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A combinatorial approach to optimal designs

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Abstract

A typical problem in experimental design theory is to find a block design in a class that is optimal with respect to some criteria, which are usually convex functions of the Laplacian eigenvalues. Although this question has a statistical background, there are overlaps with graph and design theory: some of the optimality criteria correspond to graph properties and designs considered 'nice' by combinatorialists are often optimal. In this thesis we investigate this connection from a combinatorial point of view.

We extend a result on optimality of some generalized polygons, in particular the generalized hexagon and octagon, to a third optimality criterion. The E-criterion is equivalent with the graph theoretical problem of maximizing the algebraic connectivity. We give a new upper bound for regular graphs and characterize a class of E-optimal regular graph designs (RGDs). We then study generalized hexagons as block designs and prove some properties of the eigenvalues of the designs in that class. Proceeding to higher-dimensional geometries, we look at projective spaces and find optimal designs among two-dimensional substructures. Some new properties of Grassmann graphs are proved. Stepping away from the background of geometries, we study graphs obtained from optimal graphs by deleting one or several edges. This chapter highlights the currently available methods to compare graphs on the A- and D-criteria. The last chapter is devoted to designs to which a number of blocks are added. Cheng showed that RGDs are A- and D-optimal if the number of blocks is large enough for which we give a bound and characterize the best RGDs in terms of their underlying graphs. We then present the results of an exhaustive computer search for optimal RGDs for up to 18 points. The search produced examples supporting several open conjectures.

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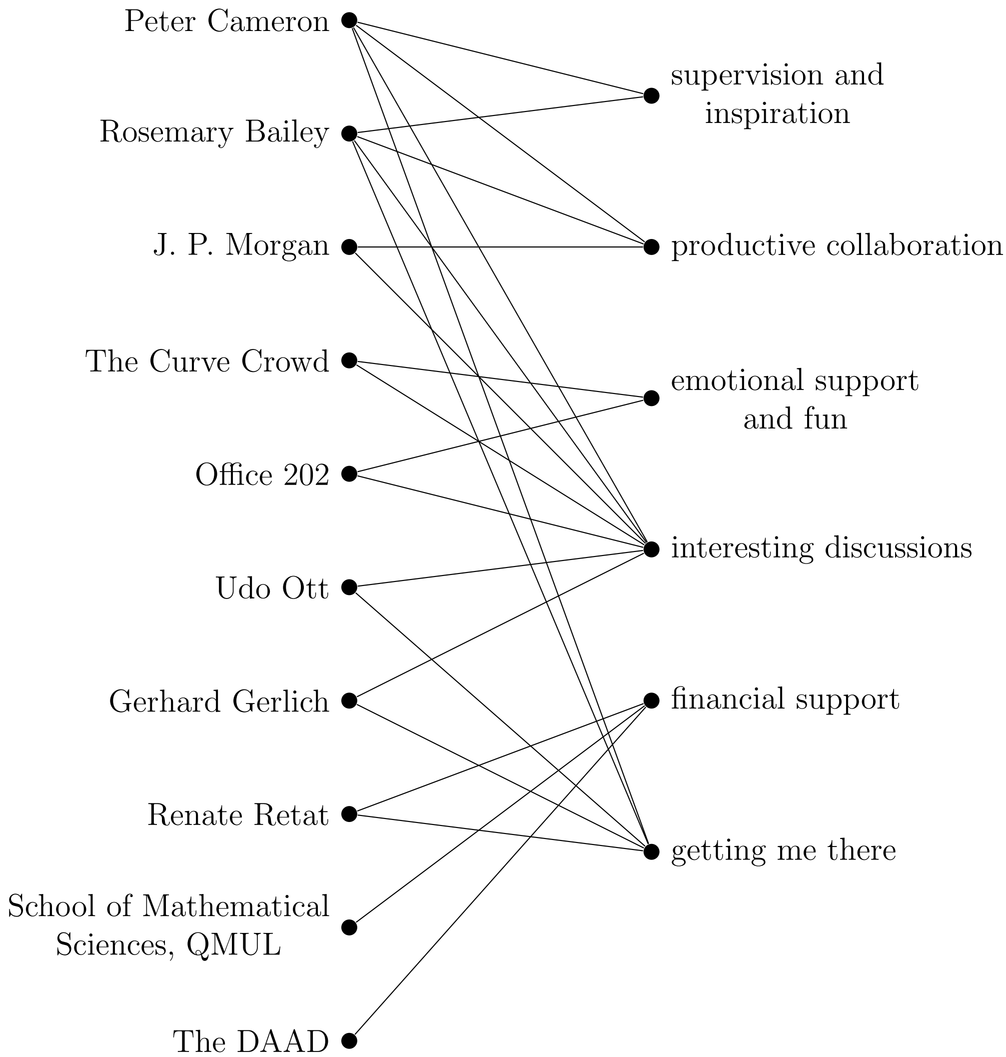

Une mer calme n'a jamais fait un bon marin (Breton proverb).

Sera Aylin Cakiroglu

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Contents

Chapter 1

Introduction

1.1 An Overview

Suppose we are to design the following statistical experiment: there are v treatments to be compared on a number of experimental units that can be partitioned into b blocks of size k with $k < v$. Typically, the blocks might differ systematically but all units in a block are assumed to be alike. For fixed v, b and k, how should the treatments be allocated to the units to get as much information as possible from the available data? The latter often means the estimate of the unknown parameters with the least possible variance. If there are several parameters, this is a multidimensional problem and a design can be 'good' in many ways, for example minimizing the average variance or minimizing the maximum variance. If every treatment is allowed to occur at most once per block, we can describe this experiment as a combinatorial block design.

The optimality criteria can be expressed as convex functions of the nontrivial Laplacian eigenvalues of the designs or their adjacency graphs. Although this question has a statistical background, there are suprising connections to graph and design theory: Some of the most popular optimality criteria can be translated into graph properties that have been studied by graph theorists independently. For example, the D-criterion is equivalent with maximizing the number of spanning trees and the E-criterion is maximizing the algebraic connectivity among graphs with v vertices and a fixed number of edges. There is another optimality criterion, the A-criterion, that can be defined in terms of graph properties as maximizing the ratio of the number of spanning trees and the number of thickets of a graph.

Not only do some of the most popular optimality criteria correspond to graph properties, but also optimal designs are often found among designs considered 'nice' by combinatorialists: for example, one of the earliest result on optimal designs is by Kiefer, who showed that 2-designs are optimal among all designs under the most general criterion ([Kie75]). But it is not clear which designs to choose if no 2-designs exist. One class of designs that have been suggested to be good candidates are the regular graph designs (RGDs). These are designs that are 'close' to 2-designs in the sense that every point occurs in the same number of blocks and any pair of points occurs in either λ or $\lambda + 1$ blocks for some integer $\lambda \geq 0$. The RGDs owe their name to the fact that their adjacency graph is obtained from a simple regular graph $\mathcal{G}$ by adding λ edges between any two pairs of vertices, for which we will write $\mathcal{G} + \lambda * K_v$. In 1977, John and Mitchell posed the following conjecture.

Conjecture 1.1 ([JM77])**.** *If an incomplete block design is D-optimal (or A-optimal or E-optimal), then it is an RGD (if an RGD exists).*

It took 30 years for the conjecture to be proven to be wrong in general ([Bai07]), but it holds if the number of blocks is big enough ([Che92]). Even for a small number of blocks, RGDs occur as optimal designs in a lot of results. For example, Cheng proved in [Che81a] that the design with block size 2 with adjacency graph $K_{n,n} + \lambda * K_{2n}$ is optimal with respect to a range of criteria (see Theorem 2.28). Therefore, regular graphs as so called underlying graphs of

RGDs are of interest in optimal design theory.

Another class of optimal designs arise from finite geometries: the generalized polygons. Finite geometries and in particular generalized polygons have been studied famously in the attempt to classify the semi-simple Lie groups. Some of the results in optimal design theory imply that some finite geometries are optimal designs. For example, a generalized triangle is a 2-design and as such optimal by Kiefer's result. Bailey and Cheng show in [CB91] that partial geometries and in particular generalized quadrangles are optimal with regards to several criteria among a certain class of designs. Finite rank-two geometries and in particular generalized polygons seem therefore to be good candidates for optimal designs.

That means statisticians and graph and design theorists have been studying similar problems. In this thesis we want to explore the connections between experimental designs and graphs and finite geometries from the combinatorial point of view, where studying finite geometries in the setting of optimal design theory is a so far completely new angle.

The results in Chapter 3 are a good example for overlapping results in optimal design and graph theory and finite geometries. If there exists a finite generalized $2N$-gon with parameters $s = 1$ and $t > 1$, then by a well-known result (see Theorem 3.5), its adjacency graph is isomorphic to the point-line incidence graph of a finite generalized N-gon with parameters (t, t). The adjacency graph of its dual design is therefore the line graph of the incidence graph of the N-gon. There exist experimental designs that can be constructed in a similar way: In [PW75], Patterson and Williams show that the adjacency graph of a connected two-replicate resolvable design is the line graph of the point-block incidence graph of a unique symmetric design, that is a design with $v = b$. Williams, Patterson and John showed that such a design is A-optimal among connected two-replicate resolvable designs if and only if the corresponding symmetric design is A-optimal among symmetric designs ([WPJ76, PW75]). Combining results from graph theory on the number of spanning trees of the line graph of a regular graph (see Proposition 2.15) and a result by Gaffke ([Gaf82], see Proposition 3.3) on the number of spanning trees of the incidence graph of a block design, we can extend their result to D-optimality (Corollary 3.4). In particular, it follows that the duals of generalized $2N$-gons whose adjacency graph is isomorphic to the point-line incidence graph of a generalized N-gon are A- and D-optimal for $N = 3, 4$ in certain classes of designs (Corollary 3.6 and Corollary 3.7).

In Chapter 4 we study E-optimal strongly regular graphs and their designs. The E-criterion is a wonderful example of a criterion that has a strong background in graph theory. The criterion corresponds to maximizing the smallest non-trivial Laplacian eigenvalue of the adjacency graph of a block design. This eigenvalue is also known as the algebraic connectivity of the graph. This definition is due to Fiedler ([Fie75]) who proved as one of the first results that the vertex connectivity is an upper bound for the eigenvalue. There has been wide interest on the bounds on this eigenvalue, whose importance is '[...] difficult to overemphasize', since the larger the algebraic connectivity of a graph $\mathcal{G}$, '[...] the more difficult it is to cut $\mathcal{G}$ into pieces, and the more $\mathcal{G}$ expands' as B. Bollobas writes in [Bol98], p. 269. This gives the motivation for this chapter in which we derive a new upper bound for the algebraic connectivity of a regular graph (Proposition 4.6) using the Higman-Sims technique introduced in [Hae80]. Together with a new result on the connectivity of the neighbourhood graph of strongly regular graphs (Proposition 4.2) our result gives a characterization of some E-best RGDs with strongly regular underlying graph (Theorem 4.8 and Corollary 4.9). Moreover, Proposition 4.2 implies a result on the size of the connected components of any neighbourhood graph which we apply to generalized quadrangles and the triangular graph (Corollary 4.3, Proposition

4.4 and Proposition 4.12). As an application we prove the known facts that the complete regular multipartite graph and the partial geometries are E-optimal among all RGDs (Proposition 4.10 and Proposition 4.14). Further we show that the triangular graph gives rise to an E-optimal design for a certain block size (Proposition 4.13).

In Chapter 5 we want to go back to looking at finite geometries, in particular the generalized hexagons. A famous theorem by Feit and Higman asserts that, unless there are only 2 points on a line, there exist generalized N-gons only if $N = 3, 4, 6, 8, 12$ (see for example [Ron89]). By the above mentioned results by Kiefer, Bailey and Cheng, generalized triangles and quadrangles are optimal with regards to several criteria among a certain class of designs. In view of this, we study generalized hexagons in the setting of optimal block designs. There are some facts that suggest that they might be good candidates for optimal designs. For one, we have shown in Chapter 3 that the generalized hexagons whose adjacency graphs are isomorphic to the line graph of the point-line incidence graph of a projective plane are A-, D- and E-optimal among certain designs. And secondly, we found by computer search (see Chapter 8) that the generalized hexagon with 14 points is A- and D-optimal among RGDs with block size 2 and replication 3. The generalized hexagons are only known for some parameters and naturally a good understanding of the structure of the adjacency graph is very important. The generalized hexagons have a distance regular adjacency graph and we start by computing the eigenvalues and their multiplicities (Proposition 5.1). Although the spectrum is known and a proof using results on distance regular graphs can be found for example in [BCN89], we want to prove this result using the combinatorial properties of the generalized hexagon to give an insight to the structure of the graph. The last results in this chapter narrow down what kind of designs are in the class of the generalized hexagon (Proposition 5.3 and Proposition 5.4) and we find a condition that forces a design to have the same

Laplacian spectrum (Corollary 5.5).

In Chapter 6, we want to move to higher-dimensional structures and search for optimal designs among rank-two substructures. Since projective planes are 2-designs and therefore optimal, the projective space is a good place to start. Let $\mathcal{V}$ be a vector space of dimension $n+1$ over GF(q) for any prime or prime power q. Call any two subspaces U_1 and U_2 incident if and only if $U_1 \leq U_2$ or $U_2 \leq U_1$. The set of proper subspaces together with this incidence relation is the n-dimensional projective space over GF(q). The subspaces of $\mathcal{V}$ of dimension i are also called the elements of type i of the projective space for $i \in \{1, \ldots, n\}$. The set of the elements of the same type are also called Grassmannians. This definition has its origin in the example of the projective space but can be generalized to other classical geometries. The Grassmannians of the projective space give rise to so called 'Grassmann graphs' with the elements of type $i \in \{2, \ldots, n\}$ as vertices, any two of which are joined by an edge if they intersect in a subspace of dimension $i-1$. This is equivalent to the two subspaces of dimension i being contained in a subspace of dimension $i+1$. More generally, the design with the elements of type $i \in \{1, \ldots, n\}$ as vertices and the blocks being the subspaces of dimension j for some $i < j \leq n$ are called the $\{i, j\}$-truncations and these are the designs we want to study in this chapter.

But before we study the projective space we start with a simpler substructure: Suppose we have a maximal chain of n mutually distinct nested subspaces. There are many ways to exchange the subspaces of dimension i while keeping the structure of a nested chain. Now, let $\mathcal{E}$ be a fixed basis of $\mathcal{V}$ and consider only the chains of mutually distinct subspaces that all admit a basis which is a proper subset of $\mathcal{E}$. Among these, there are exactly two possible subspaces of dimension i to complete a chain of $n-1$ given nested subspaces. The adjacency graph of the restriction of the $\{i, j\}$-truncation to these kind of subspaces is a subgraph of a Grassmann graph. Since the $\{1, j\}$-truncations are 2-designs, they are universally optimal (Proposition 6.3) and their duals are optimal among certain designs (Corollary 6.4). We prove a structural property of the adjacency graph of the $\{2, 3\}$-truncation of this incidence structure (Proposition 6.5) and we discuss the optimality of these designs by presenting the results of an exhaustive computer search for optimal designs having the same adjacency graph as the lines and planes (see also Chapter 8). We conclude this section investigating the $\{i, j\}$-truncations with $1 < i < j < n + 1$.

Then we proceed to the truncations of the projective space over $\mathrm{GF}(q)$. Again, the $\{1, j\}$-truncations are universally optimal block designs (Proposition 6.11) and their duals are optimal among certain designs (Corollary 6.12). There has been interest in characterizing the Grassmann graphs in hope to show their uniqueness, see [BCN89]. For characterizations see for example [Num90, Num85, Spr78]. In view of this, we are also interested in the structure of these graphs: It is known that they are distance regular graphs (see for example [BCN89], p. 269). We give an easy proof that the adjacency graph of the $\{2, 3\}$-truncation is strongly regular and compute its parameters (Proposition 6.13). Moreover, we prove that if the dimension is odd, any connected strongly regular graph with the same number of vertices and same degree must have the same parameters (Proposition 6.14). Further we show some structural properties of the neighbourhood graphs (Proposition 6.15 and Corollary 6.17) and that the graph is isomorphic to the Grassmann graph of the hyperplanes, i.e. the elements of type n (Proposition 6.16). We then move on to the general setting of the graphs with vertices being the subspaces of dimension i intersecting in a subspace of dimension j for some constant $j < i$. These graphs are distance regular and we compute the parameters.

Stepping away from the background of geometries in Chapter 7, we then study graphs obtained from optimal graphs by deleting one or several edges. We show the known fact that the graph obtained from the complete graph by deleting mutually disjoint edges maximizes the number of spanning trees. This is the motivation for Proposition 7.3, where we show that this is also true for A-optimality and two deleted edges. By a result by Cheng ([Che81a], see also Theorem 2.28), the complete regular bipartite graph is D-optimal and a more recent result by Petingi and Rodriguez shows that this is also true for the complete almost-regular multipartite graph among simple graphs ([PR02], see also Proposition 2.33). In particular, in their paper Petingi and Rodriguez characterize the D-best simple graphs as being as regular as possible and minimizing the number of V-subgraphs (these are subgraphs of size three with exactly two edges) in their complement. We investigate this correspondence by comparing the complete bipartite graph $K_{n-1,n+1}$ and the graph $K_{n,n} \setminus \{f\}$ obtained from the complete regular bipartite graph $K_{n,n}$ by deleting an edge f. As the complement of a union of cliques, the graph $K_{n-1,n+1}$ minimizes the number of V-subgraphs in its complement, whereas the complement of $K_{n,n} \setminus \{f\}$ has a lot of V-subgraphs but its vertex degrees differ by at most 1. In this case, the graph $K_{n,n} \setminus \{f\}$ being closer to a regular graph beats $K_{n-1,n+1}$ on both the A- and D-criterion (Corollary 7.7). In fact we can even show the stronger result that the Laplacian eigenvalues of $K_{n,n} \setminus \{f\}$ are majorized by the Laplacian eigenvalues of $K_{n-1,n+1}$ (Proposition 7.5) which implies that $K_{n,n} \setminus \{f\}$ performs better on any criterion that is Schur-concave or Schur-convex than any graph with the same degree sequence as $K_{n-1,n+1}$ (Corollary 7.8). Of course, it can not be true that deleting any edge from an optimal graph will always result in an optimal graph, because the performance especially on the D-criterion depends highly on which edge has been deleted. A good example is deleting an edge from the complete almost-regular graph $K_{n,n+1}$. Although deleting an edge from $K_{n,n} \setminus \{f\}$ results in a graph that performs well on the D-criterion and $K_{n,n+1}$ being D-optimal (by Proposition 2.32), it is easy to find a graph that beats $K_{n,n+1}$ with a deleted edge on the D-criterion. We give an example for $n = 3$.

In the last chapter we want to look at designs in large systems, that means designs with a large number of blocks. Suppose Λ is the Laplacian matrix of any connected design with v points and b blocks of size k and $\tilde{d}$ is a 2-$(v, k, \tilde{\lambda})$-design on v points and block size k with $\tilde{b}$ blocks. Then for $y \in \mathbb{N}$ the matrix $\Lambda[y] = \Lambda + y\tilde{\Lambda}$ is the Laplacian matrix of a design on v points, replication $r + y\frac{\tilde{\lambda}(v-1)}{k-1}$ and $b + y\tilde{b}$ blocks of size k. The motivation for Chapter 8 lies in the following question posed by J.P. Morgan.

Question 1.2 ([Mor11])**.** *Suppose there exists a 2-design $\tilde{d}$ on v points and $\tilde{b}$ blocks and an A-best RGD d on v points and b blocks. Can then an A-best RGD with $b + y\tilde{b}$ blocks be found by just adding y copies of the 2-design $\tilde{d}$ to d?*

We start by investigating the question, what kind of Laplacian matrices Λ produce optimal designs with Laplacian matrices $\Lambda[y]$. A first observation is that the majorization of eigenvalues is preserved by adding blocks to the design. More precisely, we can show that if the eigenvalues of a matrix Λ are majorized by another matrix Λ', then $\Lambda[y]$ beats $\Lambda'[y]$ on any Schur-concave or Schur-convex criterion (Proposition 8.1). This implies for example that the graph $K_{n,n} \setminus \{f\} + y * K_{2n}$ (obtained from $K_{n,n} \setminus \{f\}$ by adding y copies of the edges of the complete graph) beats $K_{n-1,n+1} + y * K_{2n}$ on any Schur-concave criterion, in particular on the A- and D-criteria, for all $y \geq 0$ (Proposition 8.3). Proposition 8.1 also implies that Schur-optimal designs stay D-optimal under this transformation (Corollary 8.2). We introduce a pre-order on the Laplacian matrices that is given by the values of the elementary symmetric polynomials on the non-trivial eigenvalues. The reason for this is that the D-value of $\Lambda[y]$ can be written as a polynomial in y whose coefficients are determined by the elementary symmetric polynomials on the non-trivial eigenvalues of Λ (equation 8.1.2) and the A-value is determined by the D-value divided by its derivative in y (equation 8.1.3). We show that this pre-order is exactly the order corresponding to the performance of the matrices of the form $\Lambda[y]$ on the A- and D-criterion if y is large enough (Theorem 8.5). It follows directly from this result that to find Laplacian matrices Λ that produce optimal designs with Laplacian matrices $\Lambda[y]$ for large y, we have to find matrices Λ that maximize the first and second symmetric elementary polynomials on their non-trivial eigenvalues. This implies a result by Constantine ([Con86], see also Corollary 8.6) and a special case of a result by Cheng ([Che92]). While Cheng showed that RGDs are optimal in large systems with regard to a general type of optimality, we prove it here for A- and D-optimality (Proposition 8.13 and Proposition 8.15). Our approach lets us give lower bounds for y such that RGDs are optimal and that A- and D-optimality of a design are equivalent for large y (Corollary 8.16). We apply the results on the cycle on v edges (as design with block size 2) and can show with results from Bailey ([Bai07]) and Stevanovic and Ilic ([SI09]) that $Cycle(v) + y * K_v$ is D-optimal for all $y \geq 0$ and A-optimal for large y among all graphs $\mathcal{G}$ with v vertices and v edges (Proposition 8.9).

Therefore, if the dominating RGD for big y can be identified, then this is also the best design among all designs in the class for large y. This is the motivation for the rest of the chapter where we show that comparing RGDs in this setting is equivalent to compairing their underlying graphs. In this way we obtain lower bounds for y such that the RGDs characterized in Constantine's result are optimal (Proposition 8.17 and Proposition 8.18). A new result on the correspondence of the symmetric elementary polynomials of the Laplacian eigenvalues of a simple graph and its complement (Proposition 8.19) gives rise to a new characterization of optimal RGDs for large y in terms of their underlying graph (Theorem 8.21). With this we can extend Cheng's result that the complete regular multipartite graph $K_{\alpha,\dots,\alpha}$ is A- and D-optimal among all simple graphs to the graph $K_{\alpha,\dots,\alpha} + y * K_{\alpha m}$ being A- and D-optimal among all graphs $\mathcal{G} + y * K_{\alpha m}$ where $\mathcal{G}$ is any connected (multi-)graph in the class of $K_{\alpha,\dots,\alpha}$ for large y

(Corollary 8.22). In the last section of the chapter, we present the results of an exhaustive computer search for optimal RGDs with connected underlying graph for up to 18 points. In [JM77], John and Mitchell provided a list of the best RGDs with $v \leq 12$, $r \leq 10$ and $v \leq b$. To identify the best RGD for big y, we are taking the same approach as John and Mitchell and start by generating all connected simple δ-regular graphs with $5 \leq v \leq 13$ and $2 \leq \delta \leq 9$, $v = 14$ and $2 \leq \delta \leq 5$, $v = 15$ and $\delta = 4$, $v = 16, 18$ and $\delta = 3$. The other cases were too extensive to handle. There are two main things we do differently. For one, we only consider connected regular graphs and secondly instead of calculating the A- and D-values of the designs for particular values of y, we compute the values as polynomials in y. In this way, we could extend the list of best RGDs to $v \leq 14$ for all admissible block sizes and $v \leq 18$ for block size 2. In particular, we were able to get some improvements on results of John and Mitchell on the designs with $v = 11$ and $y = 0$. We have restricted our calculations on A- and D-optimality since there is a list of binary connected E-optimal designs up to 15 points without the restriction to RGDs by J.P. Morgan which can be found on www.designtheory.org. For the A- and D-values as polynomials in y, we find the smallest values y_0^A and y_0^D such that the ordering of the graphs according to their A- and D-values (evaluated in y_0^A, y_0^D) stabilize. The main observation is that in all of our cases $y_0^D \leq y_0^A \leq \delta+1$ and the order of the graphs according to their A-value evaluated in y_0^A and the order according to their D-value evaluated in y_0^D are the same (Observation 8.26 and Observation 8.28). A list of all designs can be found in the appendix. The search produced examples supporting several open conjectures, such as the optimality of the generalized hexagon or of the complete multipartite regular graphs after adding the edges of a complete graph any number of times. Moreover, we have found an example where A- and D-optimality are not equivalent for small y even among RGDs.

1.2 Original Content

Throughout this text any result that is not my own has been clearly labelled as such and a reference is given. Below is a summary of my main results in this thesis.

- **Chapter 3**: Corollary 3.4, Corollary 3.6 and Corollary 3.7
- **Chapter 4**: Proposition 4.2, Corollary 4.3, Proposition 4.6, Theorem 4.8, Corollary 4.9 and Proposition 4.13
- **Chapter 5**: Proposition 5.3, Proposition 5.4, Corollary 5.5
- **Chapter 6**: Proposition 6.3, Corollary 6.4, Proposition 6.11, Corollary 6.12, Proposition 6.14, Proposition 6.15 and Corollary 6.17
- **Chapter 7**: Proposition 7.3, Proposition 7.5, Corollary 7.7 and Corollary 7.8
- **Chapter 8**: Proposition 8.3, Theorem 8.5, Proposition 8.9, Proposition 8.13, Proposition 8.15, Proposition 8.17, Proposition 8.18, Proposition 8.19, Theorem 8.21, Corollary 8.22 and the results of the computer search on order stabilization.
- **Appendix**: All optimal designs with $v \geq 12$ and $\delta \geq 5$, where δ is the degree of the underlying regular graph.

Chapter 2

Preliminaries

Throughout this text let v, m and n be positive integers.

2.1 Eigenvalues of Real Matrices

The eigenvalues of certain matrices associated with graphs or designs will play a key role in this thesis. This is why we want to provide here the basic notation, properties and some general results on eigenvalues of real square matrices that we will need later. All matrices in this thesis are assumed to have only real entries.

Throughout this text we denote by $\mathbb{I}_n$ the $n \times n$ identity matrix and by $\mathbb{J}_{m\times n}$ the $m \times n$ matrix with all entries equal to 1. If $m = n$ we also write $\mathbb{J}_n$. For a $m \times n$ matrix with entries a_{ij} for $i = 1, \dots, m$ and $j = 1, \dots, n$ we write (a_{ij}). Further, the $n \times n$ matrix (a_{ij}) with diagonal elements $a_{ii} = x_i$ and $a_{ij} = 0$ whenever $i \neq j$ is denoted by $\mathrm{diag}(x_1, \dots, x_n)$. The transpose of a matrix M is denoted by M^T. If there exist $a, b \in \mathbb{Q}$ such that $M = a\mathbb{I}_n + b\mathbb{J}_n$, then M is called *completely symmetric*. We denote by $\chi_M(x) = \det(x\mathbb{I}_n - M)$ the characteristic polynomial of a real $n \times n$ matrix M. The derivative of $\chi_M(x)$ in x will be denoted by $\chi'_M(x)$. By the Cayley–Hamilton theorem (see for example [CDS79],

p. 20), any $n \times n$ square matrix M satisfies its own characteristic equation, i.e. $\chi_M(M) = 0$, and the degree of $\chi_M(x)$ is equal to n. The *minimal polynomial* $\bar{\chi}_M(x)$ is the monic polynomial with the smallest degree such that $\bar{\chi}_M(M) = 0$.

Lemma 2.1 ([CDS79], p. 61)**.** *Let L be an $m \times n$ matrix. Then*

$$x^n \chi_{LL^T}(x) = x^m \chi_{L^T L}(x).$$

The multiplicities of the eigenvalues of an $n \times n$ matrix M as roots of the characteristic polynomial $\chi_M(x)$ are called the *algebraic multiplicities*. The set of eigenvectors belonging to an eigenvalue $\nu(M)$ along with the zero-vector form the *eigenspace* to $\nu(M)$ and its dimension is called the *geometric multiplicity* of $\nu(M)$. Throughout this text the *multiplicity of an eigenvalue* will mean the algebraic multiplicity, but in the case of symmetric matrices the multiplicities are the same:

Theorem 2.2 ([CDS79], p. 17)**.** *The geometric and algebraic multiplicities of an eigenvalue of a real symmetric matrix are equal.*

Let $\nu_1(M) > \cdots > \nu_l(M)$ be the distinct eigenvalues of an $n \times n$ matrix M, i.e. the distinct roots of $\chi_M(x)$, with multiplicities $m_1, \ldots, m_l$, where $l \leq n$. Since M is a real symmetric matrix, all eigenvalues of M are real and we denote the *spectrum* of M by

$$\mathrm{Spec}(M) = (\nu_l(M)^{m_l}, \ldots, \nu_1(M)^{m_1}).$$

Proposition 2.3 ([CDS79], p. 20)**.** *If M is a symmetric $n \times n$ matrix with spectrum*

$$\mathrm{Spec}(M) = (\nu_l(M)^{m_l}, \ldots, \nu_1(M)^{m_1}),$$

then
$$\bar{\chi}_M(x) = \prod_{i=1}^{l}(x - \nu_i(M)).$$

Suppose L is an $m \times m$ matrix with $m \geq n$, having eigenvalues $\nu_1(L) \geq \cdots \geq \nu_m(L)$. If
$$\nu_i(L) \geq \nu_i(M) \geq \nu_{m-n+i}(L)$$

for all $i \in \{1, \ldots, n\}$, then we say that the eigenvalues of M *interlace* the eigenvalues of L. If there exists an integer $j \in \{0, \ldots, n\}$ such that

$$\nu_i(L) = \nu_i(M) \text{ for } i = 1, \ldots, j$$

and
$$\nu_{m-n+i}(L) = \nu_i(M) \text{ for } i = j+1, \ldots, n,$$

then the interlacing is called *tight*.

Theorem 2.4 ([Hae80], p. 9)**.** *Let L be a symmetric real square matrix partitioned as follows*
$$L = \begin{pmatrix} L_{11} & \cdots & L_{1m} \\ \vdots & & \vdots \\ L_{m1} & \cdots & L_{mm} \end{pmatrix}$$
such that L_{ii} is square for $i = 1, \ldots, m$. Let M be the $m \times m$ matrix whose ij-entry is the average row sum of L_{ij} for $i, j = 1, \ldots, m$.

1. *The eigenvalues of M interlace the eigenvalues of L.*

2. *If the interlacing is tight, then L_{ij} has constant row and column sums for $i, j = 1, \ldots, m$.*

3. *If for all $i, j \in \{1, \ldots, m\}$ the matrix L_{ij} has constant row and column sums, then any eigenvalue of M is also an eigenvalue of L with not smaller a multiplicity.*

2.2 Graphs: Basic Concepts and Notation

In this section we will give the basic definitions and results on graphs used in this thesis. However, a good introduction to graph theory and graph spectra and more details can be found for example in [Bol98, Big93, CDS79].

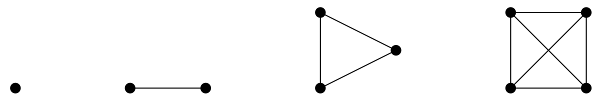
In this thesis, a graph $\mathcal{G}$ is always finite and can have multiple edges, but has no loops and no orientation. If the graph has no multiple edges, we will say the graph is *simple*. Unless we explicitly say that $\mathcal{G}$ is simple, we always assume that $\mathcal{G}$ is a multigraph. The set of vertices of a graph $\mathcal{G}$ is denoted by $V(\mathcal{G})$ and the set of edges by $E(\mathcal{G})$. If $u, w \in V(\mathcal{G})$ are joined by an edge f, we say they are *adjacent* and we write $f = \{u, w\}$. If $f = \{u, w\}$ for $f \in E(\mathcal{G})$, we also say that u and w are incident with f. A set of graphs with v vertices and e edges with or without any additional attributes is called a *class* of graphs. For the rest of this section, let $\mathcal{G}$ be a graph with v vertices and e edges and $V(\mathcal{G}) = \{u_1, \ldots, u_v\}$ and $E(\mathcal{G}) = \{f_1, \ldots, f_e\}$.

For $u \in V(\mathcal{G})$, the number of edges incident with u is called the *degree* of the vertex and we write $\delta(u)$. The sequence $(\delta(u_1), \ldots, \delta(u_v))$ is called *degree sequence* of $\mathcal{G}$ and we write $(\delta_1, \ldots, \delta_v)$. If $\delta = \delta(u)$ for all $u \in V(\mathcal{G})$ and a $\delta \in \mathbb{N}$, the graph is called *δ-regular* and if $\delta(u) \in \{\delta, \delta + 1\}$ for all $u \in V(\mathcal{G})$, the graph is called *almost-regular*.

A graph $\mathcal{G}'$ is a *subgraph* of $\mathcal{G}$ if $V(\mathcal{G}') \subseteq V(\mathcal{G})$ and $E(\mathcal{G}') \subseteq E(\mathcal{G})$. The *size* of a subgraph $\mathcal{G}'$ is the number of vertices in $V(\mathcal{G}')$. For a subgraph $\mathcal{G}'$ of $\mathcal{G}$ the graph with vertex set $V(\mathcal{G})$ and edges $E(\mathcal{G}) \setminus E(\mathcal{G}')$ is denoted by $\mathcal{G} \setminus \mathcal{G}'$. For a graph $\mathcal{G}'$ with $V(\mathcal{G}') \subseteq V(\mathcal{G})$ the graph with vertex set $V(\mathcal{G})$ and edges $E(\mathcal{G}) \cup E(\mathcal{G}')$ is denoted by $\mathcal{G} + \mathcal{G}'$.

Any finite sequence $w_1, \ldots, w_n$ of vertices of $\mathcal{G}$ such that w_i and w_{i+1} are joined by an edge for $i = 1, \ldots, n-1$ is called a *walk*. A *path* of length $n-1$ in $\mathcal{G}$ is a simple subgraph with n vertices $w_1, \ldots, w_n$ and $n-1$ edges such that w_i

and w_{i+1} are joined by an edge for $i = 1, \ldots, n-1$; that is a path is a walk with distinct vertices. The graph is called *connected* if any two vertices can be joined by a path. The *distance* in $\mathcal{G}$ between vertices u and w, denoted by $\mathrm{dist}(u, w)$, is the length of the shortest path joining u and w. The *diameter* of $\mathcal{G}$, denoted by $\mathrm{diam}(\mathcal{G})$, is the maximal distance between any two vertices in $\mathcal{G}$. A *closed path* or a *cycle* of length $n-1$ in $\mathcal{G}$ is a path of length n with vertices $w_1, \ldots, w_n = w_1$. For example, a cycle of length 1 is a loop, a cycle of length 2 is a pair of parallel edges and a cycle of length 3 is a triangle. The *girth* of $\mathcal{G}$, denoted by $\mathrm{girth}(\mathcal{G})$, is the length of the shortest closed path in $\mathcal{G}$. A *V-subgraph* of $\mathcal{G}$ is subgraph of $\mathcal{G}$ of size three with exactly two edges. A *clique* in $\mathcal{G}$ is a simple subgraph of pairwise adjacent vertices. For example, the following graphs are cliques of size 1, 2, 3 and 4:



For a vertex u let $\mathcal{G}_u$ be the subgraph of $\mathcal{G}$ with vertex set

$$V(\mathcal{G}_u) = \{w \in V(\mathcal{G}) \setminus \{u\} | \{u, w\} \in E(\mathcal{G})\}$$

and edge set

$$E(\mathcal{G}_u) = \{f \in E(\mathcal{G}) | f = \{w_1, w_2\}, w_1, w_2 \in V(\mathcal{G}_u)\}.$$

Then $\mathcal{G}_u$ is called the *neighbourhood graph* of u.

If $\mathcal{G}$ is a clique of size v, the graph is called *complete* and we write K_v. For $y \in \mathbb{N}_{>0}$, we denote by $y * K_v$ the graph on v vertices where any pair of vertices is joined by exactly y edges. A *tree* is a simple connected graph with v vertices and $v - 1$ edges that contains no cycle.

A simple graph is called *multipartite* or *m-partite* if the vertex set $V(\mathcal{G})$ can be partitioned into sets $V_1(\mathcal{G}), \ldots, V_m(\mathcal{G})$ such that $\{u, w\} \notin E(\mathcal{G})$ whenever $u, w \in V_i(\mathcal{G})$ for $i \in \{1, \ldots, m\}$. Let $\alpha_i = |V_i(\mathcal{G})|$ for $i = 1, \ldots, m$. In the case $m = 2$ the graph is called *bipartite* with the convention that $\alpha_1 \leq \alpha_2$. If $\mathcal{G}$ is bipartite and if there exist positive integers δ_1, δ_2 such that each vertex $u \in V_i(\mathcal{G})$ has degree δ_i for $i = 1, 2$, then $\mathcal{G}$ is called *semiregular* of degrees δ_1, δ_2. If $\mathcal{G}$ is a multipartite graph and $\{u, w\} \in E(\mathcal{G})$ whenever u and w are in different parts of the vertex partition, then $\mathcal{G}$ is called *complete multipartite* and we denote $\mathcal{G}$ by $K_{\alpha_1,\ldots,\alpha_m}$. The *line graph* of $\mathcal{G}$ is the graph $L(\mathcal{G})$ with vertex set $V(L(\mathcal{G})) = E(\mathcal{G})$ where any two vertices are joined by a single edge if and only if the corresponding edges in $\mathcal{G}$ have exactly one vertex in common and by a double edge if both of their vertices coincide.

Example. We want to illustrate the above definitions with the example of the bipartite graph $\mathcal{G} = K_{m,n}$ with $1 \leq m \leq n$. Suppose the vertices in $V(\mathcal{G})$ are labelled such that $V_1(K_{m,n}) = \{u_1, \ldots, u_m\}$ and $V_2(K_{m,n}) = \{u_{m+1}, \ldots, u_{m+n}\}$. Any vertex in $V_1(K_{m,n})$ has degree n and any vertex in $V_2(K_{m,n})$ has degree m, that means $K_{m,n}$ is semiregular with degrees $\delta_1 = n$ and $\delta_2 = m$ and the degree sequence of $K_{m,n}$ is

$$(\overbrace{n, \ldots, n}^{m\text{-times}}, \overbrace{m, \ldots, m}^{n\text{-times}}).$$

If $n = m$, then $K_{m,n}$ is regular. For $i = 1, 2$, any two vertices in $V_i(K_{m,n})$ can be joined by a path of length 2 and therefore $\operatorname{diam}(K_{m,n}) = 2$. Since $K_{m,n}$ is a simple graph and only vertices in different parts are adjacent, the shortest cycle has length 4. For a vertex $u \in V_1(K_{m,n})$, the neighbourhood graph $\mathcal{G}_u$ contains all vertices in $V_2(K_{m,n})$ but no other. Because any vertex in $V_2(K_{m,n})$ is only adjacent to vertices in $V_1(K_{m,n})$, the graph $\mathcal{G}_u$ is the union of n isolated vertices.

The line graph of the complete bipartite graph $K_{m,n}$ has the vertices

$$\{\{x, y\} \in V_1(K_{m,n}) \times V_2(K_{m,n})\}$$

and any two vertices $\{x, y\}$ and $\{w, z\}$ are joined by an edge if and only if $\{x, y\} \neq \{w, z\}$ and either $x = w$ or $y = z$. That means, $L(K_{m,n})$ is $(m+n-2)$-regular.

Proposition 2.5 ([CDS79], p. 31)**.** *The line graph $L(\mathcal{G})$ of a connected graph $\mathcal{G}$ is regular if and only if $\mathcal{G}$ is regular or semiregular.*

Example. We want to give another example for the line graph of a graph. Let $n \geq 2$ and $\mathcal{G}$ be the complete graph on $n+1$ vertices, that is $\mathcal{G} = K_{n+1}$. Suppose we have labelled the vertices such that $V(\mathcal{G}) = \{1, \dots, n+1\}$. The vertices of the line graph of $\mathcal{G}$ are the subsets of size 2 of $V(\mathcal{G})$, any two of which are joined by an edge if and only if they have a non-trivial intersection. This defines a well-known graph, the *triangular graph*, which we will denote by $T(n+1)$.

There are several matrices associated with a graph and they are the main tools for studying graphs. A straightforward way to express the adjacency relations of the vertices and edges of a graph in a matrix is the following: the *vertex-edge incidence matrix* of $\mathcal{G}$ is the $v \times e$ matrix $\mathbb{A}_{v,e}(\mathcal{G})$ whose ij-entry is 1 if vertex u_i is incident with edge f_j and 0 otherwise, for all $i \in \{1, \dots, v\}$ and $j \in \{1, \dots, e\}$. The ij-entry of the matrix $\mathbb{A}_{v,e}(\mathcal{G})\mathbb{A}_{v,e}(\mathcal{G})^T$ is for $i \neq j$ the number of edges between vertices u_i and u_j and for $i = j$ the number of edges that are incident with u_i, that is $\delta(u_i)$. The *adjacency matrix* of $\mathcal{G}$ is the $v \times v$ matrix

$$\mathbb{A}(\mathcal{G}) = \mathbb{A}_{v,e}(\mathcal{G})\mathbb{A}_{v,e}(\mathcal{G})^T - \operatorname{diag}(\delta_1, \dots, \delta_v).$$

The *complement* of a simple graph $\mathcal{G}$ is the graph $\bar{\mathcal{G}}$ with adjacency matrix $\mathbb{J}_v - \mathbb{I}_v - \mathbb{A}(\mathcal{G})$.

Lemma 2.6 ([CDS79], p. 44)**.** *The number of walks of length n in $\mathcal{G}$ from u_i to u_j is the ij-entry of the matrix $\mathbb{A}(\mathcal{G})^n$.*

Apart from the adjacency matrix, there is another important matrix whose eigenvalues will play a key role in this thesis: the Laplacian matrix of $\mathcal{G}$,

$$\Lambda(\mathcal{G}) = \operatorname{diag}(\delta_1, \ldots, \delta_v) - \mathbb{A}(\mathcal{G}).$$

We call the eigenvalues and the spectrum of $\mathbb{A}(\mathcal{G})$ also the *eigenvalues* and the *spectrum* of $\mathcal{G}$ and we write $\operatorname{Spec}(\mathcal{G})$. The *Laplacian eigenvalues* and *Laplacian spectrum* of $\mathcal{G}$ are the eigenvalues and the spectrum of $\Lambda(\mathcal{G})$. Note that since both $\mathbb{A}(\mathcal{G})$ and $\Lambda(\mathcal{G})$ are real symmetric matrices, all of their eigenvalues are real. Throughout this text, we order the eigenvalues $\nu_1(\mathcal{G}), \ldots, \nu_v(\mathcal{G})$ of $\mathcal{G}$ and the Laplacian eigenvalues $\rho_1(\mathcal{G}), \ldots, \rho_v(\mathcal{G})$ of $\mathcal{G}$ in weakly decreasing order, that is $\nu_1(\mathcal{G}) \geq \cdots \geq \nu_v(\mathcal{G})$ and $\rho_1(\mathcal{G}) \geq \cdots \geq \rho_v(\mathcal{G})$. The Laplacian matrix is positive definite, hence $\rho_v(\mathcal{G}) \geq 0$. Since the row and column sums of the Laplacian matrix of any graph $\mathcal{G}$ are all equal to zero, the Laplacian matrix $\Lambda(\mathcal{G})$ has the eigenvector $(1, \ldots, 1)^T$ with eigenvalue $\rho_v(\mathcal{G}) = 0$. Due to this, we say that the other Laplacian eigenvalues $\rho_1(\mathcal{G}) \geq \cdots \geq \rho_{v-1}(\mathcal{G})$ are the *non-trivial* Laplacian eigenvalues of $\mathcal{G}$ and we write $\rho_{\mathcal{G}}$ for the vector $(\rho_1(\mathcal{G}), \cdots, \rho_{v-1}(\mathcal{G}))$.

Example. The bipartite graph $K_{m,n}$ has the adjacency matrix that can be partitioned into blocks in the following way:

$$\mathbb{A}(K_{m,n}) = \left(\begin{array}{c|c} 0 & \mathbb{J}_{m\times n} \\ \hline \mathbb{J}_{n\times m} & 0 \end{array}\right).$$

The characteristic polynomial of $\mathbb{A}(K_{m,n})$ is ([Big93], p. 53)

$$\chi_{\mathbb{A}(K_{m,n})}(x) = x^{m+n-2}(x^2 - mn)$$

and therefore the spectrum of $K_{m,n}$ is

$$\mathrm{Spec}(K_{m,n}) = ((-\sqrt{mn})^1, 0^{m+n-2}, (\sqrt{mn})^1)$$

The Laplacian matrix of $K_{m,n}$ is therefore

$$\Lambda(K_{m,n}) = \left(\begin{array}{c|c} n\mathbb{I}_m & -\mathbb{J}_{m\times n} \\ \hline -\mathbb{J}_{n\times m} & m\mathbb{I}_n \end{array}\right).$$

and

$$\mathrm{Spec}(\Lambda(K_{m,n})) = (0^1, m^{n-1}, n^{m-1}, (m+n)^1).$$

Proposition 2.7 ([CDS79], p. 30)**.** *Let $\mathcal{G}$ be a δ-regular graph. Then*

$$\chi_{\mathbb{A}(\mathcal{G})}(\delta) = 0.$$

Proposition 2.8 ([Big93], p. 43)**.** *Let $\mathcal{G}$ be a simple graph and $\bar{\mathcal{G}}$ its complement, then*

$$\Lambda(\bar{\mathcal{G}}) = v\mathbb{I}_v - \mathbb{J}_v - \Lambda(\mathcal{G})$$

and

$$(v-x)\chi_{\Lambda(\bar{\mathcal{G}})}(x) = (-1)^{v-1} x \chi_{\Lambda(\mathcal{G})}(v-x).$$

Corollary 2.9 ([CRS10], p. 185)**.** *Let $\mathcal{G}$ be a simple graph and $\bar{\mathcal{G}}$ its complement, then*

$$\rho_v(\mathcal{G}) = 0 \text{ and } \rho_i(\bar{\mathcal{G}}) = v - \rho_{v-i}(\mathcal{G}) \text{ for } i = 1, \ldots, v-1.$$

If the graph is regular then there is a direct correspondence between the ordinary spectrum and the Laplacian spectrum of $\mathcal{G}$. The following proposition can be found for simple graphs in [Big93], p. 29, but we want to show that it also holds for graphs with multiple edges.

Proposition 2.10. *If $\mathcal{G}$ is a δ-regular graph with eigenvalues $\nu_1(\mathcal{G}) \geq \cdots \geq \nu_v(\mathcal{G})$, then $\rho_{v+1-i}(\mathcal{G}) = \delta - \nu_i(\mathcal{G})$ for $i = 1, \ldots, v$.*

Proof. The Laplacian matrix of $\mathcal{G}$ is $\Lambda(\mathcal{G}) = \delta\mathbb{I}_v - \mathbb{A}(\mathcal{G})$ and for all $i \in \{1, \ldots, v\}$ and any eigenvector $X \in \mathbb{R}^v$ of $\mathbb{A}(\mathcal{G})$ with eigenvalue $\nu_i(\mathcal{G})$ we have

$$\Lambda(\mathcal{G})X = (\delta\mathbb{I}_v - \mathbb{A}(\mathcal{G}))X = \delta\mathbb{I}_v X - \nu_i(\mathcal{G})X = (\delta - \nu_i(\mathcal{G}))X.$$

□

Corollary 2.11. *If $\mathcal{G}$ is a δ-regular graph, then $\nu_1(\mathcal{G}) = \delta$.*

The Laplacian matrix gives us information on the important class of spanning subgraphs: a *spanning forest* F of $\mathcal{G}$ with $j \in \{1, \ldots, v\}$ components is a disjoint union of j trees T_i that are subgraphs of $\mathcal{G}$ with n_i vertices for $i = 1, \ldots, j$, such that $\sum_{i=1}^{j} n_i = v$ for some $j \in \{1, \ldots, v\}$. A *spanning tree* is a subgraph of $\mathcal{G}$ with v vertices that is a tree. The number of spanning trees of $\mathcal{G}$ is denoted by $\kappa(\mathcal{G})$ and is also called the *tree number* of $\mathcal{G}$. The following theorem is a well-known result connecting the tree number and the Laplacian matrix of a graph.

Theorem 2.12 (Matrix–Tree–Theorem, [CDS79], p. 38)**.** *Let $j \in \{1, \ldots, v\}$ and let $\Lambda(\mathcal{G})_j$ denote the matrix obtained from $\Lambda(\mathcal{G})$ by deleting row j and column j. Then*

$$\kappa(\mathcal{G}) = \det(\Lambda(\mathcal{G})_j).$$

Before we can state the main results for the Laplacian spectrum of a graph, we need to study the characteristic polynomial of $\Lambda(\mathcal{G})$. In fact, the polynomial already provides us with some information on the structure of the graph. For any non-empty set $J \subseteq V(\mathcal{G})$ we denote by $\mathcal{G}._J$ the multigraph obtained from $\mathcal{G}$ by identifying the vertices in J, thereby replacing the set of vertices J with a single vertex. By this process multiple edges and loops may be created. Note that loops have no impact on the tree number of a graph and can be deleted. Therefore, we can define $\kappa(\mathcal{G}._J)$ as the number of spanning trees of $\mathcal{G}._J$ after deleting any loops, where we define $\kappa(\mathcal{G}._\emptyset) = 0$.

Theorem 2.13 ([CDS79], p. 38)**.** *Let*

$$\chi_{\Lambda(\mathcal{G})}(x) = x^v - c_1 x^{v-1} + \ldots + (-1)^i c_i x^{v-i} + \ldots + (-1)^{v-1} c_{v-1} x + (-1)^v c_v,$$

then $c_v = 0$ *and for* $i = 0, \ldots, v-1$

$$c_i = \sum_{\substack{J \subset V(\mathcal{G}) \\ |J| = v-i}} \kappa(\mathcal{G}._J)$$

Let $\mathcal{F}_j$ *be the set of all spanning forest of* $\mathcal{G}$ *with* j *components. Then*

$$c_i = \sum_{F \subset \mathcal{F}_{v-i}} \gamma(F),$$

where $\gamma(F) = \prod_{k=1}^{j} n_k$ *for all* $F \in \mathcal{F}_j$*. Here,* F *is a disjoint union of trees on* $n_1, \ldots, n_j$ *vertices.*

Corollary 2.14 ([CDS79], p. 39)**.** *Let* $\chi_{\Lambda(\mathcal{G})}(x) = \sum_{i=0}^{v} (-1)^i c_i x^{v-i}$*. Then*

1. $c_v = 0$ *and*

2. $$\kappa(\mathcal{G}) = \frac{c_{v-1}}{v} = \frac{1}{v} \prod_{i=1}^{v-1} \rho_i(\mathcal{G}).$$

In particular, if $\mathcal{G}$ *is connected, then* $\rho_1(\mathcal{G}), \ldots, \rho_{v-1}(\mathcal{G}) > 0$.

Example. The complete graph K_v has spectrum ([Big93], p. 17)

$$\mathrm{Spec}(K_v) = ((-1)^{v-1}, (v-1)^1)$$

and degree $v-1$, therefore the only non-trivial Laplacian eigenvalue is $\delta+1 = v$ with multiplicity $v-1$. Hence,

$$\kappa(K_v) = \tfrac{1}{v} v^{v-1} = v^{v-2}.$$

Example. The number of spanning trees of $K_{m,n}$ is

$$\frac{1}{v}\prod_{i=1}^{v-1} \rho_i(K_{m,n}) = \frac{(m+n)m^{n-1}n^{m-1}}{m+n} = m^{n-1}n^{m-1}.$$

We can also compute the number of spanning trees of a δ-regular graph $\mathcal{G}$ with the following equation

$$\kappa(\mathcal{G}) = \frac{1}{v}\chi'_{\mathbb{A}(\mathcal{G})}(\delta) \tag{2.2.1}$$

which can be found in [CDS79], p. 39. We want to compute the number of spanning trees of the line graphs of regular graphs. The following proposition can be found in [Big93], p. 40, for simple graphs, but we want to show that it also holds for graphs with multiple edges.

