sqrt{-g}\,g^{00}\,(G_{zz}\dot{z} + G_{z\bar{z}}\dot{\bar{z}})\end{aligned} \tag{F.9}$$

and the Wess-Zumino piece:

$$\omega_{\mathrm{WZ}} = \frac{2N}{\pi^2}\int_\Sigma \mathrm{d}^3\sigma\sqrt{g^{S^3}}\,(1-z\bar{z})\frac{\delta\bar{z}\wedge\delta z}{2i} \tag{F.10}$$

The S^5 is now represented as a S^3 fibered over a unit disk in z, so

$$(\mathrm{d}s^2)_G = -\mathrm{d}t^2 + \frac{\bar{z}^2\mathrm{d}z^2 + 2(2-z\bar{z})\mathrm{d}z\mathrm{d}\bar{z} + z^2\mathrm{d}\bar{z}^2}{4(1-z\bar{z})} + (1-z\bar{z})(\mathrm{d}s^2)_{S^3} \tag{F.11}$$

and the relevant components:

$$G_{zz} = \frac{\bar{z}^2}{4(1-z\bar{z})},\quad G_{z\bar{z}} = \frac{2-z\bar{z}}{4(1-z\bar{z})} \tag{F.12}$$

The induced metric on the unperturbed solution is

$$(\mathrm{d}s^2)_g = -(1-|c_0|^2)\mathrm{d}t^2 + (1-|c_0|^2)(\mathrm{d}s^2)_{S^3} \tag{F.13}$$

and so

$$\sqrt{-g} = (1-|c_0|^2)^2\sqrt{g^{S^3}} \tag{F.14}$$

The bit that requires some work is the evaluation of δp_z in (F.9) under the deformation

δz. We need to vary all components:

$$\begin{aligned}\delta p_z &= \delta(\sqrt{-g})\, g^{00}\,(G_{zz}\dot{z} + G_{z\bar{z}}\dot{\bar{z}}) \\ &\quad + \sqrt{-g}\,\delta g^{00}\,(G_{zz}\dot{z} + G_{z\bar{z}}\dot{\bar{z}}) \\ &\quad + \sqrt{-g}\,g^{00}\,(\delta G_{zz}\dot{z} + \delta G_{z\bar{z}}\dot{\bar{z}}) \\ &\quad + \sqrt{-g}\,g^{00}\,(G_{zz}\delta\dot{z} + G_{z\bar{z}}\delta\dot{\bar{z}})\end{aligned} \tag{F.15}$$

First we re-express $\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\,g^{\mu\nu}\delta g_{\mu\nu}$ and $\delta g^{00} = -(g^{00})^2\delta g_{00}$. Then we need variations of the induced metric:

$$\begin{aligned}\delta g_{00} &= \delta\left(\dot{z}\dot{z}G_{zz} + 2\dot{z}\dot{\bar{z}}G_{z\bar{z}} + \dot{\bar{z}}\dot{\bar{z}}G_{\bar{z}\bar{z}}\right) \\ \delta g_{ij} &= -(g^{S^3})_{ij}\delta(z\bar{z})\end{aligned} \tag{F.16}$$

And the δG_{zz}, $\delta G_{z\bar{z}}$ are calculated by varying (F.12). Putting everything together we find a simple result:

$$\delta p_z = -\sqrt{g^{S^3}}\left(\frac{1}{2}\delta\dot{\bar{z}} + i|c_0|^2\delta\bar{z} + \frac{i}{2}\bar{c}_0\,\delta z\right) \tag{F.17}$$

Now we can evaluate (F.9):

$$\omega_{\mathrm{BI}} = \frac{2N}{\pi^2}\int_{S^3}\mathrm{d}^3\sigma\,\left(|c_0|^2\,\frac{\delta\bar{z}\wedge\delta z}{2i} - \frac{\delta\dot{\bar{z}}\wedge\delta z}{8} + \frac{\delta\bar{z}\wedge\delta\dot{z}}{8}\right) \tag{F.18}$$

We have dropped the explicit measure on the unit sphere $\sqrt{g^{S^3}}$ and consider it part of the definition of $\int_{S^3}\mathrm{d}^3\sigma$. Finally, combining this with ω_{WZ} we arrive at the final result

$$\boxed{\omega = \frac{2N}{\pi^2}\int_{S^3}\mathrm{d}^3\sigma\,\left(\frac{\delta\bar{z}\wedge\delta z}{2i} - \frac{\delta\dot{\bar{z}}\wedge\delta z}{8} + \frac{\delta\bar{z}\wedge\delta\dot{z}}{8}\right)} \tag{F.19}$$

where the integral $\mathrm{d}^3\sigma$ is now over unit S^3 with its standard volume form.

Now let us use (F.19) to evaluate the symplectic form in a particular basis of world-volume perturbations:

$$P(z) = z - c_0 + \sum_{m,n\geq 0}\delta b_{m,n}\,x^m y^n \tag{F.20}$$

With time dependence as $P(e^{-it}x, e^{-it}y, e^{-it}z) = 0$ it is

$$z = e^{it}c_0 - \sum_{m,n\geq 0}\delta b_{m,n}e^{(1-m-n)it}x^m y^n\,. \tag{F.21}$$

We have $x = \sqrt{1-|c_0|^2}\,x_0$ and $y = \sqrt{1-|c_0|^2}\,y_0$ as in (F.4). To first order in $\delta b_{m,n}$ it remains unchanged and so:

$$z = e^{it}c_0 - \sum_{m,n\geq 0}\delta b_{m,n}e^{(1-m-n)it}(1-|c_0|)^{(m+n)/2}x_0^m y_0^n \tag{F.22}$$

That gives us the variation in z and $\dot{z}$:

$$\begin{aligned} \delta z &= -\sum_{m,n\geq 0} \delta b_{m,n} e^{(1-m-n)it} (1-|c_0|)^{(m+n)/2} x_0^m y_0^n \\ \delta \dot{z} &= i \sum_{m,n\geq 0} \delta b_{m,n} (m+n-1) e^{(1-m-n)it} (1-|c_0|)^{(m+n)/2} x_0^m y_0^n \end{aligned} \tag{F.23}$$

Plugging this in (F.19) we find

$$\omega = \frac{2N}{2\pi^2} \int_{S^3} \mathrm{d}^3\sigma \sum_{m,n\geq 0} (m+n+1)|x_0|^{2m}|y_0|^{2n}(1-|c_0|^2)^{m+n} \frac{\delta \bar{b}_{m,n} \wedge \delta b_{m,n}}{2i} \tag{F.24}$$

We have already dropped the cross-terms which depend on x_0, y_0 and not only on $|x_0|, |y_0|$, since the integral of such terms on S^3 is 0. The remaining terms are time-independent. The integral is easy to do:

$$\int_{S^3} \mathrm{d}^3\sigma \, |x_0|^{2m}|y_0|^{2n} = 2\pi^2 \frac{m!\, n!}{(m+n+1)!} \tag{F.25}$$

Note that we never needed the explicit choice of the coordinate σ^i on the sphere. The final symplectic form evaluated at $P(z) = z - c_0$ is thus

$$\boxed{\omega = 2N \sum_{m,n\geq 0} \frac{m!\, n!}{(m+n)!} (1-|c_0|^2)^{m+n} \frac{\delta \bar{b}_{m,n} \wedge \delta b_{m,n}}{2i}} \tag{F.26}$$

