HARDY INEQUALITIES ON METRIC MEASURE SPACES, III: THE CASE $q \leq p < 0$ AND APPLICATIONS

AIDYN KASSYMOV, MICHAEL RUZHANSKY, AND DURVUDKHAN SURAGAN

ABSTRACT. In this paper, we obtain a reverse version of the integral Hardy inequality on metric measure space with two negative exponents. Also, as for applications we show the reverse Hardy-Littlewood-Sobolev and the Stein-Weiss inequalities with two negative exponents on homogeneous Lie groups and with arbitrary quasi-norm, the result which appears to be new already in the Euclidean space. This work further complements the ranges of p and q (namely, $q \leq p < 0$) considered in [35] and [36], where one treated the cases $1 < p \leq q < \infty$ and $p > q$, respectively.

1. INTRODUCTION

In the famous work [19], G.H. Hardy showed the following (direct) integral inequality:

$$\int_a^\infty \frac{1}{x^p}\left(\int_a^\infty f(t)dt\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_a^\infty f^p(x)dx, \tag{1.1}$$

where $f \geq 0$, $p > 1$, and $a > 0$. The subject of the Hardy inequalities has been extensively investigated and we refer to the book [27].

We refer to direct inequalities [7, 9, 10, 16, 27, 26, 28, 32] and to the reverse inequalities [2, 17, 25, 29, 34].

The main goal of this paper is to extend the reverse Hardy inequalities to general metric measure space with two negative exponents. More specifically, we consider metric spaces $\mathbb{X}$ with a Borel measure dx allowing for the following *polar decomposition* at $a \in \mathbb{X}$: we assume that there is a locally integrable function $\lambda \in L^1_{loc}$ such that for all $f \in L^1(\mathbb{X})$ we have

$$\int_{\mathbb{X}} f(x)dx = \int_0^\infty \int_{\Sigma_r} f(r,\omega)\lambda(r,\omega)d\omega_r dr, \tag{1.2}$$

for some set $\Sigma_r = \{x \in \mathbb{X} : d(x,a) = r\} \subset \mathbb{X}$ with a measure on it denoted by $d\omega$, and $(r,\omega) \to a$ as $r \to 0$.

The condition (1.2) is rather general (see [35]) since we allow the function λ to depend on the whole variable $x = (r,\omega)$. Since $\mathbb{X}$ does not necessarily have a differentiable structure, the function $\lambda(r,\omega)$ can not be in general obtained as the Jacobian of the polar change of coordinates. However, if such a differentiable structure exists on $\mathbb{X}$, the condition (1.2) can be obtained as the standard polar

2020 *Mathematics Subject Classification.* 22E30, 43A80.

Key words and phrases. Reverse Hardy inequality, metric measure space, Reverse Hardy-Littlewood-Sobolev inequality, Reverse Stein-Weiss inequality.

The first and second authors were supported in parts by the FWO Odysseus 1 grant no. G.0H94.18N: Analysis and Partial Differential Equations, by the Methusalem programme of the Ghent University Special Research Fund (BOF) (grant no. 01M01021) and by the EPSRC grant EP/R003025/2. Also, this research has been funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No.AP19676031) and partially supported by the collaborative research program "Qualitative analysis for nonlocal and fractional models" from Nazarbayev University.

decomposition formula. In particular, let us give several examples of $\mathbb{X}$ for which the condition (1.2) is satisfied with different expressions for $\lambda(r,\omega)$:

(I) Euclidean space $\mathbb{R}^n$: $\lambda(r,\omega)=r^{n-1}$.

(II) Homogeneous groups: $\lambda(r,\omega)=r^{Q-1}$, where Q is the homogeneous dimension of the group. Such groups have been consistently developed by Folland and Stein [14], see also an up-to-date exposition in [12] and [41].

(III) Hyperbolic spaces $\mathbb{H}^n$: $\lambda(r,\omega)=(\sinh r)^{n-1}$.

(IV) Cartan-Hadamard manifolds: Let K_M be the sectional curvature on (M,g). A Riemannian manifold (M,g) is called *a Cartan-Hadamard manifold* if it is complete, simply connected and has non-positive sectional curvature, i.e., the sectional curvature $K_M\leq 0$ along each plane section at each point of M. Let us fix a point $a\in M$ and denote by $\rho(x)=d(x,a)$ the geodesic distance from x to a on M. The exponential map $\exp_a: T_aM\to M$ is a diffeomorphism, see e.g. Helgason [21]. Let $J(\rho,\omega)$ be the density function on M, see e.g. [15]. Then we have the following polar decomposition:

$$\int_M f(x)dx=\int_0^\infty\int_{\mathbb{S}^{n-1}} f(\exp_a(\rho\omega))J(\rho,\omega)\rho^{n-1}d\rho d\omega,$$

so that we have (1.2) with $\lambda(\rho,\omega)=J(\rho,\omega)\rho^{n-1}$.

In [35] and [36], the (direct) integral Hardy inequality on metric measure spaces was established with applications to homogeneous Lie groups, hyperbolic spaces, Cartan-Hadamard manifolds with negative curvature and on general Lie groups with Riemannian distance for $1<p\leq q<\infty$ and $p>q$, respectively. Also, in [23], the authors showed the integral Hardy inequality for $p\in(0,1)$ and $q<0$ on metric measure space. In this paper, we continue the investigation of the integral Hardy inequality on a metric measure space, i.e., we show the reverse integral Hardy inequality with negative exponents.

In [20], Hardy and Littlewood considered the one dimensional fractional integral operator on $(0,\infty)$ given by

$$T_\lambda u(x)=\int_0^\infty \frac{u(y)}{|x-y|^\lambda}dy,\quad 0<\lambda<1, \tag{1.3}$$

where they also showed the following L^q-L^p boundedness of this operator T_λ:

Theorem 1.1. *Let $1<p<q<\infty$ and $u\in L^p(0,\infty)$ with $\frac{1}{q}=\frac{1}{p}+\lambda-1$. Then*

$$\|T_\lambda u\|_{L^q(0,\infty)}\leq C\|u\|_{L^p(0,\infty)}, \tag{1.4}$$

where C is a positive constant independent of u.

The multi-dimensional analogue of (1.3) can be represented by the formula:

$$I_\lambda u(x)=\int_{\mathbb{R}^N}\frac{u(y)}{|x-y|^\lambda}dy,\quad 0<\lambda<N. \tag{1.5}$$

In [42], Sobolev generalised Theorem 1.1 for multi-dimensional case in the following form:

Theorem 1.2. *Let $1<p<q<\infty$, $u\in L^p(\mathbb{R}^N)$ with $\frac{1}{q}=\frac{1}{p}+\frac{\lambda}{N}-1$. Then*

$$\|I_\lambda u\|_{L^q(\mathbb{R}^N)}\leq C\|u\|_{L^p(\mathbb{R}^N)}, \tag{1.6}$$

where C is a positive constant independent of u.

In [43], Stein and Weiss obtained the following radially weighted Hardy-Littlewood-Sobolev inequality, which is known as the Stein-Weiss inequality.

Theorem 1.3. *Let* $0 < \lambda < N$, $1 < p < \infty$, $\alpha < \frac{N(p-1)}{p}$, $\beta < \frac{N}{q}$, $\alpha + \beta \geq 0$ *and* $\frac{1}{q} = \frac{1}{p} + \frac{\lambda+\alpha+\beta}{N} - 1$. *If* $1 < p \leq q < \infty$, *then*

$$\||x|^{-\beta} I_\lambda u\|_{L^q(\mathbb{R}^N)} \leq C \||x|^\alpha u\|_{L^p(\mathbb{R}^N)}, \tag{1.7}$$

where C *is a positive constant independent of* u.

To the best of our knowledge, the Hardy-Littlewood-Sobolev inequality on the Heisenberg group was proved by Folland and Stein in [13] and the best constants of the Hardy-Littlewood-Sobolev inequality, in the Euclidean space and Heisenberg group were obtained in [30] and [11], respectively. Also, in [18], [41] and [22], the authors studied the Hardy-Littlewood-Sobolev and the Stein-Weiss inequalities on Heisenberg and homogeneous Lie groups. Note that systematic studies of different functional inequalities on general homogeneous (Lie) groups were initiated by the papers [33, 37, 39, 40].