Proposition 2.15. *Let $\mathcal{G}$ be a δ-regular graph on v vertices and e edges. Then the number of spanning trees of its line graph $L(\mathcal{G})$ is*

$$\kappa(L(\mathcal{G})) = 2^{e-v+1}\delta^{e-v-1}\kappa(\mathcal{G}).$$

Proof. By Proposition 2.5 the line graph is regular. A vertex $\{u,w\} \in V(L(\mathcal{G}))$ of the line graph has all the existing edges $\{u,u'\}, \{w',w\} \in E(\mathcal{G})$ as neighbours that are distinct from $\{u,w\}$. That means, if $\mathcal{G}$ is δ-regular, the line graph is $2(\delta-1)$-regular and we can compute the number of spanning trees of $L(\mathcal{G})$ with equation 2.2.1 as

$$\kappa(L(\mathcal{G})) = \frac{1}{e}\chi'_{\mathbb{A}(L(\mathcal{G}))}(2(\delta-1)).$$

We have the following relations between the vertex-edge incidence matrix of $\mathcal{G}$ and the adjacency matrices of $\mathcal{G}$ and $L(\mathcal{G})$ ([CDS79], p. 16):

$$\begin{aligned}\mathbb{A}_{v,e}(\mathcal{G})\mathbb{A}_{v,e}(\mathcal{G})^T &= \mathbb{A}(\mathcal{G}) + \delta\mathbb{I}_v \\ \mathbb{A}_{v,e}(\mathcal{G})^T\mathbb{A}_{v,e}(\mathcal{G}) &= \mathbb{A}(L(\mathcal{G})) + 2\mathbb{I}_v.\end{aligned}$$

From Lemma 2.1 follows that

$$x^e \det(x\mathbb{I}_v - \mathbb{A}(\mathcal{G}) - \delta\mathbb{I}_v) = x^v \det(x\mathbb{I}_v - \mathbb{A}(L(\mathcal{G})) - 2\mathbb{I}_v)$$

and therefore

$$\chi_{\mathbb{A}(L(\mathcal{G}))}(x) = (x+2)^{e-v}\chi_{\mathbb{A}(\mathcal{G})}(x - \delta + 2).$$

The derivative is

$$\chi'_{\mathbb{A}(L(\mathcal{G}))}(x) = (e-v)(x+2)^{e-v-1}\chi_{\mathbb{A}(\mathcal{G})}(x-\delta+2) + (x+2)^{e-v}\chi'_{\mathbb{A}(\mathcal{G})}(x-\delta+2).$$

With $\chi_{\mathbb{A}(\mathcal{G})}(\delta) = 0$ (Proposition 2.7) and setting $x = 2(\delta - 1)$ this gives

$$\chi'_{\mathbb{A}(L(\mathcal{G}))}(2(\delta-1)) = (2\delta)^{e-v}\chi'_{\mathbb{A}(\mathcal{G})}(\delta).$$

From equation 2.2.1 follows that $\chi'_{\mathbb{A}(L(\mathcal{G}))}(2(\delta-1)) = e\kappa(L(\mathcal{G})) = \frac{v\delta}{2}\kappa(L(\mathcal{G}))$ and $\chi'_{\mathbb{A}(\mathcal{G})}(\delta) = v\kappa(\mathcal{G})$.

□

Example ([Big93], p. 41)**.** The number of spanning trees of the triangular graph $T(n)$ is

$$2^{\frac{1}{2}(n^2-3n+2)}(n-1)^{\frac{1}{2}(n^2-3n-2)}n^{n-2}.$$

The following bound is given in [DB05] for weighted graphs, that is simple graphs where every edge has been assigned a positive weight. Of course, we can view a multigraph as a weighted simple graph, where the weights are the multiplicities of the edges. Here, the degree of a vertex is just the sum over the weights of the edges incident with that vertex.

Theorem 2.16 ([DB05]). *Let $\mathcal{G}$ be a connected graph, then*

$$\rho_1(\mathcal{G}) \leq \max\{\delta_u + \delta_w | (u,w) \in E(\mathcal{G})\}$$

with equality if and only if $\mathcal{G}$ is a bipartite regular or semiregular graph.

But the Laplacian eigenvalues are not only good for computing the number of spanning trees. They also give information on other subgraphs as the following propositions show.

Proposition 2.17 ([PR02]). *Let $\mathcal{G}$ be a simple graph with degree sequence $(\delta_1, \ldots, \delta_v)$ and non-trivial Laplacian eigenvalues $\rho_1(\mathcal{G}) \geq \cdots \geq \rho_{v-1}(\mathcal{G})$, then*

1. $$\sum_{i=1}^{v-1} \rho_i(\mathcal{G}) = \sum_{i=1}^{v} \delta_i;$$

2. $$\sum_{i=1}^{v-1} \rho_i(\mathcal{G})^2 = \sum_{i=1}^{v} \delta_i(\delta_i + 1);$$

3. $$\sum_{i=1}^{v-1} \rho_i(\mathcal{G})^3 = \sum_{i=1}^{v} \delta_i(\delta_i + 1)^2 + \eta(\mathcal{G}),$$

   where $\eta(\mathcal{G})$ is the number of V-subgraphs of $\mathcal{G}$.

Proposition 2.18 ([Hae80], p. 17). *Let $\mathcal{G}$ be a δ-regular simple graph on v vertices and let $\nu_1(\mathcal{G}) \geq \cdots \geq \nu_v(\mathcal{G})$ denote the eigenvalues of $\mathcal{G}$. The size of the largest clique of $\mathcal{G}$ is bounded from above by $v\frac{1+\nu_2(\mathcal{G})}{v-\delta+\nu_2(\mathcal{G})}$. This bound is called the* Hoffman bound.

Example. The second largest eigenvalue of $K_{n,n}$ is 0. Since $v = 2n$ it follows that the largest clique in $K_{n,n}$ has at most $\frac{2n}{n} = 2$ vertices. Hence, the complete bipartite regular graph is triangle free for all $n > 0$. In fact, more generally a graph $\mathcal{G}$ is bipartite if and only if $\mathcal{G}$ does not contain any odd cycles ([ADH98], p. 8).

Among regular simple graphs, there are classes of graphs with high symmetry. One of the most well-known classes is the following: a *strongly regular graph* with parameters (δ, λ, μ) is a connected simple δ-regular graph such that

1. each pair of adjacent vertices has exactly λ common neighbours;

2. each pair of non-adjacent vertices has exactly $\mu \geq 1$ common neighbours.

Example. The regular complete bipartite graph $K_{n,n}$ is strongly regular: any two vertices that are joined by an edge must be in different parts of the vertex partition and cannot have a common neighbour, that is $\lambda = 0$. On the other hand, if two vertices are not adjacent, that means they are both in the same part and every vertex in the other part is a common neighbour and therefore $\mu = n$.

Example. For $n \neq 7$, the triangular graph $T(n+1)$ is the unique strongly regular graph on $\frac{n(n+1)}{2}$ points, degree $2(n-1)$ and parameters $\lambda = n-1$ and $\mu = 4$ ([Con58, Shr59, Hof60]). For $n = 7$, there exist precisely three other graphs with the same parameters, known as the Chang graphs ([Cha60]).

Lemma 2.19 ([BCN89], p. 11)**.** *The parameters λ and μ of a strongly regular graph on v vertices and degree δ satisfy*

$$\delta(\delta - \lambda - 1) = (v - \delta - 1)\mu.$$

The following proposition tells us exactly what the adjacency matrix and its spectrum look like if the graph is strongly regular.

Proposition 2.20 ([BCN89], p. 8)**.** *The adjacency matrix $\mathbb{A}(\mathcal{G})$ of a strongly regular graph $\mathcal{G}$ with parameters (δ, λ, μ) satisfies the equation*

$$\mathbb{A}(\mathcal{G})^2 + (\mu - \lambda)\mathbb{A}(\mathcal{G}) + (\mu - \delta)\mathbb{I}_v = \mu \mathbb{J}_v.$$

Any eigenvalue other than δ is a solution to the equation

$$x^2 + (\mu - \lambda)x + (\mu - \delta) = 0.$$

The eigenvalues of $\mathcal{G}$ are δ with multiplicity 1 and

$$\nu_{1,2}(\mathcal{G}) = \frac{1}{2}\left[(\lambda - \mu) \pm \sqrt{(\lambda - \mu)^2 + 4(\delta - \mu)}\right]$$

with multiplicities

$$m_{1,2} = \frac{1}{2}\left[(v-1) \mp \frac{2\delta + (v-1)(\lambda - \mu)}{\sqrt{(\lambda - \mu)^2 + 4(\delta - \mu)}}\right].$$

Example. We can check with the above proposition that our previous calculation of the spectrum of the regular complete bipartite graph $K_{n,n}$ is correct: indeed we have $\delta = n$ and

$$\nu_{1,2}(K_{n,n}) = \frac{1}{2}\left[-n \pm \sqrt{n^2 + 4(n-n)}\right] = \frac{-n \pm n}{2}.$$

A strongly regular graph is a special case of a *distance regular* graph. This is a simple regular graph of diameter $\mathrm{diam}(\mathcal{G})$ that allows integers $\beta_i, \gamma_i, \alpha_i$ for $i = 0, \ldots, \mathrm{diam}(\mathcal{G})$ such that for any two points u, w at distance i, there are precisely $\gamma_i, \beta_i, \alpha_i$ neighbours of w at distance $i-1, i+1, i$ from u respectively. The numbers $\beta_i, \gamma_i, \alpha_i$ for $i = 0, \ldots, \mathrm{diam}(\mathcal{G})$ are called *intersection numbers*. In particular, $\mathcal{G}$ is regular with degree $\delta = \beta_0$. As for strongly regular graphs, we know exactly what the spectrum of a distance regular graph looks like.

Proposition 2.21 ([BCN89], pp. 126, 131)**.** *Let $\mathcal{G}$ be a distance regular graph with intersection numbers $\beta_i, \gamma_i, \alpha_i$ for $i = 0, \ldots, \operatorname{diam}(\mathcal{G})$. Define the polynomials*

$$\omega_0(x) = 1,\ \omega_1(x) = x \text{ and}$$

$$\omega_{i+1}(x) = (x - \alpha_i)\omega_i(x) - \gamma_i\beta_{i-1}\omega_{i-1}(x) \text{ for } i = 1, \ldots, \operatorname{diam}(\mathcal{G}).$$

The eigenvalues of $\mathbb{A}(\mathcal{G})$ are exactly the roots of $\omega_{\operatorname{diam}(\mathcal{G})+1}(x)$. The multiplicity $m(\theta)$ of an eigenvalue θ is given by

$$m(\theta) = \frac{v}{\sum_{i=0}^{\operatorname{diam}(\mathcal{G})} \frac{(\omega_i(\theta))^2 h_i}{(\beta_0 \cdots \beta_{i-1})^2}} \quad ,$$

where

$$h_0 = 1,\ h_1 = \beta_0 = \delta \text{ and } h_{i+1} = \frac{h_i\beta_i}{\gamma_{i+1}} \text{ for } i = 1, \ldots, \operatorname{diam}(\mathcal{G}) - 1.$$

2.3 Block Designs

Let $\mathcal{P}$ and $\mathcal{B}$ be non-empty disjoint sets; we say that $\mathcal{P}$ is the set of points and $\mathcal{B}$ is the set of blocks. Further, let $\mathcal{I}$ be a binary symmetric incidence relation on $\mathcal{P} \cup \mathcal{B}$ such that for any $\omega_1, \omega_2 \in \mathcal{P}$ or $\omega_1, \omega_2 \in \mathcal{B}$ with $\omega_1 \mathcal{I} \omega_2$ we have $\omega_1 = \omega_2$. Then $d = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is called a *block design.* The size of the blocks is assumed to be a constant k. Although it is possible to have block designs with non-constant block size, this case will not occur in this text. The number r_i of blocks incident with point i is called the *replication number* of i. For the rest of the section, let $d = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ denote a block design on v points and b blocks of size k and replication numbers $r_1, \ldots, r_v$. The design is called *complete* if $k = v$ and every point occurs once in each block and *incomplete* if $k < v$. Unless otherwise stated, we will always assume that the designs are incomplete. A *class* $\mathcal{D}$ of designs is a set of designs on v points and b blocks of size k with or without any additional attributes.

The *point-block incidence matrix* of d is the $v \times b$ matrix $\mathbb{A}_{v,b}(d)$ whose ij-entry is 1 if point i occurs in block j and 0 otherwise, for $i = 1, \ldots, v$ and $j = 1, \ldots, b$. The *adjacency matrix* of d is the $v \times v$ matrix

$$\mathbb{A}(d) = \mathbb{A}_{v,b}(d)\mathbb{A}_{v,b}(d)^T - \mathrm{diag}(r_1, \ldots, r_v).$$

That is for $i \neq j$, the ij-entry a_{ij} of $\mathbb{A}(d)$ is the number of blocks containing both points i and j. The graph $\mathcal{G}(d)$ with vertex set $V(\mathcal{G}) = \mathcal{P}$ where any two vertices $v_i, v_j \in V(\mathcal{G})$ are joined by exactly a_{ij} edges for $i \neq j$, is called the *adjacency graph* of d. We say that two points are *adjacent* if they are joined by an edge in $\mathcal{G}$. Of course, the graph can have multiple edges. The design is called *connected* if the adjacency graph is connected. We will always assume that all designs are connected. The design is called *equireplicate*, if the replication numbers of all points are constant; note that this is only possible if $\frac{bk}{v} \in \mathbb{N}$. We denote by $\mathcal{D}_{v,b,k}$ the class of connected equireplicate designs on v points with b blocks of size k and replication $r = \frac{bk}{v}$. For a design in $\mathcal{D}_{v,b,k}$ with replication r, the adjacency graph is an $r(k-1)$-regular graph. In the following, we will describe a design by writing the b blocks as rows of the k points.

Example. The design given by

$$\begin{array}{ccccc} 1\ 2 & 1\ 3 & 1\ 4 & 1\ 5 & 2\ 3 \\ 2\ 4 & 2\ 5 & 3\ 4 & 3\ 5 & 4\ 5 \end{array}$$

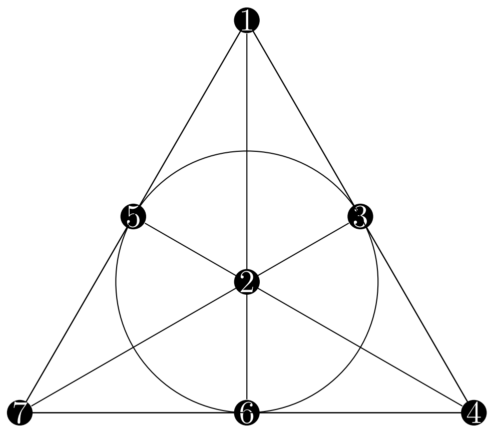
has point and block sets

$$\mathcal{P} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$$

$$\mathcal{B} \;=\; \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}\}.$$

The design has block size 2 and the replication of each point is 4. Since any two points are contained in exactly one block, the design is connected. In fact, the adjacency graph is the complete graph on 5 points and the design belongs to the class $\mathcal{D}_{5,10,2}$.

Example. Another example of an equireplicated design is the *Fano plane*:



The points of the design are the points of the plane; here numbered from 1 to 7. The blocks correspond to the lines of the plane, which include the circle through 3, 5, 6. That means, the design is given by

1	5	7	1	3	4	1	2	6	2	4	5
2	3	7	3	5	6	4	6	7			

Therefore $k = 3$ and $r = 3$ and the design belongs to the class $\mathcal{D}_{7,7,3}$.

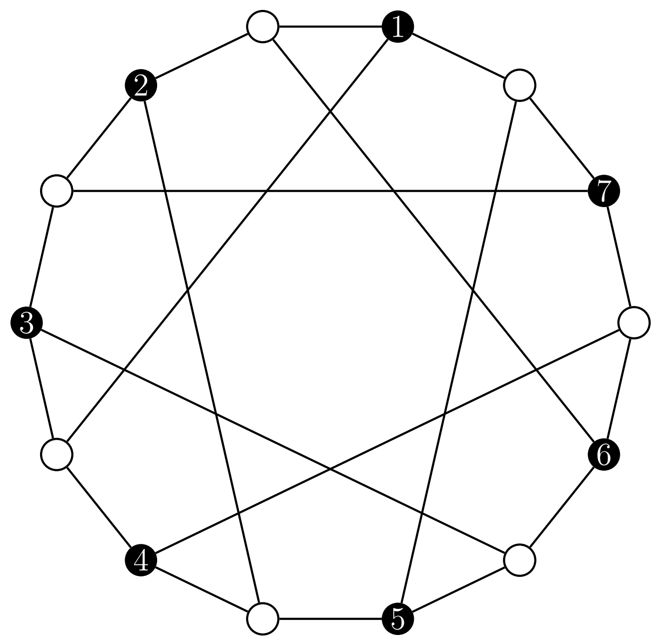
The Fano plane is a good example for a special class of designs: a design is said to be *symmetric* if $v = b$.

The *Laplacian matrix* of the design d is the Laplacian matrix of the adjacency graph, that is

$$\begin{aligned}\Lambda(d) = \Lambda(\mathcal{G}(d)) &= (k-1)\,\mathrm{diag}(r_1, \ldots, r_v) - \mathbb{A}(d) \qquad (2.3.1)\\ &= k\,\mathrm{diag}(r_1, \ldots, r_v) - \mathbb{A}_{v,b}(d)\mathbb{A}_{v,b}(d)^T\end{aligned}$$

and we can apply all the conventions and results from the previous section to $\Lambda(d)$. We will call the Laplacian eigenvalues of $\mathcal{G}(d)$ also the *Laplacian eigenvalues* of d and write $\rho_i(d)$ for $\rho_i(\mathcal{G}(d))$, $i = 1, \ldots, v$, and ρ_d for the vector of the non-trivial Laplacian eigenvalues $\rho_{\mathcal{G}(d)}$. There is another graph associated with the design d: the *incidence graph* or *Levi graph*. This is the graph $\Gamma(d)$ that has the points and blocks of the design as vertices of which any two are joined by an edge if they are an incident point-block pair. Since no two points and no two blocks are joined by and edge, we can partition the vertex set $V(\Gamma(d))$ into $V_1(\Gamma(d)) = \mathcal{P}$ and $V_2(\Gamma(d)) = \mathcal{B}$, hence the graph is bipartite.

Example. The incidence graph of the Fano plane, called the *Heawood graph*, is the following graph:



Here, the black vertices correspond to the points and the white vertices correspond to the blocks of the Fano plane.

We can see in the Heawood graph, that we also could have interchanged the colouring of the vertices, i. e. saying the black vertices correspond to blocks and the white vertices correspond to points. In general however, it does matter which colour we have assigned to vertices or blocks, but we could switch the meaning of the colours for points and blocks to obtain a new design. The resulting design is the *dual design* $d^* = (\mathcal{B}, \mathcal{P}, \mathcal{I}^*)$ of d, where $B\mathcal{I}^*u$ if and only if $u\mathcal{I}B$ for any $u \in \mathcal{P}$ and $B \in \mathcal{B}$. For convenience we also write $\mathcal{I}$ for $\mathcal{I}^*$. Suppose $d \in \mathcal{D}_{v,b,k}$, then any vertex in $V_1(\Gamma(d))$ has exactly r neighbours in $V_2(\Gamma(d))$ and any vertex in $V_2(\Gamma(d))$ has exactly k neighbours in $V_1(\Gamma(d))$. That means $d^* \in \mathcal{D}_{b,v,r}$. A *morphism* between designs d and d' is a map from d to d' mapping points to points and blocks to blocks preserving the incidence relations. If a morphism is bijective, then it is called an *isomorphism*.

Example. Let $k = 2$. Then every edge of the adjacency graph $\mathcal{G}(d)$ corresponds to a block. Therefore $\mathbb{A}_{v,b}(d) = \mathbb{A}_{v,e}(\mathcal{G}(d))$. The dual design d^* has adjacency graph $\mathcal{G}(d^*)$ with vertex set $E(\mathcal{G}(d))$. Any two vertices are joined by an edge if they have a vertex in $\mathcal{G}(d)$ in common. That means, $\mathcal{G}(d^*)$ is the line graph $L(\mathcal{G}(d))$ of $\mathcal{G}(d)$.

2.4 Optimality of Block Designs

The motivation for studying designs in the way it is presented in this thesis comes from the background of statistical designs of experiments. We want to give a short introduction to this following [BC09] and [BC13] where more details and a good overview can be found. For more details see [Bai08] and [SS89]. A statistical experimental design describes the allocation of v treatments to experimental units. In this setting it is more convenient to think of the set of units to be partitioned into b blocks of size k together with a function Ξ from the set of units to the set of v treatments specifying which treatment is allocated to which unit (the parameters v, b, k are always given). That is, for a unit ω the image $\Xi(\omega)$ is the treatment allocated to ω. Although units in different blocks might differ systematically, all units in a block are assumed to be alike. Our aim is to find out about the treatments and their differences.

For example, suppose we want to test fertilizers on different farms each of which has several plots. Here, the units are the plots, the blocks correspond to the farms and of course, the treatments are the fertilizers. We can now partition the set of units into blocks regarding to which farm the plot belongs. Now, the farms might be far apart such that the weather or the soil might have an impact on how well the crops grow, that means the blocks differ systematically, but the weather or soil on one farm is the same on all its plots.

We can recover our combinatorial definition of a block design as follows: assume that any treatment is contained at most once per block; the statistical experimental design is then called *binary*. Now we can take the v treatments as points and take the blocks to be sets of points. However, the previous definition of the Laplacian matrix stays valid for non-binary designs: define the point-block incidence matrix $\mathbb{A}_{v,b}(d)$ of a non-binary design d to have the number of times point i occurs in block j as the ij-entry. Then the Laplacian matrix $\Lambda(d)$ can be defined as in equation 2.3.1. The ij-entry of $\Lambda(d)$ counts the number of occurrences of the point pair $\{i, j\}$ in blocks according to multiplicity. Note that the adjacency graph might now have loops but we can define a non-binary design to be connected as before if the adjacency graph is. There can be good reasons for allowing a treatment to occur more than once per block, for example if self-interaction is to be studied. But unless otherwise stated, we will assume that the designs are binary. This does not mean however, that non-binary designs are not of interest, for example see [BC09]. But most of the designs we are interested in happen to be binary.

Now, suppose we have conducted an experiment where we applied treatments to units in blocks where we allow for now the design to be non-binary. For each unit ω we measure the response Y_ω. If $\omega \in B$ and for some block $B \in \mathcal{B}$, we assume that

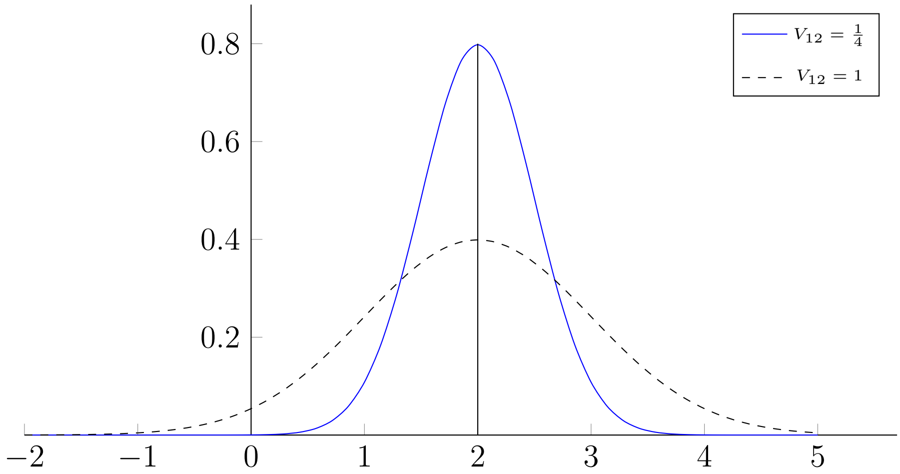
$$Y_\omega = \tau_{\Xi(\omega)} + \beta_B + \epsilon_\omega, \tag{2.4.1}$$

where τ_i is a constant depending on treatment i, β_B is a constant depending on block B and ϵ_ω is a random variable with expectation 0 and variance σ^2. Moreover, if $\omega_1 \neq \omega_2$ then ϵ_{ω_1} and ϵ_{ω_2} are uncorrelated. Of course, we can add a constant to all treatment parameters and substract it from all block parameters without changing equation 2.4.1 and therefore it is impossible to estimate the individual treatment parameters. But if the design is connected, we can estimate any linear combination of the form $\sum_i x_i \tau_i$ where $\sum_i x_i = 0$. In particular, we can estimate all pairwise differences $\tau_i - \tau_j$ and get information on which of the treatments (compared to all other treatments) is best. A *linear unbiased estimator* is a linear function of the responses Y_ω (and therefore is itself a random variable) such that its expectation is equal to the true value. The *best* linear unbiased estimator is the one with the least variance.

We want to study a simple example: let $v = 2$ and all units form a single block. Suppose we have allocated the two treatments t_1 and t_2 to r_1 and r_2 units. In this case, the best linear unbiased estimator for the difference $\tau_1 - \tau_2$ of the parameters of the treatments t_1, t_2 is the difference of the average responses of the treatments t_1 and t_2. Its variance V_{12} is

$$V_{12} = \left(\frac{1}{r_1} + \frac{1}{r_2}\right)\sigma^2,$$

which is minimized if $r_1 = r_2$. If the design is binary, then necessarily $k = 2$ and $r_1 = r_2 = 1$ and therefore $V_{12} = 2\sigma^2$. The following picture shows the likelihood that the estimator with expectation 2 takes on a particular value for different variances if the responses are normally distributed.



We see that the smaller V_{12}, the more likely it is that the estimate is close to the expected value. Moreover, for $V_{12} = 1$ the estimator is giving a negative value and even wrongly suggests that treatment t_2 is better than t_1 in about 2.3% of the cases. That means we want to minimize the variance of the estimator in order to be more likely to choose the right treatment as the best one. The 95% confidence interval of the estimator, that is the interval that will contain the true value in 19 cases out of 20, is the interval of length $t(r_1 + r_2 - 2)\sqrt{V_{12}}$ centered at the true value, where $t(n)$ is the 97.5th percentile of the t-distribution on n degrees of freedom. The function $t(n)$ decreases as n increases, with a limiting value of 1.96, which is near-enough achieved by $n = 30$. The length of the interval can therefore be decreased by increasing $r_1 + r_2$ or decreasing $|r_1 - r_2|$ or σ^2. In the above picture, for $V_{12} = \frac{1}{4}$ this is the interval $[1, 3]$ and for $V_{12} = 1$ it is $[0, 4]$. The smaller V_{12}, the smaller the confidence interval and the more likely it is that our estimate is close to the true value.

But if $v > 2$ and $k < v$, things are more complicated. To deal with this case, we will need the *Penrose inverse* of the Laplacian matrix $\Lambda(d)$ of a connected design d, which is the following matrix:

$$\Lambda(d)^- = (\Lambda(d) + \tfrac{1}{v}\mathbb{J}_v)^{-1} - \tfrac{1}{v}\mathbb{J}_v.$$

Note that we need the assumption that d is connected for $\Lambda(d) + \frac{1}{v}\mathbb{J}_v$ to be invertible. Now, the variances of the estimators of the pairwise differences can be computed in the following way.

Theorem 2.22 ([BC13]). *Let $\Lambda(d)$ be the Laplacian matrix of a connected block design d with Penrose inverse $\Lambda(d)^- = (\Lambda^-_{ij})$. If $\sum_i x_i = 0$, then the variance of the best linear unbiased estimator of $\sum_i x_i \tau_i$ is equal to $(x^T \Lambda(d)^- x) k\sigma^2$. In particular, the variance V_{ij} of the best linear unbiased estimator of the simple difference $\tau_i - \tau_j$ is given by $V_{ij} = \left(\Lambda^-_{ii} + \Lambda^-_{jj} - 2\Lambda^-_{ij}\right) k\sigma^2$.*

Using the above theorem, we obtain the variance V_{ij} for every treatment pair $\{t_i, t_j\}$ with $i \neq j$ and we have now a multidimensional problem. In this setting, a statistical experimental design can be 'good' in different ways. For one, minimizing the average of the variances V_{ij} would be one thing to wish for, this is called the *A-criterion*. Or minimizing the volume of the multidimensional analog to the confidence interval, the confidence ellipsoid, is another criterion called the *D-criterion*. For $v = 2$ these criteria are equivalent with minimizing the variance, but things are different in the multidimensional case. Still, for both of the just mentioned criteria, the value that we want to minimize can be expressed as a function of the non-trivial eigenvalues of the Laplacian matrix:

Proposition 2.23 ([BC09]). *Let $\Lambda(d)$ be the Laplacian matrix of a connected block design d with non-trivial eigenvalues $\rho_1(d), \ldots, \rho_{v-1}(d)$. Let V_{ij} be the variance of the best linear unbiased estimator of the difference $\tau_i - \tau_j$, $i, j = 1, \ldots, v$ with $i \neq j$ as in Theorem 2.22. Then*

$$\overline{V} = \frac{1}{v(v-1)} \sum_i \sum_{j \neq i} V_{ij} = \frac{2k\sigma^2}{v-1} \sum_{i=1}^{v-1} \frac{1}{\rho_i(d)},$$

and the volume of the confidence ellipsoid is proportional to the reciprocal of $\sqrt{\prod_{i=1}^{v-1} \rho_i(d)}$.

This is the motivation to define general optimality criteria as functions of these eigenvalues. More precisely, let $\mathbb{M}_v$ be the set of all $v \times v$ real symmetric matrices with row and column sums zero. An *optimality criterion* is a function $\Psi : \mathbb{M}_v \to \mathbb{R}$ if it has the following properties.

1. Ψ is convex;
2. for every $\Lambda \in \mathbb{M}_v$ the function $\alpha \mapsto \Psi(\alpha\Lambda)$ is monotonic non-increasing for non-negative α;
3. Ψ is invariant under any simultaneous permutation of rows and columns by the same permutation – that is, re-labelling the points does not affect Ψ.

Now a design in some class $\mathcal{D}$ of designs is said to be Ψ-*optimal* if it minimizes the value of $\Psi(\Lambda)$ over all Laplacian matrices Λ of designs in $\mathcal{D}$. A design is said to be *universally optimal* in some class if it minimizes every Ψ satisfying the above conditions. Of course, there are a lot of different optimality criteria. The most popular fall under the umbrella of the following optimality criterion. For $p \in (0, \infty)$, a design is Φ_p-*optimal* if it minimizes

$$\left(\frac{\sum_{i=1}^{v-1} \rho_i(d)^{-p}}{v-1}\right)^{\frac{1}{p}}$$

among all designs d in a class. For $p = 1$ we get the A-*criterion*; the limit as $p \to 0$ is equivalent to the above definition of the D-*criterion*; the limit as $p \to \infty$ is called the E-*criterion* ([BC09], p. 13). The reason that A-, D- and E-optimality are the most popular optimality criteria is the application to statistical experimental designs. But we will see later that all of these three criteria correspond to interesting graph properties; in particular D- and E-optimality have been studied by graph theorists extensively in their own right.

However, we will work mostly with the following equivalent definitions for A-, D- and E-optimality:

Remark 2.24 ([BC09], pp. 12)**.** A block design is A-optimal if it maximizes the harmonic mean of the non-trivial Laplacian eigenvalues, which we will also call the *A-value*. D-optimality corresponds to maximizing the product of the non-trivial Laplacian eigenvalues, the *D-value*, and E-optimality is maximizing the smallest non-trivial Laplacian eigenvalue, the *E-value*. We will sometimes denote the A-, D- and E-values of a design d by $A(d)$, $D(d)$ and $E(d)$ respectively.

We want to define another optimality criterion that will be used in this thesis: let $\mathcal{D}$ be a class of designs on v points and define $\Psi_f : \mathcal{D} \to \mathbb{R}$ by

$$\Psi_f(d) = \sum_{i=1}^{v-1} f(\rho_i(d)) \text{ for all } d \in \mathcal{D},$$

where $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ satisfies the following conditions:

1. $f''(x) > 0$, $f'''(x) < 0$ for $x > 0$;
2. $\lim_{x \to 0^+} f(x) = \infty$.

Note that these criteria cover the standard criteria: for D-optimality, take $f(x) = -\log(x)$ and for A-optimality take $f(x) = x^{-1}$. The E-criterion is also covered as a pointwise limit of criteria derived from functions satisfying the conditions in the above definition ([CB91]).

As in the case of graphs, there are classes of designs with high symmetry. One of the most popular are the t-(v, k, λ)-designs, these are designs where each subset of $\mathcal{P}$ of size t is contained in a constant number $\lambda > 0$ of blocks. If $k < v$, a 2-(v, k, λ)-design is also called a balanced incomplete block design, in short BIBD. An example is the Fano plane which is a 2-$(7, 3, 1)$-design. BIBDs are the designs with the highest possible symmetry and we will see later that they are optimal in the most general way. Let Λ be the Laplacian matrix of a not necessarily binary design. We want to know when the trace of Λ is maximized. Let m_{iB} be the number of occurrences of point i in block B. Then $\Lambda_{ij} = \sum_B m_{iB} m_{jB}$ and with $\sum_i m_{iB} = k$ we have

$$\begin{aligned}
\text{Trace}(\Lambda) &= \sum_i \sum_{j \neq i} \Lambda_{ij} \\
&= \sum_i \sum_B m_{iB} \sum_{j \neq i} m_{jB} \\
&= \sum_i \sum_B m_{iB}(k - m_{iB}) \\
&= bk^2 - \sum_B \sum_i m_{iB}^2.
\end{aligned}$$

To compute the optimal solution we will need the following lemma.

Lemma 2.25 ([GP70])**.** *Let $m, n \in \mathbb{N}$, then the optimization problem*

$$\begin{aligned}
\textit{minimize} \quad & \sum_{i=1}^{n} x_i^2 \\
\textit{subject to} \quad & \sum_{i=1}^{n} x_i = m, \\
& x_1, \dots, x_n \in \mathbb{N}_{\geq 0}
\end{aligned}$$

has the unique optimal solution

$$\begin{aligned}
x_i &= \lfloor \tfrac{m}{n} \rfloor + 1, \qquad i = 1, \dots, m - n\lfloor \tfrac{m}{n} \rfloor \\
x_i &= \lfloor \tfrac{m}{n} \rfloor, \qquad\quad i = m - n\lfloor \tfrac{m}{n} \rfloor + 1, \dots, n.
\end{aligned}$$

Since $\sum_i m_{iB} = bk$, the optimal solution to minimizing $\sum_B \sum_i m_{iB}^2$ is given by Lemma 2.25 with $m = bk$ and $n = vb$ as taking $b(k - v\lfloor \frac{k}{v} \rfloor)$ occurrences m_{iB} to be $\lfloor \frac{k}{v} \rfloor + 1$ and the rest to be $\lfloor \frac{k}{v} \rfloor$. This proves the following theorem.

Theorem 2.26 ([BC09], p. 8)**.** *A design d in a class $\mathcal{D}$ of not necessarily binary designs maximizes the trace of $\Lambda(d)$ among all designs in $\mathcal{D}$ if and only if each point occurs $\lfloor \frac{k}{v} \rfloor$ or $\lceil \frac{k}{v} \rceil$ times in each block. If $k < v$, the traces of the Laplacian matrices are maximized by the binary designs; the maximum value is $bk(k-1)$.*

This leads to the following characterization of optimal designs which is one of the earliest results and is due to Kiefer ([Kie75]).

Theorem 2.27 ([BC09], p. 14)**.** *Let $\Psi : \mathbb{M}_v \to \mathbb{R}$ be an optimality criterion. If there is a design d in a class $\mathcal{D}$ of not necessarily binary designs for which $\Lambda(d)$ is completely symmetric and has maximum trace, then it is universally optimal in $\mathcal{D}$. In particular, if $\mathcal{D}_{v,b,k}$ contains 2-designs, then the minimum value of $\Psi(\Lambda)$ over Laplacian matrices of designs in $\mathcal{D}_{v,b,k}$ is attained by the 2-designs.*

But for most values of v, b, k there exist no 2-designs. To deal with these cases, one main approach that has been suggested for $k < v$ are the binary designs where the adjacency matrix has only entries λ and $\lambda + 1$ off the diagonal for some $\lambda \in \mathbb{N}$. Drawing an edge between two vertices if the corresponding entry is $\lambda + 1$, produces a simple graph, the *underlying graph.* These designs are called *nearly balanced incomplete block designs*, in short NBDs. If the design is equireplicate with replication r, then d has Laplacian matrix $\Lambda(d)$ that we can write with $\lambda = \lfloor \frac{r(k-1)}{v-1} \rfloor$ as

$$\Lambda(d) = (r(k-1) + \lambda)\mathbb{I}_v - \lambda \mathbb{J}_v - \mathbb{A}(\mathcal{G}),$$

where $\mathcal{G}$ is a simple, regular graph of degree $\delta = r(k-1) - \lambda(v-1)$. For this reason the design is in this case called *regular graph design* (in short RGD). If $\mathcal{M}(v, \delta)$ denotes the set of all simple, not necessarily connected, δ-regular graphs on v points then any RGD on v points and b blocks of size k corresponds to a $\mathcal{G} \in \mathcal{M}(v, \delta)$. If the underlying graph is strongly regular, the design is also called *strongly regular graph design*, in short SRGD.

It has been conjectured that if there exist RGDs in a class, then an A-optimal or D-optimal or E-optimal design is among them ([JM77]). This conjecture has been proven to be wrong in general ([Bai07]), but in some special cases RGDs are indeed optimal.

Theorem 2.28 ([Che81a]). *Let* $\mathcal{G} = K_{n,n} + y * K_{2n}$ *for some* $n, y \in \mathbb{N}$ *and* $n > 0$. *Then* $\mathcal{G}$ *is the unique graph on* $2n$ *vertices and* $n^2 + yn(2n-1)$ *edges that minimizes any criterion* Ψ_f. *In particular,* $\mathcal{G}$ *is the unique graph that has the maximum number of spanning trees among all graphs with* $2n$ *vertices and* $e = n(n + y(2n-1))$ *edges.*

In the same paper, Cheng partially extended Theorem 2.28 to regular complete multipartite graphs with the following theorem.

Theorem 2.29 ([Che81a]). *Let* $\mathcal{G}$ *be the regular complete* m*-partite graph* $K_{\alpha,\ldots,\alpha}$ *on* αm *vertices. Then* $\mathcal{G}$ *is the unique simple graph with* αm *vertices and* $\frac{\alpha^2 m(m-1)}{2}$ *edges that minimizes any criterion* Ψ_f.

Theorem 2.30 ([CB91]). *If* $\mathcal{D}_{v,b,k}$ *contains a connected strongly regular graph design* d *whose Laplacian matrix has eigenvalue* rk, *then* d *is optimal over* $\mathcal{D}_{v,b,k}$ *with respect to any criterion* Ψ_f. *In particular,* d *is* A-, D- *and* E*-optimal. Moreover, the dual design* d^* *is also optimal over* $\mathcal{D}_{b,v,r}$ *with respect to the same criteria.*

The last part of the above theorem can be generalized as follows: suppose d is any design in $\mathcal{D}_{v,b,k}$; then d has the Laplacian matrix

$$\Lambda(d) = rk\mathbb{I}_v - \mathbb{A}_{v,b}(d)\mathbb{A}_{v,b}(d)^T$$

and its dual design d^* has the Laplacian matrix

$$\Lambda(d^*) = rk\mathbb{I}_b - \mathbb{A}_{v,b}(d)^T\mathbb{A}_{v,b}(d).$$

The eigenvalues of $\mathbb{A}_{v,b}(d)\mathbb{A}_{v,b}(d)^T$ and $\mathbb{A}_{v,b}(d)^T\mathbb{A}_{v,b}(d)$ are the same including the multiplicities apart from $|b-v|$ extra zeros. It follows that the last part of the above theorem can also be written as follows.

Proposition 2.31 ([BC09], p. 25). *A given design $d \in \mathcal{D}_{v,b,k}$ is Φ_p-optimal over $\mathcal{D}_{v,b,k}$ if and only if the dual design is Φ_p-optimal over $\mathcal{D}_{b,v,r}$.*

2.5 Optimality and Graphs

In this section we want to go into detail in what way A-, D- and E-optimality correspond to graph properties. Let $\mathcal{G}$ be a graph on v vertices and e edges and let $\rho_1(\mathcal{G}) \geq \cdots \geq \rho_{v-1}(\mathcal{G})$ be its non-trivial Laplacian eigenvalues.

D-optimality We have seen that for the number of spanning trees the following equation holds:

$$\kappa(\mathcal{G}) = \frac{c_{v-1}}{v} = \frac{1}{v}\prod_{i=1}^{v-1}\rho_i(\mathcal{G}).$$

Since a design is D-optimal if it maximizes the product of its Laplacian eigenvalues, its adjacency graph maximizes the number of spanning trees among all graphs that give rise to a design with the appropriate parameters. Therefore, we call a graph D-optimal if it maximizes the number of spanning trees among all graphs on v points and e edges. The number of spanning trees has been studied widely by graph theorists and has a number of applications, see for example [Shi74, NP06, PBS98, Che81a, CM85]. For instance, consider a graph as a network. If there is a spanning tree, then any two vertices can communicate and maximizing the number of spanning trees therefore maximizes the reliability of the network.

The next proposition gives an upper bound for the tree number that is reached if and only if the graph is the complement of a union of cliques, that is a complete graph or a complete multipartite graph.

Proposition 2.32 ([PR02]). *For any simple graph $\mathcal{G}$ on v vertices with degree sequence $(\delta_1, \ldots, \delta_v)$ and tree number $\kappa(\mathcal{G})$ we have*

$$\kappa(\mathcal{G}) \leq v^{v-2} \exp\left(\tfrac{-2\eta(\bar{\mathcal{G}})}{3v^3}\right) \prod_{i=1}^{v} (1 - \frac{1+\delta_i}{v})^{\frac{\delta_i}{\delta_i+1}},$$

where $\eta(\bar{\mathcal{G}})$ is the number of V-subgraphs of the complement of $\mathcal{G}$. The above inequality is an equality if and only if $\bar{\mathcal{G}}$ is a disjoint union of cliques.

Using the above proposition, Petingi and Rodriguez show in the same paper the following special case of Theorem 2.28.

Proposition 2.33 ([PR02]). *The almost-regular complete multipartite graph $K_{\alpha_1,\ldots,\alpha_m}$ where $|\alpha_i| \in \{n, n+1\}$ for $i = 1, \ldots, m$ for some $n \in \mathbb{N}_{>0}$, maximizes the number of spanning trees among all simple graphs with the same number of vertices and edges.*

Recall that we denote by $y * K_v$ the complete multigraph on v vertices where any pair of vertices is joined by exactly y edges. If we add a subgraph of $y * K_v$ to itself, the following proposition might give in some cases an easier method to compute the tree number of the resulting graph.

Proposition 2.34 ([NP06]). *Suppose $\mathcal{G}$ is a subgraph of $y * K_v$ on $l \leq v$ vertices and $y \geq 1$. Then*

$$\kappa(y * K_v \pm \mathcal{G}) = y(vy)^{v-l-2} \det(vy\mathbb{I}_l \pm \Lambda(\mathcal{G})).$$

***A*-optimality** Let $\chi_{\Lambda(\mathcal{G})}(x) = \sum_{j=0}^{v}(-1)^j c_j x^{v-j}$ be the characteristic polynomial of $\Lambda(\mathcal{G})$. Theorem 2.13 states that

$$c_j = \sum_{\substack{J \subset V(\mathcal{G}) \\ |J| = v-j}} \kappa(\mathcal{G}_{\cdot J}).$$

For $u, w \in V(\mathcal{G})$, a spanning tree of $\mathcal{G}._{\{u,w\}}$ corresponds to a spanning forest in $\mathcal{G}$ with two parts, one containing u and the other containing w. Such a spanning forest is called a *thicket* separating u and w. On the other hand, the coefficients can be computed ([Bro06]) as

$$c_j = \sum_{J \subseteq I, |J|=j} \prod_{i \in J} \rho_i(\mathcal{G})$$

and in particular

$$c_{v-2} = \sum_{j=1}^{v-1} \frac{\prod_{i=1}^{v-1} \rho_i(\mathcal{G})}{\rho_j(\mathcal{G})}.$$

Since A-optimality of a design with adjacency graph $\mathcal{G}$ is maximizing the harmonic mean of the non-trivial Laplacian eigenvalues, A-optimality is equivalent to maximizing

$$\frac{c_{v-1}}{c_{v-2}} = \frac{v\kappa(\mathcal{G})}{\sum_{u,w \in V(\mathcal{G})} \kappa(\mathcal{G}._{\{u,w\}})} \tag{2.5.1}$$

among all graphs giving rise to a design with the appropriate parameters.

A-optimality has an important application to electrical networks: suppose the graph $\mathcal{G}$ represents an electrical network where we assigned each edge a resistance of one-ohm. Given any two vertices i and j, the effective resistance R_{ij} between them is the voltage of a battery which, when connected to the two vertices, causes a current of 1 ampere to flow. There is a connection between the effective resistance, thickets and spanning trees:

$$R_{ij} = \frac{\text{number of thickets separating } i \text{ and } j}{\kappa(\mathcal{G})};$$

for more details see [BC09], pp. 29, and [Bol98], pp. 39, 296. Now, with the A-value defined as in equation (2.5.1), we see by taking the sum over all pairs $i \neq j$ on both sides that the A-value is inversely proportional to the sum over all effective resistances.

E-optimality E-optimality corresponds to maximizing the smallest non-trivial Laplacian eigenvalue of $\mathcal{G}$. As with D-optimality, this too is an area that has been studied extensively by graph theorists. In particular, there has been wide interest on the bounds on this eigenvalue, which is also called *algebraic connectivity* of the graph. This definition has its reason in the correspondence between the eigenvalue and the connectivity of the graph. For example, the graph is connected if and only if the algebraic connectivity is strictly positive ([Fie75], see also Corollary 2.14) and the vertex connectivity (i.e. the minimum number of vertices that need to be deleted to disconnect the graph) is an upper bound if the graph is not complete. We will also need the following main results on the algebraic connectivity.

Theorem 2.35 ([Fie75]). *Let δ_{min} be the smallest degree in a simple graph $\mathcal{G}$. Then*

$$\rho_{v-1}(\mathcal{G}) \leq \frac{v\delta_{min}}{v-1}.$$

Let U be a subset of the vertex set $V(\mathcal{G})$ of a simple graph $\mathcal{G}$. Then the *edge-boundary* $\mathcal{S}(U)$ is the set of edges of $\mathcal{G}$ from U to $V(\mathcal{G}) \setminus U$.

Theorem 2.36 ([Bol98], p. 270). *Let $\mathcal{G}$ be a simple graph. For $U \subset V(\mathcal{G})$ we have*

$$|\mathcal{S}(U)| \geq \frac{\rho_{v-1}(\mathcal{G})|U||V(\mathcal{G}) \setminus U|}{v}.$$

Theorem 2.37 (Exercise 50 in [Bol98], p. 289). *Let δ_{max} be the maximal degree of a simple graph $\mathcal{G}$ and let*

$$\alpha = \frac{2\rho_{v-1}(\mathcal{G})}{\delta_{max} + 2\rho_{v-1}(\mathcal{G})}.$$

Then for every $U \subset V(\mathcal{G})$ with $|U| \leq \frac{v}{2}$ there are at least $\alpha|U|$ vertices not in U

that are joined to vertices in U.

This means, the larger the algebraic connectivity the harder it is to disconnect the graph. For example, if the graph represents an electrical network this means the larger the algebraic connectivity the more robust that network is to cut wires, or generally to flow failing to go through edges.

There are many more results on upper bounds in terms of various properties of the graph, see for example [Kir01, Kir00, LLT05, GB06].

2.6 Majorization and Schur-Optimality

For this section let $I = \{1, \ldots, v-1\}$. A stronger optimality criterion than Φ_p-optimality arises from the theory of majorization. All of the following definitions and an extensive overview can be found in [MO79].

Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ such that $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$. We say that the sequence x is *majorized* by y, if

$$\sum_{i=1}^{l} x_i \leq \sum_{i=1}^{l} y_i \text{ for } l = 1, \ldots, n-1.$$

and

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i.$$

Let $\mathcal{A} \subset \mathbb{R}^n$. A function $\Phi : \mathcal{A} \to \mathbb{R}^m$ such that, if x is majorized by y then $\Phi(x) \leq \Phi(y)$, is called *Schur-convex.* If, in addition, $\Phi(x) < \Phi(y)$ whenever x is not a permutation of y, then Φ is called *strictly Schur-convex.* The function Φ such that, if x is majorized by y then $\Phi(x) \geq \Phi(y)$, is called *Schur-concave* and *strictly Schur-concave* if in addition $\Phi(x) > \Phi(y)$ whenever x is not a permutation of y. A design is called *Schur-optimal* if its non-trivial Laplacian eigenvalues majorize the non-trivial Laplacian eigenvalues of any competing design. A Schur-optimal design minimizes any Schur-convex function and maximizes any

Schur-concave function of the non-trivial Laplacian eigenvalues. An example for Schur-concave functions are the *elementary symmetric polynomials*

$$S_j(z) = \sum_{J\subseteq I, |J|=j} \prod_{i\in J} z_i.$$

for $1 \leq j \leq v-1$ and $S_0(z) = 1$ for any vector $z = (z_1, \ldots, z_{v-1}) \in \mathbb{R}^{v-1}$.

Proposition 2.38 ([MO79], p. 78)**.** *For $j = 1, \ldots, v-1$, the function S_j is increasing and Schur-concave on $\mathbb{R}^{v-1}_{\geq 0}$. If $j \neq 1$, then S_j is strictly Schur-concave on $\mathbb{R}^{v-1}_{>0}$.*

Since the D-value is the product over all non-trivial Laplacian eigenvalues, the D-value is $S_{v-1}(\rho_d)$ and is therefore Schur-concave. Since we can write the A-value as the ratio $(v-1)\frac{S_{v-1}(\rho_d)}{S_{v-2}(\rho_d)}$, by the following proposition the A-value is a Schur-concave function as well.

Proposition 2.39 ([MO79], p. 80)**.** *The ratio $\frac{S_j}{S_{j-1}}$ is Schur-concave on $\mathbb{R}^{v-1}_{>0}$ for $j = 1, \ldots, v-1$. If $j \neq 1$, then the ratio $\frac{S_j}{S_{j-1}}$ is strictly Schur-concave on $\mathbb{R}^{v-1}_{>0}$.*

Suppose there exists a design d whose non-trivial Laplacian eigenvalues are majorized by the Laplacian eigenvalue of any other design. By the above propositions we know that d is A- and D-optimal and as the following proposition asserts, this is also true for the Φ_p-criterion.

Proposition 2.40 ([BC09], p. 14)**.** *If a design is Schur-optimal within any class of designs then it is also Φ_p-optimal for all p, in particular A-, D- and E-optimal.*

We will need the following fact about elementary symmetric polynomials which can be found in [Mac95], p. 21.

$$jS_j(z) = \sum_{l=1}^{j} (-1)^{l-1} S_{j-l}(z) \sum_{i=1}^{v-1} z_i^l \quad \text{(Newton's Identities).} \tag{2.6.1}$$

The symmetric polynomial omitting one coordinate will be denoted by

$$S_{j;l}(z) := \sum_{J\subset I\setminus\{l\},|J|=j} \prod_{i\in J} z_i,$$

or generally omitting all coordinates z_i for $i \in J \subset I$ by $S_{j;J}(z)$. We have ([BB65], p. 34)

$$\sum_{l=1}^{v-1} S_{j;l}(z) = (v-1-j)S_j(z). \tag{2.6.2}$$

The elementary symmetric polynomials occur as coefficients of the characteristic polynomial of a matrix:

Proposition 2.41 ([Bro06])**.** *Let M be a $v\times v$ matrix with eigenvalues $\nu_1,\ldots,\nu_v$, then*

$$\chi_M(x) = \sum_{j=0}^{v}(-1)^j S_j(\nu_1,\ldots,\nu_v)x^{v-j}.$$

Corollary 2.42. *Let Λ be a Laplacian matrix of a design with Laplacian eigenvalues $\rho_1(d),\ldots,\rho_v(d)$. Then*

$$S_j(\rho_1(d),\ldots,\rho_v(d)) = S_j(\rho_1(d),\ldots,\rho_{v-1}(d)) \in \mathbb{N}$$

for $j = 1,\ldots,v-1$ and $S_v(\rho_1(d),\ldots,\rho_v(d)) = 0$.

