Some computations in the Monster group and related topics

Adeel Farooq

A Thesis submitted for the degree of Doctor of Philosophy

written under the supervision of

Prof. R. A. Wilson

School of Mathematical Sciences
Queen Mary, University of London.

2012

Declaration by candidate

I hereby declare that this thesis is my own work while a student at Queen Mary University of London and that it has not been submitted anywhere for any award. Where other sources of information have been used, they have been acknowledged.

Adeel Farooq

Contents

Acknowledgements

I am heartily thankful to my supervisor, Prof. R. A. Wilson. It has been an honor to be his Ph.D student. His encouragement, patience and supervision from the preliminary to the concluding level enabled me to develop an understanding of the subject and complete this thesis. It would not have been possible to write this thesis without his guidance and constant encouragement. Throughout the time he was the source of inspiration to me.

I would like to thank Dr. John Bray for his conversations on technical matters and also for his representation theory seminars which helped me understand the subject.

I am grateful to Petra Holmes for her Monster programs which enabled me to do the computations regarding my thesis.

I would like to thank my teacher Prof. Kramat Hussain Dar who introduced me to group theory and also like to thank my previous teachers who have taught me mathematics at any level.

I wish to thank my friends specially Manjula Samarasinghe and M. Akram for their emotional support and helping me get through the difficult times.

Lastly, and most importantly, I wish to thank my parents. To them I dedicate this thesis.

Abstract

Simon Norton has produced a presentation for the Monster group induced by the action of $L_3(3)$ on a projective plane of order 3. The generators given by Norton can be found inside the involution centralizer of the Monster.

R. A. Wilson and Petra Holmes has already given a 2-local construction of the Monster and this implies that it is possible to give a computational proof of the existence of the Monster.

In this thesis we construct the generators given by Norton and also verify the new presentation given by him. Thus we give a new existence proof of the Monster which does not depend on the original proof given by Griess.

Chapter 1

Introduction

The Monster is the largest of the family of 26 sporadic simple groups having order $\approx 10^{54}$. The smallest matrix representations of the Monster have dimension 196882 in characteristics 2 and 3, and dimension 196883 in all other characteristics. Due to its immense size it was not possible to give a computational existence proof at the time its idea was conceived.

R. A. Wilson with his collaborators gave a 3-local construction of the Monster [14]. Despite its success, these 3-local subgroups were too small to contain many useful subgroups. After some time, R. A. Wilson and Petra Holmes gave a 2-local construction of the Monster [8]. They were able to compute inside the Monster and found new maximal subgroups as well.

Simon Norton on the other hand produced a presentation of the Monster on generators closely related to the 2-local subgroups used in [8]. This offered an opportunity to verify this presentation and give a computational proof of existence of the Monster independent of Griess's original proof [7].

In this thesis we use Monster programs written by Petra Holmes [8] for our computations.

Throughout this thesis we will use ATLAS [4] notation and conventions.

Now we give a brief overview of the chapters.

In chapter 1 we will discuss some background material which can be taken as a reference in the remaining part of the thesis.

In chapter 2 we construct Coxeter generators of the Baby monster and this chapter discusses various techniques which are used later.

Chapter 3 discusses the construction of Norton's generators in detail. Our main theorem which asserts the existence of computational proof of the Monster for first time is given in chapter 4.

In Chapter 5 we discuss how one can construct the Coxeter generators for the Bimonster using the Norton's generators.

Since Norton's presentation is unpublished to this date, we have given Norton's preprint "Transforming the Monster presentation" as an appendix to this thesis.

We have attached a dvd with this thesis which contains programs to construct coxeter generators for Baby monster, and Norton's generators. The programs to verify the Norton's presentation have also been included.

1.1 Bray's method

Before John Bray came up with his method of constructing involution centralizer, group theorists were using a method due to Richard Parker to find involution centralizer which relied on the fact that two involutions generate a dihedral group.

We briefly describe John Bray's method [2] here.

Let g be an involution of a finite group G whose centralizer we wish to determine and h be a random element of G.

- If $[g,h]$ has even order say $2k$ then $[g,h]^k$ and $[g,h^{-1}]^k \in C_G(g)$.

  To see why this is true, observe that $g[g,h]^{-1} = [g,h]g$. By induction we have $g[g,h]^{-n} = [g,h]^n g,$ for all $n \in \mathbb{N}$.

  Hence if $[g,h]$ has even order say $2k$ then $g[g,h]^k = [g,h]^{-k}g = [g,h]^k g$. This implies that $[g,h]^k \in C_G(g)$.
- If $[g,h]$ has odd order say $2k+1$ then $h[g,h]^k \in C_G(g)$.

  Suppose that $[g,h]$ has order $2k+1$. Then $gh[g,h]^k = hg[g,h]^{k+1} = hg[g,h]^{-k} = h[g,h]^k g$. This implies that $h[g,h]^k \in C_G(g)$.

It has been observed by Richard Parker that elements of centralizer produced in odd order case are uniformly distributed over $C_G(g)$. To see this, fix h with $[g,h]$ of order $2k+1$. Now let c run through the elements of $C_G(g)$. Observe that for $c \in C_G(g)$, we have $[g,ch] = [g,h]$. Now consider the coset $C_G(g)h$. We see that $ch[g,ch]^k = c(h[g,h]^k)$ runs through the elements of $C_G(g)$. This means that if h is a random element of G such that $[g,h]$ has odd order then $h[g,h]^k$ is a random element of $C_G(g)$.

Usually three or four such elements arising from the odd order case generate the whole centralizer.

Lemma 1.1.1 (Conway, Parker and Norton) *If G is a finite group, $K \lhd G$ an elementary abelian 2-group, $G/K \cong 3$ and $G = \langle x, z\rangle$ where x and z both have order 3 in the same coset of K, then $x^{zx} = z$ and $z^{xz} = x$.*

This lemma which is better known by the name of "the formula" is an extremely useful tool in conjugating elements of order 3 which are already conjugate modulo the elementary abelian 2-group. We have an immediate corollary.

Corollary 1.1.1 *If G is a finite group, $K \lhd G$ an elementary abelian 2-group, $G = \langle x, y\rangle$ where x has order 3 and $\langle x, x^y\rangle K/K \cong 3$ then $[x, yxx^y] = 1$.*

Letting $z = x^y$ in the above lemma gives us $(x^y)^{xx^y} = x$ and hence
$[x, yxx^y] = 1$. □

There are various generalizations of the above lemma and one such is as follows: if x has order $2k + 1$ in the above lemma then $[x, y(xx^y)^k] = 1$.

1.2 An overview of the 2-local computational construction of the Monster

The detailed 2-local computational construction of the $\mathbb{M}$ is given in [8]. This construction is based on the Griess's strategy [7] which was further simplified by Conway [3].

A brief overview of the construction is as follows:

- The smallest matrix representation of $\mathbb{M}$ over characteristic 2 or 3 has dimension 196882. So, restrict the 3 modular irreducible representation of $\mathbb{M}$ to $2^{1+24}_{+}{}^{\cdot}Co_1$ as **98304** ⊕ **298** ⊕ **98280**.

  The module **98304** is a tensor product of **24** and **4096**. Over $GF(3)$, the modules **24** and **298** represents the double cover of Co_1 and quotient Co_1 respectively. Whereas the module **98280** is a monomial permutation representing $2^{24} \cdot Co_1$. The module **4096** represents $2^{1+24} \cdot Co_1$. Despite its shape, this group is not the involution centralizer of $\mathbb{M}$. We give here the following explanation:

The group $2^{24} \cdot Co_1$ has Schur multiplier 2^2. Its universal cover is $(2^{1+24} \times 2) \cdot Co_1 \cong 2^2.2^{24}Co_1$. We denote its centre 2^2 by $\{1, x, y, z\}$.

For the group $2^{1+24} \cdot Co_1$ which has representation **4096**, y acts trivially while x and z act as -1.

For the group $2 \cdot Co_1$ having representation **24**, x acts trivially while y and z act as -1.

Now the group of shape $2^{1+24} \cdot Co_1$ having representation $\mathbf{4096} \otimes \mathbf{24}$, z acts trivially while x and y act as -1. This group is the involution centralizer of $\mathbb{M}$.

- Now construct the involution centralizer $2_+^{1+24} \cdot Co_1$ so that we can calculate in this group and also its action on 196882 dimensional module over $GF(3)$.
- Then construct a "triality" element, which normalizes a subgroup of shape $2^2.2^{11}.2^{22}.M_{24}$ which has index 2 in $2^{1+24}2^{11}M_{24}$. In Conway's construction [3], one can see the action of triality element which is permuting three copies of $2_+^{1+24} \cdot Co_1$.

  For the subgroup $2^2.2^{11}.2^{22}.M_{24}$, we see that

  98304 decomposes as $\mathbf{49152_a} \oplus \mathbf{49152_b}$.

  98280 decomposes as $\mathbf{49152_c} \oplus \mathbf{48576} \oplus \mathbf{276_a} \oplus \mathbf{276_b}$.

  298 decomposes as $\mathbf{276_c} \oplus \mathbf{22}$.

  The action of triality element on a vector can be computed, see [8] for more details.

1.2.1 Basic calculations in $\mathbb{M}$

We can treat every element of $2_+^{1+24\cdot}Co_1$ as a generator of $\mathbb{M}$. Also, every element of $\mathbb{M}$ which is not in $2_+^{1+24\cdot}Co_1$ can be calculated as a word $g_1T^{\pm}g_2T^{\pm}\dots$, where T is the triality element, T^{-1} is its inverse and $g_i \in 2_+^{1+24\cdot}Co_1$. To calculate the order of such an element, we take a random vector v in the underlying module, the chances are extremely good that it lies in a regular orbit under the $\mathbb{M}$. Thus the divisor of the order of an element g is equal to the smallest positive integer n such that $vx^n = v$ and in most cases it turns out to be the actual order of g. In 4.1 (or see [14], [28]) we will describe how to calculate the exact order by taking two specially constructed vectors instead of one.

1.3 Projective plane, Y groups and the Bimonster

1.3.1 Projective plane

Let V be vector space over field F having rank $n+1$. The subspaces of V apart from V itself and zero subspace $\{0\}$ form a *projective space*. We call these subspaces as "objects" of the projective space and denote it by $PG(n, F)$. Each object is assigned a dimension which is one less than its rank, and we use geometric terminology, so that *points*, *lines*, *planes* are the objects of dimension 0, 1 and 2 (that is, rank 1, 2 and 3 respectively). A hyperplane is a subspace of codimension 1. Two objects are *incident* if one contains the other.

If $n > 1$, $PG(n, F)$ contains objects of different dimensions, and the relation of incidence gives it a non-trivial structure.

A projective plane is a projective space consisting of objects of dimension 0 called points and objects of dimension 1 called lines and a relation of incidence between them, having the following properties:

- Given any two distinct points, there is exactly one line incident with both of them.
- Given any two distinct lines, there is exactly one point incident with both of them.
- There are four points such that no line is incident with more than two of them.

A projective plane has the same number of lines as it has points. A finite projective plane of order n has:

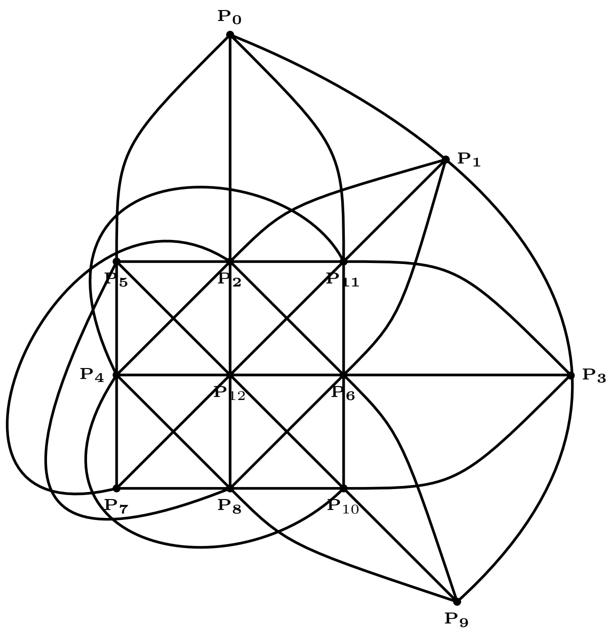
- $n^2 + n + 1$ points.
- $n^2 + n + 1$ lines.
- $n + 1$ points on each line.
- $n + 1$ lines through each point.

In this thesis, we will be interested in projective plane of order 3 only. The points and lines of this projective plane act as generators of the Bimonster group.

1.3.1.1 Incidence graph

We can construct an incidence graph on 26 vertices corresponding to 13 points and 13 lines as follows:

We join P_i to L_j if $i + j \equiv 0$, 1, 3 or 9 (mod 13). Hence we label the points



Projective plane of order 3

by $P_0, P_1, \ldots, P_{12}$ and lines $L_0, L_1, \ldots, L_{12}$ (mod 13) such that $L_{-j} = \{P_j, P_{j+1}, P_{j+3}, P_{j+9}\}$.

1.3.1.2 Automorphisms of projective plane of order 3

We define the following automorphisms of projective plane of order 3:

1. $\alpha : P_j \mapsto P_{j+1},\ L_j \mapsto L_{j-1}$
2. $\beta : P_j \mapsto P_{3j},\ L_j \mapsto L_{3j}$
3. $\gamma : (P_1, P_3)(P_2, P_6)(P_{12}, P_{10})(P_8, P_{11}),\ (L_{12}, L_{10})(L_{11}, L_8)(L_2, L_6)(L_1, L_3)$

We can also coordinatise the the projective plane by identifying the points as follows: $P_7 = \langle(001)\rangle$, $P_8 = \langle(101)\rangle$, $P_{10} = \langle(201)\rangle$, $P_4 = \langle(011)\rangle$, $P_{12} = \langle(111)\rangle$, $P_6 = \langle(211)\rangle$, $P_5 = \langle(021)\rangle$, $P_2 = \langle(121)\rangle$, $P_{11} = \langle(221)\rangle$, $P_0 = \langle(010)\rangle$, $P_1 = \langle(110)\rangle$, $P_3 = \langle(100)\rangle$ and $P_9 = \langle(120)\rangle$. We can write the matrices over $\mathbb{F}_3$ corresponding to above transformations. The matrices corresponding to α, β and γ are $\begin{pmatrix} 0 & 2 & 2 \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ respectively. We see that the automorphism group of the projective plane of order 3 is $L_3(3)$.

1.3.2 Coxeter group

A Coxeter group is generated by involutions corresponding to the nodes of a Coxeter diagram. The product of two generators has order three if the nodes are joined by a single edge, and order two if they are not joined. We use the standard notation A_n, D_n, E_n for Dynkin diagrams of these types, and also for the corresponding Coxeter or Weyl groups and $\tilde{A_n}, \tilde{D_n}, \tilde{E_n}$ for the corresponding extended Dynkin diagrams

1.3.3 The Bimonster and Y-groups

The Bimonster is defined to be the group $\mathbb{M} \wr 2$ (Wreath product of $\mathbb{M}$ and $\mathbb{Z}_2$). Hence it is generated by $\mathbb{M} \times \mathbb{M}$ and an involution which is swapping two copies of the Monster.

1.3.3.1 From projective plane to Y_{555}

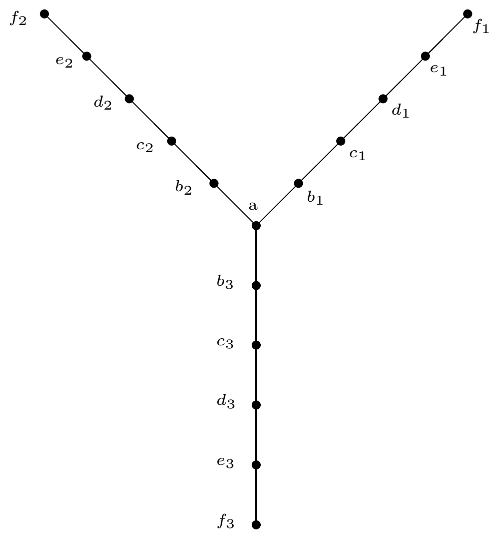
The Y-subgroups are the subgroups of $\mathbb{M} \wr 2$ generated by a and

- the first p terms of the sequence b_1, c_1, d_1, e_1, f_1
- the first q terms of the sequence b_2, c_2, d_2, e_2, f_2
- the first r terms of the sequence b_3, c_3, d_3, e_3, f_3

Norton, Conway and Soicher has shown that $\mathbb{M} \wr 2 \cong Y_{555}$ [5].

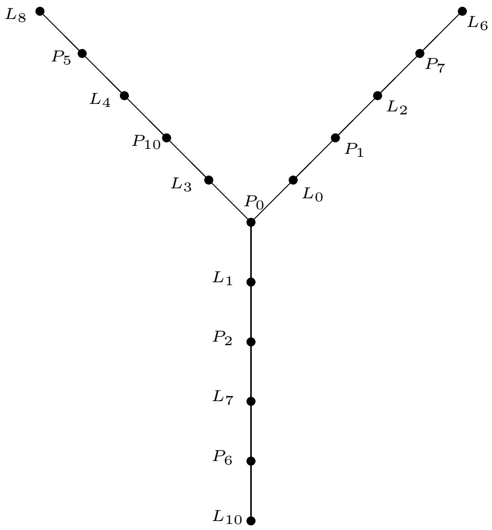
The group Y_{555} contains 26 involutions, which can be taken as generators of the Bimonster, satisfying the Coxeter relations and an additional relator $(ab_2c_2ab_1c_1ab_3c_3)^{10}$. These 26 involutions can be identified with the points and lines of the projective plane of order 3. The nodes f_1, f_2 and f_3 are actually redundant since $Y_{444} \cong Y_{555}$ [5]. This means that we only need 13 generators for the Bimonster.

Defining relations. Following [4], we define $f_{ij} = (ab_ib_jb_kc_id_i)^9$, whenever $\{i, j, k\} = \{1, 2, 3\}$. Inside Y_{555}, we also have the following relations:
$f_1 = f_{12}$ or f_{13}, $f_2 = f_{23}$ or f_{21}, $f_3 = f_{31}$ or f_{32} [4].



The Group Y_{555}

Now we identify the generators of Y_{555} with points and lines of the projective plane of order 3.



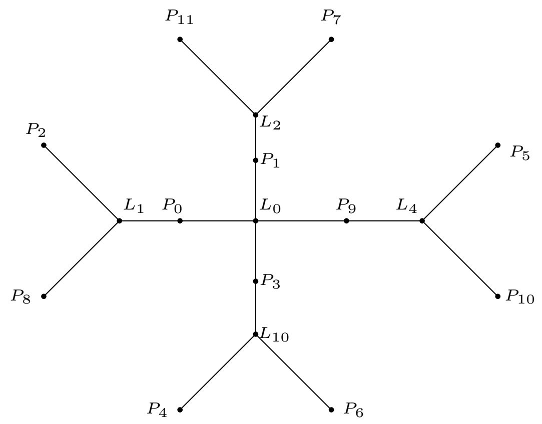
1.3.4 The group X_{3333}

A 17-node subgraph of the incidence graph is shown in the following figure. We denote its 12 points and 5 lines as shown; the thirteenth point is called P_{12} and the other eight lines are unnamed. The group generated by this subgraph is defined to be X_{3333} and it turns out that $X_{3333} \cong (2^{10+16} \times 2^{10+16}).O_{10}^{+}(2){:}2$ [18].

Let P_i, P_j be two points and l be the (unique) line containing them. We define *cog* to be an element $(P_iP_j)^l$. It is shown in [18] that two *cogs* determine a Conway group.

Lemma 1.3.1 ([18]) *The centre of the group generated by a D_4 diagram depends only on the (unique) node that completes this diagram to a $\tilde{D}_4$.*

Notation The central involution of a D_4 diagram completed to a $\tilde{D}_4$ by a node N will be called N^*. We may call elements of this type point-stars or



line-stars according as N is a point or a line. We will refer point-stars as *stars.*

Lemma 1.3.2 ([18]) *If A and B are two distinct points in $\{P_0, \dots, P_{12}\}$, then A^* and $[A, A^*] = [B, B^*]$ has order 2. Also $[A, B^*] = [A^*, B^*] = I$.*

It follows from the above lemma that $[A, A^*]$ is the same for points A. We call this element π.

For proof of above lemma, consider a diagram of type X_{3111} which can be described as a union of disjoint (but not disconnected) subdiagrams of type A_2 and $\tilde{D}_4$, such as $\{L_6, P_7, L_9, P_0, L_0, L_3, L_1\}$. This contains the three D_6- subdiagrams, obtainable in this case by suppressing the nodes L_0, L_3 and L_1 in turn. The non-trivial elements of the centres of the groups generated by these D_6's are each product of two points and and therefore commute with the suppressed nodes., giving three relations. Coset enumerations can be used to identify the group presented by these relations and the Coxeter relations of X_{3111}, showing that the subgroup of X_{3111} (with these relations) generated by $\tilde{D}_4$ has order 2^6 times that of D_4 itself. This determines the group. There is a (unique) $\tilde{D}_4$ containing any two points A and B, and by working inside this group (see the appendix) we can prove all the relations required.