The reverse Stein-Weiss inequality in Euclidean setting has the following form:

Theorem 1.4 ([5], Theorem 1)**.** *For* $n \geq 1, p \in (0,1), q < 0, \lambda > 0, 0 \leq \alpha < -\frac{n}{q}$, *and* $0 \leq \beta < -\frac{n}{p'}$ *satisfying* $\frac{1}{p} + \frac{1}{q'} - \frac{\alpha+\beta+\lambda}{n} = 2$, *there is a constant* $C = C(n, \alpha, \beta, \lambda, p, q) > 0$ *such that for any non-negative functions* $f \in L^{q'}(\mathbb{R}^n)$ *and* $0 < \int_{\mathbb{R}^n} g^p(y)dy < \infty$, *we have*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x|^\alpha |x-y|^\lambda f(x)g(y)|y|^\beta dydx \geq C\left(\int_{\mathbb{R}^n} f^{q'}(x)dx\right)^{\frac{1}{q'}}\left(\int_{\mathbb{R}^n} g^p(y)dy\right)^{\frac{1}{p}}, \tag{1.8}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$.

Note, we obtain the reverse Hardy-Littlewood-Sobolev inequality if $\alpha = \beta = 0$. Improved Stein-Weiss inequality was obtained in [4] on the Euclidean upper half-space and in [24] on homogeneous Lie groups. For more results about the reverse Hardy–Littlewood–Sobolev inequality in Euclidean space, we refer the reader to [3] [6], [8], [31] and the references therein. Note that the reverse Hardy-Littlewood-Sobolev and Stein-Weiss inequalities were shown in [24] for the case $p \in (0,1)$ and $q < 0$. In this paper, we show the reverse Hardy-Littlewood-Sobolev and Stein-Weiss inequalities with two negative exponents i.e., $q < p < 0$, which is also new in the Euclidean space.

2. Main result

Firstly, let us denote by $B(a,r)$ a ball in $\mathbb{X}$ with centre a and radius r, i.e.,

$$B(a,r) := \{x \in \mathbb{X} : d(x,a) < r\},$$

where d is the metric on $\mathbb{X}$. Once and for all let us fix some point $a \in \mathbb{X}$, and denote

$$|x|_a := d(a,x). \tag{2.1}$$

Let us recall briefly the reverse Hölder inequality.

Theorem 2.1 ([1], Theorem 2.12, p. 27)**.** *Let* $p < 0$, *so that* $p' = \frac{p}{p-1} > 0$. *If non-negative functions satisfy* $0 < \int_{\mathbb{X}} f^p(x)dx < +\infty$ *and* $0 < \int_{\mathbb{X}} g^{p'}(x)dx < +\infty$, *we have*

$$\int_{\mathbb{X}} f(x)g(x)dx \geq \left(\int_{\mathbb{X}} f^p(x)dx\right)^{\frac{1}{p}}\left(\int_{\mathbb{X}} g^{p'}(x)dx\right)^{\frac{1}{p'}}. \tag{2.2}$$

As the main results of this section, we show the reverse integral Hardy inequality as well as its conjugate.

Theorem 2.2. *Assume that $p, q < 0$ are such that $q \leq p < 0$. Let $\mathbb{X}$ be a metric measure space with a polar decomposition at $a \in \mathbb{X}$. Suppose that $u, v \geq 0$ are locally integrable functions on $\mathbb{X}$. Then the inequality*

$$\left[\int_{\mathbb{X}} \left(\int_{B(a,|x|_a)} f(y)dy\right)^q u(x)dx\right]^{\frac{1}{q}} \geq C_1(p,q)\left(\int_{\mathbb{X}} f^p(x)v(x)dx\right)^{\frac{1}{p}} \tag{2.3}$$

holds for all non-negative real-valued measurable functions f, if and only if

$$0 < D_1 = \inf_{x\neq a} \mathcal{D}_1(|x|_a) = \inf_{x\neq a}\left[\left(\int_{B(a,|x|_a)} u(y)dy\right)^{\frac{1}{q}}\left(\int_{B(a,|x|_a)} v^{1-p'}(y)dy\right)^{\frac{1}{p'}}\right], \tag{2.4}$$

and $\mathcal{D}_1(|x|_a)$ is non-decreasing. Moreover, the largest constant $C_1(p,q)$ in (2.3) satisfies

$$D_1 \geq C_1(p,q) \geq |p|^{\frac{1}{q}}(p')^{\frac{1}{p'}} D_1, \tag{2.5}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Remark 2.3. *In (2.5), by simple calculation, we have that the for the case $q \leq p < 0$*

$$|p|^{\frac{1}{q}}(p')^{\frac{1}{p'}} \leq 1. \tag{2.6}$$

Proof of Theorem 2.2. Let us divide a proof of this theorem to 2 steps.

Step 1. Firstly, let us denote

$$F(s) := \int_{\Sigma_s} \lambda(s,\sigma) f^p(s,\sigma)v(s,\sigma)d\sigma, \tag{2.7}$$

$$V(s) := \int_{\Sigma_s} \lambda(s,\sigma)v^{1-p'}(s,\sigma)d\sigma, \tag{2.8}$$

$$h(t) := \left(\int_0^t \int_{\Sigma_s} \lambda(s,\sigma)v^{1-p'}(s,\sigma)d\sigma ds\right)^{\frac{1}{pp'}}, \tag{2.9}$$

$$H_1(t) := \int_0^t \int_{\Sigma_s} \lambda(s,\sigma)v^{-\frac{p'}{p}}(s,\sigma)h^{-p'}(s)d\sigma ds, \tag{2.10}$$

$$U_1(s) := \int_{\Sigma_s} \lambda(s,\sigma)u(s,\sigma)d\sigma. \tag{2.11}$$

By using the reverse Hölder inequality (2.2) with the polar decomposition, we compute

$$\begin{aligned}
\int_{B(a,|x|_a)} f(y)dy &= \int_{B(a,|x|_a)} [f(y)v^{\frac{1}{p}}(y)h(y)][v^{\frac{1}{p}}(y)h(y)]^{-1}dy \\
&\geq \left(\int_{B(a,|x|_a)} (f(y)v^{\frac{1}{p}}(y)h(y))^p dy\right)^{\frac{1}{p}} \left(\int_{B(a,|x|_a)} (v^{\frac{1}{p}}(y)h(y))^{-p'} dy\right)^{\frac{1}{p'}} \\
&= \left(\int_0^{|x|_a} \int_{\Sigma_s} h^p(s)\lambda(s,\sigma) f^p(s,\sigma) v(s,\sigma) d\sigma ds\right)^{\frac{1}{p}} \\
&\times \left(\int_0^{|x|_a} \int_{\Sigma_s} v^{-\frac{p'}{p}}(s,\sigma) h^{-p'}(s)\lambda(s,\sigma) d\sigma ds\right)^{\frac{1}{p'}} \\
&= \left(\int_0^{|x|_a} h^p(s)F(s)ds\right)^{\frac{1}{p}} H_1^{\frac{1}{p'}}(|x|_a).
\end{aligned} \tag{2.12}$$

Let us calculate $H_1(t)$:

$$\begin{aligned}
H_1(t) &= \int_0^t \int_{\Sigma_s} \lambda(s,\sigma) v^{-\frac{p'}{p}}(s,\sigma) h^{-p'}(s) d\sigma ds \\
&\overset{(2.8)}{=} \int_0^t h^{-p'}(s)V(s)ds \\
&\overset{(2.9)}{=} \int_0^t \left(\int_0^s \int_{\Sigma_z} \lambda(z,\omega) v^{1-p'}(z,\omega) dz d\omega\right)^{-\frac{1}{p}} V(s)ds \\
&\overset{(2.8)}{=} \int_0^t \left(\int_0^s V(z)dz\right)^{-\frac{1}{p}} V(s)ds \\
&= \int_0^t \left(\int_0^s V(z)dz\right)^{-\frac{1}{p}} d_s\left(\int_0^s V(z)dz\right) \\
&= p'\left(\int_0^s V(z)dz\right)^{\frac{1}{p'}} \Big|_0^t \\
&\overset{\frac{1}{p'}>0}{=} p'\left(\int_0^t V(z)dz\right)^{\frac{1}{p'}} \\
&= p' h^p(t).
\end{aligned} \tag{2.13}$$

By combining (2.13) and (2.12), we get

$$\begin{aligned}
\int_{B(a,|x|_a)} f(y)dy &\geq \left(\int_0^{|x|_a} h^p(s)F(s)ds\right)^{\frac{1}{p}} H_1^{\frac{1}{p'}}(|x|_a) \\
&\overset{(2.13)}{=} (p')^{\frac{1}{p'}} \left(\int_0^{|x|_a} h^p(s)F(s)ds\right)^{\frac{1}{p}} h^{\frac{p}{p'}}(|x|_a).
\end{aligned} \tag{2.14}$$