Note from (F.20) that $\delta b_{0,0}$ is in fact the variation of c_0, that is, $\mathrm{d}c_0 = -\delta b_{0,0}$. Thus we can use the requirement that the symplectic form be closed

$$\mathrm{d}\omega = 0 \tag{F.27}$$

to complete ω to an exact expression in c_0 and up to quadratic order in other $b_{m,n}$. The result is:

$$\begin{aligned} \omega = 2N \Bigg[\Bigg(1 &- \sum_{m+n>0} \frac{m!\, n!}{(m+n-1)!} (1-|c_0|^2)^{m+n-1} |b_{m,n}|^2 \\ &+ \sum_{m+n>0} \frac{m!\, n!}{(m+n-2)!} |c_0|^2 (1-|c_0|^2)^{m+n-2} |b_{m,n}|^2 \Bigg) \frac{\mathrm{d}\bar{c}_0 \wedge \mathrm{d}c_0}{2i} \\ &+ \sum_{m+n>0} \frac{m!\, n!}{(m+n)!} (1-|c_0|^2)^{m+n} \frac{\mathrm{d}\bar{b}_{m,n} \wedge \mathrm{d}b_{m,n}}{2i} \\ &- \sum_{m+n>0} \frac{m!\, n!}{(m+n-1)!} (1-|c_0|^2)^{m+n-1} \frac{c_0 \bar{b}_{m,n}\, \mathrm{d}\bar{c}_0 \wedge \mathrm{d}b_{m,n} + b_{m,n}\bar{c}_0\, \mathrm{d}\bar{b}_{m,n} \wedge \mathrm{d}c_0}{2i} \Bigg] + O(|b|^4) \end{aligned} \tag{F.28}$$

This is the full symplectic form at any point c_0 expanded for small $b_{m,n}$.

Appendix G

Assorted calculations

G.1 Proofs of quiver character identities

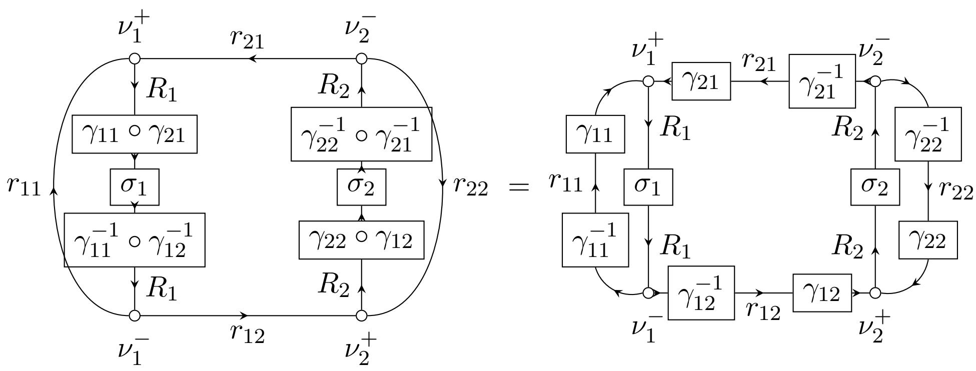
Here we prove the identities (B.10), (B.11), (B.12), (B.13) obeyed by the restricted quiver characters $\chi_Q(\boldsymbol{L}, \boldsymbol{\sigma})$.

Invariance of $\chi_Q(\boldsymbol{L}, \boldsymbol{\sigma})$

Here we show that restricted quiver characters $\chi_Q(\boldsymbol{L}, \boldsymbol{\sigma})$ obey (B.10), invariance under $\boldsymbol{\sigma} \to \mathrm{Adj}_{\boldsymbol{\gamma}}(\boldsymbol{\sigma})$.

It is easiest to see from a diagram. For example, if we take simplified version of (B.8) with only single flavor, we have:

$\chi_Q(\boldsymbol{L}, \mathrm{Adj}_{\boldsymbol{\gamma}}(\boldsymbol{\sigma})) \sim$



$= \chi_Q(\boldsymbol{L}, \boldsymbol{\sigma})$

(G.1)

This follows from the property (A.16) of the branching coefficients, which allows to pull γ's through and cancel with each other

This procedure can be written explicitly for the general case (3.95):

$$\begin{aligned}
\chi_Q(\boldsymbol{L},\mathrm{Adj}_{\boldsymbol{\gamma}}(\boldsymbol{\sigma})) &= \prod_a D^{R_a}_{i_a i'_a}(\otimes_{b,\alpha}\gamma_{ba;\alpha}) D^{R_a}_{i'_a j'_a}(\sigma_a) D^{R_a}_{j'_a j_a}(\otimes_{b,\alpha}\gamma^{-1}_{ab;\alpha}) \\
&\quad \times B^{R_a\to\bigcup_{b,\alpha} r_{ba;\alpha},\nu_a^+}_{i_a\to\bigcup_{b,\alpha} l_{ba;\alpha}} B^{R_a\to\bigcup_{b,\alpha} r_{ab;\alpha},\nu_a^-}_{j_a\to\bigcup_{b,\alpha} l_{ab;\alpha}} \\
&= \prod_a D^{R_a}_{i_a j_a}(\sigma_a) B^{R_a\to\bigcup_{b,\alpha} r_{ba;\alpha},\nu_a^+}_{i_a\to\bigcup_{b,\alpha} l'_{ba;\alpha}} B^{R_a\to\bigcup_{b,\alpha} r_{ab;\alpha},\nu_a^-}_{j_a\to\bigcup_{b,\alpha} l''_{ab;\alpha}} \\
&\quad \times \left(\prod_{b,\alpha} D^{r_{ba;\alpha}}_{l_{ba;\alpha} l'_{ba;\alpha}}(\gamma_{ba;\alpha})\right)\left(\prod_{b,\alpha} D^{r_{ab;\alpha}}_{l''_{ab;\alpha} l_{ab;\alpha}}(\gamma^{-1}_{ab;\alpha})\right) \\
&= \prod_a D^{R_a}_{i_a j_a}(\sigma_a) B^{R_a\to\bigcup_{b,\alpha} r_{ba;\alpha},\nu_a^+}_{i_a\to\bigcup_{b,\alpha} l'_{ba;\alpha}} B^{R_a\to\bigcup_{b,\alpha} r_{ab;\alpha},\nu_a^-}_{j_a\to\bigcup_{b,\alpha} l''_{ab;\alpha}} \\
&\quad \prod_{a,b,\alpha} D^{r_{ab;\alpha}}_{l_{ab;\alpha} l'_{ab;\alpha}}(\gamma_{ab;\alpha}) D^{r_{ab;\alpha}}_{l''_{ab;\alpha} l_{ab;\alpha}}(\gamma^{-1}_{ab;\alpha}) \\
&= \prod_a D^{R_a}_{i_a j_a}(\sigma_a) B^{R_a\to\bigcup_{b,\alpha} r_{ba;\alpha},\nu_a^+}_{i_a\to\bigcup_{b,\alpha} l_{ba;\alpha}} B^{R_a\to\bigcup_{b,\alpha} r_{ab;\alpha},\nu_a^-}_{j_a\to\bigcup_{b,\alpha} l_{ab;\alpha}} \\
&= \chi_Q(\boldsymbol{L},\boldsymbol{\sigma})
\end{aligned} \tag{G.2}$$

