# Algebraic number-theoretic properties of graph and matroid polynomials

Adam Stuart Bohn

*School of Mathematical Sciences*

*Queen Mary, University of London*

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*To Mum, who would have been proud*

**Abstract**

This thesis is an investigation into the algebraic number-theoretical properties of certain polynomial invariants of graphs and matroids.

The bulk of the work concerns chromatic polynomials of graphs, and was motivated by two conjectures proposed during a 2008 Newton Institute workshop on combinatorics and statistical mechanics. The first of these predicts that, given any algebraic integer, there is some natural number such that the sum of the two is the zero of a chromatic polynomial (chromatic root); the second that every positive integer multiple of a chromatic root is also a chromatic root. We compute general formulae for the chromatic polynomials of two large families of graphs, and use these to provide partial proofs of each of these conjectures. We also investigate certain correspondences between the abstract structure of graphs and the splitting fields of their chromatic polynomials.

The final chapter concerns the much more general multivariate Tutte polynomials—or Potts model partition functions—of matroids. We give three separate proofs that the Galois group of every such polynomial is a direct product of symmetric groups, and conjecture that an analogous result holds for the classical bivariate Tutte polynomial.

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# Contents

# List of Figures

# Declaration of collaboration

The results presented in Chapter 7 of this thesis were obtained collaboratively: this chapter is essentially an extended version of [3], co-authored with Prof. Peter J. Cameron and Prof. Peter Müller. I was the corresponding author for this paper, and responsible for the editing, revising and final write-up. However, the main result of this chapter is likely to not to have been achieved at all without the timely intervention of Prof. Müller, who contributed those of Lemmas 3.2—3.7 which constitute original material, as well as the first proof of Theorem 3.1. Additionally, while the idea for the third proof was my own, the details were largely provided by Prof. Müller. As my supervisor, Prof. Cameron was the source of much advice and guidance.

# Chapter 1

# Introduction

## 1.1 Chromatic roots as algebraic integers

The work in the first few chapters of this thesis is largely inspired by the question of which algebraic integers are zeros of chromatic polynomials of graphs. More generally, we are interested in which algebraic extensions of the rational numbers are splitting fields of chromatic polynomials, and ways in which these fields correspond to the structure of the graphs they are derived from.

We begin by providing a brief analysis of why we chose to study the chromatic polynomial, and a survey of related work on its zeros.

### 1.1.1 The chromatic polynomial

Quite apart from its applicability to the wide range of practical problems that can be couched in graph colouring terms, the chromatic polynomial is a powerful tool for the study of graphs, encoding deep graph-theoretic properties with a degree of structure and order high enough to enable methodical and rigorous mathematical investigation. Furthermore, its significance outside of combinatorics extends much further than practical map-colouring and timetabling problems: its complex zeros are of some importance in physics in relation to the Potts model (see, for example, [42]), and a proof was recently given by Huh [20] of Welsh's long-standing conjecture [49, Exercise 5, p2661] that the absolute values of a chromatic polynomial's coefficients form a log-concave sequence[1], which exploits deep links with branches of topology and algebraic geometry.

The non-integer zeros of chromatic polynomials have been the subject of quite intensive study, and our understanding of their analytic distribution is essentially complete. In the case of real chromatic roots this is largely due to the combined efforts of Jackson [21] and Thomassen [47], who between them classified every zero-free interval for the chromatic polynomial. As for complex roots, it was conjectured as recently as 1980 by Farrell [17] that there were no chromatic roots with negative real part. This was a commonly held view: small graphs are misleading in this context, and the difficulty of computing the chromatic polynomial meant that for a long time it was not possible to directly compute larger examples. However, a counterexample was found 8 years later by Read and Royle [37], and a series of papers in the last decade of the 20th century made incremental progress in revealing the ubiquity of complex chromatic roots, culminating in Sokal's proof [43] that they are, in fact, dense in the whole complex plane. Adding to the impact of this revelation was the fact that the graphs used in the proof comprise a relatively small and special family of graphs (a significant generalisation of these is studied in §3.2.1).

### 1.1.2 Two conjectures on chromatic roots

Sokal's result provides us with good reason to devote significant attention to the study chromatic roots, however it does not actually provide us with a single such number. Indeed, our knowledge of which algebraic integers are chromatic roots is still paltry at best (as some marker of how ignorant we are in this regard, it is not yet even known if $i = \sqrt{-1}$ is a chromatic root). This issue is a large motivating factor of the work presented here, although as we shall explain we approach it quite indirectly.

The work presented in Chapters 3–6 was directly motivated by two conjectures, which were proposed at a 2008 Newton workshop on combinatorics and statistical mechanics, and were stated for the first time in [8].

[1] Welsh's well-known conjecture was actually formulated for the more general characteristic polynomial of a matroid. Its proof settles an even older conjecture of Read [34] that, in the special case of the chromatic polynomial, this sequence is unimodal

The "$\alpha + n$ conjecture" suggests that, given any algebraic integer $\alpha$, we can always find some $n \in \mathbb{N}$ such that $\alpha + n$ is a chromatic root; while the "$n\alpha$ conjecture" asserts that any positive integer multiple of a chromatic root is also a chromatic root. Of these two, the first is perhaps of more significance, as a proof would imply that every algebraic integer—and thus every number field—is contained in the splitting field of a chromatic polynomial.

In order to address these conjectures, we first compute the chromatic polynomials of some large families of graphs, resulting in general formulae which may be of independent interest to researchers in this area. We use one of these families—complements of bipartite graphs that we refer to as bicliques—to prove the first two non-trivial cases of the $\alpha + n$ conjecture, showing that it holds for both quadratic and cubic integers. We go on to analyse members of this family whose chromatic polynomials have the same splitting field, rediscovering an old result in the process, and showing some interesting connections with other combinatorial objects such as matchings and rook polynomials.

The second family of chromatic polynomials we investigate is, to our knowledge asymptotically the largest for which we now have a closed general formula. It contains as subfamilies both the generalised theta graphs used by Sokal to prove his density result, and another family known as "rings of cliques", whose chromatic polynomials have also been the subject of some interest ([36, 14]). We show that the chromatic polynomials of these "clique-theta graphs" have zeros satisfying the hypotheses of the $n\alpha$ conjecture, by proving that the set of chromatic roots of these graphs is closed under multiplication by positive integers. Combined with Sokal's result, this implies that the set of chromatic roots for which this conjecture holds forms a dense subset of the complex plane.

## 1.2 Galois groups of Tutte polynomials

In Chapter 7 of this thesis we study the Galois groups of multivariate Tutte polynomials of connected matroids. Although this at first appears to be quite a departure from the study of zeros of chromatic polynomials, there are nevertheless many links between the two subjects: Tutte polynomials of matroids are direct generalisations of chromatic polynomials of graphs, and we retain a similar focus on algebraic number-theoretical aspects of these polynomials.

This work in fact began with consideration of which groups which appear as Galois groups of the chromatic polynomial. It has been observed that, in the case of small graphs, a wide range of groups appear in this context ([30, 8]). In an effort to learn more about the Galois theory of the chromatic polynomial [2], we performed similar computations for the Tutte polynomial. To our surprise, we found that the groups appearing for graphs with biconnected components of order 10 or less are always direct products of symmetric groups of corresponding degrees. This genericity implies something quite special about the Tutte polynomial, and we conjectured that only direct products of symmetric groups can appear.

This conjecture is still open, however we were able to prove an analogous result for a further generalisation, which essentially encodes all the information about a graph: this polynomial is known to combinatorialists as the multivariate Tutte polynomial, and to physicists as the Potts model partition function. We were able to extend our proof (that only direct products of symmetric groups appear as Galois groups) to all matroids, and also to verify that our results hold regardless of the characteristic of the ground field. This is certainly notable: it proves this polynomial is truly "generic", which is in some sense to be expected, but surprising nonetheless, given the sheer number and variety of them (no two matroids have the same multivariate Tutte polynomial).

[2]as the Tutte polynomial is a bivariate generalization of the chromatic polynomial, its Galois groups contain those of the corresponding chromatic polynomials as subgroups

# Chapter 2

# Definitions and notation

There are a number of mathematical subfields which contribute material to this thesis; we shall address each in turn.

## 2.1 Graph Theory

Graphs are the central object of study in this thesis. A graph $G$ consists of a *vertex-set* $V(G)$ and an *edge-set* $E(G)$, where elements of the latter are simply subsets of $V(G)$ of size 2. We will normally write $G = (V, E)$, and, when it is otherwise clear which graph we are referring to, we shall denote sets of vertices and edges by $V$ and $E$ respectively. It is implicit in our definition that graphs as we define them are *simple*, that is they do not have loops (edges whose endpoints are identified) or multiple edges (edges incident to the same pair of vertices); however it should be noted that graphs having both loops and multiple edges will implicitly be included in any discussion of the Tutte or multivariate Tutte polynomials. The glossary of graph-theoretic terminology is enormous, and notation varies considerably between sources. So here we give only the most fundamental definitions; others will be defined as and when they are required. For a general source, we refer the reader to [4].

### 2.1.1 Basic definitions

Let $G = (V, E)$ be a graph. The *order* and *size* of $G$ are, respectively, the number of vertices $|V|$ and the number of edges $|E|$. Unless specifically assigned a label, an edge of $G$ will be represented by the juxtaposition of the two vertices it connects. Two vertices are *adjacent* if they are joined by an edge, so we have that $u, v \in V$ are adjacent if and only if $uv \in E$. We say that the edge $uv$ is *incident* to the vertices $u$ and $v$ (and vice-versa).

The *neighbours* of $v \in V$ comprise the set of vertices adjacent to $v$, and the *degree* of $v$ is the cardinality of this set. A *clique* of $G$ is a subset of $V$ whose elements are pairwise adjacent, and a $k$-*clique* is a clique of cardinality $k$. If $G$ contains all $\binom{n}{2}$ possible edges then $G$ is the *complete graph* $K_n$ (we may also write that $G$ is an $n$-clique.)

A *subgraph* of $G$ is a graph $G' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$. In the case that $V' = V$ we say that $G'$ is a *spanning subgraph*[1] of $G$, or else that $G'$ *spans* $V$. Given some $W \subseteq V$, the *induced subgraph* $G[W]$ of $G$ has vertex set $W$ and edge-set $\{uv \in E : u, v \in W\}$. A subgraph of $G$ may also be induced by a subset of $F \subseteq E$, in which case its vertex set consists of every vertex $v \in V$ which is incident to at least one $e \in F$.

The *complement* $\bar{G} = (V, \bar{E})$ of $G$ (where $\bar{E} = E(K_n) \setminus E$) is the graph obtained by replacing all edges with non-edges, and vice versa. For example, the complement of $K_n$ is the *null graph* $N_n$ having $n$ vertices and no edges.

### 2.1.2 Connectivity

There are various notions of connectedness for graphs; we will not discuss these here, aside from remarking that the definitions we give are by no means universal (in particular, we are concerned with *vertex*-connectivity; for brevity we will omit the qualifier). A *path* between two vertices $v_1, v_k \in V$ is a sequence $v_1, v_2, \ldots, v_k$ of distinct vertices such that $v_i v_{i+1} \in E$ for all $1 \leq i \leq k - 1$. The number of vertices in a path, including endpoints, will be referred to as the *length* of the path (note that in some of the literature this term is used in reference to edges, of which there are of course one less than vertices).

In the case that $v_1$ is adjacent to $v_k$, such a sequence is known as a

[1] The notion of "spanning" has nothing to do with edges: some vertices (so-called "singletons") in a spanning subgraph may have no neighbours, and thus no incident edges.

$k$-*cycle*, or when length is unspecified, simply as a *cycle*. Any additional edge $v_iv_j$ with $j$ not equivalent to $i \pm 1$ modulo $k$ is known as *chord*, and a cycle with no chords is a *simple cycle*. As with cliques, we use the terms path and cycle to refer both to subgraphs of a larger one, and stand-alone graphs; this is discussed below.

A graph is *connected* if there exists a path between any two of its vertices (otherwise it is *disconnected*); and *biconnected* if any two vertices are contained in some simple cycle (or equivalently, if it cannot be made disconnected by the removal of a single vertex). If we consider connectivity and biconnectivity to be shorthand for, respectively, 1- and 2-connectivity, then we can extend this definition arbitrarily: a graph is $k$-*connected* if it cannot be made disconnected by the removal of $k-1$ vertices (by convention, the complete graph $K_n$ is considered to be $(n-1)$-connected). Clearly then, a $k$-connected graph is also $j$-connected for all $j < k$. If $G$ is at most $(k-1)$-connected, but has one or more vertex-induced subgraphs which are $k$-connected, then we refer to these as the $k$-connected *components* of $G$ (note that for $k > 1$ these components are not necessarily disjoint). The biconnected parts of a connected graph are known as *blocks*, and when not otherwise qualified a *component* of $G$ is a connected component (and $G$ is by implication disconnected).

### 2.1.3 Joins, separations and alterations

There are various ways in which to join, separate or otherwise alter graphs. *Deletion* of an edge $e$ of $G$ simply results in the graph $G - e = (V, E \setminus \{e\})$; if $e \notin E$ then the *addition* of $e$ gives the graph $G + e = (V, E \cup \{e\})$. Deletion or removal of vertices is similarly self-explanatory, although note that when we remove a vertex $v$ from $G$ we implicitly also delete all edges incident to $v$.

Now suppose that $e = uv$ for some $u, v \in V$. Then the *contraction* of $e$ is defined to be the identification of $u$ and $v$ and deletion of any resulting loops or multiple edges (equivalently, the replacement of $u$ and $v$ by a single new vertex $w$ whose set of neighbours is the union of those of $u$ and $v$). We denote the graph produced by this operation as $G/e$. The concepts of contraction and deletion are important in the study of graph polynomials, as we shall see.

If there are graphs $G_1$ and $G_2$ such that $V(G_1) \cup V(G_2) = V(G)$, and the subgraph of $G$ induced by $V(G_1) \cap V(G_2)$ is an $r$-clique for some $r \in \mathbb{N}$, then we say that $G$ is a *$K_r$-sum* of $G_1$ and $G_2$. So, for example, a $K_2$-sum comprises two graphs which intersect in a single edge. When it is not necessary to specify the size of the clique, we may simply write that such a graph is *clique-separable*. Finally, if a graph $G$ consists of two subgraphs $G_1$ and $G_2$, with every possible edge between them, then we write that $G$ is the *join* of $G_1$ and $G_2$.

There are a bewildering number of named graphs; we shall only require the very simplest of these. After the complete and null graphs, probably the easiest to describe is an *$n$-path* $P_n$, which is simply a path of $n$ vertices (and thus $n - 1$ edges). Adding a single extra edge between the endpoint vertices of $P_n$ produces an *$n$-cycle* $C_n$, and a generalisation of a path is a *tree*: a connected graph containing no cycles. Finally, a *bipartite graph* is one whose vertex set can be partitioned into two subsets, each of which has no internal edges.

### 2.1.4 Labellings and isomorphisms

The ease of representing graphs simply by connecting dots can make it quite easy to lose sight of their abstract nature. It is natural for us to conflate a graph with some representation, which is in fact just one of uncountably many.

Indeed, it seems quite unfortunate that graphs have been left with the name they have: to non-mathematicians a graph is invariably a graphical representation of the range of some function; while to mathematicians, a graph is a purely abstract construction. Having said this, graphs have the property of being both easily approachable and immediately applicable, something that is perhaps testament to our intuitive understanding of networks.

In order to demonstrate quite how intangible a graph is, it is useful to consider for a moment what a vertex actually *is*. Given that the vertices of a graph are considered to form a set, there must be some way to distinguish between them, yet the only distinguishing property they have is their interconnectivity, a property which can only be described in two ways: either as a set of adjacent vertices, or as a set of of incident edges. Both of these lead to recursive definitions, and anyway do not address the problem of a graph having two distinct vertices with the same set of neighbours. We also need a convenient way to refer to specific vertices, and the only sensible way to deal with these problems is to label the vertices arbitrarily, for example as $v_1$ up to $v_n$.

However, given our assertion that a vertex is defined entirely by its adjacency relations, we must be careful to keep in mind that these labels are just reference points, and not intrinsic properties of vertices. Indeed, a *labelled graph* is quite a different object from the (unlabelled) graphs we shall be considering; especially as regards enumeration (as a simple example, there are 3 possible ways in which to label the vertices of a path of length 3, and thus 3 labelled versions of $P_3$). It is, of course, highly impractical to study unlabelled graphs without some way to refer to individual vertices. In practice we get around this by assigning a different "placeholder" label to each vertex, and representing the vertex set as the set of these labels; thus if two vertices share the same set of neighbours then their labels are interchangeable.

In order to make this system rigorous, we need the following definition:

**Definition 2.1.** Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there exists a bijective map $f : V_1 \to V_2$ such that $uv \in E_1$ if and only if $f(u)f(v) \in E_2$. If this is the case then $f$ is a *graph isomorphism*, and we write $G_1 \cong G_2$. The set of all graphs isomorphic to $G$ is known as the *isomorphism class* of $G$.

Now we can review our previous statement, and give a more precise definition of the graphs we shall be working with.

**Definition 2.2.** Let $V = \{1, 2, \dots, n\}$ and $E$ be some set of subsets of $V$ of size two. The *unlabelled graph* $G = (V, E)$ is the isomorphism class of all labelled graphs having vertex set $V$ and edge-set $E$.

This implies that if a property of a labelled graph is invariant under isomorphism, then it can be considered to be a property of the underlying unlabelled graph.

All graphs discussed in this thesis will be unlabelled; any labels subsequently given to vertices or edges are thus simply placeholders. This include the indeterminates assigned to each edge by the multivariate Tutte polynomial, which is the subject of 7: as these are by definition transcendental over the ground field, they are essentially interchangeable, in that any permutation of them will not affect any of the properties of the corresponding multivariate Tutte polynomial.

## 2.2 Matroid Theory

### 2.2.1 Introduction

Matroids are abstractions of the notions of rank and dependence which appear in association with many different combinatorial and algebraic structures. In particular, they are usually associated with vector spaces and matrices—for which rank/dimension and linear dependence need no introduction—and graph theory, where similar notions are often implicit, if not stated outright. As such, their terminology is mainly taken from these two subjects, and is in many cases self-explanatory for a graph theorist. However, matroids are much less intuitive than graphs, and certainly they are less amenable to representation (indeed some are not representable at all—currently some of the most important work currently being done in the field is towards characterisations of those that are, see for example [52] and [53]).

Graph theoretic results can often be generalised to matroids with very little effort, and this is essentially what we do in Chapter 7. No other part of this thesis explicitly uses matroid theory, and thus we shall not dwell long on the associated terminology. In particular the only special subfamily we will address are the *graphic matroids* of graphs.

For background and a thorough treatment of matroid theory we refer the reader to Oxley's book [32], while Chapter 6 of [50] provides a useful reference on the matroid theoretic Tutte polynomial which is the object of our focus.

### 2.2.2 Basic definitions

**Definition 2.3.** A *matroid* is an ordered pair $M = (E, \mathfrak{I})$, where $E$ is some finite set, and $\mathfrak{I}$ is a collection of subsets of $E$[2] satisfying the following conditions:

1. $\emptyset \in \mathfrak{I}$
2. If $I \in \mathfrak{I}$, then so too is every subset of $I$
3. If $I_1, I_2 \in \mathfrak{I}$, and $|I_1| > |I_2|$, then there is some $e \in I_1 \setminus I_2$ such that $I_2 \cup e \in \mathfrak{I}$

We refer to $E$ as the *ground set*, and write that $M$ is a finite matroid on the set $E$. The elements of $\mathfrak{I}$ are the *independent sets* of $M$, and any subset of $E$ not contained in $\mathfrak{I}$ is said to be *dependent*. These conditions can be seen to capture in set-theoretic language the essence of a mathematical dependence relations. Unless otherwise mentioned, for the remainder of this section $M$ will be assumed to be some finite matroid with ground-set $E$.

As mentioned above, the terminology of matroid theory largely comes from linear algebra and graph theory, thus it will help to see an example from each subject before continuing.

**Example 2.4.** Let $E$ be a subset of any vector space $V$, and let the independent subsets of $E$ be those which are linearly independent in $V$. The first two axioms of Definition 2.3 are obvious, and the third follows from the Steinitz Exchange Lemma; the resulting matroid $M(E)$ is known as a *vector matroid*. Note that a vector matroid can be defined on any set of vectors of given dimension over some specified field. Thus by dispensing with all structure but that endowed by linear dependence, we arrive at a significant generalisation of vector spaces.

The most important family for our purposes, however, comes from graph theory:

[2]Note that we are concerned here only with subsets of $E$, rather than individual elements. However, for ease of expression and notation it is common practice to equate any subset of size one with its content, for example by writing $e \in E$ as opposed to $\{e\} \subseteq E$, and by referring to subsets of cardinality 1 as elements of $E$.

**Example 2.5.** The *cycle matroid* $M(G)$ of a graph $G = (V, E)$ takes as its ground-set the edge-set $E$. In this case a subset of $E$ is defined to be independent if it contains no cycles. The family of *graphic matroids* comprises every matroid which is the cycle matroid of some graph.

The remaining terminology becomes more easily comprehensible in light of these examples. Let $M = (E, \mathfrak{I})$ be a matroid. The *rank* $r(M)$ of $M$ is defined to be the cardinality of the largest independent set of $M$, and similarly the *rank function* of $M$ is the map $r_M : 2^E \to \mathbb{N}$ whose value on each $F \subseteq E$, is the cardinality of the largest independent set of $F$. With this notation we have that $r_M(E) = r(M)$. Continuing to follow the terminology of vector spaces, we write that a set $B$ is a *basis* of $M$ if it is a minimal spanning set (that is, if $|B| = r_M(B) = r(M)$).

### 2.2.3 Graph-theoretic analogues

For the next few definitions it is convenient to keep the example of a graph in mind. Firstly, a *circuit* is a minimally dependent set, that is any set $C \subseteq E$ satisfying $r_M(C) = |C| - 1 = r(M)$. Clearly, a circuit of the cycle matroid of a graph corresponds to a cycle of the graph. We define a *loop* to be any element $e \in E$ which is contained in no independent set, and a *coloop* to be any element contained in no circuit; equivalently $e \in E$ is a loop if $r_M(\{e\}) = 0$, and a coloop if for all $F \subseteq E$ we have $r_M(F \cup \{e\}) = r_M(F) + 1$. A loop of a graph corresponds to a loop in its cycle matroid; the graph-theoretic analogue for a coloop is an edge between biconnected components. Although we disallow loops in graphs in the majority of this thesis, they will be always be allowed in any discussion of matroids. The *restriction* $M|A$ of $M$ to some subset $A \subseteq E$ is itself a matroid, each of whose independent sets is simply the intersection of $A$ with an independent set of $M$, that is, those independent sets of $M$ contained in $A$ (similarly circuits of $M|A$ are circuits of $M$ contained in $A$). This can be viewed as analogous to the subgraph induced by a subset of edges, or equivalently to that produced by the deletion of the elements of $E \setminus A$ from $E$.

Thus restriction to a subset corresponds to deletion of that subset's complement. It will often be the case that we wish to delete one element only from a matroid; in this case we differ from graph theoretic notation in order to emphasis that in reality we are removing a subset of cardinality one, so that the *deletion* of an element $e \in E$ from $M$ will be written as $M \setminus e$[3], and is the same thing as the restriction of $M$ to $M \setminus \{e\}$.

It is more difficult to visualise the contraction of an element of a matroid, and we must define it more abstractly: the *contraction* $M/A$ of $M$ by a subset $A \subseteq E$ is the matroid which is defined on the same ground-set $E \setminus A$ as the deletion, but whose independent sets are defined instead to be all those sets $B \subseteq E \setminus A$ which are such that $B \cup A$ is independent in $M$. As before, we will denote the contraction of $M$ by a one-element set $\{e\}$ as $M/e$. (In practice, we will use the restriction notation $M|A$ when deleting more than one edge, and otherwise write $M \setminus e$ and $M/e$ to respectively represent deletion and contraction of one element. We will have no need to contract by more than one element at once.)

We need one more definition, which this time differs somewhat from graph theory: $M$ is said to be *connected* if there is at least one circuit containing any given pair of elements of $E$. Thus, rather confusingly, connectivity in matroids corresponds directly to biconnectivity in graphs (note that the cycle matroid does not distinguish between components and blocks of a graph).

Given our subject matter, we would be amiss to end our discussion of matroids without mentioning one further example.

**Example 2.6.** An *algebraic matroid* $M = M(L/K)$ can be defined on any (transcendental) field extension $L/K$. The ground set is the set of subsets of $L$ containing $K$, and a given set $K \subseteq S \subseteq L$ is said to be independent if its elements are algebraically independent over $K$. Thus the rank $r_M(S)$ of $S$ is the transcendence degree of $S$ over $K$. Note that any algebraic extension is therefore associated with a trivial algebraic matroid.