Multiplying by u, integrating over $\mathbb{X}$ with $q<0$ and by using (direct) Minkowski's inequality with $\frac{q}{p}\geq 1$ (see [1], Theorem 2.9, p.26), we compute

$$
\begin{aligned}
&\int_{\mathbb{X}}\left(\int_{B(a,|x|_a)} f(y)dy\right)^q u(x)dx\\
&=\int_0^\infty\int_{\Sigma_r} u(z,\omega)\lambda(z,\omega)\left(\int_0^{|x|_a}\int_{\Sigma_s}\lambda(s,\sigma)f(s,\sigma)dsd\sigma\right)^q dzd\omega\\
&\overset{(2.11)}{=}\int_0^\infty U_1(z)\left(\int_0^z\int_{\Sigma_s}\lambda(s,\sigma)f(s,\sigma)dsd\sigma\right)^q dz\\
&\overset{q<0,\,(2.14)}{\leq}(p')^{\frac{q}{p'}}\int_0^\infty U_1(z)\left(\int_0^z h^p(s)F(s)ds\right)^{\frac{q}{p}} h^{\frac{qp}{p'}}(z)dz\\
&=(p')^{\frac{q}{p'}}\int_0^\infty U_1(z)\left(\int_0^\infty \chi_{[0,z]}h^p(s)F(s)ds\right)^{\frac{q}{p}} h^{\frac{qp}{p'}}(z)dz\\
&\leq(p')^{\frac{q}{p'}}\left[\int_0^\infty h^p(s)F(s)\left(\int_s^\infty U_1(z)h^{\frac{qp}{p'}}(z)dz\right)^{\frac{p}{q}}ds\right]^{\frac{q}{p}},
\end{aligned}
\tag{2.15}
$$

where $\chi_{[0,r]}$ is the cut-off function. At the same time, one can also estimate

$$
\begin{aligned}
h^{\frac{pq}{p'}}(t)&=\left[\left(\int_0^t\int_{\Sigma_s}\lambda(s,\sigma)v^{1-p'}(s,\sigma)dsd\sigma\right)^{\frac{q}{p'}}\right]^{\frac{1}{p'}}\\
&\overset{(2.8)}{=}\left[\left(\int_0^t V(s)ds\right)^{\frac{q}{p'}}\right]^{\frac{1}{p'}}\\
&=\left[\left(\int_0^t V(s)ds\right)^{\frac{q}{p'}}\left(\int_0^t U_1(s)ds\right)\left(\int_0^t U_1(s)ds\right)^{-1}\right]^{\frac{1}{p'}}\\
&=\mathcal{D}_1^{\frac{q}{p'}}(|t|_a)\left(\int_0^t U_1(s)ds\right)^{-\frac{1}{p'}},
\end{aligned}
\tag{2.16}
$$

where $\mathcal{D}_1(|t|_a) := \left(\int_0^t V(s)ds\right)^{\frac{1}{p'}}\left(\int_0^t U_1(s)ds\right)^{\frac{1}{q}}$. By using this fact and since $\mathcal{D}_1(|x|_a)$ is non-decreasing, we get

$$
\begin{aligned}
&\int_{\mathbb{X}}\left(\int_{B(a,|x|_a)} f(y)dy\right)^q u(x)dx\\
&\overset{(2.15)}{\leq}(p')^{\frac{q}{p'}}\left[\int_0^\infty h^p(s)F(s)\left(\int_s^\infty U_1(r)h^{\frac{qp}{p'}}(r)dr\right)^{\frac{p}{q}}ds\right]^{\frac{q}{p}}
\end{aligned}
$$

$$\overset{\frac{p}{q}>0,\,(2.16)}{\leq} (p')^{\frac{q}{p'}} \left[\int_0^\infty h^p(s) F(s) \mathcal{D}_1^{\frac{p}{p'}}(s) \left(\int_s^\infty U_1(r) \left(\int_0^r U_1(z) dz \right)^{-\frac{1}{p'}} dr \right)^{\frac{p}{q}} ds \right]^{\frac{q}{p}}$$

$$= (p')^{\frac{q}{p'}} \left[\int_0^\infty h^p(s) F(s) \mathcal{D}_1^{\frac{p}{p'}}(s) \left(\int_s^\infty d_r \left[p \left(\int_0^r U_1(z) dz \right)^{\frac{1}{p}} \right] \right)^{\frac{p}{q}} ds \right]^{\frac{q}{p}}$$

$$= (p')^{\frac{q}{p'}} \left[\int_0^\infty h^p(s) F(s) \mathcal{D}_1^{\frac{p}{p'}}(s) \left(p \left(\int_0^\infty U_1(z) dz \right)^{\frac{1}{p}} - p \left(\int_0^s U_1(z) dz \right)^{\frac{1}{p}} \right)^{\frac{p}{q}} ds \right]^{\frac{q}{p}}$$

$$\overset{p<0}{\leq} (-p)(p')^{\frac{q}{p'}} \left[\int_0^\infty h^p(s) F(s) \mathcal{D}_1^{\frac{p}{p'}}(s) \left(\int_0^s U_1(z) dz \right)^{\frac{1}{q}} ds \right]^{\frac{q}{p}} \tag{2.17}$$

$$\overset{(2.9)}{=} (-p)(p')^{\frac{q}{p'}} \left[\int_0^\infty F(s) \mathcal{D}_1^{1+\frac{p}{p'}}(s) ds \right]^{\frac{q}{p}}$$

$$= (-p)(p')^{\frac{q}{p'}} \left[\int_0^\infty F(s) \mathcal{D}_1^{p}(s) ds \right]^{\frac{q}{p}}$$

$$\overset{p<0}{\leq} (-p)(p')^{\frac{q}{p'}} D_1^q \left[\int_0^\infty F(s) ds \right]^{\frac{q}{p}}$$

$$\overset{(2.7)}{=} (-p)(p')^{\frac{q}{p'}} D_1^q \left(\int_{\mathbb{X}} f^p(x) v(x) dx \right)^{\frac{q}{p}}$$

$$= |p|(p')^{\frac{q}{p'}} D_1^q \left(\int_{\mathbb{X}} f^p(x) v(x) dx \right)^{\frac{q}{p}}.$$

Finally, we obtain

$$\left(\int_{\mathbb{X}} \left(\int_{B(a,|x|_a)} f(y) dy \right)^q u(x) dx \right)^{\frac{1}{q}} \geq |p|^{\frac{1}{q}} (p')^{\frac{1}{p'}} D_1 \left(\int_{\mathbb{X}} f^p(x) v(x) dx \right)^{\frac{1}{p}}. \tag{2.18}$$

Hence, it follows that (2.3) holds with $C_1(p,q) \geq |p|^{\frac{1}{q}} (p')^{\frac{1}{p'}} D_1$, proving one of the relations in (2.5).

Step 2. Now it remains to show that (2.3) yields (2.4). Let us fix $t > 0$ and denote the following function:

$$f(x) := \begin{cases} v^{1-p'}(x), & \text{if } |x|_a \leq t, \\ \alpha f_1(x), & \text{if } |x|_a > t, \end{cases} \tag{2.19}$$

where f_1 is any function satisfying $\int_{B(a,|x|_a)} f_1(y) dy < \infty$ and $\int_{|x|_a \geq t} v(x) f_1^p(x) dx < \infty$, and $\alpha > 0$. Then we compute

$$C_1(p,q) \leq \left[\int_{\mathbb{X}} \left(\int_{|y|_a \leq |x|_a} f(y) dy \right)^q u(x) dx \right]^{\frac{1}{q}} \left[\int_{\mathbb{X}} f^p(y) v(y) dy \right]^{-\frac{1}{p}}$$

$$= \left[\int_{\mathbb{X}} \left(\int_{|y|_a \le |x|_a} f(y)dy\right)^q u(x)dx\right]^{\frac{1}{q}} \left[\int_{|y|_a\le t} v^{1-p'}(y)dy + \alpha^p \int_{|y|_a>t} v(y)f_1^p(y)dy\right]^{-\frac{1}{p}}$$

$$\overset{q<0}{\le} \left[\int_{|x|_a\le t} \left(\int_{|y|_a \le |x|_a} f(y)dy\right)^q u(x)dx\right]^{\frac{1}{q}} \left[\int_{|y|_a\le t} v^{1-p'}(y)dy + \alpha^p \int_{|y|_a>t} v(y)f_1^p(y)dy\right]^{-\frac{1}{p}}$$

$$= \left[\int_{|x|_a\le t} \left(\int_{|y|_a \le |x|_a} v^{1-p'}(y)dy\right)^q u(x)dx\right]^{\frac{1}{q}} \left[\int_{|y|_a\le t} v^{1-p'}(y)dy + \alpha^p \int_{|y|_a>t} v(y)f_1^p(y)dy\right]^{-\frac{1}{p}}$$

$$\overset{q<0}{\le} \left[\int_{|x|_a\le t} \left(\int_{|y|_a \le t} v^{1-p'}(y)dy\right)^q u(x)dx\right]^{\frac{1}{q}} \left[\int_{|y|_a\le t} v^{1-p'}(y)dy + \alpha^p \int_{|y|_a>t} v(y)f_1^p(y)dy\right]^{-\frac{1}{p}}$$

$$= \left[\int_{|x|_a\le t} u(x)dx\right]^{\frac{1}{q}} \left[\int_{|y|_a\le t} v^{1-p'}(y)dy\right] \left[\int_{|y|_a\le t} v^{1-p'}(y)dy + \alpha^p \int_{|y|_a>t} v(y)f_1^p(y)dy\right]^{-\frac{1}{p}}.$$