Proof of orthogonality in $\boldsymbol{L}$ of $\chi_Q(\boldsymbol{L},\boldsymbol{\sigma})$

Here we will prove (B.13), of which (B.11) is a special case. Expanding the definition of $\chi_Q(\boldsymbol{L},\boldsymbol{\sigma})$:

$$\begin{aligned}
\sum_{\tilde{\boldsymbol{\sigma}}} \chi_Q(\boldsymbol{L},\boldsymbol{\sigma}\tilde{\boldsymbol{\sigma}})\chi_Q(\tilde{\boldsymbol{L}},\tilde{\boldsymbol{\sigma}}) &= \sum_{\tilde{\boldsymbol{\sigma}}}\prod_a D^{R_a}_{i_a j_a}(\sigma_a\tilde{\sigma}_a) D^{\tilde{R}_a}_{\tilde{i}_a\tilde{j}_a}(\tilde{\sigma}_a) \\
&\times B^{R_a\to\bigcup_{b,\alpha} r_{ba;\alpha},\nu_a^+}_{i_a\to\bigcup_{b,\alpha} l_{ba;\alpha}} B^{R_a\to\bigcup_{b,\alpha} r_{ab;\alpha},\nu_a^-}_{j_a\to\bigcup_{b,\alpha} l_{ab;\alpha}} B^{\tilde{R}_a\to\bigcup_{b,\alpha} \tilde{r}_{ba;\alpha},\tilde{\nu}_a^+}_{\tilde{i}_a\to\bigcup_{b,\alpha} \tilde{l}_{ba;\alpha}} B^{\tilde{R}_a\to\bigcup_{b,\alpha} \tilde{r}_{ab;\alpha},\tilde{\nu}_a^-}_{\tilde{j}_a\to\bigcup_{b,\alpha} \tilde{l}_{ab;\alpha}}
\end{aligned} \tag{G.3}$$

We apply identity (A.5) to do the sum in each product term

$$\sum_{\tilde{\sigma}_a} D^{R_a}_{i_a j_a}(\sigma_a\tilde{\sigma}_a) D^{\tilde{R}_a}_{\tilde{i}_a\tilde{j}_a}(\tilde{\sigma}_a) = \frac{n_a!}{d(R_a)}\delta_{R_a\tilde{R}_a} D^{R_a}_{i_a\tilde{i}_a}(\sigma_a)\delta_{j_a\tilde{j}_a} \tag{G.4}$$

Now contract a pair of branching coefficients with $\delta_{j_a\tilde{j}_a}$, applying (A.17)

$$B^{R_a\to\bigcup_{b,\alpha} r_{ab;\alpha},\nu_a^-}_{j_a\to\bigcup_{b,\alpha} l_{ab;\alpha}} B^{R_a\to\bigcup_{b,\alpha} \tilde{r}_{ab;\alpha},\tilde{\nu}_a^-}_{j_a\to\bigcup_{b,\alpha} \tilde{l}_{ab;\alpha}} = \delta_{\nu_a^-\tilde{\nu}_a^-}\prod_{b,\alpha}\delta_{r_{ab;\alpha}\tilde{r}_{ab;\alpha}}\delta_{l_{ab;\alpha}\tilde{l}_{ab;\alpha}} \tag{G.5}$$

Since this appears in (G.3) under $\prod_a$, we effectively get a delta on all $\nu_a^-, r_{ab;\alpha}, l_{ab;\alpha}$. So the sum is

$$\sum_{\tilde{\boldsymbol{\sigma}}} \chi_Q(\boldsymbol{L},\boldsymbol{\sigma}\tilde{\boldsymbol{\sigma}})\chi_Q(\tilde{\boldsymbol{L}},\tilde{\boldsymbol{\sigma}}) = \delta_{\boldsymbol{R}\tilde{\boldsymbol{R}}}\delta_{\boldsymbol{r}\tilde{\boldsymbol{r}}}\delta_{\boldsymbol{\nu}^-\tilde{\boldsymbol{\nu}}^-}\prod_a \frac{n_a!}{d(R_a)} D^{R_a}_{i_a\tilde{i}_a}(\sigma_a) B^{R_a\to\bigcup_{b,\alpha} r_{ba;\alpha},\nu_a^+}_{i_a\to\bigcup_{b,\alpha} l_{ba;\alpha}} B^{R_a\to\bigcup_{b,\alpha} r_{ba;\alpha},\tilde{\nu}_a^+}_{\tilde{i}_a\to\bigcup_{b,\alpha} l_{ba;\alpha}} \tag{G.6}$$

which is (B.13).

Proof of orthogonality in $\boldsymbol{\sigma}$ conjugacy class of $\chi_Q(\boldsymbol{L},\boldsymbol{\sigma})$

Here we show (B.12).

Consider the product of quiver characters $\chi_Q(\boldsymbol{L},\boldsymbol{\sigma})\chi_Q(\boldsymbol{L},\boldsymbol{\tau})$:

$$\begin{aligned}&\chi_Q(\boldsymbol{L},\boldsymbol{\sigma})\chi_Q(\boldsymbol{L},\boldsymbol{\tau})\\ &= \prod_a D^{R_a}_{i_a j_a}(\sigma_a) B^{R_a\to\bigcup_{b,\alpha} r_{ab;\alpha},\nu_a^-}_{j_a\to\bigcup_{b,\alpha} l_{ab;\alpha}} B^{R_a\to\bigcup_{b,\alpha} r_{ba;\alpha},\nu_a^+}_{i_a\to\bigcup_{b,\alpha} l_{ba;\alpha}} D^{R_a}_{\tilde{j}_a\tilde{i}_a}(\tau_a^{-1}) B^{R_a\to\bigcup_{b,\alpha} r_{ab;\alpha},\nu_a^-}_{\tilde{j}_a\to\bigcup_{b,\alpha}\tilde{l}_{ab;\alpha}} B^{R_a\to\bigcup_{b,\alpha} r_{ba;\alpha},\nu_a^+}_{\tilde{i}_a\to\bigcup_{b,\alpha}\tilde{l}_{ba;\alpha}}\end{aligned} \tag{G.7}$$

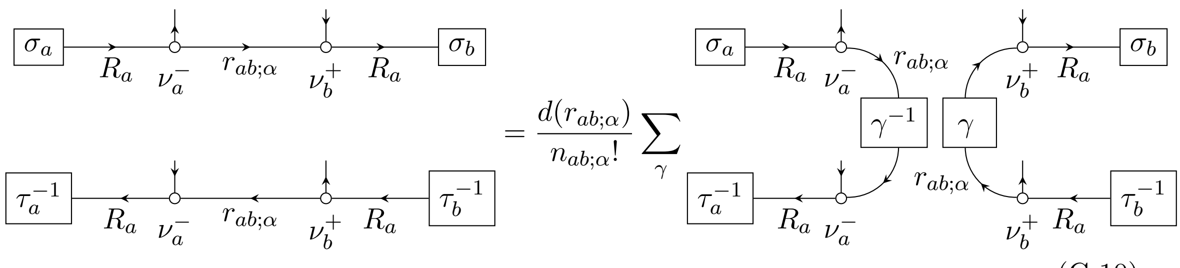
We flipped $D^R_{ij}(\tau) = D^R_{ji}(\tau^{-1})$ in the second character for later convenience. Each index $l_{ab;\alpha}$ appears once in a branching coefficient with ν^+ and once with ν^-, which are contracted together (and same for $\tilde{l}_{ab;\alpha}$). Next we "reconnect" the branching coefficients by inserting

$$\delta_{i_{ab;\alpha}j_{ab;\alpha}}\delta_{\tilde{i}_{ab;\alpha}\tilde{j}_{ab;\alpha}} = \frac{d(r_{ab;\alpha})}{n_{ab;\alpha}!}\sum_{\gamma_{ab;\alpha}} D^{r_{ab;\alpha}}_{\tilde{i}_{ab;\alpha}i_{ab;\alpha}}(\gamma_{ab;\alpha}) D^{r_{ab;\alpha}}_{j_{ab;\alpha}\tilde{j}_{ab;\alpha}}(\gamma^{-1}_{ab;\alpha}) \tag{G.8}$$

for each $l_{ab;\alpha}, \tilde{l}_{ab;\alpha}$:

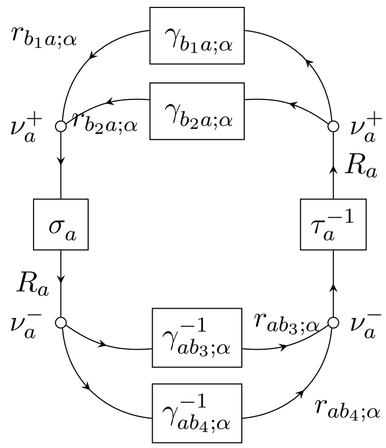
$$\begin{aligned}&\chi_Q(\boldsymbol{L},\boldsymbol{\sigma})\chi_Q(\boldsymbol{L},\boldsymbol{\tau})\\ &= \prod_a D^{R_a}_{i_a j_a}(\sigma_a) B^{R_a\to\bigcup_{b,\alpha} r_{ab;\alpha},\nu_a^-}_{j_a\to\bigcup_{b,\alpha} j_{ab;\alpha}} B^{R_a\to\bigcup_{b,\alpha} r_{ba;\alpha},\nu_a^+}_{i_a\to\bigcup_{b,\alpha} i_{ba;\alpha}} D^{R_a}_{\tilde{j}_a\tilde{i}_a}(\tau_a^{-1}) B^{R_a\to\bigcup_{b,\alpha} r_{ab;\alpha},\nu_a^-}_{\tilde{j}_a\to\bigcup_{b,\alpha}\tilde{j}_{ab;\alpha}} B^{R_a\to\bigcup_{b,\alpha} r_{ba;\alpha},\nu_a^+}_{\tilde{i}_a\to\bigcup_{b,\alpha}\tilde{i}_{ba;\alpha}}\\ &\quad\times\left(\prod_{a,b,\alpha}\frac{d(r_{ab;\alpha})}{n_{ab;\alpha}!}\sum_{\gamma_{ab;\alpha}} D^{r_{ab;\alpha}}_{\tilde{i}_{ab;\alpha}i_{ab;\alpha}}(\gamma_{ab;\alpha}) D^{r_{ab;\alpha}}_{j_{ab;\alpha}\tilde{j}_{ab;\alpha}}(\gamma^{-1}_{ab;\alpha})\right)\end{aligned} \tag{G.9}$$

After this, $i_{ba;\alpha}, \tilde{i}_{ba;\alpha}$ appear in a matrix element of $\gamma_{ba;\alpha}$, hence they link, via branching coefficients, to σ_a, τ_a^{-1}. Likewise $j_{ba;\alpha}, \tilde{j}_{ba;\alpha}$ appear in a matrix element of $(\gamma_{ba;\alpha})^{-1}$ and, via branching coefficients, link σ_b, τ_b^{-1}. This reconnection step can be understood diagrammatically, for each $r_{ab;\alpha}$:



(G.10)

Performing reconnection for all legs, the group factors disconnect into pieces like



(G.11)

Here $r_{b_1a;\alpha}$, $r_{b_2a;\alpha}$ represent fields incoming to a, and $r_{ab_3;\alpha}$, $r_{ab_4;\alpha}$ represent fields outgoing from a. The full expression (G.9) is just a product of such factors over a.

We can move $D(\gamma)$ and $D(\gamma^{-1})$ through branching coefficients next to $D(\sigma)$

$$
\begin{aligned}
&\chi_Q(\boldsymbol{L},\boldsymbol{\sigma})\chi_Q(\boldsymbol{L},\boldsymbol{\tau}) \\
&= \frac{\prod d(r_{ab;\alpha})}{\prod n_{ab;\alpha}!} \sum_{\boldsymbol{\gamma}} \prod_a D^{R_a}_{i_a j_a}((\otimes_{b,\alpha}\gamma_{ba;\alpha})\sigma_a(\otimes_{b,\alpha}\gamma^{-1}_{ab;\alpha}))D^{R_a}_{\tilde{j}_a\tilde{i}_a}(\tau_a^{-1}) \\
&\qquad \times B^{R_a\to\bigcup_{b,\alpha} r_{ab;\alpha},\nu_a^-}_{j_a\to\bigcup_{b,\alpha} j_{ab;\alpha}} B^{R_a\to\bigcup_{b,\alpha} r_{ab;\alpha},\nu_a^-}_{\tilde{j}_a\to\bigcup_{b,\alpha} j_{ab;\alpha}} B^{R_a\to\bigcup_{b,\alpha} r_{ba;\alpha},\nu_a^+}_{\tilde{i}_a\to\bigcup_{b,\alpha} i_{ba;\alpha}} B^{R_a\to\bigcup_{b,\alpha} r_{ba;\alpha},\nu_a^+}_{i_a\to\bigcup_{b,\alpha} i_{ba;\alpha}}
\end{aligned}
\tag{G.12}
$$

Now the branching coefficients are contracted in a way to make projectors, which we can sum over, using (A.18)

$$
\sum_{\{r_{ab,\alpha}\},\nu_a^-} B^{R_a\to\bigcup_{b,\alpha} r_{ab;\alpha},\nu_a^-}_{j_a\to\bigcup_{b,\alpha} j_{ab;\alpha}} B^{R_a\to\bigcup_{b,\alpha} r_{ab;\alpha},\nu_a^-}_{\tilde{j}_a\to\bigcup_{b,\alpha} j_{ab;\alpha}} = \sum_{r_{ab},\nu_a^-} P^{R_a\to\bigcup_{b,\alpha} r_{ab;\alpha},\nu_a^-}_{j_a\tilde{j}_a} = \delta_{j_a\tilde{j}_a} \tag{G.13}
$$

Performing this for both pairs of branching coefficients we arrive at

$$
\sum_{\boldsymbol{L}} \frac{\prod n_{ab;\alpha}!}{\prod d(r_{ab;\alpha})}\chi_Q(\boldsymbol{L},\boldsymbol{\sigma})\chi_Q(\boldsymbol{L},\boldsymbol{\tau}) = \sum_{R_a}\sum_{\boldsymbol{\gamma}}\prod_a \chi_{R_a}((\otimes_{b,\alpha}\gamma_{ba;\alpha})\sigma_a(\otimes_{b,\alpha}\gamma^{-1}_{ab;\alpha})\tau_a^{-1}) \tag{G.14}
$$