1.4 Norton's Presentation

We denote the elements of the Bimonster corresponding to the points and lines of the projective plane by P_i and L_i where $0 \leq i \leq 12$, and the point P_i is incident with the line L_j if and only if $i + j \equiv 0, 1, 3$ or $9 \pmod{13}$.

Following Norton [17] we define the following generators of the Bimonster:

1. s is the element of $L_3(3)$ that acts as

$$(P_1, P_2, P_5, P_9, P_8, P_7)(P_3, P_{12}, P_4)(P_{10}, P_{11})$$

   on the points. It can be seen that its action on the lines is $(L_0, L_1, L_9)(L_2, L_{12}, L_{11}, L_4, L_5, L_6)(L_7, L_8)$.

   As in 1.3.1.2, the matrix corresponding to the action of s is

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 2 & 0 \end{pmatrix}$$

2. t is the element of $L_3(3)$ that acts as

$$(P_0, P_{12}, P_3)(P_1, P_2, P_4)(P_5, P_7, P_{11})(P_6, P_9, P_8)$$

   on the points and which may be seen to have action

$(L_0, L_1, L_{10})(L_2, L_{11}, L_9)(L_3, L_4, L_6)(L_5, L_8, L_7)$ on the lines.

We give the corresponding matrix for t which is:

$$\begin{pmatrix} 0 & 2 & 0 \\ 2 & 2 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

From s and t we may derive $u = sts^2t^2$ which acts as $z \mapsto z - 1$ on the points and $z \mapsto z + 1$ on the lines(where all numbers are taken mod 13).

3. v is the product of all the points with non-zero subscript.
4. x is the product P_0L_0.
5. α is the product of all thirteen points (note that α lies outside the subgroup $\mathbb{M} \times \mathbb{M}$ of the Bimonster).

After defining these generators, Norton then gives the following relations to achieve a presentation for $\mathbb{M}$ (see appendix) and we give some comments after each relation:

$$s^6 = t^3 = (st)^4 = (s^2t)^4 = (s^3t)^3 = [s^2, (ts^2t)^2] = 1 \tag{1.1}$$

This gives a presentation for $L_3(3)$.

$$[v, ut^{-1}] = [v, u^3su^{-2}] = v^2 = [v, v^t] = (vu)^{13} = 1 \tag{1.2}$$

It is easy to see that the subgroup $\langle ut^{-1}, u^3su^{-2}\rangle$ of $L_3(3)$ stabilizes P_0. The first two relators say that v commutes with $\langle ut^{-1}, u^3su^{-2}\rangle$. The third relator says that v has order 2 and the fourth relator says that v commutes with all images of v under $L_3(3)$.

$$[\alpha, s] = [\alpha, t] = [\alpha, v] = \alpha^2 = 1 \tag{1.3}$$

The first three relators say that α centralizes $2^{12}L_3(3)$.

$$[vx, ut^{-1}] = [vx, s^{u^3}] = 1 \tag{1.4}$$

Again, we observe that the subgroup $\langle ut^{-1}, s^{u^3} \rangle$ of $L_3(3)$ stabilizes L_0. This relation says that vx commutes with $\langle ut^{-1}, s^{u^3} \rangle$.

$$x^3 = (v^u vx)^2 = (x^{-1}x^s)^2 = 1 \tag{1.5}$$

The first relator says that order of P_0L_0 is 3. The second relator says that the order of $P_{12}L_0$ is 2 and the third one says that the order of $L_0^{-1}L_1$ is 2. The significance of second and third relators is that we can deduce by the action of $L_3(3)$, the order of product of any point and a line and also order of quotient of any two lines.

$$(\alpha vx)^2 = 1 \tag{1.6}$$

Since L_0 is defined to be αvx [17], this relation says that lines are involutions.

$$(xt^{-1})^{12} = 1 \tag{1.7}$$

This relation reduces to $(P_0L_0P_{12}L_1P_3L_{10})^4 = 1$. This relation reduces the affine Weyl group $\tilde{A}_5$ to $3^4{\cdot}S_6$ as can be checked by coset enumeration. The relations (1.1) – (1.7) gives presentation for $\mathbb{M} \wr 2 \times L_3(3)$.

$$(u^{-6}xu^6s)^6(sux^{-1}u^{-1})^6s^{-1} = 1 \tag{1.8}$$

This relation gets rid of $L_3(3)$ and now the relations (1.1), (1.2), (1.4), (1.5), (1.7), (1.8) define a presentation for $\mathbb{M} \times \mathbb{M}$.

$$((xv^{u^4}v^{u^{10}})^3u)^{13} = 1 \tag{1.9}$$

First note that $(xv^{u^4}v^{u^{10}})^3$ turns out to be $(P_0L_0P_3P_9)^3$. This element is the central involution of the Weyl group of the graph of type D_4 generated by P_0, L_0, P_3 and P_9. As in [19] we denote $(P_0L_0P_3P_9)^3$ by P_1^*. Also, the product of thirteen images of P_1^* under $L_3(3)$ is the projection of $\pi = [P_1^*, P_1]$ to one of the factors of $\mathbb{M} \times \mathbb{M}$. Hence if we adjoin the above relation to relations (1.1), (1.2), (1.4), (1.5), (1.7), and (1.8) we get a presentation for $\mathbb{M}$ [17].

In the following we give some remarks which are due to Norton.

Remarks:

1. The central involution of extra-special group 2^{1+26} which is denoted by π can be obtained from projective plane generators by taking a D_4 diagram consisting of 3 points and a line, considering its centre and commutating that with the point that extends the D_4 to an affine D_4 [18]. This element π is independent of which D_4 we start with so it commutes with $L_3(3)$ generated by elements s and t.

2. One can obtain an element λ by starting with a D_4 diagram that consists of 3 lines and a point, and again this centralizes $L_3(3)$. The element λ is the dual of π and can be obtained by swapping points and lines.
3. The elements π and λ can both be found in the $O_8(3) \wr 2$ that is the centralizer in the Bimonster of a 3^2 which permutes the projective plane. This 3^2 lies inside $L_3(3)$ and since both π and λ commute with $L_3(3)$, so they also commute with this 3^2. The matrices for the generators of $O_8(3) \wr 2$ are given in [16].
4. The product of π and λ has order 13. This can be shown as follows: $f^* = ab_1b_2b_3$, $\pi = (ff^*)^2$, $a^* = (fe_{123})^3$, and $\lambda = (aa^*)^2$, where the matrices for $a, a^*, b_1, b_2, b_3, f, f^*$ and e_{123} can be found in [16]. These matrices can be loaded into GAP and it can be seen that $(\lambda\pi)^{13} = 1$. Hence the (projection to either of the Monsters of the) $L_3(3)$ must lie in the monstralizer [1] of this element, which (ignoring the 13 element itself) also monstralizes an element of order 3.
5. The $L_3(3)$ generated by s and t centralizes 13:6. As there are two classes of $L_3(3)$ inside $2^{1+24} \cdot Co_1$, this remark is crucial in finding the right $L_3(3)$. More details can be found in 3.9.

[1]Monstralizer of the subgroup G of $\mathbb{M}$ is the centralizer $C_{\mathbb{M}}(G)$

Chapter 2

Coxeter generators for the Baby monster

2.1 Introduction

The Baby Monster $\mathbb{B}$ is a sporadic simple group having order

$$2^{41}.3^{13}.5^{6}.7^{2}.11.13.17.19.23.31.47.$$

Regarding the size of $\mathbb{B}$, it is second only to the Monster group in the "family" of 26 sporadic simple groups. It was discovered by B. Fischer and J. S. Leon. It was constructed as a permutation group on $13,571,955,000$ points by C. C. Sims [12].

From [4], we see that $\mathbb{B}$ has a $4371-$dimensional real orthogonal representation and reducing it modulo 2 we obtain a $4370-$dimensional self dual representation over $GF(2)$.

The Coxeter diagram Y_{433}, with an added relation, represents the group $2 \times 2^{\cdot}\mathbb{B}$ [4]. Ivanov has shown [10] that the presentation for $2 \times 2^{\cdot}\mathbb{B}$ given in [4] is correct, on the assumption that $\mathbb{B}$ exists. He needed this assumption in order to show that the presentation does not collapse to a group of order 2. R. A. Wilson has found matrices which satisfy the above presentation and do not collapse to a group of order 2 [27]. Hence the assumption that $\mathbb{B}$ exists can be removed from [10] and [27]. Thus a new existence proof of $\mathbb{B}$ has been obtained which is independent of Leon and Sims [12] and it is even not a consequence of Griess's construction of the Monster [7].

In this chapter, we will construct the Coxeter generators of the baby monster as words in the standard generators of $\mathbb{B}$ which are given in [26] as 4370×4370 matrices.

The purpose of this construction is to elaborate various techniques and methods which are used later.

2.2 The group Y_{533}

Since $Y_{433} \cong Y_{533}$ [19], we add an extra generator to Y_{433} and proceed to find the generators. The *ATLAS* [4] contains the presentation of $2 \times 2^{\cdot}\mathbb{B}$, in terms of Y_{533}. The generators (nodes) of this diagram satisfy the Coxeter relations.

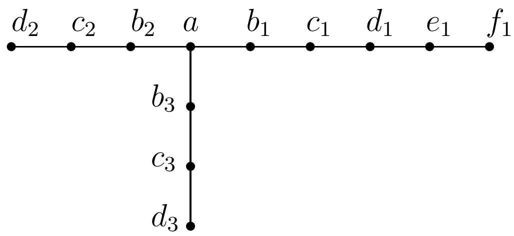


Figure 2.1: The Y_{533} diagram

All the generators are involutions and product of the adjacent nodes has order 3 otherwise their product has order 2. In addition, the following relation is also satisfied:

$$f_1 = (ab_1b_2b_3c_1c_2d_1)^9$$

Since the Weyl group of the Coxeter diagram of type A_9 is S_{10}, so we have

$$\langle d_2, c_2, b_2, a, b_1, c_1, d_1, e_1, f_1 \rangle \cong S_{10}.$$

Next we identify some other subgroups of the group Y_{533} which are relevant in our case.

$Y_{431} \cong 2 \times S_8(2)$, $Y_{432} \cong 2 \times Fi_{23}$, $Y_{333} \cong 2 \times 2^2.{}^2E_6(2)$ and $Y_{332} \cong 2 \times 2{}^{\cdot}Fi_{22}$ [9]. Further we have $Y_{432} \cong Y_{532}$ and $Y_{431} \cong Y_{531}$ [9]. We choose the following chain of subgroups

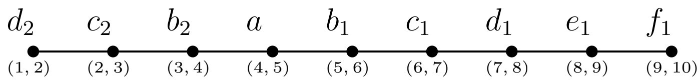
$$2 \times 2{}^{\cdot}\mathbb{B} > 2 \times Fi_{23} > 2 \times S_8(2) > S_{10}$$

corresponding to the chain of Y-diagrams

$$Y_{433} \supset Y_{432} \supset Y_{431} \supset Y_{430} = A_9.$$

2.2.1 Coxeter generators for S_{10}

Our aim in this section is to find generators d_2, c_2, b_2, a, b_1, c_1, d_1, e_1 and f_1 satisfying the Coxeter relations of Fig. 2.1. Let z_1 and z_2 be the standard generators of S_{10} so that $z_1 = (1,2)$ and $z_2 = (2,3,4,5,6,7,8,9,10)$. We can write the Coxeter generators of S_{10} as



It is easy to see that

$$d_2 = z_1, \qquad c_2 = z_1^{z_2 z_1}, \qquad b_2 = z_1^{z_2 z_1 z_2},$$
$$a = z_1^{z_2 z_1 z_2^2}, \qquad b_1 = z_1^{z_2 z_1 z_2^3}, \qquad c_1 = z_1^{z_2 z_1 z_2^4},$$
$$d_1 = z_1^{z_2 z_1 z_2^5}, \qquad e_1 = z_1^{z_2 z_1 z_2^6} \quad \text{and} \quad f_1 = z_1^{z_2 z_1 z_2^7}.$$

2.2.2 Coxeter generators for $S_8(2)$

The standard generators of $S_8(2)$ are w_1 and w_2 where w_1 is in class $2B$, w_2 is in class $5B$, w_1w_2 has order 17 and $w_1w_2^2$ has order 21 [26]. We used the permutation representation on 120 points for ease of calculations.

First we find the standard generators of S_{10} expressed as words in generators of $S_8(2)$. In this way we will lift the Coxeter generators for S_{10} inside $S_8(2)$ and then we have to find just one more generator i.e. b_3.

From [4], we note by restricting characters that $c \in 2C$ in S_{10} fuses to $2A$ in $S_8(2)$ and $d \in 9A/B$ in S_{10} fuses to $9B$ in $S_8(2)$. Furthermore, $cd \in 10C$ in S_{10} fuses to $10D$ in $S_8(2)$. So we adopt the following strategy.

First we find elements of classes $2A$ and $9B$ inside $S_8(2)$. An element of class $2A$ can be constructed by taking 7^{th} power of an element of order 14 [4]. There are two classes of elements of order 9. From the character table [4], we see that $9A$ elements have 3 fixed points and $9B$ elements have no fixed point in the 120 points permutation representation and with this information, using MAGMA citemagma, we find the desired element.

Let $x \in 2A$ and $y \in 9B$ inside $S_8(2)$. Now we conjugate x and y at random until we find g and h such that $\langle x^g, y^h \rangle \cong S_{10}$, where g and h are random elements of $S_8(2)$. To calculate the probability of finding such a pair of x^g and y^h, we first prove the following lemma.

Lemma 2.2.1 *Let H be a subgroup of a finite group G such that $C_G(H) = 1$. Then the probability of finding a pair (x, y) such that x, y are standard generators for H is at least* $\dfrac{|C_G(x)||C_G(y)|}{|G|}$.

Proof. Let x and y be representatives of conjugacy classes which generate

H. We also assume that x_0 and y_0 are standard generators for H. We are searching for a pair (x^g, y^h), $g, h \in G$ such that x^g and y^h generate H and are conjugate to x_0 and y_0 respectively. The space of all possibilities in this case is

$$\Omega = \{(x^g, y^h)\} = \{x^g | g \in G\} \times \{y^h | h \in G\}$$

and

$$|\Omega| = |\{x^g\}||\{y^g\}|.$$

Now we count the pairs (x^g, y^h) which are conjugate to a fixed pair (x_0, y_0). It is clear that since $C_G(H) = 1$, hence

$$(x_0, y_0)^r = (x_0, y_0) \text{ implies } r = 1.$$

We have $|\{(x^g, y^h)\}| = |\{(x_0^r, y_0^r)\}| = |\{(x_0, y_0)^r\}| = |G|, \ r \in G$.
Hence the probability of finding a pair (x^g, y^h) such that $\langle x^g, y^h \rangle \cong H$ is at least

$$\frac{|G|}{|x^g||y^h|} = \frac{|C_G(x)||C_G(y)|}{|G|} \qquad \square$$

Applying the above lemma to the case when $H \cong S_{10}$, $G \cong S_8(2)$ and noting that $C_G(H) = 1$, we see that the probability of finding a pair (x^g, y^h) such that $\langle x^g, y^h \rangle \cong S_{10}$ is approximately 1 in 10.

We define $x = (w_2^4 w_1 w_2^2 w_1 w_2^4 w_1 w_2^4)^7$, $y = w_2^3 w_1 w_2^3 w_1 w_2^4 w_1 w_2^2 w_1$, $g = w_2^{w_1 w_2^2 w_1}$, and $h = w_2^{w_1}$. Then $z_1 = x^g$ and $z_2 = y^h$ are standard generators of S_{10}.

To find b_3, we adopt the following strategy:

Step 1. Find $C_1 = C_{S_8(2)}(e_1)$

Step 2. Find $C_2 = C_{C_1}(c_1)$

Step 3. Find $C_3 = C_{C_2}(b_2)$

Step 4. Find $C_4 = C_{C_3}(d_2)$

We now give the details of the above mentioned strategy.

The generators for C_1 are found to be:

$t_1 = w_1w_2^3[e_1, w_1w_2^3]$, $t_2 = w_1w_2w_1[e_1, w_1w_2w_1]$, and $t_3 = w_1w_2^3w_1w_2^3[e_1, w_1w_2^3w_1w_2^3]$.

We also give some other elements of C_1:

$t_4 = t_1t_2$, $t_5 = t_1t_2t_1$, and $t_6 = t_5t_4$.

The generators for C_2 are found to be:

$u_1 = t_4[c_1, t_4]$, $u_2 = t_5[c_1, t_5]$, $u_3 = t_6[c_1, t_6]$, $u_4 = u_1u_2u_1$, $u_5 = u_4u_3u_2$, $u_6 = u_4^2u_3u_1$, $u_7 = u_4u_3u_1$ and $u_8 = u_3u_5$.

The generators for C_3 are:

$v_1 = [b_2, u_1]$, $v_2 = [b_2, u_3]$, $v_3 = [b_2, u_4]$, $v_4 = u_5[b_2, u_5]$, $v_5 = [b_2, u_6]$, $v_6 = u_7[b_2, u_7]$ and $v_7 = u_8[b_2, u_8]$.

We also give some elements of C_3 which will be useful in computation:

$v_8 = v_7v_3$, $v_9 = v_5v_3v_1$, $v_{10} = v_4v_3v_1v_8$, $v_{11} = v_5v_6v_2v_9$, $v_{12} = v_9v_{10}$, $v_{13} = v_{10}v_8v_{11}v_{12}$ and $v_{14} = v_5v_8v_{11}v_{12}v_{13}$.

The generators for C_4 are:

$y_1 = v_6[d_2, v_6]$, $y_2 = v_7[d_2, v_7]$, $y_3 = v_3[d_2, v_3]$, $y_4 = [d_2, v_8]$, $y_5 = v_9[d_2, v_9]$, $y_6 = v_{10}[d_2, v_{10}]$, $y_7 = v_{11}[d_2, v_{11}]$, $y_8 = v_{12}$, $y_9 = [d_2, v_{13}]$, and $y_{10} = [d_2, v_{14}]$.

It turns out that the order of C_4 is 2^{10}. We search inside C_4 and find a word for b_3 which is:

$$b_3 = y_3y_6y_4$$

2.2.3 Coxeter generators for Fi_{23}

Here we first take generators of $S_8(2)$ as words in standard generators of Fi_{23} from [26] as permutations on 31671 points and use MAGMA [1] for calculations. These generators of $S_8(2)$ are non-standard, so first we have to find the standard generators expressed as words in standard generators of Fi_{23}. From [26], we have $m = x$, and $n = xyxy(xyy)^3xy$, where x and y are standard generators of Fi_{23} and m, n are generators of $S_8(2)$. By random search, we find elements of order 12 and their 6^{th} power gives us elements of the class $2B$ in $S_8(2)$. There are two classes of elements of order 15 and both power up to class $5A$. Searching randomly, we find elements of order 15 and raising them to the 3^{rd} power gives us elements of class $5A$. Then we check whether these elements are standard generators of $S_8(2)$ by verifying the presentation

$$w_1^2 = w_2^5 = (w_1w_2)^{17} = [w_1, w_2]^3 = [w_1, w_2]^4 = [w_1, w_2w_1w_2]^3$$
$$= [w_1, w_2^2w_1w_2w_1w_2^2]^2 = [w_1, w_2^2w_1w_2^2]^3 = [w_1, (w_1w_2^2)^4]^2 = [w_1, w_2w_1w_2w_1w_2]^3$$
$$= (w_1w_2w_1w_2w_1w_2^2w_1w_2^{-1}w_1w_2^{-2})^4 = 1.$$

Eventually we found the standard generators. The words are:
$w_1 = (mn)^6$ and $w_2 = (mn^3mnmn^2mnmn^2mn^3)^3$. Now w_1 and w_2 are the standard generators of $S_8(2)$. Now the result of 2.2.2 gives the required generators $d_2, c_2, b_2, a, b_1, c_1, d_1, e_1, f_1$ and b_3.

Next we have to the generator c_3. We adopt the same strategy of working inside the involution centralisers as above.

Step 1. Construct $C_1 = C_{Fi_{23}}(e_1) \cong Fi_{22}.2$

Step 2. Construct $C_2 = C_{C_1}(c_1)$

Step 3. Construct $C_3 = C_{C_2}(a)$

Step 4. Construct $C_4 = C_{C_3}(c_2)$

Let r_0, r_1, r_2 and r_3 be random elements of Fi_{23} defined below:

$r_0 = x$, $r_1 = e_1x$, $r_2 = xyy$ and $r_3 = xyxy^2$.

Now we give generators for C_1:

$o_1 = [e_1, r_1]$, $o_2 = [e_1, r_0]$, $o_3 = r_2[e_1, r_2]$, $o_4 = o_3o_2$, $o_5 = r_3[e_1, r_3]$ and $o_6 = o_5o_2o_4$.