[3]In order to simplify our expressions we often use "the deletion" and "the contraction" as proper nouns, referring to the matroid produced by the given action. Thus we employ the phrases "deletion of $e$ from $M$" and "matroid obtained from $M$ by deleting e" interchangeably, for example.

## 2.3 Graph polynomials

Graph polynomials are polynomial invariants of graphs, and thus to varying degrees encode the abstract structure of graphs. They are usually constructed in order to enumerate certain graph-theoretic properties, however it is often the case that the polynomial can be made to yield much more data than was originally intended. The universality of polynomials in mathematics, and the ease with which they can be manipulated makes them a useful proxy object for for graph theorists to study, and they are central to algebraic graph theory. A wide variety of graph polynomials are currently studied, and new ones appear frequently.

We are concerned in particular with a special family of graph polynomials, which are arguably the most important in graph theory, due to the surprising amount of graph-theoretical data they encode, the ease with which they are defined, and the convenient recursive identities which they satisfy. The defining feature of a family belonging to this class of polynomials is that they satisfy some kind of recurrence relation of the following kind.

**Definition 2.7.** Let $\mathfrak{F}$ be a family of graph polynomials, and write $f_G(x)$ for the member of this family which is assigned to the graph $G$. We say that $\mathfrak{F}$ satisfies a *deletion-contraction recurrence relation* if, for all graphs $G = (V, E)$ in $\mathfrak{F}$, and all $e \in E$ we can express $f_G(x)$ (linearly) in terms of $f_{G-e}(x)$ and $f_{G/e}(x)$.

We shall discuss three families of polynomials, all of which satisfy some form of deletion-contraction recursion, and which have the property that any given pair of families comprises one which is a specialisation (perhaps up to multiplication by some prefactor) of the other. These are in some sense the true “characteristic polynomials” of graphs, encoding—as they each do—a huge amount of graph-theoretic information, increasing with the number of variables.

We will work our way backwards in time, and start with the most general of the three.

### 2.3.1 The multivariate Tutte polynomial

Let $G = (V, E)$ be a finite graph, for each $e \in E$ let $v_e$ be a variable, and let $\mathbf{v}$ be the collection of these variables. We will refer to the members of this set as *edge-variables*[4]. For each subset of edges $A \subset E$, we denote by $\mathbf{v}_A$ the set $\{v_e\}_{e \in A}$.

Now, let $q$ be another indeterminate. Following [44] we define the *multivariate Tutte polynomial* of $G$ to be:

$$Z_G(q, \mathbf{v}) = \sum_{A \subseteq E} q^{k(A)} \prod_{e \in A} v_e. \tag{2.1}$$

Essentially this is a generating function for the spanning subgraphs of $G$, but one which encodes not only the precise edge-content of each such subgraph (as a product of corresponding edge-variables), but also its number of connected components (as an exponent of $q$). This data is enough not only to uniquely characterise a graph, but to encode all of its structure; moreover variations are very important in statistical mechanics, in which it is referred to as the Potts model.

The multivariate Tutte polynomial is multi-affine in its edge-variables—that is, it has maximum degree 1 in each variable (this follows immediately from the spanning subgraph interpretation of each monomial, as clearly no one edge can appear twice in any given subgraph). This makes it ideal for inductive proofs on the number of edges in a graph, and indeed we shall present a proof using an analogous idea for matroids in Chapter 7.

The deletion-contraction recurrence for this polynomial is as follows:

$$Z_G(q, \mathbf{v}) = Z_{G-e}(q, \mathbf{v}_{\hat{E}}) + v_e Z_{G/e}(q, \mathbf{v}_{\hat{E}}), \tag{2.2}$$

where $\hat{E}$ is shorthand for $E \backslash \{e\}$. The proof of this identity amounts to little more than noting that spanning subgraphs of $G - e$ correspond to those of $G$ which do not contain $e$; and that spanning subgraphs of $G/e$ are in one-to-one correspondence with those of $G$ containing $e$. The simplicity of this identity adds to the convenience of the multivariate Tutte polynomial for inductive proofs based around iterative addition or removal of edges from a graph.

[4]For consistency we shall retain this terminology in the context of matroids

We will study the algebraic properties of this polynomial in Chapter 7, in particular showing it always has symmetric Galois group over the field of functions in the edge variables; and that this is true additionally for more general matroids. Note that adaptations of all three of the polynomials in this family are important in matroid theory[5]. We shall define the matroid theoretic version when it is required in the following chapter.

Now, by making the simple substitution $v_e \leftarrow -1$ for all $e \in E$ we obtain the chromatic polynomial $P_G(q)$ of $G$ (see 2.3.3 below). Along with the desirable properties described above, this has led to the multivariate Tutte polynomial being used to great effect in the study of its more widely known specialisations, especially as regards the location of chromatic roots [22, 43, 42].

We discuss next the original bivariate Tutte polynomial, which is a specialisation of the multivariate case, and a generalisation of the univariate chromatic polynomial.

### 2.3.2 The Tutte polynomial

The ideas behind the Tutte polynomial first appeared in a 1947 paper [48] by its namesake. The original name given to it—the dichromatic polynomial—betrays its roots in the study of the chromatic polynomial, and the attempts being made around that time to prove the Four Colour Conjecture (as it then was). Tutte was interested in deletion-contraction recurrences, and had purposefully set out to create a bivariate generalisation of the chromatic polynomial. Although it was originally presented in a quite different way, the polynomial is now usually defined as follows.

Let $G = (V, E)$ be a graph, and $x$ and $y$ be indeterminates. The *Tutte polynomial of* $G$ is the bivariate polynomial in $\mathbb{Z}[X]$ of the form

$$T_G(x, y) = \sum_{A \subseteq E} (x-1)^{k(A)-k(E)}(y-1)^{k(A)+|A|-|V|}, \qquad (2.3)$$

where $k(A)$ is the number of connected components in the subgraph induced

[5]In fact, these are direct generalisations: the matroid theoretic polynomials corresponding to graphic matroids are essentially the same in all three cases as their graph theoretic counterparts

by the edge subset $A$.

The Tutte polynomial of $G$ is obtained from its multivariate generalisation $Z_G(q, \mathbf{v})$ by making the following substitutions:

$$q \leftarrow (x-1)(y-1)$$
$$v_e \leftarrow y-1, \text{all } e \in E,$$

and multiplying by a prefactor $(x-1)^{-k(E)}(y-1)^{k(A)-|V|}$. Thus $T_G$ is essentially equivalent to a special case of $Z_G$ in which the same variable is assigned to every element of $E$.

Evaluating $T_G(x, y)$ at different small integer values of $x$ and $y$ produces a range information about $G$, such as the number of forests, spanning forests, spanning subgraphs and acyclic orientations. In fact, it has recently been shown [9] that there is a combinatorial interpretation for its evaluation at *any* pair of positive integers.

As the Tutte polynomial is not a direct object of study in this thesis, we shall not attempt to describe it further, or even provide one reference in particular; the polynomial is so important that whole chapters of graph and matroid theory texts are devoted to it.

The final polynomial we describe is the oldest, and most specialised. As it is our main object of study we shall give a thorough introduction.

### 2.3.3 The chromatic polynomial

Recall that, for some positive integer $q$, a proper $q$-colouring of a graph $G$ is a function from the vertices of $G$ to a set of $q$ colours, with the property that adjacent vertices receive different colours. The *chromatic polynomial* $P_G(x)$[6] of $G$ is the unique monic polynomial whose evaluation at each $q \in \mathbb{N}$ is the number of proper $q$-colourings of $G$. A *chromatic root* of $G$ is a zero of $P_G(x)$. This polynomial was originally introduced in 1912 by Birkhoff [1] in an attempt to prove what is now the Four-Colour Theorem (equivalent

[6]Due to the algebraic slant of our work, we vary from the usual convention of representing the argument of the chromatic polynomial by $q$ or $\lambda$, meaning that it will occasionally be convenient to view $x$ as some arbitrary number of colours rather than an indeterminate. This means that $P_G(x)$ can be both a univariate polynomial or a natural number, depending on the context.

to the statement $P_G(4) > 0$ for all planar graphs $G$). A summary of these attempts can be found in his 1946 collaboration with Lewis [2].

### Chromatic roots

Chromatic roots have been the subject of much study, especially as regards their location and distribution on the real line and in the complex plane. As mentioned in the introduction, the most important results in this area can be summarised by the following conditions, which are due, respectively, to Jackson [21], Thomassen [47] and Sokal [43].

**Theorem 2.8.** *(i) There are no negative real chromatic roots, and none in either of the intervals* $(0, 1)$ *or* $(1, 32/27]$

*(ii) Chromatic roots are dense in the remainder of the real line*

*(iii) Chromatic roots are dense in the whole complex plane*

Surprisingly, there is relatively little known about which specific algebraic integers can or cannot be chromatic roots. We do not even know if, for example, $i$ is a chromatic root or not. A major open problem is to determine which univariate polynomials are chromatic, and various necessary conditions are known; some simple examples of these are that the coefficients of such a polynomial must alternate (this follows from the fact that there are no negative real roots), that the constant term must be zero, and that, except in the case of the null graph, we must always have $P_G(1) = 0$. Many more such conditions are known; see in particular [54], and the recent proof of the long-standing conjecture that the absolute values of the coefficients of any chromatic polynomial form a logarithmically concave sequence [20].

### Combinatorial interpretations

There are a number of combinatorial interpretations for the coefficients of $P_G(x)$ when written in various polynomial bases. One of the most important expansions is as sums of falling factorials: it can be easily shown simply by counting the number of possible colourings of each vertex that the chromatic polynomial of the complete graph on $n$ vertices is

$$P_{K_n}(x) = (x)_n, \tag{2.4}$$

where $(x)_n = x(x-1)\cdots(x-n+1)$.

and this reasoning can be generalised as follows. Let $a_i$ be the number of isomorphism classes of proper $i$-colourings of a graph $G$ (that is, the number of partitions of the vertices of $G$ into $i$ parts such that no part contains any two adjacent vertices). Then the number of ways to properly colour $G$ with $i$ colours is

$$a_i(x)_i = a_i P_{K_i}(x).$$

Thus we can count the total number of proper colourings of $G$ by summing these terms over $i$, giving us

$$P_G(x) = \sum_{i=0}^{n} a_i P_{K_i}(x), \tag{2.5}$$

where $n$ is the order of $G$ (note that we will always have $a_n = 1$).

Another important combinatorial interpretation was found by Whitney, who showed that the coefficients $\{b_i\}$ in the more familiar polynomial basis

$$P_G(x) = \sum_{i=0}^{n} b_i x^i$$

count what he refers to as "broken circuits" of $G$. We shall not directly study these in the present work, and refer the reader instead to the original paper [51], and to [5] for a number of other combinatorial interpretations of chromatic polynomials.

As regards its relation to the other polynomials discussed in this section, the chromatic polynomial can be obtained from the by multivariate Tutte polynomial by specialising all the edge-variables to $-1$ (note that this directly leads us to another combinatorial interpretation for the coefficients of the chromatic polynomial: the coefficient of $x^i$ in $P_G(x)$ is simply the signed sum of spanning subgraphs of $G$ having $i$ connected components, where each subgraph $(V, A)$ of $G = (V, E)$ contributes a summand $(-1)^{|A|}$).

### Identities

Identities for manipulating chromatic polynomials will be used frequently in this thesis. The most important of these is the following basic deletion-contraction recurrence, which we can most conveniently obtain from 2.2 by specialisation[7]

$$P_G(x) = P_{G-e}(x) - P_{G/e}(x). \tag{2.6}$$

If $e \notin E(G)$, then we can express this recurrence instead as:

$$P_G(x) = P_{G+e}(x) + P_{G+e/e}(x); \tag{2.7}$$

we refer to this form of the identity as *addition-contraction*.

We will also make use of the following identity concerning joins of graphs with complete graphs:

**Proposition 2.9.** *Let $H$ be the join of some graph $G$ with $K_n$. Then $P_H(x) = (x)_n P_G(x-n)$.*

*Proof.* There are $(x)_n$ ways to properly $x$-colour the vertices of the copy of $K_n$. As each of these vertices is connected to all of the vertices of $G$, this leaves $x - n$ colours with which to properly $x$-colour the vertices of $G$. □

Another useful feature of the chromatic polynomial allows us to decompose chromatic polynomials of clique-separable graphs as a product of factors corresponding to the separable subgraphs. We shall sometimes refer to this as the "clique-sum property".

**Proposition 2.10.** *Suppose $G$ is a $K_n$-sum of two subgraphs $H_1$ and $H_2$ for some $n \in \mathbb{N}$ (that is, $H_1 \cup H_2 = G$ and $H_1 \cap H_2 = K_n$). Then*

$$P_G(x) = \frac{P_{H_1}(x) P_{H_2}(x)}{(x)_n}.$$

[7] Although it is not difficult to prove this identity directly, specialisation from the multivariate Tutte polynomial reduces the proof almost to a triviality; this is a good example of the somewhat counter-intuitive way in which consideration of this more general object can simplify proofs of results about the chromatic polynomial.

*Proof.* The number of ways of properly colouring the $K_n$ subgraph common to $H_1$ and $H_2$ with $x$ colours is $(x)_n$. For any fixed colouring of the $n$ vertices of this subgraph, there are $P_{H_i}(x)/(x)_n$ ways to colour the remaining vertices of $H_i$ for $i = 1, 2$. Hence there are:

$$\frac{P_{H_1}(x)P_{H_2}(x)}{(x)_n(x)_n}$$

ways to $x$-colour $G$ for any fixed colouring of the $K_n$ subgraph. As there are $(x)_n$ such colourings of $K_n$, multiplying by this factor gives the total number of possible ways to properly $x$-colour $G$. □

We will need one further quite specialised identity, which follows from an iterative application of the deletion-contraction identity, and which we will apply later when constructing families of chromatic polynomials.

**Proposition 2.11.** *Let $n_1$ and $n_2$ be natural numbers greater than 1, let $H_1$ and $H_2$ be any graphs, and for each $1 \leq i \leq 2$ let $G_i$ be the join of $H_i$ with a copy of the complete graph $K_{n_i}$. Choose some vertex $v$ in the copy of $K_{n_2}$ in $G_2$, and add every possible edge between it and the vertices of $K_{n_1}$ in $G_1$. Let $G$ be the resulting graph. Then we have:*

$$P_G(x) = (1 - \frac{n_1}{x})P_{G_1}(x)P_{G_2}(x).$$

*Proof.* Label the edges between $v$ and the copy of $K_{n_1}$ in $G_1$ as $e_1, e_2, \ldots e_{n_1}$, and let $\mathbf{e}$ be the set of these edges. Note that contracting any element $e_j$ of $\mathbf{e}$ produces the same graph. Moreover, deleting a number of edges before contracting one of the remaining ones does not affect the outcome: we still end up with a 1-vertex sum of $G_2$ with $G_1$ (the vertex in question being contained in $K_{n_1}$). By Proposition 2.10, this graph has chromatic polynomial:

$$P_{G/e_j}(x) = \frac{1}{x}P_{G_1}(x)P_{G_2}(x). \tag{2.8}$$

Deleting all of the elements of $\mathbf{e}$ results again in a combination of $G_2$ and $G_1$, except instead of sharing a vertex they are now disjoint. The whole graph therefore simply has chromatic polynomial:

$$P_{G-\mathbf{e}}(x) = P_{G_1}(x)P_{G_2}(x)$$

Therefore applying the deletion-contraction identity $n_1$ times, and repeatedly substituting using 2.8 gives us:

$$\begin{aligned} P_G(x) &= P_{G-\mathbf{e}}(x) - n_1 P_{G/e_1}(x) \\ &= P_{G_1}(x)P_{G_2}(x) - \frac{n_1}{x} P_{G_1}(x)P_{G_2}(x) \\ &= (1 - \frac{n_1}{x}) P_{G_1}(x)P_{G_2}(x), \end{aligned}$$

as claimed. □

For more information on the chromatic polynomial, a brief yet comprehensive introduction can be found in [34]; for a more in-depth reference see [13], or for a relatively recent survey specifically concerning chromatic roots we refer the reader to [23].

## 2.4 Basic algebraic number theory

We only use the most basic aspects of the theory of algebraic numbers in this thesis. In particular, we are interested in polynomials, the field extensions their roots generate, and the basic Galois theory that provides a powerful tool with which to study these extensions. We shall assume familiarity with much of this, and refer the reader to, for example [46] for Galois theory, and [27] for more advanced algebraic number theory. (Note however, that notation varies considerably from source to source, and ours is not consistent with any one in particular).

We will briefly revise the theory of field extensions, before addressing a couple of less well-known concepts which appear in Chapter 7.

### 2.4.1 Field extensions

Let $L$ be field which contains the field $K$; then $K$ is a *subfield* of $L$, and $L$ is an *extension* of $K$. More commonly, we express this setup by simply writing that $L/K$ is a *field extension*. The larger field $L$ is a vector space over $K$; its dimension is denoted by $[L : K]$, and is referred to as the *degree* of the extension. If $[L : K]$ is finite, then the extension is *algebraic*, and elements of $L$ are said to be *algebraic over $K$*. Two extensions $L_1 : K$ and

$L_2 : K$ are said to be *isomorphic* if there exists a bijective homomorphism $\theta : L_1 \to L_2$ which is the identity map on elements of $K$ (such a map is known as an *isomorphism of field extensions*).

Algebraic elements of a field extension are normally defined in terms of polynomials: we write that $\alpha \in L$ is *algebraic over* $K$ if there is some irreducible polynomial $f(X) \in K[X]$ such that $f(\alpha) = 0$. Now suppose that $L = K(\alpha)$; then we say that $\alpha$ is a *primitive element* of the extension $L/K$, and that $f(X)$ is the *minimal polynomial* of $L/K$ (or that it *generates* $L$ *over* $K$). If $L$ contains every root of some polynomial $f(X) \in K[X]$ then it is said to be a *splitting field* for $f(X)$ (as $f(X)$ *splits* in $L$—that is, it factorises into linear factors in $L[X]$). More commonly, we will refer to *the* splitting field of a polynomial $f(X)$; by this we mean the minimal splitting field of $f(X)$. As we will be studying these in some depth in Part II—particularly in Chapter 6—we will use the shorthand *chromatic splitting field* to denote the splitting field of a chromatic polynomial.

Now, a field extension must satisfy two important conditions if it is to be studied by classical Galois theory. The first of these is separability: an irreducible polynomial $f(X) \in K[X]$ is *separable* if it has no repeated roots; in this case adjoining one or more roots of $f(X)$ to $K$ produces a separable extension.

The second property of field extensions which is fundamental to Galois theory is normality: a field extension $L/K$ is *normal* if $L$ is the splitting field of some irreducible polynomial in $K[X]$. If this is not the case, then we can extend $L/K$ to a normal extension by defining the *normal closure* of $L$ over $K$ to be the minimal normal extension of $K$ containing $L$.

If an algebraic field extension is both normal and separable the we refer to it as a *Galois extension.* However, note that any algebraic extension of a field of characteristic zero is separable. Thus when working exclusively over such fields Galois extensions and normal extensions are one and the same, and the terminology may be used interchangeably.

### 2.4.2 Galois groups

Finally we are able to define one of the central objects of this work, the Galois group of an algebraic extension. The elements of these groups are *field automorphisms*, which are simply isomorphisms from a field to itself. A *fixed-field automorphism* is a special type of field automorphism which is the identity map on elements of a subfield. The automorphisms of any given field form a group, of which the groups of fixed-field automorphisms are subgroups. If $L/K$ is an algebraic extension; then a $K$-*automorphism* of $L$ is simply a fixed-field automorphism of $L$ for which $K$ is the "fixed field" in question.

If $L/K$ is a Galois extension, then it can be shown that the order of the group of $K$-automorphisms of $L$ is precisely $[L : K]$; we call this group the *Galois group of* $L/K$, and denote it by $\mathrm{Gal}(L/K)$. Given that $L/K$ is a Galois extension, there must be an irreducible polynomial $f(X) \in K[X]$ whose roots generate the extension. The smaller (and not necessarily normal) extensions which are each obtained by adjoining one individual root of $f(X)$ to $K$ are pairwise isomorphic, and the isomorphisms between them are the elements of $\mathrm{Gal}(L/K)$. The induced permutations comprise a transitive action of $\mathrm{Gal}(L/K)$ on the set $\{K(\alpha) : f(\alpha) = 0\}$; thus $\mathrm{Gal}(L/K)$ acts transitively by permutation on the set of all roots of $f(X)$.

In certain contexts (Chapter 7 in particular) it is convenient to be able to discuss the Galois group that acts on the roots of the irreducible polynomial $f(X) \in K[X]$ without reference to field extensions. In this case we use the notation $\mathrm{Gal}(f/K)$, and refer to the *Galois group of the polynomial* $f(X)$; this is precisely the same object as the group $\mathrm{Gal}(L/K)$, where $L$ is the splitting field of $f(X)$.

### 2.4.3 Transcendental extensions

An element $\alpha \in L \setminus K$ is said to be *algebraically independent* over $K$ if it does not satisfy any polynomial equation having coefficients in $K$. If such an element exists then it is said to be *transcendental over* $K$; its existence implies that $[L : K]$ is not finite, and we write that $L/K$ is a *transcendental extension* (as opposed to an algebraic one). More generally, a subset $S$ of $L$ of cardinality $n$ is algebraically independent over $K$ if its elements do not satisfy any polynomial equation in $n$ variables having coefficients in $K$. The cardinality of the largest such subset of $L$ is known as the *transcendence degree* of $L/K$.

Any extension $L/K$ having transcendence degree $n$ is isomorphic to the function field $K(X_1, \ldots, X_n)$, where each $X_i$ is an indeterminate. Thus there is essentially only one unique extension of any given field having transcendence degree $0 \neq n \in \mathbb{N}$.

# Chapter 3

# Some families of chromatic polynomials

In the not-so-distant past, the study of graphs with small order informed the commonly-held view that all chromatic roots had positive real part. This conjecture was disproved by a series of discoveries of chromatic roots with increasingly large negative real part (see, for example, [7],[6],[19]), culminating with Sokal's proof [43] that chromatic roots are in fact dense in the whole complex plane. Many of these discoveries made use of the large family of generalised theta graphs: a general formula for the chromatic polynomials of these graphs is known, and it has the salient feature of varying according to parameters which appear in the exponents of the argument [6]. This enables the study of closely related chromatic polynomials having arbitrarily high degree; clearly a desirable property when studying the distribution of chromatic roots.

We shall follow this principle of studying large families of graphs for which a general chromatic polynomial formula is known. However, as we will be considering algebraic properties of chromatic roots—as opposed to their distribution in $\mathbb{C}$ or $\mathbb{R}$—we wish to find families of graphs which provide us with a range of examples of chromatic polynomials of fixed degree, whose general formulae thus have parameters appearing as coefficients rather than exponents. This way a range of chromatic roots and splitting fields of the same degree can be considered and compared.

Below we describe a number of families of such graphs, and in each case derive a formula for the chromatic polynomial of an arbitrary member. All of these graphs will be referenced at different points in this chapter, with the family of bicliques in particular being a central object of study.

## 3.1 Bicliques

*Bicliques* are complements of bipartite graphs; as such each consists of two cliques joined by a number of edges[1]. When we need to be more specific, we shall refer to a biclique in which the two cliques are of size $j$ and $k$ as a $(j, k)$*-biclique.* By convention, $k$ will be greater than or equal to $j$, and we shall refer to the edges between the two cliques as *bridging edges.*

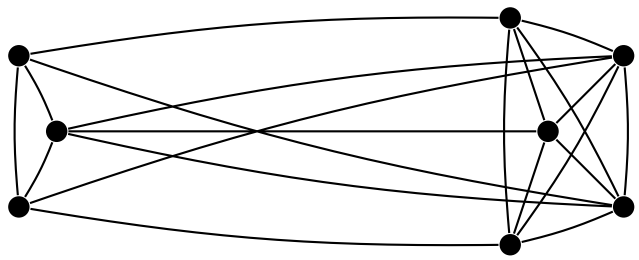

Figure 3.1: A $(3, 5)$-biclique

We will give two quite different constructions for the chromatic polynomial of a general biclique, each resulting in a form of the polynomial which will be useful for certain applications. For the first construction we will require the following definition.

**Definition 3.1.** An $i$*-matching* of a graph $G$ is a set of $i$ edges of $G$, no two of which are incident to a common vertex. Two graphs are said to be *matching equivalent* if they have the same number of $i$-matchings for all non-negative integers $i$ (by convention, every graph is assumed to have a single 0-matching).