Finally, the sum over R_a can be done for each group factor using (A.3), if we include a factor $\frac{d(R_a)}{n_a!}$

$$
\begin{aligned}
\sum_{\boldsymbol{L}} \frac{\prod n_{ab;\alpha}!}{\prod d(r_{ab;\alpha})}\frac{\prod d(R_a)}{\prod n_a!}\chi_Q(\boldsymbol{L},\boldsymbol{\sigma})\chi_Q(\boldsymbol{L},\boldsymbol{\tau}) &= \sum_{\boldsymbol{\gamma}}\prod_a\sum_{R_a}\frac{\prod d(R_a)}{\prod n_a!}\chi_{R_a}((\otimes_{b,\alpha}\gamma_{ba;\alpha})\sigma_a(\otimes_{b,\alpha}\gamma^{-1}_{ab;\alpha})\tau_a^{-1}) \\
&= \sum_{\boldsymbol{\gamma}}\prod_a \delta((\otimes_{b,\alpha}\gamma_{ba;\alpha})\sigma_a(\otimes_{b,\alpha}\gamma^{-1}_{ab;\alpha})\tau_a^{-1})
\end{aligned}
\tag{G.15}
$$

Thus we arrive at (B.12).

G.2 Derivation of two-point function

Here we show (3.102), the two-point function of operators $\mathcal{O}_Q(\boldsymbol{n},\boldsymbol{\sigma})$ defined in (3.79), which is used to show the orthogonality of restricted basis in Section 3.3.2.

The conjugated operator is:

$$\begin{aligned}
\mathcal{O}_Q(\boldsymbol{n},\boldsymbol{\sigma})^\dagger &= \prod_{a,b,\alpha}\left(\bar{\Phi}_{ab;\alpha}^{\otimes n_{ab;\alpha}}\right)^{\boldsymbol{I}_{ab;\alpha}}_{\boldsymbol{J}_{ab;\alpha}} \prod_a (\sigma_a)^{\bigcup_{b,\alpha}\boldsymbol{J}_{ba;\alpha}}_{\bigcup_{b,\alpha}\boldsymbol{I}_{ab;\alpha}} \\
&= \prod_{a,b,\alpha}\left(\Phi_{ab;\alpha}^{\dagger\,\otimes n_{ab;\alpha}}\right)^{\boldsymbol{J}_{ab;\alpha}}_{\boldsymbol{I}_{ab;\alpha}} \prod_a (\sigma_a^{-1})^{\bigcup_{b,\alpha}\boldsymbol{I}_{ab;\alpha}}_{\bigcup_{b,\alpha}\boldsymbol{J}_{ba;\alpha}}
\end{aligned} \tag{G.16}$$

In the first line, since $\mathcal{O}_Q$ is a scalar, conjugation is simply a complex conjugation of the fields $\bar{\Phi}$. In the second line we convert it to Hermitian conjugate by transposing both $(\bar{\Phi})^i_j = (\Phi^\dagger)^j_i$ and $(\sigma)^i_j = (\sigma^{-1})^j_i$. The appearance of σ^{-1} indicates reversal of cyclic order, so that e.g. $\mathrm{tr}(XYZ)^\dagger = \mathrm{tr}(Z^\dagger Y^\dagger X^\dagger)$. The two point function for two fields is

$$\left\langle (\Phi_{ab;\alpha})^i_j (\Phi^{\dagger\,\alpha}_{ab})^k_l \right\rangle = \delta^i_l \delta^k_j \tag{G.17}$$

The Wick contraction between $n_{ab;\alpha}$ fields generate

$$\left\langle \left(\Phi_{ab;\alpha}^{\otimes n_{ab;\alpha}}\right)^{\boldsymbol{I}_{ab;\alpha}}_{\boldsymbol{J}_{ab;\alpha}} \left(\Phi_{ab;\alpha}^{\dagger\,\otimes n_{ab;\alpha}}\right)^{\tilde{\boldsymbol{J}}_{ab;\alpha}}_{\tilde{\boldsymbol{I}}_{ab;\alpha}} \right\rangle = \sum_{\gamma\in S_{n_{ab;\alpha}}} \delta^{\gamma(\boldsymbol{I}_{ab;\alpha})}_{\tilde{\boldsymbol{I}}_{ab;\alpha}} \delta^{\tilde{\boldsymbol{J}}_{ab;\alpha}}_{\gamma(\boldsymbol{J}_{ab;\alpha})} = \sum_{\gamma\in S_{n_{ab;\alpha}}} (\gamma^{-1})^{\boldsymbol{I}_{ab;\alpha}}_{\tilde{\boldsymbol{I}}_{ab;\alpha}} (\gamma)^{\tilde{\boldsymbol{J}}_{ab;\alpha}}_{\boldsymbol{J}_{ab;\alpha}} \tag{G.18}$$

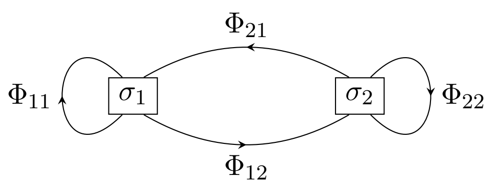
So the two point function, combining (3.79), (G.16) and (G.18):

$$\begin{aligned}
\left\langle \mathcal{O}_Q(\boldsymbol{n},\boldsymbol{\sigma})\mathcal{O}_Q(\boldsymbol{n},\tilde{\boldsymbol{\sigma}})^\dagger \right\rangle &= \sum_{\boldsymbol{\gamma}}\prod_{a,b,\alpha} (\gamma^{-1}_{ab;\alpha})^{\boldsymbol{I}_{ab;\alpha}}_{\tilde{\boldsymbol{I}}_{ab;\alpha}} (\gamma_{ab;\alpha})^{\tilde{\boldsymbol{J}}_{ab;\alpha}}_{\boldsymbol{J}_{ab;\alpha}} \prod_a (\sigma_a)^{\bigcup_{b,\alpha}\boldsymbol{J}_{ba;\alpha}}_{\bigcup_{b,\alpha}\boldsymbol{I}_{ab;\alpha}} (\tilde{\sigma}_a^{-1})^{\bigcup_{b,\alpha}\tilde{\boldsymbol{I}}_{ab;\alpha}}_{\bigcup_{b,\alpha}\tilde{\boldsymbol{J}}_{ba;\alpha}} \\
&= \sum_{\boldsymbol{\gamma}}\prod_a \mathrm{tr}\left(\sigma_a \left(\otimes_{b,\alpha}\gamma^{-1}_{ab;\alpha}\right) \tilde{\sigma}_a^{-1} \left(\otimes_{b,\alpha}\gamma_{ba;\alpha}\right)\right) \\
&\equiv \sum_{\boldsymbol{\gamma}}\prod_a \mathrm{tr}\left(\mathrm{Adj}_{\boldsymbol{\gamma}}(\sigma_a)\tilde{\sigma}_a^{-1}\right)
\end{aligned} \tag{G.19}$$

which gives (3.102).