Next we give generators for C_2:

$x_1 = o_1[c_1, o_1]$, $x_2 = o_2[c_1, o_2]$ and $x_3 = o_3[c_1, o_3]$.

We also give some elements of this group:

$x_4 = o_4[c_1, o_4]$, $x_5 = x_4x_3x_2$, $x_6 = x_4^2x_3x_1$, $x_7 = x_4x_3x_1$ and $x_8 = x_3x_5$.

Now we generate centraliser of a inside C_2 by the following elements:

$p_1 = x_4[a, x_4]$, $p_2 = [a, x_3]$ and $p_3 = x_5[a, x_5]$.

Now we generate centraliser of c_2 inside C_3 by the following elements:

$q_1 = p_1[c_2, p_1]$, $q_2 = p_2[c_2, p_2]$, $q_3 = [c_2, p_3]$ and $q_4 = p_3p_1[c_2, p_3p_1]$.

Searching inside this subgroup of order $2^{15}.3^2$, we found c_3 satisfying the relations

$$c_3^2 = (c_3a)^2 = (c_3b_1)^2 = (c_3c_1)^2 = (c_3d_1)^2 = 1$$
$$(c_3e_1)^2 = (c_3f_1)^2 = (c_3b_2)^2 = (c_3c_2)^2 = (c_3d_2)^2 = 1$$
$$\text{and } (c_3b_3)^3 = 1.$$

Now we give a word for c_3 which is:

$$c_3 = (q_1^4 q_3 q_1 q_3 q_1^2)^3$$

2.2.4 Coxeter generators for $\mathbb{B}$

In the Baby Monster case, we first express all the Coxeter generators we have found, as words in the standard generators of $\mathbb{B}$ in matrix representation of degree 4370 over $\mathbb{F}_2$. The CPU time for multiplication of two matrices in this case is almost $2s$ on Intel Pentium 4, CPU 3.20GHz.

We first observe that the generator d_3 lies in the centraliser of f_1 (inside $\mathbb{B}$) which is $2 \times 2^2.{}^2E_6(2)$. This means that we have to search the required element within $2^{38}.3^9.5^2.7^2.11.13.17.19$ elements. As explained earlier in case of other generators, we make the following strategy of working within centralisers.

Strategy:

Step 1. Find $C_1 = C_{\mathbb{B}}(f_1) \cong 2 \times 2^2.{}^2E_6(2)$

Step 2. Find $C_2 = C_{C_1}(d_1) = C_{\mathbb{B}}(f_1, d_1) \cong 2^7(2 \times S_6(2))$

Step 3. Find $C_3 = C_{C_2}(b_1) = C_{\mathbb{B}}(f_1, d_1, b_1)$.

Step 4. Find $C_4 = C_{C_3}(b_2) = C_{\mathbb{B}}(f_1, d_1, b_1, b_2)$

Step 5. Find $C_5 = C_{C_4}(b_3) = C_{\mathbb{B}}(f_1, d_1, b_1, b_2, d_2, b_3)$

Step 7. Perform an exhaustive search inside C_5

We can expect C_6 to be reasonably small so that we can search inside for the element d_3. Now instead of working with 4370 degree representation, we use MEATAXE [15] to chop this representation to 78 degree representation of C_1. The element d_3 lies inside $C_{\mathbb{B}}(f_1) \cong 2 \times 2^2.{}^2E_6(2)$, so we give the generators

$d_1,\ c_1,\ b_1,\ b_2,\ c_2,\ d_2,\ d_1,\ a,\ b_3,\ c_3,$ and an element of the centralizer of f_1 as arguments to the chop command. This yields three representations, **1702a**, **572a** and **78a**. We choose to work in 78 dimensions and load the '**atlasrep**' package into GAP [6]. We also make use of 'ScanMeatAxeFile' to load our matrices into GAP.

Now we give details of the calculation performed. Let x_1 and x_2 be standard generators for $\mathbb{B}$. We first make the following generators for the centralizer of f_1.

$d_4 = [f_1, x_1x_2^2x_1]^2$, $z_1 = ab_1c_1$, $z_2 = d_1c_2$, $z_3 = d_4d_2$, $z_4 = z_3z_1$, $z_5 = b_3z_2$, $z_6 = z_5c_3$, $z_7 = b_2z_3$, $z_8 = z_4z_7$, $z_9 = z_1z_7z_6z_5$, $z_{10} = z_2z_9z_3$, $z_{11} = (z_9z_{10}z_2)^2$, $z_{12} = (z_{11}z_8z_6)^5$, $z_{13} = z_{10}z_9z_{12}z_5$, $z_{14} = (z_{11}z_8z_6)$, $z_{15} = z_{14}z_{13}$ and $z_{16} = z_{13}z_{12}$.

Following elements can be taken as generators for C_2.

$u_1 = z_4[d_1, z_4]$, $u_2 = z_5$, $u_3 = z_7[d_1, z_7]$, $u_4 = [d_1, z_8]^2$, $u_5 = u_4u_3$, $u_6 = u_5u_2u_3$, $u_7 = u_5u_3u_4u_6u_3u_1$, $u_8 = u_1^2u_5u_1u_3u_1^3u_4^2u_2u_5u_2u_1$, $u_9 = z_9[d_1, z_9]$, $u_{10} = z_{11}[d_1, z_{11}]$, $u_{11} = z_{13}[d_1, z_{13}]$, $u_{12} = z_{15}[d_1, z_{15}]$ and $u_{13} = [d_1, z_{16}]^2$.

Now we form generators for C_3.

$v_1 = [b_1, u_1]$, $v_2 = u_2$, $v_3 = [b_1, u_4u_3u_1]^2$, $v_4 = [b_1, u_5]$, $v_5 = [b_1, u_7]^2$, $v_6 = u_3$, $v_7 = u_8[b_1, u_8]$, $v_8 = v_2v_3$, $v_9 = v_2v_3v_5$, $v_{10} = u_9[b_1, u_9]$, $v_{11} = [b_1, u_{10}]^2$, $v12 = [b_1, u_{11}]^2$ and $v_{13} = u_{12}[b_1, u_{12}]$.

We make the generators for C_4 as follows

$s_1 = [b_2, v_1]$, $s_2 = v_2[b_2, v_2]$, $s_3 = v_7[b_2, v_7]$, $s_4 = v_3$, $s_5 = [b_2, v_4]$, $s_6 = v_5$, $s_7 = v_6$, $s_8 = v_8[b_2, v_8]$, $s_9 = v_9[b_2, v_9]$, $s_{10} = v_{10}[b_2, v_{10}]$, $s_{11} = [b_2, v_{11}]$ and $s_{12} = [b_2, v_{12}]$.

Finally we give the following generators for C_5.

$t_1 = [b_3, s_1]$, $t_2 = s_3[b_3, s_3]$, $t_3 = [b_3, s_4$, $t_4 = [b_3, s_5]$, $t_5 = [b_3, s_6]$, $t_6 = s_7$,
$t_7 = [b_3, s_8]$, $t_8 = s_{10}[b_3, s_{10}]$, $t_9 = [b_3, s_{11}]$, $_{t10} = [b_3, s_{12}]$,
$t_{11} = s_{10}s_8s_{11}s_{10}s_{12}[b_3, s_{10}s_8s_{11}s_{10}s_{12}]$, $t_{12} = s_8s_{10}[b_3, s_8s_{10}]$,
$t_{13} = [b_3, s_3s_{11}s_{10}s_{12}]$ and $t_{14} = [b_3, s_3s_8s_3s_{10}s_8s_9s_7]$.

We find that C_5 is a group of order $2^{13}.3^2$ and it is still too big for an exhaustive search. So, we also make use of the fact that all the generators of a Y_{533} diagram are conjugate. In particular, d_3 is a conjugate of c_3 . Now we give a word for d_3 which is

$$d_3 = c_3^{(t_{12}t_{13}t_{14}t_{12}^2t_{13}t_{14})^4}$$

As a final step, we verify all relations of Y_{533} diagram in 4370 dimensions. Hence we have found the generators for the baby monster which occur in the Y_{533} diagram.

The programs for all the computations performed can be found inside the dvd, attached with this thesis.

Chapter 3

Norton's Generators

Our aim in this chapter is to construct Norton's generators $s,\ t,\ u,\ v$ and x, see section 1.4.

The construction of generators for 2_{+}^{1+24} is discussed in 3.1 as they are much needed for later calculations.

Next we need to find an $L_3(3)$ inside $3{}^{\cdot}Suz$ that intersects with $2^{11}{}^{\cdot}M_{24}$ in $3^{1+2}2^2$. The words for subgroups of $3{}^{\cdot}Suz$ are given in 3.2 and words for $2^{11}{}^{\cdot}M_{24}$ are discussed in 3.3. The standard generators for $3^{1+2}D_8$ are given in 3.5.

Sections 3.6, 3.7 and 3.8 give details of conjugating a group of type $3^{1+2}2^2$ inside $3{}^{\cdot}Suz$ with one inside M_{24}. After this conjugation we can choose T for our generator x.

The later chapter deals with finding generators s and t. We first find generators of $L_3(3)$ satisfying the presentation given in [4], relabel the points and lines so that action of generators of $L_3(3)$ on points and lines of the projective plane of order 3 is exactly same as given by Norton [17] and finally conjugate the generators by the above mentioned conjugation. At this stage the generators s and t are approximately correct.

We still need to conjugate candidates for s and t by elements of the centralizer of $\langle s, t\rangle$ inside $2^{1+24}\cdot Co_1$. But to fit the pieces of this puzzle together we have to conjugate not only candidates for s and t but also the possible candidates for the generator v as well at the same time.

For the generator v, we see that v is an involution in 2^{1+24} centralised by $3^2{:}2\cdot S_4$. After calculating the centralizer, we work out possible candidates for v by conjugating with the above mentioned conjugation. We conjugate generators of $L_3(3)$ and possible candidates for v by the elements of the centralizer of $3^{1+2}{:}2^2$, and find s, t and v.

From now on we denote the group $2^{1+24}\cdot Co_1$ by G.

3.1 Standard generators for 2_+^{1+24}

Let $E = 2_+^{1+24}$, $\bar{E} = E/Z(E)$ and $Z = Z(E)$. As a vector space, $\bar{E}$ is isomorphic to $\mathbb{F}_2^{24}$. We want a particular basis for $\bar{E}$ that is, we want to find a set of 24 involutions $\mu_1, \mu_2, \ldots, \mu_{12},\ \nu_1, \nu_2, \ldots, \nu_{12}$ in E such that

$$\mu_i^2 = \nu_i^2 = 1, \quad \forall i$$

$$[\mu_i, \mu_j] = [\nu_i, \nu_j] = 1, \quad \forall i, j \text{ and}$$

$$[\mu_i, \nu_j] = \begin{cases} 1 & \text{if } i \neq j \\ -1 & \text{if } i = j \end{cases}$$

We also know that E being an extra-special group is isomorphic to the central product of dihedral groups of order 8, where the central involutions are identified at each step. Thus

$$2_+^{1+24} \cong D_8 \circ D_8 \circ \cdots \circ D_8.$$

Following is the strategy to find generators of E, which is based on the idea that the commutators in the group E correspond to inner products in the orthogonal space $\bar{E}$.

To begin with, we look for elements of orders 44, 46, 48, 52, 66 and 92 inside G. Since there are no elements of these orders in the quotient Co_1, these elements power up to involutions in E. Moreover, we also obtain some elements of order 4 in E by powering up elements of orders 88 and 92.

Let $\langle \mathsf{a}, \mathsf{b} \rangle$ be the first D_8, c be the new element and suppose first that a, b and c all have order 2. We now process these elements of E one at a time to build up the generating set.

- Calculate the orders of ac and bc. If these both have order 2, then c already commutes with a and b.
- If ac has order 4, and bc has order 2, replace c by bc.
- If bc has order 4 and ac has order 2, replace c by ac.
- If both ac and bc have order 4, replace c by abc.
- If c has order 4, swap 2 and 4 in this strategy.

Then we apply the above mentioned strategy and obtain the following words for the generators of E, where ϕ_1 to ϕ_{157} are as defined below. First we give words for ψ's which are used to define ϕ's. Here ψ_1 and ψ_2 are generators for G.

$\psi_3 = \psi_1\psi_2$, $\psi_4 = \psi_3\psi_2$, $\psi_5 = \psi_4\psi_1$,

$\psi_6 = \psi_2\psi_3$, $\psi_7 = \psi_6\psi_5$, $\psi_8 = \psi_7\psi_6$,

$\psi_9 = \psi_8\psi_7$, $\psi_{10} = \psi_9\psi_5$, $\psi_{11} = \psi_{10}\psi_9$,

$\psi_{12} = \psi_{11}\psi_8$, $\psi_{13} = \psi_{12}\psi_{11}$,

$\psi_{14} = \psi_{13}\psi_{12}$, $\psi_{15} = \psi_{14}\psi_{13}$,

$\psi_{16} = \psi_{15}\psi_{14}$, $\psi_{17} = \psi_{16}\psi_{12}$, $\psi_{18} = \psi_{17}\psi_{15}$,

$\psi_{19} = \psi_{18}\psi_{14}$, $\psi_{20} = \psi_{19}\psi_{13}$,

$\psi_{21} = \psi_{20}\psi_{17}$, $\psi_{22} = \psi_{21}\psi_{13}$,

$\psi_{23} = [\psi_{20}, \psi_{15}]$,

$\psi_{24} = \psi_{19}^{\psi_7}$, $\psi_{25} = \psi_{24}\psi_{22}$,

$\psi_{26} = \psi_{25}\psi_8$, $\psi_{27} = \psi_{23}\psi_{17}$,

$\psi_{28} = \psi_{26}\psi_{27}$, $\psi_{29} = \psi_{12}\psi_{24}$,

$\psi_{30} = \psi_{29}\psi_{23}$, $\psi_{31} = \psi_{11}\psi_{30}$,

$\psi_{32} = \psi_{31}\psi_9$, $\psi_{33} = \psi_{32}\psi_{31}$,

$\psi_{34} = \psi_{33}\psi_{25}$, $\psi_{35} = \psi_{33}\psi_{34}$,

$\psi_{36} = \psi_{33}\psi_{30}$, $\psi_{37} = \psi_{36}\psi_{11}$,

$\psi_{38} = \psi_{11}\psi_{37}$, $\psi_{39} = \psi_{36}\psi_{21}$,

$\psi_{40} = \psi_{39}\psi_{18}$, $\psi_{41} = \psi_{11}\psi_{40}$,

$\psi_{42} = [\psi_{33}, \psi_{12}]$, $\psi_{43} = [\psi_{42}, \psi_{22}]$,

$\psi_{44} = \psi_{43}\psi_{42}$, $\psi_{45} = [\psi_{44}, \psi_{13}]$,

$\psi_{46} = \psi_{45}^{\psi_{42}}$, $\psi_{47} = \psi_{45}^{\psi_{33}}$,

$\psi_{48} = \psi_{45}^{\psi_{22}}$, $\psi_{49} = \psi_{45}^{\psi_{43}}$,

$\psi_{50} = \psi_{44}^{\psi_{42}}$, $\psi_{51} = \psi_{45}^{\psi_{50}}$,

$\psi_{52} = \psi_{45}^{\psi_{44}}$, $\psi_{53} = \psi_{45}^{\psi_{36}}$,

$\psi_{54} = \psi_{45}^{\psi_{26}}$, $\psi_{55} = \psi_{45}^{\psi_{28}}$,

$\psi_{56} = \psi_{45}^{\psi_{40}}$, $\psi_{57} = \psi_{45}^{\psi_{15}}$,

$\psi_{58} = \psi_{45}^{\psi_{27}}$, $\psi_{59} = \psi_{45}^{\psi_{30}}$,

$\psi_{60} = \psi_{45}^{\psi_{40}}$, $\psi_{61} = [\psi_{33}, \psi_{14}]$,

$\psi_{62} = [\psi_{33}, \psi_{14}]$, $\psi_{63} = [\psi_{25}, \psi_{38}]$,

$\psi_{64} = [\psi_{35}, \psi_{59}]$, $\psi_{65} = [\psi_{27}, \psi_{59}]$,

$\psi_{66} = [\psi_{42}, \psi_{17}]$, $\psi_{67} = \psi_{66}\psi_{32}$,

$\psi_{68} = \psi_{67}\psi_{45}$, $\psi_{69} = \psi_{68}\psi_{23}$,

$\psi_{70} = \psi_{69}\psi_{66}$, $\psi_{71} = \psi_{70}\psi_{11}$,

$\psi_{72} = \psi_{71}\psi_{9}$, $\psi_{73} = \psi_{72}^{\psi_{34}}$,

$\psi_{74} = [\psi_{73}, \psi_{40}]$, $\psi_{75} = \psi_{74}\psi_{54}$,

$\psi_{76} = \psi_{75}^{\psi_{22}}$, $\psi_{77} = \psi_{76}\psi_{25}$,

$\psi_{79} = \psi_{74}^{\psi_{64}}$, $\psi_{80} = [\psi_{79}, \psi_{30}]$,

$\psi_{81} = \psi_{80}\psi_{63}$, $\psi_{82} = \psi_{69}\psi_{59}$,

$\psi_{83} = \psi_{82}^{\psi_{31}}$, $\psi_{84} = \psi_{83}\psi_{71}$,

$\psi_{85} = \psi_{84}\psi_{60}$, $\psi_{86} = \psi_{85}\psi_{42}$,

$\psi_{87} = \psi_{86}\psi_{57}$, $\psi_{88} = \psi_{87}\psi_{47}$,

$\psi_{89} = \psi_{88}\psi_{15}$, $\psi_{90} = \psi_{82}\psi_{38}$,

$\psi_{91} = \psi_{90}\psi_{67}$, $\psi_{92} = \psi_{91}\psi_{12}$,

$\psi_{93} = \psi_{19}^{\psi_{92}}$, $\psi_{94} = [\psi_{85}, \psi_{62}]$,

$\psi_{95} = \psi_{93}\psi_{94}$, $\psi_{96} = \psi_{64}^{\psi_{95}}$,

$\psi_{97} = \psi_{96}\psi_{20}$, $\psi_{98} = \psi_{97}\psi_{38}$,

$\psi_{99} = [\psi_{98}, \psi_{35}]$, $\psi_{100} = \psi_{99}\psi_{13}$,

$\psi_{101} = [\psi_{96}, \psi_{95}]$, $\psi_{102} = [\psi_{98}, \psi_{86}]$,

$\psi_{103} = \psi_{102}\psi_{56}$, $\psi_{104} = \psi_{103}\psi_{81}$,

$\psi_{105} = [\psi_{103}, \psi_{104}]$, $\psi_{106} = [\psi_{102}, \psi_{86}]$,

$\psi_{107} = [\psi_{72}, \psi_{94}]$, $\psi_{108} = \psi_{105}^{\psi_{107}}$,

$\psi_{109} = \psi_{107}\psi_{32}$, $\psi_{110} = \psi_{109}\psi_{66}$,

$\psi_{111} = [\psi_{100}, \psi_{15}]$, $\psi_{112} = [\psi_{104}, \psi_{99}]$,

$\psi_{113} = \psi_{91}^{\psi_{49}}$, $\psi_{114} = [\psi_{112}, \psi_{113}]$,

$\psi_{115} = [\psi_{113}, \psi_{114}]$, $\psi_{116} = \psi_{115}\psi_{107}$,

$\psi_{117} = \psi_{116}\psi_{74}$, $\psi_{118} = \psi_{117}\psi_{47}$,

$\psi_{120} = \psi_{119}\psi_{112}$, $\psi_{121} = [\psi_{120}, \psi_{94}]$,

$\psi_{122} = \psi_{121}\psi_{54}$, $\psi_{123} = \psi_{122}\psi_{106}$,

$\psi_{124} = [\psi_{123}, \psi_{93}]$, $\psi_{125} = \psi_{101}^{\psi_{124}}$,

$\psi_{126} = \psi_{95}^{\psi_{110}}$, $\psi_{127} = \psi_{101}^{\psi_{54}}$,

$\psi_{128} = \psi_{95}^{\psi_{67}}$, $\psi_{129} = \psi_{105}^{\psi_{119}}$,

$\psi_{130} = \psi_{105}^{\psi_{106}}$, $\psi_{131} = \psi_{105}^{\psi_{99}}$,

$\psi_{132} = \psi_{105}^{\psi_{79}}$, $\psi_{133} = \psi_{108}^{\psi_{115}}$,

$\psi_{134} = \psi_{95}^{\psi_{122}}$, $\psi_{135} = [\psi_{134}, \psi_{16}]$,

$\psi_{136} = \psi_{135}\psi_{34}$, $\psi_{137} = [\psi_{136}, \psi_{67}]$,

$\psi_{138} = \psi_{137}\psi_{88}$, $\psi_{139} = \psi_{136}\psi_{29}$,

$\psi_{140} = \psi_{135}\psi_{77}$, $\psi_{141} = [\psi_{138}, \psi_{43}]$,

$\psi_{142} = [\psi_{139}, \psi_{34}]$, $\psi_{143} = [\psi_{140}, \psi_{78}]$,

$\psi_{144} = \psi_{125}^{\psi_{141}}$, $\psi_{145} = \psi_{105}^{\psi_{142}}$,

$\psi_{146} = \psi_{126}^{\psi_{143}}$, $\psi_{147} = \psi_{105}^{\psi_{144}}$,

$\psi_{148} = \psi_{52}^{\psi_{137}}$, $\psi_{149} = \psi_{53}^{\psi_{140}}$,

$\psi_{150} = \psi_{110}^{\psi_{124}}$, $\psi_{151} = \psi_{60}^{\psi_{115}}$,

$\psi_{152} = \psi_{110}^{\psi_{139}}$, $\psi_{153} = \psi_{59}^{\psi_{124}}$,

$\psi_{154} = \psi_{56}^{\psi_{136}}$, $\psi_{155} = \psi_{55}^{\psi_{145}}$,

$\psi_{156} = \psi_{54}^{\psi_{134}}$, $\psi_{157} = \psi_{125}^{\psi_{107}}$,

$\psi_{158} = \psi_{126}^{\psi_{127}}$, $\psi_{159} = \psi_{126}^{\psi_{111}}$.