We will use the notation $m_G^i$ for the number of $i$-matchings of $G$; thus two graphs $G$ and $H$ are matching equivalent if and only if $m_G^i = m_H^i$ for all $i$.

[1]Please note that our definition varies from much of the literature, in which a biclique is considered to be a complete bipartite graph.

**Construction 1** For some positive integers $j$ and $k$, let $G$ be a $(j,k)$-biclique, and let $\bar{G}$ be the complement of $G$ (obtained by replacing edges of $G$ with non-edges, and vice-versa). Then $\bar{G}$ is a subgraph of the complete bipartite graph $K_{j,k}$. We shall construct the chromatic polynomial of $G$ by considering matchings of $\bar{G}$.

Given some matching of $\bar{G}$, partition the vertices of $G$ such that two vertices are contained in the same part if and only if the corresponding vertices of $\bar{G}$ are joined by an element of the matching. Then, by assigning a different colour to each part of this partition, we obtain a proper colouring of $G$. Conversely, any proper colouring of $G$ corresponds to a partition induced by some matching of $\bar{G}$. Thus we can compute the chromatic polynomial of $G$ by counting $x$-colourings of partitions induced by matchings of $\bar{G}$, as follows.

If each part of such a partition receives a different colour, then there are $(x)_{j+k-i}$ ways of assigning $x$ colours to a partition induced by an $i$-matching of $\bar{G}$ (as any such partition consists of $j+k-i$ parts). Thus by the same reasoning we used in §2.3.3 to obtain (2.5) of the chromatic polynomial, we get the expression:

$$P_G(x) = \sum_M (x)_{j+k-|M|}, \tag{3.1}$$

where the sum is over all possible matchings $M$ of $\bar{G}$.

Now suppose that, for some $1 \leq p \leq j$, there are $p$ vertices in the $j$-clique of $G$ which are adjacent to every vertex of the $k$-clique. Then these $p$ vertices are each adjacent to every other vertex of the graph. Thus, counting proper $x$-colourings of $G$, we have that there are $(x)_p$ ways in which to colour these $p$ vertices, and $x-p$ colours remaining with which to colour the remaining vertices. So the chromatic polynomial of $G$ will be of the form:

$$P_G(x) = (x)_p P_H(x-p),$$

where $H$ is the $(j-p,k)$-biclique obtained from $G$ by deleting each of the $p$ vertices and all incident edges. A similar situation arises if some vertices of the $k$-clique are adjacent to every vertex of the $j$-clique. As we are concerned with algebraic properties of the chromatic polynomial, we shall discount these cases, and assume that no vertex of $G$ is connected to every other vertex of the graph.

It is not difficult to see that no matchings of size larger than $j$ are possible. Hence we have:

$$P_G(x) = \sum_{i=0}^{j} m^i_{\bar{G}}(x)_{j+k-i},$$

where $m^0_{\bar{G}} = 1$, and in general $m^i_{\bar{G}}$ is a non-negative integer. Thus the chromatic polynomial of $G$ is a product of $(x)_k$ with a degree $j$ factor $g(x)$ of the form:

$$g(x) = \sum_{i=0}^{j} m^i_{\bar{G}}(x-k)_{j-i}. \tag{3.2}$$

When all but one of the factors in a general formula for a family of chromatic polynomials are linear, we shall refer to the non-linear factor as the *interesting factor* of the polynomial. As we are chiefly concerned with algebraic properties of chromatic polynomials it is these interesting factors that will be our main focus. Note that for certain assignments of the parameters of a formula the interesting factor may not be irreducible.

The factor $g(x)$ above is the interesting factor of $P_G(x)$. The next construction gives us a different way in which to formulate this factor. The derivation is much more complicated, but has the advantage of producing $2^j - 2$ parameters by which to define a $(j,k)$-biclique; these will become necessary in what follows.

**Construction 2** As just established, the chromatic polynomial of a $(j,k)$-biclique $G$ is a product of $(x)_k$ with a degree $j$ interesting factor $g(x)$. Observe that $(x)_k$ is the number of ways to properly $x$-colour the $k$-clique of $G$; thus we can view $g(x)$ as an expression for the number of proper $x$-colourings of the $j$-clique. We can construct this expression independently of the rest of the polynomial using Möbius inversion, as follows.

Label the vertices of the $j$-clique $1, 2, \ldots, j$, and let $X$ be the set of these $j$ vertices. Let $A_i$ be the set of neighbours of vertex $i$ in the $k$-clique. We will represent by $a_{\{i_1,i_2,\ldots,i_s\}}$ the number of vertices lying in $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_s}$ but in none of the other sets. So, for example, $a_{\{1,2\}}$ is the cardinality of $(A_1 \cap A_2) \setminus \bigcup_{i \neq 1,2} A_i$. Note that any vertex lying in the intersection of all the $A_i$ is connected to every other vertex of the graph. As in the previous construction, we will assume that there is no such vertex, so that

$$A_1 \cap A_2 \cap \cdots \cap A_j = \emptyset,$$

that is, $a_{\{1,2,\ldots,j\}} = 0$.

Now, for some partition $\sigma$ of $X$, we define $f_\sigma(x)$ to be the number of ways of colouring the vertices of $X$ with $x$ colours, such that the members of any given part have the same colour, and such that no part has the same colour as any of its neighbouring vertices in the $k$-clique. We also define $g_\sigma(x)$ to be the number of these colourings in which every part is assigned a different colour. Write $\sigma \leq \tau$ if $\tau$ is another partition of $X$ which is coarser than $\sigma$, so that every part of $\sigma$ is contained in a part of $\tau$. It is easy to see that

$$f_\sigma(x) = \sum_{\sigma \leq \tau} g_\tau(x),$$

where the sum ranges over all partitions of $X$ which are coarser than $\sigma$. By the Möbius Inversion Theorem, this gives

$$g_\sigma(x) = \sum_{\sigma \leq \tau} \mu(\sigma, \tau) f_\tau(x),$$

where $\mu$ is the Möbius function on the poset of partitions. This is known (see, for example, [39]) to be:

$$\mu(\sigma, \tau) = (-1)^{p-q}(2!)^{q_3}(3!)^{q_4} \cdots ((j-1)!)^{q_j},$$

where $p$ is the number of parts in the finer partition $\sigma$, $q$ is the number of parts of the coarser partition $\tau$, and $q_i$ is the number of parts of $\tau$ which contain exactly $i$ parts of $\sigma$.

If $\hat{\sigma}$ is the finest partition of $X$ (in which every vertex lies in a different part), then the interesting factor of $P_G(x)$ is precisely $g_{\hat{\sigma}}(x)$. So by the above, in order to formulate this interesting factor we need to find $f_\tau(x)$ for every partition $\tau$ of $X$.

Let $\tau$ be a partition of $X$, let $\tau_1, \tau_2, \ldots, \tau_t$ be the parts of $\tau$, and let $f_{\tau_i}(x)$ be the number of colours available with which to colour the vertices in $\tau_i$. Then $f_\tau(x) = f_{\tau_1}(x) f_{\tau_2}(x) \cdots f_{\tau_t}(x)$. Now, for a given $\tau_i$, we obtain $f_{\tau_i}(x)$

by subtracting the number of neighbours in the $k$-clique of the vertices in $\tau_i$ from $x$ (because each such neighbour has a different colour, and none of these colours can be used to colour the vertices in $\tau_i$).

So for example, suppose $j = 3$, and $\tau$ is the partition $\{\{1,2\},\{3\}\}$. Then

$$f_\tau(x) = (x - a_{\{1\}} - a_{\{2\}} - a_{\{1,2\}} - a_{\{1,3\}} - a_{\{2,3\}})(x - a_{\{3\}} - a_{\{1,3\}} - a_{\{2,3\}}).$$

In general, for a given part $\tau_i$ of a partition $\tau$,

$$f_{\tau_i}(x) = x - \sum_{S \cap \tau_i \neq \emptyset} a_S,$$

and so

$$f_\tau(x) = \prod_{\tau_i \in \tau} \Big( x - \sum_{S \cap \tau_i \neq \emptyset} a_S \Big).$$

Finally, using Möbius inversion, we have that the interesting factor of $P_G(x)$ is:

$$g(x) = g_{\hat{\sigma}}(x) = \sum_{\hat{\sigma} \leq \tau} \mu(\hat{\sigma}, \tau) \prod_{\tau_i \in \tau} \Big( x - \sum_{S \cap \tau_i \neq \emptyset} a_S \Big), \tag{3.3}$$

where $\mu$ is the Möbius function of the partition poset, and $\hat{\sigma} = \{\{1\}, \ldots, \{j\}\}$.

## 3.2 Clique-graphs

Clique-graphs (referred to as *clan-graphs* in Read's pioneering study [36]) consist of a graph of which each vertex has been replaced by a clique (possibly of size 1), and the vertices of each clique have been joined to all those of its neighbouring cliques (where "neighbourhood" is used with respect to the original graph connections). We shall discuss three families of these graphs, giving a general form for the chromatic polynomial of an arbitrary member in each case. Although the second of these families is of some significance itself, our aim is largely to develop the necessary framework for constructing the chromatic polynomials of what we call *clique-theta graphs*, which shall be used later on to show that the set of algebraic integers for which the $n\alpha$ conjecture holds is dense in the complex plane.

### 3.2.1 Clique-paths

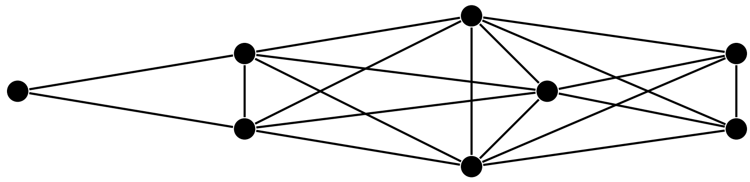

Figure 3.2: The clique-path $L(1, 2, 3, 2)$

Aside from disjoint complete graphs, which are clique-graphs based on a null graph, the simplest family of clique-graphs have underlying structure a path; we shall refer to these as *clique-paths*.

Let $L(a_1, \ldots, a_n)$ denote a path of length $n$, in which the $i$th vertex has been blown up into a clique of size $a_i$. Applying the clique-sum property proved in Proposition 2.10 iteratively gives us the following formula for its chromatic polynomial:

$$P_{L(a_1,\ldots,a_n)}(x) = \frac{(x)_{a_1+a_2} \ldots (x)_{a_{n-1}+a_n}}{(x)_{a_2} \ldots (x)_{a_{n-1}}}$$

A similar formula holds for clique-graphs based on more complicated trees. Algebraically, however, the chromatic polynomials of general clique-trees hold little interest for us, factorising as they do completely into linear factors. Thus we omit any further discussion, and note that a specific formula is given here for clique-paths simply because it will be required later.

## Rings of cliques

Aside from a tree, the simplest underlying structure a clique graph can have is a simple cycle; we refer to these cyclic clique graphs as *rings of cliques*.

Let $R(a_1, a_2, \ldots, a_n)$ be an $n$-cycle in which, for each $1 \leq i \leq n$, the $i$th vertex has been blown up into an $a_i$-clique, and every vertex of this clique has been joined to each of those of its neighbouring $a_{i-1}$- and $a_{i+1}$-cliques (taking indices modulo $n$).

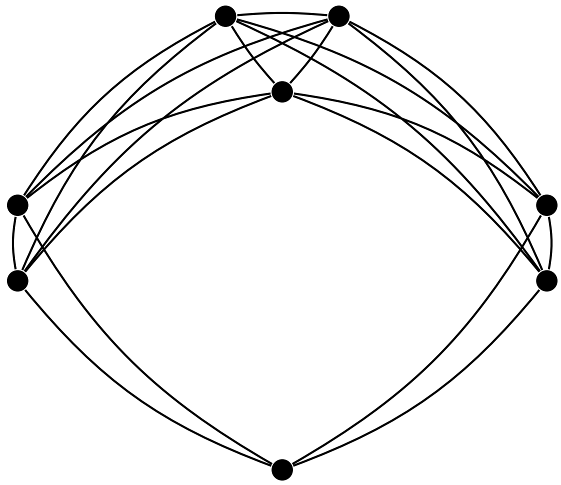

Figure 3.3: The ring of cliques $R(1,2,3,2)$

In [36] Read gives the following general formula for its chromatic polynomial:

$$P_{R(a_1,a_2,\ldots,a_n)}(x) = (x)_{a_1+a_2}\ldots(x)_{a_n+a_1}\sum_{k=1}^{n}(-1)^{nk}v_k(x)\left(\prod_{i=1}^{n}\frac{-(a_i)_k}{(x)_{a_i+k}}\right),$$

where $v_k(x) = \binom{x}{k} - \binom{x}{k-1}$.

It is not immediately obvious that this is indeed a polynomial, but on examination the terms in the denominator of the summation are seen to be cancelled by some of the preceding linear factors. Interestingly, a permutation of the $\{a_i\}$ may change the linear factors, but does not affect the final more complicated factor. This is a desirable property for our purposes: as has been mentioned, our focus on algebraic properties will often lead us to concentrate solely on the interesting factor of a chromatic polynomial.

Now, suppose that $a_1 = 1$. The chromatic polynomial of $R(1, a_2, \ldots, a_n)$ reduces to the following considerably simpler expression:

$$P_{R(1,a_2,\ldots,a_n)}(x) = x(x-1)_{a_{n-1}+a_n-1}\left(\prod_{i=2}^{n-2}(x-a_{i+1}-1)_{a_i-1}\right)r(1,a_2,\ldots,a_n), \tag{3.4}$$

where

$$r(1, a_2, \ldots, a_n) = \frac{1}{x}\left(\prod_{i=2}^{n}(x - a_i) - \prod_{i=2}^{n}(-a_i)\right)$$

is the interesting factor of the polynomial. This specialisation of the previous formula was discovered, but not published, by Read [33] (a separate construction is given in [15]).

A special case of a ring of cliques is, of course, the cycle graph $C_n$ on $n$ vertices, in which $a_i = 1$ for all $i$. Chromatic polynomials of rings of cliques have been shown to have various interesting properties. For example, in [15], Dong *et al* prove that there are non-chordal rings of cliques having only integer roots, a property which at one time it had been thought was restricted to chordal graphs. Furthermore, in [14] it is shown that the real part of a non-integer chromatic root of a ring of four cliques is dependent only on the number of vertices in the graph.

We note in passing that the graph $R(a, b, c, d)$ is an $(a+b, c+d)$-biclique.

## Clique-theta graphs

It is natural to next consider clique graphs with underlying structure having more than one fundamental cycle. Given some natural numbers $n, s_1, \ldots, s_n$, each greater than 2, we define the *generalised theta graph* $\Theta_{s_1,\ldots,s_n}$ to be the graph consisting of $n$ disjoint paths of lengths $s_1, \ldots, s_n$ whose endpoints are identified[2]. These graphs are a natural generalisation of cycles, and their importance in the study of the location of chromatic roots has been noted.

In the same vein as above, we can now define a *clique-theta graph* to be a generalised theta graph whose vertices have been blown up into cliques. As might be expected given the complicated formula for rings of cliques, the chromatic polynomials of these graphs are difficult to construct. However, we can considerably simplify this task by specifying that one endpoint vertex remains fixed as a singleton; in fact we will see that this condition is actually necessary to guarantee that the corresponding chromatic polynomials have certain desirable algebraic properties.

[2]Note that this is the "most general" definition of a generalised theta graph; sometimes members of this family are described as having all paths of the same length

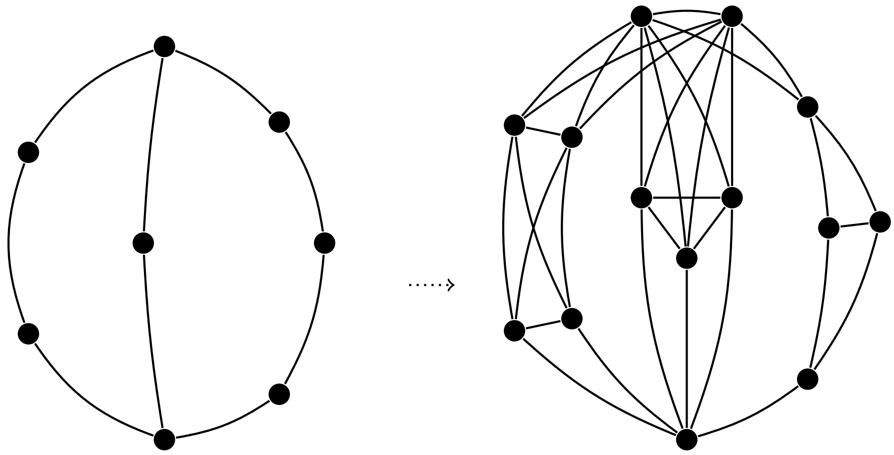

Figure 3.4: The clique-theta graph $T(1,(2,2),(3),(1,2,1),2)$, and its underlying generalised theta graph $\Theta_{4,3,5}$

Clique-theta graphs (with one vertex fixed) are likely to be, in an informal sense, the largest family of graphs for which a general chromatic polynomial formula is now known. Sokal was able to prove his density result using slight modifications of only those theta graphs whose paths between endpoints are all of the same length (thus using just 2 parameters), and the addition of extra parameters for both path length and clique-size makes this family an incredibly diverse source of chromatic roots.

Formally, clique-theta graphs can be defined in the following way: let $p$ be a positive integer, and let $S_1,\ldots,S_k$ be $k$ non-empty ordered sets of positive integers with $S_i=(a_{i(1)},a_{i(2)},\ldots,a_{i(m_i)})$. For each set $S_i$, let $L(1,S_i,p)$ be a clique path with a single vertex at one end, a $p$-clique at the other end, and clique sizes otherwise determined by the elements of the sets $S_i$. The clique-theta graph $T(1,S_1,\ldots,S_k,p)$ is the graph obtained by identifying the single vertices at one end of these clique-paths, and the $p$-cliques at the other.

**Proposition 3.2.** *The chromatic polynomial of the clique-theta graph $T(1,S_1,S_2,\ldots,S_k,p)$ is:*

$$\left[(x)_{a_{k(m_k)}+p}\left(\prod_{i=1}^{k-1}(x-p-1)_{a_{i(m_i)}-1}\right)\left(\prod_{i=1}^{k}\prod_{l=1}^{m_i-1}(x-a_{i(l+1)}-1)_{a_{i(l)}-1}\right)\right]$$
$$\times\left[\left(p(x-p)^{k-1}\prod_{i=1}^{k}r(1,a_{i(1)},\ldots,a_{i(m_i)})\right)+\left(\prod_{i=1}^{k}r(1,a_{i(1)},\ldots,a_{i(m_i)},p)\right)\right],$$

*where* $r(1, a_{i(1)}, \ldots, a_{i(m_i)})$ *is the interesting factor from the chromatic polynomial of the ring of cliques* $R(1, a_{i(1)}, \ldots, a_{i(m_i)})$*.*

We will need the following lemma:

**Lemma 3.3.** *For some* $1 \leq i \leq k$ *let* $S_i = (a_{i(1)}, \ldots, a_{i(m_i)})$*, and let* $\bar{S}_i = (a_{i(2)}, \ldots, a_{i(m_i)})$*. Then:*

$$\begin{aligned} P_{T(1,S_1,\ldots,S_k,p)}(x) = & \frac{P_{T(1,S_1,..,\hat{S}_i,..,S_k,p)}(x) P_{L(a_{i(1)},\ldots,a_{i(m_i)},p)}(x)}{(x)_p} \\ & - a_{i(1)} \frac{(x)_{a_{i(1)}+a_{i(2)}} P_{T(1,S_1,..,\bar{S}_i,..,S_k,p)}(x)}{(x)_{a_{i(2)}+1}}, \end{aligned}$$

*where* $\hat{S}_i$ *indicates that* $S_i$ *has been omitted.*

*Proof.* This follows from a simple application of the deletion-contraction rule. If we let $v$ be the singleton endpoint vertex of the clique-theta graph, then deleting the $a_{i(1)}$ edges between $v$ and the $a_{i(1)}$-clique will produce a $K_p$-sum of $T(1, S_1, .., \hat{S}_i, .., S_k, p)$ and $L(a_{i(1)}, \ldots, a_{i(m_i)}, p)$. Contracting one of these edges produces a $K_{a_{i(2)}+1}$-sum of $T(1, S_1, .., \bar{S}_i, .., S_k, p)$ and $K_{a_{i(1)}+a_{i(2)}}$. As the contraction of any edge produces the same graph, the chromatic polynomial of the latter appears with multiplicity $a_{i(1)}$ (see Proposition 2.11). □

We can now proceed by induction on the sizes of the sets $S_i$.

*Proof of Proposition 3.2.* First suppose that the size of each set is 1, so that $S_i = (a_{i(1)})$ for all $i$. Let $v$ be the single endpoint vertex attached to all other vertices apart from the $p$-clique. Note that contracting any added edge between $v$ and the $p$-clique produces a $K_p$-sum of $(a_{i(1)} + p)$-cliques. This graph has chromatic polynomial:

$$f(x) = \frac{\prod_{i=1}^{k} (x)_{a_{i(1)+p}}}{(x)_p^{k-1}},$$

Also note that adding all edges between $v$ and the $p$-clique gives a $K_{p+1}$-sum of $(a_{i(1)} + p + 1)$-cliques, having chromatic polynomial:

$$g(x) = \frac{\prod_{i=1}^{k} (x)_{a_{i(1)+p+1}}}{(x)_{p+1}^{k-1}}.$$

We now apply the contraction-addition identity $p$ times, where "addition" consists of adding an edge between $v$ and the $p$-clique, and "contraction" consolidates the two vertices between which the new edge is to be added. At every stage the consolidation of these two vertices will produce the graph with chromatic polynomial $f(x)$, and so, again using Proposition 2.11, our final graph will have chromatic polynomial which is a sum of $g(x)$ with $p$ copies of $f(x)$, that is:

$$P_{T(1,S_1,\ldots,S_k,p)}(x) = p\frac{\prod_{i=1}^{k}(x)_{a_{i(1)}+p}}{(x)_p^{k-1}} + \frac{\prod_{i=1}^{k}(x)_{a_{i(1)}+p+1}}{(x)_{p+1}^{k-1}}$$
$$= \left(p(x)_{a_{k(1)}+p}\prod_{i=1}^{k-1}(x-p)_{a_{i(1)}}\right) + \left((x)_{a_{k(1)}+p+1}\prod_{i=1}^{k-1}(x-p-1)_{a_{i(1)}}\right)$$
$$= \left((x)_{a_{k(1)}+p}\prod_{i=1}^{k-1}(x-p-1)_{a_{i(1)}-1}\right)\left(p(x-p)^{k-1}+\prod_{i=1}^{k}(x-a_{i(1)}-p)\right).$$

Note that $r(1, a_{i(1)}) = 1$ and $r(1, a_{i(1)}, p) = x - a_{i(1)} - p$ for all $i$. Hence Proposition 3.2 holds when $|S_i| = 1$ for all $i$.

These graphs suffice as the base case for the induction, as we can build up any clique-theta graph by starting with one having $|S_i| = 1$ for all $i$, and systematically increasing the length of the clique-paths. Note that reordering the $S_i$ does not alter the graph, so for ease of notation we can assume that at each stage the path to which we are adding a new clique is $S_1 = (a_{1(1)}, a_{1(2)}, \ldots, a_{1(m_1)})$. In a similar way, at each stage we can shift the labelling of the individual cliques up one, so that the new element begin added to $S_1$ is always $a_{1(1)}$.