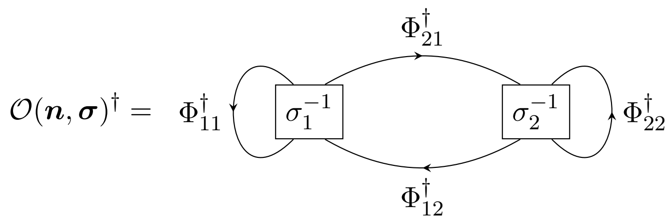
This calculation can also be understood diagrammatically. As an example let us take a simplified $\mathbb{C}^3/\mathbb{Z}_2$ quiver, with only a single flavor of Φ_{12} and Φ_{21}

$$\mathcal{O}(\boldsymbol{n},\boldsymbol{\sigma}) =$$



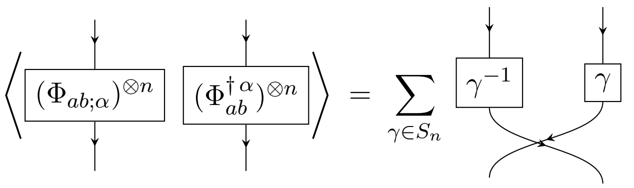
(G.20)

Conjugate operator (G.16) is represented by



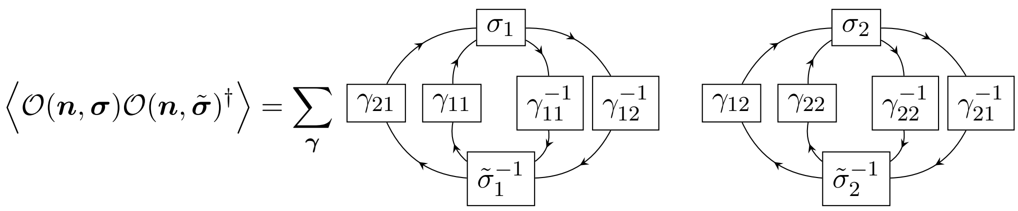
(G.21)

Our convention is that outgoing arrow corresponds to lower index, and incoming to upper index, so the reversed arrows indicate transposed indices in the second line of (G.16). The Wick contraction between blocks of conjugate fields (G.18) is, diagrammatically



(G.22)

Applying this rule to the product of diagrams (G.20) and (G.21) we find



(G.23)

It is easy to see that in general quivers will break up into separate factors for each group, with σ_a and $\tilde{\sigma}_a^{-1}$ connected by $\gamma^{-1}_{ab;\alpha}$ and $\gamma_{ba;\alpha}$. This reproduces (G.19).

G.3 Derivation of chiral ring structure constants

Here we explain the formulae corresponding to the diagrammatic derivation of (3.115).

We can write (3.112) as

$$\begin{aligned}
G(\boldsymbol{L}^{(1)},\boldsymbol{L}^{(2)};\boldsymbol{L}^{(3)}) &= \tilde{f}^{\boldsymbol{L}^{(3)}}_{\boldsymbol{L}^{(1)}\boldsymbol{L}^{(2)}} \frac{1}{\prod_a n_a^{(1)}! n_a^{(2)}!} \sum_{\boldsymbol{\sigma}^{(1)},\boldsymbol{\sigma}^{(2)}} \\
&\quad \prod_a \left(\prod_{p=1}^{3} B^{R_a^{(p)} \to \cup_{b,\alpha} r_{ba;\alpha}}_{i_a^{(p)} \to \cup_{b,\alpha} l_{ba}^{(p)\alpha}} \right) \left(\prod_{p=1}^{3} B^{R_a^{(p)} \to \cup_{b,\alpha} r_{ab;\alpha}}_{i_a^{(p)} \to \cup_{b,\alpha} l_{ab}^{(p)\alpha}} \right) \\
&\quad \times D^{R_a^{(1)}}_{i_a^{(1)} j_a^{(1)}}(\sigma_a^{(1)}) D^{R_a^{(2)}}_{i_a^{(2)} j_a^{(2)}}(\sigma_a^{(2)}) D^{R_a^{(3)}}_{i_a^{(3)} j_a^{(2)}}(\sigma_a^{(1)} \circ \sigma_a^{(2)}) \\
&= \tilde{f}^{\boldsymbol{L}^{(3)}}_{\boldsymbol{L}^{(1)}\boldsymbol{L}^{(2)}} \frac{1}{\prod_a n_a^{(1)}! n_a^{(2)}!} \sum_{\boldsymbol{\sigma}^{(1)},\boldsymbol{\sigma}^{(2)}} \\
&\quad \prod_a \left(\prod_{p=1}^{3} B^{R_a^{(p)} \to \cup_{b,\alpha} r_{ba;\alpha}}_{i_a^{(p)} \to \cup_{b,\alpha} l_{ba}^{(p)\alpha}} \right) \left(\prod_{p=1}^{3} B^{R_a^{(p)} \to \cup_{b,\alpha} r_{ab;\alpha}}_{i_a^{(p)} \to \cup_{b,\alpha} l_{ab}^{(p)\alpha}} \right) \\
&\quad \times D^{R_a^{(1)}}_{i_a^{(1)} j_a^{(1)}}(\sigma_a^{(1)}) D^{R_a^{(2)}}_{i_a^{(2)} j_a^{(2)}}(\sigma_a^{(2)}) D^{R_a^{(3)}}_{j_a^{(3)} i_a^{(3)}}((\sigma_a^{(1)})^{-1} \circ (\sigma_a^{(2)})^{-1})
\end{aligned} \tag{G.24}$$

Next we do the sum over the $\sigma_a^{(1)}, \sigma_a^{(2)}$, expressing the answer in terms of branching coefficients as in (3.121).

$$\begin{aligned}
&\sum_{\sigma^{(1)} \in S_{n^{(1)}}} \sum_{\sigma^{(2)} \in S_{n^{(2)}}} D^{R^{(1)}}_{i_1 j_1}(\sigma^{(1)}) D^{R^{(2)}}_{i_2 j_2}(\sigma^{(2)}) D^{R^{(3)}}_{i_3 j_3}(\sigma^{(1)} \circ \sigma^{(2)}) \\
&= \sum_{\sigma^{(1)},\sigma^{(2)}} \sum_{S^{(1)},S^{(2)}} \sum_{\nu} D^{R^{(1)}}_{i_1 j_1}(\sigma^{(1)}) D^{R^{(2)}}_{i_2 j_2}(\sigma^{(2)}) B^{R^{(3)} \to S^{(1)},S^{(2)};\nu}_{i_3 \to k_1,k_2} D^{S^{(1)}}_{k_1 m_1}(\sigma^{(1)}) D^{S^{(2)}}_{k_2 m_2}(\sigma^{(2)}) B^{R^{(3)} \to S^{(1)},S^{(2)};\nu}_{j_3 \to m_1,m_2} \\
&= \sum_{S^{(1)},S^{(2)}} \sum_{\nu} \frac{n^{(1)}!}{d(R^{(1)})} \frac{n^{(2)}!}{d(R^{(2)})} \delta_{R^{(1)},S^{(1)}} \delta_{R^{(2)},S^{(2)}} \delta_{i_1 k_1} \delta_{j_1 m_1} \delta_{i_2 k_2} \delta_{j_2 m_2} B^{R^{(3)} \to S^{(1)},S^{(2)};\nu}_{i_3 \to k_1,k_2} B^{R^{(3)} \to S^{(1)},S^{(2)};\nu}_{j_3 \to m_1,m_2} \\
&= \frac{n^{(1)}!}{d(R^{(1)})} \frac{n^{(2)}!}{d(R^{(2)})} \sum_{\nu} B^{R^{(3)} \to R^{(1)},R^{(2)};\nu}_{i_3 \to i_1,i_2} B^{R^{(3)} \to S^{(1)},S^{(2)};\nu}_{j_3 \to j_1,j_2}
\end{aligned} \tag{G.25}$$