Summarising above facts with $q \le p < 0$ and taking limit as $\alpha \to 0$, we obtain

$$C_1(p,q) \le \left[\int_{|y|_a\le t} v^{1-p'}(y)dy\right]^{\frac{1}{p'}} \left[\int_{|x|_a\le t} u(x)dx\right]^{\frac{1}{q}}. \tag{2.20}$$

Finally, we get $C_1(p,q) \le D_1$. □

Now let us prove the conjugate integral Hardy inequality.

Theorem 2.4. *Assume that $p, q < 0$ such that $q \le p < 0$. Let $\mathbb{X}$ be a metric measure space with a polar decomposition at $a \in \mathbb{X}$. Suppose that $u, v \ge 0$ are locally integrable functions on $\mathbb{X}$. Then the inequality*

$$\left[\int_{\mathbb{X}} \left(\int_{\mathbb{X}\setminus B(a,|x|_a)} f(y)dy\right)^q u(x)dx\right]^{\frac{1}{q}} \ge C_2(p,q) \left(\int_{\mathbb{X}} f^p(x)v(x)dx\right)^{\frac{1}{p}} \tag{2.21}$$

holds for all non-negative real-valued measurable functions f, if and only if

$$0 < D_2 = \inf_{x\ne a} \mathcal{D}_2(|x|_a) = \inf_{x\ne a} \left[\left(\int_{\mathbb{X}\setminus B(a,|x|_a)} u(y)dy\right)^{\frac{1}{q}} \left(\int_{\mathbb{X}\setminus B(a,|x|_a)} v^{1-p'}(y)dy\right)^{\frac{1}{p'}}\right], \tag{2.22}$$

and $\mathcal{D}_2(|x|_a)$ is non-increasing. Moreover, the largest constant $C_2(p,q)$ satisfies

$$D_2 \ge C_2(p,q) \ge |p|^{\frac{1}{q}} (p')^{\frac{1}{p'}} D_2, \tag{2.23}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. The main idea of the proof of this theorem is similar to that of Theorem 2.2 with the only difference that $\mathcal{D}_2(|x|_a)$ is non-increasing, so we omit the details. □

3. Consequences on homogeneous groups

In this section, we consider several consequences of the main results for the reverse integral Hardy, Hardy-Littlewood-Sobolev and Stein-Weiss inequalities on homogeneous groups.

Let us recall that a Lie group (on $\mathbb{R}^n$) $\mathbb{G}$ with the dilation

$$D_\lambda(x) := (\lambda^{\nu_1}x_1, \dots, \lambda^{\nu_n}x_n),\ \nu_1, \dots, \nu_n > 0,\ D_\lambda : \mathbb{R}^n \to \mathbb{R}^n,$$

which is an automorphism of the group $\mathbb{G}$ for each $\lambda > 0$, is called a *homogeneous (Lie) group*. For simplicity, throughout this paper we use the notation λx for the dilation $D_\lambda(x)$. The homogeneous dimension of the homogeneous group $\mathbb{G}$ is denoted by $Q := \nu_1 + \ldots + \nu_n$. Also, in this paper we denote a homogeneous quasi-norm on $\mathbb{G}$ by $|x|$, which is a continuous non-negative function

$$\mathbb{G} \ni x \mapsto |x| \in [0, \infty), \tag{3.1}$$

with the following properties

i) $|x| = |x^{-1}|$ for all $x \in \mathbb{G}$,
ii) $|\lambda x| = \lambda |x|$ for all $x \in \mathbb{G}$ and $\lambda > 0$,
iii) $|x| = 0$ if and only if $x = 0$.

Let us also recall the following well-known fact about quasi-norms.

Proposition 3.1 (e.g. [12], Proposition 3.1.38 and [38], Proposition 1.2.4)**.** *If $|\cdot|$ is a homogeneous quasi-norm on $\mathbb{G}$, there exists $C > 0$ such that for every $x, y \in \mathbb{G}$, we have*

$$|xy| \leq C(|x| + |y|). \tag{3.2}$$

The following polarisation formula on homogeneous Lie groups will be used in our proofs: there is a (unique) positive Borel measure σ on the unit quasi-sphere $\mathfrak{S} := \{x \in \mathbb{G} : |x| = 1\}$, so that for every $f \in L^1(\mathbb{G})$ we have

$$\int_{\mathbb{G}} f(x)dx = \int_0^\infty \int_{\mathfrak{S}} f(ry) r^{Q-1} d\sigma dr. \tag{3.3}$$

We refer to [14] for the original appearance of such groups, to [12] and to [38] for a recent comprehensive treatment. Let us define quasi-ball centered at x with radius r in the following form:

$$B(x, r) := \{y \in \mathbb{G} : |x^{-1}y| < r\}. \tag{3.4}$$

3.1. **Reverse integral Hardy inequality.**

In this sub-section we show the reverse integral Hardy inequality on homogeneous Lie groups.

Theorem 3.2. *Let $\mathbb{G}$ be a homogeneous Lie group of homogeneous dimension Q with a quasi-norm $|\cdot|$. Assume that $q \leq p < 0$ and $\alpha, \beta \in \mathbb{R}$. Then the reverse integral Hardy inequality*

$$\left[\int_{\mathbb{G}} \left(\int_{B(0,|x|)} f(y)dy\right)^q |x|^\alpha dx\right]^{\frac{1}{q}} \geq C_1 \left(\int_{\mathbb{G}} f^p(x)|x|^\beta dx\right)^{\frac{1}{p}}, \tag{3.5}$$

holds for some $C_1 > 0$ and for all non-negative measurable functions f, if $\alpha + Q > 0$, $\beta(1-p') + Q > 0$ and $\frac{Q+\alpha}{q} + \frac{Q+\beta(1-p')}{p'} = 0$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, the biggest constant C_1 for (3.4) satisfies

$$\left(\frac{|\mathfrak{S}|}{\alpha + Q}\right)^{\frac{1}{q}} \left(\frac{|\mathfrak{S}|}{Q + \beta(1-p')}\right)^{\frac{1}{p'}} \geq C_1 \geq |p|^{\frac{1}{q}} (p')^{\frac{1}{p'}} \left(\frac{|\mathfrak{S}|}{\alpha + Q}\right)^{\frac{1}{q}} \left(\frac{|\mathfrak{S}|}{Q + \beta(1-p')}\right)^{\frac{1}{p'}}.$$

Proof. Let us show that the condition (2.4) is satisfied with $u(x) = |x|^\alpha$ and $v(x) = |x|^\beta$. We calculate the first integral in (2.4):

$$\int_{B(0,|x|)} u(y)dy = \int_{B(0,|x|)} |y|^\alpha dy \overset{(3.2)}{=} \int_0^{|x|} \int_{\mathfrak{S}} r^\alpha r^{Q-1} dr d\sigma = \frac{|\mathfrak{S}|}{Q + \alpha} |x|^{Q+\alpha}, \tag{3.6}$$

where $|\mathfrak{S}|$ is the area of the unit quasi-sphere in $\mathbb{G}$. Then,

$$\begin{aligned}\int_{B(0,|x|)} v^{1-p'}(y)dy &= \int_{B(0,|x|)} |y|^{\beta(1-p')}dy \\ &\stackrel{(3.2)}{=} \int_0^{|x|}\int_{\mathfrak{S}} r^{\beta(1-p')}r^{Q-1}drd\sigma \\ &= \frac{|\mathfrak{S}|}{Q+\beta(1-p')}|x|^{Q+\beta(1-p')}.\end{aligned}$$

Finally, by using above facts and $\frac{Q+\alpha}{q}+\frac{Q+\beta(1-p')}{p'}=0$, we have

$$\mathcal{D}_1(|x|) = \left(\frac{|\mathfrak{S}|}{\alpha+Q}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}}\left[|x|^{\frac{Q+\alpha}{q}+\frac{Q+\beta(1-p')}{p'}}\right] = \left(\frac{|\mathfrak{S}|}{\alpha+Q}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}},$$

which shows that $\mathcal{D}_1(|x|)$ is a non-decreasing function. Then

$$D_1 = \inf_{x\neq a}\mathcal{D}_1(|x|) = \left(\frac{|\mathfrak{S}|}{\alpha+Q}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}} > 0.$$

Therefore, by (2.5) we have

$$D_1 \geq C_1 \geq |p|^{\frac{1}{q}}(p')^{\frac{1}{p'}}D_1,$$

where $D_1 = \left(\frac{|\mathfrak{S}|}{\alpha+Q}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}}$ thereby, completing the proof. ☐

Now we obtain the conjugate reverse integral Hardy inequality on homogeneous Lie groups.