Thus, by Lemma 3.3, we need only show that, if Proposition 3.2 holds for $T(1, \hat{S}_1, \ldots, S_k, p)$ and $T(1, \bar{S}_1, \ldots, S_k, p)$, then it holds too for $T(1, S_1, .., S_k, p)$. So assume that $T(1, \hat{S}_1, \ldots, S_k, p)$ and $T(1, \bar{S}_1, \ldots, S_k, p)$ have chromatic polynomials of the stated form, and let:

$$f(x) = (x)_{a_{k(m_k)}+p}\left(\prod_{i=1}^{k-1}(x-p-1)_{a_{i(m_i)}-1}\right)\left(\prod_{i=1}^{k}\prod_{l=1}^{m_i-1}(x-a_{i(l+1)}-1)_{a_{i(l)}-1}\right).$$

Then removing $f(x)$ as a factor from the expressions

$$\frac{1}{(x)_p}\left(P_{T(1,\hat{S_1},\ldots,S_k,p)}(x)P_{L(a_{1(1)},\ldots,a_{1(m_1)},p)}(x)\right)$$

and

$$\frac{1}{(x)_{a_{1(2)}+1}}\left((x)_{a_{1(1)}+a_{1(2)}}(x)_{a_{1(2)}+1}P_{T(1,\bar{S_1},\ldots,S_k,p)}(x)\right)$$

leaves us with, respectively:

$$\left(\prod_{l=2}^{m_1}(x-a_{1(l)})\right)\left(p(x-p)^{k-1}\prod_{i=2}^{k}r(1,a_{i(1)},\ldots,a_{i(m_i)})\right) + (x-p)\left(\prod_{l=2}^{m_1}(x-a_{1(l)})\right)\left(\prod_{i=2}^{k}r(1,a_{i(1)},\ldots,a_{i(m_i)},p)\right) \tag{3.5}$$

and

$$p(x-p)^{k-1}r(1,a_{1(2)},\ldots,a_{1(m_1)})\left(\prod_{i=2}^{k}r(1,a_{i(1)},\ldots,a_{i(m_i)})\right) + r(1,a_{1(2)},\ldots,a_{1(m_1)},p)\left(\prod_{i=2}^{k}r(1,a_{i(1)},\ldots,a_{i(m_i)},p)\right). \tag{3.6}$$

By Lemma 3.3, the interesting factor of the chromatic polynomial of $T(1,S_1,\ldots,S_k,p)$ is obtained by subtracting (3.6) $a_{1(1)}$ times from (3.5). This gives us a complicated expression which would be not illuminating to reproduce here. However we may simplify this using the identity

$$\left(\prod_{l=2}^{m_1}(x-a_{1(l)})\right) - a_{1(1)}r(1,a_{1(2)},\ldots,a_{1(m_1)}) = r(1,a_{1(a)},\ldots,a_{1(m_1)});$$

on doing so it remains simply to rearrange the resulting expression to produce our desired formula.

□

# Chapter 4

# Factorisation of chromatic polynomials

A fundamental algebraic property of a polynomial is its factorisability. As with much of the algebraic theory of chromatic polynomials, little is known about the way in which they may factorise. In this section we will survey the current knowledge and present some new results on this topic.

## 4.1 Background

It follows from the definition of a chromatic polynomial that, for any graph $G$, $P_G(x)$ is always divisible by the linear factors

$$x, (x-1), \ldots, (x-\chi(G)+1),$$

where $\chi(G)$ is the *chromatic number* of $G$; that is, the smallest number of colours required in order to properly colour $G$. However, these are not the only linear factors which may occur in a factorisation of a chromatic polynomial. As the chromatic polynomial of a graph consisting of more than one connected component is the product of those of the components, the multiplicity of the factor $x$ in $P_G(x)$ corresponds to the number of connected components of $G$. Similarly, the multiplicity of $(x-1)$ gives the number of blocks of $G$.

Moreover, other linear factors can appear in connected graphs. For example, chordal graphs have only integer chromatic roots, and so their chromatic polynomials factorise completely into linear factors. It was at one point conjectured that these were the only graphs having purely integral chromatic roots, but Read [35] found a counterexample in 1975. As mentioned in §3.2.1, Dong *et al* then found various other examples of integral-root chromatic polynomials [15].

By Proposition 2.10, clique-separable chromatic polynomials factorise into polynomials corresponding to the subgraphs involved in the separation. Farr and Morgan [16] showed furthermore that there exist graphs which are not clique separable, but which factorise in the same way as clique-separable graphs. They identified all such graphs of order at most 10, and showed that each of these are chromatically equivalent to a clique-separable graph. There also exist chromatic polynomials which have more than one

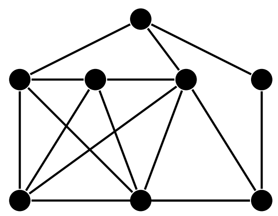

Figure 4.1: The smallest graph with a chromatic polynomial factorisation not corresponding to clique-separability

non-linear factor, and whose factorisations do not correspond to clique-separations (that is, they have at least one irreducible non-linear factor which is not a factor of a chromatic polynomial of lower degree). One of the two smallest examples found by Morgan [29] is the graph displayed in Fig 4.1, which has the following chromatic polynomial

$$x(x-1)(x-2)(x^2-4x+6)(x^3-8x^2+23x-23).$$

It can be easily verified that there is no chromatic polynomial of degree less than 8 having $x^2-4x+6$ as a factor, which means that this factorisation cannot correspond to a simple clique separation.

The question of which polynomials can actually occur as factors of chromatic polynomials is of course the same as that of which algebraic integers are chromatic roots, and it is this aspect of the subject of chromatic factorisation that we will focus on here.

## 4.2 Conditions on coefficients

We begin this section with a conjecture on the form of factors of chromatic polynomials, which we will refer to as *chromatic factors* for brevity. It is well known that the coefficients of chromatic polynomials alternate in sign, however this by no means implies that the same property holds for chromatic factors. A proof of the following conjecture would dramatically increase our knowledge of which algebraic integers can appear as chromatic roots.

**Conjecture 4.1.** *The coefficients of a chromatic factor are always alternating in sign.*

Extensive computer searches [40] have not produced any examples of non-alternating chromatic factors. However, as mentioned previously, due to the complexity of computing chromatic polynomials of graphs of high order, little is known about them, and history tells us that conjectures based on the properties of chromatic polynomials of low degree should always be made tentatively.

It is unclear how to approach a proof of Conjecture 4.1, however we can verify it for some very special cases. The following proposition rules out the existence of chromatic factors which are alternating in all but the constant coefficient.

**Proposition 4.2.** *There are no chromatic factors whose constant term is the same sign as its $x$ term, and whose coefficients are otherwise alternating in sign.*

*Proof.* Let $G$ be a graph of order $n$ with chromatic polynomial

$$P_G(x) = x^n - a_{n-1}x^{n-1} + \ldots \pm a_1 x \mp a_0.$$

Note that, as the coefficients of $P_G(x)$ alternate, those of $(-1)^n P_G(-x)$ are all positive. Let $g(x) = \sum_{i=0}^{m} b_i x^i$, where $b_m > 0$, $b_0 < 0$ and $b_i \geq 0$ for $m > i > 0$. We will show that $g$ cannot be a factor of $(-1)^n P_G(-x)$.

Suppose $h(x) = \sum_{i=0}^{p} c_i x^i$ is such that $g(x)h(x) = (-1)^n P_G(-x)$. Then $a_0 = b_0 c_0$, and so $c_0 < 0$. Now suppose that $c_k < 0$ for all $0 < k < r$. Every summand of

$$a_r = \sum_{i=0}^{r} b_i c_{r-i}$$

other than $b_0 c_r$ is non-positive. But $a_r \geq 0$, and so $c_r < 0$. So by induction, every coefficient $c_i$ of $h$ is negative. But $a_n = b_m c_p$, and $b_m > 0$. So $c_p$ must be positive, and we have a contradiction.

Thus we have shown that $(-1)^n P_G(-x)$ has no factor with negative constant coefficient and all other coefficients non-negative. It follows that $P_G(x)$ has no factor whose constant term and $x$-coefficient have the same sign, and whose coefficients are otherwise alternating. □

Furthermore, if we impose the condition that a chromatic polynomial has only one non-linear factor (this does appear to be the case for the vast majority of chromatic polynomials), then we can show that the coefficient of the term of second-highest degree in this factor must be negative.

**Proposition 4.3.** *Suppose a chromatic polynomial $P_G(x) = \sum_{i=0}^{n} a_i x^i$ factorises as $x(x-1)\ldots(x-n+k+1)f(x)$, where $f(x) = \sum_{j=0}^{k} b_j x^j$. Then $b_{k-1} \leq 0$.*

*Proof.* We will use that fact that $a_{n-1}$ is the number of edges of $G$ (this is not difficult to prove; see, for example [13, Theorem 2.2.1]). By the factorisation of $P_G(x)$,

$$a_{n-1} = 1 + 2 + \ldots + (n-k-1) - b_{k-1} = \binom{n-k}{2} - b_{k-1}.$$

The chromatic number $\chi(G)$ of $G$ is $n-k$, and in an optimal colouring of $G$, there must be at least one edge between every pair of colour classes. So

$$a_{n-1} \geq \binom{n-k}{2}.$$

Hence $b_{k-1} \leq 0$ (note that if $f(x)$ is irreducible we must have $b_{k-1} = \binom{n-k}{2} - |E(G)|$). □

Now, as there are no negative real chromatic roots, a necessary and sufficient condition for the zeros of an irreducible chromatic factor $f(x) = x^2 + bx + c$ to be complex with negative real part is for $b$ to be positive.

A special case is if all of the other factors of the chromatic polynomial in question are linear; then by Proposition 4.3 $b \leq 0$, from which we may draw the conclusion that if a chromatic polynomial $P_G(x)$ factors into linear factors and one quadratic factor then it has no zeros with negative real part.

Of course, a proof of Conjecture 4.1 would imply that there are no quadratic chromatic roots with negative real part at all, and this would generalise to higher degrees: in [10] the authors prove that a positive polynomial (one having strictly non-negative coefficients) of degree $n$ has no zeros $\alpha$ with $|\arg(\alpha)| \leq \pi/n$. Given that a zero of an alternating polynomial is of the form $-\alpha$, where $\alpha$ is a zero of some positive polynomial, this implies that an alternating polynomial of degree $n$ has no zeros $\alpha$ with $|\arg(\alpha)| \geq (n-1)\pi/n$. Thus a verification of Conjecture 4.1 would imply that for each $n \in \mathbb{N}$ the region $(n-1)\pi/n \leq |\arg(z)|$ contains no chromatic roots of degree less than $n+1$.

## 4.3 Quadratic factors

We will begin this section by presenting a conjecture on the relative positions of coefficients of quadratic factors of chromatic polynomials in the real line, and commenting on its potential implication for the location of quadratic chromatic roots.

**Conjecture 4.4.** *Suppose $p(x) = x^2 - bx + c$ divides a chromatic polynomial. Then $b > 0$, $b \leq c$ and $c \leq (b^2 + b)/2$.*

The evidence for this conjecture is, admittedly, rather weak. However, we have verified it for all chromatic polynomials of degree less than 11, as well as a number of polynomials of much higher degree belonging to certain special families such as theta graphs.

In the case that $b^2 - 4c > 0$, the first two assertions of Conjecture 4.4 follow from known facts about the location of chromatic roots in the real line: firstly, there are no negative real chromatic roots, so $b$ must be positive. Secondly, there are no chromatic roots in the interval $(0, 1)$: as one of the zeros of $p(x)$ is $(b - \sqrt{b^2 - 4c})/2$, this means that we must have $\sqrt{b^2 - 4c}/2 < b/2 - 1$. That is, $\sqrt{b^2 - 4c} < b - 2$. Squaring both sides of the inequality and rearranging gives $c > b - 1$.

Now, let $p(x) = x^2 - bx + c$ divide a chromatic polynomial, where $b^2 - 4c < 0$, and let $\alpha$ be a zero of $p(x)$. If we were to suppose that the first and third of the assertions of the conjecture are true, we would have the following:

$$\begin{aligned}
|\Im(\alpha)| &= |\sqrt{b^2 - 4c}/2| \\
&= \sqrt{|b^2 - 4c|}/2 \\
&\leq \sqrt{|-b^2 - 2b|}/2 \\
&= \sqrt{|b^2 + 2b|}/2 \\
&< \sqrt{|b^2 + 2b + 1|}/2 \\
&= (b+1)/2 \\
&= \Re(\alpha) + 1/2.
\end{aligned}$$

That is, the entire region of the complex plane defined by $|\Im(z)| < \Re(z) + 1/2$ could be shown to contain no quadratic chromatic roots.

Now we present a result which gives a lower bound on the order of a graph having chromatic polynomial which is divisible by certain quadratic factors.

**Theorem 4.5.** *If $b > 4$ and $0 < c < 2b - 4$ then $x^2 - bx + c$ does not divide any chromatic polynomial of degree less than $b$.*

The proof of this theorem depends on the fact that every chromatic polynomial is a sum of positive multiples of falling factorials. In fact it could be more generally formulated for any polynomial which can be written as such a sum. The proof is quite long, especially given that fact that the theorem is not of great consequence. However, the techniques used are quite novel, and could potentially be useful for other applications.

Given a quadratic polynomial $p(x)$ and another polynomial $f(x)$ we will represent by $[f(x)]_{p(x)}$ the coefficient vector of $f(x) \mod p(x)$, so that we have:

$$f(x) \equiv [f(x)]_{p(x)}^T \begin{bmatrix} 1 \\ x \end{bmatrix} \mod p(x).$$

We will require the following lemma.

**Lemma 4.6.** *Let $p(x) = x^2 - bx + c$, for some integers $b$ and $c$. Then, for all $i \geq 2$:*

$$[(x)_i]_{p(x)} = \begin{bmatrix} 1-n & -c \\ 1 & 1-i+b \end{bmatrix} [(x)_{i-1}]_{p(x)}.$$

*Proof.* Note that

$$(x)_i = (x-i+1)(x)_{i-1} = x(x)_{i-1} + (1-i)(x)_{i-1},$$

and let

$$C(p(x)) = \begin{bmatrix} 0 & -c \\ 1 & b \end{bmatrix}$$

be the companion matrix of $p(x)$. Then we have:

$$\begin{aligned} [(x)_i]_{p(x)} &= [x(x)_{i-1}]_{p(x)} + [(1-i)(x)_{i-1}]_{p(x)} \\ &= C(p(x))[(x)_{i-1}]_{p(x)} + (1-i)I[(x)_{i-1}]_{p(x)} \\ &= \begin{bmatrix} 1-i & -c \\ 1 & 1-i+b \end{bmatrix} [(x)_{i-1}]_{p(x)}. \end{aligned}$$

□

We are now ready to proceed with the proof of the theorem.

*Proof of Theorem 4.5.* Let $p(x) = x^2 - bx + c$, where $b > 4$ and $0 < c < 2b - 4$. Recall that the chromatic polynomial $P_{K_i}(x)$ of the complete graph on $i$ vertices is the falling factorial $(x)_i = x(x-1)\cdots(x-i+1)$, and that any chromatic polynomial is a sum of positive multiples of chromatic polynomials of complete graphs. This means that if $G$ has order $n$, then we have:

$$P_G(x) = P_{K_n}(x) + a_{n-1}P_{K_{n-1}}(x) + \cdots + a_2 P_{K_2}(x) + a_1 x,$$

and so:

$$[P_G(x)]_{p(x)} = [(x)_n]_{p(x)} + a_{n-1}[(x)_{n-1}]_{p(x)} + \cdots + a_2[(x)_2]_{p(x)} + a_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

where each $a_i$ is non-negative. If we set

$$A_i = \begin{bmatrix} 1-i & -c \\ 1 & 1-i+b \end{bmatrix},$$

then from Lemma 4.6 we obtain:

$$[P_G(x)]_p = A_n A_{n-1} \cdots A_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + a_{n-1} A_{n-1} \cdots A_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \cdots + a_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (4.1)$$

Let $\alpha_1$ and $\alpha_2$ be the two roots of $p(x) = 0$. As $A_i = C(p(x)) + (1-i)I$ for any $i \geq 2$, the eigenvalues of $A_i$ are

$$\lambda_{1_i} = 1 - i + \alpha_2$$

$$\lambda_{2_i} = 1 - i + \alpha_1,$$

and eigenvectors corresponding to these are, respectively, $\begin{bmatrix} -\alpha_1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -\alpha_2 \\ 1 \end{bmatrix}$.

Let $A = A_j A_{j-1} \ldots A_2$ for some $2 \leq j \leq n$. As the eigenvectors of the $A_i$ are independent of $i$, they are also eigenvectors for $A$, and the corresponding eigenvalues $\lambda_1$ and $\lambda_2$ of $A$ are then the product of the eigenvalues of the $A_i, 2 \leq i \leq j$, namely:

$$\begin{aligned} \lambda_1 &= \prod_{i=2}^{j} \lambda_{1_i} = \prod_{i=2}^{j} (1 - i + \alpha_2) = (\alpha_2 - 1)_{j-1} \\ \lambda_2 &= \prod_{i=2}^{j} \lambda_{2_i} = \prod_{i=2}^{j} (1 - i + \alpha_1) = (\alpha_1 - 1)_{j-1}. \end{aligned} \quad (4.2)$$

Thus we can diagonalise $A$ in the standard way. Let

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad B = \begin{bmatrix} -\alpha_1 & -\alpha_2 \\ 1 & 1 \end{bmatrix}.$$

Then $D = B^{-1}AB$, and so

$$A = BDB^{-1} = \frac{1}{\alpha_1 - \alpha_2} \begin{bmatrix} \lambda_1\alpha_1 - \lambda_2\alpha_2 & \alpha_1\alpha_2(\lambda_1 - \lambda_2) \\ \lambda_2 - \lambda_1 & \lambda_2\alpha_1 - \lambda_1\alpha_2 \end{bmatrix}$$

As $A$ multiplies $\begin{bmatrix}0\\1\end{bmatrix}$ in (4.1), we will only be concerned with its second column. Consider the $(1,2)$-entry $(A)_{1,2}$ of $A$. Using the identities of (4.2), this becomes:

$$(A)_{1,2} = \frac{\alpha_1\alpha_2}{\alpha_1 - \alpha_2}[(\alpha_2 - 1)_{j-1} - (\alpha_1 - 1)_{j-1}]. \tag{4.3}$$

Now suppose that $\alpha_1 = (b - \sqrt{b^2 - 4c})/2$, and that therefore $\alpha_2 = (b + \sqrt{b^2 - 4c})/2$. As $b > 4$ and $0 < c < 2b - 4$, it is not difficult to see that $\alpha_1$ and $\alpha_2$ are real, and that we have with

$$0 < \alpha_1 < 2 < b - 2 < \alpha_2 < b.$$

This means that every factor of $(\alpha_2 - 1)_{b-2}$, and hence every factor of $(\alpha_2 - 1)_{j-1}$ for $2 \le j < b$, is positive. Clearly then, for all $2 \le j < b$:

$$(\alpha_2 - 1)_{j-1} > (\alpha_1 - 1)_{j-1}. \tag{4.4}$$

Now, consider the expression given in (4.3) for the $(1,2)$ entry $(A)_{1,2}$ of $A = A_jA_{j-1}\ldots A_2$. Looking at the constituent parts of this formula, we have the following:

- $\alpha_1\alpha_2 = c > 0$;
- $\alpha_1 - \alpha_2 = -\sqrt{b^2 - 4c} < -b + 4 < 0$;
- $(\alpha_2 - 1)_{j-1} - (\alpha_1 - 1)_{j-1} > 0$.

Hence $(A)_{1,2}$ is negative.

To complete the proof, we suppose that $p(x)$ divides a chromatic polynomial $P_G(x)$ of degree $n < b$. Then using the previous notation for the coefficient vector of the residue of a polynomial modulo $p(x)$, we have that $[P_G(x)]_{p(x)} = \begin{bmatrix}0\\0\end{bmatrix}$. So (4.1) gives us a pair of linear equations in the $\{a_i\}$, $1 \le i \le n-1$, represented by:

$$[P_G(x)]_{p(x)} = A_nA_{n-1}\cdots A_2\begin{bmatrix}0\\1\end{bmatrix} + a_{n-1}A_{n-1}\cdots A_2\begin{bmatrix}0\\1\end{bmatrix} + \cdots + a_1\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}, \tag{4.5}$$

where each $a_i$ is the coefficient of $(x)_i$ in the representation of $P_G(x)$ as a sum of falling factorials.

For each $0 \leq j \leq n$, let $x_j$ be the $(1, 2)$-entry of the matrix $A_j A_{j-1} \ldots A_2$. Then the first of the two equations in (4.5) can be written:

$$x_n + a_{n-1}x_{n-1} + a_{n-2}x_{n-2} + \ldots + a_2 x_2 = 0. \qquad (4.6)$$

However, as we have established, each $x_j$ is negative. So in any non-zero solution to (4.6) at least one of the $a_i$ must be negative, which contradicts the fact that every chromatic polynomial is the sum of positive multiples of falling factorials. □

Now, there have been a number of conjectures in the past as to which graphs may or may not have chromatic roots in the interval $(1, 2)$ (see, for example, [41, 12]). Theorem 4.5 has the following corollary regarding quadratic chromatic roots in this interval.

**Corollary 4.7.** *Let $G$ be a graph. Then any quadratic chromatic root of $G$ having rational part greater than $|V(G)|/2$ lies outside the interval $(1, 2)$.*

*Proof.* Suppose there is a real quadratic zero $\alpha$ of $P_G(x)$ with rational part greater than $|V(G)|/2$ lying in the interval $(1, 2)$. Let $p(x) = x^2 - bx + c$ be the corresponding irreducible factor of $P_G(x)$; then the rational part of $\alpha$ is $b/2$. Any chromatic root of a graph on less than 4 vertices is an integer, and so $|V(G)| \geq 4$, which implies that $b > 4$. This means that $\alpha$ is of the form $(b - \sqrt{b^2 - 4c})/2$, and so we have $(b - \sqrt{b^2 - 4c})/2 < 2$, giving $b - 4 < \sqrt{b^2 - 4c}$. Squaring both sides and rearranging we find that $8b - 16 > 4c$, and so $c < 2b - 4$. Finally note that $c > 0$, as it is the product of two positive real numbers.

So we have shown that if $G$ has a chromatic root $\alpha$ with rational part greater than $|V(G)|/2$ lying in the interval $(1, 2)$ then $P_G(x)$ is divisible by $p(x) = x^2 - bx + c$, where $b > 4$ and $0 < c < 2b - 4$. By Theorem 4.5 then, $G$ must have order at least $b$. This contradicts the assertion that the rational part of $\alpha$ is greater than $|V(G)|/2$. □

It should be noted that the above corollary is not a new result. Let $\beta = (108 + 12\sqrt{93})^{1/3}$. In [11] Dong showed that for $6 \leq n \leq 8$ and

$n \geq 9$, the largest non-integer real chromatic roots of graphs of order $n$ are, respectively, $n - 4 + \beta/6 - 2/\beta$ and $(n - 1 + \sqrt{(n-3)(n-7)})/2$. If a real quadratic integer lies in the interval $(1, 2)$ and has rational part greater than $n/2$ then its conjugate will be greater than $n - 2$. in this sense Corollary 4.7 is implied by Dong's result.

We finish this section by mentioning one final implication of the above results on the location of chromatic roots of certain dense graphs.

**Corollary 4.8.** *Suppose a graph $G$ of order $n$ has chromatic number $\chi(G) = n - 2$. Then, if $|E(G)| > (n^2 - 3n + 6)/2$, $P_G(x)$ has no real zeros in the interval $(1, 2)$.*

*Proof.* If $G$ has chromatic number $n - 2$, then $P_G(x) = x(x-1)\dots(x - n + 3)(x^2 - bx + c)$. As mentioned previously, the coefficient of $x^{n-1}$ in $P_G(x)$ is precisely $-|E(G)|$. Hence $|E(G)| = 1 + 2 + \cdots + (n-3) + b = (n-3)(n-2)/2 + b$. So if $|E(G)| > (n^2 - 3n + 6)/2$, then $b > (n^2 - 3n + 6)/2 - (n-3)(n-2)/2 = n$. So the rational part $b/2$ of the zeros of $x^2 - bx + c$ is greater than $n/2$, and by Corollary 4.7, the zeros of $x^2 - bx + c$—and hence all zeros of $P_G(x)$—lie outside the interval $(1, 2)$. □

It is not difficult to construct graphs satisfying the hypotheses of Corollary 4.8; indeed the subfamily of bicliques discussed in §6.3 provide many such examples. To see this, let $G$ be a $(2, n-2)-$biclique, with $m > n - 1$ bridging edges. Then $G$ has chromatic number $n - 2$, and

$$|E(G)| = \binom{n-2}{2} + m + 1 = \frac{n^2 - 5n + 2m + 8}{2} > \frac{n^2 - 3n + 6}{2}.$$

# Chapter 5

# The $\alpha + n$ conjecture

In [8] it was conjectured that for every algebraic integer $\alpha$ there is a natural number $n$ such that $\alpha + n$ is a chromatic root. The authors proved this conjecture for quadratic integers, but it has remained unresolved for algebraic numbers of higher degree. In this chapter we present an alternative proof of the quadratic case, and then show how a similar technique can be used to prove the conjecture for cubic integers.

Our methods are constructive: given any quadratic or cubic integer $\alpha$, we not only show that there is, respectively, a $(2, k)$-biclique or $(3, k)$-biclique having a chromatic root which is an integer translation of $\alpha$, but detail an explicit construction which produces a large (in most cases infinite) family of graphs satisfying the hypothesis.

Indeed, an interesting question raised by this work is that of *how many* graphs there are having a chromatic root which is a shift of each given algebraic integer. This question is equivalent to that of how often different number fields arise as splitting fields of factors of chromatic polynomials: once we know that every possible such field arises in this way, the natural next question is as to whether or not some are more prevalent than others, and why. This is beyond the scope of the current work, and would be likely to require the application of probabilistic and enumerative methods. However, in the next chapter we do approach this question obliquely, by investigating operations on graphs which preserve such splitting fields.

