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Algebraic Structure of Topological and Conformal Field Theories

Rajath Krishna Radhakrishnan

Supervisor

Dr Matthew Buican

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Centre for Theoretical Physics
School of Physical and Chemical Sciences
Queen Mary University of London

Dedicated to my parents and Deepthi.

Declaration

I, Rajath Krishna Radhakrishnan, confirm that the research included within this thesis is my own work or that where it has been carried out in collaboration with, or supported by others, that this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below.

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Details of collaboration and publications:

This thesis describes research carried out with my supervisor Matthew Buican, which was published in [1], [2], [3], [4] and [5]. The research published in [2] and [3] was carried out also in collaboration with Linfeng Li. The research published in [5] was carried out also in collaboration with Anatoly Dymarsky.

Where other sources have been used, they are cited in the bibliography.

Abstract

Quantum field theories (QFTs) are geometric and analytic in nature. With enough symmetry, some QFTs may admit partial or fully algebraic descriptions. Topological and conformal field theories are prime examples of such QFTs. In this thesis, the algebraic structure of $2+1$D Topological Quantum Field Theories (TQFTs) and associated Conformal Field Theories (CFTs) is studied. The line operators of $2+1$D TQFTs and their correlation functions are captured by an algebraic structure called a Modular Tensor Category (MTC). A basic property of line operators is their operator product expansion. This is captured by the fusion rules of the MTC. We study the existence and consequences of special fusion rules where two line operators fuse to give a unique outcome.

There is a natural action of a Galois group on MTCs which allows us to jump between points in the space of TQFTs. We study how the physical properties of a TQFT like its symmetries and gapped boundaries transform under Galois action. We also study how Galois action interacts with other algebraic operations on the space of TQFTs like gauging and anyon condensation. Moreover, we show that TQFTs which are invariant under Galois action are very special. Such Galois invariant TQFTs can be constructed from gauging symmetries of certain simple abelian TQFTs.

TQFTs also admit gapless boundaries. In particular, 1+1D Rational CFTs (RCFTs) and $2+1$D TQFTs are closely related. Given a chiral algebra, the consistent partition functions of an RCFT are classified by surface operators in the bulk $2+1$D TQFT. On the other hand, Narain RCFTs can be constructed from quantum error-correcting codes (QECCs). We give a general map from Narain RCFTs to QECCs. We explore the role of topological line operators of the RCFT in this construction and use this map to give a quantum code theoretic interpretation of orbifolding.

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Contents

Chapter 1

Introduction

Quantum field theory (QFT) is used in describing a vast variety of physical phenomena. From fundamental particles and their interactions, condensed matter physics to applications to mathematics, QFTs play a crucial role. To understand the landscape of QFTs, it is interesting to study the relationships between different QFTs. However, since we lack a mathematical definition of a QFT, it is hard to make progress in general. Moreover, a Lagrangian-based approach to QFTs is riddled with dualities. While dualities reveal beautiful connections between different descriptions of physical phenomena, they show that our description of nature has huge redundancies.

A Lagrangian-independent approach to QFT is more universal, and it can sometimes make relations between different descriptions of a QFT more apparent. The idea is to extract the operator content and correlation functions of a QFT and encode them in natural mathematical structures. Vertex operator algebras (and their representations) are one such structure for $1+1$D CFTs, while, for $1+1$D TQFTs, Frobenius algebras play a similarly important role. For TQFTs in general spacetime dimensions, n-categories play a central role [6]. In the case of $1+1$D CFTs and TQFTs, symmetries play a crucial role in enabling a non-perturbative description.

Symmetries provide a non-perturbative way to constrain the dynamics of a quantum field theory (QFT). Depending on the spacetime dimension and the symmetry under consideration, one may be able to, in principle, solve many or all consistent QFTs with that symmetry. A well-known example uses infinite conformal symmetry to bootstrap certain $1+1$D conformal field theories (CFTs) [7–11]. While conformal symmetry tightly constrains the space of CFTs, solving the conformal bootstrap equations is, in general, highly non-trivial. The subspace of Rational Conformal Field Theories (RCFTs) is easier to tackle. While a full classification is still hard, there are several general results that can be proven. In particular, given a chiral algebra, a classification of consistent partition functions is known.

If we enlarge the symmetry from conformal to full metric independence[1], we get Topological Quantum Field Theories (TQFTs). TQFTs are constrained so much so that the corresponding bootstrap equations can be solved in lower dimensions. For example, in $1+0$D, TQFTs are fully determined by a finite-dimensional Hilbert space. In $1+1$D, the TQFT bootstrap equations can be solved to show that they are determined by Frobenius algebras. [6, 13–15]. Topological quantum field theories (TQFTs) lie at the heart of important physical [16], mathematical [12], and computational [17] systems and constructions. From a high-energy physics perspective, TQFTs may seem like trivial QFTs. However, they play a crucial role in the description of symmetries and anomalies of general quantum field theories.

TQFTs by definition have topological operators. These are operators whose correlation functions are independent of the metric. In recent years, it has been understood that topological operators play a crucial role in describing symmetries of a QFT [18]. Given a QFT with its spectrum of local and extended operators, the symmetries of the QFT are given by the subset of topological local and extended operators. Topological operators of codimension-1 implement ordinary (0-form) symmetries while those of higher codimension implement higher-form symmetries. Moreover, since topological operators may not always obey a group law under fusion, it naturally leads to the concept of non-invertible symmetries [19–23]. These new notions of symmetries have been used to constrain the dynamics of QFTs [24] [25] [22]. Moreover, through anomaly inflow, d-dimensional invertible TQFTs capture the anomalies of a $(d-1)$-dimensional QFT [26]. More generally, the symmetries of a d-dimensional QFT as well as their anomalies are captured by a generically non-invertible $(d+1)$-dimensional TQFT [27] [28]. Therefore, studying topological operators and their algebraic structure is a study of general symmetry structures of QFTs. TQFTs provide the simplest setting to explore this structure.

In this thesis, we will mainly focus on $2+1$D TQFTs and $1+1$D RCFTs.[2] The algebraic structure describing a $2+1$D TQFT is called a Modular Tensor Category (MTC) [29] [30]. MTCs encode essential physical data of a TQFT without additional redundancies like the choice of a gauge group in a Lagrangian description.[3] Indeed, the fact that the same MTC can be realized by Lagrangians based on different gauge groups makes it clear that gauge groups are, as is well known, not generally duality invariant. Topological symmetry is powerful enough to give us a set of constraints known as the Pentagon and Hexagon equations whose solutions give us all possible consistent MTCs.

[1] Note that even if the classical Lagrangian is metric independent, the correlation functions of the quantized theory may not be strictly a topological invariant. For example, in Chern-Simons theory the correlation functions have a framing dependence [12].

[2] Throughout, we will study non-spin TQFTs and RCFTs (i.e., TQFTs and RCFTs that do not depend on a choice of spin structure).

[3] Although MTCs also have redundancies related to points where particle worldlines fuse.

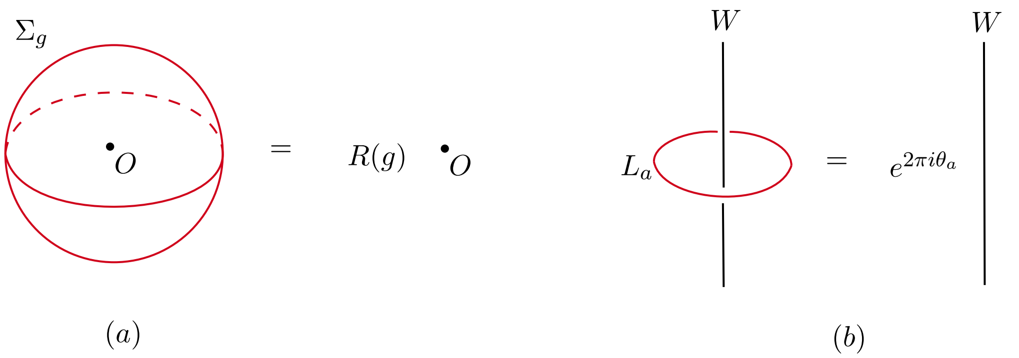


Figure 1.1: (a) In $2+1$D, a 0-form symmetry G is implemented by a topological surface operator Σ_g acting on a local operator O, for some $g \in G$. (b) In $2+1$D, a 1-form symmetry $\mathcal{A}$ is implemented by a topological line operator L_a acting on a line operator W, for some $a \in \mathcal{A}$.

Upon making a further discrete choice, one obtains a corresponding 2+1D TQFT with a fully-specified set of line operators [31]. Therefore, finding a consistent $2+1$D TQFT in this sense essentially follows from finding the zeros of some multivariable polynomials. The Pentagon and Hexagon equations are the bootstrap equations for $2+1$D TQFTs. However, they are hard to solve in general.

The basic data that goes into defining a $2+1$D TQFT is a choice of line operators and their operator product expansion (OPE). The OPE is captured abstractly by the fusion rules of the MTC. While a full description of the TQFT requires us to solve the Pentagon and Hexagon equations, the fusion rules themselves contain crucial information about the TQFT. Abelian/invertible line operators are those which fuse with other line operators to give a unique outcome, while non-abelian/non-invertible line operators fuse with other operators, in general, to give multiple outcomes. In particular, given a non-abelian line a, we can consider the line $\overline{a}$ with the opposite orientation and their fusion, denoted $a \otimes \overline{a}$, results in multiple outcomes. However, sometimes, two non-invertible lines can fuse to give a unique outcome. In other words, some TQFTs admit non-invertible lines which can be written as a fusion of two other non-invertible lines. In this thesis, we explore the existence and physical consequences of such fusion rules.

Almost always, the physics of a system is described by differential equations. In contrast, TQFTs are determined by Pentagon and Hexagon equations, which are multivariable polynomial equations. This allows us to study TQFTs through the lens of Galois theory. Using the data that defines a TQFT, we can construct a field extension whose Galois group acts on the TQFT to give us another TQFT [32] [33]. Therefore, Galois actions allow us to jump between different points in the space of TQFTs. While the origin of this map between TQFTs may seem abstract, it relates TQFTs with various common properties. In this thesis, we study the Galois orbits of various well-known

TQFTs. We study how various physical properties of a TQFT, like its symmetries and gapped boundaries, change under Galois action. We show how Galois actions interact with other operations on a TQFT like gauging and anyon condensation. To gain further physical insight into Galois action, we study how Galois actions act on a notion of entanglement entropy defined for links in TQFTs.

$2+1$D TQFTs can also have gapless boundaries. In particular, there is a close relationship between $1+1$D RCFTs and $2+1$D TQFTs [12]. Given a $1+1$D RCFT with chiral algebra V, the representations of V form a modular tensor category [34] [35], which is precisely the data defining a TQFT. Conversely, given a TQFT with MTC C, there are an infinite number of RCFTs with a chiral algebra whose representations form C. However, once we fix a chiral algebra, then the possible choices of consistent partition functions are in one-to-one correspondence with surface operators in the TQFT [36] [37].

There is another construction of $1+1$D CFTs which starts from a quantum error-correcting code (QECC) [38]. In this framework, a Narain lattice is constructed from the QECC. The Siegel-Theta function of the Narain lattice gives the partition function of the CFT. The final part of this thesis is aimed at relating these two methods of constructing $1+1$D CFTs. We define a general map from RCFTs whose primaries form an abelian group under fusion to quantum stabilizer codes. We also study the role played by topological operators implementing 0-form symmetries of the CFT in this map. This allows us to study various operations on a CFT, like orbifolding, at the level of the stabilizer code.

The structure of this thesis is as follows. In Chapter 2, we review several aspects of TQFTs which are important for the rest of the thesis. In particular, we emphasize on an algebraic approach and explain how a Modular Tensor Category captures the observables in a TQFT and their correlation functions. In Chapter 3 we study fusion of non-invertible line operators to give a unique non-invertible line operator in $2+1$D TQFTs. In Chapter 4 we introduce Galois actions on TQFTs and study its various properties. In Chapter 5 we further study how Galois action acts on entanglement entropy of links in TQFTs. Finally, in Chapter 6, we introduce an explicit map from RCFTs to quantum stabilizer codes and study how properties of the bulk TQFT and orbifolding are captured by the stabilizer code.

This thesis also includes three appendices. Appendix A contains several examples complementing the discussion in Chapter 3. It also contains the GAP codes used for various explicit calculations mentioned in Chapter 3. Appendix B contains calculations of the entanglement entropy of some hyperbolic and satellite links. It also contains proof of a lemma crucial for the results in Chapter 5. Finally, Appendix C discusses various properties of orbifolded RCFTs. It also contains a discussion on Verlinde subgroups introduced in Chapter 6.

Chapter 2

Topological Quantum Field Theory

Topological quantum field theories are characterized by the fact that the correlation functions of all operators in a TQFT are independent of the metric on the manifold. A well-known class of TQFTs is Chern-Simons theories [12]. For a choice of 3-manifold $\mathcal{M}$ and simple Lie group G, Chern-Simons theory is defined by the action

$$S = \frac{k}{4\pi} \int_{\mathcal{M}} \mathrm{tr}(A \wedge dA + A \wedge A \wedge A) \ , \tag{2.1}$$

where A is the gauge field valued in the Lie algebra of G. k is the level of the theory. Note that the action is an integral of a 3-form over a 3-manifold and is defined without reference to a metric. The classical equation of motion is

$$F = dA = 0 \ . \tag{2.2}$$

Therefore, A is a flat connection. Since all gauge-invariant local operators are constructed from F, the equation of motion implies that we do not have any gauge-invariant local operators. However, we can use the gauge field to construct Wilson line operators given by

$$W_R(\gamma) = \mathrm{Tr}_R \mathcal{P} e^{i \oint_\gamma A} \ , \tag{2.3}$$

where the trace is taken in the representation R of the gauge group G, γ is a closed curve and $\mathcal{P}$ denotes the path-ordering of the exponential.

This description of TQFTs in terms of actions and gauge fields has huge redundancies. These manifest themselves as non-trivial dualities between Chern-Simons theories based on different gauge groups. For example, the $Spin(16)_1$ Chern-Simons theory is dual to a discrete gauge theory with gauge group $\mathbb{Z}_2$. In $3+1$D, one can construct TQFTs based on various higher-form gauge fields, but all of them are equivalent to discrete gauge theories. Dual theories have the same spectrum of line operators and correlation functions. Therefore, it is desirable to capture this information in a mathematical structure which makes these dualities manifest. In the rest of this thesis, we will focus on 2+1D TQFTs. In the following section, we will introduce Modular Tensor Categories (MTCs), which capture data contained in the line operators in the TQFT and their correlation functions.

2.1 Modular Tensor Categories: The Algebra of Line Operators

In this thesis, we will only consider TQFTs with no local operators. In such a $2+1$D TQFT we have line and surface operators. From a general theorem in [6], the absence of local operators implies that the surface operators in 2+1D TQFT can be constructed from its line operators.[4] Therefore, if we want to capture the minimal data required to define a $2+1$D TQFT, we only need to keep track of the line operators and their correlation functions. In the following, we will assume that the TQFT has a finite number of line operators. The line operators of such a TQFT and their correlation functions are captured by an algebraic object called a Modular Tensor Category (MTC).

2.1.1 Fusion

An MTC consists of a finite set of labels, $\{a, b, \cdots\}$. They satisfy the fusion rules

$$a \otimes b = \sum_c N_{ab}^c c \ , \quad N_{ab}^c \in \mathbb{Z}_{\geq 0} \ . \tag{2.4}$$

The labels denote the different line operators in the TQFT and their fusion rules capture the position-independent operator product expansion (OPE) of these operators. These line operators are simple, in the sense that they cannot be written as a sum of other line operators. Among the labels, there is a distinguished label, $\mathbf{1}$, which denotes the trivial line operator (sometimes, in an additive notation for abelian theories, the trivial line is labelled $\mathbf{0}$). Since MTCs describe topological phases of matter, we can also interpret the labels as charges of the quasiparticles in the topological phase. In this language, the label $\mathbf{1}$ denotes the vacuum. The fusion rules describe the ways in which these particles combine to form new ones. The morphisms in this category denote the topological local operators sitting at junctions of line operators. There are no local operators between any two distinct simple line operators.

[4] In the modern physics language, all topological surface operators are obtained by higher gauging 1-form symmetries on surfaces. [39]

The non-negative integers, N_{ab}^c, count the different ways in which a and b combine to form c. Note that the fusion $a \otimes b = c$ is allowed if and only if $N_{ab}^c > 0$. In fact, the N_{ab}^c fusion coefficient is the dimension of the V_{ab}^c fusion Hilbert space. This is the Hilbert space of all local operators at a trivalent junction of lines a, b, c[5]. More generally, the fusion space corresponding to the anyons $a_1, \cdots, a_n$ fusing to give anyon b is written as $V_{a_1 a_2 \cdots a_n}^b$.

Given the fusion rules, we can define the Frobenius-Perron dimension of an anyon a, denoted FPdim(a), as the maximal non-negative eigenvalue of the matrix N_a, where $(N_a)_{b,c} := N_{ab}^c$. The Frobenius-Perron dimension of the MTC C is defined as

$$\text{FPdim(C)} := \sum_a \text{FPdim}(a)^2 \ . \tag{2.5}$$

An MTC is called integral if $\text{FPdim}(a) \in \mathbb{Z}\ \forall a$. An MTC is called weakly integral if $\text{FPdim}(C) \in \mathbb{Z}$.

Fusion of three anyons is associative. This implies that the fusion space $V_{abc}^d = \sum_f V_{ab}^f \otimes V_{fc}^d$ can also be decomposed as $V_{abc}^d = \sum_e V_{bc}^e \otimes V_{ea}^d$. The F matrix is the linear map associated with the isomorphism $\sum_f V_{ab}^f \otimes V_{fc}^d \cong \sum_e V_{bc}^e \otimes V_{ea}^d$ (see Fig. 2.1).

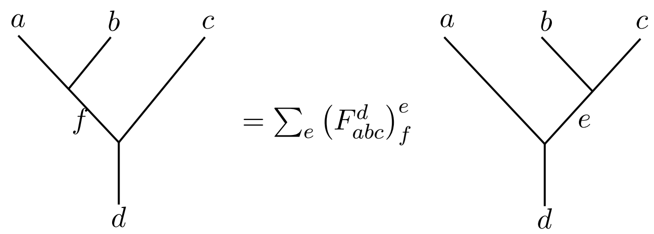


Figure 2.1: Pictorial definition of the F-matrix

From this discussion, we see that

$$F_{abc}^d : \sum_f V_{ab}^f \otimes V_{fc}^d \to \sum_e V_{bc}^e \otimes V_{ea}^d \ , \quad R_{ab}^c : V_{ab}^c \to V_{ba}^c \ . \tag{2.6}$$

Next, from the action of F on V_{abcd}^e, the "Pentagon" consistency equation follows

$$\left(F_{a,b,k}^e\right)_i^l \left(F_{i,c,d}^e\right)_j^k = \sum_m \left(F_{b,c,d}^l\right)_m^k \left(F_{a,m,d}^e\right)_j^l \left(F_{a,b,c}^j\right)_i^m \ . \tag{2.7}$$

[5]Note that even though there are no local operators between two simple lines a and b, there are local operators at a trivalent junction of lines.

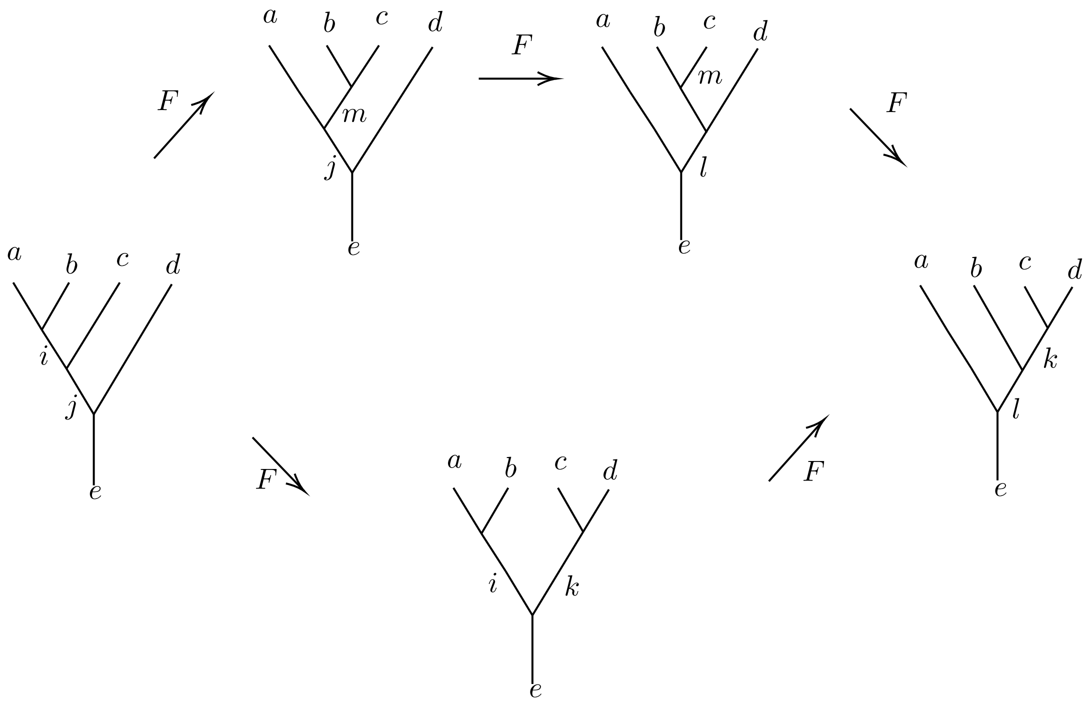


Figure 2.2: Pentagon equations

2.1.2 Braiding

The fusion of two anyons is commutative. This implies the existence of an isomorphism, $V_{ab}^{c} \cong V_{ba}^{c}$, and the associated linear map corresponding to this isomorphism is called the R matrix (see Fig. 2.3).

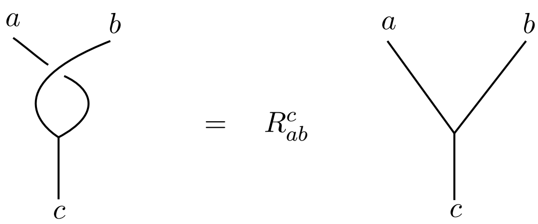


Figure 2.3: Pictorial definition of the R-matrix

Moreover, the braiding of anyons captured by the R matrix should be consistent with the associativity of the fusion rules. In other words, the action of the R and F matrices on V_{abc}^{d} should be consistent. This requirement leads to two "Hexagon" equations. The first takes the form

$$R_{a,c}^{k}\left(F_{a,c,b}^{d}\right)_{j}^{k} R_{b,c}^{j}=\sum_{i}\left(F_{c,a,b}^{d}\right)_{k}^{i} R_{i,c}^{d}\left(F_{a,b,c}^{d}\right)_{i}^{j}\ , \tag{2.8}$$

and the second is

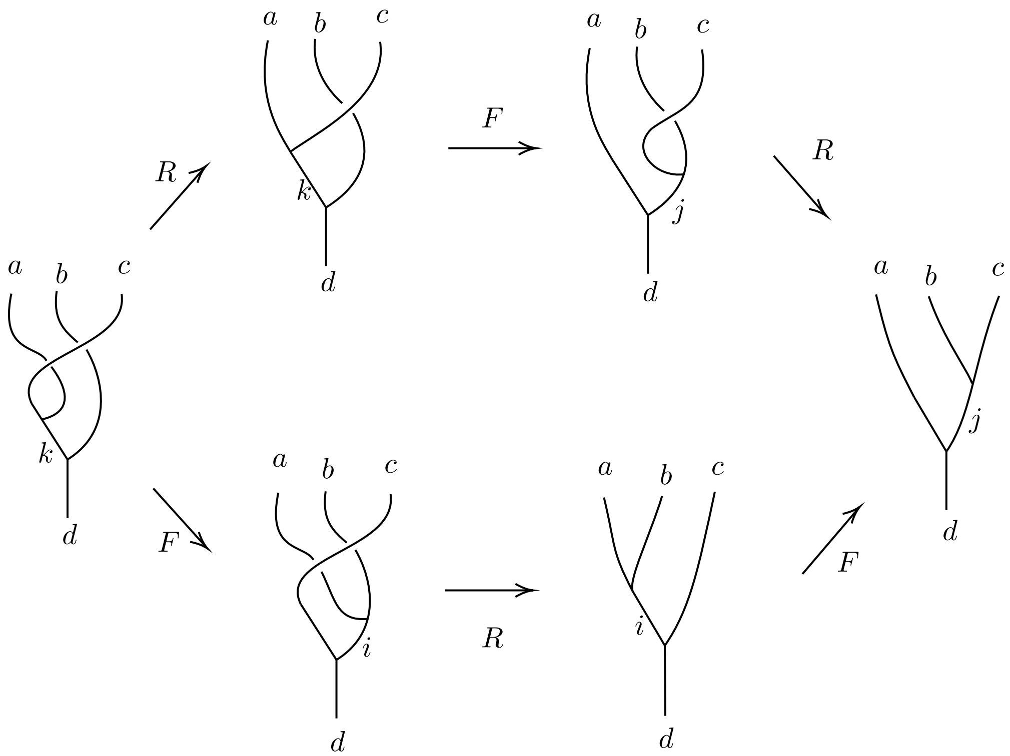


Figure 2.4: Hexagon Equations 1

$$\left(R^{k}_{c,a}\right)^{-1}\left(F^{d}_{a,c,b}\right)^{k}_{j}\left(R^{j}_{c,b}\right)^{-1}=\sum_{i}\left(F^{d}_{c,a,b}\right)^{i}_{k}\left(R^{d}_{c,i}\right)^{-1}\left(F^{d}_{a,b,c}\right)^{j}_{i}\ , \tag{2.9}$$

Suppressing all indices, we will refer to solutions of (2.7), (2.8), and (2.9) simply as F and R. Even though one can start with any set of labels and fusion rules, a consistent MTC exists only if (2.7), (2.8), and (2.9) are satisfied [29,30,40,41].

If we wish to calculate F and R explicitly, we have to choose a basis for the fusion spaces, V^c_{ab}. The solutions to the Hexagon and Pentagon equations obtained by choosing different sets of basis vectors should be considered equivalent. This equivalence is known as the "gauge freedom" in defining F and R. The Pentagon and Hexagon equations have at most a finite number of inequivalent solutions [42, 43]. To summarize, we have captured the line operators and their OPEs via the labels and fusion rules. The commutativity and associativity of the fusion rules lead to the Pentagon and Hexagon equations. At this level of structure, we have defined a braided fusion category.

To add more structure, note that for every anyon a, there is a dual anyon, $\bar{a}$, such that $a\otimes\bar{a}$ involves the vacuum. In other words, $\bar{a}$ is the anti-particle of a, and $\bar{\bar{a}}=a$. To capture this fact in our algebraic construction, we need to define a ribbon structure on the braided fusion category by defining isomorphisms from a to $\bar{\bar{a}}$. These isomorphisms

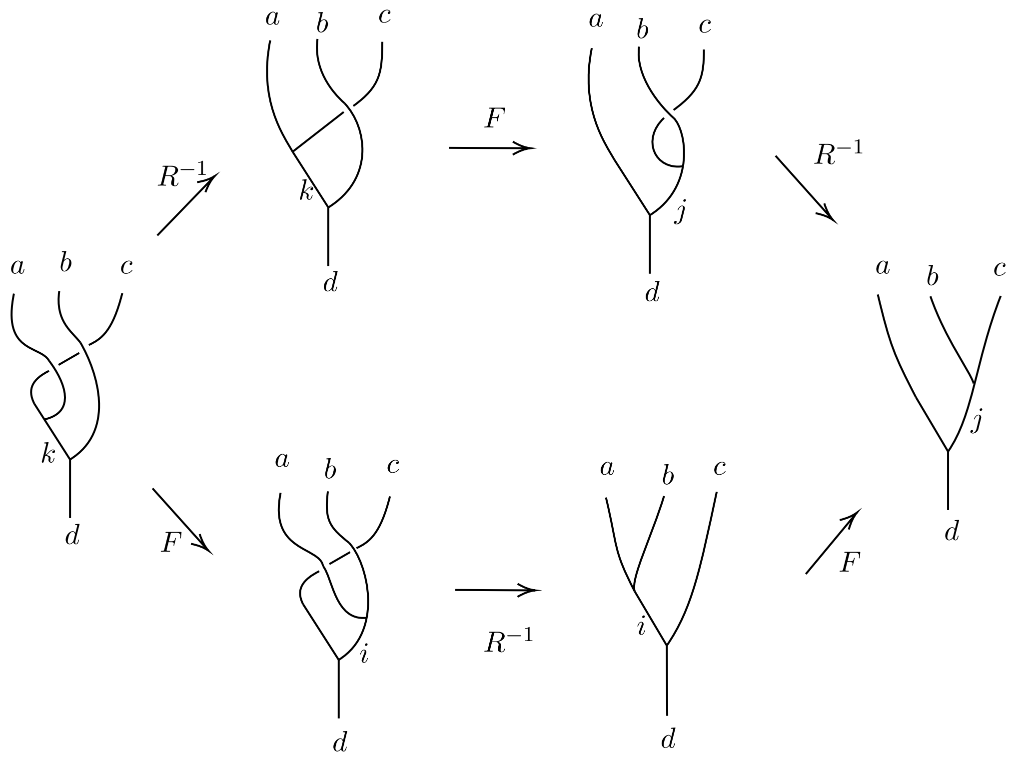


Figure 2.5: Hexagon Equations 2

are captured by phases, ϵ_a, for each label a, satisfying the constraint

$$\epsilon_a^{-1}\epsilon_b^{-1}\epsilon_c = (F^{\mathbf{1}}_{a,b,\bar{c}})^{\bar{a}}_{c}(F^{\mathbf{1}}_{b,\bar{c},a})^{\bar{a}}_{a}(F^{\mathbf{1}}_{\bar{c},a,b})^{\bar{b}}_{b} \ . \tag{2.10}$$

We also require there to be a gauge in which $\epsilon_a \in \{\pm 1\}\ \forall a$. In general, if there is a solution to these constraints, it need not be unique, though the number of distinct solutions is always finite and has been classified [44]. Using these, we can define the quantum dimension of an anyon a as follows

$$d_a := (\epsilon_a (F^{a}_{a\bar{a}a})^{\mathbf{1}}_{\mathbf{1}})^{-1} \ . \tag{2.11}$$

This expression is valid only in a particular basis as chosen in Lemma 3.4 of [33]. d_a is the S^3 link invariant of an unknot labelled by a. Note that d_a depends on several choices and it is not, in general, equal to the Frobenius-Perron dimension of an anyon. In fact, FPdim(a) is always positive, while d_a can be negative for certain choices of solutions ϵ_a to (2.10). In a unitary TQFT, the quantum dimensions are required to be positive, and in this case, there is a unique unitary[6] ribbon structure such that

[6]The phases ϵ_a along with the R matrices can be used to define the topological twist θ_a (described in the next subsection). A ribbon structure is called unitary if θ_a is unitary.

d_a =FPdim(a) $\forall a$ [44]. The total quantum dimension of the TQFT is defined as

$$\mathcal{D} := \sqrt{\sum_a d_a^2} \ , \tag{2.12}$$

where we picked a particular sign that is necessary for the TQFT to be unitary.

2.1.3 Modular Data

At this level of structure, we have defined a ribbon fusion category. We want an MTC to describe systems with no transparent anyons. That is, all non-trivial anyons should braid non-trivially with at least one anyon. This condition is captured by the invertibility of the matrix

$$S_{ab} = \frac{1}{\mathcal{D}} \sum_c d_c Tr(R^c_{ab} R^c_{ba}) = \frac{1}{\mathcal{D}} \tilde{S}_{ab} \ . \tag{2.13}$$

Here, $\tilde{S}_{ab}$ is the invariant of the Hopf link, which captures the creation of two anyon-anti-anyon pairs, their braiding and their annihilation. The T matrix is defined as

$$T_{aa} = d_a^{-1} \sum_c d_c R^c_{aa} = \theta_a \ , \tag{2.14}$$

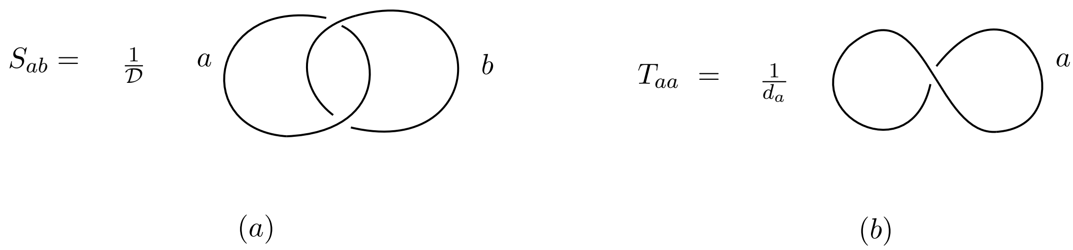


Figure 2.6: (a) The S matrix is the invariant of the Hopf link up to a normalization factor. (b) The T matrix is the invariant of the unknot with a twist.

Recall the description of SL$(2, \mathbb{Z})$ in terms of the generators

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \ t = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \tag{2.15}$$

satisfying relations $(st)^3 = s^2, s^4 = 1$. S and T defined above gives rise to a unitary (projective) representation of the modular group, $SL(2, \mathbb{Z})$, where the generators s and t are represented by S and T, respectively[7]. Indeed, these quantities obey the following

[7]The unitarity of this representation does not imply unitarity of the TQFT.

equations

$$(ST)^3 = \Theta C \ , \quad S^2 = C \ , \quad C^2 = I \ , \tag{2.16}$$

where $\Theta = \frac{1}{\sqrt{\sum_c d_c^2}} \sum_a d_a^2 T_{aa}$, and C is the charge conjugation matrix. Note that we can rescale the T matrix to get a true representation of $\mathrm{SL}(2,\mathbb{Z})$.

The fusion coefficients, N_{ab}^c, are determined by the S matrix elements via the Verlinde formula

$$N_{ab}^c = \sum_e \frac{S_{ae} S_{be} S_{ec^*}}{S_{0e}} \ . \tag{2.17}$$

The central charge of a TQFT, c, is given in terms of Θ through the relation

$$e^{\frac{2\pi i c}{8}} = \Theta \ . \tag{2.18}$$

The solutions to (2.7) and (2.8) admit a cohomological interpretation, where the relevant coboundaries capture the gauge freedom. For example, in the case of abelian MTCs, (F, R) are valued in abelian group cohomology. Given a collection of labels and fusion rules, a $2+1$D TQFT with non-trivial labels/anyons is a cohomologically non-trivial solution to these polynomial equations.[8] We will refer to the collection, (N_{ab}^c, R, F), as the "MTC data", and to the (S, T) pair (or, depending on the context, the $(\tilde{S}, T)$ pair) as the "modular" data.

Finally, note that we can take the total quantum dimension to be

$$\mathcal{D}^{(-)} = -\sqrt{\sum_a d_a^2} \tag{2.19}$$

In this case, the expression for the normalized S matrix changes by a sign. In fact, given the modular data (S, T) of an MTC, there also exists an MTC realizing the modular data $(-S, T)$. Unless otherwise stated, we will use the definition of the total quantum dimension with a positive sign.

[8] In particular, the space of consistent $2+1$D TQFTs satisfying the MTC axioms is discrete [43].

Chapter 3

Irreducible Fusion of Simple Lines

3.1 Introduction

In Chapter 2, we saw that TQFTs can be fully characterized by solving a set of polynomial consistency conditions [40, 41, 45], However, proceeding in this way is often quite difficult as a practical matter (however, see [46, 47] for examples of some results; see also [48] for a potentially very different approach). More generally, it is interesting to understand aspects of the global structure of a TQFT and its symmetries without the need to fully solve the theory (e.g., see [49]).

Proceeding in this way, we will study anyonic fusions $a \times b$ that have a unique product anyon, c

$$a \times b = c \ , \quad a,\ b,\ c \in \mathcal{T} \ , \tag{3.1}$$

in a general $2+1$ dimensional TQFT, $\mathcal{T}$.[9] Our main questions is: 1. When do fusion of simple line operators result in a simple line operator? 2. What do such fusions (see (3.1)) tell us about the global structure of $\mathcal{T}$ and its symmetries?

For invertible a and b (i.e., a and b are abelian anyons), fusion rules of the form (3.1) describe the abelian 1-form symmetry group of the theory [18] (the closely related modular S matrix characterizes its 't Hooft anomalies [50]). In the case in which, say, a is abelian and b is non-abelian,[10] the equation (3.1) gives the fixed points of the fusion of anyons in the theory with the one-form generator, a. Such equations have important consequences for anyon condensation / one-form symmetry gauging in TQFT [50, 51] as well as for orbifolding and coset constructions in closely related 2D rational conformal

[9] Throughout what follows, we only consider non-spin TQFTs. These are theories that do not require a spin structure in order to be well-defined.

[10] In this case, b is non-invertible, and the fusion $b \times \bar{b} = 1 + \cdots$, where $\bar{b}$ is the anyon conjugate to b, necessarily contains at least one more anyon in the ellipses.

field theories (RCFTs) (e.g., see [52, 53]).

Although these cases will play a role below, we will be more interested in the situation in which both a and b are non-abelian

$$a \times b = c \ , \quad d_a \ , \ d_b > 1 \ . \tag{3.2}$$

Here $d_{a,b}$ denote the quantum dimensions of a and b (given they are larger than one, neither a nor b are invertible). Since both a and b are non-abelian, one typically finds that the right-hand side of (3.2) has multiple fusion products. For example, fusions as in (3.2) do not occur in $SU(2)_k$ Chern-Simons (CS) theory for any value of $k \in \mathbb{N}$.[11] Moreover, for discrete gauge theories based on non-abelian simple groups, such fusions are highly constrained. As we will see, when fusions of non-abelian a and b do have a unique outcome, there are consequences for the global structure of $\mathcal{T}$.

The most trivial case in which a fusion of the type (3.2) occurs is when $\mathcal{T}$ factorizes (not necessarily uniquely) as

$$\mathcal{T} = \mathcal{T}_1 \boxtimes \mathcal{T}_2 \ , \tag{3.3}$$

with $\mathcal{T}_1$ and $\mathcal{T}_2$ two separate TQFTs that have trivial mutual braiding, $a \in \mathcal{T}_1$, and $b \in \mathcal{T}_2$.[12] Here "$\boxtimes$" denotes a categorical generalization of the direct product called a "Deligne product" that respects some of the additional structure present in TQFT.

As we will discuss in section 3.5, precisely such a situation arises in the modular tensor categories (MTCs) related to unitary A-type Virasoro minimal models with $c > 1/2$.[13] MTCs are mathematical descriptions of TQFTs, and, for the theories in question, they encapsulate the topological properties of the Virasoro primary fields. One may think of the, say, left-movers in these RCFTs as arising at a 1+1 dimensional interface between 2+1 dimensional CS theories with gauge groups $SU(2)_1 \times SU(2)_k$ and $SU(2)_{k+1}$. In the minimal models, we have

$$\varphi_{(r,1)} \times \varphi_{(1,s)} = \varphi_{(r,s)} \ , \tag{3.4}$$

where $2 \leq r < p-2$ and $2 \leq s < p-1$ are Kac labels that give Virasoro primaries with non-abelian fusion rules (here we have $(r,s) \sim (p-1-r, p-s)$, and $p > 4$ is an integer labeling the unitary minimal model).[14] Thinking in terms of cosets, we will see that (3.4) arises because the Virasoro MTC factorizes as in (3.3).[15]

[11] In section 3.5, we will discuss the situation for more general G_k CS theories.

[12] Note that $\mathcal{T}_{1,2}$ may factorize further. Moreover, a may contain an abelian component in $\mathcal{T}_2$, and b may contain an abelian component in $\mathcal{T}_1$.

[13] Note that in the case of the Ising model ($c = 1/2$), at least one of the anyons in the fusion $a \times b = c$ is abelian (and the corresponding MTC does not factorize). We thank I. Runkel for drawing our attention to the $a \times b = c$ fusion rules for non-abelian fields in Virasoro minimal models.

[14] The abelian field $\varphi_{(p-2,1)} \sim \varphi_{(1,p-1)}$ satisfies the fusion rule $\varphi_{(1,p-1)} \times \varphi_{(1,p-1)} = \varphi_{(1,1)} = 1$.

[15] Note that this factorization does not extend to one of the RCFT.

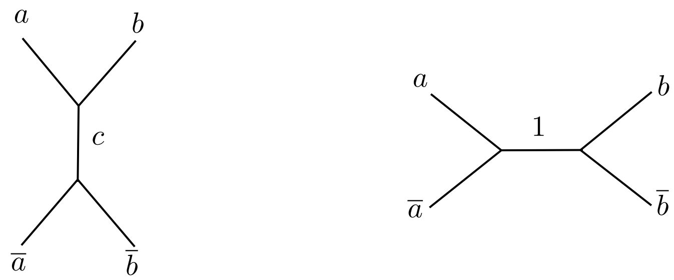


Figure 3.1: The fusions $a \times b$ and $\bar{a} \times \bar{b}$ have unique outcomes c and $\bar{c}$ respectively. In the left diagram, we connect the corresponding fusion vertices. To get to the diagram on the right, we perform an $F_{\bar{b}}^{\bar{a}ab}$ transformation. Just as the left diagram has a unique internal line, so too does the diagram on the right (in this latter case, the internal line must be the identity).

To gain further insight into more general situations in which (3.2) occurs, it is useful to imagine connecting a fusion vertex involving the a, b, c anyons with a fusion vertex involving the $\bar{a}$, $\bar{b}$, and $\bar{c}$ anyons via a c internal line as in the left diagram of figure 3.1. Using associativity of fusion (via the $F_{\bar{b}}^{\bar{a}ab}$ symbol) we arrive at the right diagram of figure 3.1. The relation between these two diagrams can be thought of as a change of basis on the space of internal states. Since, by construction, the left diagram in figure 3.1 can only involve a c internal line, the right diagram in figure 3.1 can also only involve a single internal line. On general grounds, this line must be the identity.[16] This result can also be derived by looking at decomposition of fusion spaces. Consider the fusion space $V_{ba\bar{a}}^{b}$. It can be decomposed in the following different ways

$$V_{ba\bar{a}}^{b} \simeq \sum_{c} V_{ba}^{c} \otimes V_{c\bar{a}}^{b} \simeq \sum_{x} V_{bx}^{b} \otimes V_{a\bar{a}}^{x} \simeq \sum_{x} V_{b\bar{b}}^{\bar{x}} \otimes V_{a\bar{a}}^{x}, \tag{3.5}$$

where, in the last equality above, we have used the fusion space isomorphism, $V_{bx}^{b} \simeq V_{b\bar{b}}^{\bar{x}}$. If we have the fusion rule $a \times b = c$, then (3.5) simplifies to

$$V_{ba\bar{a}}^{b} \simeq V_{ba}^{c} \otimes V_{c\bar{a}}^{b} \simeq \sum_{x} V_{b\bar{b}}^{\bar{x}} \otimes V_{a\bar{a}}^{x} \tag{3.6}$$

Moreover, we know that V_{ba}^{c} and $V_{c\bar{a}}^{b}$ are 1-dimensional. Hence, the dimension of direct sum of fusion spaces $\sum_x V_{b\bar{b}}^{\bar{x}} \otimes V_{a\bar{a}}^{x}$ should be 1-dimensional. It follows that the sum should be over a single element and that the fusion spaces $V_{a\bar{a}}^{x}$ and $V_{b\bar{b}}^{\bar{x}}$ should be 1-dimensional. Since the trivial anyon 1 is always an element in the fusions $a \times \bar{a}$ and $b \times \bar{b}$, we have

$$V_{ba\bar{a}}^{b} \simeq V_{ba}^{c} \otimes V_{c\bar{a}}^{b} \simeq V_{b\bar{b}}^{1} \otimes V_{a\bar{a}}^{1} \tag{3.7}$$

[16] By rotating the $\bar{a}$, $\bar{b}$, and $\bar{c}$ vertex, we see that $a \times b = c$ is equivalent to requiring $a \times \bar{b} = d$ and $\bar{a} \times b = \bar{d}$ (see figure 3.2). This logic also explains why, for non-abelian a, it is impossible to have $a \times a = c$ even if $a \neq \bar{a}$.

Therefore, we learn that a fusion rule of the form (3.2) is equivalent to the following

$$a \times \bar{a} = 1 + \sum_{a_i \neq 1} N_{a\bar{a}}^{a_i}\ a_i\ , \quad b \times \bar{b} = 1 + \sum_{b_j \neq 1} N_{b\bar{b}}^{b_j}\ b_j\ ,$$
$$b_j \in b \times \bar{b}\ \Rightarrow\ b_j \notin a \times \bar{a}\ , \quad a_i \in a \times \bar{a}\ \Rightarrow\ a_i \notin b \times \bar{b} \quad \forall\ i, j\ . \tag{3.8}$$

In other words, the fusion of $a \times b$ has a unique outcome if and only if the only fusion product that $a \times \bar{a}$ and $b \times \bar{b}$ have in common is the identity.

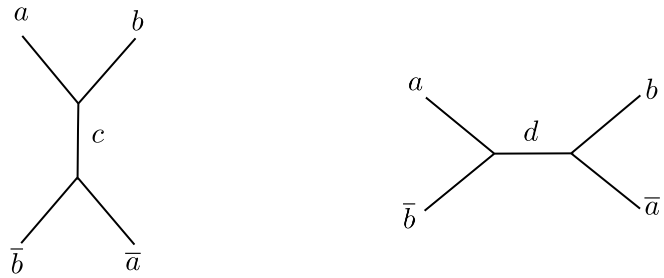


Figure 3.2: By rotating the bottom vertex in the left diagram of figure 3.1, we arrive at the above diagram on the left. Again, we have a single internal line labeled by c. We get to the diagram on the right by performing an $F_{\bar{a}}^{\bar{b}ab}$ transformation. Just as the left diagram has a unique internal line, so too does the diagram on the right.

Reformulating the problem as in (3.8) immediately suggests scenarios in which fusions of the form (3.2) occur beyond cases in which $\mathcal{T}$ factorizes into prime TQFTs. For example, if $a \in \mathcal{C}_1 \subset \mathcal{T}$ and $b \in \mathcal{C}_2 \subset \mathcal{T}$ lie in non-modular fusion subcategories of $\mathcal{T}$, $\mathcal{C}_{1,2}$, with trivial intersection (i.e., $\mathcal{C}_1 \cap \mathcal{C}_2 = 1$ only contains the trivial anyon), then we have (3.2) and $\mathcal{T}$ need not factorize.[17] More generally, when $a \in \mathcal{C} \subset \mathcal{T}$ is a member of a non-modular subcategory that does not include b (i.e., $b \notin \mathcal{C}$), we expect it to be more likely to find fusions of the form (3.8) and (3.2) since $a \times \bar{a} \in \mathcal{C}$, but $b \times \bar{b}$ will generally include elements outside $\mathcal{C}$. In fact, we will see that we can often say more when the fusion of a non-abelian Wilson line carrying charge in an unfaithful representation of a discrete gauge group is involved.

Another scenario in which we can imagine (3.8)—and therefore (3.2)—arising is one in which zero-form symmetries act non-trivially on a (i.e., $g(a) \neq a$ for some zero-form generator $g \in G$, where G is the zero-form group) and the $a_i \neq 1$ but not on b.[18] In this case, combinations of a_i that do not form full orbits under G are forbidden from

[17] In other words, fusion of anyons in $\mathcal{C}_i$ is closed. Moreover, the $\mathcal{C}_i$ inherit associativity and braiding from $\mathcal{T}$, but the Hopf link evaluated on anyons in these subcategories is degenerate (as a matrix). By modularity, the $\mathcal{C}_i$ will have non-trivial braiding with some anyons $x_A \notin \mathcal{C}_{1,2}$ (where A is an index running over such anyons). On the other hand, if the Hopf links for the $\mathcal{C}_i$ are non-degenerate, Müger's theorem [49] guarantees that they will in fact be separate TQFTs and so we are back in the situation of (3.3).

[18] By definition, the symmetry also acts non-trivially on $\bar{a}$ so that $\overline{g(a)} = g(\bar{a}) \neq \bar{a}$. On the other hand, note that one-form symmetry will act trivially on the product $a \times \bar{a}$.

appearing in $b \times \bar{b}$. Given a particular G, this argument may suffice to show that, for all i, $a_i \notin b \times \bar{b}$. More generally, symmetries constrain what can appear as fusion products of $a \times \bar{a}$ and $b \times \bar{b}$. The more powerful these symmetries, the more likely to find fusion rules of the type (3.8).

Interestingly, there is a close connection between the existence of symmetries and the existence of subcategories in TQFT. For example, as we will discuss further in section 3.4.2, for TQFTs corresponding to discrete gauge theories [54, 55], certain "quantum symmetries" or electric-magnetic self-dualities arise when we have particular non-modular subcategories $\mathcal{C}_i \subset \mathcal{T}$ (see [56] for a general theory of such symmetries and [57] for the case of S_3 discrete gauge theory).

We will also find various other, more subtle, connections between symmetries and fusion rules of the form (3.8) and (3.2). Moreover, we will see that symmetry is ubiquitous: in all the theories with fusion rules of the form (3.8) and (3.2) we analyze, either there is a zero-form symmetry present or else there is, at the very least, a symmetry of the modular data that exchanges anyons (in cases where this action does not lift to the full TQFT, we call these symmetries "quasi zero-form symmetries").

We will study fusions of the above type in two typically very different classes of 2+1D TQFTs:[19] discrete gauge theories and cosets built out of CS theories with continuous gauge groups (we will refer to these latter theories simply as "cosets"). Discrete gauge theories are always non-chiral, whereas Chern-Simons theories and their associated cosets are typically chiral.[20]

In the context of discrete gauge theories, whenever we have a (full) zero-form symmetry present, we will see that fusion rules of the type (3.8) and (3.2) have simple interpretations in certain parent theories gotten by gauging the zero-form symmetry, G_0. We go from the parent theories back to the original theories by gauging a "dual" one-form symmetry, G_1, that is isomorphic (as a group) to G_0 (see [58] for a more general review of this procedure). In this reverse process, we produce the $a \times b = c$ fusion rules of the corresponding discrete gauge theories via certain fusion fixed points of the one-form symmetry generators in the parent theories.

Similarly, in the context of our coset theories, we will see that fusion rules of the form $a \times b = c$ arise due to certain fixed points in the coset construction (though these fixed points do not generally involve a, b, and c). Cosets corresponding to the Virasoro minimal models lack such fixed points and so, as discussed above, they factorize. On the other hand, more complicated cosets do sometimes have such fixed points, and we will construct an explicit example of such a prime TQFT that has fusion rules of the form (3.8) and (3.2).

To summarize, this discussion leads us to the following questions we will answer in

[19] Note that there are sometimes dualities between theories in these two classes.

[20] By a chiral TQFT, we mean one in which the topological central charge satisfies $c_{\rm top} \neq 0 \pmod 8$.

subsequent sections:

1. When do fusions of the type (3.2) occur in TQFTs?

2. Does (3.2) imply a factorization of TQFTs

$$\mathcal{T} = \mathcal{T}_1 \boxtimes \mathcal{T}_2 \ , \tag{3.9}$$

with $a \in \mathcal{T}_1$ and $b \in \mathcal{T}_2$? As has been hinted at above, we will see in sections 3.3, 3.4 and 3.5 that the answer is generally no.

3. Does (3.2) imply that a belongs to one fusion subcategory and b to another and that the intersection of these subcategories is trivial? In other words, do we have

$$a \in \mathcal{C}_1 \subset \mathcal{T} \ , \quad b \in \mathcal{C}_2 \subset \mathcal{T} \ , \quad \mathcal{C}_1 \cap \mathcal{C}_2 = 1 \ ? \tag{3.10}$$

As we will see in section 3.4, the answer is generally no, even if we relax the requirement of trivial intersection. However, we will explicitly construct such examples (with non-modular $\mathcal{C}_{1,2} \subset \mathcal{T}$, where $\mathcal{T}$ is prime) in the case of discrete gauge theories.

4. Does (3.2) imply that a is in some subcategory $\mathcal{C} \subset \mathcal{T}$ that b is not a member of? In other words, do we have

$$a \in \mathcal{C} \subset \mathcal{T} \ , \quad b \notin \mathcal{C} \ ? \tag{3.11}$$

As we will see in section 3.4, the answer is generally no. However, we will argue that such constructions are quite easy to engineer in the context of discrete gauge theories, and we will explain when they arise. We will see that these constructions often have interesting interactions with symmetries.

5. Given a and b as in (3.2), do they have trivial mutual braiding? In other words, do we have

$$\frac{S_{ab}}{S_{0b}} = d_a \ , \tag{3.12}$$

where S is the modular S-matrix? This is true in the context of discrete gauge theories with a simple gauge group [2]. However, non-trivial braiding does arise naturally in the context of the fusion of non-abelian electrically charged lines with non-abelian magnetically charged lines.

6. Given a and b as in (3.2), does $\mathcal{T}$ have a non-trivial zero-form symmetry acting on either a or b? Does the TQFT have a zero-form symmetry that acts more generally? We will see in section 3.4 the answer to both these questions is no.

However, in cases in which this is true, it seems to always be related to the existence of a certain fusion fixed point of one-form symmetry generators in a parent TQFT. Of the infinitely many examples of untwisted discrete gauge theories we study, only gauge theories based on the Mathieu groups M_{23} and M_{24} fail to have zero-form symmetries.

7. Given a and b as in (3.2), does $\mathcal{T}$ have a non-trivial symmetry of the modular data? As we will see in sections 3.4 and 3.5, the answer seems to be yes. Clearly, it would be interesting to see if it is possible to define parents of such theories that generalize the relationship in (6). Note that the Mathieu gauge theories discussed in the previous point do have symmetries of their modular data (however, these symmetries do not lift to symmetries of the full TQFTs).

As we will see, many of these questions have simpler answers when studying discrete gauge theories. The reason is that powerful statements in these TQFTs can often be deduced from simple reasoning in the underlying theory of discrete groups. On the other hand, intuition one gains from taking products of representations in various continuous groups, like $SU(N)$, turns out to be somewhat misleading for our questions above.

The plan of this chapter is as follows. In section 3.2, we will introduce discrete gauge theories. Using the general structure of discrete gauge theories, in section 3.3 we will discuss the existence of fusions of the form (3.2) in discrete gauge theories with non-abelian simple gauge groups. We will explain how this problem is closely related to the Arad-Herzog conjecture in finite group theory. In section 3.4, we will discuss discrete gauge theories with general gauge groups and explain how intuition in the theory of finite groups leads us to various answers to the questions listed above. Along the way, we prove various theorems about discrete gauge theories and fusion rules of the form (3.2) and (3.8). Moreover, we discuss the role that subcategories and symmetries of discrete gauge theories play in such fusion rules. In the final part of the chapter (section 3.5), we go to continuous groups and discuss coset theories. We tie the existence of fusion rules of type (3.2) and (3.8) to certain fixed points in the coset construction. We then finish with some conclusions and summary of results.

3.2 Discrete Gauge Theories

One modern perspective on how to go from a group, G, to a $2+1$-dimensional discrete gauge theory is to start from a G-symmetry-protected topological phase (G-SPT) and gauge G [58]. At the same time, it may be useful to keep in mind that many of the results we will need in this section predate this perspective and follow from the classic work [54].

The starting point is a set of surface defects in one-to-one correspondence with the elements $g \in G$. For simplicity, we label these defects by group elements as well. Fusion of these defects obeys the usual group multiplication law of G, so $g \times h = gh$. One may also consider deforming the associativity of defect fusion via a 3-cocycle

$$\omega(g,h,k) \in H^3(G,U(1)) \ . \tag{3.13}$$

The $H^3(G,U(1))$ cohomology group then labels the distinct $2+1$-dimensional G-SPTs.

Gauging G corresponds to constructing conjugacy classes, $[g_i]$, for a set of representative $g_i \in G$ and pairing this data with an irreducible representation, $\pi^{\omega}_{g_i}$, of the centralizer of each g_i, N_{g_i}. These are, respectively, the magnetic and electric charges of the discrete gauge theory. The 3-cocycle in (3.13) is the Dijkgraaf-Witten twist (when $\omega = 0$ in cohomology we have an untwisted gauge theory).

In this way, lines bounding the G-SPT surface operators are liberated and become anyons in the—depending on ω—twisted or untwisted G discrete gauge theory. These latter objects are given by the pair $([g], \pi^{\omega}_g)$, where the square brackets around g are there to emphasize that we are dealing with a conjugacy class (for any representative in $[g]$, the corresponding centralizers are isomorphic). A discrete gauge theory specified by the data $G, \omega \in H^3(G,U(1))$ will de denoted as $\mathcal{Z}(\mathrm{Vec}^{\omega}_G)$.[21]

The question of whether the electric charge, π^{ω}_g, is projective is determined by the reduction of ω to N_g

$$\eta_g(h,k) := \frac{\omega(g,h,k)\omega(h,k,g)}{\omega(h,g,k)} \in H^2(N_g,U(1)) \ , \tag{3.14}$$

where $h,k \in N_g$. Indeed, this is the phase that appears in

$$\pi^{\omega}_g(h)\pi^{\omega}_g(k) = \eta_g(h,k)\pi^{\omega}_g(hk) \ . \tag{3.15}$$

If η_g is trivial in $H^2(N_g,U(1))$ the representation is linear [22]. For example, the group $PSL(2,4)$ has $\mathbb{Z}_3$ as the centralizer of elements in its length twenty conjugacy class. Since $H^2(\mathbb{Z}_3,U(1)) = \mathbb{Z}_1$, the corresponding η_g is trivial regardless of the choice of $\omega \in H^3(PSL(2,4),U(1)) \simeq \mathbb{Z}_6 \times \mathbb{Z}_{10}$. More generally, if ω is cohomologically non-trivial, then π^{ω}_g is typically projective.

In light of the discussion in the introduction, the most important thing for us to understand is the fusion of two anyons, $([g], \pi^{\omega}_g)$ and $([h], \pi^{\omega}_h)$. Intuitively, we have to

[21] This notation originates from the Drinfeld centre construction of discrete gauge theories.

[22] If $\eta_g(h,k)$ is a non-trivial 2-coboundary, the projective representations obtained will be in one-to-one correspondence with linear representations. These projective factors can be removed using a symmetry gauge transformation as detailed in [58]. On the other hand, if $\eta_g(h,k)$ is non-trivial in cohomology, then the corresponding representations of N_g must be higher dimensional. To see this statement, suppose this were not the case. Then, solving (3.15) for $\eta_g(h,k)$ implies that η_g is a 2-coboundary.

fuse both the conjugacy classes as well as the representations that the anyons depend on. This involves identifying the conjugacy classes of the elements obtained by multiplying the elements in $[g]$ and $[h]$. The product of elements in the conjugacy classes $[g]$ and $[h]$ can be written as

$$\{lgl^{-1}mhm^{-1}|l \in G/N_g, m \in G/N_h\} \tag{3.16}$$

We have to decompose this set of elements into conjugacy classes. Suppose $l' = pl$ and $m' = pm$, then we have

$$l'gl'^{-1}m'hm'^{-1} = p(lgl^{-1}mhm^{-1})p^{-1} \tag{3.17}$$

Therefore, the decomposition of the elements in the product of conjugacy classes $[g]$ and $[h]$ into conjugacy classes can be found by acting on the representatives g and h with the coset of diagonal left multiplication on $G/N_g \times G/N_h$. This is precisely the double coset $N_g\backslash G/N_h$. Also, we have to consistently decompose the product $\pi_g^\omega \otimes \pi_h^\omega$ into irreducible representations of centralizers of G. The precise way to carry out these steps is given by [54, 58]

$$\begin{aligned} N_{([g],\pi_g^\omega),([h],\pi_h^\omega)}^{([k],\pi_k^\omega)} \quad = \sum_{(t,s)\in N_g\backslash G/N_h} & m(\pi_k^\omega|_{N_{^tg}\cap N_{^sh}\cap N_k}, {}^t\pi_g^\omega|_{N_{^tg}\cap N_{^sh}\cap N_k} \\ & \otimes {}^s\pi_h^\omega|_{N_{^tg}\cap N_{^sh}\cap N_k} \otimes \ \pi^\omega_{(^tg,^sh,k)}) \ , \end{aligned} \tag{3.18}$$

where ${}^t\pi_g^\omega|_{N_{^tg}\cap N_{^sh}\cap N_k} \otimes {}^s\pi_h^\omega|_{N_{^tg}\cap N_{^sh}\cap N_k} \otimes \pi^\omega_{(^tg,^sh,k)}$ and $\pi_k^\omega|_{N_{^tg}\cap N_{^sh}\cap N_k}$ are (in general reducible) representations of $N_{^tg}\cap N_{^sh}\cap N_k$ (${}^t\pi_g^\omega$, ${}^s\pi_h^\omega$, and π_k^ω are representations of $N_{^tg}$, $N_{^sh}$, and N_k which are then restricted to the intersection subgroup). Here we define ${}^tg := t^{-1}gt$. The projectivity of the ${}^t\pi_g^\omega$, ${}^s\pi_h^\omega$, and π_k^ω representations is determined by the corresponding cohomology as in (3.14). The representation $\pi^\omega_{(^tg,^sh,k)}$ is one dimensional (it is a representation of the action of symmetries on the one-dimensional $V^k_{^tg^sh}$ fusion space in the G-SPT) and satisfies

$$\pi^\omega_{(^tg,^sh,k)}(\ell)\pi^\omega_{(^tg,^sh,k)}(m) \quad = \quad \frac{\eta_k(\ell,m)}{\eta_{^tg}(\ell,m)\eta_{^sh}(\ell,m)} \cdot \pi^\omega_{(^tg,^sh,k)}(\ell m) \ . \tag{3.19}$$

These projective factors guarantee that the two arguments of the $m(\cdot,\cdot)$ function can be meaningfully compared. Roughly speaking, this $m(\cdot,\cdot)$ function computes the inner products of the representations appearing as arguments (see [58] for further details). Finally, let us note that the sum in (3.18) is over the double coset, $N_g\backslash G/N_h$.

Another closely related quantity of interest is the modular data of a (twisted or untwisted) discrete gauge theory. It is given by [59]

$$\begin{aligned} S_{([g],\pi_g^\omega),([h],\pi_h^\omega)} &= \frac{1}{|G|}\sum_{\substack{k\in[g],\ \ell\in[h],\\ k\ell=\ell k}} \chi^k_{\pi_g^\omega}(\ell)^* \chi^\ell_{\pi_h^\omega}(k)^* \quad , \\ \theta_{([g],\pi_g^\omega)} &= \frac{\chi_{\pi_g^\omega}(g)}{\chi_{\pi_g^\omega}(e)} \ , \end{aligned} \tag{3.20}$$

where $\chi^h_{\pi_g^\omega}(\ell)$ is defined through the relation

$$\chi^{xgx^{-1}}_{\pi_g^\omega}(xhx^{-1}) := \frac{\eta_g(x^{-1},xhx^{-1})}{\eta_g(h,x^{-1})}\chi_{\pi_g^\omega}(h) \ . \tag{3.21}$$

Here, θ is the topological spin, and S is the modular S matrix. From these definitions, one can check that the quantum dimensions of the anyons are

$$d_{([g],\pi_g^\omega)} = \frac{S_{([g],\pi_g^\omega)([1],1)}}{S_{([1],1)([1],1)}} = |[g]| \cdot \deg\, \pi_g^\omega \ , \tag{3.22}$$

where $|[g]|$ is the size of $[g]$, and deg π_g^ω is the dimension of π_g^ω. Non-abelian anyons have $d_{([g],\pi^\omega)} > 1$ and necessarily satisfy

$$([g],\pi_g^\omega)\times([g^{-1}],\overline{\pi}_g^\omega) = ([1],1)+\cdots \ , \tag{3.23}$$

where the ellipses necessarily contain additional terms, 1 is the trivial representation of G, and $(([g^{-1}],\overline{\pi}_g^\omega)$ is the charge conjugate of $([g],\pi_g^\omega)$. Here, the representation $\overline{\pi}_g^\omega(k) := \frac{\omega(g,g^{-1},k)\omega(k,k^{-1}gk,k^{-1}g^{-1}k)}{\omega(g,k,k^{-1}g^{-1}k)}(\pi_g^\omega(k))^*$.

Anyons $([g],\pi_g^\omega)$ and $([h],\pi_h^\omega)$ that fuse to give a unique outcome satisfy the following condition with respect to the S matrix

$$|S_{([g]\pi_g^\omega),([h],\pi_h^\omega)}| = \frac{1}{|G|}d_{([g],\pi_g^\omega)}d_{([h],\pi_h^\omega)} \ . \tag{3.24}$$

Let us explore the consequences of this relation. To that end, using (3.22), we have $d_{([g],\pi_g^\omega)}d_{([h],\pi_h^\omega)} = |[g]||[h]|\cdot \deg\, \pi_g^\omega \cdot \deg\, \pi_h^\omega$. Substituting in (3.24) and using (3.20), we have

$$\begin{aligned}
\tfrac{1}{|G|} \quad & |[g]||[h]| \cdot \deg\, \pi_g^\omega \cdot \deg\, \pi_h^\omega \\
= \quad & \left| \frac{1}{|G|} \sum_{\substack{k\in[g],\ \ell\in[h], \\ k\ell=\ell k}} \chi^k_{\pi_g^\omega}(\ell)^* \chi^\ell_{\pi_h^\omega}(k)^* \right| \\
\leq \quad & \frac{1}{|G|} \sum_{\substack{k\in[g],\ \ell\in[h], \\ k\ell=\ell k}} |\chi^k_{\pi_g^\omega}(\ell)||\chi^\ell_{\pi_h^\omega}(k)| \\
\leq \quad & \frac{|[g]||[h]|}{|G|} \cdot \deg\, \pi_g^\omega \cdot \deg\, \pi_h^\omega
\end{aligned} \tag{3.25}$$

In the last inequality above, we have used (3.21) as well as the fact that projective characters satisfy $|\chi_{\pi_g^\omega}| \leq \deg\, \pi_g^\omega$ [23]. It is clear that (3.24) is satisfied if and only if the conjugacy classes $[g]$ and $[h]$ commute element-wise and the projective characters satisfy

$$|\chi_{\pi_g^\omega}(l)| = deg\ \pi_g^\omega \ \ and\ |\chi_{\pi_h^\omega}(k)| = deg\ \pi_h^\omega \tag{3.26}$$

$\forall\ l \in [h], k \in [g]$. This result is a generalization of lemma 3.4 of [60] and will be crucial in our discussions in the following section.

Let us introduce the following notation for non-abelian Wilson lines, flux lines, and dyons.

$$\begin{aligned}
\mathcal{W}_{\pi_1} \quad &\leftrightarrow \quad ([1], \pi_1)\ , \quad |\pi_1| > 1\ , \\
\mu_{[g]} \quad &\leftrightarrow \quad ([g], 1_g^\epsilon)\ , \quad |[g]| > 1\ , \\
\mathcal{L}_{([h],\pi_h^\omega)} \quad &\leftrightarrow \quad ([h], \pi_h^\omega)\ , \quad |[h]| \cdot |\pi_h^\omega| > 1\ .
\end{aligned} \tag{3.27}$$

We have dropped the ω superscript from π_1 in order to emphasize that the Wilson lines always transform under linear representations of G. We attach the ϵ superscript on the trivial representation of the flux line because these objects only exist when the relevant η_g in (3.14) is trivial in cohomology, and hence of the form $\eta_g(h,k) = \frac{\epsilon_g(h)\epsilon_g(k)}{\epsilon_g(h\cdot k)}$. Finally, 1_g^ϵ is the irreducible projective representation of N_g whose character is proportional to the trivial representation of N_g.

As a final comment, we note that, from the above modular data, it is easy to show that

$$\theta_{\mathcal{W}_\pi} = 1\ , \quad \frac{S_{\mathcal{W}_\pi \mathcal{W}_{\pi'}}}{S_{\mathcal{W}_1 \mathcal{W}_{\pi'}}} = 1\ , \tag{3.28}$$

where $\mathcal{W}_1 = ([1], 1)$ is the trivial Wilson line. In other words, the Wilson lines are all

[23] This statement is guaranteed as long as the projection factors defining the representations are roots of unity, which is satisfied in our case. Indeed, the 3-cocycle $\omega \in H^3(G, U(1))$ can be chosen to be valued in roots of unity without loss of generality.

bosons and have trivial mutual braiding with each other (they have non-trivial braiding with other lines in the theory).

3.3 $\mathcal{Z}(\mathbf{Vec}_G^\omega)$ with non-abelian simple G

In this section, we will study non-abelian simple lines fusing to give simple lines in discrete gauge theories with a non-abelian simple gauge group. Such a discrete gauge theory is a prime TQFT [60]. In other words, it does not decompose into other discrete gauge theories. Therefore, a fusion of the form

$$\mathcal{L}_{([g],\pi_g^\omega)} \otimes \mathcal{L}_{([h],\pi_h^\omega)} = \mathcal{L}_{([k],\pi_k^\omega)} \tag{3.29}$$

for non-abelian lines does not obviously exist. In fact, we will show below that fusion of the type (3.29) are very special in discrete gauge theories with non-abelian simple gauge group. The existence of such fusions are closely related to the Arad-Herzog Conjecture.

Conjecture (Arad-Herzog): *Consider a non-abelian finite simple group, G, and non-trivial elements $g, h \in G$. Then,*

$$[g] \cdot [h] \neq [gh] \ , \tag{3.30}$$

where $[g]$, $[h]$, and $[gh]$ are conjugacy classes of g, h, and gh respectively [61].

More pithily, Arad and Herzog (AH) conjectured that in non-abelian finite simple groups, the product of non-trivial conjugacy classes cannot be a single conjugacy class.

As we will argue in section 3.3.1, this conjecture has the following implication (which we then prove in section 3.3.2):

Theorem 3.3.1 : *In a (twisted or untwisted) $2+1$-dimensional discrete gauge theory with a non-abelian finite simple gauge group, the fusion of any two lines carrying non-trivial magnetic flux (as in Figure 3.3) cannot have a unique fusion outcome.*

In other words, theorem 3.3.1 asserts we cannot have

$$\mathcal{L}_{([g],\pi_g^\omega)} \times \mathcal{L}_{([h],\pi_h^\omega)} = \mathcal{L}_{([k],\pi_k^\omega)} \ , \quad g, h \neq 1 \ , \tag{3.31}$$

where, generically, all lines (denoted by $\mathcal{L}$) are non-abelian dyons [24]. We will think of this theorem as a first cousin of the AH conjecture.

[24]Of course, we may allow for pure fluxes to appear in (3.31).

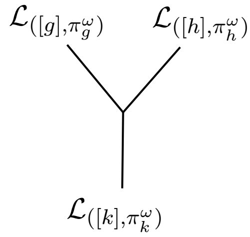


Figure 3.3: Fusion of dyons

So far, we have avoided discussing the fusion of Wilson lines. However, in light of (3.31), it is interesting to ask if we can fuse non-abelian Wilson lines $\mathcal{W}_\pi$ and $\mathcal{W}_{\pi'}$ (as in Figure 3.4) to obtain a unique outcome

$$\mathcal{W}_\pi \times \mathcal{W}_{\pi'} = \mathcal{W}_{\pi''} \ . \tag{3.32}$$

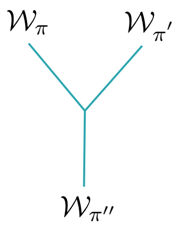


Figure 3.4: Fusion of Wilson lines

As we will briefly explain in section 3.3.1, (3.32) is equivalent to demanding that, at the level of group theory

$$\chi_\pi \cdot \chi_{\pi'} = \chi_{\pi''} \ , \tag{3.33}$$

where $\chi_\pi, \chi_{\pi'}$, and $\chi_{\pi''}$ are, respectively, the characters of irreducible linear representations, π, π', and π'', of G with dimension greater than 1. Although it might seem strange that (3.33) is possible (especially if one thinks of taking products of irreducible representations in $SU(N)$), it turns out that products of irreducible representations of finite simple groups can be irreducible [62].

The corresponding (twisted or untwisted) discrete gauge theory then has a product of Wilson lines as in (3.32). One simple example of this phenomenon in theories with a non-abelian simple gauge group involves the fusion of a Wilson line carrying charge in the 8-dimensional representation of A_9 with a Wilson line carrying charge in either of the 21-dimensional representations. Intriguingly, the discrete gauge theories based on finite simple groups are prime [60], so they do not consist of separate TQFTs with trivial mutual braiding. Therefore, (3.32) corresponds to some other structural properties of the A_9 discrete gauge theory. We will discuss these properties more generally in section 3.4.

Therefore, we learn that a version of the AH conjecture for characters alone cannot hold. However, our physical discussion above suggests studying one more type of fusion with a unique outcome (Figure 3.5)

$$\mathcal{W}_\pi \times \mathcal{L}_{([g],\pi_g^\omega)} = \mathcal{L}_{([h],\pi_h^\omega)} \ , \quad g \neq 1 \ , \tag{3.34}$$

where $\mathcal{W}_\pi$ is a non-abelian Wilson line, and the remaining anyons are non-abelian dyons. As a slightly simpler fusion, we may study the following fusion with a unique outcome

$$\mathcal{W}_\pi \times \mu_{[g]} = \mathcal{L}_{([h],\pi_h^\omega)} \ , \quad g \neq 1 \ , \tag{3.35}$$

where we have replaced the dyon on the left-hand side of (3.34) with a non-abelian flux line. Here we have implicitly assumed that the flux line also exists in the theory (depending on the twist, this assumption may or may not hold).

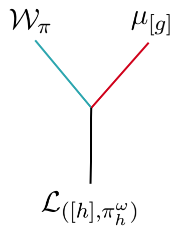


Figure 3.5: Fusion of a Wilson line with a magnetic flux line

This observation brings us to our second cousin of the AH conjecture:

Theorem 3.3.2 : *In any (twisted or untwisted) discrete gauge theory based on a non-abelian finite simple group, G, fusion of the types in (3.34) and (3.35) is forbidden.*

Intuition: One heuristic intuition behind this theorem is the following. As a consequence of theorem 3.3.1, theorem 3.3.2 implies that in discrete gauge theories based on non-abelian simple groups, the only allowed fusions with unique outcomes involving non-abelian anyons are those in (3.32). Wilson lines have trivial braiding amongst themselves [25]. Therefore, even though the fusion in (3.32) does not arise from a factorization of the TQFT into separate theories with trivial mutual braiding, the Wilson lines themselves have trivial mutual braiding.

Just as theorem 3.3.1 follows from the AH conjecture, so too theorem 3.3.2 follows from a more basic theorem on finite simple groups which we refer to as the third cousin of the AH conjecture:

[25] Physically, this last statement is clear from the fact that Wilson lines do not carry magnetic flux. In the language of category theory, this statement follows from the well-known fact that Wilson lines form a symmetric fusion subcategory. In fact, it is a Lagrangian subcategory isomorphic to the category of finite dimensional representations of G over $\mathbb{C}$, $\text{Rep}(G)$.

Theorem 3.3.3 : *Consider any non-abelian finite simple group, G, any irreducible linear representation, π, of G having dimension greater than one, and the centralizer, N_g, of any $g \neq 1$. The restricted representation, $\pi|_{N_g}$, is reducible.*

We refer to theorems 3.3.1, 3.3.2 and 3.3.3 as "cousins" of the AH conjecture since they are all related by TQFT.

Note that the above discussion is not relevant for abelian simple groups since these groups do not have conjugacy classes of length larger than one or representations of dimension larger than one. In other words, their fusion rules are those of a discrete finite group. As a result, we focus on non-abelian finite simple groups.

Duality: It is also interesting to understand how our above picture is compatible with a type of electric/magnetic duality that often features in discrete gauge theories. For example, the S_3 discrete gauge theory has a duality that exchanges the Wilson line charged under the 2-dimensional representation with the line having flux in the 3-cycle conjugacy class [56, 57]. More general examples have been discussed in [56, 63, 64]. Clearly, theorems 3.3.1, 3.3.2, and 3.3.3 can only be compatible with such dualities if the Wilson lines participating in (3.32) are not exchanged with lines carrying non-abelian flux. In fact, no such dualities exist in theories based on non-abelian finite simple gauge groups (**Proof:** apply theorem 5.8 of [63] noting that non-abelian simple groups have no non-trivial abelian normal subgroups). This fact is a non-trivial check of the above picture and is a check of the AH conjecture (this latter claim holds since, if theorem 3.3.1 were not true, then the AH conjecture would be false) [26].

3.3.1 From fusion to theorem 3.3.1 and a relation between theorems 3.3.2 and 3.3.3

Given the construction in section 3.3, we will first explain why the AH conjecture implies that, in (twisted and untwisted) discrete gauge theories based on simple groups, the fusion of any two lines carrying magnetic flux must have more than one fusion outcome (i.e., theorem 3.3.1). After explaining this fact, we will explain the relation between theorems 3.3.2 and 3.3.3.

To understand the connection between (twisted and untwisted) discrete gauge theories and the AH conjecture, recall the fusion formula in (3.18). Since the arguments of the $m(\cdot,\cdot)$ function are representations of $N_{{}^tg} \cap N_{{}^sh} \cap N_k$, we can decompose them in terms of irreducible representations, $\pi^{\omega(i)}$, of this group

$$
{}^t\pi_g^\omega|_{N_{{}^tg}\cap N_{{}^sh}\cap N_k} \otimes {}^s\pi_h^\omega|_{N_{{}^tg}\cap N_{{}^sh}\cap N_k} \otimes \pi^\omega_{({}^tg,{}^sh,k)} = \sum_i \alpha_i \pi^{\omega(i)} \ ,
$$

[26]This discussion does not preclude a duality between a twisted discrete gauge theory with a finite simple gauge group and some other type of TQFT (although presumably the dualities should live inside the class of theories considered in [65]).

$$\pi_k^\omega|_{N_{t_g} \cap N_{s_h} \cap N_k} = \sum_i \alpha_i^{'} \pi^{\omega(i)} \ , \tag{3.36}$$

for some non-negative integers $\alpha_i, \alpha_i^{'}$. Then the definition of $m(\cdot,\cdot)$ in [58] implies

$$m(\pi_k^\omega|_{N_{t_g} \cap N_{s_h} \cap N_k}, {}^t\pi_g^\omega|_{N_{t_g} \cap N_{s_h} \cap N_k} \otimes {}^s\pi_h^\omega|_{N_{t_g} \cap N_{s_h} \cap N_k} \otimes \ \pi^\omega_{({}^tg,{}^sh,k)}) = \sum_i \alpha_i \alpha_i^{'} \tag{3.37}$$

We know that π_k^ω is an irreducible representation of N_k. Also, $N_{t_g} \cap N_{s_h} \cap N_k$ is a subgroup of N_k. According to the Frobenius reciprocity theorem for projective representations of finite groups [66] [27], we know that, given any irreducible representation, $\pi^{\omega(i)}$, of $N_{t_g} \cap N_{s_h} \cap N_k$, there is always an irreducible representation, π_k^ω, of N_k such that the decomposition of $\pi_k^\omega|_{N_{t_g} \cap N_{s_h} \cap N_k}$ into irreducible representations of $N_{t_g} \cap N_{s_h} \cap N_k$ contains $\pi^{\omega(i)}$. This reasoning shows that, given ${}^t\pi_g^\omega|_{N_{t_g} \cap N_{s_h} \cap N_k} \otimes {}^s\pi_h^\omega|_{N_{t_g} \cap N_{s_h} \cap N_k} \otimes \pi^\omega_{({}^tg,{}^sh,k)}$, there is always some irreducible representation, π_k^ω, such that $m(\pi_k^\omega|_{N_{t_g} \cap N_{s_h} \cap N_k}, {}^t\pi_g^\omega|_{N_{t_g} \cap N_{s_h} \cap N_k} \otimes {}^s\pi_h^\omega|_{N_{t_g} \cap N_{s_h} \cap N_k} \otimes \pi^\omega_{({}^tg,{}^sh,k)})$ is non-zero. It follows that once we choose some conjugacy class, $[k]$, such that $[k] \in [g] \cdot [h]$, there is always some π_k^ω such that $N^{([k],\pi_k^\omega)}_{([g],\pi_g^\omega)([h],\pi_h^\omega)} \neq 0$. Here, $[g] \cdot [h]$ are the conjugacy classes obtained from taking a product of anyons with magnetic charges in $[g]$ and $[h]$.

Hence, in order to have a fusion rule of the type

$$([g], \pi_g^\omega) \times ([h], \pi_h^\omega) = ([k], \pi_k^\omega) \ , \quad g, h \neq 1 \ , \tag{3.38}$$

where all magnetic fluxes on the LHS are non-trivial, we need the fusion of the orbits $[g] \cdot [h]$ to contain only a single orbit $[k]$ (note that $|[k]|$ need not be equal to $|[g]||[h]|$ [28]). Moreover, commutativity of the fusion rules requires $[k] = [h] \cdot [g]$. Hence, the double coset $N_g \backslash G / N_h$ should have only a single element. (Since the double coset is trivial, we will remove the t, s dependence in the expressions below). We also require that the decomposition of representations $\pi_k^\omega|_{N_g \cap N_h \cap N_k}$ and $\pi_g^\omega|_{N_g \cap N_h \cap N_k} \otimes \pi_h^\omega|_{N_g \cap N_h \cap N_k} \otimes \pi^\omega_{(g,h,k)}$ into irreps of $N_g \cap N_h \cap N_k$ to have only a single irrep (of multiplicity one) in common. That is, if

$$\begin{aligned} \pi_g^\omega|_{N_g \cap N_h \cap N_k} \otimes \pi_h^\omega|_{N_g \cap N_h \cap N_k} \otimes \pi^\omega_{(g,h,k)} &= \sum_i \alpha_i \pi^{\omega(i)} \\ \pi_k^\omega|_{N_g \cap N_h \cap N_k} &= \sum_i \alpha_i^{'} \pi^{\omega(i)} \ , \end{aligned} \tag{3.39}$$

then there should be only one $i = i_0$ for which $\alpha_{i_0} = \alpha_{i_0}^{'} \neq 0$. Furthermore, we require that $\alpha_{i_0} = 1$.

[27] We use this theorem in the twisted case; in the untwisted case we use the usual theorem for linear representations.

[28] In the case of the fusion of pure fluxes, we do require $|[k]| = |[g]||[h]|$.

So, in order to have a fusion of the type (3.38), we have two constraints:

1. $[g] \cdot [h] = [k] = [h] \cdot [g]$
2. $\exists!\ \pi_k^\omega$ *such that* $m(\pi_k^\omega|_{N_g\cap N_h\cap N_k}, \pi_g^\omega|_{N_g\cap N_h\cap N_k} \otimes \pi_h^\omega|_{N_g\cap N_h\cap N_k} \otimes \pi^\omega_{(g,h,k)}) = 1$

The first constraint is on the conjugacy classes involved, and the second one is on the representations. The AH conjecture immediately implies that (1) is impossible for finite simple groups. Therefore, we see that

$$\text{AH conjecture} \Rightarrow \text{no fusions as in (3.38) for simple } G\ .$$

In particular, as claimed in the introduction, we see that

$$\mathcal{L}_{([g],\pi_g^\omega)} \times \mathcal{L}_{([h],\pi_h^\omega)} \neq \mathcal{L}_{([k],\pi_k^\omega)}\ , \tag{3.40}$$

where $\mathcal{L}_{([g],\pi_g^\omega)} = ([g],\pi_g^\omega)$, $\mathcal{L}_{([h],\pi_h^\omega)} = ([h],\pi_h^\omega)$, and $\mathcal{L}_{([k],\pi_k^\omega)} = ([k],\pi_k^\omega)$. So, in that language

$$\text{AH conjecture} \Rightarrow \text{Theorem 3.3.1}\ .$$

Of course, this does not prove theorem 3.3.1 since the AH conjecture has not been proven. However, it is a non-trivial consistency check of the AH conjecture. We will prove theorem 3.3.1 in the next section.

Next, let us show how theorem 3.3.3 implies theorem 3.3.2. To understand this point, let us specialize the general fusion in (3.18) to the product of a non-abelian Wilson line, $\mathcal{W}_{\pi_1} = ([1],\pi_1)$, with a non-abelian flux line, $\mu_{[h]} = ([h], 1_h^\epsilon)$. In order to have such a flux line in our theory we should, as discussed in section 3.2, either consider an untwisted discrete gauge theory or else a theory in which ω is such that $\eta_h \in H^2(N_h, U(1))$ is cohomologically trivial.