Theorem 3.3. *Let $\mathbb{G}$ be a homogeneous Lie group of homogeneous dimension Q with a quasi-norm $|\cdot|$. Assume that $q \leq p < 0$ and $\alpha, \beta \in \mathbb{R}$. Then the reverse conjugate integral Hardy inequality*

$$\left[\int_{\mathbb{G}}\left(\int_{\mathbb{G}\setminus B(0,|x|)} f(y)dy\right)^q |x|^\alpha dx\right]^{\frac{1}{q}} \geq C_2\left(\int_{\mathbb{G}} f^p(x)|x|^\beta dx\right)^{\frac{1}{p}}, \tag{3.7}$$

holds for some $C_2 > 0$ and for all non-negative measurable functions f, if $\alpha+Q<0$, $\beta(1-p')+Q<0$ and $\frac{Q+\alpha}{q}+\frac{Q+\beta(1-p')}{p'}=0$. Moreover, the biggest constant C_2 for (3.6) satisfies

$$\left(\frac{|\mathfrak{S}|}{|\alpha+Q|}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{|Q+\beta(1-p')|}\right)^{\frac{1}{p'}} \geq C_2 \geq |p|^{\frac{1}{q}}(p')^{\frac{1}{p'}}\left(\frac{|\mathfrak{S}|}{|\alpha+Q|}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{|Q+\beta(1-p')|}\right)^{\frac{1}{p'}}.$$

Proof. Proof of this theorem is similar to the previous case, where we use Theorem 2.4 instead of of Theorem 2.2. ☐

3.2. **The reverse Hardy-Littlewood-Sobolev inequality and Stein-Weiss inequality.** In this subsection we obtain the reverse Hardy-Littlewood-Sobolev inequality and Stein-Weiss inequality on Euclidean space and homogeneous Lie groups.

Let us introduce the Riesz operator on homogeneous Lie groups in the following form:

$$I_{\lambda,|\cdot|}u(x) = |x|^\lambda * u = \int_{\mathbb{G}} |y^{-1}x|^\lambda u(y)dy, \quad \lambda < 0, \tag{3.8}$$

where $*$ is the convolution. Hence, by taking $\mathbb{G} = (\mathbb{R}^n, +)$, $Q = n$ and $|\cdot| = |\cdot|_E$ ($|\cdot|_E$ is the Euclidean distance), we get the Riesz operator on Euclidean space:

$$I_{\lambda,|\cdot|_E} u(x) = |x|_E^{\lambda} * u = \int_{\mathbb{G}} |x-y|_E^{\lambda} u(y) dy, \quad \lambda < 0. \tag{3.9}$$

Firstly, let us present the Hardy-Littlewood-Sobolev inequality on Euclidean space.

Theorem 3.4 (The reverse Hardy-Littlewood-Sobolev inequality on $\mathbb{R}^n$)**.** *Assume that $n \geq 1$, $q < p < 0$, $\lambda < 0$ such that $\frac{1}{p'} + \frac{1}{q} + \frac{\lambda}{n} = 0$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Then for all non-negative functions $f \in L^{q'}(\mathbb{R}^n)$ and $0 < \int_{\mathbb{R}^n} h^p(x)dx < \infty$, we get*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)|x-y|_E^{\lambda} h(y) dx dy \geq C \left(\int_{\mathbb{R}^n} f^{q'}(x) dx \right)^{\frac{1}{q'}} \left(\int_{\mathbb{R}^n} h^p(x) dx \right)^{\frac{1}{p}}, \tag{3.10}$$

where C is a positive constant independent of f and h.

Proof. By using the reverse Hölder inequality with $\frac{1}{q} + \frac{1}{q'} = 1$, we calculate

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)|x-y|_E^{\lambda} h(y) dy dx \overset{(2.2)}{\geq} \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |x-y|_E^{\lambda} h(y) dy \right)^q dx \right)^{\frac{1}{q}} \|f\|_{L^{q'}(\mathbb{R}^n)}.$$

Thus for (3.9), it is enough to show that

$$\left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |x-y|_E^{\lambda} h(y) dy \right)^q dx \right)^{\frac{1}{q}} \geq C \left(\int_{\mathbb{R}^n} h^p(x) dx \right)^{\frac{1}{p}}.$$

By direct calculation, we have

$$\left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |x-y|_E^{\lambda} h(y) dy \right)^q dx \right)^{\frac{1}{q}} \overset{q<0}{\geq} \left(\int_{\mathbb{R}^n} \left(\int_{B_E(0,|x|_E)} |x-y|_E^{\lambda} h(y) dy \right)^q dx \right)^{\frac{1}{q}}, \tag{3.11}$$

where $B_E(0, |x|_E)$ is the Euclidean ball centered at 0 with radius $|x|_E$. By using $|y|_E \leq |x|_E$, we get

$$|x-y|_E \leq |x|_E + |y|_E \leq 2|x|_E. \tag{3.12}$$

Then for any $\lambda < 0$, we have

$$\begin{aligned} \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |x-y|_E^{\lambda} h(y) dy \right)^q dx \right)^{\frac{1}{q}} &\overset{q<0}{\geq} \left(\int_{\mathbb{R}^n} \left(\int_{B_E(0,|x|_E)} |x-y|_E^{\lambda} h(y) dy \right)^q dx \right)^{\frac{1}{q}} \\ &\geq 2^{\lambda} \left(\int_{\mathbb{R}^n} |x|_E^{\lambda q} \left(\int_{B_E(0,|x|_E)} h(y) dy \right)^q dx \right)^{\frac{1}{q}}. \end{aligned} \tag{3.13}$$

If condition (2.4) in Theorem 2.2 with $u(x) = |x|^{\lambda q}$ and $v(x) = 1$ in (2.3) is satisfied, then we have

$$\left(\int_{\mathbb{R}^n} |x|_E^{\lambda q} \left(\int_{B_E(0,|x|_E)} h(y) dy \right)^q dx \right)^{\frac{1}{q}} \geq C \left(\int_{\mathbb{R}^n} h^p(x) dx \right)^{\frac{1}{p}}.$$

Let us show that the condition (2.4) is satisfied. From the assumption, we have

$$0=\frac{1}{p'}+\frac{1}{q}+\frac{\lambda}{n}\overset{\frac{1}{p'}>0}{>}\frac{1}{q}+\frac{\lambda}{n}, \tag{3.14}$$

which means $n+\lambda q>0$. By using this fact, we obtain

$$\begin{aligned}\int_{B_E(0,|x|_E)}u(y)dy&=\int_{B(0,|x|_E)}|y|_E^{\lambda q}dy\\ &\overset{(3.2)}{=}\int_0^{|x|_E}\int_{\mathfrak{S}}r^{\lambda q}r^{n-1}drd\sigma\\ &=\frac{|\mathfrak{S}|}{n+\lambda q}|x|_E^{n+\lambda q},\end{aligned} \tag{3.15}$$

and

$$\int_{B_E(0,|x|_E)}v^{1-p'}(y)dy=\int_{B_E(0,|x|_E)}1dy=|\mathfrak{S}||x|_E^n. \tag{3.16}$$

Finally, by using the assumption $\frac{1}{p'}+\frac{1}{q}+\frac{\lambda}{n}=0$,

$$\mathcal{D}_1(|x|_E)=\left(\frac{|\mathfrak{S}|}{n+\lambda q}\right)^{\frac{1}{q}}(|\mathfrak{S}|)^{\frac{1}{p'}}|x|_E^{\frac{n}{p'}+\frac{n+\lambda q}{q}}=\left(\frac{|\mathfrak{S}|}{n+\lambda q}\right)^{\frac{1}{q}}|\mathfrak{S}|^{\frac{1}{p'}}, \tag{3.17}$$

which implies, $\mathcal{D}_1(|x|_E)$ is a non-decreasing function. Thus,

$$D_1=\inf_{x\neq 0}\mathcal{D}_1(|x|_E)=\left(\frac{|\mathfrak{S}|}{n+\lambda q}\right)^{\frac{1}{q}}|\mathfrak{S}|^{\frac{1}{p'}}>0,$$

completing the proof. □

Remark 3.5. *Inequality (3.10) seems to be new even in the Euclidean space.*

Also, let us now present the reverse Hardy-Littlewood-Sobolev inequality on $\mathbb{G}$.