Referring back to §3.1, we recall the useful fact that the chromatic polynomial of any given biclique factorises into one "interesting factor"and otherwise linear factors. Note that we need only concern ourselves with these interesting factors when considering algebraic properties.

We will write that a polynomial of degree $d$ is *reduced* if its second-highest degree term has coefficient lying in the set $\{0, 1, \ldots, d-1\}$. The following simple observation is key to our proof method.

**Lemma 5.1.** *For any irreducible polynomial $f(x) \in \mathbb{Z}[x]$ there is some integer $m$ such that $f(x-m)$ is reduced.*

*Proof.* Let

$$f(x) = x^d + a_{d-1}x^{d-1} + \ldots + a_1 x + a_0$$

be some polynomial in $\mathbb{Z}[x]$. Then setting $m = \lfloor \frac{a_{d-1}}{d} \rfloor$ will give us $0 \leq [x^{d-1}]f(x-m) \leq d-1$, as required. □

Our proof method is as follows: given some algebraic integer $\alpha$, we wish to show that there is some $n \in \mathbb{N}$ such that $\alpha + n$ is a chromatic root. By definition, there exists a minimal polynomial of $\alpha$; that is, a monic, irreducible polynomial $f(x) \in \mathbb{Z}[X]$ of degree $d$ such that $f(\alpha) = 0$. And by Lemma 5.1, there is a unique integer $m$ such that $f(x-m)$ is reduced. Clearly, $\alpha + m$ is a zero of this reduced polynomial.

Suppose now that we can show there is a chromatic factor $g(x)$ and integer $k$ such that $f(x-m) = g(x+k)$. This means that $f(x) = g(x+k+m)$, and setting $n = k+m$, we have that $\alpha$ is a zero of $g(x+n)$, implying that $\alpha + n$ is a zero of $g(x)$, and hence a chromatic root. By dispensing with direct consideration of algebraic integers, and focusing instead on their minimal polynomials, the above reasoning leads us to the following much more approachable statement of the $\alpha + n$ conjecture.

**Conjecture 5.2.** *For every reduced polynomial $f(x)$ there is a chromatic factor $g(x)$ and integer $n$ such that $g(x+n) = h(x)$.*

As mentioned above, we will show that the chromatic polynomials of bicliques furnish all the chromatic factors necessary to prove the quadratic and cubic cases of this conjecture.

## 5.1 Quadratic integers

The interesting factor of a $(2, k)$-biclique is, simply:

$$g_{a,b}(x) = (x - a)(x - b) - (x - a - b), \tag{5.1}$$

where $a$ and $b$ represent the numbers of neighbours in the $k$-clique of each of the vertices of the 2-clique (this follows from a basic application of the Möbius inversion method of §3.1). In order to prove our result it suffices to show that, given any reduced quadratic polynomial $h(x)$, there are natural numbers $a, b$ and $n$ such that $h(x) = g_{a,b}(x + n)$.

To begin with, suppose the $x$-coefficient of $h(x)$ is zero, so $h(x) = x^2 + a_0$ for some $a_0 \in \mathbb{Z}$. Let

$$a = -1/2 + n - (\sqrt{4n - 4a_0 - 3})/2$$
$$b = -1/2 + n + (\sqrt{4n - 4a_0 - 3})/2.$$

Then $g_{a,b}(x) = x^2 - 2nx + n^2 + a_0 = h(x - n)$, as desired. By choosing $n$ high enough, and such that $4n - 4a_0 - 3$ is a perfect square, we can always ensure that $a$ and $b$ are non-negative integers.

Note that if $a_0$ is positive here, we can simply choose $a = a_0$ and $b = a_0 + 1$; this would give $g_{a,b}(x) = h(x - a_0 - 1)$.

The second case we need to consider is where the $x$-coefficient of $h(x)$ is 1, that is, where: $h(x) = x^2 + x + a_0$ for some $a_0 \in \mathbb{Z}$. We can approach this in a similar way: this time let

$$a = -1 + n - \sqrt{n - a_0 - 1}$$
$$b = -1 + n + \sqrt{n - a_0 - 1}.$$

Then $g_{a,b}(x) = x^2 + (1 - 2n)x + n^2 + a0 - n = h(x - n)$. Again, by choosing $n$ high enough, and such that $n - a_0 - 1$ is a perfect square, we can ensure that $a$ and $b$ are non-negative integers.

In an analogous way to the previous case, if $a_0$ is positive there is a particularly simple solution: choose $a = a_0$ and $b = a_0 + 2$, then we have $g_{a,b}(x) = h(x - a_0 - 2)$.

So we have shown that, given any quadratic reduced polynomial $h(x)$, we can find non-negative integers $a, b$ and $n$ such that $g_{a,b}(x+n) = h(x)$, thus proving the quadratic case of the $\alpha + n$ conjecture.

## 5.2 Cubic integers

Let $G$ be a $(3, k)$-biclique. Label the three vertices of the 3-clique $v_1, v_2$ and $v_3$; let $a, b$ and $c$ represent, respectively, the number of neighbours of $v_1, v_2$ and $v_3$ in the $k$-clique; and let $d, e$ and $f$ represent the number of vertices in the $k$-clique joined to both $v_2$ and $v_3$, both $v_1$ and $v_3$, and both $v_1$ and $v_2$ respectively. Then we can use the second construction of the chromatic polynomials of bicliques given in §3.1 to construct the interesting factor of the chromatic polynomial of $G$ as follows:

$$\begin{aligned} g(x) = {} & (x-a-e-f)(x-b-d-f)(x-c-d-e) \\ & -(x-a-b-d-e-f)(x-c-d-e) \\ & -(x-a-c-d-e-f)(x-b-d-f) \\ & -(x-b-c-d-e-f)(x-a-e-f) + 2(x-a-b-c-d-e-f). \end{aligned} \tag{5.2}$$

As in the previous section, it suffices to show that, given any reduced cubic polynomial $h(x)$, there is an interesting factor $g(x)$ and natural number $n$ such that $h(x) = g(x+n)$.

We will proceed with each of the three types of reduced polynomial in turn, showing that for each type, and for every choice of the $x-$coefficient and constant term, the parameters $a, \ldots, f$ can be chosen in such a way as to produce the desired chromatic polynomial. There are no doubt many possible ways in which to correctly choose the parameters; in each case we will mention just one.

**Case 1:** $a_2 = -1$

Let $h(x) = x^3 - x^2 + a_1 x + a_0$, and let $i$ represent any number. Assign the below values to the parameters $a, b, c, d, e, f$:

$$\begin{aligned}
a &= (2n + a_0)^2 - 11a_0 + 35 + a_1 - (8a_0 - 45)i - (16i + 24)n + 16i^2 \\
b &= -2i + n - 3 \\
c &= (2n + a_0)^2 - 13a_0 + 46 + a_1 - (8a_0 - 53)i - (16i + 28)n + 16i^2 \\
d &= i + 1 \\
e &= -(2n + a_0)^2 + 12a_0 - 41 - a_1 + (8a_0 - 50)i + (16i + 27)n - 16i^2 \\
f &= i
\end{aligned}$$

Let $g(x)$ be the polynomial obtained by substituting these values into (5.2). Then we have

$$g(x) = x^3 + (-3n - 1)x^2 + (3n^2 + 2n + a_1)x - n^3 - n^2 - a_1 n + a_0 = h(x - n),$$

as desired. It remains to show that, for any $a_0$ and $a_1$, appropriate values for $i$ and $n$ can be found such that each of the above parameters are non-negative integers. From the expressions for $b, d$ and $f$, $i$ must be non-negative and $n$ must satisfy $n \geq 2i + 3$. We introduce a new variable $t$ by making the substitution

$$n = -a_0/2 + 2i + t,$$

giving us new expressions for $a, c$ and $e$:

$$\begin{aligned}
a &= a_0 + 35 + a_1 - 3i - 24t + 4t^2 \\
c &= a_0 + 46 + a_1 - 3i - 28t + 4t^2 \\
e &= -3a_0/2 - 41 - a_1 + 4i + 27t - 4t^2
\end{aligned}$$

Requiring that all these be non-negative then gives us the three inequalities:

$$3i \leq a_0 + 35 + a_1 - 24t + 4t^2 \tag{5.3}$$

$$3i \leq a_0 + 46 + a_1 - 28t + 4t^2 \tag{5.4}$$

$$4i \geq 3a_0/2 + 41 + a_1 - 27t + 4t^2 \tag{5.5}$$

Let $t$ be an integer that is greater than 3, greater than $a_0/2 + 3$, and otherwise large enough to satisfy:

$$\frac{a_0 + 46 + a_1 - 28t + 4t^2}{3} \geq \frac{3a_0/2 + 41 + a_1 - 27t + 4t^2}{4} + 1.$$

There is at least one integer between the expression on the left and the quotient on the right. Choose $i$ to be such an integer; then the chosen values for $i$ and $t$ satisfy (5.4) and (5.5). Because $t \geq 3$, (5.4) implies (5.3). Finally set $n = \lceil -a_0/2 \rceil + 2i + t$. Because $t > a_0/2 + 3$, we then have that $n$ satisfies the condition $n \geq 2i + 3$.

The remaining two cases are similar, and so will be more briefly described.

## Case 2: $a_2 = 0$

Let $h(x) = x^3 + a_1 x + a_0 x$, and again let $i$ be any number. This time set:

$$\begin{aligned}
a &= (n + a_0)^2 + a_1 + 14 + 19i + 9i^2 - (6i + 8)n - (6i + 6)a_0 \\
b &= -2i + n - 3 \\
c &= (n + a_0)^2 + a_1 + 20 + 25i + 9i^2 - (6i + 10)n - (6i + 8)a0 \\
d &= i + 1 \\
e &= -(n + a_0)^2 - a_1 - 18 - 23i - 9i^2 + (6i + 10)n + (6i + 7)a_0 \\
f &= i
\end{aligned}$$

Let $g(x)$ be the polynomial obtained by substituting these values into (5.2). Then

$$g(x) = x^3 - 3nx^2 - (3n^2 - a_1 + 3n^2)x - n^3 - a_1 n + a_0 = h(x - n).$$

Now make the substitution

$$n = -a_0 + 3i + t.$$

This gives us the following expressions for $a, c$ and $e$:

$$\begin{aligned}
a &= t^2 + a_1 + 14 - 5i + 2a_0 - 8t \\
c &= t^2 + a_1 + 20 - 5i + 2a_0 - 10t \\
e &= -t^2 - a_1 - 18 + 7i - 3a_0 + 10t,
\end{aligned}$$

leading to the inequalities:

$$\begin{aligned}
5i &\leq t^2 + a_1 + 14 + 2a_0 - 8t \\
5i &\leq t^2 + a_1 + 20 + 2a_0 - 10t \\
7i &\geq t^2 + a_1 + 18 + 3a_0 + 10t.
\end{aligned}$$

Again, by choosing $t$ to be very large, a positive value for $i$ can be found to satisfy these for any $a_0, a_1$.

## Case 3: $a_2 = 1$

Let $h(x) = x^3 + x^2 + a_1x + a_0x$, and set:

$$\begin{aligned}
a &= a_0^2 + 5 - a_0 + a1 + (3 - 4a_0)i - 2n + 4i^2 \\
b &= -2i + n - 3 \\
c &= a_0^2 + 6 - 3a_0 + a_1 + (7 - 4a_0)i - 2n + 4i^2 \\
d &= i + 1 \\
e &= -a_0^2 - 7 + 2a_0 - a_1 - (6 - 4a_0)i + 3n - 4i^2 \\
f &= i
\end{aligned}$$

Substituting into (5.2) we obtain

$$g(x) = x^3 + (1 - 3n)x^2 + (3n^2 - 2n + a1)x - n^3 + n^2 - a1n + a0 = h(x - n).$$

We now express $i$ in terms of a new parameter $t$, by setting:

$$i = a_0/2 - t.$$

This gives us

$$\begin{aligned}a &= 5 + a_0/2 + a_1 - 3t - 2n + 4t^2\\ c &= 6 + a_0/2 + a_1 - 7t - 2n + 4t^2\\ e &= -7 - a_0 - a_1 + 6t + 3n - 4t^2,\end{aligned}$$

and so we must satisfy

$$\begin{aligned}2n &\leq 5 + a_0/2 + a_1 - 3t + 4t^2\\ 2n &\leq 6 + a_0/2 + a_1 - 7t + 4t^2\\ 3n &\geq 7 + a_0 + a_1 - 6t + 4t^2.\end{aligned}$$

This time we need to choose a large negative value for $t$. If it is large enough then $d$ and $f$ will be non-negative, and we can easily find a positive $n$ to satisfy the three inequalities, as well as the requirement $n \geq 2i + 3$.

Thus we have given a means to construct a $(3, k)$-clique with a chromatic root $\alpha + n$ for any cubic integer $\alpha$, thereby proving the cubic case of the $\alpha + n$ conjecture.

**Remark 5.3.** Given the exponential increase in the number of $(j, k)$-cliques as $j$ increases (constructed as in §3.1, a $(j, k)$-clique has $2^j - 2$ parameters), it seems entirely plausible that they might satisfy the general conjecture. Unfortunately the increase in parameters leads to difficulties in finding correct specialisations in the manner of the two cases proved so far, and it seems likely that a different method from that used here would need to be found for algebraic numbers of higher degree.

# Chapter 6

# Chromatic splitting field-equivalence

The main motivation behind this chapter could perhaps best be summed up by the simple question: how is the the abstract structure of a graph related to the algebraic properties of its chromatic roots?

Recall that the splitting field of a polynomial is the smallest field extension of $\mathbb{Q}$ in which that polynomial factorises entirely into linear factors (equivalently, the smallest field extension containing every zero of the polynomial), and that we refer to the splitting field of the chromatic polynomial of a graph as its *chromatic splitting field*. An obvious way to investigate correspondences between graphs' structures and the algebraic properties of their chromatic roots is to look for families of graphs having the same chromatic splitting field. A number of such families will be presented and studied in this chapter.

As a means of constructing these families, we will be interested in certain graph operations—by which we simply mean sequences of additions and removals of edges and vertices—which preserve chromatic splitting fields. Some types of chromatic splitting field-preserving operation can be performed on any graph. For example, let $G$ be a graph, and let $H$ be the new graph formed by adding a single vertex $v$ to $G$ connected to only one of the existing vertices. Then for any proper $x$-colouring for $G$ there are additionally $x - 1$ choices of colour for $v$, and so $P_H(x) = (x - 1)P_G(x)$; thus $H$ clearly has the same chromatic splitting-field as $G$. Similarly, if $G$

contains a $k$-clique, then connecting a new vertex to every vertex of this clique will give us a new graph with chromatic polynomial $(x-k)P_G(x)$.

In Proposition 2.9 we proved that the join of any graph $G$ with an $n$-clique has chromatic polynomial $(x)_n P_G(x-n)$. This gives a slightly less trivial chromatic splitting field-preserving operation, and one which has the following interesting consequence:

**Corollary 6.1.** *If $\alpha$ is a chromatic root, then so too is $\alpha+n$ for all $n \in \mathbb{N}$.*

We shall discuss some much more unexpected chromatic splitting field-preserving operations which can be performed on some of the graphs from Chapter 3. The first of these enables us to prove that a multiplicative analogue of Corollary 6.1 holds for certain chromatic roots.

## 6.1 The $n\alpha$ conjecture

The following was proposed by Cameron and Morgan in [8]:

**Conjecture 6.2** (The $n\alpha$ conjecture)**.** *If $\alpha$ is a chromatic root, then so too is $n\alpha$ for any natural number $n$.*

This conjecture is currently very far from being resolved either way. However, in connection with it the authors did make the following observation (recall from §3.2.1 that $R(a_1, \ldots, a_k)$ represents a ring of cliques):

**Proposition 6.3.** *If $\alpha$ is a non-integer chromatic root of $R(1, a_2, \ldots, a_k)$, then $n\alpha$ is a chromatic root of $R(1, na_2, \ldots, na_k)$.*

*Proof.* Any non-integer zero of the chromatic polynomial of $R(1, a_2, \ldots, a_k)$ is a zero of the interesting factor:

$$\frac{1}{x}\left(\prod_{i=2}^{k}(x-a_i) - \prod_{i=2}^{k}(-a_i)\right), \tag{6.1}$$

as all other factors are linear. The corresponding factor of the chromatic polynomial of $R(1, na_2, \ldots, na_k)$ is

$$\frac{1}{x}\left(\prod_{i=2}^{k}(x-na_i) - \prod_{i=2}^{k}(-na_i)\right),$$

and dividing through by $n^{k-1}$ gives

$$\frac{1}{x}\left(\prod_{i=2}^{k}(x/n-a_i)-\prod_{i=2}^{k}(-a_i)\right). \tag{6.2}$$

If $\alpha$ is a zero of (6.1), then $n\alpha$ is a zero of (6.2). □

This means that increasing the sizes of all but one of the cliques in a ring of cliques by some fixed factor preserves the chromatic splitting field of the graph. Now, as rings of cliques are a special case of clique-theta graphs, it is natural to speculate as to whether we can use a similar method to show that the $n\alpha$ conjecture holds for this more general family. As we shall see, this is indeed the case, and a direct implication is that there exists a set of chromatic roots which is both dense in the complex plane, and closed under multiplication by positive integers.

Recall that in Proposition 3.2 we presented a general formula for the chromatic polynomials of clique-theta graphs. In what follows, given a set of numbers $S$ and an integer $a$, we shall denote by $aS$ the set produced by multiplying every element of $S$ by $a$.

**Theorem 6.4.** *Suppose $\alpha$ is a non-integer chromatic root of the clique-theta graph $T(1,S_1,S_2,\ldots,S_k,p)$; then $n\alpha$ is a chromatic root of $T(1,nS_1,nS_2,\ldots,nS_k,np)$.*

*Proof.* Again we need only consider the interesting factors of the relevant chromatic polynomials. For $T(1,S_1,S_2,\ldots,S_k,p)$ this is, by Proposition 3.2:

$$\left[p(x-p)^{k-1}\prod_{i=1}^{k}r(1,a_{i(1)},\ldots,a_{i(m_i)})\right]+\left[\prod_{i=1}^{k}r(1,a_{i(1)},a_{i(2)},\ldots,a_{i(m_i)},p)\right]. \tag{6.3}$$

Expanding the interesting factors of the rings of cliques, this becomes:

$$\left[p(x-p)^{k-1}\prod_{i=1}^{k}\frac{1}{x}\left(\prod_{l=1}^{m_i}(x-a_{i(l)})-\prod_{l=1}^{m_i}(-a_{i(l)})\right)\right]$$
$$+\left[\prod_{i=1}^{k}\frac{1}{x}\left((x-p)\prod_{l=1}^{m_i}(x-a_{i(l)})+p\prod_{l=1}^{m_i}(-a_{i(l)})\right)\right]. \tag{6.4}$$

For $T(1, nS_1, nS_2, \ldots, nS_k, np)$, we have:

$$\left[np(x-np)^{k-1}\prod_{i=1}^{k}\frac{1}{x}\left(\prod_{l=1}^{m_i}(x-na_{i(l)})-\prod_{l=1}^{m_i}(-na_{i(l)})\right)\right] \\ +\left[\prod_{i=1}^{k}\frac{1}{x}\left((x-np)\prod_{l=1}^{m_i}(x-na_{i(l)})+np\prod_{l=1}^{m_i}(-na_{i(l)})\right)\right]. \tag{6.5}$$

Let $s = \sum_{i=1}^{k}(m_i + 1)$ Then dividing (6.5) by $n^s$ gives:

$$\left[p(x/n-p)^{k-1}\prod_{i=1}^{k}\frac{1}{x}\left(\prod_{l=1}^{m_i}(x/n-a_{i(l)})-\prod_{l=1}^{m_i}(-a_{i(l)})\right)\right] \\ +\left[\prod_{i=1}^{k}\frac{1}{x}\left((x/n-p)\prod_{l=1}^{m_i}(x/n-a_{i(l)})+p\prod_{l=1}^{m_i}(-a_{i(l)})\right)\right]. \tag{6.6}$$

If $\alpha$ is a zero of (6.4), then $n\alpha$ is a zero of (6.6). □

As we shall see, not much more work will be required to show that 6.4 implies the following, which is the most important consequence of this section.

**Corollary 6.5.** *The set of chromatic roots satisfying the $n\alpha$ conjecture is dense in the complex plane. Thus there exists a set of chromatic roots which is both dense in the complex plane, and closed under multiplication by positive integers.*

By [43], the chromatic roots of those generalised theta graphs having paths of one fixed length are dense in the whole complex plane, with the exception of the disc $\{z \in \mathbb{C} : |z-1| < 1\}$. As a generalised theta graph is simply a clique-theta graph with all cliques of size one, Corollary 6.5 follows from Theorem 6.4 for this region. However, we still need to provide a proof for the excluded disc.

Sokal was able to prove that chromatic roots are dense inside this disc simply by allowing joins with copies of $K_2$ (see Corollary 1.3 in [43]). We can follow this approach, but need to modify it slightly.

*Proof for the disc $|z-1| < 1$.* Let $z \in \mathbb{C}$ be such that $|z-1| < 1$, and let $\epsilon > 0$ be arbitrarily small (in particular we may assume that $\epsilon < |2-$

$\Re(z)|$). We wish to show that there is a chromatic root $\alpha$ which satisfies the following two conditions:

*a*) $|\alpha - z| < \epsilon$

*b*) For all natural numbers $n$ there exists a graph having chromatic root $n\alpha$

Let $w = z - 2$. Then, by Sokal's result, there exists some generalised theta graph $G_1$ having a chromatic root $\beta$ such that $|\beta - w| < \epsilon$. Now let $G_2$ be the join of $G_1$ with $K_2$, and let $\alpha = \beta + 2$. Then $\alpha$ is a chromatic root of $G_2$ by Proposition 2.9, and we have:

$$|\alpha - z| = |\beta - w| < \epsilon,$$

thus proving part *(a)*.

For part *(b)*, let $n$ be any natural number, and let $G_n$ be the clique-theta graph obtained from $G_1$ by blowing up all but one endpoint vertex into a clique of size $n$. By Theorem 6.4, $G_n$ then has a chromatic root $n\beta$, and thus the join of $G_n$ with $K_{2n}$ has a chromatic root:

$$n\beta + 2n = n(\beta + 2) = n\alpha$$

□

## 6.2 Complementary bicliques

We will now discuss a splitting field-preserving operation that can be performed on any biclique. The proof of the main result regarding bicliques that we present in this section is quite technical: a more enlightening (and more general) one appears in §6.4, where we examine the relation in more detail.

Let $G$ be a $(j, k)$-biclique. As we saw in §3.1, the chromatic polynomial of $G$ is of the form:

$$P_G(x) = (x)_k g(x),$$

where $(x)_k$ denotes the falling factorial $x(x-1)(x-2)\dots(x-k+1)$, and $g$ is the interesting factor of degree $j$. Let $H$ be the graph consisting of

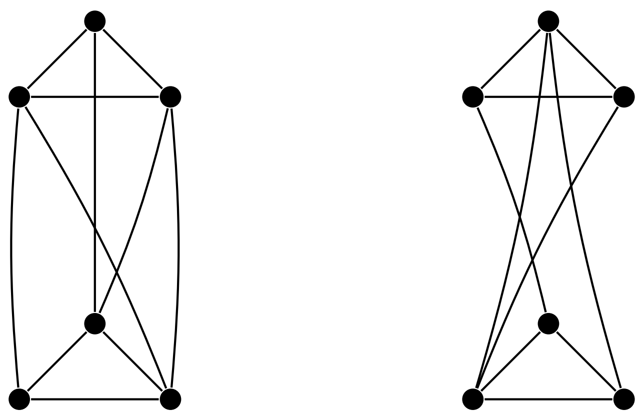

Figure 6.1: A complementary $(3, 3)$-clique pair

the same two cliques, but with edges between them which complement those of $G$. That is, to form $H$ we replace all edges in $G$ between the two cliques by non-edges, and vice-versa. We will refer to such a pair of graphs as *complementary* $(j, k)$*-bicliques*. Being another $(j, k)$-biclique, $H$ has a chromatic polynomial of the same form as $G$:

$$P_H(x) = (x)_k h(x),$$

where $h(x)$ has degree $n$. In terms of the parameters given by the second construction of §3.1, $h(x)$ is obtained from $g(x)$ by switching each $a_S$ with its "complement" $a_{X\setminus S}$.