To that end, we find

$$N^{([h],\pi_h^\omega)}_{([1],\pi_1),([h],1_h^\epsilon)} = \sum_{(t,s)\in G\backslash G/N_h} m(\pi_h^\omega, {}^t\pi_1|_{N_h} \otimes {}^s1_h^\epsilon \otimes\ \pi^\omega_{(1,h,h)}|_{N_h})\ . \tag{3.41}$$

In this case, the double coset $G\backslash G/N_h$ is trivial. Hence, we have

$$N^{([h],\pi_h^\omega)}_{([1],\pi_1),([h],1_h^\epsilon)} = m(\pi_h, \pi_1|_{N_h} \otimes 1_h^\epsilon \otimes \pi^\omega_{(1,h,h)}|_{N_h})\ . \tag{3.42}$$

In fact, the representation $\pi^\omega_{(1,h,h)}$ is trivial (this follows from the fixed nature of the V^h_{1h} fusion space in the G-SPT [58]). So the product of representations $\pi_1|_{N_h}\otimes 1_h^\epsilon \otimes \pi^\omega_{(1,h,h)}|_{N_h}$

is isomorphic to $\pi_1|_{N_h} \otimes 1_h^\epsilon$. Therefore, the expression above simplifies to

$$N_{([1],\pi_1),([h],1_h^\epsilon)}^{([h],\pi_h^\omega)} = m(\pi_h^\omega, \pi_1|_{N_h} \otimes 1_h^\epsilon) \ . \tag{3.43}$$

Note that π_1 is an irreducible representation of G. Its restriction to N_h is in general reducible. So $m(\pi_h, \pi_1|_{N_h} \otimes 1_h^\epsilon)$ gives the multiplicity of the irreducible representation, π_h, in the decomposition of the representation, $\pi_1|_{N_h} \otimes 1_h^\epsilon$, into irreducible representations of N_h. If $\pi_1|_{N_h}$ is irreducible, $m(\pi_h, \pi_1|_{N_h} \otimes 1_h^\epsilon) = \delta_{\pi_h, \pi_1|_{N_h} \otimes 1_h^\epsilon}$. Hence, we have the following fusion rules

$$([1], \pi_1) \otimes ([h], 1_h) = ([h], \pi_1|_{N_h} \otimes 1_h^\epsilon) \ , \tag{3.44}$$

if and only if $\pi_1|_{N_h}$ is an irreducible representation of N_h.

As a result, theorem 3.3.3 implies that we have more than one channel in the fusion

$$\mathcal{W}_{\pi_1} \times \mu_{[h]} = \mathcal{L}_{([h],\pi_h^\omega)} + \cdots \ . \tag{3.45}$$

In fact, we may take the flux, $([h], 1_h^\epsilon)$, and replace it with a dyon, $([h], \pi_h^\omega)$. Note that, in some theories, such a dyon may exist while the flux line does not. We then find that the right-hand side of (3.43) becomes $m(\tilde{\pi}_h^\omega, \pi_1|_{N_h} \otimes \pi_h^\omega)$. Clearly, if the fusion in (3.45) requires more terms on the right-hand side, so too will the fusion with the dyon replacing the flux. This is the content of theorem 3.3.2.

Similarly, by the logic of this section, if we satisfy theorem 3.3.2 for the untwisted discrete G gauge theory, we then have that, for any irreducible linear representation, π_1, of G having dimension greater than one, $\pi_1|_{N_h}$ is reducible. This is the content of theorem 3.3.3. In conclusion, we have

Theorem 3.3.3 $\Leftrightarrow$ Theorem 3.3.2 .

Let us also note that we have chosen π_1 to be an irreducible representation of G with dimension > 1 so that $([1], \pi_1)$ is non-abelian. Hence, for the above fusion rule to be consistent, $\pi_1|_{N_h}$ should be an irreducible representation of N_h of the same dimension.

What remains is to prove at least one of theorems 3.3.2 or 3.3.3. In the next section we give independent proofs of theorems 3.3.2 and 3.3.3. Proceeding through theorem 3.3.2 first gives us a more TQFT-flavored proof. Proceeding through theorem 3.3.3 first gives us a more group theory-flavored proof. We then conclude the next section by proving theorem 3.3.1 as well.

3.3.2 Proofs of the cousin theorems

From the discussion in the previous section, to prove theorems 3.3.2 and 3.3.3 we need only prove one of them. However, each route has its own merits, so we give independent proofs of each. We follow by proving theorem 3.3.1 (which is logically independent of the others).

Let us first prove theorem 3.3.2. To that end, suppose we have a fusion of the form given in (3.35), which we reproduce below for ease of reference

$$\mathcal{W}_\pi \times \mu_{[g]} = \mathcal{L}_{([h],\pi_h^\omega)} \ , \quad g \neq 1 \ . \tag{3.46}$$

In section 3.3.1, we argued that, if such a fusion exists, the electric charge of the dyon on the right-hand side is given by a reduction of an irreducible representation of the gauge group G (i.e., $\pi_h^\omega = \pi|_{N_g} \otimes 1_h^\epsilon$) and $h = g$. Next, we note that the S-matrix satisfies [41]

$$S_{\mathcal{W}_\pi \bar{\mu}_{[g^{-1}]}} = \frac{1}{|G|}\frac{\theta_{\mathcal{L}_{([g],\pi_g^\omega)}}}{\theta_{\mathcal{W}_\pi}\theta_{\mu_{[g]}}} d_{\mathcal{L}_{([g],\pi_g)}} = \frac{1}{|G|}\frac{\theta_{\mathcal{L}_{([g],\pi_g^\omega)}}}{\theta_{\mathcal{W}_\pi}\theta_{\mu_{[g]}}} d_{\mathcal{W}_\pi} d_{\mu_{[g]}} \ , \tag{3.47}$$

where $\bar{\mu}_{[g^{-1}]}$ is the conjugate of $\mu_{[g]}$. Therefore,

$$|S_{\mathcal{W}_\pi \mu_{[g]}}| = \frac{1}{|G|} d_{\mathcal{W}_\pi} d_{\mu_{[g]}} \ . \tag{3.48}$$

Using (3.26), we know that (3.48) implies $|\chi_\pi(g)| = \deg\,\chi_\pi$, where χ_π is the character corresponding to the Wilson line's charge, and $\deg\,\chi_\pi = |\pi| > 1$ is the dimension of π.

A standard argument in representation theory then implies that $\pi(g) = c \cdot \mathbb{1}_{|\pi|}$, where $\mathbb{1}_{|\pi|}$ is the $|\pi| \times |\pi|$ unit matrix, and c is an n^{th} root of unity (the twist of the dyon). Next, choose some $k \in [G,g] := \langle \ell g \ell^{-1} g^{-1} | \ell \in G \rangle$. Clearly,

$$\begin{aligned} \pi(k) &= \pi(\ell g \ell^{-1} g^{-1}) = \pi(\ell) \cdot c \cdot \mathbb{1}_{|\pi|} \cdot \pi(\ell)^{-1} \cdot c^{-1} \cdot \mathbb{1}_{|\pi|} \\ &= \mathbb{1}_{|\pi|} \ . \end{aligned} \tag{3.49}$$

Since G is a simple group, we can choose $k \neq 1$. As a result, π is an unfaithful representation of G. Therefore, the kernel, $\ker(\pi)$, is a non trivial normal subgroup. Since G is simple, we must have $\ker(\pi) = G$. But then, π cannot be an irreducible representation. Note that we may repeat this proof verbatim by taking $\mathcal{L}_{([g],\pi_g^\omega)}$ instead of the flux line. Therefore fusion of the form in (3.34) is also forbidden. □

By the discussion in section 3.3.1, we have also proved theorem 3.3.3. Although this proof is mathematical, it also has a distinctly TQFT-flavor: notice the prominent role of the modular S matrix (and also, to a lesser extent, the twists).

Alternatively, we may also give a direct group theoretical proof of theorem 3.3.3

(and therefore of theorem 3.3.2 via section 3.3.1) as follows:

Since G is a non-abelian simple group, its irreducible representations of dimension larger than one must be faithful (otherwise their kernels would be non-trivial normal subgroups). Now, consider some faithful non-abelian representation, π. Furthermore, take some $g \in G$ such that $g \neq 1$ and consider the centralizer, N_g.

Suppose the restriction $\pi|_{N_g}$ is irreducible. Clearly g is central in N_g. As a result, by Schur's lemma

$$\pi|_{N_g}(g) = c \cdot \mathbb{1}_{|\pi|} \ , \tag{3.50}$$

where c is an n^{th} root of unity. Since this is a restriction of a representation of G, we must also have in the parent group that

$$\pi(g) = c \cdot \mathbb{1}_{|\pi|} \ , \tag{3.51}$$

and so it follows that

$$\pi(hgh^{-1}g^{-1}) = \mathbb{1}_{|\pi|} \ . \tag{3.52}$$

Since the group is simple, $g \neq 1$ cannot be in the (trivial) center of G. As a result, there exists h such that $hgh^{-1}g^{-1} \neq 1$. The result in (3.52) contradicts the fact that π is faithful. $\square$

Let us now prove theorem 3.3.1. We reproduce the forbidden (3.31) for ease of reference

$$\mathcal{L}_{([g],\pi_g^\omega)} \times \mathcal{L}_{([h],\pi_h^\omega)} = \mathcal{L}_{([k],\pi_k^\omega)} \ , \quad g, h \neq 1 \ , \tag{3.53}$$

where, according to the discussion in the previous section, $[k] = [gh]$. Similarly to the case of theorem 3.3.2, we have that

$$\begin{aligned} S_{\mathcal{L}_{([g],\pi_g^\omega)}\mathcal{L}_{([h^{-1}],(\pi_h^\omega)^*)}} &= \frac{1}{|G|}\frac{\theta_{\mathcal{L}_{([gh],\pi_{gh}^\omega)}}}{\theta_{\mathcal{L}_{([g],\pi_g^\omega)}}\theta_{\mathcal{L}_{([h],\pi_h^\omega)}}} d_{\mathcal{L}_{([gh],\pi_{gh}^\omega)}} \\ &= \frac{1}{|G|}\frac{\theta_{\mathcal{L}_{([gh],\pi_{gh}^\omega)}}}{\theta_{\mathcal{L}_{([g],\pi_g^\omega)}}\theta_{\mathcal{L}_{([h],\pi_h^\omega)}}} \cdot \\ &\quad \cdot d_{\mathcal{L}_{([g],\pi_g^\omega)}} d_{\mathcal{L}_{([h],\pi_h^\omega)}} \ , \end{aligned} \tag{3.54}$$

where $\mathcal{L}_{([h^{-1}],(\pi_h^\omega)^*)}$ is the conjugate of $\mathcal{L}_{([h],\pi_h^\omega)}$. Therefore,

$$|S_{\mathcal{L}_{([g],\pi_g^\omega)}\mathcal{L}_{([h],\pi_h^\omega)}}| = \frac{1}{|G|} d_{\mathcal{L}_{([g],\pi_g^\omega)}} d_{\mathcal{L}_{([h],\pi_h^\omega)}} \ . \tag{3.55}$$

This last result allows us, as in the case of theorem 3.3.2, to use (3.26). We then conclude that for any $\ell \in [g]$ and $m \in [h]$, $\ell m = m\ell$ (i.e., that the two conjugacy classes $[h]$ and $[g]$ commute element-by-element).

Now, consider the product of conjugacy classes

$$[g] \cdot [g] = \sum_{[a]} N_{[g][g]}^{[a]} [a] \ , \quad N_{[g][g]}^{[a]} \in \mathbb{Z}_{\geq 0} \ . \tag{3.56}$$

Clearly, we have that all elements on the left hand side commute with all elements of $[h]$. Therefore, the same is true of all elements in the conjugacy classes $[a]$. Now, consider taking pairwise products of all the $[a]$'s with themselves and with $[g]$ and so on. Eventually, we will come to a set of conjugacy classes closed under multiplication. This defines a normal subgroup $K \trianglelefteq G$ in which each element commutes with $[h]$. Since G is simple, we must have that $K = G$. However, this means that $[h]$ commutes with all elements of the group and so we have a non-trivial center. This is a contradiction. □

3.4 $\mathcal{Z}(\mathbf{Vec}_G^\omega)$ with general G

We would like to recast the problem of constructing discrete gauge theories with fusion rules (3.2) and (3.8) in terms of the closely related problem of finding irreducible products of irreducible finite group representations. To make this connection as direct as possible, it is useful to focus on Wilson lines of the discrete gauge theories we are studying. Indeed, by specializing (3.18) to Wilson lines, we find

$$N_{(1,\pi),(1,\pi')}^{(1,\pi'')} = m(\pi'', \pi \otimes \pi') = \frac{1}{|G|} \sum_{g \in G} \chi_{\pi''}(g) \chi_\pi^*(g) \chi_{\pi'}^*(g) = \langle \chi_{\pi''}, \chi_\pi \chi_{\pi'} \rangle \tag{3.57}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on characters. Therefore, the Wilson lines form a closed fusion subcategory of the discrete gauge theory, $\mathcal{C}_\mathcal{W}$. Moreover, the fusion rules of the Wilson lines are those of the representation semiring of the gauge group.[29] Note that $\mathcal{C}_\mathcal{W}$ is, in some sense, the "least anyonic" part of the theory: it is easy to check from (3.20) that the Wilson lines are bosonic, so $\theta_{\mathcal{W}_i} = 1$, and that the braiding of Wilson lines amongst themselves is trivial,[30] so $S_{\mathcal{W}_1 \mathcal{W}_2} = d_{\mathcal{W}_1} d_{\mathcal{W}_2} / \mathcal{D}$ (here $\mathcal{D} = \sqrt{\sum_{i=1}^N d_i^2}$, and the sum is over all the anyons).[31] To summarize, we see that if we can find representations of some group, G, satisfying

$$\chi_\pi \cdot \chi_{\pi'} = \chi_{\pi''} \ , \quad |\pi|, \ |\pi'|, \ |\pi''| > 1 \ , \tag{3.58}$$

[29] In fact, we have $\mathcal{C}_\mathcal{W} \simeq \mathrm{Rep}(G)$, where $\mathrm{Rep}(G)$ is the category of finite dimensional representations of G over $\mathbb{C}$.

[30] The Wilson lines braid non-trivially with other anyons in the theory (more formally: the Wilson line subcategory is Lagrangian and so the Müger center of $\mathcal{C}_\mathcal{W}$ is $\mathcal{C}_\mathcal{W}$ itself).

[31] In fact, [67] guarantees that any such subcategory is equivalent to $\mathrm{Rep}(H)$ for some group H.

where π, π', and π'' are irreducible, then, in the corresponding G discrete gauge theory, we will have non-abelian Wilson lines satisfying

$$\mathcal{W}_\pi \times \mathcal{W}_{\pi'} = \mathcal{W}_{\pi''} \ . \tag{3.59}$$

Since, by Cayley's theorem, every finite group is isomorphic to a subgroup of the symmetric group, S_N, (for some N) it is natural to start our discussion with S_N. In particular, to check whether π'' is irreducible, we want to perform the group theory analog of the F transformation discussed in the introduction (see figure 3.1)

$$\langle \chi_\pi \cdot \chi_{\pi'}, \chi_\pi \cdot \chi_{\pi'} \rangle = \langle \chi_\pi^2, \chi_{\pi'}^2 \rangle \ , \tag{3.60}$$

where we have used the fact that S_N is ambivalent (g and g^{-1} are in the same conjugacy class for all $g \in S_N$) so that the characters are real. A theorem of Zisser [62] shows that $\chi_{[N-2,2]} \in \chi_\alpha^2$, where $[N-2,2]$ is a partition of N labeling the corresponding representation of S_N, and α is any irreducible representation of dimension larger than one, $|\alpha| > 1$. Moreover, since S_N is ambivalent, this means that $\chi_{[N]} \in \chi_\alpha^2$, where $\chi_{[N]}$ is the trivial representation of S_N. As a result, we see that the analog of (3.8) yields

$$\chi_\pi \cdot \chi_\pi = \chi_{[N]} + \chi_{[N-2,2]} + \cdots \ , \ \ \chi_{\pi'} \cdot \chi_{\pi'} = \chi_{[N]} + \chi_{[N-2,2]} + \cdots \ \Rightarrow \ \langle \chi_\pi \cdot \chi_{\pi'}, \chi_\pi \cdot \chi_{\pi'} \rangle > 1 \ , \tag{3.61}$$

and so products of non-abelian representations of S_N are never irreducible. Therefore, we cannot have (3.59) in S_N discrete gauge theory.

Discrete gauge theories of finite simple groups

Since we have $A_N \lhd S_N$ (i.e., the alternating group, A_N, is a normal subgroup of S_N), it is natural to consider A_N discrete gauge theories as the next possibility for realizing (3.58) [62] and hence (3.59). Moreover, since A_N is simple, only pure Wilson lines can be involved in fusions of the form (3.2) [2], and the A_N discrete gauge theories are guaranteed to be prime [60] (we will return to the question of primality in greater generality in section 3.4.1). Therefore, finding an example of (3.59) in A_N discrete gauge theories is sufficient to answer question (1) from the introduction in the negative.

To understand if going to A_N is a fruitful direction, we note that there are two types of characters that arise in going from S_N to A_N:

(A) Characters that are restrictions of S_N characters satisfying $\chi_\lambda \neq \chi_{[1^N]} \cdot \chi_\lambda$, where $\chi_{[1^N]}$ corresponds to the sign representation of S_N. Let us call these "type A" characters: $\tilde{\chi}_\lambda := \chi_\lambda|_{A_N}$.

(B) Characters that descend from S_N characters satisfying $\chi_\rho = \chi_{[1^N]} \cdot \chi_\rho$. As representations of A_N, they split into two representations of the same dimension, $\lambda_\pm$.

Let us call these "type B" characters: $\chi^{(B)}_\rho = \chi_{\rho+} + \chi_{\rho-} = \chi_\rho|_{A_N}$.

In going from S_N to A_N, we perform a group-theoretical version of gauging the "one-form symmetry" generated by $\chi_{[1^N]}$: we identify characters related by multiplication with $\chi_{[1^N]}$, and we split characters that are invariant under multiplication with $\chi_{[1^N]}$. Clearly, products of type A characters cannot be irreducible since they will always contain $\chi^{(A)}_{[N]}$ and $\chi^{(A)}_{[N-2,2]}$ after performing the F-transformation and computing (3.61).[32]

A little more work in [62] shows that we can obtain (3.58) for A_N if and only if $N = k^2 \geq 9$ by taking the product of the following type A and type B representations

$$\tilde{\chi}_{[N-1,1]} \cdot \chi_{[k^k]_\pm} = \tilde{\chi}_{[k^{k-1},k-1,1]} \ . \tag{3.62}$$

Moreover, the $\mathbb{Z}_2$ outer automorphism of A_N acts on the type B characters as

$$g\left(\chi_{[k^k]_\pm}\right) = \chi_{[k^k]_\mp} \ , \quad 1 \neq g \in \mathrm{Out}(A_N) \simeq \mathbb{Z}_2 \ . \tag{3.63}$$

Therefore, at the level of the non-abelian Wilson lines in the corresponding A_N discrete gauge theory, we learn that

$$\mathcal{W}_{[N-1,1]} \times \mathcal{W}_{[k^k]_\pm} = \mathcal{W}_{[k^{k-1},k-1,1]} \ . \tag{3.64}$$

Finally, $\mathrm{Out}(A_N)$ lifts to a full zero-form symmetry of the discrete gauge theory [56], since, according to corollary 7.8 of [56]

$$\mathrm{Aut}^{\mathrm{br}}(\mathcal{Z}(\mathrm{Vec}_{A_N})) \simeq H^2(A_N, U(1)) \rtimes \mathrm{Out}(A_N) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \ , \tag{3.65}$$

where the group on the left hand side is the group of braided tensor auto-equivalences of the MTC underlying the discrete gauge theory, $\mathcal{Z}(\mathrm{Vec}_{A_N})$. As a result, we learn that the symmetries of the discrete gauge theory exchange the $\mathcal{W}_{[k^k]_\pm}$ lines

$$g\left(\mathcal{W}_{[k^k]_\pm}\right) = \mathcal{W}_{[k^k]_\mp} \ , \quad 1 \neq g \in \mathrm{Out}(A_N) \triangleleft \mathrm{Aut}^{\mathrm{br}}(\mathcal{Z}(\mathrm{Vec}_{A_N}) \ . \tag{3.66}$$

In other words, we have found that, in an infinite number of prime theories, fusion rules of the type (3.2) are generated in pairs related by symmetries of the discrete gauge theory. This discussion shows that TQFTs with fusions of the form (3.2) need not factorize and so the answer to question (2) in the introduction is "no."

Let us now drive home the importance of symmetries in arriving at (3.64) and, at the same time, gain insight that will be useful later. To that end, let us consider gauging

[32] In this discussion, we have implicitly assumed that $N \neq 4$ (although, for $N = 3$, we should take $[N-2,2] \to [2,1]$ to conform to usual conventions). For $N = 4$, we have $\chi^{(A)}_{[N-2,2]} \to \chi^{(B)}_{[N-2,2]} = \chi_{[N-2,2],+} + \chi_{[N-2,2],-}$.

the $\mathbb{Z}_2$ outer automorphism symmetry of the A_N discrete gauge theory. Note that this gauging is allowed since the "defectification" obstruction described physically in [58] is trivial here: $H^4(\mathbb{Z}_2, U(1)) = \mathbb{Z}_1$. Moreover, since A_N is simple, the discrete gauge theory has no non-trivial abelian anyons (i.e., $\mathcal{A} = \mathcal{W}_{[N]}$) and so $H^3(\mathbb{Z}_2, \mathcal{A}) = \mathbb{Z}_1$. Therefore, (3.65) is a genuine zero-form symmetry group (as opposed to being a 2-group).

More abstractly, let us consider a generalization of the fusion rules in (3.18) to the case of gauging a zero form group, H, of a more general G-crossed braided theory, $\mathcal{T}_{G^\times}$ (as worked out in [58])

$$N_{([a],\pi_a),([b],\pi_b)}^{([c],\pi_c)} = \sum_{(t,s)\in N_a\backslash H/N_b} m(\pi_c|_{N_{t_a}\cap N_{s_b}\cap N_c}, {}^t\pi_a|_{N_{t_a}\cap N_{s_b}\cap N_c} \otimes {}^s\pi_b|_{N_{t_a}\cap N_{s_b}\cap N_c} \otimes \pi^\omega_{({}^ta,{}^sb,c)}) \ , \quad (3.67)$$

where $a, b, c \in \mathcal{T}_{G^\times}$, $[a] := \{h(a),\ \forall h \in H\}$, $N_a := \{h \in H | h(a) = a\}$, and π_a is a representation of N_a.

In our case at hand, $\mathcal{T}_{G^\times} = \mathcal{Z}(\mathrm{Vec}_{A_N})_{H^\times}$ is the A_N discrete gauge theory extended by surface defects implementing the $H = \mathbb{Z}_2$ global symmetry. Moreover, $a = \mathcal{W}_{[N-1,1]}$, $b = \mathcal{W}_{[k^k]_\pm}$, $N_a = \mathbb{Z}_2$, and $N_b = \mathbb{Z}_1$. As a result, $t = s = 1$, the summation in (3.67) is trivial, the various representations are all restricted to the trivial subgroup, and $\pi^\omega_{a,b,c} = 1$ (this latter statement follows from the fact that the action of H on the V_{ab}^c fusion space via $U_1(a, b, c)$ is trivial). In particular, we have

$$N_{([\mathcal{W}_{[N-1,1]}],\pm),([\mathcal{W}_{[k^k]_\pm}],+)}^{([\mathcal{W}_{[k^{k-1},k-1,1]}],\pm)} = m(\pm|_{\mathbb{Z}_1}, \pm|_{\mathbb{Z}_1} \otimes +|_{\mathbb{Z}_1}) = m(1,1) = 1 \ , \qquad (3.68)$$

where $\pm$ denote the two representations of $\mathbb{Z}_2$. Therefore, we learn that when we gauge the outer automorphism group of A_N, we have

$$([\mathcal{W}_{[N-1,1]}], \pm) \times ([\mathcal{W}_{[k^k]_\pm}], +) = ([\mathcal{W}_{[k^{k-1},k-1,1]}], +) + ([\mathcal{W}_{[k^{k-1},k-1,1]}], -) \ , \qquad (3.69)$$

which is the TQFT version of the lift of (3.62) to S_N. This is what we expect, since we can always fix our choice of parameters so that gauging $\mathbb{Z}_2$ yields [68]

$$\mathcal{Z}(\mathrm{Vec}_{A_N})_{\mathbb{Z}_2^\times} \xrightarrow{\text{gauge}} \mathcal{Z}(\mathrm{Vec}_{A_N\rtimes\mathbb{Z}_2}) = \mathcal{Z}(\mathrm{Vec}_{S_N}) \ , \qquad (3.70)$$

where we have used the fact that $S_N \simeq A_N \rtimes \mathbb{Z}_2$.

Finally, from the general rules above, it is not hard to check that the trivial Wilson line in the A_N theory lifts to a $\mathbb{Z}_2$ one-form symmetry in the S_N gauge theory. The resulting non-trivial one-form symmetry generator acts as

$$\begin{aligned} ([\mathcal{W}_{[N]}], -) \times ([\mathcal{W}_{[N-1,1]}], \pm) &= ([\mathcal{W}_{[N-1,1]}], \mp) \ , \\ ([\mathcal{W}_{[N]}], -) \times ([\mathcal{W}_{[k^k]_\pm}], +) &= ([\mathcal{W}_{[k^k]_\pm}], +) \ , \end{aligned}$$

$$([\mathcal{W}_{[N]}],-)\times([\mathcal{W}_{[k^{k-1},k-1,1]}],\pm) \quad = \quad ([\mathcal{W}_{[k^{k-1},k-1,1]}],\mp)\ , \tag{3.71}$$

where $([\mathcal{W}_{[N]}],-)=\mathcal{W}_{[1^N]}$.

To summarize, we learn that, in order to generate the fusion rule (3.64), we can gauge a $\mathbb{Z}_2$ one-form symmetry in the S_N (with $N=k^2\geq 9$) discrete gauge theory with fusion rules (3.69) and (3.71). Crucially, we need a fixed point of the one-form symmetry (as in the second line in (3.71)) in order to generate the fusion rule of the form (3.64) in the A_N discrete gauge theory. We will return to the existence of fixed points of various kinds repeatedly throughout this chapter.

One may wonder if zero-form gaugings always resolve fusion rules of the form $a\times b=c$ into fusion rules with multiple outcomes. Taking $G=O(5,3)$, one can see the answer is no.[33] Indeed, in this theory, one can check that we have the following analogs of (3.64)

$$\mathcal{W}_{5_i}\times\mathcal{W}_6=\mathcal{W}_{30_i}\ ,\quad i=1,2\ , \tag{3.72}$$

where 5_i are the two five-dimensional representations of $O(5,3)$, 6 is the unique six-dimensional representation, and 30_i are the two complex thirty-dimensional representations (there is also a third, real, thirty-dimensional representation that does not appear in (3.72)). As in the previous case, $\mathrm{Out}(O(5,3))=\mathbb{Z}_2$ and it acts non-trivially on the Wilson lines involved in the fusion above. In particular, we have

$$\mathcal{W}_{5_1}\leftrightarrow\mathcal{W}_{5_2}\ \text{ and }\ \mathcal{W}_{30_1}\leftrightarrow\mathcal{W}_{30_2} \tag{3.73}$$

under the action of the non-trivial element in $\mathrm{Out}(O(5,3))$. This symmetry lifts to a symmetry of the discrete gauge theory that we can gauge. Doing so, we can choose parameters such that

$$\mathcal{Z}(\mathrm{Vec}_{O(5,3)})_{\mathbb{Z}_2^\times}\xrightarrow{\text{gauge}}\mathcal{Z}(\mathrm{Vec}_{O(5,3)\rtimes\mathbb{Z}_2})\ . \tag{3.74}$$

We may again apply (3.67) to find

$$N^{([\mathcal{W}_{30_i}],+)}_{([\mathcal{W}_{5_i}],+),(\mathcal{W}_6,\pm)}=m(+|_{\mathbb{Z}_1},+|_{\mathbb{Z}_1}\otimes\pm|_{\mathbb{Z}_1})=m(1,1)=1\ , \tag{3.75}$$

and conclude

$$([\mathcal{W}_{5_i}],+)\times(\mathcal{W}_6,\pm)=([\mathcal{W}_{30_i}],+)\ . \tag{3.76}$$

Such a situation arises whenever $N_c=\mathbb{Z}_1=N_a\cap N_b$. This equality is special since, more generally, we have $N_a\cap N_b\subseteq N_c$.

Before moving on to discuss other phenomena, let us note that the above discrete

[33] This is the group $O(5)$ over the field $\mathbb{F}_3$. It has order 25920 and is the smallest simple group whose discrete gauge theory has a fusion of non-abelian Wilson lines with a unique outcome.

gauge theories based on simple groups also provide answers to questions (3) and (4) from the introduction. Indeed, as we will see in greater detail in section 3.4.1, a discrete gauge theory with a simple gauge group has no non-trivial proper fusion subcategories except the subcategory of Wilson lines. Therefore, our above examples are enough to answer questions (3) and (4) generally in the negative (although we will see interesting examples of some of these ideas below).

Non-simple groups and unfaithful higher-dimensional representations

Let us now consider discrete gauge theories with unfaithful higher-dimensional (i.e., non-abelian) representations. The corresponding gauge groups are necessarily non-simple because the kernel of a non-trivial unfaithful representation is a non-trivial proper normal subgroup. As we will explain at a more pedestrian level below (and in a somewhat more sophisticated way in section 3.4.1), these examples illustrate the appearance of non-trivial fusion subcategories in the Wilson line sector. As a result, they demonstrate some of the ideas—described in the introduction—behind constraints from subcategory structure leading to fusion rules of the type (3.2). In particular, these theories provide examples where ideas in questions (3) and (4) of the introduction are realized.

To that end, let us consider some unfaithful higher-dimensional irreducible representation of the gauge group, $\pi \in \mathrm{Irrep}(G)$. Since π is unfaithful, it has a non-trivial kernel, $\mathrm{Ker}(\pi) \triangleleft G$. Let us also define the set of characters whose kernel includes $\mathrm{Ker}(\pi)$ as follows

$$K_\pi = \left\{ \chi_\rho : \ \chi_\rho|_{\mathrm{Ker}(\pi)} = \deg \chi_\rho \right\} \ , \tag{3.77}$$

where $\deg \chi_\rho = |\rho|$ is the degree of the character. Now, consider $\chi_\lambda, \chi_{\lambda'} \in K_\pi$. We claim $\chi_\lambda \cdot \chi_{\lambda'} \in K_\pi$. To see this, let us study

$$\chi_\lambda|_{\mathrm{Ker}(\pi)} \cdot \chi_{\lambda'}|_{\mathrm{Ker}(\pi)} = \deg \chi_\lambda \cdot \deg \chi_{\lambda'} = \sum_{\lambda''} \chi_{\lambda''}|_{\mathrm{Ker}(\pi)} \leq \sum_{\lambda''} \left| \chi_{\lambda''}|_{\mathrm{Ker}(\pi)} \right| \ . \tag{3.78}$$

Evaluating this expression on the identity element shows that $\deg \chi_\lambda \cdot \deg \chi_{\lambda'} = \sum_{\lambda''} \deg \chi_{\lambda''}$. Therefore, we have $\chi_{\lambda''}|_{\mathrm{Ker}(\pi)} = \deg \chi_{\lambda''}$, and $\lambda'' \in K_\pi$. In particular, we see that

$$\chi_\lambda \cdot \chi_{\lambda'} = \sum_{\lambda'' \in K_\pi} \chi_{\lambda''} \ . \tag{3.79}$$

As a result, the Wilson lines with charges in K_π form a closed fusion subcategory[34]

$$\mathcal{W}_\lambda \times \mathcal{W}_{\lambda'} = \sum_{\lambda'' \in K_\pi} \mathcal{W}_{\lambda''} \in \mathcal{C}_{K_\pi} \simeq \mathrm{Rep}(G/\mathrm{Ker}(\pi)) \ . \tag{3.80}$$

If we now consider the fusion of $\mathcal{W}_\pi \in \mathcal{C}_{K_\pi}$ with a non-abelian Wilson line $\mathcal{W}_\gamma \notin \mathcal{C}_{K_\pi}$, we see that the subcategory structure makes it more likely to find a unique outcome. Indeed, $\mathcal{W}_\pi \times \mathcal{W}_{\bar{\pi}} \in \mathcal{C}_{K_\pi}$ whereas $\mathcal{W}_\gamma \times \mathcal{W}_{\bar{\gamma}}$ will typically include lines not in $\mathcal{C}_{K_\pi}$.

In fact, we can go further if we take $\gamma|_{\mathrm{Ker}(\pi)}$ to be an irreducible representation of $\mathrm{Ker}(\pi)$. Since we are assuming that γ is a higher-dimensional representation, irreducibility of $\gamma|_{\mathrm{Ker}(\pi)}$ implies that $\mathrm{Ker}(\pi)$ is a non-abelian group. Invoking Gallagher's theorem (e.g., see corollary 6.17 of [70]), we see that, for $\gamma, \pi \in \mathrm{Irrep}(G)$, $\gamma \otimes \pi$ is an irreducible representation of G if the restriction $\gamma|_{\mathrm{Ker}(\pi)}$ is irreducible. Then, we are guaranteed to have the following fusion rule of non-abelian Wilson lines

$$\mathcal{W}_\pi \times \mathcal{W}_\gamma = \mathcal{W}_{\pi\gamma} \ . \tag{3.81}$$

To understand this statement, let us first prove that $\gamma \notin K_\pi$. Suppose this were not the case: then we arrive at a contradiction since $|\gamma| > 1$ would imply that $\gamma|_{\mathrm{Ker}(\pi)}$ is reducible. As a result, $\mathcal{W}_\gamma \notin \mathcal{C}_{K_\pi}$. Let us now consider the product

$$\chi_\gamma \cdot \chi_{\overline{\gamma}} = \chi_1 + \sum_i \chi_{\alpha_i} \ , \tag{3.82}$$

where α_i are irreps of G. Then we have

$$(\chi_\gamma \cdot \chi_{\overline{\gamma}})|_{\mathrm{Ker}(\pi)} = \chi_1|_{\mathrm{Ker}(\pi)} + \sum_i \chi_{\alpha_i}|_{\mathrm{Ker}(\pi)} \ . \tag{3.83}$$

Here, $\chi_1|_{\mathrm{ker}(\pi)}$ corresponds to the trivial irreducible representation of $\mathrm{Ker}(\pi)$, $\chi_{\alpha_i}|_{\mathrm{Ker}(\pi)}$ corresponds to an, in general, reducible representation of $\mathrm{Ker}(\pi)$. Suppose that $\alpha_i|_{\mathrm{Ker}(\pi)}$ contains the trivial irreducible representation of $\mathrm{Ker}(\pi)$ for some i, then we will have at least two copies of the trivial character of $\mathrm{Ker}(\pi)$ on the right hand side of (3.83). However, we know that $(\gamma \otimes \overline{\gamma})|_{\mathrm{Ker}(\pi)} = \gamma|_{\mathrm{Ker}(\pi)} \otimes \overline{\gamma}|_{\mathrm{Ker}(\pi)}$. Therefore, we cannot have more than one copy of the trivial character in the decomposition (3.83). Hence, $\alpha_i|_{\mathrm{Ker}(\pi)}$ cannot contain the trivial representation for any i. It follows that $\alpha_i|_{\mathrm{Ker}(\pi)}(h)$ is non-trivial for at least some $h \in \mathrm{Ker}(\pi)$. Therefore, it is clear that $\mathrm{Ker}(\pi)$ cannot be in the

[34] Such Wilson lines recently played an interesting role in [69]. Indeed, when one adds non-topological matter charged under these representations, the corresponding Wilson lines can end on a point. Magnetic flux lines or dyons with flux supported in $\mathrm{Ker}(\pi)$ remain topological while lines carrying other fluxes do not.

kernel of the representations α_i for any i. This shows that

$$\mathcal{W}_{\alpha_i} \in \mathcal{W}_\gamma \times \mathcal{W}_{\bar{\gamma}} \;\Rightarrow\; \mathcal{W}_{\alpha_i} \notin \mathcal{C}_{K_\pi} \ . \tag{3.84}$$

As a result, the subcategory structure guarantees (3.81).

To better understand the above general discussion (as well as the continuing role of symmetries), let us consider some examples. Note that these results give explicit realizations of the idea in question (4) in the introduction. The simplest discrete gauge theories realizing the above discussion are based on gauge groups of order forty-eight. Interestingly, the existence of subcategory structure in the Wilson line sector, $\mathcal{C}_\mathcal{W} \simeq \text{Rep}(G)$, explains the large ratio of orders, Δ_{gap}, between these groups and the smallest simple group, $O(5,3)$, with unique non-abelian fusion outcomes

$$\Delta_{\text{gap}} = \frac{25920}{48} = 540 \gg 1 \ . \tag{3.85}$$

In this section, we will discuss the examples of the binary octahedral group (BOG) and the very closely related general linear group of 2×2 matrices with elements in the finite field $\mathbb{F}_3$, $GL(2,3)$. In appendix A.1 we will consider the remaining cases at order forty-eight.

Let us begin with BOG. In this case, we have that 2_1 is an unfaithful (real) two-dimensional representation and that the restrictions of the other (real and faithful) two-dimensional irreducible representations to $\text{Ker}(2_1) = Q_8 \triangleleft BOG$, $2_{2,3}|_{\text{Ker}(2_1)}$, are irreducible. As expected from the general discussion above we have the following Wilson line fusions

$$\mathcal{W}_{2_1} \times \mathcal{W}_{2_2} = \mathcal{W}_{2_1} \times \mathcal{W}_{2_3} = \mathcal{W}_4 \ . \tag{3.86}$$

Similarly to the simple discrete gauge theories discussed in the previous subsection, BOG's $\mathbb{Z}_2$ outer automorphism again lifts to a non-trivial symmetry of the TQFT, and the non-trivial element $g \neq 1$ acts as follows: $g(\mathcal{W}_{2_2}) = \mathcal{W}_{2_3}$.

Let us note that in this case, the role of symmetries is even more pronounced. Indeed, one can check that

$$\begin{aligned} \mathcal{W}_{2_1} \times \mathcal{W}_{2_1} &= \mathcal{W}_1 + \mathcal{W}_{1_2} + \mathcal{W}_2 \in \mathcal{C}_{K_{2_1}} \simeq \text{Rep}(BOG/Q_8) \simeq \text{Rep}(S_3) \ , \\ \mathcal{W}_{2_2} \times \mathcal{W}_{2_2} &= \mathcal{W}_{2_3} \times \mathcal{W}_{2_3} = \mathcal{W}_1 + \mathcal{W}_{3_2} \ , \end{aligned} \tag{3.87}$$

where 1_2 is a non-trivial one-dimensional irreducible representation, and 3_2 is a real three-dimensional irreducible representation.[35] This latter representation satisfies $\chi_{1_2} \cdot \chi_{3_2} = \chi_{3_1}$ (and similarly $\chi_{1_2} \cdot \chi_{3_1} = \chi_{3_2}$). Therefore, we see that $\mathcal{W}_{1_2}$ generates a

[35]Note that since $2_{2,3}$ are faithful representations, a result of Burnside [71] generalized to Wilson lines shows that there exist $n_{1,2} \in \mathbb{N}$ such that $W_{2_{2,3}}^{\times n_1} \supset \mathcal{W}_{1_2}$ and $W_{2_{2,3}}^{\times n_2} \supset \mathcal{W}_{2_1}$. Our discussion implies $n_{1,2} > 2$.

non-trivial one-form symmetry in the BOG discrete gauge theory and that $\mathcal{W}_{3_{1,2}}$ and $\mathcal{W}_{2_{2,3}}$ form doublets under fusion with this generator while $\mathcal{W}_{2_1}$ is fixed

$$\mathcal{W}_{1_2} \times \mathcal{W}_{3_2} = \mathcal{W}_{3_1} \ , \quad \mathcal{W}_{1_2} \times \mathcal{W}_{2_2} = \mathcal{W}_{2_3} \ , \quad \mathcal{W}_{1_2} \times \mathcal{W}_{2_1} = \mathcal{W}_{2_1} \ . \tag{3.88}$$

This non-trivial orbit structure then implies that $\mathcal{W}_{3_2} \notin \mathcal{W}_{2_1} \times \mathcal{W}_{2_1}$ on symmetry grounds alone. Hence, in this example, both the subcategory structure and the symmetries guarantee the fusion rules (3.86).

Before finishing this example, we should check that $\mathcal{Z}(\mathrm{Vec}_{BOG})$ is indeed prime. After we discuss more formal aspects of subcategory structure in section 3.4.1, we will have more tools to use when answering this type of question. For now, let us prove that the Wilson lines must all lie in the same TQFT factor.[36] To that end, write down the Wilson lines of the BOG discrete gauge theory

$$\begin{aligned}&\mathcal{W}_1 \ , \ \ \mathcal{W}_{1_2} \ , \ \ \mathcal{W}_{2_1} \ , \ \ \mathcal{W}_{2_2} \ , \ \ \mathcal{W}_{2_3} = \mathcal{W}_{2_2} \times \mathcal{W}_{1_2} \ , \ \ \mathcal{W}_{3_1} \ , \ \ \mathcal{W}_{3_2} = \mathcal{W}_{3_1} \times \mathcal{W}_{1_2} \ , \\ &\mathcal{W}_4 = \mathcal{W}_{2_1} \times \mathcal{W}_{2_2} = \mathcal{W}_{2_1} \times \mathcal{W}_{2_3} \ .\end{aligned} \tag{3.89}$$

We can consider two cases: (1) $\mathcal{W}_{3_1}$ is in the same TQFT factor as $\mathcal{W}_{1_2}$ (call this factor $\mathcal{T}_0$) or (2) $\mathcal{W}_{3_1}$ is not in the same TQFT factor as $\mathcal{W}_{1_2}$.

Let us consider case (1) first. From the fusion equation involving $\mathcal{W}_{3_2}$, we immediately see that $\mathcal{W}_{3_2}$ is also in $\mathcal{T}_0$. Note that $\mathcal{W}_{2_1}$ cannot be written as the fusion product of two other Wilson lines. Since there is no Wilson line of quantum dimension six, we also have $\mathcal{W}_{2_1} \in \mathcal{T}_0$. Now, we must clearly have that either $\mathcal{W}_{2_{2,3}} \in \mathcal{T}_0$ or $\mathcal{W}_{2_{2,3}} \notin \mathcal{T}_0$. However, in the latter case we will again have a Wilson line of quantum dimension six. Therefore, we have that $\mathcal{W}_{2_{2,3}} \in \mathcal{T}_0$. Therefore, by the $\mathcal{W}_4$ fusion rule in (3.89), all Wilson lines are in the same TQFT factor.

Let us now consider case (2). Let $\mathcal{W}_{3_1} \in \mathcal{T}_0$ and $\mathcal{W}_{1_2} \in \mathcal{T}_1$ with $\mathcal{Z}(\mathrm{Vec}_{BOG}) = \mathcal{T}_0 \boxtimes \mathcal{T}_1$. As in case (1), $\mathcal{W}_{2_1}$ cannot be written as the fusion product of two other Wilson lines, and, since there is no Wilson line of quantum dimension six, we have $\mathcal{W}_2 \in \mathcal{T}_0$. However, this leads to a contradiction because then $\mathcal{W}_2 \times \mathcal{W}_1' \neq \mathcal{W}_2$. As a result, we conclude that all Wilson lines must lie in the same factor of $\mathcal{Z}(\mathrm{Vec}_{BOG})$.

Let us conclude with a brief discussion of the $GL(2,3)$ discrete gauge theory. This gauge group is quite similar to BOG. For the purposes of the above discussion, the only difference is that $2_{2,3}$ become complex conjugate two-dimensional irreducible representations (otherwise, the remaining representations and remaining parts of the character tables are the same). Therefore, (3.86) and (3.88) apply to $\mathcal{Z}(\mathrm{Vec}_{GL(2,3)})$ as well (by identifying these Wilson lines with their relatives in $\mathcal{Z}(\mathrm{Vec}_{GL(2,3)})$). The only change is that in the second line of (3.87), we should take $\mathcal{W}_{2_{2,3}} \times \mathcal{W}_{2_{2,3}} \to \mathcal{W}_{2_2} \times \mathcal{W}_{2_3}$. In

[36] The same pedestrian arguments used below can be extended to the full set of lines in the theory to prove that $\mathcal{Z}(\mathrm{Vec}_{BOG})$ is prime.

particular, the roles of subcategory structure (again $\mathrm{Rep}(S_3) \subset \mathrm{Rep}(GL(2,3))$) as well as outer automorphisms and one-form symmetries is the same in both the BOG and the $GL(2,3)$ discrete gauge theories.

Note that Gallagher's theorem does not exhaust all cases where representations with non-trivial kernel have irreducible products. Another interesting case is given by Gajendragadkar's theorem [72, 73]. If we have a group G which is both π-separable as well as Σ-separable, for two disjoint set of primes π and Σ, then this theorem guarantees that the product of a π-special character with a Σ-special character is irreducible. A character χ is known as π-special if $\chi(1)$ is a product of powers of primes in π (a π number) and if, for every subnormal subgroup N of G, any irreducible constituent θ of $\chi|_N$ is such that $o(\theta)$[37] is a π-number. Hence, the fusion of Wilson lines corresponding to such characters have a unique outcome. Note that, in this case, one of the characters involved in the fusion is not required to be irreducible in the kernel of the other (unlike in Gallagher's theorem).

Some general lessons and theorems

Let us conclude this section with a recapitulation of some of the main points above as well as some general theorems that amplify our discussion:

- In all of the infinitely many examples we studied so far, symmetries played an important role. For example, zero-form symmetries had a non-trivial action on Wilson lines involved in the fusion rules of interest in the A_N (with $N = k^2 \geq 9$) and $O(5,3)$ discrete gauge theories (see (3.66) and (3.73)), and similarly in theories based on BOG, $GL(2,3)$, and the other order forty-eight groups (e.g., see below (3.86) and in appendix A.1). We will revisit some of these discussions after introducing further technical tools for symmetries in section 3.4.2.

- We also saw that we could use $\mathbb{Z}_2$ one-form symmetry gauging in the S_N (with $N = k^2 \geq 9$) gauge theory to generate fusion rules involving non-abelian Wilson lines with unique outcomes in the A_N discrete gauge theories. We can constrain when such a situation arises with the following theorem:

  Theorem 3.4.1 (one-form fixed points): *Consider a TQFT, $\mathcal{T}$, with no fusion rules of the form (3.2). Suppose we can gauge a non-trivial one-form symmetry of this TQFT, H. After performing this gauging, we have fusion rules of the form (3.2) only if there are $a \in \mathcal{T}$ such that fusion with at least one of the one-form generators, $\alpha \in H$, yields $\alpha \times a = a$.*

[37] $o(\theta)$ is the order of the determinental character $\det(\chi)$ in the group of linear characters.

Proof: Let $\mathcal{T}/H$ be the TQFT obtained from gauging a 1-form symmetry H of $\mathcal{T}$ and suppose we have a fusion $a \times b = c$ for some anyons $a, b, c \in \mathcal{T}/H$. To go from $\mathcal{T}/H$ to $\mathcal{T}$ we can gauge a dual 0-form symmetry $\hat{H}$ of $\mathcal{T}/H$. Suppose the 1-form symmetry H of $\mathcal{T}$ does not have any fixed points under fusion with the anyons. Then, all anyons in $\mathcal{T}$ are organized into full length orbits under fusion with the one-form symmetry generators. This implies that the 0-form symmetry $\hat{H}$ acts trivially on the anyons a, b, c in $\mathcal{T}/H$. On gauging $\hat{H}$ we get anyons $(a,+), (b,+)$ where $+$ denotes the trivial representation of $\hat{H}$. Using (3.67), we find the fusion

$$(a,+) \times (b,+) = \sum_{\pi_c} m(\pi_c|_{\hat{H}}, +|_{\hat{H}} \otimes +|_{\hat{H}} \otimes \pi_{(a,b,c)})\ (c, \pi_c) \tag{3.90}$$

where the sum is over irreducible representations of $\hat{H}$, $\pi_{(a,b,c)}$ is a representation of H acting on the fusion space V_{ab}^c. Note that since V_{ab}^c is 1-dimensional (because $a \times b = c$), $\pi_{(a,b,c)}$ is a 1-dimensional representation. Therefore, it is clear that the only non-zero term in the sum above is $(c, \pi_{(a,b,c)})$. This contradicts the assumption that $\mathcal{T}$ has no fusion of the form (3.2). Therefore, the 1-form symmetry H of $\mathcal{T}$ has fixed points. □

As we will see, this theorem will have echoes in the coset theories we describe in the second half of this chapter.

- In the case of $O(5,3)$ discrete gauge theories, we saw that we could gauge the outer automorphisms and have fusion rules of form (3.2) in this gauged theory as well. This discussion inspires the following theorem:

Theorem 3.4.2 (zero-form fixed points): *Consider a TQFT, $\mathcal{T}$, and suppose we can gauge a non-trivial zero-form symmetry of this TQFT, H. After performing this gauging, we have fusion rules of the form (3.2) only if there are non-trivial $a_i \in \mathcal{T}$ such that at least one of the non-trivial elements of the zero-form group fixes a_i.*

Proof: Suppose that all non-trivial elements of the discrete gauge theory leave all the non-trivial anyons unfixed. Now consider anyons $a, b, c \in \mathcal{T}$ such that $c \in a \times b$. From the general discussion around (3.67), we see that $N_{^t a} \cap N_{^s b} \cap N_c = \mathbb{Z}_1$ and $N_a \backslash H / N_b = H$. Moreover, since the stabilizers are trivial, $\pi_a = \pi_b = \pi_c = 1$ are the trivial representations. We then have

$$N_{([a],1),([b],1)}^{([c],1)} = |H| \cdot m(1,1) = |H| > 1\ . \tag{3.91}$$

Therefore, we cannot produce fusion rules of the desired type. □

Our discussion of the $O(5,3)$ theory also suggests the following theorem

Theorem 3.4.3 *Consider a TQFT, $\mathcal{T}$, with a fusion rule of the form $a \times b = c$ and a zero-form symmetry, H. If at least one of $\{a, b, c\}$ is unfixed by H, then the only way for $a \times b = c$ to map to a fusion rule with unique outcome in the gauged theory is for c to be unfixed by H.*

Proof: If c is unfixed by H, then $N_c = N_a \cap N_b = \mathbb{Z}_1$. If either a or b are unfixed then $N_a \cap N_b = \mathbb{Z}_1$ as well (although we need not have $N_c = \mathbb{Z}_1$). In any case, (3.67) becomes

$$N_{([a],\pi_a),([b],\pi_b)}^{([c],\pi_c)} = \sum_{(t,s)\in N_a\backslash H/N_b} m(\pi_c|_{\mathbb{Z}_1}, {}^t\pi_a|_{\mathbb{Z}_1} \otimes {}^s\pi_b|_{\mathbb{Z}_1} \otimes \pi^{\omega}_{({}^ta,{}^sb,c)}) \ . \tag{3.92}$$

We have two cases: **(1)** $N_a\backslash H/N_b \neq \mathbb{Z}_1$ or **(2)** $N_a\backslash H/N_b = \mathbb{Z}_1$. Consider case **(1)** first. In this case, all resulting fusion rules will have multiplicity $|N_a\backslash H/N_b| > 1$. Next, consider case **(2)**. If c is fixed by some element of H, then we have at least two possible π_c (one is the trivial representation). This results in a fusion rules with non-unique outcomes. □

- In the case of the BOG and $GL(2,3)$ discrete gauge theories we saw that both one-form symmetries and subcategory structure offered an explanation of the existence of the fusion rules (3.86). The following theorem further explains and generalizes this connection between symmetries and subcategories of the Wilson line sector:

Theorem 3.4.4 (subcategories and symmetries): *Consider a finite group, G, with an unfaithful higher-dimensional irreducible representation, π. Moreover, suppose there are one-dimensional representations, π_i, with* $\mathrm{Ker}(\pi_i) \trianglerighteq \mathrm{Ker}(\pi)$*. Then, in the corresponding (twisted or untwisted) discrete gauge theory, Wilson lines charged under representations, γ, that have $\gamma|_{\mathrm{Ker}(\pi)}$ irreducible transform non-trivially under fusion with the abelian Wilson lines, $\mathcal{W}_{\pi_i}$.*

Proof: We have that $\mathcal{W}_{\pi_i} \in \mathcal{C}_{K_\pi}$, where $\mathcal{C}_{K_\pi}$ was defined around (3.80) as the subcategory of Wilson lines charged under representations whose kernels contain $\mathrm{Ker}(\pi)$ (see (3.77)). Therefore, we see that the abelian Wilson lines $\mathcal{W}_{\pi_i} \in \mathcal{C}_{K_\pi}$.

By the discussion around (3.84), we also see that all non-identity lines $\mathcal{W}_{\alpha_i} \in \mathcal{W}_\gamma \times \mathcal{W}_{\bar{\gamma}}$ are not elements of $\mathcal{C}_{K_\pi}$. As a result, $\mathcal{W}_{\pi_i} \notin \mathcal{W}_\gamma \times \mathcal{W}_{\bar{\gamma}}$. On the other hand, the trivial line is clearly in $\mathcal{W}_\gamma \times \mathcal{W}_{\bar{\gamma}}$. This logic implies

$$\mathcal{W}_{\pi_i} \times \mathcal{W}_\gamma \times \mathcal{W}_{\bar{\gamma}} \neq \mathcal{W}_\gamma \times \mathcal{W}_{\bar{\gamma}} \ , \tag{3.93}$$

from which the claim in the theorem trivially follows. □

This result tells us that the $\mathcal{W}_\gamma$ must transform under fusion with the one-form symmetry generators while $\mathcal{W}_\pi$ need not. In the case of the BOG and $GL(2,3)$ discrete gauge theories, precisely this mechanism gave a symmetry explanation for the $\mathcal{W}_\pi \times \mathcal{W}_\gamma = \mathcal{W}_{\pi\gamma}$ fusion rule in (3.86). Here we see it is somewhat more general.

- Note that the results of this section answer questions (2)-(4) of the introduction negatively in general. Still, we saw that in the BOG and $GL(2,3)$ discrete gauge theories, the ideas in (4) and (3.11) do apply in some cases. We will return to a proposal for constructing a theory satisfying (3.10) in question (3) in section 3.4.1.

3.4.1 Subgroups, subcategories, and primality

In sections 3.4, we saw the important role subcategories play in generating fusion rules involving non-abelian Wilson lines with unique outcomes (e.g., they explained the hierarchy in (3.85)). Moreover, understanding the subcategory structure is crucial to resolving the question of whether a particular discrete gauge theory is prime or not. In the case of theories with simple gauge groups (see section 3.4), we used results from [60]. In the case of the examples of discrete gauge theories with non-simple groups we studied, we used an argument that does not easily generalize. Therefore, in this section, we review some of the more general results of [60] on subcategories of discrete gauge theories. We then apply these results to generate some useful theorems that will serve us in subsequent sections.

The main power of the results in [60] is that they rephrase questions about subcategories in discrete gauge theories in terms of data of the underlying gauge group. In particular, we have:

Theorem 3.4.5 [60]: *Fusion subcategories of discrete gauge theories with finite group G are in bijective correspondence with triples, (K, H, B). Here $K, H \trianglelefteq G$ are normal subgroups that centralize each other (i.e., they commute element-by-element), and $B : K \times H \to \mathbb{C}^\times$ is a G-invariant bicharacter. If we have a non-trivial twist, ω, then the same conditions hold except that we demand that B is a G-invariant ω-bicharacter.*

Proof: See proofs of Theorems 1.1 and 1.2 (though they are phrased using different, but equivalent, terminology) of [60]. □

Since B is a bicharacter, we have

$$B(k_1k_2, h) = B(k_1, h) \cdot B(k_2, h) \ , \quad B(k, h_1h_2) = B(k, h_1) \cdot B(k, h_2) \ . \tag{3.94}$$

Here G invariance means that $B(g^{-1}kg, g^{-1}hg) = B(k,h)$ for all $k \in K$, $h \in H$, and $g \in G$. In fact, [60] also give a way to construct the subcategory, $\mathcal{S}(K,H,B)$, in question given the above data:

$$\mathcal{S}(K,H,B) := \text{gen}\left((a,\chi)|\left\{a \in K \cap R\ ,\ \chi \in \text{Irr}(N_a) \text{ s.t. } \chi(h) = B(a,h) \deg\chi\ ,\ \forall h \in H\right\}\right)\ , \tag{3.95}$$

where R is a set of representatives of conjugacy classes, $\text{Irr}(N_a)$ is the set of characters of irreducible representations of the centralizer N_a, and "gen$(\cdots)$" means that the category is generated by the simple objects inside the parenthesis. A normal subgroup is a union of conjugacy classes. Hence, K specifies all the conjugacy classes labelling the anyons in the subcategory $\mathcal{S}(K,H,B)$. Also, all the Wilson lines in $\mathcal{S}(K,H,B)$ are such that the corresponding representations have kernels which contain H.

If we have non-trivial twist, then (3.94) and G-invariance become [60]

$$B(k_1k_2,h) = \eta_h(k_1,k_2)\cdot B(k_1,h)\cdot B(k_2,h)\ , \tag{3.96}$$
$$B(k,h_1h_2) = \eta_k^{-1}(h_1,h_2)\cdot B(k,h_1)\cdot B(k,h_2)\ ,$$
$$B(g^{-1}kg,h) = \frac{\eta_k(g,h)\eta_k(gh,g^{-1})}{\eta_k(g,g^{-1})}B(k,ghg^{-1})\ , \tag{3.97}$$

where

$$\eta_g(h,k) := \frac{\omega(g,h,k)\cdot\omega(h,k,k^{-1}h^{-1}ghk)}{\omega(h,h^{-1}gh,k)}\ , \tag{3.98}$$

is a generalization of (3.14). For non-trivial twist, we also have that (3.95) becomes

$$\mathcal{S}(K,H,B) := \text{gen}\left((a,\chi)|\left\{a \in K \cap R\ ,\chi \in \text{Irr}_\omega(N_a) \text{ s.t. } \chi(h) = B(a,h) \deg\chi\ ,\forall h \in H\right\}\right)\ , \tag{3.99}$$

where the ω in $\text{Irr}_\omega(N_a)$ is a reminder that we should consider characters with projectivity phase given by (3.14) or (3.98).

We can now immediately see how the subcategories we studied in previous sections arose: $\mathcal{S}(G,\mathbb{Z}_1,1) \simeq \mathcal{Z}(\text{Vec}_G^\omega)$ is the full discrete gauge theory, $\mathcal{S}(\mathbb{Z}_1,G,1)$ is the trivial subcategory, and $\mathcal{S}(\mathbb{Z}_1,\mathbb{Z}_1,1) \simeq \text{Rep}(G) \simeq \mathcal{C}_\mathcal{W}$ is the full subcategory of Wilson lines. In the case of simple discrete gauge theories, we see that, as claimed in section 3.4, these are the *only* subcategories. However, in the case of the $\mathcal{Z}(\text{Vec}_{BOG}^\omega)$, $\mathcal{Z}(\text{Vec}_{GL(2,3)}^\omega)$, and other gauge theories based on gauge groups with unfaithful irreducible representations, π, we find additional subcategories: $\mathcal{S}(\mathbb{Z}_1,\text{Ker}(\pi),1) \simeq \text{Rep}(G/\text{Ker}(\pi))$ and $\mathcal{S}(\text{Ker}(\pi),\mathbb{Z}_1,1)$. Using Lemma 3.11 of [60], we have that $\mathcal{S}(\text{Ker}(\pi),\mathbb{Z}_1,1)$ is the Müger center of $\mathcal{S}(\mathbb{Z}_1,\text{Ker}(\pi),1)$.

Since we will study flux lines and dyons below, it is interesting to ask what the above theorems imply for such operators. One immediate consequence is that magnetic flux lines behave very differently from Wilson lines. For example:

Theorem 3.4.6 : *The set of magnetic flux lines, $\mathcal{M}$, of a discrete gauge theory (both untwisted and twisted) with non-abelian gauge group, G, do not form a fusion subcategory. In particular, $\mathcal{M} \not\simeq \text{Rep}(G)$.*

Proof: Suppose the full set of flux lines form a subcategory. Then, we need K to include at least one element of each conjugacy class in order to include all of $\mathcal{M}$ in $\mathcal{S}$. However, since K is a normal subgroup, it must consist of full conjugacy classes. Therefore, $K = G$. Using Theorem 3.4.5, we can label this putative subcategory as $\mathcal{S}(G, H, B)$. Since H has to commute with all elements in G, it has to be a subgroup of the center of the group $Z(G)$. Suppose the group has trivial center. This forces $B = 1$, and $\mathcal{S}(G, \mathbb{Z}_1, 1)$ is the full discrete gauge theory, which means we also include objects with charge. This is a contradiction.

Suppose H is a non-trivial subgroup of $Z(G)$. We know that the function B, being a bicharacter, satisfies $B(e, h) = 1 \ \forall h \in H$. So the Wilson line $([e], \pi) \in \mathcal{S}(G, H, B)$ if π has H in its kernel. Recall that the irreducible representations of G/H are in one-to-one correspondence with irreducible representations of G with H in its kernel. Since G is non-abelian, $Z(G) \neq G$. Hence, G/H is a non-trivial group. It follows that there is at least one non-trivial irreducible representation π' of G such that H is in its kernel. Hence, the Wilson line $([e], \pi')$ belongs to the subcategory $\mathcal{S}(G, H, B)$ for any B. A contradiction. □

The fact that $\mathcal{M} \not\simeq \text{Rep}(G)$ has consequences in section 3.4.2. In particular, it explains why electric-magnetic self-dualities are non-trivial to engineer in theories with non-abelian gauge groups and trivial centers.[38] If such a duality exists and involves magnetic flux lines, then they will necessarily be in a $\text{Rep}(G)$-like subcategory with objects carrying electric charge (e.g., see the S_3 discrete gauge theory self-duality [57], where the dimension-two flux line is in a $\text{Rep}(S_3)$ subcategory with both dimension one Wilson lines).

Now, we turn to the question of primality. Here the following theorem of [60] is useful

Theorem 3.4.7 [60]: *A discrete gauge theory with gauge group, G, is a prime TQFT if and only if there is no triple (K, H, B) with $K, H \triangleleft G$ normal subgroups centralizing each other, $HK = G$, $(G, \mathbb{Z}_1) \neq (K, H) \neq (\mathbb{Z}_1, G)$, and B is a G-invariant bicharacter on $K \times H$ such that $BB^{\text{op}}|_{(K \cap H) \times (K \cap H)}$ is non-degenerate. In the case of non-trivial twisting, ω, the previous conditions still hold, but B is also a G-invariant ω-bicharacter.*

[38]In any untwisted abelian gauge theory, this is not an issue as $\mathcal{M} \simeq \text{Rep}(G)$ and there is a canonical electric/magnetic duality.

Proof: See proof of theorem 1.3 (though it is phrased using different, but equivalent, terminology) in [60]. □

Note that in the statement of theorem 3.4.7, $B^{\mathrm{op}}(h,k) := B(k,h)$ for all $k \in K$ and $h \in H$.

Given this theorem, we may prove the following result that will be useful to us in section 3.4.4:

Theorem 3.4.8 : *If G is a non-direct product group with trivial center, then the corresponding (twisted or untwisted) gauge theory is a prime TQFT.*

Proof: We have a non-direct product group G with trivial center. Let us assume that $\mathcal{Z}(\mathrm{Vec}_G^{\omega})$ has a modular subcategory. Then, there exists two normal subgroups, K and H, commuting with each other and satisfying $KH = G$. So, every element of G is a product of an element of K with an element of H. Hence, any element in $K \cap H$ has to commute with all elements of G. Since the center of G is trivial by choice, $K \cap H = \mathbb{Z}_1$. It follows that G has to be a direct product of K and H. A contradiction. Hence, for non-direct product groups G with trivial center, $\mathcal{Z}(\mathrm{Vec}_G^{\omega})$ is prime. □

A simple set of examples subject to this theorem include the S_N discrete gauge theories analyzed above and the $\mathbb{Z}_{15} \rtimes \mathbb{Z}_4$ discrete gauge theory we will analyze further in section 3.4.4.

Finally, we conclude with a proposal for engineering an example of a theory of the type envisioned in question (2) in the introduction. In particular, consider a $G \times G$ discrete gauge theory, $\mathcal{Z}(\mathrm{Vec}_{G\times G}^{\omega})$. Clearly, for trivial twisting this is a non-prime theory since $\mathcal{Z}(\mathrm{Vec}_{G\times G}) = \mathcal{Z}(\mathrm{Vec}_G) \boxtimes \mathcal{Z}(\mathrm{Vec}_G)$. Indeed, by theorem 3.4.7, we can take $K = G \times \mathbb{Z}_1$, $H = \mathbb{Z}_1 \times G$, and $B = 1$. However, if we turn on a twist, $\omega \in H^3(G \times G, U(1))$, we might be able to generate a prime theory. In particular, if we can find G such that ω is non-trivial and does not factorize, then we would have an example of a prime theory with Wilson lines in $\mathrm{Rep}(G \times G) = \mathrm{Rep}(G) \boxtimes \mathrm{Rep}(G)$. Choosing one Wilson line in each $\mathrm{Rep}(G)$ factor and fusing would give a unique fusion outcome.[39] It would be interesting to see if this proposal can be realized. For example, we would like to see if there is an obstruction at the level of the existence of a G-invariant ω-bicharacter (all other requirements of theorem 3.4.7 can be satisfied). A concrete example of a theory of the type discussed in question (2) is studied in section 3.4.4.

3.4.2 Zero-form symmetries

In sections 3.4 and 3.4 we saw that zero-form symmetries played an important role in generating fusions rules of the form (3.2). In this section we review some relevant

[39] We thank D. Aasen for suggesting the basis for this idea.

results of [56] and prove a theorem that will be useful to us in section 3.4.4.

In three spacetime dimensions, zero-form symmetries are implemented by dimension two topological defects (recall that one-form symmetries are generated by abelian lines). These defects act on lines that pierce them as in figure 3.6. We will say the

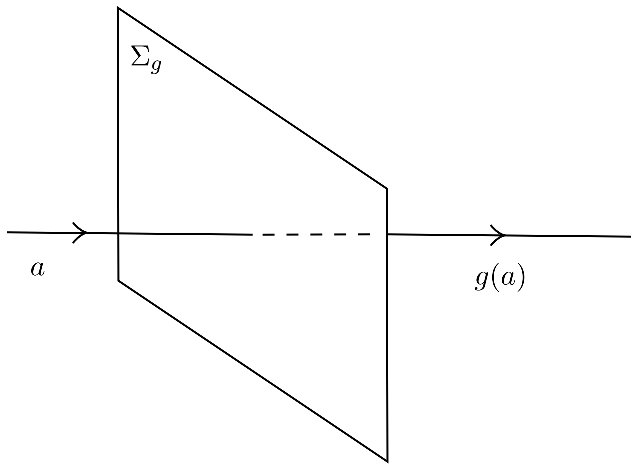


Figure 3.6: The symmetry defect Σ_g, labelled by a zero-form symmetry group element g, acts on an anyon a.

corresponding symmetry group, H, is non-trivial iff it has a generator, $h \in H$, such that there is an anyon $a \in \mathcal{T}$ satisfying $h(a) \neq a$.

Note that the automorphisms of the gauge group G, $\text{Aut}(G)$, are a natural source of symmetries. Indeed, in the context of the G-SPT that we gauge to generate the discrete gauge theory, these automorphisms permute the symmetry defects. Therefore, we expect they will play a role in the discrete gauge theory. To be more precise, recall that we can distinguish between the inner automorphisms $\text{Inn}(G) \trianglelefteq \text{Aut}(G)$, generated by conjugations of the form gxg^{-1} for $x, g \in G$, and outer automorphisms, $\text{Out}(G) := \text{Aut}(G)/\text{Inn}(G)$. Since the discrete gauge theory involves magnetic charges labeled by conjugacy classes and electric charges labeled by representations of centralizers, it is clear that inner automorphisms will act trivially on the discrete gauge theory (conjugacy classes are invariant under $\text{Inn}(G)$ and the normalizers of different elements in a conjugacy class are isomorphic). Therefore, we can at best expect $\text{Out}(G)$ to lift to a symmetry of the TQFT. Indeed, this is precisely what happens.

More formally, we have that, in a discrete gauge theory $\text{Out}(G)$ lifts to a part of the group of braided autoequivalences of the discrete gauge theory, $\text{Aut}^{\text{br}}(\mathcal{Z}(\text{Vec}_G))$:

Theorem 3.4.9 [56]**:** *The subgroup of braided autoequivalences that fix the Wilson lines* $\text{Stab}(\text{Rep}(G)) \leq \text{Aut}^{\text{br}}(\mathcal{Z}(\text{Vec}_G))$ *takes the form*

$$\text{Stab}(\text{Rep}(G)) \simeq H^2(G, U(1)) \rtimes \text{Out}(G) \ . \tag{3.100}$$

Proof: See the proof of Corollary 6.9 (though it is phrased using different, but equivalent, terminology) in [56]. □

Note that Out(G) generally acts non-trivially on the conjugacy classes. Therefore, it will also generally act non-trivially on the Wilson lines. However, in certain more exotic cases, all of Out(G) preserves conjugacy classes.[40] In such cases, the Wilson lines are fixed. Note that elements $\zeta \in H^2(G, U(1))$ always leave the Wilson lines invariant since they act as follows [56]

$$\zeta(([g], \pi_g)) = ([g], \pi_g \rho_g) \ , \quad \rho_g(x) := \frac{\zeta(x,g)}{\zeta(g,x)} \ . \tag{3.101}$$

In particular, $g = 1$ for Wilson lines. Note that $\rho_g(x)$ depends only on the cohomology class of ζ (it is invariant under shifts by a 2-coboundary).

A second set of symmetries involves the exchange of electric and magnetic degrees of freedom. These are electric/magnetic self-dualities and are inherently quantum mechanical in nature. These symmetries are closely related to the existence of Lagrangian subcategories. As we briefly mentioned at the beginning of section 3.4, a Lagrangian subcategory, $\mathcal{L}$, is a collection of bosons with trivial mutual braiding that is equal to its Müger center (e.g., like the subcategory of Wilson lines, $\mathcal{C}_{\mathcal{W}} \simeq \text{Rep}(G)$) . This latter condition simply means that the only objects that braid trivially with *every* element of $\mathcal{L}$ are elements of that subcategory.

To find the set of these symmetries, it turns out to be useful to construct the categorical Lagrangian Grassmannian, $\mathbb{L}(G)$. This is the collection of all Lagrangian subcategories. Each such subcategory, $\mathcal{L}_{(N,\mu)} \simeq \text{Rep}(G_{(N,\mu)})$ with $|G_{(N,\mu)}| = |G|$, is labeled by a normal abelian subgroup, $N \triangleleft G$, and a G-invariant $\mu \in H^2(N, U(1))$ (the Wilson line subcategory is $\mathcal{L}_{1,1}$). For the purposes of understanding these symmetries, the important subcategory is [56]

$$\mathbb{L} \supseteq \mathbb{L}_0 := \{\mathcal{L} \in \mathbb{L}(G) | \mathcal{L} \simeq \text{Rep}(G)\} \ . \tag{3.102}$$

In particular, we have

Theorem 3.4.10 [56]: *The action of* $\text{Aut}^{\text{br}}(\mathcal{Z}(\text{Vec}_G))$ *on* $\mathbb{L}_0(G)$ *is transitive. Moreover,*

$$|\text{Aut}^{\text{br}}(\mathcal{Z}(\text{Vec}_G))| = |H^2(G, U(1))| \cdot |\text{Out}(G)| \cdot |\mathbb{L}_0(G)| \ . \tag{3.103}$$

Proof: See proposition 7.6 and corollary 7.7 of [56]. □

Examples of such dualities appear in the S_3 discrete gauge theory [57] and beyond [64].

[40] The smallest group that has this feature has order 2^7 [74]. See [75] for an application of groups that have at least some class-preserving outer automorphisms to quantum doubles.

Let us now apply this theorem to prove a result that will be useful for us below

Theorem 3.4.11 : *If $G \simeq N \rtimes K$, where N is an abelian group, then the corresponding untwisted discrete gauge theory has an electric-magnetic self-duality.*

Proof: By theorem 3.4.10, in order to find a self-duality, we need to find a normal abelian subgroup $N \triangleleft G$ and a G-invariant 2-cocycle, $\mu \in H^2(N, U(1))$. Moreover, we need to find a corresponding $G_{(N,\mu)} \simeq G$. In particular, from remark 7.3 of [56], when μ is trivial, we have that $G_{(N,1)} \simeq \widehat{N} \rtimes G/N$, where $\widehat{N}$ is the character group of N. For an abelian group, $\widehat{N} \simeq N$. Therefore, we have that $G_{(\tilde{N},1)} \simeq N \rtimes K = G$ as desired. □

This theorem will be useful in our symmetry searches in section 3.4.4. Note that one immediate consequence of the above discussion is that none of the examples discussed above have self-dualities. Indeed, theories with simple gauge groups have no non-trivial normal abelian subgroups. On the other hand, theories like BOG and $GL(2,3)$ have $H^2(BOG, U(1)) \simeq H^2(GL(2,3), U(1)) \simeq \mathbb{Z}_1$ (and similarly for all normal abelian subgroups). Since these groups are not semi-direct products, we conclude they lack self-dualities.

3.4.3 Quasi-zero-form symmetries

In the previous subsections, we have seen that zero-form symmetries play an important role in generating fusion rules for non-abelian anyons with unique outcomes. However, since our interest is simply in the existence of such fusion rules, it is natural that we should generalize our notion of symmetry to include symmetries of the modular data (and hence, by Verlinde's formula, automorphisms of the fusion rules) that don't necessarily lift to symmetries of the TQFT.[41] The basic reason such "quasi zero-form symmetries" as we will call them exist is that the modular data does not define a TQFT (see [76] for a consequence of this fact). In particular, the underlying F and R symbols may not be invariant (up to an allowed gauge transformation) under a quasi zero-form symmetry even if S and T are.

In fact, such "quasi-zero-form symmetries" are common, with charge conjugation being a particular example [77]. Indeed, even in the A_N (with $N = k^2 \geq 9$) theories we discussed in section 3.4, such quasi-charge conjugation symmetries exist. These symmetries are in addition to the genuine zero-form symmetries we described when analyzing these examples. In appendix A.2, we study the particular case of A_9 discrete gauge theory in more detail and explicitly disentangle the quasi-symmetries from the genuine symmetries.

[41] In fact, most generally, we might expect automorphisms of the fusion rules that are not even symmetries of the modular data (e.g., as studied recently in [1]).

More generally, there are theories that have no genuine symmetries. One set of examples include discrete gauge theories based on the Mathieu groups. These are simple groups with trivial Out(G) and $H^2(G, U(1))$. Moreover, since these groups have no non-trivial normal abelian subgroups, $\mathbb{L}(G) = \mathbb{L}_0(G) \simeq \text{Rep}(G)$, and so there are no non-trivial self-dualities.

The largest Mathieu groups, M_{23} and M_{24} are of particular interest to us since their discrete gauge theories have non-abelian Wilson lines that fuse together to produce a unique outcome.[42] Moreover, of the theories with fusions of type (3.2), these are the only untwisted discrete gauge theories that have no modular symmetries that lift to symmetries of the full TQFTs.

For M_{23} it is not hard to check that

$$\mathcal{W}_{22} \times \mathcal{W}_{45_1} = \mathcal{W}_{990_1} \ , \quad \mathcal{W}_{22} \times \mathcal{W}_{45_2} = \mathcal{W}_{990_2} \ , \tag{3.104}$$

where 22 is the real twenty-two dimensional representation, $45_{1,2}$ are two forty five dimensional complex representations, and $990_{1,2}$ are two nine hundred and ninety dimensional representations. Under charge conjugation

$$\mathcal{W}_{45_1} \leftrightarrow \mathcal{W}_{45_2} \ , \quad \mathcal{W}_{990_1} \leftrightarrow \mathcal{W}_{990_2} \ . \tag{3.105}$$

For M_{24}, we have a particularly rich set of fusions[43]

$$\begin{aligned} \mathcal{W}_{23} \times \mathcal{W}_{45_1} &= \mathcal{W}_{1035_2} \ , \ \mathcal{W}_{23} \times \mathcal{W}_{45_2} = \mathcal{W}_{1035_3} \ , \ \mathcal{W}_{23} \times \mathcal{W}_{231_1} = \mathcal{W}_{5313} \\ \mathcal{W}_{23} \times \mathcal{W}_{231_2} &= \mathcal{W}_{5313} \ , \ \mathcal{W}_{45_1} \times \mathcal{W}_{231_1} = \mathcal{W}_{10395} \ , \ \mathcal{W}_{45_2} \times \mathcal{W}_{231_1} = \mathcal{W}_{10395} \ , \\ \mathcal{W}_{45_1} \times \mathcal{W}_{231_2} &= \mathcal{W}_{10395} \ , \ \mathcal{W}_{45_2} \times \mathcal{W}_{231_2} = \mathcal{W}_{10395} \ . \end{aligned} \tag{3.106}$$

where 23 is a real twenty-three dimensional representation, $45_{1,2}$ are complex forty-five dimensional representations, $231_{1,2}$ are two-hundred and thirty-one dimensional complex representations, and $1035_{2,3}$ are complex one-thousand and thirty-five dimensional representations, 5313 is a real five-thousand three-hundred and thirteen dimensional representation, and 10395 is a real ten-thousand three-hundred and ninety-five dimensional representation. Under charge conjugation, we have

$$\mathcal{W}_{45_1} \leftrightarrow \mathcal{W}_{45_2} \ , \quad \mathcal{W}_{231_1} \leftrightarrow \mathcal{W}_{231_2} \ , \quad \mathcal{W}_{1035_2} \leftrightarrow \mathcal{W}_{1035_3} \ . \tag{3.107}$$

While we have seen similar actions in previous sections, but here the novelty is that charge conjugation is a quasi-symmetry.

[42] By the results of [2], these theories cannot have such fusions involving lines that carry magnetic flux.

[43] It would be interesting to know if our results here have any connection with moonshine phenomena observed involving M_{24} as in [78–80].

More generally, as we will discuss in greater detail below, all other examples of TQFTs that we have found with fusion rules involving non-abelian anyons with unique outcome have at least quasi zero-form symmetries.

Finally, let us conclude this section by discussing how twisting affects the quasi-zero-form symmetries. When the quasi-symmetry is charge conjugation and the group has complex representations, the quasi-symmetry lifts to an action on Wilson lines (see appendix A.2 for a discussion in a concrete example). In this case, the quasi-symmetry persists regardless of the twisting.

As a more complicated example, let us consider the case of BOG first discussed in section 3.4. This theory only has real conjugacy classes and representations. However, there is still a non-trivial charge conjugation acting on certain dyons since elements in BOG have centralizer groups $\mathbb{Z}_4$, $\mathbb{Z}_6$, and $\mathbb{Z}_8$. These latter groups admit complex representations. However, unlike the spectrum of Wilson lines, the spectrum of dyons generally changes as we change the twist. Therefore, we might imagine that the charge conjugation quasi symmetry can be twisted away.

In fact, this is not the case. The main point is that any twisting $\omega \in H^3(BOG, U(1)) \simeq \mathbb{Z}_{48}$ of the BOG discrete gauge theory is "cohomologically trivial" in the following sense: the $\eta_g(h,k) \in H^2(N_g, U(1))$ phases defined in (3.14) are all trivial. Indeed, this statement follows from the fact that $H^2(N_g, U(1)) = \mathbb{Z}_1$ for all $g \in BOG$. Therefore, none of the anyons are lifted by the twisting, and the characters of BOG change as follows

$$\chi_{\pi_g^\omega}(h) \to \epsilon_g(h) \cdot \chi_{\pi_g^\omega}(h) \tag{3.108}$$

where ϵ_g is a 1-cochain that gives the 2-coboundary, η_g. It is not too hard to check that all choices of the twisting leave us with complex characters. Therefore, the charge conjugation quasi-symmetry persists (here it would be more accurate to term it a "modular symmetry" since it is a priori possible—though we have not checked—that charge conjugation becomes a symmetry of the theory for certain choices of ω).[44]

3.4.4 Beyond Wilson lines

So far, we have only constructed fusion rules of the form (3.2) using Wilson lines. In the case of gauge theories with simple groups, this is all we can do [2]. However, when we

[44] One may also wonder about the fate of the genuine $\mathrm{Out}(BOG) \simeq \mathbb{Z}_2$ zero-form symmetry under twisting. First, consider ω corresponding to the order 2 element in $\mathbb{Z}_{48}$. Since $\mathrm{Out}(BOG)$ acts on $H^3(BOG, U(1))$ through $\mathrm{Aut}(H^3(BOG, U(1)))$, ω should be fixed under it. Hence, it seems plausible that the twisted discrete gauge theory corresponding to this choice of ω has $\mathrm{Out}(BOG)$ as a subgroup of its symmetries (while theorem 3.4.9 has nothing to say on this point since it assumes untwisted theories, we view the existence of a symmetry in this case as a plausible assumption). In fact, more generally, if the action of $\mathrm{Out}(G)$ leaves $\omega \in H^3(G, U(1))$ invariant up to a 3-coboundary, then it can be shown that this is a symmetry of the modular data of the twisted theory. It would be interesting to understand what happens for other twists as well.

have non-simple gauge groups, the existence of self-dualities discussed in section 3.4.2 as well as the possibility of electric-magnetic dualities between theories with different gauge groups and Dijkgraaf-Witten twists [63, 64] suggests that we should also be able to involve non-abelian anyons carrying flux. Indeed, we will see this is the case.

To that end, let us study a fusion of the form

$$\mathcal{L}_{([g],\pi_g^\omega)} \times \mathcal{L}_{([h],\pi_h^\omega)} = \mathcal{L}_{([k],\pi_k^\omega)} \ , \quad g,h \neq 1 \ , \tag{3.109}$$

From our general analysis in Section 3.3.1, we know that this fusion rule is satisfied if and only if the following constraints are satisfied.

1. $[g] \cdot [h] = [k] = [h] \cdot [g]$
2. $\exists! \ \pi_k^\omega$ such that $m(\pi_k^\omega|_{N_g \cap N_h \cap N_k}, \pi_g^\omega|_{N_g \cap N_h \cap N_k} \otimes \pi_h^\omega|_{N_g \cap N_h \cap N_k} \otimes \pi_{(g,h,k)}^\omega) = 1$

We will apply these constraints in what follows.

For an untwisted discrete gauge theory based on a group G with a non-trivial center $Z(G)$, the constraints above implies that if we have a fusion of Wilson lines giving a unique outcome

$$\mathcal{W}_\pi \times \mathcal{W}_\gamma = \mathcal{W}_{\pi\gamma} \ , \tag{3.110}$$

then we have a fusion of dyons of the form

$$\mathcal{L}_{([g],\pi)} \times \mathcal{L}_{([h],\gamma)} = \mathcal{L}_{([gh],\pi\gamma)} \ , \tag{3.111}$$

where for any $g, h \in Z(G)$. Hence, we can dress the Wilson lines with fluxes from the center of the group to obtain fusion rules involving dyons with unique outcomes. For example, we have already seen that the discrete gauge theories corresponding to BOG and $GL(2,3)$ have Wilson lines fusing to give a unique outcome. Since these two groups have a non-trivial center (isomorphic to $\mathbb{Z}_2$), the above discussion immediately implies the existence of dyonic fusions where the dyons are labelled by the non-trivial element of the centre. In fact, these two types of fusions exhaust all $a \times b = c$ type fusions in both $\mathcal{Z}(\text{Vec}_{BOG})$ and $\mathcal{Z}(\text{Vec}_{GL(2,3)})$.

In the case of the fusion of non-abelian Wilson lines with a unique outcome, we saw that we were not guaranteed to find fusion subcategories beyond the three universal subcategories present in any discrete gauge theory (the theory itself, the trivial TQFT, and the Wilson line sector, $\mathcal{C}_\mathcal{W} \simeq \text{Rep}(G)$). On the other hand, when we have fusions of non-abelian anyons carrying flux with a unique outcome, we are guaranteed to have fusion subcategories. When the gauge group has a non-trivial center, $Z(G)$, this statement is trivial.[45] The following theorems guarantee this fact more generally:

[45] The discussion in section 3.4.1 guarantees that $\mathcal{S}(Z(G), \mathbb{Z}_1, 1)$ and $\mathcal{S}(\mathbb{Z}_1, Z(G), 1)$ are non-trivial subcategories.