$\phi_1 = \psi_{45}^{23}\psi_{46}^{23}$,

$\phi_4 = \psi_{45})^{23}(\psi_{48}^{23}$,

$\phi_6 = \psi_{47}^{23}\psi_{48}^{23}$,

$\phi_7 = \psi_{45}^{23}\psi_{49}^{23}$,

$\phi_8 = \psi_{46}^{23}\psi_{49}^{23}$,

$\phi_{11} = \psi_{45}^{23}\psi_{51}^{23}$,

$\phi_{16} = \phi_6\phi_7$,

$\phi_{17} = \phi_6\phi_{11}$,

$\phi_{31} = \psi_{45}^{23}\psi_{52}^{23}$,

$\phi_{36} = \psi_{51}^{23}\psi_{52}^{23}$,

$\phi_{38} = \phi_{31}\phi_{36}$,

$\phi_{42} = \psi_{53}^{23}$,

$\phi_{43} = \phi_{42}\phi_{38}$,

$\phi_{44} = \phi_8\phi_4\phi_{17}$,

$\phi_{45} = \psi_{54}^{23}$,

$\phi_{49} = \phi_1\phi_8\phi_4\phi_{44}\phi_{43}\phi_{45}$,

$\phi_{52} = (\psi_{45}^{23})^{\psi_{42}}$,

$\phi_{53} = (\psi_{47}^{23})^{\psi_{42}}$,

$\phi_{54} = \phi_4\phi_{52}$,

$\phi_{55} = \phi_8\phi_{16}\phi_{44}\phi_{53}$,

$\phi_{57} = \phi_{55}\psi_{53}^{23}$,

$\phi_{58} = \phi_1\phi_4\phi_{16}\phi_{43}\phi_{44}\phi_{57}$,

$\phi_{59} = (\psi_{48}^{23})^{\psi_{42}}$,

$\phi_{64} = \phi_8\phi_4\phi_{43}\phi_{58}\phi_{59}$,

$\phi_{65} = \phi_{49}\phi_{54}$,

$\phi_{69} = \psi_{56}^{23}$,

$\phi_{70} = \phi_1\phi_8\phi_4\phi_{43}\phi_{44}\phi_{65}\phi_{69}$,

$\phi_{71} = \psi_{57}^{23}$,

$\phi_{72} = \phi_8\phi_4\phi_{16}\phi_{44}\phi_{49}\phi_{58}\phi_1\phi_8\phi_{64}\phi_{65}\phi_{71}$,

$\phi_{73} = \phi_1\phi_8\phi_{64}$,

$\phi_{75} = \phi_{73}\phi_{65}$,

$\phi_{76} = \phi_{73}\phi_{72}$,

$\phi_{79} = \phi_{72}\phi_{70}$

$\phi_{88} = \psi_{59}^{23}\psi_{60}^{23}$,

$\phi_{90} = \phi_1\phi_8\phi_4\phi_{44}\phi_{43}\phi_{49}\phi_{76}\phi_{88}$,

$\phi_{93} = \psi_{64}^{22}(\psi_{64}^{22})^{\psi_{50}}$,

$\phi_{96} = \phi_{58}\phi_{43}\phi_{75}\phi_{93}$,

$\phi_{98} = \psi_{105}^{23}(\psi_{105}^{23})^{\psi_{101}}$,

$\phi_{100} = (\psi_{105}^{23})^{\psi_{101}}(\psi_{105}^{23})^{\psi_{95}}$,

$\phi_{101} = \phi_8\phi_{16}\phi_4\phi_{43}\phi_{44}\phi_{49}\phi_{58}\phi_{75}\phi_{98}$,

$\phi_{103} = \phi_8\phi_4\phi_{43}\phi_{49}\phi_{101}\phi_{100}$,

$\phi_{106} = (\psi_{129}^{23})^{\psi_{125}}(\psi_{130}^{23})^{\psi_{126}}$,

$\phi_{107} = (\psi_{129}^{23})^{\psi_{125}}(\psi_{131}^{23})^{\psi_{127}}$,

$\phi_{108} = (\psi_{129}^{23})^{\psi_{125}}(\psi_{132}^{23})^{\psi_{128}}$,

$\phi_{112} = \phi_8\phi_4\phi_{16}\phi_{49}\phi_{58}\phi_{75}\phi_{76}\phi_{106}$,

$\phi_{113} = \phi_{75}\phi_{79}\phi_{90}$,

$\phi_{114} = \phi_{16}\phi_{44}\phi_{90}\phi_{112}\phi_{107}$,

$\phi_{115} = \phi_4\phi_{49}\phi_{75}\phi_{108}$,

$\phi_{117} = (\psi_{147}^{23})^{\psi_{144}}(\psi_{145}^{23})^{\psi_{146}}$,

$\phi_{118} = (\psi_{147}^{23})^{\psi_{144}}(\psi_{148}^{23})^{\psi_{128}}$,

$\phi_{120} = \phi_1\phi_{16}\phi_{43}\phi_{75}\phi_{76}\phi_{113}\phi_{114}\phi_{117}$,

$\phi_{121} = \phi_8\phi_{16}\phi_{58}\phi_{90}\phi_{112}\phi_{118}$,

$\phi_{126} = \psi_{151}^{23}(\psi_{151}^{23})^{\psi_{152}}$,

$\phi_{127} = (\psi_{149}^{23})^{\psi_{150}}(\psi_{151}^{23})^{\psi_{152}}$,

$\phi_{134} = (\psi_{153})^{23}(\psi_{154})^{23}$,

$\phi_{153} = (\psi_{155}^{23})^{\psi_{158}}(\psi_{156})^{23})^{\psi_{159}}$,

$\phi_{154} = \phi_8\phi_4\phi_{76}\phi_{90}\phi_{101}\phi_{113}\phi_{114}\phi_{120}\phi_{153}$,

$\phi_{155} = \phi_{101}\phi_{113}\phi_{114}\phi_{120}\phi_{96}$

$\phi_{157} = \phi_{16}\phi_{49}\phi_{76}\phi_{90}\phi_{101}\phi_{113}\phi_{115}\phi_{121}\phi_{129}\phi_{154}\phi_{155}\phi_{134}$.

$\mu_1 = \phi_1$, $\mu_2 = \phi_4$, $\mu_3 = \phi_8\phi_4\phi_{17}$,

$\mu_4 = \phi_1\phi_8\phi_4\phi_{44}\phi_{43}\phi_{45}$,

$\mu_5 = \phi_{73}\phi_{72}$,

$\mu_6 = \phi_8\phi_{16}\phi_4\phi_{43}\phi_{44}\phi_{49}\phi_{58}\phi_{75}\phi_{98}$,

$\mu_7 = \phi_8\phi_4\phi_{16}\phi_{49}\phi_{58}\phi_{75}\phi_{76}\phi_{106}$,

$\mu_8 = \phi_4\phi_{49}\phi_{75}\phi_{108}$,

$\mu_9 = \phi_8\phi_{16}\phi_{58}\phi_{90}\phi_{112}\phi_{118}$,

$\mu_{10} = \phi_4\phi_{44}\phi_{76}\phi_{90}\phi_{101}\phi_{112}\phi_{115}\phi_{120}\phi_{127}$,

$\mu_{11} = \phi_8\phi_4\phi_{76}\phi_{90}\phi_{101}\phi_{113}\phi_{114}\phi_{120}\phi_{153}$,

$\mu_{12} = \phi_1\phi_4\phi_{58}\phi_{76}\phi_{90}\phi_{115}\phi_{126}$,

$\nu_1 = \phi_8$,

$\nu_2 = \phi_{16}$, $\nu_3 = \phi_8\phi_4\phi_{17}\phi_{42}\phi_{38}$,

$\nu_4 = \phi_1\phi_8\phi_4\phi_{44}\phi_{43}\phi_{45}\phi_1\phi_4\phi_{16}\phi_{43}\phi_{44}\phi_{57}$,

$\nu_5 = \phi_{73}\phi_{72}\phi_{73}\phi_{65}$,

$\nu_6 = \phi_8\phi_{16}\phi_4\phi_{43}\phi_{44}\phi_{49}\phi_{58}\phi_{75}\phi_{98}\phi_1\phi_8\phi_4\phi_{44}\phi_{43}\phi_{49}\phi_{76}\phi_{88}$,

$\nu_7 = \phi_8\phi_4\phi_{16}\phi_{49}\phi_{58}\phi_{75}\phi_{76}\phi_{106}\phi_{75}\phi_{79}\phi_{90}$,

$\nu_8 = \phi_4\phi_{49}\phi_{75}\phi_{108}\phi_{16}\phi_{44}\phi_{90}\phi_{112}\phi_{107}$,

$\nu_9 = \phi_8\phi_{16}\phi_{58}\phi_{90}\phi_{112}\phi_{118}\phi_1\phi_{16}\phi_{43}\phi_{75}\phi_{76}\phi_{113}\phi_{114}\phi_{117}$,

$\nu_{10} = \phi_4\phi_{44}\phi_{76}\phi_{90}\phi_{101}\phi_{112}\phi_{115}\phi_{120}\phi_{127}\phi_{112}\phi_{115}\phi_{121}\phi_{103}$,

$\nu_{11} = \phi_{101}\phi_{113}\phi_{114}\phi_{120}\phi_{96}$ and

$\nu_{12} = \phi_{43}\phi_{44}\phi_{157}$.

The dvd attached with this thesis contains a folder "monster/gens" inside where the above words can be found.

3.2 Words for some subgroups of $3{\cdot}Suz$

Let $\mathtt{x}$ and $\mathtt{y}$ be the standard generators for Co_1 [30]. Consider the chain of subgroups:

$Co_1 > 3{\cdot}Suz{:}2 > 3{\cdot}Suz > L_3(3){:}2 > L_3(3)$.

The words for maximal subgroups of Co_1 in above chain are given in [26]. For the sake of completeness, we give the words here as well.

Let $\mathfrak{s}_1 = (\mathtt{xy})^{-2}\mathtt{yxy}$ and $\mathfrak{s}_2 = (\mathtt{xy}^2)^{-2}(\mathtt{xyxyxy}^2\mathtt{xyxyxy}^2\mathtt{xy})^8(\mathtt{xy}^2)^2$.

Then $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are standard generators for $3{\cdot}Suz{:}2$.

Now let $s_1 = (\mathfrak{s}_1\mathfrak{s}_2^2)^{-2}(\mathfrak{s}_1\mathfrak{s}_2)^{14}(\mathfrak{s}_1\mathfrak{s}_2^2)^2$ and $s_2 = \mathfrak{s}_2$.

Then s_1, s_2 are standard generators for $3{\cdot}Suz$.

Now let $\mathfrak{l}_1 = (s_1s_2^2)^{-2}s_1(s_1s_2^2)^2$ and $\mathfrak{l}_2 = (s_1s_2)^{-8}(((s_1s_2^2)^2s_1s_2^2)^2s_1s_2^2)^2(s_1s_2)^8$.

Then $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are generators for $L_3(3) : 2$.

We take $l_1 = (\mathfrak{l}_1\mathfrak{l}_2\mathfrak{l}_1)^2$ and $l_2 = (\mathfrak{l}_2\mathfrak{l}_1)^3\mathfrak{l}_2^2\mathfrak{l}_1\mathfrak{l}_2^3\mathfrak{l}_1\mathfrak{l}_2\mathfrak{l}_1\mathfrak{l}_2^3$.

Then l_1 and l_2 are standard generators for $L_3(3)$. We found the words for l_1 and l_2 by searching inside $L_3(3){:}2$ and we used MAGMA for this computation.

Lifting these subgroups to G, now we have the following chain of subgroups: $2^{1+24}{}^{\cdot}Co_1 > 2^{1+24}3{}^{\cdot}Suz{:}2 > 2^{1+24}6{}^{\cdot}Suz > 2 \times (L_3(3){:}2{}^{\cdot}2) > 2 \times L_3(3) > L_3(3)$.

We can apply the Parker's magic formula 1.1.1 to get rid of the 2-group beneath $6{}^{\cdot}Suz{:}2$. Instead we see that after using the same words for $L_3(3)$ as in the case of quotient Co_1 and multiplying the generators by the central involution we can still get $L_3(3)$ without using the above mentioned formula. The word for the central involution was found to be $((\mathfrak{l}_1\mathfrak{l}_2^2)^4\mathfrak{l}_2)^2$.

We also give words for the groups $M_{12}{:}2$ and M_{12} expressed as words in generators of $3{}^{\cdot}Suz$.

Generators for $M_{12}{:}2$ are $k = (s_1s_2)^{-2}s_2s_1s_2$ and $l = (s_1s_2^2)^{-6}s_2(s_1s_2^2)^6$ and generators for M_{12} are $m_1 = ((kl)^2kl^2kl)^3$ and $m_2 = (kl^2)^{-3}(kl)^4(kl^2)^3$ [26]. When we lift these groups to G, we have a double cover of M_{12} instead of M_{12}.

3.3 Words for $2^{11}{}^{\cdot}M_{24}$

We want to identify the generator $x = P_0L_0$ with the triality element T. Now T centralizes $2^{11}{}^{\cdot}M_{24}$ in G in the 2-local construction of $\mathbb{M}$. In this section, we follow [8] and use same words for generators of $2^{11}{}^{\cdot}M_{24}$ (with corrections) which have been described there.

We start by describing the restriction of the module for G to $2^{2+11+11+11}{}^{\cdot}M_{24}$, which is,

276a $\oplus$ **22** $\oplus$ **276b** $\oplus$ **276c** $\oplus$ **48576** $\oplus$ **49152a** $\oplus$ **49152b** $\oplus$ **49152c**.

Here, **22** is a representation of the quotient group M_{24} , **276a**, **276b** and **276c** are representations of three isomorphic quotients $2^{11}{:}M_{24}$ and **48576** is a representation of $2^{11+11+11}\cdot M_{24}$. The three isomorphic quotients $2^{1+11+11+11}\cdot M_{24}$ are represented by **49152a**, **49152b** and **49152c**. For further details we refer to [8].

Since the generators $\mathfrak{b}$ and $\mathfrak{c}$ of G are preimages for standard generators of Co_1, so we can look at words for maximal subgroups for Co_1. We start with the pair of words giving generators for $2^{11}{:}M_{24}$ [26]. The words are:

$e_1 = ((\mathfrak{b}\mathfrak{c}\mathfrak{b}\mathfrak{c}^2)^3)^{(\mathfrak{b}\mathfrak{c})^6}$, $f_1 = ((((\mathfrak{b}\mathfrak{c})^2\mathfrak{b}\mathfrak{c}^2(\mathfrak{b}\mathfrak{c})^2\mathfrak{b}\mathfrak{c}^2\mathfrak{b}\mathfrak{c})^4)^{(\mathfrak{b}\mathfrak{c}^2)^5}$.

Here, e_1 and f_1 generate the group $2^{2+11+11+11}M_{24}.2$.

Again, following [8] we give words

$a_3 = ((e_1f_1)(e_1f_1e_1f_1^2)^2)^5$, and

$b_3 = (e_1f_1((e_1f_1)^2(e_1f_1e_1f_1^2)^2e_1f_1^2)e_1f_1^2e_1f_1e_1f_1^2e_1f_1^2)^8$, generating the group M_{24} in **276a**.

However, in the representation **552** $\cong$ **276b** $\oplus$ **276c**, it was found that a_3 and b_3 generate $2^{11}{:}M_{24}{:}2$ modulo the subgroup 2^{2+11}.

Next we move into the subgroup of index 2 by taking the generators

$a_2 = (e_1f_1)^{23}a_3$, and $b_2 = b_3$.

Now we obtain a complement to the subgroup of order 2^{11} by using the following words

$a_1 = (a_2b_2((a_2b_2)^2(a_2b_2a_2b_2^2)^2a_2b_2^2)a_2b_2^2a_2b_2a_2b_2^2)^6$ and

$b_1 = ((a_2b_2)^2(a_2b_2a_2b_2^2)^2a_2b_2^2a_2b_2^2)^2$. In representation of degree 49152, these generators generate $2 \times 2^{11}\cdot M_{24}$. Now we can multiply these generators by the central involution to go down to $2^{11}\cdot M_{24}$. The word for central involution is found to be $z = (a_1b_1)^{23}$. Hence generators for $2^{11}\cdot M_{24}$ are

$a = a_1z$ and $b = b_1$.

3.4 Strategy for finding the generator x

We know that $x = P_0L_0$ is conjugate to the triality element T inside $\mathbb{M}$, and the centralizer of x in $L_3(3)$ is $3^{1+2}{:}2^2$. The centralizer of T in G is $2^{11}{\cdot}M_{24}$ [8]. Thus we would like to conjugate our generators for $L_3(3)$ so that it intersects $2^{11}{\cdot}M_{24}$ in $3^{1+2}{:}2^2$ and we can choose $x = T$. But we need to show that $2^{11}{\cdot}M_{24}$ and $3{\cdot}Suz{:}2$ has a non-trivial intersection which we discuss below.

3.4.1 Intersections of $2^{11}{\cdot}M_{24}$ and $3{\cdot}Suz{:}2$ in Co_1

We let $\mathcal{A} = 2^{11}{:}M_{24}$ and $\mathcal{B} = 3{\cdot}Suz{:}2$. We note that $\mathcal{A}$ and $\mathcal{B}$ both contain a Sylow 11- subgroup of Co_1 so one intersection of a conjugated $\mathcal{A}$ and a conjugate of $\mathcal{B}$ has order divisible by 11.

We know that $[Co_1{:}\mathcal{A} \cap \mathcal{B}] \leq [Co_1{:}\mathcal{A}][Co_1{:}\mathcal{B}]$ and it follows that $|\mathcal{A} \cap \mathcal{B}| \geq 3 \times 10^5$. Now we consider various candidates for the intersection by looking at maximal subgroups of $\mathcal{B}$ containing an 11 element.

- $3 \times U_5(2)$. This group has order $2^{10}.3^6.5.11$. To lie in the intersection, this group can have a Sylow 3-subgroup of order at most 3^3. Hence we can discard this possibility.
- $3^5{:}M_{11}$. This group has order $2^4.3^7.5.11$. This group too can have a Sylow 3-subgroup of order at most 3^3 if it is in the intersection of $\mathcal{A}$ and $\mathcal{B}$. We can eliminate this possibility as well.
- $2 \times M_{12}{:}2$. This group is a candidate to lie in the intersection of $\mathcal{A}$ and $\mathcal{B}$.
- $2^{4+6}{:}3{\cdot}S_6$. This group also lies in the intersection of suitable conjugates

of $\mathcal{A}$ and $\mathcal{B}$, as we shall see below.

Now we calculate permutation characters of Co_1 on the cosets of $\mathcal{A}$ and $\mathcal{B}$ and see that they have inner product equal to 2. This implies that there are two orbits of $\mathcal{A}$ on the cosets of $\mathcal{B}$. If $2 \times M_{12}{:}2$ is the intersection then the other orbit length doesn't divide the order of Co_1. This implies that $M_{12}{:}2$ is one intersection.
Similarly one can show that $2^{4+6}{:}3{^\cdot}S_6$ is the other intersection.

Now we discuss a strategy to achieve the above mentioned conjugation.

3.4.2 Strategy

Step 1. Find $M_{12} < 3{^\cdot}Suz$ and conjugate $L_3(3)$ inside $3{^\cdot}Suz$ so that it intersects this M_{12} in $3^{1+2}{:}2^2$

Step 2. Conjugate this M_{12} in Co_1 into M_{24}.

Step 3. Lift to G.

We give some details of the above mentioned strategy.