**Proposition 6.6.** *For some positive integers $j$ and $k$ with $j \leq k$ let $G$ and $H$ be complementary $(j, k)$-bicliques, and let $g(x)$ and $h(x)$ be the interesting factors of $P_G(x)$ and $P_H(x)$ respectively. Then:*

$$g(x + j + k - 1) = (-1)^j h(-x).$$

Whenever there exists some integer $c$ such that two degree $j$ polynomials $g(x)$ and $h(x)$ satisfy $g(x) = (-1)^j h(-x + c)$ we will write that they are *reflections* of each other. In the next section we will see that Proposition 6.6 is in fact a special case of a more general result on reflections between interesting factors of chromatic polynomials of bicliques. It is perhaps the most striking special case however, and for now we will restrict ourselves to proving it directly.

For the purposes of the proof it is more convenient to express Proposition 6.6 in a different way. Note that

$$P_G(x+j+k-1) = (x+j+k-1)_k g(x+j+k-1),$$

and

$$\begin{aligned}(-1)^k P_H(-x) &= (-1)^k (-x)_k h(-x)\\ &= (x+k-1)_k h(-x).\end{aligned}$$

Thus we can rewrite Proposition 6.6 as:

$$P_G(x+j+k-1) = (-1)^{j+k} \frac{(x+j+k-1)_k}{(x+k-1)_k} P_H(-x). \tag{6.7}$$

If the chromatic polynomials of any two bicliques $G$ and $H$ are related in this way, then we will write that the pair satisfies (6.7).

We will require the following lemma:

**Lemma 6.7.** *Let $G$ and $H$ be complementary $(j,k)$-bicliques, and let $e$ be a bridging edge of $G$. Suppose that (6.7) holds for complementary $(j-1,k-1)$-bicliques. Then, if (6.7) holds for $G$ and $H$, we have the following similar relationship for $G/e$ and $H/e$:*

$$P_{G/e}(x+j+k-1) = (-1)^{j+k-1} \frac{(x+j+k-1)_k}{(x+k-1)_k} P_{H/e}(-x).$$

*Proof.* Suppose $e$ joins the vertices $u$ and $v$. Bringing these two vertices together in $G$ or $H$ is the same as first deleting them, along with all adjacent edges, and then adding a new vertex $w$ joined to every remaining vertex. Let $G\backslash uv$ represent the graph formed by deleting $u$ and $v$ from $G$. Then:

$$P_{G/e}(x) = xP_{G\backslash uv}(x-1) \tag{6.8}$$

and

$$P_{H/e}(x) = xP_{H\backslash uv}(x-1). \tag{6.9}$$

Deleting $u$ and $v$ from $G$ and $H$ leaves us with a pair of complementary

$(j-1,k-1)$-bicliques $G\backslash uv$ and $H\backslash uv$. By hypothesis, (6.7) holds for this pair. That is:

$$P_{G\backslash uv}(x+j+k-3) = (-1)^{j+k-2}\frac{(x+j+k-3)_{k-1}}{(x+k-2)_{k-1}}P_{H\backslash uv}(-x). \quad (6.10)$$

So we have:

$$\begin{aligned}
P_{G/e}(x+j+k-1) &= (x+j+k-1)P_{G\backslash uv}(x+j+k-2)\\
&= (x+j+k-1)P_{G\backslash uv}((x+1)+j+k-3)\\
&= (-1)^{j+k-2}(x+j+k-1)\frac{((x+1)+j+k-3)_{k-1}}{((x+1)+k-2)_{k-1}}P_{H\backslash uv}(-(x+1))\\
&= (-1)^{j+k-2}(x+j+k-1)\frac{(x+j+k-2)_{k-1}}{(x+k-1)_{k-1}}P_{H\backslash uv}(-x-1)\\
&= (-1)^{j+k-2}\frac{(x+j+k-1)_{k}}{(x+k-1)_{k-1}}P_{H\backslash uv}(-x-1)\\
&= (-1)^{j+k-2}\frac{(x+j+k-1)_{k}}{(x+k-1)_{k-1}}\frac{1}{-x}P_{H/e}(-x)\\
&= (-1)^{j+k-1}\frac{(x+j+k-1)_{k}}{(x+k-1)_{k}}P_{H/e}(-x)
\end{aligned}$$

Here the first equality follows from (6.8), the third from (6.10), and the sixth from (6.9).

□

**Lemma 6.8.** *Let $G_0$ and $H_0$ be complementary $(j,k)$-bicliques with the property that $G_0$ has every possible edge between the two cliques, and $H_0$ has none. Then (6.7) holds for $G_0$ and $H_0$.*

*Proof.* Note that $G_0$ is a $(j+k)$-clique, and $H_0$ is the disjoint union of an $j$-clique and a $k$-clique. We have:

$$P_{G_0}(x+j+k-1) = (x+j+k-1)_{j+k},$$

and

$$\begin{aligned}
(-1)^{j+k}P_{H_0}(-x) &= (-1)^{j+k}(-x)_j(-x)_k\\
&= (x+j-1)_j(x+k-1)_k.
\end{aligned}$$

So (6.7) clearly holds for $(G_0, H_0)$. $\square$

We are now ready to proceed with the proof of the proposition, which is by induction on $j$. First, let $G'$ and $H'$ be complementary $(1, k)$-bicliques, in which $G'$ has $m$ edges between the clique and the singleton, and $H'$ therefore has $k - m$ such edges. We will show that $G'$ and $H'$ satisfy (6.7). We have:

$$P_{G'}(x+k) = (x+k)_k(x+k-m),$$

and

$$\begin{aligned}(-1)^{1+k}P_{H'}(-x) &= (-1)^{1+k}(-x)_k(-x-k+m)\\ &= (x+k-1)_k(x+k-m).\end{aligned}$$

Hence

$$P_{G'}(x+k) = (-1)^{1+k}\frac{(x+k)_k}{(x+k-1)_k}P_{H'}(-x) \tag{6.11}$$

as desired.

Now assume that $j \geq 2$, suppose the result is true for any pair of complementary $(j-1, k-1)$-bicliques, and let $G$ and $H$ be complementary $(j, k)$-bicliques, with $H$ having $m$ edges bridging the two cliques. Label the bridging edges of $H$ as $\{e_1, \ldots, e_m\}$. We will construct a series of graphs $\{G_i\}$, starting with $G_0$ (the graph having all possible bridging edges) and culminating in $G_m = G$, by removing edges $e_i$ from $G_0$ one at a time. That is:

$$G_1 = G_0 \setminus e_1, G_2 = G_1 \setminus e_2, \ldots, G = G_{m-1} \setminus e_m.$$

By the addition-contraction identity (2.7), for all $1 \leq i \leq m$ we have:

$$P_{G_i}(x) = P_{G_{i-1}}(x) + P_{G_{i-1}/e_i}(x). \tag{6.12}$$

At the same time, we can construct another series of graphs $\{H_i\}$, this time starting with $H_0$ (the graph with no bridging edges) and adding the edges $e_i$ one at a time, so that:

$$H_1 = H_0 + e_1, H_2 = H_1 + e_2, \ldots, H = H_{m-1} + e_m.$$

Now, inserting negative arguments into (2.6) gives us, for all $1 \le i \le m$:

$$(-1)^{j+k} P_{H_i}(x) = (-1)^{j+k} P_{H_{i-1}}(x) + (-1)^{j+k-1} P_{H_{i-1}/e_1}(x). \qquad (6.13)$$

Note that $G_i$ and $H_i$ are complementary $(j,k)$-bicliques for all $i$.

By Lemma 6.8, condition (6.7) holds for $G_0$ and $H_0$. Assume that, for some $i$, (6.7) holds for some pair $G_i$ and $H_i$, so that:

$$P_{G_i}(x+j+k-1) = (-1)^{j+k} \frac{(x+j+k-1)_k}{(x+k-1)_k} P_{H_i}(-x).$$

Then, by Lemma 6.7, it holds too for $G_i/e_{i+1}$ and $H_i/e_{i+1}$, that is

$$P_{G_i/e_{i+1}}(x+j+k-1) = (-1)^{j+k-1} \frac{(x+j+k-1)_k}{(x+k-1)_k} P_{H_i/e_{i+1}}(-x).$$

Using (6.12) we have:

$$\begin{aligned} P_{G_{i+1}}(x+j+k-1) &= P_{G_i}(x+j+k-1) + P_{G_i/e_{i+1}}(x+j+k-1) \\ &= \frac{(x+j+k-1)_k}{(x+k-1)_k} \left( (-1)^{j+k} P_{H_i}(-x) + (-1)^{j+k-1} P_{H_i/e_{i+1}}(-x) \right) \\ &= (-1)^{j+k} \frac{(x+j+k-1)_k}{(x+k-1)_k} P_{H_{i+1}}(-x), \end{aligned}$$

where the last equality follows from (6.13). Hence (6.7) holds for $G_{i+1}$ and $H_{i+1}$, and so by induction for $G$ and $H$. Thus we have shown that, if condition (6.7) holds for complementary $(j-1,k-1)$-bicliques, then it holds too for complementary $(j,k)$-bicliques. As $k$ was arbitrary in the initial case of a $(1,k)$-clique pair, the result follows by further induction on $j$.

## 6.3 $(2,k)$-bicliques

Recall that any $(2,k)$-biclique is uniquely defined by two integers $a$ and $b$, which represent the number of neighbours in the $k$-clique of each of the vertices in the 2-clique (as usual, we are assuming that no vertex is adjacent to every other vertex of the graph ). As mentioned in §5.1, the chromatic polynomial of such a graph is a product of linear factors and the interesting factor:

$$g_{a,b}(x) = (x-a)(x-b) - (x-a-b).$$

We will use this fact to classify every $(2,k)$-biclique according to its chromatic splitting field.

**Proposition 6.9.** *Let $a$ and $b$ be non-negative integers, and let $g_{a,b}(x)$ be the interesting factor from the chromatic polynomial of the $(2,k)$-biclique defined by $a$ and $b$. Let $z$ be some integer. Then:*

1. $\Delta(g_{a,b}(x)) = -4z$ *if and only if* $a = z + i^2$ *and* $b = z + (i+1)^2$ *for some* $i \in \mathbb{Z}$.

2. $\Delta(g_{a,b}(x)) = -4z + 1$ *if and only if* $a = z + i^2 + i$ *and* $b = z + (i+1)^2 + (i+1)$ *for some* $i \in \mathbb{Z}$.

*Proof.* Recall that the discriminant of $g_{a,b}(x)$ is:

$$(a-b)^2 - 2(a+b) + 1.$$

In order to prove sufficiency for the first statement we simply need to note that evaluating this expression at $a = z + i^2, b = z + (i+1)^2$ gives $-4z$. Similarly, evaluating at $a = z+i^2+i, b = z+(i+1)^2+(i+1)$ gives $-4z+1$.

The proof of the converse is slightly trickier. First let $a = z + c$, where $c$ is not a perfect square. Suppose that there is some $b \in \mathbb{N}$ such that $\Delta(g_{a,b}(x)) = -4z$. Then we have:

$$(z+c-b)^2 - 2(z+c+b) + 1 = -4z.$$

Expanding this equation and solving for $b$ gives $b = z+c+1 \pm 2\sqrt{c}$, which contradicts the assertion that $b$ is a natural number. This means that in order for the discriminant of $g_{a,b}(x)$ to equal $-4z$, both $a$ and $b$ must be a sum of $z$ with a perfect square.

Now let $j$ be any integer, let $a = z + j^2$, and suppose that $b$ is another integer such that $g_{a,b}(x)$ has discriminant $-4z$. Then we have:

$$(z+j^2-b)^2 - 2(z+j^2+b) + 1 = -4z.$$

Solving this for $b$ gives that $b = z + (j+1)^2$ or $b = z + (j-1)^2$. Thus if $\Delta(g_{a,b}(x)) = -4z$ then $a = z + i^2$ and $b = z + (i+1)^2$ for some $i \in \mathbb{Z}$.

We can use a similar method to prove necessity in the second statement. Let $a = z + c$, where $c$ is not of the form $i^2 + i$ for some integer $i$. Suppose that there is some $b \in \mathbb{N}$ such that $\Delta(g_{a,b}(x)) = -4z + 1$. Then we have:

$$(z + c - b)^2 - 2(z + c + b) + 1 = -4z + 1.$$

Solving for $b$ gives $b = z + c + 1 \pm \sqrt{4c+1}$. As $4c + 1$ is only a perfect square if $c = i^2 + i$ for some $i \in \mathbb{Z}$, we have a contradiction.

Now let $j$ be any integer, let $a = z + j^2 + j$, and suppose that $b$ is another integer such that $g_{a,b}(x)$ has discriminant $-4z + 1$. Then we have:

$$(z + j^2 + j - b)^2 - 2(z + j^2 + j + b) + 1 = -4z + 1.$$

Solving this for $b$ we see that either:

$$b = z + (j+1)^2 + (j+1),$$

or:

$$b = z + (j-1)^2 + (j-1).$$

Thus if $\Delta(g_{a,b}(x)) = -4z+1$ then $a = z+i^2+i$ and $b = z+(i+1)^2+(i+1)$ for some $i \in \mathbb{Z}$. $\square$

**Corollary 6.10.** *Let $a, b, c$ and $d$ be non-negative integers. Then the $(2,k)$-biclique defined by $a$ and $b$ has the same discriminant as that defined by $c$ and $d$ if and only if there is some integer $k$ such that:*

$$c = (k+2)(k+1+a-b) + b$$
$$d = (k+1)(k+a-b) + b$$

*Proof.* Sufficiency is proved simply by noting that, with the given assignments for $c$ and $d$:

$$\Delta(g_{a,b}(x)) = (a-b)^2 - 2(a+b) + 1 = \Delta(g_{c,d}(x)),$$

for any integer $k$.

For the converse, suppose that $\Delta(g_{a,b}(x)) = \Delta(g_{c,d}(x)) = -4z$ for some $z \in \mathbb{Z}$. Then by Proposition 6.9 there are integers $i$ and $j$ such that:

$$\begin{aligned} a &= z + i^2 \\ b &= z + (i+1)^2 \\ c &= z + (j-1)^2 \\ d &= z + j^2 \end{aligned}$$

Let $k = i - j$. Then:

$$\begin{aligned} (k+2)(k+1+a-b)+b &= (i-j+2)(-i-j)+z+(i+1)^2 \\ &= z+(j-1)^2 \\ &= c, \end{aligned}$$

and similarly, $(k+1)(k+a-b)+b = d$. This proves the result in the case that $\Delta(g_{a,b}(x))$ is congruent to $0 \pmod 4$.

Now suppose that $\Delta(g_{a,b}(x)) = \Delta(g_{c,d}(x)) = -4z+1$ for some $z \in \mathbb{Z}$. Then by Proposition 6.9 there are integers $i$ and $j$ such that:

$$\begin{aligned} a &= z + i^2 + i \\ b &= z + (i+1)^2 + (i+1) \\ c &= z + j^2 + j \\ d &= z + (j+1)^2 + (j+1) \end{aligned}$$

This time let $k = i - j - 1$. Then:

$$\begin{aligned} (k+1)(k+a-b)+b &= (i+j+2)(-i-j-2)+z+(i+1)^2+(i+1) \\ &= z+j^2+j \\ &= c, \end{aligned}$$

and similarly, $(k+1)(k+a-b)+b = d$. □

Corollary 6.10 enables us to classify $(2,k)$-bicliques according to the discriminants of the interesting factors in their chromatic polynomials. As all of the other factors in these polynomials are linear, and the discriminant of a quadratic polynomial determines its splitting field, in some cases we can use this corollary to deduce that two given $(2, k)$-bicliques have the same splitting field.

However, in order for two quadratic polynomials to generate the same splitting field it is not essential that they have the same discriminant; this also occurs when the discriminant of one is a perfect square multiple of that of the other. So there is some more work to do in order to completely classify $(2, k)$-bicliques according to their chromatic splitting field.

**Theorem 6.11.** *Let $a, b, c$ and $d$ be non-negative integers, and let*

$$\Delta(g_{a,b}(x)) = (a-b)^2 - 2(a+b) + 1$$

*be the discriminant of $g_{a,b}(x)$. Suppose, without loss of generality, that*

$$|\Delta(g_{a,b}(x))| \leq |\Delta(g_{c,d}(x))|.$$

*Then the $(2, k)$-biclique defined by $a$ and $b$ generates the same splitting field as that defined by $c$ and $d$ if and only if there are integers $k$ and $n$ such that:*

$$c = (k+2)(k+1+a-b) + b + \frac{1}{4}(1-n^2)\Delta(g_{a,b}(x))$$
$$d = (k+1)(k+a-b) + b + \frac{1}{4}(1-n^2)\Delta(g_{a,b}(x))$$

*Proof.* It can be easily verified using computer algebra software that

$$\Delta(g_{c,d}(x)) = n^2\Delta(g_{a,b}(x))$$

for both cases, thus proving sufficiency. For the converse, we must split the proof into two cases: where $\Delta(g_{a,b})$ is odd, and where it is even.

First suppose $\Delta(g_{a,b})$ is even, so that $\Delta(g_{a,b}) = -4z$ for some $z \in \mathbb{Z}$, and suppose that $g_{c,d}(x)$ generates the same splitting field as $g_{a,b}(x)$. Then

$$\Delta(g_{c,d}(x)) = n^2\Delta(g_{a,b}(x)) = -4n^2 z$$

for some integer $n$. Proposition 6.9 implies that $a = z+i^2$ and $b = z+(i+1)^2$ for some $i \in \mathbb{Z}$, and as $-4n^2 z \equiv 0 \pmod 4$ for all $n$, the same result gives that $c = n^2z + (j-1)^2$ and $d = n^2z + j^2$ for some $j \in \mathbb{Z}$.

As in the proof of Corollary 6.10, we now let $k = i - j$. Then:

$$\begin{aligned}
&(k+2)(k+1+a-b) + b + \frac{1}{4}(1-n^2)\Delta(g_{a,b}(x)) \\
&= (k+2)(k+1+a-b) + b - (1-n^2)z \\
&= (i-j+2)(-i-j) + (i+1)^2 + n^2z \\
&= n^2z + (j-1)^2 \\
&= c,
\end{aligned}$$

and

$$\begin{aligned}
&(k+1)(k+a-b) + b + \frac{1}{4}(1-n^2)\Delta(g_{a,b}(x)) \\
&= (k+1)(k+a-b) + b - (1-n^2)z \\
&= (i-j+1)(-i-j-1) + (i+1)^2 + n^2z \\
&= n^2z + j^2 \\
&= d.
\end{aligned}$$

This proves necessity in the case that $\Delta(g_{a,b})$ is even.

Now we consider the case that the discriminant of $g_{a,b}(x)$ is odd, that is: $\Delta(g_{a,b}) = -4z + 1$ for some $z \in \mathbb{Z}$. Suppose again that $g_{c,d}(x)$ generates the same splitting field as $g_{a,b}(x)$. Then

$$\Delta(g_{c,d}(x)) = n^2\Delta(g_{a,b}(x)) = -4n^2z + n^2$$

for some integer $n$. By Proposition 6.9 $a = z + i^2 + i$ and $b = z + (i+1)^2 + (i+1)$ for some $i \in \mathbb{Z}$.

We have two subcases to consider. First suppose that $n$ is odd. Then $-4n^2z + n^2 \equiv 1 \pmod 4$, in which case Proposition 6.9 tells us that

$$c = n^2z - \frac{1}{4}n^2 + \frac{1}{4} + j^2 + j$$

and

$$d = n^2z - \frac{1}{4}n^2 + \frac{1}{4} + (j+1)^2 + (j+1)$$

for some $j \in \mathbb{Z}$. Let $k = i - j - 1$. Then:

$$
\begin{aligned}
&(k+2)(k+1+a-b)+b+\frac{1}{4}(1-n^2)\Delta(g_{a,b}(x)) \\
&= (k+2)(k+1+a-b)+b+\frac{1}{4}(1-n^2)(-4z+1) \\
&= (i+j+2)(-i-j-2)+z+(i+1)^2+i+1+\frac{1}{4}(1-n^2)(-4z+1) \\
&= n^2z-\frac{1}{4}n^2+\frac{1}{4}+j^2+j \\
&= c,
\end{aligned}
$$

and

$$
\begin{aligned}
&(k+1)(k+a-b)+b+\frac{1}{4}(1-n^2)\Delta(g_{a,b}(x)) \\
&= (k+1)(k+a-b)+b+\frac{1}{4}(1-n^2)(-4z+1) \\
&= (i+j+2)(-i-j-3)+z+(i+1)^2+i+1+\frac{1}{4}(1-n^2)(-4z+1) \\
&= n^2z-\frac{1}{4}n^2+\frac{1}{4}+(j+1)^2+j+1 \\
&= d.
\end{aligned}
$$

Now suppose that $n$ is even. Then $-4n^2z+n^2 \equiv 0 \pmod 4$, in which case $c = n^2z - \frac{1}{4}n^2 + (j-1)^2$ and $d = n^2z - \frac{1}{4}n^2 + j^2$ for some $j \in \mathbb{Z}$. Substituting $i - j + \frac{1}{2}$ for $k$, along with the previously specified values for $a$ and $b$, gives:

$$
\begin{aligned}
&(k+2)(k+1+a-b)+b+\frac{1}{4}(1-n^2)\Delta(g_{a,b}(x)) \\
&= (k+2)(k+1+a-b)+b+\frac{1}{4}(1-n^2)(-4z+1) \\
&= (i-j+\frac{5}{2})(-i-j-\frac{1}{2})+z+(i+1)^2+i+1+\frac{1}{4}(1-n^2)(-4z+1) \\
&= n^2z-\frac{1}{4}n^2+(j-1)^2 \\
&= c,
\end{aligned}
$$

and

$$
\begin{aligned}
&(k+1)(k+a-b)+b+\frac{1}{4}(1-n^2)\Delta(g_{a,b}(x)) \\
&= (k+1)(k+a-b)+b+\frac{1}{4}(1-n^2)(-4z+1) \\
&= (i-j+\frac{3}{2})(-i-j-\frac{3}{2})+z+(i+1)^2+i+1+\frac{1}{4}(1-n^2)(-4z+1) \\
&= n^2z-\frac{1}{4}n^2+j^2 \\
&= d.
\end{aligned}
$$

□

Theorem 6.11 provides us with an equivalence relation on the collection of 2-sets of natural numbers, whereby two sets are equivalent if the $(2,k)$-bicliques they define have the same chromatic splitting field. Thus we have completely classified $(2,k)$-bicliques according to the splitting fields of their chromatic polynomials.

It seems likely that this technique could be used to similarly classify members of other families of graphs having chromatic polynomials with one quadratic factor and the rest linear, such as the rings of cliques $R(1,a,b,c)$. However, as Galois extensions of degree 3 and higher are no longer completely characterised by the discriminants of their minimal polynomials, different techniques are required when the interesting factors are of higher degree.

## 6.4 Shifts and reflections of chromatic roots of bicliques

Staying with the family of bicliques, but turning now to the first construction of the chromatic polynomial given in §3.1, we will show how matchings of the complements of these graphs can be used to prove some chromatic splitting field equivalences between $(j,k)$-bicliques with $j>2$.

It is clear that, in order for two bicliques to share the same chromatic splitting field, the interesting factors of their chromatic polynomials must be of the same degree. That is, there must be positive integers $j, k_G$ and

$k_H$ with $j \leq k_G \leq k_H$ such that one graph is a $(j, k_G)$-biclique and the other is a $(j, k_H)$-biclique. The simplest way in which two such graphs might share the same chromatic splitting field is if the interesting factor of one chromatic polynomial is an integer shift of that of the other, and it is easy to show that this is always the case when the graphs' complements are matching equivalent.

**Proposition 6.12.** *Let $j, k_G$ and $k_H$ be positive integers with $j \leq k_G \leq k_H$, and let $G$ and $H$ be, respectively, a $(j, k_G)$-biclique and a $(j, k_H)$-biclique. Denote by $g(x)$ and $h(x)$ the degree $j$ interesting factors of $P_G(x)$ and $P_H(x)$. If $\bar{G}$ and $\bar{H}$ are matching equivalent then $g(x) = h(x + k_H - k_G)$.*

*Proof.* Suppose that $\bar{G}$ and $\bar{H}$ are matching equivalent. As $m^i_{\bar{H}} = m^i_{\bar{G}}$ for all $0 \leq i \leq j$, we have from (3.2) that

$$g(x) = \sum_{i=0}^{j} m^i_{\bar{G}} (x - k_G)_{j-i}$$

and

$$h(x) = \sum_{i=0}^{j} m^i_{\bar{G}} (x - k_H)_{j-i},$$

so clearly $g(x) = h(x + k_H - k_G)$. □

The converse of Proposition 6.12 is not true: there exist pairs of non-matching equivalent bicliques having chromatic polynomials with interesting factors which are integer shifts of each other. Later in this section we will show, without using Proposition 2.9, that for any given integer $c$ there exist interesting factors $g(x)$ and $h(x)$ of chromatic polynomials of bicliques having non-matching equivalent complements with the property that $g(x) = h(x + c)$.