Proof. This follows from Theorem 2.13 and Proposition 2.41. Since $\rho_v(d) = 0$ and $S_v(\rho_1(d),\ldots,\rho_v(d))$ is the product over all eigenvalues of Λ, the statement follows. □

2.7 Finite Geometries

By Theorem 2.26 and Theorem 2.30 structures that are 'nice' from the combinatorial point of view are optimal under a wide range of criteria. This is the reason why we want to look more closely at other such 'nice' designs and see if they give rise to optimal designs as well.

First of all, we want to define what a geometry is. As before we have points as our basic elements, but there is no reason to stop with blocks. We can extend block designs to higher-rank incidence structures having points, blocks (then called 'lines'), planes and so on. Let Ω be a finite non-empty set and $\{\Omega_1, \ldots, \Omega_n\}$ a partition of Ω with $n \geq 2$ and $\Omega_i \neq \emptyset$ for $i = 1, \ldots, n$. We will call elements $\alpha \in \Omega_i$ *of type* i for $i = 1, \ldots, n$. Further, let $\mathcal{I}$ be a binary, reflexive and symmetric incidence relation on Ω. Then $(\Omega_1, \ldots, \Omega_n, \mathcal{I})$ is called *incidence structure* of rank n. An incidence structure $\mathbb{G} = (\Omega_1, \ldots, \Omega_n, \mathcal{I})$ is called *geometry* of rank n if

1. for any ω and ξ of the same type with $\omega \mathcal{I} \xi$ follows that $\omega = \xi$;

2. if any set of pairwise incident elements, called a *flag*, can be extended to a maximal flag of n elements.

We will deal with higher rank a little later, but for now we want to look closer at rank-2 geomtries. Of course, we can think of a rank-2 geometry as a block design and can apply all the conventions, definitions and results from the previous sections.

Example. Again, the Fano plane (see page 31) is a good example in this case. Here, we have points and lines and a flag is for example a single point. Since there is always a line incident with that point we can extend any flag to a maximal flag consisting of an incident point-line pair.

The Fano plane is the smallest example for one of the most popular geometries, the *projective planes*; a projective plane is a rank-2 geometry satisfying the following additional conditions.

1. any two points are incident with exactly one common line;

2. any two lines are incident with exactly one common point;

3. there exist four points each three of which are not incident with the same line.

Of course, the Fano plane satisfies the first two conditions. In the labelling as in the picture on page 31, the points $2, 3, 4, 6$ are four points of which at most 2 are incident with a common line.

The projective planes belong to the most famous class of rank-2 geometries; the finite *generalized N-gons* for an $N \in \mathbb{N}_{>0}$. These are geometries $\mathbb{G}$ with an incidence graph $\Gamma(\mathbb{G})$ that satisfies the following conditions:

- every vertex of $\Gamma(\mathbb{G})$ is on at least two edges;
- $\Gamma(\mathbb{G})$ is connected, bipartite and has diameter N and girth $2N$.

Example. The Fano plane is a generalized triangle. Its incidence graph, the Heawood graph, is connected, bipartite and 3-regular. Its diameter is 3 and it has girth 6.

A finite generalized polygon that allows finite constants s and t such that any point lies on $t + 1$ lines and there are $s + 1$ points on a line is said to have *parameters* s and t which we will denote by $\mathbb{G}_N(s, t)$.

Example. In the generalized 2-gon with parameters s and t every point is incident with every line and vice versa. That means, $\mathbb{G}_2(s, t)$ has the incidence graph $K_{s+1,t+1}$.

Switching the meaning of lines and points of $\mathbb{G}_N(s, t)$ gives the dual design, which is of course $\mathbb{G}_N(t, s)$.

Example. We have seen that in the Heawood graph it does not matter which colours we assign to points and blocks of the Fano plane. That means, the dual of the Fano plane is the Fano plane itself.

Generalized N-gons are very important in the theory of geometries, because they build up a special class of higher rank geometries with nice properties, called 'buildings'. These kind of geometries are a well studied area, see for example [Ron89] and [AB08] for an introduction. The next theorem is one of the most fundamental results in this area.

Theorem 2.43 ([Ron89] p. 30)**.** *Let $N \in \mathbb{N}_{>0}$. A finite generalized N-gon has either $s = t = 1$ or $N \in \{2, 3, 4, 6, 8, 12\}$.*

But not only do we know that for $(s,t) \neq (1,1)$ there exist generalized N-gons only for certain N, we can say even more about the parameters:

Theorem 2.44 ([BCN89], p. 201)**.** *Let $N \in \mathbb{N}_{>0}$. Let (s,t) be the parameters of a finite generalized N-gon. If $s > 1$ and $t > 1$ then*

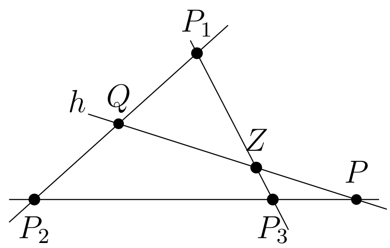
1. *$N \neq 12$;*
2. *if $N = 4$ then $s \leq t^2$ and $t \leq s^2$;*
3. *if $N = 6$, then st is a square and $s \leq t^3$ and $t \leq s^3$;*
4. *if $N = 8$, then $2st$ is a square and $s \leq t^2$ and $t \leq s^2$.*

Since we can think of rank-2 geometries as block designs, we can apply the results from the previous sections on optimality and in fact for $(s,t) \neq (1,1)$ we already know that some are optimal:

for $N = 3$, the distance between any two vertices of the incidence graph is at most 3. Therefore, any two points lie on at least one line and any two lines intersect in at least one point. We want to show that any two points are incident with exactly one line and dually, any two lines are incident with exactly two points. Assume, this is not true and for $j \in \{1,2\}$ there exist elements $\omega_1, \omega_2 \in \Omega_j$ and $\xi_1, \xi_2 \in \Omega_i$ for $i \in \{1,2\} \setminus \{j\}$, such that

$$\omega_1 \mathcal{I} \xi_1,\ \omega_2 \mathcal{I} \xi_1 \text{ and } \omega_1 \mathcal{I} \xi_2,\ \omega_2 \mathcal{I} \xi_2.$$

Then $\Gamma(\mathbb{G}_3(s,t)$ contains the closed path $(\omega_1, \xi_1, \omega_2, \xi_2, \omega_1)$ of length 4. This is a contradiction to girth$(\Gamma(\mathbb{G}_3(s,t)) = 6$. It follows, that any two points are on exactly one line and any two lines intersect in exactly one point. Hence, the generalized triangle and its dual are both 2-designs and Theorem 2.26 gives us universal optimality in this case. Note that we have shown that the first two properties for a projective plane are satisfied. For the last one we need to construct four points of which no three are on the same line. Since girth$(\Gamma(\mathbb{G}_3(s,t)) = 6$, there exist at least three distinct lines with three distinct intersection points P_1, P_2 and P_3. On every line there are at least three points, hence the lines P_1P_2 and P_1P_3 contain points Q and P. Since the intersection of any two lines is unique, Q and P are distinct from P_1, P_2 and P_3. Because any two points lie on a common line, the line QP exists and it intersects with every other line in exactly one point. In particular, QP intersects P_1P_3 in a point Z that is distinct from Q and P. The following picture shows the points and lines we have just constructed.



Now, the points P_2, P_3, Q and Z are four points of which no three are on a common line. This means, generalized triangles with at least 3 points on a line are projective planes and in fact they are exactly the projective planes ([AB08], p. 180).

Now, for $N = 4$, the adjacency graph is strongly regular on $(s+1)(st+1)$ vertices with degree $s(t+1)$ and parameters $\lambda = s-1$ and $\mu = t+1$ ([BCN89], p. 200). The eigenvalues of the adjacency matrix are $s(t+1)$, $s-1$ and $-(t+1)$ by Theorem 2.20 and the largest Laplacian eigenvalue is therefore

$s(t+1)+(t+1) = (s+1)(t+1)$. The generalized quadrangle, denoted by $\mathrm{GQ}(s,t)$ in the following, is a design with replication $t+1$ and block size $s+1$. Since the largest Laplacian eigenvalue equals rk, it follows from Theorem 2.30, that $\mathrm{GQ}(s,t)$ is A-, D- and E-optimal among equireplicate binary designs.

The generalized quadrangle is an example of an important class of geometries that all satisfy the properties of Theorem 2.30, the partial geometries: a *partial geometry* with parameters $s, t, \alpha \in \mathbb{N}$ is an incidence structure of rank 2 satisfying the following axioms:

1. any line contains $s+1$ points, any point lies on $t+1$ lines;
2. two lines meet in at most one point;
3. if the point u is not on the line h, then there are precisely α incident pairs (w, h'), where w is a point of h and h' is a line through u.

As the generalized quadrangles, the partial geometries are A-, D- and E-optimal among binary equireplicate designs by Theorem 2.30.

But we want to study higher rank geometries. First of all, here are two examples that both have the symmetric group $\mathbb{S}_{n+1}$ as automorphism group.

Example. For $i = 1, \ldots, n$ let Ω_i be the set of subsets of $\{1, \ldots, n+1\}$ of size i. Any two subsets are called incident if one is contained in the other. This defines a geometry which we will denote by $\mathbb{G}(n)$ in the following chapters.

Example. Let $q \geq 2$ be a prime or a prime power. Let $\mathcal{V}$ be an $(n+1)$-dimensional vector space over the finite field $\mathrm{GF}(q)$. For $i \in \{1, \ldots, n\}$, let $\mathrm{PG}^i(n,q)$ denote the set of subspaces of $\mathcal{V}$ of dimension i and let $\mathrm{PG}(n,q)$ be the corresponding projective space, that is the incidence structure

$$\mathrm{PG}(n,q) = (\mathrm{PG}^1(n,q), \ldots, \mathrm{PG}^n(n,q), \leq).$$

Proposition 2.45. $\mathrm{PG}(n,q)$ *is a geometry of rank* n.

Proof. The properties of a geometry are quickly verified: if two subspaces of the same dimension are incident, they have a non-trivial intersection and therefore must coincide. Any flag can be extended to a maximal flag by the Basis–Extension Theorem. □

For the rest of this chapter, let $\mathbb{G} = (\Omega_1, \dots, \Omega_n, \mathcal{I})$ be a geometry and $n > 2$. We want to apply the theory we have for optimal block designs also to $\mathbb{G}$. How can we do that? We can get back to a rank-2 structure from a geometry by just taking two different types of elements, say points and lines, and forget about the rest of the elements and use as incidence relation the relation induced by the geometry. This incidence structure is

$$\mathrm{Tr}(\mathbb{G}) = (\Omega_i, \Omega_j, \mathcal{I}'), \text{ for } i < j,$$

where $\omega \mathcal{I}' \xi$ if $\omega \mathcal{I} \xi$ for all $\omega \in \Omega_i$, $\xi \in \Omega_j$ and is called $\{i,j\}$-*truncation*. For convenience we will write $\mathcal{I}$ for $\mathcal{I}'$.

Example. Let $j > 1$. The set of points and the set of elements of type j of the geometry $\mathbb{G}(n)$ form the $\{1,j\}$-truncation. For example, for $n = 4$ and $j = 2$ this is

$$\begin{aligned} \Omega_1 &= \mathcal{P} = \{\{1\},\{2\},\{3\},\{4\},\{5\}\} \\ \Omega_2 &= \mathcal{B} = \{\{1,2\},\{1,3\},\dots,\{1,5\},\{2,3\},\dots,\{3,4\},\{3,5\},\{4,5\}\}. \end{aligned}$$

The neighbours of the point $\{1\}$ in the adjacency graph are $\{\{2\},\{3\},\{4\},\{5\}\}$. The design has block size 2 and replication 4. The adjacency graph has degree $r(k-1) = 4$.

Example. The subsets of $\{1,\dots,n+1\}$ of size 2 and 3 form the $\{2,3\}$-truncation of the geometry $\mathbb{G}(n)$. For example for $n = 4$ the design has the following points and blocks

$$\begin{aligned}
\Omega_1 = \mathcal{P} &= \{\{1,2\},\{1,3\},\ldots,\{1,5\},\{2,3\},\ldots,\{3,4\},\{3,5\},\{4,5\}\} \\
\Omega_2 = \mathcal{B} &= \{\{1,2,3\},\{1,3,4\},\{1,2,5\},\{1,4,5\},\{1,2,4\},\{1,3,5\}, \\
&\quad \{2,3,4\},\{2,3,5\},\{2,4,5\},\{3,4,5\}\}
\end{aligned}$$

and the neighbours in the adjacency graph of the point $\{1,2\}$ for example are

$$\underbrace{\{2,3\},\{1,3\}}_{\text{block }\{1,2,3\}},\underbrace{\{1,4\},\{2,4\}}_{\text{block }\{1,2,4\}},\underbrace{\{1,5\},\{2,5\}}_{\text{block }\{1,2,5\}}$$

We will see the truncations of the geometries from our examples in detail in Chapter 6.

Chapter 3

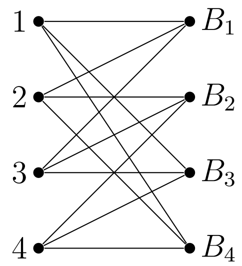
Two-replicate Resolvable Designs

A connected design d_{res} with replication $r = 2$ whose set of blocks can be partitioned into two sets such that any point is incident with one block of each set is called *two-replicate resolvable design.* Let d_{symm} denote a (possibly non-binary) symmetric equireplicate design with v points, replication r and $b = v$ blocks of size $k = r$. Patterson and Williams show in [PW75] that every two-replicate binary design d_{res} is uniquely determined by a symmetric equireplicate design d_{symm} by taking the vertices of the incidence graph $\Gamma(d_{symm})$ as blocks of the design d_{res} and the edges of $\Gamma(d_{symm})$ as points of d_{res}. The design d_{res} has vk points, $2v$ blocks, replication 2 and block size k.

Example ([WPJ76])**.** Let d_{symm} be the design with blocks being the rows of

$$\begin{aligned} B_1 &: \quad 1 \quad 2 \quad 3 \\ B_2 &: \quad 2 \quad 3 \quad 4 \\ B_3 &: \quad 3 \quad 4 \quad 1 \\ B_4 &: \quad 4 \quad 1 \quad 2. \end{aligned}$$

The incidence graph $\Gamma(d_{symm})$ of d_{symm} is



,

whose edges are the points of the design d_{res}. Any two points (x, B_i) and (y, B_j) are incident in d_{res} if either $x = y$ or $i = j$. That means, the blocks of d_{res} are the sets $\{(x, B_i)|x = 1, 2, 3, 4 \text{ and } (x, B_i) \in E(\Gamma(d_{symm}))\}$ for $i = 1, 2, 3, 4$ and the sets $\{(x, B_i)|i = 1, 2, 3, 4 \text{ and } (x, B_i) \in E(\Gamma(d_{symm}))\}$ for $x = 1, 2, 3, 4$. Any point of d_{res} is contained in exactly two blocks, either the one corresponding to the first or to the second coordinate.

Because the points of d_{res} are the edges in $\Gamma(d_{symm})$, any two points of d_{res} are in either 0, 1 or 2 blocks depending on the number of vertices on which the edges coincide. For $k < v$, the latter case occurs if and only if $\Gamma(d_{symm})$ is not simple, that is if d_{symm} is not binary.

Patterson and Williams show in [PW75] that if ρ is a Laplacian eigenvalue of d_{symm}, then $1 \pm \frac{\sqrt{1-\rho}}{2}$ are Laplacian eigenvalues of d_{res}, proving the following proposition.

Proposition 3.1 ([PW75]). *For $k = 2$, a connected binary two-replicate resolvable incomplete block design is E-optimal among all connected binary two-replicate resolvable designs iff the corresponding symmetric design is E-optimal among all symmetric designs; in particular, if the corresponding symmetric design is a BIBD.*

If d_{res} is connected then its A-value $A(d_{res})$ is the following function of v, k

and $A(d_{symm})$, the A-value of d_{symm} ([WPJ76]):

$$A(d_{res}) = \frac{vk - 1}{vk - 2v + 1 + 4(v-1)\frac{1}{A(d_{symm})}},$$

and this proves the following proposition.

Proposition 3.2 ([WPJ76]). *A connected binary two-replicate resolvable incomplete block design is A-optimal among all connected binary two-replicate resolvable designs iff the corresponding symmetric design is A-optimal among all symmetric designs; in particular, if the corresponding symmetric design is a BIBD.*

Since d_{symm} is symmetric and equireplicate of replication r, it follows that $\Gamma(d_{symm})$ is regular of degree r. Since $\mathcal{G}(d_{res})$ is the line graph of $\Gamma(d_{symm})$, Proposition 2.15 tells us that for $e = |E(\mathcal{G}(d_{symm}))|$

$$\kappa(\mathcal{G}(d_{res})) = 2^{e-v+1} r^{e-v-1} \kappa(\Gamma(d_{symm})).$$

The number of spanning trees of $\Gamma(d_{symm})$ is closely related to the number of spanning trees of $\mathcal{G}(d_{symm})$ as the following result by Gaffke shows.

Proposition 3.3 ([Gaf82]). *Let d be a connected incomplete block design on v points and b blocks of size k. Then*

$$\kappa(\Gamma(d)) = k^{b-v+1} \kappa(\mathcal{G}(d)).$$

It follows with $r = k$ that

$$\kappa(\mathcal{G}(d_{res})) = 2^{e-v+1} k^{b+e-2v} \kappa(\mathcal{G}(d_{symm})),$$

proving the following corollary.

Corollary 3.4. *A connected binary two-replicate resolvable incomplete block design is D-optimal among all connected binary two-replicate resolvable designs iff the corresponding equireplicate (possibly non-binary) symmetric design is D-optimal over all symmetric designs, in particular if it is a BIBD.*

That means the above example of a two-replicate resolvable design is also D-optimal. Moreover, the corollary shows that A- and D-optimality of designs among connected binary two-replicate resolvable incomplete block designs are equivalent. This is not true in general, there are many examples where an A-optimal design is not D-optimal (see [Bai07]).

The following theorem shows, that we can apply these results on certain generalized polygons. For the rest of this chapter let $(s, t) \neq (1, 1)$.

Theorem 3.5 ([BCN89], p. 201)**.** *Let $n \in \mathbb{N}_{>0}$ and let (s, t) be the parameters of a generalized n-gon. If $s = 1$, then n is even, say $n = 2N$, and the adjacency graph of the generalized $2N$-gon is the point-line incidence graph of a generalized N-gon with parameters (t, t).*

Proof. To highlight the correspondence between the adjacency and incidence graphs of N- and $2N$-gons we want to present a short proof of the theorem. The incidence graph of $\mathbb{G}_N(t, t)$ has diameter N and girth $2N$, hence is the adjacency graph of a generalized $2N$-gon $\mathbb{G}_{2N}(1, t)$ if we take every edge to represent a line. Note that the incidence graph of $\mathbb{G}_N(t, t)$ is bipartite, that means that the points of $\mathbb{G}_{2N}(1, t)$ can be split up into two groups such that any two points in the same group are not on a common line. Conversely, the adjacency graph of $\mathbb{G}_{2N}(1, t)$ has diameter N and girth $2N$. To show that it is precisely the incidence graph of a generalized N-gon with parameters $s = t$, we need to prove that it is bipartite. But this follows from the fact that the graph cannot contain any odd cycles ([ADH98], p. 8). □

We have mentioned before that if we have a design with block size 2, the adjacency graph of the dual design is the line graph of the adjacency graph. That means, the adjacency graph of $\mathbb{G}_{2N}(t,1)$ is the line graph of the incidence graph $\Gamma(\mathbb{G}_N(t,t))$.

Example. The generalized 2-gon $\mathbb{G}_2(2,2)$ has the regular complete bipartite graph $K_{3,3}$ as incidence graph. Therefore, $K_{3,3}$ is the adjacency graph of a generalized quadrangle $\mathbb{G}_4(1,2)$ and the line graph $L(K_{3,3})$ is the adjacency graph of $\mathbb{G}_4(2,1)$.

Example. The Fano plane is a generalized triangle with parameters $s=2$ and $t=2$. Its incidence graph, the Heawood graph (see page 32), is the adjacency graph of the generalized hexagon $\mathbb{G}_6(1,2)$. The adjacency graph of $\mathbb{G}_6(2,1)$ is the line graph of the Heawood graph, which is also known as the $(2,3,7)$-Bower graph. We can speak of 'the' generalized hexagon in these cases, because the ones with parameters $(1,q)$ and $(q,1)$ for any prime power $q \leq 8$ are unique ([BCN89], p. 204).

For the other cases, we want to use Proposition 3.2. By [BCN89], p. 203, a $2N$-gon has the parameters $r=t+1,\ k=s+1$ and

1. if $N=2$ then $v=k(st+1)$ and $b=(t+1)(st+1)$;
2. if $N=3$ then $v=k(1+st+s^2t^2)$ and $b=(1+t)(1+st+s^2t^2)$;
3. if $N=4$ then $v=k(1+st)(1+s^2t^2)$ and $b=(1+t)(1+st)(1+s^2t^2)$.

That means $\mathbb{G}_{2N}(s,1)$ satisfies the conditions of Proposition 3.2 and Corollary 3.4 for $N=2,3,4$, and $\mathbb{G}_{2N}(s,1)$ is an A- and D-optimal design among resolvable designs if the corresponding $\mathbb{G}_N(s,s)$ is A- and D-optimal among all symmetric designs. For $N=3$, we have shown earlier that the corresponding $\mathbb{G}_3(s,s)$ is a 2-design and is as such universally optimal. Thus we have the following corollary.

Corollary 3.6. *The generalized hexagons with $t = 1$ and $s > 1$ are A- and D-optimal among binary resolvable connected designs with replication $r = 2$.*

In particular, the generalized hexagon with the $(2, 3, 7)$-Bower graph as adjacency graph is A- and D-optimal among binary two-resolvable designs.

For $N = 4$, a generalized octagon $\mathbb{G}_8(s, 1)$ is determined by a generalized quadrangle $\mathbb{G}_4(s, s)$. In this case we can not directly apply Proposition 3.2 and Corollary 3.4, because we only know that $\mathbb{G}_4(s, s)$ are A- and D-optimal among connected binary equireplicate designs (see Proposition 2.30). This means we have to restrict the class of two-resolvable binary designs to designs that correspond to a binary symmetric design. We have seen that these are the two-resolvable binary designs where any two points occur in at most one block.

Corollary 3.7. *The generalized octagons with $t = 1$ and $s > 1$ are A- and D-optimal among binary two-resolvable connected designs with simple adjacency graph.*

Chapter 4

On the Algebraic Connectivity of Regular Graphs

Throughout this chapter, let $\mathcal{G}$ be a simple connected regular graph on v vertices with degree δ, where $1 < \delta < v - 1$. Recall that we order the eigenvalues of $\mathcal{G}$ in decreasing order, that is $\delta = \nu_1(\mathcal{G}) \geq \nu_2(\mathcal{G}) \geq \cdots \geq \nu_v(\mathcal{G})$.

4.1 On the Neighbourhood Graphs of Strongly Regular Graphs

Lemma 4.1. *Let $\mathcal{G}$ be a strongly regular graph with degree δ and parameters λ and μ. The neighbourhood graph $\mathcal{G}_u$ for any vertex $u \in V(\mathcal{G})$ is regular and has degree λ.*

Proof. Any vertex in $\mathcal{G}_u$ has exactly λ common neighbours with u since $\mathcal{G}$ is strongly regular. □

Proposition 4.2. *Let $\mathcal{G}$ be a strongly regular graph with degree δ and parameters λ and μ. If*

$$\lambda > \tfrac{1}{2}\left(\lambda - \mu + \sqrt{(\lambda - \mu)^2 + 4(\delta - \mu)}\right),$$

then the neighbourhood graph $\mathcal{G}_u$ of any vertex $u \in V(\mathcal{G})$ is connected.

Proof. Suppose there is a vertex u such that $\mathcal{G}_u$ is not connected and let $\mathcal{C}(\mathcal{G}_u)$ be a connected component of $\mathcal{G}_u$ of size $\gamma < \delta$. We divide the adjacency matrix of $\mathcal{G}$ into 16 block matrices M_{ij} according to u and the vertices of $\mathcal{C}(\mathcal{G}_u)$, $\mathcal{G}_u \setminus \mathcal{C}(\mathcal{G}_u)$ and $\mathcal{G} \setminus \{u, \mathcal{G}_u\}$. Let $\bar{m}_{ij}$ denote the average row sum of M_{ij}. Then the matrix $M = (\bar{m}_{ij})$ is of the form

$$M = \begin{pmatrix} 0 & \gamma & \delta - \gamma & 0 \\ 1 & \lambda & 0 & \delta - (\lambda + 1) \\ 1 & 0 & \lambda & \delta - (\lambda + 1) \\ 0 & \frac{\gamma(\delta - (\lambda+1))}{v - (\delta+1)} & \frac{(\delta-\gamma)(\delta-(\lambda+1))}{v-(\delta+1)} & \delta - \frac{\delta(\delta-(\lambda+1))}{v-(\delta+1)} \end{pmatrix}.$$

The characteristic polynomial of M is

$$\chi_M(x) = (\lambda - x)(\delta - x)\left(x^2 + \left(\frac{\delta(\delta-1) - \lambda(v-1)}{v - \delta - 1}\right)x + \frac{\delta(2\delta - \lambda - v)}{v - \delta - 1}\right)$$

and the vector $(1,1,1,1)^T$ is an eigenvector of M with eigenvalue δ. Since by Lemma 2.19 the parameters of a strongly regular graph satisfy

$$\mu(v - \delta - 1) = \delta(\delta - \lambda - 1),$$

it follows that

$$\frac{\delta(\delta-1) - \lambda(v-1)}{v - \delta - 1} = \frac{\delta(\delta - \lambda - 1)}{(v - \delta - 1)} - \lambda = \mu - \lambda$$

and

$$\delta\frac{2\delta - \lambda - v}{v - \delta - 1} = \frac{\delta(\delta - \lambda - 1)}{(v - \delta - 1)} - \delta = \mu - \delta.$$

Therefore,

$$\chi_M(x) = (\lambda - x)(\delta - x)(x^2 + (\mu - \lambda)x + \mu - \delta)$$

and the eigenvalues of M are

$$\delta, \lambda, \tfrac{1}{2}\left(\lambda - \mu \pm \sqrt{(\lambda-\mu)^2 + 4(\delta-\mu)}\right).$$

By Theorem 2.4, the eigenvalues of M interlace the eigenvalues of the adjacency matrix of $\mathcal{G}$ and it follows directly that

$$\nu_2(\mathcal{G}) = \tfrac{1}{2}\left(\lambda - \mu + \sqrt{(\lambda-\mu)^2 + 4(\delta-\mu)}\right) \geq \lambda.$$

□

Corollary 4.3. *Let $\mathcal{G}$ be a strongly regular graph with degree δ and parameters λ and μ. If $\lambda = \nu_2(\mathcal{G})$, then the size of every connected component of $\mathcal{G}_u$ is divisible by $\lambda + 1$.*

Proof. Since $\nu_2(\mathcal{G}) = \frac{1}{2}\left(\lambda - \mu + \sqrt{(\lambda-\mu)^2 + 4(\delta-\mu)}\right)$, if $\lambda = \nu_2(\mathcal{G})$ then $\lambda = \frac{\delta-\mu}{\mu}$, that is $\delta = \mu(\lambda+1)$. Let u be a vertex of $\mathcal{G}$ and $\mathcal{G}_u$ the neighbourhood graph. Now, if $\mathcal{G}_u$ is connected, then the size of $\mathcal{G}_u$ is δ which is divisible by $\lambda+1$. Suppose $\mathcal{G}_u$ is not connected. Then we can divide the adjacency matrix of $\mathcal{G}$ into 16 block matrices M_{ij} according to u and the vertices of a connected component $\mathcal{C}(\mathcal{G}_u)$ of $\mathcal{G}_u$ of size $\gamma < \delta$, $\mathcal{G}_u \setminus \mathcal{C}(\mathcal{G}_u)$ and $\mathcal{G} \setminus \{u, \mathcal{G}_u\}$. The 4×4 matrix of the average row sums of these block matrices is exactly the matrix M in the proof of Proposition 4.2. Again, the eigenvalues of M interlace the eigenvalues of the adjacency matrix of $\mathcal{G}$. Since λ is an eigenvalue of $\mathcal{G}$, the interlacing is tight. It follows with Theorem 2.4 that all matrices M_{ij} have constant row sums, in particular

$$\frac{\gamma(\delta-\lambda-1)}{v-\delta-1} \in \mathbb{N}.$$

From Lemma 2.19 it follows that

$$\frac{\gamma(\delta-\lambda-1)}{v-\delta-1} = \gamma\frac{\mu}{\delta} = \frac{\gamma}{\lambda+1} \in \mathbb{N},$$

hence γ is divisible by $\lambda + 1$. □

Proposition 4.4. *Let $\mathcal{G}$ be a strongly regular graph on $(s+1)(st+1)$ vertices with degree $s(t+1)$ and parameters $\lambda = s-1$, $\mu = t+1$. The neighbourhood graph $\mathcal{G}_u$ has at most $t+1$ connected components whose size is divisible by s.*

Proof. Since $\lambda = s - 1 = \nu_2(\mathcal{G})$, by Corollary 4.3 the size of any connected component of $\mathcal{G}_u$ is divisible by $\lambda + 1 = s$. Because $\mathcal{G}_u$ has $s(t+1)$ vertices, there are at most $t+1$ components of size s. □

Example. Let $t > 1$. The generalized quadrangle $\mathrm{GQ}(s,t)$ has a strongly regular adjacency graph with parameters satisfying the conditions in the above proposition. For any vertex u, the neighbourhood graph $\mathcal{G}_u$ contains the $t+1$ cliques of size s corresponding to the points (distinct from u) on the lines through u. This is the maximal size of a clique, since the size of the largest clique is bounded from above by the Hoffmann bound (see Proposition 2.18); in this case for $t > 1$ the bound is

$$v\frac{1+\nu_2(\mathcal{G})}{v-\delta+\nu_2(\mathcal{G})} = v\frac{1+\lambda}{v-\delta+\lambda} = s+1+\frac{s+1}{st} < s+2.$$

Because $\mathcal{G}_u$ has exactly $s(t+1)$ vertices, the neighbourhood graph consists of exactly $t+1$ cliques of size s. Note that if a strongly regular graph with these parameters has the property that $\mathcal{G}_u$ is a union of $t+1$ cliques of size s for any vertex u, then it is the point graph of a generalized quadrangle ([BCN89], pp. 29).

4.2 A New Bound on the Algebraic Connectivity of Regular Graphs

Lemma 4.5. *For $x \in \mathbb{R}_{\geq 0}$, $1 < \delta < v - 1$ and $v \geq 3$*

$$(x(v-1) - \delta(\delta-1))^2 + 4(v-\delta-1)\delta(v-2\delta+x) > 0.$$

Proof. For $x \geq 0$ and $2\delta \leq v$ the statement is of course true.

Now, let $v < 2\delta$ and

$$f(x) = (x(v-1) - \delta(\delta-1))^2 + 4(v-\delta-1)\delta(v-2\delta+x).$$

The derivative of $f(x)$ is

$$f'(x) = 2\left[(v-1)^2 x + \delta(3(v-1) - \delta(v+1))\right]$$

and has root

$$x_0 = \delta\left(\frac{\delta(v+1) - 3(v-1)}{(v-1)^2}\right).$$

For $v < 2\delta$ and $v \geq 3$ we have

$$\begin{aligned}
\delta(v+1) - 3(v-1) &> \tfrac{v}{2}(v+1) - 3(v-1) \\
&= \frac{v^2+v}{2} - 3v + 3 \\
&= \frac{v^2 - 5v + 6}{2} \\
&= \frac{(v-3)^2 + v - 3}{2} \\
&\geq 0.
\end{aligned}$$

Therefore, $x_0 > 0$ and since $f(x)$ is a quadratic polynomial with positive leading coefficient it follows that $f(x)$ attains its minimum at x_0. Since $v > \delta - 1$,

$$f(x_0) = \frac{4v\delta(v-\delta-1)^3}{(v-1)^2} > 0$$

and therefore

$$f(x) > 0 \text{ for all } x \in \mathbb{R}_{\geq 0}.$$

$\square$

Proposition 4.6. *Let $v \geq 3$ and let $F : \mathbb{R}_{\geq 0} \to \mathbb{R}$,*

$$F(x) = \frac{x(v-1)-\delta(\delta-1)+\sqrt{(x(v-1)-\delta(\delta-1))^2+4(v-\delta-1)\delta(v-2\delta+x)}}{2(v-\delta-1)}.$$

Further, let $\mathcal{G}$ be a regular graph with degree δ. For a vertex $u \in V(\mathcal{G})$ let $\mathcal{G}_u$ denote the neighbourhood graph. If $\mathcal{G}_u$ is not connected, let $\bar{\delta}_{\mathcal{C}(\mathcal{G}_u)}$ denote the average degree of the connected component $\mathcal{C}(\mathcal{G}_u)$ and

$$\eta_u = \max_{\mathcal{C}(\mathcal{G}_u)} \left\{ \max\left\{ \bar{\delta}_{\mathcal{C}(\mathcal{G}_u)}, F(\bar{\delta}_{\mathcal{C}(\mathcal{G}_u)}) \right\} \right\}.$$

If $\mathcal{G}_u$ is connected, let $\bar{\delta}_u$ denote the average degree of $\mathcal{G}_u$ and

$$\xi_u = F(\bar{\delta}_u).$$

Then

$$\varrho(\mathcal{G}) = \max_{u \in V(\mathcal{G})} \left\{ \{\eta_u | \mathcal{G}_u \text{ not connected } \} \cup \{\xi_u | \mathcal{G}_u \text{ connected } \} \right\}$$

is a lower bound for $\nu_2(\mathcal{G})$ and therefore $\delta - \varrho(\mathcal{G})$ is an upper bound for the algebraic connectivity of $\mathcal{G}$.

Proof. Case 1: $\mathcal{G}_u$ is connected

We divide the adjacency matrix of $\mathcal{G}$ into 9 block matrices M_{ij} according to

u and the vertices of $\mathcal{G}_u$ and $\mathcal{G} \setminus \{u, \mathcal{G}_u\}$. With $\bar{\delta}_u$ denoting the average degree of $\mathcal{G}_u$, the matrix of average row sums is

$$M = \begin{pmatrix} 0 & \delta & 0 \\ 1 & \bar{\delta}_{\mathcal{G}_u} & \delta - \bar{\delta}_{\mathcal{G}_u} - 1 \\ 0 & \frac{\delta(\delta-(\bar{\delta}_{\mathcal{G}_u}+1))}{v-(\delta+1)} & \delta - \frac{\delta(\delta-(\bar{\delta}_{\mathcal{G}_u}+1))}{v-(\delta+1)} \end{pmatrix}$$

and has the characteristic polynomial

$$\chi_M(x) = (\delta - x)\left(x^2 + \left(\frac{\delta(\delta-1) - \bar{\delta}_{\mathcal{G}_u}(v-1)}{v-\delta-1}\right)x + \frac{\delta(2\delta - \bar{\delta}_{\mathcal{G}_u} - v)}{v-\delta-1}\right).$$

The eigenvalues of M are δ and

$$\frac{\bar{\delta}_u(v-1)-\delta(\delta-1)\pm\sqrt{(\bar{\delta}_u(v-1)-\delta(\delta-1))^2+4(v-\delta-1)\delta(v-2\delta+\bar{\delta}_u)}}{2(v-\delta-1)}.$$

Note that by Lemma 4.5 all eigenvalues of M are real. By Theorem 2.4 the eigenvalues of M interlace the eigenvalues of the adjacency matrix of $\mathcal{G}$, hence

$$\nu_2(\mathcal{G}) \geq \frac{\bar{\delta}_u(v-1)-\delta(\delta-1)+\sqrt{(\bar{\delta}_u(v-1)-\delta(\delta-1))^2+4(v-\delta-1)\delta(v-2\delta+\bar{\delta}_u)}}{2(v-\delta-1)} = F(\bar{\delta}_u).$$

This is true for any vertex u with connected neighbourhood graph, in particular if the right hand side of the above inequality is maximized.

Case 2: $\mathcal{G}_u$ is not connected

For any connected component $\mathcal{C}(\mathcal{G}_u)$ we divide the adjacency matrix of $\mathcal{G}$ into 16 block matrices M_{ij} according to u and the vertices of $\mathcal{C}(\mathcal{G}_u)$, $\mathcal{G}_u \setminus \mathcal{C}(\mathcal{G}_u)$ and $\mathcal{G} \setminus \{u, \mathcal{G}_u\}$. Let γ denote the size of $\mathcal{C}(\mathcal{G}_u)$ and $\bar{\delta}_{\mathcal{C}(\mathcal{G}_u)}$ denote the average row sum of M_{ij}. Then the matrix of average row sums is

$$M = \begin{pmatrix} 0 & \gamma & \delta-\gamma & 0 \\ 1 & \bar{\delta}_{\mathcal{C}(\mathcal{G}_u)} & 0 & \delta - (\bar{\delta}_{\mathcal{C}(\mathcal{G}_u)}+1) \\ 1 & 0 & \bar{\delta}_{\mathcal{C}(\mathcal{G}_u)} & \delta - (\bar{\delta}_{\mathcal{C}(\mathcal{G}_u)}+1) \\ 0 & \frac{\gamma(\delta-(\bar{\delta}_{\mathcal{C}(\mathcal{G}_u)}+1))}{v-(\delta+1)} & \frac{(\delta-\gamma)(\delta-(\bar{\delta}_{\mathcal{C}(\mathcal{G}_u)}+1))}{v-(\delta+1)} & \delta - \frac{\delta(\delta-(\bar{\delta}_{\mathcal{C}(\mathcal{G}_u)}+1))}{v-(\delta+1)} \end{pmatrix}$$

and has the characteristic polynomial

$$\chi_M(x) = (\bar{\delta}_{\mathcal{C}(\mathcal{G}_u)} - x)(\delta - x)\left(x^2 + \left(\frac{\delta(\delta-1)-\bar{\delta}_{\mathcal{C}(\mathcal{G}_u)}(v-1)}{v-\delta-1}\right)x + \delta\frac{2\delta-\bar{\delta}_{\mathcal{C}(\mathcal{G}_u)}-v}{v-\delta-1}\right).$$

By Theorem 2.4 the eigenvalues of M interlace the eigenvalues of the adjacency matrix of $\mathcal{G}$, hence

$$\nu_2(G) \geq \max\left\{\bar{\delta}_{\mathcal{C}(\mathcal{G}_u)}, F(\bar{\delta}_{\mathcal{C}(\mathcal{G}_u)})\right\}.$$

This is true for any connected component $\mathcal{C}(\mathcal{G}_u)$, in particular if the right hand side of the above inequality is maximized. □

The following lemma tells us how the function $F(x)$ in Proposition 4.6 behaves and what the values for η_u are in the different cases.

Lemma 4.7. *Let $v \geq 3$, $1 < \delta < v-1$ and $F : \mathbb{R}_{\geq 0} \to \mathbb{R}$,*

$$F(x) = \frac{x(v-1)-\delta(\delta-1)+\sqrt{(x(v-1)-\delta(\delta-1))^2+4(v-\delta-1)\delta(v-2\delta+x)}}{2(v-\delta-1)}.$$

Then

- *F is increasing on $\mathbb{R}_{\geq 0}$*
- *F is strictly convex on $\mathbb{R}_{\geq 0}$*
- *if $\delta = 2$ then*
  - *if $v = 4$, $F(x) \leq x$ if $x = 0$ and $F(x) > x$ else;*

- *if* $v < 4$, $F(x) \leq x$ *if* $x \in [0, \sqrt{4-v}]$ *and* $F(x) > x$ *else;*
- *if* $v > 4$, $F(x) > x$ *for all* $x \geq 0$.

- *if* $2 < \delta < v - 1$ *then*
  - *if* $v \leq 2\delta$ *then*

$$F(x) \leq x \text{ for } x \in \left[0, \tfrac{1}{2}(\delta - 2 + \sqrt{(\delta+2)^2 - 4v})\right]$$

and $F(x) > x$ *else;*

  - *if* $2\delta < v < \frac{1}{4}(\delta+2)^2$ *then* $F(x) \leq x$ *for*

$$x \in \left[-\tfrac{1}{2}(\delta - 2 - \sqrt{(\delta+2)^2 - 4v}), \tfrac{1}{2}(\delta - 2 + \sqrt{(\delta+2)^2 - 4v})\right]$$

and $F(x) > x$ *else;*

  - *if* $v = \frac{1}{4}(\delta+2)^2$ *then*

$$F(x) \leq x \text{ for } x = \tfrac{1}{2}(\delta - 2)$$

and $F(x) > x$ *else;*

  - *if* $v > \frac{1}{4}(\delta+2)^2$ *then* $F(x) > x$ *for all* $x \geq 0$.

Proof. Since

$$(x(v-1) - \delta(\delta-1))^2 + 4(v-\delta-1)\delta(v - 2\delta + x) > 0$$

for $1 < \delta < v-1$ and $x \geq 0$, and with Lemma 4.5 we can compute the derivative of F as

$$F'(x) = \tfrac{v-1}{2(v-1-\delta)} + \tfrac{(v-1)((v-1)x-(\delta-1)\delta)+2\delta(v-\delta-1)}{2(v-\delta-1)\sqrt{(x(v-1)-\delta(\delta-1))^2+4(v-\delta-1)\delta(v-2\delta+x)}}.$$

Suppose there exists $x_1 > 0$ such that $F'(x_1) \leq 0$, then

$$(v-1)\sqrt{(x_1(v-1)-\delta(\delta-1))^2+4(v-\delta-1)\delta(v-2\delta+x_1)}$$
$$\leq (v-1)((v-1)x_1-(\delta-1)\delta)+2\delta(v-\delta-1).$$

Squaring both sides gives

$$(v-1)^2(x_1(v-1)-\delta(\delta-1))^2+4(v-\delta-1)\delta(v-2\delta+x_1)$$
$$\leq ((v-1)((v-1)x_1-(\delta-1)\delta)+2\delta(v-\delta-1))^2.$$

The left hand side of the inequality is

$$(v-1)^2[(v-1)^2x_1^2+2\delta(3(v-1)-\delta(v+1))x_1+\delta(6\delta^2+\delta^3-3\delta(4v-3)+4(v-1)v)]$$

and the right hand side equals

$$(v-1)^2\left[(v-1)^2x_1^2+2\delta(3(v-1)-\delta(v+1))x_1\right]-\delta^2(3(v-1)-\delta(v+1))^2.$$

It follows

$$-4\delta v(\delta+1-v)^3 \leq 0,$$

a contradiction to $\delta < v-1$. Therefore, $F'(x) > 0$ for all $x \in \mathbb{R}_{>0}$ and the map $F(x)$ is strictly increasing for all $x \geq 0$.

The second derivative $F''(x)$ is

$$F''(x) = \frac{2v\delta(\delta+1-v)^2}{((x(v-1)-\delta(\delta-1))^2+4(v-\delta-1)\delta(v-2\delta+x))^{\frac{2}{3}}}.$$

Since the numerator and denominator are strictly positive for all $x \geq 0$ by Lemma 4.5, we have $F''(x) > 0$ for all $x > 0$ and therefore $F(x)$ is strictly convex on $\mathbb{R}_{\geq 0}$.

We want to solve the equation $F(x) = x$. This equation is satisfied if

$$\sqrt{(x(v-1) - \delta(\delta-1))^2 + 4(v-\delta-1)\delta(v-2\delta+x)} = x(v-1-2\delta) + \delta(\delta-1).$$

Squaring both sides and subtracting the right hand side gives

$$4\delta(\delta+1-v)\left(x^2 - (\delta-2)x - (2\delta-v)\right) = 0. \tag{4.2.1}$$

The equation has the solutions

$$\tfrac{1}{2}\left(\delta - 2 \pm \sqrt{(\delta+2)^2 - 4v}\right).$$

First of all we note that for $v > \frac{1}{4}(\delta+2)^2$ and $\delta \geq 2$ there are no solutions of equation 4.2.1 in $\mathbb{R}$, and therefore $F(x) > x$ for all $x \in \mathbb{R}_{\geq 0}$ in this case.

Suppose $4v \leq (\delta+2)^2$. We start with the case $\delta = 2$: here, there are no solutions for $v > 4$. If $v = 4$, then the only solution of equation 4.2.1 is $x = 0$ and indeed $F(0) = 0$. If $v < 4$, then

$$\tfrac{1}{2}\left(\delta - 2 - \sqrt{(\delta+2)^2 - 4v}\right) = -\tfrac{1}{2}\sqrt{4^2 - 4v} < 0$$

and the positive solution of equation 4.2.1 is $\sqrt{4-v}$ and indeed we have

$$F(\sqrt{4-v}) = \sqrt{4-v}.$$

Now let $\delta > 2$. If $v \leq 2\delta$, then $(\delta+2)^2 - 4v \geq (\delta-2)^2$ and therefore

$$\tfrac{1}{2}\left(\delta - 2 - \sqrt{(\delta+2)^2 - 4v}\right) \leq 0.$$

Hence, the positive solution of equation 4.2.1 is

$$\tfrac{1}{2}\left(\delta-2+\sqrt{(\delta+2)^2-4v}\right).$$

If $2\delta < v < \frac{1}{4}(\delta+2)^2$, then $(\delta+2)^2-4v < (\delta-2)^2$ and therefore

$$\tfrac{1}{2}\left(\delta-2-\sqrt{(\delta+2)^2-4v}\right) > 0.$$

Hence, the positive solutions of equation 4.2.1 are

$$\tfrac{1}{2}\left(\delta-2-\sqrt{(\delta+2)^2-4v}\right) \text{ and } \tfrac{1}{2}\left(\delta-2+\sqrt{(\delta+2)^2-4v}\right).$$

If $v = \frac{1}{4}(\delta+2)^2$, then $\sqrt{(\delta+2)^2-4v} = 0$ and the only solution to equation 4.2.1 is $x = \frac{1}{2}(\delta-2)$.

It remains to show that $\frac{1}{2}\left(\delta-2\pm\sqrt{(\delta+2)^2-4v}\right)$ are solutions to the equation $F(x)=x$. In fact, we show that $\frac{1}{2}(\delta-2+y)$ are solutions to $F(x)=x$ for $y \in \{\pm\sqrt{(\delta+2)^2-4v}\}$. As in Lemma 4.5, let

$$f(x) = (x(v-1)-\delta(\delta-1))^2 + 4(v-\delta-1)\delta(v-2\delta+x).$$

Then

$$\begin{aligned}
4f(\tfrac{1}{2}(\delta-2+y)) &= (v-1)^2y^2 + 2(v-\delta-1)(v(\delta-2)+\delta+2)y \\
&\quad +(2+5\delta+2\delta^2)^2 + (4+\delta(\delta+12))v^2 \\
&\quad -2(4+16\delta+\delta^2(2\delta+15))v \\
&= (v-1-2\delta)^2y^2 + 2(v-\delta-1)(v(\delta-2)+\delta+2)y \\
&\quad +4\delta(v-1-\delta)y^2 + (v(\delta-2)+\delta+2)^2 \\
&\quad -4\delta(v-\delta-1)(\delta+2)^2-4v)
\end{aligned}$$

$$
\begin{aligned}
&= ((v-1-2\delta)y + v(\delta-2) + \delta + 2)^2 \\
&\quad +4\delta(v-1-\delta)(y^2 - (\delta+2)^2 - 4v)).
\end{aligned}
$$

It follows for $y \in \{\pm\sqrt{(\delta+2)^2 - 4v}\}$ that

$$
4f(\tfrac{1}{2}(\delta - 2 + y)) = [(v-1-2\delta)y + v(\delta-2) + \delta + 2]^2,
$$

and therefore

$$
\begin{aligned}
& F\left(\tfrac{1}{2}(\delta-2+y)\right) \\
= & \frac{\frac{1}{2}(v-1)(\delta-2+y) - \delta(\delta-1) + \sqrt{f(\frac{1}{2}(\delta-2+y))}}{2(v-\delta-1)} \\
= & \frac{\frac{1}{2}(v-1)(\delta-2+y) - \delta(\delta-1) + \frac{1}{2}((v-1-2\delta)y + v(\delta-2) + \delta + 2)}{2(v-\delta-1)} \\
= & \frac{\frac{1}{2}(v-1)(\delta-2+y) - \delta(\delta-2) - \delta - \delta y + \frac{1}{2}((v-1)y + v\delta - 2v + \delta + 2)}{2(v-\delta-1)} \\
= & \frac{\frac{1}{2}(v-1)(\delta-2+y) - \delta(\delta-2) - \delta - \delta y + \frac{1}{2}((v-1)(\delta-2+y) + 2\delta)}{2(v-\delta-1)} \\
= & \frac{(v-1-\delta)(\delta-2+y) - \delta + \frac{1}{2}(2\delta)}{2(v-\delta-1)} \\
= & \tfrac{1}{2}(\delta-2+y).
\end{aligned}
$$

Since F is increasing and convex on $\mathbb{R}_{\geq 0}$, for solutions $x_1 \leq x_2$ of $F(x_i) = x_i$ for $i = 1, 2$, it follows $F(x) \leq x$ for $x \in [x_1, x_2]$ and $F(x) > x$ else. For $v \leq 2\delta$ there are solutions $x_1 < 0$ and $x_2 \geq 0$ for $F(x) = x$. But since F is convex, it follows that $F(x) \leq x$ for $x \in [0, x_2]$. $\square$

4.3 A Class of E-optimal SRGDs

Theorem 4.8. *Let $v \geq 3$. Let $\mathcal{G}$ be a strongly regular graph with degree δ and parameters λ and μ such that*

$$\lambda \geq \tfrac{1}{2}(\lambda - \mu + \sqrt{(\lambda-\mu)^2 + 4(\delta-\mu)}).$$

If λ minimizes the average degree of the connected components of all neighbourhood graphs in a class of δ-regular graphs on v vertices, then $\mathcal{G}$ maximizes the algebraic connectivity among all graphs in that class.

Proof. Let $u \in V(\mathcal{G})$. Any connected component of $\mathcal{G}_u$ has average degree λ, since any neighbour of u has exactly λ common neighbours. With Proposition 4.6 it follows that for all $u \in V(\mathcal{G})$

$$\begin{aligned} \xi_u &= \tfrac{\lambda(v-1)-\delta(\delta-1)+\sqrt{(\lambda(v-1)-\delta(\delta-1))^2+4(v-\delta-1)\delta(v-2\delta+\lambda)}}{2(v-\delta-1)} \\ &= \tfrac{1}{2}(\lambda-\mu+\sqrt{(\lambda-\mu)^2+4(\delta-\mu)}) \end{aligned}$$

is a lower bound for the second largest eigenvalue of $\mathcal{G}$, and in fact we have equality. Let $F : \mathbb{R}_{\geq 0} \to \mathbb{R}$,

$$F(x) = \tfrac{x(v-1)-\delta(\delta-1)+\sqrt{(x(v-1)-\delta(\delta-1))^2+4(v-\delta-1)\delta(v-2\delta+x)}}{2(v-\delta-1)}.$$

From Lemma 4.7 follows that $F(x)$ is increasing for all $x \geq 0$ and if λ is the minimal average degree of the neighbourhood graph of any δ-regular graph in the class, then $F(x)$ attains its minimum at $x = \lambda$ among all regular graphs in the class. Note, that $F(\lambda) = \nu_2(\mathcal{G}) \leq \lambda$ by assumption.

Let $\mathcal{G}'$ be any δ-regular graph and $\varrho(\mathcal{G}')$ the lower bound for the second largest eigenvalue $\nu_2(\mathcal{G}')$ of $\mathcal{G}'$ from Proposition 4.6. Then $\varrho(\mathcal{G}')$ is either an average degree $x \geq \lambda$ or $\varrho(\mathcal{G}') = F(x)$. Since $x \geq \lambda \geq F(\lambda)$ and since $F(x) \geq F(\lambda)$ for all $x \geq \lambda$, the eigenvalue $\nu_2(\mathcal{G}')$ is at least as large as $F(\lambda) = \nu_2(\mathcal{G})$, the second largest eigenvalue of $\mathcal{G}$. Therefore, $\mathcal{G}$ maximizes $\delta - \nu_2(\mathcal{G})$ among all δ-regular graphs in the class. □

Corollary 4.9. *Suppose there exists a strongly regular graph $\mathcal{G}$ on $v \geq 3$ vertices with degree δ and parameters λ, μ. If there exist an SRGD with underlying graph $\mathcal{G}$, then its adjacency matrix and the adjacency matrix of any competing RGD are the adjacency matrices of $\mathcal{G}$ and of a δ-regular simple graph shifted by the matrix $a(\mathbb{J}_v - \mathbb{I}_v)$ for some $a \in \mathbb{N}$ and this preserves the order of the eigenvalues. Therefore, if λ and μ satisfy the conditions of Theorem 4.8, then*

- *if there exists an SRGD with underlying graph $\mathcal{G}$ with block size $k \leq \lambda + 2$, then it is E-optimal among all RGDs on v points and block size k whose underlying regular graph has no neighbourhood graph whose minimum average degree is in the interval $[0, \lambda)$;*
- *if an SRGD for block size $k = \lambda + 2$ with underlying graph $\mathcal{G}$ exists, then it is E-optimal among all RGDs.*

In [CB91], R. A. Bailey and C.-S. Cheng show Ψ_f-optimality of SRGDs whose adjacency matrix has eigenvalue $-r$ over $\mathcal{D}_{v,b,k}$ (see also Theorem 2.30). The above corollary recovers these cases for E-optimality: if we have the equality $\lambda = \nu_2(\mathcal{G})$ in Theorem 4.8, then $\lambda = \frac{\delta - \mu}{\mu}$. That means $\delta = \mu(\lambda + 1)$ and the adjacency matrix of $\mathcal{G}$ has eigenvalue μ. For $k = \lambda + 2$ and $r = -\mu$, these are exactly the SRGDs in the result of Bailey and Cheng. The above corollary shows that for $k = \lambda + 2$ the SRGD is E-optimal even if r is not an eigenvalue of $\mathcal{G}$. The first statement of Corollary 4.9 is not trivial either. There can be different reasons why the neighbourhood graph of any competing graph has not a smaller average degree than λ. The next proposition is a good example.