1. First we conjugate the group $3^{1+2}{:}2^2$ inside $L_3(3)$ with a group of similar type sitting inside $3^{1+2}{:}D_8$.
2. Then, inside $3{^\cdot}Suz$, we conjugate two copies of groups of type $3^{1+2}{:}D_8$ one sitting inside $L_3(3){:}2$ and the other inside $M_{12}{:}2$.
3. After this, we conjugate a copy of $3^{1+2}{:}2^2$ inside $3{^\cdot}Suz$ to one inside M_{24}.
4. We lift all subgroups to G.
5. We apply Parker's formula to get rid of 2-groups (decorations) where necessary.

Before proceeding further, we would like to find standard generators for the group of type $3^{1+2}{:}D_8$.

3.5 Standard generators for $3^{1+2}{:}D_8$

The group $3^{1+2}{:}D_8$ has the following presentation:

$\langle \mathsf{x}, \mathsf{y}, \mathsf{z}, \mathsf{r}, \mathsf{s} \mid \mathsf{x}^3 = \mathsf{y}^3 = \mathsf{z}^3 = [\mathsf{x}, \mathsf{z}] = [\mathsf{y}, \mathsf{z}] = 1, \mathsf{z} = [\mathsf{x}, \mathsf{y}],$

$\mathsf{r}^2 = \mathsf{s}^2 = (\mathsf{rs})^4 = 1, \mathsf{x}^\mathsf{r} = \mathsf{x}, \mathsf{y}^\mathsf{r} = \mathsf{y}^{-1}, \mathsf{x}^\mathsf{s} = \mathsf{y}\rangle$ in which $\langle \mathsf{x}, \mathsf{y}, \mathsf{z}\rangle$ is the normal 3^{1+2} and $\langle \mathsf{r}, \mathsf{s}\rangle$ is a complementary D_8.

We give a definition for a generating triple.

Definition 3.5.1 *A (p, q, r) generating triple is a triple of generators $x, y, z = xy$ of orders p,q and r respectively.*

A $(2, 6, 12)$ generating triple for $3^{1+2}{:}D_8$, without loss of generality can be taken as $(\mathsf{r},\ \mathsf{x}^\mathsf{a}\mathsf{y}^\mathsf{b}\mathsf{z}^\mathsf{c}\mathsf{s},\ \mathsf{r}\mathsf{x}^\mathsf{a}\mathsf{y}^\mathsf{b}\mathsf{z}^\mathsf{c}\mathsf{s})$, $\mathsf{a}, \mathsf{b}, \mathsf{c} \in \{-1, 0, 1\}$. It is easy to see that

$\mathsf{s}^\mathsf{x} = \mathsf{x}^{-1}\mathsf{s}\mathsf{x} = \mathsf{s}\mathsf{y}^{-1}\mathsf{x} = \mathsf{x}^{-1}\mathsf{y}\mathsf{s}$,

$\mathsf{y}^\mathsf{x} = \mathsf{x}^{-1}\mathsf{y}\mathsf{x} = \mathsf{y}\mathsf{y}^{-1}\mathsf{x}^{-1}\mathsf{y}\mathsf{x} = \mathsf{y}\mathsf{z}^{-1}$ and

$(\mathsf{x}^\mathsf{a}\mathsf{y}^\mathsf{b}\mathsf{z}^\mathsf{c})^\mathsf{x} = \mathsf{x}^\mathsf{a}\mathsf{y}^\mathsf{b}\mathsf{z}^{\mathsf{c}-\mathsf{b}}$.

The centralizer of r is $\langle \mathsf{r}, (\mathsf{rs})^2, \mathsf{x}\rangle$. We conjugate $\mathsf{x}^\mathsf{a}\mathsf{y}^\mathsf{b}\mathsf{z}^\mathsf{c}$ by elements of the centralizer of r.

$(\mathsf{rs})^2 : \mathsf{x}^\mathsf{a}\mathsf{y}^\mathsf{b}\mathsf{z}^\mathsf{c} \mapsto \mathsf{x}^{-\mathsf{a}}\mathsf{y}^{-\mathsf{b}}\mathsf{z}^\mathsf{c}$.

$\mathsf{x} : \mathsf{x}^\mathsf{a}\mathsf{y}^\mathsf{b}\mathsf{z}^\mathsf{c}\mathsf{s} \mapsto (\mathsf{x}^\mathsf{a}\mathsf{y}^\mathsf{b}\mathsf{z}^{\mathsf{c}-\mathsf{b}})\mathsf{x}^{-1}\mathsf{y}\mathsf{s} = (\mathsf{x}^{\mathsf{a}-1}\mathsf{y}^{\mathsf{b}+1}\mathsf{z}^\mathsf{c})\mathsf{s}$.

By conjugating by x successively, we may assume that $\mathsf{b} = 0$. Then $\mathsf{a} \neq 0$ for if $\mathsf{a} = 0$ the elements r, $\mathsf{z}^{\mathsf{c}\mathsf{s}}$ do not generate the whole group. So without loss of generality we have $\mathsf{a} = 1$. This implies that we have $\mathsf{x}\mathsf{z}^\mathsf{c}\mathsf{s}$ as our second generator.

Now we have three cases to consider, owing to three values of c:
$(\mathsf{r},\ \mathsf{xs},\ \mathsf{rxs})$, $(\mathsf{r},\ \mathsf{xzs},\ \mathsf{rxzs})$ and $(\mathsf{r},\ \mathsf{xz}^{-1}\mathsf{s},\ \mathsf{rxz}^{-1}\mathsf{s})$.

A fingerprint is a list of element orders for various words in the generators. The exact nature of the fingerprint depends on the orders of the generators. We check orders of random words generated by the above three triples and observe that there are three distinct fingerprints.

We let $\mathsf{x}_1 = \mathsf{r}$ and $\mathsf{x}_2 = \mathsf{rxs}$. Then $(\mathsf{x}_1\mathsf{x}_2)^3\mathsf{x}_1$ has order 12. Now we define $\mathsf{y}_1 = \mathsf{r}$ and $\mathsf{y}_2 = \mathsf{rxzs}$. Then we see that $(\mathsf{y}_1\mathsf{y}_2)^3\mathsf{y}_1$ has order 4. Now let $\mathsf{z}_1 = \mathsf{r}$ and $\mathsf{z}_2 = \mathsf{rxz}^{-1}\mathsf{s}$. Then $(\mathsf{z}_1\mathsf{z}_2)^3\mathsf{z}_1$ has order 6. Hence the above three cases are different.

3.6 Conjugating a copy of $3^{1+2}{:}2^2$ inside $L_3(3)$ to one inside $3^{1+2}{:}D_8$

Let l_1 and l_2 be generators for $L_3(3)$ and $\mathfrak{l}_1$ and $\mathfrak{l}_2$ be generators for $L_3(3){:}2$ as defined in section 3.2. Let y_1 and y_2 be generators for the group $3^2{:}2S_4$ inside $L_3(3)$. Then $y_1 = l_1$ and $y_2 = l_2^{l_1 l_2}$.

Using MAGMA, the generators for the subgroup $3^{1+2}{:}2^2$ are found to be $\mathtt{r}_1 = y_1$ and $\mathtt{s}_1 = y_1 y_2^{y_1 y_2 y_1} y_1 y_2^2 y_1^{y_2 y_1 y_1} y_2$.

Now consider the group $3^{1+2}{:}D_8$ inside $L_3(3){:}2$. The generators for $3^{1+2}{:}D_8$ are found to be $w_1 = \mathfrak{l}_1$ and $w_2 = (\mathfrak{l}_1\mathfrak{l}_2^3\mathfrak{l}_1\mathfrak{l}_2^2\mathfrak{l}_1\mathfrak{l}_2)^{\mathfrak{l}_1\mathfrak{l}_2\mathfrak{l}_1(\mathfrak{l}_1\mathfrak{l}_2^3\mathfrak{l}_1\mathfrak{l}_2^2\mathfrak{l}_1\mathfrak{l}_2)^4}$. Then $\mathtt{r}_1^{'} = w_1 w_2^{w_1 w_2 w_1}(w_1^{w_2 w_1 w_2})^2 w_2^4)$ and $\mathtt{s}_1^{'} = w_1^{w_2 w_1} w_2^{w_1 w_2^2 w_1}(w_1^{w_2 w_1 w_2^2})^2 w_2^4$ are generators for $3^{1+2}{:}2^2$. We find the element $\mathtt{conj}_1 = \mathfrak{l}_1\mathfrak{l}_2(\mathfrak{l}_1^{\mathfrak{l}_2^2\mathfrak{l}_1\mathfrak{l}_2}\mathfrak{l}_2^{\mathfrak{l}_1\mathfrak{l}_2\mathfrak{l}_1}\mathfrak{l}_1^2)^3$ which conjugates $\langle \mathtt{r}_1, \mathtt{s}_1 \rangle$ to $\langle \mathtt{r}_1^{'}, \mathtt{s}_1^{'} \rangle$. We also give generators for the group 3^{1+2} inside $L_3(3)$ which are

$\mathtt{t}_1 = \mathtt{s}_1^2$, $\mathtt{t}_2 = (\mathtt{r}_1\mathtt{s}_1\mathtt{r}_1\mathtt{s}_1\mathtt{r}_1\mathtt{s}_1^2)^2$ and $\mathtt{t}_3 = (\mathtt{r}_1\mathtt{s}_1\mathtt{r}_1\mathtt{s}_1^2)$.

Next we conjugate two copies of $3^{1+2}{:}D_8$ one inside $L_3(3){:}2$ the other one inside M_{12} and both these copies are inside $3^{\cdot}Suz$.

3.7 Conjugating two copies of $3^{1+2}{:}D_8$

Consider the chain of subgroups

$$Co_1 > 3^{\cdot}Suz > L_3(3) : 2 > 3^{1+2}{:}D_8 = A$$

and

$$Co_1 > 3^{\cdot}Suz > M_{12}{:}2 > 3^{1+2}{:}D_8 = B.$$

The generators for the group A are :

$q_1 = k$ and $q_2 = kl^2kl(kl^2)^2(kl)^3$ [26].

Let w_1, w_2 be generators for B as in 3.6 satisfying the following relation:

$$(w_2)^{12} = (w_1)^2 = (w_1w_2)^6 = (q_2)^{12} = (q_1)^2 = (q_1q_2)^6 = 1$$

It follows from 3.5 that we have to raise w_2 to the 5^{th} (or 7^{th}) power, so that (w_1, w_2^5) and (q_1, q_2) are standard generators for two copies of $3^{1+2}{:}D_8$. We set $w_2 := w_2^5$. The strategy to conjugate the groups A and B inside the Co_1 is discussed below. We start by proving the following lemma:

Lemma 3.7.1 *If a, b are conjugate involutions and if ab has order $2n+1$,* *then* $b^{(ab)^n} = a$.

Proof. We see that

$$\begin{aligned} b^{(ab)^n} &= (b^{-1}a^{-1})^n b(ab)^n \\ &= (ba)^n b(ab)^n \\ &= b(ab)^{2n} \\ &= a^{-1}(ab)^{2n+1} \\ &= a. \end{aligned}$$

3.7.1 Conjugate q_2^6 to w_2^6.

We use the above lemma to conjugate q_2^6 to w_2^6. It is well known that two involutions always generate a dihedral group D_{2n} and there is only one conjugacy class of involutions if n is odd. We take conjugates of q_2^6 by random elements and we do the same thing for w_2^6. Then we look for odd order products between these two sets of involutions. Hence q_2^6 is conjugated to w_2^6.
Let g_1 be the element which conjugates q_2^6 to w_2^6.
Define $a_1 = w_2^6$, and $b_1 = q_2^6$. We also take $z_1 = s_1$ and $z_2 = s_2$ as defined in section 3.2. Now we give a word for the above conjugation which is:

$$g_1 = (b_1 b_1^{(z_1 z_2)^2 z_2 z_1 z_2^2 (z_1 z_2)^2 z_2})^2 (b_1^{z_1 z_2 (z_1 z_2^2)^2 (z_1 z_2)^2 z_2} a_1)^3$$

We let $q_2' = q_2^{g_1}$ and it follows that $(q_2')^6 = ((q_2)^{g_1})^6 = (q_2^6)^{g_1} = w_2^6$.

3.7.2 Find the centralizer of the involution w_2^6.

We used Bray's method [2] to find the centralizer of the involution $a_1 = w_2^6$, in $3^{\cdot}Suz$ which turns out to be $3 \times 2_+^{1+6\cdot}U_4(2)$ [4], and then work inside the involution centralizer from this point.

We define $z_3 = z_1z_2$, $z_5 = z_3z_4^2$, $z_6 = z_3z_4$, $z_7 = z_6z_4z_6$, $z_8 = z_7z_6z_4$ and $z_9 = z_5z_6$, $d_1 = z_1z_2(z_1z_3)^2$, $d_2 = (z_1z_2(z_1z_3)^2z_1z_2z_3)^3$.

$d_3 = ((z_1z_2(z_1z_3)^2(z_1z_2)^2z_2)^2z_1z_2(z_1z_3)^2)^{13}$,

$d_4 = (z_1z_2(z_1z_3)^2(z_1z_2)^2z_2(z_1z_2(z_1z_3)^2((z_1z_2)^2z_2)^2z_1z_2^2)^3(z_3)^5(z_1z_2(z_3)^2)^2)^{11}$,

$d_5 = z_9(z_9z_6z_4)^3z_4^5z_5z_9^2z_5(z_9(z_9z_6z_4)^3)^{z_9(z_5(z_6^2z_4)^3z_4^5z_5}z_9(z_5(z_6^2z_4)^3z_4^5z_5$,

$d_6 = q_1^{z_7(z_5z_6^2z_4)^3}$ and $d_7 = ((w_1^{(z_7z_8^3)^{z_7z_8^3z_4^5z_6z_4}})^{z_7z_8^2})^{d_5}$.

The generators for the group $3 \times 2_+^{1+6\cdot}U_4(2)$ turn out to be:

$c_1 = d_1[a_1, d_1]^2$, $c_2 = d_2[a_1, d_2]^2$, $c_3 = [a_1, d_7]^2$, $c_4 = [a_1, d_3]^3$,

$c_5 = d_5[a_1, d_5]^2$, $c_6 = d_4[a_1, d_4]^2$ and $c_7 = [a_1, d_6]^3$.

3.7.3 Conjugate $(q_2')^4$ to w_2^4 (and hence $(q_2')^2$ to w_2^2)

Let S denote the group $3^{\cdot}Suz$. Since $(q_2')^4$ and (w_2^4) both lie inside the $C_S(w_2^6) = 3 \times 2_+^{1+6\cdot}U_4(2)$, we seek an element $g_2 \in C_S(w_2^6)$ such that

$$g_2 : (q_2')^4 \mapsto w_2^4$$

The strategy is as follows:

1. Let $\langle w_2^4, q \rangle$ be some group X, where q is some conjugate of q_2^4.
2. Check if the order of X is less than the order of the involution cen-

tralizer.

3. If not, go to step 1.
4. Repeat this process until the order of X is small enough so that we can use brute force.

Let o_1, o_2, o_3 and o_4 be elements of the involution centralizer where $o_1 = c_1c_2$, $o_2 = c_3c_4$, $o_3 = c_4c_2$, $o_4 = o_3c_1$ and $o_5 = o_4c_1$. We define $b_2 = b_1'$, $w_3 = w_2^4$ and $b_3 = (q_2')^4$.

We find by using MAGMA that the group $\langle w_2^4, b_3^{o_5}\rangle$ has order 12 and searching inside this small group, we give a word for the desired conjugation which is:

$$g_2 = (o_1o_2^2o_3^4o_1^2)^3$$

We let $q_2'' = (q_2')^{g_2}$. We have $(q_2'')^4 = w_2^4$, $(q_2'')^6 = (q_2')^6 = w_2^6$ and therefore $(q_2'')^2 = w_2^2$.

3.7.4 Find $C(w_2^2)$ inside $3 \times 2_+^{1+6}{\cdot}U_4(2)$

From [4], we see that the order of the centralizer of w_2^2 inside $3{\cdot}Suz$ is $2^4.3^2$. To construct this centralizer, we look at various subgroups generated by w_2^2 and a random involution inside $3 \times 2_+^{1+6}{\cdot}U_4(2)$. Using MAGMA, we found following elements of the centralizer of w_2^2:

$c_8 = c_5^6$, $c_9 = c_6^6$, $h_1 = (c_8w_2c_8w_2w_2^2)^5$, $h_2 = (c_8w_2c_8w_2^7w_2^2)^5$, $h_3 = (c_8w_2^6c_8w_2^5)$, $h_4 = (c_8w_2^3c_8w_2^3c_8w_2)^3$, $h_5 = (c_9w_2c_9w_2c_9w_2^2)^4$ and $h_6 = (c_9w_2c_9w_2w_2^5)^2$.

We see that $C_{3\times 2_+^{1+6}\cdot U_4(2)}(w_2^2) = \langle w_2, h_6\rangle$. The words for c_5 and c_6 can be found in 3.7.2.

3.7.5 Conjugate q_2'' to w_2.

Here we search inside the group $\langle w_2, h_6 \rangle$ to find an element g_3 inside the centralizer of w_2^2 such that

$$g_3 : q_2'' \mapsto w_2$$

Now we give a word for the conjugation which is:

$$g_3 = (h_6 w_2 h_6 w_2 h_6^2 w_2^7 h_6)^2$$

Hence the desired conjugation from q_2 to w_2 is $g_4 = g_1 g_2 g_3$.

3.7.6 Conjugate q_1''' to w_1

We generate the centralizer of w_2 and the generators for the centralizer found are w_2 and $t_1 = (c_1 c_2 c_1 c_2 c_1 c_2^3 c_1^3 c_2^7 c_1^3)^5$. Inside this centralizer we found an element g_5 such that $g_5 : q_1''' \mapsto w_1$. Let $q_1''' = (q_1)^{g_4}$. We now give the word which is:

$$g_5 = (t_1 w_2 t_1 w_2 t_1)^8.$$

Hence an element which conjugates A to B is $g = g_4 g_5$.

So, we have $A^g = B$.

3.8 Conjugating a copy of $3^{1+2}{:}2^2$ inside $3^{1+2}{:}D_8$ to one inside M_{12}

Let m_1 and m_2 be generators for M_{12} as in 3.2. The words for generators for $3^2{:}2S_4$ are:

$y_1' = m_2^{-1}m_1m_2$ and $y_2' = (m_1m_2^2)^{-3}m_2(m_1m_2^2)^3$.

The generators for the group $3^{1+2}{:}2^2$ inside $\langle y_1', y_2'\rangle$ are $r_1'' = y_1'$ and $s_1'' = (y_2')^2((y_2')^{y_1'(y_2')^2y_1'y_2'})^2y_1'$.

Now let q_1 and q_2 are generators for $3^{1+2}{:}D_8$ (which is inside $M_{12}{:}2$) as in 3.7. The generators for $3^{1+2}{:}2^2$ inside $\langle q_1, q_2\rangle$ are

$\mathtt{r}_1''' = q_1q_2^{q_1q_2q_1}(q_1^{q_2q_1q_2})^2q_2^4)$ and $\mathtt{s}_1''' = q_1^{q_2q_1}q_2^{q_1q_2^2q_1}$. We find that

$\mathtt{conj}_2 = (kl^2(k^{l^2kl})^2(l^{klk})^3k^2)^{-10}$ conjugates $\langle \mathtt{r}_1''', \mathtt{s}_1'''\rangle$ to $\langle \mathtt{r}_1'', \mathtt{s}_1''\rangle$.

3.8.1 Conjugating a copy of $3^{1+2}{:}2^2$ inside $3{\cdot}Suz$ to one inside M_{24}

We have two copies of M_{12} one inside the $3{\cdot}Suz$ and the other one inside the M_{24} and we want to conjugate two copies of $3^{1+2}{:}2^2$ within these M_{12}'s. For M_{12} sitting inside the $3{\cdot}Suz$, the generators are m_1 and m_2 as in 3.2 and for the M_{12} inside the $M_{24} = \langle a, b\rangle$ (see section 3.3), we let

$z_1 = a$ and $z_2 = ((ab)^{12}(ababb)^5)^{-1}b(ab)^{12}(ababb)^5$. Then

$m_1' = (z_1z_2z_1z_2z_1z_2^2z_1z_2)^3$ and $m_2' = (z_1z_2^2)^{-3}(z_1z_2)^4(z_1z_2^2)^3$ [26] are the generators for M_{12} inside $M_{24} = \langle a, b\rangle$.

The generators for the group $3^2{:}2S_4$ inside $\langle m_1', m_2'\rangle$ are $y_1'' = m_1'$ and

$y_2'' = (m_1'(m_2')^2)^{-3}m_2'(m_1'(m_2')^2)^3$.

The generators for the group $3^{1+2}{:}2^2$ inside $\langle m_1^{'}, m_2{'}\rangle$ are

$\mathtt{r}_1^{''''} = y_1^{''}$ and $\mathtt{s}_1^{''''} = (y_2^{''})^2((y_2^{''})^{(}y_1^{''}(y_2^{''})^2y_1^{''}y_2^{''}))^2y_1^{''})$.