Now, let $g(x)$ and $h(x)$ be polynomials of equal degree, and suppose there exists some integer $c$ such that $g(x) = (-1)^j h(-x + c)$. As previously mentioned, we shall refer to such polynomials as reflections of each other. There turns out to be many more pairs of bicliques having chromatic polynomials which are related by a reflection than the complementary graphs discussed in §6.2. The following theorem gives a necessary and sufficient condition for two bicliques to be related in this way.

**Theorem 6.13.** *Let $j, k_G$ and $k_H$ be positive integers satisfying $j \le k_G \le k_H$, and let $G$ and $H$ be, respectively, a $(j, k_G)$-biclique and a $(j, k_H)$-biclique, having chromatic polynomials $P_G(x) = (x)_{k_G} g(x)$ and $P_H(x) = (x)_{k_H} h(x)$. Then $g(x) = (-1)^j h(-x + c)$ for some integer $c$ if and only if*

$$m^i_{\bar{G}} = \sum_{l=0}^{i} (-1)^l m^l_{\bar{H}} \binom{j-l}{j-i} (k_G + k_H + j - c - l - 1)_{i-l}, \tag{6.14}$$

*for all $0 \le i \le j$.*

*Proof.* Suppose that the stated condition holds. From (3.2), we have that:

$$g(x) = \sum_{i=0}^{j} m^i_{\bar{G}} (x - k_G)_{j-i},$$

and so substituting for $m^i_{\bar{G}}$ we get:

$$\begin{aligned}
g(x) =& \sum_{i=0}^{j} \sum_{l=0}^{i} (-1)^l m^l_{\bar{H}} \binom{j-l}{j-i} (k_G + k_H + j - c - l - 1)_{i-l} (x - k_G)_{j-i} \\
=& \sum_{l=0}^{j} (-1)^l m^l_{\bar{H}} \sum_{i=0}^{j-l} \binom{j-l}{j-i-l} (k_G + k_H + j - c - l - 1)_{i} (x - k_G)_{j-i-l} \\
=& \sum_{l=0}^{j} (-1)^l m^l_{\bar{H}} \sum_{i=0}^{j-l} \binom{j-l}{i} (k_G + k_H + j - c - l - 1)_{i} (x - k_G)_{j-i-l} \\
=& \sum_{l=0}^{j} (-1)^l m^l_{\bar{H}} (x + k_H + j - c - l - 1)_{j-l} \\
=& \sum_{l=0}^{j} (-1)^l m^l_{\bar{H}} (-1)^{j-l} (-x - k_H + c)_{j-l} \\
=& (-1)^j \sum_{l=0}^{j} m^l_{\bar{H}} (-x - k_H + c)_{j-l} \\
=& (-1)^j h(-x + c).
\end{aligned}$$

The converse is proved by simply reversing these steps.

□

Theorem 6.13 enables us to give another proof of Proposition 6.6. We will require the following lemma, which is essentially a specialisation or rephrasing of similar results by, among others, Riordan [38], Farrell and

Whitehead [18] and Zaslavsky [55]. For the sake of completion, we shall include a full proof here.

**Lemma 6.14.** *Let $\bar{G}$ be a spanning subgraph of the complete bipartite graph $K_{j,k}$, where $j \leq k$, and let $\bar{H}$ be the complement of $G$ in $K_{j,k}$. Then:*

$$m_{\bar{G}}^i = \sum_{l=0}^{i} (-1)^l m_{\bar{H}}^l \binom{j-l}{j-i} (k-l)_{i-l}.$$

*Proof.* Given some $0 \leq i \leq j$, let $\mathcal{X}^i$ be the set of all $i$-matchings of $K_{j,k}$, and for each edge $e$ of $K_{j,k}$ let $\mathcal{A}_e^i \subset \mathcal{X}^i$ be the collection of those $i$-matchings containing $e$. Label the edges of $\bar{H}$ as $\{1, 2, \ldots, m\}$. The $i$-matchings of $\bar{G}$ are simply those $i$-matchings of $K_{j,k}$ not containing any edge of $\bar{H}$, and so the number of $i$-matchings of $\bar{G}$ is:

$$m_{\bar{G}}^i = \left| \mathcal{X}^i \setminus \bigcup_{e=1}^{m} \mathcal{A}_e^i \right|.$$

By the Principle of Inclusion-Exclusion, the right-hand side of this equation is precisely:

$$\sum_{I \subseteq \{1,\ldots,m\}} (-1)^{|I|} \left| \bigcap_{e \in I} \mathcal{A}_e^i \right|. \tag{6.15}$$

Now, note that the $i$-matchings contained in $\bigcap_{e \in I} \mathcal{A}_e^i$ are precisely those $i$-matchings of $K_{j,k}$ containing every $e \in I$. Furthermore, if $I \subseteq \{1, \ldots, m\}$ is not a matching, or else has size greater than $i$, then the number of $i$-matchings of $K_{j,k}$ containing every $e \in I$ is zero; and if $I$ is a matching of size less than or equal to $i$ then the number of $i$-matchings of $K_{j,k}$ containing every $e \in I$ depends only on the cardinality of $I$ (not on its precise edge-content). Thus (6.15) is equivalent to the alternating sum, over all $0 \leq l \leq i$, of the product of the number of $i$-matchings of $K_{j,k}$ containing a given $l$-matching, with the number of possible $l$-matchings of $\bar{H}$.

We can count the $i$-matchings of $K_{j,k}$ containing a given $l$-matching as follows: the $l$ edges of the $l$-matching join $l$ vertices on the "$j$-side" of $K_{j,k}$ to $l$ vertices on the "$k$-side". From the remaining $j-l$ vertices on the $j$-side, we have a choice of $i-l$ to be incident to the extra edges in our desired $i$-matching. For each such choice of $i-l$ vertices we then have $(k-l)_{i-l}$

ways to choose their neighbours on the $k$-side. So we have that the number of $i$-matchings of $K_{j,k}$ containing a given $l$-matching is:

$$\binom{j-l}{i-l}(k-l)_{i-l} = \binom{j-l}{j-i}(k-l)_{i-l}.$$

The number of possible $l$-matchings of $\bar{H}$ is simply $m^l_{\bar{H}}$, and so putting this all together we obtain from (6.15) the desired expression:

$$m^i_{\bar{G}} = \sum_{l=0}^{i}(-1)^l m^l_{\bar{H}}\binom{j-l}{j-i}(k-l)_{i-l}. \tag{6.16}$$

□

Now, note that if $G$ and $H$ are complementary $(j,k)$-bicliques, then $\bar{G}$ and $\bar{H}$ complement each other inside the complete bipartite graph $K_{j,k}$. Hence, by Lemma 6.14:

$$m^i_{\bar{G}} = \sum_{l=0}^{i}(-1)^l m^l_{\bar{H}}\binom{j-l}{j-i}(k-l)_{i-l}.$$

This expression is simply (6.14) with $k_G = k_H = k$ and $c = j+k-1$, and so by Theorem 6.13 we have that

$$g(x) = (-1)^j h(-x+j+k-1),$$

or equivalently:

$$g(x+j+k-1) = (-1)^j h(-x).$$

Thus we have an alternative proof of Proposition 6.6.

Theorem 6.13 in fact enables us to show that there are many pairs of bicliques other than complementary pairs having chromatic polynomials with reflected interesting factors. We will give some examples of these for the case $j = 3$.

We will use the parametrisation of a $(3,k)$-clique $G$ in the same way as described at the beginning of §5.2. To recap: we label the three vertices in the 3-clique of $G$ as $v_1, v_2$ and $v_3$; let $a, b$ and $c$ represent the number of neighbours of $v_1, v_2$ and $v_3$ respectively in the $k$-clique; and let $d, e$ and $f$ represent the number of vertices in the $k$-clique joined to both

$v_2$ and $v_3$, both $v_1$ and $v_3$, and both $v_1$ and $v_2$ respectively. The 6-tuple $(a,b,c,d,e,f)$ then completely describes $G$, and we will simply write that $G=(a,b,c,d,e,f)$.

It will be helpful for what follows to point out some properties of the complement of this graph. So note that the order of $\bar{G}$ is

$$|V(\bar{G})| = a+b+c+d+e+f+3;$$

that the number of edges of $\bar{G}$ is

$$|E(\bar{G})| = 2a+2b+2c+d+e+f;$$

and that the degrees of $v_1, v_2$ and $v_3$ in $\bar{G}$ are $(b+c+d)$, $(a+c+e)$ and $(a+b+f)$ respectively.

**Proposition 6.15.** *Let $r,s,t,u$ and $v$ be non-negative integers satisfying $u+v=4t-r-s+3$, and let $G=(r,s,t,t,t,u)$ and $H=(r,s,t,t,t,v)$ be $(3,k)$-bicliques, having chromatic polynomials with interesting factors $g(x)$ and $h(x)$ respectively. Then:*

$$g(x) = -h(-x+6t+4).$$

*Proof.* First note that the number of vertices connected only to $v_1, v_2$ and $v_3$ in $\bar{G}$ are $t,t$ and $u$ respectively, and the corresponding values for $\bar{H}$ are $t,t$ and $v$. Furthermore, in both $\bar{G}$ and $\bar{H}$ the number of vertices connected to both $v_2$ and $v_3$, both $v_1$ and $v_3$, and both $v_1$ and $v_2$ are $r,s$ and $t$ respectively.

Now, it is clear that $m^0_{\bar{G}} = m^0_{\bar{H}} = 1$. The matching numbers $m^1_{\bar{G}}$ and $m^1_{\bar{H}}$ can also be easily found, as they are simply the numbers of edges of $\bar{G}$ and $\bar{H}$, that is:

$$m^1_{\bar{G}} = 2r+2s+4t+u$$

and

$$m^1_{\bar{H}} = 2r+2s+4t+v.$$

Now, let $B$ be the subgraph of $\bar{G}$ resulting from the removal of those edges incident only to $v_3$ and no other $v_i$, and let $A$ be the subgraph of $B$ obtained by removing $v_3$ and all remaining incident edges. For $l=2$

or $l = 3$, we can split each $l$-matching of $\bar{G}$ into one of two groups: those containing one of the $u$ edges incident to $v_3$ and no other $v_i$, and those not containing such an edge. The former consists of every $l$-matching which is a union of an $(l-1)$-matching of $A$ with one of the $u$ edges in question; the latter are simply the $l$-matchings of $B$. So we have, for $l = 2$ or 3:

$$m^l_{\bar{G}} = um^{l-1}_A + m^l_B. \tag{6.17}$$

By enumerating the edges of $A$ we immediately find that $m^1_A = r+s+4t$. The 2-matchings of $A$ can be counted by multiplying the number of vertices adjacent to $v_1$ by the number adjacent to $v_2$, and subtracting $t$ (as there are $t$ vertices adjacent to both $v_1$ and $v_2$), giving us:

$$m^2_A = (r + 2t)(s + 2t) - t = rs + 2rt + 2st + 4t^2 - t.$$

It remains to find $m^2_B$ and $m^3_B$. The 2-matchings can be found by subtracting the number of pairs of edges of $B$ which are incident to a common vertex from the total number of pairs of edges of $B$. The total number of pairs of edges of $B$ is:

$$\binom{|E(B)|}{2} = \binom{2r + 2s + 4t}{2} = \frac{1}{2}(2r + 2s + 4t)(2r + 2s + 4t - 1),$$

and if we represent by $d(v)$ the degree of the vertex $v$, then the number of pairs which are incident to a common vertex is:

$$\begin{aligned}\sum_{v\in V(B)} \binom{d(v)}{2} &= \frac{1}{2} \sum_{v\in V(B)} \Big(d(v)^2 - d(v)\Big) \\ &= -|E(B)| + \frac{1}{2} \sum_{v\in V(B)} d(v)^2 \\ &= s^2 + 2st + 4t^2 + r^2 + 2rt + rs - t.\end{aligned}$$

Thus:

$$\begin{aligned}m_B^2 &= \frac{1}{2}(2r+2s+4t)(2r+2s+4t-1) \\ &\quad - (s^2+2st+4t^2+r^2+2rt+rs-t) \\ &= r^2+3rs+6rt-r+s^2+6st-s+4t^2-t.\end{aligned}$$

To count 3-matchings, note that the total number of choices of 3 edges such that one is incident to each of the $v_i$ is

$$(s+2t)(r+2t)(r+s).$$

From this we will need to subtract:

1. the $t(r+s)$ 3-matchings in which the chosen edges incident to $v_1$ and $v_2$ share a common endpoint;
2. the $s(r+2t)$ 3-matchings in which the chosen edges incident to $v_1$ and $v_3$ share a common endpoint; and:
3. the $r(s+2t)$ 3-matchings in which the chosen edges incident to $v_2$ and $v_3$ share a common endpoint.

This gives us:

$$m_B^3 = (s+2t)(r+2t)(r+s) - t(r+s) - s(r+2t) - r(s+2t).$$

Finally, substituting all of these into (6.17), we have:

$$m_{\bar{G}}^2 = u(r+s+4t) + r^2+3rs+6rt-r+s^2+6st-s+4t^2-t,$$

and

$$\begin{aligned}m_{\bar{G}}^3 &= u(rs+2rt+2st+4t^2-t) \\ &\quad + (s+2t)(r+2t)(r+s) - t(r+s) - s(r+2t) - r(s+2t).\end{aligned}$$

The matching numbers of $\bar{H}$ can now be derived by simply substituting $v$ for $u$ in these expressions, giving:

$$m_{\bar{H}}^2 = v(r+s+4t) + r^2+3rs+6rt-r+s^2+6st-s+4t^2-t,$$

and

$$
\begin{aligned}
m^3_{\bar{H}} &= v(rs + 2rt + 2st + 4t^2 - t) \\
&\quad + (s + 2t)(r + 2t)(r + s) - t(r + s) - s(r + 2t) - r(s + 2t).
\end{aligned}
$$

It is now simple, if rather tedious, to verify that for each $0 \leq i \leq 3$:

$$
m^i_{\bar{G}} = \sum_{l=0}^{i} (-1)^l m^l_{\bar{H}} \binom{3-l}{3-i} (4t + r + s + 1 - l)_{i-l}.
$$

This equation is simply (6.14) with the substitutions:

$$
\begin{aligned}
k_G &= r + s + 3t + u \\
k_H &= r + s + 3t + v \\
j &= 3 \\
c &= 6t + 4,
\end{aligned}
$$

which means that $G$ and $H$ satisfy the conditions of Theorem 6.13, and

$$
g(x) = -h(-x + 6t + 4).
$$

□

An alternative proof of Proposition 6.15 can be found by substituting the appropriate 6 parameters into the construction (5.2) and writing it in the falling factorial basis (this can be easily done using Stirling numbers of the second kind). For each $0 \leq i \leq 3$ the number of $i$-matchings of $\bar{G}$ will then be the coefficient of $(x)_i$ in this basis.

The following two results can be proved in exactly the same way as Proposition 6.15, and so we will spare the reader the details. In both cases $G$ and $H$ are $(3, k)$-bicliques having chromatic polynomials with interesting factors $g(x)$ and $h(x)$ respectively.

**Proposition 6.16.** *Let $r, s, t, u$ and $v$ be non-negative integers satisfying $u + v = 4t - 2r + 4$. If $G = (r, r + s - 1, t, t, s + t, u)$ and $H = (r, r + s -$*

$1, t, t, s+t, v)$, *then:*

$$g(x) = -h(-x + 2s + 6t + 4).$$

**Proposition 6.17.** *Let $r, s, t, u$ and $v$ be non-negative integers satisfying $u + v = 4s - 2r + t^2 + 2t + 4$. If $G = (r, r, s, s + \binom{t+1}{2}, s + \binom{t+2}{2}, u)$ and $H = (r, r, s, s + \binom{t+1}{2}, s + \binom{t+2}{2}, v)$, then:*

$$g(x) = -h(-x + 6s + 2t^2 + 4t + 6).$$

This is just a sample of reflection relations between bicliques; there are likely to be more such relations for the relatively simple case $j = 3$, and no doubt many more for larger $j$. Even the few examples we have presented do however suggest patterns which invite further investigation; it would be desirable to find a more graph-theoretic classification of pairs of bicliques having chromatic polynomials which are related by a reflection than that given by Theorem 6.13. In addition, note that the pivotal relation between the chromatic polynomials of these graphs and matchings of their complements in fact holds for any triangle-free graph (see [18] for a proof), raising the possibility that our results might generalise to larger classes of graphs.

There is an interesting link between proper colourings and acyclic orientations of pairs of bicliques of the same order that have chromatic polynomials related by a reflection, which is as follows: for some positive integers $j$ and $k$ satisfying $j \leq k$ let $G$ and $H$ be $(j, k)$-bicliques having chromatic polynomials with interesting factors $g(x)$ and $h(x)$ respectively, and suppose that $g(x) = (-1)^j h(-x + c)$ for some integer $c$. Then:

$$g(x + c) = (-1)^j h(-x),$$

and so

$$P_G(x + c) = (-1)^j \frac{(x + c)_k}{(-x)_k} P_H(-x) = (-1)^{j+k} \frac{(x + c)_k}{(x + k - 1)_k} P_H(-x).$$

Evaluating this equation at $x = 1$ gives:

$$P_G(c+1) = \binom{c+1}{k}(-1)^{j+k}P_H(-1). \tag{6.18}$$

Now, Stanley [45] showed that $(-1)^n P_G(-1)$ is the number of acyclic orientations of an $n$-vertex graph $G$. Thus (6.18) implies that the number of proper $(c+1)$-colourings of $G$ is $\binom{c+1}{k}$ times the number of acyclic orientations of $H$. In particular, Proposition 6.6 implies that the number of proper $(j+k)$-colourings of a $(j,k)$-biclique is $\binom{j+k}{k}$ times the number of acyclic orientations of its complementary partner. It seems likely that a combinatorial proof of this result could be found, which might shed some more light on the results we have presented in this section.

Finally, we return to our previous assertion, that any integer can be realised as the shift between two interesting factors of chromatic polynomials of bicliques having non-matching equivalent complements.

**Corollary 6.18.** *Given any $c \in \mathbb{Z}$, there exists a pair of $(3,k)$-bicliques having non-matching equivalent complements and chromatic polynomials with cubic interesting factors $g(x)$ and $h(x)$ which satisfy $g(x) = h(x+c)$.*

*Proof.* The condition $g(x) = h(x+c)$ holds if and only if $h(x) = g(x-c)$, and so without loss of generality we may assume that $c > 0$. Let $G$ be the $(3, 3c+3)$-biclique defined by the parameters $(1,1,c,c,c,1)$, and let $H$ be the $(3, 7c+2)$-biclique defined by $(c,c,4c,1,1,c)$. Let $g(x)$ be the interesting factor of $P_G(x)$. By Proposition 6.15 the graph $J = (1,1,c,c,c,4c)$ has a chromatic polynomial with interesting factor $j(x) = -g(-x+6c+4)$. This graph is the complementary partner of $H$, and so by Proposition 6.6 we have that the interesting factor of $P_H(x)$ is:

$$h(x) = -j(-x+7c+4) = g((x-7c-4)+6c+4) = g(x-c).$$

Clearly $\bar{G}$ and $\bar{H}$ are not matching equivalent, as they have, respectively, $4c+5$ and $13c+2$ edges. Thus $G$ and $H$ are our desired graphs. □

# Chapter 7

# Galois groups of Tutte polynomials

The work in this chapter was inspired by the relatively recent discovery by Merino, de Mier and Noy [28] that the reducibility of the Tutte polynomial $T_M(x, y)$ of any matroid $M$ corresponds precisely to the connectedness of $M$. It is not difficult to show that, if $M$ is not connected, then $T_M(x, y)$ is a product of the Tutte polynomials of its connected components. However, the authors of the above paper were able to show that connectedness of $M$ is in fact sufficient to guarantee irreducibility of $T_M(x, y)$ over $K(y)$, where $K$ is any field of characteristic zero. A consequence of this theorem is that the degree of the Galois group of $T_M(x, y)$ over $\mathbb{Q}(y)$ depends only on that of the Tutte polynomial of $M$, and thus only on the rank of $M$. This led us to speculate as to whether there exist any other clear correspondences between properties of matroids and those of the Galois groups of their Tutte polynomials.

We computed the Galois groups of the Tutte polynomials of all connected graphic matroids having rank less than ten in the case $K = \mathbb{Q}$, and discovered that every such group was the full symmetric group of degree corresponding to the rank of the matroid. It would be quite remarkable if this were true in general, but we were not able to prove this, and it remains an open problem. However the malleability afforded by extra variables enabled us to prove an analogous result for the multivariate Tutte polynomial, and one which holds for any coefficient field $K$.

We will provide three separate self-contained proofs of this result, each of which provides some illumination as to why such a wide-ranging object is restricted in this way. It is necessary to first show that the multivariate Tutte polynomial of a connected matroid is irreducible over fields of arbitrary characteristic; this follows from the original paper of Merino et al. for fields of characteristic zero, but, as the the authors themselves note, the Tutte polynomial is in fact reducible over some finite fields. Our proof only holds for the multivariate version, however it is significantly more efficient than that given in [28].

The bivariate Tutte polynomial will be addressed in §7.2, where we discuss the results of our computation, and note some interesting implications for the chromatic polynomial which would follow from a proof of our original conjecture.

We shall begin by defining the multivariate Tutte polynomial of a matroid, and explaining its relation to the previously defined graph-theoretical version (§2.3.1). Note that any discussion of graphs in this chapter will include those with loops and multiple edges.

## 7.1 The multivariate Tutte polynomial

The multivariate Tutte polynomial for matroids appears to have been discovered a number of times; the first appearance we are aware of is in [25], where the author denotes it the "Tugger polynomial" and credits its discovery to one R.T Tugger[1]. However, for the sake of continuity, we shall continue to follow [44].

Let $M$ be a finite matroid of positive rank on the set $E$ (our results are trivial for a matroid having zero rank). As previously, we assign a variable $v_e$ to each element $e \in E$, write $\mathbf{v}$ for the collection of all these variables and $\mathbf{v}_A$ for the set $\{v_e\}_{e \in A}$, and define $q$ to be another indeterminate. Sokal gives the following definition for the multivariate Tutte polynomial of a

[1]For the benefit of those readers who may otherwise follow us in searching fruitlessly for this author's original work, we should note that "R.T. Tugger" is in fact Kung's cat, whose full name is Rum Tum Tugger [24].

matroid $M$:

$$\tilde{Z}_M(q,\mathbf{v}) = \sum_{A\subseteq E} q^{-r_M(A)} \prod_{e\in A} v_e.$$

It can easily be seen that $\tilde{Z}_M(q,\mathbf{v})$ is therefore a polynomial in $\frac{1}{q}$ with coefficients in $\mathbb{Z}[\mathbf{v}]$. For our purposes it is more convenient to use the following minor modification:

$$\hat{Z}_M(q,\mathbf{v}) = q^{r(M)}\tilde{Z}_M(q,\mathbf{v}).$$

We then have:

$$\hat{Z}_M(q,\mathbf{v}) = \sum_{A\subseteq E} q^{r(M)-r_M(A)} \prod_{e\in A} v_e. \tag{7.1}$$

which is closer to the graph theoretical version defined in §2.3.1. Note that $\hat{Z}_M(q,\mathbf{v})$ is a polynomial of degree $r(M)$ in $q$, which is monic if $M$ contains no loops (note that this is certainly the case whenever $M$ is connected). Analogously to the previous definition for graphs, $\hat{Z}_M(q,\mathbf{v})$ is a generating function for the content and rank of the subsets of $E$, and thus encodes all of the information about $M$[2]

As matroids are generalisations of graphs, the results of this section of course apply directly to these objects. In fact, by simply substituting, for example, "biconnected graph" for "connected matroid," each proof can be easily translated into graph theoretical terms whilst retaining exactly the same structure. With this in mind, to avoid repetition we shall henceforth dispense with graphs and concentrate on the stronger matroid formulations.

Our main result is the following:

[2]To see precisely how to derive the graph-theoretical multivariate Tutte polynomial from (7.1), let $M(G)$ be the cycle matroid of a graph $G = (V,E)$. Then for any subset $A \subseteq E$ of the edge-set of $G$, we have $r_{M(G)}(A) = |V| - k(A)$, where $k(A)$ is the number of connected components of the subgraph $(V,A)$. Hence we can define the multivariate Tutte polynomial of $G$ to be:

$$\hat{Z}_G(q,\mathbf{v}) = \sum_{A\subseteq E} q^{k(A)-k(G)} \prod_{e\in A} v_e.$$

Multiplication by the prefactor $q^{k(G)}$ then produces the previous definition of $Z_G(q,\mathbf{v})$ given in (2.1).