Theorem 3.4.12 : *Let G be a non-simple finite non-abelian group. If we have a fusion rule involving two dyons or fluxes giving a unique outcome in the (twisted or untwisted) G gauge theory, then $\mathcal{S}(M_g, \mathbb{Z}_1, 1)$ and $\mathcal{S}(M_h, \mathbb{Z}_1, 1)$ (along with $\mathcal{S}(\mathbb{Z}_1, M_g, 1)$ and $\mathcal{S}(\mathbb{Z}_1, M_h, 1)$) are proper fusion subcategories of the theory. Here, g and h are elements labelling the non-trivial conjugacy classes (of length > 1) involved in the fusion. M_g is the normal subgroup generated by the elements in $[g]$.*

Proof: We have an $a \times b = c$ type fusion rule involving the non-trivial conjugacy classes $[g]$ and $[h]$. Let M_g be the normal subgroup generated by $[g]$. In fact, it has to be a proper normal subgroup. To see this, suppose $M_g = G$. From Lemma 3.4 of [60], we know that $[g]$ and $[h]$ commute element-wise. Hence, $[h]$ commutes with all elements in $M_g = G$. It follows that $[h]$ should be a subset of the elements in $Z(G)$. However, elements of $Z(G)$ form single element conjugacy classes. A contradiction. Hence, M_g has to be a proper normal subgroup of G. Since $g \neq e$, it is clear that M_g is not the trivial subgroup either. We can use the same argument to show that M_h is also a proper non-trivial normal subgroup of G. Therefore, by theorem 3.4.5, we have fusion subcategories corresponding to the choices $\mathcal{S}(M_g, \mathbb{Z}_1, 1)$ and $\mathcal{S}(M_h, \mathbb{Z}_1, 1)$ (and similarly $\mathcal{S}(\mathbb{Z}_1, M_g, 1)$ and $\mathcal{S}(\mathbb{Z}_1, M_h, 1)$). □

Note that we have, $\mathcal{L}_{([g],\pi_g^\omega)} \in \mathcal{S}(M_g, \mathbb{Z}_1, 1)$ and $\mathcal{L}_{([h],\pi_h^\omega)} \in \mathcal{S}(M_h, \mathbb{Z}_1, 1)$. Generically, we also expect $\mathcal{L}_{([g],\pi_g^\omega)} \notin \mathcal{S}(M_h, \mathbb{Z}_1, 1)$ and $\mathcal{L}_{([h],\pi_h^\omega)} \notin \mathcal{S}(M_g, \mathbb{Z}_1, 1)$. In such situations we have, in the spirit of section 3.4, an "explanation" for the fusion rule.

In fact, the reasoning in the proof to theorem (3.4.12) immediately implies that if $[h]$ has at least one element $h' \in [h]$ such that $[h', h] \neq 1$, then $\mathcal{L}_{([g],\pi_g^\omega)}$ and $\mathcal{L}_{([h],\pi_h^\omega)}$ lie in different subcategories

Corollary 3.4.12.1 : *Given the conditions in theorem 3.4.12, if there exists $h' \in [h]$ such that $[h', h] \neq 1$, $\mu_{[g]} \in \mathcal{S}(M_g, \mathbb{Z}_1, 1)$, $\mathcal{L}_{([g],\pi_g^\omega)} \notin \mathcal{S}(M_h, \mathbb{Z}_1, 1)$, and similarly for $h \leftrightarrow g$.*

For $a \in M_g$ the fusion subcategory $\mathcal{S}(M_g, \mathbb{Z}_1, 1)$ contains anyons $([a], \pi_a)$ where π_a is any irrep of the centralizer N_a. In an untwisted discrete gauge theory, for a fusion of fluxes labelled by conjugacy classes $[g]$ and $[h]$, we can define fusion subcategories $\mathcal{S}(M_g, M_h, 1)$ and $\mathcal{S}(M_h M_g, 1)$ which have a more restricted set of elements. For $a \in M_g$, the anyon $([a], \pi_a)$ is an element of $\mathcal{S}(M_g, M_h, 1)$ if and only if $M_h \subseteq \text{Ker}(\pi_a)$. Clearly, $([g], 1_g) \in \mathcal{S}(M_g, M_h, 1)$ and $([h], 1_h) \in \mathcal{S}(M_h, M_g, 1)$. However, in general, we don't expect $([g], 1_g) \notin \mathcal{S}(M_h, M_g, 1)$ and $([h], 1_h) \notin \mathcal{S}(M_g, M_h, 1)$. We will discuss an example of this below.

If one of the operators involved in the fusion of non-abelian anyons with a unique outcome is a Wilson line, then we also have the following theorem:

Theorem 3.4.13 : *Let G be a non-simple group. If we have a fusion of a Wilson line and a dyon giving a unique outcome, then $\mathcal{S}(Ker(\chi_\pi), \mathbb{Z}_1, 1)$ and $\mathcal{S}(\mathbb{Z}_1, \mathrm{Ker}(\chi_\pi), 1)$ are proper fusion subcategories of the (twisted or untwisted) discrete gauge theory. Here, π is an irrep of G labelling the Wilson line.*

Proof: Suppose $[b]$ is the non-trivial conjugacy labelling the flux line. Let χ_π be the character of an irreducible representation, π, of G labelling the Wilson line. From note 3.5 of [60] we know that χ should be trivial on a subset of elements given by $[G, b]$. Since b is not in the center, $[G, b]$ is guaranteed to have a non-trivial element. Hence, χ_π is not a faithful representation. $\mathrm{Ker}(\chi_\pi)$ is a non-trivial normal subgroup of G. Since χ_π is not the trivial representation, $\mathrm{Ker}(\chi_\pi) \neq G$ is a non-trivial proper normal subgroup. Hence, by theorem 3.4.5, we have a fusion subcategory given by $\mathcal{S}(\mathrm{Ker}(\chi_\pi), \mathbb{Z}_1, 1)$ and $\mathcal{S}(\mathbb{Z}_1, \mathrm{Ker}(\chi_\pi), 1)$. □

Note that in this case the Wilson line is an element of $\mathcal{S}(\mathbb{Z}_1, \mathrm{Ker}(\chi_\pi), 1)$ while the magnetic flux is not. In this sense, such fusions are "natural." To illustrate the ideas above, let us consider the following examples.

$\mathcal{Z}(\mathbf{Vec}_{\mathbb{Z}_3 \rtimes Q_{16}})$

Let us consider the $\mathbb{Z}_3 \rtimes Q_{16}$ discrete gauge theory. Even though this group has many non-trivial proper normal subgroups, we have $\mathbb{Z}_3 \rtimes Q_{16} \neq HK$ for any proper normal subgroups H, K. Hence, using theorem 3.4.7, we have that $\mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_3 \rtimes Q_{16}})$ is a prime theory.

This group has a length 2 conjugacy class $[f_3]$ (here we are using the notation of GAP [81], where this group is entry $(48, 18)$ in GAP's small group library) and a 2-dimensional representation 2_3 (the third 2-dimensional representation in the character table of $\mathbb{Z}_3 \rtimes Q_{16}$ on GAP). We have the following fusion of a Wilson line and a flux line giving a unique outcome.

$$\mathcal{W}_{2_3} \times \mu_{[f_3]} = \mathcal{L}_{([f_3], 2_3|_{N_{f_3}})} , \tag{3.112}$$

where the restricted representation $2_3|_{N_{f_3}}$ is irreducible.

Since we have a prime theory, the existence of this fusion rule is not due to a Deligne product. However, it can be explained using the subcategory structure of $\mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_3 \rtimes Q_{16}})$. To that end, consider the fusion subcategory $\mathcal{S}(\mathbb{Z}_1, \mathrm{Ker}(2_3)), 1)$. This fusion subcategory contains only Wilson lines. A Wilson line $\mathcal{W}_\pi$ belongs to this subcategory only if $\mathrm{Ker}(2_3)$ is in $\mathrm{Ker}(\pi)$. From the character table of $\mathbb{Z}_3 \rtimes Q_{16}$, we find three representations satisfying this constraint: 1, 1_3 and 2_3. Here 1 is the trivial representation and 1_3 is the third 1-dimensional representation in the character table. Hence, the anyons contained in the fusion subcategory $\mathcal{S}(\mathbb{Z}_1, \mathrm{Ker}(2_3), 1)$ are the Wilson lines $\mathcal{W}_1$, $\mathcal{W}_{1_3}$ as well as $\mathcal{W}_{2_3}$. Moreover, we can check the following

$$1_3 \times 1_3 = 1; \quad 1_3 \times 2_3 = 2_3; \quad 2_3 \times 2_3 = 1 + 1_3 + 2_3. \tag{3.113}$$

Now let us consider a fusion subcategory corresponding to the triple $\mathcal{S}(M_{f_3}, \mathrm{Ker}(1_2), 1)$ where M_{f_3} is the normal subgroup generated by the elements of the conjugacy class $[f_3]$ and 1_2 is the second 1 dimensional representation in the character table of $\mathbb{Z}_3 \rtimes Q_{16}$. We have $M_{f_3} = \{e, f_3, f_4, f_3 \cdot f_4\}$. A Wilson line $\mathcal{W}_\pi$ belongs to the set of generators of this subcategory only if $\mathrm{Ker}(1_2)$ is in $\mathrm{Ker}(\pi)$. Using the character table we can check that there are only two representations which satisfy this constraint: 1 and 1_2. Moreover, we have $1_2 \times 1_2 = 1$. Hence, the Wilson lines in $\mathcal{S}(M_{f_3}, \mathrm{Ker}(1_2), 1)$ are $\mathcal{W}_1$ and $\mathcal{W}_{1_2}$. Note that the flux line $\mu_{[f_3]}$ belongs to this subcategory.

Hence, we have two fusion subcategories $\mathcal{S}(\mathbb{Z}_1, \mathrm{Ker}(2_3)), 1)$ and $(M_{f_3}, \mathrm{Ker}(1_2), 1)$ with the following structure

$$\begin{aligned} \mathcal{W}_{2_3} \in \mathcal{S}(\mathbb{Z}_1, \mathrm{Ker}(2_3)), 1); \quad \mu_{[f_3]} \in (M_{f_3}, \mathrm{Ker}(1_2), 1); \\ \mathcal{S}(\mathbb{Z}_1, \mathrm{Ker}(2_3)), 1) \cap \mathcal{S}(M_{f_3}, \mathrm{Ker}(1_2), 1) = \{\mathcal{W}_1\} \end{aligned} \tag{3.114}$$

Therefore, the fusions $\mathcal{W}_{2_3} \times \overline{\mathcal{W}}_{2_3}$ and $\mu_{[f_3]} \times \overline{\mu}_{[f_3]}$ have only $\mathcal{W}_1$ in common. This trivial intersection explains the fusion (3.112) and gives an example of the idea behind question (3) in the introduction.

$\mathcal{Z}(\mathbf{Vec}_{\mathbb{Z}_{15} \rtimes \mathbb{Z}_4})$

Let us consider the $\mathbb{Z}_{15} \rtimes \mathbb{Z}_4$ discrete gauge theory. Since the center of the gauge group is trivial and the group involves a semi-direct product, we know from theorem 3.4.8 that this gauge theory is prime.

This group has a length 5 conjugacy class labelled by the element f_2 and a length 2 conjugacy class labelled by the element f_3 (here we are using the notation of GAP, where this group is entry $(60, 7)$ in GAP's small group library). We also have a length 10 conjugacy class labeled by $f_2 f_3$. It is therefore clear that we have a fusion of flux lines giving a unique outcome corresponding to these conjugacy classes

$$\mu_{[f_2]} \times \mu_{[f_3]} = \mu_{[f_2 f_3]} \ . \tag{3.115}$$

Based on our discussion above, let us consider the groups M_{f_2} and M_{f_3} generated by the elements in the corresponding conjugacy class. It is not too hard to show that

$$M_{f_2} = [e] \cup [f_2] \tag{3.116}$$

$$M_{f_3} = [e] \cup [f_3] \cup [f_4] \tag{3.117}$$

Hence, the fusion subcategories $\mathcal{S}(M_{f_2}, M_{f_3}, 1)$ and $\mathcal{S}(M_{f_3}, M_{f_2}, 1)$ can only have Wilson lines as common elements. The trivial Wilson line $\mathcal{W}_1$ is of course a common element. As we saw in section 3.4.1, a Wilson line, $\mathcal{W}_\pi$, is a member of the fusion subcategory, $\mathcal{S}(M_{f_2}, M_{f_3}, 1)$, only if the condition

$$\chi_\pi(h) := B(e,h) \deg \chi_\pi = \deg \chi_\pi \ , \quad \forall\, h \in M_{f_3} \ , \tag{3.118}$$

is satisfied. Hence, M_{f_3} should be in the kernel of χ_π. Similarly, a Wilson line $\mathcal{W}_{\pi'}$, is a member of $(M_{f_3}, M_{f_2}, 1)$ only if M_{f_2} is in the kernel of $\chi_{\pi'}$. Therefore, the common elements of the two fusion subcategories are given by the Wilson lines $W_{\tilde{\pi}}$ for which M_{f_2} and M_{f3} are in the kernel of $\chi_{\tilde{\pi}}$. Using the character table of $\mathbb{Z}_{15} \rtimes \mathbb{Z}_2$, we find that there is only one representation π_{1_2}, which satisfies this constraint.

Consider the fusions

$$\begin{aligned} \mu_{[f_2]} \times \mu_{[f_2^{-1}]} &= \mathcal{W}_1 + \cdots \ , & (3.119)\\ \mu_{[f_3]} \times \mu_{[f_3{}^{-1}]} &= \mathcal{W}_1 + \cdots \ . & (3.120)\end{aligned}$$

We know $\mu_{[f_2]}$ and $\mu_{[f_3]}$ belong to the fusion subcategories $(M_{f_2}, M_{f_3}, 1)$ and $(M_{f_3}, M_{f_2}, 1)$. Therefore, the only anyons common to both fusions above are $\mathcal{W}_1$ and $\mathcal{W}_{1_2}$. We would like to know whether the Wilson line, $\mathcal{W}_{1_2}$, appears on the right hand side of these fusions. To that end, consider the fusion

$$\mathcal{W}_{1_2} \times \mu_{[f_3]} = \mathcal{L}_{([f_3], 1_2|_{N_{f_3}})} \ . \tag{3.121}$$

It turns out that $1_2|_{N_{f_3}}$ is the trivial representation of N_{f_3}. Hence, $\mu_{[f_3]}$ is fixed under fusion with the one-form symmetry generator, $\mathcal{W}_{1_2}$. So it is clear that $\mathcal{W}_{1_2}$ should appear in the fusion $\mu_{[f_3]} \times \mu_{[f_3{}^{-1}]}$. Similarly, consider the fusion

$$\mathcal{W}_{1_2} \times \mu_{[f_2]} = \mathcal{L}_{([f_2], 1_2|_{N_{f_2}})} \ . \tag{3.122}$$

It is easy to check that $1_2|_{N_{f_2}}$ is a non-trivial representation of N_{f_2}. Hence, $\mu_{[f_2]}$ is not fixed under the fusion with $\mathcal{W}_{1_2}$. Since $\mathcal{W}_{1_2}$ is an order two anyon, it cannot appear in the fusion $\mu_{[f_2]} \times \mu_{[f_2{}^{-1}]}$ (because if $\mathcal{W}_{1_2} \subset \mu_{[f_2]} \times \mu_{[f_2{}^{-1}]}$, then multiplying both sides on the left with $\mathcal{W}_{1_2}$ implies that $\mathcal{L}_{([f_2], 1_2)}$ is the inverse of $\mu_{[f_2]}$ which is clearly false).

We have that the fusions $\mu_{[f_2]} \times \mu_{[f_2^{-1}]}$ and $\mu_{[f_3]} \times \mu_{[f_3^{-1}]}$ only have the trivial anyon in common. Hence, the combination of subcategory structure and one-form symmetry explains the fusion rule

$$\mu_{[f_2]} \times \mu_{[f_3]} = \mu_{[f_2 f_3]} \ . \tag{3.123}$$

It is interesting to note that this discussion parallels the one for Wilson lines in section 3.4.

This example is additionally illuminating because this theory also has a fusion involving a Wilson and a flux line with unique outcome. Indeed, we have two 2-dimensional representations 2_1 and 2_2 of $\mathbb{Z}_{15} \rtimes \mathbb{Z}_4$ whose restriction to the centralizer $N_{f_2} = \mathbb{Z}_3 \rtimes \mathbb{Z}_4$ are irreducible. Hence, we have the fusion rules

$$\mathcal{W}_{2_1} \times \mu_{[f_2]} = \mathcal{L}_{([f_2],2_1|_{N_{f_2}})} , \quad \mathcal{W}_{2_2} \times \mu_{[f_2]} = \mathcal{L}_{([f_2],2_2|_{N_{f_2}})} . \tag{3.124}$$

Do we have trivial braiding between the anyons involved in this fusion? This question is equivalent to whether the dyons are bosons are not. For $\mathcal{L}_{([f_2],2_i|_{N_{f_2}})}$ to be a boson, we want f_2 to be in the kernel of $2_i|_{f_2}$, which is equivalent to the condition that f_2 be in the kernel of 2_i. Using this condition, we can easily check to see that the anyons $\mathcal{W}_{2_1}$ and $\mu_{[f_2]}$ braid non-trivially with each other, while $\mathcal{W}_{2_2}$ and $\mu_{[f_2]}$ braid trivially with each other.

Moreover, this theory has several fusions involving dyons which give a unique output. For example, consider the dyons $\mathcal{L}_{([f_2],\tilde{1}_{f_2})}$ and $\mathcal{L}_{([f_3],\tilde{1}_{f_3})}$, where $\tilde{1}_{f_2}$ and $\tilde{1}_{f_3}$ are the unique non-trivial real 1-dimensional representations of $N_{f_2} = \mathbb{Z}_3 \rtimes \mathbb{Z}_4$ and $N_{f_3} = \mathbb{Z}_3 \times D_{10}$, respectively. We have the fusion

$$\mathcal{L}_{([f_2],\tilde{1}_{f_2})} \times \mathcal{L}_{([f_3],\tilde{1}_{f_3})} = \mathcal{L}_{([f_2 f_3],\tilde{1}_{f_2 f_3})} \tag{3.125}$$

where $\tilde{1}_{f_2 f_3}$ is the unique non-trivial 1-dimensional representation of $N_{f_2 f_3} = \mathbb{Z}_6$.

Let us also explore the zero-form symmetry of this theory. We have $\mathrm{Out}(\mathbb{Z}_{15} \rtimes \mathbb{Z}_4) = \mathbb{Z}_2$ and $H^2(\mathbb{Z}_{15} \rtimes \mathbb{Z}_4) = \mathbb{Z}_1$. From theorem 3.4.11, we know that this theory features non-trivial self-duality. In fact, the group $\mathbb{Z}_{15} \rtimes \mathbb{Z}_4$ has three non-trivial normal abelian subgroups $\mathbb{Z}_3, \mathbb{Z}_5, \mathbb{Z}_{15}$ all of which have trivial 2nd cohomology group. So we have the Lagrangian subcategories

$$\{\mathcal{L}_{(\mathbb{Z}_1,1)}, \mathcal{L}_{(\mathbb{Z}_3,1)}, \mathcal{L}_{(\mathbb{Z}_5,1)}, \mathcal{L}_{(\mathbb{Z}_{15},1)}\} \tag{3.126}$$

Using remark 7.3 in [56], we have

$$\mathcal{L}_{(N,1)} \simeq \mathrm{Rep}((\mathbb{Z}_{15} \rtimes \mathbb{Z}_4)_{(N,1)}) \simeq \hat{N} \rtimes (\mathbb{Z}_{15} \rtimes \mathbb{Z}_4)/\hat{N} \tag{3.127}$$

where $\hat{N}$ is the group of representations of N and $N = \mathbb{Z}_3, \mathbb{Z}_5, \mathbb{Z}_{15}$. Also, we have the isomorphisms

$$\mathbb{Z}_{15} \rtimes \mathbb{Z}_4 \simeq \mathbb{Z}_3 \rtimes (\mathbb{Z}_5 \rtimes \mathbb{Z}_4) \simeq \mathbb{Z}_5 \rtimes (\mathbb{Z}_3 \rtimes \mathbb{Z}_4) \tag{3.128}$$

Hence, all Lagrangian subcategories above are isomorphic to $\mathrm{Rep}(\mathbb{Z}_{15} \rtimes \mathbb{Z}_4)$. Hence, $|\mathbb{L}_0(\mathbb{Z}_{15} \rtimes \mathbb{Z}_4)| = 4$. From theorem 3.4.10, we know that $\mathrm{Aut}^{\mathrm{br}}(\mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_{15} \rtimes \mathbb{Z}_4}))$ should act transitively on $|\mathbb{L}(\mathbb{Z}_{15} \rtimes \mathbb{Z}_4)|$. In fact, we can use proposition 7.11 of [56] to show that $H^2(\mathbb{Z}_{15} \rtimes \mathbb{Z}_4, U(1)) \rtimes \mathrm{Out}(\mathbb{Z}_{15} \rtimes \mathbb{Z}_4) \simeq \mathbb{Z}_2$ acts trivially on $|\mathbb{L}_0(\mathbb{Z}_{15} \rtimes \mathbb{Z}_4)|$. Using theorem 3.4.10, we have $|\mathrm{Aut}^{\mathrm{br}}(\mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_{15}\rtimes\mathbb{Z}_4}))| = 8$.

Finally, since $\mathbb{Z}_{15}\rtimes\mathbb{Z}_4$ has complex characters, $\mathcal{Z}(\mathrm{Vec}^{\omega}_{\mathbb{Z}_{15}\rtimes\mathbb{Z}_4})$ has a non-trivial quasi-zero-form symmetry given by charge conjugation.

3.4.5 Symmetry and quasi-symmetry searches

We have used the software GAP to search for groups for which the corresponding untwisted discrete gauge theories have fusions rules with unique outcomes. We present our results below. The relevant GAP code is given in Appendix A.3.

Fusion of Wilson lines

Irreducible representations of a direct product of groups are the product of representations of the individual groups. Hence, it is natural that the first example with two Wilson lines fusing to give a unique Wilson line is the quantum double of $S_3 \times S_3$ (however, this fusion arises because the discrete gauge theory factorizes; this follows from theorem 3.4.7). More interesting (non-direct-product) groups with this property only appear at order 48 (see Appendix A.1). For groups of order less than or equal to 639 (except orders 384, 512, 576)[46] we have verified that whenever the corresponding untwisted discrete gauge theory has a fusion Wilson lines giving a unique outcome, $\mathrm{Aut}^{\mathrm{br}}\mathcal{Z}(\mathrm{Vec}_G)$ is non-trivial. In this set of groups, there are two which have a trivial automorphism group. They are $S_3 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$ and $(((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$. However, $H^2(S_3 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4), U(1)) = \mathbb{Z}_2$ leading to non-trivial $\mathrm{Aut}^{\mathrm{br}}(\mathcal{Z}(\mathrm{Vec}_{S_3\times(\mathbb{Z}_5\rtimes\mathbb{Z}_4)}))$. The group $(((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ has trivial $H^2(G, U(1))$. So the theory $\mathcal{Z}(\mathrm{Vec}_{(((\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes Q_8)\rtimes\mathbb{Z}_3)\rtimes\mathbb{Z}_2})$ doesn't have classical symmetries. $(((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ has only one abelian normal subgroup $N = \mathbb{Z}_3 \times \mathbb{Z}_3$. Moreover, we have $(((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2 \simeq N \rtimes K$ where $K = GL(2, 3)$. Therefore, using theorem 3.4.11, we know that this theory has non-trivial electric-magnetic duality. The groups $S_3 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$ and $(((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ have complex characters, hence the corresponding discrete gauge theories have quasi-zero-form symmetries.

Fusion of flux lines

The simplest example of an untwisted discrete gauge theory with a fusion of two flux lines giving a single outcome is $\mathcal{Z}(\mathrm{Vec}_{S_3\times S_3})$. The conjugacy classes of a direct product is a product of conjugacy classes of the individual groups. Hence, it follows that quantum doubles of direct products naturally have such fusions. As mentioned above, it follows from theorem 3.4.7 that discrete gauge theories based on direct product groups

[46] We have not checked order 384, 512, 576 due to the huge number of groups (up to isomorphism) with these orders.

are non-prime. Therefore, the fusion rules with unique outcome in this case are a consequence of the Deligne product. Since $\mathrm{Out}(S_3 \times S_3) = \mathbb{Z}_2$, $\mathcal{Z}(\mathrm{Vec}_{S_3 \times S_3})$ has non-trivial zero-form symmetry.

After $S_3 \times S_3$, we have several groups of order 48 with flux fusions giving unique outcome. The examples discussed in Appendix A.1 (except BOG and $GL(2,3)$) exhaust all such groups of order 48. All of these groups have non-trivial automorphism group, and hence the corresponding discrete gauge theory has non-trivial symmetries. In fact, for groups of order less than or equal to 639 (except orders 384, 512, 576) we have verified that whenever the corresponding untwisted discrete gauge theory has a fusion of flux lines with a unique outcome, $\mathrm{Aut}^{\mathrm{br}}(\mathcal{Z}(\mathrm{Vec}_G))$ is non-trivial. In fact, the only group with a trivial automorphism group in this set is $S_3 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$. We already discussed above that this theory has non-trivial zero-form symmetries as well as non-trivial quasi-zero-form symmetries.

Fusion of a Wilson line with a flux line

The simplest example with a fusion of a Wilson line and a flux line giving a single outcome is $\mathcal{Z}(\mathrm{Vec}_{S_3 \times S_3})$. Then we have more examples in order 48. The examples discussed in Appendix A.1 (except BOG and $GL(2,3)$) exhausts all such groups of order 48. For groups of order less than or equal to 639 (except orders 384, 512, 576) we have verified that whenever the corresponding untwisted discrete gauge theory has a fusion of a Wilson line with a flux line giving a unique outcome, $\mathrm{Aut}^{\mathrm{br}}\mathcal{Z}(\mathrm{Vec}_G)$ is non-trivial. In this set of groups, there are three which have a trivial automorphism group. They are $S_3 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes QD_{16}$ (where QD_{16} is the semi-dihedral group of order 16) and $(((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$. We discussed the groups $S_3 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$ and $(((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ above. The group $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes QD_{16}$ has trivial $H^2(G, U(1))$. So the theory $\mathcal{Z}(\mathrm{Vec}_{(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes QD_{16}})$ doesn't have classical symmetries. However, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes QD_{16}$ has one abelian normal subgroup $N = \mathbb{Z}_3 \times \mathbb{Z}_3$. Moreover, we have $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes QD_{16} \simeq N \rtimes K$ where $K = QD_{16}$. Therefore, using theorem 3.4.11, we know that the corresponding untwisted discrete gauge theory has non-trivial electric-magnetic self-duality.

The group $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes QD_{16}$ has complex characters, hence the corresponding discrete gauge theory has quasi-zero-form symmetries.

Fusion of general dyons

Being a Deligne product, $\mathcal{Z}(\mathrm{Vec}_{S_3 \times S_3})$ also has fusions involving dyons, and this is the smallest rank theory with such fusions. The next example is in order 48. The examples discussed in Appendix A.1 exhausts all such groups of order 48. For groups of order less than or equal to 100 we have verified that whenever the corresponding untwisted discrete gauge theory has a fusion of two dyons giving a unique outcome, $\mathrm{Aut}^{\mathrm{br}}\mathcal{Z}(\mathrm{Vec}_G)$ is non-trivial. In fact, every group in this set has non-trivial automorphism group. Hence, they all have non-trivial classical 0-form symmetries.

3.5 Chern-Simons Theories and Cosets

In this section, we turn our attention to a (generally) very different set of theories: TQFTs based on G_k Chern-Simons (CS) theories and cosets thereof (here G is a compact simple Lie group). Unlike the theories discussed in section 3.4, the theories we discuss here are typically chiral (i.e., $c_{\rm top} \neq 0 \pmod 8$).

In order to gain a sense of what such theories allow us to do in constructing TQFTs with fusion rules of the form (3.2) and (3.8), it is useful to recall the basic representation theory of $SU(2)$. Somewhat surprisingly, this intuition will be quite useful for more general $SU(N)_k$ CS theories. To that end, consider the textbook matter of the fusion of $SU(2)$ spin j_1 and j_2 representations

$$j_1 \otimes j_2 = \sum_{j=|j_1-j_2|}^{j_1+j_2} j \ . \tag{3.129}$$

As in the case of the finite groups in the previous section, we would like to understand if we can have $j_1 \otimes j_2 = j_3$ for $j_1, j_2 > 0$ and fixed j_3 spin. Clearly this is impossible, since we would have $j_1 + j_2 > |j_1 - j_2|$ and the sum (3.129) will have at least two contributions.

While this result is rather trivial, it is useful to recast it using the group theory analog of the F-transformation described in the introduction (as well as in section 3.4 for the case of discrete groups). To that end, we wish to consider

$$j_1 \otimes j_1 = \sum_{j=0}^{2j_1} j \ , \quad j_2 \otimes j_2 = \sum_{k=0}^{2j_2} k \ , \quad |j_{1,2}| > 1 \ , \tag{3.130}$$

where $|j_{1,2}|$ are the dimensions of the representations. In particular, we see that (since $j_{1,2} > 0$) both products in (3.130) must always contain the trivial representation and the adjoint representation. This observation also implies that $j_1 \otimes j_2 \neq j_3$ for fixed j_3 spin.

The discussion around (3.130) easily generalizes to arbitrary compact simple Lie group, G. In particular, let us consider

$$\alpha \otimes \bar{\alpha} = 1 + \sum_{\gamma \neq 1} N_{\alpha\bar{\alpha}}^{\gamma}\, \gamma \ , \quad \beta \otimes \bar{\beta} = 1 + \sum_{\delta} N_{\beta\bar{\beta}}^{\delta}\, \delta \ , \quad |\alpha|, \ |\beta| > 1 \ , \tag{3.131}$$

where α, β and $\bar{\alpha}, \bar{\beta}$ are conjugate higher-dimensional irreducible representations of G, $\mathrm{Irr}(G)$. The number of times the adjoint appears in the product $\alpha \otimes \bar{\alpha}$ is [82]:

$$N_{\alpha\bar{\alpha}}^{\mathrm{adj}} = \left| \left\{ \lambda_j^{(\alpha)} \neq 0 \right\} \right| \geq 1 \ , \tag{3.132}$$

where $\lambda_j^{(\alpha)}$ are the Dynkin labels of α. Therefore, we learn that for all higher-dimensional representations of G

$$\alpha \otimes \beta \neq \gamma \ , \ \forall \ |\alpha|, \ |\beta| > 1 \ , \ \alpha, \beta, \gamma \in \mathrm{Irr}(G) \ , \tag{3.133}$$

Of course, our interest is in the fusion algebra of G_k. From this perspective, the above discussion is in the limit $k \to \infty$. As we will prove in the next section, taking $G_k = SU(N)_k$ and imposing finite level does not lead to fusions of the form (3.2) or (3.8).

3.5.1 G_k CS theory

Let us now consider the finite-level deformation of the fusion rules discussed in the previous section. These are the fusion rules of Wilson lines in G_k CS theory. We first consider $SU(2)_k$ as it is rather illustrative. We will then generalize to $SU(N)_k$ and comment on more general G_k.

In the case of $SU(2)_k$, (3.129) becomes [83, 84]

$$j_1 \otimes j_2 = \sum_{j=|j_1-j_2|}^{\min(j_1+j_2, k-j_1-j_2)} j \ . \tag{3.134}$$

In addition to truncating the spectrum to the spins $\{0, 1/2, 1, \cdots, k/2\}$, the above deformation abelianizes the spin $k/2$ representation (since $k/2 \otimes k/2 = 0$). However, these changes do not alter the conclusion from the previous section: we cannot write $j_1 \otimes j_2 = j_3$ for j_3 non-abelian irreducible $j_{1,2,3}$. Indeed, consider

$$j_1 \otimes j_1 = \sum_{j=0}^{\min(2j_1, k-2j_1)} j \ , \quad j_2 \otimes j_2 = \sum_{j=0}^{\min(2j_2, k-2j_2)} j \ , \quad j_{1,2} \neq 0, \ \frac{k}{2} \ . \tag{3.135}$$

The conditions $j_{1,2} \neq 0$, $\frac{k}{2}$ are to ensure that the representation is non-abelian. In particular, we again see that the adjoint representation appears in (3.135).

While the fusion rules discussed in [83, 84] apply to more general groups, they are rather difficult to implement. Instead, using proposals suggested in [85, 86] and finally proven in [87], the authors of [88] show that for α an irreducible representation of G_k

(with G a compact simple Lie group), we have

$$^{(k)}N^{\text{adj}}_{\alpha\bar{\alpha}} = \left|\left\{\hat{\lambda}^{(\alpha)}_j \neq 0\right\}\right| - 1 \ , \tag{3.136}$$

where $\hat{\lambda}^{\alpha}_j$ are the associated affine Dynkin labels.

In particular, for $SU(N)_k$, if $|\alpha| > 1$, then $^{(k)}N^{\text{adj}}_{\alpha\bar{\alpha}} \geq 1$.[47] Indeed, the abelian representations, γ_i, satisfy a $\mathbb{Z}_N$ fusion algebra and are characterized by $\hat{\lambda}^{(\gamma_i)}_j = k\delta_{ij}$, where $i \in \{0, 1, ..., N-1\}$. On the other hand, all non-abelian representations have at least two non-zero Dynkin labels. As a result, we learn that

$$\alpha \otimes \beta \neq \gamma \ , \quad \forall \alpha, \beta, \gamma \in \text{Irr}(SU(N)_k) \ , \quad |\alpha|, |\beta| > 1 \ . \tag{3.137}$$

Therefore, we see that we have the following fusions for non-abelian Wilson lines in $SU(N)_k$ CS TQFT

$$\mathcal{W}_\alpha \times \mathcal{W}_\beta = \mathcal{W}_\gamma + \cdots \ , \quad |\alpha|, |\beta| > 1 \ , \tag{3.138}$$

where the ellipses necessarily include additional Wilson lines. This statement is more generally true in any G_k CS theory (with G a compact and simple Lie group) for which the lines in question correspond to affine representations with at least two non-zero Dynkin labels.

Note that for certain G_k, non-abelian representations can have a single non-vanishing Dynkin label. For example, consider the $(E_7)_2$ CS theory.[48] It has Wilson lines $\mathcal{W}_\tau$ and $\mathcal{W}_\sigma$ with quantum dimensions $\frac{1+\sqrt{5}}{2}$ and $\sqrt{2}$, respectively, and they fuse to give a unique outcome. The existence of this fusion rule follows from the fact that $(E_7)_2$ is not a prime TQFT. In fact, it resolves into the product of prime theories Fib $\boxtimes$ Ising$'$, where Fib is the Fibonacci anyon theory and Ising$'$ is a TQFT with the same fusion rules as the the Ising model.

We can apply the above arguments to learn about global properties of G_k CS theory. For example, we can ask if G_k CS theory is prime or not. The answer is no in general. Indeed, consider the case $G = SU(2)$. For $k \in \mathbb{N}_{\text{even}}$, $SU(2)_k$ is prime. However, for $k \in \mathbb{N}_{\text{odd}}$, the abelian anyon generating the $\mathbb{Z}_2$ one-form symmetry forms a modular subcategory. By Müger's theorem [49] (see also [52] for a discussion at the level of RCFT), it then decouples and the theory resolves into a product of two prime theories

$$SU(2)_k \simeq \begin{cases} SU(2)_1 \boxtimes SU(2)^{\text{int}}_k \ , & \text{if } k = 1 \ (\text{mod} 4) \\ \overline{SU(2)_1} \boxtimes SU(2)^{\text{int}}_k \ , & \text{if } k = 3 \ (\text{mod} 4) \ . \end{cases} \tag{3.139}$$

where $SU(2)^{\text{int}}_k$ is a TQFT built out of the integer spin $SU(2)_k$ representations. Here

[47]Here we define $|\alpha|$ to be the quantum dimension.

[48]We thank a referee for pointing out this example.

$\overline{SU(2)_1}$ is the TQFT conjugate to $SU(2)_1$ (these TQFTs are sometimes called the anti-semion and semion theories in the condensed matter literature).

While G_k CS theory is not prime in general, our arguments above readily prove the following:

Claim 3.5.1 *Non-abelian Wilson lines in $SU(N)_k$ CS theory must all lie in the same prime TQFT factor. For more general G_k CS theory (with G compact and simple), all Wilson lines corresponding to affine representations with at least two non-zero Dynkin labels must be part of the same prime TQFT factor.*

Proof: Suppose this were not the case. Then, we would find fusion rules of the form (3.138) with *no* Wilson lines in the ellipses. □

Clearly, to produce fusion rules of the form (3.2) for non-abelian Wilson lines in the same prime TQFT, we will need to go beyond $SU(N)_k$ CS theory. One way to proceed is to consider coset theories and use some intuition from section 3.4. Indeed, since cosets can have fixed points (which we will describe below), it is natural to think they can lead to fusion rules of the form (3.2).

3.5.2 Virasoro minimal models and some cosets without fixed points

We begin with a discussion of the Virasoro minimal models, as these are simple examples of theories that are related to cosets. While these cosets do not have fixed points, they turn out to produce factorized TQFTs that are nonetheless illustrative. In the next section, we will focus on cosets that have fixed points, and we will see how to engineer fusion rules of the form (3.2).

One way to construct the Virasoro minimal models is to take a three-dimensional spacetime $\mathbb{R}\times\Sigma$ and place $SU(2)_{k-1}\times SU(2)_1$ CS theory on $I\times\Sigma$, where I is an interval in $\mathbb{R}$. We can place $SU(2)_k$ CS theory outside this region. At the two $1+1$ dimensional interfaces between the CS theories (which form two copies of Σ, call them $\Sigma_{1,2}$), we obtain the left and right movers of the RCFT. Here the chiral (anti-chiral) primaries lie where endpoints of Wilson lines from the $SU(2)_k$ and $SU(2)_{k-1}\times SU(2)_1$ theories meet on Σ_1 (Σ_2).

Another way to think about the Wilson lines related to the Virasoro minimal models is to start with $SU(2)_{k-1}\times SU(2)_1$ CS theory and change variables to make an $SU(2)_k$ subsector manifest [89]. Integrating this sector out leaves an effective coset TQFT.

The end result is that the TQFT we are interested in is[49]

$$\mathcal{T}_p = \frac{SU(2)_{p-2} \boxtimes SU(2)_1}{SU(2)_{p-1}} \ , \quad p \geq 3 \ . \tag{3.141}$$

Here, the natural number $p \geq 3$ labels the corresponding Virasoro minimal model (so, for example, $p = 3$ for the Ising model).[50] We may construct the MTC data underlying the RCFT and the coset TQFT by taking products (e.g., see [91])

$$F_{\mathcal{T}_p} = F_{SU(2)_{p-2}} \cdot F_{SU(2)_1} \cdot \bar{F}_{SU(2)_{p-1}} \ , \quad R_{\mathcal{T}_p} = R_{SU(2)_{p-2}} \cdot R_{SU(2)_1} \cdot \bar{R}_{SU(2)_{p-1}} \ . \tag{3.142}$$

In order to make (3.142) precise, we need to explain how the states in $\mathcal{T}_p$ are related to those in the individual $SU(2)_k$ theories that make up the coset. Let us denote the $SU(2)_{p-2}$, $SU(2)_1$, and $SU(2)_{p-1}$ weights as λ, μ, and ν. Then, to build the coset we should identify Wilson lines as follows

$$\mathcal{W}_{\{\lambda,\mu,\nu\}} := \mathcal{W}_\lambda \times \mathcal{W}_\mu \times \mathcal{W}_\nu \simeq (\mathcal{W}_{p-2} \times \mathcal{W}_\lambda) \times (\mathcal{W}_1 \times \mathcal{W}_\mu) \times (\mathcal{W}_{p-1} \times \mathcal{W}_\nu) \ , \tag{3.143}$$

where $\mathcal{W}_{p-2}$, $\mathcal{W}_1$, and $\mathcal{W}_{p-1}$ are abelian Wilson lines transforming in the weight $p-2$ (spin $(p-2)/2$), weight 1 (spin $1/2$), and weight $p-1$ (spin $(p-1)/2$) representations of the different TQFT factors.[51] Moreover, in order to be a valid Wilson line in $\mathcal{T}_p$, we should demand that our Wilson lines satisfy

$$\mathcal{W}_{\{\lambda,\mu,\nu\}} \in \mathcal{T}_p \ \Leftrightarrow \ \lambda + \mu - \nu \in Q \ \Leftrightarrow \ \lambda + \mu + \nu = 0 \ (\mathrm{mod}\ 2) \ , \tag{3.145}$$

where Q is the $SU(2)$ root lattice. This relation guarantees that all lines that remain have trivial braiding with $\mathcal{W}_{\{p-2,1,p-1\}}$ (which is a boson that is in turn identified with the vacuum). It is in terms of these degrees of freedom that (3.142) should be understood.

Before proceeding, let us stop and note that the fusion in (3.143) has no fixed points. Indeed, this statement readily follows from the fact that $SU(2)_1$ is an abelian TQFT, and abelian theories cannot have fixed points since their fusion rules are those of a

[49] This is the TQFT analog of the classic result [90] for the corresponding affine algebras:

$$\mathrm{Vir}_p \simeq \frac{\widehat{\mathfrak{su}}(2)_{p-2} \times \widehat{\mathfrak{su}}(2)_1}{\widehat{\mathfrak{su}}(2)_{p-1}} \ . \tag{3.140}$$

[50] In writing (3.141), we have used the Deligne product to emphasize the fact that the $SU(2)_{p-2} \times SU(2)_1$ CS theory is a product TQFT.

[51] At the level of the corresponding affine algebras, this is the statement that [84]

$$\left\{\hat{\lambda}, \hat{\mu}, \hat{\nu}\right\} \simeq \left\{a\hat{\lambda}, a\hat{\mu}, a\hat{\nu}\right\} \ , \tag{3.144}$$

where the hat denotes affine weights, and a is the generator of the (diagonal) $\mathcal{O}(\widehat{\mathfrak{su}}(2))$ outer automorphism.

finite abelian group (in this case $\mathbb{Z}_2$).

Given this groundwork, we claim that $\mathcal{T}_p$ factorizes as follows

$$\mathcal{T}_p \simeq \begin{cases} (SU(2)_{p-2} \boxtimes SU(2)_1)^{\text{int}} \boxtimes SU(2)^{\text{int}}_{p-1} \ , & \text{if } p = 0 \pmod 2 \\ SU(2)^{\text{int}}_{p-2} \boxtimes SU(2)^{\text{conj}}_{p-1} \ , & \text{if } p = 1 \pmod 2 \ . \end{cases} \tag{3.146}$$

The various TQFTs appearing in (3.146) are

$$\begin{aligned} (SU(2)_{p-2} \boxtimes SU(2)_1)^{\text{int}} &:= \text{gen}\left(\{\mathcal{W}_{\{\lambda,\mu\}} \in SU(2)_{p-2} \boxtimes SU(2)_1 |\ \lambda + \mu = 0 \pmod 2\}\right) \ , \\ SU(2)^{\text{int}}_{p-1} &:= \text{gen}\left(\{\mathcal{W}_\nu \in SU(2)_{p-1} |\ \nu = 0 \pmod 2\}\right) \ , \\ SU(2)^{\text{conj}}_{p-1} &:= \text{gen}\left(\{\mathcal{W}_{\{\lambda,\mu,\nu\}} |\ \lambda + \mu + \nu = 0 \pmod 2 \ , \ \mathcal{W}_\lambda, \mathcal{W}_\mu \text{ abelian}\}\right) \ , \\ SU(2)^{\text{int}}_{p-2} &:= \text{gen}\left(\{\mathcal{W}_\lambda \in SU(2)_{p-2} |\ \lambda = 0 \pmod 2\}\right) \ , \end{aligned} \tag{3.147}$$

where "gen$(\cdots)$" means that the TQFT is generated by the Wilson lines enclosed. Notice that in the case that p is even, $p-1$ is odd and $SU(2)^{\text{int}}_{p-1}$ is precisely the decoupled TQFT factor required by Müger's theorem in (3.139) containing integer spins (even Dynkin labels). Similar logic applies to $SU(2)^{\text{int}}_{p-2}$ in the case that p is odd. The TQFT $SU(2)^{\text{conj}}_{p-1}$ has the same fusion rules as $SU(2)_{p-1}$, but it is a different TQFT. Finally, for the case that $p = 3$ (i.e., the Ising model), we see that $\mathcal{T}_3$ does not factorize.[52]

Our strategy to prove the factorization in (3.146) is to construct the various factors and then argue that they are well-defined TQFTs by Müger's theorem [49]. Although we will not pursue it, this same approach leads to interesting generalizations for cosets built out of groups other than $SU(2)$.

To that end, let us first take the case of $p \geq 3$ odd. Using the result in (3.142), we have that the modular S matrix also takes a product form

$$S_{\{\lambda,\mu,\nu\}\{\lambda',\mu',\nu'\}} = S^{(p-2)}_{\lambda\lambda'} \cdot S^{(1)}_{\mu\mu'} \cdot S^{(p-1)}_{\nu\nu'} \ , \quad \theta_{\{\lambda,\mu,\nu\}\{\lambda',\mu',\nu'\}} = \theta^{(p-2)}_{\lambda\lambda'} \cdot \theta^{(1)}_{\mu\mu'} \cdot \bar{\theta}^{(p-1)}_{\nu\nu'} \ , \tag{3.148}$$

where the superscripts on the righthand sides of the above equations refer to the corresponding factors in the coset (3.141). From the S matrix, Verlinde's formula yields (see also the discussion in [84])

$$N^{\{\lambda'',\mu'',\nu''\}}_{\{\lambda,\mu,\nu\}\{\lambda',\mu',\nu'\}} = N^{(p-2)\lambda''}_{\lambda\lambda'} \cdot N^{(1)\mu''}_{\mu\mu'} \cdot N^{(p-1)\nu''}_{\nu\nu'} \ , \tag{3.149}$$

where, again, the superscripts on the righthand side denote the different coset factors in (3.141). The factor $SU(2)^{\text{int}}_{p-2}$ in the second line of (3.146) is clearly closed under fusion. So too is $SU(2)^{\text{conj}}_{p-1}$. To have factorization of the TQFT, we need only show that

[52] Note also that Ising shares the same fusion rules as $SU(2)_2$, though they are not the same TQFTs. For example, the σ fields have different twists.

all Wilson lines can be written in this way and, by Müger's theorem, that one of these factors is modular. The second part is trivial: we have already seen that $SU(2)^{\rm int}_{p-2}$ is modular in the discussion surrounding (3.139). We can confirm this statement by looking at the modular S-matrix for $SU(2)_{p-2}$

$$S^{(p-2)}_{\lambda\lambda'} = \sqrt{\frac{2}{p}}\sin\left(\frac{(\lambda+1)(\lambda'+1)\pi}{p}\right) \ . \tag{3.150}$$

and taking the submatrix involving the integer spins (even weights).

Therefore, we need only check that all states in the coset (3.141) can be expressed in this way. To that end, we can see that

$$|SU(2)^{\rm int}_{p-2}| = \frac{p-1}{2} \ , \ |SU(2)^{\rm conj}_{p-1}| = p \ , \tag{3.151}$$

where the norm denotes the number of simple elements within. Therefore, we see that we have $|\mathcal{T}_p| = p(p-1)/2$, which is precisely the number of states in the coset (3.141) (note that in these computations we have used (3.143) and (3.145)) and the corresponding A-type Virasoro minimal model.

To make contact with the fusion rules in (3.4), we need to explain precisely how coset lines map onto the Virasoro primaries. The results above allow us to realize the, say, Virasoro left-movers as states on the boundary of the bulk TQFT, $\mathcal{T}_p \simeq SU(2)^{\rm int}_{p-2} \boxtimes SU(2)^{\rm conj}_{p-1}$ with p odd. Now, we need to see how we can map boundary endpoints of lines in this theory to Virasoro primaries, $\varphi_{(r,s)}$. To that end, by comparing the S-matrix for $\mathcal{T}_p \simeq SU(2)^{\rm int}_{p-2} \boxtimes SU(2)^{\rm conj}_{p-1}$ with the corresponding expressions for those of the Virasoro minimal models, we have that the labels of the Virasoro primary, $\varphi_{(r,s)}$ map as follows (see also [84])

$$r = \lambda + 1 \ , \quad s = \nu + 1 \ . \tag{3.152}$$

In particular, we see that the $\varphi_{(r,1)}$ primaries are endpoints of lines in $SU(2)^{\rm int}_{p-2}$ while the $\varphi_{(1,s)}$ are endpoints of lines in $SU(2)^{\rm conj}_{p-1}$. This reasoning explains the fact that non-abelian Virasoro primaries of these types have unique fusion outcomes[53]

$$\varphi_{(r,1)} \times \varphi_{(1,s)} = \varphi_{(r,s)} \ , \tag{3.153}$$

discussed in the introduction (at least for p odd). As an example, we have $\mathcal{T}_3 \simeq$ Ising (i.e., the TQFT is the Ising MTC), which does not factorize. On the other hand, for $p=5$, we have

$$\mathcal{T}_5 = (G_2)_1 \boxtimes SU(2)^{\rm conj}_4 \ , \tag{3.154}$$

[53]Though, again, we stress that this factorization is not a factorization of RCFT correlators.

where $(G_2)_1$ is the so-called "Fibnonacci" TQFT, and $SU(2)_4^{\text{conj}}$ is a TQFT with the same fusion rules and S-matrix as $SU(2)_4$.

Let us now consider $p \geq 4$ even. The modular data and fusion rules still take a product form as in (3.148) and (3.149). Now, however, we should examine the first line in (3.146). Using (3.149), it is again easy to see that both $SU(2)_{p-1}^{\text{int}}$ and $(SU(2)_{p-2} \boxtimes SU(2)_1)^{\text{int}}$ are separately closed under fusion. Moreover, just as before, we can use the discussion around (3.139) and Müger's theorem to conclude that $SU(2)_{p-1}^{\text{int}}$ is indeed a decoupled TQFT as claimed in (3.146).

We should again check that all states in (3.141) can be reproduced. To that end, we have

$$|SU(2)_{p-1}^{\text{int}}| = \frac{p}{2} \ , \ |(SU(2)_{p-2} \boxtimes SU(2)_1)^{\text{int}}| = p-1 \ . \tag{3.155}$$

As a result, we have $|\mathcal{T}_p| = p(p-1)/2$, which is the correct number of states in the coset (3.141) and the corresponding A-type Virasoro minimal model.

Our mapping is again as in (3.152), but now $\varphi_{(r,1)}$ primaries are endpoints of lines in $(SU(2)_{p-2} \boxtimes SU(2)_1)^{\text{int}}$, and $\varphi_{1,s}$ are endpoints of lines in $SU(2)_{p-1}^{\text{int}}$. This again explains the fusion outcomes in (3.153) for the case of p even as well. As an example, note that

$$\mathcal{T}_4 = \text{Ising}' \boxtimes (F_4)_1 \ , \tag{3.156}$$

where the first factor is a rank three TQFT with the same fusion rules as Ising (and $SU(2)_2$), and the second factor is the time reversal of the Fibonacci theory in (3.154).

As a result, we conclude that, although the TQFTs discussed in this section do have non-abelian anyons fusing to give a unique outcome, this is due to the fact that the corresponding TQFTs factorize.

3.5.3 Beyond Virasoro: cosets with fixed points

In section 3.4 we saw that fixed points of various kinds gave rise to fusion rules of the form (3.8) (in particular, see theorem 3.4.1 of section 3.4). In the context of cosets, we can also naturally engineer fixed points under the action of fusion with abelian anyons generating identifications of fields. In the case of Virasoro, this didn't happen (see (3.143)). Indeed, this statement followed from the fact that we had an abelian factor in the coset (3.141).

The simplest way to get around this obstacle and generate fixed points is to consider instead

$$\widehat{\mathcal{T}}_p = \frac{SU(2)_{p-2} \boxtimes SU(2)_2}{SU(2)_p} \ , \tag{3.157}$$

where $p \geq 3$ (we should take $p \geq 4$ to avoid the problem of abelian factors). By further identifying some of these coset fields, we get theories related to the $\mathcal{N} = 1$ super-Virasoro minimal models [90,92]. Note that the case of $p = 3$ corresponds to the

$\mathcal{T}_4$ case discussed previously (i.e., to the TQFT related to the tri-critical Ising model).

For the theories in (3.157), we find the following generalization of the identification condition in (3.143)[54]

$$\begin{aligned}\mathcal{W}_{\{\lambda,\mu,\nu\}} &:= \mathcal{W}_\lambda \times \mathcal{W}_\mu \times \mathcal{W}_\nu \simeq (\mathcal{W}_{p-2}\times\mathcal{W}_\lambda)\times(\mathcal{W}_2\times\mathcal{W}_\mu)\times(\mathcal{W}_p\times\mathcal{W}_\nu)\\ &= \mathcal{W}_{p-2-\lambda}\times\mathcal{W}_{2-\mu}\times\mathcal{W}_{p-\nu}\ ,\end{aligned} \tag{3.158}$$

In particular, if $\lambda=(p-2)/2$, $\mu=1$, and $\nu=p/2$, we can have a fixed point[55]. Of course, if p is odd, we don't have a fixed point. In this case, we can again run logic similar to that used in the Virasoro case to argue that the TQFT factorizes.

However, if p is even, then we need to properly define the coset. In particular, we should resolve the fixed point Wilson line as follows (see [53, 93] for the dual RCFT discussion)

$$\mathcal{W}_{\{(p-2)/2,1,p/2\}} \to \mathcal{W}^{(1)}_{\{(p-2)/2,1,p/2\}} + \mathcal{W}^{(2)}_{\{(p-2)/2,1,p/2\}}\ . \tag{3.159}$$

Let us consider what turns out to be the simplest interesting case, $p=6$

$$\widehat{\mathcal{T}}_6 = \frac{SU(2)_4\boxtimes SU(2)_2}{SU(2)_6}\ . \tag{3.160}$$

The fixed point resolution in (3.159) becomes $\mathcal{W}_{\{2,1,3\}}\to\mathcal{W}^{(1)}_{\{2,1,3\}}+\mathcal{W}^{(2)}_{\{2,1,3\}}$. As in the cases of one-form gauging with fixed points discussed in section 3.4, it is natural that there should be a zero-form symmetry exchanging $\mathcal{W}^{(1)}_{\{2,1,3\}}\leftrightarrow\mathcal{W}^{(2)}_{\{2,1,3\}}$.

As a first step to better understand the theory after resolving the fixed point, note that $\widehat{\mathcal{T}}_6$ has the following number of lines

$$|\widehat{\mathcal{T}}_6| = 28\ . \tag{3.161}$$

Of these fields, twenty-six come from identifying full length-two orbits in (3.158) while two come from resolving the fixed point. In what follows, $\{\lambda,\mu,\nu\}$ will denote fields in full orbits, while labels of the form $\{2,1,3\}^{(i)}$ (with $i=1,2$) will denote the fixed point lines.

To understand the fusion rules and the question of primality after fixed point resolution, we can compute the S matrix using the algorithm discussed in [93] (let us denote the result by $\tilde{S}$). It takes the form

$$\tilde{S}_{\{\lambda,\mu,\nu\}\{\lambda',\mu',\nu'\}} = 2S_{\{\lambda,\mu,\nu\}\{\lambda',\mu',\nu'\}}\ ,\quad \tilde{S}_{\{2,1,3\}^{(i)}\{\lambda',\mu',\nu'\}} = S_{\{2,1,3\}\{\lambda',\mu',\nu'\}}\ ,$$

[54]We also require that $\lambda+\mu+\nu=0 \pmod 2$ so that the lines in the coset theory have trivial braiding with the bosonic line $\mathcal{W}_{\{p-2,2,p\}}$. This line is in turn identified with the vacuum.

[55]Note that the fixed points discussed in section 3.4 are fixed points under 1-form and 0-form symmetry action. In the coset examples studied here, fixed points refer to field identification fixed points.

Wilson lines	Quantum dimensions
$\mathcal{W}_{\{0,0,0\}}, \mathcal{W}_{\{4,0,0\}}, \mathcal{W}_{\{0,2,0\}}, \mathcal{W}_{\{0,0,6\}}$	1
$\mathcal{W}_{\{0,0,2\}}, \mathcal{W}_{\{0,0,4\}}, \mathcal{W}_{\{4,0,2\}}, \mathcal{W}_{\{4,0,4\}}$	$\cot\left(\frac{\pi}{8}\right)$
$\mathcal{W}_{\{1,0,1\}}, \mathcal{W}_{\{1,0,5\}}, \mathcal{W}_{\{3,0,1\}}, \mathcal{W}_{\{3,0,5\}}$	$\sqrt{\frac{3}{2}}\csc\left(\frac{\pi}{8}\right)$
$\mathcal{W}_{\{0,1,3\}}, \mathcal{W}_{\{2,1,3\}^{(1)}}, \mathcal{W}_{\{2,1,3\}^{(2)}}$	$\sqrt{2}\csc\left(\frac{\pi}{8}\right)$
$\mathcal{W}_{\{1,0,3\}}, \mathcal{W}_{\{3,0,3\}}$	$\sqrt{3}\csc\left(\frac{\pi}{8}\right)$
$\mathcal{W}_{\{2,0,0\}}, \mathcal{W}_{\{2,0,6\}}$	2
$\mathcal{W}_{\{0,1,1\}}, \mathcal{W}_{\{0,1,5\}}$	$\csc\left(\frac{\pi}{8}\right)$
$\mathcal{W}_{\{1,1,0\}}, \mathcal{W}_{\{1,1,6\}}$	$\sqrt{6}$
$\mathcal{W}_{\{2,0,2\}}, \mathcal{W}_{\{2,0,4\}}$	$2\cot\left(\frac{\pi}{8}\right)$
$\mathcal{W}_{\{1,1,2\}}, \mathcal{W}_{\{1,1,4\}}$	$\sqrt{6}\cot\left(\frac{\pi}{8}\right)$
$\mathcal{W}_{\{2,1,1\}}$	$2\csc\left(\frac{\pi}{8}\right)$

Table 3.1: The twenty-eight Wilson lines and associated quantum dimensions in the $\widehat{\mathcal{T}}_6$ TQFT.

$$\tilde{S}_{\{2,1,3\}^{(i)}\{2,1,3\}^{(j)}} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} , \tag{3.162}$$

where

$$S_{\{\lambda,\mu,\nu\}\{\lambda',\mu',\nu'\}} = S^{(p-2)}_{\lambda\lambda'} \cdot S^{(2)}_{\mu\mu'} \cdot S^{(p)}_{\nu\nu'} , \tag{3.163}$$

is the naive generalization of (3.148) to the cosets at hand. Note that the fusion rules we obtain from $\tilde{S}$ for fields not involving $\{2,1,3\}^{(i)}$ are the naive ones we get from S via the restrictions and identifications described above.

The above discussion is sufficient to prove that $\widehat{\mathcal{T}}_6$ is prime. Indeed, we see from (3.162) that the fields that come from identifying length-two orbits have the quantum dimensions they inherit from S. The fixed point resolution fields, on the other hand, have half the quantum dimension of the fixed point field. We therefore have the following four abelian anyons generating a $\mathbb{Z}_2 \times \mathbb{Z}_2$ fusion algebra

$$\mathcal{W}_{\{0,0,0\}} \simeq \mathcal{W}_{\{4,2,6\}} \ , \ \mathcal{W}_{\{4,0,0\}} \simeq \mathcal{W}_{\{0,2,6\}} \ , \ \mathcal{W}_{\{0,2,0\}} \simeq \mathcal{W}_{\{4,0,6\}} \ , \ \mathcal{W}_{\{0,0,6\}} \simeq \mathcal{W}_{\{4,2,0\}} \ . \tag{3.164}$$

By (3.162), we see that the braiding amongst abelian anyons is not affected by taking $S \to \tilde{S}$. As a result, we see that the four abelian anyons all braid trivially. Therefore, they cannot form a decoupled TQFT.

Given this discussion, what could a putative factorized theory look like? Since $\widehat{\mathcal{T}}_6$ has order $28 = 7 \cdot 2^2$, we see that the only way to have a non-trivial factorization is to have a factorization of the form $\tilde{\mathcal{T}}_{14} \boxtimes \tilde{\mathcal{T}}_2$ into prime TQFTs with rank fourteen and rank two, or $\tilde{\mathcal{T}}_7 \boxtimes \tilde{\mathcal{T}}_4$ with prime TQFTs of rank seven and four, or $\tilde{\mathcal{T}}_7 \boxtimes \tilde{\mathcal{T}}_2 \boxtimes \tilde{\mathcal{T}}'_2$ with prime TQFTs of rank seven, two, and two.

Let us consider the first factorization first. Since the abelian anyons (and any subset thereof) cannot form a separate TQFT factor (this factor would be non-modular), the classification in [46] implies that we have either $\tilde{\mathcal{T}}_2 \simeq (G_2)_1$ or $\tilde{\mathcal{T}}_2 \simeq (F_4)_1$. In any case, the non-trivial anyon in $\tilde{\mathcal{T}}_2$ has quantum dimension $d_\tau = (1+\sqrt{5})/2$. It is easy to check that no such quantum dimension can be produced from products of quantum dimensions in the different coset factors (and so restrictions cannot produce them either). Moreover, one can check that the resolved fixed point fields cannot have this quantum dimension either. This same logic applies to the $\tilde{\mathcal{T}}_7 \boxtimes \tilde{\mathcal{T}}_2 \boxtimes \tilde{\mathcal{T}}_2'$ factorization as well.

Therefore, it only remains to consider $\tilde{\mathcal{T}}_7 \boxtimes \tilde{\mathcal{T}}_4$. The other factor, $\tilde{\mathcal{T}}_4$, has four anyons. By [46], this theory is either $(G_2)_2$ or its time reversal. In either case, we cannot produce the requisite $d_\alpha = 2\cos(\pi/9)$ quantum dimension from our coset. Therefore, we conclude that $\widehat{\mathcal{T}}_6$ is indeed a prime TQFT.

Moreover, we find the following fusion rules of non-abelian Wilson lines with unique outcome

$$\begin{aligned}
\mathcal{W}_{\{2,0,0\}} \times \mathcal{W}_{\{0,0,2\}} &= \mathcal{W}_{\{2,0,2\}} \ , \ \mathcal{W}_{\{2,0,0\}} \times \mathcal{W}_{\{0,0,4\}} = \mathcal{W}_{\{2,0,4\}} \ , \\
\mathcal{W}_{\{1,1,0\}} \times \mathcal{W}_{\{0,0,2\}} &= \mathcal{W}_{\{1,1,2\}} \ , \ \mathcal{W}_{\{1,1,0\}} \times \mathcal{W}_{\{0,0,4\}} = \mathcal{W}_{\{1,1,4\}} \ , \\
\mathcal{W}_{\{0,1,1\}} \times \mathcal{W}_{\{2,0,0\}} &= \mathcal{W}_{\{2,1,1\}} \ . \qquad (3.165)
\end{aligned}$$

We can obtain additional such fusion rules by taking a product with some of the abelian lines in (3.164).

Just as in the case of discrete gauge theories with fusion rules of the above type, our theory also has a non-trivial symmetry of the modular data. Indeed, from (3.162), it is clear that the $\tilde{S}$-matrix has a $\mathbb{Z}_2$ symmetry under the interchange

$$g\left(\mathcal{W}_{\{2,1,3\}^{(1)}}\right) = \mathcal{W}_{\{2,1,3\}^{(2)}} \ , \quad 1 \neq g \in \mathbb{Z}_2 \ . \qquad (3.166)$$

Note that this symmetry is not charge conjugation since $\tilde{S}$ is manifestly real. Moreover, since we don't change the twists, this action lifts to a symmetry of the modular data (additionally, it should lift to a symmetry of the full TQFT).

If we wish to make contact with the $\mathcal{N}=1$ minimal model, then we should note that the fermionic $\mathcal{W}_{\{0,2,0\}}$ line corresponds to the supercurrent of the SCFT. We can then organize the Neveu-Schwarz (NS) sector into supermultiplets under fusion with this operator. Doing so (and paying careful attention to the fields in the resolution of the fixed point), we find nine NS sector fields and nine Ramond sector fields as required.

There are many ways to generalize the example we have given here. Indeed, when there are fixed points in the coset construction we expect to often be able to generate fusion rules of the form (3.2). A deeper understanding of these theories and some more general methods to characterize whether the cosets are prime (along the lines of the general criteria we have in the case of discrete gauge theories) would be useful. In any case, we see that, as in the case of discrete gauge theories, symmetry fixed points and zero-form (quasi) symmetries are deeply connected with fusion rules of the form (3.2).

3.6 Conclusion

In this chapter, we have seen that the existence of fusions of non-abelian anyons having a unique outcome is intimately connected with the global structure of the corresponding TQFT.

Let us summarize our results for continuous gauge groups (and continuous groups more generally):

- Building on the well-known fact that $SU(2)$ spin addition / fusion of two non-abelian representations (i.e., higher-dimensional / spin non-singlet representations) is reducible (i.e., has multiple outcomes with different total spin), we argued that a similar result holds in all compact simple Lie groups.

- We argued that the result in the previous bullet point on classical groups can be extended to a theorem constraining $SU(N)_k$ CS theory: fusions of non-abelian Wilson lines in these theories do not have unique outcomes. More generally, Wilson lines corresponding to affine representations with at least two non-vanishing Dynkin labels in any G_k CS theory (for G a compact simple Lie group) do not have unique outcomes. These results have implications for the global structure of these theories (claim 3.5.1): the Wilson lines discussed here must all lie in the same prime factor (although G_k CS theories are not prime in general).

- We showed that one way to produce $a \times b = c$ fusions involving non-abelian a and b is to consider cosets. In the case of TQFTs underlying Virasoro minimal models we argued that (as in the $(E_7)_2$ case) such rules arise from factorizations of the TQFTs into multiple prime factors. On the other hand, if we include cosets with fixed points, we can obtain prime theories with such fusion rules.

Next, let us summarize our results for discrete gauge groups (and discrete groups more generally):

- We have argued that $2+1$-dimensional discrete gauge theory is useful for putting conjectures and ideas involving finite simple groups into a broader context and unifying various relevant objects. Using this approach, we proved three theorems that TQFT relates to the AH conjecture.

  In fact, we may also generalize the discussion in section 3.3 and show that the AH conjecture implies that, for any twisted or untwisted discrete gauge theory based on a non-abelian finite simple group, fusions of the form

$$\mathcal{L}_{([g],\pi^{\omega}_{g})} \times \mathcal{L}_{([h],\pi^{\omega}_{h})} = \sum_{\pi^{\omega}_{gh}} \mathcal{L}_{([gh],\pi^{\omega}_{gh})} \ , \quad g,h \neq 1 \ , \tag{3.167}$$

are not allowed.

- We argued that Zisser's construction of irreducible products of higher-dimensional irreducible A_N representations [62] can be lifted to fusions of non-abelian Wilson lines with unique outcomes in A_N discrete gauge theory. From the perspective of the closely related S_N group and corresponding discrete gauge theory, the A_N result requires certain 1-form symmetry fixed points (where we define "one-form symmetry" in the S_N group to correspond to the $\mathbb{Z}_2 \subset \text{Rep}(\text{S}_\text{N})$ generated by the sign representation). We then derived theorem 3.4.1 that generalizes this relation between the A_N and S_N discrete gauge theories to other TQFTs.

- Going to the S_N discrete gauge theory by gauging the $\mathbb{Z}_2$ 0-form outer automorphism symmetry of the A_N discrete gauge theory resolves the $a \times b = c$ non-abelian fusion rule into fusion rules not of this type. However, we saw that in the case of $O(5,3)$ discrete gauge theory such resolutions do not always occur via automorphism gauging. On the other hand, a symmetry fixed point again plays a role: in the resulting $O(5,3) \rtimes \mathbb{Z}_2$ discrete gauge theory, there is a 0-form symmetry fixed point. We then proved theorem 3.4.2, which explains why this phenomenon occurs in more general theories. In fact, the $O(5,3) \rtimes \mathbb{Z}_2$ discrete gauge theory relative of the $a \times b = c$ fusion equations in the $O(5,3)$ TQFT described in (3.76) also has a 1-form symmetry fixed point for the anyon appearing on the right hand side. In the original $O(5,3)$ TQFT this latter anyon becomes a set of two anyons related by the 0-form symmetry. Our theorem 3.4.3 generalizes this observation to other TQFTs.

- We showed that one can lift Gallagher's theorem to a statement on the fusion of non-abelian Wilson lines involving unfaithful representations with a unique outcome in TQFT. Moreover, we elucidated the roles that subcategory structure and symmetries play in this result for various specific TQFTs. We then proved theorem 3.4.4 that generalizes these observations to a broader set of theories. We also argued that this subcategory structure helps explain the large ratio of group orders in (3.85).

- To gain a sense of how magnetic fluxes behave in general discrete gauge theories, we proved theorem 3.4.6. In particular, we showed that in discrete gauge theories with a non-abelian gauge group, G, the magnetic fluxes do not form a fusion subcategory. This result immediately places constraints on electric-magnetic self-dualities / quantum symmetries that constrain our symmetry searches later in section 2.

- At a more constructive level, we also proved theorem 3.4.11. This result gives infinitely many generalizations of the well-known electric-magnetic self-duality of the S_3 discrete gauge theory.

- In order to better understand which discrete gauge theories are prime, we proved theorem 3.4.8. This result allowed us to more easily analyze which prime discrete gauge theories have fusions of non-abelian anyons with unique outcomes.

- In order to get a handle on the structure of discrete gauge theories with fusion rules of our desired type involving anyons carrying non-trivial flux, we proved theorem 3.4.12 and corollary 3.4.12.1. These results give the subcategory structure that arises when such fusions occur. In turn, this structure gives an explanation of these fusion rules. Theorem 3.4.13 then partially extends these results to the case in which one of the non-abelian anyons involved is a Wilson line.

- The software GAP was used to analyze the fusion rules of hundreds of untwisted discrete gauge theories. In all the cases we checked, we find that discrete gauge theories with $a \times b = c$ type fusion rules have quasi-zero-form symmetries. This suggests that symmetries of the modular data are a characteristic feature of such fusion rules.

Chapter 4

Galois Conjugation of TQFTs

4.1 Introduction

In Chapter 2, we explored how 2+1D TQFTs are determined by Pentagon and Hexagon equations. Though these constraints are often too complicated to be solved exactly, a general mathematical result (the Ocneanu rigidity theorem) states that there are only a finite number of inequivalent solutions to the Pentagon and Hexagon equations for a given set of fusion rules [42]. This fact allows us to define a Galois group which permutes the solutions to these polynomials. In other words, Galois conjugation is a systematic way to move around the space of TQFTs. Moreover, Galois conjugation is useful in practice: it has played a role in the classification of low-rank TQFTs [46], proving the rank finiteness theorem [94], finding modular isotopes [76], studying low-dimensional lattice models [95–97], in connections between TQFT and other types of QFTs [48, 98–102], and in the study of gapped boundaries [103].

Galois conjugate TQFTs share many important properties. In particular, they have the same fusion rules. However, other observables, like the expectation values of Wilson loop operators, can change under Galois action. Therefore, Galois conjugate TQFTs are typically not dual theories. Still, one can define quantities like multi-boundary entanglement entropy, which are invariant under Galois conjugation in abelian TQFTs,

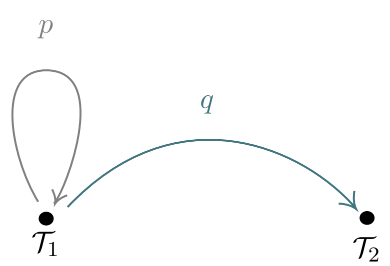


Figure 4.1: Galois conjugation of TQFT $\mathcal{T}_1$ by elements p, q in the Galois group. $\mathcal{T}_1$ is invariant under Galois action by p, but it transforms non-trivially to $\mathcal{T}_2$ under Galois action by q.

and in an infinite set of links in non-abelian TQFTs [1].

In this chapter, we will show that Galois conjugate TQFTs share a lot more structure. More precisely, we will argue that Galois conjugate TQFTs have isomorphic 0-form, 1-form, and 2-group symmetry structure (up to a mild assumption, this result also holds for anti-unitary symmetries). Moreover, there is a well-defined map between the gapped boundaries of Galois conjugate TQFTs. These results show that, compared to other procedures relating distinct TQFTs like gauging, condensation, etc., there is a sense in which Galois conjugation is a particularly mild change to the TQFT.

On the other hand, unlike gauging, Galois actions can map a unitary TQFT to a non-unitary one. While non-unitary TQFTs (and more general non-unitary QFTs) are interesting in their own right, one of the motivations for our work is to better understand when unitary TQFTs are related by a Galois action. In other words, we would like to ask: Given a unitary TQFT, when is a Galois conjugation guaranteed to land on another unitary TQFT? One common way in which this can happen is if we consider a unitary theory without a time-reversal symmetry. In this case, applying time reversal takes us to a different theory that should also be unitary (examples of such phenomena include $SU(2)_1 \leftrightarrow (E_7)_1$ and $SU(3)_1 \leftrightarrow (E_6)_1$ in Chern-Simons theory). This procedure gives a simple example of a Galois action that preserves unitarity, but we will see that the story is more complex and interesting.

Another motivation for our work comes from the observation that several important low-rank unitary TQFTs like the Toric Code, Double Semion, and the 3-Fermion Model are Galois invariant.[56] These examples illustrate that, while most TQFTs transform under a Galois action, a potentially important subset are Galois invariant. This discussion begs the question of what this more general set of unitary "Galois fixed point TQFTs" looks like. As we will see, this set is substantially simpler than its non-unitary counterpart.[57] It also leads to questions of whether this Galois invariance is preserved under other operations like gauging and anyon condensation. We will see that, while Galois invariance is generally preserved under anyon condensation (which includes 1-form symmetry gauging as a special case), it can be violated when 0-form symmetries are gauged. We will prove some general statements about when such anomalous violation is allowed.

The plan of this chapter is as follows. In the next section, we define Galois conjugation of a TQFT and study unitary Galois orbits. We continue with an analysis of Galois actions on various classes of unitary theories: abelian TQFTs, discrete gauge theories, and certain weakly integral MTCs. In section 4.3 we study theories with

[56] Note that by Galois invariant, we do not mean that all the data of the TQFT is invariant. For example, in the Double Semion, the anyon, s, has its twist $\theta_s = i$ Galois conjugated to $g(\theta_s) = -i$. However, the anyon, $\tilde{s}$, has its twist $\theta_{\tilde{s}} = -i$ Galois conjugated to $g(\theta_{\tilde{s}}) = i$. Therefore, this Galois action can be compensated by the time reversal symmetry that exchanges $s \leftrightarrow \tilde{s}$.

[57] Perhaps this relative simplicity hints at even deeper simplifications in the space of unitary TQFTs.

gapped boundaries and explain how Galois conjugation relates gapped boundaries of Galois conjugate TQFTs. In section 4.4 we discuss the relationship between symmetries of Galois conjugate TQFTs. Following this, we look at how Galois conjugation interacts with gauging 0-form symmetries and anyon condensation. We use these results to characterize Galois invariant TQFTs. Section 4.5 contains several additional examples of Galois conjugation of TQFTs which concretely illustrate our ideas and compliment our discussion. Finally, we conclude with some comments.

4.2 Galois Conjugation of TQFTs

Let us consider a $2+1$D TQFT, $\mathcal{T}$, corresponding to an MTC, C. We will think of C as being determined through the action of the F symbols, R symbols, and pivotal coefficients, $\epsilon_a \in \{\pm 1\}$ [58], on the trivalent fusion vertices of the simple objects / anyons [17]. An important subtlety to keep in mind is that this data only determines the total quantum dimension, $\mathcal{D} := \pm\sqrt{\sum_a d_a^2}$, and hence the normalized S in (2.13) up to an overall sign. On the other hand, $\mathcal{D}$ is an important quantity in $\mathcal{T}$. For example, $\mathcal{D} > 0$ is a necessary condition for a unitary TQFT (as follows from positivity of the TQFT inner product [31]). In particular, for a given C, there are two TQFTs, $\mathcal{T}_\pm$, that differ by $\mathcal{D} \to -\mathcal{D}$, $S \to -S$, and $c \to c+4 \pmod 8$ (at least one of these TQFTs must be non-unitary). When the distinction between the two TQFTs is clear from the context or does not matter, we will simply write $\mathcal{T}$.

From this discussion, we can describe the number fields that enter our analysis and set the stage for the appearance of Galois groups. To that end, first construct a field extension, $K'_C = \mathbb{Q}(F, R)$, from the adjunction of the elements of F and R to the field of rational numbers [33] [104]. Using the gauge freedom alluded to above, the authors of [33] showed that there is a gauge in which K'_C is particularly simple: it is a finite field extension.

To understand how this finite field arises, let us consider the case of a system of multivariable polynomial equations over the rational numbers, $p_1(x_1, \cdots, x_n) = \cdots = p_k(x_1, \cdots, x_n) = 0$, with a finite number of solutions. Any solution of this system belongs to a finite extension of $\mathbb{Q}$.[59] On the other hand, the Pentagon and Hexagon equations are multivariable polynomials over $\mathbb{Q}$ with an infinite number of solutions (because of the gauge freedom). Therefore, in this case, we have an algebraic variety, V, in which some points do not belong to a finite field extension of $\mathbb{Q}$. However,

[58] For unitary MTCs, the pivotal coefficients are fixed. Therefore, in this case the F and R symbols completely determine the unitary MTC.

[59] One way to show this statement involves proving that the ideal, I, generated by $p_1, \cdots,\ p_k$ in the polynomial ring $\mathbb{Q}[x_1, \cdots, x_n]$ is zero dimensional. Given a solution $a_1, \cdots, a_n$ to the set of polynomial equations $p_1 = 0, \cdots, p_k = 0$, one then shows that there exists some polynomial $r_i(x_i) \in \mathbb{Q}[x_i]$, $1 \leq i \leq n$, such that $r_i(a_i) = 0$. For more details, see [105].

algebraic points of a complex affine algebraic variety defined over $\overline{\mathbb{Q}}$ are dense in the Zariski topology [106] ($\bar{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$).[60] The upshot is that, given a set of multivariable polynomials with coefficients in $\bar{\mathbb{Q}}$, we can always find solutions that are algebraic. Therefore, there are solutions to the Pentagon and Hexagon equations that are algebraic. To show that all MTCs allow a gauge in which F and R are algebraic, the authors of [33] showed that the gauge freedom acts on V as an algebraic group and that each orbit of this action has an algebraic point.

Next let us discuss how Galois groups enter our story. Recall that, given a number field, we can study its automorphisms. As a simple example, consider the polynomial equation $x^2 = 2$. This is a polynomial over $\mathbb{Q}$, but its solutions are $\pm\sqrt{2} \notin \mathbb{Q}$. To describe these solutions, we can construct the field extension $\mathbb{Q}(\sqrt{2})$, which consists of elements of the form $a + b\sqrt{2}$ where $a, b \in \mathbb{Q}$. Note that the field $\mathbb{Q}(\sqrt{2})$ has an automorphism given by $\sqrt{2} \to -\sqrt{2}$. In particular, any algebraic equation involving the elements of $\mathbb{Q}(\sqrt{2})$ does not change under the exchange $\sqrt{2} \to -\sqrt{2}$. Note that this action permutes the two roots of the polynomial $x^2 = 2$ we started with. In this simple case, this is the only non-trivial permutation of the roots. However, in more general cases, the automorphisms of the number field obtained from the roots of a polynomial may not exhaust all possible permutations of the roots.

Throughout this chapter, we will work in a gauge in which F and R belong to a number field. Now, any finite field extension over $\mathbb{Q}$ is separable. However, it need not be normal. Since normal closures have useful algebraic properties, let us consider the normal closure of K'_C, and call it K_C. Because K_C is normal and separable, it is a Galois field, and we will refer to it as *the defining number field of* C. Note that K_C need not contain the total quantum dimension, $\mathcal{D}$, and therefore need not contain the normalization of the S matrix in (2.13) (it does contain $\mathcal{D}^2$ and $\tilde{S}$).[61]

The authors of [33] conjectured that there is a gauge in which the defining number field of an MTC is cyclotomic (in other words, the number field can be obtained by appending a primitive n^{th} root of unity to $\mathbb{Q}$).[62] Note that this claim does not hold for general fusion categories. For example, the fusion category obtained from the principal even part of the Haagerup subfactor does not admit a gauge in which the defining number field is cyclotomic [107].

Given the above construction, we can act on K_C with some element, q, of the Galois group, $\text{Gal}(K_C)$. Since F and R are elements of K_C, they get acted on by q; we denote the result as $q(F)$ and $q(R)$ respectively. Recall that the automorphisms of the field, K_C, preserve all algebraic equations involving the elements of K_C. The

[60] This result will play an important role in our analysis.

[61] For example, in abelian TQFTs with $\mathbb{Z}_3$ fusion rules (see Table (4.1) for the explicit MTC data), F and R can be chosen to belong to the cyclotomic field $\mathbb{Q}(\xi_3)$, while $\mathcal{D} = \sqrt{3} \notin \mathbb{Q}(\xi_3)$ is only an element of $\mathbb{Q}(\xi_{12})$. Here, ξ_n is a primitive n^{th} root of unity.

[62] To the best of our knowledge, there are no known counterexamples to this conjecture.

Pentagon and Hexagon equations are algebraic equations satisfied by some elements of K_C. Therefore, they are preserved under a Galois action. That is, if F and R satisfies the Pentagon and Hexagon equations, so do $q(F)$ and $q(R)$! Therefore, $q(F)$ and $q(R)$ defines an MTC, which we denote as $q(C)$.

Definition: *We define the Galois action on TQFTs through the Galois action on the defining MTC data, F and R. In particular, we choose not to act to reverse the sign of the total quantum dimension, $\mathcal{D}$, or, equivalently, the sign of the normalization of S (this choice amounts to working with the $(\tilde{S}, T)$ modular pair in (2.13), (2.14)). We lose no generality since, after performing such a Galois action, we can, in principle, consider TQFTs with either sign of $\mathcal{D}$ and normalization of S.*

Various authors have established that the modular data of a TQFT is always contained in a cyclotomic field extension [32,108–111]. In the language of these references, the modular data is given by the pair, $(S, \varphi \cdot T)$, where $\varphi := \exp(-\pi i c/24)$ (here c can essentially be thought of as the central charge of the associated 2D RCFT[63]). Let this cyclotomic extension be $\mathbb{Q}(\xi_{N'})$, where $\xi_{N'}$ is a primitive N'^{th} root of unity. Since our MTC is insensitive to the sign of $\mathcal{D}$, and since we do not consider Galois actions that take $(\mathcal{D}, S) \to (-\mathcal{D}, -S)$, it is more natural to work with the modular data field $\mathbb{Q}(\xi_N)$ (with $N \leq N'$) for $(\tilde{S}, T)$ in (2.13), (2.14).[64]

Let us now connect this discussion with the defining number field. To that end, note that $\mathbb{Q}(\xi_N) \subset K_C$ is a subfield. Now, every element of $q \in \mathbb{Z}_N^\times = \text{Gal}(\mathbb{Q}(\xi_N))$ acts on the modular data to give potentially new modular data, $q(\tilde{S}), q(T)$. Then, for every $q \in \mathbb{Z}_N^\times$ acting on the modular data, we have some $\sigma \in \text{Gal}(K_C)$ such that $\sigma|_{\mathbb{Q}(\xi_N)} = q$. This statement holds because K_C is normal.[65]

As we have seen from the above discussion, Galois conjugation permutes the solutions of the pentagon and hexagon equations. Hence, it relates distinct TQFTs with the same fusion rules. However, as the following example illustrates, there may not be a Galois conjugation relating any two solutions of the Pentagon and Hexagon equations for particular fixed fusion rules:

Example: Consider the Toric Code (a.k.a. $\mathbb{Z}_2$ discrete gauge theory), the 3-Fermion Model (a.k.a. $\text{Spin}(8)_1$ Chern-Simons theory), and Double Semion (a.k.a. $SU(2)_1 \boxtimes (E_6)_1$ Chern-Simons theory or twisted $\mathbb{Z}_2$ discrete gauge theory). All these theories have $\mathbb{Z}_2 \times \mathbb{Z}_2$ fusion rules. For abelian theories (i.e., theories whose

[63] Although, see [111] for a more RCFT-independent discussion.