Theorem 3.6 (The reverse Hardy-Littlewood-Sobolev inequality on $\mathbb{G}$)**.** *Let $\mathbb{G}$ be a homogeneous Lie group of homogeneous dimension $Q\geq 1$ with arbitrary quasi-norm $|\cdot|$. Assume that $q<p<0$, $\lambda<0$ such that $\frac{1}{p'}+\frac{1}{q}+\frac{\lambda}{Q}=0$, where $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$. Then for all non-negative functions $f\in L^{q'}(\mathbb{G})$ and $0<\int_{\mathbb{G}}h^p(x)dx<\infty$, we get*

$$\int_{\mathbb{G}}\int_{\mathbb{G}}f(x)|y^{-1}x|^{\lambda}h(y)dxdy\geq C\left(\int_{\mathbb{G}}f^{q'}(x)dx\right)^{\frac{1}{q'}}\left(\int_{\mathbb{G}}h^p(x)dx\right)^{\frac{1}{p}},$$

where C is a positive constant independent of f and h.

Proof. The proof of this theorem is similar to Theorem 3.4, but here we use Proposition 3.1 and the polar decomposition formula (3.3). □

Let us now show the reverse Stein-Weiss inequality on $\mathbb{R}^n$.

Theorem 3.7 (The reverse Stein-Weiss inequality on $\mathbb{R}^n$)**.** *Assume that $n \geq 1$, $q \leq p < 0$, $\lambda < 0$, and $\frac{1}{p'} + \frac{1}{q} + \frac{\alpha+\beta+\lambda}{n} = 0$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Then for all non-negative functions $f \in L^{q'}(\mathbb{R}^n)$ and $0 < \int_{\mathbb{R}^n} h^p(x)dx < \infty$, we have*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x|_E^{\alpha} f(x)|x-y|_E^{\lambda} h(y)|y|_E^{\beta} dxdy \geq C\left(\int_{\mathbb{R}^n} f^{q'}(x)dx\right)^{\frac{1}{q'}}\left(\int_{\mathbb{R}^n} h^p(x)dx\right)^{\frac{1}{p}}, \tag{3.18}$$

if one of the following conditions is satisfied:

(a) $\beta > -\frac{n}{p'}$;
(b) $\alpha > -\frac{n}{q}$.

Proof. Similarly to Theorem 3.3, by using the reverse Hölder inequality with $\frac{1}{q}+\frac{1}{q'} = 1$, we calculate

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x|_E^{\alpha} f(x)|x-y|_E^{\lambda} h(y)|y|_E^{\beta} dydx &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} |x|_E^{\alpha}|x-y|_E^{\lambda} h(y)|y|_E^{\beta} dy\right) f(x)dx\\ &\geq \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} |x|_E^{\alpha}|x-y|_E^{\lambda} h(y)|y|_E^{\beta} dy\right)^q dx\right)^{\frac{1}{q}} \|f\|_{L^{q'}(\mathbb{R}^n)}.\end{aligned}$$

Thus for (3.18), it is enough to show that

$$\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} |x|_E^{\alpha}|x-y|_E^{\lambda} h(y)|y|_E^{\beta} dy\right)^q dx\right)^{\frac{1}{q}} \geq C\left(\int_{\mathbb{R}^n} h^p(x)dx\right)^{\frac{1}{p}},$$

and by substituting $z(y) = h(y)|y|_E^{\beta}$, this is equivalent to

$$\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} |x|_E^{\alpha}|x-y|_E^{\lambda} z(y)dy\right)^q dx \leq C\left(\int_{\mathbb{R}^n} |y|_E^{-\beta p} z^p(x)dx\right)^{\frac{q}{p}}.$$

We have that

$$\int_{\mathbb{R}^n} |x|_E^{\alpha}|x-y|_E^{\lambda} z(y)dy \geq \int_{B_E(0,|x|_E)} |x|_E^{\alpha}|x-y|_E^{\lambda} z(y)dy,$$

then

$$\left(\int_{\mathbb{R}^n} |x|_E^{\alpha}|x-y|_E^{\lambda} z(y)dy\right)^q \overset{q<0}{\leq} \left(\int_{B_E(0,|x|_E)} |x|_E^{\alpha}|x-y|_E^{\lambda} z(y)dy\right)^q.$$

Therefore, we obtain

$$\begin{aligned}&\left(\int_{\mathbb{R}^n} |x|_E^{\alpha q}\left(\int_{\mathbb{R}^n} |x-y|_E^{\lambda} z(y)dy\right)^q dx\right)^{\frac{1}{q}}\\ &\qquad \overset{q<0}{\geq} \left(\int_{\mathbb{R}^n} |x|_E^{\alpha q}\left(\int_{B_E(0,|x|_E)} |x-y|_E^{\lambda} z(y)dy\right)^q dx\right)^{\frac{1}{q}} := I_1^{\frac{1}{q}}.\end{aligned} \tag{3.19}$$

Similarly to (3.19), we have

$$(3.20)\quad \left(\int_{\mathbb{R}^n} |x|_E^{\alpha q}\left(\int_{\mathbb{R}^n} |x-y|_E^{\lambda} z(y)dy\right)^q dx\right)^{\frac{1}{q}}$$

$$\overset{q<0}{\geq} \left(\int_{\mathbb{R}^n} |x|_E^{\alpha q}\left(\int_{\mathbb{R}^n\setminus B_E(0,|x|_E)} |x-y|_E^{\lambda} z(y)dy\right)^q dx\right)^{\frac{1}{q}} := I_2^{\frac{1}{q}}.$$

From (3.19)-(3.20), we obtain

$$\left(\int_{\mathbb{R}^n} |x|_E^{\alpha q}\left(\int_{\mathbb{R}^n} |x-y|_E^{\lambda} z(y)dy\right)^q dx\right)^{\frac{1}{q}} \geq I_1^{\frac{1}{q}}, \tag{3.21}$$

and

$$\left(\int_{\mathbb{R}^n} |x|_E^{\alpha q}\left(\int_{\mathbb{R}^n} |x-y|_E^{\lambda} z(y)dy\right)^q dx\right)^{\frac{1}{q}} \geq I_2^{\frac{1}{q}}. \tag{3.22}$$

Step 1. Let us prove (a) for (3.21). By using $|y|_E \leq |x|_E$, we get

$$|x-y|_E \leq |x|_E + |y|_E \leq 2|x|_E. \tag{3.23}$$

Then for any $\lambda < 0$, we have

$$2^{\lambda}|x|_E^{\lambda} \leq |x-y|_E^{\lambda}.$$

Therefore, we get

$$I_1 = \int_{\mathbb{R}^n} |x|_E^{\alpha q}\left(\int_{B_E(0,|x|_E)} |x-y|_E^{\lambda} z(y)dy\right)^q dx \leq 2^{\lambda q}\int_{\mathbb{R}^n} |x|_E^{(\alpha+\lambda)q}\left(\int_{B_E(0,|x|_E)} z(y)dy\right)^q dx.$$

If condition (2.4) in Theorem 2.2 with $u(x) = |x|_E^{(\alpha+\lambda)q}$ and $v(y) = |y|_E^{-\beta p}$ in (2.3) is satisfied, then we have

$$I_1 \leq C\int_{\mathbb{R}^n}\left(\int_{B_E(0,|x|_E)} z(y)dy\right)^q |x|_E^{(\alpha+\lambda)q}dx \leq C\left(\int_{\mathbb{R}^n} |y|_E^{-\beta p} z^p(y)dy\right)^{\frac{q}{p}}.$$