Applying this at each node, gives two extra branching coefficients at each node of the quiver Q, leading to:

$$G(\boldsymbol{L}^{(1)},\boldsymbol{L}^{(2)};\boldsymbol{L}^{(3)}) = \frac{\tilde{f}^{\boldsymbol{L}^{(3)}}_{\boldsymbol{L}^{(1)}\boldsymbol{L}^{(2)}}}{\prod_a d(R_a^{(1)})d(R_a^{(2)})}\sum_{\{\nu_a\}}$$
$$\prod_a B^{R_a^{(1)}\to\cup_{b,\alpha} r^{(1)}_{ba;\alpha};\ \nu_a^{(1)+}}_{i_a^{(1)}\to\cup_{b,\alpha} l^{(1)}_{ba;\alpha}} B^{R_a^{(2)}\to\cup_{b,\alpha} r^{(2)}_{ba;\alpha};\ \nu_a^{(2)+}}_{i_a^{(2)}\to\cup_{b,\alpha} l^{(2)}_{ba;\alpha}} B^{R_a^{(3)}\to R_a^{(1)},R_a^{(2)};\nu_a^+}_{i_a^{(3)}\to i_a^{(1)},i_a^{(2)}} B^{R_a^{(3)}\to\cup_{b,\alpha} r^{(3)}_{ba;\alpha};\nu_a^{(3)+}}_{i_a^{(3)}\to\cup_{b,\alpha} l^{(3)}_{ba;\alpha}}$$
$$\times B^{R_a^{(3)}\to\cup_{b,\alpha} r^{(3)}_{ab;\alpha};\nu_a^{(3)-}}_{j_a^{(3)}\to\cup_{b,\alpha} l^{(3)}_{ab;\alpha}} B^{R_a^{(3)}\to R_a^{(1)},R_a^{(2)};\nu_a^-}_{j_a^{(3)}\to j_a^{(1)},j_a^{(2)}} B^{R_a^{(1)}\to\cup_{b,\alpha} r^{(1)}_{ab;\alpha};\ \nu_a^{(1)-}}_{j_a^{(1)}\to\cup_{b,\alpha} l^{(1)}_{ab;\alpha}} B^{R_a^{(2)}\to\cup_{b,\alpha} r^{(2)}_{ab;\alpha};\ \nu_a^{(2)-}}_{j_a^{(2)}\to\cup_{b,\alpha} l^{(2)}_{ab;\alpha}} \tag{G.26}$$
$$= \frac{\tilde{f}^{\boldsymbol{L}^{(3)}}_{\boldsymbol{L}^{(1)}\boldsymbol{L}^{(2)}}}{\prod_a d(R_a^{(1)})d(R_a^{(2)})}\sum_{\{\nu_a\}}\prod_a B^{R_a^{(3)}\to R_a^{(1)},R_a^{(2)};\nu_a^+}_{i_a^{(3)}\to i_a^{(1)},i_a^{(2)}} B^{R_a^{(3)}\to R_a^{(1)},R_a^{(2)};\nu_a^-}_{j_a^{(3)}\to j_a^{(1)},j_a^{(2)}}$$
$$\left(\prod_{p=1}^{3} B^{R_a^{(p)}\to\cup_{b,\alpha} r^{(p)\alpha}_{ba};\ \nu_a^{(p)+}}_{i_a^{(p)}\to\cup_{b,\alpha} l^{(p)\alpha}_{ba}}\right)\left(\prod_{p=1}^{3} B^{R_a^{(p)}\to\cup_{b,\alpha} r^{(p)\alpha}_{ab};\nu_a^{(3)-}}_{j_a^{(3)}\to\cup_{b,\alpha} l^{(p)\alpha}_{ab}}\right)$$

The label ν_a is summed over the Littlewood-Richardson multiplicity $g(R_a^{(1)},R_a^{(2)};R_a^{(3)})$ for the reduction of the irrep $R_a^{(3)}$ of $S_{n_a^{(3)}}$ to irrep $R^{(1)}\otimes R_a^{(2)}$ of $S_{n_a^{(1)}}\times S_{n_a^{(2)}}$. By Schur-Weyl duality, this is also the multiplicity of the $U(N_a)$ representation $R_a^{(3)}$ in the tensor product of $R_a^{(1)}\otimes R_a^{(2)}$.

The next step is to exploit the invariance, under the action of $\times_{a,b,\alpha} S_{n^{(1)}_{ab;\alpha}}\times S_{n^{(2)}_{ab;\alpha}}$, of the branching coefficients in (G.26) labelled by $\nu_a^{(1)-},\nu_a^{(2)-}\nu_a^{(3)-}$ (we could equally well have chosen to work with the $\nu_a^{(1)+},\nu_a^{(2)+}\nu_a^{(3)+}$) as indicated in (3.123).

$$G(\boldsymbol{L}^{(1)},\boldsymbol{L}^{(2)};\boldsymbol{L}^{(3)}) = \tilde{f}^{\boldsymbol{L}^{(3)}}_{\boldsymbol{L}^{(1)}\boldsymbol{L}^{(2)}}\frac{1}{\prod_a d(R_a^{(1)})d(R_a^{(2)})}\frac{1}{\prod_{a,b,\alpha} n^{(1)}_{ab;\alpha}!n^{(2)}_{ab;\alpha}!}\sum_{\{\nu_a\}}\sum_{\gamma^{(1)}_{ab;\alpha},\gamma^{(2)}_{ab;\alpha}}$$
$$\prod_a B^{R_a^{(3)}\to R_a^{(1)},R_a^{(2)};\nu_a^+}_{i_a^{(3)}\to i_a^{(1)},i_a^{(2)}} B^{R_a^{(3)}\to R_a^{(1)},R_a^{(2)};\nu_a^-}_{j_a^{(3)}\to j_a^{(1)},j_a^{(2)}} \left(\prod_{p=1}^{3} B^{R_a^{(p)}\to\cup_{b,\alpha} r^{(p)\alpha}_{ba};\ \nu_a^{(p)+}}_{i_a^{(p)}\to\cup_{b,\alpha} l^{(p)\alpha}_{ba}}\right)$$
$$\left(\prod_{a,b,\alpha} D^{r^{(1)}_{ab}\alpha}_{k^{(1)}_{ab;\alpha} l^{(1)}_{ab;\alpha}}(\gamma^{(1)\alpha}_{ab}) D^{r^{(2)}_{ab}\alpha}_{k^{(2)}_{ab;\alpha} l^{(2)}_{ab;\alpha}}(\gamma^{(2)\alpha}_{ab}) D^{r^{(3)}_{ab}\alpha}_{k^{(3)}_{ab;\alpha} l^{(3)}_{ab;\alpha}}((\gamma^{(1)\alpha}_{ab})^{-1}\circ(\gamma^{(2)\alpha}_{ab})^{-1}) \prod_{p=1}^{3} B^{R_a^{(p)}\to\cup_{b,\alpha} r^{(p)\alpha}_{ab};\nu_a^{(3)-}}_{j_a^{(3)}\to\cup_{b,\alpha} k^{(p)\alpha}_{ab}}\right) \tag{G.27}$$