[64] We have $\mathcal{D}^2 \in \mathbb{Q}(\xi_N)$ since the quantum dimensions are in $\tilde{S}$ but, in general, $\mathcal{D}, \varphi \notin \mathbb{Q}(\xi_N)$ (this last fact follows from the observation in [110,111] that the elements of $\varphi \cdot T$ determine the cyclotomic extension of the modular data).

[65] This discussion explains why it is better to work with the normal field K_C instead of K'_C itself.

fusion rules are abelian groups), it turns out that all the defining data discussed above—the F and R symbols—can be determined in terms of the twists of the anyons, θ_i. Moreover, for abelian theories, we can choose a gauge in which K_C is the number field determined by the twists.[a] For Toric Code, we have anyons $1, e, m, \epsilon$ (where $\epsilon = e \times m$) with twists

$$\theta_1 = \theta_e = \theta_m = 1 \ , \quad \theta_\epsilon = -1 \ , \tag{4.1}$$

while the 3-Fermion Model has anyons $1, f_1, f_2, f_3$ (where $f_3 = f_1 \times f_2$) with twists

$$\theta_1 = 1 \ , \quad \theta_{f_1} = \theta_{f_2} = \theta_{f_3} = -1 \ , \tag{4.2}$$

and Double Semion has anyons $1, s, \tilde{s}, d$ (where $d = s \times \tilde{s}$) with twists

$$\theta_1 = \theta_d = 1 \ , \quad \theta_s = i \ , \quad \theta_{\tilde{s}} = -i \ . \tag{4.3}$$

In the first two cases, $K_C = \mathbb{Q}$, and the corresponding Galois group is trivial. Therefore Toric Code and the 3-Fermion Model are Galois invariant and are not related to each other by a Galois action. In the Double Semion case (4.3), we see that $K_C = \mathbb{Q}(i)$ and so $\text{Gal}(K_C) = \mathbb{Z}_2$ has a non-trivial element implementing complex conjugation. However, complex conjugation exchanges the twists $\theta_s \leftrightarrow \theta_{\tilde{s}}$ while leaving the rest of the data invariant. Therefore, the Double Semion theory is mapped to itself. In summary, all three of these theories share the same fusion rules, but they are unrelated by a Galois action.

[a] This statement follows from (2.17) and (2.18) of [112] along with the fact that the twists are valued in a cyclotomic (and hence Galois) field.

Sometimes, even if a Galois action is non-trivial, it may act as a gauge transformation on the F and R symbols, leaving the theory invariant.

Example: In the previous example, we saw that Galois action acts trivially on the Toric code. Consider a gauge in which the F and R symbols of the Toric code takes the values

$$\begin{aligned} R^1_{\epsilon,\epsilon} &= R^\epsilon_{m,e} = R^e_{m,\epsilon} = R^m_{\epsilon,e} = -1 \ , \quad R^\epsilon_{e,m} = R^1_{e,e} = R^1_{m,m} = 1 \ , \\ F^1_{\epsilon,m,e} &= F^1_{\epsilon,e,m} = -F^1_{m,e,\epsilon} = -F^1_{e,m,\epsilon} = i \ , \end{aligned} \tag{4.4}$$

with all other $F = 1$. Clearly, the Galois group is $\mathbb{Z}_2$ and acts via complex conjugation. However, this action is trivial in abelian group cohomology. Indeed, by rotating the basis vector $\psi \in V^1_{\epsilon,\epsilon}$ as $\psi \to -i\psi$, we find that all $F, R = \pm 1$. Therefore, in this gauge, the Galois group is trivial and so the original $\mathbb{Z}_2$ Galois group leaves the theory invariant.

More generally, we will have theories that transform non-trivially under Galois actions.

Example: $\mathbb{Z}_N$ TQFT (N odd). Let us consider abelian TQFTs with $\mathbb{Z}_N$ fusion rules and N odd. One set of solutions to the hexagon and pentagon equations is

$$F_{j_1,j_2,j_3} = 1 \ , \quad R_{j_1,j_2} = \exp\left(\frac{2\pi i j_1 j_2}{N}\right) \ , \tag{4.5}$$

where $j_i \in \mathbb{Z}_N$. From these quantities, we can build the modular data

$$T_{j_1,j_2} = \delta_{j_1,j_2} \exp\left(\frac{2\pi i j_1 j_2}{N}\right) \ , \quad S_{j_1,j_2} = \frac{1}{\sqrt{N}} R_{j_1,j_2} R_{j_2,j_1} = \frac{1}{\sqrt{N}} \exp\left(\frac{4\pi i j_1 j_2}{N}\right) \ . \tag{4.6}$$

Clearly, for any N, the Galois action is non-trivial (i.e., the above solution always lies in a non-trivial Galois orbit).

As a particularly simple example, consider $N = 3$. In this case, the F and R matrices are all 3rd roots of unity. Therefore, they lie in the number field $\mathbb{Q}(e^{\frac{2\pi i}{3}})$ with Galois group $\mathbb{Z}_3^\times \simeq \mathbb{Z}_2$. The solution (4.5), (4.6), with $N = 3$ plugged in, corresponds to $SU(3)_1$ Chern-Simons theory. The Galois element $2 \in \mathbb{Z}_3^\times$ implements time reversal and produces $(E_6)_1$.[a] Note that $SU(3)_1$ and $(E_6)_1$ are unitary theories and correspond to abelian CS theories.

[a]Note that our definition of the defining number field does not involve the normalization of the S-matrix. Also, Galois action on the TQFT does not involve an action on the normalization of the S matrix. If we include the normalization of the S-matrix in our discussion, then noting that $\sqrt{3} = \exp(2\pi i/12) + \exp(-2\pi i/12)$ makes it clear that the Galois group is $\mathbb{Z}_{12}^\times \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. Acting with $11 \in \mathbb{Z}_{12}^\times$ takes $(S, T) \to (S^*, T^*)$, while the remaining elements $(5, 7 \in \mathbb{Z}_{12}^\times)$ also flip the sign of the normalization of the S matrix.

Example: Fibonacci TQFT $\simeq (G_2)_1$ Chern-Simons. Here we consider a non-abelian example. Let us suppose that there are two simple elements, $\{1, \tau\}$, and that the only non-trivial fusion rule is

$$\tau \otimes \tau = 1 + \tau \ . \tag{4.7}$$

The Fibonacci MTC, which gives rise to $(G_2)_1$ Chern-Simons theory, solves the pentagon and hexagon equations with these fusion rules. The corresponding non-trivial MTC data[a] is

$$\begin{aligned} F^\tau_{\tau\tau\tau} &= \begin{pmatrix} \varphi^{-1} & \varphi^{-1/2} \\ \varphi^{-1/2} & -\varphi^{-1} \end{pmatrix} \ , \quad R^1_{\tau\tau} = \xi^2 \ , \\ R^\tau_{\tau\tau} &= \xi^{-\frac{3}{2}} \ , \ \varphi = \frac{1}{2}(1+\sqrt{5}) = \xi^{-1} + 1 + \xi \ , \ \xi = \exp\left(\frac{2\pi i}{5}\right) \ . \end{aligned} \tag{4.8}$$

From these quantities, one can construct the modular data

$$S = \frac{1}{\sqrt{2+\varphi}}\begin{pmatrix} 1 & \varphi \\ \varphi & -1 \end{pmatrix} \quad , \quad T = \mathrm{diag}\left(1, \exp\left(\frac{4\pi i}{5}\right)\right) \ . \tag{4.9}$$

Since all MTCs with Fibonacci type fusion rules are determined by its modular data, without losing generality, we can look at the number field containing the modular data and find its Galois conjugates. The entries in the S and T matrices lie in the number field $\mathbb{Q}(e^{\frac{2\pi i}{5}})$ with Galois group $\mathbb{Z}_5^\times \simeq \mathbb{Z}_4$. Acting with $4 \in \mathbb{Z}_5^\times$ takes $\xi \to \xi^4 = \bar{\xi}$ while leaving S invariant and corresponds to time reversal. This transformation takes us to $(F_4)_1$. On the other hand, acting with $3, 2 \in \mathbb{Z}_5^\times$ gives the Lee-Yang and conjugate Lee-Yang MTCs respectively.[b]

[a] All MTC data not explicitly mentioned is equal to 1.

[b] If we include the normalization of the S-matrix in the defining number field, we get the following Galois orbits. Writing $\sqrt{2+\varphi} = \exp(2\pi i/20) - \exp(2\pi i 9/20)$, we see that $\mathbb{Z}_{20}^\times$ Galois group acts on the modular data. The elements of $\mathbb{Z}_{20}^\times$ are

$$\mathbb{Z}_{20}^\times \simeq \mathbb{Z}_4 \times \mathbb{Z}_2 = \{1, 11\} \times \{1, 3, 7, 9\} \tag{4.10}$$

Acting with $19 \in \mathbb{Z}_{20}^\times$ takes $\xi \to \xi^{19} = \bar{\xi}$ while leaving S invariant and corresponds to time reversal. This transformation takes us to $(F_4)_1$. On the other hand, acting with $13, 7 \in \mathbb{Z}_4 \times \mathbb{Z}_2$ gives the Lee-Yang and conjugate Lee-Yang MTCs respectively (the remaining transformations give other theories related to the ones mentioned here by $S \to -S$).

In fact, it is possible that Galois actions supplemented by some other procedure act transitively on the solutions of the Pentagon and Hexagon equations. For example, Galois conjugations along with a particular change of F symbols act transitively on all MTCs with the same fusion rules as $SU(N)_k$ Chern-Simons theory [33].

Given our discussion of Galois conjugation, we would like to study how these operations interact with global properties of TQFT. In particular, our immediate goal is to understand how Galois conjugation affects the subcategory structure of C (and therefore $\mathcal{T}$).

To that end, note that a proper subset of anyons in C may close under fusion and therefore form a braided fusion subcategory whose F and R symbols are given by the restriction of the F and R symbols of C onto that subcategory. Since Galois conjugation preserves fusion rules, it is clear that it preserves the braided fusion subcategory structure of a modular tensor category. This observation will play a crucial role in our analysis of discrete gauge theories.

The braiding in a subcategory may or may not be degenerate (i.e., the corresponding modular $\tilde{S}$ matrix may or may not be degenerate). If it is non-degenerate, then a general result of Müger [49] guarantees that the subcategory factorizes from the rest of the theory (i.e., anyons in the subcategory braid trivially with anyons outside the subcategory). In this case, our TQFT has a product structure

$$\mathcal{T} = \boxtimes_i \mathcal{T}_i \ , \tag{4.11}$$

where each $\mathcal{T}_i$ has a corresponding MTC C_i.[66] The decomposition in (4.11) is called a "prime decomposition" into prime factors $\mathcal{T}_i$ (each factor braiding trivially with other factors).

A basic question is then to understand the Galois action on the prime factors:

Theorem 4.2.1 *The space of prime TQFTs is closed under Galois action.*

Proof: Consider a non-prime TQFT, $\mathcal{T}$. The associated MTC, C, has a modular subcategory, K. The set of anyons in K label a modular sub-matrix, $\tilde{S}_K$, of the $\tilde{S}$ matrix of C. Suppose $\mathbb{Q}(\xi_N)$ is the cyclotomic field containing the elements of the $\tilde{S}$ matrix, where $\xi_N := \exp(2\pi i/N)$. The cyclotomic field $\mathbb{Q}(\xi_N)$ has a cyclotomic subfield $\mathbb{Q}(\xi_{N_K})$ which contains the elements of the matrix $\tilde{S}_K$. Any element of $\mathrm{Gal}(\mathbb{Q}(\xi_N))$ restricts to a Galois action on $\mathbb{Q}(\xi_{N_K})$. Hence, under a Galois conjugation of the $\tilde{S}$ matrix, the modular sub-matrix $\tilde{S}_K$ gets transformed into another modular matrix. Therefore, the set of anyons in K forms a modular subcategory of the Galois-conjugated theory. As a result, Galois conjugation of a non-prime TQFT results in a non-prime TQFT. From invertibility of the Galois action it is clear that the space of prime TQFTs is closed under Galois conjugation.[67] □

As a result, the Galois action on $\mathcal{T}$ in (4.11) can be obtained from the Galois action on the prime theories, $\mathcal{T}_i$. The notion of primeness is independent of whether the TQFT is unitary or not. Note that the prime factorization of TQFTs into Deligne products described above is not always unique. Even the number of prime TQFTs in a prime factorization is not always unique. For example, Toric Code ⊠ Semion = Semion ⊠ $\overline{\text{Semion}}$ ⊠ Semion.

If a TQFT, $\mathcal{T}$, transforms non-trivially under Galois action, then it is clear that at least one of its prime factors should transform non-trivially under it. However, non-trivial Galois transformation of the prime factors of a TQFT may act trivially on the full TQFT. Indeed, this is the case in the Double Semion example discussed previously since it turns out that

$$\text{Double Semion} = \text{Semion} \boxtimes \overline{\text{Semion}} \ . \tag{4.12}$$

As we saw above, both the Semion and $\overline{\text{Semion}}$ models transform non-trivially under Galois action (the twists $\theta_s = i$ from the Semion model and $\theta_{\tilde{s}} = -i$ from $\overline{\text{Semion}}$ are

[66] The symbol "⊠" denotes the so-called "Deligne" product and is an appropriate categorical generalization of a direct product.

[67] This result can also be seen from the fact that Galois conjugations preserve invertibility of a matrix.

complex conjugated). However, since these two factors transform into each other under Galois action, the Double Semion model is invariant under all Galois conjugations. As we will see in Section 4.3, the Galois invariance of Double Semion model can also be explained using the Galois invariance of its gapped boundary.

4.2.1 Unitary Galois Orbits

The preceding discussion was very general and applies to all types of Galois transformations, including those that transform unitary theories into non-unitary ones (and vice-versa). However, on physical grounds, it is important to understand the conditions under which Galois transformations preserve unitarity. To that end, in this section we will obtain a sufficient condition for unitarity preservation.

Let us begin by building up to the extra constraints that a unitary MTC should satisfy. We will call a fusion category unitary if there exists a gauge in which the F symbols are unitary.[68] A braided fusion category is unitary if there exists a gauge in which both the F and R symbols are unitary. The condition on R is not an extra constraint, since any set of consistent R symbols obtained from unitary F symbols is unitary [44]. Therefore, every braided fusion category defined over a unitary fusion category is unitary. Note that to define quantum dimensions, we need to add a ribbon structure to a braided fusion category. In general, there is more than one inequivalent choice for the ribbon structure. However, there is a unique choice which guarantees that all the quantum dimensions are positive [44]. A ribbon fusion category is called unitary if there exists a gauge in which the F and R symbols are unitary and if all quantum dimensions are positive.

Given this discussion, we see that a sufficient and necessary condition for an MTC, C, to be unitary is that it has a gauge in which the F and R symbols are unitary and the quantum dimensions satisfy $d_a > 0$, $\forall a \in C$ (here a is an anyon of the TQFT or a simple object of C) [115].[69]

From a unitary MTC, we can always construct a unitary TQFT by choosing the total quantum dimension to be positive (i.e., $\mathcal{D} > 0$). Indeed, the corresponding TQFT inner product is then positive definite [31]. On the other hand, starting from a non-unitary MTC, we cannot construct a unitary TQFT.

Since making unitarity manifest requires choosing a particular gauge, it is useful to

[68] There is a gauge-independent definition of unitarity of a fusion category. But we will use the definition in terms of the F matrices since both are equivalent [113]. Given a set of labels and its fusion rules, a necessary condition for a unitary fusion category with these fusion rules to exist is given in [114].

[69] In an MTC without unitarity, the F and R symbols are defined only up to gauge transformations. In a unitary MTC, the unitary F and R matrices are defined only up to unitary gauge transformations. Moreover, if the F and R matrices can be made unitary in two different gauges, then they are unitarily gauge equivalent [116]. Therefore, the unitary structure on an MTC is unique. Since all anyons have a dual, $d_a > 0 \implies d_a \geq 1$

check that this choice does not clash with the gauge choice required for the MTC data to belong to a finite field extension, K_C. In fact, the authors of [115] showed that there is a gauge in which both can be achieved simultaneously.

Now, given a finite field extension, K_C, in which the F symbols of C are unitary, determining whether a Galois conjugation of a unitary MTC results in a unitary MTC depends on how the F symbols and quantum dimensions get transformed under the Galois action. We will call a Galois action which takes a unitary MTC to a unitary MTC a unitarity-preserving Galois action. This construction also maps a unitary TQFT to a unitary TQFT since we can always supplement our MTC action with a choice of $\mathcal{D} > 0$. Before looking at the F symbols, let us study the action of a unitarity-preserving Galois action on the quantum dimensions.

Lemma 4.2.2 *A unitarity-preserving Galois action acts trivially on the quantum dimensions.*

Proof: Consider a unitary TQFT, $\mathcal{T}$, with associated unitary MTC, C, having defining number field K_C. Let $q(C)$ be a unitary MTC (with corresponding unitary TQFT, $q(\mathcal{T})$), where $q(C)$ is the Galois conjugate of C with respect to some $q \in \mathrm{Gal}(K_C)$. Since C is unitary, the quantum dimension, d_a, of an anyon $a \in C$ is equal to the corresponding Frobenius-Perron dimension and is positive. We denote $q(d_a)$ as the quantum dimension of the corresponding anyon in $q(C)$. Since $q(C)$ is unitary, $q(d_a)$ are also positive. By proposition 3.3.4 of [43],

$$|q(d_a)| \leq d_a \ . \tag{4.13}$$

Suppose $q(d_a) < d_a$ for some anyon a. Using the inverse Galois action, we have $\bar{q}(d_a) > d_a$. This contradicts (4.13). Therefore, we must have

$$q(d_a) = d_a \ \forall a \ . \tag{4.14}$$

□

As a result of the above lemma, invariance of the quantum dimensions under Galois action is necessary for preserving unitarity. However, this is not sufficient. To see this, let us study how Galois conjugation changes the F symbols. In general, Galois conjugation does not preserve unitarity of a matrix. To understand this statement, suppose we have some unitary matrix, U, such that the elements of the matrix belong to an algebraic number field, K. U satisfies $U^\dagger U = I$. Galois conjugating this relation which respect to some $q \in \mathrm{Gal}(K)$ gives

$$q(U^\dagger U = I) \implies q(U^\dagger)q(U) = I \ . \tag{4.15}$$

If complex conjugation commutes with q, then the above equation simplifies to

$$q(U)^{\dagger} q(U) = I \; . \tag{4.16}$$

Therefore, the Galois conjugated matrix is still unitary. However, it often happens that complex conjugation does not commute with the Galois action. In this case, $q(U)$ is non-unitary.

All MTCs conjecturally have a gauge in which the defining number field is cyclotomic [33], and in this case any Galois conjugation commutes with complex conjugation. However, a unitary TQFT in such a gauge may not have unitary F symbols. The simplest example of this is Galois conjugation of the Fibonacci model to get the Yang-Lee model. In the Fibonacci model, there is a basis in which F and R are unitary. However, in this basis, F and R symbols belong to a field extension which has a non-abelian Galois group. On the other hand, if we choose a gauge in the F and R symbols are in a cyclotomic field, F symbols become non-unitary. This example is studied in detail in [104, 115].

It is clear from this discussion that if there is a gauge in which the unitary F and R symbols of a unitary TQFT are real, then any Galois conjugation will result in unitary F and R matrices. In this case, the defining number field, K_C, is called "totally real." A more general statement holds if the defining number field is a CM field (note: all cyclotomic fields are CM fields, although the converse is not true). A CM field is a quadratic extension of a totally real field. In other words, a CM field, K, is of the form $H(\alpha)$, where H is a totally real field such that K is complex (i.e., it cannot be embedded as a subfield of $\mathbb{R}$). A simple example is the cyclotomic field (appearing in the Double Semion discussed above), $\mathbb{Q}(i)$, which contains numbers of the form $a + ib$ where $a, b \in \mathbb{Q}$. A CM field has the property that complex conjugation is in the center of the Galois group. In fact, any number field with complex conjugation in the center of the Galois group should either be a totally real field (in which case complex conjugation acts trivially) or a CM field [117]. This discussion leads to the following result:

Theorem 4.2.3 *Let C be a unitary MTC, and let K_C be its defining number field. Let K_F be the Galois field obtained from the normal closure of the F symbols added to the rationals. If there is a gauge in which the F symbols are unitary and K_F is a totally real field or a CM field, then any Galois conjugation which acts trivially on the quantum dimensions results in a unitary TQFT.*

Proof: If K_F is a totally real field, then complex conjugation acts trivially on the F symbols. Any Galois conjugation $q \in \mathrm{Gal}(K_C)$ takes unitary F symbols to unitary F symbols. Therefore, the Galois conjugate TQFT has unitary F symbols. From [44], the R matrices of the Galois conjugate TQFT should be unitary. Now suppose the Galois conjugation acts trivially on the quantum dimensions, then the Galois conjugate TQFT has positive quantum dimensions. It follows that the resulting TQFT is unitary.

If K_F is a CM field, then complex conjugation is in the center of the Galois group $\mathrm{Gal}(K_F)$. Therefore, the unitarity of the F symbols is preserved under Galois action with respect to any $q \in \mathrm{Gal}(K_C)$. If the quantum dimensions are invariant under Galois action, then the Galois conjugate TQFT has positive quantum dimensions. It follows that the resulting MTC is unitary, and we can therefore also take the corresponding TQFT to be unitary (we must choose positive total quantum dimension). □

For TQFTs described by integral MTCs, the quantum dimensions are integers and hence Galois invariant. Therefore, any Galois action which preserves the unitarity of the F symbols gives us a unitary Galois conjugate MTC and hence (by a choice of $\mathcal{D}$) a unitary TQFT. For example, in abelian TQFTs, there exists a gauge in which the F symbols belong to $\{\pm 1\}$ [112]. Therefore, in this case $K_F = \mathbb{Q}$ is a totally real field and any such Galois action preserves unitarity.

In [17], Wang conjectures that a ribbon fusion category (and hence an MTC) with positive quantum dimensions is unitary. If this conjecture is true, then our lemma 4.2.2 alone is enough to characterize unitary Galois orbits. That is, any Galois action of the type we consider which acts trivially on the quantum dimensions of a unitary TQFT results in a unitary TQFT.

In the next subsection, we will study Galois actions on abelian TQFTs. These provide the simplest example of unitary Galois orbits.

4.2.2 Abelian TQFTs and Unitary Galois Orbits

In this section we study abelian TQFTs (i.e., theories whose fusion rules are those of a finite abelian group) and the corresponding Galois orbits. As is well-known, a TQFT is abelian if and only if the quantum dimensions of all anyons are equal to 1. Since Galois conjugation preserves integers, abelian TQFTs are closed under this action. Moreover, since the F and R matrices of an abelian theory are phases [112], any abelian MTC is unitary and has a cyclotomic defining number field (by choosing $\mathcal{D} > 0$ as described above, we restrict our attention to unitary abelian TQFTs). Therefore, Galois conjugation of such an abelian TQFT always preserves unitarity, and so we will leave the unitary nature of these theories implicit in what follows.

Our strategy below consists of noting that general abelian TQFTs can be written as Deligne products of prime abelian TQFTs. Galois conjugation of an abelian TQFT can thus be reduced to describing the Galois conjugation of prime TQFTs. Moreover, by the discussion in footnote *a* and the surrounding text, for abelian theories the defining number field can be taken to be the cyclotomic field of the twists.

Table 4.1 gives the classification of prime abelian TQFTs [118] in one particular description of the underlying defining twist data. As we will see when we study Galois actions on these theories, there can be dual descriptions as well.

Theory	Fusion rules	Twists
A_{p^r}	$\mathbb{Z}_{p^r}$	$\theta_a = e^{\frac{2\pi i 2a^2}{p^r}}$
B_{p^r}	$\mathbb{Z}_{p^r}$	$\theta_a = e^{\frac{2\pi i a^2}{p^r}}$
A_{2^r}	$\mathbb{Z}_{2^r}$	$\theta_a = e^{\frac{2\pi i a^2}{2^{r+1}}}$
B_{2^r}	$\mathbb{Z}_{2^r}$	$\theta_a = e^{\frac{-2\pi i a^2}{2^{r+1}}}$
C_{2^r}	$\mathbb{Z}_{2^r}$	$\theta_a = e^{\frac{2\pi i 5a^2}{2^{r+1}}}$
D_{2^r}	$\mathbb{Z}_{2^r}$	$\theta_a = e^{\frac{-2\pi i 5a^2}{2^{r+1}}}$
E_{2^r}	$\mathbb{Z}_{2^r} \times \mathbb{Z}_{2^r}$	$\theta_{(m,n)} = e^{\frac{2\pi i mn}{2^r}}$
F_{2^r}	$\mathbb{Z}_{2^r} \times \mathbb{Z}_{2^r}$	$\theta_{(m,n)} = e^{\frac{2\pi i(m^2+n^2+mn)}{2^r}}$

Table 4.1: Classification of prime abelian TQFTs.

Since we know that the space of abelian TQFTs and the space of prime TQFTs is closed under Galois action (theorem 4.2.1), a prime abelian TQFT should either be invariant or get transformed into another prime abelian TQFT under a Galois conjugation. Consider the A_{p^r} prime abelian TQFT. Since the twists are p^r-roots of unity, the cyclotomic field containing the data of this theory is $\mathbb{Q}(\xi_{p^r})$ with Galois group $\mathbb{Z}_{p^r}^\times$ (the multiplicative group of integers mod p^r).

Under a Galois action corresponding to some $q \in \mathbb{Z}_{p^r}^\times$ there are two possibilities, A_{p^r} remains invariant or $A_{p^r} \to B_{p^r}$. We can consider two cases. Suppose q mod $p^r = \alpha^2$ for some α. Then,

$$\theta_a = e^{\frac{2\pi i 2a^2}{p^r}} \to e^{\frac{2\pi i 2\alpha^2 a^2}{p^r}} = e^{\frac{2\pi i 2(\alpha a)^2}{p^r}} = \theta_{\alpha a \bmod p^r} \ . \tag{4.17}$$

Hence, the Galois conjugation in this case can be interpreted as a permutation of anyons in the theory given by αa mod p^r. Moreover, this permutation preserves the fusion rules. Therefore, this is a dual description of the same theory. Now suppose q is not a quadratic residue mod p^r, then it is clear that $4q$ is also not a quadratic residue mod p^r. As a result, we have

$$\theta_a = e^{\frac{2\pi i 2a^2}{p^r}} \to e^{\frac{2\pi i 2qa^2}{p^r}} \ , \tag{4.18}$$

where the resulting twists are those of the B_{p^r} theory (since any integer which is not a quadratic residue mod p^r defines the same theory). So we can summarize the Galois action on A_{p^r} as follows

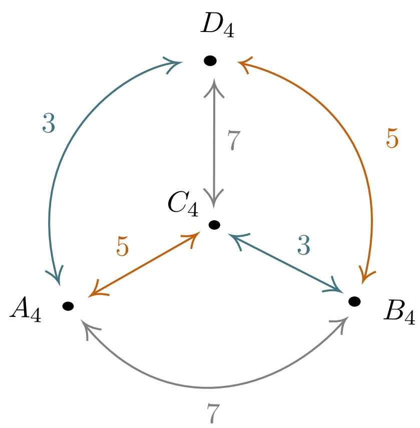
$$\alpha^2 = q \bmod p^r : A_{p^r} \to A_{p^r} \ , \quad \alpha^2 \neq q \bmod p^r : A_{p^r} \to B_{p^r} \ , \tag{4.19}$$

for some integer α.

Example: For the A_5 theory, we have Galois conjugations corresponding to $q = 1, 2, 3, 4$ (the Galois group is $\mathbb{Z}_4$). $q = 4$ is a duality, while for $q = 2, 3$ we have $A_5 \to B_5$.

Now let us consider Galois conjugation of the A_{2^r} theory. Since the roots are 2^{r+1}-roots of unity, the cyclotomic field containing all the data of the theory is $\mathbb{Q}(\xi_{2^{r+1}})$ with Galois group $\mathbb{Z}^\times_{2^{r+1}} = \{\text{all odd integers} < 2^{r+1}\}$. Before discussing the general pattern of Galois action on A_{2^r} theory, let us discuss an example.

Example: Consider the A_4 theory. We have Galois conjugations corresponding to $q = 1, 3, 5, 7$ constituting the Klein four-group. We have the following transformations forming the edges of a tetrahedron:



The Galois action on a general A_{2^r} theory by some $q \in \mathbb{Z}^\times_{2^{r+1}}$ depends on the nature of $q \bmod 2^{r+1}$. Suppose $\alpha^2 = q \bmod 2^{r+1}$. Using Hensel's lemma, this has a solution if and only if $q = 1 \bmod 8$. Since q is odd, α has to be odd, and $\gcd(\alpha, 2^r) = 1$. Therefore, if $\alpha^2 = q \bmod 2^{r+1}$, then this Galois action acts as an automorphism of the fusion rules $\mathbb{Z}_{2^r}$ given by $a \to \alpha a \bmod 2^r$, $a \in \mathbb{Z}_{2^r}$. Similarly, if $-1\alpha^2 = q \bmod 2^{r+1}$, then the Galois action transforms the twists of A_{2^r} to B_{2^r} family of prime abelian theories up to an automorphism of the fusion rules given by α.

Most generally, Galois conjugations permute the prime theories $A_{2^r}, B_{2^r}, C_{2^r}$, and

D_{2^r} as follows:

$$\alpha^2 = q \bmod 2^{r+1} : A_{2^r} \to A_{2^r}, B_{2^r} \to B_{2^r}, C_{2^r} \to C_{2^r}, D_{2^r} \to D_{2^r} \ , \tag{4.20}$$

$$-1\alpha^2 = q \bmod 2^{r+1} : A_{2^r} \to B_{2^r}, B_{2^r} \to A_{2^r}, C_{2^r} \to D_{2^r}, D_{2^r} \to C_{2^r} \ , \tag{4.21}$$

$$5\alpha^2 = q \bmod 2^{r+1} : A_{2^r} \to C_{2^r}, B_{2^r} \to D_{2^r}, C_{2^r} \to A_{2^r}, D_{2^r} \to B_{2^r} \ , \tag{4.22}$$

$$-5\alpha^2 = q \bmod 2^{r+1} : A_{2^r} \to D_{2^r}, B_{2^r} \to C_{2^r}, C_{2^r} \to B_{2^r}, D_{2^r} \to A_{2^r} \ . \tag{4.23}$$

Now let us consider Galois action on E_{2^r} theories. From the twists, it is clear that the defining number field is $\mathbb{Q}(\xi_{2^r})$ with Galois group $\mathbb{Z}_{2^r}^\times = \{\text{all odd integers} < 2^r\}$. Under a Galois action corresponding to q, we have

$$\theta_{(m,n)} = e^{\frac{2\pi i m n}{2^r}} \to e^{\frac{2\pi i q m n}{2^r}} = e^{\frac{2\pi i (qm) n}{2^r}} = \theta_{(qm \bmod 2^r, n)} \ . \tag{4.24}$$

So Galois conjugation with respect to any q can be interpreted as a permutation of the anyons given by $(m,n) \to (qm \bmod 2^r, n)$. In fact, this is an automorphism of the fusion rules, $\mathbb{Z}_{2^r} \times \mathbb{Z}_{2^r}$. Therefore, we see that the Galois conjugate of E_{2^r} corresponds to a dual description of the same theory.

The Galois invariance of E_{2^r} can also be deduced from the existence of a Lagrangian subcategory. To understand this statement, first note that, given an abelian group G, we can construct an abelian TQFT with fusion rules $G \times \hat{G}$. Here the anyons are labelled by (g, χ) where $g \in G$, and $\chi \in \hat{G}$ is a character of an irreducible representation belonging to the character group $\hat{G}$ of G. The twist of the anyon (g,χ) is $\theta_{(g,\chi)} = \chi(g)$. In fact, this construction gives the untwisted discrete gauge theory based on the abelian group G. Indeed, the E_{2^r} family of prime abelian TQFTs are untwisted $\mathbb{Z}_{2^r}$ discrete gauge theories.

Now, the Lagrangian subcategory arises as follows: we have the anyons $(0, g)$ for any $g \in \mathbb{Z}_{2^r}$ which are all bosons. These form a subcategory of E_{2^r} equivalent to the symmetric tensor category $\text{Rep}(\mathbb{Z}_{2^r})$ (a symmetric subcategory is characterized by completely trivial braiding). Moreover, note that $\dim(\text{Rep}(\mathbb{Z}_{2^r}))^2 = \dim(E_{2^r})$ (a subcategory of bosons satisfying this constraint is called a Lagrangian subcategory). A Galois conjugation of E_{2^r} must result in a prime TQFT with a $\text{Rep}(\mathbb{Z}_{2^r})$ Lagrangian subcategory. However, the E_{2^r} TQFTs are the only prime abelian TQFTs with a $\text{Rep}(\mathbb{Z}_{2^r})$ Lagrangian subcategory. Hence, E_{2^r} TQFTs are mapped to themselves under Galois conjugation (i.e., they are unitary Galois fixed point TQFTs).

It is clear that F_{2^r} theories are also unitary Galois fixed point TQFTs. This invariance follows from the fact that the only possibility for F_{2^r} to transform to another prime theory is for it to get transformed into E_{2^r} theory. However, E_{2^r} and F_{2^r} have different numbers of bosons. Indeed, an anyon, (m,n), of F_{2^r} theory is a boson if and only if it satisfies

$$\theta_{(m,n)} = 1 \implies m^2 + n^2 + mn = 0 \text{ mod } 2^r \; . \tag{4.25}$$

It is clear that if $m^2 = 0 \text{ mod } 2^r$, then $(0, m)$ and $(m, 0)$ are bosons. In fact, (m, n) is a boson in F_{2^r} if and only if $m^2 = 0 \text{ mod } 2^r$ and $n^2 = 0 \text{ mod } 2^r$. Note that for (m, n) to be a boson, both m and n should be even. Let $m = 2m_1$ and $n = 2n_2$ for some integers m_1, n_1. If (m, n) is boson, then

$$m^2 + n^2 + mn = 0 \text{ mod } 2^r \implies m_1^2 + n_1^2 + m_1 n_1 = 0 \text{ mod } 2^{r-2} \; . \tag{4.26}$$

This constraint is satisfied only if m_1 and n_1 are even. Therefore, we can choose $m_1 = 2m_2$ and $n_1 = 2n_2$ for some integers m_2, n_2. Iterating this process, we find that both m^2 and n^2 should be multiples of 2^r. Given a boson (m, n) of F_{2^r} TQFT, note that it is also a boson of E_{2^r} theory, since $mn =$ mod 2^r. However, there are clearly more bosons in E_{2^r} than in F_{2^r}. For example, $(0, m)$ for any m is a boson in E_{2^r} TQFT while this is true for F_{2^r} only if $m^2 = 0 \text{ mod } 2^r$. Since Galois conjugations preserve the number of bosons, F_{2^r} cannot transform into an E_{2^r} theory. Note that a boson (m, n) in F_{2^r} theory has order (under fusion) strictly less than 2^r. Hence, F_{2^r} does not have a $\text{Rep}(\mathbb{Z}_{2^r})$ Lagrangian subcategory.

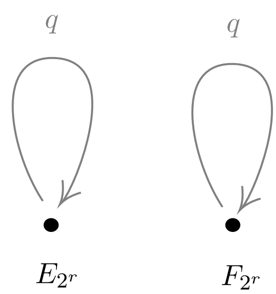


Figure 4.2: The E_{2^r} and F_{2^r} families of prime abelian TQFTs are invariant under Galois conjugation.

Therefore, we find that E_{2^r} and F_{2^r} are the only prime abelian TQFTs which are invariant under Galois conjugation. We found that the Galois invariance of these theories can be explained using their bosonic substructure. Discrete gauge theories, generalizing the abelian E_{2^r} cases, are another class of TQFTs largely determined by their bosonic substructure. In the next section, we will explore Galois actions on these theories. We will find that, similarly to the E_{2^r} and F_{2^r} prime abelian TQFTs, the transformation of discrete gauge theories under Galois conjugation is heavily constrained by the presence of certain bosons.

4.2.3 Discrete Gauge Theories

Since Galois conjugation fixes rational numbers, it is clear that the space of integral MTCs (i.e., theories whose anyons all have integer quantum dimensions) is closed under it. An important class of integral MTCs are (twisted) discrete gauge theories (see [2,3] for a recent discussion of these theories, their subcategory structure, and their fusion rules). Since there are integral MTCs that are not (twisted) discrete gauge theories [119], we might naively imagine that these theories mix with discrete gauge theories under Galois conjugation. We will argue below that this is not the case and that the space of discrete gauge theories is therefore closed under Galois conjugation.

Before discussing discrete gauge theories, let us recall some notions which will be useful for our discussion. Two anyons a and b are said to centralize each other if $S_{ab} = \frac{1}{\mathcal{D}} d_a d_b$. This is the statement that the braiding between a and b is trivial (the Hopf link can be replaced by two disjoint circles labelled by a and b). This notion can be used to define the centralizer for a fusion subcategory D as the fusion subcategory, $D^{'}$, where any $b \in D'$ centralizes any $a \in D$. It is clear that the fusion subcategory D is symmetric (i.e., has completely trivial braiding) if and only if $D \subseteq D'$. A fusion subcategory is called isotropic if all its anyons are bosons. An isotropic subcategory $D \subset C$ is called Lagrangian if $D^{'} = D$, or equivalently $\dim(D)^2 = \dim(C)$.

A twisted discrete gauge theory, $\mathcal{Z}(\mathrm{Vec}_G^{\omega})$, with Dijkgraaf-Witten twist, $\omega \in H^3(G, U(1))$, has $\mathrm{Rep}(G)$ as a fusion subcategory. $\mathrm{Rep}(G)$ has the irreducible representations of G as its simple objects, the representation semi-ring of G as the fusion rules, and F symbols given by the $6j$ symbols. $\mathrm{Rep}(G)$ is a symmetric fusion subcategory where we have

$$R^c_{ab} R^c_{ba} = 1; \quad S_{ab} = \frac{1}{\mathcal{D}} d_a d_b; \quad \theta_a = 1 \quad \forall\, a, b, c \in \mathrm{Rep}(G)\ . \tag{4.27}$$

In fact, if a fusion subcategory only has bosons in it, it is a symmetric fusion category and is gauge equivalent to $\mathrm{Rep}(H)$ for some group H [67]. An important property of $\mathrm{Rep}(G)$ that will be crucial for our discussion is that it is Lagrangian. We have $\dim(\mathrm{Rep}(G))^2 = |G|^2 = \dim(\mathcal{Z}(\mathrm{Vec}_G^{\omega}))$. It is clear that under Galois conjugation of $\mathcal{Z}(\mathrm{Vec}_G^{\omega})$, $\mathrm{Rep}(G)$ will transform into a symmetric fusion category. Given anyons a and b in a modular tensor category C that centralize each other, the corresponding anyons in the Galois conjugate theory also centralize each other. Therefore, Galois conjugation of a discrete gauge theory results in a modular tensor category that has a Lagrangian subcategory. Hence, we have the following result:

Lemma 4.2.4 *The space of twisted discrete gauge theories is closed under Galois conjugation.*

This result holds because an MTC corresponds to a discrete gauge theory if and only if it has a Lagrangian subcategory [43]. The invertibility of Galois conjugation then implies that the Galois conjugation of a non-discrete gauge theory (for example, those originating from quantum groups) should result in a non-discrete gauge theory.[70] To determine how Galois conjugation affects the gauge group and twist of a discrete gauge theory, we have to study the behavior of Rep(G) under Galois action.

Galois Conjugation of Rep(G)

The discussion in the previous subsection shows that the Galois action on Rep(G) results in a symmetric tensor category, Rep(H), for some finite group H. We will find that $G \cong H$ follows from the algebraic nature of the Tannaka-Krein reconstruction theorem.

In Tannaka-Krein reconstruction, the group is reconstructed from a subgroup of the group of endomorphisms of the vector spaces on which the representations act. More precisely, consider a fiber functor (i.e., a monoidal functor to Vec)

$$F : \mathrm{Rep}(G) \to \mathrm{Vec} \ , \tag{4.28}$$

$$\pi \to V_{\pi'} \ , \tag{4.29}$$

where π is a representation of G (in general, reducible). This map can be thought of as forgetting all information about the category Rep(G) except for the vector spaces on which the irreducible representations act. F is a monoidal functor. That is, there exists a natural transformation

$$\mu_{\pi,\pi'} : F(\pi) \otimes F(\pi') \to F(\pi \otimes \pi') \ \forall \ \pi, \pi' \in \mathrm{Rep}(G) \ , \tag{4.30}$$

which is consistent with associativity of Rep(G). $\mu_{\pi,\pi'}$ are simply the basis transformation matrices between the isomorphic vector spaces $V_\pi \otimes V_{\pi'}$ and $V_{\pi\otimes\pi'}$. The former has a natural tensor product basis, and the latter has a basis given by the decomposition of $\pi \otimes \pi'$ into irreducible representations. In other words, $\mu_{\pi,\pi'}$ are determined by the 3j symbols.

Recall that two functors can be related by a natural transformation. A natural automorphism is a natural isomorphism between the same functor; it can be seen as a symmetry of the functor. Automorphisms of the functor, F, defined above are given by a collection of invertible matrices, $\{U_\pi\}$, that act on the vector spaces, V_π. These actions should commute with any intertwiners between V_π and $V_{\pi'}$. This requirement implies that U_π is completely specified by its action on the vector spaces of the irreps

[70] There are some discrete gauge theories that have a dual description in terms of a Chern-Simons theory with a continuous Lie gauge group. For example, the Toric code, which is a $\mathbb{Z}_2$ discrete gauge theory, can also be described as Spin(16)$_1$ Chern-Simons theory. By a non-discrete gauge theory, we mean a TQFT which is not equivalent to a (twisted) discrete gauge theory.

of G. Therefore, the symmetry group of the monoidal functor F is

$$\mathrm{Aut}(F) = \prod_i GL(V_{\pi_i}) \ , \tag{4.31}$$

where V_{π_i} are the vector spaces corresponding to the π_i irreps of G.

The finite group G can be reconstructed from $\mathrm{Rep}(G)$ by picking a particular subgroup of $\mathrm{Aut}(F)$. This subgroup is specified by the following extra condition on the $\{U_\pi\}$

$$\begin{array}{ccc} F(\pi\otimes\pi') & \xrightarrow{U_{\pi\otimes\pi'}} & F(\pi\otimes\pi') \\ \downarrow{\scriptstyle \mu_{\pi,\pi'}} & & \downarrow{\scriptstyle \mu_{\pi,\pi'}} \\ F(\pi)\otimes F(\pi') & \xrightarrow{U_\pi\otimes U_{\pi'}} & F(\pi)\otimes F(\pi') \end{array} \ ,$$

which is same as the constraint

$$\begin{array}{ccc} \bigoplus_i F(\pi_i) & \xrightarrow{\bigoplus_i U_{\pi_i}} & \bigoplus_i F(\pi_i) \\ \downarrow{\scriptstyle \mu_{\pi,\pi'}} & & \downarrow{\scriptstyle \mu_{\pi,\pi'}} \\ F(\pi)\otimes F(\pi') & \xrightarrow{U_\pi\otimes U_{\pi'}} & F(\pi)\otimes F(\pi') \end{array} \ , \tag{4.32}$$

where π_i are the irreducible representations into which the representation $\pi\otimes\pi'$ decomposes. The matrices $\{U_\pi\}$ which satisfy this constraint are called, "tensor-preserving automorphisms." We can also define a conjugation operation on U_π given by

$$\overline{U}_\pi(x) := \overline{U_{\overline{\pi}}(\overline{x})} \ , \tag{4.33}$$

where $\overline{\pi}$ is the conjugate representation of π, $x \in V_\pi$ and $\bar{x} \in V_{\overline{\pi}}$. Let $\mathrm{Aut}^{\otimes}(F) \subset \mathrm{Aut}(F)$ be the set of self-conjugate ($\overline{U}_\pi = U_\pi$) tensor-preserving automorphisms. It is clear that, given some element $g \in G$, there is a canonical map

$$L: \qquad G \to \mathrm{Aut}^{\otimes}(F) \ , \tag{4.34}$$

$$g \to U^{(g)} \ , \tag{4.35}$$

where $U^{(g)}$ acts on the vector space V_π through $\pi(g)$. The non-trivial result of Tannaka-Krein is that the canonical map L defined above is in fact an isomorphism [120]. Therefore, the automorphisms of the fiber functor F, along with the tensor-preserving and self-conjugation constraints, give us all the representations of the group. We can then reconstruct the group.

Consider $\mathrm{Rep}(G)$ defined over some finite field extension $K_{\mathrm{Rep}(G)}$.[71] In other words, the $3j, 6j$ symbols, and the R-matrices belong to $K_{\mathrm{Rep}(G)}$. In particular, the matrices

[71] In fact, from Brauer's Theorem [121], we can choose $K_{\mathrm{Rep}(G)}$ to be cyclotomic, but the exact nature of the field won't be important for our argument.

$\mu_{\pi,\pi'}$ belong to $K_{\mathrm{Rep}(G)}$. Therefore, (4.32) gives a set of polynomial constraints on the elements of the U_π matrices. The coefficients of the polynomial belong to the field $K_{\mathrm{Rep}(G)}$. Therefore, these polynomials are defined over $\bar{\mathbb{Q}}$, and hence there exists a solution belonging to a number field (up to gauge choices). Therefore, every Galois action with respect to some element of $\mathrm{Gal}(K_{\mathrm{Rep}(G)})$ induces a Galois action on U_π. If

$$U_\pi^{(g)} U_\pi^{(h)} = U_\pi^{(k)} , \tag{4.36}$$

for some g, h, k, then this relation does not change under a Galois action on $U_\pi^{(g)}$. As a result, the group $\mathrm{Aut}^{\otimes}(F)$ is invariant under Galois action. Hence, the representation category of a group is invariant under Galois conjugation.

In a discrete gauge theory $\mathcal{Z}(\mathrm{Vec}_G^\omega)$, $\mathrm{Rep}(G)$ is a Lagrangian subcategory. Moreover, since there is a gauge in which the data of $\mathcal{Z}(\mathrm{Vec}_G^\omega)$ is in a finite field extension, the data of the subcategory $\mathrm{Rep}(G)$ is in the same finite field extension. Therefore, our discussion above applies, and we find that the gauge group of a discrete gauge theory is invariant under Galois conjugation.

All that is left to study is how the Galois group acts on the Dijkgraaf-Witten twist. The cyclotomic field containing the elements of the S and T matrices of the discrete gauge theory, $\mathcal{Z}(\mathrm{Vec}_G^\omega)$, is $\mathbb{Q}(\xi_{ne(G)})$, where n is the order of the 3-cocycle $\omega \in H^3(G, U(1))$, $e(G)$ is the exponent of the group G [122], and $\xi_{ne(G)}$ is a primitive $ne(G)^{\text{th}}$ root of unity. In particular, the 3-cocycle ω is contained in this cyclotomic field. Suppose K_C is the defining number field containing the full data of $\mathcal{Z}(\mathrm{Vec}_G^\omega)$ in some gauge. $\mathbb{Q}(\xi_n)$ is a cyclotomic subfield of K_C. If $q \in \mathrm{Gal}(K_C)$ acts on $\mathcal{Z}(\mathrm{Vec}_G^\omega)$, then it acts on the 3-cocycle ω as $q|_{\mathbb{Q}(\xi_n)}(\omega)$. Moreover, since K_C and $\mathbb{Q}(\xi_n)$ are Galois extensions, for every Galois action $q' \in \mathrm{Gal}(\mathbb{Q}(\xi_n))$ there exists a $q \in \mathrm{Gal}(K_C)$ such that $q|_{\mathbb{Q}(\xi_n)} = q'$. Therefore, any Galois conjugation of the MTC $\mathcal{Z}(\mathrm{Vec}_G^\omega)$ acts as a Galois conjugation on the 3-cocycle ω. We get the following results:

Theorem 4.2.5 *Let K_C be the number field containing the MTC data of $\mathcal{Z}(\mathrm{Vec}_G^\omega)$. Galois conjugation with respect to $q \in \mathrm{Gal}(K_C)$ results in the discrete gauge theory $\mathcal{Z}(\mathrm{Vec}_G^{q|_{\mathbb{Q}(\xi_n)}(\omega)})$.*

Corollary 4.2.5.1 *The untwisted discrete gauge theory $\mathcal{Z}(\mathrm{Vec}_G)$ is invariant under Galois conjugation.*

Corollary 4.2.5.2 *Every Galois conjugation of $\mathcal{Z}(\mathrm{Vec}_G^\omega)$ acts as a Galois conjugation on the gapped boundary described by Vec_G^ω.*

Suppose we have the fusion category Vec_G^ω. The cyclotomic field containing the MTC data of this category is $\mathbb{Q}(\xi_n)$, where n is the order of $\omega \in H^3(G, U(1))$. Therefore, after a Galois conjugation, we get $\mathrm{Vec}_G^{\omega^q}$ for some q co-prime to n. Taking the Drinfeld center before and after the Galois conjugation gives us the discrete gauge theories, $\mathcal{Z}(\mathrm{Vec}_G^{\omega})$ and $\mathcal{Z}(\mathrm{Vec}_G^{\omega^q})$, respectively. Since these discrete gauge theories are related by a Galois conjugation, we see that Galois conjugation commutes with taking the Drinfeld center of Vec_G^{ω}. In Section 4.3, we will generalize this result to Drinfeld centers of general spherical fusion categories.

Note that a TQFT can have multiple Lagrangian subcategories. In particular, if a TQFT has Lagrangian subcategories $\mathrm{Rep}(G)$ and $\mathrm{Rep}(H)$, where G is not isomorphic to H, then it can be seen as a discrete gauge theory based on the gauge group G or the gauge group H. That is, the gauge group is not duality invariant.[72]

Therefore, Galois invariance of the gauge group of the discrete gauge theory is more precisely stated as follows: Given a discrete gauge theory $\mathcal{Z}(\mathrm{Vec}_G^{\omega})$, all of its Galois conjugates are G gauge theories up to dualities. In fact, the dualities of a discrete gauge theory are determined by the Lagrangian subcategories in the theory and they have been classified in [63, 123]. Since the number of Lagrangian subcategories does not change under Galois conjugation, the duality structure of Galois conjugate discrete gauge theories is the same.

While Galois conjugation of a discrete gauge theory $\mathcal{Z}(\mathrm{Vec}_G^{\omega})$ may act non-trivially on the 3-cocycle ω, the resulting discrete gauge theory is not guaranteed to be distinct. This is because for a given group G, distinct 3-cocycles in $H^3(G, U(1))$ can give the same TQFT. In the next subsection, we will explore how this happens for discrete gauge theories with abelian gauge groups.

Discrete gauge theories with abelian gauge group

In this subsection we will study (twisted) discrete gauge theories with abelian gauge groups more carefully (we already encountered many of these theories when we discussed abelian TQFTs previously). This discussion will help us to understand Galois action on the 3-cocycle ω implied by theorem 4.2.5 more explicitly.

Note that discrete gauge theories based on abelian gauge groups need not be abelian. Indeed, the quantum dimension of an anyon $([g], \pi_g^{\omega})$ in a general discrete gauge theory with gauge group G and 3-cocycle ω is

$$d_{([g],\pi_g^{\omega})} = |[g]|\dim(\pi_g^{\omega}) , \tag{4.37}$$

where $[g]$ is a conjugacy class in G, and π_g^{ω} is a projective representation of the cen-

[72] This is different from 3 + 1D discrete gauge theories whose gauge group is invariant under such dualities. This is because in 3 + 1D *all* line operators braid trivially with each other, and they are described by $\mathrm{Rep}(G)$, where G is the gauge group of the 3 + 1D discrete gauge theory. If there were a dual description based on a gauge group H, then the line operators would be described by $\mathrm{Rep}(H)$. However, $\mathrm{Rep}(G) \cong \mathrm{Rep}(H)$ if and only if $G \cong H$ from Tannaka-Krein reconstruction.

tralizer of g, say N_g, determined by the 2-cocycle

$$\gamma_g(h,k) = \frac{\omega(g,h,k)\omega(h,k,g)}{\omega(h,g,k)} \ . \tag{4.38}$$

In an abelian discrete gauge theory, all anyons have quantum dimension 1. Therefore, all conjugacy classes should have only a single element. Hence, the gauge group G should be abelian. Therefore, the centralizer of each element is G itself. Moreover, we also require the representation π_g^ω to be 1-dimensional. Since projective representations are necessarily higher dimensional, for an abelian discrete gauge theory, the 3-cocycle ω should be such that $\gamma_g(h,k) \in H^2(G,U(1))$ is trivial for all $g \in G$. Therefore, a discrete gauge theory is abelian if and only if the gauge group is abelian with CT (cohomologically trivial) twisting [122]. This is a stronger constraint than having abelian gauge groups.

For an abelian group G, a general 3-cocycle $\omega \in H^3(G,U(1))$ is generated by the following 3-cocycles [124]

$$\omega^{(i)}(\vec{g},\vec{h},\vec{k}) = e^{\frac{2\pi i p^{(i)}}{n_i^2}(g_i(h_i+k_i-(h_i+k_i) \bmod n_i))} ,1 \le i \le n \ , \tag{4.39}$$

$$\omega^{(i,j)}(\vec{g},\vec{h},\vec{k}) = e^{\frac{2\pi i p^{(i,j)}}{n_i n_j}(g_i(h_j+k_j-(h_j+k_j) \bmod n_j))} ,1 \le i < j \le n \ , \tag{4.40}$$

$$\omega^{(i,j,l)}(\vec{g},\vec{h},\vec{k}) = e^{\frac{2\pi i p^{(i,j,l)}}{\gcd(n_i,n_j,n_l)}(g_i h_j k_l)} ,1 \le i < j < l \le n \ , \tag{4.41}$$

where $G \cong \mathbb{Z}_{n_1} \otimes \cdots \otimes \mathbb{Z}_{n_N}$, $\vec{g},\vec{h},\vec{k} \in G$. Here $p^{(i)}$ is an integer defined modulo n_i, $p^{(i,j)}$ is an integer defined modulo $\gcd(n_i,n_j)$, and $p^{(i,j,l)}$ is an integer defined modulo $\gcd(n_i,n_j,n_l)$. We will refer to these as Type I,II, and III generators respectively.

Consider the action of $\alpha \in \text{Aut}(G)$ on the group G. This induces an action on $\omega(\vec{g},\vec{h},\vec{k})$ as $\omega(\vec{g},\vec{h},\vec{k}) \to \omega(\alpha(\vec{g}),\alpha(\vec{h}),\alpha(\vec{k}))$. Suppose α acts on the group elements through an $N \times N$ matrix M. We have $(M \cdot \vec{g})_i := \sum_a M_{ia} g_a \bmod n_i$. Since this is a group automorphism, we have

$$M \cdot (\vec{g} + \vec{h}) = M \cdot \vec{g} + M \cdot \vec{h} \ . \tag{4.42}$$

Using the explicit expressions for the 3-cocycle generators above, we get

$$\omega^{(i)}(\alpha(\vec{g}),\alpha(\vec{h}),\alpha(\vec{k})) = e^{\frac{2\pi i p^{(i)}}{n_i^2}(\sum_{a,b} M_{ia} M_{ib} g_a (h_b+k_b-(h_b+k_b) \bmod n_i))} = \prod_{a,b} (\omega^{(a,b)}(\vec{g},\vec{h},\vec{k}))^{M_{ia}M_{ib}} \ . \tag{4.43}$$

Note that we have taken the matrix M_{ib} out of the brackets using (4.42). Similarly, we get

$$\omega^{(i,j)}(\alpha(\vec{g}),\alpha(\vec{h}),\alpha(\vec{k})) = \prod_{a,b}(\omega^{(a,b)}(\vec{g},\vec{h},\vec{k}))^{M_{ia}M_{jb}} , \tag{4.44}$$

$$\omega^{(i,j,l)}(\alpha(\vec{g}),\alpha(\vec{h}),\alpha(\vec{k})) = \prod_{a,b,c}(\omega^{(a,b,c)}(\vec{g},\vec{h},\vec{k}))^{M_{ia}M_{jb}M_{lc}} . \tag{4.45}$$

Note that the discrete gauge theory $\mathcal{Z}(\mathrm{Vec}_G^{\omega})$ is uniquely specifed by the fusion category Vec_G^{ω} [125]. Since Vec_G^{ω} and $\mathrm{Vec}_G^{\alpha(\omega)}$ are equivalent as fusion categories when $\alpha \in \mathrm{Aut}(G)$, the discrete gauge theories $\mathcal{Z}(\mathrm{Vec}_G^{\omega})$ and $\mathcal{Z}(\mathrm{Vec}_G^{\alpha(\omega)})$ are also equivalent.

The modular data of a discrete gauge theory lies in the cyclotomic field $\mathbb{Q}(\xi_{ne})$ where n is the order of ω and e is the exponent of G. Consider a Galois action corresponding to some $q \in \mathbb{Z}_{ne}^{\times}$. Then the 3-cocycle generators transform as

$$\begin{aligned} \omega^{(i)}(\vec{g},\vec{h},\vec{k}) &\to (\omega^{(i)}(\vec{g},\vec{h},\vec{k}))^q , \\ \omega^{(i,j)}(\vec{g},\vec{h},\vec{k}) &\to (\omega^{(i,j)}(\vec{g},\vec{h},\vec{k}))^q , \\ \omega^{(i,j,l)}(\vec{g},\vec{h},\vec{k}) &\to (\omega^{(i,j,l)}(\vec{g},\vec{h},\vec{k}))^q . \end{aligned} \tag{4.46}$$

Consider a general 3-cocycle ω

$$\omega = \prod_{a=1}^{N_I}\omega^{(i_a)}\prod_{b=1}^{N_{II}}\omega^{(j_b,l_b)}\prod_{c=1}^{N_{III}}\omega^{(m_c,r_c,o_c)} , \tag{4.47}$$

with N_I type I generators, N_{II} type II generators, and N_{III} type III generators. Here $i_a, j_b, l_b, m_c, r_c, o_c$ are all integers valued in $\{1, ..., N\}$. Without loss of generality we can assume that i_a is distinct for each a in the product (and similarly for (j_b, l_b) and (m_c, r_c, o_c)).

This Galois action coincides with the transformation of the 3-cocycle under an automorphism of G if the following conditions are satisfied

$$M_{i_a x} = 0 \text{ for } i_a \neq x \text{ and } M_{i_a i_a}^2 = q \bmod n_{i_a} \ \forall a , \tag{4.48}$$

$$M_{j_b x}M_{l_b y} = 0 \text{ for } j_b \neq x \text{ or } l_b \neq y \text{ and } M_{j_b j_b}M_{l_b l_b} = q \bmod n_{j_b}n_{l_b} \ \forall b , \tag{4.49}$$

$$M_{m_c x}M_{r_c y}M_{o_c z} = 0 \text{ for } m_c \neq x \text{ or } r_c \neq y \text{ or } o_c \neq z \text{ and} \tag{4.50}$$

$$M_{m_c m_c}M_{r_c r_c}M_{o_c o_c} = q \bmod \gcd(n_{m_c}, n_{r_c}, n_{o_c}) . \tag{4.51}$$

If these conditions are satisfied, the Galois action is an automorphism of the gauge group G, and the discrete gauge theory is Galois invariant. However, all Galois actions (4.46) need not correspond to automorphisms of the gauge group.

Example: Consider the $\mathbb{Z}_N$ discrete gauge theory with some twist $\omega \in H^3(\mathbb{Z}_N, U(1))$. In this case, ω has the explicit expression

$$\omega(g,h,k) = e^{\frac{2\pi i p}{N^2}(g(h+k-h+k \bmod N))} \ . \tag{4.52}$$

Since $H^2(\mathbb{Z}_N, U(1))$ is trivial, $\mathbb{Z}_N$ discrete gauge theory for any twist ω is abelian. Consider the action of $\alpha \in \mathrm{Aut}(\mathbb{Z}_N) \cong \mathbb{Z}_N^\times$ given by $g \to \alpha g \bmod N$, $g \in \mathbb{Z}_N$. Then ω transforms as

$$\omega_p(g,h,k) \to \omega_p(\alpha g, \alpha h, \alpha k) = \omega_p(g,h,k)^{\alpha^2} = \omega_{\alpha^2 p}(g,h,k) \ . \tag{4.53}$$

A Galois conjugation with respect to some q coprime to N transforms the 3-cocycle as

$$\omega_p(g,h,k) \to \omega_p(g,h,k)^q = \omega_{qp}(g,h,k) \ . \tag{4.54}$$

Therefore, a Galois conjugation w.r.t. q is an automorphism of the gauge group G only if $\alpha^2 = q \bmod N$.
As a particularly concrete example, consider $N = 5$ and the 3-cocycle with $p = 1$. Then $2 \bmod 5 \neq \alpha^2$ for any $\alpha \in \mathrm{Aut}(\mathbb{Z}_5) \cong \mathbb{Z}_5^\times = \{1,2,3,4\}$. Therefore, Galois conjugation w.r.t. to 2 takes us from the discrete gauge theory $\mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_5}^{\omega_1})$ to $\mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_5}^{\omega_2})$. In fact, $\mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_5}^{\omega_1})$ is the prime abelian theory B_{25} and $\mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_5}^{\omega_2})$ is the prime abelian theory A_{25}. From our discussion in section 4.2.2, we know that there are non-trivial Galois conjugations which take us between these theories, and we know that our discussion in this section is consistent.

Note that while automorphisms of the group naturally lead to equivalence of discrete gauge theories based on different twists, this is not the only way in which equivalences arise. Even after taking the automorphisms of the gauge group G and its action on the 3-cocycle into account, labelling discrete gauge theories by the gauge group and the orbits of the automorphism group action on $H^3(G, U(1))$ is not faithful. For example, consider the group $\mathbb{Z}_2 \times D_8$. There exists two 3-cocycles for this group, not related by group automorphisms, which give the same discrete gauge theory [126].[73]

4.2.4 Weakly Integral Modular Categories

Until now, we studied theories that only have integer quantum dimensions. We saw that in these theories any Galois action on a unitary TQFT results in a unitary TQFT. Now we look at TQFTs described by weakly integral MTCs. These theories have quantum dimensions of the form $d_a = \sqrt{n_a}$, for some integer n_a. As a result such MTCs have

[73] In [127], the authors conjecture that equivalence classes of $3+1$D discrete gauge theories based on gauge group G are classified by $H^4(G, U(1))$ up to group automorphisms. We note that the $2+1$D version of this conjecture is not true because of this counter-example.

Galois conjugations that take a unitary TQFT to a non-unitary one. Before looking at the general case, let us consider the specific case of the Ising model and its Galois conjugates.

The Ising$^{(\nu)}$ Model

The Ising$^{(\nu)}$ family of theories is specified by the following data. There are three anyons $\{I, \sigma, \psi\}$ satisfying the fusion rules

$$\psi \otimes \psi = I, \;\; \sigma \otimes \sigma = I + \psi \;, \tag{4.55}$$

where I is an boson, ψ is a fermion, and σ is an anyon with twist $e^{\frac{2\pi i \nu}{16}}$. They have quantum dimensions $d_I = 1, d_\psi = 1, d_\sigma = \sqrt{2}$. Here ν is an odd integer modulo 16. The Ising model corresponds to $\nu = 1$. Note that the ν parameter here only classifies the unitary MTCs with the same fusion rules as the Ising model.

The full MTC data belongs to the cyclotomic field $\mathbb{Q}(\xi_{16})$. Therefore, we have the Galois group $\mathbb{Z}_{16}^\times = \{1, 3, 5, 7, 9, 11, 13, 15\} \cong \mathbb{Z}_4 \times \mathbb{Z}_2$. If we start with any of the above family of Ising$^{(\nu)}$ models, a unitarity preserving Galois action should not change the quantum dimension of σ to $-\sqrt{2}$. Therefore, the unitarity preserving Galois actions correspond to $q = 1, 7, 9, 15$. These form the Klein four-group. Under these Galois actions, the Ising$^{(\nu)}$ family of models transform as

$$\text{Ising}^{(\nu)} \to \text{Ising}^{(q\nu \text{ mod } 16)} \;. \tag{4.56}$$

Metaplectic Modular Categories

Let us now discuss unitary Galois orbits in the more general family of metaplectic modular categories (of which Ising$^{(\nu)}$ are examples). These are categories for which the fusion rules are the same as those of the Spin$(N)_2$ theories. In general, metaplectic categories have strictly weakly integral anyons. However, certain metaplectic categories are integral (e.g., Spin$(8)_2$). As shown in [128], integral metaplectic categories are group theoretical; hence, they belong to the class of theories discussed in the previous section.

Therefore, we can focus on strictly weakly integral metaplectic categories. Even though it is an extremely hard problem to solve the Pentagon and Hexagon equations for large rank theories, amazingly, for metaplectic categories, the MTC data can be found. Moreover, they play an important role (along with discrete gauge theories) in the classification of weakly integral categories.

By examining the explicit expressions for the F and R matrices for Spin$(N)_2$ metaplectic modular categories for odd N in [129], we find the following cyclotomic field extensions

$$\begin{aligned} K_{\tilde{S},T} &= \mathbb{Q}(\xi_{\text{lcm}(2N,8)}) \ , & (4.57) \\ K_R &= \mathbb{Q}(\xi_{\text{lcm}(2N,16)}) \ , & (4.58) \\ K_{F,R} &= \mathbb{Q}(\xi_{\text{lcm}(2N,16)}) \ . & (4.59) \end{aligned}$$

Since the metaplectic modular categories are multiplicity free, the R matrices are all phases. Some are $2N^{\text{th}}$ roots of unity while others are 16^{th} roots of unity. The F-matrices consists of $2N^{\text{th}}$ roots of unity, $\sqrt{2}$, and $\sqrt{N}$. Note that $\sqrt{N}$ belongs to the cyclotomic field $\mathbb{Q}(\xi_{\text{lcm}(N,4)}) \subset \mathbb{Q}(\xi_{\text{lcm}(2N,16)})$. Therefore, the F and R symbols belong to the cyclotomic field $\mathbb{Q}(\xi_{\text{lcm}(2N,16)})$.

The F matrices given in [129] are real and unitary. As a result, Galois conjugation is guaranteed to result in unitary F matrices. Also, since the R matrices are phases, they remain unitary under Galois conjugation. However, the quantum dimensions need not remain positive. Therefore, the resulting theory need not be unitary. This is a generalization of what happens in Ising$^{(\nu)}$ models that we discussed above. However, we know from [44] that braided fusion categories with unitary F and R symbols have a unique spherical structure which makes it a unitary MTC. Therefore, even though Galois conjugation of a metaplectic theory need not land us on a unitary TQFT, we can always choose a spherical structure to make the theory unitary (we must also choose $\mathcal{D} > 0$). This statement is in fact true for any weakly group theoretical modular tensor category from the following result

Theorem 4.2.6 [130]: *Every weakly group theoretical fusion category is unitary.*

Therefore, any Galois conjugate of a given unitary weakly group-theoretical modular tensor category can be made unitary by the choice of a unique spherical structure. All known weakly integral categories are weakly group theoretical. If all weakly integral categories can be shown to be weakly group theoretical, then any Galois conjugate of a unitary weakly integral modular tensor category can be made unitary by the choice of a unique spherical structure.

Let us discuss another example of a metaplectic modular category that we will come back to in our further discussions. The Spin$(5)_2$ Chern-Simons theory has 6 anyons labelled by $\{1, \epsilon, \phi_1, \phi_2, \psi_+, \psi_-\}$ with quantum dimensions $\{1, 1, 2, 2, \sqrt{5}, \sqrt{5}\}$, respectively. The fusion rules are given by

$$\begin{aligned} \epsilon \otimes \epsilon &= 1 \ , \quad \epsilon \otimes \phi_i = \phi_i \ , \quad \epsilon \otimes \psi_\pm = \psi_\mp \ , \quad \phi_i \otimes \phi_i = 1 \oplus \epsilon \oplus \phi_{\min(2i,5-2i)} \ , \\ \phi_1 \otimes \phi_2 &= \phi_1 \oplus \phi_2 \ , \quad \phi_i \otimes \psi_\pm = \psi_\pm \oplus \psi_\mp \ , \quad \psi_\pm \otimes \psi_\pm = 1 \oplus \phi_1 \oplus \phi_2 \ , \qquad (4.60) \\ & \psi_\pm \otimes \psi_\mp = \epsilon + \phi_1 \oplus \phi_2 \ , \end{aligned}$$

where $i = 1, 2$. The twists of the anyons are

$$\theta_\epsilon = 1,\ \theta_{\phi_1} = e^{\frac{4\pi i}{5}},\ \theta_{\phi_2} = e^{-\frac{4\pi i}{5}},\ \theta_{\psi_\pm} = \pm i\ . \tag{4.61}$$

Therefore, the twists belong to the cyclotomic field $\mathbb{Q}(\xi_{20})$. All MTCs with the same fusion rules as $\text{Spin}(5)_2$ Chern-Simons theory can be distinguished using the T matrix alone [129]. Therefore, we only need to consider the Galois action on the twists to study the Galois action on the whole theory. The Galois group acting on the twists is $\mathbb{Z}_{20}^\times = \{1, 3, 7, 9, 11, 13, 17, 19\}$. For unitary Galois orbits, we should consider Galois actions which leave $d_{\psi_\pm} = \sqrt{5}$ invariant. These are $\{1, 9, 11, 19\}$. Under the action of 9 we get the twists

$$\theta_\epsilon = 1\ ,\ \theta_{\phi_1} = e^{-\frac{4\pi i}{5}}\ ,\ \theta_{\phi_2} = e^{\frac{4\pi i}{5}}\ ,\ \theta_{\psi_\pm} = \pm i\ . \tag{4.62}$$

This theory is the same as $\text{Spin}(5)_2$ under the permutation of the anyons $\phi_1 \leftrightarrow \phi_2$. Under the action of 19 we get the twists

$$\theta_\epsilon = 1\ ,\ \theta_{\phi_1} = e^{-\frac{4\pi i}{5}}\ ,\ \theta_{\phi_2} = e^{\frac{4\pi i}{5}}\ ,\ \theta_{\psi_\pm} = \mp i\ . \tag{4.63}$$

Therefore, acting with 19 complex conjugates the theory. This Galois action can be inverted using the permutation of the anyons $\phi_1 \leftrightarrow \phi_2$ and $\psi_+ \leftrightarrow \psi_-$. This is because $\text{Spin}(5)_2$ Chern-Simons theory is time-reversal invariant. Under the action of 11 we get the twists

$$\theta_\epsilon = 1\ ,\ \theta_{\phi_1} = e^{\frac{4\pi i}{5}}\ ,\ \theta_{\phi_2} = e^{-\frac{4\pi i}{5}}\ ,\ \theta_{\psi_\pm} = \mp i\ . \tag{4.64}$$

It is clear that this theory is same as $\text{Spin}(5)_2$ because of the time-reversal symmetry and the symmetry of the fusion rules under $\phi_1 \leftrightarrow \phi_2$.

Therefore, we find that the $\text{Spin}(5)_2$ Chern-Simons theory is invariant under all unitarity preserving Galois actions (recall we fix $\mathcal{D} > 0$).

4.3 Gapped Boundaries and Galois Conjugation

In section 4.2.3, we found that Galois conjugation of the gapped boundary of a discrete gauge theory induces a Galois action on the bulk TQFT and vice-versa. In this section, we will explore this connection further. First, we will revisit discrete gauge theories using the classification of bosonic gapped boundaries. Then we will look at bosonic gapped boundaries of more general TQFTs by studying the properties of their Lagrangian algebras. Finally, we will discuss how Galois conjugation and taking the Drinfeld center of a spherical fusion category interact with each other. This will give us a general result relating the Galois action of the bosonic gapped boundary and the bulk TQFT.

4.3.1 Gapped boundaries of discrete gauge theories

An abelian TQFT is described by a so-called "pointed" MTC. As a fusion category, a pointed MTC is equivalent to Vec_G^ω for some abelian group G and some $\omega \in H^3(G, U(1))$. The bosonic gapped boundaries of this theory correspond to Lagrangian subgroups of G [131, 132]. A Lagrangian subgroup of $L \subset G$ is a subgroup such that the fusion subcategory $\mathrm{Vec}_L^{\omega|_L}$ is Lagrangian. This discussion immediately implies that in order for an abelian theory to have bosonic gapped boundaries, it should necessarily originate from a discrete gauge theory. Since we argued that the number of Lagrangian subcategories is invariant under Galois conjugation, it is clear that Galois conjugate abelian theories have the same number of bosonic gapped boundaries.[74]

Now let us study a non-abelian discrete gauge theory, $\mathcal{Z}(\mathrm{Vec}_G^\omega)$. The gapped boundaries of this theory are classified by the pair (L, η), where L is a subgroup up to conjugation of G such that $\omega|_L$ is trivial in cohomology, and $\eta \in H^2(L, U(1))$ [134]. From our previous discussion, we know that the Galois conjugate of $\mathcal{Z}(\mathrm{Vec}_G^\omega)$ is $\mathcal{Z}(\mathrm{Vec}_G^{q|_{\mathbb{Q}(\xi_n)}(\omega)})$ for some $q \in \mathrm{Gal}(K_C)$. Moreover, if $\omega|_L$ is cohomologically trivial, so is $(q|_{\mathbb{Q}(\xi_n)}(\omega))|_L$. Therefore, the number of gapped boundaries of Galois conjugate twisted discrete gauge theories is the same.

4.3.2 Gapped boundaries of general TQFTs

In a general TQFT described by a modular tensor category C, a gapped boundary corresponds to the condensation of a subset of anyons in C which admits the structure of a Lagrangian algebra [132]. Therefore, in order to study the nature of bosonic gapped boundaries of Galois conjugate TQFTs, we have to study the behavior of a Lagrangian algebra under Galois conjugation. To that end, consider the following two theorems that characterize a Lagrangian algebra.

Theorem 4.3.1 **[132]:** *$\mathcal{A}$ is a commutative algebra in a modular tensor category C if and only if the object $\mathcal{A}$ decomposes as $\mathcal{A} = \oplus n_i a_i$ into simple objects $a_i \in C$ and $\theta_{a_i} = 1$ for all i such that $n_i \neq 0$.*

Theorem 4.3.2 **[132]:** *A commutative connected algebra $\mathcal{A} = \oplus n_i a_i$ in a unitary modular tensor category C with dim$(\mathcal{A})^2 =$ dim(C) is Lagrangian if and only if*

$$n_i n_j \leq N_{ik}^k n_k \ , \tag{4.65}$$

[74] Bosonic TQFTs can have gapped boundaries sensitive to the spin structure; such boundaries are obtained from fermion condensation [133]. We only discuss gapped boundaries obtained from condensation of bosons.

for all i, j, k.

It is clear that if a set of anyons satisfies the constraints in theorem 4.3.1 and 4.3.2 in an MTC, then the same holds after a unitarity-preserving Galois conjugation. Hence, if a set of anyons form a Lagrangian algebra, then the same set of anyons form a Lagrangian algebra in the Galois conjugate unitary theory. Therefore, a set of condensable anyons remains condensable under unitarity preserving Galois conjugation.

Suppose we have a bulk excitation a. Under condensation, this anyon "splits" into several anyons to give excitations on the boundary.

$$a = \sum_x W_{ax} x \ , \tag{4.66}$$

where W_{ax} is an integer matrix. Note that even though this is called "splitting" in the literature (for example see [135]), it may be that the anyon a gets identified with other anyons to produce some boundary excitation. The matrix W determines the relationship between bulk and boundary excitations. The W matrix plays a crucial rule in determining the fusion rules of boundary excitations. From [135], we have

$$n_{xy}^z = \sum_w \frac{V_{xw} V_{yw} V_{wz}^{-1}}{S_{0w}} \ , \tag{4.67}$$

where $V_{xw} := \sum_a \overline{S_{ax}} W_{aw}$ and S_{ax} is the bulk S-matrix (note that in this formula, the normalization of S does not matter). If we substitute for the V_{xw} matrix in (4.67), it becomes an equation in n_{xy}^z, W, and the bulk S-matrix S. If the W matrix is invariant under Galois conjugation, then it is clear that the integer n_{xy}^z being a combination of S and W is also invariant under Galois conjugation. Note that even though the S matrix can change non-trivially under Galois conjugation, n_{xy}^z is an integer given by a combination of S matrix elements, and hence it is preserved under Galois conjugation.[75]

For the above picture to hold, we have to show that the integer matrix W is invariant under Galois conjugation. Given a Lagrangian algebra $\mathcal{A}$ in an MTC C, the relationship between bulk and boundary excitations is found by constructing the pre-quotient category $\tilde{Q} = C/\mathcal{A}$. The simple objects of the canonical idempotent completion, Q, of $C/\mathcal{A}$ are the boundary excitations. The details of the construction of these categories are not relevant to our discussion. The crucial point is that the construction of the simple elements of Q and their relationship to bulk anyons depend only on the fusions rules of the bulk theory and the choice of the anyons forming the Lagrangian algebra $\mathcal{A}$ [132,136]. Hence, it follows that the W matrix is invariant under Galois conjugation. As a result, we have the following theorem:

[75]Moreover, since the S-matrix belongs to a cyclotomic field, any Galois conjugation acting on the S-matrix commutes with complex conjugation.

Theorem 4.3.3 *The fusion rules of the boundary excitations are invariant under a unitarity-preserving Galois conjugation of the bulk TQFT.*

The full data of the gapped boundary is encoded in a spherical fusion category. The above theorem guarantees that the fusion rules of this spherical fusion category are invariant under Galois conjugation. However, the F matrices of the boundary theory can change.

Suppose we have a discrete gauge theory $\mathcal{Z}(\text{Vec}_G^\omega)$. This theory always allows for a gapped boundary described by Vec_G^ω whose fusion rules are simply the group multiplication in G. The invariance of the fusion rules of the boundary excitations under Galois conjugation implies that, under a Galois conjugation, the gapped boundary described by Vec_G^ω changes at most by a difference in the twist ω. That is, Galois conjugation of the bulk theory, $\mathcal{Z}(\text{Vec}_G^\omega)$, results in a new theory with gapped boundary described by $\text{Vec}_G^{\omega'}$ for some ω' which may not be equal to ω. After Galois conjugation, the bulk theory is given by $\mathcal{Z}(\text{Vec}_G^{\omega'})$. This statement agrees with our discussion of Galois conjugation of discrete gauge theories.

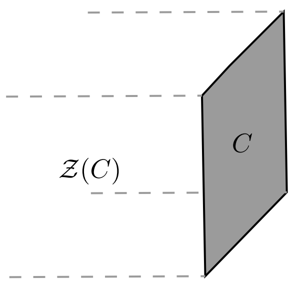


Figure 4.3: The bulk TQFT is the Drinfeld center of the spherical fusion category describing the boundary excitations.

4.3.3 Galois Conjugation and the Drinfeld Center

In this section we will explore how the Galois action on a spherical fusion category affects its Drinfeld center. To that end, suppose we have a spherical fusion category C. Using the F symbols of C, we can construct an algebraic field extension, $\mathbb{Q}(F)$, by adjoining the elements of the F symbols to the rationals. Let K_C be the Galois closure of $\mathbb{Q}(F)$. This is the defining number field of C that we will work with. The Galois group, $\text{Gal}(K_C)$, acting on C gives us other spherical fusion categories.

Now consider the Drinfeld center, $\mathcal{Z}(C)$, of C, which, on general grounds, is an MTC [137]. Let $K_{\mathcal{Z}(C)}$ be the Galois closure of the number field obtained by adjoining the F and R symbols of $\mathcal{Z}(C)$ to the rationals. We can then act on $\mathcal{Z}(C)$ with the elements of $\text{Gal}(K_{\mathcal{Z}(C)})$ to get other MTCs.

If x is an object in C, the objects of $\mathcal{Z}(C)$ are of the form (x, e_x) where $e_x(y) \in$

$\text{Hom}(xy, yx)$ is a half-braiding which satisfies the constraint [138]

$$\alpha^{-1}(y,z,x) \circ (1 \otimes e_x(z)) \circ \alpha_{y,x,z} \circ (e_x(y) \otimes 1) \circ \alpha^{-1}_{x,y,z} = e_x(yz) \ , \tag{4.68}$$

where $e_x(1)$ is normalized to be the identity map, and $\alpha_{x,y,z}$ is the associativity map of the spherical fusion category. The Hom spaces, tensor product of objects, and braiding of the resulting modular tensor category are given by [137]

$$\text{Hom}((x,e_x),(y,e_y)) = \{f \in \text{Hom}(x,y) | 1 \otimes f \circ e_x(z) = e_y(z) \circ f \otimes 1 \ \forall \ z \in C\} \tag{4.69}$$

$$(x,e_x) \otimes (y,e_y) = (x \otimes y, e_{xy}), \ \text{where} \ e_{xy} = (e_x \otimes id_y) \circ (id_x \otimes e_y) \ , \tag{4.70}$$

$$c((x,e_x),(y,e_y)) = e_x(y) \ . \tag{4.71}$$

Therefore, we see that the braidings in the bulk are determined by the half-braidings. Note that given a simple object, $(x,e_x) \in \mathcal{Z}(C)$, $x \in C$ need not be simple. Indeed, we have to use (4.69) to identify the simple objects in the bulk using the fact that

$$\text{Hom}((x,e_x),(x,e_x)) \simeq \mathbb{C} \ , \tag{4.72}$$

if and only if (x,e_x) is simple.

Note that the MTC data of $\mathcal{Z}(C)$ is determined by the data of C along with the half-braidings. We can choose a basis for the fusion spaces and solve for the half-braidings by solving some multi-variable polynomials with coefficients in the field $\mathbb{Q}(F)$ obtained by adding the F symbols of C to the rationals (the constraints are given explicitly in equation (48) of [139]). Also, determining the full data of C describing the boundary of the bulk TQFT corresponding to $\mathcal{Z}(C)$ involves a series of steps. First we have to determine the multiplication of the Lagrangian algebra in $\mathcal{Z}(C)$ corresponding to the gapped boundary. Representations of this algebra form the fusion category C. Therefore, to determine the boundary F symbols, we have to find the $6j$ symbols for these representations [125, 133]. Though tedious, the constraints to be solved are algebraic in the data defining the bulk and boundary theory.

Given a Galois action on C by some element of $q \in \text{Gal}(K_C)$, we have some corresponding Galois action $q' \in \text{Gal}(K_{\mathcal{Z}(C)})$ obtained as follows.[76] Let $g_1, \cdots, g_n$ be a basis of K_C as a vector space, where n is the finite degree of the field extension. The Galois action by some element $q \in \text{Gal}(K_C)$ on K_C is completely specified by its action on the g_i. Similarly, let $h_1, \cdots, h_{n'}$ be a basis of $K_{\mathcal{Z}(C)}$ as a vector space, where n' is the finite degree of the field extension $K_{\mathcal{Z}(C)}$. Then we can choose some $q' \in \text{Gal}(K_{\mathcal{Z}(C)})$ such that the action of q and q' on $\{g_1, \cdots, g_n\} \cap \{h_1, \cdots, h_{n'}\}$ agree.[77] If K_C and $K_{\mathcal{Z}(C)}$

[76]Since the F symbols of C belong to a number field K_C, the equations which define the data of $\mathcal{Z}(C)$ are polynomials over K_C. Therefore, we can always find a solution to these polynomials which belongs to a number field (up to gauge choices).

[77]These can be thought of as Galois actions on the composite extension obtained from K_C and

are distinct, this choice is not unique. Galois action by q' on the F and R symbols of $\mathcal{Z}(C)$ results in the MTC which is the Drinfeld center of the spherical fusion category obtained by Galois conjugating C with respect to q. This leads to the following result:

Theorem 4.3.4 *Corresponding to every Galois action, $q(C)$, on a spherical fusion category, C, where $q \in K_C$, there exists a Galois action $q' \in K_{\mathcal{Z}(C)}$ such that*

$$\mathcal{Z}(q(C)) = q'(\mathcal{Z}(C)) \ , \tag{4.73}$$

and vice-versa.

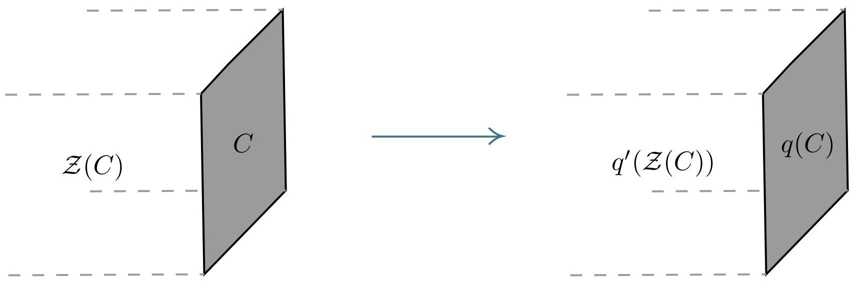


Figure 4.4: Galois conjugation on the bulk induces a Galois action on the boundary and vice-versa.