Our aim is now to conjugate $\langle \mathtt{r}_1^{''''}, \mathtt{s}_1^{''''}\rangle$ and $\langle \mathtt{r}_1^{'''}, \mathtt{s}_1^{'''}\rangle$. We first form the involution centralizer inside $Co_1 = \langle \mathtt{x}, \mathtt{y}\rangle$ and work within it. Using Bray's method [2], the following elements can be taken as generators for the involution centralizer, which is a group of the form $2^{1+8}{\cdot}O_8^+(2)$. We take $z_1 = \mathtt{x}$, $z_2 = \mathtt{y}$, $z_3 = z_1z_2$, $z_4 = z_3z_2$, $z_5 = z_4^5$, $z_6 = z_3z_5$, $z_7 = z_3z_4$, $z_8 = z_7z_4$, $z_9 = z_8z_7$, $z_{10} = z_9z_8$ and $z_{11} = z_9z_{10}$.

Now the generators for $2^{1+8}{\cdot}O_8^+(2)$ are:

$\varrho_1 = [\mathtt{r}_1^{''''}, z_5]^3$, $\varrho_2 = z_6[\mathtt{r}_1^{''''}, z_6]^2$, $\varrho_3 = [\mathtt{r}_1^{''''}, z_7]^3$, $\varrho_4 = [\mathtt{r}_1^{''''}, z_8]^3$ and $\varrho_5 = z_{11}[\mathtt{r}_1^{''''}, z_{11}]^2$.

It turns out that ϱ_2 and ϱ_5 generate the full $2^{1+8}{\cdot}O_8^+(2)$.

We now use the dihedral trick to conjugate $(\mathtt{s}_1^{''''})^3$ and $(\mathtt{s}_1^{'''})^3$ inside the involution centralizer. We now give words for some elements of the involution centralizer which are used in below.

$\varepsilon_1 = \varrho_2$, $\varepsilon_2 = \varrho_5$, $\varepsilon_3 = \varepsilon_1\varepsilon_2$, $\varepsilon_4 = \varepsilon_3\varepsilon_2$, $\varepsilon_5 = \varepsilon_4\varepsilon_3$, $\varepsilon_6 = \varepsilon_5\varepsilon_4$, $\varepsilon_7 = \varepsilon_6\varepsilon_5$, $\varepsilon_8 = \varepsilon_5\varepsilon_7$ and $\varepsilon_9 = \varepsilon_8\varepsilon_7$. Let $i_1 = (\mathtt{s}_1^{''''})^3$ and $i_2 = (\mathtt{s}_1^{'''})^3$ then we see that $\delta_1 = (i_1 i_1^{\varepsilon_9})^3(i_1^{\varepsilon_9}i_2)^3$ conjugates i_1 to i_2 so that we have $i_1^{\delta_1} = i_2$.

We now conjugate $((\mathtt{s}_1^{''''})^{\delta_1})^2$ to $(\mathtt{s}_1^{'''})^2$ inside the Centraliser of $((\mathtt{s}_1^{''''})^{\delta_1})^3$. Let $i_3 = ((\mathtt{s}_1^{''''})^{\delta_1})^2$ and $i_4 = (\mathtt{s}_1^{'''})^2$. We form centralizer of $i_c = ((\mathtt{s}_1^{''''})^{\delta_1})^3$ using Bray's method [2]

$\rho_1 = z_5[i_c, z_5]^2$, $\rho_2 = [i_c, z_6]^3$ and $\rho_3 = z_7[i_c, z_7]^2$. We find that ρ_1 and ρ_2 generate the full centralizer.

Now searching inside a group of order $2^{21}.3^5.5^2.7$ even with the help of MAGMA can be quite cumbersome. So we searched inside various subgroups of the of involution centralizer and then further into their subgroups to find the desired element.

Following are some random elements which were used to form various random subgroups of $2^{1+8}{}^{\cdot}O_8^+(2)$.

$\rho_4 = \rho_3\rho_2$, $\rho_5 = \rho_4\rho_3$, $\rho_6 = \rho_5\rho_4$ $\rho_8 = \rho_5\rho_7$, $\rho_9 = \rho_8\rho_7$, $\rho_{10} = \rho_9\rho_7$, $\rho_{11} = \rho_{10}^{\rho_7}$, $\rho_{12} = \rho_4\rho_{10}$, $\rho_{13} = \rho_{12}\rho_9$, $\rho_{14} = \rho_1^{(\rho_1\rho_3\rho_1\rho_3\rho_1\rho_3^3)^2}$, $\rho_{15} = \rho_3^{(\rho_1\rho_3\rho_1\rho_3\rho_1\rho_3^3)^4}$, $\rho_{16} = \rho_{14}^{(\rho_{14}\rho_{15}\rho_{14}\rho_{15}\rho_{14}^4\rho_{15}^4)^5}$, $\rho_{17} = \rho_{15}^{(\rho_{14}\rho_{15}\rho_{14}\rho_{15}\rho_{14}^4\rho_{15}^4)^{10}}$, $\rho_{18} = (\rho_{16}\rho_{17}\rho_{16}\rho_{17}\rho_{16}\rho_{17}^6)^2$, $\rho_{19} = \rho_{17}\rho_{16}\rho_{17}\rho_{16}\rho_{17}\rho_{16}^6$, $\rho_{20} = (\rho_{18}\rho_{19}\rho_{18}\rho_{19}\rho_{18}^3\rho_{19})$, $\rho_{21} = \rho_{19}\rho_{18}\rho_{19}\rho_{18}\rho_{19}^3\rho_{18}$, $\rho_{22} = \rho_{20}\rho_{21}\rho_{20}\rho_{21}\rho_{20}^2\rho_{21}^3$, $\rho_{23} = \rho_{21}\rho_{20}\rho_{21}\rho_{20}\rho_{20}^2\rho_{21}^3$ and $\rho_{24} = \rho_{13}\rho_{11}$.

The element which conjugates i_3 to i_4 is given by:

$\delta_2 = (\rho_{22}^{\rho_{23}\rho_{22}^6\rho_{23}^2}\rho_{23}^{\rho_{22}^4\rho_{23}\rho_{22}^8}\rho_{22}\rho_{23}\rho_{22}^9\rho_{23}^4)^5$.

We now let $\delta_3 = \delta_1\delta_2$.

Now we give words for generators for the group 3^{1+2}.

We define $z_1 = (\mathtt{s}_1^{''''})^2$, $z_2 = (\mathtt{r}_1^{''''}\mathtt{s}_1^{''''}\mathtt{r}_1^{''''}\mathtt{s}_1^{''''}\mathtt{r}_1^{''''}(\mathtt{s}_1^{''''})^2)^2$ and $z_3 = (\mathtt{r}_1^{''''}\mathtt{s}_1^{''''}\mathtt{r}_1^{''''}(\mathtt{s}_1^{''''})^3)$ then generators for 3^{1+2} are:

$\mathtt{u}_1 = (z_2z_1z_3^2z_2)$, $\mathtt{u}_2 = z_2^2$ and $\mathtt{u}_3 = (z_2z_3z_2^2)^2$.

3.8.2 Lifting to G

We now lift these subgroups to G. We have the following chain of subgroups

$$G > 2^{1+24}6{}^{\cdot}Suz{:}2 > 2^{1+24}L_3(3){:}2 > 2^{1+24}L_3(3) > 2^{1+24}3^{1+2}{:}2^2 = C$$

and

$$G > 2^{1+24}2^{11}{}^{\cdot}M_{24} > 2^{1+24}{}^{\cdot}M_{12}{:}2 > 2^{1+24}{}^{\cdot}M_{12} > 2^{1+24}3^{1+2}.2^2 = D.$$

Many of the words are taken from [8] or [26] (with some corrections). In fact, these words did not always give groups containing the whole 2^{1+24}, and we were able to simplify some of the later calculations by using the fact that in the last two terms of the first chain of subgroups above we actually had only $2 \times L_3(3)$ and $2 \times 3^{1+2}.2^2$. After lifting, we observe that there is a 2-group beneath the group D since the order of $\mathtt{u}_1$ is 6 now.First we raise $\mathtt{u}_1$, $\mathtt{u}_2$ and $\mathtt{u}_3$ to their 4^{th} powers so that they stay same modulo the 2-group. Next we apply Parker's formula to the central 3 first so that we have $\mathtt{u}_2^{\delta_3 \mathtt{t}_2 \mathtt{u}_2^{\delta_3}} = \mathtt{t}_2$. Let's put $\alpha = \delta_3 \mathtt{t}_2 \mathtt{u}_2^{\delta_3}$, so that α conjugates $\mathtt{u}_2$ to $\mathtt{t}_2$. Now it turns out that α also conjugates $\mathtt{u}_1$ to $\mathtt{t}_1$. Now applying the formula again on the third generator, we have, $\mathtt{u}_3^{\alpha \mathtt{t}_3 \mathtt{u}_3^{\alpha}} = \mathtt{t}_3$. Let $\gamma = \alpha \mathtt{t}_3 \mathtt{u}_3^{\alpha}$. Now α already works to get $\mathtt{u}_2$ and $\mathtt{u}_3$ correct and now we just need a new version of γ which does not destroy $\mathtt{u}_2$ when it corrects $\mathtt{u}_1$. So we try $\beta = \mathtt{u}_3\gamma$ as it has the same effect on $\mathtt{u}_3$ as γ does. Hence we have $C^{\beta} = D$ and thus β is the desired conjugation. After this conjugation we can use T for our generator x.

3.9 Candidates for generators s and t

Before we begin, it is useful to decide which of the two conjugacy classes of $L_3(3)$ inside $2^{1+24}Co_1$ is the correct one to use. We show that the $L_3(3)$ described above has Monstralizer 13:6, and therefore lies in a subgroup $6Suz$ of the Monster. First note that the elements of order 13 in $\langle s, t\rangle$ must be in class $13B$ because they lie in $2^{1+24}.Co_1$. Therefore the monstralizer of this $L_3(3)$ lies in $13^{1+2}.2A_4$. Two involutions in this centralizer are $\pi = [P_0, P_0^*]$ which by [19] is independent of which suffix we choose, and its dual

$\lambda = [L_0, L_0^*]$. Clearly they can't be the same, as π is conjugated by L_0 into $\pi.P_0^*P_1^*P_3^*P_9^*$ but λ commutes with all the lines. So as they both lie in $13^{1+2}.A_4$ their product must have order 13. Therefore we have an element of order 13 centralizing $\langle s, t\rangle$. So it's the type whose monstralizer is 13:6.

We know that s is an element of $L_3(3)$ which acts as

$$(L_1, L_2, L_5, L_9, L_8, L_7)(L_3, L_{12}, L_4)(L_{10}, L_{11})$$

on lines and t is an element of $L_3(3)$ which has action

$$(L_0, L_1, L_9)(L_2, L_{12}, L_{11}, L_4, L_5, L_6)(L_7, L_8)$$

on lines **[17]**. We need to have exactly the same numbering of points and lines as in [17] and also that $\langle \mathtt{r}_1, \mathtt{s}_1\rangle$ fixes P_0 and L_0. It should also centralize x, so that $x = P_0L_0$ or its inverse.

We label the points so that l_1 is the permutation

$$(P_{10}, P_{12})(P_1, P_3)(P_2, P_6)(P_8, P_{11})$$

and l_2 is the permutation

$$(P_0, P_{12}, P_3)(P_1, P_2, P_4)(P_6, P_9, P_8)(P_5, P_7, P_{11})$$

Words for l_1 and l_2 are given in section 3.2.

Now we can set $s^* = (l_2l_1l_2l_2((l_1l_2)^2l_1l_2^2)^4)^3(((l_1l_2)^2l_2)^2(l_1l_2l_1l_2l_1l_2^2)^4)^2$and $t^* = l_2$. We find that s^* and t^* are the same permutations that are given in

1.4. Moreover, with respect to this numbering, $\mathtt{r}_1 = l_1 = (P_{10}, P_{12})(P_1, P_3)(P_2, P_6)(P_8, P_{11})$ and $\mathtt{s}_1 = (P_1, P_9)(P_2, P_{11}, P_5)(P_4, P_8, P_6, P_7, P_{12}, P_{10})$ so both fix P_0 and $L_0 = \{P_0,\ P_1,\ P_3,\ P_9\}$. Next we conjugate s^* and t^* by β so that we have $s^{**} = (s^*)^{\beta}$ and $t^{**} = (t^*)^{\beta}$. Now we would like to test our elements s^{**} and t^{**} which are candidates for s and t. Since we haven't calculated v yet, so we pick one of Norton's relations which does not involve v. We see that the relation $(xt^{-1})^{12} = 1$ is not satisfied. This means that we have to conjugate s^{**} and t^{**} by an element which centralizes x. We will calculate generators s and t in 3.11.

3.9.1 Cases

Let's define $H_1 = L_3(3)$ and $H_2 = 2^{11}{\cdot}M_{24}$. Now H_1 and H_2 intersect in $H_3 = 3^{1+2}.2^2$ and all of them are sitting inside a copy of G. Using Norton's Monstralizer list [16] we get:
$C_{\mathbb{M}}(G) = 2$, $C_{\mathbb{M}}(H_1) = 13 : 6$ (a Frobenius group of order 78), $C_{\mathbb{M}}(H_2) = A_4$ and $C_{\mathbb{M}}(H_3) = G_2(3)$.
Hence the centralizers in G are as follows:
$C_G(G) = 2$, $C_G(H_1) = 6$, $C_G(H_2) = 2^2$ and $C_G(H_3) = 2^{1+4}.3^2.2$.

We decide to search inside $C_G(H_3)$ first.

Different copies of $L_3(3)$ that need to be considered are given by the double cosets of $C_G(H_1)$ and $C_G(H_2)$ in $C_G(H_3)$. That is to say, if we conjugate by an element of the form abc, where $a \in C_G(H_1)$, b is a double coset representative and $c \in C_G(H_2)$, then (by definition of double cosets) we have considered all cases, and $H_1^{abc} = H_1^{bc}$, and the pair (H_1^{bc}, H_2) is conjugate to the pair $(H_1^b, H_2^{c^{-1}}) = (H_1^b, H_2)$. In other words we only need to consider the cases H_1^b where b runs over a set of double coset representatives. There are just 96 cosets of the cyclic 6 (=centre of $6Suz$) inside $2^{1+4}.3^2.2$, so at most 96 cases to test.

3.9.2 Generators for $C_G(H_3)$

For the ease of calculations, we work inside Co_1 and then lift our group to G. First we find a word for a central 3 inside $3^{\cdot}Suz$. Let s_1 and s_2 be standard generators for $3^{\cdot}Suz$ as in 3.2. We look for an element of order 33 and its 11^{th} power will give us a central 3 element [4]. Hence a word for a central 3 is

$\mathtt{y}_1 = ((s_1 s_2)^3 s_1^2 s_2^2 (s_1^{s_2} s_2^{s_1 s_2 s_1 s_2^{s_1}} s_1^{s_2 s_1} s_2^{s_1 s_2^{s_1} s_1} s_1^{s_2 s_1 s_2 s_1} s_2 s_1)^2)^{11}$.

Next we construct generators for the centralizer of $\mathtt{r}_1$ inside Co_1 using Bray's trick [2] which are:

$\epsilon_1 = z_4[\mathtt{r}, z_4]^2$, $\epsilon_2 = [\mathtt{r}, z_5]^3$ and $\epsilon_3 = [\mathtt{r}_1, z_6]^3$. Words for z_4, z_5 and z_6 can be found in 3.8.1.

Our task is now to find the centralizer of $\mathtt{s}_1$ inside the above mentioned involution centralizer. Several methods are available for achieving this. One way is to look at various subgroups generated by an involution and $\mathtt{s}_1$ [13]. Since we are working with the permutation representation of Co_1 on 98280 points, so we decide to make use of the backtrack search utility of MAGMA. We make the following elements:

$\zeta_1 = \epsilon_1$, $\zeta_2 = \epsilon_2$, $\zeta_3 = \zeta_1\epsilon_3$ $\zeta_4 = \zeta_3\zeta_2$, $\zeta_5 = \zeta_4\zeta_3$, $\zeta_6 = \zeta_5\zeta_4$, $\zeta_7 = \zeta_6\zeta_5$, $\zeta_8 = \zeta_5\zeta_7$, $\zeta_9 = \zeta_8\zeta_7$, $\zeta_{10} = \zeta_9\zeta_7$, $\zeta_{11} = \zeta_{10}^{\zeta_7}$, $\zeta_{12} = \zeta_4\zeta_{10}$, $\zeta_{13} = \zeta_{12}\zeta_9$, $\zeta_{14} = \zeta_1^2\zeta_2^7\zeta_1^2\zeta_2^6\zeta_1$,

$\zeta_{15} = \zeta_6^2\zeta_{10}^7\zeta_6^2\zeta_1^6\zeta_6$, $\zeta_{16} = \zeta_{14}\zeta_{15}$, $\zeta_{17} = \zeta_{16}\zeta_{15}\zeta_{14}$, $\zeta_{18} = \zeta_{14}\zeta_{15}^2\zeta_{14}^5\zeta_{15}^8\zeta_{14}$, $\zeta_{19} = \zeta_{16}\zeta_{17}^2\zeta_{16}^5\zeta_{17}^8\zeta_{16}$, $\zeta_{20} = \zeta_{18}\zeta_{19}$, $\zeta_{21} = \zeta_{18}\zeta_{19}\zeta_{18}$, $\zeta_{22} = \zeta_{18}^3\zeta_{19}\zeta_{18}^6\zeta_{19}^2\zeta_{18}^8$, $\zeta_{23} = \zeta_{20}^3\zeta_{21}\zeta_{20}^6\zeta_{21}^2\zeta_{20}^8$, $\zeta_{24} = \zeta_{23}\zeta_{22}\zeta_{23}$, $\zeta_{25} = \zeta_{22}\zeta_{23}\zeta_{22}$, $\zeta_{26} = \zeta_{22}\zeta_{23}\zeta_{22}\zeta_{23}^6\zeta_{22}^6)$, $\zeta_{27} = \zeta_{24}\zeta_{25}\zeta_{24}\zeta_{25}^6\zeta_{24}^6$, $\zeta_{28} = \zeta_{18}\zeta_{19}\zeta_{18}\zeta_{19}^3\zeta_{18}^2$, $\zeta_{29} = \zeta_{20}\zeta_{21}\zeta_{20}\zeta_{21}^3\zeta_{20}^2$, $\zeta_{30} = \zeta_{28}\zeta_{29}\zeta_{28}$, $\zeta_{31} = \zeta_{29}\zeta_{28}\zeta_{29}$, $\zeta_{32} = \zeta_{28}\zeta_{29}\zeta_{28}^2\zeta_{29}\zeta_{28}$ and $\zeta_{33} = \zeta_{30}\zeta_{31}\zeta_{30}^2\zeta_{31}\zeta_{30}$.

Now we find a chain of subgroups

$G_1 = \langle \epsilon_1, \epsilon_2, \epsilon_3 \rangle > G_2 = \langle \zeta_{14}, \zeta_{15} \rangle > G_3 = \langle \zeta_{18}, \zeta_{19} \rangle > G_4 = \langle \zeta_{22}, \zeta_{23} \rangle > G_5 = \langle \zeta_{26}, \zeta_{27} \rangle$.

Inside G_5, we found the second generator $\mathtt{y}_2 = \zeta_{26}^3\zeta_{27}^2\zeta_{26}^3\zeta_{27}\zeta_{26}^8$.

We continue with our chain of subgroups

$G_5 > G_6 = \langle \zeta_{28}, \zeta_{29} \rangle > G_7 = \langle \zeta_{32}, \zeta_{33} \rangle$.

Inside G_7, we found our third generator which is given by:

$\mathtt{y}_3 = (\zeta_{32}\zeta_{33}^{\zeta_{32}\zeta_{33}\zeta_{32}}\zeta_{32}^2\zeta_{33}^2(\zeta_{32}^{\zeta_{33}\zeta_{32}^3\zeta_{33}})^4\zeta_{33}^2\zeta_{32}^4)^2$.

Now we have $C_{Co_1}(H_3) = \langle \mathtt{y}_1, \mathtt{y}_2, \mathtt{y}_3 \rangle$ and has order $3^{1+1}.2$. Next we lift it to G and now we have a 2-group (or a decoration) beneath $C_G(H_3)$. It turns out that it is enough to centralize 3^{1+2}.

We take two non-commuting elements of order 3 in $\langle \mathtt{r}_1, \mathtt{s}_1 \rangle$. So let $\omega_1 = \mathtt{s}_1^2$ and $\omega_2 = (\mathtt{r}_1\mathtt{s}_1)^2$ be two such elements of order 3 and z be the central involution. The word for z is given in **3.3**. We apply Parker's formula to ω_1 and $\mathtt{y}_3$ so that $\omega_3 = \mathtt{y}_3\omega_1\omega_1^{\mathtt{y}_3}z$ commutes with ω_1. Since ω_3 now commutes with ω_1, so we take $\omega_4 = \omega_3\omega_2\omega_2^{\omega_3}\omega_2 z$ which commutes with ω_1 and ω_2.