Proposition 4.10. *Let $\alpha, m \in \mathbb{N}$ such that $\alpha \geq 2$ and $\alpha m \geq 4$. The complete regular multipartite graph $K_{m,m,\ldots,m}$ with α parts of size m maximizes the the algebraic connectivity among all $(\alpha-1)m$-regular graphs on αm vertices.*

Proof. The graph $K_{m,m,\ldots,m}$ is a strongly regular graph of degree $(\alpha-1)m$ and parameters

$$\lambda = (\alpha-2)m,\ \ \mu = (\alpha-1)m.$$

Since

$$\lambda > 0 = \tfrac{1}{2}(\lambda - \mu + \sqrt{(\lambda-\mu)^2 + 4(\delta-\mu)}),$$

the complete regular multipartite graph satisfies the conditions of Theorem 4.8. All we need to show is that for any graph $\mathcal{G}$ on $v = \alpha m$ vertices and degree $(\alpha-1)m$ and for any vertex $u \in V(\mathcal{G})$ the degree of any vertex in $\mathcal{G}_u$ is at least $(\alpha-2)m$.

Suppose there is a vertex w in $\mathcal{G}_u$ that has degree $\delta_u(w) < (\alpha-2)m$ in $\mathcal{G}_u$. Then w has

$$\delta - \delta_u(w) > (\alpha-1)m - (\alpha-2)m = m$$

neighbours in $\mathcal{G} \setminus \mathcal{G}_u$. But $\mathcal{G}_u$ contains already $\delta = v - m$ vertices, and w can have at most m neighbours in $\mathcal{G} \setminus \mathcal{G}_u$.

□

As a corollary we get the following result by Takeuchi who proved optimality among $\mathcal{D}_{v,b,k}$ ([Tak61]).

Corollary 4.11. *Let $v \geq 4$. If an SRGD with block size 2 whose underlying graph is a complete regular multipartite graph exists, then it is E-optimal among RGDs.*

Proof. The adjacency matrix of any competing RGD is the adjacency matrix of a regular simple graph shifted by the matrix $a(\mathbb{J}_v - \mathbb{I}_v)$ for some $a \in \mathbb{N}$ and this preserves the order of the eigenvalues. The proposition follows from Proposition 4.10. □

Recall, that we denote by $T(n+1)$ the triangular graph, which is a a strongly regular graph on $\frac{n(n+1)}{2}$ points, degree $2(n-1)$ and parameters $\lambda = n-1$ and $\mu = 4$ (see page 19). The vertices correspond to the subsets of $\{1, \ldots, n+1\}$ of size 2 any two of which are joined by an edge if they have a non-empty intersection.

Proposition 4.12. *For $n > 2$, the largest clique in $T(n+1)$ has size n and any vertex is contained in exactly two maximal cliques.*

Proof. All subsets of $\{1, \ldots, n+1\}$ of size 2 with a common subset of just one element correspond to a clique of size n. Therefore, any vertex lies in at most two cliques of at least size n. The second largest eigenvalue of $\mathcal{G}$ is $\nu_2(\mathcal{G}) = n-2$. The size of a clique is bounded by the Hoffman bound (Proposition 2.18)

$$v\frac{1+\nu_2(\mathcal{G})}{v-\delta+\nu_2(\mathcal{G})} = v\frac{n-2}{v-\delta+n-2} = n\frac{n-1}{n-1} = n$$

and therefore any vertex lies in exactly two cliques of size n. □

Proposition 4.13. *If an RGD on $\frac{n(n+1)}{2}$ points, block size n and replication 2 with underlying graph $T(n+1)$ exists, then it is E-optimal among RGDs.*

Proof. The adjacency matrix of any competing RGD is the adjacency matrix of a regular simple graph shifted by the matrix $a(\mathbb{J}_v - \mathbb{I}_v)$ for some $a \in \mathbb{N}$ and this preserves the order of the eigenvalues. Since

$$\lambda > n-2 = \tfrac{1}{2}(\lambda - \mu + \sqrt{\lambda - \mu)^2 + 4(\delta - \mu)}),$$

the triangular graph satisfies the conditions of Theorem 4.8. The proposition follows with Proposition 4.12 and Corollary 4.9. □

In fact, this result is not surprising. The triangular graph is the line graph of the complete graph. The complete graph is a BIBD with block size 2, hence the design in the above proposition with blocks being all sets of size 2 with a common element is just the dual design of a BIBD and is E-optimal among all equireplicate binary connected designs by Theorem 2.26.

The triangular graph is also the adjacency graph of a design with block size 3 and replication $n-1$ where the blocks correspond to sets of size 3. Since the map $F(x)$ in the proof of Theorem 4.8 is increasing for $x > 0$, if there is a regular graph design where the maximal average degree of the connected components of the neighbourhood graphs is strictly less than $n-1$, this design will beat any existing design with the triangular graph as adjacency graph. In fact, we give a design that is better for $n = 5$ in Section 6.1.2.

Proposition 4.14. *Let* $\frac{(s+1)(st+\alpha)}{\alpha} \geq 4$. *If an RGD on* $\frac{(s+1)(st+\alpha)}{\alpha}$ *vertices, block size* $s+1$ *and replication* $t+1$ *whose underlying graph is the adjacency graph of a partial geometry with parameters* s, t, α *exists, then it is* E*-optimal among RGDs.*

Proof. The adjacency graph of a partial geometry is a strongly regular graph on $\frac{(s+1)(st+\alpha)}{\alpha}$ vertices and degree $s(t+1)$ with parameters $\lambda = s-1+t(\alpha-1)$ and $\mu = (t+1)\alpha$ (see [CvL91], p. 92). The eigenvalues are $s(t+1)$, $s-\alpha$ and $-(t+1)$. Therefore, the parameters of a partial geometry satisfy $\lambda \geq s-\alpha, -(t+1)$ and we can apply Theorem 4.8. It follows that the partial geometry maximizes the algebraic connectivity among all graphs with minimum average degree λ of the neighbourhood graphs. In particular, the partial geometries give rise to E-optimal designs with block size $s+1$. □

Note that the above proposition is again a special case of Theorem 2.30 since $-(t+1) = -r$ is an eigenvalue of the adjacency graph of the partial geometry.

Chapter 5

The Generalized Hexagon

Recall that a geometry $\mathbb{G} = (\Omega_1, \Omega_2, \mathcal{I})$ is a generalized hexagon if its incidence graph $\mathcal{G}(\mathbb{G})$ is a simple connected bipartite graph, the degree of any vertex is at least 2, $\text{diam}(\mathcal{G}(\mathbb{G})) = 6$ and $\text{girth}(\mathcal{G}(\mathbb{G})) = 12$. Every line in Ω_2 contains exactly $s+1$ points for some $s \in \mathbb{N}_{>0}$ and every point in Ω_1 lies on exactly $t+1$ lines for some $t \in \mathbb{N}_{>0}$. In these terms, a generalized hexagon is a 1-$(v, s+1, t+1)$-design where

$$v = |\Omega_1| = s^3t^2 + s^2t(t+1) + s(t+1) + 1.$$

The generalized hexagon is a binary, equireplicate block design with $b = (t+1)(s^2t^2+st+1)$ blocks of size $k = s+1$ and replication number $r = t+1$. The adjacency matrix $\mathbb{A}(\mathbb{G}) = (a_{ij})$ is given by

$$a_{ij} = \begin{cases} 0 & \text{if } i = j, \\ 1 & \text{if points } i \text{ and } j \text{ are on the same line,} \\ 0 & \text{else.} \end{cases}$$

Our aim is to understand the adjacency graphs of the designs in the class of the generalized hexagon and for this a good understanding of the structure of the adjacency graph is very important. We want to compute the spectrum of the generalized hexagon. This has been done in [BCN89], p. 203, using results on distance regular graphs, but we want to give a proof that gives an insight in the incidence relation.

Proposition 5.1 ([BCN89], p. 203)**.** *The adjacency graph of the generalized hexagon is a distance regular graph of diameter* 3 *with intersection numbers*

$$\begin{array}{lll} \alpha_0 = 0 & \beta_0 = s(t+1) & \gamma_0 = 0 \\ \alpha_1 = s-1 & \beta_1 = st & \gamma_1 = 1 \\ \alpha_2 = s-1 & \beta_2 = st & \gamma_2 = 1 \\ \alpha_3 = (t+1)(s-1) & \beta_3 = 0 & \gamma_3 = t+1. \end{array}$$

The eigenvalues of the adjacency matrix $\mathbb{A}(\mathbb{G})$ *of the generalized hexagon with parameters* $s, t \geq 1$ *are*

$$-t-1, s-1-\sqrt{st}, s-1+\sqrt{st} \text{ and } s(t+1).$$

Proof. Let $\mathbb{A}(\mathbb{G}) = (a_{ij})$. We start by computing the il-entry of $\mathbb{A}(\mathbb{G})^2$:

$$\begin{aligned} \sum_{j=1}^{v} a_{ij}a_{jl} &= \begin{cases} \sum_{j=1}^{v} a_{ij}^2 & \text{if } i = l, \\ \sum_{j=1, j\neq i,l}^{v} a_{ij}a_{jl} & \text{else} \end{cases} \\ &= \begin{cases} s(t+1) & \text{if } i = l, \\ s-1 & \text{if } i \text{ and } l \text{ are on the same line}, \\ 1 & \text{if } \operatorname{dist}(i,l) = 2 \\ 0 & \text{else.} \end{cases} \end{aligned}$$

It follows that

$$\mathbb{A}(\mathbb{G})^2 = (s-1)\mathbb{A}(\mathbb{G}) + s(t+1)\mathbb{I}_v + X,$$

where the ij-entry x_{ij} of the matrix X is equal to 1 if and only if the point j is at distance 2 from i and equals 0 anywhere else for all $i, j \in \{1, \ldots, v\}$. Hence

$$\mathbb{A}(\mathbb{G})^3 = (s-1)\mathbb{A}(\mathbb{G})^2 + s(t+1)\mathbb{A}(\mathbb{G}) + X\mathbb{A}(\mathbb{G}).$$

The il-entry of the product $X\mathbb{A}(\mathbb{G})$ is

$$\sum_{j=1}^{v} x_{ij}a_{jl} = \begin{cases} \sum_{j=1}^{v} x_{ij}a_{ji} & \text{if } i = l, \\ \sum_{j=1, j\neq i,l}^{v} x_{ij}a_{jl} & \text{else.} \end{cases}$$

For fixed i, either $x_{ij} = 0$ or $a_{ij} = 0$ for all $j \in \{1, \ldots, v\}$ and therefore $\sum_{j=1}^{v} x_{ij}a_{ji} = 0$ for $i = l$. If i and l lie on the same line, the second row equals the number of vertices j that are at distance 2 from i and can be reached via l. There are st such vertices. Suppose l is at distance 2 from i and let iml be the path of length 2 from i to l. Then for every point j on the same line as m and l, the product $x_{ij}a_{jl}$ equals 1 for every $j \neq m, l$ and 0 for all other $j \neq m, l$. Therefore, $\sum_{j=1, j\neq i,l}^{v} x_{ij}a_{jl} = s-1$ in this case. If l is not at distance 2 from i and does not lie on the same line as i, then l has to be at distance 3 from i. In this case the sum $\sum_{j=1, j\neq i,l}^{v} x_{ij}a_{jl}$ equals the number of vertices j that are at distance 2 from i and are on a path from i to l. Because the adjacency graph has girth 6, the $s(t+1)$ points at distance 1 from i and the $s^2t(t+1)$ points at distance 2 from i are distinct points. Any point at distance 2 from i has st neighbours that are not neighbours of i. Therefore, there are $st(s^2t(t+1))$ (not distinct) points at distance 3 from i. Since there are $v = s^3t^2 + s^2t(t+1) + s(t+1) + 1$ points in total, there must be $t+1$ different paths from i to any point l at distance 3.

Therefore, $\sum_{j=1,j\neq i,l}^{v} x_{ij}a_{jl} = t+1$ in this case. Hence

$$\sum_{j=1}^{v} x_{ij}a_{jl} \;=\; \begin{cases} 0 & \text{if } i = l, \\ st & \text{if } i \text{ and } l \text{ are on the same line}, \\ s-1 & \text{if } i \text{ and } l \text{ are at distance 2}, \\ t+1 & \text{else} \end{cases}$$

and therefore

$$\mathbb{A}(\mathbb{G})^3 = (s-1)\mathbb{A}(\mathbb{G})^2 + (s(t+1)+st-t-1)\mathbb{A}(\mathbb{G}) + (t+1)\mathbb{J}_v + (s-2-t)X - (t+1)\mathbb{I}_v.$$

Replacing X by $\mathbb{A}(\mathbb{G})^2 - (s-1)\mathbb{A}(\mathbb{G}) - s(t+1)\mathbb{I}_v$ yields

$$\begin{aligned} \mathbb{A}(\mathbb{G})^3 \;=\; & (2s-t-3)\mathbb{A}(\mathbb{G})^2 - (s^2-4s-3st+2t+3)\mathbb{A}(\mathbb{G}) \\ & +(t+1)\mathbb{J}_v - (s^2-2s-st+1)(t+1)\mathbb{I}_v. \end{aligned}$$

The matrix $\mathbb{J}_v$ has the vector $(1,\ldots,1)^T$ as an eigenvector with eigenvalue v. The row sum of $\mathbb{A}(\mathbb{G})$ is $s(t+1)$ and therefore, the vector $(1,\ldots,1)^T$ is an eigenvector of $\mathbb{A}(\mathbb{G})$ with eigenvalue $s(t+1)$. All other eigenvalues of $\mathbb{J}_v$ are 0 and any eigenvalue $\nu(\mathbb{A}(\mathbb{G}))$ not corresponding to the vector $(1,\ldots,1)^T$ of $\mathbb{A}(\mathbb{G})$ satisfies the equation

$$x^3 = (2s-t-3)x^2 - (s^2-4s-3st+2t+3)x - (s^2-2s-st+1)(t+1).$$

The solutions of this equation are

$$\nu(\mathbb{A}(\mathbb{G}))_3 = -(t+1),\; \nu(\mathbb{A}(\mathbb{G}))_{1,2} = s-1 \pm \sqrt{st}.$$

□

Proposition 5.2 ([BCN89], p.203)**.** *The spectrum of the adjacency matrix* $\mathbb{A}(\mathbb{G})$ *of the generalized hexagon with parameters* $s, t \geq 1$ *is*

$$\mathrm{Spec}(\mathbb{A}(\mathbb{G})) = ((-t-1)^{m_0}, (s-1-\sqrt{st})^{m_1}, (s-1+\sqrt{st})^{m_2}, (s(t+1))^1)$$

where

$$m_0 = \frac{vs^3}{s^3 + s^2(t+1) + st(t+1) + t^2},$$

$$m_1 = \frac{\frac{1}{2}st(1+t)v}{s(t-1)^2 + t(s-1)^2 + 3st - (s-1)(t-1)\sqrt{st}}$$

and

$$m_2 = \frac{\frac{1}{2}st(1+t)v}{s(t-1)^2 + t(s-1)^2 + 3st + (s-1)(t-1)\sqrt{st}}.$$

Proof. We follow [BCN89], and compute the multiplicities by using Proposition 2.21 and need the following parameters and polynomials:

$$h_0 = 1,\ h_1 = s(t+1),\ h_2 = s^2t(t+1),\ h_3 = s^3t^2$$

and

$$\omega_2(x) = x^2 - (s-1)x - s(t+1),$$

$$\omega_3(x) = x^3 - 2(s-1)x^2 - (2st + 3s - s^2 - 1)x + (s^2 - s)(t+1),$$

$$\omega_4(x) = (x - (s-1)(t+1))\omega_3(x) - st(t+1)\omega_2(x).$$

We can check now our calculations above, since the eigenvalues of the adjacency graph are the zeros of $\omega_4(x)$, which indeed are

$$-(t+1), s(t+1), s-1 \pm \sqrt{st}$$

with multiplicities

$$m(-(t+1)) = \frac{vs^3}{s^3 + s^2(t+1) + st(t+1) + t^2},$$

$$m(s(t+1)) = \frac{v}{s^3t^2 + s^2t(t+1) + s(t+1) + 1} = 1$$

and

$$m(s - 1 \pm \sqrt{st}) = \frac{\frac{1}{2}st(1+t)v}{s(t-1)^2 + t(s-1)^2 + 3st \pm (s-1)(t-1)\sqrt{st}}.$$

Note that Theorem 2.44 states that either the product st is a square or $(s-1)(t-1) = 0$. □

Proposition 5.3. *Any equireplicate binary connected design in the class of the Generalized Hexagon has at least* 3 *distinct non-trivial Laplacian eigenvalues.*

Proof. Let $d \in \mathcal{D}_{v,b,s+1}$ where $v = s^3t^2 + s^2t(t+1) + s(t+1) + 1$ and $b = (t+1)(s^2t^2 + st + 1)$ and suppose d has only one distinct non-zero Laplacian eigenvalue. As d is equireplicate, the adjacency graph has two distinct eigenvalues, hence must be a complete graph on v vertices ([YYY07]). Then the degree of each vertex is $v - 1$, a contradiction.

Now suppose, d has two distinct Laplacian eigenvalues other than 0. Because d is equireplicate, the adjacency matrix $\mathbb{A}(d)$ has two distinct eigenvalues other than δ. Then there exists a monic polynomial $h(x)$ of degree 2 such that $h(\mathbb{A}(d)) = \beta\mathbb{J}_v$ for some $\beta \in \mathbb{N}$. By Lemma 2.6 the ij-entry $\mathbb{A}(d)^2$ is the number of walks of length 2 between the vertices i and j. Hence, the adjacency graph $\mathcal{G}(d)$ has diameter 2. Let u be a point in d. There are less than $s(t+1)$ distinct points at distance 1 and less than $st^2(t+1)$ distinct points at distance 2 from u. As the graph has diameter 2, it follows that there are less than $s(t+1) + st^2(t+1) + 1$ distinct points in total, a contradiction. □

Proposition 5.4. *Suppose* $(s,t) \neq (1,1)$. *Any equireplicate binary connected design in the class of the generalized hexagon with exactly* 3 *distinct non-trivial Laplacian eigenvalues satisfying the condition*

(*) *for any vertex* u *there is no block that contains only vertices that are at distance* 3 *from* u *in* $\mathcal{G}(d)$

has a distance regular adjacency graph that allows the intersection numbers of the generalized hexagon.

Proof. Let d be a design satisfying all conditions of the proposition and let $\mathbb{A}(d)$ be its adjacency matrix. Since $\mathbb{A}(d)$ has three distinct eigenvalues other than $s(t+1)$, then there exists a monic polynomial $h(x)$ of degree 3 such that $h(\mathbb{A}(d)) = \beta\mathbb{J}_v$ for some $\beta \in \mathbb{N}$. By Lemma 2.6 the ij-entry $\mathbb{A}(d)^3$ is the number of walks of length 3 between the vertices i and j. If two points are at distance 3, then the corresponding entry of $\mathbb{A}(d)^2$ and $\mathbb{A}(d)$ equals 0. Therefore, β equals the number of paths between any two points at distance 3.

Let u be a vertex in $\mathcal{G}(d)$. As d satisfies the condition **(*)**, any of the $t+1$ blocks containing point w at distance 3 from u contains at least one point at distance 2 from u. Hence, $\beta \geq t+1$. Let $\mathcal{X}$ be the set of vertices of $\mathcal{G}(d)$ at distance 1 or 2 from u that can be reached via at least 2 paths of length at most 2. Let $m \geq 1$ denote the minimum number of paths of length 2 to such points. Then there are at most $s^2t(t+1) - |\mathcal{X}|$ vertices at distance 2 from u. Vertices at distance 3 from u can be reached via vertices at distance 2 that either allow exactly 1 or at least m paths of length 2 to u. In the first case, there can be reached at most $s^3t^2(t+1) - mst|\mathcal{X}|$ (not distinct) vertices, because every vertex reached via one of the vertices in $\mathcal{X}$ at distance 2 has at least m paths going back to the starting vertex u. Via these other vertices in $\mathcal{X}$ there can be γ vertices reached at distance 3 for some integer $\gamma \leq st|\mathcal{X}|$. In total, there are at most

$$s^3t^2(t+1) - mst|\mathcal{X}| + \gamma$$

(not distinct) vertices at distance 3. Hence,

$$v \leq 1 + s(t+1) + s^2t(t+1) - |\mathcal{X}| + \frac{s^3t^2(t+1) - mst|\mathcal{X}| + \gamma}{\beta}.$$

With

$$v = s^3t^2 + s^2t(t+1) + s(t+1) + 1$$

it follows

$$s^3t^2 + |\mathcal{X}| \leq \frac{s^3t^2(t+1) - mst|\mathcal{X}| + \gamma}{\beta}$$

and therefore

$$\begin{aligned} \beta &\leq \frac{s^3t^2(t+1) - mst|\mathcal{X}| + \gamma}{s^3t^2 + |\mathcal{X}|} \\ &= t + 1 - \frac{(t+1+mst)|\mathcal{X}| - \gamma}{s^3t^2 + |\mathcal{X}|}. \end{aligned}$$

For $|\mathcal{X}| > 0$, it follows $(t+1+mst)|\mathcal{X}| > st|\mathcal{X}| \geq \gamma$ and therefore $b < t+1$, a contradiction. Hence, $|\mathcal{X}| = \gamma = 0$ and $\beta = t+1$, so $\mathcal{G}(d)$ contains no closed paths of length 3 or 4. If there were $|\mathcal{X}|$ vertices at distance 2 from u that can also be reached by a path of length 3 starting at u, then there are at most $s^3t^2(t+1) - |\mathcal{X}|$ (not distinct) points at distance 3. With the same arguments as above it follows that

$$\beta \leq t + 1 - \frac{|\mathcal{X}|}{s^3t^2},$$

a contradiction for $|\mathcal{X}| > 0$.

Overall, $\mathcal{G}(d)$ contains no closed paths of length $3, 4$ or 5 and is distance regular with the same intersection numbers as the generalized hexagon. □

Note that a distance regular graph with the same intersection numbers as the generalized hexagon is not necessarily a generalized hexagon, see [BCN89], p.205.

Corollary 5.5. *Any equireplicate binary connected design satisfying the conditions of Proposition 5.4 has the same Laplacian spectrum as the generalized hexagon.*

Proof. Follows directly from the fact that the eigenvalues and their multiplicities of the adjacency graph are uniquely determined by the intersection numbers (Theorem 2.21). □

For a discussion of A- and D-optimality of the generalized hexagon see Section 8.5.4 in Chapter 8.

Chapter 6

Truncations of the Projective Space over a Finite Field and a related Incidence Structure

For $i \in \{1, \ldots, n\}$, let $\mathrm{PG}^i(n,q)$ denote the set of i-dimensional subspaces of the vector space $\mathcal{V}$ and let $\mathrm{PG}(n,q)$ be the corresponding projective space, that is the geometry

$$\mathrm{PG}(n,q) = (\mathrm{PG}^1(q,n), \ldots, \mathrm{PG}^n(q,n), \leq).$$

6.1 Truncations of a related Incidence Structure

Before we look at the truncations of the projective space, we want to start with a substructure that is easier to handle. For this section, let $\mathcal{E} = \{E_1, \ldots, E_{n+1}\}$ be a fixed basis of $\mathcal{V}$. Let $\Sigma_i \subset \mathrm{PG}^i(n,q)$ denote the set of all subspaces of dimension i that allow a basis that is a proper subset of $\mathcal{E}$. The substructure

$$\mathbb{G}_n^{\mathcal{E}} = (\Sigma_1, \ldots, \Sigma_n, \mathcal{I})$$

of $\mathrm{PG}(n,q)$ is a geometry of rank n. For $1 \leq i < j \leq n$ any two elements $U_i \in \Sigma_i$ and $U_j \in \Sigma_j$ are incident if and only if there is a permutation $\pi \in \mathbb{S}_{n+1}$ such that

$$U_i = \langle E_{\pi(1)}, \ldots, E_{\pi(i)} \rangle \subset \langle E_{\pi(1)}, \ldots, E_{\pi(i)}, E_{\pi(i+1)}, \ldots, E_{\pi(j)} \rangle = U_j.$$

For a fixed basis $\mathcal{E}$, the only information that is needed to study this geometry are therefore the indices of the basis vectors and $\mathbb{G}_n^{\mathcal{E}}$ is isomorphic to the geometry $\mathbb{G}(n) = (\Omega_1, \ldots, \Omega_n, \subseteq)$ from Example 2.7 where Ω_i is the set of all subsets of $\{1, \ldots, n+1\}$ of size i for $i = 1, \ldots, n$.

We want to study the rank-2 truncations

$$\mathrm{Tr}_{i,j}(n) = (\Omega_i, \Omega_j, \mathcal{I})$$

for $1 \leq i < j \leq n$ of $\mathbb{G}(n)$. The adjacency graph of $\mathrm{Tr}_{i,j}(n)$ has all subsets of $\{1, \ldots, n+1\}$ of size i as points and any two are adjacent if they are contained in a set of size j. This is equivalent with the subsets of size i meeting in a subset of size $\geq \max\{0, 2i-j\}$. If $j = i+1$, the truncation is well-known: the *Johnson graph* $J(n+1, i)$ has the set of all i-subsets of $\{1, \ldots, n+1\}$ as vertex set and two sets are adjacent if they intersect in a set of size $i-1$. We will need the following lemma.

Lemma 6.1 ([BCN89], p. 255)**.** *The graphs $J(n+1, i)$ and $J(n+1, n+1-i)$ are isomorphic.*

For any $i \in \{2, \ldots, n\}$, the notion of the Johnson graph can be extended to the *generalized Johnson graph* $J(n+1, i, l)$ with the sets of size i as vertex set any two of which are adjacent if they meet in a set of size $i-l$, for $l = 0, \ldots, i$. If $i \leq \frac{j}{2}$, then $\max\{0, 2i-j\} = 0$. The adjacency graph of $\mathrm{Tr}_{i,j}(n)$ has the generalized Johnson graphs $J(n+1, i, l)$ for $l = \max\{0, 2i-j\}, \ldots, i$ as subgraphs. Moreover, the edge set of the adjacency graph $\mathcal{G}$ of $\mathrm{Tr}_{i,j}(n)$ is the disjoint union of the edge sets of the generalized Johnson graphs:

$$E(\mathcal{G}) = \dot{\bigcup}_{l=\max\{0,2i-j\}}^{i} E(J(n+1,i,l)).$$

This means we can colour the edges of $\mathcal{G}$ with $i - \max\{0, 2i-j\} + 1$ colours according to which generalized Johnson graph they correspond.

Optimality of the $\mathrm{Tr}_{1,i}(n)$- and $\mathrm{Tr}_{j,n}(n)$-truncations

Lemma 6.1 gives the following proposition for $i = 2$, but we want to prove it here for all $i \geq 2$.

Proposition 6.2. *Let $i \in \{2, \ldots, n\}$. The truncation $\mathrm{Tr}_{n-i+1,n}(n)$ is isomorphic to $\mathrm{Tr}^*_{1,i}(n)$, the dual of $\mathrm{Tr}_{1,i}(n)$.*

Proof. The dual of $\mathrm{Tr}_{1,i}(n)$ has the subsets of size i as points and any two points are adjacent if they intersect in a set of size ≥ 1.

Consider the map $\phi : \Omega_i \to \Omega_{n-i+1}$ where for a set $J \in \Omega_i$ of size i gets mapped to its complement $\phi(J) = \{1, \ldots, n+1\} \setminus J$. The map ϕ defines a bijective map $\Phi : \mathrm{Tr}^*_{1,i}(n) \to \mathrm{Tr}_{n-i+1,n}(n)$ given by the action of ϕ on the points of $\mathrm{Tr}^*_{1,i}(n)$. Let $J' \in \Omega_i$ be a set such that $|J \cap J'| \geq 1$. Then $J \cap J' \not\subset \phi(J) \cap \phi(J')$ and therefore $|\phi(J) \cup \phi(J')| \leq n + 1 - |J \cap J'| \leq n$ and $\phi(J)$ and $\phi(J')$ are both contained in a set of size n and Φ is an isomorphism of block designs. □

Proposition 6.3. *Let $i \in \{2, \ldots, n\}$. The truncations $\mathrm{Tr}_{1,i}(n)$ of the geometry $\mathbb{G}(n)$ are universally optimal among all designs (binary or not) on $n+1$ points, replication $\binom{n+1}{i}$ and block size i.*

Proof. The truncation $\mathrm{Tr}_{1,i}(n)$ is a binary equireplicate block design on $n+1$ points. Since any two points are contained in $\lambda_i = \binom{n+1}{i-2}$ subsets of size i, the truncation is a 2-$(n+1, i, \lambda_i)$-design. With Theorem 2.26 the proposition follows. □

Corollary 6.4. *Let $i \in \{1, \ldots, n-1\}$. The truncations $\mathrm{Tr}_{i,n}(n)$ are optimal on the Φ_p-criterion among all equireplicate designs.*

Proof. This follows directly from Proposition 6.2, Proposition 6.3 and Proposition 2.31. □

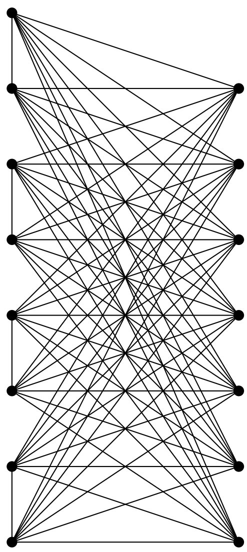
The Truncation $\mathrm{Tr}_{2,3}(n)$

Proposition 6.5. *The adjacency graph of $\mathrm{Tr}_{2,3}(n)$ and the adjacency graph of $\mathrm{Tr}^*_{1,2}(n)$ are isomorphic.*

Proof. Two points of the design $\mathrm{Tr}_{2,3}(n)$, that is two sets of size 2, are adjacent if they are contained in a set of size 3. This is the case if and only if they meet in a set of size 1. On the other hand, in the design $\mathrm{Tr}^*_{1,2}(n)$ the points are the sets of size 2 and two of them are adjacent if and only if they share the same set of size 1. Hence, sets of size 2 are adjacent in $\mathrm{Tr}_{2,3}(n)$ if and only if they are adjacent in $\mathrm{Tr}_{1,2}(n)$. □

The adjacency graph of $\mathrm{Tr}_{2,3}(n)$ is the triangular graph $T(n+1)$. We already know that this graph is the line graph of the complete graph K_{n+1} which is a binary 2-$(n+1, 2, 1)$-design and as such universally optimal (Theorem 2.26). As the dual design, the triangular graph is A-, D- and E-optimal among all binary equireplicate designs on $\frac{n(n+1)}{2}$ points, block size n and replication 2 by Proposition 2.31.

John and Mitchell found for $n = 4$ an A- and D-better design for block size 2 and 3 ([JM77]). But by the above discussion, the adjacency graph of $\mathrm{Tr}_{2,3}(n)$ is also the adjacency graph of an optimal design for block size 4. What can we say about other block sizes for $n > 4$? Proposition 4.12 gives us n as a bound on the maximal block size for a design with adjacency graph $T(n+1)$ if $n > 2$. For $n = 5$, the Laplacian spectrum of the triangular graph is $(0^1, 6^5, 10^9)$, the D-value is $518,400,000,000 \times 15$ and the A-value is $\frac{105}{13}$. A pair (r, k) of replication and block size is feasible, if $8 = r(k-1)$, that is $k = 2, 3$ or 5. For $k = 2$, we found the following graph that beats $T(6)$ on the A- and D-criteria.



The Laplacian spectrum of this graph is $(0^1, 7^3, 8^6, 9^4, 15^1)$, the D-value is $589,934,886,912 \times 15$ and the A-value is $\frac{17640}{2129}$. For block size 3, the following design ([Bai]) beats $T(6)$ on all three criteria.

00	11	22	01	12	23	02	13	24	03	14	20	04	10	21
00	12	24	01	13	20	02	14	21	03	10	22	04	11	23
00	13	21	01	14	22	02	10	23	03	11	24	04	12	20
00	14	23	01	10	24	02	11	20	03	12	21	04	13	22

The Laplacian spectrum of this design is $(0^1, 7^8, 10^4, 12^2)$, the D-value is $553,420,896,000 \times 15$ and the A-value is $\frac{2940}{359}$. For block size 5 the graph is again the adjacency graph of the dual of a 2-design and as such optimal among equireplicate designs.

In the case of $n = 7$, the triangular graph has degree 12. The Laplacian spectrum of $T(8)$ is $(0^1, 8^7, 14^{20})$, the A-value is $\frac{504}{43}$ and the D-value is $6,266,608,900,058,816,862,941,609,984 \times 28$. A pair (r, k) of replication and block size is feasible, if $12 = r(k-1)$, that is $k = 2, 3, 4$ or 7. For block size 2, the graph obtained from the complete bipartite graph $K_{14,14}$ by deleting seven disjoint 4-cycles has Laplacian spectrum $(0^1, 10^6, 12^{14}, 14^6, 24^1)$, the A-value is $\frac{22680}{1879}$ and the D-value is $8,286,265,971,330,338,783,232,000,000 \times 28$.

For block size 3, the graph is the adjacency graph of $\mathrm{Tr}_{2,3}(7)$. We do not know whether this design is optimal or not. Because $T(8)$ has already 28 vertices there is no hope of doing any calculations with the computer. For block size 4 there is no design with adjacency graph $T(8)$ (tested with the DESIGN package in GAP). For block size 7, the graph $T(8)$ is the adjacency graph of the dual of a BIBD.

With the triangular graph we have found a graph that, depending on how we partition the neighbours of a vertex into cliques, gives a non-optimal or an optimal design.

Remark 6.6. Since $J(5,3) \simeq J(5,2) = T(5)$ the Johnson graph will not be optimal for all block sizes and all v, i.

6.2 The Truncations of the Projective Space over a Finite Field

Before we can study the truncations of the projective space $\mathrm{PG}(n, q)$ we will need the following results.

Theorem 6.7 ([Hir79], p.65)**.** *Let $i \in \{1, \ldots, n\}$.*

1.

$$|\mathrm{PG}^i(n,q)| = \frac{(q^{n+1}-1)(q^{n+1}-q)\ldots(q^{n+1}-q^{i-1})}{(q^i-1)(q^i-q)\ldots(q^i-q^{i-1})}.$$

2. The number of j spaces through an i space for $1 \leq i < j \leq n$ is

$$\frac{\prod_{l=n-i}^{j-i+1}(q^l - 1)}{\prod_{l=1}^{n-j}(q^l - 1)}.$$

Corollary 6.8. *The projective space* $\mathrm{PG}(n, q)$ *contains* $\frac{q^{n+1}-1}{q-1}$ *points,* $\frac{(q^{n+1}-1)(q^n-1)}{(q^2-1)(q-1)}$ *lines and* $\frac{(q^{n+1}-1)(q^n-1)(q^{n-1}-1)}{(q^3-1)(q^2-1)(q-1)}$ *planes.*

Let $i \in \{1, \ldots, n\}$ and U_1, $U_2 \in \mathrm{PG}^i(n, q)$. The smallest subspace of $\mathcal{V}$ that contains both U_1 and U_2 is the sum $U_1 + U_2$ of dimension $j = 2i - \dim(U_1 \cap U_2)$.

Proposition 6.9. *Let* $i \in \{1, \ldots, n\}$ *and* U_1, $U_2 \in \mathrm{PG}^i(n, q)$ *and let* $j = 2i - \dim(U_1 \cap U_2)$.

There are

$$|\mathrm{PG}^{n-j-1}(n - i - 1, q)| = \frac{(q^{n-i} - 1)(q^{n-i} - q) \ldots (q^{n-i} - q^{n-j-2})}{(q^{n-j-1} - 1)(q^{n-j-1} - q) \ldots (q^{n-j-1} - q^{n-j-2})}$$

elements of $\mathrm{PG}^j(n, q)$ *that contain both* U_1 *and* U_2

Proof. This follows directly from Theorem 6.7. □

For $1 \leq i < j \leq n$

$$\mathrm{Tr}_{i,j}(n, q) = (\mathrm{PG}^i(n, q), \mathrm{PG}^j(n, q), \mathcal{I})$$

denotes the truncation containing all elements of $\mathrm{PG}^i(n, q)$ and $\mathrm{PG}^j(n, q)$. In the case that $j = i + 1$, the truncation is also called a *Grassmannian.*

The follwing proposition is known for $i = 2$, see for example [BCN89], p. 268, but we want to prove it here for all $i \geq 2$.

Proposition 6.10. *Let* $i \in \{2, \ldots, n\}$. *The truncation* $\mathrm{Tr}_{n-i+1,n}(n, q)$ *is isomorphic to* $\mathrm{Tr}^*_{1,i}(n, q)$*, the dual design of* $\mathrm{Tr}_{1,i}(n, q)$.

Proof. Consider the map $\phi : \mathrm{PG}^i(n,q) \to \mathrm{PG}^{n-i+1}(n,q)$, where for a subspace $U_i \in \mathrm{PG}^i(n,q)$ the image $\phi(U_i) \in \mathrm{PG}^{n-i+1}(n,q)$ is defined to be the $n-i+1$-dimensional space U_{n-i+1} with

$$U_i \oplus U_{n-i+1} = \mathcal{V}.$$

The map ϕ defines a map Φ from the dual design $\mathrm{Tr}^*_{1,i}(n,q)$ of $\mathrm{Tr}_{1,i}(n,q)$ to $\mathrm{Tr}_{n-i+1,n}(n,q)$ given by the action of ϕ on $\mathrm{PG}^i(n,q)$. We want to show that Φ is an isomorphism. Since $\dim \mathcal{V} = n+1$ the map Φ is bijective. Now, suppose U_i and U'_i meet in a subspace U with $\dim U \geq 1$. It follows for $\phi(U_i) = U_{n-i+1}$ and $\phi(U'_i) = U'_{n+i-1}$ that $U \not\subset U_{n-i+1} \cap U'_{n-i+1}$. Therefore,

$$\dim(U_{n-i+1} \cup U'_{n-i+1}) = \dim \mathcal{V} - \dim U \leq n$$

and U_{n-i+1} and U'_{n-i+1} are both contained in a common n-dimensional space. Hence Φ is preserving the adjacency relation. □

Optimality of $\mathrm{Tr}_{1,j}(n,q)$ and $\mathrm{Tr}_{i,n}(n,q)$

Let $1 < j \leq n$. The truncation

$$\mathrm{Tr}_{1,j}(n,q) = (\mathrm{PG}^1(n,q), \mathrm{PG}^j(n,q), \mathcal{I})$$

has $v_j = \frac{q^{n+1}-1}{q-1}$ points and any point is contained in

$$r_j = \frac{(q^n-1)(q^n-q)\dots(q^n-q^{j-2})}{(q^{j-1}-1)(q^{j-1}-q)\dots(q^{j-1}-q^{j-2})}$$

elements in $\mathrm{PG}^j(n,q)$.

An element in $\mathrm{PG}^j(n,q)$ contains

$$k_j = \frac{q^j - 1}{q - 1}$$

points and any two points are contained in

$$\lambda_j = \frac{(q^j - 1)(q^{j-1} - q)}{(q^2 - 1)(q^2 - q)}$$

elements in $\mathrm{PG}^j(n,q)$ ([BCN89], p.269).

Proposition 6.11. *The truncations* $\mathrm{Tr}_{1,j}(n,q)$ *are universally optimal among all designs (binary or not) on* v_j *points, replication* r_j *and block size* k_j*.*

Proof. The truncation $\mathrm{Tr}_{1,j}(n,q)$ is a binary equireplicate block design with $\mathrm{PG}^1(n,q)$ as points and $\mathrm{PG}^j(n,q)$ as blocks. Since any two points are contained in λ_j blocks, the truncation is a 2-(v_j, k_j, λ_j)-design. With Theorem 2.26 the proposition follows. □

Corollary 6.12. *The truncations* $\mathrm{Tr}_{i,n}(n,q)$ *are optimal on the* Φ_p*-criterion among all binary equireplicate designs.*

Proof. Since the truncation $\mathrm{Tr}_{i,n}(n,q)$ is the dual of $\mathrm{Tr}_{1,n-i+1}(n,q)$ by Proposition 6.10, this follows directly from Proposition 6.11 and Proposition 2.31. □

The Truncation $\mathrm{Tr}_{2,3}(n,q)$

There are

$$v = |\mathrm{PG}^2(n,q)| = \frac{(q^{n+1} - 1)(q^n - 1)}{(q^2 - 1)(q - 1)}$$

lines in $\mathrm{PG}(n,q)$. Every line lies in

$$r = \frac{q^{n-1} - 1}{q - 1}$$

planes and any two lines are contained in either no or exactly one plane. In fact, two distinct lines lie in the same plane if and only if they intersect in a single point. Any plane contains $q^2 + q + 1$ distinct lines. Thus, the adjacency graph $\mathcal{G}(\mathrm{Tr}_{2,3}(n, q))$ of $\mathrm{Tr}_{2,3}(n, q)$ is regular and has v vertices and degree

$$\delta = \frac{q^{n-1} - 1}{q - 1} \left(q^2 + q\right).$$

Of course, for any fixed basis $\mathcal{E}$, the graph $\mathcal{G}(\mathrm{Tr}_{2,3}(n, q))$ contains the subgraph given by the truncation of $\mathbb{G}_n^{\mathcal{E}}$, i.e. the triangular graph $T(n)$. In fact, for any two adjacent vertices in $\mathcal{G}(\mathrm{Tr}_{2,3}(n, q))$ we can always find a basis $\mathcal{E}$ such that the vertices are points in the truncation of the corresponding $\mathbb{G}_n^{\mathcal{E}}$.

The following proposition (in fact its generalization to all Grassmannians) can also be found in [BCN89], p. 269.

Proposition 6.13. *The graph $\mathcal{G}(\mathrm{Tr}_{2,3}(n, q))$ is strongly regular with parameters*

$$\begin{aligned} \lambda &= q^{n-1} + q^{n-2} + \ldots + q^3 + 2q^2 + q - 1 \\ \mu &= (q+1)^2. \end{aligned}$$

The spectrum of $\mathcal{G}(\mathrm{Tr}_{2,3}(n, q))$ is

$$\mathrm{Spec}(\mathcal{G}(\mathrm{Tr}_{2,3}(n, q))) = ((-(q+1))^{m_2}, (\Delta - (q+1))^{m_1}, \delta^1),$$

where

$$\begin{aligned} m_1 &= q(\Delta + 1) \\ m_2 = &= v - 1 - m_1. \end{aligned}$$

and

$$\Delta = \sqrt{(\lambda - \mu)^2 + 4(\delta - \mu)} = q^{n-1} + q^{n-2} + \ldots + q.$$

Proof. We start by computing the parameter λ which counts the number of common neighbours of any adjacent pair of vertices. Let h_1, $h_2 \in \mathrm{PG}^2(n, q)$ be two distinct lines and $X \in \mathcal{V}$ a vector such that

$$h_1 \cap h_2 = \langle X \rangle_{\mathrm{GF}(q)}.$$

We want to compute the number of lines that lie in a plane with both h_1 and h_2. Any line other than h_1, h_2 containing the subspace $\langle X \rangle_{\mathrm{GF}(q)}$ is in the same plane as h_1 and as h_2. There are $\frac{q^n-1}{q-1} - 2$ of these, this is the number of one-dimensional subspaces in the $\mathrm{GF}(q)$-vector space $\mathcal{V}/\langle X \rangle$ other than $h_1/\langle X \rangle_{\mathrm{GF}(q)}$ and $h_2/\langle X \rangle_{\mathrm{GF}(q)}$. Of course, all $\frac{(q^3-q)(q^3-q^2)}{(q^2-1)(q^2-q)}$ lines in the plane spanned by the basis vectors of h_1 and h_2 whose spans do not contain the subspace $\langle X \rangle_{\mathrm{GF}(q)}$ are also in the same plane as h_1 and h_2. In total, there are

$$\begin{aligned} \lambda &= \frac{q^n - 1}{q - 1} - 2 + \frac{(q^3 - q)(q^3 - q^2)}{(q^2 - 1)(q^2 - q)} \\ &= q^{n-1} + q^{n-2} + \ldots + q^3 + 2q^2 + q - 1 \end{aligned}$$

distinct lines that lie on a plane with both h_1 and h_2.

To compute the parameter μ, that is the number of common neighbours of any non-adjacent pair of vertices, let h_1, h_2 be any two skew lines. The number of lines h that are in a plane with both h_1 and h_2 must satisfy $\dim(h \cap h_2) = \dim(h \cap h_1) = 1$. Since $\dim(h_1, h_2) = 4$, this is the number of ways of choosing a one-dimensional subspace from both h_1 and h_2.

Hence, there are

$$\mu = \frac{(q^2 - 1)(q^2 - 1)}{(q - 1)(q - 1)} = (q + 1)^2$$

lines in a plane with both h_1 and h_2.

From Proposition 2.20 it follows that all eigenvalues of the adjacency matrix

$\mathcal{G}(\mathrm{Tr}_{2,3}(n,q))$ are δ with multiplicity 1 and

$$\nu_1 = \frac{1}{2}\left[(\lambda-\mu) + \sqrt{(\lambda-\mu)^2 + 4(\delta-\mu)}\right] = \Delta - (q+1)$$

with multiplicity

$$m_1 = \frac{1}{2}\left[(v-1) - \frac{2\delta + (v-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2 + 4(\delta-\mu)}}\right] = q(\Delta+1)$$

and

$$\nu_2 = \frac{1}{2}\left[(\lambda-\mu) - \sqrt{(\lambda-\mu)^2 + 4(\delta-\mu)}\right] = -(q+1)$$

with multiplicity

$$m_2 = \frac{1}{2}\left[(v-1) + \frac{2\delta + (v-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2 + 4(\delta-\mu)}}\right] = v - 1 - q(\Delta+1).$$

□

There has been interest in characterizing Grassmann graphs in terms of their parameters (an overview of this can be found in [BCN89],pp. 268). The following result shows that in the case of the lines of the projective space for odd n, there are no other possible parameters for a strongly regular graph.

Proposition 6.14. *For any odd $n \geq 3$ and any $q \geq 2$ not necessarily a prime power, any strongly regular graph on*

$$v = \frac{(q^{n+1}-1)(q^n-1)}{(q^2-1)(q-1)}$$

vertices and degree

$$\delta = \frac{q^{n-1}-1}{q-1}(q^2+q)$$

has the parameters

$$\begin{aligned}\lambda &= q^{n-1} + q^{n-2} + \ldots + q^3 + 2q^2 + q - 1 \\ \mu &= (q+1)^2.\end{aligned}$$

Proof. By Lemma 2.19, the parameters $\lambda \geq 0$ and $\mu \geq 1$ of any strongly regular graph on v vertices with degree δ satisfy the equation

$$\delta(\delta - \lambda - 1) = (v - \delta - 1)\mu.$$

and therefore

$$\lambda = \delta - 1 - \frac{v-1}{\delta}\mu + \mu \geq 0$$

and

$$\mu \equiv 0 \mod g \text{ for } \frac{v-1}{\delta} = \frac{f}{g} \text{ with } \gcd(f, g) = 1.$$

Therefore,

$$\begin{aligned}\frac{v-1}{\delta} &= (q-1)\frac{(q^{n+1}-1)(q^n-1)-(q^2)(q-1)}{(q^2-1)(q-1)(q+1)q(q^{n-1}-1)} \\ &= \frac{q^{2n+1} - q^{n+1} - q^n + 1 - q^3 + q^2 + q}{(q^2-1)(q+1)q(q^{n-1}-1)}.\end{aligned}$$

From

$$(q^{n+1} - 1 + q^2 - q)(q^n - q) = q^{2n+1} - q^{n+1} - q^n + 1 - q^3 + q^2 + q$$

follows that

$$\frac{v-1}{\delta} = \frac{q^{n+1} - 1 + q^2 - q}{(q-1)(q+1)^2}$$

$$
\begin{aligned}
&= \frac{(q-1)(q^n+q^{n-1}+\ldots+q^2+q+1)+(q-1)q}{(q-1)(q+1)^2} \\
&= \frac{q^n+q^{n-1}+\ldots+q^2+2q+1}{(q+1)^2}.
\end{aligned}
$$

Since $\lambda \in \mathbb{N}$, we have $\frac{v-1}{\delta} \in \mathbb{N}$. Suppose $q+1$ and $q^n+q^{n-1}+\ldots+q^2+2q+1$ are not coprime, say p is a common factor. Hence

$$q \equiv -1 \mod p.$$

But then for odd n

$$
\begin{aligned}
0 &\equiv (-1)^n+(-1)^{n-1}+\ldots+(-1)^2+2(-1)+1 \mod p \\
&\equiv (-1)^n+(-1)^{n-1}+\ldots+(-1)^2+(-1) \mod p \\
&\equiv -1 \mod p.
\end{aligned}
$$

This is only true for $p=1$. It follows, that $(q+1)^2$ divides μ, that is $\mu=(q+1)^2x$ for some non-negative integer x.

As

$$
\begin{aligned}
\lambda &= \delta-1-\frac{v-1}{\delta}\mu+\mu \\
&= \frac{q^{n+1}+q^n-q^2-2q+1}{q-1} - x\frac{q^{n+1}-q^3}{q-1} \\
&\geq 0,
\end{aligned}
$$

it follows for $n \geq 3$ and $q \geq 2$ that

$$\frac{q^{n+1}+q^n-q^2-2q+1}{q^{n+1}-q^3} = 1+\frac{q^n+q^3-q^2-2q+1}{q^{n+1}-q^3} \geq x.$$

But $\frac{q^n+q^3-q^2-2q+1}{q^{n+1}-q^3} < 1$ for $n \geq 3$ and $q \geq 2$. Since x is a non-negative integer, either $x = 0$ or $x = 1$. Therefore, either $\mu = 0$ and $\lambda = \delta - 1$ or

$$\mu = (q+1)^2$$

and

$$\lambda = q^{n-1} + q^{n-2} + \ldots + q^3 + 2q^2 + q - 1.$$

Note that in the case that $\mu = 0$, by our definition the graph is as a disconnected union of cliques not a strongly regular graph.

□

In terms of the structure of the graph we have the following result:

Proposition 6.15. *The neighbourhood graph of any vertex of a strongly regular graph with the same parameters as the adjacency graph of* $\mathrm{Tr}_{2,3}(n,q)$ *is connected.*

Proof. Since

$$\begin{aligned} \lambda &= q^{n-1} + q^{n-2} + \ldots + q^3 + 2q^2 + q - 1 \\ &> q^{n-1} + q^{n-2} + \ldots + q^3 + q^2 \\ &= \frac{1}{2}\left((\lambda - \mu) + \sqrt{(\lambda-\mu)^2 + 4(\delta - \mu)}\right), \end{aligned}$$

the parameters satisfy the conditions of Proposition 4.2. □

We have shown in Proposition 6.10 that the truncation $\mathrm{Tr}_{n-1,n}(n,q)$ is the dual of $\mathrm{Tr}_{1,2}(n,q)$ and that they are both optimal block designs. Although we do not know whether $\mathrm{Tr}_{2,3}(n,q)$ is optimal, we can show that its adjacency graph is the adjacency graph of an optimal design:

Proposition 6.16. *The adjacency graph of* $\mathrm{Tr}_{n-1,n}(n,q)$ *is isomorphic to the adjacency graph of* $\mathrm{Tr}_{2,3}(n,q)$*.*

Proof. The truncation $\mathrm{Tr}_{n-1,n}(n,q)$ is the dual of $\mathrm{Tr}_{1,2}(n,q)$, therefore in its adjacency graph the vertices correspond to lines and an edge to a pair of lines meeting in a point. But any two lines meet in a point if and only if they lie in a common plane, which is the adjacency relation of $\mathrm{Tr}_{2,3}(n,q)$. □

Corollary 6.17. *The largest clique in the adjacency graph of* $\mathrm{Tr}_{2,3}(n,q)$ *has size* $\frac{q^n-1}{q-1}$ *and the vertices of the neighbourhood graph of any vertex can be partitioned into* $q+1$ *cliques of size* $\frac{q^n-1}{q-1}-1$.