Note that it is possible for K_C to be a non-abelian field extension and $K_{\mathcal{Z}(C)}$ to be abelian. For example, the data of the fusion category, H, obtained from the principal even part of the Haagerup subfactor, cannot be contained in a cyclotomic field [107]. Therefore, by the Kronecker-Weber theorem, K_H for this category is necessarily a non-abelian extension. It is also known that the MTC data of the Drinfeld center $\mathcal{Z}(H)$ belongs to a cyclotomic number field. Therefore, we can choose $K_{\mathcal{Z}(H)}$ to be an abelian extension.

We immediately get an application of (4.73) as follows. Recall that the Drinfeld center of a spherical fusion category is unique. Moreover, Morita equivalent spherical fusion categories have the same Drinfeld center. Therefore, (4.73) implies:

Corollary 4.3.4.1 *The number of distinct Galois conjugates of $\mathcal{Z}(C)$ is a lower bound on the number of non-Morita equivalent Galois conjugates of C.*

Corollary 4.3.4.2 *The number of distinct Galois conjugates of C is an upper bound on the number of distinct Galois conjugates of $\mathcal{Z}(C)$.*

As a result, if C is Galois invariant, so is $\mathcal{Z}(C)$. That is, the Galois invariance of the 1+1D boundary implies that the bulk TQFT is Galois invariant. Similarly, if $\mathcal{Z}(C)$ is Galois invariant, all Galois conjugates of C should be Morita equivalent to C.

$K_{\mathcal{Z}(C)}$.

It follows that given a bulk TQFT $\mathcal{Z}(C)$ with boundary described by the spherical fusion category C, the Galois conjugate $q'(\mathcal{Z}(C))$ admits the boundary condition $q(C)$. This agrees with our result above that Galois action on a Lagrangian algebra results in a Lagrangian algebra. Using Galois actions arising from the cyclotomic field containing the modular data, this was argued recently in [103].

4.4 Symmetries, Gauging and Galois Fixed Point TQFTs

Symmetries are, of course, a duality-invariant feature of quantum field theories. However, there is a priori, no guarantee that they are also Galois invariant. We therefore wish to study the question of how symmetries transform under Galois actions.

To that end, recall that the main observables in $2+1$D TQFTs are line operators. These naturally lead to 1-form symmetries. One can also define surface operators which act on these line operators and permute them. These are 0-form symmetries. Sometimes the 0-form symmetry and 1-form symmetry can form a 2-group. Therefore, $2+1$D TQFTs have a rich symmetry structure.[78] In this section, we will study the relationship between the symmetries of Galois conjugate TQFTs. The case of abelian TQFTs is the simplest to analyse. After that we will study symmetries of non-abelian Galois conjugate TQFTs. Following this, we will look at gauging the 0-form symmetry and how Galois conjugation of the TQFT affects the gauging procedure.

4.4.1 Symmetries of a TQFT

Given the set of anyons, $\{a, b, \cdots\}$, of a TQFT, the subset of abelian anyons corresponds to some abelian group. This is the 1-form symmetry group, $\mathcal{A}$, of a TQFT. Moreover, we can define an automorphism group of the set of anyons, G, which preserves the MTC data (up to conjugation for anti-unitary symmetries). This is the 0-form symmetry group of the TQFT. These symmetries can lead to a natural 2-Group structure. For a given MTC, there are certain permutations of the anyons that leave all the gauge-invariant data unchanged. These form the intrinsic symmetry of the TQFT. The gauge-invariant data is left unchanged up to a conjugation for anti-unitary symmetries.

Given an MTC, the 2-group structure is define by the quadruple $(G, \mathcal{A}, \rho, [\beta])$. Here, ρ is the action of the 0-form symmetry group on the 1-form charges, $\rho : G \to \mathrm{Aut}(\mathcal{A})$, and $[\beta] \in H^3_\rho(G, \mathcal{A})$. To understand how this 2-group structure arises, let us define how the 0-form symmetry acts on the MTC. Let $g \in G$. As alluded to before, G acts on the anyons through a permutation. Hence, $g(a) = a'$. For it to be symmetry, the

[78] If we allow for topological point operators, then we can also have 2-form symmetries.

gauge-invariant quantities should be invariant under it. For example:

$$\begin{aligned} g(N_{ab}^{c}) &= N_{g(a)g(b)}^{g(c)} = N_{ab}^{c} \ , & (4.74)\\ g(\theta_a) &= K^g \theta_a K^g \ , & (4.75)\\ g(S_{ab}) &= K^g S_{g(a)g(b)} K^g \ , & (4.76)\end{aligned}$$

where K^g is an operator which complex conjugates the quantity in between if g is an anti-unitary symmetry. The gauge-dependent quantities should change only up to a gauge transformation. Since G acts on all anyons, its restriction to the abelian anyons, $\mathcal{A}$, specifies the map, $\rho : G \to \mathrm{Aut}(\mathcal{A})$.

The action of g on the fusion space is

$$g(|a, b, c; \mu\rangle) = \left|a^{'}, b^{'}, c^{'}; \mu\right\rangle \ . \tag{4.77}$$

For our convenience, we would like to define a map which leaves even the gauge-dependent quantities invariant. For this, we will redefine the action of the above map on the fusion space as

$$g(|a, b, c; \mu\rangle) = \sum_{\mu^{'}} U_g(a^{'}, b^{'}, c^{'})_{\mu,\mu^{'}} K^g \left|a^{'}, b^{'}, c^{'}; \mu^{'}\right\rangle \ , \tag{4.78}$$

where $U_g(a^{'}, b^{'}; c^{'})_{\mu,\mu^{'}}$ is a unitary matrix, and K^g is an operator introduced above so that the quantities sandwiched between two K^g's are complex conjugated if g is an anti-unitary symmetry. This changes the F and R-matrices as follows

$$\begin{aligned} & U_g(g(b), g(a), g(c)) R_{g(a)g(b)}^{g(c)} U_g(g(a), g(b), g(c))^{-1} & (4.79)\\ & U_g(g(a), g(b), g(e)) U_g(g(e), g(c), g(d)) (F_{g(a)g(b)g(c)}^{g(d)})_{g(e)}^{g(f)} & \\ & \qquad \times U_g(g(b), g(c), g(f))^{-1} U_g(g(a), g(f), g(d))^{-1} \ , & (4.80)\end{aligned}$$

where $a \otimes b = e$, $b \otimes c = f$, and we have supressed the indices labelling the basis vectors of the fusion spaces. For $g \in G$ to be a symmetry, we require

$$g(R_{ab}^{c}) = U_g(g(b), g(a), g(c)) R_{g(a)g(b)}^{g(c)} U_g(g(a), g(b), g(c))^{-1} = K^g R_{ab}^{c} K^g \ , \tag{4.81}$$

$$\begin{aligned} g((F_{abc}^{d})_{e}^{f}) &= U_g(g(a), g(b), g(e)) U_g(g(e), g(c), g(d)) (F_{g(a)g(b)g(c)}^{g(d)})_{g(e)}^{g(f)} \\ & \quad \times U_g(g(b), g(c), g(f))^{-1} U_g(g(a), g(f), g(d))^{-1} = K^g (F_{abc}^{d})_{e}^{f} K^g \ , \end{aligned} \tag{4.82}$$

where $a \otimes b = e$, and $b \otimes c = f$. This definition of g ensures the invariance of even gauge dependent quantities under its action. Hence, the action of g on a category can be seen as a permutation of the anyons along with a gauge transformation. Among such maps, there are those that act on the labels and fusion spaces as follows

$$\Upsilon(a) = a; \qquad \Upsilon(|a,b,c;\mu\rangle) = \frac{\gamma_a\gamma_b}{\gamma_c}\,|a,b,c;\mu\rangle \ , \tag{4.83}$$

for some phases, γ_a. By definition, such maps don't permute the anyons, and they leave all the data invariant. The Υ are called natural isomorphisms. Note that these are gauge transformations, where the unitary gauge transformation matrix acting on the fusion space is $\frac{\gamma_a\gamma_b}{\gamma_c}\delta_{\mu\mu'}$. The 0-form symmetry group of the theory, G, is the set of maps, g, modulo natural isomorphisms. Hence, the group elements are equivalence classes, $[g]$. For $[g],[h],[k]\in G$ the group multiplication is given by

$$[g]\cdot[h] = [k] \Longleftrightarrow \Upsilon_1\cdot g\cdot\Upsilon_2\cdot h = \Upsilon_3\cdot k \implies k = \kappa_{g,h}\cdot g\cdot h \ , \tag{4.84}$$

where $\kappa_{g,h} = \Upsilon_3^{-1}\cdot\Upsilon_1\cdot g\cdot\Upsilon_2\cdot g^{-1}$. Here $\kappa_{g,h}$ is a natural isomorphism which can be written in terms of phases as

$$\kappa_{g,h}(a,b,c)_{\mu\nu} = \frac{\gamma_a(g,h)\gamma_b(g,h)}{\gamma_c(g,h)}\delta_{\mu,\nu} \ . \tag{4.85}$$

The phases in $\gamma_a(g,h)$ look arbitrary, but they obey some consistency conditions. In fact, they can be extracted from the TQFT data. In the language of symmetry defects, $U_g(a,b,c)$ represents the action of a symmetry defect on a fusion vertex, and the $\gamma_a(g,h)$ phases represent the difference in the action of g and then h on an anyon compared to the action of $g\cdot h$ (see Fig. 4.5 and Fig. 4.6).

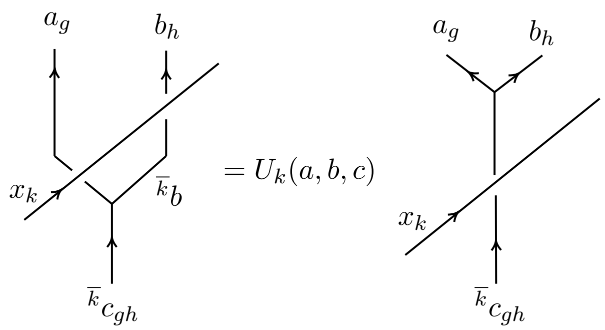


Figure 4.5: Diagrammatic definition of $U_k(a,b,c)$

To respect the freedom to add or remove identity lines, we should impose

$$\gamma_1(g,h) = \gamma_a(e,h) = \gamma_a(g,e) = 1 \ , \tag{4.86}$$

$$U_e(a,b,c) = U_g(1,b,c) = U_g(a,1,c) = 1 \ , \tag{4.87}$$

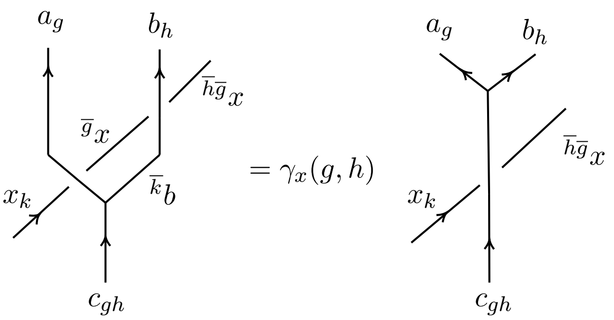


Figure 4.6: Diagrammatic definition of $\gamma_k(g,h)$

where 1 represents the vacuum, and e is the identity in G. Using $\kappa_{g,h} g \cdot h = k$ and the action of the symmetries in the fusion spaces, we get the following expression for $\kappa_{g,h}$.

$$\kappa_{g,h} = \frac{\gamma_a(g,h)\gamma_b(g,h)}{\gamma_c(g,h)} = U_g(a,b,c)^{-1}(K^g U_h^{-1}(g^{-1}(a), g^{-1}(b), g^{-1}(c))K^g)U_{gh}(a,b,c) \ . \tag{4.88}$$

Let us look at the 0-form symmetry group element $g \cdot h \cdot k$

$$g \cdot h \cdot k = \kappa_{g,hk} \cdot g \cdot (h \cdot k) \tag{4.89}$$
$$= \kappa_{g,hk} \cdot g \cdot \kappa_{h,k} \cdot h \cdot k \tag{4.90}$$
$$= \kappa_{g,hk} \cdot g \cdot \kappa_{h,k} \cdot g^{-1} \cdot g \cdot h \cdot k \ . \tag{4.91}$$

We also have

$$g \cdot h \cdot k = \kappa_{gh,k} \cdot (g \cdot h) \cdot k \tag{4.92}$$
$$= \kappa_{gh,k} \cdot \kappa_{g,h} \cdot g \cdot h \cdot k \ . \tag{4.93}$$

Hence, we find the following consistency condition

$$\kappa_{g,hk} \cdot g \cdot \kappa_{h,k} \cdot g^{-1} = \kappa_{gh,k} \cdot \kappa_{g,h} \ . \tag{4.94}$$

Action of $\kappa_{g,hk}$ on the fusion spaces gives

$$K^g \frac{\gamma_{g^{-1}(a)}(h,k)\gamma_{g^{-1}(b)}(h,k)}{\gamma_{g^{-1}(c)}(h,k)} K^g \frac{\gamma_a(g,hk)\gamma_b(g,hk)}{\gamma_c(g,hk)} = \frac{\gamma_a(gh,k)\gamma_b(gh,k)}{\gamma_c(gh,k)} \frac{\gamma_a(g,h)\gamma_b(g,h)}{\gamma_c(g,h)} \ . \tag{4.95}$$

Sometimes the 0-form and 1-form symmetries form a non-trivial 2-group. This is determined by a 3-cocycle, $[\beta]$, sometimes called the "Postnikov class," and it belongs to the cohomology group $H^3_{[\rho]}(G, \mathcal{A})$, where $\rho : G \to \mathrm{Aut}(\mathcal{A})$ specifies the action of the 0-form symmetry group G on the 1-form symmetry group $\mathcal{A}$. To determine this class, let us define the phase

$$\Omega_a(g,h,k) := \frac{K^g\gamma_{g^{-1}(a)}(h,k)K^g\gamma_a(g,hk)}{\gamma_a(gh,k)\gamma_a(g,h)} \ . \tag{4.96}$$

From this definition, it follows that

$$\frac{K^g\Omega_{g^{-1}(a)}(h,k,l)K^g\Omega_a(g,hk,l)\Omega_a(g,h,k)}{\Omega_a(gh,k,l)\Omega_a(g,h,kl)} = 1 \ . \tag{4.97}$$

This result can be shown by brute-force substitution and simplification. Using (4.95), we can show that

$$\Omega_a(g,h,k)\Omega_b(g,h,k) = \Omega_c(g,h,k) \ , \tag{4.98}$$

whenever $N_{ab}^c \neq 0$. Then,

$$d_a\Omega_a(g,h,k)d_b\Omega_b(g,h,k) = \sum_c N_{ab}^c d_c\Omega_c(g,h,k) \ . \tag{4.99}$$

Hence, $d_a\Omega_a(g,h,k)$ forms a 1-dimensional representation of the fusion rules and should be equal to $\frac{S_{ae}}{S_{1e}}$ for some charge e. As a result, we have

$$\Omega_a(g,h,k) = \frac{S_{ae}S_{11}}{S_{1e}S_{1a}} = M_{ae}^* \ . \tag{4.100}$$

Since, for a given e, $\Omega_a(g,h,k)$ is a phase for all a, the label e is abelian in the sense that its quantum dimension satisfies $d_e = 1$. This fact can be shown using the following argument.

$$d_e^2 = \sum_b \left|\frac{d_e d_b}{D}\right|^2 = \sum_b \left|\frac{d_e d_b}{D}M_{be}\right|^2 = \sum_b \left|\frac{d_e d_b}{D}\frac{S_{be}S_{00}}{S_{0e}S_{0b}}\right|^2 = \sum_b |S_{be}|^2 = 1 \ . \tag{4.101}$$

Hence, $e = \beta(g,h,k)$ is a map $\beta(g,h,k) : G\times G\times G \to \mathcal{A}$. It is a 3-cochain $\beta(g,h,k) \in C^3(G,\mathcal{A})$. Let us use (4.100) to simplify (4.97).

$$\begin{aligned}
1 &= \frac{\Omega_{g^{-1}(a)}(h,k,l)\Omega_a(g,hk,l)\Omega_a(g,h,k)}{\Omega_a(gh,k,l)\Omega_a(g,h,kl)} \\
1 &= M^*_{g^{-1}(a)\beta(h,k,l)}M^*_{a\beta(g,hk,l)}M^*_{a\beta(g,h,k)}M_{a\beta(gh,k,l)}M_{a\beta(g,h,kl)} \\
&= M^*_{ag(\beta(h,k,l))}M^*_{a\beta(g,hk,l)}M^*_{a\beta(g,h,k)}M^*_{a\overline{\beta(gh,k,l)}}M^*_{a\overline{\beta(g,h,kl)}} \\
&= M^*_{ag(\beta(h,k,l))\cdot\beta(g,hk,l)\cdot\beta(g,h,k)\cdot\overline{\beta(gh,k,\varphi_4)}\cdot\overline{\beta(g,h,kl)}} \ .
\end{aligned} \tag{4.102}$$

Since this logic holds for all a, we have

$$g(\beta(h,k,l))\cdot\beta(g,hk,l)\cdot\beta(g,h,k)\cdot\overline{\beta(gh,k,\varphi_4)}\cdot\overline{\beta(g,h,kl)} = 0 \ . \tag{4.103}$$

This shows that β is a 3-cocycle. In particular, $\beta \in Z^3_{[\rho]}(G, \mathcal{A})$, where the subscript, ρ, indicates a twisted cohomology group due to the non-trivial action of G on $\mathcal{A}$.

In fact, we can say more. Indeed, there is some freedom in decomposing natural isomorphisms in terms of phases (in (4.85)). More specifically, we have the freedom to choose

$$\gamma_a(g,h) \quad \text{or} \quad v_a(g,h)\gamma_a(g,h) \ , \tag{4.104}$$

where v_a are phases that satisfy $v_a v_b = v_c$ whenever $N^c_{ab} \neq 0$. It is easy to see that either choice leads to the same $\kappa_{g,h}$ in (4.85). However, the latter will change $\beta \in Z^3_{[\rho]}(G, \mathcal{A})$ by an exact cocycle. Hence, what defines a 2-group are actually equivalence classes $[\beta] \in H^3_{[\rho]}(G, \mathcal{A})$.

Example: Recall the Spin$(5)_2$ Chern-Simons theory that we discussed previously. This theory has a time reversal symmetry given by the permutation $\phi_1 \leftrightarrow \phi_2$ and $\psi_+ \leftrightarrow \psi_-$. Hence, it has a $\mathbb{Z}_2 = \{e, z\}$ 0-form symmetry. The modular data can be used to fix the possible values for the Postnikov class. For a non-unitary symmetry, we have

$$\Omega_a(g,h,k) = \frac{K^g \gamma_{g^{-1}(a)}(h,k) K^g \gamma_a(g,hk)}{\gamma_a(gh,k)\gamma_a(g,h)} \ , \tag{4.105}$$

where K^g is an operator which complex conjugates the element in between if g is a non-unitary symmetry. The only non-trivial $\Omega_a(g,h,k)$ in our case is

$$\Omega_a(z,z,z) = \frac{\gamma^*_{z(a)}(z,z)}{\gamma_a(z,z)} \ . \tag{4.106}$$

From, (4.100) we know that the only non-trivial $\beta(\cdots)$ is given by $\beta(z,z,z)$. Since the relevant cohomology group is $H^3(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$, $\beta(z,z,z)$ should be an order 2 abelian anyon. The only options are $\beta(z,z,z) = 1, \epsilon$. The relation (4.100) is trivially satisfied for $\beta(z,z,z) = 1$. For, $\beta(z,z,z) = \epsilon$ we have

$$\frac{\gamma^*_{z(a)}(z,z)}{\gamma_a(z,z)} = \frac{S_{a\epsilon}}{S_{a1}} \ . \tag{4.107}$$

Using this equation, we can derive some relations among the $\gamma_a(z,z)$ phases. In particular

$$\gamma_e(z,z) = \gamma^*_e(z,z) \ , \quad \gamma_{\phi_1}(z,z) = \gamma^*_{\phi_2}(z,z) \ , \quad \gamma_{\psi_+}(z,z) = -\gamma^*_{\psi_-}(z,z) \ . \tag{4.108}$$

If these relations are satisfied, then $\beta(z,z,z) = \epsilon$ is a valid choice. Note that since ϵ is not a quadratic residue, this choice corresponds to a non-trivial Postnikov class.

On the other hand, for $\beta(z,z,z)=1$, the quantities in (4.108) satisfy

$$\gamma_e(z,z)=\gamma_e^*(z,z)\ ,\quad \gamma_{\phi_1}(z,z)=\gamma_{\phi_2}^*(z,z)\ ,\quad \gamma_{\psi_+}(z,z)=\gamma_{\psi_-}^*(z,z)\ . \tag{4.109}$$

Now, the values for the F and R matrices for Spin$(5)_2$ can be used to constrain $U_z(a,b,c)$, which, in turn, will put several constraints on $\gamma_a(z,z)$. Using (4.88) we have the equation

$$\frac{\gamma_a(z,z)\gamma_{a^*}(z,z)}{\gamma_1(z,z)}=U_z(a,a^*,1)^{-1}U_z(z(a),z(a^*),1)\ . \tag{4.110}$$

It follows that

$$\gamma_{a^*}(z,z)=\gamma_a^*(z,z)\ , \tag{4.111}$$

for anyon, a, satisfying $z(a)=a$. Also, since all anyons in this theory are self conjugate, (4.111) implies that $\gamma_a(z,z)$ with $z(a)=a$ are real. This discussion restricts the quantities in (4.111) to be ±1.

Now let us make the choice $a=\psi_+, b=\psi_-, c=\epsilon$ in (4.88)

$$\frac{\gamma_{\psi_+}(z,z)\gamma_{\psi_-}(z,z)}{\gamma_\epsilon(z,z)}=U_z(\psi_+,\psi_-,\epsilon)^{-1}U_z(\psi_-,\psi_+,\epsilon)\ . \tag{4.112}$$

We would like to substitute for $\gamma_\epsilon(z,z)$ to write $\gamma_{\psi_+}(z,z)\gamma_{\psi_-}(z,z)$ purely in terms of $U_z(a,b,c)$ phases. Let us choose $a=\epsilon, b=\phi_1, c=\phi_1$ in (4.88)

$$\begin{aligned}\frac{\gamma_\epsilon(z,z)\gamma_{\phi_1}(z,z)}{\gamma_{\phi_1}(z,z)} &= U_z(\epsilon,\phi_1,\phi_1)^{-1}U_z(\epsilon,\phi_2,\phi_2) && (4.113)\\ \implies \gamma_\epsilon(z,z) &= U_z(\epsilon,\phi_1,\phi_1)^{-1}U_z(\epsilon,\phi_2,\phi_2)\ . && (4.114)\end{aligned}$$

Then we get,

$$\gamma_{\psi_+}(z,z)\gamma_{\psi_-}(z,z)=U_z(\psi_+,\psi_-,\epsilon)^{-1}U_z(\psi_-,\psi_+,\epsilon)U_z(\epsilon,\phi_1,\phi_1)^{-1}U_z(\epsilon,\phi_2,\phi_2)\ . \tag{4.115}$$

The R-matrix, $R^\epsilon_{\psi_+\psi_-}$, transforms under the symmetry in the following way

$$z(R^\epsilon_{\psi_+\psi_-})=U_z(\psi_-,\psi_+,\epsilon)R^\epsilon_{\psi_-\psi_+}U_z(\psi_+,\psi_-,\epsilon)^{-1}=(R^\epsilon_{\psi_+\psi_-})^*\ . \tag{4.116}$$

From the MTC data of Spin$(5)_2$ (see [129] for the full MTC data), we have $R^\epsilon_{\psi_+\psi_-}=R^\epsilon_{\psi_-\psi_+}=1$. It follows that

$$U_z(\psi_-,\psi_+,\epsilon)U_z(\psi_+,\psi_-,\epsilon)^{-1}=1\ . \tag{4.117}$$

Also, the F-matrix $(F^{\phi_1}_{\epsilon\phi_2\phi_1})^{\phi_1}_{\phi_2}$ transforms under the symmetry action as

$$U_z(\epsilon,\phi_1,\phi_1)U_z(\phi_1,\phi_2,\phi_2)(F^{\phi_2}_{\epsilon\phi_1\phi_2})^{\phi_2}_{\phi_1}U_z(\phi_1,\phi_2,\phi_2)^{-1}U_z(\epsilon,\phi_2,\phi_2)^{-1} = ((F^{\phi_1}_{\epsilon\phi_2\phi_1})^{\phi_1}_{\phi_2})^* \ . \tag{4.118}$$

Using, $(F^{\phi_1}_{\epsilon\phi_2\phi_1})^{\phi_1}_{\phi_2} = -1$ and $(F^{\phi_2}_{\epsilon\phi_1\phi_2})^{\phi_2}_{\phi_1} = 1$, we have

$$U_z(\epsilon,\phi_1,\phi_1)U_z(\epsilon,\phi_2,\phi_2)^{-1} = -1 \ . \tag{4.119}$$

From these relations, we have

$$\gamma_{\psi_+}(z,z)\gamma_{\psi_-}(z,z) = -1 \ . \tag{4.120}$$

This agrees with the constraints on $\gamma_a(z,z)$ set by $\beta(z,z,z) = \epsilon$. Hence, Spin(5)$_2$ MTC has a non-trivial Postnikov class.[a]

[a] This theory and its non-trivial Postnikov class are discussed in [140] with the equivalent name USp(4)$_2$ Chern-Simons theory.

4.4.2 Symmetries of Abelian TQFTs and Galois Conjugation

In abelian TQFTs, all anyons are abelian and hence the 1-form symmetry group coincides with the fusion rules. We know that Galois conjugation relates different solutions of the Pentagon and Hexagon equations. Hence, it preserves the fusion rules. Thus, the 1-form symmetry group is invariant under Galois conjugation. We would like to find the relationship between 0-form symmetries of Galois conjugate abelian theories. To that end, let $\mathcal{A}$ be the set of anyons of the theory. As alluded to previously, an abelian TQFT is determined completely by this set and the topological spin function

$$\theta : \mathcal{A} \to U(1) \ . \tag{4.121}$$

The automorphism group of $\mathcal{A}$, denoted Aut($\mathcal{A}$), is a subset of the permutation group acting on $\mathcal{A}$. For it to be a symmetry, G, of the TQFT, it has to preserve the topological spins. That is, if $g \in G$ we require

$$\theta_{g(a)} = \theta_a \ \text{(up to conjugation for anti-unitary symmetries)} \ . \tag{4.122}$$

The symmetry group, G, is a subgroup of Aut($\mathcal{A}$). The topological spins are of the form $\theta_a = e^{2\pi i h_a}$, where h_a is a rational number. Hence, the topological spins are roots of unity, and we can write $h_a = \frac{f(a)}{N}$ for some integer N. The condition for g to be a symmetry can be written as

$$h_{g(a)} = \pm \quad h_a \quad \text{mod } 1 \ \text{(minus sign for anti-unitary symmetries)} \tag{4.123}$$

$$\implies \quad f(g(a)) = \pm f(a) \text{ mod } N \ . \tag{4.124}$$

Under Galois conjugation by some q coprime to N,

$$h_a \to q h_a \ . \tag{4.125}$$

We can see that the condition (4.123) becomes

$$q f(g(a)) = \pm q f(a) \text{ mod } N \implies f(g(a)) = \pm f(a) \text{ mod } N \ . \tag{4.126}$$

Since the set of labels $\mathcal{A}$ do not change under Galois conjugation, neither does the automorphism group. We have seen above that the condition (4.123) which restricts the symmetry group G to a subgroup of Aut($\mathcal{A}$) also doesn't change under Galois conjugation. Hence, the 0-form symmetry group is Galois invariant.

In summary, we have found that Galois conjugate symmetries have the same 0-form and 1-form symmetries. What about the 2-group structure? It is widely believed that all abelian TQFTs have trivial 2-group. This is in agreement with all known cases. A proof for unitary symmetries was given in [141]. Assuming this result extends to all 0-form symmetries, it is trivially true that the 2-group symmetry is invariant under Galois conjugation. However, let us give an alternate proof which does not rely on this conjecture.

For an abelian TQFT, the F and R symbols are phases. Moreover, the $U_g(a,b)$ can be taken to belong to a cyclotomic field. This statement follows from the fact that for abelian TQFTs, we can choose a gauge in which all the F symbols are valued in ± 1 [112] . For abelian TQFTs, the symmetry transformation of the F symbols (4.82) can be written as

$$U_g(g(a),g(b))U_g(g(a)+g(b),g(c))U_g(g(b),g(c))^{-1}U_g(g(a),g(b)+g(c))^{-1}F(g(a),g(b),g(c)) = K^g F(a,b,c) K^g \ . \tag{4.127}$$

If we are in a gauge in which the F symbols are valued in ± 1, we get

$$U_g(g(a),g(b))^2 U_g(g(a)+g(b),g(c))^2 U_g(g(b),g(c))^{-2} U_g(g(a),g(b)+g(c))^{-2} = 1 \ . \tag{4.128}$$

This result shows that the phases $U_g(g(a),g(b))^2$ should form a 2-cocycle. Moreover, since $U_g(g(a),g(b))^2$ is defined only up to symmetry gauge transformations, $U_g(g(a),g(b))^2$ should be an element of $H^2(G,U(1))$. These quantities can always be chosen to be $|G|^{\text{th}}$

roots of unity. Therefore, the phases $U_g(g(a), g(b))$ are at most $2|G|^{\text{th}}$ roots of unity.

Therefore, a Galois conjugation of F and R induces a Galois conjugation on $U_g(a, b)$ which acts on these phases as

$$F(a,b,c) \to F(a,b,c)^q \ , \quad R(a,b) \to R(a,b)^q \ , \quad U(a,b) \to U(a,b)^q \ , \tag{4.129}$$

for some integer q coprime to the order of the cyclotomic field. As a result, in the Galois conjugate theory, the symmetry acts on the fusion spaces as $U_g(a, b)^q$. The phases satisfying (4.88) are $\gamma_a(g, h)^q$. Therefore, in the Galois conjugate theory, using (4.96) and (4.100), we get

$$\Omega_a(g,h,k)^q = \frac{(S_{a\beta(g,h,k)})^q}{(S_{a1})^q} \ , \tag{4.130}$$

where S^q_{ab} is an element of the S-matrix of the Galois conjugate theory. Hence, the Galois conjugate theory has the same Postnikov class.

4.4.3 Symmetries of non-abelian TQFTs and Galois conjugation

As a more gentle starting point, we first consider the case of multiplicity free non-abelian TQFTs (i.e., theories with non-invertible anyons where each fusion product appears at most once in a given fusion) with a cyclotomic defining number field. We then proceed to the general non-abelian case.

$N^c_{ab} = 0, 1$ and cyclotomic defining number field

Let us consider a multiplicity-free MTC, C, with MTC data denoted by R^c_{ab}, F^d_{abc}. Let K_C be the defining number field of C. Before considering the possibility of a more general defining number field, it is useful to consider the case when K_C is a cyclotomic field.

In this case, the Galois action on the MTC data, as well as its effect on the $U_g(a, b, c)$ and $\gamma_a(g, h)$ phases, can be described explicitly. Therefore, let $K_C = \mathbb{Q}(\xi_N)$, where N is some integer. Let us consider the MTC, $q(C)$, which is obtained by Galois conjugating this data with respect to some $q \in \text{Gal}(\mathbb{Q}(\xi_N))$. All quantities in $q(C)$ will have a hat on top, and so the MTC data of the Galois conjugated theory is $\widehat{R}^c_{ab}, \widehat{F}^d_{abc}$. C has a 1-form symmetry group $\mathcal{A}$ and 0-form symmetry group G. G acts on the anyons in the theory as $g(a)$ and permutes them.

The gauge-invariant quantities of the theory should be invariant under the symmetry action. For example, we have $S_{ab} = S_{g(a)g(b)}$. This relation hold holds even after Galois conjugation. Therefore $\widehat{S}_{ab} = \widehat{S}_{g(a)g(b)}$. As a result, the zero-form symmetry group, G, of the initial TQFT, $\mathcal{T}_1$ is isomorphic to the symmetry group of the gauge-invariant data of the Galois-conjugated TQFT, $\mathcal{T}_2$.

The symmetry acts on the fusion spaces of C through the unitary matrix, U_g. Because we have a multiplicity free theory, the U_g's are just phases. By definition, we have the following equalities

$$U_g(g(b),g(a),g(c))R^{g(c)}_{g(a)g(b)}U_g(g(a),g(b),g(c))^{-1} = K^g R^c_{ab} K^g \ , \tag{4.131}$$

$$U_g(g(a),g(b),g(e))U_g(g(e),g(c),g(d))(F^{g(d)}_{g(a)g(b)g(c)})^{g(f)}_{g(e)} \times U_g(g(b),g(c),g(f))^{-1}U_g(g(a),g(f),g(d))^{-1} = K^g(F^d_{abc})^f_e K^g \ . \tag{4.132}$$

From these equations we have

$$U_g(g(b),g(a),g(c))U_g(g(a),g(b),g(c))^{-1} = K^g R^c_{ab} K^g (R^{g(c)}_{g(a)g(b)})^{-1} \ , \tag{4.133}$$

$$U_g(g(a),g(b),g(e))U_g(g(e),g(c),g(d))U_g(g(b),g(c),g(f))^{-1}U_g(g(a),g(f),g(d))^{-1} = K^g(F^d_{abc})^f_e K^g((F^{g(d)}_{g(a)g(b)g(c)})^{g(f)}_{g(e)})^{-1} \tag{4.134}$$

Since $R^{g(c)}_{g(a)g(b)}$ and $(F^{g(d)}_{g(a)g(b)g(c)})^{g(f)}_{g(e)}$ belong to $\mathbb{Q}(\xi_N)$, $U_g(g(b),g(a),g(c))U_g(g(a),g(b),g(c))^{-1}$ and $U_g(g(a),g(b),g(e))U_g(g(e),g(c),g(d))U_g(g(b),g(c),g(f))^{-1}U_g(g(a),g(f),g(d))^{-1}$ are both phases in $\mathbb{Q}(\xi_N)$. Note that even though the above combinations of the U_g phases are guaranteed to be in the cyclotomic field of the MTC data, we do not assume that the individual phases themselves belong to a cyclotomic field. Galois conjugating both sides of the above equations by $q \in \mathrm{Gal}(\mathbb{Q}(\xi_N))$, we get

$$\begin{aligned} q(U_g(g(b),g(a),g(c))U_g(g(a),g(b),g(c))^{-1}) &= q(K^g R^c_{ab} K^g (R^{g(c)}_{g(a)g(b)})^{-1}) &(4.135)\\ &= K^g (R^c_{ab})^q K^g (R^{g(c)}_{g(a)g(b)})^{-q} &(4.136)\\ &= K^g \widehat{R}^c_{ab} K^g (\widehat{R}^{g(c)}_{g(a)g(b)})^{-1}) \ . &(4.137) \end{aligned}$$

In writing down the equations above, we used the fact that the R^c_{ab} are phases for a multiplicity-free theory and that $\widehat{R}^c_{ab} = (R^c_{ab})^q$. Also, since $U_g(g(b),g(a),g(c))U_g(g(a),g(b),g(c))^{-1}$ is a phase in $\mathbb{Q}(\xi_N)$, Galois conjugating it by q amounts to taking its q^{th} power. Note that we can commute the complex conjugation and Galois conjugation operation on the RHS of the above equation since we have chosen a gauge in which the MTC data is in a cyclotomic field (the Galois group in this case is abelian). If we had chosen another basis in which the MTC data belongs to a field extension with non-abelian Galois group, complex conjugation might not commute with a general Galois conjugation. We have

$$(U_g(g(b),g(a),g(c))U_g(g(a),g(b),g(c))^{-1})^q \widehat{R}^{g(c)}_{g(a)g(b)} = K^g \widehat{R}^c_{ab} K^g \ . \tag{4.138}$$

Following the same arguments, from the action of the symmetry on the F^d_{abc}, we obtain

$$(U_g(g(a),g(b),g(e))U_g(g(e),g(c),g(d))U_g(g(b),g(c),g(f))^{-1}U_g(g(a),g(f),g(d))^{-1})^q \times (\widehat{F}^{g(d)}_{g(a)g(b)g(c)})^{g(f)}_{g(e)} = K^g(\widehat{F}^d_{abc})^f_e K^g \ . \tag{4.139}$$

Note that since the F matrix elements need not be phases, their Galois conjugation does not usually correspond to taking a q^{th} power. However, we have only used $\widehat{F}^d_{abc} = q(F^d_{abc})$ in writing down the above equations.

Let us define phases $\widehat{U}_g(g(a),g(b),g(c))$ as follows

$$\widehat{U}_g(g(a),g(b),g(c)) := U_g(g(a),g(b),g(c))^q \ . \tag{4.140}$$

Then, we have

$$\widehat{U}_g(g(b),g(a),g(c))\widehat{U}_g(g(a),g(b),g(c))^{-1}\widehat{R}^{g(c)}_{g(a)g(b)} = K^g\widehat{R}^c_{ab}K^g \ , \tag{4.141}$$

and

$$\widehat{U}_g(g(a),g(b),g(e))\widehat{U}_g(g(e),g(c),g(d))\widehat{U}_g(g(b),g(c),g(f))^{-1}\widehat{U}_g(g(a),g(f),g(d))^{-1} \times (\widehat{F}^{g(d)}_{g(a)g(b)g(c)})^{g(f)}_{g(e)} = K^g(\widehat{F}^d_{abc})^f_e K^g \ . \tag{4.142}$$

This argument shows that $q(C)$, with MTC data $\widehat{R}^c_{ab}$, $\widehat{F}^d_{abc}$, has an isomorphic symmetry group, G, which acts on its anyons as $g(a)$, but now with an action on the fusion spaces given by $\widehat{U}_g(a,b,c)$. This discussion implies that Galois conjugation preserves the 0-form symmetry of the theory.[79]

To understand what happens to the Postnikov class, let us also define the phases $\widehat{\gamma}_a(g,h)$

$$\widehat{\gamma}_a(g,h) := (\gamma_a(g,h))^q \ , \tag{4.143}$$

where $\gamma_a(g,h)$ are phases satisfying (4.88). It is clear that we have,

$$\frac{\widehat{\gamma}_a(g,h)\widehat{\gamma}_b(g,h)}{\widehat{\gamma}_c(g,h)} = \widehat{U}_g(a,b,c)^{-1}(K^g\widehat{U}_h^{-1}(g^{-1}(a),g^{-1}(b),g^{-1}(c))K^g)\widehat{U}_{gh}(a,b,c) \ , \tag{4.144}$$

If $\beta(g,h,k)$ is the Postnikov class of C, it satisfies (from (4.100))

$$\Omega_a(g,h,k) = \frac{S_{a\beta(g,h,k)}}{S_{a1}} \ . \tag{4.145}$$

Here $\frac{S_{a\beta(g,h,k)}}{S_{a1}}$ is a phase for an abelian anyon, $\beta(g,h,k)$. Hence, Galois conjugation by

[79]More precisely, what we have shown is that the 0-form symmetry of C maps to a subgroup of that of $q(C)$. But, using the invertibility of the Galois action, we can run the above argument starting from $q(C)$ proving that their 0-form symmetry groups are indeed isomorphic.

q corresponds to taking its q^{th} power. So we have,

$$\frac{\widehat{S}_{a\beta(g,h,k)}}{\widehat{S}_{a1}} = q\left(\frac{S_{a\beta(g,h,k)}}{S_{a1}}\right) = \left(\frac{S_{a\beta(g,h,k)}}{S_{a1}}\right)^q . \tag{4.146}$$

Also, from the relation between $\widehat{\gamma}_a(g,h)$ and $\gamma_a(g,h)$, we have

$$\widehat{\Omega}_a(g,h,k) = (\Omega_a(g,h,k))^q , \tag{4.147}$$

where $\widehat{\Omega}_a(g,h,k)$ is defined similarly to (4.96), but now with $\widehat{\gamma}_a(g,h)$.

Using (4.145) we have,

$$\widehat{\Omega}_a(g,h,k) = \frac{\widehat{S}_{a\beta(g,h,k)}}{\widehat{S}_{a1}} . \tag{4.148}$$

Hence, the Postnikov class, $\widehat{\beta}(g,h,k)$, of $q(C)$ is the same as that of C. This discussion shows that Galois conjugation preserves the complete 2-group symmetry of a multiplicity-free TQFT.

In the next subsection, we will extend the argument in this section to TQFTs with multiplicity in its fusion rules.

General TQFTs

Let us consider a general MTC, C, with defining number field, K_C (i.e., we do not impose a restriction on multiplicity or take K_C to necessarily be cyclotomic). In this case, the transformation laws for the F and R matrices under the symmetry action are more complicated.

$$\sum_{\mu'\nu'}[U_g(g(b),g(a),g(c))]_{\mu\mu'}(R^{g(c)}_{g(a)g(b)})_{\mu'\nu'}[U_g(g(a),g(b),g(c))^{-1}]_{\nu'\nu} = K^g(R^c_{ab})_{\mu\nu}K^g , \tag{4.149}$$

$$\sum_{\alpha'\beta',\mu',nu'}[U_g(g(a),g(b),g(e))]_{\alpha\alpha'}[U_g(g(e),g(c),g(d))]_{\beta\beta'}(F^{g(d)}_{g(a)g(b)g(c)})^{(g(f),\mu',\nu')}_{(g(e),\alpha',\beta')}$$
$$\times[U_g(g(b),g(c),g(f))^{-1}]_{\mu'\mu}[U_g(g(a),g(f),g(d))^{-1}]_{\nu'\nu} = K^g(F^d_{abc})^{(f,\mu,\nu)}_{(e,\alpha,\beta)}K^g . \tag{4.150}$$

Note that the above equations form a set of polynomial equations for $U_g(a,b,c)$ with coefficients belonging to K_C. Hence, if the $U_g(a,b,c)$'s belong to a finite field extension, then it has to be an extension over K_C. The following Lemma shows that the $U_g(a,b,c)$'s belong to a finite field extension:

Lemma 4.4.1 [106]: *Algebraic points of a complex affine algebraic variety defined over $\overline{\mathbb{Q}}$ are dense in the Zariski topology.*

We know that there is a gauge in which F and R matrices are given in an algebraic number field. Any algebraic number field is a subfield of $\overline{\mathbb{Q}}$. Hence, $U_g(a,b,c)$ are solutions of polynomials with coefficients in $\overline{\mathbb{Q}}$. Using the Lemma above, it is clear that there is a gauge in which $U_g(a,b,c)$ belongs to an algebraic field, say K'_U. Let K_U be the normal closure of K'_U. This procedure defines a Galois field, and K_U is, in general, a field extension of K_C.

We expect the equations (4.82) and (4.81) to give a unique solution up to symmetry gauge transformations.[80] Hence, any element $p \in \text{Gal}(K_U/K_C)$ acts on $U_g(a,b,c)$ to relate it to another set of solutions which is gauge equivalent to the one we started with.

The existence of the Galois field K_U shows that we have an action of $\text{Gal}(K_U)$ on F, R, and U_g. Therefore, we have a map from MTC data with symmetry g and symmetry action U_g on the fusion spaces to another such system. Consider the Galois action on the F and R matrices corresponding to some $q \in \text{Gal}(K_C)$. We know that there exists some $\sigma \in \text{Gal}(K_U)$ such that the restriction of the action of σ to K_C is equal to q. Hence, $\sigma(U_g(a,b,c))$ is a solution for the equations (4.150) and (4.149) where the F and R matrices are replaced by $\sigma(F) = q(F)$ and $\sigma(R) = q(R)$.

Note that the equations (4.150) and (4.149) are not algebraic. For anti-unitary symmetries, we have a complex conjugation action on the F and R symbols which may not commute with the Galois action. If F and R belongs to a CM field, then we know that any Galois conjugation commutes with complex conjugation. Therefore, we get the following result:

Theorem 4.4.2 *A TQFT and its Galois conjugates have isomorphic unitary and anti-unitary 0-form symmetries if there is a gauge in which the F and R symbols of the TQFT belong to a CM field.*

For unitary symmetries, the equations (4.150) and (4.149) are algebraic. Therefore, we get the corollary

Corollary 4.4.2.1 *A TQFT and its Galois conjugates have isomorphic 0-form unitary symmetries.*

In order to check whether the whole 2-group is invariant under Galois conjugation, we have to show that the Postnikov class remains invariant under it. In order to find the Postnikov class, we have to solve the constraint

$$\frac{\gamma_a(g,h)\gamma_b(g,h)}{\gamma_c(g,h)}\delta_{\mu\nu} = \sum_{\alpha,\beta}[U_g(a,b;c)^{-1}]_{\mu\alpha}K^{q(g)}[U_h[\bar{g}(a),\bar{g}(b),\bar{g}(c)]_{\alpha\beta}K^{q(g)}[U_{gh}(a,b,c)]_{\beta\nu}\,. \tag{4.151}$$

[80] This statement has been proven in the case with no multiplicity [141], but it is an open problem in the general case.

Using the same arguments as we used in analyzing the Galois action on the $U_g(a,b,c)$, we can define a Galois field, K_γ, containing $\gamma_a(g,h)$, that is, in general, a field extension of K_U. Corresponding to every element $q \in \text{Gal}(K_C)$, where K_C is the Galois field containing the F and R symbols, we have some $\sigma \in \text{Gal}(K_\gamma)$ such that $\sigma|_{K_C} = q$. The phases $\sigma(\gamma_a(g,h))$ satisfy the constraint (4.151) with $U_g(a,b,c)$ replaced by $\sigma(U_g(a,b,c))$ if Galois action on the U_g matrices commutes with complex conjugation.

Therefore, we find that if g is a unitary symmetry of an MTC, C, with symmetry action phases $U_g(a,b,c)$ and $\gamma_a(g,h)$ satisfying (4.151), then the Galois conjugate theory $q(C)$ for some $q \in \text{Gal}(K_C)$ has symmetry g with symmetry action phases $\sigma(U_g(a,b,c))$ and $\sigma(\gamma_a(g,h))$ where $\sigma \in \text{Gal}(K_\gamma)$ and $\sigma|_{K_C} = q$. If q is anti-unitary, then the same is true if K_C and K_U are CM fields.

If K_U is a cyclotomic field extension, we can show that the $\gamma_a(g,h)$ also belong to a cyclotomic field. Indeed, suppose we have $K_U = \mathbb{Q}(\xi_M)$ for some integer M to which U_g belongs to. Since the RHS of (4.151) is a phase, it should have an order which divides M. Hence, we have

$$\left(\frac{\gamma_a(g,h)\gamma_b(g,h)}{\gamma_c(g,h)}\right)^M = \frac{\gamma_a(g,h)^M\gamma_b(g,h)^M}{\gamma_c(g,h)^M} = 1 \ , \tag{4.152}$$

whenever $N_{ab}^c \neq 0$. Therefore, we can perform the ν-gauge transformation

$$\gamma_a(g,h)^M \to \gamma_a(g,h)^M \nu_a(g,h) \ , \tag{4.153}$$

where $\nu_a(g,h) = \gamma_a(g,h)^{-M}$ to set $\gamma_a(g,h)^M = 1$ for all anyons a and $g,h \in G$. This shows that there exists a ν-gauge in which the phases $\gamma_a(g,h)$ all belong to $\mathbb{Q}(\xi_M)$. Hence, given U_g matrices, the solutions to (4.151) belong to $\mathbb{Q}(\xi_M)$.

To complete our discussion, note that we have the relation

$$\Omega_a(g,h,k) = \frac{S_{a\beta(g,h,k)}}{S_{a1}} \ , \tag{4.154}$$

where $\Omega_a(g,h,k)$ is defined in (4.96). Under the action of any $\sigma \in \text{Gal}(K_\gamma)$, we have

$$\sigma(\Omega_a(g,h,k)) = \sigma\left(\frac{S_{a\beta(g,h,k)}}{S_{a1}}\right) \ . \tag{4.155}$$

Since $\widehat{\Omega}_a(g,h,k) = \sigma(\Omega_a(g,h,k))$ and $\widehat{S}_{ab} = \sigma(S_{ab})$ are the respective quantities in the Galois conjugate theory, we have

$$\widehat{\Omega}_a(g,h,k) = \frac{\widehat{S}_{a\beta(g,h,k)}}{\widehat{S}_{a1}} \ . \tag{4.156}$$

The actions of such σ's exhaust all possible Galois conjugations of F and R.

In summary, we have the following result:

Theorem 4.4.3 *A TQFT and its Galois conjugates have isomorphic 2-group symmetry if there is a gauge in which the F and R symbols as well as the $U_g(a, b, c)$ belong to a CM field.*

For unitary symmetries, all the constraints involved are algebraic. Therefore, we get the corollary

Corollary 4.4.3.1 *A TQFT and its Galois conjugates have the same unitary 2-group symmetry.*

By a unitary 2-group symmetry, we mean a 2-group symmetry in which the 0-form symmetry is a group of unitary symmetries. Note that the set of TQFTs with the same fusion rules shares the same 1-form symmetry group. However, they may not share the same 0-form, and consequently the same 2-group symmetry. For example, the Toric code has $\mathbb{Z}_2$ 0-form symmetry while the 3-fermion model has an S_3 0-form symmetry group. However, our results above show that Galois orbits should contain TQFTs with the same 0-form and 2-group symmetries (up to a mild assumption for anti-unitary symmetries).

4.4.4 Gauging and Galois Conjugation

In previous sections, we studied how Galois conjugation acts on the space of TQFTs and how it acts on specific families of TQFTs within it. Gauging is another way to move through the space of TQFTs.

If a TQFT $\mathcal{T}$ has a 0-form symmetry given by some finite group G, it can be gauged to obtain a new TQFT, $\mathcal{T}/G$. We will somewhat unconventionally refer to $\mathcal{T}$ and $\mathcal{T}/G$ as the *magnetic and electric theories*, respectively.[81] The $\mathcal{T}/G$ TQFT does not have the 0-form symmetry G. Instead, it has a Rep(G) fusion subcategory. We can go from $\mathcal{T}/G$ to $\mathcal{T}$ by condensing Rep(G) [58]. When G is abelian, this condensation is the same as gauging the 1-form symmetry, Rep(G) [50]. Therefore, Rep(G) condensation is the inverse of G gauging.

We would like to understand how gauging a 0-form symmetry interacts with Galois conjugation. Our discussion of the Galois action on Rep(G) reveals the following result:

[81] Let C^G be the MTC corresponding to the TQFT $\mathcal{T}/G$, where C is the MTC corresponding to $\mathcal{T}$. We will use the notation C^G and $\mathcal{T}/G$ more or less interchangeably to denote the electric theory.

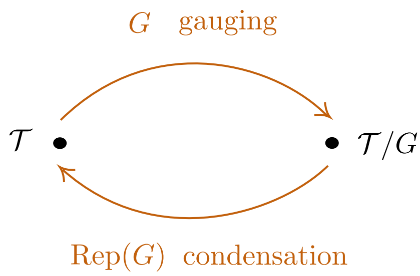


Figure 4.7: Gauging and condensation are inverses.

Theorem 4.4.4 *If a TQFT, $\mathcal{T}$, is obtained from gauging a symmetry G of another TQFT, so are all of its Galois conjugates.*

Proof: Since $\mathcal{T}$ is obtained from another TQFT by gauging G, it contains a fusion subcategory, Rep(G). Under a Galois conjugation of $\mathcal{T}$, the resulting theory, $\mathcal{T}'$, also has a fusion subcategory Rep(G). This statement holds because Rep(G) is invariant under Galois conjugation. Now, we can condense Rep$(G) \subset \mathcal{T}'$ to obtain another TQFT with 0-form symmetry G. Therefore, $\mathcal{T}'$ is also obtained from gauging a symmetry G of some TQFT. □

In fact, Galois conjugation of a TQFT, C^G, with a Rep(G) subcategory can be related to Galois conjugation of the TQFT, C, obtained by condensing the Rep(G) subcategory. To obtain C from C^G through condensation of Rep(G), we don't have to keep track of all the anyons in C^G. In fact, the anyons in C correspond to the anyons in the subcategory, $L \subset C^G$, which braid trivially with all condensing anyons. That is, L is the centralizer of Rep(G) in C^G

$$L = \{c \in C^G | S_{ca} = \frac{1}{\mathcal{D}} d_c d_a \; \forall a \in \text{Rep}(G)\} \; . \tag{4.157}$$

The twists of the anyons in C are completely determined by the twists of the anyons in L. Moreover, the quantum dimensions of the anyons in C are the same as those in L, up to some integer factors. Therefore, the cyclotomic field containing the $(\tilde{S}, T)$ modular data of $L \subset C^G$, $\mathbb{Q}(\xi_M)$, is the same as the cyclotomic field containing the corresponding modular data of the anyons in C, K_M.[82] Also, from theorem 4.3.4, we know that the fusion rules of the boundary excitations are invariant under Galois action of the bulk TQFT. Condensation of anyons is a more general procedure, where we have a domain wall between two phases instead of a gapped boundary. In fact, the TQFT obtained after condensation is described by a modular subcategory of the category of

[82]This statement follows from re-writing the un-normalized S matrix as [41]

$$\tilde{S}_{ab} = \sum_c N_{a\bar{b}}^c \frac{\theta_c}{\theta_a \theta_b} d_c \; . \tag{4.158}$$

representations of the connected commutative separable algebra describing the boundary excitations [125]. Therefore, the fusion rules of C obtained after condensation are invariant under Galois conjugation of C^G. Since the (un-normalized) S matrix of C is determined by the twists and quantum dimensions of L along with the fusion rules of C^G, we have the following result:

Theorem 4.4.5 *Galois conjugation of C^G with respect to $q \in Gal(K_{C^G})$, where K_{C^G} is the defining Galois field of C^G, induces a Galois action on the modular data of C by $q|_{K_M}$, where K_M is the subfield containing the $(\tilde{S}, T)$ modular data of C.*

Proof: Let K_{C^G} be the algebraic field extension containing the data of the MTC C^G. Consider the Galois conjugation of C^G by some $q \in \mathrm{Gal}(K_{C^G})$. The cyclotomic field K_M is a subfield of K_{C^G}, which is a normal extension of $\mathbb{Q}$. Therefore, the restriction $q|_{K_M}$, where $q \in \mathrm{Gal}(K_{C^G})$ acts on K_M as Galois action on the field and this restriction is surjective. Since the modular data of C is determined by twists and quantum dimensions of L, as well as the fusion rules of C^G, $q \in \mathrm{Gal}(K_{C^G})$ action on C^G induces a $q|_{K_M}$ action on the modular data of C. □

In fact, the results above can be generalized due to the algebraic nature of 0-form symmetry gauging. To understand this statement, consider a fusion category, C, with a G-action. Gauging G amounts to constructing the category of G-equivariant objects. A G-equivariant object is a pair, (x, u_g), for all $g \in G$ and x an object in C. Here, u_g are isomorphisms, $u_g : g(x) \to x$, such that the following constraint is satisfied for all $g, h \in G$

$$u_{gh} \circ \gamma_a(g, h) = u_g \circ g(u_h) \ , \tag{4.159}$$

where $\gamma_a(g, h)$ is the isomorphism $g(h(a)) \to gh(a)$. This discussion is analogous to how we go from a global symmetry acting on a Hilbert space, which acts non-trivially on the states, to a gauged theory where the physical states are invariant under the gauge group. In the G-equivariant object (x, u_g), u_g is the isomorphism which tells us that $g(x)$ is the same as x. Since C is a tensor category, we also have isomorphisms, $\psi_g(a, b) : g(a) \otimes g(b) \to g(a \otimes b)$. The morphisms between G-equivariant objects are

$$\mathrm{Hom}((x, u_g), (y, v_g)) = \{f \in \mathrm{Hom}(x, y) | v_g \circ g(f) = f \circ u_g \ \forall g \in G\} \ . \tag{4.160}$$

The tensor product of objects is

$$(x, u_g) \otimes (y, v_g) = (x \otimes y, w_g) \ , \tag{4.161}$$

where $w_g = u_g v_g \circ \psi_g^{-1}(a, b)$. The G-equivariant objects form a fusion category C^G (for a detailed discussion of this construction see [142, 143]).

Given a TQFT with a 0-form symmetry G, we have a corresponding MTC, C, with a G action. Provided that certain obstructions vanish, we can construct a G-crossed braided category, C_G, from C with a G-crossed braiding [144]

$$c_{x,y} : x \otimes y \to g(y) \otimes x \text{ where } x \in C_g, g \in G, y \in C \ . \tag{4.162}$$

This amounts to adding the data of the symmetry defects. In a somewhat more field theoretical language, this step can be thought of as coupling the theory to background gauge fields prior to gauging [141]. Gauging the symmetry, G, then amounts to constructing the category of G-equivariant objects of C_G. The G-crossed braiding in C_G can be used to endow C^G with a braiding as follows [142]

$$b_{(x,u_g),(y,v_g)} = (v_g \otimes id_{x_g}) \circ c_{x_g,y} \ , \tag{4.163}$$

where $x = \oplus_g x_g$. Note that the braided fusion category, C^G, is modular if and only if C is modular, and the grading in C_G is faithful (recall that since C is modular, it has a spherical structure, so C^G is also spherical) [145].

Since the data of C^G and C_G are related algebraically, every Galois action on C_G leads to a Galois conjugated C^G and vice-versa. We can also use our discussion on Galois action and Drinfeld center to obtain this result. Indeed, suppose we have some MTC, C, with 0-form symmetry, G. Let us also suppose that the obstructions to gauging vanishes and we have a G-crossed braided category, C_G. Let C^G be the TQFT obtained after gauging the symmetry G. These theories are related in the following way [68]

$$C \boxtimes \bar{C}^G = \mathcal{Z}(C_G) \ . \tag{4.164}$$

Here, $\bar{C}^G$ is a modular tensor category with braiding given by $\bar{c}_{x,y} = c^{-1}_{y,x}$, where $c_{x,y}$ is the braiding of C^G.

To understand this relation, it is useful to examine two special cases. When C is the trivial TQFT, then C_G is equivalent to Vec^ω_G, and the above relation becomes $\bar{C}^G = \mathcal{Z}(\text{Vec}^\omega_G)$, which is the familiar result that taking the Drinfeld center of Vec^ω_G is the same as gauging a natural isomorphism of the trivial TQFT (up to inverted braiding). When the group G is trivial, this relation becomes $C \boxtimes \overline{C} = Z(C)$, which shows that the Drinfeld center of an MTC is a Deligne product of that MTC with itself up to inverted braiding. Equation (4.164) implies that the MTC data of the various TQFTs appearing in (4.164) are related via

$$F_C \otimes F_{\bar{C}^G} = F_{\mathcal{Z}(C_G)}, \ R_C \otimes R_{\bar{C}^G} = R_{\mathcal{Z}(C_G)} \ . \tag{4.165}$$

Therefore, the MTC data of C^G can be determined in terms of the data of C and $\mathcal{Z}(C_G)$.

Suppose we Galois conjugate C_G w.r.t. some $q \in \mathrm{Gal}(K_{C_G})$. C is a modular subcategory of C_G. Therefore, q acts on C. From (4.73) we have some $q' \in \mathrm{Gal}(K_{\mathcal{Z}(C_G)})$ such that

$$\mathcal{Z}(q(C_G)) = q'(\mathcal{Z}(C_G)) \ . \tag{4.166}$$

Hence, we get

$$\mathcal{Z}(q(C_G)) = q'(\mathcal{Z}(C_G)) = q'(C \boxtimes \bar{C}^G) = q(C) \boxtimes q''(C^G) \ , \tag{4.167}$$

where $q'' \in \mathrm{Gal}(K_{C^G})$, and in the last equality above we have used the fact that any Galois action on a Deligne product can be written as a Galois action on the individual TQFTs. We have also used the fact that q acts on C when q acts on C_G. Therefore, we find that Galois action on the G-crossed braided theory induces a Galois action on the gauged theory.

As a consequence, similarly to theorem 4.3.4, we obtain the following:

Theorem 4.4.6 *Corresponding to every $q \in Gal(K_{C_G})$, there exists a $q' \in Gal(K_{C^G})$ such that*

$$(q(C_G))^G = q'(C^G) \ , \tag{4.168}$$

where $(q(C_G))^G$ denotes gauging the G symmetry after Galois action on C_G.[a]

[a] See [146] for a similar result in the context of gauging symmetries of certain VOAs.

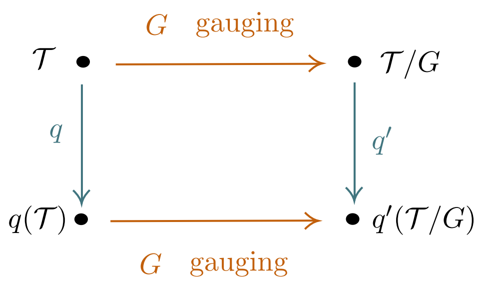


Figure 4.8: Galois conjugation of $\mathcal{T}$ induces a Galois action on $\mathcal{T}/G$ and vice-versa.

Given an MTC, C, with symmetry, G, there is a cohomological classification of unitary G-crossed braided categories that can be constructed from C [58,144]. We can use this classification to describe the Galois action on C_G more explicitly. To gauge a symmetry G of C, we should have trivial Postnikov class. This is because a non-trivial Postnikov class leads to a coupling between gauge transformations of the 1-form and 0-form symmetry background gauge fields [141]. Therefore, the 0-form symmetry alone cannot be gauged, though if the 0-form and 1-form 't Hooft anomaly vanishes, the full 2-group can be gauged. If the Postnikov class vanishes, then the classification follows from a choice of the fractionalization class, η, which forms a torsor over $H^2_{[\rho]}(G, A)$, where $[\rho]$ indicates that we have a twisted cohomology group due to the G group action on the abelian anyons, A, in C. Given a fractionalization class, it determines element of the group $H^4(G, U(1))$ which is the 't Hooft anomaly of the symmetry G (which is also sometimes called the defectification obstruction). If the 't Hooft anomaly vanishes, we can gauge the symmetry G.

However, before gauging the symmetry, we have the freedom to stack an SPT. That is, given the G-crossed braided theory C_G, we can form the Deligne product

$$C_G \boxtimes_G \mathrm{Vec}_G^{\omega} \ , \tag{4.169}$$

where $\omega \in H^3(G, U(1))$. The subscript G on the Deligne product indicates that we should take a product of the C_g sector of C_G with the g anyon in Vec_G^{ω}. We will denote a G-crossed braided theory obtained from these choices as $C_G(\eta, \alpha)$. The phase $\eta_a(g,h)$, which is the fractionalization class when a is a genuine anyon, enters into the Heptagon equations; these equations need to be solved in order to construct C_G. Therefore, a Galois action on C_G by some $q \in \mathrm{Gal}(K_{C_G})$ should act on η as[83]

$$\eta_a(g,h) \to q(\eta_a(g,h)) \ . \tag{4.170}$$

Similarly, since $\omega \in H^3(G, U(1))$ enters into the gauging procedure through stacking by an SPT, under Galois conjugation we get[84]

$$\omega(g,h,k) \to q(\omega(g,h,k)) \ . \tag{4.171}$$

Therefore, Galois conjugations which take a unitary G-crossed braided MTC to a unitary G-crossed braided MTC are completely specified by their action on C, $\eta_a(g,h)$, and $\omega(g,h,k)$. In particular, if C, $\eta_a(g,h)$, and $\omega(g,h,k)$ are invariant under Galois action, then there are no unitarity preserving non-trivial Galois actions on C_G. Therefore, the corresponding gauged theory, C^G, is not related through Galois conjugations to other unitary theories.

Example: Spin$(k)_2$ Chern-Simons Theory

We have already seen that the Spin$(5)_2$ theory has a non-trivial Postnikov class. This theory can be obtained from the A_5 abelian TQFT by gauging the charge-conjugation symmetry. The A_5 theory also has a time-reversal symmetry given

[83] If $\eta_a(g,h)$ is a root of unity, then the Galois action will act on it by raising it to a power co-prime to the order of $\eta_a(g,h)$.

[84] Since $\omega(g,h,k)$ can always be chosen to be a root of unity, the Galois action on it can also be written as $\omega(g,h,k) \to \omega(g,h,k)^p$ where p is an integer co-prime to the order of $\omega(g,h,k)$ specified by the restriction of $q \in \mathrm{Gal}(K_C)$ to the cyclotomic field containing ω.

by

$$T : j \to 2j \ , \tag{4.172}$$

where $T^2 = C$, and C is the charge-conjugation symmetry. We can generalize this procedure to generate an infinite family of theories with non-trivial Postnikov class and then explicitly analyze the Galois action.

Let us consider a general abelian TQFT with fusion rules forming the group $\mathbb{Z}_k$. For $\mathbb{Z}_k$ fusion rules, there are several gauge-inequivalent solutions to the Pentagon and Hexagon equations labelled by $p = 0, \cdots, k-1$. The twists of the anyons in the $\mathbb{Z}_k$ MTC, corresponding to a choice of p, are

$$\theta_a = e^{\frac{2\pi i p a^2}{k}} \ . \tag{4.173}$$

We have the set of anyons $0, 1, \ \cdots, \ k-1$. Irrespective of k, we always have the charge conjugation symmetry

$$C : j \to -j \text{ mod } k \ . \tag{4.174}$$

However, the TQFT has a time-reversal symmetry if and only if k satisfies $1 + l^2 = 0$ mod k for some integer l [147]. We will assume that k is odd. The time-reversal symmetry is given by

$$T : j \to lj \text{ mod } k \ . \tag{4.175}$$

It is clear that $T^2 = C$. Hence, we have a $\mathbb{Z}_4 = \{e, z, c, cz\}$ time-reversal symmetry and a $\mathbb{Z}_2 = \{e, c\}$ charge conjugation symmetry. The idea is to gauge this charge conjugation symmetry. To that end, we have to first construct the $\mathbb{Z}_2$-crossed braided category $(\mathbb{Z}_k)_{\mathbb{Z}_2}$. We have

$$(\mathbb{Z}_k)_{\mathbb{Z}_2} = C_e \oplus C_c \ , \tag{4.176}$$

where C_e contains the anyons $0, ..., k-1$. For odd k, the vacuum is the only element invariant under charge conjugation. Hence, C_c contains only a single defect ψ. Along with the fusion rules of the anyons in $\mathbb{Z}_k$, the $\mathbb{Z}_2$-crossed braided theory has the fusion rules

$$\psi \otimes j = \psi \ , \quad \psi \otimes \psi = 0 \oplus ... \oplus k-1 \ , \tag{4.177}$$

which implies that $d_\psi = \sqrt{k}$.

It is easy to verify that the $H^2_{[\rho]}(\mathbb{Z}_2, \mathbb{Z}_2)$ group is trivial. Therefore, there is a unique fractionalization class. Moreover, $H^4(\mathbb{Z}_2, U(1)) \cong \mathbb{Z}_1$, and the $\mathbb{Z}_2$ charge conjugation symmetry does not have a 't Hooft anomaly. As a result, this symmetry can be gauged. To obtain the anyons in the gauged theory, we need the $\mathbb{Z}_2$ orbits and their stabilizers. We have the following orbits: $[0]$, $[1]$, $\cdots$, $[\frac{k-1}{2}]$, $[\psi]$. The $[1]$, $\cdots$, $[\frac{k-1}{2}]$ orbits have trivial stabilizers, while $[0]$ and $[\psi]$ have a $\mathbb{Z}_2$ stabilizer group. The representations of $\mathbb{Z}_2$ can be labelled by $[+], [-]$, where $[+]$ is the trivial representation. We have the following anyons in the gauged theory

$$([0],[+])\ ,\ ([0],[-])\ ,\ ([1],\mathbb{1})\ ,\cdots\ ,\ \left(\left[\frac{k-1}{2}\right],\mathbb{1}\right)\ ,\ ([\psi],[+])\ ,\ ([\psi],[-])\ . \quad (4.178)$$

We will denote the first two anyons as $1, \epsilon$, the last two as ψ_+, ψ_-, and the rest by ϕ_j. For different p, the fusion rules of the gauged theory remain the same, however the MTC data of the gauged theory changes. For $p = \frac{k-1}{2}$, it was shown in [58] that the resulting gauged theory has the fusion rules and MTC data of $\text{Spin}(k)_2$ Chern-Simons theory. For other values of p, we get theories with the same fusion rules, but different MTC data. In the discussion below, we will choose the value of p to be $\frac{k-1}{2}$.

The topological twists of the anyons are[a]

$$\theta_1 = \theta_\epsilon = 1\ ,\quad \theta_{\phi_j} = e^{\frac{2\pi i(k-1)j^2}{2k}}\ ,\quad \theta_{\psi_\pm} = \pm\theta_\psi = \pm e^{\frac{2\pi i(k-1)}{16}}\ . \quad (4.179)$$

Note that the topological twist of the symmetry defect is not invariant under gauge-transformations of the symmetry action. However, the twist of $\psi_\pm$ is given by

$$\theta_{\psi_\pm} = \theta_\psi \chi(\pi_\pm)\ , \quad (4.180)$$

where $\chi(\pi_a)$ is the projective character of π_a. In the gauge $\eta_a(g,h) = 1\ \forall g,h$ we have $\chi(\pi_\pm) = \pm 1$. A symmetry action gauge transformation changes θ_ψ and $\chi(\pi_a)$ by opposite phases, resulting in gauge-invariant twists $\theta_{\psi_\pm}$.

If k is such that $\theta_{\psi_\pm}$ are complex conjugates of each other (so $k = 5 \bmod 8$), then we can define a time-reversal symmetry for this theory which acts on the anyons as follows

$$T : \phi_j \to \phi_{qj}\ ,\quad T : \psi_+ \to \psi_-\ . \quad (4.181)$$

Since $\text{Spin}(k)_2$ MTC is self-dual, it is clear that this time-reversal symmetry is a $\mathbb{Z}_2$ symmetry. Similar to our analysis of the $\text{Spin}(5)_2$ theory, we can use the explicit MTC data of $\text{Spin}(k)_2$ in [129] to show that this time-reversal symmetry, along with the $\mathbb{Z}_2$ 1-form symmetry generated by the anyon ϵ forms a non-trivial 2-group.

The authors of [141] describe a much simpler way to show there is a non-trivial 2-group using the sufficient conditions in [140]. Following this procedure, let us assume that the theory has a trivial Postnikov class and show that this leads to a contradiction. If the theory has a trivial Postnikov class, it is realizable at the boundary of a 4D SPT phase. The $\mathbb{RP}^4$ partition function of this 4D SPT phase is given by [140]

$$Z(\mathbb{RP}^4) = \sum_{a,a=T(a)} S_{1a}\theta_a\eta_a \ , \tag{4.182}$$

where η_a is the fractionalization class corresponding to the time-reversal symmetry of the Spin$(k)_2$ theory and $T(a)$ denotes the time-reversal symmetry action on the anyon a. For Spin$(k)_2$, where k satisfies $1 + l^2 = 0 \bmod k$ for some integer l and $k = 5 \bmod 8$, we can calculate this as

$$Z(\mathbb{RP}^4) = S_{11}\theta_1\eta_1 + S_{1\epsilon}\theta_\epsilon\eta_\epsilon = \mathcal{D}(1+\eta_\epsilon) \neq \pm 1 \ . \tag{4.183}$$

However, it is known that the partition function of a time-reversal invariant 4D SPT on $\mathbb{RP}^4$ is valued in ± 1. This shows that the Spin$(k)_2$ theory ($k = 5 \bmod 8$) cannot be realized at the surface of a 4D SPT. Hence, the Postnikov class of the theory is non-trivial. Note that if k is such that $\theta_{\psi_\pm}$ is real, then $\psi_\pm$ are also invariant under the symmetry. Hence, they will contribute to the above partition function. In fact, for these theories the partition function is valued in ± 1. Indeed, in this case the Postnikov class is trivial.[b]

Now that we have explored the 2-group structure of Spin$(k)_2$ Chern-Simons theory, let us show that the Postnikov class is invariant under Galois actions on this theory. Recall that the Spin$(5)_2$ TQFT was invariant under all unitarity-preserving Galois actions. We showed this explicitly by studying the Galois action on the T matrix. Alternatively, this can also be seen from the fact that Spin$(5)_2$ TQFT is obtained from gauging a $\mathbb{Z}_2$ symmetry of the A_5 abelian TQFT. Indeed, we know that the fractionalization class is trivial, and the SPT stacking is determined by $\omega \in H^3(\mathbb{Z}_2, U(1))$ (which is valued in ± 1). Therefore, unitarity-preserving Galois actions on $(A_5)_{\mathbb{Z}_2}$ cannot change the G-crossed braided structure. We also know that the A_5 TQFT has four Galois actions corresponding to $\mathbb{Z}_5^\times = \{1,2,3,4\}$. The only non-trivial Galois action which preserves the unitarity of $(A_5)_{\mathbb{Z}_2}$ (i.e, which doesn't flip the sign of $d_\psi = \sqrt{5}$) is 4. We also know that A_5 is invariant under Galois action by 4. Therefore, we find that $(A_5)_{\mathbb{Z}_2}$ is invariant under all unitarity-preserving Galois actions. Therefore, using theorem 4.4.6, we find that the electric theory Spin$(5)_2$ is invariant under all unitarity-preserving Galois actions.

[a]We can stack a non-trivial $\mathbb{Z}_2$ SPT before gauging. $\theta_{\psi_\pm}$ of the resulting gauged theory is same as the twists obtained without SPT stacking up to a factor of -1.

[b]Note that $Z(\mathbb{RP}^4)$ being valued in ± 1 does not guarantee that the Postnikov class is trivial. It is only a necessary condition. But it can be checked that whenever $\psi_\pm$ is fixed under the symmetry action, then (4.100) forces the Postnikov class to be trivial.

More generally, the unitarity-preserving Galois actions on $(\mathbb{Z}_k)_{\mathbb{Z}_2}$ are those Galois actions of $\mathbb{Z}_k$ which preserve the quantum dimensions of all anyons and defects in $(\mathbb{Z}_k)_{\mathbb{Z}_2}$. Using theorem 4.4.6, these Galois actions correspond to unitarity-preserving Galois action on Spin$(k)_2$ TQFT. Indeed, a unitarity-preserving Galois action on $(\mathbb{Z}_k)_{\mathbb{Z}_2}$ with respect to some q co-prime to k can be see as changing our choice of $p = \frac{k-1}{2}$ to $p = \frac{q(k-1)}{2}$. The twists of the resulting gauged theory then becomes

$$\theta_1 = \theta_\epsilon = 1 \ , \quad \theta_{\phi_j} = e^{\frac{2\pi i q(k-1)j^2}{2k}} \quad , \quad \theta_{\psi_\pm} = \pm\theta_\psi = \pm e^{\frac{2\pi i q(k-1)}{16}} \ . \tag{4.184}$$

If θ_{ψ_+} and θ_{ψ_-} are complex conjugates before Galois action, the same is true after Galois action. Therefore, at the level of the T matrix, Spin$(5)_2$ and its Galois conjugates have the same time-reversal symmetry structure. This is in agreement with our theorem 4.4.2. Moreover, if the $Z(\mathbb{RP}^4)$ is not valued in ± 1 before Galois action, the same is true after Galois action. Therefore, the Postnikov class is non-trivial before and after Galois action. Similarly, if the symmetry acts trivially on $\psi_\pm$ before Galois action, then we know that the Postnikov class is trivial. This result is also true after Galois action. These observations agree with our theorem 4.4.3.

4.4.5 Galois Invariance and Gauging

Suppose C is a Galois-invariant theory with symmetry G. It is then natural to ask if this invariance is preserved under gauging 0-form symmetries, 1-form symmetries, and more general anyon condensation. We expect any lack of invariance in the gauged / condensed theory to be due to a kind of generalized mixed 't Hooft anomaly between the Galois action and the symmetry / condensation in question. On the other hand, there may be subtler effects due to such an anomaly that we do not study here, and so the preservation of Galois invariance alone may not be sufficient to conclude that there is no generalized 't Hooft anomaly.[85] Therefore, all we can say is that that there is a non-trivial Galois action-0-form mixed 't Hooft anomaly if gauging the symmetry G results in a Galois non-invariant theory. Similarly, suppose we have an MTC, C, with a 1-form symmetry, A. We can say that there is a non-trivial Galois action-1-form mixed anomaly if gauging the symmetry A results in a Galois non-invariant theory. More generally, we can say there is a Galois action-anyon condensation anomaly by replacing A with a general connected commutative separable algebra and finding a non-invariant condensed theory.

[85] Indeed, in the more standard case of mixed 't Hooft anomalies between 0-form symmetries, gauging part of the 0-form symmetry group can sometimes lead to non-trivial 2-groups and other phenomena [148]. As a result, one may wonder if there is a generalization of this story involving Galois actions as well. As another possibility, recall that a mixed 0-form / 1-form 't Hooft anomaly can result in a non-trivial group extension for the 0-form symmetry after 1-form symmetry gauging [148, 149]. It would be interesting to study whether there is a generalization of this story to Galois group extensions under 1-form symmetry gauging / anyon condensation.

Let us study the behavior of Galois invariance under gauging more carefully. To that end, suppose C is Galois invariant. Using theorem 4.4.6, we find that C^G is Galois invariant if and only if C_G is Galois invariant. This follows from the one-to-one relation between G-crossed braided categories and modular tensor categories with a Rep(G) subcategory [43]. We therefore get the following result:

Lemma 4.4.7 *Starting from a Galois invariant MTC, C, with 0-form symmetry, G, we obtain a Galois invariant theory, C^G, after gauging if and only if C_G is Galois invariant.*

In more field theoretical language, the above lemma amounts to the statement that the Galois invariance of the gauged TQFT can be determined by turning on background fields for G and studying the Galois invariance of the TQFT prior to gauging. In the examples section, we will study particular TQFTs where 0-form gauging preserves the Galois invariance as well as cases where 0-form gauging violates the Galois invariance.

Next let us discuss how Galois invariance interacts with anyon condensation. To that end, suppose C^G is Galois invariant, then it follows from (4.73) that C is Galois invariant if and only if C_G is Galois invariant. Suppose C_G is not Galois invariant. Then there exists some $q \in \mathrm{Gal}(K_{C_G})$ such that $q(C_G)$ is inequivalent to C_G. We have some $q' \in \mathrm{Gal}(K_{C^G})$ such that $(q(C_G))^G = q'(C^G)$. Since $q(C_G)$ is inequivalent to C_G, $q'(C^G)$ has to be different from C^G. This contradicts the assumption that C^G is Galois invariant. Therefore, C_G should be Galois invariant. We get the result:

Lemma 4.4.8 *If we start from a Galois-invariant theory, then the theory after anyon condensation is also Galois invariant.*

More generally, lemma 4.4.8 implies that, for every element of the Galois group that leaves the electric theory invariant, there is a (not necessarily unique) Galois action on the magnetic theory that leaves it invariant.[86] For example, consider a TQFT invariant under complex conjugation. If the TQFT has real MTC data this is of course trivially true. But if the MTC data is complex, then there exists a combination of gauge transformations and a map between the anyons of the TQFT and its complex conjugate preserving the fusion rules. Sometimes, such a map along with a gauge transformation arises from the time-reversal symmetry of the TQFT. However, there may not be a unqiue way to lift the complex conjugation Galois action to a time-reversal symmetry.

For example, consider the A_5 TQFT. The complex conjugation Galois action can be reversed using a permutation of the anyons $T(a) = 2a \bmod 5$, which is a time-reversal symmetry. However, T^3 is also a time-reversal symmetry. Therefore, the

[86] In particular, this statement is true even if there are other elements of the Galois group that do not leave the electric theory invariant.

complex conjugation Galois action can be reversed using T or T^3. Note that complex conjugation Galois action is always order two, while time-reversal symmetry may not be order two (it is order four in the A_5 example). This discrepancy is due to the fact that Galois conjugation acts directly on the MTC data, and an order two permutation of the anyons reversing this Galois action may not preserve the fusion rules of the MTC.

We know that the $\mathrm{Spin}(5)_2$ Chern-Simons theory can be obtained by gauging the $\mathbb{Z}_2$ charge-conjugation symmetry of A_5 TQFT. In this case both the electric and magnetic theories are invariant under the complex conjugation Galois action. In the electric theory, the Galois invariance can be lifted to an order-two time-reversal symmetry, while on the magnetic side, it can be lifted only to an order-four time-reversal symmetry. The origin of this order-four time-reversal symmetry is due to the non-trivial mixed 't Hooft anomaly between the order-two time-reversal symmetry and the $\mathbb{Z}_2$ 1-form symmetry in the electric theory. The magnetic theory then has a $\mathbb{Z}_4$ time-reversal symmetry which arises from a group extension of the $\mathbb{Z}_2$ time-reversal symmetry of the electric theory by the $\mathbb{Z}_2$ charge conjugation symmetry of the magnetic theory [148].

4.4.6 Galois Fixed Point TQFTs

In this section, our goal is to better elucidate generalizations of the basic unitary Galois fixed point TQFTs we encountered earlier (i.e., the 3-Fermion Model, Toric Code, Double Semion, and various other more complicated (twisted) discrete gauge theories). Of course, most TQFTs transform non-trivially under Galois conjugation. For example, consider a theory which is not integral. Such a TQFT should have at least one anyon, say a, with a real irrational quantum dimension, $d_a \notin \mathbb{Q}$. Then there exists a Galois conjugation which acts non-trivially on d_a and results in a different TQFT. More generally, we have the following theorem:

Lemma 4.4.9 *All unitary Galois-invariant TQFTs have only integer quantum dimensions.*

Proof: Consider a unitary MTC, C. Recall from lemma 4.2.2 that a unitarity-preserving Galois conjugation by an element, g, must satisfy

$$g(d_a) = d_a \ , \tag{4.185}$$

for all d_a. As a result, $d_a \in \mathbb{Q}\ \forall a \in C$. Since quantum dimensions are algebraic integers, the rational root theorem guarantees that all $d_a \in \mathbb{Z}$. □

Note that this result does not hold for non-unitary Galois fixed point theories. Indeed, consider the following TQFT

$$\mathcal{T} = \boxtimes_{q \in \mathrm{Gal}(K_{C_0})} q(\mathcal{T}_0) \ , \tag{4.186}$$

where $\mathcal{T}_0$ is a TQFT with at least one irrational quantum dimension, and C_0 is the associated MTC. In (4.186), we take a product over the full Galois orbit of $\mathcal{T}_0$ (thereby rendering $\mathcal{T}$ Galois-invariant). Since there is an irrational quantum dimension, the product TQFT, $\mathcal{T}$, will contain at least one non-unitary factor and hence will be non-unitary. As an example, we can take $\mathcal{T}_0$ to be the Fibonacci theory (then there will be anyons with quantum dimension $(1 \pm \sqrt{5})/2$ in $\mathcal{T}$). Finally, note that not all integral theories are Galois invariant. For example, consider the Semion TQFT. Therefore, unitary Galois invariant TQFTs should lie in the subspace of integral TQFTs.[87]

Interestingly, all known examples of integral TQFTs are also weakly group theoretical (the converse does not hold). These latter TQFTs are under good control since they all have a Tannakian subcategory that comes from gauging a symmetry of a weakly anisotropic abelian TQFT [150]. Weakly anisotropic pointed categories are classified in [145]. The upshot is that any weakly anisotropic abelian TQFT is of the form $D \boxtimes A$, where D is the discrete gauge theory, $\mathcal{Z}(\text{Vec}_G)$, where G is an abelian group consisting of a direct sum of cyclic groups of prime orders, and A is an anisotropic abelian TQFT.[88] These latter theories are:

1. A_p TQFT
2. B_p TQFT
3. $A_p \boxtimes A_p = B_p \boxtimes B_p$ TQFT
4. Semion and $\overline{\text{Semion}}$.
5. Semion $\boxtimes$ Semion and $\overline{\text{Semion}} \boxtimes \overline{\text{Semion}}$
6. 3-Fermion Model
7. $\mathbb{Z}_4$ TQFT and Galois conjugates.
8. $\mathbb{Z}_4 \boxtimes$ Semion TQFT, $\mathbb{Z}_4 \boxtimes \overline{\text{Semion}}$ TQFT and Galois conjugates.