Let us verify that the condition (2.4) holds. By using the assumption $\beta > -\frac{n}{p'}$, we obtain

$$0 = \frac{1}{p'} + \frac{1}{q} + \frac{\alpha+\beta+\lambda}{n} > \frac{1}{q} + \frac{\alpha+\lambda}{n},$$

that is, $\frac{n+(\alpha+\lambda)q}{nq} < 0$, or $n+(\alpha+\lambda)q > 0$,. Then, we get

$$\left(\int_{B_E(0,|x|_E)} u(y)dy\right)^{\frac{1}{q}} = \left(\int_{B_E(0,|x|_E)} |y|_E^{(\alpha+\lambda)q}dy\right)^{\frac{1}{q}} = \left(\frac{|\mathfrak{S}|}{n+(\alpha+\lambda)q)}\right)^{\frac{1}{q}}|x|^{\frac{n+(\alpha+\lambda)q}{q}}.$$

Since $\beta > -\frac{n}{p'}$, we have

$$-\beta p(1-p') + n = \beta p' + n > 0.$$

Thus $-\beta p(1-p')+n>0$. Then, a direct computation gives

$$\left(\int_{B_E(0,|x|_E)} v^{1-p'}(y)dy\right)^{\frac{1}{p'}} = \left(\int_{B_E(0,|x|_E)} |y|_E^{-\beta p(1-p')}dy\right)^{\frac{1}{p'}} = \left(\frac{|\mathfrak{S}|}{\beta p'+n}\right)^{\frac{1}{p'}} |x|_E^{\frac{\beta p'+n}{p'}}. \tag{3.24}$$

Therefore by using $\frac{1}{p'}+\frac{1}{q}+\frac{\alpha+\beta+\lambda}{n}=0$, we have

$$\mathcal{D}_1(|x|_E) = \left(\int_{B_E(0,|x|_E)} u(y)dy\right)^{\frac{1}{q}} \left(\int_{B_E(0,|x|_E)} v^{1-p'}(y)dy\right)^{\frac{1}{p'}} = \left(\frac{|\mathfrak{S}|}{n+(\alpha+\lambda)q}\right)^{\frac{1}{q}} \left(\frac{|\mathfrak{S}|}{\beta p'+n}\right)^{\frac{1}{p'}},$$

which means $\mathcal{D}_1(|x|_E)$ is a non-decreasing function. Therefore,

$$D_1 = \inf_{x\neq 0} \mathcal{D}_1(|x|_E) = \left(\frac{|\mathfrak{S}|}{n+(\alpha+\lambda)q}\right)^{\frac{1}{q}} \left(\frac{|\mathfrak{S}|}{\beta p'+n}\right)^{\frac{1}{p'}} > 0. \tag{3.25}$$

Then by using (2.3), we obtain

$$I_1^{\frac{1}{q}} \geq C\left(\int_{\mathbb{R}^n} |y|_E^{-\beta p} z^p(y)dy\right)^{\frac{1}{p}} = C\left(\int_{\mathbb{R}^n} h^p(y)dy\right)^{\frac{1}{p}}. \tag{3.26}$$

Step 2. Let us prove (b) for (3.22). From $|x|_E \leq |y|_E$, we calculate

$$|x-y|_E \leq |x|_E + |y|_E \leq 2|y|_E,$$

then

$$|x-y|_E^{\lambda} \geq C|y|_E^{\lambda},$$

where $C>0$. Then, if condition (2.22) with $u(x)=|x|_E^{\alpha q}$ and $v(y)=|y|_E^{-(\beta+\lambda)p}$ is satisfied, then we have

$$I_2 \leq C\int_{\mathbb{R}^n} |x|_E^{\alpha q}\left(\int_{\mathbb{R}^n\setminus B_E(0,|x|_E)} z(y)|y|_E^{\lambda}dy\right)^q dx \leq C\left(\int_{\mathbb{R}^n} |y|_E^{-\beta p} z^p(y)dy\right)^{\frac{q}{p}}.$$

Now let us check that the condition (2.22) holds. We have

$$\left(\int_{\mathbb{R}^n\setminus B_E(0,|x|_E)} u(y)dy\right)^{\frac{1}{q}} = \left(\int_{\mathbb{R}^n\setminus B_E(0,|x|_E)} |y|_E^{\alpha q}dy\right)^{\frac{1}{q}} = \left(\int_{|x|_E}^{\infty}\int_{\mathfrak{S}} r^{\alpha q} r^{n-1} dr d\sigma\right)^{\frac{1}{q}} = \left(\frac{|\mathfrak{S}|}{|n+\alpha q|}\right)^{\frac{1}{q}} |x|_E^{\frac{n+\alpha q}{q}},$$

where $n+\alpha q<0$. From $\alpha > -\frac{n}{q}$, we have $0=\frac{1}{p'}+\frac{1}{q}+\frac{\alpha+\beta+\lambda}{n} > \frac{1}{p'}+\frac{\beta+\lambda}{n}$, then

$$(\beta+\lambda)p'+n<0. \tag{3.27}$$

By using this fact, we have

$$\left(\int_{\mathbb{R}^n\setminus B_E(0,|x|_E)} v^{1-p'}(y)dy\right)^{\frac{1}{p'}} = \left(\int_{\mathbb{R}^n\setminus B_E(0,|x|)} |y|_E^{-(\beta+\lambda)(1-p')p}dy\right)^{\frac{1}{p'}}$$
$$= \left(\frac{|\mathfrak{S}|}{|n+(\beta+\lambda)p'|}\right)^{\frac{1}{p'}} |x|_E^{\frac{n+(\beta+\lambda)p'}{p'}}.$$

Then by using $\frac{1}{p'}+\frac{1}{q}+\frac{\alpha+\beta+\lambda}{n}=0$, we get

$$\mathcal{D}_2(|x|_E) = \left(\frac{|\mathfrak{S}|}{|n+\alpha q|}\right)^{\frac{1}{q}} \left(\frac{|\mathfrak{S}|}{|n+(\beta+\lambda)p'|}\right)^{\frac{1}{p'}}, \tag{3.28}$$

which means $\mathcal{D}_2(|x|_E)$ is a non-increasing function. Therefore, we have

$$D_2 = \inf_{x\neq 0} \mathcal{D}_2(|x|_E) = \left(\frac{|\mathfrak{S}|}{|n+\alpha q|}\right)^{\frac{1}{q}} \left(\frac{|\mathfrak{S}|}{|n+(\beta+\lambda)p'|}\right)^{\frac{1}{p'}} > 0. \tag{3.29}$$

Then, we have

$$I_2^{\frac{1}{q}} \geq C\left(\int_{\mathbb{R}^n} |y|_E^{-\beta p} z^p(y)dy\right)^{\frac{1}{p}} = C\left(\int_{\mathbb{R}^n} h^p(y)dy\right)^{\frac{1}{p}}. \tag{3.30}$$

□

Remark 3.8. *Inequality (3.18) seems to be new even in the Euclidean space.*

Let us now show the reverse Stein-Weiss inequality $\mathbb{G}$.

Theorem 3.9 (The reverse Stein-Weiss inequality on $\mathbb{G}$)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q \geq 1$ and let $|\cdot|$ be an arbitrary homogeneous quasi-norm on $\mathbb{G}$. Assume that $q \leq p < 0$, $\lambda < 0$, and $\frac{1}{p'}+\frac{1}{q}+\frac{\alpha+\beta+\lambda}{Q}=0$, where $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$. Then for all non-negative functions $f \in L^{q'}(\mathbb{G})$ and $0 < \int_{\mathbb{G}} h^p(x)dx < \infty$, we have*

$$\int_{\mathbb{G}}\int_{\mathbb{G}} |x|^{\alpha} f(x)|y^{-1}x|^{\lambda} h(y)|y|^{\beta} dxdy \geq C\left(\int_{\mathbb{G}} f^{q'}(x)dx\right)^{\frac{1}{q'}} \left(\int_{\mathbb{G}} h^p(x)dx\right)^{\frac{1}{p}}, \tag{3.31}$$

if one of the following conditions is satisfied:

(a) $\beta > -\frac{Q}{p'}$;
(b) $\alpha > -\frac{Q}{q}$.

Proof. The proof of similar to the previous theorem, but here we use Proposition 3.1 and the polar decomposition formula (3.3). □

References

[1] R. A. Adams and J. J. F. Fournier. Sobolev spaces. Vol. 140. Elsevier, 2003.
[2] P. R. Beesack and H. P. Heinig. Hardy's inequalities with indices less than 1. *Proc. Amer. Math. Soc.,* 83(3):532–536, 1981.
[3] W. Beckner. Functionals for multilinear fractional embedding. *Acta Math. Sin. (Engl.Ser.,)* 31(1):1–28, DOI 10.1007/s10114-015-4321-6, 2015.