Proof. All lines through a fixed point produce a clique of size $\frac{q^n-1}{q-1}$ in the adjacency graph. Any line contains $q+1$ points, so for any of these points there is a corresponding clique of $\frac{q^n-1}{q-1}-1$ lines sharing this point, thus the second statement of the Corollary follows. Because of the Hoffman bound (Proposition 2.18), the size of the largest clique in the graph is bounded from above by

$$v\frac{1+\nu_2}{v-\delta+\nu_2}=\frac{q^n-1}{q-1},$$

where ν_2 is the second largest eigenvalue of the graph. □

The Truncations $\mathrm{Tr}_{2,j}(n,q)$, $j>3$

Now, let $j\in\{4,\dots,n\}$ and

$$v=|\mathrm{PG}^2(n,q)|=\frac{(q^{n+1}-1)(q^n-1)}{(q^2-1)(q-1)}.$$

The adjacency matrix $\mathbb{A}(\mathrm{Tr}_{2,j})=(a_{lm})$ has the entries

$$a_{lm}=\begin{cases}\alpha_j \text{ if } l\neq m \text{ and lines } l \text{ and } m \text{ lie on the same plane}\\ \beta_j \text{ else}\end{cases}$$

where α_j denotes the number of j-dimensional subspaces of $\mathcal{V}$ that contain a fixed plane and β_j the number of j-dimensional subspaces containing a fixed 4-dimensional subspace.

Therefore we can write the adjacency matrix as

$$\mathbb{A}(\mathrm{Tr}_{2,j}) = \alpha_j \mathbb{A}(\mathrm{Tr}_{2,3}) + \beta_j \mathbb{J}_{v_j} = \alpha_j \mathbb{A}(\mathrm{Tr}_{n-1,n}) + \beta_j \mathbb{J}_{v_j}$$

or

$$\mathbb{A}(\mathrm{Tr}_{2,j}) = \frac{\alpha_j}{\alpha_n} \mathbb{A}(\mathrm{Tr}_{2,n}) + (\beta_j - \frac{\alpha_j}{\alpha_n}\beta_n)\mathbb{J}_{v_j}.$$

Since

$$\alpha_{j+1} = \frac{q^{n-3} - q^{j-4}}{q^{j-2} - 1}\alpha_j \text{ and } r_j = \frac{q^{j-1}(q^j - 1)}{q^n - q^{j-1}},$$

we can compute the parameters recursively as

$$\alpha_j = \alpha_n \prod_{l=j}^{n-1} \frac{q^{l-3}(q^{l-1} - 1)}{q^{n-1} - q^{l-2}} = \frac{q^{n-2} - 1}{q - 1} \prod_{l=j}^{n-1} \frac{q^{l-3}(q^{l-1} - 1)}{q^{n-1} - q^{l-2}}$$

and

$$r_j = r_n \prod_{l=j}^{n-1} \frac{q^{l-1}(q^l - 1)}{q^n - q^{l-1}} = \frac{q^{n-1} - 1}{q - 1} \prod_{l=j}^{n-1} \frac{q^{l-1}(q^l - 1)}{q^n - q^{l-1}}.$$

The Truncations $\mathrm{Tr}_{i,j}(n, q)$, $3 \leq i < j$

There are

$$v = |\mathrm{PG}^i(n, q)| = \frac{(q^{n+1} - 1)(q^{n+1} - q) \ldots (q^{n+1} - q^{i-1})}{(q^i - 1)(q^i - q) \ldots (q^i - q^{i-1})}.$$

subspaces of dimension i of $\mathcal{V}$. Let $\mathrm{PG}^i(n, q) = \{U_1, \ldots, U_v\}$.

Let $j = i + 1$. In this case, the truncation has the adjacency matrix

$\mathbb{A}(\mathrm{Tr}_{i,i+1}) = (a_{ml})$ with the following entries:

$$a_{ml} = \begin{cases} 0 \text{ if } \dim(U_m \cap U_l) < i-1 \text{ or } m = l \\ 1 \text{ if } m \neq l \text{ and } \dim(U_m \cap U_l) = i-1 \end{cases}$$

Now, let $j > i+1$. The adjacency matrix $\mathbb{A}(\mathrm{Tr}_{i,j}) = (a_{ml})$ of the truncation has the entries

$$a_{ml} = \begin{cases} 0 \text{ if } \dim(U_m \cap U_l) < 2i-j \text{ or } m = l \\ \beta_0 = 1 \text{ if } m \neq l \text{ and } \dim(U_m \cap U_l) = 2i-j \\ \beta_1 \text{ if } m \neq l \text{ and } \dim(U_m \cap U_l) = 2i-j+1 \\ \beta_2 \text{ if } m \neq l \text{ and } \dim(U_m \cap U_l) = 2i-j+2 \\ \vdots \\ \beta_{j-i+1} \text{ if } m \neq l \text{ and } \dim(U_m \cap U_l) = i-1 \end{cases}$$

where β_r denotes the number of j-dimensional subspaces containing a fixed $j-l$-dimensional subspace for $l = 0, \ldots, j-i+1$.

It follows for $l = 1, \ldots, n-1$

$$\beta_{l+1} = \frac{q^l - 1}{q^{n+1-(j-l)} - 1}\beta_l,$$

hence for $l = 2, \ldots, n$

$$\beta_l = \beta_1 \prod_{t=2}^{t-1} \frac{q^t - 1}{q^{n+1-(j-t)} - 1} = \frac{q^{n+1-(j-1)} - 1}{q - 1} \prod_{t=2}^{t-1} \frac{q^t - 1}{q^{n+1-(j-t)} - 1}$$

where β_1 is the number of points in the $n+2-j$-dimensional projective space over $\mathcal{V}/(U_m + U_l)$.

Of course, for any fixed basis $\mathcal{E}$, the graph $\mathcal{G}(\mathrm{Tr}_{i,j}(n,q))$ contains the subgraph given by the truncation of $\mathbb{G}_n^{\mathcal{E}}$, i.e. the union of edges of certain generalized Johnson graphs. Note that for any two adjacent vertices in $\mathcal{G}(\mathrm{Tr}_{i,j}(n,q))$ we can always find a basis $\mathcal{E}$ such that the vertices are points in the truncation of the corresponding $\mathbb{G}_n^{\mathcal{E}}$.

Chapter 7

Robustness - Deleting Edges from an Optimal Graph

Throughout this chapter we only consider graphs with at least two vertices, i.e. binary designs with block size $k = 2$ and $v \geq 2$. We want to compare graphs on v vertices and e edges on the A- and D-value.

7.1 Deleting Edges from the Complete Graph

The following proposition is a known result. We want to prove it here to illustrate the use of the technique introduced in [PR02].

Proposition 7.1 ([Shi74]). *Let $\mathcal{G}^m$ be the graph obtained from the complete graph K_v by deleting m mutually disjoint edges (for $2m < v$). Then*

$$\kappa(\mathcal{G}^m) = v^{v-2} \left(1 - \frac{2}{v}\right)^m$$

and $\mathcal{G}^m$ maximizes the number of spanning trees among all simple graphs on v vertices and $\frac{v(v-1)}{2} - m$ edges.

Proof. Since the complement of $\mathcal{G}^m$ is a union of cliques, we have equality in

Proposition 2.32 and

$$\kappa(\mathcal{G}^m) = v^{v-2}\prod_{i=1}^{v}(1-\frac{1+\delta_i}{v})^{\frac{\delta_i}{\delta_i+1}} = v^{v-2}\left(1-\frac{2}{v}\right)^m,$$

where $(\delta_1,\dots,\delta_v)$ denotes the degree sequence of the complement of $\mathcal{G}^m$. The product on the right hand side is maximized by the degree sequence ([PR02])

$$(\underbrace{l,\dots,l}_{v-g},\underbrace{l+1,\dots,l+1}_{g}) \text{ where } \sum_{i=1}^{v}\delta_i = vl+g,\ g<v.$$

In the case of $\sum_{i=1}^{v}\delta_i = 2m$ this is $(\underbrace{0,\dots,0}_{v-2m},\underbrace{1,\dots,1}_{2m})$, which is the degree sequence of $\mathcal{G}^m$. □

Let $y*K_v$ denote the complete multigraph on v vertices, i.e. any two vertices are joined by exactly y edges.

Proposition 7.2. *Let $\mathcal{G}$ be the graph obtained from $y*K_v$ by deleting m mutually disjoint edges (for $2m<v$). Then*

$$\kappa(\mathcal{G}) = y(yv)^{v-m-2}(yv-2)^m.$$

Proof. The Laplacian matrix of the matching given by the $2m$ removed edges from $y*K_v$ is the $2m\times 2m$-matrix

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 1 & 0 & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \ddots & \cdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 1 & -1 \\ 0 & \cdots & \cdots & \cdots & \cdots & -1 & 1 \end{pmatrix}.$$

We use Proposition 2.34 to obtain

$$\begin{aligned}\kappa(\mathcal{G}) &= yv(yv)^{v-2m-2}\left(\det\begin{pmatrix} yv-1 & 1 \\ 1 & yv-1 \end{pmatrix}\right)^m \\ &= y(yv)^{v-2m-2}(y^2v^2-2yv)^m.\end{aligned}$$

□

Proposition 7.3. *Let $v \geq 4$. The graph $\mathcal{G}^2$ is A-optimal among simple graphs on v vertices and $\frac{v(v-1)-1}{2}$ edges.*

Proof. We compute the A-value of a graph $\mathcal{G}$ as

$$\frac{\kappa(\mathcal{G})}{\sum_{u_1,u_2\in V(\mathcal{G})}\kappa(\mathcal{G})_{.\{u_1,u_2\}}},$$

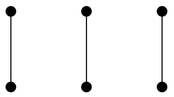
which we can write

$$\begin{aligned}\sum_{u_1,u_2\in V(\mathcal{G})}\kappa(\mathcal{G})_{.\{u_1,u_2\}} &= \sum_{f\in E(\mathcal{G})}\kappa(\mathcal{G})_{.f}+\sum_{f\notin E(\mathcal{G})}\kappa(\mathcal{G})_{.f} \\ &= \sum_{f\in E(\mathcal{G})}(\kappa(\mathcal{G})-\kappa(\mathcal{G}\setminus\{f\}) \\ &\quad+\sum_{f\notin E(\mathcal{G})}(\kappa(\mathcal{G}\cup\{f\})-\kappa(\mathcal{G})) \\ &= (|E(\mathcal{G})|-|E(\bar{\mathcal{G}})|)\kappa(\mathcal{G}) \\ &\quad+\sum_{f\notin E(\mathcal{G})}\kappa(\mathcal{G}\cup\{f\})-\sum_{f\in E(\mathcal{G})}\kappa(\mathcal{G}\setminus\{f\}).\end{aligned}$$

Let $f_1, f_2 \in E(K_v)$ such that $\mathcal{G}^2 = K_v \setminus \{f_1, f_2\}$. For $i = 1, 2$ Proposition 7.1 yields

$$\kappa(\mathcal{G}^2\cup\{f_i\}) = \kappa(\mathcal{G}^1) = v^{v-2}(1-\tfrac{2}{v}).$$

For any edge $f = \{u_1, u_2\} \in E(\mathcal{G}^2)$ the complement of $\mathcal{G}^2 \setminus \{f\}$ is one of the following graphs

1. 

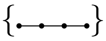
2.

3.

In the first case, we have $v-4$ vertices to choose u_1 from and $v-5$ vertices to choose u_2 from. That means, there are $\frac{(v-4)(v-5)}{2}$ choices for f and

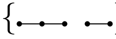
$$\kappa(\mathcal{G}^2 \setminus \{f\}) = \kappa(\mathcal{G}^3) = v^{v-2}(1-\tfrac{2}{v})^3.$$

For the last two cases, we compute the tree number with Proposition 2.34. In case 2, there are 4 possible ways to connect f_1 and f_2 with an edge and

$\kappa(K_v \setminus \{$  $\})$

$$\begin{aligned} &= v^{v-6} \det \left(v\mathbb{I}_4 + \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \right) \\ &= v^{v-2} - 6v^{v-3} + 10v^{v-4} - 4v^{v-5}. \end{aligned}$$

In the last case, there are $v-4$ vertices to choose u_1 or u_2 from and 4 ways to attach f to either f_1 or f_2. Therefore, this case occurs $4(v-4)$ times and

$\kappa(K_v \setminus \{$  $\})$

$$\begin{aligned} &= v^{v-7} \det \left(v\mathbb{I}_5 + \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \right) \\ &= v^{v-2} - 6v^{v-3} + 11v^{v-4} - 6^{v-5}. \end{aligned}$$

In total this is

$$\sum_{u_1,u_2\in V(\mathcal{G}^2)} \kappa(\mathcal{G}^2)_{.\{u_1,u_2\}} = \left(\tfrac{v(v-1)}{2} - 4\right)\kappa(\mathcal{G}^2) + 2\kappa(\mathcal{G}^1) - \tfrac{(v-1)(v-5)}{2}\kappa(\mathcal{G}^3)$$
$$-4(v^{v-2} - 6v^{v-3} + 10v^{v-4} - 4v^{v-5})$$
$$-4(v-4)(v^{v-2} - 6v^{v-3} + 11v^{v-4} - 6^{v-5})$$
$$= v^{v-5}(v-2)^2(v-1)(v+4)$$
$$-4v(v-2)(v^2-4v+2).$$

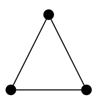
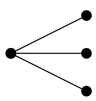
Now, suppose we are deleting two joined edges $f_1, f_2 \in E(K_v)$ from K_v, then

$$\kappa(K_v \setminus \{\wedge\}) = v^{v-5}\det\left(v\mathbb{I}_3 + \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}\right)$$
$$= v^{v-5}(v^3 - 4v^2 + 3v)$$

and adding f_1 or f_2 yields

$$\kappa(\mathcal{G}^1) = v^{v-2}(1 - \tfrac{2}{v}).$$

Removing another edge $f = \{u_1, u_2\} \in E(K_v \setminus \{\wedge\})$, the complement is one of the following graphs:

1. 
2.
3. 
4.

In the first case we have

$$\begin{aligned}
\kappa(K_v \setminus \{\triangle\}) &= v^{v-5} \det \left(v\mathbb{I}_3 + \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \right) \\
&= v^{v-5}(v^3 - 6v^2 + 9v)
\end{aligned}$$

and in the third case

$$\begin{aligned}
\kappa(K_v \setminus \{\lessdot\}) &= v^{v-6} \det \left(v\mathbb{I}_4 + \begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \right) \\
&= v^{v-6}(v^4 - 6v^3 + 9v^2 - 3v - 1).
\end{aligned}$$

The second and fourth cases are the same as above. The first case occurs exactly once. In the second case, we can choose any of the remaining $v-3$ vertices as u_1 or u_2 and there are exactly two ways to join f to f_1 or f_2, therefore the second case occurs $2(v-3)$ times. The third case occurs $v-3$ times since there are again $v-3$ choices for u_1 or u_2 and only one way of joining f with f_1 and f_2 in the one vertex. The last case occurs $\frac{(v-3)(v-4)}{2}$ times since we have $v-3$ ways to choose u_1 and $v-4$ ways to choose u_2. In total this is

$$\sum_{u_1,u_2 \in V(K_v \setminus \{\wedge\})} \kappa(K_v \setminus \{\wedge\})_{.\{u_1,u_2\}} = v^{v-4}(v^3 - 5v^2 + 11v - 9).$$

That means

$$\begin{aligned}
& \frac{\kappa(\mathcal{G}^2)}{\sum_{u_1,u_2 \in V(\mathcal{G}^2)} \kappa(\mathcal{G}^2)_{.\{u_1,u_2\}}} - \frac{\kappa(K_v \setminus \{\wedge\})}{\sum_{u_1,u_2 \in V(K_v \setminus \{\wedge\})} \kappa(K_v \setminus \{\wedge\})_{.\{u_1,u_2\}}} \\
= \; & \frac{v^{v-2}(1-\frac{2}{v})^2}{v^{v-4}(v^3 - 5v^2 + 12v - 12)} - \frac{v^{v-5}(v^3 - 4v^2 + 3v)}{v^{v-4}(v^3 - 5v^2 + 11v - 9)}
\end{aligned}$$

$$= \frac{2v}{(v^2-3v+6)(v^3-5v^2+11v-9)}$$

For $v \geq 4$, the denominator of the above fraction is strictly positive and therefore the A-value is maximized if the removed edges are disjoint. $\square$

7.2 Deleting an Edge from the Complete Bipartite Graph

Let $n \geq 2$ and let $K_{n,n} \setminus \{f\}$ be the graph obtained from the regular complete bipartite graph $K_{n,n}$ by removing one edge f. Then

$$|E(K_{n,n} \setminus \{f\})| = n^2 - 1 = (n-1)(n+1) = |E(K_{n-1,n+1})|$$

and $K_{n,n} \setminus \{f\}$ and $K_{n-1,n+1}$ are in the same class of binary designs with block size 2 and we can compare the two graphs on the A- and D-criterion. Proposition 2.32 gives an upper bound on the number of spanning trees that depends on the number of V-subgraphs in the complement and the degree sequence of a graph. The graph $K_{n,n} \setminus \{f\}$ is almost-regular but its complement contains many V-subgraphs. On the other hand, the complement of $K_{n-1,n+1}$ is a union of cliques and therefore minimizes the number of V-subgraphs, but the degree of the vertices can differ by 2. We want to compare these two graphs on the A- and D-criterion; for this we compute the Laplacian eigenvalues of $K_{n,n} \setminus \{f\}$ first.

Lemma 7.4. *Let $n \geq 2$, then*

$$\mathrm{Spec}(\Lambda(K_{n,n} \setminus \{f\})) = \left(0^1, \left(\tfrac{(3n-2-\sqrt{n^2+4n}-n)}{2}\right)^1, n^{2n-3}, \left(\tfrac{(3n-2+\sqrt{n^2+4n}-n)}{2}\right)^1\right).$$

Proof. The Laplacian matrix of $K_{n,n} \setminus \{f\}$ can be written as

$$\Lambda(K_{n,n} \setminus \{f\}) = \left(\begin{array}{c|ccc|ccc|c} n-1 & 0 & \cdots & 0 & -1 & \cdots & -1 & 0 \\ \hline 0 & & & & & & & -1 \\ \vdots & & n\mathbb{I}_{n-1} & & & -\mathbb{J}_{n-1} & & \vdots \\ 0 & & & & & & & -1 \\ \hline -1 & & & & & & & 0 \\ \vdots & & -\mathbb{J}_{n-1} & & & n\mathbb{I}_{n-1} & & \vdots \\ -1 & & & & & & & 0 \\ \hline 0 & -1 & \cdots & -1 & 0 & \cdots & 0 & n-1 \end{array}\right).$$

For a vector $X = (x_1, \dots, x_v) \in \mathbb{R}^v$ we have

$$\Lambda(K_{n,n} \setminus \{f\})X = \begin{pmatrix} (n-1)x_1 - \sum_{j=n+1}^{2n-1} x_j \\ nx_2 - \sum_{j=n+1}^{2n} x_j \\ \vdots \\ nx_n - \sum_{j=n+1}^{2n} x_j \\ nx_{n+1} - \sum_{j=1}^{n} x_j \\ \vdots \\ nx_{2n-1} - \sum_{j=1}^{n} x_j \\ (n-1)x_{2n} - \sum_{j=2}^{n} x_j \end{pmatrix}.$$

Of course, the vector $(1, 1, \dots, 1)^T \in \mathbb{R}^v$ is an eigenvector of $\Lambda(K_{n,n} \setminus \{f\})$ with eigenvalue 0.

Now, fix $l, p \in \{2, \dots, n\}$ with $l < p$. The vector with coordinates $x_l = 1$, $x_p = -1$ and $x_j = 0$ for all $j = 1, \dots, 2n$ with $j \neq l, p$ is an eigenvector with eigenvalue n:

$$(n-1)x_1 - \sum_{j=n+1}^{2n-1} x_j \;=\; 0,$$

$$
\begin{aligned}
nx_l - \sum_{j=n+1}^{2n} x_j &= n \\
nx_p - \sum_{j=n+1}^{2n} x_j &= -n \\
nx_i - \sum_{j=n+1}^{2n} x_j &= 0 \text{ for } i = 2, \ldots, n,\ i \neq l, p \\
nx_i - \sum_{j=1}^{2n} x_j &= -(x_l + x_p) = 0 \text{ for } i = n+1, \ldots, 2n,\ i \neq l, p \\
(n-1)x_{2n} - \sum_{j=2}^{2n} x_j &= -(x_l + x_p) = 0.
\end{aligned}
$$

The same is true if l and p are fixed with $l, p \in \{n+1, \ldots, 2n-1\}$. There are in total $2n-4$ linearly independent vectors of this form.

The vector with coordinates $x_1 = x_{2n} = 1$ and $x_2 = x_{n+1} = -1$ and $x_j = 0$ for $j = 3, \ldots, n, n+2, \ldots, 2n-1$ is also an eigenvector with eigenvalue n:

$$
\begin{aligned}
(n-1)x_1 - \sum_{j=n+1}^{2n-1} x_j &= (n-1)x_1 - x_{n+1} = n, \\
nx_2 - \sum_{j=n+1}^{2n-1} x_j &= nx_2 - x_{n+1} - x_{2n} = -n \\
nx_i - \sum_{j=n+1}^{2n} x_j &= -x_{n+1} - x_{2n} = 0 \text{ for } i = 3, \ldots, n \\
nx_{n+1} - \sum_{j=n+1}^{2n-1} x_j &= nx_{n+1} - x_1 - x_2 = -n \\
nx_i - \sum_{j=1}^{2n} x_j &= -x_1 - x_2 = 0 \text{ for } i = n+2, \ldots, 2n-1, \\
(n-1)x_{2n} - \sum_{j=2}^{2n} x_j &= -(n-1)x_{2n} - x_2 = n = 0.
\end{aligned}
$$

Let $s \in \{\frac{-n\pm\sqrt{n^2+4n-4}}{2(n-1)}\}$, then

$$-(n-1)(s-1)s = s(2n-1) - 1.$$

Therefore, the vector with coordinates $x_1 = -1$, $x_{2n} = 1$ and $x_i = s$ if $i \in \{2, \ldots, n\}$ and $x_j = -s$ if $j \in \{n+1, \ldots, 2n-1\}$ is an eigenvector with eigenvalue $-(n-1)(s-1)$:

$$\begin{aligned}
(n-1)x_1 - \sum_{j=n+1}^{2n-1} x_j &= -(n-1) + (n-1)s = (n-1)(s-1), \\
nx_i - \sum_{j=n+1}^{2n} x_j &= ns + (n-1)s - 1 = s(2n-1) - 1 \text{ for } i = 2, \ldots, n \\
nx_i - \sum_{j=1}^{2n} x_j &= -(s(2n-1) - 1) \text{ for } i = n+1, \ldots, 2n-1, \\
(n-1)x_{2n} - \sum_{j=2}^{2n} x_j &= (n-1) - (n-1)s = -(n-1)(s-1).
\end{aligned}$$

We have found $2n = v$ linearly independent eigenvectors and their eigenvalues. From Theorem 2.2 we know that the algebraic and geometric multiplicities of the eigenvalues are the same and therefore we have found all eigenvalues.

With

$$\tfrac{1}{2}(3n - 2 + \sqrt{n^2 + 4n - 4}) \geq \tfrac{1}{2}(3n - 2 + n) = 2n - 2 \geq n$$

for $n \geq 2$ it follows, that the Laplacian spectrum with the eigenvalues in increasing order is

$$\mathrm{Spec}(\Lambda(K_{n,n} \setminus \{f\})) = \left(0^1, \left(\tfrac{(3n-2-\sqrt{n^2+4n-4})}{2}\right)^1, n^{2n-3}, \left(\tfrac{(3n-2+\sqrt{n^2+4n-4})}{2}\right)^1\right).$$

□

Proposition 7.5. *The non-trivial Laplacian eigenvalues of $K_{n,n} \setminus \{f\}$ are ma-*

jorized by the non-trivial Laplacian eigenvalues of $K_{n-1,n+1}$.

Proof. We already know the Laplacian Spectrum of $K_{m,n}$ with $m \leq n$ (see page 20), for $K_{n-1,n+1}$ it is

$$\mathrm{Spec}(\Lambda(K_{n-1,n+1})) = (0^1, (n-1)^n, (n+1)^{n-2}, (2n)^1).$$

Since $n^2 + 4n - 4 < (n+2)^2$ it follows that

$$\rho_1(K_{n-1,n+1}) = 2n = \tfrac{1}{2}(4n) > \tfrac{1}{2}(3n - 2 + \sqrt{n^2 + 4n - 4}) = \rho_1(K_{n,n} \setminus \{f\}),$$

and for $j = 1, \ldots, n-2$

$$\begin{aligned}
\sum_{i=1}^{j+1} \rho_i(K_{n-1,n+1}) &= 2n + j(n+1) \\
&> \tfrac{1}{2}(3n - 2 + \sqrt{n^2 + 4n - 4}) + jn \\
&= \sum_{i=1}^{j+1} \rho_i(K_{n,n} \setminus \{f\}).
\end{aligned}$$

With $2n > \frac{1}{2}(3n - 2 + \sqrt{n^2 + 4n - 4})$ it follows for $j = 1, \ldots, n-1$

$$\begin{aligned}
\sum_{i=1}^{n-1+j} \rho_i(K_{n-1,n+1}) &= 2n + (n-2)(n+1) + j(n-1) \\
&= 2n + (n-2+j)n + n - 2 - j \\
&\geq 2n - 1 + (n-2+j)n \\
&\geq \tfrac{1}{2}(3n - 2 + \sqrt{n^2 + 4n - 4}) + (n-2+j)n \\
&= \sum_{i=1}^{n-1+j} \rho_i(K_{n,n} \setminus \{f\}).
\end{aligned}$$

Since

$$\tfrac{1}{2}(3n - 2 - \sqrt{n^2 + 4n - 4}) + \tfrac{1}{2}(3n - 2 + \sqrt{n^2 + 4n - 4}) = 3n - 2,$$

we have

$$\begin{aligned}
\sum_{i=1}^{2n-1} \rho_i(K_{n-1,n+1}) &= 2(n^2-1) \\
&= \tfrac{1}{2}(3n-2+\sqrt{n^2+4n-4}) + (2n-3)n \\
&\quad + \tfrac{1}{2}(3n-2-\sqrt{n^2+4n-4}) \\
&= \sum_{i=1}^{2n-1} \rho_i(K_{n,n} \setminus \{f\}).
\end{aligned}$$

□

Corollary 7.6. *For the vectors of the non-trivial Laplacian eigenvalues $\rho_{K_{n,n}\setminus\{f\}}$ and $\rho_{K_{n-1,n+1}}$ and $j = 2, \ldots, 2n-1$ the following holds:*

1. $S_j(\rho_{K_{n,n}\setminus\{f\}}) > S_j(\rho_{K_{n-1,n+1}})$;
2. $\frac{S_j(\rho_{K_{n,n}\setminus\{f\}})}{S_{j-1}(\rho_{K_{n,n}\setminus\{f\}})} > \frac{S_j(\rho_{K_{n-1,n+1}})}{S_{j-1}(\rho_{K_{n-1,n+1}})}$.

Proof. Follows directly from Lemma 7.4 and Proposition 2.38, Proposition 2.39 and Proposition 7.5. □

Since the D-value of a graph $\mathcal{G}$ is $S_{v-1}(\rho_{\mathcal{G}})$ and since we can write the A-value of $\mathcal{G}$ as the ratio $(v-1)\frac{S_{v-1}(\rho_{\mathcal{G}})}{S_{v-2}(\rho_{\mathcal{G}})}$ we have the following corollary.

Corollary 7.7. *The graph $K_{n,n}\setminus\{f\}$ beats $K_{n-1,n+1}$ on the A- and D-criterion.*

Note that since the complement of $K_{n-1,n+1}$ is a union of cliques, $\kappa(K_{n-1,n+1})$ is by Proposition 2.32 an upper bound for any graph with the same degree sequence, and the next corollary follows.

Corollary 7.8. *The graph $K_{n,n}\setminus\{f\}$ beats any simple graph with degree sequence*

$$(\overbrace{n+1, n+1, \ldots, n+1}^{n-1}, \overbrace{n-1, n-1, \ldots, n-1}^{n+1})$$

on the D-criterion

In fact, we could not find a connected graph that beats $K_{n,n} \setminus \{f\}$ on the A- or D-criterion for $n = 3$, 4 or 5 where we searched among all simple connected graphs on $2n$ vertices and $n^2 - 1$ edges which we generated with B. McKay's program nauty ([McK81]).

7.3 Counterexamples

We have seen that deleting one or two edges from an optimal graph can have no impact on its optimality within the class. But this is not generally true and we want to give some counter examples. Of course, the performance of a graph obtained by deleting an edge on the D-criterion depends on which edge has been deleted. Here is an example.

Example. Let $\mathcal{G}$ be the graph obtained from the complete graph K_4 by deleting any edge. Then $\kappa(\mathcal{G}) = 8$. Now, there are two kind of edges that can be deleted: either the graph becomes a four-cycle or a three-cycle with a pendant edge. In the first case, the number of spanning trees is 4 and it is 3 in the latter.

Example. By Proposition 2.33, the complete bipartite graph $K_{n,n+1}$ is D-optimal among all simple graphs. We have seen that $K_{n,n} \setminus \{f\}$ is a good candidate for the D-criterion. The complement of $K_{n,n+1}$ is again a union of cliques and contains no V-subgraphs. But in this case it is easy to find a graph that performs better on the D-criterion than the graphs obtained by removing an edge $f = \{v, w\} \in E(K_{n,n+1})$ from $K_{n,n+1}$.

The spanning trees of $K_{n,n+1} \setminus \{f\}$ are all the spanning trees of $K_{n,n+1}$, except the ones containing f. The number of these is precisely the number of spanning forests $T_{u,w}$ with two parts such that u, w belong to different parts.

Let $V_1(K_{m,n})$ and $V_2(K_{m,n})$ denote the two parts of vertices of the complete bipartite graph $K_{m,n}$ with $|V_1(K_{m,n})| = m$ and $|V_2(K_{m,n})| = n$. By [JL04], for

$(s,l) \neq (0,0)$ the number of spanning forests of $K_{m,n}$ with $l+s$ parts of which different parts contain $\{a_1,\dots,a_l\} \subset V_1(K_{m,n})$ and $\{b_1,\dots,b_s\} \subset V_2(K_{m,n})$ is

$$f[n,s;m,l] = n^{m-s-1} m^{n-l-1}(lm + sn - sl), \text{ for } s,l \geq 0.$$

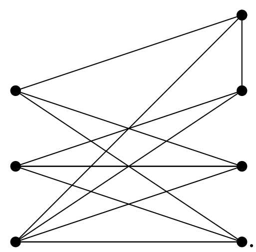
Hence $T_{u,w} = f[n,1;n+1,1] = n^{n-1}(n+1)^{n-2}2n$ and with $\kappa(K_{m,n}) = m^{n-1}n^{m-1}$ (see page 24) it follows that

$$\begin{aligned} \kappa(K_{n,n+1} \setminus \{f\}) &= \kappa(K_{n,n+1}) - f[n,1,n+1,1] \\ &= (n+1)^{n-1}n^n - n^{n-1}(n+1)^{n-2}2n. \end{aligned}$$

For $v = 2n+1 = 7$ this is

$$\kappa(K_{3,4} \setminus \{f\}) = 216.$$

But the following graph $\mathcal{G}$ beats with 231 spanning trees (computed with Mathematica) the above graph on the D-criterion:



Chapter 8

Optimal Designs in Large Systems

8.1 A- and D-optimality in Large Systems

Let $\mathcal{L}(v, b, k)$ be the set of the Laplacian matrices of all connected designs (binary or not) with v points and b blocks of size k. For $\Lambda \in \mathcal{L}(v, b, k)$ we will write ρ_Λ for the vector of the non-trivial eigenvalues $(\rho_1(\Lambda), \ldots, \rho_{v-1}(\Lambda))$. Let $\tilde{d}$ be a 2-$(v, k, \tilde{\lambda})$-design on v points and block size k with $\tilde{b}$ blocks, where $\tilde{\lambda}$ is minimal such that a 2-$(v, k, \tilde{\lambda})$-design exists. Then for $\Lambda \in \mathcal{L}(v, b, k)$ and $y \in \mathbb{N}$ the matrix

$$\Lambda[y] = \Lambda + y\tilde{\Lambda} = \Lambda + y(v\mathbb{I}_v - \mathbb{J}_v) \tag{8.1.1}$$

is the Laplacian matrix of a design on v points, replication $r + y\frac{\tilde{\lambda}(v-1)}{k-1}$ and $b + y\tilde{b}$ blocks of size k. The non-trivial eigenvalues of $\Lambda[y]$ are $vy + \rho_1(\Lambda), \ldots, vy + \rho_{v-1}(\Lambda)$.

Proposition 8.1. *For given v, b, k, let $\Lambda, \Lambda' \in \mathcal{L}(v, b, k)$ such that the eigenvalues of Λ' majorize the eigenvalues of Λ. If designs d and d' with Laplacian matrices $\Lambda[y]$ and $\Lambda'[y]$ exist for an $y \geq 0$, then d is A- and D-better than d'.*

Proof. Suppose $\Lambda, \Lambda' \in \mathcal{L}(v, b, k)$ are Laplacian matrices such that the eigenvalues of Λ' majorize the eigenvalues of Λ. Then of course the eigenvalues of $\Lambda[y]$ are also majorized by the eigenvalues of $\Lambda'[y]$. Since the A- and D-value are Schur-concave functions (Proposition 2.38 and Proposition 2.39) the statement follows. □

Corollary 8.2. *For given v, b, k, let $\Lambda \in \mathcal{L}(v, b, k)$ be the Laplacian matrix of a Schur-optimal design. If a design d with Laplacian matrix $\Lambda[y]$ exist for an $y \geq 0$, then d is A- and D-optimal among all designs with Laplacian matrices $\Lambda'[y]$ where $\Lambda' \in \mathcal{L}(v, b, k)$.*

As an application of this corollary we have the following proposition.

Proposition 8.3. *Let $K_{n,n} \setminus \{f\}$ denote the regular complete bipartite graph with a deleted edge f. The eigenvalues of $\Lambda(K_{n,n} \setminus \{f\})[y]$ are majorized by the eigenvalues of $\Lambda(K_{n-1,n+1})[y]$ for all $y \geq 0$ and a design with Laplacian matrix $\Lambda(K_{n,n} \setminus \{f\})[y]$ performs better on both the A- and D-criterion than any design with Laplacian matrix $\Lambda(K_{n-1,n+1})[y]$.*

Proof. By Proposition 7.5, the eigenvalues of $\Lambda(K_{n,n} \setminus \{f\})$ are majorized by the eigenvalues of $\Lambda(K_{n-1,n+1})$. □

The D-value of $\Lambda[y]$ is the product of all non-trivial eigenvalues of $\Lambda[y]$, that is

$$D(\rho_\Lambda, y) = \prod_{i=1}^{v-1} (vy + \rho_i(\Lambda)),$$

which we can write as polynomial in y ([Bro06]):

$$D(\rho_\Lambda, y) = \sum_{j=0}^{v-1} (vy)^{v-1-j} S_j(\rho_\Lambda). \tag{8.1.2}$$

The A-value of $\Lambda[y]$ is the harmonic mean over all non-trivial eigenvalues, that is

$$A(\rho_\Lambda, y) = (v-1)\frac{1}{\sum_{i=1}^{v-1}\frac{1}{vy+\rho_i(\Lambda)}} = \frac{D(\rho_\Lambda, y)}{\sum_{l=1}^{v-1}\prod_{i=1,i\neq l}^{v-1}(vy+\rho_i(\Lambda))}.$$

With equation 2.6.2 we have

$$\begin{aligned}\sum_{l=1}^{v-1}\prod_{i=1,i\neq l}^{v-1}(vy+\rho_i(\Lambda)) &= \sum_{l=1}^{v-1}\sum_{j=0}^{v-2}(vy)^{v-2-j}S_{j;l}(\rho_\Lambda)\\ &= \sum_{j=0}^{v-2}(vy)^{v-2-j}(v-1-j)S_j(\rho_\Lambda).\end{aligned}$$

Let $D_y(\rho_\Lambda, y)$ denote the derivative of $D(\rho_\Lambda, y)$ in y, then

$$A(\rho_\Lambda, y) = v(v-1)\frac{D(\rho_\Lambda, y)}{D_y(\rho_\Lambda, y)}. \tag{8.1.3}$$

Because the performance of a matrix $\Lambda \in \mathcal{L}(v, b, k)$ on the A- and D-criteria is in this way closely related to the elementary symmetric polynomials of the non-trivial eigenvalues of Λ, we want to order the matrices accordingly. We will take the lexicographic ordering corresponding to the elementary symmetric polynomials of the non-trivial eigenvalues of the Laplacian matrices. That means, for two Laplacian matrices $\Lambda, \Lambda' \in \mathcal{L}(v, b, k)$ with $\mathrm{Spec}(\Lambda) \neq \mathrm{Spec}(\Lambda')$, we will write $\Lambda' \prec \Lambda$ if there exists an $l \in \{1, \ldots, v-1\}$ such that

$$S_j(\rho_\Lambda) = S_j(\rho_{\Lambda'}),\ j = 1, \ldots, l-1 \text{ and } S_l(\rho_{\Lambda'}) < S_l(\rho_\Lambda).$$

This is a transitive relation which we will call the *stable order* on $\mathcal{L}(v, b, k)$. Note that matrices are indistinguishable in this order if they have the same spectrum. From equation 8.1.1 it is immediately clear that if $\Lambda' \prec \Lambda$, then there exists a y_0 such that $D(\rho_{\Lambda'}, y) \leq D(\rho_\Lambda, y)$ for $y \geq y_0$ and we want to show that this is also true for the A-value.

Lemma 8.4. *Let* $\Lambda, \Lambda' \in \mathcal{L}(v,b,k)$ *and*

$$P(\Lambda,\Lambda',y) = \sum_{i=1}^{2v-3} p_i y^i = D(\rho_\Lambda, y)D_y(\rho_{\Lambda'}, y) - D(\rho_{\Lambda'}, y)D_y(\rho_\Lambda, y).$$

Then the leading coefficients of $P(\Lambda,\Lambda',y)$ *are*

$$p_{2v-3} = 0,\ p_{2v-4} = v^{2v-4}\left(S_1(\rho_\Lambda) - S_1(\rho_{\Lambda'})\right),\ p_{2v-5} = 2v^{2v-5}\left(S_2(\rho_\Lambda) - S_2(\rho_{\Lambda'})\right).$$

If $\Lambda' \prec \Lambda$ *such that* l *is the smallest index with* $S_l(\rho_\Lambda) - S_l(\rho_{\Lambda'}) > 0$*, then the first non-vanishing coefficient of* $P(\Lambda,\Lambda',y)$ *is*

$$p_{2v-3-l} = v^{2v-3-l} l\left(S_l(\rho_\Lambda) - S_l(\rho_{\Lambda'})\right).$$

Proof. Since

$$\begin{aligned} D(\rho_\Lambda, y)D_y(\rho_{\Lambda'}, y) &= \sum_{i=0}^{v-1}\sum_{j=0}^{v-1}(vy)^{2v-3-j-i}(v-1-j)S_j(\rho_{\Lambda'})S_i(\rho_\Lambda) \\ &= \sum_{i=0}^{v-1}(vy)^{2v-3-i}\sum_{j=0}^{i}(v-1-j)S_j(\rho_{\Lambda'})S_{i-j}(\rho_\Lambda) \end{aligned}$$

the polynomial $P(\Lambda,\Lambda',y)$ can be written as

$$P(\Lambda,\Lambda',y) = \sum_{i=0}^{v-1}(vy)^{2v-3-i}\sum_{j=0}^{i}(v-1-j)\left(S_j(\rho_{\Lambda'})S_{i-j}(\rho_\Lambda) - S_j(\rho_\Lambda)S_{i-j}(\rho_{\Lambda'})\right)$$

and with $S_0 \equiv 1$, its first coefficient is

$$\begin{aligned} p_{2v-3} &= v^{2v-3}\sum_{j=0}^{0}(v-1-j)\left(S_j(\rho_{\Lambda'})S_{0-j}(\rho_\Lambda) - S_j(\rho_\Lambda)S_{0-j}(\rho_{\Lambda'})\right) \\ &= v^{2v-3}(v-1)\left(S_0(\rho_{\Lambda'})S_0(\rho_\Lambda) - S_0(\rho_\Lambda)S_0(\rho_{\Lambda'})\right) \\ &= 0. \end{aligned}$$

The coefficient of y^{2v-4} is

$$\begin{aligned}
p_{2v-4} &= v^{2v-4}\sum_{j=0}^{1}(v-1-j)\left(S_j(\rho_{\Lambda'})S_{1-j}(\rho_\Lambda)-S_j(\rho_\Lambda)S_{1-j}(\rho_{\Lambda'})\right)\\
&= v^{2v-4}(v-1)(S_0(\rho_{\Lambda'})S_1(\rho_\Lambda)-S_0(\rho_\Lambda)S_1(\rho_{\Lambda'}))\\
&\quad +v^{2v-4}(v-2)(S_1(\rho_{\Lambda'})S_0(\rho_\Lambda)-S_1(\rho_\Lambda)S_0(\rho_{\Lambda'}))\\
&= v^{2v-4}\left(S_0(\rho_{\Lambda'})S_1(\rho_\Lambda)-S_1(\rho_\Lambda)S_0(\rho_{\Lambda'})\right)\\
&= v^{2v-4}(S_1(\rho_\Lambda)-S_1(\rho_\Lambda))
\end{aligned}$$

and the coefficient of of y^{2v-5} is

$$\begin{aligned}
p_{2v-5} &= v^{2v-5}\sum_{j=0}^{2}(v-1-j)\left(S_j(\rho_{\Lambda'})S_{2-j}(\rho_\Lambda)-S_j(\rho_\Lambda)S_{2-j}(\rho_{\Lambda'})\right)\\
&= v^{2v-5}(v-1)(S_0(\rho_{\Lambda'})S_2(\rho_\Lambda)-S_0(\rho_\Lambda)S_2(\rho_{\Lambda'}))\\
&\quad +v^{2v-5}(v-2)(S_1(\rho_{\Lambda'})S_1(\rho_\Lambda)-S_1(\rho_\Lambda)S_1(\rho_{\Lambda'}))\\
&\quad +v^{2v-5}(v-3)(S_2(\rho_{\Lambda'})S_0(\rho_\Lambda)-S_2(\rho_\Lambda)S_0(\rho_{\Lambda'}))\\
&= 2v^{2v-5}(S_0(\rho_{\Lambda'})S_2(\rho_\Lambda)-S_0(\rho_\Lambda)S_2(\rho_{\Lambda'}))\\
&= 2v^{2v-5}(S_2(\rho_\Lambda)-S_2(\rho_{\Lambda'}).
\end{aligned}$$

If $\Lambda' \prec \Lambda$, then $S_j(\rho_\Lambda)=S_j(\rho_{\Lambda'})$ for $1 \leq j < l$; the first non-zero coefficient is

$$\begin{aligned}
p_{2v-3-l} &= v^{2v-3-l}\sum_{j=0}^{l}(v-1-j)\left(S_j(\rho_{\Lambda'})S_{l-j}(\rho_\Lambda)-S_j(\rho_\Lambda)S_{l-j}(\rho_{\Lambda'})\right)\\
&= v^{2v-3-l}(v-1-l)\left(S_l(\rho_{\Lambda'})S_0(\rho_\Lambda)-S_l(\rho_\Lambda)S_0(\rho_{\Lambda'})\right)\\
&\quad +v^{2v-3-l}(v-1)\left(S_l(\rho_\Lambda)S_0(\rho_{\Lambda'})-S_l(\rho_{\Lambda'})S_0(\rho_\Lambda)\right)\\
&= v^{2v-3-l}l\left(S_l(\rho_\Lambda)-S_l(\rho_{\Lambda'})\right).
\end{aligned}$$

□

We can now prove the following theorem.

Theorem 8.5. *For given v, b, k, there exists a y_0 such that, if designs with Laplacian matrix $\Lambda[y]$ exist for some $\Lambda \in \mathcal{L}(v, b, k)$ and $y \geq y_0$, then their ordering under the A- or D-criterion is the stable ordering of the matrices Λ.*

Proof. Let $\Lambda, \Lambda' \in \mathcal{L}(v, b, k)$ with $\Lambda' \prec \Lambda$. Then there exists an $l \in \{1, \ldots, v-1\}$ such that $S_j(\rho_\Lambda) = S_j(\rho_{\Lambda'})$ for $j = 1, \ldots, l-1$ and $S_l(\rho_\Lambda) > S_l(\rho_{\Lambda'})$. That means, with Lemma 8.4 that the first non-vanishing coefficient of the polynomial

$$D(\rho_\Lambda, y)D_y(\rho_{\Lambda'}, y) - D(\rho_{\Lambda'}, y)D_y(\rho_\Lambda, y)$$

is

$$v^{2v-3-l}l(S_l(\rho_\Lambda) - S_l(\rho_{\Lambda'})) > 0.$$

The difference of the D-values $D(\rho_\Lambda, y) - D(\rho_{\Lambda'}, y)$ has of course the first non-vanishing coefficient $v^{v-1-l}(S_l(\rho_\Lambda) - S_l(\rho_{\Lambda'})) > 0$. It follows that there exists a y_0, such that

$$A(\rho_\Lambda, y) > A(\rho_{\Lambda'}, y) \text{ and } D(\rho_\Lambda, y) > D(\rho_{\Lambda'}, y) \text{ for } y \geq y_0.$$

□

For $i = 1, \ldots, v-1$, we will denote by $\mathcal{L}_i(v, b, k)$ the set of Laplacian matrices $\Lambda \in \mathcal{L}(v, b, k)$ such that

$$S_j(\rho_\Lambda) = \max\{S_j(\rho_{\Lambda'}) | \Lambda' \in \mathcal{L}(v, b, k)\} \text{ for } j = 1, \ldots, i.$$

That means, that for $i \in \{1, \ldots, v-2\}$ and for $\Lambda, \Lambda' \in \mathcal{L}_i(v, b, k)$ with $\Lambda' \notin \mathcal{L}_{i+1}(v, b, k)$ we have $\Lambda' \prec \Lambda$. We obtain the result by Constantine that is given in [Con86] in terms of the traces of the Laplacian matices.

Corollary 8.6 ([Con86])**.** *For given v, b, k and $3 \leq j \leq v$, if designs with Laplacian matrix $\Lambda[y]$ where $\Lambda \in \mathcal{L}_j(v, b, k)$ exist for some $y \geq y_0$, then they* *are A- and D-optimal among all designs with Laplacian matrices* $\Lambda'[y]$ *with* $\Lambda' \in \mathcal{L}(v,b,k) \setminus \mathcal{L}_j(v,b,k)$.

We want to characterize the Laplacian matrices that are in $\mathcal{L}_j(v,b,k)$ for a $j \geq 1$.

Proposition 8.7. *For given* v, b, k*, if a Schur-optimal design with Laplacian matrix* $\Lambda \in \mathcal{L}(v,b,k)$ *exists, then* $\Lambda \in \mathcal{L}_{v-1}(v,b,k)$.

Proof. If Λ is the Laplacian matrix of a Schur-optimal design, then the eigenvalues of Λ are majorized by any $\Lambda' \in \mathcal{L}(v,b,k)$. Since the symmetric functions are Schur-concave by Proposition 2.38, the matrix Λ then maximizes $S_j(\rho_\Lambda)$ for any $j = 1, \ldots, v-1$ and it follows $\Lambda \in \mathcal{L}_{v-1}(v,b,k)$. □

To characterize the matrices in $\mathcal{L}_1(v,b,k)$ and $\mathcal{L}_2(v,b,k)$ we will need the following results on the elementary symmetric functions on the eigenvalues for $\Lambda \in \mathcal{L}(v,b,k)$.

Proposition 8.8. *For given* v, b, k *and* $\Lambda = (\Lambda_{ij}) \in \mathcal{L}(v,b,k)$*, the following holds:*

1. $$S_1(\rho_\Lambda) \leq bk(k-1)$$

   with equality if and only if the corresponding design is binary;

2. *if* $S_1(\rho_\Lambda) = bk(k-1)$*, then* $S_2(\rho_\Lambda)$ *is maximized if and only if*

$$\begin{aligned} \Lambda_{ii} &\in \{\lfloor \tfrac{bk(k-1)}{v} \rfloor, \lfloor \tfrac{bk(k-1)}{v} \rfloor + 1\} \text{ and} \\ \Lambda_{ij} &\in \{\lfloor \tfrac{bk(k-1)}{v(v-1)} \rfloor, \lfloor \tfrac{bk(k-1)}{v(v-1)} \rfloor + 1\} \end{aligned}$$

   for $i, j, l, m \in \{1, \ldots, v\}$ *where* $i \neq j$ *and* $l \neq m$*;*

3. *if* $S_1(\rho_\Lambda) = bk(k-1)$ *and* $\frac{bk}{v} = r \in \mathbb{N}$, *then*

$$2S_2(\rho_\Lambda) \leq (vr(k-1))^2 - v(r(k-1))^2 - vr(k-1)$$

with equality if and only if

$$\begin{aligned} \Lambda_{ii} &= r(k-1) \text{ and} \\ \Lambda_{ij} &\in \{\lfloor \tfrac{r(k-1)}{v-1} \rfloor, \lfloor \tfrac{r(k-1)}{v-1} \rfloor + 1\} \end{aligned}$$

for $i, j \in \{1, \ldots, v\}$ *where* $i \neq j$.

Proof. Since $S_1(\rho_\Lambda) = \mathrm{Trace}(\Lambda)$, the first statement follows with Theorem 2.26. Now, let $\Lambda \in \mathcal{L}(v, b, k)$ such that $S_1(\rho_\Lambda) = bk(k-1)$. Since

$$\begin{aligned} 2S_2(\rho_\Lambda) &= \left(\sum_{i=1}^{v-1} \rho_i(\Lambda)\right)^2 - \mathrm{Trace}(\Lambda^2) \\ &= (S_1(\rho_\Lambda))^2 - \mathrm{Trace}(\Lambda^2) \\ &= (bk(k-1))^2 - \mathrm{Trace}(\Lambda^2), \end{aligned}$$

maximizing $S_2(\rho_\Lambda)$ is equivalent with minimizing $\mathrm{Trace}(\Lambda^2)$, which we can write as

$$\mathrm{Trace}(\Lambda^2) = \sum_{i=1}^{v} \Lambda_{ii}^2 + \sum_{i=1}^{v} \sum_{j \neq i} (\Lambda_{ij})^2.$$

The vector $(1, \ldots, 1)^T \in \mathbb{R}^v$ is an eigenvector of Λ with eigenvalue 0, and therefore

$$\sum_{j \neq i} \Lambda_{ij} = -\Lambda_{ii} \text{ for all } i = 1, \ldots, v.$$

Because $\sum_{i=1}^{v} \Lambda_{ii} = bk(k-1)$, it follows that

$$\sum_{i=1}^{v} \sum_{j \neq i} (-\Lambda_{ij}) = \sum_{i=1}^{v} \Lambda_{ii} = bk(k-1)$$

is a constant as well.

Lemma 2.25 gives for $n = v$ and $m = bk(k-1)$ the optimal solution for the diagonal entries Λ_{ii} is to take $bk(k-1) - v\lfloor \frac{bk(k-1)}{v} \rfloor$ entries to be $\lfloor \frac{bk(k-1)}{v} \rfloor + 1$ and the rest to be $\lfloor \frac{bk(k-1)}{v} \rfloor$. For $n = v(v-1)$ and $m = bk(k-1)$, the optimal solution for the entries Λ_{ij} with $i \neq j$ is to take $bk(k-1) - v(v-1)\lfloor \frac{bk(k-1)}{v(v-1)} \rfloor$ entries to be $\lfloor \frac{bk(k-1)}{v(v-1)} \rfloor + 1$ and the rest to be $\lfloor \frac{bk(k-1)}{v(v-1)} \rfloor$.