Now we take $\omega_5 = \mathtt{y}_2\omega_1\omega_1^{\mathtt{y}_2}\omega_1 z$ which commutes with ω_1. Again, since ω_5 commutes with ω_1, so we let $\omega_6 = \omega_5\omega_2\omega_2^{\omega_5}\omega_2 z$ so that ω_6 commutes with ω_1 and ω_2. Hence the generators for $C_G(H_3)$ after conjugating by β are given by:

$\sigma_1 = (\mathrm{y}_1 z)^{\beta}$, $\sigma_2 = \omega_5^{\beta}$ and $\sigma_3 = \omega_6^{\beta}$. Here β is the conjugation in 3.8.2.

Now $C_G(H_3)$ is a soluble group of order $2^{1+4}.3^2.2$ so every element of $C_G(H_3)$ can be expressed as $x_1^{i_1}x_2^{i_2}x_3^{i_3}x_4^{i_4}\sigma_2^{i_5}\sigma_3^{i_6}$,
where x_1, x_2, x_3, x_4 are generators for the group 2^{1+4}, $i_1, i_2, i_3, i_4, i_6 \in \{0, 1\}$ while $i_5 \in \{0, 1, 2\}$.

Now we proceed to find generators for 2^{1+4}. We adopt the same strategy as in **3.1**. We define the following elements:
$\sigma_5 = \sigma_1\sigma_1$, $\sigma_6 = \sigma_2\sigma_2$, $\sigma_7 = \sigma_1\sigma_2$, $\sigma_8 = \sigma_1\sigma_7$, $\sigma_9 = \sigma_6\sigma_2$, $\sigma_{10} = \sigma_6\sigma_7$, $\sigma_{11} = \sigma_1\sigma_3$, $\sigma_{12} = \sigma_6\sigma_3$, $\sigma_{13} = \sigma_2\sigma_3$, $\sigma_{14} = \sigma_7\sigma_3$, $\sigma_{15} = \sigma_8\sigma_3$, $\sigma_{16} = \sigma_9\sigma_3$, $\sigma_{17} = \sigma_{10}\sigma_3$, $\sigma_{18} = \sigma_{11}\sigma_3$, $\sigma_{19} = \sigma_{14}\sigma_{14}$, $\alpha_1 = \sigma_{11}\sigma_{11}$, $\alpha_2 = \sigma_{14}\sigma_{14}$, $\alpha_3 = \alpha_1\alpha_2$, $\alpha_4 = \sigma_{15}\sigma_{15}$, $\alpha_5 = \alpha_2\alpha_4$, $\alpha_6 = \sigma_{19}\sigma_{19}$, $\alpha_7 = \alpha_5\alpha_6$, $\alpha_8 = \sigma_7^3$, $\alpha_9 = \alpha_2\alpha_8$ and $\alpha_{10} = \alpha_9\alpha_3$.
We find that $\langle\alpha_2, \alpha_3\rangle$ and $\langle\alpha_5, \alpha_{10}\rangle$ are the commuting D_8's and can be taken as generators for 2^{1+4}. Hence every element of $C_G(H_3)$ can be written as $\alpha_2^{i_1}\alpha_3^{i_2}\alpha_5^{i_3}\alpha_{10}^{i_4}\sigma_2^{i_5}\sigma_3^{i_6}$, where i's are as before.

3.10 Candidates for the generator v

The generator v is defined to be the product of all points except P_0 [17]. According to the second relation of Norton, v is fixed by the stabilizer of the point P_0 in $L_3(3)$. It turns out that $v = P_1P_2P_3 \ldots P_{12}$ is an involution in 2^{1+24} centralised by $3^2{:}2{\cdot}S_4$. So in order to find v, we have to find the centralizer of the $3^2{:}2{\cdot}S_4$ (inside the centralizer of 3^2). The strategy to find the generator v is as follows:
Inside $3^2{:}2{\cdot}S_4$ we found generators for the normal 3^2:

$k_1 = ((h_1h_2)^3h_2)^2$ and $k_2 = k_1^{(h_1h_2)^5}$. Next we conjugate k_1 and k_2 by β so that $k_1^* = k_1^\beta$ and $k_2^* = k_2^\beta$. We let $\langle k_1^*, k_2^* \rangle = K$ and now proceed to find the centralizer of K in the next subsection.

3.10.1 The Centralizer of K

We can generate the centralizer of K inside 2^{1+24} by the following method. Let $\mathtt{x} \in 2^{1+24}$ then $(\mathtt{x}k_i^*)^3$ commutes with k_i^*. This implies that
$((\mathtt{x}k_1^*)^3k_2^*)^3 \in C(K) \cong 2^{1+8}$. In the following $u_1, \ldots, u_7, v_2$ and v_5 are generators for 2^{1+24} and are defined in section refgens. $\mathtt{c}_1 = ((u_1k_1^*)^3k_2^*)^3$, $\mathtt{c}_2 = ((u_2k_1^*)^3k_2^*)^3$, $\mathtt{c}_3 = ((u_3k_1^*)^3k_2^*)^3$, $\mathtt{c}_4 = ((u_4k_1^*)^3k_2^*)^3$, $\mathtt{c}_5 = ((u_5k_1^*)^3k_2^*)^3$, $\mathtt{c}_6 = ((u_6k_1^*)^3k_2^*)^3$, $\mathtt{c}_7 = ((u_7k_1^*)^3k_2^*)^3$, $\mathtt{c}_8 = ((v_2k_1^*)^3k_2^*)^3$, $\mathtt{c}_9 = ((v_5k_1^*)^3k_2^*)^3$ and $\mathtt{c}_{10} = ((u_{11}k_1^*)^3k_2^*)^3$.
We adopt the same strategy as in **3.1** to find generators for 2^{1+8}. First we form the following words: $\mathtt{d}_1 = \mathtt{c}_1$, $\mathtt{d}_2 = \mathtt{c}_6$, $\mathtt{d}_3 = \mathtt{c}_3$, $\mathtt{d}_4 = \mathtt{c}_8$, $\mathtt{d}_5 = \mathtt{d}_4\mathtt{d}_3$, $\mathtt{d}_6 = \mathtt{c}_7$, $\mathtt{d}_7 = \mathtt{c}_2$, $\mathtt{d}_9 = \mathtt{d}_4\mathtt{d}_8$, $\mathtt{d}_{10} = \mathtt{d}_9\mathtt{d}_2$, $\mathtt{d}_{11} = \mathtt{d}_{10}\mathtt{d}_6$, $\mathtt{d}_{13} = \mathtt{d}_{11}\mathtt{d}_{12}$, $\mathtt{d}_{14} = \mathtt{d}_{13}\mathtt{d}_2$, $\mathtt{d}_{15} = \mathtt{c}_{11}$, $\mathtt{d}_{16} = \mathtt{d}_{15}\mathtt{d}_{11}$, $\mathtt{d}_{17} = \mathtt{d}_5\mathtt{d}_{16}$ and $\mathtt{d}_{18} = \mathtt{d}_{17}\mathtt{d}_4$.
We find that following are the commuting D_8's:
$\langle \mathtt{d}_1, \mathtt{d}_2 \rangle$, $\langle \mathtt{d}_4, \mathtt{d}_5 \rangle$, $\langle \mathtt{d}_{10}, \mathtt{d}_{11} \rangle$ and $\langle \mathtt{d}_{14}, \mathtt{d}_{18} \rangle$. Hence $\mathtt{d}_1$, $\mathtt{d}_2$, $\mathtt{d}_4$, $\mathtt{d}_5$, $\mathtt{d}_{10}$, $\mathtt{d}_{11}$, $\mathtt{d}_{14}$ and $\mathtt{d}_{18}$ can be taken as generators for 2^{1+8}.

3.10.2 Cases for v

We run through the elements of $C_{2^{1+24}}(K)$ and check whether relations $[v, ut^{-1}] = 1$ and $[v, u^3su^{-2}] = 1$ are satisfied. The group $\langle ut^{-1}, u^3su^{-2} \rangle \cong 3^2{:}2{\cdot}S_4$ is the point stabilizer of $L_3(3)$.

We let $j = \alpha_3\sigma_2$ where α_3 and σ_2 are as in 3.9.2. We also give the following words:

$\eta_3 = \mathtt{d}_1\mathtt{d}_2$, $\eta_5 = \mathtt{d}_1\mathtt{d}_4$, $\eta_8 = \mathtt{d}_1\mathtt{d}_5$, $\eta_{11} = \eta_3\mathtt{d}_5$, $\eta_{13} = \eta_5\mathtt{d}_5$, $\eta_{19} = \eta_{13}\mathtt{d}_{10}$, $\eta_{27} = \eta_{11}\mathtt{d}_{10}$, $\eta_{40} = \eta_8\mathtt{d}_{11}$, $\eta_{19} = \eta_{13}\mathtt{d}_{10}$, $\eta_{51} = \eta_{19}\mathtt{d}_{11}$, $\eta_{19} = \eta_{13}\mathtt{d}_{10}$,$\eta_{91} = \eta_{27}\mathtt{d}_{14}$, $\eta_{19} = \eta_{13}\mathtt{d}_{10}$, $\eta_{104} = \eta_{40}\mathtt{d}_{14}$, $\eta_{179} = \eta_{51}\mathtt{d}_{18}$, $\eta_{19} = \eta_{13}\mathtt{d}_{10}$ and $\eta_{219} = \eta_{91}\mathtt{d}_{18}$.

We find that there are eighteen cases for v which satisfy the above mentioned relations:

$v_1 = (\eta_{104})^j$, $v_2 = (\eta_{179})^j$, $v_3 = (\eta_{219})^j$, $v_4 = zv_1$, $v_5 = zv_2$, $v_6 = zv_3$, $v_7 = (v_1)^{\sigma_2}$, $v_8 = (v_2)^{\sigma_2}$, $v_9 = (v_3)^{\sigma_2}$, $v_{10} = (zv_1)^{\sigma_2}$, $v_{11} = (zv_2)^{\sigma_2}$, $v_{12} = (zv_3)^{\sigma_2}$, $v_{13} = (v_1)^{\sigma_3}$, $v_{14} = (v_2)^{\sigma_2}$, $v_{15} = (v_3)^{\sigma_2}$, $v_{16} = (zv_1)^{\sigma_3}$, $v_{17} = (zv_2)^{\sigma_2}$ and $v_{18} = (zv_3)^{\sigma_3}$.

3.11 Generators s, t and v

In section 3.9, we calculated s^{**} and t^{**} and in 3.9.2 we have calculated the centralizer of $L_3(3)$. Now we conjugate s^{**} and t^{**} by elements of $C_G(H_3)$ and also try the above eighteen candidates for v. We give the following words:

$j_3 = \alpha_2\alpha_3$, $j_7 = j_3\alpha_5$, $j_{23} = j_7\sigma_2$ and $j_{71} = j_{23}\sigma_3$. Here σ's and α's are as in 3.9.2.

We find that $s = (s^{**})^{j_{71}}$, $t = (t^{**})^{j_{71}}$, $u = sts^2t^2$, $v = v_{11}$ and $x = T$ satisfy all relations of the Monster presentation. Moreover, this is the only case which works.

All these relations have been checked using a single vector. Actually this will give us a divisor of the order of the element. To prove that this is in fact the actual order, we need two vectors whose joint stabilizer is the trivial group. In the next chapter we will construct these special vectors and verify our relations. Our next step will be to test the relations on a basis for the 196882 dimensional space. After the successful testing of relations, Ivanov's theorem [11] now tells us that we have a new proof of the existence of the Monster group which is independent of Griess's original proof [7].

Chapter 4

Existence of the Monster

In this chapter we will test the presentation for the Monster using two special vectors and then we will verify the relations on a basis for the 196882 space. We then give our main theorem which claims a new existence proof of the Monster.

4.1 Verifying the relations

So far, we have verified the Monster presentation by using only one vector but to actually prove it we need two vectors whose joint stabilizer is trivial. Now we will discuss how to construct these two vectors.

First we find elements of orders 71 and 94 say y_1 and y_2 respectively. We find that ψ_{16} and ψ_{93} are two such elements which are defined in section **3.1**.

Let r be a random vector. We define

$$\begin{aligned} v_1 &= r(1 + y_1 + y_2^2 + \cdots + y_1^{70}) \\ &= r + ry_1 + (ry_1)y_1 + ((ry_1)y_1)y_1 + \cdots + ((ry_1)y_1)y_1 \ldots y_1. \\ v_2 &= r(1 + y_2^2 + y_2^4 + \cdots + y_2^{92}) \\ &= r + (ry_2)y_2 + (((ry_2)y_2)y_2)y_2 + \cdots + ((ry_2)y_2)y_2 \ldots y_2. \end{aligned}$$

4.1.1 Non-triviality check:

We first check that $v_1 \neq 0$ and $v_2 y_2 \neq v_2$.

This implies that $y_1 \in Stab_{\mathbb{M}}(v_1)$ hence $Stab_{\mathbb{M}}(v_1)$ can be

71, 71:5, 71:7, 71:35 or $L_2(71)$ [26].

Now $y_2^2 \in Stab_{\mathbb{M}}(v_2)$ and $y_2 \notin Stab_{\mathbb{M}}(v_2)$ implies that

$Stab_{\mathbb{M}}(v_2) = 47$ or 47:23 [29].

This implies that $Stab_{\mathbb{M}}(v_1) \cap Stab_{\mathbb{M}}(v_2) = \{1\}$.

We now check that the Monster presentation is indeed verified by using these two vectors.

4.2 A new proof

Next we verify the Monster presentation on a full basis for the 196882 dimensional space. The calculations were performed on a high performance cluster (HPC) provided by QMUL known as "Taurus" which is a 74× Dual Dual-core Opteron (270) and 15×4 socket 12-core AMD Magny Cours (6172). Its queueing system is grid engine $6.2u4$. Parallel Libraries were of type Open-

MPI.

It took almost 50 hours to perform the computation.

We now come to our main theorem but before stating it we state without proof an important theorem due to Ivanov [11]. The crucial point to note is that this theorem does not assume the existence of Monster.

Theorem 4.2.1 *Let H be a subgroup satisfying the following properties:*

(a) *It is generated by subgroups H_1, H_2, H_3 of shapes $2^{1+24}{\cdot}Co_1$, $2^{2+11+22}{\cdot}(M_{24} \times S_3)$ and $2^{3+6+12+18}{\cdot}(3{\cdot}S_6 \times L_3(2))$ respectively. In H_2 and H_3, the elementary abelian normal subgroups of orders 2^2 and 2^3 are fully normalized.*

(b) *$H_1 \cap H_2$ has index 3 in H_2.*

(c) *$H_1 \cap H_3$ and $H_2 \cap H_3$ both have index 7 in H_3, corresponding to the points and lines of a projective plane of order 2 acted on by the composition factor of $L_3(2)$ of H_3.*

Then H is a group of Monster type.

Theorem 4.2.2 *The elements s, t, u, v and x constructed in 3.11 generate a group of monster type. Moreover, this construction of $\mathbb{M}$ is independent of the construction given in [7].*

Proof. The verification of the presentation given by Norton in **[17]** on the full basis of 196882 space shows that the elements s, t, u, v and x generate the same group as that defined by the projective plane presentation. It has been shown by Norton that Y_{555} is indeed the Bimonster by showing that the configuration given in 4.2.1 holds in relevant subgroups of the Bimonster

[18]. In fact his argument shows that if the Monster does not exist, then this presentation defines the group of order 2. This means that the group $\langle s, t, u, v, x\rangle$ satisfying the presentation

$$
\begin{aligned}
s^6 = t^3 = (st)^4 = (s^2t)^4 = (s^3t)^3 = [s^2, (ts^2t)^2] = 1\\
[v, ut^{-1}] = [v, u^3su^{-2}] = v^2 = [v, v^t] = (vu)^{13} = 1\\
[vx, ut^{-1}] = [vx, s^{u^3}] = 1\\
x^3 = (v^u vx)^2 = (x^{-1}x^s)^2 = 1\\
(xt^{-1})^{12} = 1\\
(u^{-6}xu^6s)^6(sux^{-1}u^{-1})^6s^{-1} = 1\\
((xv^{u^4}v^{u^{10}})^3u)^{13} = 1.
\end{aligned}
$$

is $\mathbb{M}$. The argument that the matrices s, t, u, v and x generate $\mathbb{M}$ depends on 4.2.1 without appealing to the Griess's construction [7]. This completes the proof. $\square$

Chapter 5

Conclusions and suggestions for further work

5.1 Conclusions

We have now constructed Norton's generators. We have also produced a computational proof of the existence of the Monster group by verifying the presentation of the Monster on the full basis of 196882 dimensional space .

5.2 Further work

This project is complete in its own. However, for future activity, we can construct the Bimonster using Norton's generators. Some details are given below.

The elements of the Bimonster are either (m_1, m_2) or $(m_1, m_2)\alpha$ where $m_i \in \mathbb{M}$ and α as in 1.4. We can construct a data structure to store these and then multiply them together as $(m_1, m_2)\alpha.(m_3, m_4) = (m_1.m_4, m_2.m_3)\alpha$. We just need a bit somewhere which says whether α is there or not. Say a file containing either 0 or 1, where 0 means the identity and 1 means α.

To construct the Coxeter generators for the Bimonster, we see that action of $L_3(3)$ on $L_0 = \alpha vx$ and $P_0 = \alpha v$ gives us

$L_i = \alpha(vx)^{u^i}$ and $P_i = \alpha v^{u^{-i}}$. Hence to compute inside the Bimonster we have $L_i = \alpha((vx)^{u^i}, (vx)^{u^i})$ and $P_i = \alpha(v^{u^{-i}}, v^{u^{-i}})$. It is possible to construct these 26 generators of the Bimonster.

Once this task is accomplished, we can also have some nice generators for the 67 groups mentioned in [5].

Appendix

Transforming the Monster presentation

Simon Norton

1 Oct 2002

Abstract

Here we transform the "projective plane" presentation of the Monster into a new form which is convenient for direct verification on its 2-local matrix construction.

The "projective plane" formulation for the Bimonster $\mathbb{M} \wr 2$, where $\mathbb{M}$ denotes the Fischer-Griess Monster, was first described in the author's joint paper with J. H. Conway and L. H. Soicher [5], and later in the Atlas of Finite Groups [4]. In this formulation there are 26 involutory generators corresponding to the points and lines of a projective plane of order 3. (For the sake of convenience we identify the points and lines with the generators.) All points commute with one another, as do all lines; a point commutes with a line except when they are incident, when their product has order 3.

Other relations satisfied by these generators were described in [5], but it was later shown by Soicher [22] that they could all be deduced from a single further relation. Let us take the three vertices and edges of a triangle in the projective plane. With the above mentioned relations only, these would generate the affine Weyl group $\tilde{A}_5$, i.e. the extension of a free abelian group of rank 5 by S_6. We adjoin a relation which collapses this group to the finite group $3^4.S_6$. Soicher shows that a single such relation is enough to imply a similar relation for every triangle in the projective plane, as well as all the other relations given in [5].

Soicher also showed [22] that with this extra relation the group generated by the projective plane is isomorphic to the wreath square of a group defined by a related presentation.

Some time afterwards, A. A. Ivanov [11] showed that any group in which the subgroups $G_1 = 2^{1+24}.Co_1$, $G_2 = 2^{2+11+22}.(M_{24} \times S_3)$ and $G_3 = 2^{3+6+12+18}.(3.S_6 \times L_3(2))$ intersect as in the Monster, i.e. where $G_1 \cap G_2$ has index 3 in G_2, and $G_1 \cap G_3$, $G_2 \cap G_3$ and $G_1 \cap G_2 \cap G_3$ have respective indexes 7, 7 and 21 in G_3, must be isomorphic to the Monster. The author was then able [5] to find the above configuration in the group defined by the related presentation referred to above, thereby showing that this group is the Monster and that the projective plane presentation defines the Bimonster.

More recently, various people in Birmingham [8] have developed techniques for making computations with 196882×196882 matrices over $GF(3)$ that generate the Monster. This offers an opportunity to provide a *direct* proof that these matrices generate the same group as that defined by the projective plane presentation.

The technique involves storing generators for the subgroup $2^{1+24}.Co_1$ in smaller representations, from which the 196882 dimensional representation can in theory be built. This can be used to work out the action of any element of this subgroup on any vector. At the same time, one can devise a means of working out the action on such a vector of an extra "triality" element of order 3 which together with the subgroup generates the whole Monster.