Now we do the sum over the permutations $\{\gamma^{(1)}_{ab;\alpha},\gamma^{(2)}_{ab;\alpha}\}$ which introduces branching coefficients for $r^{(3)}_{ab;\alpha}\to r^{(1)}_{ab;\alpha}\otimes r^{(2)}_{ab;\alpha}$

$$
\begin{aligned}
G(\boldsymbol{L}^{(1)}, \boldsymbol{L}^{(2)}; \boldsymbol{L}^{(3)}) &= \tilde{f}^{\boldsymbol{L}^{(3)}}_{\boldsymbol{L}^{(1)}\boldsymbol{L}^{(2)}} \frac{1}{\prod_a d(R_a^{(1)}) d(R_a^{(2)})} \frac{1}{\prod_{a,b,\alpha} d(r_{ab;\alpha}^{(1)}) d(r_{ab;\alpha}^{(2)})} \\
&\sum_{\{\nu_a, \nu_{ab;\alpha}\}} \prod_a \left(B^{R_a^{(3)} \to R_a^{(1)}, R_a^{(2)}; \nu_a^+}_{i_a^{(3)} \to i_a^{(1)}, i_a^{(2)}} B^{R_a^{(3)} \to R_a^{(1)}, R_a^{(2)}; \nu_a^-}_{j_a^{(3)} \to j_a^{(1)}, j_a^{(2)}} \right) \\
&\prod_a \left(\prod_{p=1}^{3} B^{R_a^{(p)} \to \cup_{b,\alpha} r_{ba}^{(p)\alpha};\ \nu_a^{(p)+}}_{i_a^{(p)} \to \cup_{b,\alpha} l_{ba}^{(p)\alpha}} \right) \left(\prod_{p=1}^{3} B^{R_a^{(p)} \to \cup_{b,\alpha} r_{ab}^{(p)\alpha}; \nu_a^{(p)-}}_{j_a^{(p)} \to \cup_{b,\alpha} k_{ab}^{(p)\alpha}} \right) \\
&\prod_{a,b,\alpha} B^{r_{ab;\alpha}^{(3)} \to r_{ab;\alpha}^{(1)}, r_{ab;\alpha}^{(2)}; \nu_{ab;\alpha}}_{l_{ab;\alpha}^{(3)} \to l_{ab;\alpha}^{(1)}, l_{ab;\alpha}^{(2)}} B^{r_{ab;\alpha}^{(3)} \to r_{ab;\alpha}^{(1)}, r_{ab;\alpha}^{(2)}; \nu_{ab;\alpha}}_{k_{ab;\alpha}^{(3)} \to k_{ab;\alpha}^{(1)}, k_{ab;\alpha}^{(2)}}
\end{aligned}
\tag{G.28}
$$

We now see that there is a factorization between state indices for Young diagrams associated branching coefficients carrying ν^- indices and those for Young diagrams associated branching coefficients carrying ν^- indices, which corresponds to the factorized form in the diagram (3.113)

$$
\begin{aligned}
G(\boldsymbol{L}^{(1)}, \boldsymbol{L}^{(2)}; \boldsymbol{L}^{(3)}) &= \tilde{f}^{\boldsymbol{L}^{(3)}}_{\boldsymbol{L}^{(1)}\boldsymbol{L}^{(2)}} \frac{1}{\prod_a d(R_a^{(1)}) d(R_a^{(2)})} \frac{1}{\prod_{a,b,\alpha} d(r_{ab;\alpha}^{(1)}) d(r_{ab;\alpha}^{(2)})} \\
&\sum_{\{\nu_a, \nu_{ab;\alpha}\}} \prod_a \left(B^{R_a^{(3)} \to R_a^{(1)}, R_a^{(2)}; \nu_a^+}_{i_a^{(3)} \to i_a^{(1)}, i_a^{(2)}} \prod_{p=1}^{3} B^{R_a^{(p)} \to \cup_{b,\alpha} r_{ba}^{(p)\alpha};\ \nu_a^{(p)+}}_{i_a^{(p)} \to \cup_{b,\alpha} l_{ba}^{(p)\alpha}} \prod_{b,\alpha} B^{r_{ab;\alpha}^{(3)} \to r_{ab;\alpha}^{(1)}, r_{ab;\alpha}^{(2)}; \nu_{ab;\alpha}}_{l_{ab;\alpha}^{(3)} \to l_{ab;\alpha}^{(1)}, l_{ab;\alpha}^{(2)}} \right) \\
&\prod_a \left(B^{R_a^{(3)} \to R_a^{(1)}, R_a^{(2)}; \nu_a^-}_{j_a^{(3)} \to j_a^{(1)}, j_a^{(2)}} \prod_{p=1}^{3} B^{R_a^{(p)} \to \cup_{b,\alpha} r_{ab}^{(p)\alpha}; \nu_a^{(p)-}}_{j_a^{(p)} \to \cup_{b,\alpha} k_{ab}^{(p)\alpha}} \prod_{b,\alpha} B^{r_{ab;\alpha}^{(3)} \to r_{ab;\alpha}^{(1)}, r_{ab;\alpha}^{(2)}; \nu_{ab;\alpha}}_{k_{ab;\alpha}^{(3)} \to k_{ab;\alpha}^{(1)}, k_{ab;\alpha}^{(2)}} \right)
\end{aligned}
\tag{G.29}
$$

This is the factorized result, where we have a factor for each gauge group, and for each gauge group there is a factorization separating the ν^+ branching coefficients from the ν^- branching coefficients. The close connection between the final formula and the diagrammatic moves means that we can interpret the process of constructing the final answer diagrammatically. Start with the original quiver and modify it to produce the split-node version with R_a lines joining the plus and minus nodes. Now cut this split-node quiver along all the edges, thus separating it into a collection of nodes labelled ν_a^+, ν_a^-. The ν_a^+ nodes have a collection of directed lines carrying labels $R_a, r_{ba;\alpha}$. The ν_a^- nodes have outgoing directed lines labelled $R_a, r_{ab;\alpha}$. Doing this cutting procedure for the three labelled quivers, to produce nodes $(\nu_a^{(I)+}, \nu_a^{(I)-})$ (for $I = 1, 2, 3$) with dangling lines labelled $R_a^{(I)}, r_{ba}^{(I)\alpha}$. Link up the nodes $\nu_a^{(I)+}$ using new nodes μ_a^+ for $(R_a^{(1)}, R_a^{(2)}) \to R_a^{(3)}$, and new nodes $\mu_{ab;\alpha}$ for the $r_{ba;\alpha}^{(1)}, r_{ba;\alpha}^{(2)} \to r_{ba;\alpha}^{(3)}$. This gives a graph for each gauge group labelled a, with nodes labelled by $\{\cup_I \nu_a^{(I)+}, \mu_a, \mu_{ab;\alpha}\}$. Repeating the same procedure for the minus nodes gives another set of graphs for each gauge group, with nodes labelled $\{\cup_I \nu_a^{(I)-}, \mu_a, \mu_{ba;\alpha}\}$. So the result for the chiral ring structure constants can be obtained by cutting and gluing of the split-node quivers labelled $\boldsymbol{L}_1, \boldsymbol{L}_2, \boldsymbol{L}_3$. This is an illustration of the power of quivers as calculators.

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