**Theorem 7.1.** *Let $M$ be a finite connected matroid with positive rank $n = r(M)$, and let $\hat{Z}_M(q, \mathbf{v})$ be as defined above. Let $K$ be an arbitrary field. Then the Galois group of $\hat{Z}_M(q, \mathbf{v})$ over $K(\mathbf{v})$ is the symmetric group on the $n$ roots of $\hat{Z}_M(q, \mathbf{v})$.*

All three of the proofs we present here depend fundamentally on the fact that $\hat{Z}_M(q, \mathbf{v})$ is linear in $v_e$ for all $e \in E$. The first and second proofs are inductive, and both are based on a well known result linking specialisations of polynomials with Galois subgroups. However, they are quite different: in the first we use techniques from algebraic number theory to prove the result from the deletion-contraction recurrence, effectively using induction on individual elements of the ground set $E$ of $M$; in the second we employ group-theoretic arguments, and our induction is on connected "sub-matroids" (that is, connected restrictions of $M$ to subsets of $E$). In the final proof, which is by far the simplest of the three (and was inevitably the last we discovered), we prove directly that $\hat{Z}_M(q, \mathbf{v})$ is a "generic" polynomial, in that its set of coefficients (viewed as elements of $K(\mathbf{v})$) form an algebraically independent set over $K$.

As discussed previously (§2.2.3), for any $e \in E$ the deletion $M \setminus e$ and the contraction $M/e$ of $e$ are both matroids on the set $E \setminus e$. The essential tool for our first proof is a theorem of Tutte (see [32, Theorem 4.3.1]), which says that connectivity of $M$ implies that of at least one of the matroids $M \setminus e$ or $M/e$. Since $M$ is connected, $e$ is not a coloop, so $r(M \setminus e) = r_M(E \setminus e) = r_M(E) = r(M)$. By [32, Prop. 3.1.6] we have that $r(M/e) = r_M(E) - r_M(e)$. Now $r_M(e) = 1$, since $e$ is not a loop. So $r(M/e) = r(M) - 1$.

The proof will be based on some lemmas.

**Lemma 7.2.** *Let $M$ be a finite connected matroid and $e \in E$. Then*

$$\hat{Z}_M = \hat{Z}_{M\setminus e} + v_e \hat{Z}_{M/e}.$$

*Proof.* Since $M$ is connected, $e$ is neither a loop nor a coloop. By [44, (4.18a)] $\tilde{Z}_M = \tilde{Z}_{M\setminus e} + \frac{v_e}{q}\tilde{Z}_{M/e}$, hence

$$\hat{Z}_M = q^{r(M)-r(M\setminus e)}\hat{Z}_{M\setminus e} + q^{r(M)-r(M/e)}\frac{v_e}{q}\hat{Z}_{M/e}.$$

The claim then follows from the previous determination of the ranks of $E \setminus e$ and $E/e$. $\square$

As an intermediate step in the proof of the theorem, we need to know that $\hat{Z}_M$ is irreducible over $K(\mathbf{v})$. As $T_M$ is essentially a specialisation of $\hat{Z}_M$, this would follow from [28] in the case where $K$ has characteristic zero. However, the multivariate case allows for a much simpler proof, and one which holds for any characteristic.

**Lemma 7.3.** *Let $M$ be a finite connected matroid. Then $\hat{Z}_M$ is irreducible over $K(\mathbf{v})$.*

*Proof.* The induction proof is most conveniently formulated by considering a counterexample $M$ where $r(M)$ is minimal. Among those counterexamples, we pick one where $|E|$ is minimal. Clearly, the result holds for $r(M) = 1$, so let $r(M) \geq 2$. Pick $e \in E$. By Lemma 7.2, $\hat{Z}_M = \hat{Z}_{M\setminus e} + v_e \hat{Z}_{M/e}$. Note that $v_e$ does not appear in $\hat{Z}_{M\setminus e}$ and $\hat{Z}_{M/e}$. If $M \setminus e$ is connected, then $\hat{Z}_{M\setminus e}$ is irreducible by minimality of $|E|$. As $\hat{Z}_M$ and $\hat{Z}_{M\setminus e}$ have the same degree, setting $v_e = 0$ shows that $\hat{Z}_M$ is irreducible, a contradiction. So $M \setminus e$ is not connected, which by Tutte's theorem means that $M/e$ is connected. So $r(M/e) \geq 1$ (because $r(M) \geq 2$), and $\hat{Z}_{M/e}$ is monic. Note also that because $M$ is loopless, so too is $M \setminus e$, and hence $\hat{Z}_{M\setminus e}$ is also monic.

Now, consider a non-trivial factorization of $\hat{Z}_M$. Since $\hat{Z}_M$ is monic and linear in $v_e$, we can write $\hat{Z}_M = (U + v_e V)W$, where $U, V, W$ are polynomials in $K[\mathbf{v}][q]$ in which $v_e$ does not appear, and where each factor has positive degree in $q$.

So $(U + v_e V)W = \hat{Z}_{M\setminus e} + v_e \hat{Z}_{M/e}$. Comparing coefficients with respect to $v_e$ gives $UW = \hat{Z}_{M\setminus e}$ and $VW = \hat{Z}_{M/e}$. By minimality of the counterexample, $\hat{Z}_{M/e}$ is irreducible. But $W$ has positive degree in $q$, so $V = 1$ and $W = \hat{Z}_{M/e}$. Thus $U\hat{Z}_{M/e} = \hat{Z}_{M\setminus e}$. Now, $\hat{Z}_{M/e}$ and $\hat{Z}_{M\setminus e}$ are monic of degrees $r(M) - 1$ and $r(M)$ respectively. So $U = q + \beta$ for some $\beta \in K[\mathbf{v}]$. Let $\bar{\mathbf{v}} = \mathbf{v} \setminus \{v_e\}$, and note that

$$\hat{Z}_{M\setminus e}(1, \bar{\mathbf{v}}) = \prod_{i \in E\setminus e} (1 + v_i) = \hat{Z}_{M/e}(1, \bar{\mathbf{v}}),$$

so $\beta = 0$. Now setting $q = 0$ gives $\hat{Z}_{M\setminus e}(0, \bar{\mathbf{v}}) = 0$. This means that the only basis of $M \setminus e$ is the empty set, which is only possible if every element of $E \setminus e$ is a loop. So we have a contradiction. □

In order to prove the theorem, we need more precise information about how Galois groups of multivariate polynomials behave under specialisations of variables. The next result is well-known, although it is what one might describe as a "folklore" result, in that it is surprisingly difficult to find a proof of it. The only direct and well-explained one we know of appears in the unlikely setting of a paper about rigidity in planar graphs [31]; otherwise, it follows from the much more general [26, Theorem IX.2.9].

**Proposition 7.4.** *Let $R$ be an integral domain which is integrally closed in its quotient field $F$. Let $f \in R[X]$ be monic and irreducible over $F$. Let $R \to k$, $r \mapsto \bar{r}$ be a homomorphism to a field $k$. If $\bar{f} \in k[X]$ is separable, then $\mathrm{Gal}(\bar{f}/k)$ is a subgroup of $\mathrm{Gal}(f/F)$.*

The following two lemmas can be obtained through applications of this proposition.

**Lemma 7.5.** *Let $A$ be a subset of $E$. Then $\mathrm{Gal}(\hat{Z}_{M|A}/K(\mathbf{v}_A))$ is a subgroup of $\mathrm{Gal}(\hat{Z}_M/K(\mathbf{v}))$.*

*Proof.* Let $B$ be such that $A \subset B \subseteq E$, and let $e$ be an element of $B \setminus A$. Note that removing $e$ from $B$ corresponds to specialising $v_e$ to zero in $\hat{Z}_{M|B}$. Let $R = K(\mathbf{v}_{B\setminus e})[v_e]$, and let $I$ be the maximal ideal of $R$ generated by $v_e$. The image of $\hat{Z}_M$ in the canonical homomorphism $R \to R/I$ is either $q\hat{Z}_{M|(B\setminus e)}$ or $\hat{Z}_{M|(B\setminus e)}$, depending on whether or not $e$ is a coloop. In both cases we have a separable polynomial, as factorisation into repeated non-linear factors would contradict the fact that $\hat{Z}_{M|(B\setminus e)}$ is linear in the elements of $\mathbf{v}_{B\setminus e}$. Furthermore, $R$ is integrally closed in its quotient field $K(\mathbf{v})$. So we have that $\mathrm{Gal}(\hat{Z}_{M|(B\setminus e)}/K(\mathbf{v}_{B\setminus e})) \leq \mathrm{Gal}(\hat{Z}_{M|B}/K(\mathbf{v}_B))$ by Proposition 7.4, and the result follows by induction. □

**Lemma 7.6.** *Let $y$ be a variable over the field $k$, and $U, V \in k[X]$ with $\deg V = n - 1$, and $U$ monic of degree $n$ (where $n \geq 2$). Suppose that $f(X) = U(X) + yV(X)$ is irreducible over $k(y)$ (which is equivalent to $U$ and $V$ being relatively prime). If $\mathrm{Gal}(U/k) = S_n$ or $\mathrm{Gal}(V/k) = S_{n-1}$, then $\mathrm{Gal}(f/k(y)) = S_n$.*

*Proof.* First suppose that $\mathrm{Gal}(U/k) = S_n$. Then the assertion follows immediately from Proposition 7.4 by setting $R = k[y]$ and considering the homomorphism $R \to k$, $h(y) \mapsto h(0)$.

Now assume that $\mathrm{Gal}(V/k) = S_{n-1}$. Set $t = 1/y$ and consider the polynomial

$$\hat{f}(X) = tX^n f(1/X) = X^n(tU(1/X) + V(1/X))$$

obtained by multiplying the reciprocal function of $f(X) = U(X) + \frac{1}{t}V(X)$ by $t$. Then $k(t) = k(y)$, and we have that $\mathrm{Gal}(f/k(y)) = \mathrm{Gal}(\hat{f}/k(t))$. The coefficient of $X^n$ in $\hat{f}$ is $tu + v$, where $u$ and $v$ are the constant terms of $U$ and $V$. If $v = 0$, then $V$ has the root 0. However, $V$ is irreducible, since $\mathrm{Gal}(V/k) = S_{n-1}$. So we must have that $V$ is linear; that is, $n = 2$. The result therefore holds in this case, as an irreducible polynomial of degree 2 clearly has Galois group $S_2$.

So assume $v \neq 0$. Let $R \subset k(t)$ be the localisation of $k[t]$ with respect to the ideal $(t)$, so that $R$ consists of the fractions $p(t)/q(t)$ with $q(0) \neq 0$. Note that $\frac{1}{tu+v}\hat{f}$ is monic with coefficients in $R$. Also, $R$ (as a local ring) is integrally closed in $k(t)$. Let $R \to k$ be the homomorphism given by $p(t)/q(t) \mapsto p(0)/q(0)$. Proposition 7.4 then gives $\mathrm{Gal}(\hat{f}/k(t)) \geq \mathrm{Gal}(X^n V(1/X)/k) = S_{n-1}$. Because $\mathrm{Gal}(\hat{f}/k(t))$ is transitive on the $n$ roots of $\hat{f}$, we must have $\mathrm{Gal}(\hat{f}/k(t)) = S_n$. □

We are now ready to prove Theorem 7.1.

*First proof of Theorem 7.1.* Again assume that the matroid $M$ is a counterexample with $r_M(E)$ minimal, and among these cases pick one with $|E|$ minimal. Note that the statement is trivially true if $r(M) = 1$, thus $r(M) \geq 2$ in the minimal counterexample.

Pick $e \in E$. By Lemma 7.2 $\hat{Z}_M = \hat{Z}_{M \setminus e} + v_e \hat{Z}_{M/e}$. Let $\bar{\mathbf{v}} = \mathbf{v} \setminus \{v_e\}$, and set $k = K(\bar{\mathbf{v}})$. Recall that $\hat{Z}_M$ is irreducible over $k(v_e)$ by Lemma 7.3. We have seen above that $r(M \setminus e) = r(M) = n$ and $r(M/e) = n - 1$. As established previously, either $M \setminus e$ or $M/e$ is connected. By assuming a minimal counterexample we have $\mathrm{Gal}(\hat{Z}_{M\setminus e}/k) = S_n$ or $\mathrm{Gal}(\hat{Z}_{M/e}/k) = S_{n-1}$. Theorem 7.1 then follows from Lemma 7.6. □

Our second proof of Theorem 7.1 uses induction on restrictions of matroids and subgroups. It is perhaps slightly more intuitive than the above, but requires us to prove the base case of circuits separately.

**Lemma 7.7.** *Let $C \subseteq E$ be a circuit of a finite matroid $M$. Then:*

$$\mathrm{Gal}(\hat{Z}_{M|C}/K(\mathbf{v}_C)) = S_{r_M(C)}.$$

*Proof.* The rank of any proper subset of $C$ is the same as its cardinality, and $r_M(C) = |C| - 1$, so:

$$\hat{Z}_{M|C}(q, \mathbf{v}) = q^n + \sigma_1 q^{n-1} + \sigma_2 q^{n-2} + \ldots + \sigma_{n-1} q + (\sigma_n + \sigma_{n+1}),$$

where $n = r_M(C)$ and $\sigma_i$ is the $i$th elementary symmetric polynomial in the $\{v_e\}_{e \in C}$ for each $i$. The elementary symmetric polynomials are algebraically independent, and thus so too are the coefficients of $\hat{Z}_{M|C}(q, \mathbf{v})$. It is well known that the Galois group of a polynomial with algebraically independent coefficients is the full symmetric group (see, for example, Chapter VI, §2 of [26]). □

*Second proof of Theorem 7.1.* Let $C$ be a circuit of maximum cardinality in $M$. By Lemma 7.7, $\mathrm{Gal}(\hat{Z}_{M|C}/K(\mathbf{v}_C)) = S_{r_M(C)}$. This will serve as the base case for the induction.

Now, let $A$ be any proper subset of $E$ such that $C \subseteq A$ and $M|A$ is connected, and suppose that $\mathrm{Gal}(\hat{Z}_{M|A}/K(\mathbf{v}_A)) = S_{r_M(A)}$. Identify a non-empty independent set $B \subseteq E \setminus A$ of minimal size such that $M|(A \cup B)$ is connected, and let $A' = (A \cup B)$. We will show that $\mathrm{Gal}(\hat{Z}_{M|A'}/K(\mathbf{v}_{A'})) = S_{r_M(A')}$.

By connectivity of $M|A'$, we have $r_M(A') < r_M(A) + r_M(B)$, and minimality of $B$ implies that $r_M(B) \leq r_M(C)$. Thus

$$r_M(A') < r_M(A) + r_M(C). \tag{7.2}$$

By Lemma 7.5, $S_{r_M(A)} = \mathrm{Gal}(\hat{Z}_{M|A}/K(\mathbf{v}_A)) \leq \mathrm{Gal}(\hat{Z}_{M|A'}/K(\mathbf{v}_{A'}))$. So $\mathrm{Gal}(\hat{Z}_{M|A'}/K(\mathbf{v}_{A'}))$ must contain at least one transposition. Let $H$ be the group generated by all of the transpositions in $\mathrm{Gal}(\hat{Z}_{M|A'}/K(\mathbf{v}_{A'}))$; then $H$ is a direct product of symmetric groups. As $\mathrm{Gal}(\hat{Z}_{M|A'}/K(\mathbf{v}_{A'}))$ is transitive, each of these symmetric groups must have the same degree $i$, which must therefore divide the degree of $\mathrm{Gal}(\hat{Z}_{M|A'}/K(\mathbf{v}_{A'}))$. By Lemma 7.3, $\hat{Z}_{M|A'}$ is irreducible, and its Galois group must therefore be transitive of degree $r_M(A')$. So we have that $ji = r_M(A')$ for some positive integer $j$.

Now, $S_{r_M(A)}$ contains at least one of the transpositions of $H$, so must be a subgroup of one the $S_i$, which means $r_M(A) \leq i$. So we have:

$$jr_M(A) \leq ji = r_M(A'),$$

and substituting from (7.2) we obtain

$$jr_M(A) < r_M(A) + r_M(C).$$

Now suppose that $j \geq 2$. Then $2r_M(A) < r_M(A) + r_M(C)$, and so $r_M(A) < r_M(C)$. This is impossible, as $C \subset A$. So $j = 1$, and hence $i = r_M(A')$. This means that $H$ is a direct product of symmetric groups of degree $r_M(A')$. But $H$ is a subgroup of $\mathrm{Gal}(\hat{Z}_{M|A'}/K(\mathbf{v}_{A'}))$, which is transitive of degree $r_M(A')$, and so $\mathrm{Gal}(\hat{Z}_{M|A'}/K(\mathbf{v}_{A'})) = H = S_{r_M(A')}$. □

Now, in view of the proof of Lemma 7.7, one might wonder if the coefficients of $\hat{Z}_M(q, \mathbf{v})$ are algebraically independent for *any* finite connected matroid. This does indeed turn out to be the case, leading us to our third and final proof of Theorem 7.1.

*Third proof of Theorem 7.1.* Let $M$ be a finite connected matroid of rank $r(M) = n \geq 1$, and write $\hat{Z}_M(q, \mathbf{v}) = q^n + a_{n-1}q^{n-1} + \cdots + a_1 q + a_0 \in K[\mathbf{v}][q]$, where $K$ is an arbitrary field. It suffices to show that the coefficients $a_0, a_1, \ldots, a_{n-1}$ are algebraically independent over $K$.

If $n = 1$, then $\hat{Z}_M(q, \mathbf{v}) = q - 1 + \prod_{e \in E}(v_e + 1)$, so the claim clearly holds. Thus we may assume $n \geq 2$.

Assume that $M$ is a counterexample in which $|E|$ is minimal. We will use the deletion-contraction identity $\hat{Z}_M = \hat{Z}_{M \setminus e} + v_e \hat{Z}_{M/e}$ of Lemma 7.2. First consider the case that $M \setminus e$ is connected. By the assumption of a minimal counterexample, the coefficients of $\hat{Z}_{M \setminus e}$ (excluding the leading coefficient 1) are algebraically independent over $K$. However, these coefficients arise from the coefficients $a_0, a_1, \ldots, a_{n-1}$ upon setting $v_e = 0$. Of course, an algebraic dependency relation between $a_0, a_1, \ldots, a_{n-1}$ over $K$

remains an algebraic dependency relation upon setting $v_e = 0$, a contradiction.

Thus $M \setminus e$ is not connected, so we may assume that $M/e$ is connected. For each $0 \leq i \leq n-1$, write $a_i = b_i + v_e c_i$, where $b_i$ and $c_i$ are polynomials in the elements of $\mathbf{v}_{E\setminus e}$. Each $c_j$ is then the coefficient of $q^j$ in $\hat{Z}_{M/e}$, so $c_{n-1} = 1$ (as $r(M/e) = n-1$) and $c_0, c_1, \ldots, c_{n-2}$ are algebraically independent over $K$. As $a_0, a_1, \ldots, a_{n-1}$ are algebraically dependent, there is a non-zero polynomial $P$ in $n$ variables over $K$ such that

$$P(b_0 + v_e c_0, \ldots, b_{n-2} + v_e c_{n-2}, b_{n-1} + v_e) = 0.$$

Let $Q$ be the expansion of $P$ with respect to $v_e$, so that $Q$ is a polynomial in $v_e$ with coefficients in $K[\mathbf{v}_{E\setminus e}]$. As the elements of $\mathbf{v}$ are algebraically independent, these coefficients must be identically zero. Let $d$ be the total degree of $P$. Then $Q$ has degree $d$ in $v_e$, and the $v_e^d$ term must arise from a $K$-linear sum of products of the form:

$$(b_0 + v_e c_0)^{d_0} \ldots (b_{n-2} + v_e c_{n-2})^{d_{n-2}} (b_{n-1} + v_e)^{d_{n-1}},$$

where $d_0, \ldots, d_{n-1}$ are non-negative integers which sum to $d$. This means that the coefficient of $v_e^d$ in $Q$ is a $K$-linear combination of monomials of the form $c_0^{d_0} \ldots c_{n-2}^{d_{n-2}}$, where $d_i \geq 0$ for each $i$, and $d_0 + \cdots + d_{n-2} \leq d$. The vanishing of this coefficient then implies that the set of such monomials is linearly dependent over $K$, which contradicts our assertion that $c_0, \ldots, c_{n-2}$ are algebraically dependent over $K$. □

**Remark 7.8.** Sokal showed that the the multivariate Tutte polynomial for matroids factorizes over summands (see [44, (4.4)]). That is, if $M$ is the direct sum of connected matroids $M_1, M_2$ on the sets $E_1, E_2$ respectively (where $E_1$ and $E_2$ are disjoint and $E = E_1 \cup E_2$) then:

$$\hat{Z}_M(q, \mathbf{v}) = \hat{Z}_{M_1}(q, \mathbf{v}_{E_1}) \hat{Z}_{M_2}(q, \mathbf{v}_{E_2}).$$

As $\mathbf{v}_{E_1}$ and $\mathbf{v}_{E_2}$ are disjoint, there are clearly no algebraic dependencies between the roots of $\hat{Z}_{M_1}$ and $\hat{Z}_{M_2}$, so we have that

$$\mathrm{Gal}(\hat{Z}_M / K(\mathbf{v})) = \mathrm{Gal}(\hat{Z}_{M_1} / K(\mathbf{v}_{E_1})) \times \mathrm{Gal}(\hat{Z}_{M_2} / K(\mathbf{v}_{E_2})).$$

Theorem 7.1 then implies that the Galois group of the multivariate Tutte polynomial of any matroid is a direct product of symmetric groups corresponding to the connected direct summands.

## 7.2 The bivariate Tutte polynomial

The Tutte polynomial of a matroid $M$ is usually defined to be:

$$T_M(x,y) = \sum_{A \subseteq E} (x-1)^{r(M)-r_M(A)}(y-1)^{|A|-r_M(A)}.$$

Similarly to the graph-theoretical version described in §2.3.2, it can be obtained from the multivariate Tutte polynomial $\hat{Z}_M(q,\mathbf{v})$ by making the following substitutions:

$$\begin{aligned} q &\leftarrow (x-1)(y-1) \\ v_e &\leftarrow y-1 \end{aligned}$$

for each $e \in E$, and multiplying by a prefactor $(y-1)^{-r(M)}$.

Thus $T_M$ is essentially equivalent to a special case of $\hat{Z}_M$ in which the same variable is assigned to every element of $E$. As with the multivariate Tutte polynomial, we can use the formula $r_{M(G)}(A) = |V| - k(A)$, where $M(G)$ is the cycle matroid of the graph $G = (V,E)$, to derive the following graph-theoretic formulation of the Tutte polynomial:

$$T_G(x,y) = \sum_{A \subseteq E} (x-1)^{k(A)-k(E)}(y-1)^{k(A)+|A|-|V|}.$$

The work which eventually led to Theorem 7.1 was originally inspired by a computation of the Galois groups (over $\mathbb{Q}(y)$) of the Tutte polynomials of all biconnected graphs on up to 10 vertices. We found that all were the symmetric group, which led us to conjecture that this was the case for every biconnected graph. As mentioned in the introduction to this chapter, the Tutte polynomial of any connected matroid is irreducible over fields of characteristic zero, and so it is natural to extend this conjecture to connected matroids. However, unlike the multivariate version, the Tutte polynomial is not necessarily irreducible over fields of positive characteristic. So we are led to the following most general possible form of our conjecture.

**Conjecture 7.9.** *Let $M$ be a finite connected matroid with positive rank $n = r(M)$, and let $K$ be a field of characteristic zero. Then the Galois group of the Tutte polynomial $T_M(x, y)$ over $K(y)$ is the symmetric group of degree $n$.*

The fact that the bivariate Tutte polynomial is a specialisation of the multivariate version implies that, for fields of characteristic zero, Theorem 7.1 would follow from a proof of Conjecture 7.9 (via an application of Proposition 7.4). However, without the flexibility afforded by the extra variables of the multivariate Tutte polynomial, it is difficult to see a way of proving the conjecture.

Interestingly, specialising the Tutte polynomial further produces a range of different Galois groups. For example, it was shown in [8] that all of the transitive permutation groups of degree at most 5 apart from $C_5$ appear as Galois groups of just one family of chromatic polynomials. Furthermore, Morgan [30] showed that a range of transitive groups of higher degree occur for chromatic polynomials of graphs on up to 10 vertices. If Conjecture 7.9 is true, then given any matroid $M$, Hilbert's Irreducibility Theorem implies that almost all rational specialisations $y \leftarrow a$ of $T_M(x, y)$ produce a polynomial with symmetric Galois group over $\mathbb{Q}$. Thus a proof of this conjecture would make the rich algebraic structure of the chromatic polynomial all the more surprising.

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