Returning to the projective plane, any word of even length in the points and lines belongs to the subgroup of index 2 in the Bimonster, i.e. $\mathbb{M}\times\mathbb{M}$; and if we project down to one of the factors we can make such a word correspond to an element of the Monster. It can be shown that the Monster contains a subgroup of type $L_3(3)$ which permutes such words according to the action of the automorphism group of the projective plane; and that the Monster has a unique subgroup of type $2^{1+24}.Co_1$ which contains this $L_3(3)$ and also any even product of the *points*. Furthermore, the product of an incident point and line can be taken as the triality element.

We denote the elements of the Bimonster corresponding to the points and lines of the projective plane by P_i and L_i where $0 \leq i \leq 12$, and the point P_i is incident with the line L_j if and only if $i + j \equiv 0$, 1, 3 or 9 (mod 13).

We now define the following generators:

s is the element of $L_3(3)$ that acts as $(1, 2, 5, 9, 8, 7)(3, 12, 4)(10, 11)$ on the points. It can be seen that its action on the lines is $(0, 1, 9)(2, 12, 11, 4, 5, 6)(7, 8)$.

t is the element of $L_3(3)$ that acts as $(0, 12, 3)(1, 2, 4)(5, 7, 11)(6, 9, 8)$ on the points, and which may be seen to have action $(0, 1, 10)(2, 11, 9)(3, 4, 6)(5, 8, 7)$ on the lines. From s and t we may derive $u = sts^2t^2$ which acts as $z \mapsto z - 1$ on the points and $z \mapsto z + 1$ on the lines (where all numbers are taken mod 13).

v is the product of all the points with non-zero subscript.

x is the product P_0L_0.

α is the product of all thirteen points.

With this notation, $\alpha v = P_0$ and $\alpha vx = L_0$, and as the group generated by s and t is transitive (and is in fact the full group $L_3(3)$) it follows that the group $\langle s, t, v, x, \alpha\rangle$ contains the Bimonster. In fact we may construct an endomorphism which fixes s and t, and takes v, x and α to the identity; as the image and kernel of this endomorphism are $L_3(3)$ and the Bimonster, respectively, the group can only be the direct product of the Bimonster with $L_3(3)$.

We now specify the following relators:

$$s^6 = t^3 = (st)^4 = (s^2t)^4 = (s^3t)^3 = [s^2, (ts^2t)^2] = 1 \tag{1}$$

According to the [4] , this is a presentation for $L_3(3)$, and it can easily be seen that the permutations specified above satisfy all these relations.

$$[v, ut^{-1}] = [v, u^3su^{-2}] = v^2 = [v, v^t] = (vu)^{13} = 1 \tag{2}$$

$$[\alpha, s] = [\alpha, t] = [\alpha, v] = \alpha^2 = 1 \tag{3}$$

The first two relators of (2) say that v is fixed by the subgroup $\langle ut^{-1}, u^3su^{-2}\rangle$ of $L_3(3)$; it can be seen that these elements generate the full stabilizer of P_0. In particular it contains s, a fact that we shall be needing later. The other relators of (2) say that v is an involution, that it commutes with all its images under $L_3(3)$ (using the double transitivity of this group on points), and that the product of these images is the identity.

The relators of (3) say that α is invariant under the full $L_3(3)$ and that it is an involution commuting with v.

We may now define $P_0 = \alpha v$ and $P_i = P_0^{u^{-i}}$, from which the original definitions of v and α in terms of the P_i can be recovered. It therefore follows that the group $\langle s, t, v, \alpha \rangle$ with the relations of (1)-(3) is $2^{13}.L_3(3)$.

$$[vx, ut^{-1}] = [vx, s^{u^3}] = 1 \tag{4}$$

This says that vx is invariant under the subgroup $\langle ut^{-1}, s^{u^3} \rangle$ of $L_3(3)$; it can be seen that these elements generate the full stabilizer of L_0. If we define L_0 as αvx, which is clearly equivalent to our earlier definition of x as P_0L_0, then this shows that L_0 has thirteen images L_i $(0 \leq i \leq 12)$ under the group $L_3(3)$; we may define $L_i = L_0^{u^i}$.

$$x^3 = (v^u vx)^2 = (x^{-1}x^s)^2 = 1 \tag{5}$$

The first relator says that the product P_0L_0 has order 3; the second that the product $P_{12}L_0$ has order 2, and the third that $L_0^{-1}L_1$ has order 2. By the action of $L_3(3)$ one can deduce the order of the product of any point and line, or of the quotient of any two lines.

$$(\alpha vx)^2 = 1 \tag{6}$$

This says that the lines are involutions. It therefore follows from the previous relations that they commute with one another and with points to which they are not incident. Because by (3) α is an involution which commutes with v, this relation can also be written $x^\alpha = vx^{-1}v$.

$$(xt^{-1})^{12} = 1 \tag{7}$$

This reduces to $(P_0L_0P_{12}L_1P_3L_{10})^4 = 1$, the relation that reduces the affine Weyl group $\tilde{A}_5$ to $3^4.S_6$. It therefore follows that the relations (1)-(7) define a presentation for $(\mathbb{M} \wr 2) \times L_3(3)$.

We now consider the subgroup $\langle s, t, v, x \rangle$. All four generators are conjugated into this group by α, so this subgroup is normal. As α occurs evenly many times in every relation, and is an involution, it follows that the normal subgroup must have index 2. We may therefore use the Schreier process to define a presentation for this subgroup.

The relations involving α (i.e. (3) and (6)) are just sufficient to define its action by conjugation on the other four generators, which turns out to be involutory, and to say that it is itself an involution. It therefore follows that a sufficient set of relations for the subgroup of index 2 consists of all the remaining relations (i.e. those not involving α), plus the relations obtained by conjugating the corresponding relators by α and then eliminating the α's by using its conjugation action on the other four generators.

As α commutes with s, t and v, nothing new can come from the relations which do not involve x, i.e. (1) and (2). We now deal in order with the other relations, i.e. (4), (5) and (7).

The relations of (4) say that vx commutes with two elements of $L_3(3)$. As α commutes with s and t by (3) and inverts vx by (6), the conjugate relations merely express the equivalent statement that the inverse of vx commutes with the same elements of $L_3(3)$.

The first relator of (5) is taken to $(vx^{-1}v)^3$ which is conjugate (within $\langle s, t, v, x\rangle$) to x^{-3}, so that, again, the conjugate relation is equivalent to the original.

The second relator of (5) is taken to $(v^u x^{-1} v)^2$ which, again, is conjugate to the inverse of the original, so yields an equivalent relation.

The third relator of (5) is taken to $(vxvv^s(x^{-1})^s v^s)^{12} = (vx(x^{-1})^s v)^{12}$ (using the earlier result that v commutes with s). Again, this is conjugate to an inverse of the original.

To deal with (7), we write $w = v^{u^3}$, the product of all the points except for P_{10}, and note that $vx^{-1}v = x^w$ (which is essentially saying that P_0L_0 commutes with P_{10}) and that w commutes with t, as can be seen from the permutation expression of the latter. So the conjugate of $(xt^{-1})^{12}$ by α is $(vx^{-1}vt^{-1})^{12} = ((xt^{-1})^{12})^w$ which is conjugate to the original. This completes the proof that the Schreier process adds no new relations, so the relations (1), (2), (4), (5) and (7) define a presentation for $\mathbb{M} \times \mathbb{M} \times L_3(3)$.

We now give a relation that gets rid of the $L_3(3)$.

$$(u^{-6}xu^6s)^6(sux^{-1}u^{-1})^6 = s \tag{8}$$

The left hand side of this comes out as

$$P_7L_6P_8L_5P_9L_4P_5L_{11}P_2L_{12}P_1L_2.L_2P_7L_6P_8L_5P_9L_4P_5L_{11}P_2L_{12}P_1,$$

a word in six points and six lines whose incidence graph is a dodecagon, i.e. a graph of affine Weyl type $\tilde{A}_{11}$, which is known [5] to generate an S_{12}. In fact, it is the element of this S_{12} whose action on the nodes of the dodecagon is to rotate the graph through an angle of 60^o. It is known that such an element acts as a permutation of the whole projective plane – and in this case it can be seen that the action is the same as that of s.

Finally, we go down to the Monster. The element

$$(xv^{u4}v^{u^{10}})^3 = (P_0L_0P_3P_9)^3$$

is the central involution of the Weyl group of the graph of type D_4 generated by L_0, P_0, P_3 and P_9. In [19] this element was denoted by P_1^*. Note that P_1 is the unique point incident to L_0 which is not a generator of the above group. It is known that any element of $L_3(3)$ which fixes P_1 also fixes P_1^* (hence the notation), and that the product of the thirteen images of P_1^* under $L_3(3)$ is the projection of $\pi = [P_i^*, P_i]$ (again using the notation of [19]) to one of the factors of $\mathbb{M} \times \mathbb{M}$. It therefore follows that if we adjoin the relation

$$((xv^{u4}v^{u^{10}})^3u)^{13} = 1 \tag{9}$$

to (1), (2), (4), (5), (7) and (8) we get a presentation for the Monster.

We now describe how one might locate the elements s, t, v and x inside a Monster defined by adjoining a triality element to $2^{1+24}.Co_1$. Before we do this, though, we introduce some notation and outline some results (mostly from [5] or [19]) on subgroups of the projective plane.

As stated earlier we are using the term *points* to describe the group elements associated with the P_i. We call the P_i^*'s *stars*. It follows from the description in [19] of the group generated by a $\tilde{D}_4$ subdiagram of the projective plane, which we shall reproduce later, that all points and starts commute with one another except for P_i and P_i^*, whose commutator is a fixed element which we call π as above. The group generated by even combinations of the points, and all combinations of the lines modulo complementation (which has trivial kernel because of the last relation of the presentation above), is the 2^{1+24} that is the O_2-subgroup of $C_{\mathbb{M}}(\pi) \cong 2^{1+24}.Co_1$. Any element of this O_2-subgroup modulo the centre can be expressed uniquely as a product of the points and stars. We use an abbreviated notation that can be described by saying that P_{0139} denotes $P_0P_1P_3P_9$ and P_{0139}^* denotes $P_0^*P_1^*P_3^*P_9^*$. (To avoid ambiguity we use X, E and T to denote the subscripts 10, 11 and 12 respectively.)

To work within a $\tilde{D}_4$-subdiagram such as $\langle P_0, P_1, P_3, P_9, L_0\rangle$, we write the elements as 4×4 matrices over the dihedral group D_8 with presentation $\langle a, b \mid a^2 = b^2 = (ab)^4 = 1\rangle$. We write $-1 = (ab)^2$. The group consists of monomial matrices with "determinant" (we put this word in quotes because due to the non-commutativity of the base group it is only defined up to sign) ± 1, modulo multiplication by -1. The points, stars and L_0 can be written as follows:

$$P_0 = \begin{pmatrix} 0 & a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, P_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, P_9 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & b & 0 \end{pmatrix},$$

$$P_0^* = \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, P_1^* = \begin{pmatrix} -b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, P_3^* = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & -a \end{pmatrix},$$

$$P_9^* = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, L_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is easy to check that the stars are correctly derived from the points and L_0 using the definition given earlier.

We also use the fact that a diagram of type $\tilde{D}_8$ generates a group $2^6.S_8$.

Products of two points centralize the double cover of a Baby Monster and therefore belong to conjugacy class $2A$. By looking inside the $A_5 \times A_{12}$ generated by the doubly even elements in an $A_4 + A_{11}$ subdiagram, we can deduce that the product of the six points in the symmetric difference of two lines also belongs to $2A$. The element $P_{01}^{L_0}$ conjugates P_{13} to $P_{13}P_9^*$ so the last element, and anything obtained from it by applying an automorphism of the projective plane, also belongs to $2A$.

The automorphism group of the projective plane, which is $L_3(3)$, can be described as the centralizer in the Monster of a Frobenius group of order 78 in which the 13-elements belong to class $13A$. The element π is in this Frobenius group (because it centralizes $L_3(3)$) and its centralizer in this group contains a 3-element, say ω. Now because ω centralizes π it must correspond to an action on the 2^{1+24}, and this action must be invariant under $L_3(3)$. The only such action (up to inversion) is the one that takes P_{ij} to P^*_{ij} to $P_{ij}P^*_{ij}$. We may therefore deduce consequences such as that the product of two stars belongs to class $2A$.

We are now ready to try to identify s, t, v and x in the Monster. We start by identifying the triality element with $x = P_0L_0$. We now proceed to identify a subgroup $2^{1+24}.3^{1+2}.2^2$ of $\langle s, t, v, x\rangle$ modulo its central involution π. We use standard Leech Lattice (Λ) notation for the elements of the O_2-subgroup, displaying the 24 coordinates in the standard MOG formation [4]. As $[\pi, L_0] = P^*_{0139}$ we can identify the latter with any vector of type $(8, 0^{23})$ (they all correspond to the same group element as they are congruent modulo 2Λ).

Inside the coordinate permutation group M_{24} (which should be considered as a quotient, as the relevant extension is non-split), we can identify a subgroup fixing a $(3, 9, 12)$ splitting with the $3^2.2S_4$ fixing the line L_0 in our projective plane automorphism group. The corresponding elements act monomially on the 24 coordinates. But we can choose signs such that a subgroup 3^{1+2} acts as coordinate permutations. The identification we choose is such that the element $(2, 8, T)(6, E, X)(7, 4, 5)$ takes rows 1, 2, 3 and 4 to 1, 3, 4 and 2 respectively; the element $(1, 3, 9)(6, X, E)(7, 4, 5)$ takes columns 1, 2 and 3 to 2, 3 and 1 respectively, fixes column 4, acts on column 5 as the inverse of the previous element, and acts on column 6 as the previous element; and the element $(2, 6, 7)(8, E, 4)(T, X, 5)$ does exactly the same with the six columns taken in reverse order.

The three parts of the $(3, 9, 12)$ splitting of our 24 coordinates correspond respectively to the three pairings of the four points on the line L_0, the nine points not on L_0, and the 12 other lines in the projective plane. It may be checked that the action of $3^2.2S_4$ on the 24 coordinates can be described by the following diagram:

$$\begin{array}{cccccc} 01|39 & 03|91 & 09|13 & L_1 & L_3 & L_9 \\ L_T & L_E & L_7 & P_2 & P_6 & P_7 \\ L_8 & L_6 & L_5 & P_8 & P_E & P_4 \\ L_2 & L_X & L_4 & P_T & P_X & P_5 \end{array}$$

As there is a unique $2A$-element inverting a $3A$-pure group of type 3^2, it follows that we can also identify the element interchanging the last two rows and columns with $(4, X)(5, E)(6, 7)(8, T)$, and the element interchanging the last two rows and first two columns with $(1, 3)(4, 5)(8, T)(X, E)$, as the diagram would suggest. In what follows we can use these permutations to transform our other identifications into enough to determine the image of the full 2^{1+24} group on $\Lambda/2\Lambda$.

According to the construction of the Monster, involutions in class $2A$ which invert x correspond to vectors of type $(-3, 1^{23})$. So we can make the identification

$$P_{02} = \begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -3 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array}$$

and eight others, generating a group 2^9 in which 2^8 centralizes x, e.g.

$$P_{24} = \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

which means that together with P^*_{0139} we have now identified a total of 2^{10}.

As a $2A$-element in the centralizer of x, P^*_{13} also corresponds to a vector of type $(4, -4, 0^{22})$, for which the only possibility is

$$P^*_{13} = \begin{matrix} 4 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

which takes us up to 2^{12}. The rest of the 2^{1+11} corresponding to vectors whose coordinates are divisible by 4 can be generated by P^*_{139} whose corresponding vector can be evaluated by using the fact that elements like $P^*_1 P_{24}$ are in class $2A$, viz.

$$P^*_{139} = \begin{matrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

We conclude by evaluating the vectors corresponding to P_{01} and $P^*_2 P_{8T}$, both of which can be seen to generate an A_4 with x and which therefore have coordinate shape $(2^8, 0^{16})$, viz.

$$P_{01} = \begin{matrix} 0 & 2 & 2 & 2 & 2 & 2 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

$$P_2^* P_{8T} = \begin{matrix} 2 & 2 & 2 & 2 & 0 & 0 \\ 2 & 2 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

which with their transforms by our permutation group generate the full 2^{1+24}.

This description enables us to deduce the action on $\Lambda/2\Lambda$ of any element of $L_3(3)$, and hence to identify the element modulo the O_2-subgroup. This should make it possible to obtain an exact identification by using the matrix representations of $2^{1+24}.Co_1$.

References

[1] W. Bosma, J. Cannon, and C. Playoust, The Magma algebra system. I. The user language, *J. Symbolic Comput.*, 24(3-4), 235–265, (1997).

[2] J. N. Bray, An improved method for generating the centralizer of an involution, *Arch. Math. (Basel)* **74** (2000), no. 4, 241–245.

[3] J. H. Conway, A simple construction for the Fischer-Griess monster group, *Invent. Math.* **79** (1985), 513–540.

[4] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *An ATLAS of finite groups*, Oxford University Press, 1985.

[5] J. H. Conway, S. P. Norton and L. H. Soicher, The Bimonster, the group Y_{555}, and the projective plane of order 3. *Computers in algebra (Chicago, IL*, 1985), 27–50, Lecture Notes in Pure and Appl. Math., **111**, Dekker, New York, 1988.

[6] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4.12*, 2008, (`http://www.gap-system.org`).

[7] R. L. Griess, Jr, The Friendly Giant, *Invent. Math.* **69** (1982), 1–102.

[8] P. E. Holmes and R. A. Wilson, A new computer construction of the Monster using 2-local subgroups, *J. Algebra* **251** (2002), 435–447.

[9] A. A. Ivanov, Y-groups via transitive extension, *J. Algebra,* **218**, (1999), 412–435.

[10] A. A. Ivanov, Presenting the Baby Monster, *J. Algebra* **163** (1994), no. 1, 88–108.

[11] A. A. Ivanov, A geometric characterization of the Monster, *Groups, combinatorics & geometry* (Durham, 1990), 46–62, London Math. Soc. Lecture Note Ser., **165**, Cambridge Univ. Press, Cambridge(1992).

[12] J. S Leon and C. C Sims, The existence and uniqueness of a simple group generated by {3,4}-transpositions, *Bull. Amer. Math. Soc.* **83**(1977) 1039–1040.

[13] S. A. Linton, The art and science of computing in large groups, *Computational algebra and number theory* (Sydney, 1992), 91–109, Math. Appl., **325**, Kluwer Acad. Publ., Dordrecht, 1995.

[14] S. A. Linton, R. A. Parker, P. Walsh, R. A. Wilson, Computer construction of the Monster. *J. Group Theory* 1 (1998), no. 4, 307–337.

[15] M. Ringe, The C MeatAxe, *Lehrstuhl D für Mathematik, Aachen, Germany, release 2.4.3 edition*, 2004.

[16] S. P. Norton, Anatomy of the Monster. I. *The atlas of finite groups: ten years on* (Birmingham, 1995), 198–214, London Math. Soc. Lecture Note Ser., 249, Cambridge Univ. Press, Cambridge(1998).

[17] S. P. Norton, Transforming the monster presentation, (2002), (Preprint).

[18] S. P. Norton, Constructing the Monster, *Groups, combinatorics & geometry* (Durham, 1990), 63–76, London Math. Soc. Lecture Note Ser., **165**, Cambridge Univ. Press, Cambridge, 1992.

[19] S. P. Norton, Presenting the monster? , *Bull. Soc. Math. Belg.* **62** (1990) 595–605.

[20] S. P. Norton; R. A. Wilson, Anatomy of the Monster. II. *Proc. London Math. Soc.* (3) **84** (2002), no. 3, 581–598.

[21] L. H. Soicher, More on the group Y_{555} and the projective plane of order 3, *J. Algebra* **136** (1991), no. 1, 168–174.

[22] L. H. Soicher, From the Monster to the Bimonster. *J. Algebra* **121** (1989), no. 2, 275–280.

[23] R. A. Parker, The computer calculation of modular characters (The 'Meat-Axe'), *Computational group theory,* ed. M. D. Atkinson, Academic Press, London,(1984), 267–274.

[24] R. A. Parker and R. A. Wilson, The computer construction of matrix representations of finite groups over finite fields, *J. Symbolic Comput.* **9** (1990), no. 5-6, 583–590.

[25] J. Thackray, A new meataxe, (2002).

[26] R. A. Wilson *et al.*, A world wide-web Atlas of Group Representations, http://brauer.maths.qmul.ac.uk/atlas/

[27] R. A. Wilson, A new construction of the Baby Monster and its applications, *Bull. London Math. Soc.* **25**(1993) 431–437.

[28] R. A. Wilson, Computing in the Monster, *Groups, combinatorics & geometry* (Durham, 2001), 327–335, World Sci. Publ., River Edge, NJ, 2003.

[29] R. A. Wilson, More on maximal subgroups of the Baby Monster. *Arch. Math. (Basel)* **61** (1993), no. 6, 497–507.

[30] R. A. Wilson, Standard generators for sporadic simple groups, *J. Algebra,* **184** (1996), no. 2, 505–515.