[4] L. Chen, G. Lu and C. Tao. Reverse Stein–Weiss Inequalities on the Upper Half Space and the Existence of Their Extremals. *Advanced Nonlinear Studies*, 19(3):475-494, 2019.
[5] L. Chen, Z. Liu, G. Lu, and C. Tao. Reverse Stein-Weiss inequalities and existence of their extremal functions. *Transactions of the American Mathematical Society*, 370(12):8429–8450, 2018.
[6] J. A. Carrillo, M. G. Delgadino, J. Dolbeault, R. L. Frank and F. Hoffmann. Reverse Hardy-Littlewood-Sobolev inequalities. *Journal de Mathématiques Pures et Appliquées*, 132:133–165, 2019.
[7] E. B. Davies. A review of Hardy inequalities. In *The Maz'ya anniversary collection, Vol. 2 (Rostock, 1998)*, volume 110 of *Oper. Theory Adv. Appl.*, pages 55–67. Birkhäuser, Basel, 1999.
[8] J. Dou and M. Zhu. Reversed Hardy-Littewood-Sobolev inequality. *Int. Math. Res. Not. IMRN*, 19:9696–9726, 2015.
[9] P. Drábek, H. P. Heinig and A. Kufner. *Higher-dimensional Hardy inequality, in: General Inequalities.* 7 (Oberwolfach 1995), Internat. Ser. Numer. Math. 123, Birkhauser, Basel, 3–16, 1997.
[10] D. E. Edmunds and W. D. Evans. *Hardy operators, function spaces and embeddings*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2004.
[11] R. L. Frank and E. H Lieb. Sharp constants in several inequalities on the Heisenberg group. *Ann. of Math.*, 176:349–381, 2012.
[12] V. Fischer and M. Ruzhansky. *Quantization on nilpotent Lie groups*, volume 314 of *Progress in Mathematics*. Birkhäuser. (open access book), 2016.
[13] G. B. Folland and E. M. Stein. Estimates for the $\overline{\partial_b}$ complex and analysis on the Heisenberg group. *Comm. Pure Appl. Math.*, 27:429–522, 1974.
[14] G. B. Folland and E. M. Stein. *Hardy spaces on homogeneous groups*, volume 28 of *Mathematical Notes*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1982.
[15] S. Gallot, D. Hulin, and J. Lafontaine. *Riemannian geometry*. Universitext. Springer-Verlag, Berlin, third edition, 2004.
[16] A. Gogatishvili, A. Kufner, L.-E. Persson, and A. Wedestig. An equivalence theorem for integral conditions related to Hardy's inequality. *Real Anal. Exchange*, 29(2):867–880, 2003/04.
[17] Gogatishvili, A., K. Kuliev, and G. Kulieva. Some conditions characterizing the "reverse" Hardy inequality. *Real Anal. Exchange*, 33(1): 249-258, 2008.
[18] X. Han, G. Lu and J. Zhu. Hardy-Littlewood-Sobolev and Stein-Weiss inequalities and integral systems on the Heisenberg group. *Nonlinear Analysis*, 75, 4296–4314, 2012.
[19] G. H. Hardy. Note on a theorem of Hilbert. *Math. Z.*, 6(3-4):314–317, 1920.
[20] G. H. Hardy and J. E. Littlewood. Some properties of fractional integrals. *I. Math. Z.*, 27:565–606, 1928.
[21] S. Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 34 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. Corrected reprint of the 1978 original.
[22] A. Kassymov, M. Ruzhansky and D. Suragan. Hardy-Littlewood-Sobolev and Stein-Weiss inequalities on homogeneous Lie groups. *Integral Transform. Spec. Funct.*, 30(8):643–655, 2019.
[23] A. Kassymov, M. Ruzhansky and D. Suragan. Reverse integral Hardy inequality on metric measure spaces. *Annales Fennici Mathematici*, 47(1):39–55. 2022.
[24] A. Kassymov, M. Ruzhansky and D. Suragan. Reverse Stein-Weiss, Hardy-Littlewood-Sobolev, Hardy, Sobolev and Caffarelli-Kohn-Nirenberg inequalities on homogeneous groups. *Forum Mathematicum,*, 34(5):1147–1158, 2022.
[25] A. Kufner, K. Kuliev and G. Kulieva. The Hardy inequality with one negative parameter. *Banach J. Math. Anal.,*2(2):76–84, 2008.
[26] A. Kufner and L.-E. Persson. *Weighted inequalities of Hardy type*. World Scientific Publishing Co., Inc., River Edge, NJ, 2003.
[27] A. Kufner, L. Maligranda and L.-E. Persson. *The Hardy inequality: About its history and some related results*, Vydavatelský servis, 162pp., 2007.
[28] A. Kufner, L.-E. Persson, and N. Samko. *Weighted inequalities of Hardy type*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, second edition, 2017.
[29] A. Kufner and K. Kuliev, The Hardy inequality with "negative powers". *Adv. Alg. Anal.,* 1(3): 219–228, 2006.
[30] E. H. Lieb. Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities. *Ann. of Math.*, 118:349–374, 1983.

[31] Q. A. Ngô and V. Nguyen. Sharp reversed Hardy-Littlewood-Sobolev inequality on $\mathbb{R}^n$. *Israel J. Math.*, 220:189–223, 2017.

[32] B. Opic and A. Kufner. *Hardy-type inequalities*, volume 219 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow, 1990.

[33] T. Ozawa, M. Ruzhansky, D. Suragan. L^p-Caffarelli-Kohn-Nirenberg type inequalities on homogeneous groups. *Quart. J. Math.,* 70: 305–318, 2019.

[34] D. V. Prokhorov, Weighted Hardy's inequalities for negative indices. *Publ. Mat.,* 48 :423–443, 2004.

[35] M. Ruzhansky and D. Verma. Hardy inequalities on metric measure spaces. *Proc. R. Soc. A.,* 475(2223):20180310, 2019.

[36] M. Ruzhansky and D. Verma. Hardy inequalities on metric measure spaces, II: the case $p > q$. *Proc. R. Soc. A.,* 477(2250):20210136, 2021.

[37] M. Ruzhansky and D. Suragan. Hardy and Rellich inequalities, identities, and sharp remainders on homogeneous groups. *Adv. Math.*, 317:799–822, 2017.

[38] M. Ruzhansky and D. Suragan. *Hardy inequalities on homogeneous groups*. Progress in Math. Vol. 327, Birkhäuser, 588 pp, 2019.

[39] M. Ruzhansky, D. Suragan, and N. Yessirkegenov. Extended Caffarelli-Kohn-Nirenberg inequalities, and remainders, stability, and superweights for L^p-weighted Hardy inequalities. *Trans. Amer. Math. Soc. Ser. B*, 5:32–62, 2018.

[40] M. Ruzhansky, D. Suragan, and N. Yessirkegenov. Sobolev type inequalities, Euler-Hilbert-Sobolev and Sobolev-Lorentz-Zygmund spaces on homogeneous groups. *Integr. Equ. Oper. Theory*, 90:10, 2018.

[41] M. Ruzhansky and N. Yessirkegenov. Hypoelliptic functional inequalities. *https://arxiv.org/abs/1805.01064*, 2018.

[42] S. L. Sobolev. On a theorem of functional analysis. *Mat. Sb. (N.S.)*, 4:471–479, 1938, English transl. in Amer. Math. Soc. Transl. Ser. 2, 34:39–68, 1963.

[43] E. M. Stein and G. Weiss. Fractional integrals on n-dimensional Euclidean Space. *J. Math. Mech.*, 7(4):503–514, 1958.

Aidyn Kassymov:
Department of Mathematics: Analysis, Logic and Discrete Mathematics
Ghent University, Belgium
and
Institute of Mathematics and Mathematical Modeling
125 Pushkin str.
050010 Almaty
Kazakhstan
and
Al-Farabi Kazakh National University
71 Al-Farabi avenue
050040 Almaty
Kazakhstan
E-mail address aidyn.kassymov@ugent.be and kassymov@math.kz

Michael Ruzhansky:
Department of Mathematics: Analysis, Logic and Discrete Mathematics
Ghent University, Belgium
and
School of Mathematical Sciences
Queen Mary University of London
United Kingdom
E-mail address michael.ruzhansky@ugent.be

Durvudkhan Suragan:
Department of Mathematics
School of Science and Technology, Nazarbayev University

53 Kabanbay Batyr Ave, Nur-Sultan 010000
Kazakhstan
E-mail address durvudkhan.suragan@nu.edu.kz