If $\frac{bk}{v} = r \in \mathbb{N}$, the optimal solution for the diagonal entries is then $\Lambda_{ii} = r(k-1)$ for $i = 1, \ldots, v$, and the optimal solution for the entries Λ_{ij} with $i \neq j$ is to take $vr(k-1) - v(v-1)\lfloor \frac{r(k-1)}{v-1} \rfloor$ entries to be $\lfloor \frac{r(k-1)}{v-1} \rfloor + 1$ and the rest to be $\lfloor \frac{r(k-1)}{v-1} \rfloor$. It follows that the minimal value of $\text{Trace}(\Lambda^2)$ is

$$\begin{aligned}
\text{Trace}(\Lambda^2) &= v(r(k-1))^2 + \left(vr(k-1) - v(v-1)\lfloor \tfrac{r(k-1)}{v-1} \rfloor\right) \left(\lfloor \tfrac{r(k-1)}{v-1} \rfloor + 1\right) \\
&+ \left(v(v-1) - \left(vr(k-1) - v(v-1)\lfloor \tfrac{r(k-1)}{v-1} \rfloor\right)\right) \lfloor \tfrac{r(k-1)}{v-1} \rfloor \\
&= v(r(k-1))^2 + vr(k-1).
\end{aligned}$$

and therefore the maximal value of $2S_2(\rho_\Lambda)$ is

$$\begin{aligned}
2S_2(\rho_\Lambda) &= S_1(\rho_\Lambda)^2 - \text{Trace}(\Lambda^2) \\
&= (vr(k-1))^2 - v(r(k-1))^2 - vr(k-1).
\end{aligned}$$

□

With the above proposition, it follows that $\mathcal{L}_1(v, b, k)$ is the set of the Laplacian matrices of all existing binary designs and $\mathcal{L}_2(v, b, k)$ is the set of the Laplacian matrices of any existing regular graph designs (RGDs) or nearly balanced incomplete block designs (NBDs).

Example. The cycle $Cycle(v)$ on v vertices is the simple, regular adjacency graph of a binary design with $v = b$ and $k = 2$. Bailey showed in [Bai07], that $Cycle(v)$ is D-optimal among connected graphs with v vertices and v edges for all $v \geq 2$ and we can prove that the graph $Cycle(v) + y * K_v$ is D-optimal among all graphs $\mathcal{G} + y * K_v$ where $\mathcal{G}$ has v edges for all $y \geq 0$: by Proposition 8.8, the graph $Cycle(v)$ maximizes $S_1(\rho_{\mathcal{G}})$ and $S_2(\rho_{\mathcal{G}})$ among all connected graphs $\mathcal{G}$ with v vertices and v edges. Stevanovic and Ilic ([SI09]) showed, that $Cycle(v)$ maximizes $S_j(\rho_{\mathcal{G}})$ for all $j = 2, \ldots, v-1$ among all simple connected graphs with v edges. Since all designs in the class with Laplacian matrix in $\mathcal{L}_2(v, v, 2)$ have a simple connected adjacency graph by Proposition 8.8, it follows that $\Lambda(Cycle(v)) \in \mathcal{L}_{v-1}(v, v, 2)$ and with that we have proved the next proposition.

Proposition 8.9. *Any design with Laplacian matrix $\Lambda(Cycle(v))[y]$ is D-optimal for all $y \geq 0$ among connected designs with Laplacian matrix $\Lambda[y]$ with $\Lambda \in \mathcal{L}(v, v, 2)$ and there exists a y_0 such that, if there exists a design with Laplacian matrix $\Lambda(Cycle(v))[y]$ and $y \geq y_0$, then it is A-optimal among all designs with Laplacian matrix $\Lambda[y]$ with $\Lambda \in \mathcal{L}(v, v, 2)$.*

Bailey showed in [Bai07] that that the cycle is the adjacency graph of an A-optimal binary design only for $v \leq 8$ and $v = 12$. For $9 \leq v \leq 11$, the quadrangle of which one vertex is joined to all the remaining $v - 4$ vertices is A-optimal. For $v \geq 13$ the triangle of which one vertex is joined to the remaining $v - 3$ vertices is A-optimal; we will denote this graph by $C_3(v - 3)$. For $v = 12$, both $Cycle(12)$ and $C_3(9)$ give A-best binary designs. In the case $v = 20$, the performance $C_3(17)$ on the A-criterion is a lot better than the performance of $Cycle(20)$ on the A-criterion:

$$\frac{A(\rho_{Cycle(20)}, 0)}{A(\rho_{C_3(17)}, 0)} = \frac{149}{285} = 0,522807,$$

but the graph $Cycle(20) + K_{20}$ beats $C_3(17) + K_{20}$ on the A-criterion:

$$\frac{A(\rho_{Cycle(20)}, 1)}{A(\rho_{C_3(17)}, 1)} = \frac{2014224563024993}{1979512211726735} = 1,01754.$$

The example of the cycle shows that the lower bound for y in Proposition 8.5 might not have to be very large. Our aim is to find out what 'large' means; for this we have to bound the differences of the elementary symmetric polynomials.

Lemma 8.10. *Let $\Lambda \in \mathcal{L}_2(v, b, k)$ and $\Lambda' \in \mathcal{L}(v, b, k)$ and $\Lambda' \notin \mathcal{L}_2(v, b, k)$. Then*

$$S_2(\rho_\Lambda) - S_2(\rho_{\Lambda'}) \geq 1.$$

Proof. By Corollary 2.42 $S_2(\rho_\Lambda) \in \mathbb{N}$ for all $\Lambda \in \mathcal{L}(v, b, k)$ and the statement follows. □

Lemma 8.11. *The largest eigenvalue of a matrix $\Lambda = (\Lambda_{ij}) \in \mathcal{L}_1(v, b, k)$ is bounded from above by* $\max\{(r_i + r_j)(k-1) | \Lambda_{ij} \neq 0, i \neq j\}$.

Proof. The eigenvalues of Λ are the Laplacian eigenvalues of the adjacency graph of the corresponding design. This graph is connected and has degree sequence $(r_1(k-1), \ldots, r_v(k-1))$. By Theorem 2.16, the Laplacian eigenvalues of a connected graph are bounded from above by

$$\max\{\delta_u + \delta_w | (u, w) \in E(\mathcal{G})\}.$$

It follows that the largest Laplacian eigenvalue of Λ is bounded from above by $\max\{(r_i + r_j)(k-1) | \Lambda_{ij} \neq 0, i \neq j\}$. □

Lemma 8.12. *Let $\Lambda, \Lambda' \in \mathcal{L}_1(v, b, k)$, then for $m = \lceil \frac{2v-3}{3} \rceil$*

$$|S_j(\rho_\Lambda) - S_j(\rho_{\Lambda'})| \leq (b(k-1))^j 2^m \binom{v-1}{m}.$$

Proof. By Lemma 8.11 for $\Lambda' = (\Lambda'_{ij}) \in \mathcal{L}_1(v,b,k)$

$$\rho_1(\Lambda') \leq \max\{(r_i + r_j)(k-1) | \Lambda'_{ij} \neq 0, i \neq j\}.$$

Since $r_i \leq b$ for all $i = 1, \dots, v-1$, we have

$$\begin{aligned} |S_j(\rho_\Lambda) - S_j(\rho_{\Lambda'})| &\leq \binom{v-1}{j} \left(\max\{(r_i + r_j)(k-1) | \Lambda'_{ij} \neq 0, i \neq j\}\right)^j \\ &= \binom{v-1}{j} (2b(k-1))^j. \end{aligned}$$

Since the fraction

$$\frac{2\binom{v-1}{j+1}}{\binom{v-1}{j}} = \frac{2(v-1-j)}{j+1}$$

is strictly greater than 1 if and only if $j < \frac{2v-3}{3}$, the binomial coefficient $2^j \binom{v-1}{j}$ as a function in j is strictly increasing until $j < \frac{2v-3}{3}$ and therefore the maximum is attained for $j = m = \lceil \frac{2v-3}{3} \rceil$. □

We get the following result which is given in [Che92] for more general optimality criteria but without a bound on y.

Proposition 8.13. *For given v, b, k and $y_0 = v^2 2^m (b(k-1))^{v-1} \binom{v-1}{m} + 1$, where $m = \lceil \frac{2v-3}{3} \rceil$, if there exist designs with Laplacian matrix $\Lambda[y]$ with $\Lambda \in \mathcal{L}_2(v,b,k)$ and $y \geq y_0$, then they beat any design with Laplacian matrix $\Lambda'[y]$ with $\Lambda' \in \mathcal{L}_1(v,b,k) \setminus \mathcal{L}_2(v,b,k)$ on the D-criterion.*

Proof. Let $\Lambda \in \mathcal{L}_2(v,b,k)$. By Proposition 8.8 $S_1(\rho_\Lambda) \geq S_1(\rho_{\Lambda'})$ for all $\Lambda' \in \mathcal{L}_1(v,b,k)$ and $S_2(\rho_\Lambda) > S_2(\rho_{\Lambda'})$ for all $\Lambda' \notin \mathcal{L}_2(v,b,k)$. From Corollary 8.10 we know that $S_2(\rho_\Lambda) - S_2(\rho_\Lambda) \in \mathbb{N}_{>0}$.
From Lemma 8.12 it follows for $y \geq v^2 2^m (b(k-1))^{v-1} \binom{v-1}{m} + 1$ that

$$\sum_{j=3}^{v-1} v^{v-1-j} y^{v-1-j} |S_j(\rho_\Lambda) - S_j(\rho_{\Lambda'})| < 2^m (vb(k-1))^{v-1} \binom{v-1}{m} \sum_{j=3}^{v-1} y^{v-1-j}$$

$$
\begin{aligned}
&\leq 2^m (vb(k-1))^{v-1}\binom{v-1}{m}\frac{y^{v-3}-1}{y-1} \\
&\leq v^{v-3}(y-1)\frac{y^{v-3}-1}{y-1} \\
&< v^{v-3}y^{v-3} \\
&\leq v^{v-3}y^{v-3}(S_2(\rho_\Lambda) - S_2(\rho_{\Lambda'})) \\
&\quad + v^{v-2}y^{v-2}(S_1(\rho_\Lambda) - S_1(\rho_{\Lambda'})).
\end{aligned}
$$

□

Lemma 8.14. *Let* $\Lambda, \Lambda' \in \mathcal{L}_1(v,b,k)$*, then for* $i > j$ *and* $m = \lceil \frac{v-2}{2} \rceil$

$$|S_j(\rho_\Lambda)S_{i-j}(\rho_{\Lambda'}) - S_j(\rho_{\Lambda'})S_{i-j}(\rho_\Lambda)| \leq (2b(k-1))^i \binom{v-1}{m}^2.$$

Proof. By Lemma 8.11 for $\Lambda' = (\Lambda'_{ij}) \in \mathcal{L}_1(v,b,k)$

$$\rho_1(\Lambda') \leq \max\{(r_i + r_j)(k-1) | \Lambda'_{ij} \neq 0 \, i \neq j\}.$$

Since $r_i \leq b$ for all $i = 1, \ldots, v-1$, we have

$$|S_j(\rho_\Lambda)S_{i-j}(\rho_{\Lambda'}) - S_j(\rho_{\Lambda'})S_{i-j}(\rho_\Lambda)| \leq \binom{v-1}{j}\binom{v-1}{i-j}(2b(k-1))^i.$$

Since

$$\frac{\binom{v-1}{j+1}}{\binom{v-1}{j}} = \frac{v-1-j}{j+1},$$

the binomial coefficient $\binom{v-1}{j}$ as a function in j is strictly increasing until $j < \frac{v-2}{2}$ and the maximum is attained for $j = m = \lceil \frac{v-2}{2} \rceil$. Therefore

$$|S_j(\rho_\Lambda) - S_j(\rho_{\Lambda'})| \leq (2b(k-1))^i \binom{v-1}{m}^2.$$

□

With the above lemma we get the following result which is again given in [Che92] for more general optimality criteria but without a bound on y.

Proposition 8.15. *For given* v, b, k *and* $y_0 = 2^{v-2}(b(k-1))^{v-1}(2v-5)\binom{v-1}{m}^2 + \frac{1}{v}$, *where* $m = \lceil \frac{v-2}{2} \rceil$, *if there exists a design with* $\Lambda[y]$ *with* $\Lambda \in \mathcal{L}_2(v, b, k)$ *and* $y \geq y_0$, *then* $\Lambda[y]$ *beats any design with Laplacian matrix* $\Lambda'[y]$ *with* $\Lambda' \in \mathcal{L}_1(v, b, k) \setminus \mathcal{L}_2(v, b, k)$ *on the A-criterion.*

Proof. Let $\Lambda \in \mathcal{L}_2(v, b, k)$. By Proposition 8.8 and Corollary 8.10 $S_1(\rho_\Lambda) \geq S_1(\rho_{\Lambda'})$ for all $\Lambda' \in \mathcal{L}(v, b, k)$ and $S_2(\rho_\Lambda) - S_2(\rho_\Lambda) \in \mathbb{N}_{>0}$ for all $\Lambda' \notin \mathcal{L}_2(v, b, k)$. By Lemma 8.4, the three leading coefficients of the polynomial

$$D(\rho_\Lambda, y)D_y(\rho_{\Lambda'}, y) - D(\rho_{\Lambda'}, y)D_y(\rho_\Lambda, y)$$

are

$$0,\ v^{2v-4}\left(S_1(\rho_\Lambda) - S_1(\rho_{\Lambda'})\right) \geq 0 \text{ and } 2v^{2v-5}\left(S_2(\rho_\Lambda) - S_2(\rho_{\Lambda'})\right) > 0.$$

By Lemma 8.14 for $i > j$ and $m = \lceil \frac{v-2}{2} \rceil$

$$|S_j(\rho_\Lambda)S_{i-j}(\rho_{\Lambda'}) - S_j(\rho_{\Lambda'})S_{i-j}(\rho_\Lambda)| \leq (2b(k-1))^i \binom{v-1}{m}^2.$$

It follows for $y - \frac{1}{v} \geq 2^{v-2}(b(k-1))^{v-1}(2v-5)\binom{v-1}{m}^2$ that

$$\begin{aligned}
&\sum_{i=3}^{v-1}(vy)^{2v-3-i}\sum_{j=0}^{i}(v-1-j)|S_j(\rho_\Lambda)S_{i-j}(\rho_{\Lambda'}) - S_j(\rho_{\Lambda'})S_{i-j}(\rho_\Lambda)| \\
\leq\ & (2b(k-1))^{v-1}\binom{v-1}{m}^2\left(\sum_{i=3}^{v-1}(vy)^{2v-3-i}\sum_{j=0}^{i}(v-1-j)\right) \\
=\ & (2b(k-1))^{v-1}\binom{v-1}{m}^2\left(\sum_{i=3}^{v-1}(vy)^{2v-3-i}(i+1)\frac{2(v-1)-i}{2}\right) \\
<\ & (2b(k-1))^{v-1}\binom{v-1}{m}^2\frac{v(2v-5)}{2}\sum_{i=3}^{v-1}(vy)^{2v-3-i}
\end{aligned}$$

$$
\begin{aligned}
&= (2b(k-1))^{v-1}\binom{v-1}{m}^2 \frac{v(2v-5)}{2}(vy)^{2v-6-(v-4)}\sum_{i=0}^{v-4}(vy)^i \\
&= 2^{v-2}(b(k-1))^{v-1}\binom{v-1}{m}^2 v(2v-5)(vy)^{v-2}\frac{(vy)^{v-3}-1}{vy-1} \\
&= v\left(2^{v-2}(b(k-1))^{v-1}\binom{v-1}{m}^2(2v-5)\right)(vy)^{v-2}\frac{(vy)^{v-3}-1}{vy-1} \\
&\leq (vy-1)(vy)^{v-2}\frac{(vy)^{v-3}-1}{vy-1} \\
&< (vy)^{2v-5} \\
&\leq 2(vy)^{2v-5}(S_2(\rho_\Lambda)-S_2(\rho_{\Lambda'}))+(vy)^{2v-4}\left(S_1(\rho_\Lambda)-S_1(\rho_{\Lambda'})\right).
\end{aligned}
$$

□

8.2 Comparing RGDs in Large Systems

For the rest of this chapter let $\frac{bk}{v} = r \in \mathbb{N}$, that means $\mathcal{L}_2(v,b,k)$ is the set of Laplacian matrices of any existing RGDs. Our aim in this section is to characterize the best RGDs, since if they exist, they are the best designs for large y by Corollary 8.6. In particular, if $\mathcal{L}_2(v,b,k) \neq \mathcal{L}_3(v,b,k)$ then, if designs with Laplacian matrix $\Lambda[y]$ with $\Lambda \in \mathcal{L}_3(v,b,k)$ exist for $y \geq y_0$, they beat any design with Laplacian matrix $\Lambda'[y]$ with $\Lambda' \in \mathcal{L}_2(v,b,k) \setminus \mathcal{L}_3(v,b,k)$ on the A- and D-criterion.

Corollary 8.16. *There exists an y_0 such that among designs with Laplacian matrix $\Lambda[y]$ with $\Lambda \in \mathcal{L}_2(v,b,k)$ any A-optimal design is D-optimal and vice versa for $y \geq y_0$.*

Recall that $\mathcal{M}(v,\delta)$ denotes the set of all not necessarily connected, simple δ-regular graphs. If we search only among RGDs, we are in fact searching among regular graphs in $\mathcal{M}(v,\delta)$: we can write any Laplacian matrix Λ of an RGD with

$\lambda = \lfloor \frac{r(k-1)}{v-1} \rfloor$ and $\delta = r(k-1) - \lambda(v-1)$ as

$$\begin{aligned} \Lambda &= (r(k-1)+\lambda)\mathbb{I}_v - \lambda\mathbb{J}_v - \mathbb{A}(\mathcal{G}) \\ &= (\delta + v\lambda)\mathbb{I}_v - \lambda\mathbb{J}_v - \mathbb{A}(\mathcal{G}), \end{aligned}$$

where $\mathcal{G}$ is the underlying graph of Λ, that is $\mathcal{G} \in \mathcal{M}(v, \delta)$. The A- and D-values of the design are $D(\rho_{\mathcal{G}}, \lambda)$ and $A(\rho_{\mathcal{G}}, \lambda) = v(v-1)\frac{D(\rho_{\mathcal{G}},\lambda)}{R(\rho_{\mathcal{G}},\lambda)}$ given as in equations 8.1.2 and 8.1.3, where $\rho_{\mathcal{G}}$ denotes the vector of the non-trivial Laplacian eigenvalues of the graph $\mathcal{G}$ and $R(\rho_{\mathcal{G}}, y) = D_y(\rho_{\mathcal{G}}, y)$ denotes the derivative of the D-value as polynomial in y. The Laplacian matrix $\Lambda[y]$ as in equation 8.1.1 of the design obtained from the RGD by adding y copies of the blocks of a 2-$(v, k, \tilde{\lambda})$-design is

$$\Lambda[y] = (r(k-1) + vy\tilde{\lambda} + \lambda)\mathbb{I}_v - (y\tilde{\lambda} + \lambda)\mathbb{J}_v - \mathbb{A}(\mathcal{G})$$

which we can write with $x = \lambda + y\tilde{\lambda}$ as

$$\Lambda[x] = (\delta + vx)\mathbb{I}_v - x\mathbb{J}_v - \mathbb{A}(\mathcal{G}).$$

We can view this matrix as a function in $x \in \mathbb{R}_{\geq 0}$ and the D-value of $\Lambda[x]$ is now $D(\rho_{\mathcal{G}}, x)$ and the A-value is $A(\rho_{\mathcal{G}}, x)$. Of course, for $\Lambda[x]$ being a Laplacian matrix of an existing design only some values for x will be admissible. But comparing designs with Laplacian matrix $\Lambda[x]$ with $\Lambda \in \mathcal{L}_2(v, b, k)$ is now reduced to comparing the values $D(\rho_{\mathcal{G}}, x)$ and $A(\rho_{\mathcal{G}}, x)$ among all δ-regular graphs on v vertices. We can now compute lower bounds for x such that designs with Laplacian matrix $\Lambda[x]$, where $\Lambda \in \mathcal{L}_2(v, b, k)$, characterized in Corollary 8.6 are A- and D-best.

Proposition 8.17. *Let $\delta = r(k-1) - \lambda(v-1)$. Suppose $\mathcal{L}_2(v, b, k) \neq \mathcal{L}_3(v, b, k)$.*

For given v, b, k and $x_0 = \frac{1}{v}\left((2\delta)^{v-1}\binom{v-1}{\lceil\frac{v-2}{2}\rceil} + 1\right)$, if there exist designs with Laplacian matrix $\Lambda[x]$ with $\Lambda \in \mathcal{L}_3(v, b, k)$ for $x \geq x_0$, then they maximize the D-value among all designs with Laplacian matrix $\Lambda[x]$ with $\Lambda \in \mathcal{L}_2(v, b, k)$.

Proof. Let $\Lambda \in \mathcal{L}_2(v, b, k)$ with underlying δ-regular simple graph $\mathcal{G}$ on v vertices such that $S_3(\rho_{\mathcal{G}}) \geq S_3(\rho_{\mathcal{G}'})$ for any δ-regular (not necessarily connected) simple graph $\mathcal{G}'$ on v vertices. By Corollary 2.42 $S_3(\rho_{\mathcal{G}}) - S_3(\rho_{\mathcal{G}'}) \in \mathbb{N}_{>0}$. Therefore, it is enough to show that

$$v^{v-4}y^{v-4} > \sum_{j=4}^{v-1} v^{v-1-j}x^{v-1-j}|S_j(\rho_{\mathcal{G}}) - S_j(\rho_{\mathcal{G}'})|$$

for $x \geq \frac{1}{v}\left((2\delta)^{v-1}\binom{v-1}{\lceil\frac{v-2}{2}\rceil} + 1\right)$. By Proposition 2.16, the Laplacian eigenvalues of a (not necessarily connected) δ-regular graph are bounded from above by 2δ and therefore $|S_j(\rho_{\mathcal{G}}) - S_j(\rho_{\mathcal{G}'})| \leq \binom{v-1}{j}(2\delta)^j \leq (2\delta)^j\binom{v-1}{\lceil\frac{v-2}{2}\rceil}$ for all $j = 1, \ldots, v-1$. It follows that

$$\begin{aligned}
\sum_{j=4}^{v-1} v^{v-1-j}x^{v-1-j}|S_j(\rho_{\mathcal{G}}) - S_j(\rho_{\mathcal{G}'})| &< (2\delta)^{v-1}\binom{v-1}{\lceil\frac{v-2}{2}\rceil}\sum_{j=4}^{v-1} v^{v-1-j}x^{v-1-j} \\
&= (2\delta)^{v-1}\binom{v-1}{\lceil\frac{v-2}{2}\rceil}\frac{(vx)^{v-4}-1}{vx-1} \\
&\leq (vx-1)\frac{(vx)^{v-4}-1}{vx-1} \\
&< v^{v-4}x^{v-4} \\
&\leq v^{v-4}x^{v-4}(S_3(\rho_{\mathcal{G}}) - S_3(\rho_{\mathcal{G}'})).
\end{aligned}$$

□

Proposition 8.18. *Let $\delta = r(k-1) - \lambda(v-1)$. For given v, b, k and $x_0 = (2\delta)^{v-1}\binom{v-1}{m}^2(v-3) + \frac{1}{v}$, where $m = \lceil\frac{v-2}{2}\rceil$, if there exist designs with Laplacian matrix $\Lambda[x]$ with $\Lambda \in \mathcal{L}_3(v, b, k)$ for $x \geq x_0$, then they maximize the A-value among all designs with Laplacian matrix $\Lambda[x]$ with $\Lambda \in \mathcal{L}_2(v, b, k)$.*

Proof. Let $\Lambda \in \mathcal{L}_2(v,b,k)$ with underlying δ-regular (not necessarily connected) simple graph $\mathcal{G}$ on v vertices such that $S_3(\rho_{\mathcal{G}}) \geq S_3(\rho_{\mathcal{G}'})$ for any δ-regular simple graph $\mathcal{G}'$ on v vertices. By Proposition 8.8 $S_1(\rho_{\mathcal{G}}) = S_1(\rho_{\mathcal{G}'})$ and $S_2(\rho_{\mathcal{G}}) = S_2(\rho_{\mathcal{G}'})$ for all $\mathcal{G}' \in \mathcal{M}(v, r(k-1)-\lambda(v-1))$. The RGD with Laplacian matrix $\Lambda[x]$ is A-better than the RGD with Laplacian matrix $\Lambda'[x]$ if and only if

$$D(\rho_{\mathcal{G}},x)R(\rho_{\mathcal{G}'},x) - D(\rho_{\mathcal{G}'},x)R(\rho_{\mathcal{G}},x) > 0.$$

By Lemma 8.4, the first non-vanishing coefficient of this polynomial is

$$3v^{2v-6}\left(S_3(\rho_{\mathcal{G}}) - S_3(\rho_{\mathcal{G}'})\right)$$

which is an integer by Corollary 2.42. By Proposition 2.16, the Laplacian eigenvalues of a (not necessarily connected) δ-regular graph are bounded from above by 2δ. Therefore, for $m = \lceil \frac{v-2}{2} \rceil$ we have $\binom{v-1}{m} \geq \binom{v-1}{i}$ for $i \in \{1,\dots,v-1\}$, hence

$$|S_j(\rho_{\mathcal{G}})S_{i-j}(\rho_{\mathcal{G}'}) - S_j(\rho_{\mathcal{G}'})S_{i-j}(\rho_{\mathcal{G}})| \leq (2\delta)^i \binom{v-1}{m}^2.$$

It follows for $x \geq (2\delta)^{v-1}\binom{v-1}{m}^2(v-3) + \frac{1}{v}$ that

$$\begin{aligned}
& \sum_{i=4}^{v-1}(vx)^{2v-3-i}\sum_{j=0}^{i}(v-1-j)|S_j(\rho_{\mathcal{G}})S_{i-j}(\rho_{\mathcal{G}'}) - S_j(\rho_{\mathcal{G}'})S_{i-j}(\rho_{\mathcal{G}})| \\
\leq\ & (2\delta)^{v-1}\binom{v-1}{m}^2 \sum_{i=4}^{v-1}(vx)^{2v-3-i}\sum_{j=0}^{i}(v-1-j) \\
=\ & (2\delta)^{v-1}\binom{v-1}{m}^2 \sum_{i=4}^{v-1}(vx)^{2v-3-i}(i+1)\frac{2(v-1)-i}{2} \\
<\ & (2\delta)^{v-1}\binom{v-1}{m}^2 v(v-3)\sum_{i=4}^{v-1}(vx)^{2v-3-i} \\
=\ & (2\delta)^{v-1}\binom{v-1}{m}^2 v(v-3)(vx)^{v-2}\sum_{i=0}^{v-5}(vx)^i
\end{aligned}$$

$$
\begin{aligned}
&= v\left((2\delta)^{v-1}\binom{v-1}{m}^2(v-3)\right)(vx)^{v-2}\frac{(vx)^{v-4}-1}{vx-1} \\
&\leq (vx-1)(vx)^{v-2}\frac{(vx)^{v-4}-1}{vx-1} \\
&< (vx)^{2v-6} \\
&\leq 3v^{2v-6}x^{2v-6}\left(S_3(\rho_{\mathcal{G}})-S_3(\rho_{\mathcal{G}'})\right).
\end{aligned}
$$

□

We want to characterize the simple regular graphs on v vertices that are underlying graphs of A- and D-best RGDs for large x. The following proposition relates the A- and D-values given as in equation 8.1.2 and 8.1.3 of such a graph with its complement $\bar{\mathcal{G}}$.

Proposition 8.19. *Let $\mathcal{G}$ be a simple (not necessarily connected) graph and $\bar{\mathcal{G}}$ its complement. Then*

$$S_j(\rho_{\bar{\mathcal{G}}}) = \sum_{l=0}^{j}\binom{v-l}{j-l}(-1)^l v^{j-l} S_l(\rho_{\mathcal{G}}).$$

Proof. By Proposition 2.8, the non-trivial Laplacian eigenvalues of $\bar{\mathcal{G}}$ are given by $v-\rho_i(\mathcal{G})$ for $i=1,\dots,v-1$. Therefore for $I=\{1,\dots,v-1\}$

$$
\begin{aligned}
S_j(\rho_{\bar{\mathcal{G}}}) &= S_j(v-\rho_1(\mathcal{G}),\dots,v-\rho_{v-1}(\mathcal{G})) \\
&= \sum_{\substack{J\subseteq I,\\ |J|=j}}\sum_{l=0}^{j}(-1)^l v^{j-l} S_{l;(I\setminus J)}(\rho_{\mathcal{G}}) \\
&= \sum_{l=0}^{j}(-1)^l v^{j-l}\sum_{\substack{J\subseteq I,\\ |J|=j}}\sum_{\substack{L\subseteq J,\\ |L|=l}}\prod_{i\in L}\rho_i(\mathcal{G}).
\end{aligned}
$$

In the last sum, the elementary symmetric polynomial $S_l(\rho_{\mathcal{G}})$ occurs as often as an l-set occurs in a j-set. This is the number of possible ways for choosing a

j-set with a fixed l-subset, that is $\binom{v-l}{j-l}$. Hence,

$$\sum_{\substack{J\subseteq I,\\ |J|=j}} \sum_{\substack{L\subseteq J,\\ |L|=l}} \prod_{i\in L} \rho_i(\mathcal{G}) = \binom{v-l}{j-l} S_l(\rho_{\mathcal{G}})$$

and the statement follows. □

That means, if two δ-regular (not necessarily connected) simple graphs $\mathcal{G}$ and $\mathcal{G}'$ have Laplacian matrices $\Lambda(\mathcal{G}') \prec \Lambda(\mathcal{G})$ and i is the smallest index such that $S_i(\rho_{\mathcal{G}}) > S_i(\rho_{\mathcal{G}'})$, then

$$v\mathbb{I}_v - \mathbb{J}_v - \Lambda(\mathcal{G}') \prec v\mathbb{I}_v - \mathbb{J}_v - \Lambda(\mathcal{G}) \text{ if } i \text{ is even,}$$
$$v\mathbb{I}_v - \mathbb{J}_v - \Lambda(\mathcal{G}) \prec v\mathbb{I}_v - \mathbb{J}_v - \Lambda(\mathcal{G}') \text{ if } i \text{ is odd.}$$

Example. Let $\mathcal{G}$ be the union of two disjoint triangles. We want to compare $\mathcal{G}$ with the 6-cycle $Cycle(6)$. Both graphs are adjacency graphs of RGDs with $v = b = 6$ and block size $k = 2$ and therefore $\Lambda(\mathcal{G}), \Lambda(Cycle(6)) \in \mathcal{L}_2(6, 6, 2)$. The Laplacian spectra are

$$\text{Spec}(\Lambda(\mathcal{G})) = (0^2, 3^4)$$

and

$$\text{Spec}(\Lambda(Cycle(6))) = (0^1, 1^2, 3^2, 4).$$

We can compute the elementary symmetric polynomials as follows:

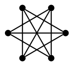
$$S_1(\rho_{\Lambda(\mathcal{G})}) = S_1(\rho_{\Lambda(Cycle(6))}) = 12,$$

$$S_2(\rho_{\Lambda(\mathcal{G})}) = S_2(\rho_{\Lambda(Cycle(6))}) = 54$$

and

$$S_3(\rho_{\Lambda(\mathcal{G})}) = 108 \text{ and } S_3(\rho_{\Lambda(Cycle(6))}) = 112.$$

That means $\Lambda(\mathcal{G}) \prec \Lambda(Cycle(6))$.

The complement of $\mathcal{G}$ is the regular complete bipartite graph $K_{3,3}$ and $Cycle(6)$ has the complement  and

$$\mathrm{Spec}(\Lambda(\bar{\mathcal{G}})) = (0^1, 3^4, 6^1)$$

and

$$\mathrm{Spec}(\Lambda(\overline{Cycle(6)})) = (0^1, 2^1, 3^2, 5^2).$$

Therefore,

$$S_1(\rho_{\Lambda(\bar{\mathcal{G}})}) = S_1(\rho_{\Lambda(\overline{Cycle(6)})}) = 18,$$

$$S_2(\rho_{\Lambda(\bar{\mathcal{G}})}) = S_2(\rho_{\Lambda(\overline{Cycle(6)})}) = 126$$

and

$$S_3(\rho_{\Lambda(\bar{\mathcal{G}})}) = 432 \text{ and } S_3(\rho_{\Lambda(\overline{Cycle(6)})}) = 428,$$

giving $\Lambda(\overline{Cycle(6)}) \prec \Lambda(\bar{\mathcal{G}})$.

Lemma 8.20. *Let $\mathcal{G}$ and $\mathcal{G}'$ be simple δ-regular graphs on v vertices. Then*

$$S_3(\rho_{\mathcal{G}}) - S_3(\rho_{\mathcal{G}'}) = \frac{1}{3}(\eta(\mathcal{G}) - \eta(\mathcal{G}')),$$

where η is the number of V-subgraphs of $\mathcal{G}$ and $\mathcal{G}'$, that is three vertices with exactly two edges.

Proof. By Proposition 2.17, if $\mathcal{G}$ and $\mathcal{G}'$ are δ-regular, then $S_1(\rho_{\mathcal{G}}) = S_1(\rho_{\mathcal{G}'})$ and $\mathrm{Trace}(\Lambda(\mathcal{G})^2) = \mathrm{Trace}(\Lambda(\mathcal{G}')^2)$. Since for any graph

$$2S_2(\rho_{\mathcal{G}}) = S_1(\rho_{\mathcal{G}})^2 - \mathrm{Trace}(\Lambda(\mathcal{G})^2),$$

it follows that $S_2(\rho_{\mathcal{G}}) = S_2(\rho_{\mathcal{G}'})$. With equation 2.6.1 and $S_0 \equiv 1$ it follows for any simple δ-regular graph $\mathcal{G}$ on v vertices that

$$\begin{aligned} 3S_3(\rho_{\mathcal{G}}) &= \sum_{i=1}^{3}(-1)^{i-1}S_{3-i}(\rho_{\mathcal{G}})\sum_{j=1}^{v-1}\rho_i(\mathcal{G})^j \\ &= S_2(\rho_{\mathcal{G}})S_1(\rho_{\mathcal{G}}) - S_1(\rho_{\mathcal{G}})\sum_{j=1}^{v-1}\rho(\mathcal{G}')_i^2 + \sum_{j=1}^{v-1}\rho_i(\mathcal{G})^3. \end{aligned}$$

By Proposition 2.17,

$$\sum_{i=1}^{v-1}\rho_i(\mathcal{G})^2 = v\delta(\delta+1)$$

and

$$\sum_{i=1}^{v-1}\rho_i(\mathcal{G})^3 = v\delta(\delta+1)^2 + \eta(\mathcal{G}),$$

where $\eta(\mathcal{G})$ is the number of V-subgraphs of $\mathcal{G}$ and it follows

$$S_3(\rho_{\mathcal{G}}) - S_3(\rho_{\mathcal{G}'}) = \frac{1}{3}(\eta(\mathcal{G}) - \eta(\mathcal{G}')).$$

□

By Proposition 8.19, among simple δ-regular graphs on v vertices that agree on $S_1(\rho_{\mathcal{G}})$ and $S_2(\rho_{\mathcal{G}})$, maximizing $S_3(\rho_{\mathcal{G}})$ is equivalent with minimizing $S_3(\rho_{\bar{\mathcal{G}}})$.

Theorem 8.21. *Suppose $\mathcal{L}_2(v,b,k) \neq \mathcal{L}_3(v,b,k)$. For given v, b, k, there exists an x_0 such that, if there exist designs with Laplacian matrix $\Lambda[x]$ with $\Lambda \in \mathcal{L}_2(v,b,k)$ for $x \geq x_0$, then the ones whose underlying graph minimizes the number of V-subgraphs in its complement are A- and D-best.*

The following corollary extends Cheng's result in [Che81a] on D-optimality of the complete regular m-partite graph $K_{\alpha,\ldots,\alpha}$ with $x = 0$ to large x.

Corollary 8.22. *There exists an x_0 such that if there exists a design with Laplacian matrix $\Lambda[x]$ where $\Lambda = \Lambda(K_{\alpha,\ldots,\alpha}) \in \mathcal{L}(\alpha m, b, k)$ and $x \geq x_0$, then it is A- and D-best among all designs with Laplacian matrix $\Lambda'[x]$ with $\Lambda' \in \mathcal{L}(\alpha m, b, k)$.*

Proof. Follows directly from Proposition 8.5, Theorem 8.21 and the fact that the complement is a union of cliques and as such V-subgraph-free. □

8.3 A- and D-best RGDs with $v \leq 18$

For the rest of this chapter, all considered designs are RGDs with connected underlying graph and we denote the set of all Laplacian matrices of such designs by $\mathcal{L}_2^c(v,b,k)$. Let $\mathcal{M}^c(v,\delta)$ denote the set of all connected δ-regular graphs on v points. Note that this is no restriction if the underlying graphs are already the adjacency graphs of the designs.

To find the best designs on v points, we follow John and Mitchell's approach and first generate all graphs in $\mathcal{M}^c(v,\delta)$ with the program genreg.exe ([Meh99]) for $\delta \leq 9$ (the restriction on the degree is made by the program genreg.exe). This gives us a list of (connected) simple regular graphs whose Laplacian eigenvalues we can now compare. Any Laplacian matrix $\Lambda \in \mathcal{L}_2^c(v,b,k)$ corresponds to a $\mathcal{G} \in \mathcal{M}^c(v,\delta)$, its underlying graph. Therefore, to order the matrices in $\mathcal{L}_2^c(v,b,k)$ corresponding to the A- and D-criterion it is enough to order all $\mathcal{G} \in \mathcal{M}^c(v,\delta)$ corresponding to the value $D(\rho_\mathcal{G},\lambda)$ as given in equation 8.1.2 and the value $A(\rho_\mathcal{G},\lambda) = v(v-1)\frac{D(\rho_\mathcal{G},\lambda)}{R(\rho_\mathcal{G},\lambda)}$, where $\rho_\mathcal{G}$ denotes the vector of the non-trivial Laplacian eigenvalues of $\mathcal{G}$ and $R(\rho_\mathcal{G},x) = D_x(\rho_\mathcal{G},x)$ denotes the derivative of the D-value as polynomial in x. We can view $A(\rho_\mathcal{G},\lambda)$ and $D(\rho_\mathcal{G},\lambda)$ as functions in λ; to make this clearer we will write $A(\rho_\mathcal{G},x)$ and $D(\rho_\mathcal{G},x)$ if we view the values as functions in a variable x and $A(\rho_\mathcal{G},\lambda)$ and $D(\rho_\mathcal{G},\lambda)$ for their evaluation in $x = \lambda$. Suppose there exist $\mathcal{G},\mathcal{G}' \in \mathcal{M}^c(v,\delta)$ and a pair r,k such that $\mathcal{G}$ and $\mathcal{G}'$ are underlying graphs of RGDs with Laplacian matrix in $\mathcal{L}_2^c(v,b,k)$ such that $A(\rho_\mathcal{G},\lambda_0) > A(\rho_{\mathcal{G}'},\lambda_0)$ where $\lambda_0 = \lfloor\frac{r(k-1)}{v-1}\rfloor$. We know from the previous sections that if λ_0 is large enough then

$$A(\rho_{\mathcal{G}}, x) > A(\rho_{\mathcal{G}'}, x) \text{ for all } x \geq \lambda_0$$

or, equivalently the polynomial

$$P(\mathcal{G}, \mathcal{G}', x) = D(\rho_{\mathcal{G}}, x)R(\rho_{\mathcal{G}'}, x) - D(\rho_{\mathcal{G}'}, x)R(\rho_{\mathcal{G}}, x)$$

has no roots bigger than λ_0, that is if $P(x) = 0$, then $x < \lambda_0$. We want to order the graphs in $\mathcal{M}^c(v, \delta)$ according to the values $A(\rho_{\mathcal{G}}, \lambda)$ for different values for λ. Let us define this formally: suppose $\mathcal{M}^c(v, \delta) = \{\mathcal{G}_1, \ldots, \mathcal{G}_M\}$. Let $r^A : \{1, \ldots, M\} \times x \to \{1, \ldots, M\}$ be a function of the indices of the graphs in $\mathcal{M}^c(v, \delta)$ such that for a value $\lambda \in \mathbb{N}$ an order on $\mathcal{M}^c(v, \delta) = \{\mathcal{G}_{r^A(i,\lambda)} | i = 1, \ldots, M\}$ is given by

$$P(\mathcal{G}_{r^A(i,\lambda)}, \mathcal{G}_{r^A(i,\lambda)+1}, \lambda) \geq 0 \text{ for all } r^A(i, \lambda) = 1, \ldots, M - 1.$$

The value we are interested in is the value λ_0 such that $r^A(i, x) = r^A(i, \lambda_0)$ for all $x \geq \lambda_0$ and $i = 1, \ldots, M$.

Here, λ_0 still depends on the choice of r and k and the existence of RGDs with underlying graph $\mathcal{G} \in \mathcal{M}^c(v, \delta)$. Instead of computing the exactly values for λ_0, we want find a value x_0^A (not depending on existence of designs) such that the following is satisfied, in which case we say that the order *stabilizes* for x_0^A.

1. There exists an $x < x_0^A$ and an $i \in \{1, \ldots, M - 1\}$ such that

   $$P(\mathcal{G}_{r^A(i,x)}, \mathcal{G}_{r^A(i,x)+1}, x) < 0;$$

   and

2. for all $x \geq x_0^A$,

$$P(\mathcal{G}_{r^A(i,x)}, \mathcal{G}_{r^A(i,x)+1}, x) \geq 0.$$

That means, if the order of the graphs stabilizes for x_0^A then for any admissible $\lambda \geq x_0^A$ we have $r^A(i,\lambda) = r^A(i, x_0^A)$. We denote by r^D and x_0^D the equivalent of r^A and x_0^A for the D-values $D(\rho_{\mathcal{G}}, x)$ with $\mathcal{G} \in \mathcal{M}^c(v,\delta)$.

In the rest of this chapter, we present the exact values for x_0^A and x_0^D found with an exhaustive computer search, for all graphs in $\mathcal{M}^c(v,\delta)$ with $5 \leq v \leq 13$ and $2 \leq \delta \leq 9$, $v = 14$ and $2 \leq \delta \leq 5$, $v = 15$ and $\delta = 4$, $v = 16, 18$ and $\delta = 3$. The other cases were too extensive to handle. We obtain the values by first guessing a value for x_0^A and x_0^D and then verifying the above two properties for all graphs in $\mathcal{M}^c(v,\delta)$. A little needs to be said of how we are computing the A- and D-values as functions in x. Of course, it would be enough to know the Laplacian eigenvalues of the graphs in $\mathcal{M}^c(v,\delta)$. But this approach leads to long computation times. There is a way to compute the exact A- and D-values more efficiently. With equations 2.14 and 2.5, we can compute $A(\rho_{\mathcal{G}}, x)$ and $D(\rho_{\mathcal{G}}, x)$ in terms of the coefficients of the characteristic polynomial of $\Lambda[x]$, where $\Lambda = \Lambda(\mathcal{G})$,

$$\chi_{\Lambda[x]}(z) = x^v - c_2 z^{v-1} \ldots + (-1)^l c_l z^{v-l+1} + \ldots + (-1)^{v-1} c_{v-1} z,$$

as

$$D(\rho_{\mathcal{G}}, x) = c_{v-1} \text{ and } A(\rho_{\mathcal{G}}, x) = \frac{c_{v-1}}{c_{v-2}}.$$

In fact, it is enough to compute $D(\rho_{\mathcal{G}}, x)$ since the A-value is determined by $\frac{D(\rho_{\mathcal{G}},x)}{R(\rho_{\mathcal{G}},x)}$. Mathematica lets us compute the coefficients of the characteristic polynomial of an integer matrix exact (depending on a variable or not). To find the best design for a block size k, we search for the first graph in this order that gives rise to a block design with block size k. To do this we use the GAP package DESIGN ([Soi06]). Because of the high number of graphs in the cases $v = 13$ and $\delta = 6$, $v = 14$ and $\delta = 4, 5$, $v = 15$ and $\delta = 4$ we only searched for designs among the best 20,000.

8.3.1 Results

A table of our results is given in tables 8.1, 8.2 and 8.3. We denote by δ the degree of the underlying regular graph. We list $\lambda = \lfloor \frac{r(k-1)}{v-1} \rfloor$ and the smallest $\tilde{\lambda}$ such that a 2-$(v, k, \tilde{\lambda})$-design exists (found with GAP). For $i \in \{A, D\}$, the values $r^i(k, 0)$ and $r^i(k, \lambda_0^i)$ listed are defined as follows: for λ let $r^i(k, \lambda)$ be the smallest index in $\{r^i(1, \lambda), \ldots, r^i(M, \lambda)\}$ such that there exist a design with Laplacian matrix in $\mathcal{L}_2(v, b, k)$ and underlying graph $\mathcal{G}_{r^i(k,\lambda)}$.

8.3.2 Observations, Remarks and Conjectures

Recall that we are computing the A- and D-values of $\Lambda[y] = \Lambda + y(v\mathbb{I}_v - \mathbb{J}_v)$ as polynomials in $x = \lambda + y\tilde{\lambda}$, where $\tilde{\lambda}$ is the smallest value such that a 2-design on v vertices and block size k exists.

Comparison with the results of John and Mitchell

For all $v \leq 12$, the optimal designs we found for $y = 0$ are isomorphic to the designs presented by John and Mitchell except for the cases $v = 11$, $k = r = 3$ and $k = r = 8$ and all designs on $v = 12$ points with underlying 5-regular graph. All the other designs listed by John and Mitchell for $y > 0$ have the same underlying graph as the designs we found.

Remark 8.23. The case $v = 10$, $k = 3$, $r = 3 + 12y$. For $b = 10 + 30y$ the optimal design we found is isomorphic to the one presented by John and Mitchell for $y = 0$. However, for $b = 40 + 30y$, which is not one of the cases John and

v	k	r	λ	$\tilde{\lambda}$	δ	$r^A(k,0)$	$r^A(k,\lambda_0)$	$r^D(k,0)$	$r^D(k,\lambda_0)$
5	2	2+4y	0	1	2	1	1	1	1
5	2	4+4y	0	1	4	1	1	1	1
5	3	3+6y	1	3	2	1	1	1	1
5	3	6+6y	2	3	4	1	1	1	1
5	4	3+4y	2	3	4	1	1	1	1
5	5	2+y	1	1	4	1	1	1	1
6	2	3+5y	0	1	3	1	1	1	1
6	2	4+5y	0	1	4	1	1	1	1
6	3	2+5y	0	2	4	1	1	1	1
6	3	4+5y	1	2	3	2	2	2	2
6	4	6+10y	3	6	3	1	1	1	1
6	4	8+10y	4	6	4	1	1	1	1
7	2	4+6y	0	1	4	1	1	1	1
7	2	6+6y	0	1	6	1	1	1	1
7	3	3+3y	0	1	6	1	1	1	1
7	4	4+4y	1	2	6	1	1	1	1
7	6	6+6y	5	5	6	1	1	1	1
7	7	2+y	1	1	6	1	1	1	1
8	2	3+7y	0	1	3	1	1	1	1
8	2	4+7y	0	1	4	1	1	1	1
8	2	5+7y	0	1	5	1	1	1	1
8	2	6+7y	0	1	6	1	1	1	1
8	3	3+21y	0	6	6	1	1	1	1
8	3	6+21y	1	6	5	1	1	1	1
8	3	9+21y	2	6	4	1	1	1	1
8	3	12+21y	3	6	3	1	1	1	1
8	4	4+7y	1	3	5	2	2	2	2
8	4	6+7y	2	3	4	1	1	1	1
8	4	8+7y	3	3	3	1	1	1	1
8	4	9+7y	3	3	6	1	1	1	1
8	5	5+35y	2	20	6	1	1	1	1
8	5	10+35y	5	20	5	1	1	1	1
9	2	2+8y	0	1	4	1	1	1	1
9	2	6+8y	0	1	6	1	1	1	1
9	3	2+4y	0	1	4	7	7	7	7
9	3	3+4y	0	1	6	1	1	1	1
9	4	4+8y	1	3	4	2	2	2	2
9	5	5+10y	2	5	4	2	2	2	2

Table 8.1: Results of computer search

v	k	r	λ	$\tilde{\lambda}$	δ	$r^A(k,0)$	$r^A(k,\lambda_0)$	$r^D(k,0)$	$r^D(k,\lambda_0)$
9	6	4+8y	2	5	4	7	7	7	7
9	6	6+8y	3	5	6	1	1	1	1
10	2	2+9y	0	1	2	1	1	1	1
10	2	3+9y	0	1	3	1	1	1	1
10	2	4+9y	0	1	4	1	1	1	1
10	2	5+9y	0	1	5	1	1	1	1
10	2	6+9y	0	1	6	1	1	1	1
10	2	7+9y	0	1	7	1	1	1	1
10	2	8+9y	0	1	8	1	1	1	1
10	3	3	0	2	6	6	6	6	6
10	3	12+9y	2	2	6	1	1	1	1
10	3	6+9y	1	2	3	1	1	1	1
10	4	2+6y	0	2	6	16	16	16	16
10	4	4+6y	1	2	3	1	1	1	1
10	4	8+6y	2	2	6	1	1	1	1
10	4	10+6y	3	2	3	1	1	1	1
10	5	2+9y	4	4	8	RGD*			
10	5	4+9y	1	4	7	RGD*			
10	5	5+9y	2	4	2	1	1	1	1
10	5	6+9y	2	4	6	1	1	1	1
10	5	8+9y	3	4	1	1	1	1	1
10	5	10+9y	4	4	6	1	1	1	1
10	6	3+9y	1	5	6	16	16	16	16
10	6	6+9y	3	5	3	1	1	1	1
10	7	7+21y	4	14	6	6	6	6	6
10	8	8+36y	6	28	2	1	1	1	1
11	2	2+10y	0	1	2	1	1	1	1
11	2	4+10y	0	1	4	1	1	1	1
11	2	6+10y	0	1	6	1	1	1	1
11	2	8+10y	0	1	8	1	1	1	1
11	3	3+15y	0	3	6	58	58	58	58
11	3	6+15y	1	3	2	1	1	1	1
11	3	9+15y	1	3	8	1	1	1	1
11	4	4+20y	1	6	2	1	1	1	1
11	4	8+20y	2	6	4	1	1	1	1
11	7	7+35y	4	21	2	1	1	1	1
11	8	8+40y	5	28	6	58	58	58	58

Table 8.2: Results of computer search (cont.)

v	k	r	λ	$\tilde{\lambda}$	δ	$r^A(k,0)$	$r^A(k,\lambda_0)$	$r^D(k,0)$	$r^D(k,\lambda_0)$
12	2	3+11y	0	1	3	1	1	1	1
12	2	4+11y	0	1	4	1	1	1	1
12	2	5+11y	0	1	5	1	1	1	1
12	2	6+11y	0	1	6	1	1	1	1
12	2	7+11y	0	1	7	1	1	1	1
12	2	8+11y	0	1	8	1	1	1	1
12	2	9+11y	0	1	9	1	1	1	1
12	3	4+11y	0	2	8	1	1	1	1
12	3	7+11y	1	2	3	1	1	1	1
12	3	9+11y	1	2	7	1	1	1	1
12	3	10+11y	1	2	9	1	1	1	1
12	3	13+11y	2	2	4	1	1	1	1
12	4	3+11y	0	3	9	1	1	1	1
12	4	5+11y	1	3	4	1	1	1	1
12	4	6+11y	1	3	7	2	2	2	2
12	5	5+55y	1	20	9	1	1	1	1
12	7	7+77y	3	42	9	1	1	1	1
12	8	6+22y	3	14	9	1	1	1	1
13	2	4+12y	0	1	4	1	1	1	1
13	2	6+12y	0	1	6	1	1	1	1
13	2	8+12y	0	1	8	1	1	1	1
13	5	5+15y	1	5	8	17	19	19	19
14	2	3+13y	0	1	3	1	1	1	1
14	2	4+13y	0	1	4	1	1	1	1
14	2	5+13y	0	1	5	1	1	1	1
14	3	9+39y	1	6	5	1	1	1	1
14	3	15+39y	2	6	4	1	1	1	1
15	2	4+14y	0	1	4	1	1	1	1
15	3	9+28y	1	1	4	1	1	1	1
16	2	3+15y	0	1	3	1	1	1	1
16	3	9+13y	1	2	3	1	1	1	1
16	4	6+5y	1	1	3	1	1	1	1
18	2	3+17y	0	1	3	1	1	1	1
18	3	10+17y	1	21	3	1	1	1	1

Table 8.3: Results of computer search (cont.)