Therefore, all weakly group theoretical integral MTCs should come from gauging a symmetry of $D \boxtimes A$ where A is one among the TQFTs listed above. Note that the discrete gauge theory, D, is invariant under Galois conjugation. A is invariant under Galois conjugation only if A is the 3-Fermion Model or $A_p \boxtimes A_p$. This discussion leads to the following theorem:

[87]This discussion shows that classifying the set of Galois-invariant unitary TQFTs should be substantially easier than classifying the set of Galois-invariant non-unitary TQFTs. Indeed, classifying this latter class is *naively* as hard as classifying the full set of non-unitary TQFTs and finding their Galois orbits! On the other hand, for unitary Galois-invariant theories, integrality is already an enormous simplification. We will soon see that there are various potential additional constraints on the unitary Galois fixed point TQFTs.

[88]Anisotropic abelian TQFTs are abelian TQFTs without any subcategories containing only bosons.

Theorem 4.4.10 *Let C^G be a Galois-invariant weakly group theoretical TQFT, then C^G can be obtained from gauging a symmetry of D, $D \boxtimes$ 3-fermion model, or $D \boxtimes A_p \boxtimes A_p$.*

Proof: Let C^G be a Galois invariant weakly group theoretical TQFT. Then it has to be integral. From lemma 4.4.8, we know that if C^G is Galois invariant, then the G-crossed braided theory C_G should be Galois invariant. In particular, the MTC, C (the C_e component of C_G), should be Galois invariant. Weakly group theoretical integral TQFT C^G comes from gauging a symmetry of $D \boxtimes A$ where A is an anisotropic TQFT. Therefore, $C = D \boxtimes A$. D is an unwisted discrete gauge theory which is invariant under Galois action. Hence, Galois invariance of C implies that A can be either the trivial MTC, the 3-fermion model, or $A_p \boxtimes A_p$. □

As a simple check of this discussion, note that gauging a $\mathbb{Z}_2 \times \mathbb{Z}_2$ natural isomorphism of the 3-Fermion Model gives the F_8 prime abelian theory (see Section 4.5.2). Both are Galois invariant.

Lemma 4.4.7 shows that the Galois invariance of C does not guarantee the Galois invariance of C^G. For example, gauging an intrinsic $\mathbb{Z}_3$ symmetry of the 3-fermion models and stacking a particular non-trivial SPT gives the $SU(3)_3$ Chern-Simons theory which has a non-trivial Galois conjugate (see Section 4.5.3). However, gauging the non-trivial $\mathbb{Z}_3$ symmetry of $SU(3)_3$ with trivial SPT stacking gives a Galois-invariant theory. This example can be generalized to the following theorem:

Theorem 4.4.11 *Let the magnetic theory, C, be Galois invariant with an integer total quantum dimension (i.e., $\mathcal{D} := \sqrt{\sum_a d_a^2} \in \mathbb{Z}$). Suppose the symmetry G acts non-trivially on all non-trivial anyons and satisfies $H^2_{[\rho]}(G, A) \cong \mathbb{Z}_1$, where A is the group of abelian anyons in C. Assuming that the obstructions to gauging vanish, and choosing the trivial SPT stacking, the electric theory obtained from gauging is Galois invariant.*

Proof: Since the symmetry, G, acts non-trivially on all non-trivial anyons, each defect sector, C_g, in the G-crossed braided extension C_G has a single defect field (i.e., a single non-genuine line operator bounding the corresponding g surface operator). The total quantum dimension of C_g is same as that of C for all g. Therefore, it is clear that the quantum dimensions of all the defects are the same as the total quantum dimension of C. The quantum dimensions of the defect are integers, and using theorem 4.2.6, we see that C_G is a unitary spherical fusion category.

Therefore, the possible G-crossed braided extensions, $C_G(\eta, \alpha)$ are classified by the fractionalization class η and possible SPT stackings determined by the 3-cocycle α. Since $H^2_{[\rho]}(G, A)$ is trivial, there is a unique fractionalization class. Let us gauge the symmetry G of $C_G(\eta, [1])$, where [1] denotes the trivial SPT. Since there is a unique fractionalization class, and since the trival SPT is Galois invariant, $C_G(\eta, [1])$ is Galois invariant. Therefore, the theory obtained from gauging G symmetry of $C_G(\eta, [1])$ is also Galois invariant. □

For example, consider the charge conjugation symmetry acting on $A_p \boxtimes A_p$. All non-trivial anyons transform non-trivially under this symmetry. By explicitly computing the twisted cohomology groups, $H^3_{[\rho]}(\mathbb{Z}_2, \mathbb{Z}_p \otimes \mathbb{Z}_p)$ and $H^2_{[\rho]}(\mathbb{Z}_2, \mathbb{Z}_p \otimes \mathbb{Z}_p)$, we can check that they are trivial. Therefore, the Postnikov class vanishes and the fractionalization class is unique. The defectification obstruction vanishes because $H^4(\mathbb{Z}_2, U(1)) \cong \mathbb{Z}_1$. Therefore, gauging the charge conjugation symmetry of the $A_p \boxtimes A_p$ TQFT produces Galois invariant TQFTs (irrespective of the SPT stacking).

In theorem 4.4.11, we considered a symmetry which acts non-trivially on all non-trivial anyons. This is to ensure that the defects have integer quantum dimensions. However, this is not a necessary constraint to get a Galois invariant TQFT by gauging non-trivial symmetries. For example, consider a $\mathbb{Z}_2$ permutation symmetry which exchanges the anyons in the two prime factors of the $C \boxtimes C$ TQFT. This symmetry is known to have trivial Postnikov class [58], and the defectification obstruction / 't Hooft anomaly vanishes since $H^4(\mathbb{Z}_2, U(1)) \cong \mathbb{Z}_1$. Also, it is known that there is a unique fractionalization class.[89] There are $|C|$ number of defects in each defect sector since all the anyons of the form (a, a) are invariant under the permutation action. The quantum dimensions of the x_a defects are given by [58]

$$d_{x_a} = |C| d_a \ . \tag{4.187}$$

If we assume that $C \boxtimes C$ is Galois invariant, then it is integral. Therefore, all the defects in the $\mathbb{Z}_2$ crossed braided theory have integer quantum dimensions. Gauging the permutation symmetry results in a Galois-invariant TQFT (irrespective of the SPT being stacked before gauging).

If every fusion category with integer Frobenius-Perron dimension is weakly-group theoretical, then any Galois invariant unitary TQFT can be obtained from gauging a symmetry of D, $D \boxtimes$ 3-Fermion Model, or $D \boxtimes A_p \boxtimes A_p$.[90] As shown in [152], any fusion category with Frobenius-Perron dimension, a natural number less than 1800 or an odd natural number less than 33075 is weakly-group theoretical. Moreover, if the Frobenius-Perron dimensions of all anyons in a TQFT are prime powers, then it is

[89] The vanishing of the Postnikov class and defectification obstruction is true even for S_n action on $C^{\boxtimes n}$. However, for $n > 2$ the fractionalization class is not unique [151].

[90] Some evidence in favor of this possibility follows from the fact that for integral theories, $c \in \mathbb{Z}$ (i.e., the topological central charge is an integer) [111]. Since topological central charge is preserved under gauging, it is easy to check that gauging the full list of weakly anisotropic abelian TQFTs above gives all possible integral central charges modulo eight.

weakly-group theoretical [153].

4.5 Examples

Let us consider several examples to explicitly see how the Galois action interacts with taking the Drinfeld center and gauging. We will use the G-crossed braided MTC data computed in [58].

4.5.1 Trivial magnetic theory

Let us first consider the simplest case of a trivial magnetic theory, Vec. In this case, gauging a natural isomorphism symmetry of G is same as taking the Drinfeld center of Vec_G^ω. The G-crossed braided theory is in fact Vec_G^ω itself. We have the Galois field $\mathbb{Q}(\omega)$ associated with this category. A Galois action by $q \in \mathrm{Gal}(\mathbb{Q}(\omega))$ changes the theory as

$$\mathrm{Vec}_G^\omega \to \mathrm{Vec}_G^{\omega^q} \ . \tag{4.188}$$

Therefore, under the action of the Galois group, the Drinfeld center (which in this case is a discrete gauge theory) changes only by $\omega \to \omega^q$.

$G = \mathbb{Z}_2$

In this case, we have two possible choices for ω. Since ω is valued in ± 1, Galois action on $\mathrm{Vec}_{\mathbb{Z}_2}^\omega$ does not change the $\mathbb{Z}_2$-crossed braided theory. Therefore, the Drinfeld center also shouldn't change under Galois action. This is indeed the case. For trivial twist, the Drinfeld center is the Toric Code which is invariant under the Galois action. For non-trivial twist, the Drinfeld center is the Double Semion model which has a complex conjugation Galois action. However, this action can be compensated by a time-reversal symmetry, and hence Double Semion is invariant under Galois action.

The example with non-trivial ω illustrates that the defining number field of the electric theory can be bigger than the G-crossed braided magnetic theory. Indeed, the G-crossed braided theory in this case is $\mathrm{Vec}_{\mathbb{Z}_2}^\omega$, whose F symbols are given by ω, and the R symbols can all be set to 1. Even though the defects in $\mathrm{Vec}_{\mathbb{Z}_2}^\omega$ have trivial twist, the twists of the electric theory have 4^{th} roots of unity in them since the gauging procedure involves representations of $\mathbb{Z}_2$ with characters valued in the 4^{th} roots of unity.

$G = \mathbb{Z}_N$

In this case, the $\mathbb{Z}_N$-crossed braided theory is $\mathrm{Vec}_{\mathbb{Z}_N}^\omega$ where

$$\omega(g,h,k) = e^{\frac{2\pi i p g}{N^2}(h+k-(h+k \bmod N))} \ , \tag{4.189}$$

and $p \in \mathbb{Z}_N$ parametrizes the different twists. Since $H^2(\mathbb{Z}_2, U(1))$ is trivial, for all values of ω the Drinfeld center is an abelian theory. The anyons of the Drinfeld center are labelled by (a,m), where $a, m \in \{0, ..N-1\}$. The fusion rules and twists of the Drinfeld center are

$$(a,m) \otimes (b,n) = \left(a+b \text{ mod } N, [m+n-\frac{2p}{N}(a+b-(a+b \text{ mod } N))] \text{ mod } N \right) , \quad (4.190)$$

and

$$\theta_{(a,m)} = e^{\frac{2\pi i}{N}am} e^{-\frac{2\pi i}{N^2}pa^2} . \quad (4.191)$$

The fusion rules form the group $\mathbb{Z}_{\gcd(2p,N)} \times \mathbb{Z}_{\frac{N^2}{\gcd(2p,N)}}$.

The Galois field of $\text{Vec}^{\omega}_{\mathbb{Z}_N}$ is the cyclotomic field $\mathbb{Q}(\xi_N)$ of N^{th} roots of unity. A Galois action on the $\mathbb{Z}_N$-crossed braided theory corresponds to changing the parameter p as follows

$$p \to qp , \quad (4.192)$$

where $\gcd(q, N) = 1$. After this Galois action, the Drinfeld center has fusion rules and twists

$$(a,m) \otimes (b,n) = \left(a+b \text{ mod } N, [m+n-\frac{2pq}{N}(a+b-(a+b \text{ mod } N))] \text{ mod } N \right) , \quad (4.193)$$

and

$$\theta_{(a,m)} = e^{\frac{2\pi i}{N}am} e^{-\frac{2\pi i}{N^2}pqa^2} . \quad (4.194)$$

It is clear that we have the same fusion rules since $\gcd(2p, N)$=$\gcd(2qp, N)$ when $\gcd(q, N) = 1$. It will be evident that the fusion rules are the same if we change the variable m to qm mod N. Then we get

$$(a, qm \text{ mod } N) \otimes (b, qn \text{ mod } N) = \left(a+b \text{ mod } N, [q(m+n-\frac{2p}{N}(a+b-(a+b \text{ mod } N)))] \text{ mod } N \right) . \quad (4.195)$$

The twists become

$$\theta_{(a,qm \text{ mod } N)} = e^{\frac{2\pi i}{N}qam} e^{-\frac{2\pi i}{N^2}pqa^2} . \quad (4.196)$$

Therefore, the twist of the anyon $(a, qm \text{ mod } N)$ in $\mathcal{Z}(\text{Vec}^{\omega^q}_{\mathbb{Z}_N})$ is the Galois conjugate of the twist of the anyon (a, m) in $\mathcal{Z}(\text{Vec}^{\omega}_{\mathbb{Z}_N})$.

4.5.2 Non-trivial magnetic theory with trivial symmetry

Let us consider some cases of a non-trivial magnetic theory with natural isomorphism symmetry.

Ising MTC with $\mathbb{Z}_2$ symmetry

From section 4.2.4, we know that the unitarity preserving Galois actions on the $\text{Ising}^{(\nu)}$ family correspond to $q = 1, 7, 9, 15$. Under these Galois actions, the $\text{Ising}^{(\nu)}$ family of models transform as

$$\text{Ising}^{(\nu)} \to \text{Ising}^{(q\nu \bmod 16)} \ . \tag{4.197}$$

The Ising model ($\nu = 1$) does not have any non-trivial intrinsic symmetries. Therefore, the $\mathbb{Z}_2$ group has to act as a natural isomorphism. We know that the Postnikov class vanishes. The fractionalization class is specified by an element in $\eta \in H^2(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2$. Since $H^4(\mathbb{Z}_2, U(1))$ is trivial, defectification obstruction vanishes. The choice of stacking a $\mathbb{Z}_2$-SPT before gauging is parametrized by an element in $\alpha \in H^3(\mathbb{Z}_2, U(1)) \cong \mathbb{Z}_2$. We get the following theories under gauging [58]

$$\eta, \alpha \text{ trivial} \quad \to \quad \text{Ising} \boxtimes \text{ Toric Code} \ , \tag{4.198}$$

$$\eta \text{ trivial } \alpha \text{ non-trivial} \quad \to \quad \text{Ising} \boxtimes \text{ Double-Semion} \ , \tag{4.199}$$

$$\eta \text{ non-trivial } \alpha \text{ trivial} \quad \to \quad \text{Ising}^{(15)} \boxtimes A_4 \ , \tag{4.200}$$

$$\eta \text{ non-trivial } \alpha \text{ non-trivial} \quad \to \quad \text{Ising}^{(3)} \boxtimes B_4 \ . \tag{4.201}$$

Since both η and α are valued in ± 1, a Galois action on the $\mathbb{Z}_2$-crossed braided structure can only affect the modular subcategory Ising. That is, let $\text{Ising}(\nu, \alpha)$ denote the $\mathbb{Z}_2$-crossed braided theory specified by η and ω. A unitarity preserving Galois action on this gives $\text{Ising}^{(q)}(\eta, \alpha)$, where q is specified by the Galois action. Therefore, the electric theories obtained above should also transform in this way.

Since the Toric code and Double-Semion model are invariant under Galois action, we find that the Galois action on (4.198) and (4.199) acts precisely as the $\mathbb{Z}_2$-crossed braided magnetic theory transforms.

Now let us focus on the electric theory, $\text{Ising}^{(15)} \boxtimes A_4$. The data of this theory belongs to the cyclotomic field $\mathbb{Q}(\xi_{16})$. The unitarity preserving Galois actions correspond to $q = 1, 7, 9, 15$. Under these Galois action we get

$$q = 7: \ \text{Ising}^{(15)} \boxtimes A_4 \to \text{Ising}^{(9)} \boxtimes B_4 \ , \tag{4.202}$$

$$q = 9: \ \text{Ising}^{(15)} \boxtimes A_4 \to \text{Ising}^{(7)} \boxtimes A_4 \ , \tag{4.203}$$

$$q = 15: \ \text{Ising}^{(15)} \boxtimes A_4 \to \text{Ising}^{(1)} \boxtimes B_4 \ . \tag{4.204}$$

Recall that $\text{Ising}^{(15)} \boxtimes A_4$ is obtained from gauging $\text{Ising}^{(1)}(\eta = -1, \alpha = +1)$. The three Galois conjugates above are obtained from the $\mathbb{Z}_2$-crossed braided categories $\text{Ising}^{(7)}(\eta = -1, \alpha = +1)$, $\text{Ising}^{(9)}(\eta = -1, \alpha = +1)$ and $\text{Ising}^{(15)}(\eta = -1, \alpha = +1)$, respectively. These are all Galois conjugates of $\text{Ising}^{(1)}(\eta = -1, \alpha = +1)$, as expected.

Similarly, we can check that the Galois conjugates of the electric theory $\text{Ising}^{(3)} \boxtimes B_4$

corresponds to Galois conjugates of the $\mathbb{Z}_2$-crossed braided theory $\text{Ising}^{(1)}(\eta = -1, \alpha = -1)$.

3-fermion model with $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry

Consider the prime abelian theory F_8. The 64 anyons are labelled by (m, n) where $m, n \in \mathbb{Z}_8$. The bosons $(0,0), (0,4), (4,0), (4,4)$ form a $\text{Rep}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ subcategory which can be condensed. The magnetic theory can be obtained by identifying the anyons which braid trivially with all anyons in $\text{Rep}(\mathbb{Z}_2 \times \mathbb{Z}_2)$. These fall into the following 4 equivalence classes of anyons under fusion with the anyons in $\text{Rep}(\mathbb{Z}_2 \times \mathbb{Z}_2)$

$$(0,0)\ ,\ (0,2)\ ,\ (2,0)\ ,\ (2,2)\ . \tag{4.205}$$

The twists of these anyons are $1, -1, -1, -1$, respectively. Therefore, the magnetic theory is the 3-fermion model. Hence, the F_8 prime abelian anyons model can be obtained from F_2 by gauging a $\mathbb{Z}_2 \times \mathbb{Z}_2$ natural isomorphism symmetry. Both the magnetic and electric theory are invariant under Galois conjugation.

For the F_2 abelian model with $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, the Postnikov class vanishes and the fractionalization class belongs to the group $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2) \cong \mathbb{Z}_2^6$. Group cohomology allows for a defectification obstruction since $H^4(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. For a given choice of the fractionalization class, if this obstruction vanishes, then the freedom to stack a $\mathbb{Z}_2 \times \mathbb{Z}_2$-SPT before gauging is parametrized by $H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) \cong \mathbb{Z}_2^3$. Therefore, we have several possible electric theories in this case based on the choice of fractionalization class and SPT stacking.

4.5.3 Non-trivial magnetic theory with non-trivial symmetry

Toric code with $\mathbb{Z}_2$ electric-magnetic symmetry

Let us consider the Toric code with a non-trivial $\mathbb{Z}_2$ symmetry which permutes the two bosons. It is known that the Postnikov class vanishes for this symmetry. We have $H^2_{[\rho]}(\mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2) \cong \mathbb{Z}_1$, $H^4(\mathbb{Z}_2, U(1)) \cong \mathbb{Z}_1$ and $H^3(\mathbb{Z}_2, U(1)) = \mathbb{Z}_2$. Therefore, there is a unique fractionalization class for which group cohomology guarantees that the defectification obstruction vanishes. We have the freedom to stack a $\mathbb{Z}_2$-SPT corresponding to $\alpha \in H^3(\mathbb{Z}_2, U(1)) \cong \mathbb{Z}_2$ before gauging. We get the following theories under gauging

$$\alpha \text{ trivial} \quad \to \quad \text{Ising}^{(1)} \boxtimes \text{Ising}^{(15)}\ , \tag{4.206}$$

$$\alpha \text{ non-trivial} \quad \to \quad \text{Ising}^{(3)} \boxtimes \text{Ising}^{(13)}\ . \tag{4.207}$$

Since the Toric code is Galois invariant, and since the $\mathbb{Z}_2$ crossed braided theory is completely rigid except for the choice of α (which is valued in ± 1), the $\mathbb{Z}_2$-crossed braided theory is invariant under all unitarity-preserving Galois actions. Therefore, the electric theories obtained above should also be invariant under all such Galois actions. Indeed, the MTCs $\text{Ising}^{(1)} \boxtimes \text{Ising}^{(15)}$ and $\text{Ising}^{(3)} \boxtimes \text{Ising}^{(13)}$ are both invariant under all unitarity preserving Galois actions.

3-fermion model with $\mathbb{Z}_2$ symmetry

Let us consider the 3-fermion model with a non-trivial $\mathbb{Z}_2$ symmetry which permutes any two of the three fermions in the theory. It is known that the Postnikov class vanishes for this symmetry. We have $H^2_{[\rho]}(\mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2) \cong \mathbb{Z}_1$, $H^4(\mathbb{Z}_2, U(1)) \cong \mathbb{Z}_1$ and $H^3(\mathbb{Z}_2, U(1)) = \mathbb{Z}_2$. Therefore, there is a unique fractionalization class for which group cohomology guarantees that the defectification obstruction vanishes. We have the freedom to stack a $\mathbb{Z}_2$-SPT corresponding to $\alpha \in H^3(\mathbb{Z}_2, U(1)) \cong \mathbb{Z}_2$ before gauging. We get the following theories under gauging

$$\alpha \text{ trivial} \quad \rightarrow \quad \text{Ising}^{(1)} \boxtimes \text{Ising}^{(7)} \ , \tag{4.208}$$

$$\alpha \text{ non-trivial} \quad \rightarrow \quad \text{Ising}^{(3)} \boxtimes \text{Ising}^{(5)} \ . \tag{4.209}$$

Since the magnetic theory is Galois invariant, and since the $\mathbb{Z}_2$ crossed braided theory is completely rigid except for the choice of α (which is valued in ± 1), the $\mathbb{Z}_2$-crossed braided theory is invariant under all unitarity preserving Galois actions. Therefore, the electric theories obtained above should also be invariant under all such Galois actions. Indeed, the MTCs $\text{Ising}^{(1)} \boxtimes \text{Ising}^{(7)}$ and $\text{Ising}^{(3)} \boxtimes \text{Ising}^{(5)}$ are both invariant under all unitarity preserving Galois actions (Galois action leads to permutations of the three anyons with $\sqrt{2}$ quantum dimensions).

3-fermion model with $\mathbb{Z}_3$ symmetry

Let us consider the 3-fermion model with a non-trivial $\mathbb{Z}_3$ symmetry which cyclically permutes the three fermions in the theory. It is known that the Postnikov class vanishes for this symmetry. We have $H^2_{[\rho]}(\mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2) \cong \mathbb{Z}_1$, $H^4(\mathbb{Z}_3, U(1)) \cong \mathbb{Z}_1$ and $H^3(\mathbb{Z}_3, U(1)) = \mathbb{Z}_3$. Therefore, there is a unique fractionalization class for which group cohomology guarantees that the defectification obstruction vanishes. We have the freedom to stack a $\mathbb{Z}_3$-SPT corresponding to $\alpha \in H^3(\mathbb{Z}_3, U(1)) \cong \mathbb{Z}_3$ before gauging. It is known that for non-trivial α and its inverse we get the MTC $SU(3)_3$ and its complex conjugate under gauging [58].

$SU(3)_3$ has only one non-trivial Galois conjugate [154], which is the complex conjugate of $SU(3)_3$. Therefore, Galois conjugation of the electric theory corresponds to changing the $\mathbb{Z}_3$-SPT being stacked before gauging ($\alpha \rightarrow \bar{\alpha}$).

For trivial α the resulting TQFT is integral and has different fusion rules than that of $SU(3)_3$ (The explicit fusion rules are given in [58]). Since the magnetic theory is Galois invariant and since the $\mathbb{Z}_3$-crossed braided theory with α trivial is invariant under Galois action, the electric theory is invariant under unitarity preserving Galois actions. Moreover, since the electric theory is integral and unitary, all Galois actions preserve unitarity (using Theorem 2.7). Therefore, the electric theory obtained for trivial α is in fact completely Galois invariant. Indeed, one can check that the modular data for this theory given in [154] is invariant under Galois conjugation (up to permutation of the anyons). This is an example of a Galois invariant non-abelian TQFT which is not a discrete gauge theory.

4.6 Conclusion

We explored several aspects of Galois actions on TQFTs and gave a sufficient condition for producing unitary Galois orbits. We also discussed how Galois conjugation of a bulk TQFT changes its gapped boundary. Using the fact that certain TQFTs are uniquely determined by their gapped boundaries, we studied how the Galois action on gapped boundaries affects the bulk TQFTs. By determining the relationship between Galois action on theories related by gauging, we showed that (assuming a conjecture in the literature) arbitrary Galois-invariant TQFTs are closely related to simple abelian Galois-invariant TQFTs.

These results show that, while Galois conjugation usually results in distinct TQFTs, the TQFTs in a Galois orbit are closely related to each other. They have the same symmetry structure (modulo mild assumptions in the defining number field of the G-crossed braided theory), and their gapped boundaries are related to each other. This situation is unlike other operations, such as gauging or condensation, which can drastically change the anyon content and symmetry structure of the theory.

Finally, we constructed the defining number field K_C of an MTC using the F and R symbols, and Galois conjugation of the TQFT acted directly on the F and R data. In general, the total quantum dimension, $\mathcal{D}$, is not an element of K_C. Moreover, we defined Galois action on the TQFT such that it doesn't change the sign of $\mathcal{D}$ (we can consider taking $\mathcal{D} \to -\mathcal{D}$ as a second step, supplementing our Galois conjugation, when exploring particular orbits).[91] Explicitly including a $\mathcal{D} \to -\mathcal{D}$ transformation leads to certain simple extensions of our results.

[91] Recall that the sign of $\mathcal{D}$ is important for TQFT unitarity.

Chapter 5

Galois Conjugation and Entanglement Entropy

5.1 Introduction

In Chapter 4, we explored several properties of Galois conjugation of TQFTs. We saw that Galois actions preserve symmetries of the TQFT. Moreover, we showed that the space of Galois invariant unitary TQFTs is very special. The purpose of this chapter is to gain additional physical insight into Galois transformations that goes beyond symmetry and fusion. In particular, we will study the effects of Galois transformations on a type of "multiboundary" topological entanglement entropy (MEE) defined in [155–157]. MEE is quite different from the more familiar entanglement entropy studied in [158,159]. Indeed, it involves first placing TQFTs on link complements, particular compact 3-manifolds that have multiple disjoint boundaries, and then tracing out Hilbert spaces associated with proper subsets of these boundaries. MEE is therefore highly non-local. Moreover, as we will see, MEE has interesting connections with knot theory, and we will phrase properties of Galois transformations in terms of the topology of knots and links.[92]

Our main claim is that, for a TQFT defined on $\mathcal{M}_{\mathcal{L}}$, the MEE we obtain by tracing out Hilbert subspaces associated with proper subsets of the disjoint boundaries is often invariant under the TQFT Galois action. In particular, we argue that the MEEs associated with any Abelian TQFT on any link complement in S^3 are invariant under the TQFT Galois action. In the case of non-abelian theories, the situation is more subtle. Building on our Abelian proof and taking into account recent results on classifications of MTCs [76, 161, 162], we argue that a natural place to look for Galois invariance of MEE in non-Abelian theories is on 3-manifolds corresponding to complements of torus links. Indeed, we then identify infinite sets of torus link complements that give rise to

[92] See [160] for an early study of Galois transformations and link invariants.

invariant MEE along Galois orbits.[93] As we will see, there is an interesting interplay between the topology of these link complements and basic modular data of the non-Abelian TQFTs living on these spaces (we highlight a simple application of this result in the conclusions).

Recall the discussion of Galois conjugation of TQFTs in Section 4.2. Since the F and R matrices are gauge-dependent, the Galois group of the full TQFT is gauge-dependent. However, we can arrive at a more gauge-invariant notion of a Galois group as follows. Recall that a TQFT computes knot and link invariants. Given a link, $\mathcal{L}$, its link invariant, $\mathcal{C}(\mathcal{L})$, is written in terms of the MTC data. If $\mathcal{C}(\mathcal{L})$ does not have self intersections, then it is independent of the choices of fusion space bases and hence is gauge invariant (see Fig. 5.1). Since we only consider observables built from such links in this chapter, all our data will be gauge invariant. This fact implies that the link data we will study is defined over some gauge-invariant subfield, $\mathcal{K} \subseteq K$. The corresponding Galois group, $\mathrm{Gal}(\mathcal{K})$, is then gauge independent.

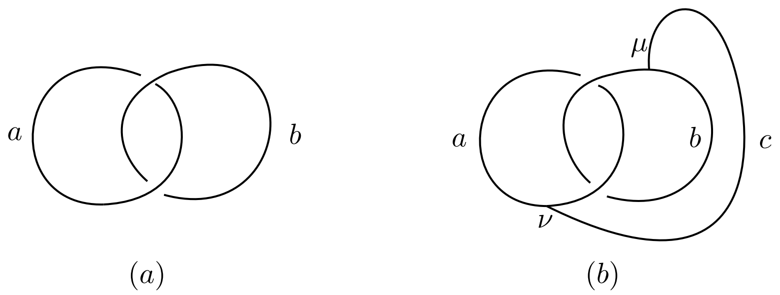


Figure 5.1: (a) The S matrix, and all the links we consider in this chapter, are gauge invariant. (b) The punctured S-matrix ($c \neq 0$) has self-intersections and is therefore gauge dependent.

As we will see, torus links will play a particularly important role in our story. These links can be constructed from words built out of the modular S and T matrices in (2.13), (2.14). For these matrices, the relevant field extension is a cyclotomic field, $\mathcal{K} = \mathbb{Q}(\xi_N)$, given by extending the rationals by powers of a primitive root of unity, $\xi_N = \exp(2\pi i/N)$ [32, 108–111]. As a result, the Galois group for the modular data is $\mathrm{Gal}(\mathbb{Q}(\xi_N)) = \mathbb{Z}_N^\times$—the multiplicative group modulo N consisting of all $n \in \{0, 1, 2, \cdots, N-1\}$ that are co-prime to N (i.e., $\gcd(n, N) = 1$).[94] In this case, we

[93] In the case of non-Abelian theories, the Galois action will, in general, take unitary theories to non-unitary ones. This fact leads to subtleties when defining what we mean by MEE in these latter theories. However, it turns out that there is in fact a natural definition of MEE even in the case of non-unitary theories.

[94] Unfortunately, the S and T matrices, along with the topological central charge, are not enough to specify an MTC [76]. As a result, we cannot take the Galois group of the S and T matrices to define a Galois group of the MTC in general. On the other hand, there may be other gauge-invariant ways to classify MTCs (e.g., see [161, 162] for preliminary results in this direction). Such a classification scheme might then allow one to assign a gauge-invariant Galois group for the full MTC.

have a trivial Galois action on $\mathbb{Q}$ and a non-trivial action on ξ

$$q(\xi) = \xi^q \ , \quad \forall q \in \mathbb{Z}_N^\times \ . \tag{5.1}$$

If the S and T matrices contain any elements not in $\mathbb{Q}$, the Galois group will take the TQFT with modular data (S, T) to a new TQFT with modular data $(q(S), q(T))$.[95]

In fact, given this discussion, one can work out the Galois action on the modular data for a given $q \in \mathbb{Z}_N^\times$ [32][96]

$$q(T_{aa}) = (T_{aa})^q \ , \quad q(S_{ab}) = \epsilon_q(a) S_{\sigma_q(a)b} = \epsilon_q(b) S_{a\sigma_q(b)} \ , \tag{5.2}$$

where $\sigma(a)$ is a permutation of the labels and $\epsilon_q(a) \in \{\pm\}$. Hence, Galois conjugation of the S matrix is a signed permutation.

We can say a bit more about how the Galois group acts on the modular data by making further contact with related results in the 2D RCFT literature [110]. In the context of RCFT, the natural normalization for the T matrix is

$$T \to \varphi \cdot T \ , \quad \varphi = \exp(\pi i c/12) \ , \tag{5.3}$$

where c is the central charge, and $\varphi^3 = \Theta$ (recall Θ was introduced in (2.16)). In this normalization, Bantay showed that the Galois group is given as follows [110]

$$\mathrm{Gal}(\varphi \cdot T, S) = \mathrm{Gal}(\mathbb{Q}(\xi_N)) \simeq \mathbb{Z}_N^\times \ , \quad (\varphi \cdot T)^N = \mathbb{1} \ , \tag{5.4}$$

where N is the "conductor"—for our purposes, the smallest $N > 0$ such that $(\varphi \cdot T)^N$ is the identity matrix. By definition, Bantay's Galois group must have a (not necessarily faithful) action on $\varphi \cdot T_{00} = \varphi$. Therefore, going back to the natural MTC normalization for T, we may conclude that

$$\mathrm{Gal}(T, S) = \mathbb{Z}_N^\times \ , \tag{5.5}$$

acts (not necessarily faithfully) on the modular data of the MTC and therefore constitutes a Galois group for the modular data. This statement does not preclude subgroups, $H \subset \mathbb{Z}_N^\times$, from acting faithfully on the modular data of the MTC.[97]

In our discussion of entanglement entropy in non-abelian theories, we will see that

[95]If S or T are not real, we can always take $(S, T) \to (S^*, T^*)$ (and similarly $(F, R) \to (F^*, R^*)$) and get a consistent TQFT related to the original one by time reversal.

[96]The Galois action on T follows from the fact that $T_{ij} = \theta_i \delta_{ij}$ is a diagonal matrix of phases. The Galois action on S follows from a careful analysis of the consequences of Verlinde's formula and the fact that the fusion rules are preserved by the Galois action.

[97]For example, in the context of the modular data of the Lee-Yang RCFT, the natural Galois group is $\mathbb{Z}_{60}^\times \simeq \mathbb{Z}_2^2 \times \mathbb{Z}_4$. This group acts unfaithfully on the modular data of the corresponding MTC. However, a $\mathbb{Z}_{20}^\times \simeq \mathbb{Z}_2 \times \mathbb{Z}_4 \subset Z_{60}^\times$ subgroup does act faithfully on this data. Note that the $\mathbb{Z}_{20}^\times$ subgroup is twice as large as the $\mathbb{Z}_5^\times \simeq \mathbb{Z}_4$ Galois group defined by the twists and quantum dimensions alone.

a slightly different notion of the conductor arises. There it is more natural to discuss an "MTC conductor" defined as the smallest $N_0 > 0$ such that

$$T^{N_0} = \mathbb{1} \ . \tag{5.6}$$

This quantity is closely related to Bantay's conductor since it turns out that $N = fN_0$, where $f \in \{1, 2, 3, 4, 6, 12\}$ [110].

While the above discussion, following [110], is tied to the existence of RCFTs realizing a particular MTC, it turns out that one may rephrase the above discussion without explicit reference to an underlying RCFT [111]. This latter approach is somewhat more mathematical, and we will not summarize it here. However, the upshot is that we will be able to make certain statements below about MTCs that need not be related to RCFTs.

The plan of this chapter is as follows. In the next section, we will review the definition of link states and MEE. With these concepts under our belt, in Sec. 5.3 we prove universal results on MEE in Abelian TQFTs. The following section is dedicated to generalizing this discussion to non-Abelian TQFTs living on torus link complements. Finally, we conclude with some comments on open problems and applications suggested by our work.

5.2 Multiboundary entanglement entropy in TQFT

In the previous section, we saw that solutions of the Pentagon and Hexagon equations can be partitioned into Galois orbits. This fact allows us to take the data of one TQFT and Galois conjugate it to get the data of another theory. The correlation functions / link invariants of the theory also get transformed under Galois conjugation.

To make this abstract discussion somewhat more physical, we will study how the Galois action affects a particular type of entanglement entropy defined in [156,157] (see also [155]). As we will see, studying this question will lead to an interesting interplay between MTC data and the topology of 3-manifolds.

To proceed, let us imagine a unitary TQFT defined on a compact 3-manifold, $\mathcal{M}_{\mathcal{L}}$, that is a link complement of some closed 3-manifold, $\mathcal{M}$. Note that we will mostly focus on the case $\mathcal{M} = S^3$ in what follows (we briefly discuss certain generalizations to other lens spaces in Sec. 5.4.5). We can construct such an $\mathcal{M}_{\mathcal{L}}$ by first drawing a non-self-intersecting n-component link, $\mathcal{L}^n = \sqcup_{i=1}^n L_i$, on S^3 and then removing a tubular neighborhood of the link, $\mathcal{N}(\mathcal{L}^n)$, from S^3. In other words

$$\mathcal{M}_{\mathcal{L}} \equiv S^3 - \mathcal{N}(\mathcal{L}^n) \ . \tag{5.7}$$

Then, it is clear that $\partial\mathcal{M}_{\mathcal{L}} = \sqcup_{i=1}^n T_i^2$. In other words, the boundary of our 3-manifold consists of a disjoint union of T^2's.

To any T^2, we can associate a Hilbert space (we will discuss subtleties related to the case of non-unitary MTCs below), $\mathcal{H}(T^2)$, whose basis states, $\{|j_a\rangle\}$, can be constructed by first filling in the T^2 to obtain a solid torus, U, with $\partial U = T^2$. The partition function of the theory on U with line j_a wrapping the non-contractible cycle of U, $Z_U(j_a)$, then defines a corresponding state, $|j_a\rangle$, on the boundary T^2. We can compute the inner product of this state with a set of dual states by first thinking of U as $U = D^2 \times S^1$, where D^2 is the 2-disk. The dual state, $\langle j_b|$, comes from studying the partition function on $U' = D^2 \times S^1$, where $\partial U' = -\partial U$,[98] with a line, j_b, inserted along the non-contractible cycle. The corresponding inner product, $\langle j_b|j_a\rangle$, can also be obtained by instead inserting the conjugate line, j_b^*, (in addition to j_a) along the non-contractible cycle of U. The partition function for $U \cup U' = S^2 \times S^1$ takes the form

$$\langle j_b|j_a\rangle \equiv Z_{S^2\times S^1}(j_b^*, j_a) = \delta_{ab} \ , \tag{5.8}$$

which follows from conservation of topological charge.

Now we can consider the boundary state, $|\mathcal{L}^n\rangle$, which belongs to the tensor product of Hilbert spaces $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ corresponding to the n T^2 boundaries. This state is defined by considering the partition function of the theory on $\mathcal{M}_\mathcal{L}$

$$|\mathcal{L}^n\rangle \equiv Z_{\mathcal{M}_\mathcal{L}} \ . \tag{5.9}$$

We may expand this state in terms of the T^2 states as follows

$$|\mathcal{L}^n\rangle = \sum_{j_1,\dots,j_n} C_{\mathcal{L}^n}(j_1, \dots, j_n)\, |j_1\rangle \otimes \cdots \otimes |j_n\rangle \ . \tag{5.10}$$

Using (5.8), we can compute the $C_{\mathcal{L}^n}(j_1, \cdots, j_n)$ by considering the inner product with $\langle j_n|\otimes\cdots\otimes\langle j_1|$. As discussed above, this operation corresponds to filling in the boundary T^2's and inserting conjugate representations, j_i^*, along the non-contractible cycles. For concreteness, let us consider a Chern-Simons theory and its Euclidean path integral on $\mathcal{M}_\mathcal{L}$. In this case, we have that the $\mathcal{C}_{\mathcal{L}^n}(j_1, \cdots, j_n)$ coefficients are just the various link invariants on S^3 computed from correlators of the Wilson lines in the conjugate representations

$$C_{\mathcal{L}^n}(j_1, \cdots, j_n) = \langle W_{j_1}^* \cdots W_{j_n}^* \rangle \ . \tag{5.11}$$

Given the link state, we can define the density matrix $\rho = |\mathcal{L}^n\rangle \langle \mathcal{L}^n|$, where the coefficients of $\langle \mathcal{L}^n|$ are $\langle W_{j_1} \cdots W_{j_n}\rangle$. We can further define reduced density matrices of the form

$$\rho_{\text{red}\,1,\dots,m} = \text{tr}_{m+1,\dots,n}(\rho) \ , \tag{5.12}$$

[98]In other words, U and U' share the same boundary with orientation reversed.

where we trace over the Hilbert subspace $\mathcal{H}_{m+1} \otimes \cdots \otimes \mathcal{H}_n$ to get a matrix defined on $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_m$. Then, one can define the MEE to be the usual von Neumann entropy of this reduced density matrix

$$S_{vN}(\rho_{\text{red}}) = -\text{tr}(\rho_{\text{red}} \ln\rho_{\text{red}}) \ . \tag{5.13}$$

This entanglement entropy is a coarse-grained form of the information contained in the link invariants $C_{\mathcal{L}^n}(j_1, ..., j_n)$.[99] Many interesting properties of this entanglement entropy were studied in [155–157] (see also [163]). In the following section, we will compute the explicit form of the link state, $|\mathcal{L}^n\rangle$, in general abelian theories and study the behavior of its entanglement entropy (after tracing out sub-links) under Galois conjugation. Note that since the entanglement entropy is invariant under local unitaries acting on the individual Hilbert spaces, we can ignore phases that come up in the calculation of $C_{\mathcal{L}^n}(j_1, ..., j_n)$ which depend purely on any one of the labels. Building on the results of the abelian discussion, we will then move on to discuss the more subtle case of non-Abelian TQFTs.

Before we continue, let us precisely define our procedure for comparing MEE under Galois transformations:

Definition (comparing MEE under Galois conjugation): *By comparing the MEE under Galois transformations, what we mean is the following. We start with some unitary TQFT, $\mathcal{T}$, and we compute the MEE. Then, we perform a Galois transformation to produce another TQFT, $\mathcal{T}'$. We then compute the MEE in $\mathcal{T}'$ and compare with the MEE in $\mathcal{T}$. This comparison can be done directly by producing $\rho_{\text{red}}(\mathcal{T}')$ from $\rho_{\text{red}}(\mathcal{T})$ via the Galois action. We then proceed iteratively along a Galois orbit, comparing MEEs for each element of the orbit. In particular, we do not apply a Galois transformation to (5.13) directly (this quantity is typically a transcendental number and does not lie in the field extension of the MTC).*

5.2.1 Subtleties for non-unitary theories

In the next section, we will discuss abelian TQFTs. These theories are all described by (unitary) Abelian CS theories.[100] In the language of axiomatic TQFT (e.g., see [164–166]), they assign Hilbert spaces to boundaries of 3-manifolds, $\partial\mathcal{M}$. In other words, to each boundary component of $\mathcal{M}_{\mathcal{L}}$, we have a complex vector space with a positive-definite norm.

[99] Indeed, at a more operational level, one may simply view the MEE as a convenient and natural means to encapsulate information about the link invariants on S^3. This information can, in principle, be reconstructed without ever introducing boundaries and associated Hilbert spaces.

[100] This statement ignores potential $S \to -S$ Galois transformations. However, we will see that our results apply to these theories as well.

On the other hand, when we discuss non-abelian TQFTs, the Galois action often takes unitary theories to non-unitary ones. Note that these non-unitary theories still have a finite number of simple objects. However, unlike unitary theories, non-unitary TQFTs have negative S^3 expectation values for some of the loops built out of the simple objects. As a result, under the standard MTC Hermitian inner product, such theories have negative norm states.

Still, even for non-unitary theories, in the case of a 3-manifold with boundary 2-tori, T_i^2, the theory assigns vector spaces, $V(T_i^2)$, with a set of vectors obeying (5.8). Indeed, the existence of this pairing follows from topological charge conservation and is independent of unitarity. Moreover, the non-unitary theories we consider lie on the same Galois orbit as at least one unitary theory, so the link invariant coefficients in (5.11) and their orientation-reversed conjugates have a natural extension to the non-unitary case under the Galois action. As a result, even for the non-unitary theories we study, we may formally construct a positive semi-definite reduced density matrix as in the discussion below (5.10) for the state defined by the path integral over $\mathcal{M}_{\mathcal{L}}$.

Readers who find this discussion disturbing are encouraged to take the definition in the previous subsection as an operational definition for comparing MEE in our theories of interest. Note that for more general states it is not immediately clear to us if one can construct a reduced density matrix in the same way. However, in the context of related non-unitary 2D CFTs, like the Lee-Yang theory, it is known that one can construct standard density matrices for other closely related measures of entanglement and define a Hilbert space with respect to a modified norm [167].[101] We suspect that assigning such a Hilbert space to the subset of non-unitary MTCs we discuss here is also possible, but we do not prove it.[102]

5.3 Abelian TQFTs

In this section we will study how the multiboundary entanglement entropy described in the previous section transforms as we perform Galois conjugation on abelian TQFTs. As discussed in Sec.4.2.2, abelian TQFTs have labels and fusion rules given by an abelian group, $\mathcal{A}$. Since the fusion rules are invariant under the Galois action, we see that the space of abelian TQFTs—and, more specifically, the space of theories with fusion rules given by $\mathcal{A}$—is closed under Galois conjugation.

As we will discuss in more detail shortly, abelian TQFTs can always be written as abelian CS theories [112, 169–171].[103] Since the main topological property encoded by

[101]These ideas have also played a role in a non-unitary proof of Zamolodchikov's c-theorem [168].

[102]This statement may be related to the fact that the primaries in 2D CFTs like Lee-Yang have positive norm, while negative norms only enter at the level of the descendants. We thank A. Konechny for discussions on this point.

[103]Here we ignore the possibility of flipping the sign of the S matrix (as discussed below (2.16)).

such theories is linking number, it is intuitively reasonable to imagine that the Galois action will lead to abelian theories with the same entanglement entropy.[104] We will indeed see this expectation is correct and that the linear transformation properties of S under the Galois action (5.2) play an important role.

To proceed, let us first discuss abelian TQFTs in more detail. Since the fusion rules are those of an abelian group, the fusion coefficients satisfy $N_{ab}^c = \delta_{a\cdot b,c}$ where $a, b, c \in \mathcal{A}$, and $a \cdot b$ is the group multiplication. Moreover, $N_{ab}^c \in \{0, 1\}$, and so all fusion spaces are one dimensional. Hence, the F and R matrices are just phases, and we will denote them as $F(a, b, c)$ and $R(a, b)$.[105]

In this case, the pentagon equation simplifies to

$$F(a,b,c\cdot d)F(a\cdot b,c,d) = F(b,c,d)F(a,b\cdot c,d)F(a,b,c) \ . \tag{5.14}$$

A function $F : \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \to U(1)$ satisfying (5.14) is called a 3-cocyle in group cohomology. Similarly, the Hexagon equations reduce to

$$\begin{aligned} R(a,c)F(b,a,c)R(a,b) &= F(b,c,a)R(a,b\cdot c)F(a,b,c) \ , \\ R(c,a)F(b,a,c)^{-1}R(b,a) &= F(b,c,a)^{-1}R(b\cdot c,a)F(a,b,c)^{-1} \ . \end{aligned} \tag{5.15}$$

The gauge-inequivalent solutions, (F, R), belong to the third abelian cohomology group, $H^3_{ab}(\mathcal{A}, U(1))$. The gauge freedom in F and R is captured by this cohomology structure [42, 43].

As reviewed in Chapter 2, to find the MTC data of a general TQFT given a set of labels and fusion rules, one finds F matrices solving the Pentagon equations and then one solves the Hexagon equations given these F matrices. However, in abelian TQFTs, the situation is much simpler, and the MTC data is fixed by the choice of a quadratic function, $\theta(a) : \mathcal{A} \to U(1)$, that gives the topological spins (i.e., the T matrix).[106]

Although much of what we said above does not depend on the existence of Lagrangians, it will be useful for us to keep them in mind in our subsequent discussion of abelian theories. Moreover, as mentioned at the beginning of this section, it turns out that we do not lose any generality in studying abelian Chern-Simons theories with gauge group $U(1)^N$ [112, 169–171] (they span the space of TQFTs with S matrices as

However, our results apply even to any MTCs of this latter type.

[104] In this case, the simpler entanglement entropy of [158, 159] is trivially invariant since it is given by the square-root of the rank of the fusion group, $\sqrt{|\mathcal{A}|}$.

[105] Note that since all fusion spaces are one dimensional, specifying a, b, c in $F^d_{a,b,c}$ automatically specifies d, and so we loose no generality in taking the F symbols to depend on three group elements. Similar reasoning shows that we loose no generality in taking the R matrices to depend on two group elements.

[106] For further details, see the recent discussion in [112].

in (2.13)). These theories have Lagrangians of the general form

$$S = \frac{iK_{ij}}{2\pi}\int_M A^i dA^j \ . \tag{5.16}$$

Here A^i are $U(1)$ gauge fields, M is a 3-manifold, and K is a symmetric even integral matrix of levels.[107] The fusion rules of the theory are given by the abelian group, $\mathbb{Z}^N/K\mathbb{Z}^N$, and anyons are labelled by a set of basis vectors for this lattice. The fact that K is an integer matrix means it has a Smith normal form, K_S, which we denote by

$$K_S = \begin{pmatrix} n_1 & 0 & \dots & 0 \\ 0 & n_2 & & \\ \vdots & & \ddots & \\ 0 & \dots & 0 & n_N \end{pmatrix} \ . \tag{5.17}$$

There exists integer matrices U and W which are invertible over the integers such that $K_S = UKW$. Two vectors $\vec{a}, \vec{b}$ in $\mathbb{Z}^N/K\mathbb{Z}^N$ are equivalent if they satisfy

$$\vec{a} = \vec{b} + K\vec{\alpha} = \vec{b} + U^{-1}K_S W^{-1}\vec{\alpha} \tag{5.18}$$

for some integer vector $\vec{\alpha}$. Let us define a new integer vector $\vec{\alpha}' = W^{-1}\vec{\alpha}$. We have

$$\vec{a} = \vec{b} + U^{-1}K_S\vec{\alpha}' \tag{5.19}$$

After a change of basis using U, we have

$$\vec{a}' = \vec{b}' + K_S\vec{\alpha}' \tag{5.20}$$

where $\vec{a}' = U\vec{a}, \vec{b}' = U\vec{b}$. From this discussion, it is clear that the abelian group $\mathbb{Z}^N/K\mathbb{Z}^N$ is isomorphic to $\mathbb{Z}_{n_1}\otimes\cdots\otimes\mathbb{Z}_{n_N}$. Clearly we can reproduce any finite abelian fusion group using such theories. As discussed above, the MTC data is specified by the topological spin. For abelian theories, it can be expressed in terms of K as follows

$$\theta(\vec{a}) = \exp\left(\pi i \vec{a} K^{-1}\vec{a}\right) \ , \tag{5.21}$$

where $\vec{a} \in \mathbb{Z}^N/K\mathbb{Z}^N$.

Next let us explicitly fix the remainder of the modular data (recall that T is given in terms of θ). To that end, we first define the braiding

$$B(\vec{a}.\vec{b}) = \frac{\theta(\vec{a}+\vec{b})}{\theta(\vec{a})\theta(\vec{b})} = \exp\left(2\pi i \vec{a} K^{-1}\vec{b}\right) \ . \tag{5.22}$$

[107] In other words, we will assume that the diagonal entries in K are even integers (the remaining entries may be even or odd). Otherwise, the theory would be a spin-TQFT.

The R matrices and the braiding phase are related by [112]

$$B(\vec{a},\vec{b}) = R(\vec{a},\vec{b})R(\vec{b},\vec{a}) \ . \tag{5.23}$$

The representation of the modular group generators S and T realized by this theory is then

$$S_{\vec{a},\vec{b}} = \frac{1}{\sqrt{|\mathcal{A}|}} B(\vec{a},\vec{b}) \ , \quad T_{\vec{a},\vec{a}} = \theta(\vec{a}) \ , \tag{5.24}$$

where $|\mathcal{A}|$ is the order of the abelian group, $\mathcal{A}$.

In the next section, we will use the above data to find the link invariant for a general n-component link. Given this expression, we will then compute the entanglement entropy for general abelian theories and show the invariance claimed above under Galois transformations.

5.3.1 Link invariants in abelian TQFTs

Let us consider an n-component link in which the constituent knots are labelled by $j_1, \cdots, j_n$. Since the F and R matrices of abelian TQFTs are $U(1)$ valued, simplifying the individual structure of a knot, j_i, to give the unknot will give us phases which act on the Hilbert space, $\mathcal{H}_i$. Since these phases can be removed using a local unitary operation, the entanglement entropy is independent of these phases. Hence, as far as calculating the entanglement entropy is concerned, we are only interested in the braiding between the constituent knots. We will consider the case of a 2-component link in the next section which can then be easily generalized to give the link invariant for an n-component link.

Link invariant for a 2-link

Since abelian theories primarily capture linking number, it is reasonable to imagine that any link invariant can be written (up to unimportant local unitary transformations that will not affect our quantities of interest) in terms of the S matrix. We will see this statement is indeed true.

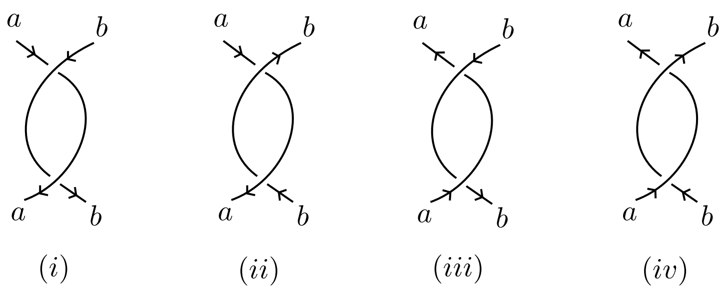


Figure 5.2: Possible braidings for oriented links.

To that end, consider a 2-component link in which the two knots are labelled a and b. There must be an even number of braids between them. As a result, the braids can be grouped into pairs. In an oriented link, four types of pairs are possible (see Fig. 5.2). Let us find the algebraic expression obtained from unbraiding diagram (i) in Fig. 5.2.

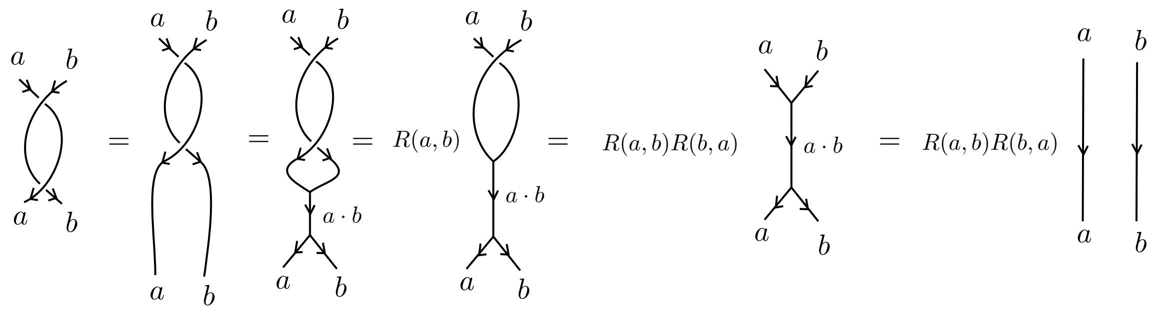


Figure 5.3: Link invariant of a braid pair. Diagrams 1-6 from left.

To understand this unbraiding, consider Fig. 5.3. In going from diagram 2 to 3 (from left) of Fig. 5.3, we have used the decomposition of the identity, which has a unique channel in an abelian TQFT. In diagrams 4 and 5, we remove the braiding by adding appropriate R matrix factors. Finally we again use the decomposition of the identity to go from diagram 5 to 6. As a result, we find that the braid pair can be replaced by the identity acting on the two anyons if we include the factor $R(a, b)R(b, a)$.

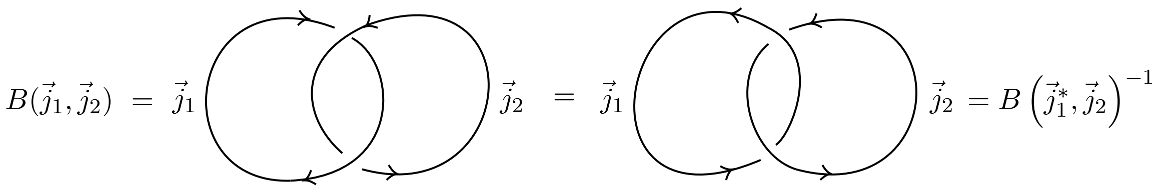


Figure 5.4: The relation in (5.26) follows from the equality of the above TQFT diagrams.

In the K matrix formalism, the knots are labelled by anyonic vectors $\vec{j}_1, \vec{j}_2 \in \mathbb{Z}^N/K\mathbb{Z}^N$, and we think of the anyons as elements of the corresponding additive group. In this notation, the algebraic expression corresponding to the diagram (i) is $R(\vec{j}_1, \vec{j}_2)R(\vec{j}_2, \vec{j}_1)$. From (5.24), we know that this is just the braiding phase $B(\vec{j}_1, \vec{j}_2)$. Repeating the above calculation for diagrams (ii), (iii) and (iv) we get $B(\vec{j}_1, \vec{j}_2^*)$, $B(\vec{j}_1^*, \vec{j}_2)$, and $B(\vec{j}_1^*, \vec{j}_2^*)$, respectively. If there are n_1 braid pairs of type (i), n_2 of type (*ii*), n_3 of type (*iii*), and n_4 of type (*iv*), the total link invariant is given by

$$(B(\vec{j}_1,\vec{j}_2))^{n_1}(B(\vec{j}_1,\vec{j}_2^*))^{n_2}(B(\vec{j}_1^*,\vec{j}_2))^{n_3}(B(\vec{j}_1^*,\vec{j}_2^*))^{n_4} \ . \tag{5.25}$$

Moreover, (5.22) implies the following relations hold

$$B(\vec{j}_1,\vec{j}_2^*) = (B(\vec{j}_1,\vec{j}_2))^{-1} \ , \quad B(\vec{j}_1^*,\vec{j}_2) = (B(\vec{j}_1,\vec{j}_2))^{-1} \ , \tag{5.26}$$

since $\vec{j}_i^* \sim -\vec{j}_i$, where "$\sim$" means, "up to vectors of the form $K \cdot \vec{\omega}_i$" (i.e., up to a K-trivial vector). In fact, (5.26) holds without the need to appeal to a K matrix, since the TQFT diagrams in Fig. 5.4 are equal.

Hence, the link invariant simplifies to

$$B(\vec{j}_1,\vec{j}_2)^{l_{12}} \sim S(\vec{j}_1,\vec{j}_2)^{l_{12}} \ , \tag{5.27}$$

where $l_{12} = n_1 + n_4 - n_2 - n_3$ is the linking number, and "$\sim$" means, "up to an overall normalization." This simple calculation shows that the MTC data of abelian TQFTs can be used to compute the linking number of a link and that the result can be expressed through the S matrix alone. Next, we generalize this argument to an n-link.

Link invariant for an n-link

For a link made up of n knots, we should repeat the calculation in Sec. 5.3.1 for each pair of knots, (j_i, j_k), where $1 \leq i < k \leq n$, and $j_{i,k}$ are the labels of the corresponding knots. Proceeding in this way, we find

$$(B(\vec{j}_i,\vec{j}_k))^{\ell_{ik}} \ , \tag{5.28}$$

where ℓ_{ik} is the linking number between the knots labelled j_i and j_k in the link. The total link invariant will be the product of these factors. As a result, the link invariant for an n-link in an abelian TQFT is

$$\prod_{i<k}(B(\vec{j}_i,\vec{j}_k))^{\ell_{ik}} \ . \tag{5.29}$$

5.3.2 Galois conjugation of entanglement entropy

Using the link invariants computed in the previous subsection, we can now find the associated entanglement entropy defined in Sec. 5.2 and study its behavior under the Galois action.

Let us again specialize to a 2-link before discussing the general $n > 2$ case. To that end, using (5.27), we have the normalized link state

$$\left|\mathcal{L}^2\right\rangle = \frac{1}{|\mathcal{A}|}\sum_{\vec{j}_1,\vec{j}_2}(B(\vec{j}_1,\vec{j}_2))^{l_{12}}\left|j_1,j_2\right\rangle = \sum_{\vec{j}_1,\vec{j}_2}|\mathcal{A}|^{\frac{l_{12}}{2}-1}\left(S_{\vec{j}_1,\vec{j}_2}\right)^{l_{12}}\left|j_1,j_2\right\rangle \ . \tag{5.30}$$

Tracing out the Hilbert space of the second link yields the following reduced density matrix

$$\rho_{red} \ = \ \sum_{\vec{j}_1,\vec{h}_1}\sum_{\vec{m}}|\mathcal{A}|^{l_{12}-2}\left(S_{\vec{j}_1\vec{m}}\right)^{l_{12}}\left(S_{\vec{h}_1\vec{m}}\right)^{-l_{12}}\left|\vec{j}_1\right\rangle\left\langle\vec{h}_1\right| \ . \tag{5.31}$$

Next we may use (5.2) to perform a Galois transformation and read off the reduced density matrix in the conjugated theory

$$\begin{aligned}\rho_{red} \ &= \ |\mathcal{A}|^{l_{12}-2}\sum_{\vec{j}_1,\vec{h}_1}\sum_{\vec{m}}\left(\epsilon_p(m)S_{\vec{j}_1\sigma_p(\vec{m})}\right)^{l_{12}}\left(\epsilon_p(m)S_{\vec{h}_1\sigma_p(\vec{m})}\right)^{-l_{12}}\left|\vec{j}_1\right\rangle\left\langle\vec{h}_1\right| \\ &= \ |\mathcal{A}|^{l_{12}-2}\sum_{\vec{j}_1,\vec{h}_1}\sum_{\vec{m}}\left(S_{\vec{j}_1\sigma_p(\vec{m})}\right)^{l_{12}}\left(S_{\vec{h}_1\sigma_p(\vec{m})}\right)^{-l_{12}}\left|\vec{j}_1\right\rangle\left\langle\vec{h}_1\right| \ . \end{aligned}\tag{5.32}$$

Since $\vec{m}$ is summed over, the reduced density matrix is invariant under Galois conjugation. As a result, the entanglement entropy for a 2-link computed in an abelian TQFT and the Galois conjugated theory are equal.

The generalization to an n-link is straightforward. Indeed, using (5.29), the link state is given (up to a normalization factor) by

$$\left|\mathcal{L}^n\right\rangle \ = \ \sum_{\vec{j}_1,\ldots,\vec{j}_n}\prod_{i\leq k}\left(S_{\vec{j}_i,\vec{j}_k}\right)^{\ell_{ik}}\left|j_1,\ldots,j_n\right\rangle \ . \tag{5.33}$$

The density matrix for this state is then

$$\rho = \sum_{\vec{j}_1,\ldots,\vec{j}_n}\sum_{\vec{h}_1,\ldots,\vec{h}_n}\left(\prod_{i<k}\left(S_{\vec{j}_i\vec{j}_k}\right)^{\ell_{ik}}\right)\left(\prod_{z<w}\left(S_{\vec{h}_z\vec{h}_w}\right)^{-\ell_{zw}}\right)\left|\vec{j}_1,\ldots,\vec{j}_n\right\rangle\left\langle\vec{h}_1,\ldots,\vec{h}_n\right| \ . \tag{5.34}$$

Without loss of generality, we can trace over the last $n-q$ links to get a reduced density matrix over the first q links. Up to overall constant factors and phases which can be removed by applying unitaries on the first q copies of the Hilbert space (which again don't affect the entanglement entropy), the components of the reduced density matrix can be written as

$$\rho_{\text{red}\,\vec{j}_1\ldots\vec{j}_q,\vec{h}_1,\ldots,\vec{h}_q} \ = \ \prod_{k=q+1}^{n}\sum_{\vec{m}}\prod_{i=1}^{q}\left(S_{\vec{j}_i,\vec{m}}\right)^{\ell_{ik}}\left(S_{\vec{h}_i,\vec{m}}\right)^{-\ell_{ik}} \ . \tag{5.35}$$

Galois conjugation of this reduced density matrix will only result in a permutation of the vectors $\vec{m}$. Since there is a sum over $\vec{m}$, the reduced density matrix is invariant under Galois conjugation. Note that this invariance includes any Galois transformation taking $S \to -S$.[108]

This discussion once again implies that the entanglement entropy is also invariant under Galois conjugation. Thus, even though the link invariants calculated in two abelian TQFTs related by Galois conjugation are generally different, the entanglement entropy is the same in both theories. Note that the linear behavior of the S matrix under Galois conjugation plays a crucial role in this result.

Before briefly exploring implications of these results for non-Abelian theories, let us note that we may explicitly compute the MEE following from (5.35). For simplicity, focussing on the 2-link case, we obtain (see App. B.1 for details)

$$S_{\mathrm{vN}}(\mathcal{L}^2) = \ln\left(\frac{\det(K)}{\gcd(\ell_{12}, n_1)\gcd(\ell_{12}, n_2)\cdots\gcd(\ell_{12}, n_N)}\right) , \tag{5.36}$$

where the n_i are the diagonal elements of the Smith normal form, K_s, and are therefore the ranks of the individual factors that make up the Abelian fusion group, $\mathcal{A}$.[109] Note that (5.36) shows a manifest symmetry under $\ell_{12} \to \ell_{12} + m\det(K)$ for any integer m since

$$\gcd(\ell_{12} + m\det(K), n_i) = \gcd(\ell_{12}, n_i) , \quad \forall n_i . \tag{5.37}$$

Actually, this same periodicity is already visible in (5.33). Indeed, from (5.24) and (5.22), we have (up to an unimportant normalization)

$$S_{\vec{j}_i,\vec{j}_2} \sim \exp\left(2\pi i \vec{j}_1 K^{-1} \vec{j}_2\right) = \left(2\pi i \frac{\vec{j}_1 \tilde{K} \vec{j}_2}{\det(K)}\right) , \tag{5.38}$$

where $\tilde{K}$ is the integer-valued adjugate matrix. Therefore, taking $\ell_{ik} \to \ell_{ik} + m\det(K)$ in (5.33) leaves $|\mathcal{L}^n\rangle$ and the MEE invariant. It is also straightforward to use (5.24) and (5.22) to establish that (5.33) and the MEE are invariant under arbitrary integer shifts of the linking numbers by the MTC conductor, $\ell_{ik} \to \ell_{ik} + mN_0$.

The upshot of the above discussion is that Galois transformations of abelian theories preserve the multiboundary entanglement entropy. However, this result hinges on the fact that, for any link complement, only the S matrix enters the computation. Moreover, the S matrix has linear transformation properties under the Galois action. For more general TQFTs we therefore expect a more subtle situation. For example, we

[108] In fact, we could have constructed a link state directly for such theories using diagramatic reductions analogous to those above, and we would have found the same reduced density matrix. Therefore, such theories have invariant ρ_{red} even if they are not Galois conjugates of abelian CS theories.

[109] As a consistency check, note that for a product TQFT, where K itself is a diagonal matrix, the entanglement entropy becomes the sum of the entanglement entropies of the individual theories.

expect the T matrix to play a more prominent role (i.e., that it will not just appear through S), and we have seen that both it and the entanglement entropy are sensitive to the conductor.[110]

5.4 Non-Abelian TQFTs

In this section, we generalize the abelian TQFT discussion to non-abelian theories. Before proceeding, it is worth considering what such a generalization should look like. To that end, let us make a few comments:

- In the abelian case, the density matrix can be written exclusively in terms of the S-matrix. This simplification is due to the fact that abelian theories are only sensitive to linking number. On the other hand, non-abelian theories compute more complicated invariants: the Jones polynomial, the HOMFLY polynomials, and infinitely many generalizations. Therefore, a non-abelian generalization of our discussion should depend on finer details of the topology of $\mathcal{M}_{\mathcal{L}}$. In the broadest terms, a result of Thurston [172], guarantees that $\mathcal{M}_{\mathcal{L}}$ can be either a torus link complement, a hyperbolic link complement, or a satellite link complement.[111] Torus links are naturally in correspondence with words that can be built out of S and T (the generators of the mapping class group of T^2). Therefore, this reasoning points to studying a generalization of the abelian result to torus links.

- Another reason to study torus links when searching for a non-abelian generalization of Sec. 5.3 comes from the results in [76]. There the authors showed that there are Dijkgraaf-Witten theories with gauge group $\mathbb{Z}_q \rtimes_n \mathbb{Z}_p$ such that the Galois group acts non-trivially on the corresponding MTCs but leaves the S and T matrices invariant. As a result, torus knot complements are natural places to look to find invariance of the entanglement entropy along Galois orbits. Moreover, recent work in [161, 162] suggests that hyperbolic link invariants can potentially be used to distinguish MTCs in a gauge-invariant manner. Indeed, it is easy to check that the entanglement entropy of one of the simplest non-abelian TQFTs, $su(2)_k$ Chern-Simons theory, is generically non-invariant along the corresponding Galois orbits when tracing out one of the links of the hyperbolic Whitehead link complement (see App. B.2 for details). Since Whitehead is one of the simplest hyperbolic links (it has linking number zero and is built from two unknots), this

[110] In the theories described above, the period of the link state and the MEE can be finer than the conductor (although these quantities are also periodic modulo the conductor). For example, in $\mathbb{Z}_2$ TQFT, we have $N_0 = 4$, but $\det(K) = 2$. The reason for this difference is that the link state and MEE depend on T only through the (unnormalized) S matrix.

[111] Torus links are links that can be drawn on the surface of a T^2 without self-intersection. Hyperbolic links are links whose complements admit complete hyperbolic metrics. By Thurston's results, satellite links are what remain (we will briefly encounter these links in App. B.2).

result suggests that changes in the entanglement entropy along Galois orbits of theories on hyperbolic link complements is more ubiquitous. Similar comments apply to satellite link complements (see App. B.2 for details).[112]

- On the other hand, we do not expect all torus link complements to give rise to invariant entanglement entropy along Galois orbits. Indeed, we generally expect the non-Abelian density matrix to depend explicitly on T and not just on S. As a result, we expect the topology of the torus link complement to be sensitive to the conductor of the MTC.

To better understand how to proceed, we review torus links in the next subsection. We then revisit the linear transformation properties of S that hold in Abelian and non-Abelian TQFTs alike and use them to identify a canonical class of torus links that give rise to invariant entanglement entropy along Galois orbits of general MTCs.

5.4.1 Torus links and canonical words

Let us recall some useful aspects of torus links. As discussed in previous sections, these links can be drawn on the surface of a T^2 without self intersection. They are classified by two integers, (m, n), corresponding to the basis of 1-cycles wrapped by the links. In particular, m corresponds to the number of times the link wraps the longitude of the torus and n corresponds to the number of times the link wraps the meridian.[113]

The links may be characterized by the components that make it up. In particular, we have

$$\nu(m,n) = \gcd(m,n) \ , \quad \ell(m,n) = \frac{mn}{\gcd(m,n)^2} \ , \tag{5.39}$$

where $\nu(n,m)$ is the number of components that make up the link (note that for $\gcd(n,m) = 1$, the link is a knot), and $\ell(m,n)$ is the linking number between any two component knots in the link (this is an invariant for any pairs of knots in the link). The knots that make up the link are of type $(m/\nu(m,n), n/\nu(m,n))$. For example, $(2,2)$ is the Hopf link, with $n_L = 2$ and $\ell = 1$. This link is made up of two $(1,1)$ unknots.

One crucial aspect of our discussion below is that the entanglement entropy arising in torus links does not depend on the number of knots we trace out [157]. More precisely, if our link consists of $\nu(m,n) \geq 2$ knots, the entanglement entropy is independent of the number of links, $1 \leq r \leq \nu(m,n) - 1$, we trace out.

In order to understand which torus link complements give rise to invariant entanglement entropy under Galois conjugation, it is useful to revisit the Galois transformation

[112] Although we suspect that there could be interesting generalizations of our work below to some subclasses of these links as well. For example, a natural set of satellite links to examine are connected sums of torus links.

[113] An invariant definition of these cycles can be given by imagining filling in the torus to obtain a solid torus. In the solid torus, the meridian becomes contractible while the longitude does not.

properties of the S matrix in (5.2). For each element, Q, of the Galois group of the modular data, these signed permutations can be generated by[114]

$$G_{\sigma_Q} = \varphi^{2Q+P} S^{-1} T^Q S T^P S T^Q \ , \quad Q \cdot P = 1 \ (\mathrm{mod}\ N) \ , \tag{5.40}$$

where N is the (generalization of the) conductor described around (5.4). In other words, the Galois transformation of the S-matrix is

$$Q(S) = G_{\sigma_Q}^{-1} S = S G_{\sigma_Q} \ . \tag{5.41}$$

From the perspective of the Galois group, the string of S and T matrices in (5.40) form a set of "canonical" words: the G_{σ_Q} are invariant under the Galois group, since all matrix elements are in the set $\{-1, 0, 1\} \subset \mathbb{Q}$.

Given each G_{σ_Q}, it is natural that there should be an infinite number of associated torus link complements that give rise to Galois invariant entanglement entropy for non-abelian TQFTs defined on these spaces.[115] In the next subsection, we will argue that the complements of torus links of type (M, MQ), with $M \in \mathbb{Z}$ and $\gcd(Q, N_0) = 1$, are precisely such a set of 3-manifolds (recall that N_0 is the MTC conductor defined in (5.6)). In what follows, we will refer to these spaces as $\mathcal{M}_{\mathcal{L}(M,MQ)}$.

5.4.2 Galois invariance of the entanglement entropy on $\mathcal{M}_{\mathcal{L}(M,MQ)}$

We begin by deriving an explicit expression for the link invariant of an (M, MQ) torus link from the MTC data. From (5.39), we see that this is an M component link in which the number of braidings between any two knots is $2Q$ (which follows from the mutual linking number, Q). Let us look at the braids between the knots labelled by j_i and j_k. The $2Q$ braids between these two knots are represented by the operators $(R_{j_i,j_k} R_{j_k,j_i})^Q$ (see Fig. 5.5).

The total invariant can then be found by computing the following quantum trace:

$$\widetilde{\mathrm{Tr}}\Big(\prod_{j_i, j_k} (R_{j_i,j_k} R_{j_k,j_i})^Q \Big) \ . \tag{5.42}$$

The operator within the trace acts on the fusion space $V_{j_1,\cdots,j_M}^{j_1,\cdots,j_M}$. In order to compute the quantum trace, we need to specify the operator's action on the fusion space, $V_{j_1,\cdots,j_M}^{c}$. Since we have

$$V_{j_1,\cdots,j_M}^{c} = \sum_{a_1,\cdots,a_{m-2}} V_{j_1,j_2}^{a_1} \otimes V_{a_1,j_3}^{a_2} \otimes \cdots \otimes V_{a_{m-2},j_M}^{c} \ , \tag{5.43}$$

[114] These matrices were constructed for cases with $C = \mathbb{1}$ in [109] and more generally in [110, 111].

[115] The reason we expect an infinite number of torus link complements for each G_{σ_Q} is that, for each torus knot, we can construct links with arbitrary numbers of these knots.

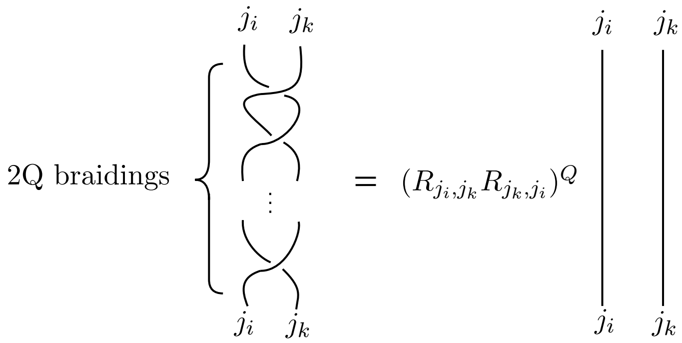


Figure 5.5: The action of $(R_{j_i,j_k}R_{j_k,j_i})^Q$ on strands of the knots labeled by j_i and j_k.

we can write the operators $\prod_{j_i,j_k}(R_{j_i,j_k}R_{j_k,j_i})^Q$ acting on $V^c_{j_1,\dots,j_M}$ as

$$\sum_{a_1,\cdots,a_{M-2}} (R^{a_1}_{j_1,j_2}R^{a_1}_{j_2,j_1})^Q(R^{a_2}_{a_1,j_3}R^{a_2}_{j_3,a_1})^Q\cdots(R^c_{a_{M-2},j_m}R^c_{j_M,a_{M-2}})^Q\ . \tag{5.44}$$

We can now evaluate (5.42) to obtain

$$\begin{aligned}
\widetilde{\mathrm{Tr}}\Big(\prod_{j_i,j_k}(R_{j_i,j_k}R_{j_k,j_i})^Q\Big) &= \sum_c d_c \sum_{a_1,\cdots,a_{M-2}} (R^{a_1}_{j_1,j_2}R^{a_1}_{j_2,j_1})^Q(R^{a_2}_{a_1,j_3}R^{a_2}_{j_3,a_1})^Q\cdots \\
&\qquad\cdots(R^c_{a_{M-2},j_M}R^c_{j_M,a_{M-2}})^Q \\
= \sum_c d_c \sum_{a_1,\cdots,a_{M-2}} &\mathrm{Tr}\Bigg(\left(\frac{\theta(a_1)}{\theta(j_1)\theta(j_2)}\right)^Q \mathrm{id}_{V^{a_1}_{j_1,j_2}} \otimes \left(\frac{\theta(a_2)}{\theta(a_1)\theta(j_3)}\right)^Q \mathrm{id}_{V^{a_2}_{a_1,j_3}} \otimes\cdots \\
\cdots \otimes &\left(\frac{\theta(c)}{\theta(a_{M-2})\theta(j_M)}\right)^Q \mathrm{id}_{V^c_{a_{M-2},j_M}}\Bigg) \\
= \sum_c d_c \sum_{a_1,\cdots,a_{M-2}} &\left(\frac{\theta(c)}{\theta(j_1)\theta(j_2)\cdots\theta(j_M)}\right)^Q \mathrm{Tr}\Big(\mathrm{id}_{V^{a_1}_{j_1,j_2}}\otimes \mathrm{id}_{V^{a_2}_{a_1,j_3}}\otimes\cdots\otimes \mathrm{id}_{V^c_{a_{M-2},j_M}}\Big) \\
= \sum_c d_c \sum_{a_1,\cdots,a_{M-2}} &\left(\frac{\theta(c)}{\theta(j_1)\theta(j_2)\cdots\theta(j_M)}\right)^Q N^{a_1}_{j_1,j_2}N^{a_2}_{a_1,j_3}\cdots N^c_{a_{M-2},j_M}\ .
\end{aligned} \tag{5.45}$$

Since the framing factors, $\theta(j_i)$, can be removed using local unitaries acting on the respective Hilbert spaces, we can ignore them. Using the Verlinde formula (2.17), we can simplify the above expression to get (up to framing factors we drop)

$$\widetilde{\mathrm{Tr}}\Big(\prod_{j_i,j_k}(R_{j_i,j_k}R_{j_k,j_i})^Q\Big) = \sum_{b_{M-1}} \frac{(ST^QS)_{0b_{M-1}}}{S_{00}}\,\frac{S_{b_{M-1}j_1}S_{b_{M-1}j_2}\cdots S_{b_{M-1}j_M}}{S^{M-1}_{0b_{M-1}}}\ . \tag{5.46}$$

Hence, the link state for an (M, MQ) link is given by

$$\begin{aligned}\left|\mathcal{L}^{(M,MQ)}\right\rangle &= \sum_{j_1,\cdots,j_M}\sum_{b_{M-1}} \frac{(ST^QS)_{0b_{M-1}}}{S_{00}} \frac{S_{b_{M-1}j_1}S_{b_{M-1}j_2}\cdots S_{b_{M-1}j_M}}{S^{M-1}_{0b_{M-1}}} |j_1,\cdots,j_M\rangle \\ &= \sum_{b_{M-1}} \frac{(ST^QS)_{0b_{M-1}}}{S_{00}S^{M-1}_{0b_{M-1}}} |b_{M-1},\cdots,b_{M-1}\rangle \ . \end{aligned} \tag{5.47}$$

From this data we can compute the eigenvalues of the unnormalized reduced density matrix. They are independent of the number of Hilbert spaces we trace over and are given by

$$\Lambda_\ell = \left|\frac{(ST^QS)_{0l}}{S_{00}S^{M-1}_{0l}}\right|^2 \ . \tag{5.48}$$

Let us now suppose that $Q \in \mathbb{Z}_N^\times$ is a Galois group element for the modular data of the MTC (i.e., we have $\gcd(Q,N) = 1$). The resulting entanglement entropy turns out to be constant along Galois orbits due to the Galois invariance of G_{σ_P} in (5.40) with $P \cdot Q = 1 \pmod N$. To understand this statement, let us consider the action of G_{σ_P} on S

$$SG_{\sigma_P} = \varphi^{2P+Q}T^PST^QST^P \ , \quad Q\cdot P = 1 \ (\mathrm{mod}\ N) \ , \tag{5.49}$$

which implies that

$$ST^QS = \varphi^{-(2P+Q)}T^{-P}SG_{\sigma_P}T^{-P} \ . \tag{5.50}$$

Taking $\lambda_\ell = S_{00}^2\Lambda_\ell$, we then have

$$\begin{aligned}\lambda_\ell = \left|\frac{(ST^QS)_{0l}}{S^{M-1}_{0l}}\right|^2 &= \frac{(ST^QS)_{0\ell}}{S^{M-1}_{0\ell}}\frac{(S^*T^{*Q}S^*)_{0\ell}}{S^{*\ M-1}_{0\ell}} \\ &= \frac{(T^{-P}SG_{\sigma_P}T^{-P})_{0\ell}}{S^{M-1}_{0\ell}}\frac{(T^{*-P}S^*G^*_{\sigma_P}T^{*-P})_{0\ell}}{S^{*\ M-1}_{0\ell}} \\ &= \sum_i \frac{S_{0i}(G_{\sigma_P})_{i\ell}T^{-P}_{\ell\ell}}{S^{M-1}_{0\ell}} \sum_j \frac{S^*_{0j}(G^*_{\sigma_P})_{j\ell}T^{*-P}_{\ell\ell}}{S^{*M-1}_{0\ell}} \\ &= \sum_i \frac{S_{0i}(G_{\sigma_P})_{i\ell}}{S^{M-1}_{0\ell}} \sum_j \frac{S^*_{0j}(G_{\sigma_P})_{j\ell}}{S^{*M-1}_{0\ell}} \ . \end{aligned} \tag{5.51}$$

In going between the first and second lines we use (5.50), and we use $T_{00} = 1$ in going between the second and third lines. Recalling that G_{σ_P} induces a signed permutation, we have

$$\lambda_\ell = \frac{S_{0\sigma_P(\ell)}}{S^{M-1}_{0\ell}}\frac{S^*_{0\sigma_P(\ell)}}{S^{*M-1}_{0\ell}} \ . \tag{5.52}$$

Clearly, this quantity transforms as a permutation under Galois conjugation by the element $r \in \mathbb{Z}_N^\times$

$$\lambda_\ell \xrightarrow{r\in\mathbb{Z}_N^\times} \frac{S_{0\sigma_r(\sigma_P(\ell))}}{S^{M-1}_{0\sigma_r(\ell)}} \frac{S^*_{0\sigma_r(\sigma_P(\ell))}}{S^{*M-1}_{0\sigma_r(\ell)}} = \frac{S_{0\sigma_P(\sigma_r(\ell))}}{S^{M-1}_{0\sigma_r(\ell)}} \frac{S^*_{0\sigma_P(\sigma_r(\ell))}}{S^{*M-1}_{0\sigma_r(\ell)}} = \lambda_{\sigma_r(\ell)} \ , \tag{5.53}$$

where, in the first equality, we used the fact that the Galois group is Abelian.

As a result, we see that the eigenvalues of the normalized reduced density matrix

$$\widehat{\lambda}_\ell = \frac{\lambda_\ell}{\sum_i \lambda_i} \ , \tag{5.54}$$

are permuted under the Galois action. Therefore, after tracing out any (proper) subset of links on the 3-manifold $\mathcal{M}_{\mathcal{L}(M,MQ)}$ with $\gcd(Q,N)=1$, the von Neumann and Rényi entanglement entropies do not change under Galois conjugation of a TQFT defined on this space.