Mitchell list, we found a design whose underlying graph performs better on the optimality criteria than the underlying graph of the former case.

Remark 8.24. The cases $v = 11$, $k = r = 3$ and $k = r = 8$. We present an A- and D-optimal design on 11 points and parameters $r = k = 3$ and $r = k = 8$ which John and Mitchell failed to find. Moreover, we also found a design which does better on the E-criterion than the design they found in these cases. The A- and D-optimal designs have in both cases the same underlying regular graph, as have the E-optimal designs.

Remark 8.25. The cases $v = 10$, $k = 5$, $r = 2 + 9y$, $b = 8 + 10y$.

The case $v = 10$, $k = 5$, $r = 2$. If there exists an RGD, then the underlying graph has degree 8. We could not find an RGD in this case. John and Mitchell list the dual of the design with blocks

$$\begin{matrix} 1 & 2 \\ 2 & 4 \end{matrix} \quad \begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix} \quad \begin{matrix} 1 & 3 \\ 3 & 4 \end{matrix} \quad \begin{matrix} 1 & 4 \\ 2 & 3 \end{matrix} \quad \begin{matrix} 1 & 4 \\ 2 & 3 \end{matrix}$$

($R3$ in Clatworthy [Cla73]), which is an RGD. The underlying graph is the 4-cycle $Cycle(4)$, the only 2-regular simple graph on 4 points (connected or not). The 4-cycle is A- and D-optimal ([Bai07]) among all binary designs with 4 points and 4 blocks of size 2. The cycle also maximizes $S_j(\rho_{\mathcal{G}})$ for all $j = 2, \ldots, v-1$ among all simple connected graphs with v edges ([SI09]). Therefore, by Proposition 8.7 the 4-cycle is D-optimal for all $x \geq 0$ among all binary designs with simple adjacency graph. By Proposition 2.31, its dual is D-optimal among all binary equireplicate designs. For $x = 0$, the dual of the $Cycle(4)$ has blocks

$$1\ 2\ 3\ 4\ 5 \qquad 1\ 2\ 6\ 7\ 8 \qquad 3\ 6\ 7\ 9\ 10 \qquad 4\ 5\ 8\ 9\ 10$$

and therefore is D-optimal. The design is not an RGD since the pair $\{1, 2\}$

occurs in 2 blocks, the pair $\{1,9\}$ in none. We do not know whether this design stays optimal after adding blocks of a BIBD.

The case $v = 10$, $k = 5$, $r = 8$. John and Mitchell list the design with blocks

$$\begin{array}{ccccc@{\qquad}ccccc@{\qquad}ccccc}
1 & 2 & 3 & 7 & 8 & 1 & 2 & 4 & 9 & 10 & 1 & 2 & 5 & 6 & 9 \\
1 & 3 & 6 & 8 & 9 & 1 & 4 & 6 & 7 & 8 & 1 & 4 & 7 & 9 & 10 \\
2 & 3 & 6 & 7 & 10 & 2 & 3 & 8 & 9 & 10 & 2 & 4 & 5 & 8 & 10 \\
2 & 5 & 6 & 7 & 9 & 3 & 4 & 5 & 6 & 10 & 3 & 4 & 5 & 7 & 9 \\
1 & 3 & 5 & 7 & 10 & 1 & 5 & 6 & 8 & 10 & 2 & 4 & 6 & 7 & 8 \\
3 & 4 & 5 & 8 & 9. & & & & & & & & & &
\end{array}$$

The underlying graph has degree 1 and is the only 1-regular simple graph on 10 points. The design is therefore optimal among RGDs for all $y \geq 0$. We did not find this design since we were only searching among connected graphs.

Order Stabilization

There are three main observations:

Observation 8.26. Except for $v = 13$, $\delta = 8$ and $v = 14$, $\delta = 5$ we have

$$r^A(k, 0) = r^A(k, \lambda_0) = r^D(k, 0) = r^D(k, \lambda_0).$$

Let $i \in \{1, \ldots, M\}$ such that $r^A(k, 0) = r^A(i, 0)$ then

$$r^A(i, 0) = r^A(k, \lambda_0),\ r^D(k, 0) = r^D(i, 0),\ r^D(k, \lambda_0) = r^D(i, \lambda_0),$$

that means in these cases the A- and D-best designs have the same underlying graph for $\lambda = 0$ and $\lambda = \lambda_0$. This is also true for $v = 13$ and $\delta = 8$ but the ranks of this graph differ corresponding to the A- and D-value for $\lambda = 0$ and $\lambda = \lambda_0$.

Observation 8.27. Except for $v = 14$, $\delta = 5$, there is always an order change from $\lambda = 0$ to $\lambda = \lambda_0$ for both the A- and D-value, but in all of the cases the best graph for $\lambda = 0$ remains best in both criteria, i.e. the best design with block size 2 stays best with growing λ. The same is true in the case of the D-criterion for $v = 14$, $\delta = 5$, but not for the A-criterion (see also Observation 8.29.

Observation 8.28. Table 8.3.2 shows that $x_0^A, x_0^D \leq \delta + 1$ and in most of the cases $x_0^A = x_0^D$. The worst case is $x_0^A = 6$ and $x_0^D = 5$, but in the most cases $x_0^A = x_0^D = 1$. These values are considerably lower than the bounds given in Proposition 8.17 and Proposition 8.15. Moreover, for $x \geq x_0^A$ the ordering for the graphs regards to the A- and D-value are the same, that is for

$$r^A(i, x_0^A) = r^D(i, x_0^D) \text{ for } i = 1, \ldots, M.$$

A- and D-optimality

Observation 8.29. By Corollary 8.16, A- and D-optimality among connected RGDs are equivalent for large y. In all the cases except for $v = 14, r = 5$, $k = 2$, the A- and D-best graphs are the same. But for $v = 14, r = 5$, $k = 2$, the D-best graph for $x = 0$ is not A-best for $x = 0$, but becomes A-best for $x \geq 1$.

Conjectures

Connected underlying graphs

John and Mitchell searched among all (not necessarily connected) regular graphs, but except for the case $v = 11$, $\delta = 6$, where we found better designs, we reproduced the same A- and D-best RGDs. That means, John and Mitchell found that in their cases the best underlying graph is connected.

v	degree	x_0^A	x_0^D
4	2	1	1
4	3	1	1
5	2	1	1
5	4	1	1
6	3	1	1
6	4	1	1
7	2	1	1
7	4	1	1
7	6	1	1
8	3	1	1
8	4	1	1
8	6	1	1
8	6	1	1
9	4	1	1
9	6	1	1
10	3	1	1
	4	1	1
	5	1	1
	6	1	1
	7	1	1
	8	1	1
11	4	1	1
	6	1	1
12	3	1	1
	4	1	1
	5	2	2
	6	2	1
	7	1	1
	8	1	1
13	4	2	2
	6	3	2
	8	4	2
14	3	1	1
	4	3	2
	5	6	5
15	4	5	4
16	3	1	1
18	3	2	1

Table 8.4: Table showing the values for which the orders stabilize.

Conjecture 8.30. *If $\mathcal{G}$ is the underlying graph of an A- or D-best RGD, then $\mathcal{G}$ is connected.*

Complete regular multipartite graphs

Observation 8.31. The graphs $K_{2,2,2}$, $K_{2,2,2,2}$, $K_{2,2,2,2,2}$, $K_{2,2,2,2,2,2}$, $K_{3,3,3}$ and $K_{3,3,3,3}$ stay optimal for all $x \geq 0$.

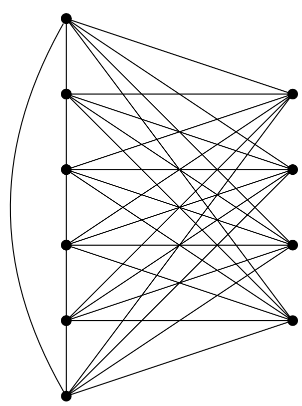
Cheng proved in [Che81a] that regular complete bipartite graphs are the unique A- and D-optimal graphs for all $x \geq 0$ (among not necessarily regular graphs). He extended his result to complete regular multipartite graphs among simple graphs. By Corollary 8.22, the complete regular multipartite graphs are A- and D-best regular graphs for big x.

Conjecture 8.32. *Complete regular multipartite graphs stay A- and D-best RGDs for all $x \geq 0$.*

Graphs with $v + 2 = 2\delta$

Conjecture 8.33 ([BC93])**.** *The A- and D-best graphs (among regular graphs) follow the pattern below.*

Take the complete bipartite graph with parts of size $\frac{v}{2} - 1$ and $\frac{v}{2} + 1$ and add on the larger part the edges of a circuit on $\frac{v}{2} + 1$ vertices, for example for $v = 10$ and $\delta = 6$:



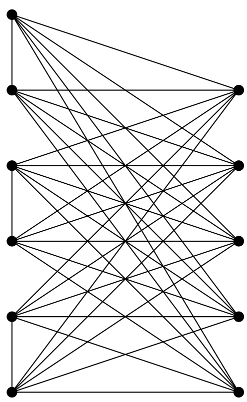
These graphs are A-, D- and E-optimal for $v < 14$ which was found by John and Mitchell [JM77] and also conjectured by Bagchi and Cheng in [BC93]. For $v = 14$ we could verify the A- and D-optimality for this graph, too. In the cases of $v = 6$ and $v = 8$ the graph is also E-optimal, in the other cases not.

Graphs with $v + 1 = 2\delta$

Note that by the Handshaking Lemma (see for example [Bol98], p. 4), which states that a simple regular graph with odd degree must have an even number of vertices, from $v + 1 = 2\delta$ follows that δ must be even and therefore $v + 1 \equiv 0 \mod 4$.

Conjecture 8.34. *The A- and D-optimal graphs (among regular graphs) follow the pattern below.*

Take the complete bipartite graph with parts of size $\frac{v-1}{2}$ and $\frac{v+1}{2}$ and add $\frac{v+1}{4}$ disjoint edges to pair up the vertices of the part of size $\frac{v+1}{2}$, for example for $v = 11$ and $\delta = 6$:



These graphs are A- and D-optimal for $v < 15$ and for $v = 11$ also E-optimal.

Note, that for $v \leq 15$ these graphs exist only for $v = 7$ and degree 4, $v = 11$ and degree 6, $v = 15$ and degree 8.

Proposition 8.35. *Let $m = \frac{v-1}{2}$ and $n = \frac{v+1}{2}$ and let $\mathcal{G}$ be the graph described above. Then*

$$\kappa(\mathcal{G}) = m^{n-1}n^{m-1}\left(1 + \sum_{l=1}^{\frac{n}{2}} 2^l \binom{\frac{n}{2}}{l} \frac{1}{m^l}\right)$$

Proof. The number of spanning trees of $\mathcal{G}$ is the sum of all spanning trees of $K_{m,n}$ and all spanning trees containing exactly l of the added edges $\{e_1, \ldots, e_t\}$ for $l = 1, \ldots, \frac{v+1}{4}$.

We claim that the number of the latter is

$$N(n, m, l) = m^{n-1}n^{m-1}2^l \binom{\frac{n}{2}}{l} \frac{1}{m^l}.$$

Suppose we have a spanning tree of $\mathcal{G}$ containing edges $\{e_1, \ldots, e_l\}$. Deleting these edges from the spanning tree results in a spanning forest of $K_{m,n}$ with $l+1$ roots. To construct a spanning tree of $K_{m,n}$ we have to add l edges, this can be done in m^l ways.

Now suppose we have a spanning tree τ in $K_{m,n}$. We construct a spanning tree in $\mathcal{G}$ containing the edges l edges from $\{e_1, \ldots, e_t\}$ in the follwing way: first, choose l edges out of $\{e_1, \ldots, e_t\}$, call this set $\mathcal{F}$. We want to split τ into l parts such that the vertices corresponding to the same edge in $\mathcal{F}$ are contained in different parts. For this, choose one vertex from each edge in $\mathcal{F}$, there are 2^l possible combinations. Let u_1 be the vertex with the smallest label among these and label the other $l-1$ vertices as $u_2, \ldots, u_l$ according to the increasing distance from u_1 in τ, i.e. $\operatorname{dist}(u_1, u_i) < \operatorname{dist}(u_1, u_{i+1})$ for $i = 2, \ldots, l-1$. If two of them have the same distance, then choose the smaller subscript for the vertex with the smaller label.

The subgraph $\tau(u_{l-1})$ of τ induced by the vertices $\{w \in V(\mathcal{G}) | \operatorname{dist}(u_1, w) = \operatorname{dist}(u_l, u_1) + \operatorname{dist}(u_l, w)\}$ is a rooted tree with root u_l that does not contain any vertices of $\{u_1, \ldots, u_{l-1}\}$. Delete $\tau(u_l)$ from the tree τ. Repeat this step with vertices $u_{l-1}, \ldots, u_2$. Then the forest $\tau, \tau(u_2), \ldots, \tau(u_l)$ is a spanning forest of $K_{m,n}$ that can be completed to a spanning tree in $\mathcal{G}$ in a unique way by adding the edges in $\mathcal{F}$. It follows that

$$m^l N(n, m, l) = m^{n-1} n^{m-1} 2^l \binom{\frac{n}{2}}{l}.$$

□

Generalized Hexagons

Recall that a point-line geometry such that the adjacency graph has diameter 3 and girth 6 is called *generalized hexagon.* A generalized hexagon has parameters s, t if there are $s + 1$ points on a line and a point lies on $t + 1$ lines; if this is the case, we write $\mathrm{GH}(s, t)$. A generalized hexagon $\mathrm{GH}(s, t)$ has a distance-regular adjacency graph on $v = s^3t^2 + s^2t(t+1) + s(t+1) + 1$ points and degree $s(t+1)$. For more details see Chapters 2 and 5.

When we compare all $s(t + 1)$-regular graphs for all valid block sizes k, we compare in the case $k = t+1$ a generalized hexagon with all other possible block designs. Is this the A- and D-best RGD for this block size?

Our computer search answers this question affirmative for $\mathrm{GH}(1, 2)$ for all $x \geq 0$. The adjacency graph is the Heawood graph (see page 32) which is the incidence graph of the Fano plane and bipartite 3-regular graph on 14 vertices. The $\mathrm{GH}(1, 2)$ as design has block size 2 and replication 3. The dual design of $\mathrm{GH}(s, t)$ is $\mathrm{GH}(t, s)$ (see page 51), in the case of $\mathrm{GH}(1, 2)$ this is $\mathrm{GH}(2, 1)$, which is a design on 21 points, block size 3 and replication 2.

What about $\mathrm{GH}(2, 2)$? Unfortunately, in this case the graph has already 63 vertices and there is no hope of solving this problem with the computer.

Appendix A

The Optimal Regular Graph Designs for $v \leq 18$

Here we give a list of the A- and D-best RGDs for $y = 0$ and $y > 0$ in the cases where they differ. For each design we list the D-value $D(y) = D(\rho_{\Lambda(d)}, \lambda + \tilde{\lambda}y)$ and the value $R(y) = R(\rho_{\Lambda(d)}, \lambda + \tilde{\lambda}y)$ where $R(\rho_{\Lambda(d)}, x) = D_x(\rho_{\Lambda(d)}, x)$. The values of the parameters λ and $\tilde{\lambda}$ can be found in the table on page 149 and following. The A-value can then be computed as the ratio $v(v-1)\frac{D(y)}{R(y)}$.

A.1 The best RGDs for $v = 5$

$v = 5,\ k = 2$

<u>$v = 5,\ k = 2,\ r = 2 + 4y,\ b = 5 + 10y$</u>

$D(y) = 25(1 + 5y + 5y^2)^2$

$R(y) = 250(2y + 1)(5y^2 + 5y + 1)$

1 2	1 3	3 5	4 5	2 4

<u>$v = 5,\ k = 2,\ r = 4 + 4y,\ b = 10 + 10y$</u>

$D(y) = 625(1 + y)^4$

$R(y) = 2500(1 + y)^3$

1 2	1 3	1 4	1 5	2 3	2 4	2 5	3 4	3 5	4 5

$v = 5,\ k = 3$

$v = 5,\ k = 3,\ r = 3 + 6y,\ b = 5 + 10y$

$D(y) = 25(45y^2 + 45y + 11)^2$

$R(y) = 750(2y + 1)(45y^2 + 45y + 11)$

1 2 3	1 2 4	2 4 5	3 4 5	1 3 5

$v = 5,\ k = 3,\ r = 6 + 6y,\ b = 10 + 10y$

$D(y) = 50625(1 + y)^4$

$R(y) = 67500(1 + y)^3$

1 2 3	1 2 4	1 2 5	1 3 4	1 3 5	1 4 5	2 3 4	2 3 5	2 4 5	3 4 5

$v = 5,\ k = 4$

$v = 5,\ k = 4,\ r = 4 + 4y,\ b = 5 + 5y$

$D(y) = 50625(1 + y)^4$

$R(y) = 67500(1 + y)^3$

1 2 3 4   1 2 3 5   1 3 4 5   2 3 4 5   1 2 4 5

$v = 5,\ k = 5$

$v = 5,\ k = 5,\ r = 2 + y,\ b = 2 + y$

$D(y) = 625(2 + y)^4$

$R(y) = 2500(2 + y)^3$

1 2 3 4 5   1 2 3 4 5

A.2 The best RGDs for $v = 6$

$v = 6,\ k = 2$

$v = 6,\ k = 2,\ r = 3 + 5y,\ b = 9 + 15y$

$D(y) = 486(y+1)(2y+1)^4$

$R(y) = 486(2y+1)^3(10y+9)$

1 2	1 3	1 4	2 5	2 6	3 5	3 6	4 5	4 6

$v = 6,\ k = 2,\ r = 4 + 5y,\ b = 12 + 15y$

$D(y) = 288(y+1)^2(3y+2)^3$

$R(y) = 288(y+1)(3y+2)^2(15y+13)$

1 2	1 3	1 4	1 5	2 3	2 4	2 6	3 5	3 6	4 5	4 6	5 6

$v = 6,\ k = 3$

$\underline{v = 6,\ k = 3,\ r = 2 + 5y,\ b = 4 + 10y}$

$D(y) = 288(2y + 1)^2(6y + 2)^3$

$R(y) = 288(2y + 1)(6y + 2)^2(30y + 13)$

$$1\ 2\ 3 \quad 1\ 4\ 5 \quad 2\ 4\ 6 \quad 3\ 5\ 6$$

$\underline{v = 6,\ k = 3,\ r = 4 + 5y,\ b = 8 + 10y}$

$D(y) = 36(3y + 2)(48y^2 + 80y + 33)^2$

$R(y) = 18(34560y^4 + 110592y^3 + 132192y^2 + 69952y + 13827)$

$$1\ 2\ 3 \quad 1\ 2\ 3 \quad 1\ 4\ 5 \quad 1\ 4\ 6 \quad 2\ 4\ 5 \quad 2\ 4\ 6 \quad 3\ 4\ 6 \quad 3\ 5\ 6$$

$v = 6,\ k = 4$

$v = 6,\ k = 4,\ r = 6 + 10y,\ b = 9 + 15y$

$D(y) = 972(3y + 2)(12y + 7)^4$

$R(y) = 1458(12y + 7)^3(20y + 13)$

1 2 3 5	1 2 3 6	1 2 4 5	1 2 4 6	1 3 4 5	1 3 4 6	2 3 5 6	2 4 5 6
3 4 5 6							

$v = 6,\ k = 4,\ r = 8 + 10y,\ b = 12 + 15y$

$D(y) = 2304(6y + 5)^2(9y + 7)^3$

$R(y) = 1152(6y + 5)(9y + 7)^2(90y + 73)$

1 2 3 4	1 2 3 5	1 2 3 6	1 2 4 5	1 2 4 6	1 3 4 5	1 3 5 6	1 4 5 6
2 3 4 6	2 3 5 6	2 4 5 6	3 4 5 6				

A.3 The best RGDs for $v = 7$

$v = 7,\ k = 2$

<u>$v = 7,\ k = 2,\ r = 4 + 6y,\ b = 14 + 21y$</u>

$D(y) = 7(7y^2 + 10y + 3)(49y^2 + 63y + 20)^2$

$R(y) = 14(50421y^5 + 168070y^4 + 220892y^3 + 143178y^2 + 45787y + 5780)$

1 2	1 3	1 4	1 5	2 3	2 4	2 5	3 6	3 7	4 6	4 7	5 6	5 7	6 7

<u>$v = 7,\ k = 2,\ r = 6 + 6y,\ b = 21 + 21y$</u>

$D(y) = 117649(y + 1)^6$

$R(y) = 705894(y + 1)^5$

1 2	1 3	1 4	1 5	1 6	1 7	2 3	2 4	2 5	2 6	2 7	3 4	3 5	3 6
3 7	4 5	4 6	4 7	5 6	5 7	6 7							

$v = 7,\ k = 3$

$\underline{v = 7,\ k = 3,\ r = 3 + 3y,\ b = 7 + 7y}$

$D(y) = 117649(y + 1)^6$

$R(y) = 705894(y + 1)^5$

1 2 3	1 4 5	1 6 7	2 4 6	3 4 7	2 5 7	3 5 6

$v = 7,\ k = 4$

$\underline{v = 7,\ k = 4,\ r = 4 + 4y,\ b = 7 + 7y}$

$D(y) = 7529536(y + 1)^6$

$R(y) = 22588608(y + 1)^5$

1 2 3 4	1 2 5 6	1 3 5 7	1 4 6 7	2 3 6 7	2 4 5 7	3 4 5 6

$v = 7,\ k = 6$

$v = 7,\ k = 6,\ r = 6 + 6y,\ b = 7 + 7y$

$D(y) = 117649(5y + 6)^6$

$R(y) = 117649(5y + 6)^6$

1 2 3 4 5 6	1 2 3 4 5 7	1 2 3 4 6 7	1 2 3 5 6 7	1 2 4 5 6 7	1 3 4 5 6 7	2 3 4 5 6 7

$v = 7,\ k = 7$

$v = 7,\ k = 7,\ r = 2 + 6y,\ b = 2 + y$

$D(y) = 117649(y + 2)^6$

$R(y) = 705894(y + 2)^5$

1 2 3 4 5 6 7	1 2 3 4 5 6 7

A.4 The best RGDs for $v = 8$

$v = 8,\ k = 2$

$v = 8,\ k = 2,\ r = 3 + 7y,\ b = 12 + 28y$

$D(y) = 64(2y+1)(128y^3 + 160y^2 + 60y + 7)^2$

$R(y) = 128(114688y^6 + 294912y^5 + 307200y^4 + 165888y^3 + 49008y^2 + 7520y + 469)$

1 2	1 3	1 4	2 5	2 6	3 5	3 7	4 6	4 8	5 8	6 7	7 8

$v = 8,\ k = 2,\ r = 4 + 7y,\ b = 16 + 28y$

$D(y) = 32768(y+1)(2y+1)^6$

$R(y) = 32768(2y+1)^5(14y+13)$

1 2	1 3	1 4	1 5	2 6	2 7	2 8	5 7	3 6	3 7	3 8	4 6	4 7	4 8
5 6	5 8												

$\underline{v = 8,\ k = 2,\ r = 5 + 7y,\ b = 20 + 28y}$

$D(y) = 8(y + 1)(512y^3 + 1024y^2 + 672y + 145)^2$

$R(y) = 8(1835008y^6 + 7864320y^5 + 13926400y^4 + 13045760y^3 + 6819840y^2 + 1886848y + 215905)$

1 2	1 3	1 4	1 5	1 6	2 3	7 8	2 4	2 5	2 7	3 6	3 7	3 8	6 8
4 6	4 7	4 8	5 6	5 7	5 8								

$\underline{v = 8,\ k = 2,\ r = 6 + 7y,\ b = 24 + 28y}$

$D(y) = 8192(y + 1)^3(4y + 3)^4$

$R(y) = 8192(y + 1)^2(4y + 3)^3(28y + 25)$

1 2	1 3	1 4	1 5	1 6	1 7	2 3	2 4	2 5	2 6	2 8	3 4
3 5	3 7	3 8	4 6	4 7	4 8	5 6	5 7	5 8	6 7	6 8	7 8

$v = 8,\ k = 3$

$\underline{v = 8,\ k = 3,\ r = 3 + 21y,\ b = 8 + 56y}$

$D(y) = 8192(6y + 1)^3(24y + 3)^4$

$R(y) = 8192(6y + 1)^2(24y + 3)^3(168y + 25)$

1 2 3	1 4 6	1 5 7	2 4 8	2 5 6	3 4 7	3 5 8	6 7 8

$\underline{v = 8,\ k = 3,\ r = 6 + 21y,\ b = 16 + 56y}$

$D(y) = 16(3y + 1)(110592y^3 + 92160y^2 + 25536y + 2353)^2$

$R(y) = 24(28538044416y^6 + 48922361856y^5 + 34893987840y^4 + 13254819840y^3 + 2828206080y^2 + 321402112y + 15198027)$

1 2 3	1 2 3	1 4 6	1 4 6	1 5 7	1 5 8	2 4 5	2 4 8	2 5 7	5 6 8
2 6 7	3 4 7	3 5 6	3 6 8	4 7 8	3 7 8				

$\underline{v = 8,\ k = 3,\ r = 9 + 21y,\ b = 24 + 56y}$

$D(y) = 98304(2y + 1)(12y + 5)^6$

$R(y) = 32768(12y + 5)^5(84y + 41)$

1 2 3	1 2 3	1 2 8	1 3 5	5 7 8	1 4 6	1 4 6	1 4 7	1 5 7	4 5 8
1 5 8	2 4 7	2 4 7	2 5 6	4 5 6	2 5 6	2 6 8	2 7 8	3 4 8	3 6 8
3 4 8	3 5 7	3 6 7	3 6 7						

$\underline{v = 8,\ k = 3,\ r = 12 + 21y,\ b = 32 + 56y}$

$D(y) = 64(12y + 7)(27648y^3 + 47232y^2 + 26856y + 5083)^2$

$R(y) = 128(114688(6y + 3)^6 + 294912(6y + 3)^5 + 307200(6y + 3)^4 + 165888(6y + 3)^3 + 49008(6y + 3)^2 + 7520(6y + 3) + 469)$

1 2 7	1 2 7	1 2 7	1 2 8	1 3 4	1 3 6	1 3 6	1 3 6	1 4 5	1 4 5
1 4 8	1 5 8	2 3 5	2 3 5	2 3 5	2 4 6	2 4 6	2 4 6	2 5 8	2 6 8
3 4 7	3 4 7	3 5 8	3 7 8	3 7 8	4 5 8	4 6 8	4 7 8	5 6 7	5 6 7
5 6 7	6 7 8								

$v = 8,\ k = 4$

$v = 8,\ k = 4,\ r = 4 + 7y,\ b = 8 + 14y$

$D(y) = 192(2y+1)(3456y^3 + 6048y^2 + 3516y + 679)^2$

$R(y) = 128(83607552y^6 + 286654464y^5 + 408913920y^4 + 310652928y^3 + 132560496y^2 + 30124896y + 2848405)$

1 2 3 5   1 2 4 7   1 3 4 6   1 5 6 8   2 3 4 8   2 5 6 7   3 6 7 8   4 5 7 8

$v = 8,\ k = 4,\ r = 6 + 7y,\ b = 12 + 14y$

$D(y) = 98304(y+1)(6y+5)^6$

$R(y) = 32768(6y+5)^5(42y+41)$

1 2 3 6   1 2 3 7   1 2 4 8   1 3 5 8   1 4 5 6   1 4 5 7   2 4 6 7   2 5 6 8
2 5 7 8   3 4 6 8   3 4 7 8   3 5 6 7

$\underline{v = 8,\ k = 4,\ r = 8 + 7y,\ b = 16 + 14y}$

$D(y) = 64(6y + 7)(3456y^3 + 11808y^2 + 13428y + 5083)^2$

$R(y) = 128(83607552(y + 1)^6 + 71663616(y + 1)^5 + 24883200(y + 1)^4 + 4478976(y + 1)^3 + 441072(y + 1)^2 + 22560(y + 1) + 469)$

1 2 3 5	1 2 3 5	1 2 4 6	1 2 4 6	1 3 6 7	1 3 7 8	1 4 5 8	4 5 6 7
1 4 7 8	2 3 4 7	2 5 6 8	2 5 7 8	2 6 7 8	3 4 5 8	3 4 6 8	3 5 6 7

$\underline{v = 8,\ k = 4,\ r = 9 + 7y,\ b = 18 + 14y}$

$D(y) = 663552(3y + 4)^3(4y + 5)^4$

$R(y) = 221184(3y + 4)^2(4y + 5)^3(84y + 109)$

1 2 3 4	1 2 3 4	1 2 5 6	1 2 5 6	1 3 5 7	1 3 5 7	1 4 6 8	1 4 7 8
1 6 7 8	2 3 6 8	2 3 7 8	2 4 5 8	2 4 6 7	2 5 7 8	3 4 5 8	3 4 6 7
3 5 6 8	4 5 6 7						

$v = 8,\ k = 5$

$v = 8,\ k = 5,\ r = 5 + 35y,\ b = 8 + 56y$

$D(y) = 8192(20y + 3)^3(80y + 11)^4$

$R(y) = 8192(20y + 3)^2(80y + 11)^3(560y + 81)$

1 2 3 4 5	1 2 4 6 7	1 2 5 6 8	1 3 4 7 8	1 3 5 6 7	2 3 4 6 8
2 3 5 7 8	4 5 6 7 8				

$v = 8,\ k = 5,\ r = 10 + 35y,\ b = 16 + 56y$

$D(y) = 400(10y + 3)(819200y^3 + 696320y^2 + 197248y + 18621)^2$

$R(y) = 40(9 + 32y)(2069 + 160y(91 + 160y))(684849 + 640y(11151 + 320y(121 + 140y)))$

1 2 3 4 6	1 2 3 4 6	1 2 3 5 7	1 2 3 5 7	1 2 4 5 8	1 2 6 7 8
1 3 4 7 8	1 3 5 6 8	1 4 5 6 7	1 4 5 6 8	2 3 4 7 8	2 3 5 6 8
2 4 5 6 7	2 4 5 7 8	3 4 6 7 8	3 5 6 7 8		

A.5 The best RGDs for $v = 9$

$v = 9,\ k = 2$

$v = 9,\ k = 2,\ r = 2 + 8y,\ b = 18 + 36y$

$D(y) = 27(27y^2 + 21y + 4)^2(2187y^4 + 5346y^3 + 4779y^2 + 1848y + 260)$

$R(y) = 324(1062882y^7 + 3720087y^6 + 5491557y^5 + 4435965y^4 + 2119365y^3 + 599346y^2 + 92958y + 6104)$

1 2	1 3	1 4	1 5	2 3	2 6	2 7	3 8	3 9	4 5	4 6	4 7	5 8
5 9	6 8	6 9	7 8	7 9								

$v = 9,\ k = 2,\ r = 6 + 8y,\ b = 27 + 36y$

$D(y) = 59049(y + 1)^2(3y + 2)^6$

$R(y) = 118098(y + 1)(3y + 2)^5(12y + 11)$

1 2	1 3	1 4	1 5	1 6	1 7	2 3	2 4	2 5	2 8	2 9	3 6	6 9	3 7
3 8	3 9	4 6	4 7	4 8	7 8	4 9	5 6	5 7	5 8	5 9	6 8	7 9	

$v = 9,\ k = 3$

$v = 9,\ k = 3,\ r = 2 + 4y,\ b = 6 + 12y$

$D(y) = 6561(9y^2 + 9y + 2)^4$

$R(y) = 236196(2y + 1)(9y^2 + 9y + 2)^3$

1 2 3   1 4 5   2 6 7   3 8 9   4 6 8   5 7 9

$v = 9,\ k = 3,\ r = 3 + 4y,\ b = 9 + 12y$

$D(y) = 59049(y + 1)^2(3y + 2)^6$

$R(y) = 118098(y + 1)(3y + 2)^5(12y + 11)$

1 2 3   1 4 6   1 5 7   2 4 8   2 5 9   3 6 9   3 7 8   4 7 9   5 6 8

$v = 9,\ k = 4$

$v = 9,\ k = 4,\ r = 4 + 8y,\ b = 9 + 18y$

$D(y) = 81(59049y^4 + 118098y^3 + 88209y^2 + 29169y + 3604)^2$

$R(y) = 486(26244y^3 + 39366y^2 + 19602y + 3241)(59049y^4 + 118098y^3 + 88209y^2 + 29169y + 3604)$

1 2 4 7   1 2 5 6   1 3 4 8   1 3 5 9   2 3 6 9   2 3 7 8   4 5 6 8   4 6 7 9

5 7 8 9

$v = 9,\ k = 5$

$v = 9,\ k = 5,\ r = 5 + 10y,\ b = 9 + 18y$

$D(y) = 81(455625y^4 + 911250y^3 + 682425y^2 + 226815y + 28231)^2$

$R(y) = 486(121500y^3 + 182250y^2 + 90990y + 15121)(455625y^4 + 911250y^3 + 682425y^2 + 226815y + 28231)$

1 2 3 4 6   1 2 3 5 8   1 2 3 7 9   1 4 5 6 9   1 4 5 7 8   2 4 6 7 8

2 5 6 7 9   3 4 7 8 9   3 5 6 8 9

$v = 9,\ k = 6$

$\underline{v = 9,\ k = 6,\ r = 4 + 8y,\ b = 6 + 12y}$

$D(y) = 6561(225y^2 + 225y + 56)^4$

$R(y) = 1180980(2y + 1)(225y^2 + 225y + 56)^3$

1 2 3 4 6 8	1 2 3 5 7 9	1 2 4 5 6 7
1 3 4 5 8 9	2 3 6 7 8 9	4 5 6 7 8 9

$\underline{v = 9,\ k = 6,\ r = 6 + 8y,\ b = 9 + 12y}$

$D(y) = 59049(5y + 4)^2(15y + 11)^6$

$R(y) = 118098(5y + 4)(15y + 11)^5(60y + 47)$

1 2 3 4 6 8	1 2 3 4 7 9	1 2 3 5 6 9	1 2 4 5 7 8	1 3 5 6 7 8
1 4 5 6 7 9	2 3 5 7 8 9	2 4 5 6 8 9	3 4 6 7 8 9	

A.6 The best RGDs for $v = 10$

$v = 10, \ k = 2$

$v = 10, \ k = 2, \ r = 2 + 9y, \ b = 10 + 45y$

$D(y) = 50(5y + 2)(2000y^4 + 1600y^3 + 420y^2 + 40y + 1)^2$

$R(y) = 250(36000000y^8 + 64000000y^7 + 47600000y^6 + 19200000y^5 + 4550000y^4 + 640640y^3 + 51480y^2 + 2112y + 33)$

1 2	1 3	2 4	3 5	4 6	6 8	7 9	8 10	9 10	5 7

$v = 10, \ k = 2, \ r = 3 + 9y, \ b = 15 + 45y$

$D(y) = 20000(2y + 1)^4(5y + 1)^5$

$R(y) = 60000(2y + 1)^3(5y + 1)^4(30y + 11)$

1 2	1 3	1 4	2 5	2 6	3 7	3 8	
5 7	5 9	6 8	6 10	7 10	8 9	4 9	4 10

$\underline{v = 10,\ k = 2,\ r = 4 + 9y,\ b = 20 + 45y}$

$D(y) = 12800(5y + 2)^5(5y^2 + 5y + 1)^2$

$R(y) = 64000(5y + 2)^4(5y^2 + 5y + 1)(45y^2 + 43y + 9)$

1 2	1 3	1 4	1 5	2 6	2 7	2 8	3 8	4 6	4 9
4 10	5 6	5 9	5 10	3 7	8 10	7 10	8 9	3 6	7 9

$\underline{v = 10,\ k = 2,\ r = 5 + 9y,\ b = 25 + 45y}$

$D(y) = 3906250(y + 1)(2y + 1)^8$

$R(y) = 3906250(2y + 1)^7(18y + 17)$

1 2	1 3	1 4	1 5	1 6	2 7	2 8	3 7	3 8	3 9	3 10	4 7	4 8
4 9	5 8	5 9	5 10	6 7	6 8	6 9	6 10	2 9	2 10	4 10	5 7	

$\underline{v = 10,\ k = 2,\ r = 6 + 9y,\ b = 30 + 45y}$

$D(y) = 4000(5y + 3)^3(5y^2 + 9y + 4)(20y^2 + 24y + 7)^2$

$R(y) = 12000(5y + 3)^2(30000y^6 + 124000y^5 + 211400y^4 + 190320y^3 + 95447y^2 + 25286y + 2765)$

1 2	1 3	1 4	1 5	1 6	1 7	2 3	2 4	2 6	2 8	3 7	3 8	3 9
4 7	4 8	4 10	5 7	5 8	5 9	5 10	6 7	6 8	6 9	2 5	7 9	4 9
6 10	9 10	3 10	8 10									

$\underline{v = 10,\ k = 2,\ r = 7 + 9y,\ b = 35 + 45y}$

$D(y) = 800(5y + 3)(10y + 7)^4(5y^2 + 9y + 4)^2$

$R(y) = 800(10y + 7)^3(5y^2 + 9y + 4)(2250y^3 + 5225y^2 + 3985y + 998)$

1 2	1 3	1 4	1 5	1 6	1 7	1 8	2 3	2 5	2 6	2 7	2 8	3 4
3 6	3 9	4 7	4 8	4 9	4 10	5 7	5 8	5 9	5 10	6 8	6 9	6 10
7 10	8 9	8 10	9 10	2 4	3 10	6 7	7 9	3 5				

$\underline{v = 10,\ k = 2,\ r = 8 + 9y,\ b = 40 = 45y}$

$D(y) = 320000(y+1)^4(5y+4)^5$

$R(y) = 320000(y+1)^3(5y+4)^4(45y+41)$

1 2	1 3	1 4	1 5	1 6	1 7	1 8	1 9	2 4	2 5	2 6	2 7	2 8
3 4	3 5	3 7	3 9	3 10	4 5	4 6	4 8	4 9	4 10	5 8	5 9	5 10
6 7	6 8	6 9	6 10	7 8	2 3	7 10	3 6	8 9	5 7	8 10	7 9	9 10
2 10												

$v = 10,\ k = 3$

$v = 10,\ k = 3,\ r = 3,\ b = 10$

$$D(y) = 40(3+10y)(1280000000y^8 + 3456000000y^7 + 4051200000y^6 + 2692800000y^5 + 1110024000y^4 + 290568000y^3 + 47166180y^2 + 4340554y + 173377)$$

$$R(y) = 160(5y(2880000000y^7 + 7680000000y^6 + 8904000000y^5 + 5862240000y^4 + 2397330000y^3 + 623575200y^2 + 100752435y + 9245204) + 1844429)$$

1 2 5	1 3 6	1 4 7	2 3 4	2 8 9	4 9 10	5 6 10	5 7 8	6 7 9	3 8 10

$v = 10,\ k = 3,\ r = 6+9y,\ b = 20+30y$

$$D(y) = 640000(4y+3)^4(5y+3)^5$$

$$R(y) = 960000(4y+3)^3(5y+3)^4(60y+41)$$

1 2 7	1 2 8	1 3 5	1 3 6	1 4 9	1 4 10	2 3 4	2 5 6	2 5 10
2 6 9	3 7 8	3 7 10	3 8 9	4 5 7	4 6 8	4 9 10	5 7 9	5 8 9
6 7 10	6 8 10							

$v=10,\ k=3,\ r=12+9y,\ b=40+30y$

$D(y)=8000(10y+13)^3(10y^2+29y+21)(80y^2+208y+135)^2$

$R(y)=12000(10y+13)^2(1920000y^6+15488000y^5+52022400y^4+93132160y^3+93723868y^2+50271428y+11228085)$

1 2 3	1 2 3	1 2 3	1 4 7	1 4 7	1 4 7	1 5 6	1 5 10	1 6 9
1 8 10	2 4 8	2 4 8	2 5 9	2 5 10	2 5 10	2 6 7	2 6 7	3 4 5
3 6 8	3 6 8	3 7 9	3 7 9	3 8 10	3 9 10	4 6 9	4 6 10	4 9 10
5 7 8	5 7 9	5 8 9	6 7 10	6 8 10	1 5 6	2 6 9	3 7 10	5 7 8
1 8 9	3 4 5	4 9 10	2 4 8					

$v=10,\ k=4$

$v=10,\ k=4,\ r=2+6y,\ b=5+15y$

$D(y)=640000(4y+1)^4(5y+2)^5$

$R(y)=960000(4y+1)^3(5y+2)^4(60y+19)$

1 2 3 4	1 5 6 7	2 5 8 9	3 6 8 10	4 7 9 10

$v = 10,\ k = 4,\ r = 4 + 6y,\ b = 10 + 15y$

$D(y) = 640000(4y+3)^4(5y+3)^5$

$R(y) = 960000(4y+3)^3(5y+3)^4(60y+41)$

1 2 3 4	1 2 5 6	1 3 7 8	1 4 9 10	2 5 7 9	2 6 8 10	3 5 7 10
4 5 8 9	4 6 7 10	3 6 8 9				

$v = 10,\ k = 4,\ r = 8 + 6y,\ b = 20 + 15y$

$D(y) = 8000(10y+13)^3(10y^2+29y+21)(80y^2+208y+135)^2$

$R(y) = 12000(10y+13)^2(1920000y^6+15488000y^5+52022400y^4+93132160y^3+93723868y^2+50271428y+11228085)$

1 2 3 4	1 2 3 5	1 2 3 7	1 4 6 7	1 5 7 8	1 5 9 10	1 6 9 10
2 5 6 8	2 5 6 9	2 6 7 10	3 4 9 10	3 6 7 9	3 6 8 10	3 7 8 9
1 4 6 8	2 4 8 10	3 5 8 10	4 5 7 10	2 4 8 9	4 5 7 9	

$v = 10,\ k = 4,\ r = 10 + 6y,\ b = 25 + 15y$

$D(y) = 640000(4y + 7)^4(5y + 8)^5$

$R(y) = 960000(4y + 7)^3(5y + 8)^4(60y + 101)$

1 2 3 4	1 2 3 4	1 2 5 6	1 2 5 6	1 3 7 8	1 4 9 10	1 4 9 10
2 3 9 10	2 4 7 8	2 5 7 9	2 5 7 9	2 6 8 10	3 4 5 6	3 5 7 10
3 6 8 9	4 5 8 9	4 5 8 9	4 6 7 10	1 3 7 8	1 6 7 9	2 6 8 10
4 6 7 10	3 6 8 9	3 5 7 10	1 5 8 10			

$v = 10,\ k = 5$

$\underline{v = 10,\ k = 5,\ r = 5 + 9y,\ b = 10 + 18y}$

$$D(y) = 200(5y+3)(512000y^4 + 1126400y^3 + 928320y^2 + 339680y + 46561)^2$$

$$R(y) = 250(36000000(4y+2)^8 + 64000000(4y+2)^7 + 47600000(4y+2)^6 + 19200000(4y+2)^5 + 4550000(4y+2)^4$$
$$+ 640640(4y+2)^3 + 51480(4y+2)^2 + 4224(2y+1) + 33)$$

1 2 3 4 9	1 2 3 5 10	1 2 4 6 7	1 3 5 7 8	1 6 8 9 10	2 4 5 6 8
2 7 8 9 10	3 4 6 8 10	3 5 6 7 9	4 5 7 9 10		

$\underline{v = 10,\ k = 5,\ r = 6 + 9y,\ b = 12 + 18y}$

$$D(y) = 8000(20y+13)^3(40y^2 + 58y + 21)(320y^2 + 416y + 135)^2$$

$$R(y) = 12000(20y+13)^2(122880000y^6 + 495616000y^5 + 832358400y^4 + 745057280y^3 + 374895472y^2 + 100542856y + 11228085)$$

1 2 3 4 7	1 2 3 5 8	1 2 4 6 9	1 3 6 7 10	1 4 5 8 10	1 5 6 7 9
2 3 5 9 10	2 4 6 8 10	2 5 6 7 8	3 4 7 8 9	3 6 8 9 10	4 5 7 9 10

$\underline{v = 10,\ k = 5,\ r = 10 + 9y,\ b = 20 + 18y}$

$D(y) = 16000(20y + 23)^3(20y^2 + 49y + 30)(320y^2 + 736y + 423)^2$

$R(y) = 12000(20y{+}23)^2(30000(4y{+}4)^6{+}124000(4y{+}4)^5{+}54118400(y{+}1)^4{+}12180480(y{+}1)^3{+}1527152(y{+}1)^2{+}101144(y{+}1){+}2765)$

1 2 3 4 6	1 2 3 4 6	1 2 5 6 9	1 2 7 8 10	1 2 8 9 10	1 3 4 7 10
1 3 5 7 10	1 3 5 8 9	1 4 5 6 8	1 4 5 7 9	2 3 5 7 9	2 3 6 7 8
2 4 5 8 10	2 4 7 8 9	2 5 6 7 10	3 4 8 9 10	3 5 6 8 10	3 6 7 8 9
4 5 6 9 10	4 6 7 9 10				

$v = 10,\ k = 6$

$\underline{v = 10,\ k = 6,\ r = 3 + 9y,\ b = 20 + 15y}$

$D(y) = 20000(10y + 3)^4(25y + 9)^5$

$R(y) = 60000(10y + 3)^3(25y + 9)^4(150y + 49)$

1 2 3 5 6 8	1 2 4 5 7 9	1 3 4 6 7 10	2 3 4 8 9 10	5 6 7 8 9 10

$\underline{v = 10,\ k = 6,\ r = 6 + 9y,\ b = 10 + 15y}$

$D(y) = 20000(10y + 7)^4(25y + 16)^5$

$R(y) = 60000(10y + 7)^3(25y + 16)^4(150y + 101)$

1	2	3	4	5	7	1	2	3	5	8	9	1	2	3	6	7	10	1	3	4	6	8	10	1	4	5	7	9	10
2	4	5	6	9	10	3	4	7	8	9	10	3	5	6	7	8	9	1	2	4	6	8	9	2	5	6	7	8	10

$v = 10,\ k = 7$

$v = 10,\ k = 7,\ r = 7 + 21y,\ b = 10 + 30y$

$$D(y) = 40(25000000(14y+4)^9 + 150000000(14y+4)^8 + 397500000(14y+4)^7 + 610650000(14y+4)^6 + 599332500(14y+4)^5 + 6235752000(7y+2)^4 + 1343365800(7y+2)^3 + 184904080(7y+2)^2 + 14755432(7y+2) + 520131)$$

$$R(y) = 160(56250000(14y+4)^8 + 300000000(14y+4)^7 + 695625000(14y+4)^6 + 915975000(14y+4)^5 + 11986650000(7y+2)^4 + 3117876000(7y+2)^3 + 503762175(7y+2)^2 + 46226020(7y+2) + 1844429)$$

1	2	3	4	5	6	7	1	2	3	4	5	8	10	1	2	3	4	6	9	10	1	2	3	4	7	8	9
1	3	5	6	7	8	10	1	4	5	6	7	9	10	2	3	5	6	8	9	10	2	4	5	7	8	9	10
3	4	6	7	8	9	10	1	2	5	6	7	8	9														

$v = 10,\ k = 8$

$v = 10,\ k = 8,\ r = 8 + 36y,\ b = 10 + 40y$

$$D(y) = 200(35y+8)(2000(28y+6)^4 + 1600(28y+6)^3 + 420(28y+6)^2 + 80(14y+3) + 1)^2$$

$$R(y) = 250(36000000(28y+6)^8 + 64000000(28y+6)^7 + 47600000(28y+6)^6 + 19200000(28y+6)^5 + 4550000(28y+6)^4 + 640640(28y+6)^3 + 51480(28y+6)^2 + 4224(14y+3) + 33)$$

1	2	3	4	5	6	7	8	1	2	3	4	5	6	7	9	1	2	3	4	5	7	9	10	1	2	3	4	6	8	9	10
1	2	4	6	7	8	9	10	1	3	5	6	7	8	9	10	3	4	5	6	7	8	9	10	1	2	3	4	5	6	8	10
1	2	3	5	7	8	9	10	2	4	5	6	7	8	9	10																

A.7 The best RGDs for $v = 11$

$v = 11,\ k = 2$

$v = 11,\ k = 2,\ r = 2 + 10y,\ b = 11 + 55y$

$$D(y) = 121(14641y^5 + 14641y^4 + 5324y^3 + 847y^2 + 55y + 1)^2$$

$$R(y) = 2662(6655y^4 + 5324y^3 + 1452y^2 + 154y + 5)(14641y^5 + 14641y^4 + 5324y^3 + 847y^2 + 55y + 1)$$

1 2	1 3	2 4	3 5	4 6	5 7	6 8	7 9	8 10	9 11	10 11

$v = 11,\ k = 2,\ r = 4 + 10y,\ b = 22 + 55y$

$$D(y) = 121(14641y^5 + 29282y^4 + 22627y^3 + 8470y^2 + 1540y + 109)^2$$

$$R(y) = 2662(6655y^4 + 10648y^3 + 6171y^2 + 1540y + 140)(14641y^5 + 29282y^4 + 22627y^3 + 8470y^2 + 1540y + 109)$$

1 2	1 3	1 4	1 5	2 6	2 7	2 8	3 6	3 9	4 6	4 8
5 11	3 7	8 9	8 11	9 10	10 11	6 11	7 10	5 7	5 9	4 10

$v = 11,\ k = 2,\ r = 6 + 10y,\ b = 33 + 55y$

$D(y) = 11(y+1)(11y+5)^2(11y+6)^4(11y+7)^3$

$R(y) = 22(11y+5)(11y+6)^3(11y+7)^2(6655y^3 + 13552y^2 + 8789y + 1832)$

1	2	1	3	1	4	1	5	1	6	1	7	2	3	2	6	2	7	3	8	3	9	3	10	3	11
4	11	5	8	5	9	5	10	5	11	6	8	6	9	6	11	7	8	7	9	7	10	7	11	8	9
2	4	2	5	4	9	6	10	4	10	4	8	10	11												

$v = 11,\ k = 2,\ r = 8 + 10y,\ b = 44 + 55y$

$D(y) = 121(11y+8)^4(121y^3 + 308y^2 + 258y + 71)^2$

$R(y) = 242(11y+8)^3(121y^3 + 308y^2 + 258y + 71)(6655y^3 + 16456y^2 + 13442y + 3626)$

1	2	1	3	1	4	1	5	1	6	1	7	1	8	1	9	2	4	2	5	2	6	2	7
3	4	3	5	3	9	3	10	3	11	4	7	4	8	4	9	4	10	4	11	5	8	5	9
5	10	5	11	6	7	6	8	6	9	6	10	7	9	7	10	7	11	8	9	8	10	8	11
9	11	10	11	2	3	3	6	6	11	2	8	2	10										

$v = 11,\ k = 3$

$\underline{v = 11,\ k = 3,\ r = 3 + 15y,\ b = 11 + 55y}$

$$D(y) = 11(139234453205859y^{10} + 278468906411718y^9 + 248934325428657y^8 + 130947103248786y^7 + 44874167894127y^6 + 10464654662154y^5 + 1681245486558y^4 + 183681841266y^3 + 13055420268y^2 + 544897296y + 10136867)$$

$$R(y) = 66(77352474003255y^9 + 139234453205859y^8 + 110637477968292y^7 + 50923873485639y^6 + 14958055964709y^5 + 2906848517265y^4 + 373610108124y^3 + 30613640211y^2 + 1450602252y + 30272072)$$

1 2 5	1 3 6	1 4 7	2 3 4	2 8 9	3 8 10
4 9 11	5 6 11	5 7 10	6 9 10	7 8 11	

$\underline{v = 11,\ k = 3,\ r = 6 + 15y,\ b = 22 + 55y}$

$$D(y) = 121(3557763y^5 + 7115526y^4 + 5678046y^3 + 2259675y^2 + 448470y + 35509)^2$$

$$R(y) = 7986(179685y^4 + 287496y^3 + 172062y^2 + 45650y + 4530)(3557763y^5 + 7115526y^4 + 5678046y^3 + 2259675y^2 + 448470y + 35509)$$

1 2 5	1 2 6	1 3 4	1 3 7	1 8 9	1 10 11	2 4 7	2 4 9	2 8 11
3 5 6	3 5 8	3 9 11	4 6 8	4 6 11	5 7 9	5 7 11	6 7 9	6 8 10
2 3 10	4 5 10	7 8 10	9 10 11					