In fact, we can prove a stronger statement. Indeed, we have proven a result in terms of the conductor, $N = fN_0$ (where f is a positive integer dividing twelve). The natural conductor in TQFT is N_0. In particular, let us consider Q such that $\gcd(Q,N_0)=1$. If $\gcd(Q,f)=1$, then we have $\gcd(Q,N)=1$, and we are back to the discussion above. On the other hand, suppose $\gcd(Q,f)\neq 1$. In this case, we can always take positive integers, $f_{1,2}$, such that $f = f_1 f_2$, where $\gcd(Q,f_2)=\gcd(f_1,f_2)=1$, and all prime factors of f_1 divide Q (of course, it may be that $f_2=1$). By construction, we must have $\gcd(N_0,f_1)=1$. Now, consider the integer

$$Q' = Q + N_0 \cdot f_2 \ . \tag{5.55}$$

Clearly, we have that $\gcd(Q',N_0)=\gcd(Q',f_2)=\gcd(Q',f_1)=1$. As a result, $\gcd(Q',f)=\gcd(Q',fN_0)=\gcd(Q',N)=1$. Now, consider the signed permutation matrix

$$G_{\sigma_{P'}} = \varphi^{2P'+Q'} S^{-1} T^{P'} S T^{Q'} S T^{P'} \ , \quad Q'\cdot P' = 1 \ (\mathrm{mod}\ N) \ . \tag{5.56}$$

From the definition of the MTC conductor (5.6), we see that $T^{Q'} = T^Q$ and so

$$G_{\sigma_{P'}} = \varphi^{2P'+Q'} S^{-1} T^{P'} S T^{Q} S T^{P'} \ . \tag{5.57}$$

Following the logic beginning in (5.51), we find the following result:

Theorem 5.4.1 *The TQFT MEE (and also the associated Rényi entropies) obtained by tracing out the Hilbert subspaces associated with any (proper) subset of linking boundary tori on the 3-manifold, $\mathcal{M}_{\mathcal{L}(M,MQ)}$, with Q co-prime to the MTC conductor, N_0, are invariant under the action of the TQFT Galois group. Im-*

plicit in this discussion is the assumption that the non-unitary theories that arise lie on the Galois orbit of at least one unitary theory.[a] *Note that, by a modular transformation, the same results apply to* $\mathcal{M}_{\mathcal{L}(MQ,M)}$.

[a]In particular, the Hilbert space in the statement of the theorem refers to the Hilbert space of a unitary member of this orbit.

In the next section we will introduce knot operators. As we will see, properties of these operators combined with the results of this section lead to a vast generalization of Theorem 1 in the case of non-Abelian Chern-Simons theories and their Galois partners.

5.4.3 Galois transformations, entanglement entropy, and more general Torus links

To find a more general class of link complements giving rise to invariant entanglement entropy along TQFT Galois orbits, it is useful to introduce the concept of knot operators. Using these operators, it is a relatively simple matter to find link invariants for general torus links [173–175]. The basic idea is to decompose a 3-manifold, M, containing Wilson lines by gluing two solid tori, U_1 and U_2, at their T^2 boundaries such that no Wilson line is cut (in this sense we consider "local" Wilson lines). The set of manifolds which can be obtained from gluing two solid tori with a boundary homeomorphism given by an element of $SL(2,\mathbb{Z})$ are called lens spaces. For $\mathbb{1} \in SL(2,\mathbb{Z})$, we get the manifold $S^2 \times S^1$ and for the S matrix we get S^3.

The expectation value of the Wilson lines in M can be recast as an inner product of states in a Hilbert space, where the states are found by performing a path integral over the two solid tori. In this formalism, the knot invariant of an (M,N) torus knot is given by the expectation value

$$\langle W_j^{(M,N)} \rangle_{S^3} = \frac{\langle 0| S W_j^{(M,N)} |0\rangle}{\langle 0| S |0\rangle} \ . \tag{5.58}$$

The vector $|0\rangle$ represents the empty solid torus, U_1. The action of $W_j^{(M,N)}$ on this state creates the (M,N) torus knot in representation j on its T^2 boundary. Applying an S transformation at the torus boundary and gluing in the other solid torus, U_2, gives the expectation value of the knot in S^3.[116]

For Chern-Simons theory with an arbitrary simple gauge group G at level k, the action of the torus knot operator, $W_j^{(M,N)}$, on a state is given by [176]

$$W_j^{(M,N)} |p\rangle = \sum_{\ell \in \Lambda_j} \exp\left(2\pi i \frac{MN}{\psi^2(2yk+g^\vee)} \ell^2 + 4\pi i \frac{N}{\psi^2(2yk+g^\vee)} (p\cdot \ell) \right) |p + M\ell\rangle \ . \tag{5.59}$$

[116]The denominator, $\langle 0| S |0\rangle$, is a normalization factor.

Here Λ_j is the set of weights of the irreducible representation V_j, y is the Dynkin index of the fundamental representation, ψ^2 is the length squared of the longest simple root, k is the level, $g^\vee$ is the dual Coxeter number, and $p \in \Lambda_W$ is a vector in the weight lattice. For example, the $W^{(1,0)}$ torus knot operator acts as

$$W_j^{(1,0)} |p\rangle = \sum_{\ell \in \Lambda_j} |p + \ell\rangle \ . \tag{5.60}$$

In terms of fusion matrices, we have

$$W_j^{(1,0)} |p\rangle = \sum_{\ell} N_{jp}^{\ell} |\ell\rangle \ . \tag{5.61}$$

For finite k, the set of states, $|p + M\ell\rangle$, that arise in (5.59) are subject to relations such that they lie within the class of integrable representations at level k. For example, in the case of $su(2)_k$ Chern-Simons theory on $T^2 \times \mathbb{R}$ (where space is a T^2), the states of the Hilbert space are given in terms of combinations of theta functions. These states are subject to the identifications $|-\ell\rangle = -|\ell\rangle$ and $|\ell\rangle = |\ell - 2(k+2)\rangle$. The first identification follows form a Weyl reflection, and the second identification follows from a periodicity property of the theta functions involving shifts by the simple root. Using these identifications, we can always reduce the sum in (5.59) to a sum over states corresponding to the integrable representations. Similar comments apply to more general gauge groups (again, only signs appear in the identification of states).[117]

In what follows, it will be useful for us to understand more carefully how the T matrix can enter general link invariants. The key is to first phrase the Rosso-Jones formula [177] in terms of torus knot operators in the large k limit [175][118]

$$\langle 0| W_j^{(M,N)} |0\rangle = \sum_{\ell} C(M)_j^{\ell} T_{\ell,\ell}^{\frac{N}{M}} \langle 0|\ell\rangle \ , \tag{5.62}$$

where the sum is over some integrable representations, the $C(M)_j^k \in \mathbb{Z}$ are independent of N, and $T_{\ell,\ell}^{\frac{N}{M}}$ is the fractional twist. For convenience, we have labeled the vacuum as $|0\rangle$.[119] Furthermore, for large k, the $C(M)_j^{\ell}$ are specified by the so-called Adams operation,[120] and $\langle 0|\ell\rangle = \delta_{0,\ell}$ [175]. Therefore, at large k, $\langle 0| W_j^{(M,N)} |0\rangle = C(M)_j^0$.

For general k (not necessarily large compared to the quantum numbers of the knot

[117] Here we will get a larger number of Weyl reflections and more complicated periodicity structure for the relevant theta functions (again these shifts are in one-to-one correspondence with the simple roots).

[118] We mean that k is large compared to the quantum numbers of the $W_j^{(M,N)}$ knot operator.

[119] However, when substituting (5.59) into (5.62), one should take $|0\rangle \to |\rho\rangle$, where $\rho = \sum_i \lambda^{(i)}$ is the sum over the fundamental weights (similar comments apply to all other appearances of $|0\rangle$ below).

[120] For example, in the case of $su(n)$, the Adams operation is defined as follows. Consider the $su(n)$ Schur polynomials, $\chi_j(z_1, \cdots, z_{n-1})$, where i is an irreducible representation of $su(n)$, and raise the $su(n)$ fugacities to the M^{th} power. Writing the result in terms of the Schur polynomials without

operator), the story is more complicated, since some of the $|\ell\rangle$ appearing in (5.62) should be identified with the vacuum representation. For example, as discussed above in $su(2)_k$ CS theory, $-|2k+2\rangle = |0\rangle$. In any case, (5.62) allows us to control the non-linearities arising in the Galois action on T.

Another crucial property of the knot operators is that they satisfy fusion rules

$$W_i^{(M,N)} W_j^{(M,N)} = \sum_i N_{ij}^k W_k^{(M,N)} \ . \tag{5.64}$$

Hence, they are generalized Verlinde operators. Using this property, we can write the torus link operator for a Q-component torus link (QM, QN) in terms of the knot operators as

$$W_{j_1,\cdots,j_Q}^{(QM,QN)} = N_{j_1,\cdots,j_Q}^{\ell} W_\ell^{(M,N)} \ . \tag{5.65}$$

Moreover, any torus knot operator can be obtained from the unknot by the action of an $SL(2,\mathbb{Z})$ element

$$W_j^{(M,N)} = F^{(M,N)^{-1}} W_j^{(1,0)} F^{(M,N)} \ , \tag{5.66}$$

where $F^{(M,N)} \in SL(2,\mathbb{Z})$. This statement is natural since a torus knot can be put on the surface of a torus without self intersections, and we can obtain such a knot from the unknot by a sequence of Dehn twists and S tranformations.

A straightforward generalization of the argument in [157] shows that the eigenvalues of the reduced density matrix of the (QM, QN) torus link are given by

$$\Lambda_\ell = \frac{1}{S_{0\ell}^{2Q-2}} \left| \sum_i S_{\ell i} \langle W_i^{(M,N)} \rangle_{S^3} \right|^2 \ . \tag{5.67}$$

We can massage this expression into a more useful form as follows:

Lemma 5.4.2 *The eigenvalues of the reduced density matrix of the (QM, QP) torus link are given by*

$$\Lambda_\ell = \frac{1}{(S_{0\ell})^{2Q-2} S_{00}^2} \sum_i S_{\ell i} \langle W_i^{(M,P)} \rangle_{S^2\times S^1} \sum_j S_{\ell j} \langle W_j^{(-P,M)} \rangle_{S^2\times S^1} \ . \tag{5.68}$$

Proof: See App. B.3.

In this lemma, $\langle W_i^{(M,P)} \rangle_{S^2\times S^1} = \langle 0 | W_i^{(M,P)} | 0 \rangle$ is the knot invariant of the (M, P) knot in $S^2 \times S^1$. As a result, we can write the entanglement entropy of links in S^3 as

transforming the fugacities, we obtain the $C(M)_j^\ell$

$$\chi_j(z_1^M, \cdots, z_{n-1}^M) = \sum_l C(M)_j^\ell \cdot \chi_\ell(z_1, \cdots, z_{n-1}) \ . \tag{5.63}$$

a product of linear combinations of knot invariants in $S^2 \times S^1$ with S matrix elements as coefficients.

We may now make use of the above lemma to gain a better understanding of Galois transformation properties of torus links in Chern-Simons theory. To that end, first consider the special case of eigenvalues of the reduced density matrix for (MQ, Q) torus links described in Sec. 5.4.2. Using (5.68), we have

$$\Lambda_\ell = \frac{1}{(S_{0\ell})^{2Q-2} S_{00}^2} \sum_i S_{\ell i} \langle W_i^{(M,1)} \rangle_{S^2 \times S^1} \sum_j S_{\ell j} \langle W_j^{(-1,M)} \rangle_{S^2 \times S^1} \ . \tag{5.69}$$

We may simplify the second summation in (5.69), since

$$\langle W_j^{(-1,M)} \rangle_{S^2 \times S^1} = \langle W_j^{(-1,0)} \rangle_{S^2 \times S^1} = \delta_{0,j} \ . \tag{5.70}$$

The second equality follows from (5.61) and charge conjugation, while the first equality follows from the fact that T acts trivially on the vacuum and so

$$\langle W_j^{(P,M+AP)} \rangle_{S^2 \times S^1} = \langle T^A W_j^{(P,M)} T^{-A} \rangle_{S^2 \times S^1} \ . \tag{5.71}$$

As a result, the eigenvalues of the reduced density matrix in (5.69) simplify to

$$\Lambda_\ell = \frac{1}{(S_{0\ell})^{2Q-3} S_{00}^2} \sum_i S_{\ell i} \langle W_i^{(M,1)} \rangle_{S^2 \times S^1} \ . \tag{5.72}$$

Now, suppose that M is co-prime to the MTC conductor. From Sec. 5.4.2, we know that the normalized eigenvalues of the reduced density matrices for such links are permuted under Galois transformations. Hence, Galois conjugating (5.72) gives

$$G_P\left(\Lambda_\ell S_{00}^2\right) = \frac{1}{(S_{0\sigma(\ell)})^{2Q-3}} \sum_i S_{\sigma(\ell) i} G_P\left(\langle W_i^{(M,1)} \rangle_{S^2 \times S^1} \right) \ , \tag{5.73}$$

where $G_P(\cdots)$ denotes the action of the Galois group element corresponding to $P \in \mathbb{Z}_{N_0}^\times$. From the invertibility of the S matrix, it follows that

$$G_P\left(\langle W_i^{(M,1)} \rangle_{S^2 \times S^1} \right) = \langle W_i^{(M,1)} \rangle_{S^2 \times S^1} \in \mathbb{Q} \ , \tag{5.74}$$

and, from (5.62), we also have

$$G_P\left(\langle W_i^{(M,1)} \rangle_{S^2 \times S^1} \right) = \langle W_i^{(M,P)} \rangle_{S^2 \times S^1} \in \mathbb{Q} \ , \tag{5.75}$$

where we have used the fact that the only source of non-rational numbers in (5.62) is from the fractional twists, and we must further assume that $\gcd(P, M) = 1$ in order to have a well-defined Galois action on the fractional twists. Therefore, $\langle W_i^{(M,P)} \rangle_{S^2 \times S^1}$ is

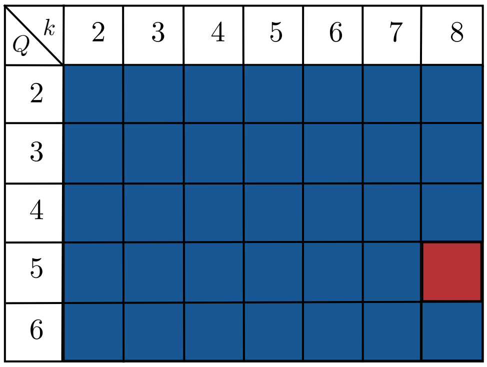


Figure 5.6: Sparseness of Galois non-invariance for the MEE of $(2, 2Q)$ torus link complements after tracing out one of the boundaries in $su(2)_k$ theories. The y-axis corresponds to Q and the x-axis corresponds to k. Blue squares correspond to theories and topologies with Galois invariant MEE, while the red square does not.

invariant under Galois conjugation.

Following the arguments above, we can also show that $\langle W_i^{(-P,M)} \rangle_{S^2\times S^1} \in \mathbb{Q}$ is invariant under Galois conjugation. Hence, it follows from lemma 5.4.2 that the normalized eigenvalues of (QM, QP) link for M, P coprime to the conductor and to themselves are permuted under Galois conjugation. Therefore, we have the following theorem:

Theorem 5.4.3 *The Chern-Simons MEE (and associated Rényi entropies) obtained by tracing out the Hilbert subspaces associated with any (proper) subset of linking boundary tori on the 3-manifold, $\mathcal{M}_{\mathcal{L}(QM,QP)}$, with M, P co-prime to the Chern-Simons conductor, N_0, and to each other are invariant under the action of the TQFT Galois group.*

For Chern-Simons theories and their Galois conjugates, this result generalizes theorem 5.4.1 in Sec. 5.4.2. However, the proof in Sec. 5.4.2 was obtained directly using the MTC data without referring to a specific realization of the TQFT, while the above proof depends on the realization of the TQFT as a Chern-Simons theory and (5.59), which was constructed for simple gauge groups. The authors of [30, 178] conjectured that every 3D TQFT is a Chern-Simons theory with some gauge group. If this conjecture is true, the results in this section might extend to all 3D unitary TQFTs and their Galois conjugates.

5.4.4 Example: $su(2)_k$

As a concrete example to illustrate the above discussion, consider $su(2)_k$ CS theory. The modular data for this theory is

$$S_{ab} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi(a+1)(b+1)}{k+2}\right) \ , \quad T_{ab} = \delta_{ab} \exp\left(\frac{2\pi i a(a+2)}{4(k+2)}\right) = \delta_{ab}\theta(a) \ , \tag{5.76}$$

where $a \in \{0, 1, \cdots, k\}$. In particular, the MTC conductor is generically $N_0 = 4(k+2)$. In Fig. 5.6, we present a set of results for the entanglement entropy after tracing one of the knots in $(2, 2Q)$ torus links for levels $2 \leq k \leq 8$ and $2 \leq Q \leq 6$. The results are completely consistent with the above theorems. In fact, we see various "accidental" invariances not guaranteed by our theorems.

The simplest Galois non-invariant entanglement entropy occurs in the $(2, 10)$ link of the $su(2)_8$ CS theory. Note that the MTC conductor in this case is $N_0 = 40$, and $M = 1$, $P = 5$. Clearly, $(P, N_0) = 5 \neq 1$, and so this lack of invariance is consistent with theorem 5.4.3.

5.4.5 Torus links in lens spaces

Let us briefly comment on the generalization of theorem 5.4.3 to more general lens spaces. The expectation value of knot operators in a lens space, $\mathcal{M}_F$, is given by

$$\langle W_j^{(m,n)} \rangle_F = \frac{\langle 0| F W_j^{(m,n)} |0\rangle}{\langle 0| F |0\rangle} , \tag{5.77}$$

where $F \in SL(2, \mathbb{Z})$ is the homeomorphism between the two tori which produces the corresponding lens space. Following the procedure in Sec. 5.4.3, the eigenvalues for torus links in a lens space specified by $F \in SL(2, \mathbb{Z})$ is given by

$$\Lambda_l = \frac{1}{(S_{0l})^{2Q-2} F_{00}^2} \sum_i S_{li} \langle W_i^{(m,n)} \rangle_{S^2 \times S^1} \sum_j S_{lj} \langle F W_j^{(m,n)} F^{-1} \rangle_{S^2 \times S^1} . \tag{5.78}$$

Therefore, the eigenvalues of torus links in a general lens space can be written as a linear combination of knot invariants in $S^2 \times S^1$ with S matrix elements as coefficients. A sufficient condition for the Galois invariance of entanglement entropy of a torus link (QM, QN) in $\mathcal{M}_F$ is the Galois invariance of knot invariants $\langle W_i^{(M,N)} \rangle_{S^2 \times S^1}$ and $\langle F W_j^{(M,N)} F^{-1} \rangle_{S^2 \times S^1}$.

5.5 Conclusion

We have argued that, in addition to preserving fusion rules (and 1-form symmetries) of TQFTs, Galois conjugation also preserves MEE in broad classes of theories. In particular, we showed that putting any Abelian TQFT on any link complement in S^3 and tracing out Hilbert spaces on any subset of the links leads to an invariant MEE along Galois orbits. We then argued that this theorem generalizes to non-Abelian TQFTs living on infinite classes of torus link complements.

The fact that the invariants of the Galois action include both fusion and, on certain torus link complements, MEE is suggestive of a deeper relation between entanglement, fusion, and modular data. In fact, recent work [179,180] suggests that the entanglement entropy of [158, 159] can be used to reconstruct the fusion rules and modular data of a TQFT. It would clearly be interesting to understand how MEE fits more precisely into this story.

Finally, let us conclude with an interesting potential application of our results. Our non-Abelian results involve number theory, and it would be interesting to find applications to this field. Here we begin by recalling that, in 300 BC, Euclid found an algorithm for computing the greatest common divisor of two natural numbers (see [181] for a modern discussion). In a similar spirit, we can potentially use our theorem 5.4.1 to give a TQFT-based algorithm to check co-primeness of two natural numbers. Indeed, we have seen that, for any TQFT, invariance of the MEE on the (M, MQ) link complement is guaranteed if Q is co-prime to the MTC conductor, i.e. $\gcd(Q, N_0) = 1$. On the other hand, when Q is not co-prime to the MTC conductor, this invariance is not guaranteed.[121] It would be interesting to try to find an infinite family of TQFTs with infinitely many different conductors that have invariant MEE if and only if Q is co-prime to the conductor. In this case, if we wish to check $(a, b) = 1$, we set $Q = a$, $N_0 = b$, and check the Galois invariance of the MEE on the (M, MQ) link complement.

[121] Note that in many theories, such as $su(2)_k$ CS theory, there is still "accidental" invariance—see Fig. 5.6.

Chapter 6

From RCFTs to Quantum Error Correcting Codes

6.1 Introduction

Quantum error correcting codes (QECCs) are integral to quantum computation. They also appear in high energy and condensed matter physics in various guises. As one important example, QECCs capture aspects of bulk reconstruction in AdS-CFT [182]. Another notable case of a QECC in physics is the Toric code, a well-known model with topological order [183]. QECCs have also unravelled the existence and properties of fractons [184]. More recently, QECCs were used to construct closed, simply connected manifolds [185].

In this chapter, we explore the relationship between conformal field theories (CFTs) in two spacetime dimensions, associated 3d Chern-Simons (CS) theories, and QECCs. The relationship between classical codes, their associated lattices, and holomorphic CFTs was originally noted by Dolan, Goddard, and Montague [186]. Recently, a quantum version of this relationship was discovered, where quantum stabilizer codes were associated with certain Narain rational CFTs (RCFTs) [38, 187]. This construction does not exhaust all Narain RCFTs and leads to several natural questions: (1) When do general Narain RCFTs admit a quantum code description? (2) How does one identify the n-qubit Hilbert space, the code subspace and its complement, within the CFT Hilbert space? (3) What is the physical meaning of this relation?

In this chapter we answer these questions using the general structure of Narain RCFTs.[122] Our main results are:

- Any abelian CS theory with an even-order fusion group is related to a Narain

[122] In principle, our results apply to any RCFT with abelian fusion rules (what we call an "abelian RCFT") whether it admits a Narain description or not. In what follows, we will not attempt to distinguish between Narain RCFTs and hypothetically more general abelian RCFTs.

RCFT that admits a stabilizer code description. Orbifolding this RCFT by a chiral algebra-preserving $Q \simeq \mathbb{Z}_2^k$ 0-form gauge group results in a Narain RCFT that continues to admit a stabilizer code description whenever the corresponding 3d bulk 1-form symmetry of the CS theory has vanishing 't Hooft anomaly.

- All Narain RCFTs have abelian 0-form symmetries implemented by topological defects. In the class of theories described in the previous bullet, topological defect endpoint operators can naturally be mapped to the full Pauli group. The stabilizer subgroup corresponds to genuine local CFT operators, which can be thought of as living at the end of the trivial defect.

- Under this map, the RCFT Hilbert space corresponds to the code subspace and certain defect Hilbert spaces correspond to the complement of the code subspace inside the n-qubit Hilbert space.

This chapter is organized as follows. In Section 6.2, we start with a brief review of stabilizer codes and Narain CFTs. We then show that Narain RCFTs with left and right movers paired via charge conjugation can be naturally associated with quantum stabilizer codes. We end Section 6.2 by extending this map to orbifold theories and deriving a relationship between vanishing 't Hooft anomalies and stabilizer codes; along the way, we consider various illustrative examples. In Section 6.3, we study symmetries of Narain CFTs implemented by Verlinde lines and show that operators living at the end of Verlinde lines give rise to the full Pauli group. We introduce the notion of a Verlinde subgroup and discuss its role in determining the error detection capability of CFT symmetry currents. In Section 6.4, we propose a map between the n-qubit Hilbert space and states in the CFT. We conclude with a discussion.

6.2 The Stabilizer Code / abelian RCFT Map

Let us briefly review the basics of stabilizer codes and RCFTs with abelian fusion rules. We then propose a natural map relating them.

A stabilizer code on n qubits is defined by an abelian subgroup, $\mathcal{S}_n$, of the generalized Pauli group on n qubits, $\mathcal{P}_n$. Elements of $\mathcal{P}_n$ are defined by $\vec{\alpha}, \vec{\beta} \in \mathbb{Z}_2^n$ via

$$G(\vec{\alpha}, \vec{\beta}) := \epsilon\, X^{\alpha_1} \otimes \cdots \otimes X^{\alpha_n} \circ Z^{\beta_1} \otimes \cdots \otimes Z^{\beta_n} = X^{\vec{\alpha}} \circ Z^{\vec{\beta}} \in \mathcal{P}_n \ , \tag{6.1}$$

where the i^{th} X and Z are the Pauli matrices acting on the i^{th} qubit and ϵ is valued in $\{\pm 1, \pm i\}$. In the following discussion, when we have $X \circ Z$ or $Z \circ X$ acting on a qubit, following standard notation, we replace this with a Y Pauli matrix action (where the signs and factors of i are kept in track using ϵ). This group has order 4^n and is non-abelian

$$G(\vec{\alpha}_1, \vec{\beta}_1)G(\vec{\alpha}_2, \vec{\beta}_2) = (-1)^{\epsilon}\ G(\vec{\alpha}_2, \vec{\beta}_2)G(\vec{\alpha}_1, \vec{\beta}_1)\ , \tag{6.2}$$

where $\epsilon(\vec{\alpha}_1, \vec{\beta}_1, \vec{\alpha}_2, \vec{\beta}_2) := \vec{\beta}_1 \cdot \vec{\alpha}_2 - \vec{\alpha}_1 \cdot \vec{\beta}_2$. The hallmark of a stabilizer subgroup is that any two elements commute with each other[123]. Clearly, if $G(\vec{\alpha}_1, \vec{\beta}_1), G(\vec{\alpha}_2, \vec{\beta}_2) \in \mathcal{S}_n$, then $G(\vec{\alpha}_1 + \vec{\alpha}_2, \vec{\beta}_1 + \vec{\beta}_2) \in \mathcal{S}_n$. In this sense, stabilizer codes are additive. Moreover, all elements satisfy $G(\vec{\alpha}_i, \vec{\beta}_i)^2 = 1$. The states in the n-qubit Hilbert space which are left invariant by all $G \in \mathcal{S}_n$ (i.e., $G\psi = \psi$) are special: they form the "code subspace."

The refined enumerator polynomial (REP) of an n qubit stabilizer code is defined as

$$W(x_1, x_2, x_3, x_4) := \sum_{G \in \mathcal{S}_n} x_1^{\omega_I} x_2^{\omega_X} x_3^{\omega_Y} x_4^{\omega_Z}\ , \tag{6.3}$$

where $\omega_{I/X/Y/Z}(G)$ count the number of $I/X/Y/Z$ Pauli matrices in the stabilizer group element G.

For our general construction below, it is useful to keep in mind that the description above contains redundancies. In particular, two stabilizer codes are physically equivalent if they are related by an action of the Clifford group – an outer automorphism of the Pauli group [188]. This group includes all 3! permutations of Pauli generators acting on each qubit.

The stabilizer codes that play a role in [38] are self-dual: in other words $|\mathcal{S}_n| = 2^n$, and so there is a one-dimensional code subspace. These codes are also real (in the sense that all elements of $\mathcal{S}_n$ in the representation (6.1) are real-valued), but we will relax this latter condition in our general construction. In the conventions of this chapter, the map between the CFTs and stabilizer codes introduced in [38] is related to our map by an $X \leftrightarrow Y$ code equivalence.

The mapping between stabilizer codes and CFTs associates classes of CFT operators with elements of $\mathcal{S}_n$. Since the code is additive, we consider CFTs with additive (abelian) fusion rules (i.e., those corresponding to a lattice)

$$\phi_{\vec{P}_L, \vec{P}_R} \times \phi_{\vec{K}_L, \vec{K}_R} = \phi_{\vec{P}_L + \vec{K}_L, \vec{P}_R + \vec{K}_R}\ , \tag{6.4}$$

where the pair of vector indexes label left-moving and right-moving momenta valued in a Narain lattice, Λ. We will use the terms "Narain theories" and "abelian CFTs" interchangeably. Since there are infinitely many CFT operators and finitely many elements of $\mathcal{S}_n$, we must organize the CFT operators into finitely many equivalence

[123] Note that, in general group actions on vector spaces the stabilizer group of a subspace need not be abelian. However, in our case, we are choosing a subspace of the Hilbert space which is the +1 eigenspace of some elements of the Pauli group. In the Pauli group, any two elements commute or anti-commute. Therefore, the subgroup of the Pauli group defining this subspace is necessarily abelian.

classes. In the context of abelian RCFT, this naturally happens since each $\phi_{\vec{P}_L,\vec{P}_R}$ in (6.4) satisfies

$$\phi_{\vec{P}_L,\vec{P}_R} \in (N_L, N_R) \ , \ N_L \in \text{Rep}(V_L) \ , \ N_R \in \text{Rep}(V_R) \ , \tag{6.5}$$

where N_L (N_R) are one of finitely many representations of the left (right) moving chiral algebra, V_L (V_R). For simplicity, we will only consider CFTs with $V_L = V_R = V$.

Specializing to $V_L = V_R = V$ and satisfying some additional mild assumptions detailed in [189], it turns out that any RCFT is a (generalized) orbifold of the "Cardy case" RCFT for V. This latter RCFT, $\mathcal{T}$, consists of operators built by pairing left and right movers transforming in Rep(V) that are related by charge conjugation.[124] In the case of an abelian RCFT, the orbifold is a standard group orbifold of $\mathcal{T}$ [190]. The $\mathcal{T}$ RCFT is sometimes referred to as the "charge conjugation modular invariant," and it has torus partition function[125]

$$Z_{\mathcal{T}}(q) = \sum_{\vec{p}} \chi_{\vec{p}}(q)\bar{\chi}_{\bar{\vec{p}}}(\bar{q}) \ , \quad \vec{p} + \bar{\vec{p}} = \vec{0} \ , \quad N_{\bar{\vec{p}}}, N_{\vec{p}}, N_{\vec{0}} \in \text{Rep}(V_{\mathcal{T}}) \ . \tag{6.6}$$

Here $\vec{p}$ is a vector labeling elements of Rep(V) *(not an element of* Λ *)*,[126] we sum over characters describing the operator content of the theory, and $\bar{\vec{p}}$ labels the representation conjugate to $\vec{p}$.[127]

Mathematically, Rep(V) corresponds to a modular tensor category (MTC). Physically, Rep(V) labels Wilson lines in the 3d Chern-Simons (CS) theory related to the 2d RCFT in question (see Fig. 6.1). The full set of MTCs/CS theories related to our abelian RCFTs have been classified in [169] (see also [118]). The result is that any such CS theory is a direct product of arbitrary combinations of the following factors

$$\begin{aligned} A_{2^r} &\sim \mathbb{Z}_{2^r} \ , \ A_{q^r} \sim \mathbb{Z}_{p^r} \ , \ B_{2^r} \sim \mathbb{Z}_{2^r} \ , \\ B_{q^r} &\sim \mathbb{Z}_{q^r} \ , \ C_{2^r} \sim \mathbb{Z}_{2^r} \ , \ D_{2^r} \sim \mathbb{Z}_{2^r} \ , \\ E_{2^r} &\sim \mathbb{Z}_{2^r} \times \mathbb{Z}_{2^r}, \quad F_{2^r} \sim \mathbb{Z}_{2^r} \times \mathbb{Z}_{2^r} \ , \end{aligned} \tag{6.7}$$

where the labels on the lefthand sides of (6.7) denote CS theories as in [118] with fusion rules for Wilson lines given by the abelian groups on the righthand sides, and q is an odd prime number.[128] The upshot is that we should think of $\vec{p}$ as valued in the

[124] Given V, it turns out that the charge-conjugation modular invariant CFT exists on very general grounds [77].

[125] Note that the construction in [190] takes as input left and right moving chiral algebras and produces an RCFT valid on any genus surface.

[126] We use capital $\vec{P}$ to denote lattice momentum and lower case $\vec{p}$ to denote elements of Rep(V).

[127] This latter statement means that we have fusion of the form $N_{\vec{p}} \times N_{\bar{\vec{p}}} = N_{\vec{0}}$.

[128] Strictly speaking, since a given label on the lefthand side of (6.7) only specifies the statistics of a set of line operators, it can correspond to different CS theories. Moreover, a CS theory that does not factorize in the geometry with boundaries depicted in Fig. 6.1 with M trivial can correspond to a

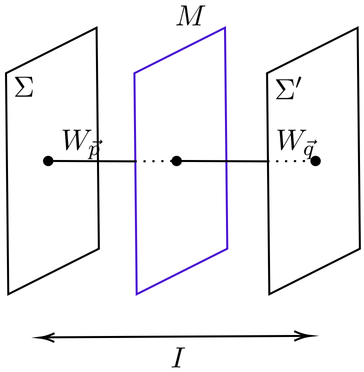


Figure 6.1: The pairing of 2d CFT left and right movers on Σ and Σ' can be specified by an abelian CS theory on $X \simeq \Sigma \times I$ with a surface operator, M, inserted in between [36, 191]. A local operator, $O_{(\vec{p},\vec{q})}$, is specified by the Wilson lines $W_{\vec{p}}$ and $W_{\vec{q}}$. Different M lead to different partition functions. Topological defects in the 2d CFT correspond to Wilson lines parallel to Σ, Σ'.

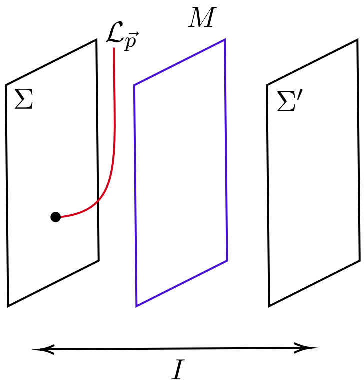


Figure 6.2: The endpoint of $\mathcal{L}_{\vec{p}}$ on Σ gives a defect endpoint operator corresponding to a state in the defect Hilbert space, $\mathcal{H}^{\text{Defect}}_{\mathcal{L}_{\vec{p}}}$. We can think of $\mathcal{L}_{\vec{p}}$ as generating a 3d 1-form symmetry or a 2d 0-form symmetry (when $\mathcal{L}_{\vec{p}}$ is pushed to lie completely on Σ).

following product group / lattice quotient

$$\vec{p} \in \prod_r \Big(\mathbb{Z}_{2^r}^{n_{A_{2^r}}} \times \mathbb{Z}_{2^r}^{n_{B_{2^r}}} \times \mathbb{Z}_{2^r}^{n_{C_{2^r}}} \times \mathbb{Z}_{2^r}^{n_{D_{2^r}}} \times \left[\mathbb{Z}_{2^r} \times \mathbb{Z}_{2^r}\right]^{n_{E_{2^r}}} \\ \times \left[\mathbb{Z}_{2^r} \times \mathbb{Z}_{2^r}\right]^{n_{F_{2^r}}} \times \prod_q \left[\mathbb{Z}_{q^r}^{n_{A_{q^r}}} \times \mathbb{Z}_{q^r}^{n_{B_{q^r}}}\right]\Big) := K \ , \tag{6.8}$$

where n_X is the number of *independent* factors of the CS theory X corresponding to the CFT in (6.6) (see Footnote 128).[129] Physically, *K is the 1-form symmetry group of the CS theory and the 0-form symmetry subgroup of the RCFT that commutes with the full left and right chiral algebras* (see Fig. 6.2).

Now we will map the pair $(\vec{\alpha}, \vec{\beta})$, which specifies a stabilizer generator from $\mathcal{S}_n$ to

product of labels (e.g., $U(1)_6$ CS theory, which corresponds to $B_2 \times B_3$). For simplicity in what follows, we will avoid this latter possibility.

[129] Here we are thinking of $\mathbb{Z}_N$ as an additive subgroup of $\mathbb{Z}$ modulo N.

a pair $(\vec{p}, \bar{\vec{p}})$ representing a family of operators contributing to $\chi_{\vec{p}}\bar{\chi}_{\bar{\vec{p}}}$ in (6.6). First we specify the dimension of $\vec{p}$: the most obvious choice is that $\vec{\alpha}$, $\vec{\beta}$, and $\vec{p}$ are n-dimensional. Moreover, in our map $\vec{\alpha}$ and $\vec{\beta}$ are linearly related to $\vec{p}$.

To begin with, let us neglect possible E_{2^r} and F_{2^r} CS theory factors. Then, $\mathcal{T}$ is a CFT with n decoupled factors having fusion rules given by the n factors in (6.8).[130] Indeed, by construction, each of the n CFT factors is closed under fusion.[131] It is therefore natural to associate such a theory with an n-fold product of one-qubit codes. Up to code equivalence, all such codes are generated by Z acting on individual qubits. Therefore, we set $\vec{\alpha} = 0$, and choose

$$\vec{\beta} = \vec{p} \ , \tag{6.9}$$

where (6.9) is the simplest natural choice.

However, note that for a CFT factor described by A_{q^r} or B_{q^r}, the simplest choice is to make the resulting code factor trivial. The reason is that the corresponding component of $\vec{p}$, p_i, has order q^s for $1 \leq s \leq r$. In this case, multiple stabilizers would correspond to the same $(\vec{p}, \bar{\vec{p}})$. We therefore ignore factors described by A_{q^r} and B_{q^r} from now on and map corresponding CFT degrees of freedom to 0-qubit codes.

In summary, we learn that linearity and code redefinitions point to the relation

$$\left\{\mathcal{O}_{\vec{p},\bar{\vec{p}}}\right\} \leftrightarrow Z^{\vec{p}} \ , \tag{6.10}$$

where we understand this map as meaning that the $Z^{\vec{p}}$ stabilizer corresponds to the collection of operators in the $(\vec{p}, \bar{\vec{p}})$ representation of the left and right moving chiral algebras (i.e., the primary and its descendants). Including factors of E_{2^r} and F_{2^r} and following logic similar to the above leads to the map

$$\left\{\mathcal{O}_{\vec{p},\bar{\vec{p}}}\right\} \leftrightarrow Z^{A\vec{p}} \ , \tag{6.11}$$

where A is block diagonal, with the following diagonal components corresponding to different CFT factors

$$A_{A_{2^r}} = A_{B_{2^r}} = A_{C_{2^r}} = A_{D_{2^r}} = 1 \ , \tag{6.12}$$

and, up to code equivalence,

$$A_{E_{2^r}} = A_{F_{2^r}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ . \tag{6.13}$$

[130] More explicitly, we have that $n = \sum_r \left(n_{A_{2^r}} + n_{B_{2^r}} + n_{C_{2^r}} + n_{D_{2^r}} + \sum_q \left(n_{A_{q^r}} + n_{B_{q^r}}\right)\right)$.

[131] If we relax the condition in Footnote 128 and allow for CS theories like $U(1)_6$, then we can also consider charge conjugation modular invariants that do not decompose into n such CFT factors.

Note that in writing (6.11), we allow for multiple families of operators to appear on the lefthand side (see Section 6.2.1 for some examples). Indeed, the exponent of Z on the RHS is only sensitive to $A\vec{p}$ modulo two. Thus in the simple case of charge conjugation modular invariant, we have the CFT to stabilizer code map

$$\mu : \mathcal{T} \longrightarrow S_{\mathcal{T}} := \text{gen}\left\{ Z^{A\vec{e}_i} \mid e_{ij} = \delta_{ij} \right\} \ , \tag{6.14}$$

where "gen$\{\cdots\}$" means that the code is generated by the enclosed Pauli operators. Note that this code is self-dual by construction. Moreover, μ is non-invertible. For example, the $SU(2)$ and E_7 WZW models at level one are distinct but map to the same code.[132]

Given the set of theories of the form (6.6), we can construct all other Narain RCFTs by orbifolding them by some non-anomalous 0-form symmetry subgroup $Q \lhd K$.[133] Here non-anomalous means that the associator of Verlinde lines implementing Q is trivial in $H^3(Q, U(1))$. [134] Therefore, if Q is non-anomalous, F is a 3-coboundary satisfying

$$F(\vec{h}_1, \vec{h}_2, \vec{h}_3) = \frac{\tau(\vec{h}_2, \vec{h}_3)\tau(\vec{h}_1, \vec{h}_2 + \vec{h}_3)}{\tau(\vec{h}_1 + \vec{h}_2, \vec{h}_3)\tau(\vec{h}_1, \vec{h}_2)} \ \forall \vec{h}_1, \vec{h}_2, \vec{h}_3 \in Q \ , \tag{6.16}$$

where τ is a 2-cochain. Then, the Q-orbifold torus partition function is

$$Z_{\mathcal{T}/Q,[\sigma]} = \sum_{\vec{g} \in Q} \sum_{\vec{p} \in B_{\vec{g}}} \chi_{\vec{p}}(q) \bar{\chi}_{\overline{\vec{p}+\vec{g}}}(\bar{q}) \ , \tag{6.17}$$

where $[\sigma]$ is an equivalence class in $H^2(Q, U(1))$ defining the discrete torsion (in the condensed matter perspective, the 2d SPT we stack when gauging Q, or the B-field in [38]), and

$$B_{\vec{g}} := \left\{ \vec{p} \;\middle|\; S_{\vec{h},\vec{p}} \ \Xi(\vec{h}, \vec{g}) = 1 \ , \ \forall \vec{h} \in Q \right\} \ , \tag{6.18}$$

[132] The reason is that in both cases, $\vec{p} = p_1$ takes values in the same group.

[133] As we will see, the theories in [38] are all orbifolds of particular theories with partition functions of the form (6.6). Note that we will only consider orbifolds with respect to symmetries which commute with the full left and right chiral algebras. Orbifolds of this type take us from a Narain CFT to another Narain CFT, while more general orbifolds may result in non-Narain CFTs.

[134] For the CFT with charge conjugation modular invariant, F can be written in terms of holomorphic scaling dimensions as

$$F(\vec{g}, \vec{h}, \vec{k}) = \prod_i \begin{cases} 1 & \text{if } h_i + k_i < n_i \\ \theta(e_i)^{g_i n_i} & \text{if } h_i + k_i \geq n_i \end{cases} \tag{6.15}$$

where e_i is a basis for the cyclic factors in (6.8), and $\vec{g} = \sum_i g_i e_i$. Here n_i is the order of the i^{th} cyclic factor, and $\theta_{\vec{p}} := \exp(2\pi i h_{\vec{p}})$, where $h_{\vec{p}}$ is the holomorphic scaling dimension of an operator in representation $\vec{p}$. The group Q is non-anomalous if and only if $\theta_{\vec{h}}^{O_{\vec{h}}} = 1 \ \forall \vec{h} \in Q$, where $O_{\vec{h}}$ is the order of $\vec{h}$ in Q [190].

where we define[135]

$$S_{\vec{h},\vec{p}} := \frac{\theta_{\vec{h}+\vec{p}}}{\theta_{\vec{h}}\theta_{\vec{p}}} \ , \ \Xi(\vec{g},\vec{h}) \ := \ R(\vec{h},\vec{g})\frac{\tau(\vec{h},\vec{g})\sigma(\vec{h},\vec{g})}{\tau(\vec{g},\vec{h})\sigma(\vec{g},\vec{h})} \ . \tag{6.19}$$

In (6.19), $\theta_{\vec{p}} := \exp(2\pi i h_{\vec{p}})$, and $h_{\vec{p}}$ is the holomorphic scaling dimension of an operator in representation $\vec{p}$.[136]

In this chapter we focus on the case

$$Q \simeq \mathbb{Z}_2^k \ . \tag{6.20}$$

Such subgroups are the most universal in the sense that they are contained in any other subgroups of K.[137] More general cases can be treated in a similar fashion.

How should we include the data of states corresponding to $\vec{g} \neq \vec{0}$ in the code? Clearly, the fields in the $\vec{g} = \vec{0}$ sector should still be captured by (6.11). Therefore, $\vec{g}$ must appear in a linear relation with $\vec{\alpha}, \vec{\beta}$ such that setting $\vec{g} = \vec{0}$ recovers terms of the form (6.11). Note that nontrivial components of any $\vec{g} \in Q$ have the form $g_i = 2^{r_i - 1} \in \mathbb{Z}_{2^{r_i}}$ (since $\vec{g} + \vec{g} = \vec{0}$). Therefore, in order to contribute to the stabilizer, $\vec{g}$ must appear through $M\vec{g}$ (M is diagonal, and $M_{ii} := 2^{1-r_i}$).

At this point, we should ask what principle requires $\vec{g}$ to contribute to the stabilizers at all. The answer is that orbifolding is an invertible procedure: when we gauge a discrete 0-form symmetry, Q, of a CFT, $\mathcal{T}$,[138] there is an isomorphic dual $Q' \simeq Q$ symmetry we can gauge in $\mathcal{T}/Q$ to return back to the original theory.[139] *We would like this invertibility to extend to the map between codes.*

If $M\vec{g}$ only appears through a factor $Z^{M\vec{g}}$, then our map between codes will not generally be invertible. The simplest and most natural possibility is the following.[140]

CFT to stabilizer operator map:

$$\left\{\mathcal{O}_{\vec{p},\overline{\vec{g}+\vec{p}}}\right\} \leftrightarrow X^{M\vec{g}} \circ Z^{A\vec{p}} := G(M\vec{g}, A\vec{p}) \ . \tag{6.21}$$

[135] Note that our S matrix differs from the unitary S matrix by an overall normalization (ours is $\sqrt{N}$ times bigger, where N is the number of Wilson lines in the CS theory associated with our RCFT).

[136] $R(\vec{h},\vec{g})$ can be written in terms of $\theta_{\vec{g}}$ as $R(\vec{h},\vec{g}) = \prod_i (\theta_{e_i})^{h_i g_i} \prod_{i<j} (S_{e_i,e_j})^{h_i g_j}$, where e_i is a basis for the cyclic factors in (6.8), and $\vec{g} = \sum_i g_i e_i$. Note that both $R(\vec{h},\vec{g})$ and $\tau(\vec{g},\vec{h})$ depend on a choice of basis in Rep(V), but $\Xi(\vec{g},\vec{h})$ is basis independent.

[137] Recall that we are ignoring CFT factors involving primaries labeled by A_{q^r} and B_{q^r}.

[138] Note that to unambiguously refer to the orbifolded theory, we should also generally specify the discrete torsion, $[\sigma]$. However, we will often be slightly imprecise and leave the discrete torsion implicit in our discussions.

[139] See [84, 192] as well as the more recent discussion in [193].

[140] We can also include an $M\vec{g}$ contribution in Z. Then we have $X^{M\vec{g}} \circ Z^{A\vec{p}+M\vec{g}} = Y^{M\vec{g}} \circ Z^{A\vec{p}}$ which is equivalent to the code $X^{M\vec{g}} \circ Z^{A\vec{p}}$. Similarly, $X^{M\vec{g}+A\vec{p}} \circ Z^{A\vec{p}}$ is code equivalent to $X^{M\vec{g}} \circ Z^{A\vec{p}}$.

In the language of (6.14), we have

$$\mu : \mathcal{T}/Q \longrightarrow S_{\mathcal{T}/Q} := \text{gen}\left\{ X^{M\vec{g}_i} Z^{A\vec{p}_J} \right\} , \tag{6.22}$$

where $\vec{g}_i$ and $\vec{p}_J$ generate Q and K respectively.

Since Z is order two, the quantum code constructed above is only sensitive to $A\vec{p}_J$ mod 2. Therefore, in general we will have multiple families of operators mapping to the same element of the stabilizer group.

Recall that the stabilizer code associated with the charge conjugation modular invariant is self-dual. Since orbifolding is invertible, the above map assigns a self-dual code to $\mathcal{T}/K$ (see Appendix C.2 for an alternate argument).

Intriguingly, given the map in (6.21), the commutation relations of elements of $\mathcal{S}_{\mathcal{T}/Q}$ are controlled by the S matrix of the RCFT. Indeed, it is a simple exercise to check that

$$\begin{aligned} G(\vec{g}_1,\vec{p}_1)G(\vec{g}_2,\vec{p}_2) &= e^{\pi i [M\vec{g}_2\cdot A\vec{p}_1 - M\vec{g}_1\cdot A\vec{p}_2]} \;\; G(\vec{g}_2,\vec{p}_2)G(\vec{g}_1,\vec{p}_1) \\ &= S_{\vec{g}_2,\vec{p}_1} S_{\vec{g}_1,\vec{p}_2} \;\; G(\vec{g}_2,\vec{p}_2)G(\vec{g}_1,\vec{p}_1) \\ &= \Xi(\vec{g}_2,\vec{g}_1)\Xi(\vec{g}_1,\vec{g}_2) \;\; G(\vec{g}_2,\vec{p}_2)G(\vec{g}_1,\vec{p}_1) \\ &= S_{\vec{g}_1,\vec{g}_2} \;\; G(\vec{g}_2,\vec{p}_2)G(\vec{g}_1,\vec{p}_1) , \end{aligned} \tag{6.23}$$

where, in the third equality, we have used (6.18). We have also used the expression for the S matrix $S_{\vec{p},\vec{q}} = e^{\frac{2\pi i}{2}\vec{p}^T M A \vec{q}}$ which follows from (6.8) [118]. Therefore, $\mathcal{S}_{\mathcal{T}/Q}$ is a stabilizer code if and only if $S_{\vec{g}_1,\vec{g}_2} = 1$. This latter statement can be reinterpreted as the vanishing of the 1-form anomaly for the Q 1-form symmetry in the bulk CS theory related to the $\mathcal{T}$ RCFT.

6.2.1 Examples

$R = 1, 2$ compact boson

The code CFTs in [38] are all orbifolds of charge conjugation modular invariants with $\text{Rep}(V) = A_4^{n_{A_4}}$, for some integer $n_{A_4} > 0$. That is, the fusion rules for the charge conjugation modular invariants are given by the abelian group, $K = (\mathbb{Z}_4)^{n_{A_4}}$ (all other n_X in (6.8) vanish). The theories discussed in [38] with non-trivial B-field correspond in our language to orbifolds of the charge conjugation theories with discrete torsion turned on (or, equivalently, a non-trivial 2D SPT in the $\mathbb{Z}_2^k \lhd \mathbb{Z}_4^{n_{A_4}}$ gauging process). As such, the CFTs in [38] are a small subset of theories discussed here.

The simplest code CFT among these is the $R = 1$ compact boson, corresponding to the choice $n_{A_4} = 1$. Let X be a $2\pi R$-periodic field describing the compact boson. We have the 2π-periodic field θ and its conjugate momentum ϕ given in terms of the left and right moving fields

$$\theta = \frac{X_L + X_R}{R};\ \phi = \frac{X_L - X_R}{R}\ . \tag{6.24}$$

θ and ϕ have a $U(1) \times U(1)$ shift symmetry. When R^2 is rational, this system has an enhanced chiral algebra (see for example [84]). We will focus on the case $R = 1, 2$ below. The extended chiral algebra has the trivial, fundamental, spinor, and conjugate spinor representations which we will denote by N_0, N_2, N_1, N_3, respectively. These form the $K = \mathbb{Z}_4$ group under fusion. The scaling dimensions of chiral primaries in these representations are

$$h_0 = 0\ ,\ h_2 = \frac{1}{2}\ ,\ h_1 = h_3 = \frac{1}{8}\ . \tag{6.25}$$

The Narain lattice for this theory is given by

$$P_L := n + \frac{m}{2}\ ,\ P_R := n - \frac{m}{2}\ , \tag{6.26}$$

where $m, n \in \mathbb{Z}$. In general, the vertex operators are given by

$$V_{(n,m)} =: e^{i\vec{p}_L \vec{X}_L} e^{i\vec{p}_R \vec{X}_R} :\ , \tag{6.27}$$

where $\vec{X}_L, \vec{X}_R$ are the left and right moving components of the field X describing the compact boson. The partition function is

$$Z_{\mathcal{T}} = \chi_0\bar{\chi}_0 + \chi_2\bar{\chi}_2 + \chi_1\bar{\chi}_3 + \chi_3\bar{\chi}_1\ , \tag{6.28}$$

which is the charge conjugation modular invariant. The scaling dimensions of the primaries are twice those in (6.25). Here χ_i is the character of N_i given by [84]

$$\chi_p(q) = \frac{1}{\eta(q)} \sum_{n\in\mathbb{Z}} q^{2(n+\frac{p}{4})^2}\ , \tag{6.29}$$

where $p = 0, 1, 2, 3$ and η is the Dedekind eta function. Note that the partition function can also be written in terms of the Narain lattice vectors as

$$Z_{\mathcal{T}}(\tau, \bar{\tau}) = \frac{1}{|\eta(\tau)|^2} \sum_{(P_L, P_R)} q^{\frac{P_L^2}{2}} \bar{q}^{\frac{P_R^2}{2}},\ q = e^{2\pi i\tau},\ \bar{q} = e^{-2\pi i\bar{\tau}} \tag{6.30}$$

The lattice vectors corresponding to a primary operator $\mathcal{O}_{p,\bar{p}}$ can be found by requiring

$$\frac{P_L^2 + P_R^2}{2} = 2h_{\vec{p}} \tag{6.31}$$

where the R.H.S. is the scaling dimension of $\mathcal{O}_{p,\bar{p}}$. In particular, the primary operators

$\mathcal{O}_{1,3}, \mathcal{O}_{3,1}$ correspond to the lattice vectors

$$(P_L, P_R) = \left(\frac{1}{2}, -\frac{1}{2}\right), \left(-\frac{1}{2}, \frac{1}{2}\right), \tag{6.32}$$

while $\mathcal{O}_{2,2}$ corresponds to[141]

$$(P_L, P_R) = (1,1) \oplus (1,-1) \oplus (-1,1) \oplus (-1,-1), \tag{6.33}$$

and $\mathcal{O}_{0,0} = \mathbb{1}$ to (0,0). We can assign each (P_L, P_R) lattice point to be in a particular $\{\mathcal{O}_{p,\bar{p}}\}$ family by considering fusions of the above operators and imposing that fusions correspond to momentum vector addition. Using (6.11), these operators map to the 1-qubit stabilizer code generated by the Z Pauli matrix via

$$I \leftrightarrow \{\mathcal{O}_{0,0}\}, \{\mathcal{O}_{2,2}\}, \quad Z \leftrightarrow \{\mathcal{O}_{1,3}\}, \{\mathcal{O}_{3,1}\}, \tag{6.34}$$

where the map includes all descendants.

A topological line operator, denoted $\mathcal{L}_2$, labelled by $\vec{p} = 2$ generates a $\mathbb{Z}_2$ 0-form symmetry. This symmetry acts by a shift $\phi \to \phi - \pi$, where $\phi := \frac{X_L - X_R}{2}$. The action on the vertex operators is

$$V_{(n,m)} \to (-1)^m V_{(n,m)} \tag{6.35}$$

In particular, the collections of operators $\{\mathcal{O}_{1,3}\}, \{\mathcal{O}_{3,1}\}$ change sign under this symmetry while $\{\mathcal{O}_{0,0}\}, \{\mathcal{O}_{2,2}\}$ remain invariant. This symmetry is non-anomalous because $h_2 = \frac{1}{2}$ [190] (see also the related discussion in [194] and Footnote 12). Taking the $\mathbb{Z}_2$-orbifold,[142] we get a dual CFT with partition function (using (6.17), (6.18))

$$Z_{\mathcal{T}/\mathbb{Z}_2} = \chi_0\bar{\chi}_0 + \chi_2\bar{\chi}_2 + \chi_1\bar{\chi}_1 + \chi_3\bar{\chi}_3 . \tag{6.36}$$

This is the partition function of the $R = 2$ compact boson, which is T-dual to the $R = 1$ compact boson. Using (6.21), the stabilizer code corresponding to this CFT is the 1-qubit code generated by Y via the map

$$I \leftrightarrow \{\mathcal{O}_{0,0}\}, \{\mathcal{O}_{2,2}\}; \; Y \leftrightarrow \{\mathcal{O}_{1,1}\}, \{\mathcal{O}_{3,3}\} . \tag{6.37}$$

T-duality between these theories is captured by the fact that the 1-qubit code generated by Y is equivalent to the code generated by Z [38] (recall that our conventions here differ from those in [38] by an $X \leftrightarrow Y$ code equivalence).

Using (6.3), we can compute the refined enumerator polynomials (REPs) for the

[141] The four states in (6.33) correspond to the fact that $\mathcal{O}_{2,2}$ transforms as a left-moving $so(2)$ vector times a right moving $so(2)$ vector.

[142] $H^2(\mathbb{Z}_2, U(1)) \cong \mathbb{Z}_1$. Therefore, there is no discrete torsion.

codes above, generated by Z and Y to get

$$\begin{aligned} W_{\text{gen}(Z)}(x_1,x_2,x_3,x_4) &= x_1+x_4 \ , \\ W_{\text{gen}(Y)}(x_1,x_2,x_3,x_4) &= x_1+x_3 \ . \end{aligned} \tag{6.38}$$

Therefore, corresponding CFT torus partition functions can be written in terms of the REPs by choosing

$$x_1=\chi_0\bar{\chi}_0+\chi_2\bar{\chi}_2 \ , \ x_4=\chi_1\bar{\chi}_3+\chi_3\bar{\chi}_1 \ , \ x_3=\chi_1\bar{\chi}_1+\chi_3\bar{\chi}_3 \ . \tag{6.39}$$

As a final note, let us comment that we obtain the same quantum codes using any RCFT with $\text{Rep}(V)=A_4^{n_{A_4}}$. For any n_{A_4} there are always infinitely many such RCFTs. For example, we can take the product of the $R=1$ compact boson with arbitrarily many E_8 WZW models at level one and trivial $\text{Rep}(V)$ (this latter theory is associated with a 0-qubit code). In this case, to get the partition function from the REP we have to input the characters $\chi_p\bar{\chi}_{\bar{p}}\chi'_0\bar{\chi}'_0$ into (6.38), where χ'_0 is the vacuum character of the E_8 WZW model at level 1 factors.

$R=\sqrt{2}$ compact boson $\sim SU(2)$ level one WZW

The compact boson at the self-dual radius, or, equivalently, the $SU(2)$ at level one WZW model has $\text{Rep}(V)=A_2$. That is, the representations of the chiral algebra are the trivial and fundamental representations, which we denote by N_0, N_1, respectively. They form a $K=\mathbb{Z}_2$ group under fusion. We have chiral primaries with scaling dimensions

$$h_0=0 \ , \ h_1=\frac{1}{4} \ . \tag{6.40}$$

The Narain lattice for this theory is given by

$$P_L:=\frac{1}{\sqrt{2}}(n+m) \ , \ P_R:=\frac{1}{\sqrt{2}}(n-m) \ , \tag{6.41}$$

where, $n,m\in\mathbb{Z}$. The vertex operators are given by (6.27) with (6.41) inserted, and the torus partition function is

$$Z_{\mathcal{T}}=\chi_0\bar{\chi}_0+\chi_1\bar{\chi}_1 \ , \tag{6.42}$$

where the characters are given by [84]

$$\chi_p(q)=\frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}}q^{\frac{(p+2n)^2}{4}} \ , \tag{6.43}$$

with $i=0,1$.

The non-trivial primary, $\mathcal{O}_{1,1}$, corresponds to the lattice vectors

$$(P_L, P_R) = \pm\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \oplus \pm\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \ , \tag{6.44}$$

where the number of states follows from the fact that the primary transforms in the fundamental representation of the left and right moving $SU(2)$. We can assign any Narain lattice vector to be a member in a $\{\mathcal{O}_{p,\bar{p}}\}$ family by considering fusions of the above primaries and imposing that they correspond to lattice vector addition. Now, using (6.11), this CFT corresponds to the 1-qubit stabilizer code generated by Z via the map

$$I \leftrightarrow \{\mathcal{O}_{0,0}\} \ , \ Z \leftrightarrow \{\mathcal{O}_{1,1}\} \ . \tag{6.45}$$

Note that this is the same quantum code as in the case of the $R = 1$ compact boson. This fact illustrates that, the map (6.21) can give the same quantum code for distinct CFTs.

The REP for this code is given by (6.38), and the torus partition function can be written in terms of W by choosing

$$x_1 = \chi_0\bar{\chi}_0 \ , \quad x_4 = \chi_1\bar{\chi}_1 \ . \tag{6.46}$$

This CFT has a $\mathbb{Z}_2$ 0-form symmetry generated by the topological line $\mathcal{L}_2$. However, this $\mathbb{Z}_2$ is anomalous [190], and hence cannot be gauged (in a purely 2d system).

Again, from our construction, we can consider arbitrary products of this theory and, when we have at least two factors, orbifolds with and without discrete torsion.

Compact boson at $R = \sqrt{\frac{2k}{\ell}}$

Let us generalize the discussion above to compact boson at $R = \sqrt{\frac{2k}{\ell}}$, where k, ℓ are co-prime integers. This RCFT has fusion rules given by the group $K = \mathbb{Z}_{2k\ell}$. The corresponding bulk CS theory is $U(1)_{2k\ell}$. Therefore, in this case Rep(V) labels the Wilson lines in the $U(1)_{2k\ell}$ CS theory. Rep(V) decomposes as follows

$$\text{Rep}(V) \simeq X_{2^s} \times \prod_i (Y_i)_{q_i^{r_i}} \ , K = \mathbb{Z}_{2^s} \times \prod_i \mathbb{Z}_{q_i^{r_i}} \ . \tag{6.47}$$

where the q_i's are distinct odd primes, $X \in \{A, B, C, D\}$, and $Y_i \in \{A, B\}$. Here the labels must be chosen so that the topological central charge is equal to 1 modulo 8. Note that this does not imply that the $U(1)_{2k\ell}$ CS theory or the associated CFT itself factorizes. The decomposition (6.47) is an algebraic property of the set of representations of the chiral algebra Rep(V).

As discussed above, the odd factors contribute trivially to the code. For simplicity, we will therefore consider $\ell = 2^{s-1}$ and $k = 1$ for some integer $s > 0$. In this case

$$U(1)_{2^s}\ \text{CS} \simeq A_{2^s}\ , \quad K = \mathbb{Z}_{2^s}\ . \tag{6.48}$$

The representations of the chiral algebra are denoted by integers $p \in \mathbb{Z}_{2^s}$. The scaling dimensions for these chiral primaries are given by $h_p = \frac{p^2}{2^{s+1}}$ if $p \leq 2^{s-1}$ and $h_p = \frac{\bar{p}^2}{2^{s+1}}$ if $p > 2^{s-1}$.

The Narain lattice for this theory is given by

$$P_L := \frac{n}{R} + \frac{mR}{2}\ ,\ P_R := \frac{n}{R} - \frac{mR}{2}\ ,\ R = 2^{\frac{2-s}{2}}\ , \tag{6.49}$$

where $m, n \in \mathbb{Z}$. The vertex operators are given by (6.27). The torus partition function is

$$Z_{\mathcal{T}} = \sum_{p\in\mathbb{Z}_{2^s}} \chi_p \bar{\chi}_{\bar{p}}\ , \tag{6.50}$$

which is the charge conjugation modular invariant. The characters, $\chi_p(q)$, are given by [84]

$$\chi_p(q) = \frac{1}{\eta(g)} \sum_{n\in\mathbb{Z}} q^{2^{s-1}\left(n+\frac{h_p}{2^s}\right)^2}\ . \tag{6.51}$$

Non-trivial primaries, $\mathcal{O}_{p,\bar{p}}$, with $p < 2^{s-1}$ correspond to lattice vectors satisfying

$$\frac{1}{2}(P_L^2 + P_R^2) = 2h_p\ , P_L > P_R\ , \tag{6.52}$$

while the charge conjugate corresponds to lattice vectors of the above type with $P_R > P_L$. Finally, the non-trivial primary $\mathcal{O}_{2^{s-1},2^{s-1}}$ corresponds to the lattice vectors satisfying

$$\frac{1}{2}(P_L^2 + P_R^2) = 2^{s-2}\ . \tag{6.53}$$

The quantum code corresponding to this CFT is the 1-qubit quantum code generated by Z, where the operators are mapped to the code as

$$\begin{aligned} I &\leftrightarrow \{\mathcal{O}_{p,\bar{p}}\}\ ,\ p = 0 \mod 2\ , \\ Z &\leftrightarrow \{\mathcal{O}_{p,\bar{p}}\}\ ,\ p = 1 \mod 2\ . \end{aligned} \tag{6.54}$$

A topological line operator, denoted $\mathcal{L}_{2^{s-1}}$, labelled by $\vec{p} = 2^{s-1}$ generates a $\mathbb{Z}_2$ 0-form symmetry. This symmetry acts by a shift $\phi \to \phi - \pi$, where $\phi := \frac{R(X_L - X_R)}{2}$. The action on the vertex operators is

$$V_{(n,m)} \to (-1)^m V_{(n,m)}\ . \tag{6.55}$$

This symmetry is non-anomalous and can be gauged. Taking the $\mathbb{Z}_2$-orbifold we get the the orbifold CFT with the partition function

$$Z_{\mathcal{T}/\mathbb{Z}_2} = \sum_{p=0 \text{ mod } 2, p\in\mathbb{Z}_{2^s}} \chi_p\bar{\chi}_{\bar{p}} + \chi_p\bar{\chi}_{\overline{2^{s-1}+p}} \ , \tag{6.56}$$

Using (6.21), the operators in this CFT can be mapped to the stabilizer code generated by X as

$$I \leftrightarrow \{\mathcal{O}_{p,\bar{p}}\}, \ X \leftrightarrow \{\mathcal{O}_{p,\overline{2^{s-1}+p}}\} \tag{6.57}$$

Note that the quantum code corresponding to the $\mathbb{Z}_2$ orbifold of the $R = 1$ compact boson CFT is gen(Y) while that for the $\mathbb{Z}_2$ orbifold of the $R = 2^{\frac{2-s}{2}}$ compact boson CFT for $s > 1$ is gen(X). This difference is because, for $s > 1$, the chiral primary $p = 2^{s-1}$ is bosonic while, for $s = 1$, it is fermionic.

The REPs for the codes obtained above are

$$\begin{aligned} W_{\text{gen}(Z)}(x_1, x_2, x_3, x_4) &= x_1 + x_4 \ , \\ W_{\text{gen}(X)}(x_1, x_2, x_3, x_4) &= x_1 + x_2 \ . \end{aligned} \tag{6.58}$$

Therefore, the partition functions considered above can be written in terms of the REPs by choosing

$$\begin{aligned} x_1 &= \sum_{p=0 \text{ mod } 2} \chi_p\bar{\chi}_{\bar{p}} \ , \\ x_4 &= \sum_{p=1 \text{ mod } 2} \chi_p\bar{\chi}_{\bar{p}} \ , \\ x_3 &= \sum_{p=0 \text{ mod } 2} \chi_p\bar{\chi}_{\overline{2^{s-1}+p}} \ . \end{aligned} \tag{6.59}$$

$\widehat{Spin(16)}_1$ CFT

The Spin(16)$_1$ CFT has Rep(V) $= E_2$ (the "toric code" MTC). We denote the representations of the chiral algebra by $N_{(0,0)}, N_{(0,1)}, N_{(1,0)}, N_{(1,1)}$, and they form a $K = \mathbb{Z}_2\times\mathbb{Z}_2$ group under fusion. We have chiral primaries with scaling dimensions

$$h_{(0,0)} = 0 \ , \ h_{(0,1)} = h_{(1,0)} = 1 \ , \ h_{(1,1)} = \frac{1}{2} \ . \tag{6.60}$$

The Narain lattice is

$$\{(\vec{P}_L, \vec{P}_R) \in \Lambda_W \times \Lambda_W, \vec{P}_L - \vec{P}_R \in \Lambda_R\} \tag{6.61}$$

where $\Lambda_W = \{\sum_i n_i\lambda_i, n_i \in \mathbb{Z}\}$ is the weight lattice, λ_i are the fundamental weights

$$\lambda_i = (1, \cdots, 1, 0, \cdots, 0), \quad 1 \leq r \leq 6 \ (1 \text{ repeated } i \text{ times})$$

$$\lambda_7 = (1,1,1,1,1,1,1,1),\ \lambda_8 = (1,1,1,1,1,1,1,-1)\ ,$$

and $\Lambda_R = \{\sum_i n_i\alpha_i, n_i \in \mathbb{Z}\}$ where α_i are the simple roots

$$\alpha_i = e_i - e_{i+1}\ 1 \le i \le 7,\ \alpha_8 = e_8 + e_7\ . \tag{6.62}$$

Here e_i is the vector with components $(e_i)_j = \delta_{i,j}$. It is easy to check that Λ_R is the set of 8-component vectors such that the sum of its components is even.

The partition function is

$$Z_{\mathcal{T}} = \chi_{(0,0)}\bar{\chi}_{(0,0)} + \chi_{(0,1)}\bar{\chi}_{(0,1)} + \chi_{(1,0)}\bar{\chi}_{(1,0)} + \chi_{(1,1)}\bar{\chi}_{(1,1)}\ , \tag{6.63}$$

where the characters are given by [84]

$$\chi_{(0,0)} = \frac{(\theta_3^8+\theta_4^8)}{2\eta^8}\ ,\ \chi_{(0,1)} = \chi_{(1,0)} = \frac{\theta_2^8}{2\eta^8}\ ,\ \chi_{(1,1)} = \frac{(\theta_3^8-\theta_4^8)}{2\eta^8}\ . \tag{6.64}$$

Here $\theta_2, \theta_3, \theta_4$ are Jacobi-Theta functions. The Dynkin labels for the representations $N_{(0,0)}, N_{(0,1)}, N_{(1,0)}$ and $N_{(1,1)}$ are $(0,0,0,0,0,0,0,1), (0,0,0,0,0,0,1,0), (0,0,0,0,0,1,0,0)$ and $(1,0,0,0,0,0,0,0)$, respectively. Therefore, the primary operators $\mathcal{O}_{(0,0),(0,0)}$, $\mathcal{O}_{(0,1),(0,1)}$, $\mathcal{O}_{(1,0),(1,0)}$ and $\mathcal{O}_{(1,1),(1,1)}$, in turn, correspond to the lattice vectors

$$(\lambda_8,\lambda_8)\ ,\ (\lambda_7,\lambda_7)\ ,\ (\lambda_6,\lambda_6)\ ,\ (\lambda_1,\lambda_1)\ . \tag{6.65}$$

Using (6.11), this CFT corresponds to the two-qubit stabilizer code generated by $I \otimes Z, Z \otimes I$ via the map

$$I \otimes Z \leftrightarrow \{\mathcal{O}_{(1,0),(1,0)}\},\ Z \otimes I \leftrightarrow \{\mathcal{O}_{(0,1),(0,1)}\}\ . \tag{6.66}$$

This CFT has three non-anomalous $\mathbb{Z}_2$ 0-form symmetries, Q_1, Q_2, Q_3, corresponding to the topological lines $\mathcal{L}_{(0,1)}, \mathcal{L}_{(1,0)}$, and $\mathcal{L}_{(1,1)}$. These symmetries act on the primary operators (and the corresponding Narain lattice vectors) as

$$\begin{aligned}
&\mathcal{L}_{(0,1)} : \mathcal{O}_{(1,0)} \to -\mathcal{O}_{(1,0)},\ \mathcal{O}_{(1,1)} \to -\mathcal{O}_{(1,1)}\ ,\\
&\mathcal{L}_{(1,0)} : \mathcal{O}_{(0,1)} \to -\mathcal{O}_{(0,1)},\ \mathcal{O}_{(1,1)} \to -\mathcal{O}_{(1,1)}\ ,\\
&\mathcal{L}_{(1,1)} : \mathcal{O}_{(0,1)} \to -\mathcal{O}_{(0,1)},\ \mathcal{O}_{(1,0)} \to -\mathcal{O}_{(1,0)}\ .
\end{aligned} \tag{6.67}$$

Actions of the symmetries on primaries not mentioned above are trivial. Orbifolding by Q_1, Q_2, Q_3, we get CFTs with partition functions (using (6.17), (6.18))

$$\begin{aligned}
Z_{\mathcal{T}/Q_1} &= \chi_{(0,0)}\bar{\chi}_{(0,0)} + \chi_{(0,0)}\bar{\chi}_{(0,1)} + \chi_{(0,1)}\bar{\chi}_{(0,0)} + \chi_{(0,1)}\bar{\chi}_{(0,1)}\ ,\\
Z_{\mathcal{T}/Q_2} &= \chi_{(0,0)}\bar{\chi}_{(0,0)} + \chi_{(0,0)}\bar{\chi}_{(1,0)} + \chi_{(1,0)}\bar{\chi}_{(0,0)} + \chi_{(1,0)}\bar{\chi}_{(1,0)}\ ,
\end{aligned}$$

$$Z_{\mathcal{T}/Q_3} = \chi_{(0,0)}\bar{\chi}_{(0,0)} + \chi_{(1,1)}\bar{\chi}_{(1,1)} + \chi_{(0,1)}\bar{\chi}_{(1,0)} + \chi_{(1,0)}\bar{\chi}_{(0,1)} \ , \tag{6.68}$$

respectively. Using (6.21), these CFTs can be mapped, in turn, to the stabilizer codes specified by $\text{gen}(Z \otimes I, I \otimes X)$, $\text{gen}(I \otimes Z, X \otimes I)$, and $\text{gen}(Z \otimes Z, Y \otimes X)$.

We can also orbifold by the full $Q_1 \times Q_2$ symmetry of the CFT. We get partition functions (using (6.17), (6.18))

$$\begin{aligned} Z_{\mathcal{T}/Q_1\times Q_2,[1]} &= \chi_{(0,0)}\bar{\chi}_{(0,0)} + \chi_{(0,1)}\bar{\chi}_{(0,0)} + \chi_{(0,0)}\bar{\chi}_{(1,0)} + \chi_{(0,1)}\bar{\chi}_{(1,0)} \ , \\ Z_{\mathcal{T}/Q_1\times Q_2,[\sigma]} &= \chi_{(0,0)}\bar{\chi}_{(0,0)} + \chi_{(0,0)}\bar{\chi}_{(0,1)} + \chi_{(1,0)}\bar{\chi}_{(0,0)} + \chi_{(1,0)}\bar{\chi}_{(0,1)} \ , \end{aligned} \tag{6.69}$$

where $[1]$ and $[\sigma]$ are the trivial and non-trivial elements of $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1))$, respectively. Using (6.21), these CFTs can be mapped, in turn, to subgroups of the Pauli group specified by $\text{gen}(Z \otimes X, X \otimes I)$ and $\text{gen}(X \otimes Z, I \otimes X)$.

The subgroup of the Pauli group generated by these elements is clearly not a stabilizer code since it is non-abelian. For example, $Z \otimes X$ and $X \otimes I$ anti-commute with each other. This is expected from out general arguments above since Q_1 and Q_2 are related to 1-form symmetries of the bulk $\text{Spin}(16)_1$ Chern-Simons theory which have a mixed 't Hooft anomaly.

6.3 Errors and the Full Pauli Group from Defects

In the context of quantum codes, the elements of the Pauli group, $\mathcal{P}_n$, that are not in the stabilizer subgroup, $\mathcal{S}_n$, are either called "logical operators" or "errors," depending, respectively, on whether they preserve the code subspace or map states from the code subspace to its complement. Since our codes are self-dual, we have no (non-trivial) logical operators,[143] and all elements of $\mathcal{P}_n$ that are not in $\mathcal{S}_n$ correspond to errors.

How can we see these errors in the CFT? An intuitive picture is provided by the toric code [195]. There one finds that error operations correspond to string operators (defects) that create anyonic pairs.[144] When the anyons annihilate, the system returns to the code subspace, implementing a logical operation. While the gapped toric code system is very different from the CFTs considered in this chapter, as we will see below, this geometric picture of errors is still informative.

A more direct way to understand errors is to look at the fields in $\mathcal{T}/Q$ that contribute the terms with $\vec{g} \neq 0$ in (6.17). In the orbifolding procedure, we gauge Q in the charge-conjugation modular invariant theory, $\mathcal{T}$. The $\vec{g} \neq \vec{0}$ bulk fields of $\mathcal{T}/Q$ then come from certain fields living at the end of Q topological defects of $\mathcal{T}$. Therefore, the X-dependent Pauli stabilizers of the $\mathcal{T}/Q$ theory appearing in (6.21) correspond to error operations in the $\mathcal{T}$ theory. This discussion suggests error operations of the code related

[143] Note that the elements in $\mathcal{S}_n$, are sometimes called "trivial" logical operators.
[144] For a pedagogical discussion, see section 11.3 of [196].

to $\mathcal{T}$ are given by defect endpoint operators of the Q symmetries of $\mathcal{T}$. In the language of quantum codes, such orbifolding exchanges certain errors with stabilizers in an n-qubit self-dual code to produce a new n-qubit self-dual code, see e.g. the examples in section 6.3.2.

With the motivation above, we are now ready to identify the full set of error operations, i.e., to reconstruct the full Pauli group, from the defect fields. Since Q consists of order-two defects which commute with the vacuum module, this suggests that we associate error operations with fields living at the ends of such defects. Through a slight abuse of terminology, we will refer to these and any other defects that preserve the maximal chiral algebra of a theory as "Verlinde lines" (for further discussion of such lines, see for example [25, 197–200]).

To understand the spectrum of defect endpoint fields in the most general case, we eventually want to consider CFTs in which the pairing of characters is given by

$$Z_{\mathcal{T}_{\mathcal{M}}} = \sum_{\vec{p},\vec{q}} \mathcal{M}_{\vec{p}\vec{q}} \chi_{\vec{p}}(q) \bar{\chi}_{\vec{q}}(\bar{q}) \ , \tag{6.70}$$

where $\mathcal{M}$ is a matrix commuting with S and T.[145] As a technically simpler starting point, let us first consider the case when $\mathcal{M}_{\vec{p}\vec{q}}$ is a permutation on the set of vectors. Such modular invariants are called "permutation modular invariants," and charge conjugation corresponds to the case $\mathcal{M}_{\vec{p},\vec{q}} = \delta_{\vec{p},\bar{\vec{q}}}$. To avoid confusion below, we call theories of this type "maximal" permutation modular invariants (MPMIs).[146] As we will see, we can reconstruct the Pauli group from Verlinde lines alone in any MPMI admitting a code description.

In MPMIs, we define Verlinde lines via[147]

$$\mathcal{L}_{(\vec{p},\vec{p}_{\mathcal{M}})} = \sum_{\vec{\ell}} \frac{\bar{S}_{\vec{p}\vec{\ell}}}{\bar{S}_{\vec{0}\vec{\ell}}} |\vec{\ell}, \vec{\ell}_{\mathcal{M}}\rangle\langle\vec{\ell}, \vec{\ell}_{\mathcal{M}}| \ , \tag{6.71}$$

where each $|\vec{\ell}, \vec{\ell}_{\mathcal{M}}\rangle\langle\vec{\ell}, \vec{\ell}_{\mathcal{M}}|$ is a projector on the primary state labeled by $(\vec{\ell}, \vec{\ell}_{\mathcal{M}})$ together with its descendants. Since this operator is a multiple of the identity within each representation of the left and right chiral algebras, it commutes with the chiral algebras and is topological (by construction, it commutes with the Virasoro sub-algebras). For convenience, we denote $\mathcal{L}_{(\vec{p},\vec{p}_{\mathcal{M}})}$ simply as $\mathcal{L}_{\vec{p}}$ since the right-moving label is determined by $\vec{p}$. Using the Verlinde formula, it is easy to check that these lines satisfy the fusion rules of the RCFT

$$\mathcal{L}_{\vec{p}} \times \mathcal{L}_{\vec{q}} = \mathcal{L}_{\vec{p}+\vec{q}} \ . \tag{6.72}$$

[145] Here, we have $T_{\vec{p},\vec{q}} := e^{-\pi i (c/12)} \theta_{\vec{p}} \delta_{\vec{p}\vec{q}}$.
[146] More general permutation modular invariants will play a role below.
[147] In (6.71) and bellow, $\vec{\ell}_{\mathcal{M}} = \vec{k}$ is the unique vector such that $\mathcal{M}_{\vec{\ell}\vec{k}} \neq 0$.

When $\vec{p}$ is order two, we have

$$\vec{p}+\vec{p}=\vec{0} \;\Rightarrow\; S_{\vec{p}\vec{\ell}}/S_{\vec{0}\vec{\ell}} \in \{\pm 1\} \ . \tag{6.73}$$

To proceed, we insert $\mathcal{L}_{\vec{g}}$ in the torus partition function (i.e., we wrap it on the spatial cycle of the torus) and perform a modular transformation so that it wraps time

$$\begin{aligned} Z_{\mathcal{T}_{\mathcal{M}}}(\mathcal{L}_{\vec{\ell}}) &= \sum_{\vec{p},\vec{q}} \frac{\bar{S}_{\vec{\ell}\vec{p}}}{\bar{S}_{\vec{0}\vec{\ell}}} \mathcal{M}_{\vec{p}\vec{q}} \chi_{\vec{p}}(q) \bar{\chi}_{\vec{q}}(\bar{q}) \\ &\to \sum_{\vec{p},\vec{q},\vec{r},\vec{s}} \frac{\bar{S}_{\vec{\ell}\vec{p}}}{\bar{S}_{\vec{0}\vec{\ell}}} \mathcal{M}_{\vec{p}\vec{q}} S_{\vec{p}\vec{r}} \bar{S}_{\vec{q}\vec{s}} \chi_{\vec{r}}(q) \bar{\chi}_{\vec{s}}(\bar{q}) \\ &= \sum_{\vec{q},\vec{r},\vec{s}} N_{\vec{r}\bar{\vec{q}}}^{\vec{\ell}} \mathcal{M}_{\vec{q}\vec{s}} \chi_{\vec{r}}(q) \bar{\chi}_{\vec{s}}(\bar{q}) := Z^{\vec{\ell}}_{\mathcal{T}_{\mathcal{M}}}(q,\bar{q}) \ , \end{aligned} \tag{6.74}$$

where, in the last line, we have arrived at a definition for the partition function of fields living at the end of the defect labeled by $\vec{\ell}$. In the second to last equality, we use the Verlinde formula. In light of (6.72), we can simplify the fusion coefficients as $N_{\vec{r}\bar{\vec{q}}}^{\vec{\ell}} = \delta^{\vec{\ell}}_{\vec{r}-\vec{q}}$. Therefore, we have

$$Z^{\vec{\ell}}_{\mathcal{T}_{\mathcal{M}}}(q,\bar{q}) = \sum_{\vec{r},\vec{s}} \mathcal{M}_{\vec{r}-\vec{\ell}\ \vec{s}} \chi_{\vec{r}}(q) \bar{\chi}_{\vec{s}}(\bar{q}) = \sum_{\vec{p},\vec{q}} \mathcal{M}_{\vec{p}\vec{q}} \chi_{\vec{p}+\vec{\ell}}(q) \bar{\chi}_{\vec{q}}(\bar{q}) \ . \tag{6.75}$$

Specializing to the case of the charge conjugation modular invariant, we obtain

$$Z^{\vec{\ell}}_{\mathcal{T}}(q,\bar{q}) = \sum_{\vec{p}} \chi_{\vec{p}+\vec{\ell}}(q) \bar{\chi}_{\bar{\vec{p}}}(\bar{q}) \ . \tag{6.76}$$

When $\vec{\ell} \in Q \simeq \mathbb{Z}_2^k$, we get, using $2\vec{\ell}=\vec{0}$,

$$Z^{\vec{\ell}}_{\mathcal{T}}(q,\bar{q}) \;=\; \sum_{\vec{p}} \chi_{\vec{p}+\vec{\ell}}(q) \bar{\chi}_{\overline{\vec{\ell}+\vec{p}+\vec{\ell}}}(\bar{q}) \ . \tag{6.77}$$

As expected, these are equivalent to the contributions in (6.21), only here they correspond to defect operators in $\mathcal{T}$ rather than bulk operators in $\mathcal{T}/Q$. Therefore, consistency with the map in (6.21) demands

$$\left\{ \mathcal{O}^{\vec{\ell}}_{\vec{p}+\vec{\ell},\bar{\vec{p}}} \right\} \leftrightarrow X^{M\vec{\ell}} \circ Z^{A(\vec{p}+\vec{\ell})} \ , \tag{6.78}$$

where $\left\{ \mathcal{O}^{\vec{\ell}}_{\vec{p}+\vec{\ell},\bar{\vec{p}}} \right\}$ should be understood as an $\vec{\ell}$-defect primary operator and its associated descendants. If $Q \simeq \mathbb{Z}_2^n$, then (6.78) gives rise to the full Pauli group. More generally, we can consider cases in which $Q \not\simeq \mathbb{Z}_2^n$ and some of the order-two Verlinde lines correspond to $\vec{\ell} \notin Q$ (e.g., see the $SU(2)$ at level one WZW model example in section 6.3.2). In this case we also obtain the full Pauli group: $\vec{\ell}$ in (6.78) is any order-two element, and $\vec{p}$ is any representation in the Narain theory. Therefore, the charge-conjugation modular invariant knows about the full set of operations acting on the quantum code: *the genuine local operators correspond to stabilizers and the defect endpoint operators correspond to the errors.*

It is straightforward to extend this picture to the most general MPMIs when these CFTs admit a quantum code description. Clearly, to be an MPMI, we need every possible $\vec{p}$ and $\overline{\vec{g}+\vec{p}}$ to appear exactly once in (6.17). Therefore, as we sum over $\vec{g}$ and take all $\vec{p} \in B_{\vec{g}}$, we produce all possible $\vec{p} \in K$. As a result, in the code we generate via (6.21), we get all possible powers of Z. The powers of X are restricted since $\vec{g} \in Q$, and Q is a proper subgroup of K.