$\underline{v = 11,\ k = 3,\ r = 9 + 15y,\ b = 33 + 55y}$

$D(y) = 121(33y + 19)^4(3267y^3 + 6039y^2 + 3711y + 758)^2$

$R(y) = 726(33y + 19)^3(3267y^3 + 6039y^2 + 3711y + 758)(59895y^3 + 109263y^2 + 66319y + 13393)$

1	2	3	1	2	3	1	4	9	1	4	9	1	5	7	1	5	7	1	6	8	1	10	11	2	4	8
2	5	10	2	5	10	2	6	7	2	9	11	3	4	5	3	4	11	3	5	9	3	6	10	3	7	9
4	7	10	4	7	11	5	6	9	5	8	11	6	9	11	7	8	9	7	10	11	8	9	10	1	6	8
3	6	11	5	8	11	4	6	10	2	4	8	3	8	10	2	6	7									

$v = 11, \; k = 4$

$v = 11, \; k = 4, \; r = 4 + 20y, \; b = 11 + 55y$

$$D(y) = 121(14641(6y+1)^5 + 14641(6y+1)^4 + 5324(6y+1)^3 + 847(6y+1)^2 + 55(6y+1) + 1)^2$$

$$R(y) = 15972(1437480y^4 + 1149984y^3 + 344124y^2 + 45650y + 2265)(14641(6y+1)^5 + 14641(6y+1)^4 + 5324(6y+1)^3 + 847(6y+1)^2 + 55(6y+1) + 1)$$

1 2 3 8	1 2 4 10	1 3 5 6	1 7 9 11	2 5 7 9	3 4 5 7	3 9 10 11
2 4 6 11	5 8 10 11	6 7 8 10	4 6 8 9			

$\underline{v = 11,\ k = 4,\ r = 8 + 20y,\ b = 22 + 55y}$

$$D(y) = 121(14641(6y+1)^5 + 14641(6y+1)^4 + 5324(6y+1)^3 + 847(6y+1)^2 + 55(6y+1) + 1)^2$$

$$R(y) = 15972(1437480y^4 + 1149984y^3 + 344124y^2 + 45650y + 2265)(14641(6y+1)^5 + 14641(6y+1)^4 + 5324(6y+1)^3$$
$$+ 847(6y+1)^2 + 55(6y+1) + 1)$$

1 2 4 8	1 2 4 8	1 2 7 9	1 3 5 11	1 3 9 10	1 4 5 10	1 5 6 11
2 5 6 9	2 6 10 11	2 7 10 11	3 4 6 7	3 8 9 10	4 5 7 10	4 6 8 11
5 8 9 11	7 8 10 11	1 3 6 7	2 3 6 8	3 4 9 11	5 7 8 9	2 3 5 7
4 6 9 10						

$v = 11,\ k = 7$

$v = 11,\ k = 7,\ r = 7 + 35y,\ b = 11 + 55y$

$$D(y) = 121(14641(6y+2)^5 + 29282(6y+2)^4 + 22627(6y+2)^3 + 8470(6y+2)^2 + 3080(3y+1) + 109)^2$$

$$R(y) = 10648(2156220y^4 + 3449952y^3 + 2068011y^2 + 550440y + 54892)(14641(6y+2)^5 + 29282(6y+2)^4 + 22627(6y+2)^3 + 8470(6y+2)^2 + 3080(3y+1) + 109)$$

1	2	3	4	5	8	10	1	2	3	4	6	10	11	1	2	4	6	8	9	11	1	2	5	7	9	10	11
1	3	7	8	9	10	11	2	3	4	5	7	9	11	3	5	6	8	9	10	11	4	5	6	7	8	10	11
1	2	3	5	6	7	8	1	3	4	5	6	7	9	2	4	6	7	8	9	10							

$v = 11,\ k = 8$

$v = 11,\ k = 8,\ r = 8 + 40y$

$$D(y) = 11(2357947691(28y+5)^{10} + 14147686146(28y+5)^{9} + 37941521937(28y+5)^{8} + 59875218678(28y+5)^{7}$$
$$+ 61555785863(28y+5)^{6} + 43064422478(28y+5)^{5} + 20756117118(28y+5)^{4} + 6803031158(28y+5)^{3}$$
$$+ 1450602252(28y+5)^{2} + 181632432(28y+5) + 10136867)$$

$$R(y) = 22(11789738455(28y+5)^{9} + 63664587657(28y+5)^{8} + 151766087748(28y+5)^{7} + 209563265373(28y+5)^{6}$$
$$+ 184667357589(28y+5)^{5} + 107661056195(28y+5)^{4} + 41512234236(28y+5)^{3} + 10204546737(28y+5)^{2}$$
$$+ 1450602252(28y+5) + 90816216)$$

1	2	3	4	5	6	9	10	1	2	3	4	5	7	8	11	1	2	3	4	7	8	9	10	1	2	3	5	6	7	8	10
1	3	4	5	6	7	10	11	1	5	6	7	8	9	10	11	2	4	5	7	8	9	10	11	3	4	6	7	8	9	10	11
1	2	3	4	6	8	9	11	1	2	4	5	6	7	9	11	2	3	5	6	8	9	10	11								

A.8 The best RGDs for $v = 12$

$v = 12,\ k = 2$

$v = 12,\ k = 2,\ r = 3 + 11y,\ b = 18 + 66y$

$D(y) = 48(144y^3 + 120y^2 + 29y + 2)(10368y^4 + 11232y^3 + 4248y^2 + 654y + 35)^2$

$R(y) = 48(10368y^4+11232y^3+4248y^2+654y+35)(16422912y^6+26998272y^5+17770752y^4+5975424y^3+1079784y^2+99282y+3631)$

1	2	1	3	1	4	2	5	2	6	3	7	3	8	5	7	5	9
6	8	6	10	7	11	8	12	9	12	4	9	4	10	10	11	11	12

$v = 12,\ k = 2,\ r = 4 + 11y,\ b = 24 + 66y$

$D(y) = 104976(3y + 1)(4y + 1)^6(24y^2 + 26y + 7)^2$

$R(y) = 104976(4y + 1)^5(24y^2 + 26y + 7)(3168y^3 + 4128y^2 + 1750y + 241)$

1	2	1	3	1	4	1	5	2	6	2	7	2	8	3	6	3	10	4	7	4	9	4	11
5	8	5	10	5	12	6	11	7	10	7	12	8	9	8	11	9	12	10	11	3	9	6	12

$\underline{v = 12,\ k = 2,\ r = 5 + 11y,\ b = 30 + 66y}$

$D(y) = 15925248(6y + 5)(6y^2 + 5y + 1)^5$

$R(y) = 15925248(6y^2 + 5y + 1)^4(396y^2 + 480y + 131)$

1 2	1 3	1 4	1 5	1 6	2 7	2 8	2 9	3 7	3 8	3 9	3 11
4 8	4 10	4 11	5 9	5 10	5 11	6 8	6 9	6 10	6 11	7 12	2 10
5 7	10 12	8 12	11 12	4 7	9 12						

$\underline{v = 12,\ k = 2,\ r = 6 + 11y,\ b = 36 + 66y}$

$D(y) = 725594112(y + 1)(2y + 1)^{10}$

$R(y) = 725594112(2y + 1)^9(22y + 21)$

1 2	1 3	1 4	1 5	1 6	1 7	2 8	2 11	2 12	3 8	3 9	3 10
3 12	4 10	4 11	4 12	5 8	5 9	5 10	5 11	6 9	6 10	6 11	6 12
7 9	7 10	2 9	2 10	4 8	4 9	5 12	6 8	7 11	7 12	7 8	3 11

$\underline{v = 12,\ k = 2,\ r = 7 + 11y,\ b = 42 + 66y}$

$D(y) = 12(y + 1)(12y + 7)^4(1728y^3 + 3168y^2 + 1908y + 377)^2$

$R(y) = 12(12y + 7)^3(1728y^3 + 3168y^2 + 1908y + 377)(228096y^4 + 634176y^3 + 647856y^2 + 288768y + 47447)$

1 2	1 3	1 4	1 5	1 6	1 7	1 8	2 3	2 4	2 5	2 6	2 7
3 8	3 9	3 10	3 11	3 12	4 8	4 9	4 10	4 11	4 12	5 8	5 9
5 11	5 12	6 8	6 9	6 10	6 11	6 12	7 8	7 9	7 10	7 11	7 12
9 11	10 12	11 12	2 9	8 10	5 10						

$\underline{v = 12,\ k = 2,\ r = 8 + 11y,\ b = 48 + 66y}$

$D(y) = 37748736(y + 1)^2(3y + 2)^9$

$R(y) = 37748736(y + 1)(3y + 2)^8(33y + 31)$

1 2	1 3	1 4	1 5	1 6	1 7	1 8	1 9	2 3	2 4	2 5	2 6
2 11	2 12	3 7	3 8	3 9	3 10	3 11	3 12	4 7	4 8	4 9	4 10
4 12	5 7	5 8	5 9	5 10	5 11	5 12	6 7	6 8	6 9	6 10	6 11
7 10	7 11	7 12	8 10	8 11	8 12	9 10	9 11	9 12	2 10	4 11	6 12

$v = 12,\ k = 2,\ r = 9 + 11y,\ b = 54 + 66y$

$D(y) = 11337408(y+1)^3(4y+3)^8$

$R(y) = 11337408(y+1)^2(4y+3)^7(44y+41)$

1	2	1	3	1	4	1	5	1	6	1	7	1	8	1	9	1	10	2	3	2	4	2	5
2	7	2	8	2	11	2	12	3	4	3	5	3	6	3	9	3	10	3	11	3	12	4	7
4	9	4	10	4	11	4	12	5	7	5	8	5	9	5	10	5	11	5	12	6	7	6	8
6	10	6	11	6	12	7	9	7	10	7	11	7	12	8	9	8	10	8	11	8	12	9	11
10	11	10	12	2	6	6	9	4	8	9	12												

$v = 12,\ k = 3$

<u>$v = 12,\ k = 3,\ r = 7 + 11y,\ b = 28 + 44y$</u>

$$D(y) = 48(1152y^3 + 2208y^2 + 1402y + 295)(165888y^4 + 421632y^3 + 400608y^2 + 168636y + 26537)^2$$

$$R(y) = 48(165888y^4 + 421632y^3 + 400608y^2 + 168636y + 26537)(1051066368y^6 + 4017143808y^5 + 6385692672y^4 + 5403995136y^3 + 2567827872y^2 + 649593924y + 68350057)$$

1	2	7	1	2	8	1	3	5	1	3	6	1	4	10	1	4	11	2	3	9	2	4	12	2	5	6
2	5	10	2	6	11	3	4	7	3	8	10	3	8	11	4	5	9	4	6	10	4	8	9	5	7	9
5	8	12	6	7	8	6	8	12	6	9	10	7	10	11	9	11	12	1	9	12	3	7	12	5	7	11
10	11	12																								

<u>$v = 12,\ k = 3,\ r = 4 + 11y,\ b = 16 + 44y$</u>

$$D(y) = 37748736(2y + 1)^2(6y + 2)^9$$

$$R(y) = 37748736(2y + 1)(6y + 2)^8(66y + 31)$$

1	2	3	1	4	7	1	5	8	1	6	9	2	4	10	2	5	11	2	6	12	3	7	10	3	8	11
3	9	12	4	8	12	4	9	11	5	7	12	5	9	10	6	7	11	6	8	10						

$\underline{v = 12,\ k = 3,\ r = 9 + 11y,\ b = 36 + 44y}$

$$D(y) = 24(y+1)(24y+19)^4(13824y^3 + 33408y^2 + 26856y + 7181)^2$$

$$R(y) = 12(24y+19)^3(13824y^3 + 33408y^2 + 26856y + 7181)(3649536y^4 + 12372480y^3 + 15675840y^2 + 8798784y + 1846343)$$

1	2	3	1	2	3	1	4	8	1	4	8	1	5	6	1	5	7	1	6	11	1	7	12	1	9	10
2	4	9	2	4	9	2	5	11	2	5	12	2	6	7	2	6	10	2	7	8	3	4	12	3	5	8
3	6	9	3	7	9	3	8	11	3	10	11	3	10	12	4	5	10	4	6	10	4	7	11	4	11	12
5	8	10	5	9	11	5	9	12	6	8	9	6	8	12	6	11	12	7	8	10	7	9	11	7	10	12

$v = 12,\ k = 3,\ r = 10 + 11y,\ b = 40 + 44y$

$D(y) = 90699264(y + 1)^3(8y + 7)^8$

$R(y) = 45349632(y + 1)^2(8y + 7)^7(88y + 85)$

1	2	3	1	2	3	1	4	7	1	4	7	1	5	8	1	5	8	1	6	9	1	6	9	1	10	11
1	10	12	2	4	11	2	4	11	2	5	12	2	5	12	2	6	7	2	6	7	2	8	9	2	8	10
3	4	10	3	4	10	3	5	9	3	5	9	3	6	11	3	6	12	3	7	12	3	8	11	4	5	6
4	8	9	4	8	12	4	9	12	5	7	10	5	7	11	5	10	11	6	8	10	6	8	11	6	10	12
7	8	12	7	9	10	7	9	11	9	11	12															

$v = 12,\ k = 3,\ r = 13 + 11y,\ b = 52 + 44y$

$D(y) = 104976(7 + 6y)(9 + 8y)^6(155 + 244y + 96y^2)^2$

$R(y) = 104976(9 + 8y)^5(7067535 + 28571848y + 46185296y^2 + 37314432y^3 + 15068160y^4 + 2433024y^5)$

1	2	9	1	2	9	1	2	12	1	3	7	1	3	7	1	3	11	1	4	6	1	4	10
1	5	8	1	5	11	1	10	12	2	3	5	2	4	8	2	4	8	2	6	7	2	6	7
2	10	11	2	11	12	3	4	12	3	4	12	3	6	8	3	6	9	3	6	9	3	8	10
3	10	11	4	5	9	4	5	9	4	7	10	4	7	11	4	7	11	4	9	11	5	6	12
5	7	10	5	7	10	5	8	11	5	10	12	6	8	11	6	10	11	6	11	12	7	8	12
7	9	12	8	9	10	8	9	11	1	4	6	2	3	5	2	7	8	8	9	12	1	5	8
2	6	10	3	9	10	5	6	12	7	9	12												

$v = 12,\ k = 4$

$v = 12,\ k = 4,\ r = 3 + 11y,\ b = 9 + 33y$

$$D(y) = 11337408(3y+1)^3(12y+3)^8$$

$$R(y) = 11337408(3y+1)^2(12y+3)^7(132y+41)$$

1 2 3 4	1 5 7 9	1 6 8 10	2 5 8 11	2 6 7 12	3 5 10 12	3 6 9 11
4 8 9 12	4 7 10 11					

$v = 12,\ k = 4,\ r = 5 + 11y,\ b = 15 + 33y$

$$D(y) = 104976(6y+7)(8y+9)^6(96y^2+244y+155)^2$$

$$R(y) = 104976(8y+9)^5(96y^2+244y+155)(25344y^3+92544y^2+112556y+45597)$$

1 2 4 5	1 2 6 7	1 3 5 8	1 3 9 12	1 4 10 11	2 3 7 10	2 6 8 12
3 4 6 9	3 6 10 11	4 7 8 11	4 7 9 12	5 6 11 12	5 7 10 12	5 8 9 10
2 8 9 11						

$\underline{v = 12,\ k = 4,\ r = 6 + 11y,\ b = 18 + 33y}$

$D(y) = 5159780352(2y + 1)^4(3y + 2)(9y + 5)^6$

$R(y) = 1719926784(2y + 1)^3(9y + 5)^5(594y^2 + 699y + 203)$

1 2 3 4	1 2 3 4	1 5 7 9	1 5 8 12	1 6 7 11	1 6 8 10	2 5 6 11
2 7 8 9	2 7 8 10	3 5 10 11	3 6 9 12	3 7 10 12	3 8 9 11	4 5 9 10
4 7 11 12	4 8 11 12	2 5 6 12	4 6 9 10			

$v = 12,\ k = 5$

$\underline{v = 12,\ k = 5,\ r = 5 + 55y,\ b = 12 + 132y}$

$D(y) = 11337408(20y + 4)^3(80y + 15)^8$

$R(y) = 14171760000000(5y + 1)^2(16y + 3)^7(880y + 173)$

1 2 3 4 5	1 2 6 7 8	1 3 6 9 10	1 4 7 9 11	1 5 8 10 12	2 3 6 11 12
2 4 7 10 12	2 5 8 9 11	3 4 8 9 12	3 5 7 10 11	4 6 8 10 11	5 6 7 9 12

$v = 12,\ k = 7$

$\underline{v = 12,\ k = 7,\ r = 7 + 77y,\ b = 12 + 132y}$

$D(y) = 595077871104(21y + 2)^3(56y + 5)^8$

$R(y) = 99179645184(21y + 2)^2(56y + 5)^7(1848y + 173)$

1	2	3	4	5	7	9	1	2	3	4	6	8	11	1	2	3	5	6	10	12	1	2	4	7	8	10	12
1	3	5	8	9	10	11	1	4	6	7	9	10	11	1	5	6	7	8	9	12	2	3	6	7	9	11	12
2	4	5	8	9	11	12	2	5	6	7	8	10	11	3	4	5	7	10	11	12	3	4	6	8	9	10	12

$v = 12,\ k = 8$

$\underline{v = 12,\ k = 8,\ r = 6 + 22y,\ b = 9 + 33y}$

$D(y) = 90699264(7y + 2)^3(56y + 15)^8$

$R(y) = 45349632(7y + 2)^2(56y + 15)^7(616y + 173)$

1	2	3	4	5	7	9	11	1	2	3	4	5	8	10	12	1	2	4	6	7	8	9	12
1	3	4	6	8	9	10	11	1	3	5	6	7	9	10	12	2	3	4	6	7	10	11	12
4	5	7	8	9	10	11	12	1	2	5	6	7	8	10	11	2	3	5	6	8	9	11	12

A.9 The best RGDs for $v = 13$

$v = 13,\ k = 2$

$v = 13,\ k = 2,\ r = 4 + 12y,\ b = 26 + 78y$

$D(y) = 28561(169y^3 + 169y^2 + 52y + 5)^4$

$R(y) = 1485172(39y^2 + 26y + 4)(169y^3 + 169y^2 + 52y + 5)^3$

1	2	1	3	1	4	1	5	2	6	2	7	2	8	3	6	3	7	3	9	4	8	4	10
4	11	5	9	5	12	5	13	6	10	6	12	7	11	7	13	8	12	8	13	9	10	9	11
10	13	11	12																				

$v = 13,\ k = 2,\ r = 6 + 12y,\ b = 36 + 78y$

$$D(y) = 13(13y+6)^5(4826809y^7 + 17822064y^6 + 27675609y^5 + 23433202y^4 + 11685336y^3 + 3432520y^2 + 550132y + 37128)$$

$$R(y) = 26(13y+6)^4(376491102y^7 + 1375640565y^6 + 2119711737y^5 + 1785976452y^4 + 888835896y^3 + 261347684y^2 + 42050268y + 2857056)$$

1 2	1 3	1 4	1 5	1 6	1 7	2 3	2 4	2 5	2 6	2 7	3 8
3 10	3 11	4 8	4 9	4 10	4 11	5 8	5 9	5 10	5 11	6 8	6 9
6 12	7 10	7 11	7 12	7 13	8 12	8 13	9 12	9 13	10 12	10 13	11 12
3 9	6 13	11 13									

$v = 13,\ k = 2,\ r = 8 + 12y,\ b = 52 + 78y$

$D(y) = 13(13y+8)^4(13y^2+22y+9)(2197y^3+4225y^2+2665y+553)^2$

$R(y) = 52(8+13y)^3(553+2665y+4225y^2+2197y^3)(184973+1350141y+3906604y^2+5597956y^3+3969979y^4+1113879y^5)$

1	2	1	3	1	4	1	5	1	6	1	7	1	8	1	9	2	3	2	4	2	5	2	6
2	10	2	11	3	8	3	9	3	10	3	11	3	12	3	13	4	8	4	9	4	10	4	11
4	13	5	8	5	9	5	10	5	11	5	12	5	13	6	8	6	9	6	10	6	11	6	12
7	8	7	9	7	10	7	11	7	12	7	13	8	10	8	12	9	11	9	13	2	7	4	12
6	13																						

$v = 13,\ k = 5$

$\underline{v = 13,\ k = 5,\ r = 5 + 15y,\ b = 13 + 39y}$

$\underline{v = 11,\ k = 8,\ r = 8 + 40y}$

$$D(y) = 13(65y + 21)^2(65y + 23)(1593224064453125y^9 + 4779672193359375y^8 + 6366862746562500y^7 + 4942710430531250y^6 + 2464459072562500y^5 + 818456783316250y^4 + 181048174542625y^3 + 25723435428925y^2 + 2130143363735y + 78332250103)$$

$$R(y) = 26(13y + 8)(827150951094y^{10} + 5556757671452y^9 + 16729005626268y^8 + 29724160748451y^7 + 34521337968000y^6 + 27384957298036y^5 + 15028322002447y^4 + 5634053059323y^3 + 1381022905730y^2 + 199879174196y + 12971983260)$$

1	2	3	6	9	1	2	4	7	10	1	3	4	5	13	1	5	6	8	12	1	7	8	9	11
2	4	6	11	13	2	5	7	11	12	3	4	8	11	12	3	7	9	12	13	4	6	9	10	12
6	7	8	10	13	2	3	5	8	10	5	9	10	11	13										

A.10 The best RGDs for $v = 14$

$v = 14,\ k = 2$

$v = 14,\ k = 2,\ r = 3 + 13y,\ b = 21 + 91y$

$D(y) = 235298(7y + 3)(28y^2 + 12y + 1)^6$

$R(y) = 235298(28y^2 + 12y + 1)^5(2548y^2 + 1596y + 223)$

1 2	1 3	1 4	2 5	2 6	3 7	3 8	4 9	4 10	5 11	5 12
6 13	6 14	7 11	7 13	8 12	8 14	9 11	9 14	10 12	10 13	

$v = 14,\ k = 2,\ r = 4 + 13y,\ b = 28 + 91y$

$D(y) = 224(7y + 2)^2(28y^2 + 24y + 5)(1372y^3 + 1176y^2 + 301y + 24)^3$

$R(y) = 224(7y + 2)(1372y^3 + 1176y^2 + 301y + 24)^2(3495856y^5 + 5916064y^4 + 3904712y^3 + 1254400y^2 + 195811y + 11862)$

1 2	1 3	1 4	1 5	2 6	2 7	2 8	3 6	3 9	3 10	4 7	4 9	
4 11	5 12	5 13	5 14	6 11	6 12	7 10	7 13	8 9	8 12	8 13	9 14	
10 12	10 14	11 13	11 14									

$\underline{v = 14,\ k = 2,\ r = 5 + 13y,\ b = 35 + 91y}$

The A-optimal design for $y \in [0,5]$

$$D(y) = 98(3 + 7y)(1075648y^6 + 2458624y^5 + 2283008y^4 + 1105440y^3 + 294980y^2 + 41188y + 2353)^2$$

$$\begin{aligned} R(y) = 49(&15041242058752y^{12} + 69421117194240y^{11} + 145453769359360y^{10} + 182997842944000y^9 + 154015454154240y^8 \\ &+ 91375075663872y^7 + 39194984588928y^6 + 12250310526720y^5 + 2769381094480y^4 + 441695197888y^3 \\ &+ 47184347112y^2 + 3031627648y + 88606921) \end{aligned}$$

1 2	1 3	1 4	1 5	1 6	2 7	2 8	2 9	2 10	3 7	3 8
3 11	4 7	4 8	4 10	4 12	5 7	5 9	5 11	5 13	6 8	6 10
6 14	7 14	8 13	9 12	9 14	10 11	10 13	11 12	11 14	12 13	13 14
3 9	6 12									

The A-optimal design for $y \geq 6$ and the D-optimal design for $y \geq 0$

$$D(y) = 48384(3+14y)^2(19+84y)^4(910520 + 27114599y + 345318288y^2 + 2438028369y^3 + 10305788976y^4 + 26082235872y^5 + 36593544960y^6 + 21956126976y^7)$$

$$R(y) = 576(19+84y)^3(8843198829 + 346046537284y + 6011249041581y^2 + 60840304300524y^3 + 395370395379036y^4 + 1710760664328912y^5 + 4928771153029248y^6 + 9116959703588352y^7 + 9824664225696768y^8 + 4699313768927232y^9)$$

1 2	1 3	1 4	1 5	1 6	2 7	2 8	2 9	2 10	3 7	3 8
3 10	4 7	4 8	4 11	4 12	5 7	5 9	5 11	5 12	6 7	6 10
6 14	8 13	8 14	9 13	9 14	10 11	10 12	11 13	11 14	12 13	12 14
3 9	6 13									

$v = 14,\ k = 3$

$v = 14,\ k = 3,\ r = 9 + 39y,\ b = 42 + 182y$

The A-optimal design for $y = 0$

$$D(y) = 196(5 + 21y)(7261241 + 190959048y + 2089091088y^2 + 12168721152y^3 + 39801259008y^4 + 69303693312y^5 + 50185433088y^6)^2$$

$$R(y) = 49(15041242058752(6y + 1)^{12} + 69421117194240(6y + 1)^{11} + 145453769359360(6y + 1)^{10} + 182997842944000(6y + 1)^9 + 154015454154240(6y + 1)^8 + 91375075663872(6y + 1)^7 + 39194984588928(6y + 1)^6 + 12250310526720(6y + 1)^5 + 2769381094480(6y + 1)^4 + 441695197888(6y + 1)^3 + 47184347112(6y + 1)^2 + 3031627648(6y + 1) + 88606921)$$

1	2	11	1	2	12	1	3	10	1	3	13	1	4	9	1	4	14	1	5	6	1	5	8
2	3	14	2	4	13	2	5	9	2	6	8	2	7	8	2	7	10	2	9	10	3	4	6
3	7	11	3	8	9	3	8	11	3	9	12	4	5	7	4	7	12	4	8	10	4	8	12
5	9	13	5	10	13	5	11	12	5	11	14	6	8	13	6	9	12	6	10	11	6	10	14
7	9	14	7	13	14	8	13	14	9	11	14	10	12	13	11	12	13	1	6	7	3	5	7
6	12	14	4	10	11																		

The A-optimal design for $y > 0$ and the D- optimal design for $y \geq 0$

$$D(y) = 48384(3+14y)^2(19+84y)^4(4y+1)(21y+5)(84y+17)(3111696y^4+3037608y^3+1106469y^2+178227y+10712)$$

$$R(y) = 576(19+84y)^3(14y+3)(335665269209088y^8+629833458433536y^7+516247094877696y^6+241430704885440y^5$$
$$+70462039262328y^4+13141734113718y^3+1529650139955y^2+101592758694y+2947732943)$$

1	2	11	1	2	12	1	3	13	1	3	14	1	4	9	1	4	10	1	5	6	1	5	8
2	3	6	2	4	13	2	5	9	2	7	8	2	7	14	2	8	10	2	9	10	3	4	8
3	7	9	3	7	10	3	8	9	3	10	12	4	5	11	4	6	7	4	7	11	4	8	12
5	7	12	5	7	13	5	9	14	5	10	12	6	8	13	6	9	13	6	10	11	6	10	14
8	11	14	8	13	14	9	11	14	9	12	13	10	11	13	11	12	13	1	6	7	3	5	11
6	12	14	4	12	14																		

$\underline{v = 14,\ k = 3,\ r = 15 + 39y,\ b = 70 + 182y}$

$$D(y) = 21504(21y + 8)^2(336y^2 + 272y + 55)(148176y^3 + 169344y^2 + 64407y + 8153)^3$$

$$R(y) = 3584(21y + 8)(148176y^3 + 169344y^2 + 64407y + 8153)^2(13591888128y^5 + 26486756352y^4 + 20635286112y^3 + 8034060384y^2 + 1563157449y + 121591598)$$

1	2	9	1	2	9	1	2	14	1	3	7	1	3	7	1	3	13	1	4	6	1	4	6
1	5	8	1	5	8	1	5	11	1	10	11	1	10	13	1	12	14	2	3	12	2	3	12
2	4	13	2	5	10	2	5	10	2	6	7	2	6	7	2	6	14	2	7	8	2	8	11
3	4	8	3	4	8	3	5	9	3	5	9	3	6	10	3	6	10	3	6	13	3	9	11
3	11	14	4	5	7	4	5	7	4	7	9	4	9	10	4	9	10	4	11	12	4	11	14
5	6	11	5	6	14	5	12	13	5	12	13	5	12	14	5	13	14	6	8	12	6	8	12
6	9	12	6	11	13	7	8	13	7	9	14	7	10	12	7	10	12	7	10	14	7	11	13
8	9	12	8	9	13	8	9	14	8	10	13	8	10	14	9	13	14	10	11	12	1	4	12
2	4	13	2	8	11	3	10	14	4	11	14	6	9	11	7	11	13						

A.11 The best RGDs for $v = 15$

$v = 15,\ k = 2$

$v = 15,\ k = 2,\ r = 4 + 14y,\ b = 30 + 105y$

$$D(y) = 405(1+5y)^4(16+135y+225y^2)^2(2093+43734y+365715y^2+1576800y^3+3715875y^4+4556250y^5+2278125y^6)$$

$$R(y) = 810(1+5y)^3(4036552734375y^{10}+12570978515625y^9+17345643750000y^8+13950667968750y^7$$
$$+7235552812500y^6+2526299381250y^5+600833700000y^4+96044420250y^3+9870481125y^2+588739245y+15476912)$$

1	2	1	3	1	4	1	5	2	6	2	7	2	8	3	6	3	9	3	10	4	7
4	11	4	12	5	8	5	13	5	14	6	11	6	13	7	9	7	14	8	10	8	15
9	13	9	15	10	12	10	14	11	14	11	15	12	13	12	15						

$v = 15,\ k = 3$

$v = 15,\ k = 3,\ r = 9 + 28y,\ b = 45 + 140y$

$$D(y) = 405(6+5y)^4(2278125y^6 + 18225000y^5 + 60669000y^4 + 107565300y^3 + 107125740y^2 + 56819064y + 12538592)$$

$$R(y) = 810(6+5y)^3(4036552734375y^{10} + 52936505859375y^9 + 312129323437500y^8 + 1089657372656250y^7$$
$$+ 2494206523125000y^6 + 3911414275631250y^5 + 4255853431387500y^4 + 3172451381626500y^3 + 1550541443895000y^2$$
$$+ 448680771309120y + 58373047980032)$$

1	2	9	1	2	10	1	3	7	1	3	8	1	4	6	1	4	13	1	5	11	1	5	12
2	3	11	2	4	14	2	5	15	2	6	7	2	6	12	2	7	8	2	8	13	3	4	15
3	6	10	3	6	14	3	9	10	3	12	13	4	5	8	4	7	10	4	7	11	4	9	12
5	6	8	5	7	14	5	10	13	5	13	14	6	9	13	6	11	13	6	11	15	7	9	13
7	12	15	8	9	15	8	10	12	8	10	15	8	11	14	9	11	15	10	11	14	10	12	14
12	13	15	1	14	15	4	11	12	3	5	9	7	9	14									

A.12 The best RGDs for $v = 16$

$v = 16$, $k = 2$

$v = 16$, $k = 2$, $r = 3 + 15y$, $b = 24 + 120y$

$D(y) = 16384(8y+3)(32y^2+12y+1)^3(128y^2+48y+3)^4$

$R(y) = 196608(32y^2+12y+1)^2(128y^2+48y+3)^3(40960y^4+37888y^3+12352y^2+1664y+77)$

1	2	1	3	1	4	2	5	2	6	3	7	3	8	4	9	4	10	5	11	5	12	6	13
6	14	7	11	7	13	8	12	8	15	9	12	9	14	10	13	10	15	11	16	14	16	15	16

$v = 16,\ k = 3$

$v = 16,\ k = 3,\ r = 9 + 15y,\ b = 48 + 80y$

$D(y) = 16384(16y + 11)(128y^2 + 152y + 45)^3(512y^2 + 608y + 179)^4$

$R(y) = 196608(128y^2 + 152y + 45)^2(512y^2 + 608y + 179)^3(655360y^4 + 1613824y^3 + 1487104y^2 + 607744y + 92941)$

1	2	7	1	2	8	1	3	5	1	3	6	1	4	11	1	4	12	1	9	10	1	13	14
2	3	15	2	4	16	2	5	9	2	5	10	2	6	11	2	6	13	2	12	14	3	4	13
3	7	16	3	8	11	3	8	14	3	10	12	4	5	15	4	6	9	4	7	10	4	8	9
5	6	12	5	7	13	5	8	12	5	11	14	5	11	16	6	7	14	6	8	15	6	10	13
7	8	12	7	11	13	7	11	15	8	10	15	8	13	16	9	11	12	9	12	13	9	14	15
10	11	16	10	13	15	12	15	16	3	7	9	4	10	14	6	14	16	9	14	16	1	15	16

$v = 16,\ k = 4$

$v = 16,\ k = 4,\ r = 6 + 5y,\ b = 24 + 20y$

$D(y) = 16384(8y + 11)(32y^2 + 76y + 45)^3(128y^2 + 304y + 179)^4$

$R(y) = 196608(32y^2 + 76y + 45)^2(128y^2 + 304y + 179)^3(40960y^4 + 201728y^3 + 371776y^2 + 303872y + 92941)$

1	2	3	15	1	2	7	8	1	3	5	6	1	4	10	11	1	4	12	16	1	9	13	14
2	5	11	12	2	5	13	16	2	6	10	14	3	4	8	9	3	7	10	13	3	7	11	16
4	5	7	14	4	10	13	15	5	8	11	15	5	9	10	12	6	7	11	13	6	8	12	13
7	9	12	15	8	10	15	16	9	11	14	16	2	4	6	9	3	8	12	14	6	14	15	16

A.13 The best RGDs for $v = 18$

$v = 18,\ k = 2$

$v = 18,\ k = 2,\ r = 3 + 17y,\ b = 27 + 153y$

$D(x) = 162(9y+1)(324y^2+90y+5)^4(11664y^4+10368y^3+3240y^2+404y+15)^2$

$R(x) = 162(11664y^4+10368y^3+3240y^2+404y+15)$

$(578207808y^6+636784416y^5+279551088y^4+62239104y^3+7366140y^2+437130y+10115)$

1	2	1	3	1	4	2	5	2	6	3	7	3	8	4	9	4	10	5	11	5	12
6	14	7	11	7	13	8	15	8	16	9	12	9	14	10	17	10	18	11	17	12	15
14	16	15	18	16	17	6	13	13	18												

$\underline{v = 18,\ k = 2,\ r = 10 + 17y,\ b = 60 + 102y}$

$$D(x) = 324(9y+5)(1296y^2+1476y+419)^4(186624y^4+456192y^3+417312y^2+169288y+25691)^2$$

$$R(x) = 162(1296y^2+1476y+419)^3(186624y^4+456192y^3+417312y^2+169288y+25691)(37005299712y^6+131393000448y^5 + 194185444608y^4+152899550208y^3+67649405040y^2+15946520004y+1564595801)$$

1	2	7	1	2	8	1	3	5	1	3	6	1	4	11	1	4	12	1	9	10
1	13	14	1	15	16	1	17	18	2	3	9	2	4	13	2	5	10	2	5	14
2	6	11	2	6	12	2	15	17	2	16	18	3	4	14	3	7	10	3	7	12
3	8	11	3	8	13	3	15	18	3	16	17	4	5	6	4	7	8	4	9	15
4	9	16	4	10	17	4	10	18	5	7	9	5	8	17	5	11	13	5	11	18
5	12	15	5	12	16	6	7	17	6	8	9	6	10	14	6	13	16	6	13	18
6	14	15	7	11	14	7	11	16	7	13	15	7	13	18	8	10	16	8	12	15
8	14	16	8	15	18	9	11	17	9	12	13	9	12	14	9	14	18	10	11	15
10	12	18	10	13	17	11	12	17	14	16	17									

Bibliography

[AB08] P. Abramenko and K. S. Brown. *Buildings*. Number 248 in Graduate Texts in Mathematics. Springer, 2008.

[AM85] W. N. Anderson and T. D. Morley. Eigenvalues of the Laplacian of a graph. *Linear and Multilinear Algebra*, 18:141–145, 1985.

[ADH98] A. S. Asratian, T. M. J. Denley, and R. Häggkvist. *Bipartite graphs and their applications*. Number 131 in Cambridge Tracts in Mathematics. Cambridge University Press, 1998.

[BB01] B. Bagchi and S. Bagchi. Optimality of partial geometric designs. *Annals of Statistics*, 29(2):577–594, 2001.

[BC93] S. Bagchi and C.-S. Cheng. Some optimal designs of block size two. *Journal of Statistical Planning and Inference*, 37:245–253, 1993.

[Bai99] R. A. Bailey. Factorial designs, 1999. Preprint.

[Bai04] R. A. Bailey. *Association Schemes: Designed experiments, Algebra and Combinatorics*. Number 84 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2004.

[Bai07] R. A. Bailey. Designs for two-colour microarray experiments. *Applied Statistics*, 56(4):365–394, 2007.

[Bai08] R. A. Bailey. *Design of Comparative Experiments*. Number 25 in Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2008.

[BC07] R. A. Bailey and P. J. Cameron. What is a design? How should we classify them? *Designs, Codes and Cryptography*, 44:223–238, 2007.

[BC09] R. A. Bailey and P. J. Cameron. Combinatorics of optimal designs. In *Surveys in Combinatorics 2009*, number 365 in London Mathematical Society Lecture Notes, pages 19–73. Cambridge University Press, 2009.

[BC13] R. A. Bailey and P. J. Cameron. Using graphs to find the best block designs. In *Topics in Structural Graph Theory*, pages 282–317. Cambridge University Press, 2013.

[BR97] R. A. Bailey and G. Royle. Optimal semi-latin squares with side six and block size two. *Proceedings of the Royal Society London*, 453:1903–1914, 1997.

[Bai] R.A. Bailey. private communications.

[BB65] E. F. Beckenbach and R. Bellman. *Inequalities.* Number 30 in Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer Verlag, 1965.

[Big93] N. Biggs. *Algebraic Graph Theory.* Cambridge University Press, 1993.

[Big82] N. L. Biggs. Distance-regular graphs with diameter three. *Annals of Discrete Mathematics*, 15:69–80, 1982.

[Bol98] B. Bollobas. *Modern Graph Theory.* Number 184 in Graduate Text in Mathematics. Springer, 1998.

[Bos42] R. C. Bose. A note on the resolvability of balanced incomplete block designs. *Sankhya*, 6:105–110, 1942.

[Bos47] R. C. Bose. Mathematical theory of the symmetrical factorial design. *Sankhya: The Indian Journal of Statistics*, 8(2):107–166, 1947.

[Bos63] R. C. Bose. Strongly regular graphs, partial geometries and partially balanced designs. *Pacific Journal of Mathematics*, 13(2):389–419, 1963.

[Bos75] R. C. Bose. Designs and multigraphs. *Sankhya: The Indian Journal of Statistics*, 37(3):315–333, 1975.

[Bro06] B. P. Brooks. The coefficients of the characteristic polynomial in terms of the eigenvalues and the elements of an $n \times n$ matrix. *Applied Mathematics Letters*, 19:511–515, 2006.

[Bro84] A. E. Brouwer. Distance regular graphs of diameter 3 and strongly regular graphs. *Discrete Mathematics*, 49:101–103, 1984.

[BCN89] A. E. Brouwer, A. Cohen, and A. Neumaier. *Distance-Regular Graphs*. Number 18 in Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer-Verlag, 1989.

[Bru06] B. De Bruyn. *Near Polygons*. Frontiers in Mathematics. Birkäuser Verlag, 2006.

[CvL91] P. J. Cameron and J. H. van Lint. *Designs, Graphs, Codes and their Links*. Number 22 in London Mathematical Society Student Texts. Cambridge University Press, 1991.

[Cam] P.J. Cameron. private communications.

[Cha60] L.-C. Chang. Associations of partially balanced designs with parameters $v = 28$, $n_1 = 12$, $n_2 = 15$, and $p^2_{11} = 4$. *Science Record New Series*, 4:12–18, 1960.

[CK74] V. M. Chelnokov and A. K. Kelmans. A certain polynomial of a graph and graphs with an extremal number of trees. *Journal of Combinatorial Theory*, 16:197–214, 1974.

[Che78] C.-S. Cheng. Optimality of certain asymmetrical experimental designs. *The Annals of Statistics*, 6:1239–1261, 1978.

[Che81a] C.-S. Cheng. Maximizing the total number of spanning trees in a graph: Two related problems in graph theory and optimum design theory. *Journal of Combinatorial Theory*, 31:240–248, 1981.

[Che81b] C.-S. Cheng. On the comparison of PBIB designs with two associate classes. *Annals of the Institute of Statistical Mathematics*, 33:155–164, 1981.

[Che92] C.-S. Cheng. On the optimality of (M.S)-optimal designs in large systems. *Sankhya*, 54:117–125, 1992.

[CB91] C.-S. Cheng and R. A. Bailey. Optimality of some two-associate-class partially balanced incomplete-block designs. *The Annals of Statistics*, 19(3):1667–1671, 1991.

[CM85] C.-S. Cheng and J. C. Masaro. Do nearly balanced multigraphs have more spanning trees? *Journal of Graph Theory*, 8:342–345, 1985.

[Cla73] W. H. Clatworthy. *Tables of Two-associate-class Partially Balanced Designs*. Number 63 in NBS Applied Mathematics Series. The United Stated Department of Commerce Publications, National Bureau of Standards (U.S.), 1973.

[Con58] W. S. Connor. The uniqueness of the triangular association scheme. *Annals of Mathematical Statistics*, 29:262–266, 1958.

[CH54] W. S. Connor and M. Hall. An embedding theorem for balanced incomplete block designs. *Canadian Journal of Mathematics*, 6:35–41, 1954.

[Con86] G. M. Constantine. On the optimality of block designs. *Annals of the Institute of Statistical Mathematics*, 38:161–174, 1986.

[CDS79] D. Cvetkovic, M. Doob, and H. Sachs. *Spectra of Graphs: Theory and Application.* Academic Press, 1979.

[CRS10] D. Cvetkovic, P. Rowlinson, and S. Simic. *An Introduction to the Theory of Graph Spectra.* Number 75 in London Mathematical Society Student Texts. Cambridge University Press, 2010.

[Dao12] S. N. Daoud. Some applications of spanning trees in complete and complete bipartite graphs. *American Journal of Applied Sciences*, 9(4):584–592, 2012.

[Dar64] J. N. Darroch. On the distribution of the number of successes in independent trials. *Annals of Mathematical Statistics*, 35(3):1317–1321, 1964.

[DP65] J. N. Darroch and J. Pitman. Note on a property of the elementary symmetric functions. *Proceedings of the American Mathematical Society*, 16:1132–1133, 1965.

[DB05] K. Ch. Das and R. B. Bapat. A sharp upper bound on the largest Laplacian eigenvalue of weighted graphs. *Linear Algebra and its Applications*, 409:153–165, 2005.

[DSV03] G. Davidoff, P. Sarnak, and A. Valette. *Elementary Number Theory, Group Theory and Ramanujan Graphs*. Number 55 in London Mathematical Society Student Texts. Cambridge University Press, 2003.

[Eis99] J. Eisfeld. The eigenspaces of the Bose-Mesner-Algebra of the association schemes corresponding to projective spaces and polar spaces. *Designs, Codes and Cryptography*, 17:129–150, 1999.

[Fie75] M. Fiedler. Algebraic connectivity of graphs. *Czechoslovak Mathematical Journal*, 23(2):298–305, 1975.

[FG06] M. A. Fiol and E. Garriga. On the spectrum of an extremal graph with four eigenvalues. *Discrete Mathematics*, 306:2241–2244, 2006.

[Gaf82] N. Gaffke. D-optimal block designs with at most six varieties. *Journal of Statistical Planning and Inference*, 6:183–200, 1982.

[GB06] A. Ghosh and S. Boyd. Upper bounds on algebraic connectivity via convex optimization. *Linear Algebra and its Applications*, 418:693–707, 2006.

[GW81] A. Giovagnoli and H. P. Wynn. Optimum continuous block designs. *Proceedings of the Royal Society London*, 377:405–416, 1981.

[GP70] H. J. Greenberg and W. P. Pierskalla. Symmetric mathematical programs. *Management Science*, 16(5):309–312, 1970.

[Hae80] W. H. Haemers. *Eigenvalue Techniques in Design and Graph Theory*. Number 121 in Mathematical Centre Tracts. Amsterdam: Mathematisch Centrum, 1980.

[Hae95] W. H. Haemers. Interlacing eigenvalues and graphs. *Linear Algebra and its Applications*, 227–228:593–616, 1995.

[Hir79] J. W. P. Hirschfeld. *Projective Geometries Over Finite Fields*. Clarendon Press, Oxford, 1979.

[Hof60] A. J. Hoffman. On the uniqueness of the triangular association scheme. *Annals of Mathematical Statistics*, 31:492–497, 1960.

[Jac78] M. Jacroux. On the properties of proper (M,S) optimal block designs. *The Annals of Statistics*, 6:1302–1309, 1978.

[Jac89] M. Jacroux. Some sufficient conditions for type 1 optimality with applications to regular graph designs. *Journal of Statistical Planning and Inference*, 23:193–215, 1989.

[JL04] Y. Jin and C. Liu. Enumeration for spanning forests of complete bipartite graphs. *Ars Combinatoria*, 70:135–138, 2004.

[JM76] J. A. John and T. J. Michell. Optimal incomplete block designs. *ORNL/CSD-8*, 1976. Available from the National Technical Information Service, The United States Department of Commerce, 5285 Port Royal Road, Springfield, VA. 22151.

[JM77] J. A. John and T. J. Mitchell. Optimal incomplete block designs. *Journal of the Royal Statistical Society*, 39B:39–43, 1977.

[JW82] J. A. John and E. R. Williams. Conjectures for optimal block designs. *Journal of the Royal Statistical Society*, 44B:221–225, 1982.

[JWD72] J. A. John, F. W. Wolock, and H. A. David. *Cyclic Designs*. Number 62 in NBS Applied Mathematics Series. The United States Department of Commerce Publications, National Bureau of Standards (U.S.), 1972.

[Kan86] W. M. Kantor. Generalized polygons, SCABs and GABs. In *Buildings and the Geometry of Diagrams, Como 1984*, number 1181 in Lecture Notes in Mathematics, pages 79–156. Springer, 1986.

[Kie75] J. Kiefer. Optimality and construction of generalized Youden designs. In *A Survey of Statistical Designs and Linear Models*, pages 333–354. North-Holland, Amsterdam, 1975.

[Kir00] S. Kirkland. A bound on algebraic connectivity of graphs in terms of the number of cutpoints. *Linear and Multilinear Algebra*, 47:93–103, 2000.

[Kir01] S. Kirkland. An upper bound on algebraic connectivity of graphs with many cutpoints. *Electronical Journal of Linear Algebra*, 8:94–109, 2001.

[LLT05] M. Lu, H. Liu, and F. Tian. Bounds of Laplacian spectrum of graphs based on the domination number. *Linear Algebra and its Applications*, 402:390–396, 2005.

[Mac95] I. G. MacDonald. *Symmetric Functions and Hall Polynomials.* Oxford Mathematical Monographs. Oxford Science Publications, 1995.

[Mar61] M. Marcus. Comparison theorems for symmetric functions of characteristic roots. *Journal of Research of the National Bureau of Standards*, 65B(2):113–115, 1961.

[MO79] A. W. Marshall and I. Olkin. *Inequalities: Theory of Majorization and its Applications.* Number 143 in Mathematics in Science and Engineering. Academic Press, 1979.

[McK81] B. D. McKay. Practical graph isomorphism. *Congressus Numerantium*, 30:45–87, 1981.

[Meh99] M. Mehringer. Fast generation of regular graphs and construction of cages. *Journal of Graph Theory*, 30:137–146, 1999.

[Mor07] J. P. Morgan. Optimal incomplete block designs. *Journal of the American Statistical Association*, 102:655–663, 2007.

[Mor11] J. P. Morgan. private communications, 2011.

[Neu80] A. Neumaier. Distances, graphs and designs. *European Journal of Combinatorics*, 1:163–174, 1980.

[NP06] S. D. Nikolopoulos and C. Papadopoulos. On the number of spanning trees of $K_n^m \pm G$ graphs. *Discrete Mathematics and Theoretical Computer Science*, 8:235–248, 2006.

[Num85] M. Numata. On the graphical characterization of the projective space over a finite field. *Journal of Combinatorial Theory*, 38:143–155, 1985.

[Num90] M. Numata. A characterization of Grassmann and Johnson graphs. *Journal of Combinatorial Theory*, 48:178–190, 1990.

[Pan10] M. Pankov. *Grassmannians of classical buildings*. Number 2 in Algebra and Discrete Mathematics. World Scientific, 2010.

[Pas94] A. Pasini. *Diagram Geometries*. Clarendon Press, Oxford, 1994.

[PW75] H. D. Patterson and E. R. Williams. Some theoretical results on general block designs. In *Proceedings of the 5th British Combinatorial Conference*, pages 489–496, 1975.

[PBS98] L. Petingi, F. Boesch, and C. Suffel. On the characterization of graphs with a maximum number of spanning trees. *Discrete Mathematics*, 179:155–166, 1998.

[PR02] L. Petingi and J. Rodriguez. A new technique for the characterization of graphs with a maximum number of spanning trees. *Discrete Mathematics*, 244:351–373, 2002.

[Rie12] C. Riener. On the degree and half-degree principle for symmetric polynomials. *Journal of Pure and Applied Algebra*, 216:850–856, 2012.

[Ron89] M. Ronan. *Lectures on Buildings*. Academic Press, 1989.

[SS89] K. R. Shah and B. K. Sinha. *Theory of Optimal Designs*. Number 54 in Lecture Notes in Statistics. Springer, 1989.

[Shi74] D. R. Shier. Maximizing the number of spanning trees in a graph with n nodes and m edges. *Journal of Research of the National Bureau of Standards*, 78B(4):193–196, 1974.

[Shr59] S. S. Shrikhande. On a characterization of the triangular association scheme. *Annals of Mathematical Statistics*, 30:39–47, 1959.

[SS81] S. S. Shrikhande and N. M. Singhi. Designs, adjacency multigraphs and embeddings: a survey. In *Combinatorics and Graph Theory*, number 885 in Lecture Notes in Mathematics, pages 113–132. Springer Verlag, 1981.

[Soi06] L. H. Soicher. *The DESIGN package for GAP, Version 1.4*. http://designtheory.org/software_design/, 2006.

[Sol92] L. Soltes. Regular graphs with regular neighbourhoods. *Glasgow Mathematical Journal*, 34:215–218, 1992.

[Spr78] A. P. Sprague. Characterization of projective graphs. *Journal of Combinatorial Theory*, 24:294–300, 1978.

[SI09] D. Stevanovic and A. Ilic. On the Laplacian coefficients of unicyclic graphs. *Linear Algebra and its Applications*, 430:2290–2300, 2009.

[SWY11] X.-T. Su, Y. Wang, and Y.-N. Yeh. Unimodality problems of multinomial coefficients and symmetric functions. *The Electronic Journal of Combinatorics*, 18:1–8, 2011.

[Tak61] K. Takeuchi. On the optimality of certain type of PBIB designs. *Reports of Statistical Application Research. Union of Japanese Scientists and Engineers*, 8:140–145, 1961.

[Tam77] A. Tamir. Ergodicity and symmetric mathematical programs. *Mathematical Programming*, 13:81–82, 1977.

[Tit74] J. Tits. *Buildings of Spherical Type and Finite BN-pairs*. Number 368 in Lecture Notes in Mathematics. Springer, 1974.

[Tit86] J. Tits. Immeubles de type affine. In *Buildings and the Geometry of Diagrams, Como 1984*, number 1181 in Lecture Notes in Mathematics, pages 157–190. Springer, 1986.

[vDS98] E. R. van Dam and E. Spence. Small regular graphs with four eigenvalues. *Discrete Mathematics*, 189:233–257, 1998.

[WPJ76] E. R. Williams, H. D. Patterson, and J. A. John. Resolvable designs with two replications. *Journal of the Royal Statistical Society*, 38B(3):296–301, 1976.

[YYY07] W. Yi, F. Yizheng, and T. Yingying. On graphs with three distinct Laplacian eigenvalues. *Applied Mathematics: A Journal of Chinese Universities*, 22(4):478–484, 2007.

[YK06] Y. S. Yoon and J. K. Kim. A relationship between bounds on the sum of squares of degrees of a graph. *Journal of Applied Mathematics and Computing*, 21(1-2):233–238, 2006.

[Zha11] X.-D. Zhang. The Laplacian eigenvalues of graphs: a survey. *eprint arXiv:1111.2897*, 2011.