However, the fields living at the end of the order-two Verlinde defects precisely make up the difference since (6.75) now becomes

$$Z^{\vec{\ell}}_{\mathcal{T}_{\mathcal{M}}}(q,\bar{q}) = \sum_{\vec{g}\in H}\sum_{\vec{p}\in B_{\vec{g}}} \chi_{\vec{p}+\vec{\ell}}(q)\bar{\chi}_{\overline{\vec{p}+\vec{g}}}(q) = \sum_{\vec{g}\in H}\sum_{\vec{p}\in B_{\vec{g}}} \chi_{\vec{p}+\vec{\ell}}(q)\bar{\chi}_{\overline{\vec{g}+\vec{\ell}+\vec{p}+\vec{\ell}}}(\bar{q}) \ . \tag{6.79}$$

As a result, our CFT-code map in (6.21) becomes

$$\left\{\mathcal{O}^{\vec{\ell}}_{\vec{p}+\vec{\ell},\overline{\vec{g}+\vec{p}}}\right\} \leftrightarrow X^{M(\vec{\ell}+\vec{g})} \circ Z^{A(\vec{p}+\vec{\ell})} \ . \tag{6.80}$$

Since the fusion rules in (6.72) do not depend on the nature of $\mathcal{M}$, we see that the number of order-two Verlinde defects is the same as in the charge-conjugation case. Therefore, upon including all order-two Verlinde lines, we get all possible Pauli group elements, and the corresponding errors that affect our stabilizer code.

Let us now consider the most general case (6.17), which we can always write as in (6.70) with $\mathcal{T}_M = \mathcal{T}/Q$ (and discrete torsion $[\sigma]$). Note that in (6.70), $\mathcal{M}_{\vec{p},\vec{q}}$ is a matrix with entries consisting of 0's and 1's (see Appendix C.2), and it will not generally be a permutation (i.e., the CFT will not be an MPMI).

As we will see in the next subsection, we have a smaller number of Verlinde lines when $\mathcal{T}/Q$ is not an MPMI. However, we can still define enough order-two symmetries to recover the Pauli group from the corresponding defect fields (note that invertibility of the orbifolding procedure guarantees that, for each symmetry we gauge, there is a dual symmetry in the orbifolded theory).

To construct these extra symmetries, it suffices to associate signs with the primaries compatible with fusion (then all local correlation functions are invariant). In the Verlinde line case, we did this via (6.71) and (6.73).

Since we have orbifolded in a way that respects $\mathcal{T}$'s chiral algebra, $\mathcal{T}/Q$ respects the fusion rules of $\mathcal{T}$. More precisely, if we have operators in the orbifolded theory transforming in representations $(\vec{p}_1, \overline{\vec{g}_1+\vec{p}_1})$ and $(\vec{p}_2, \overline{\vec{g}_2+\vec{p}_2})$, then we also have an operator transforming as $(\vec{p}_1+\vec{p}_2, \overline{\vec{g}_1+\vec{g}_2+\vec{p}_1+\vec{p}_2})$. Technically, this statement follows from

$$S_{\vec{h},\vec{p}_1+\vec{p}_2}\ \Xi(\vec{h},\vec{g}_1+\vec{g}_2) = S_{\vec{h},\vec{p}_1}\ \Xi(\vec{h},\vec{g}_1)\ S_{\vec{h},\vec{p}_2}\ \Xi(\vec{h},\vec{g}_2) = 1\ , \qquad \forall \vec{h} \in \mathbb{Z}_2^k\ , \tag{6.81}$$

where we have used the bicharacter property of both S and Ξ (see Appendix C.1). Therefore, $(\vec{p}, \overline{\vec{g}+\vec{p}})$ forms an abelian group under fusion (as it should since $\mathcal{T}/Q$ is a Narain theory). Let us denote this group as F.

Now, after acting with some order-two symmetry, π (i.e., inserting the corresponding topological defect, D_π, along a spatial cycle and computing the torus partition function), some of the 1 entries in $\mathcal{M}$ get flipped to -1 such that fusion is respected. Let us denote the matrix so obtained as $\mathcal{M}_\pi$.

As in (6.74), to calculate the defect partition function, we have to perform an S transformation to get $S^T\mathcal{M}_\pi\bar{S}$. All the characters that we get from the defect partition functions for all possible order-two π correspond to the non-zero entries of the matrix

$$\sum_\pi S^T\mathcal{M}_\pi\bar{S} = S^T\left(\sum_\pi \mathcal{M}_\pi\right)\bar{S} := S^T\mathcal{M}_\Sigma\bar{S}\ , \tag{6.82}$$

where the sum is over all such symmetries, π.

Assigning signs to the primaries such that the fusion is respected is the same as choosing an irreducible representation of F valued in ±1. The trivial representation acts trivially on the primaries. Therefore, for each π, we associate an irrep, sign π. In order to find the non-zero entries of $\sum_\pi \mathcal{M}_\pi$ we have to understand when

$$\sigma(x) := \sum_{\text{sign }\pi} \chi_{\text{sign }\pi}(x)\ , \tag{6.83}$$

is non-zero. Here, the sum is over the irreducible representations, sign π, of F valued in ±1, and $\chi_{\text{sign }\pi}(x)$ is the character of sign π (not to be confused with the RCFT characters appearing in the partition function!) evaluated on a given element $x \in F$ (note that each element in F represents a character combination $\chi_{\vec{p}}\bar{\chi}_{\overline{\vec{g}+\vec{p}}} \in Z_{\mathcal{T}/Q,[\sigma]}$; we will denote this combination $(\vec{p}, \overline{\vec{g}+\vec{p}})$).

To that end, suppose F has a decomposition in terms of cyclic groups given by

$$F \cong \mathbb{Z}_{n_1} \otimes \ldots \otimes \mathbb{Z}_{n_l}\ . \tag{6.84}$$

Since we are treating CFT factors related to A_{q^r} and B_{q^r} as spectators, the n_i are even.

We know that

$$\hat{F} = \hat{\mathbb{Z}}_{n_1} \otimes \ldots \otimes \hat{\mathbb{Z}}_{n_l}\ , \tag{6.85}$$

where $\hat{F}$ is the group of irreducible representations of F. In particular, the sign representations of F are given by products of $\mathbb{Z}_{n_i}$ sign representations. Choose a basis $\{e_1, \cdots, e_l\}$ for the cyclic groups; then, an element of F is of the form $(e_1^{m_1}, ..., e_l^{m_l})$ for some integers $0 \le m_i \le n_i - 1$. Consider $\sigma(x)$ for some $x = (e_1^{m_1}, ..., e_l^{m_l}) \in F$. We know that sign π = sign $\pi_1 \otimes \cdots \otimes$ sign π_l, where sign π_i is a representation of $\mathbb{Z}_{n_i}$ valued in ± 1. Therefore

$$\sigma(x) = \sum_{\text{sign } \pi_1, \cdots, \text{ sign } \pi_l} \chi_{\pi_1}(e_1^{m_1}) \dots \chi_{\pi_l}(e_1^{m_l}) = \prod_i \left(\sum_{\text{sign } \pi_i} (\chi_{\text{sign } \pi_i}(e_i))^{m_1} \right) . \tag{6.86}$$

Since the n_i are all even and sign π_i is valued in ± 1, we have $\chi_{\text{sign } \pi_i}(e_i) = \pm 1 \forall i$. Therefore, we find

$$\sigma(x) = \prod_i (1^{m_i} + (-1)^{m_i}) = \begin{cases} 2^l \ , & \text{iff } m_i \in 2\mathbb{Z} \ \forall i \\ 0 \ , & \text{otherwise} \ . \end{cases} \tag{6.87}$$

Now, suppose $x = (e_1^{m_1}, ..., e_l^{m_l})$ is an element of the group F such that all m_i are even. Then there exists some other element $y \in F$ such that $y^2 = x$. Recall that an element of F represents a character combination in the partition function denoted by $(\vec{p}, \overline{\vec{g} + \vec{p}})$. Adding this element to itself gives $(2\vec{p}, 2\bar{\vec{p}})$ (since $\vec{g}$ is order two). Therefore, if $x \in F$ has only even m_i, $x = (2\vec{p}, 2\bar{\vec{p}})$.

As a result, the matrix $\mathcal{M}_\Sigma$ defined in (6.82) is a matrix with entries valued in $\{0, 2^l\}$, where the only non-zero entries correspond to $(2\vec{p}, 2\bar{\vec{p}})$. In other words

$$\sum_\pi Z_{\mathcal{T}/Q,[\sigma]}(D_\pi) = \sum_\pi \mathcal{M}_{\pi;\vec{p},\vec{q}} \chi_{\vec{p}}(q) \bar{\chi}_{\vec{q}}(\bar{q}) = 2^l \sum_{\vec{2}p} \chi_{2\vec{p}}(q) \bar{\chi}_{\overline{2\vec{p}}}(\bar{q}) \ . \tag{6.88}$$

Note that the case $\pi = 1$ gives the partition function without a defect. As a result, $\chi_{2\vec{p}} \bar{\chi}_{\overline{2\vec{p}}}$ is a term in this partition function, and we know that $2\vec{p}$ has to satisfy (6.18) for $\vec{g} = 0$. That is, the CS Wilson line corresponding to $2\vec{p}$ should braid trivially with all $\vec{h} \in \mathbb{Z}_2^k$.

We want to show that the sum of defect partition functions $\sum_\pi Z^\pi_{\mathcal{T}/Q,[\sigma]}$ (coming from applying a modular transformation to (6.88)) contains all possible characters of the form $\chi_{\vec{p}} \bar{\chi}_{\overline{\vec{g}+\vec{p}}}$, where $\vec{g}$ is order two, so that we get the full Pauli group from it. To that end, consider

$$\begin{aligned} \sum_\pi Z^\pi_{\mathcal{T}/Q,[\sigma]} &= 2^l \sum_{2\vec{p}} \sum_{\vec{i},\vec{j}} S_{2\vec{p},\vec{i}} \bar{S}_{\overline{2\vec{p}},\bar{\vec{j}}} \chi_{\vec{i}} \bar{\chi}_{\bar{\vec{j}}} = 2^l \sum_{2\vec{p}} \sum_{\vec{i},\vec{j}} S_{2\vec{p},(\vec{i}-\vec{j})} \chi_{\vec{i}} \bar{\chi}_{\bar{\vec{j}}} \\ &= 2^l \sum_{2\vec{p}} \sum_{\vec{i},\vec{j}} S_{\vec{p},2(\vec{i}-\vec{j})} \chi_{\vec{i}} \bar{\chi}_{\bar{\vec{j}}} \ . \end{aligned} \tag{6.89}$$

It is clear that if $(\vec{i} - \vec{j})$ is order two, then $S_{\vec{p},2(\vec{i}-\vec{j})} = 1 \ \forall \vec{p}$. Therefore, the character

$\chi_{\vec{i}}\bar{\chi}_{\vec{j}}$ contributes non-trivially to the sum for any $\vec{i}, \vec{j}$ satisfying the constraint that $\vec{i}-\vec{j}$ is order two. These characters correspond to

$$X^{M(\vec{i}-\vec{j})} \circ Z^{A\vec{i}} \ . \tag{6.90}$$

Since $\vec{i}-\vec{j}$ is any order-two element, and $\vec{i}$ is arbitrary (though choosing $\vec{i}$ fixes $\vec{j}$ mod 2), we find that these defect fields give the full Pauli group. *This ends our proof and shows that all code CFTs contain all possible errors via order-two defects.*

6.3.1 Verlinde Subgroup of the Pauli Group

In this section, we define a "Verlinde subgroup" of $\mathcal{P}_n$. This subgroup can be constructed from any code RCFT. It is defined as follows.

Definition: The Verlinde subgroup, $\mathcal{V}_{\mathcal{T}/Q}$, is the subgroup of $\mathcal{P}_{\mathcal{T}/Q}$ coming from all stabilizers that are related to (1) CFT local fields and (2) fields living at the end of order-two Verlinde lines.

Note that, by construction $\mathcal{S}_{\mathcal{T}/Q} \subseteq \mathcal{V}_{\mathcal{T}/Q} \subseteq \mathcal{P}_{\mathcal{T}/Q}$. Physically, the ratio

$$r_{\mathcal{T}/Q} := 2^{-n}\frac{|\mathcal{P}_{\mathcal{T}/Q}|}{|\mathcal{V}_{\mathcal{T}/Q}|} \ , \quad 2^{-n} \leq r \leq 1 \ , \tag{6.91}$$

measures how well the continuous symmetries of the Narain CFT corresponding to an n-qubit code are able to detect an error. For example, in the charge conjugation modular invariant or any of the MPMIs, $r_{\mathcal{T}/Q} = 2^{-n}$, which is the smallest value possible. This is because the Verlinde subgroup corresponds to the full Pauli group. Any Verlinde line, $\mathcal{L}_{\vec{\ell}}$, commutes with the chiral algebra, since $\bar{S}_{\vec{\ell}\vec{0}}/\bar{S}_{\vec{0}\vec{0}} = 1$ in (6.71), and so the corresponding continuous symmetry currents are acted upon trivially by the Verlinde lines. In this sense, the continuous symmetry currents cannot detect errors associated with these defects.

What about more general theories? These theories are not MPMIs. However, it turns out that, if we enlarge the chiral algebras as much as possible, any orbifold theory we can construct using our methods above is a permutation modular invariant with respect to this larger algebra (see Appendix C.4). We can then define a Verlinde subgroup for any of our orbifold theories. Moreover, as we show in Appendix C.4, if we enlarge the chiral algebra, then, $r_{\mathcal{T}/Q} > 2^{-n}$, and the error detection ability of the continuous symmetry currents improves. In the most extreme cases, we get CFTs that are products of left moving meromorphic and right moving anti-meromorphic CFTs. These types of theories have $r_{\mathcal{T}/Q} = 1$, and their continuous symmetries are able to fully detect errors.

6.3.2 Examples

Pauli group from $R = 1$ compact boson

The $R = 1$ compact boson has a charge conjugation partition function which is an MPMI. Therefore, our general discussion on Pauli groups from MPMIs can be readily applied to this case. To that end, consider

$$Z_{\mathcal{T}} = \chi_0\bar{\chi}_0 + \chi_2\bar{\chi}_2 + \chi_1\bar{\chi}_3 + \chi_3\bar{\chi}_1 \ . \tag{6.92}$$

Recall that the bulk operators are mapped to the 1-qubit stabilizer code, gen(Z). This CFT has a $\mathbb{Z}_2$ symmetry generated by the Verlinde line, $\mathcal{L}_2$. Inserting this line in the partition function, we can calculate the defect partition function using (6.77)

$$Z_{\mathcal{T}}^{\vec{\ell}=2} = \chi_0\bar{\chi}_2 + \chi_2\bar{\chi}_0 + \chi_1\bar{\chi}_1 + \chi_3\bar{\chi}_3 \ . \tag{6.93}$$

Using (6.78), the defect operators are mapped to Pauli group elements as follows

$$X \leftrightarrow \{\mathcal{O}^{\vec{\ell}=2}_{(0,2)}\}, \{\mathcal{O}^{\vec{\ell}=2}_{(2,0)}\} \ , \ Y \leftrightarrow \{\mathcal{O}^{\vec{\ell}=2}_{(1,1)}\}, \{\mathcal{O}^{\vec{\ell}=2}_{(3,3)}\} \ . \tag{6.94}$$

Therefore, the bulk operators along with the defect operators give us the full Pauli group, $\mathcal{P}_{\mathcal{T}}$. Since the X and Y Pauli matrices correspond to defect operators living at the end of an order-two Verlinde line, the Verlinde subgroup, $\mathcal{V}_{\mathcal{T}}$, is the full Pauli group.

Pauli group from $R = \sqrt{\frac{2}{2^{s-1}}}$ compact boson

Recall that the $R = \sqrt{\frac{2}{2^{s-1}}}$ compact boson has the charge conjugation partition function

$$Z_{\mathcal{T}} = \sum_{p\in\mathbb{Z}_{2^s}} \chi_p\bar{\chi}_{\bar{p}} \ , \tag{6.95}$$

We know that the CFT local operators are mapped to the qubit stabilier code generated by Z. This CFT has a $\mathbb{Z}_2$ symmetry generated by the Verlinde line, $\mathcal{L}_{2^{s-1}}$. Inserting this line in the partition function, we can calculate the defect partition function using (6.77)

$$Z_{\mathcal{T}}^{\vec{\ell}=2^{s-1}} = \sum_{p\in\mathbb{Z}_{2^s}} \chi_{p+2^{s-1}}\bar{\chi}_{\bar{p}} \ , \tag{6.96}$$

Using (6.78), the defect operators are mapped to Pauli group elements as follows

$$\begin{aligned} X &\leftrightarrow \{\mathcal{O}^{\vec{\ell}=2^{s-1}}_{p+2^{s-1},\bar{p}}\} \ , \ p = 0 \mod 2 \ , \\ Y &\leftrightarrow \{\mathcal{O}^{\vec{\ell}=2^{s-1}}_{p+2^{s-1},\bar{p}}\} \ , \ p = 1 \mod 2 \ . \end{aligned} \tag{6.97}$$

Therefore, the local operators along with the defect operators at the end of the order-two Verlinde line $\mathcal{L}_{2^s}$ gives us the full Pauli group.

Now let us consider the CFT with partition function

$$Z_{\mathcal{T}/\mathbb{Z}_2} = \sum_{p=0 \text{ mod } 2, p\in\mathbb{Z}_{2^s}} \chi_p \bar{\chi}_{\bar{p}} + \chi_p \bar{\chi}_{\overline{2^s+p}} \ , \tag{6.98}$$

obtained from the $R = 2^{\frac{2-s}{2}}$ CFT by orbifolding the $\mathbb{Z}_2$ symmetry generated by $\mathcal{L}_{2^{s-1}}$. Recall that the genuine local operators in this CFT are mapped to the stabilizer code generated by X (for $s > 2$).

This CFT has a $\mathbb{Z}_2$ symmetry generated by a line defect, say D_π, which acts on the primary operators as follows

$$\{\mathcal{O}_{v,\bar{v}}\} \to \{\mathcal{O}_{v,\bar{v}}\}, \ \{\mathcal{O}_{v,\overline{2^{s-1}+v}}\} \to -\{\mathcal{O}_{v,\overline{2^{s-1}+v}}\} \tag{6.99}$$

Using a modular S transformation, we can find the defect partition function

$$Z_{\mathcal{T}/\mathbb{Z}_2}(D_\pi) = \sum_{p=1 \text{ mod } 2, p\in\mathbb{Z}_{2^s}} \chi_p \bar{\chi}_{\bar{p}} + \chi_p \bar{\chi}_{\overline{2^{s-1}+p}} \ , \tag{6.100}$$

Using (6.78), the defect operators are mapped to Pauli group elements as follows

$$Z \leftrightarrow \{\mathcal{O}^{D_\pi}_{p,\bar{p}}\}, \ Y \leftrightarrow \{\mathcal{O}^{D_\pi}_{p+2^{s-1},\bar{p}}\} \ , \tag{6.101}$$

where $p = 1$ mod 2. Therefore, we find that the local operators of the CFT along with the defect operators give us the full Pauli group.

Note that the partition function (6.98) is clearly not an MPMI. In this case we get the non-trivial group $E = \{0, 2^{s-1}\}$ defined in section 6.3.1. Therefore, using (C.27), we can enlarge the chiral algebra as follows.

$$\tilde{\chi}_0 = \chi_0 + \chi_{2^{s-1}}, \ \tilde{\chi}_\rho = \chi_\rho + \chi_{\rho+2^{s-1}} \tag{6.102}$$

where ρ is a a representative of the orbit $\{v, v + 2^{s-1}\}, v = 0 \text{ mod } 2, v \in \mathbb{Z}_{2^s}$. With respect to this enlarged chiral algebra, we have the partition function

$$Z_{\mathcal{T}/\mathbb{Z}_2} = \sum_\rho \tilde{\chi}_\rho \bar{\tilde{\chi}}_{\bar{\rho}} \ . \tag{6.103}$$

Therefore, we have Verlinde lines labelled by the primaries ρ. However, we don't have any non-trivial order-two Verlinde lines. Therefore, the Verlinde subgroup is same as the stabilizer group.

Pauli group from $\widehat{Spin(16)}_1$ CFT

Recall that the $\widehat{Spin(16)}_1$ CFT has the charge-conjugation partition function

$$Z_{\mathcal{T}} = \chi_{(0,0)}\bar{\chi}_{(0,0)} + \chi_{(0,1)}\bar{\chi}_{(0,1)} + \chi_{(1,0)}\bar{\chi}_{(1,0)} + \chi_{(1,1)}\bar{\chi}_{(1,1)} \ , \tag{6.104}$$

and the bulk operators are mapped to the 2-qubit stabilizer code gen$(I \otimes Z, Z \otimes I)$. This CFT has $\mathbb{Z}_2 \times \mathbb{Z}_2$ 0-form symmetry generated by the Verlinde lines $\mathcal{L}_{(0,1)}$ and $\mathcal{L}_{(1,0)}$. Inserting these lines in the partition function, we obtain the following defect partition functions via (6.77)

$$\begin{aligned} Z_{\mathcal{T}}^{(0,1)} &= \chi_{(0,1)}\bar{\chi}_{(0,0)} + \chi_{(0,0)}\bar{\chi}_{(0,1)} + \chi_{(1,1)}\bar{\chi}_{(1,0)} + \chi_{(1,0)}\bar{\chi}_{(1,1)} \ , \\ Z_{\mathcal{T}}^{(1,0)} &= \chi_{(1,0)}\bar{\chi}_{(0,0)} + \chi_{(1,1)}\bar{\chi}_{(0,1)} + \chi_{(0,0)}\bar{\chi}_{(1,0)} + \chi_{(0,1)}\bar{\chi}_{(1,1)} \ , \\ Z_{\mathcal{T}}^{(1,1)} &= \chi_{(1,1)}\bar{\chi}_{(0,0)} + \chi_{(1,0)}\bar{\chi}_{(0,1)} + \chi_{(0,1)}\bar{\chi}_{(1,0)} + \chi_{(0,0)}\bar{\chi}_{(1,1)} \ . \end{aligned} \tag{6.105}$$

Using (6.78), the defect operators are, in turn, mapped to Pauli group elements

$$Z \otimes X, I \otimes X, Z \otimes Y, I \otimes Y \ , \tag{6.106}$$

$$X \otimes Z, Y \otimes Z, X \otimes I, Y \otimes I \ , \tag{6.107}$$

$$Y \otimes Y, X \otimes Y, Y \otimes X, X \otimes X \ . \tag{6.108}$$

Therefore, the bulk operators along with the defect operators give us the full Pauli group $\mathcal{P}_{\mathcal{T}}$. Since all defect operators live at the end of order-two Verlinde lines, the Verlinde subgroup, $\mathcal{V}_{\mathcal{T}}$, is the full Pauli group.

Now let us consider the CFT with partition function

$$Z_{\mathcal{T}/Q_1} = \chi_{(0,0)}\bar{\chi}_{(0,0)} + \chi_{(0,0)}\bar{\chi}_{(0,1)} + \chi_{(0,1)}\bar{\chi}_{(0,0)} + \chi_{(0,1)}\bar{\chi}_{(0,1)} \ , \tag{6.109}$$

obtained from the $\widehat{Spin(16)}_1$ CFT by orbifolding the Q_1 symmetry generated by $\mathcal{L}_{(0,1)}$. Recall that the bulk operators are mapped to the 2-qubit stabilizer code gen$(Z \otimes I, I \otimes X)$. This CFT has order-two symmetries generated by D_{π_1} and D_{π_2}. D_{π_1} acts on the primaries as

$$\begin{aligned} \{\mathcal{O}_{(0,0),(0,1)}\} &\rightarrow -\{\mathcal{O}_{(0,0),(0,1)}\} \ , \\ \{\mathcal{O}_{(0,1),(0,0)}\} &\rightarrow -\{\mathcal{O}_{(0,1),(0,0)}\} \ , \end{aligned} \tag{6.110}$$

and trivially on $\{\mathcal{O}_{(0,0),(0,0)}\}$ and $\{\mathcal{O}_{(0,1),(0,1)}\}$. D_{π_2} acts on the primaries as

$$\begin{aligned} \{\mathcal{O}_{(0,1),(0,0)}\} &\rightarrow -\{\mathcal{O}_{(0,1),(0,0)}\} \text{ and} \\ \{\mathcal{O}_{(0,1),(0,1)}\} &\rightarrow -\{\mathcal{O}_{(0,1),(0,1)}\} \ , \end{aligned} \tag{6.111}$$

and trivially on $\{\mathcal{O}_{(0,0),(0,0)}\}$ and $\{\mathcal{O}_{(0,0),(0,1)}\}$.

Using a modular S transformation, we can find the defect partition functions

$$\begin{aligned} Z_{\mathcal{T}/Q_1}(D_{\pi_1}) &= \chi_{(1,0)}\bar{\chi}_{(1,0)} + \chi_{(1,1)}\bar{\chi}_{(1,1)} + \chi_{(1,1)}\bar{\chi}_{(1,0)} + \chi_{(1,0)}\bar{\chi}_{(1,1)} \ , \\ Z_{\mathcal{T}/Q_2}(D_{\pi_2}) &= \chi_{(1,0)}\bar{\chi}_{(0,0)} + \chi_{(1,0)}\bar{\chi}_{(0,1)} + \chi_{(1,1)}\bar{\chi}_{(0,0)} + \chi_{(1,1)}\bar{\chi}_{(0,1)} \ , \\ Z_{\mathcal{T}/Q_3}(D_{\pi_1\pi_2}) &= \chi_{(0,0)}\bar{\chi}_{(1,0)} + \chi_{(0,0)}\bar{\chi}_{(1,1)} + \chi_{(0,1)}\bar{\chi}_{(1,0)} + \chi_{(0,1)}\bar{\chi}_{(1,1)} \ . \end{aligned} \tag{6.112}$$

Using (6.90), the defect operators are, in turn, mapped to Pauli group elements

$$I \otimes Z, Z \otimes Z, X \otimes Y, I \otimes Y \ , \tag{6.113}$$

$$X \otimes Z, X \otimes Y, Y \otimes Y, Y \otimes Z \ , \tag{6.114}$$

$$X \otimes I, X \otimes X, Y \otimes X, Y \otimes I \ . \tag{6.115}$$

Therefore, the bulk fields along with the defect fields give us the full 2-qubit Pauli group.

The Verlinde subgroup in this case is the same as the stabilizer group. To understand this statement, note that the partition function (6.109) is clearly not an MPMI. In this case we get the non-trivial group $E = \{(0,0),(0,1)\}$ defined in section 6.3.1. Therefore, using (C.27), we can enlarge the chiral algebra as follows.

$$\tilde{\chi}_{\vec{0}} = \chi_{(0,0)} + \chi_{(0,1)} \ . \tag{6.116}$$

With respect to this enlarged chiral algebra, we have

$$Z_{\mathcal{T}/Q_1} = \tilde{\chi}_{\vec{0}}\bar{\tilde{\chi}}_{\vec{0}} \ . \tag{6.117}$$

We get a meromorphic RCFT times an anti-meromorphic RCFT. In this case we don't have any non-trivial Verlinde lines, and $\mathcal{V}_{\mathcal{T}/Q_1} = \mathcal{S}_2$.

6.4 The qubit Hilbert space / CFT Hilbert space map

We have constructed a map that relates the stabilizers and error operations acting on n qubits to an infinite number of genuine local and defect endpoint operators in very general Narain RCFTs. How then should we map the n-qubit Hilbert space, $\mathcal{H}_n$, to the infinite-dimensional CFT Hilbert space?

Let us first consider the code subspace, $\mathcal{C}_n \subset \mathcal{H}_n$. It is defined as the space invariant under the action of the stabilizer group. In our case it is one dimensional. To find the corresponding CFT states, we look for the space which is closed under action of genuine local CFT operators, since these operators correspond to stabilizers under the map μ

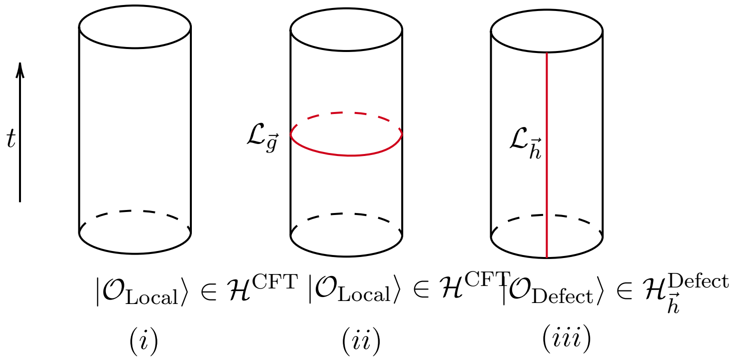


Figure 6.3: The CFT on $S^1 \times \mathbb{R}$: (i) The code subspace maps to the CFT states corresponding to genuine local operators (ii) A CFT logical operation: wrapping the spatial slice with a symmetry defect, $\mathcal{L}_{\vec{g}}$, implements the symmetry on $\mathcal{H}^{\text{CFT}}_{\text{Bulk}}$ (at the level of the code, the logical operation is trivial). (iii) The complement of the code subspace in the n-qubit Hilbert space: a state in the $\mathcal{L}_{\vec{h}}$-defect Hilbert space (here $2\vec{h} = \vec{0}$).

(6.22). By the state-operator correspondence, this is nothing but the CFT Hilbert space

$$\mu(\mathcal{H}^{\text{CFT}}) = \mathcal{C}_n \ . \tag{6.118}$$

Note that, at the level of the CFT Hilbert space, logical operations are non-trivial, but they become trivial after the action of μ (6.118).

Next, what are the $2^n - 1$ states in the complement of $\mathcal{C}_n$ inside the n-qubit Hilbert space on the CFT side? The natural choice is that these correspond to the $2^n - 1$ different defect Hilbert spaces, $\mathcal{H}_i^{\text{Defect}}$, associated with the defect endpoint fields we interpreted as errors in section 6.3,

$$\mu(\mathcal{H}_i^{\text{Defect}}) = \mathcal{C}_n^c := \mathcal{H}_n \backslash \mathcal{C}_n \ . \tag{6.119}$$

The basic property of $\mathcal{C}_n^c$ is that error operations acting on $\mathcal{C}_n$ produce states in the complement. This property is respected by μ: inserting a defect endpoint operator takes us from the bulk CFT Hilbert space to the corresponding defect Hilbert space. We illustrate our proposal (6.118) and (6.119) in Figs. 6.3 (i)-(iii).

6.5 Conclusion

We have proposed a map from very general rational Narain CFTs (including defects), and their associated CS theories, to stabilizer codes. This construction includes the theories discussed in [38,187] as a special case, and provides a CFT picture of the code space states and errors reminiscent of the toric code construction [183].

Our CFT to stabilizer map works as follows. First, we pick a Narain theory with a particular chiral algebra and construct the charge conjugation modular invariant. We then consider all orbifolds by $Q = \mathbb{Z}_2^k$ subgroups of the 0-form flavor symmetry that come from 3d CS 1-form symmetries with vanishing 't Hooft anomalies (this condition ensures the stabilizer group is abelian (6.23)) and relate genuine local operators to stabilizer generators (6.21). Under this map, operators sitting at the ends of line defects are mapped to Pauli operators acting on physical qubits. Accordingly, the whole bulk CFT Hilbert space is mapped to the code subspace (6.118), while defect Hilbert spaces are mapped to the complement of the code subspace in the n-qubit Hilbert space (6.119).

Note that, while the map is unambiguous, it can lead to the same CFT having different codes associated with it because certain CFTs can be considered rational with respect to multiple chiral algebras. For example, the $\widehat{Spin(16)}_1/\mathbb{Z}_2$ orbifolds discussed in section 6.2.1 can be interpreted as corresponding to two different chiral algebras. If we run our map with the smaller chiral algebra $V_{\min} = V_{\widehat{Spin(16)}_1}$, we produce the sequence of RCFT / code relations discussed in the text. On the other hand, if we use maximal chiral algebra, $V_{\max}$, described around (6.116), then the $\widehat{Spin(16)}_1/\mathbb{Z}_2$ orbifolds correspond to trivial 0-qubit codes, as follows from triviality of $\mathrm{Rep}(V_{\max})$, see the discussion below (6.116).

Within our construction, it is natural to ask if we can construct a CFT starting from a given stabilizer code. Since there might be different CFTs related to that code, it is clear that we need extra data. Starting from the stabilizers, we can choose a group Q, and a 2-cocycle, $\sigma \in H^2(Q, U(1))$, compatible with the code. To reconstruct the CFT requires choosing a chiral algebra such that the charge conjugation modular invariant with that chiral algebra admits a non-anomalous 0-form symmetry isomorphic to Q. Taking the Q-orbifold of this CFT with discrete torsion, σ, gives a CFT corresponding to the quantum code in question. An alternative approach is to define a Narain lattice starting from a quantum code. One particular recipe is given by the "new Construction A" of [38], which can be used to construct orbifolds of the charge conjugation modular invariant with $\mathrm{Rep}(V) = A_4^{n_{A_4}}$ for arbitrary integer n_{A_4}. There are, of course, other constructions leading to other CFTs for the same or other codes. For example, the Narain lattice (6.41) for the $SU(2)$ WZW model at level one can be generalized to yield CFTs with $\mathrm{Rep}(V) = A_2^{n_{A_2}}$ for arbitrary integer $n_{A_2} > 0$.

Chapter 7

Conclusion

In this thesis, we focused on the algebraic structure of $2+1$D TQFTs and $1+1$D RCFTs. The symmetries of these QFTs allow for a Lagrangian-independent description which is very useful for studying global aspects of these theories and in producing explicit results. More generally, identifying the algebraic structures in a QFT is often useful for explicit computations and classifications. TQFTs in general dimensions have a purely algebraic description in terms of higher fusion categories [6]. It is conceivable that general QFTs with their defects/operators of various dimensions admit a higher categorical description, though the exact structures that need to be added to the category are unclear [201]. Also, we saw that even though conformal field theories are complex-analytic in nature, obtaining consistent partition functions of an RCFT with a given chiral algebra is a purely algebraic problem [202].

While TQFTs are a rich and mathematically precise arena to explore QFTs, as mentioned in the introduction, it has been recognized that topological operators play a crucial role in describing symmetries of general QFTs. Studying the structure of topological operators of QFTs has led to very interesting generalizations of the notion of symmetry [18–23]. Extending the various algebraic properties of TQFTs studied in this thesis to topological operators of general QFTs is a very interesting problem.

More specifically, in this thesis, we first looked at fusion rules in $2+1$D TQFTs and showed that non-abelian anyons can fuse to give a unique outcome. We saw that such fusions are very special in discrete gauge theories with non-abelian simple gauge groups. One natural question that remains is to better understand to what extent ideas involving non-abelian anyons can be used to prove the AH conjecture (see [203–205] and references therein for interesting recent progress on the AH conjecture). Since discrete gauge theories feature in various physical systems, perhaps we can hope for a physics proof of this conjecture. Theorem (3.3.3) is an example of the irreducible restriction problem for simple groups, in the special case of restriction of irreducible representations to centralizers. It will be interesting to explore its relationship with the

Aschbacher-Scott program [206, 207].

For more general discrete gauge theories and Chern-Simons theories, such fusion can occur more frequently and lead to interesting consequences for the global structure of the theory. In this context, various natural questions arise:

- In the discussion around (3.85) we explained the large hierarchy between the size of simple and non-simple groups whose corresponding discrete gauge theories have non-abelian Wilson lines satisfying (3.2) by using symmetries and subcategory structure. It would be interesting to explore whether other related hierarchies can be explained in a similar way.

- We saw that in almost all the prime untwisted discrete gauge theories we studied, if there was a fusion rule of the form (3.2), then the theory had non-trivial zero-form symmetries. The only exceptions were discrete gauge theories based on the M_{23} and M_{24} Mathieu groups discussed in section 3.4.3. Here we argued that there were zero-form symmetries of the modular data that did not lift to symmetries of the full theory. It would be interesting to understand if gauge theories based on certain finite simple sporadic groups are the only prime theories with fusion rules of the form (3.2) that exhibit this phenomenon.

- In section 3.5.1, we proved that the non-abelian lines of $SU(N)_k$ CS theory don't have fusion rules of the form (3.2). While $(E_7)_2$ CS theory does have such fusion rules, we do not know of an example of such a fusion in a prime G_k CS theory with G a compact and simple Lie group. It would be interesting to either find an example of such a fusion or prove a more general theorem forbidding one. Given such fusions are common for discrete gauge theories, it would be interesting to understand how these two statements interact with each other.

- As we saw in section 3.5.3, it would be useful to develop new tools to understand primality in theories built on cosets. One promising direction is to study the role of Galois actions in such theories.

We also looked at how Galois action related TQFTs with various common properties. We showed that the 0-form, 1-form as well as the 2-group symmetries of a TQFT remain invariant under Galois action. We found that Galois invariant TQFTs are very special in that they can be constructed by gauging 0-form symmetries of very special abelian TQFTs. We showed that other algebraic operations on a TQFT, like gauging and anyon condensation have a natural interplay with Galois action. We also showed that the entanglement entropy of lines in abelian TQFTs is invariant under Galois actions. Various natural questions arise:

- Galois conjugation has played a major role in finding counter examples to the conjecture that the modular data determine a topological phase of matter [76]. A general strategy to use Galois conjugation to find modular isotopes is as follows. Let K_M be the cyclotomic field containing the components of the S and T matrices of an MTC C. Let L be another link invariant and let K_L be the Galois field containing the component of L. If K_L is not the same field extension as K_M, then there exists some element $q \in \mathrm{Gal}(K_L)$ such that the action of q on S and T is trivial, while $q(L) \neq L$. If $q(L)$ and L are not related by a permutation of the anyon labels, then the MTCs C and $q(C)$ are modular isotopes. It would be interesting to explore this direction further.

- Another interesting operation which takes us between TQFTs is Zesting [154]. Like Galois conjugation, zesting can be used to find modular isotopes [208]. The $SU(3)_3$ Chern-Simons theory and its time reversal are related by a Galois conjugation. These two theories are also related by zesting. It would be interesting to explore the relationship between Galois action and zesting, and understand when zesting produces Galois conjugate TQFTs.

- Galois invariant TQFTs are very special, and Theorem 4.12 relates them to discrete gauge theories, the 3-fermion model and $A_p \boxtimes A_p$. However, gauging an arbitrary symmetry of these theories can give us a Galois non-invariant TQFT due to a kind of Galois conjugation-0-form symmetry mixed anomaly. It would be interesting to fully define the Galois conjugation-0-form symmetry anomaly (and the Galois conjugation-anyon condensation anomaly) and give sufficient and necessary criteria for its vanishing.

- We saw that in order to argue that certain symmetries were preserved under Galois conjugation, we needed to make some mild assumptions on the underlying number fields. It would be interesting to understand if these assumptions are ever violated. If so, it would be intriguing to understand if one can think of these situations as representing certain number-theoretical anomalies.

- In 2+1D, discrete gauge theories and quantum groups form two important classes of TQFTs. In contrast, $3+1$D TQFTs are mostly governed by discrete gauge theories. For example, $3+1$D TQFTs with bosonic line operators are known to be classified by $3+1$D discrete gauge theories [209]. These are Drinfeld centres of fusion 2-categories [127], and they have many parallels with $2+1$D discrete gauge theories. This begs the question of how our results generalize to these higher dimensional TQFTs.

- Along with entanglement entropy, complexity and magic are important quantities which characterize link states [210–212]. It will be interesting to analyze the behaviour of these quantities under Galois action.

- Recall that the Witt group of TQFTs [213] may play an important role in the classification of MTCs and related structures. In this construction, two MTCs, $\mathcal{C}_1$ and $\mathcal{C}_2$, are Witt equivalent if they satisfy $\mathcal{C}_1 \boxtimes \mathcal{Z}(A_1) \simeq \mathcal{C}_2 \boxtimes \mathcal{Z}(A_2)$ (where $\mathcal{Z}(\cdots)$ is the Drinfeld center of the enclosed fusion category). It would be interesting to define and explore a notion of "Galois equivalence" of MTCs $\mathcal{C}_{1,2}$. Here we could define $\mathcal{C}_1$ and $\mathcal{C}_2$ to be Galois equivalent if $\mathcal{C}_1 \boxtimes \mathcal{C}'_1 = \mathcal{C}_2 \boxtimes \mathcal{C}'_2$ where $\mathcal{C}'_{1,2}$ are Galois invariant.

Finally, we constructed an explicit map from $1+1$D CFTs and studied how this map interplays with the properties of the corresponding bulk TQFT. We showed that the local operators of the RCFT correspond to the elements of the stabilizer group. The point operators which live at the end of line operators correspond to elements of the Pauli group. This allowed us to give a quantum code theoretic description of orbifolding. Our work opens a number of new directions to explore:

- We have emphasized that different CFTs can be associated with the same code. It is natural to ask if the space of CFTs related to a particular code admits additional structure. One possible idea is to relate these theories by RG flow, or perhaps, some other form of coarse-graining. More broadly, these theories may comprise deformation classes reminiscent of topological modular forms in 2d $\mathcal{N} = (0, 1)$ theories, see e.g. [214–216].

  An alternative idea comes from the example discussed below (6.14), where different CFTs mapping to the same code correspond to CS theories that are related by Galois conjugation [4, 99]. A natural question to ask is if more general Galois transformations always relate theories corresponding to the same code.

  Finally, when a d-dimensional QFT is invariant under gauging a $(d-2)/2$-form symmetry, one finds a non-invertible "duality" defect [22, 23]. In 2d, these defects arise when a theory is invariant under gauging a zero-form symmetry, as in the case of the $R = 1$ compact boson (see also [21]). In this theory, we saw that the codes before and after gauging the $Q = \mathbb{Z}_2$ symmetry are equivalent. The codes before and after gauging are also equivalent for $R = \sqrt{\frac{2}{k}}$ (for $k > 2$) even though the theories are not. This result begs the question of whether code equivalences correspond, in the absence of an equivalence under gauging, to the existence of more general defects.

- The construction of Chapter 6 can be extended in many possible ways. In the discussion below (6.9), the factors of A_{q^r} and B_{q^r} in (6.8) are mapped into trivial (zero qubit) codes. Quite naturally, these factors can be associated with qudit codes with $d = q$, where $d = 2$ is the qubit case [217]. Another possible generalization comes from the choice of orbifold group, Q, in (6.20) and, implicitly, a choice of stabilizer in (6.21) for RCFTs corresponding to CS theories with E_{2^r} and F_{2^r} factors. Yet another natural generalization would be to include theories with non-abelian fusion rules. In this way, one may hope to extend our construction to all RCFTs. Going in a different direction, general CFT relations to codes are likely to extend beyond RCFTs to include non-rational "finite" theories [218].

The broad program we are advocating here is to identify a generalization of codes which can be associated with general 2d CFTs.

- Relations to codes provide a powerful way to write CFT torus partition functions in terms of code enumerator polynomials. This relation applies to all CFTs discussed in Chapter 6 and can be extended to higher-genus partition functions [219]. In this way, modular bootstrap constraints can be reformulated in terms of much simpler algebraic properties of enumerator polynomials, leading to a new approach to the modular bootstrap [187]. Our work emphasized the importance of defects in the context of codes. We therefore surmise that codes will prove useful as a new tool for the program of bootstrapping CFTs with defects (e.g., see [21]). Since defects are also closely related to boundaries, we expect codes to have direct implications for bootstrapping in the presence of boundaries [220]. Intriguingly, conformal boundaries are also related to gapped boundaries of the bulk TQFT [221]. Therefore, it will be interesting to explore the role of quantum codes in describing and classifying gapped boundaries as in [103].

- The physical meaning of quantum codes outlined in Chapter 6, namely that the code subspace is related to the Hilbert space of CFT local operators, while errors correspond to defect endpoint operators, has a natural holographic interpretation. Our theories are dual to 3d CS, where the code subspace and errors have a clear geometric meaning. We raise the question of making an explicit connection with the quantum codes, which define the space of low-energy bulk states in the context of holographic quantum gravity [182].

- Finally, it will be interesting to extend our CFT to quantum stabilizer code map to fermionic CFTs. Since fermionic CFTs are related to bosonic CFTs through gauging [222] [223], this should lead to some interesting relationships between quantum codes corresponding to fermionic CFTs and quantum codes discussed in this thesis.

Appendix A

Examples of $a \times b = c$ fusion and GAP codes

A.1 Wilson line $a \times b = c$ in gauge theories with order forty-eight discrete gauge group

Let us study groups of order 48 for which the corresponding discrete gauge theories have Wilson line $a \times b = c$ type fusions[148].

(48, 15) $((\mathbb{Z}_3 \times D_8) \rtimes \mathbb{Z}_2)$;

$$\begin{aligned} \mathcal{W}_{2_2} \times \mathcal{W}_{2_4} &= \mathcal{W}_4 \ , \ \mathcal{W}_{2_2} \times \mathcal{W}_{2_5} = \mathcal{W}_4 \ , \ \mathcal{W}_{2_3} \times \mathcal{W}_{2_4} = \mathcal{W}_4 \ , \ \mathcal{W}_{2_3} \times \mathcal{W}_{2_5} = \mathcal{W}_4 \\ \mathcal{W}_{2_4} \times \mathcal{W}_{2_6} &= \mathcal{W}_4 \ , \ \mathcal{W}_{2_4} \times \mathcal{W}_{2_7} = \mathcal{W}_4 \ , \ \mathcal{W}_{2_5} \times \mathcal{W}_{2_6} = \mathcal{W}_4 \ , \ \mathcal{W}_{2_5} \times \mathcal{W}_{2_7} = \mathcal{W}_4 \end{aligned} \tag{A.1}$$

We have $\text{Out}((\mathbb{Z}_3 \times D_8) \rtimes \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2$. Let r_1 and r_2 be the generators of this group. They act on the Wilson lines involved in the fusion above as follows

$$r_1 : \mathcal{W}_{2_2} \leftrightarrow \mathcal{W}_{2_2}; \ \mathcal{W}_{2_3} \leftrightarrow \mathcal{W}_{2_3}; \ \mathcal{W}_{2_4} \leftrightarrow \mathcal{W}_{2_4}; \ \mathcal{W}_{2_5} \leftrightarrow \mathcal{W}_{2_5}; \ \mathcal{W}_{2_6} \leftrightarrow \mathcal{W}_{2_7}; \tag{A.2}$$

$$r_1 : \mathcal{W}_{2_2} \leftrightarrow \mathcal{W}_{2_2}; \ \mathcal{W}_{2_3} \leftrightarrow \mathcal{W}_{2_3}; \ \mathcal{W}_{2_4} \leftrightarrow \mathcal{W}_{2_5}; \ \mathcal{W}_{2_6} \leftrightarrow \mathcal{W}_{2_6}; \ \mathcal{W}_{2_7} \leftrightarrow \mathcal{W}_{2_7}; \tag{A.3}$$

Since this group has complex characters we also have a non-trivial quasi-zero-form symmetry given by complex conjugation. $\mathcal{Z}(\text{Vec}_{(\mathbb{Z}_3 \times D_8) \rtimes \mathbb{Z}_2})$ also has all other $a \times b = c$ type fusions (involving fluxes and dyons) discussed in this appendix.

(48, 16) $((\mathbb{Z}_3 : Q_8) \rtimes \mathbb{Z}_2)$; This has fusions identical to (A.1). The only difference is that now $\mathcal{W}_{2_4}$ and $\mathcal{W}_{2_5}$ are conjugates. The outer automorphism group and symmetry

[148] We won't discuss the direct product groups $S_3 \times S_3, D_8 \times S_3$ and $Q_8 \times S_3$ which also have such fusions (the corresponding discrete gauge theories factorize). Since we have already discussed the case of BOG and $GL(2, 3)$, we won't be discussing them here

action is identical to $\mathcal{Z}(\text{Vec}_{(\mathbb{Z}_3\times D_8)\rtimes\mathbb{Z}_2})$. Since this group has complex characters we also have a non-trivial quasi-zero-form symmetry given by complex conjugation. We additionally have all other $a \times b = c$ type fusions (involving fluxes and dyons) discussed in this appendix.

$(48, 17)$ $((\mathbb{Z}_3 \times Q_8) \rtimes \mathbb{Z}_2)$; This has identical character table to $(48, 16)$, so same fusion rules. The properties are identical to the two cases above.

$(48, 18)$ $(\mathbb{Z}_3 \rtimes Q_{16})$; Identical characters to $(48, 15)$, so shares (A.1). The discussion is identical to the case above.

$(48, 39)$ $((\mathbb{Z}_4 \times S_3) \rtimes \mathbb{Z}_2)$;

$$\begin{aligned}\mathcal{W}_{2_1} \times \mathcal{W}_{2_5} &= \mathcal{W}_4 \ , \ \mathcal{W}_{2_1} \times \mathcal{W}_{2_6} = \mathcal{W}_4 \ , \ \mathcal{W}_{2_2} \times \mathcal{W}_{2_5} = \mathcal{W}_4 \ , \ \mathcal{W}_{2_2} \times \mathcal{W}_{2_6} = \mathcal{W}_4 \\ \mathcal{W}_{2_3} \times \mathcal{W}_{2_5} &= \mathcal{W}_4 \ , \ \mathcal{W}_{2_3} \times \mathcal{W}_{2_6} = \mathcal{W}_4 \ , \ \mathcal{W}_{2_4} \times \mathcal{W}_{2_5} = \mathcal{W}_4 \ , \ \mathcal{W}_{2_4} \times \mathcal{W}_{2_6} = \mathcal{W}_4\end{aligned} \tag{A.4}$$

We have $\text{Out}((\mathbb{Z}_4 \times S_3) \rtimes \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2$. Let r_1 and r_2 be the generators of this group. They act on the Wilson lines involved in the fusion above as follows

$$r_1 : \mathcal{W}_{2_1} \leftrightarrow \mathcal{W}_{2_1};\ \mathcal{W}_{2_2} \leftrightarrow \mathcal{W}_{2_2};\ \mathcal{W}_{2_3} \leftrightarrow \mathcal{W}_{2_3};\ \mathcal{W}_{2_4} \leftrightarrow \mathcal{W}_{2_4};\ \mathcal{W}_{2_5} \leftrightarrow \mathcal{W}_{2_6}; \tag{A.5}$$

$$r_1 : \mathcal{W}_{2_1} \leftrightarrow \mathcal{W}_{2_2};\ \mathcal{W}_{2_3} \leftrightarrow \mathcal{W}_{2_3};\ \mathcal{W}_{2_4} \leftrightarrow \mathcal{W}_{2_4};\ \mathcal{W}_{2_5} \leftrightarrow \mathcal{W}_{2_5};\ \mathcal{W}_{2_6} \leftrightarrow \mathcal{W}_{2_6}; \tag{A.6}$$

Since this group has complex characters we also have a non-trivial quasi-zero-form symmetry given by complex conjugation. $\mathcal{Z}(\text{Vec}_{(\mathbb{Z}_4\times S_3)\rtimes\mathbb{Z}_2})$ also have all other $a \times b = c$ type fusions (involving fluxes and dyons) discussed in this appendix.

$(48, 41)$; $((\mathbb{Z}_4 \times S_3) \rtimes \mathbb{Z}_2)$

Fusion of Wilson lines giving unique output is same as (A.4). We have $\text{Out}((\mathbb{Z}_4 \times S_3) \rtimes \mathbb{Z}_2) = D_{12}$.

Since this group has complex characters we also have a non-trivial quasi-zero-form symmetry given by complex conjugation. $\mathcal{Z}(\text{Vec}_{(\mathbb{Z}_4\times S_3)\rtimes\mathbb{Z}_2})$ also have all other $a \times b = c$ type fusions (involving fluxes and dyons) discussed in this appendix.

A.2 Genuine zero-form symmetries and quasi-zero-form symmetries in A_9 discrete gauge theory

Recall from section 3.4 that A_9 is the simplest example of an A_N (with $N = k^2 \geq 9$) discrete gauge theory with fusion rules involving non-abelian Wilson lines having unique outcome. Here our goal is to disentangle the genuine zero form symmetries

$$\mathrm{Aut}^{\mathrm{br}}(\mathcal{Z}(\mathrm{Vec}_{A_9})) \simeq H^2(A_9, U(1)) \rtimes \mathrm{Out}(A_9) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \ , \tag{A.7}$$

from a charge conjugation quasi zero-form symmetry [77].

Let us first discuss the outer automorphisms. To that end, recall that A_9 has an outer automorphism corresponding to conjugation by odd elements of $S_9 \triangleright A_9$. Acting with the outer automorphism generated by $(89) \in S_9$, we see that the following lines are exchanged

$$\mathcal{L}_{([(123456789)],\pi_p)} \leftrightarrow \mathcal{L}_{([(123456798)],\pi_p)} \ , \quad \mathcal{L}_{([(12345)(678)],\pi_n)} \leftrightarrow \mathcal{L}_{([(12345)(679)],\pi_n)} \ , \tag{A.8}$$

where the relevant conjugacy classes are listed in table A.1, and $0 \leq p \leq 8$, $0 \leq n \leq 14$ label representations of the corresponding $\mathbb{Z}_9$ and $\mathbb{Z}_{14}$ centralizers (they are also listed in table A.1).

In fact, as described in the main text, the symmetry in (A.8) generates an action on some of the Wilson lines involved in (3.64)

$$\mathcal{W}_{[3^3]_+} \leftrightarrow \mathcal{W}_{[3^3]_-} \ . \tag{A.9}$$

This action can be read off from the character table of A_9 or, equivalently, from the braiding

$$\begin{aligned}
\frac{S_{\mathcal{W}_{[3^3]_+}\mathcal{L}_{([(12345)(678)],\pi_n)}}}{S_{\mathcal{W}_1\mathcal{L}_{([(12345)(678)],\pi_n)}}} &= \chi_{[3^3]_+}([(12345)(678)])^* = -\frac{1}{2}(1 - i\sqrt{15}) \ , \\
\frac{S_{\mathcal{W}_{[3^3]_-}\mathcal{L}_{([(12345)(678)],\pi_n)}}}{S_{\mathcal{W}_1\mathcal{L}_{([(12345)(678)],\pi_n)}}} &= \chi_{[3^3]_-}([(12345)(678)])^* = -\frac{1}{2}(1 + i\sqrt{15}) \ , \\
\frac{S_{\mathcal{W}_{[3^3]_+}\mathcal{L}_{([(12345)(679)],\pi_n)}}}{S_{\mathcal{W}_1\mathcal{L}_{([(12345)(679)],\pi_n)}}} &= \chi_{[3^3]_+}([(12345)(679)])^* = -\frac{1}{2}(1 + i\sqrt{15}) \ , \\
\frac{S_{\mathcal{W}_{[3^3]_-}\mathcal{L}_{([(12345)(679)],\pi_n)}}}{S_{\mathcal{W}_1\mathcal{L}_{([(12345)(679)],\pi_n)}}} &= \chi_{[3^3]_-}([(12345)(679)])^* = -\frac{1}{2}(1 - i\sqrt{15}) \ .
\end{aligned} \tag{A.10}$$

Note that, since the $[(12345)(678)]$ and $[12345)(679)]$ conjugacy classes are complex, we also have a non-trivial $\mathbb{Z}_2$ charge conjugation that acts on the modular data and swaps $\mathcal{W}_{[3^3]_+} \leftrightarrow \mathcal{W}_{[3^3]_-}$ and $\mathcal{L}_{([(123456789)],\pi_p)} \leftrightarrow \mathcal{L}_{([(123456798)],\pi_p)}$. Recall from the discussion in (3.101) that elements of $H^2(A_9, U(1)) \simeq \mathbb{Z}_2$ act trivially on the Wilson lines. Hence, we learn that charge conjugation cannot be a genuine symmetry of the TQFT (this statement is also confirmed by the analysis in [77]).

However, this is not a contradiction with what we have written, because $\mathrm{Out}(A_9)$ also interchanges the real conjugacy classes $[(123456789)]$ and $[(123456798)]$ along with the corresponding lines in (A.8). Since charge conjugation leaves these degrees of

Conjugacy class	Length	Centralizer
1	1	A_9
[(12)(34)]	378	SmallGroup(480, 951)
[(12)(34)(56)(78)]	945	SmallGroup(192, 1493)
[(123)]	168	SmallGroup(1080, 487)
[(123)(45)(67)]	7560	SmallGroup(24, 10) , $(D_8 \times \mathbb{Z}_3)$
[(123)(456)]	3360	SmallGroup(54, 13)
[(123)(456)(789)]	2240	SmallGroup(81, 7) , $((\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3)$
[(1234)(56)]	7560	SmallGroup(24, 5) , $(S_3 \times \mathbb{Z}_4)$
[(1234)(567)(89)]	15120	SmallGroup(12, 2) , $(\mathbb{Z}_{12})$
[(1234)(5678)]	11340	SmallGroup(16, 13) , (central product D_8, Z_4)
[(12345)]	3024	SmallGroup(60, 9)
[(12345)(67)(89)]	9072	SmallGroup(20, 5) , $(\mathbb{Z}_{10} \times \mathbb{Z}_2)$
[(12345)(678)]	12096	SmallGroup(15, 1) , $(\mathbb{Z}_{15})$
[(12345)(679)]	12096	SmallGroup(15, 1) , $(\mathbb{Z}_{15})$
[(123456)(78)]	30240	SmallGroup(6, 2) , $(\mathbb{Z}_6)$
[(1234567)]	25920	SmallGroup(7, 1) , $(\mathbb{Z}_7)$
[(123456789)]	20160	SmallGroup(9, 1) , $(\mathbb{Z}_9)$
[(123456798)]	20160	SmallGroup(9, 1) , $(\mathbb{Z}_9)$

Table A.1: The eighteen conjugacy classes of A_9, their order, and their centralizers (recall that the centralizers of elements in the same conjugacy class are isomorphic). The centralizer is labeled by its GAP ID (for sufficiently small groups) as "SmallGroup(a, b)" along with a more common name in certain cases.

freedom untouched, it is a distinct operation.

Note that in the A_9 discrete gauge theory we can also turn on a large variety of twists

$$\omega \in H^3(A_9, U(1)) \simeq \mathbb{Z}_2 \times \mathbb{Z}_3^2 \times \mathbb{Z}_4 \simeq \mathbb{Z}_6 \times \mathbb{Z}_{12} \ . \tag{A.11}$$

Since the charge conjugation quasi-symmetry is a property of the Wilson line fusion rules, it remains regardless of the twist.

A.3 GAP code

The following GAP code defines the function checkdyon() which takes in a group as an argument. It checks for $a \times b = c$ type fusions for non-abelian anyons $a, b, c \in \mathcal{Z}(\text{Vec}_G)$ and ouputs all such fusions. Moreover, if such fusions exist, it outputs $\text{Out}(G)$ as well as $H^2(G, U(1))$. Note that it requires the package HAP to function.

In order to define checkdyon() we need to first define the functions comconj() and conjprof().

```
> conjcom:=function(a,b)
> local com,i,j;
> com:=[];
> for i in [1..Size(AsList(a))] do
> for j in [i..Size(AsList(b))] do
> Append(com, [AsList(a)[i]*AsList(b)[j]*Inverse(AsList(b)[j]*AsList(a)[i])]);
> od; od;
> return DuplicateFreeList(com)=[AsList(a)[1]*Inverse(AsList(a)[1])]; end;
```

This function takes two conjugacy classes of a group G as inputs and outputs true if they commute element-wise and false otherwise. Now, let us define the function conjprod()

```
> conjprod:=function(a,b,c)
> local prod,i,j,k;
> prod:=[];
> for i in [1..Size(AsList(a))] do
> for j in [i..Size(AsList(b))] do
> for k in [1..Size(c)] do
> if AsList(a)[i]*AsList(b)[j] in AsList(c[k]) then
> Append(prod, [k]); break; fi; od; od; od;
> if Size(DuplicateFreeList(prod))=1 then
> return DuplicateFreeList(prod)[1]; else return 0; fi; end;
```

This function takes three arguments. The first two arguments a, b are two conjugacy classes of a group G and the third argument c is the set of all conjugacy classes of G. The function outputs an integer $k > 1$ if the product of two input conjugacy is a single conjugacy class (which is at position k in the list of conjugacy classes c). The function outputs 0 otherwise.

Using these two functions, we finally define the checkdyon() function.

```
checkdyon:=function(G)
> local cn,i,j,k,a,l,cen1,cen2,cen3,cenint,irrcenint,irrcen1,irrcen2,irrcen3,
cen1res,cen2res,cen3res,x,y,z,w,a1,a2,A,I,F,R;
> cn:=ConjugacyClasses(G);
> a:=0;
> for i in [1..Size(cn)] do
> for j in [i..Size(cn)] do
> if conjcom(cn[i],cn[j]) then
> k:=conjprod(cn[i],cn[j],cn);
> if k<>0 then
> cen1:=Centralizer(G,AsList(cn[i])[1]);
> cen2:=Centralizer(G,AsList(cn[j])[1]);
> cen3:=Centralizer(G,AsList(cn[k])[1]);
> cenint:=Intersection(cen1,cen2,cen3);
> irrcen1:=Irr(cen1);
> irrcen2:=Irr(cen2);
> irrcen3:=Irr(cen3);
> cen1res:=RestrictedClassFunctions(irrcen1,cenint);
> cen2res:=RestrictedClassFunctions(irrcen2,cenint);
> cen3res:=RestrictedClassFunctions(irrcen3,cenint);
> irrcenint:=Irr(cenint);
> for x in [1..Size(cen1res)] do
> for y in [1..Size(cen2res)] do
> if Size(AsList(cn[i]))*DegreeOfCharacter(cen1res[x])>1 and
Size(AsList(cn[j]))*DegreeOfCharacter(cen2res[y])>1 then
> for z in [1..Size(cen3res)] do
> a1:=[ ]; a2:=[ ];
> for w in [1..Size(irrcenint)] do
> Append(a1,[ScalarProduct(irrcenint[w],cen1res[x]*cen2res[y])]);
> Append(a2,[ScalarProduct(irrcenint[w],cen3res[z])]);
> od;
```

```
> if a1*a2=1 and
Size(AsList(cn[i]))*DegreeOfCharacter(cen1res[x])*
Size(AsList(cn[j]))*DegreeOfCharacter(cen2res[y])=
Size(AsList(cn[k]))*DegreeOfCharacter(cen3res[z]) then
> a:=1;
> Print(IdSmallGroup(G), " ", StructureDescription(G), "\n");
> Print("Anyon a: ", cn[i], " , ", irrcen1[x], "\n");
> Print("Anyon b: ", cn[j], " , ", irrcen2[y], "\n");
> Print("Anyon c: ", cn[k], " , ", irrcen3[z], "\n","\n");
> fi; od; fi; od;od; fi; fi; od; od;
> if a=1 then
> A:=AutomorphismGroup(G);
> I:=InnerAutomorphismsAutomorphismGroup(A);
> F:=FactorGroup(A,I);
> Print("Out(G): ",StructureDescription(F), "\n");
> R:=ResolutionFiniteGroup(G,3);
> Print("H2(G,U(1)): ",Homology(TensorWithIntegers(R),2),"\n");
> Print("\n","\n"); fi;
> end;
```

Appendix B

Entanglement entropy of hyperbolic links and proofs

B.1 Entanglement entropy of 2-links in abelian TQFTs

In the main text, we derived the link state for a 2-link in a general abelian CS theory using the K-matrix formalism. Here we will obtain an explicit expression for the entanglement entropy of this state as in (5.36). For the purposes of this computation, it will be useful to choose a particular basis for the lattice, $\mathbb{Z}^N/K\mathbb{Z}^N$.

Claim B.1.1 *The set of vectors $(a_1, \cdots, a_N)$ where $a_i \in \mathbb{Z}_{n_i}, 1 \leq i \leq N$, is a basis set for the lattice $\mathbb{Z}^N/UKU^T\mathbb{Z}^N$. As a result, these vectors label the anyons (we will call this basis the "Smith basis"). Here U and W are matrices which satisfy $K_S = UKW$, where K_S is the Smith normal form of K.*

Proof: Except for the zero vector, every vector of the type $\vec{a} = (a_1, \cdots, a_N)$ where $a_i \in \mathbb{Z}_{n_i}, 1 \leq i \leq N$, satisfies

$$\vec{a} \neq K_S\vec{n} \ , \tag{B.1}$$

for any $\vec{n} \in \mathbb{Z}^N/K\mathbb{Z}^N$. Let U and W be invertible matrices over the integers such that

$$K_S = UKW \ . \tag{B.2}$$

Then,

$$\begin{aligned} \vec{a} &\neq UKW\vec{n} \ , \\ \vec{a} &\neq UKU^T(U^T)^{-1}W\vec{n} \ , \\ \vec{a} &\neq UKU^T\vec{n}' \ , \end{aligned} \tag{B.3}$$

where $\vec{n}' = (U^T)^{-1}W\vec{n}$. Given that U and W are invertible over the integers, for any

$\vec{n} \in \mathbb{Z}^N$ we have a unique $\vec{n}' \in \mathbb{Z}^N$. Thus,

$$\vec{a} \neq UKU^T \vec{n}' \ , \tag{B.4}$$

for any $\vec{n}'$. □

This result means that the above choice of vectors are not linear combinations of columns of UKU^T. Since this statement is also true for differences of vectors of the above type, they are all independent and form a basis for the anyons as long as we take the level matrix to be UKU^T. Note that the TQFT corresponding to UKU^T is the same as that corresponding to K, because it corresponds to a change of gauge fields $\vec{A} \to U^T \vec{A}$ where $\vec{A}$ is the vector of gauge fields, A^i, contained in the action.

The upshot of the above argument is that the Smith basis can be used to label the anyons as long as we take UKU^T as the level matrix of the theory. Next we will see the implication for the entanglement entropy of the theory.

To that end, the reduced density matrix of a 2-link is given by

$$\rho_{\text{red}} = \frac{1}{|\mathcal{A}|^2} \sum_{\vec{j_1}, \vec{h_1}} \sum_{\vec{m}} \left(B(\vec{j_1}, \vec{m})\right)^{l_{12}} \left(B(\vec{h_1}, \vec{m})\right)^{-l_{12}} \left|\vec{j_1}\right\rangle \left\langle \vec{h_1}\right| \ . \tag{B.5}$$

Using (5.22), we can write the components of the reduced density matrix $\rho_{\text{red}_{\vec{j_1},\vec{h_1}}}$ as

$$\begin{aligned} \rho_{\text{red}_{\vec{j_1},\vec{h_1}}} &= \frac{1}{|\mathcal{A}|^2} \sum_{\vec{m}} e^{2\pi i l_{12}(\vec{j_1}-\vec{h_1})K^{-1}\vec{m}} \\ &= \frac{1}{|\mathcal{A}|^2} \sum_{\vec{m}} e^{2\pi i l_{12}(\vec{j_1}-\vec{h_1})K^{-1}\vec{m}} \cdot 1 \\ &= \frac{1}{|\mathcal{A}|^2} \sum_{\vec{m}} e^{2\pi i l_{12}(\vec{j_1}-\vec{h_1})K^{-1}\vec{m}} \cdot e^{-2\pi i l_{12}\vec{m}K^{-1}K\vec{\beta}} \\ &= \frac{1}{|\mathcal{A}|} \delta_{l_{12}(\vec{j_1}-\vec{h_1}),K\vec{\beta}} \ , \end{aligned} \tag{B.6}$$

for some vector $\vec{\beta} \in \mathbb{Z}^n$.

Let us now calculate the m^{th} Rényi entropy

$$\begin{aligned} S_m(\mathcal{L}^2) &= \frac{1}{1-m} \ln \operatorname{tr}(\rho^m) \\ &= \frac{1}{1-m} \ln\Bigg(\sum_{\vec{a_1},\vec{a_2},\cdots \vec{a_m}} \rho_{\vec{a_1},\vec{a_2}} \rho_{\vec{a_2},\vec{a_3}} \cdots \rho_{\vec{a_m},\vec{a_1}} \Bigg) \end{aligned} \tag{B.7}$$

$$= \frac{1}{1-m} \ln\Bigg(\frac{1}{|\mathcal{A}|^m} \sum_{\vec{a_1},\vec{a_2},\cdots \vec{a_m}} \delta_{l_{12}(\vec{a_1}-\vec{a_2}),K\vec{\beta}} \delta_{l_{12}(\vec{a_2}-\vec{a_3}),K\vec{\beta}} \cdots \delta_{l_{12}(\vec{a_m}-\vec{a_1}),K\vec{\beta}} \Bigg) \ .$$

In order to simplify the above expression, we have to calculate the number of vectors in the basis which satisfy $l_{12}(\vec{a_1} - \vec{a_2}) = K\vec{\beta}$ for some $\vec{\beta}$. For simplicity, let us choose the basis to be the Smith basis for which we have to take the level matrix to be UKU^T. Let us find the number of solutions of $l_{12}\vec{a} = UKU^T\vec{\beta}$, where $\vec{a}$ belongs to the Smith basis and $\vec{\beta}$ is an arbitrary vector. This equation is the same as $l_{12}\vec{a} = K_s\vec{\beta}'$, where $\vec{\beta}' = W^{-1}U^T\vec{\beta}$. The matrices U and W are uni-modular and satisfy $K_S = UKW$, K_s being the Smith normal form of K. This reasoning gives us a set of equations

$$l_{12}a_1 = n_1\beta_1^{'}; l_{12}a_2 = n_2\beta_2^{'}; \cdots ; l_{12}a_N = n_N\beta_N^{'} \ , \tag{B.8}$$

where a_i and β_i are components of $\vec{a}$ and $\vec{\beta}$, respectively, and n_i are the diagonal elements of K_s. These quantities can also be written as

$$a_1 = 0 \bmod \frac{n_1}{\gcd(l_{12}, n_1)} \ , \ a_2 = 0 \bmod \frac{n_2}{\gcd(l_{12}, n_2)} \ , \ \cdots \ , a_N = 0 \bmod \frac{n_N}{\gcd(l_{12}, n_N)} \ . \tag{B.9}$$

Since $a_i \in \{0, 1, \cdots, n_i - 1\}$, the solutions of the above equations can be parametrized as

$$a_1 = \frac{r_1 n_1}{\gcd(l_{12}, n_1)} \ , \ a_2 = \frac{r_2 n_2}{\gcd(l_{12}, n_2)} \ , \ \cdots \ , a_N = \frac{r_N n_N}{\gcd(l_{12}, n_N)} \ , \tag{B.10}$$

where $r_i \in \{0, 1, \cdots, \gcd(l_{12}, n_i)\}$. Hence, the number of $\vec{a}$ which satisfy $l_{12}\vec{a} = K_s\vec{\beta}'$ is given by $\prod_{i=1}^{N} \gcd(l_{12}, n_i)$. Similarly, for a given vector, $\vec{a_2}$, in the Smith basis, the number of $\vec{a_1}$ which satisfy $l_{12}(\vec{a_1} - \vec{a_2}) = K\vec{\beta}$ for some $\vec{\beta}$ is $\prod_{i=1}^{N} \gcd(l_{12}, n_i)$. Using this result, the n^{th} Rényi entropy can be written as

$$\begin{aligned} S_m(\mathcal{L}^2) &= \frac{1}{1-m}\ln\left(\frac{1}{|\mathcal{A}|^m}\sum_{\vec{a_1}} \delta_{\vec{a_1},\vec{a_1}}(\gcd(l_{12}, n_1)\gcd(l_{12}, n_2)\cdots\gcd(l_{12}, n_N))^{m-1}\right) \\ &= \frac{1}{1-m}\ln\left(\frac{1}{|\mathcal{A}|^{m-1}}(\gcd(l_{12}, n_1)\gcd(l_{12}, n_2)\cdots\gcd(l_{12}, n_N))^{m-1}\right) \\ &= \ln\left(\frac{\det(K)}{\gcd(l_{12}, n_1)\gcd(l_{12}, n_2)\cdots\gcd(l_{12}, n_N)}\right) . \end{aligned} \tag{B.11}$$

As a result, the entanglement entropy of a 2-link in a general abelian theory with level matrix K is given by

$$S_{\text{vN}}(\mathcal{L}^2) = \ln\left(\frac{\det(K)}{\gcd(l_{12}, n_1)\gcd(l_{12}, n_2)\cdots\gcd(l_{12}, n_N)}\right) . \tag{B.12}$$

B.2 Results for hyperbolic and satellite link complements

Knots and links are classified into three types: torus, hyperbolic, and satellite. In abelian theories, all three kinds of links have invariant entanglement entropy under the action of the Galois group. Motivated by the special role that the modular generator

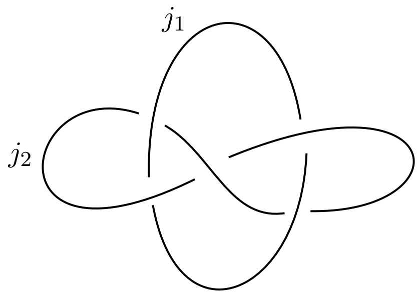


Figure B.1: Whitehead Link

S plays in this result, we looked at torus links in non-abelian TQFTs and analyzed the behavior of their entanglement entropy under Galois conjugation. Given that an infinite subset of these links have Galois invariant entanglement entropy, it is natural to ask whether similar results hold in the case of hyperbolic and satellite links.

It turns out that, in general, the entanglement entropy of hyperbolic links are different in two TQFTs related by Galois conjugation. For example, the Whitehead link is one of the simplest hyperbolic links in the sense of having just two components and minimal hyperbolic volume (for a two cusped hyperbolic manifold). Even for this link the entanglement entropy changes under Galois conjugation. We verify this statement in $su(2)_k$ CS theory for small k.

The link state for the Whitehead link in $su(2)_k$ Chern-Simons theory can be found using its link invariant [224, 225]

$$C(j_1,j_2)_{5^2_1} = \sum_{i=0}^{\min(2j_1,2j_2)} q^{-\frac{i(i+3)}{4}}(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^{3i}\frac{[2j_1+i+1]![2j_2+i+1]![i]!}{[2j_1-i]![2j_2-i]![2i+1]!}\ , \tag{B.13}$$

where $[x] = \frac{q^{\frac{x}{2}}-q^{\frac{-x}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}$, $[x]! = [x][x-1]\cdots[1]$ and $q = e^{\frac{2\pi i}{k+2}}$. In $su(2)_3$ Chern-Simons theory the entanglement entropy and its Galois conjugations are given by

$$5^2_{1\,su(2)_3} = \begin{pmatrix} 0.762866 & 0.237134 & 0 & 0 \\ 0.925325 & 0.0746746 & 0 & 0 \\ 0.925325 & 0.0746746 & 0 & 0 \\ 0.762866 & 0.237134 & 0 & 0 \end{pmatrix}. \tag{B.14}$$

The columns are labelled by the integrable representations, $0, 1, 2, 3,$[149] and the rows are labelled by the Galois conjugations corresponding to $1, 2, 3, 4 \in \mathbb{Z}_5^\times$.

Let us now consider satellite links. Examples of such links include connected sums

[149]We label representations by the Dynkin label (i.e., twice the spin).

of links. If a link $\mathcal{L}$ is a connected sum of $\mathcal{L}_1$ and $\mathcal{L}_2$, then their invariants satisfy [12]

$$C_{\mathcal{L}} \cdot C_O(i) = C_{\mathcal{L}_1} \cdot C_{\mathcal{L}_2} \ , \tag{B.15}$$

where the links $\mathcal{L}$, $\mathcal{L}_1$, and $\mathcal{L}_2$ are to be labelled in a consistent manner. $C_O(i)$ is the knot invariant of the unknot labelled by i in S^3 and i is the label of the knot which is cut to obtain $\mathcal{L}_1$ and $\mathcal{L}_2$ from $\mathcal{L}$. This implies that the entanglement entropy of most satellite links will change non-trivially under Galois conjugation. For example, let us consider the link which is obtained from a connected sum of the Trefoil knot and the Whitehead link.

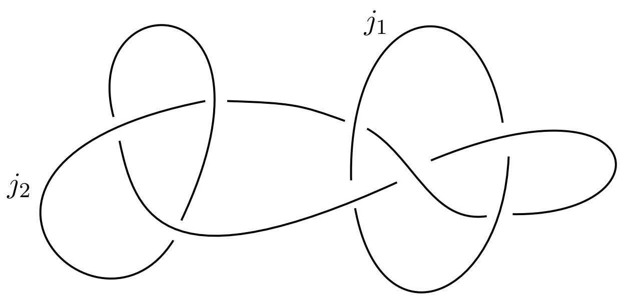


Figure B.2: Connected sum of Trefoil and Whitehead Link

The knot invariant for the Trefoil knot in $su(2)_k$ Chern-Simons theory is given by [224] [225]

$$C_{3_1}(j_1) = \sum_{i}^{2j_1} (-1)^i q^{-i(i+3)} (q - q^{-1})^{2i} \frac{[2j_1 + i + 1]!}{[2j_1 - i]!} \ , \tag{B.16}$$

where the definitions of q and $[x]$ are the same as above. Using (B.15), the link state and its entanglement entropy can be calculated for the connected sum of Trefoil and Whitehead link. In $su(2)_3$, the eigenvalues of the reduced density matrix for this link and its behaviour under Galois conjugation is given by

$$8^2_{1\,su(2)_3} = \begin{pmatrix} 0.988779 & 0.0112213 & 0 & 0 \\ 0.972184 & 0.0278156 & 0 & 0 \\ 0.972184 & 0.0278156 & 0 & 0 \\ 0.988779 & 0.0112213 & 0 & 0 \end{pmatrix} , \tag{B.17}$$

where the columns are labelled by the integrable representations $0, 1, 2, 3$, and the rows are labelled by the Galois conjugations corresponding to $1, 2, 3, 4 \in \mathbb{Z}_5^\times$.

For a few additional levels, we have checked the Galois conjugation properties of

k	2	3	4	5	6	7	8
5_1^2	blue	red	blue	red	red	red	red
8_1^2	blue	red	blue	red	red	red	red

Figure B.3: The MEEs for the 5_1^2 and 8_1^2 links are Galois non-invariant in most of the theories we checked. As in Fig. 5.6, blue squares correspond to Galois invariant MEE while red squares correspond to Galois non-invariant MEE. In contrast to Fig. 5.6, there are a lot more red squares.

the MEE in Fig. B.3. Note that there are many more non-invariant theories in this case than in the torus link case checked in Fig. 5.6.

B.3 Proof of Lemma 5.4.2

In this appendix, we prove the following Lemma:

Lemma: 5.4.2 *The eigenvalues of the reduced density matrix of the* (QM, QP) *torus link are given by*

$$\Lambda_\ell = \frac{1}{(S_{0\ell})^{2Q-2}S_{00}^2}\sum_i S_{\ell i}\langle W_i^{(M,P)}\rangle_{S^2\times S^1}\sum_j S_{\ell j}\langle W_j^{(-P,M)}\rangle_{S^2\times S^1} \tag{B.18}$$

Proof: Using (5.67) and (5.58), we have

$$\begin{aligned}\Lambda_\ell &= \frac{1}{S_{0\ell}^{2Q-2}}\left|\sum_i S_{\ell i}\frac{\langle 0|\, SW_i^{(M,P)}\,|0\rangle}{\langle 0|\, S\,|0\rangle}\right|^2 \\ &= \frac{1}{S_{0\ell}^{2Q-2}}\left|\sum_i S_{\ell i}\frac{\langle 0|\, SF^{(M,P)^{-1}}W_i^{(1,0)}F^{(M,P)}\,|0\rangle}{\langle 0|\, S\,|0\rangle}\right|^2 \qquad \text{(from (5.66))} \\ &= \frac{1}{S_{0\ell}^{2Q-2}}\left|\frac{\langle 0|\, SF^{(M,P)^{-1}}\sum_i S_{\ell i}N_iF^{(M,P)}\,|0\rangle}{\langle 0|\, S\,|0\rangle}\right|^2 \qquad \text{(from (5.61))}\ .\end{aligned} \tag{B.19}$$

Using (2.17), we can simplify this expression to obtain

$$\Lambda_\ell = \frac{1}{(S_{0\ell})^{2Q}S_{00}^2}\left|(S(F^{(M,P)})^{-1}S^{-1})_{0\ell^*}(SF^{(M,P)})_{\ell^*0}\right|^2 \tag{B.20}$$

Since S and $F^{(M,P)}$ are unitary, we have

$$\Lambda_\ell = \frac{1}{(S_{0\ell})^{2Q}S_{00}^2}(S(F^{(M,P)})^{-1}S^{-1})_{0\ell^*}(SF^{(M,P)}S^{-1})_{\ell^*0}(SF^{(M,P)})_{\ell^*0}((F^{(M,P)})^{-1}S^{-1})_{0\ell^*}\ . \tag{B.21}$$

To further simplify the above expression, note that, from (2.17)

$$\begin{aligned}\sum_i S_{\ell i} \langle 0| W_i^{(M,P)} |0\rangle &= \sum_i S_{\ell i}((F^{(M,P)})^{-1} W_i^{(1,0)} F^{(M,P)})_{00} \\ &= \frac{((F^{(M,P)})^{-1} S^{-1})_{0\ell^*} (S F^{(M,P)})_{\ell^* 0}}{S_{0\ell}} ,\end{aligned} \tag{B.22}$$

and

$$\begin{aligned}\sum_j S_{\ell j} \langle 0| W_j^{(-P,M)} |0\rangle &= \sum_j S_{\ell j} \langle 0| S W_j^{(M,P)} S^{-1} |0\rangle \\ &= \sum_j S_{\ell j} (S (F^{(M,P)})^{-1} W_j^{(1,0)} F^{(M,P)} S^{-1})_{00} \\ &= \frac{(S (F^{(M,P)})^{-1} S^{-1})_{0\ell^*} (S F^{(M,P)} S^{-1})_{\ell^* 0}}{S_{0\ell}} .\end{aligned} \tag{B.23}$$

Using these equations, we can write the expression for the eigenvalues in (B.21) as

$$\Lambda_\ell = \frac{1}{(S_{0\ell})^{2Q-2} S_{00}^2} \sum_i S_{\ell i} \langle W_i^{(M,P)} \rangle_{S^2 \times S^1} \sum_j S_{\ell j} \langle W_j^{(-P,M)} \rangle_{S^2 \times S^1} . \tag{B.24}$$

□

Appendix C

Partition function of orbifold CFTs and Verlinde subgroup

C.1 S and Ξ are bicharacters

In this appendix, we will show that both S and Ξ are bicharacters. To prove this, we need the following equations satisfied by $F(\vec{p},\vec{q},\vec{r})$ and $R(\vec{p},\vec{q})$.

$$\begin{aligned}\frac{F(\vec{q},\vec{p},\vec{r})}{F(\vec{p},\vec{q},\vec{r})F(\vec{q},\vec{r},\vec{p})} &= \frac{R(\vec{p},\vec{q}+\vec{r})}{R(\vec{p},\vec{q})R(\vec{p},\vec{r})} \ , \\ \frac{F(\vec{p},\vec{q},\vec{r})F(\vec{r},\vec{p},\vec{q})}{F(\vec{p},\vec{r},\vec{q})} &= \frac{R(\vec{p}+\vec{q},\vec{r})}{R(\vec{p},\vec{r})R(\vec{q},\vec{r})} \ .\end{aligned} \tag{C.1}$$

These are known as the Hexagon equations [45]. The modular S matrix can be written in terms of R as

$$S_{\vec{p},\vec{q}} = R(\vec{p},\vec{q})R(\vec{q},\vec{p}) \ . \tag{C.2}$$

We have

$$S_{\vec{p},\vec{q}}S_{\vec{p},\vec{r}} = R(\vec{p},\vec{q})R(\vec{q},\vec{p})R(\vec{p},\vec{r})R(\vec{r},\vec{p}) \ = R(\vec{p},\vec{q}+\vec{r})R(\vec{q}+\vec{r},\vec{p}) = S_{\vec{p},\vec{q}+\vec{r}} \ , \tag{C.3}$$

where in the second equality we used (C.1). A similar argument can be used to show that $S_{\vec{p},\vec{r}}S_{\vec{q},\vec{r}} = S_{\vec{p}+\vec{q},\vec{r}}$. This shows that the modular S matrix is a bicharacter.

Consider the expression for Ξ in terms of R, the 2-cochain τ and the 2-cocycle σ.

$$\Xi(\vec{g},\vec{h}) = R(\vec{g},\vec{h})\frac{\tau(\vec{g},\vec{h})\sigma(\vec{g},\vec{h})}{\tau(\vec{h},\vec{g})\sigma(\vec{h},\vec{g})} \ . \tag{C.4}$$

Recall that Ξ is defined on a subgroup Q of K on which F is trivial in cohomology. In fact, we can choose a gauge in which $F(\vec{g},\vec{h},\vec{k}) = 1 \ \forall \vec{g},\vec{h},\vec{k} \in Q$. Then $\tau(\vec{g},\vec{h})$ can be set to 1 for all $\vec{g},\vec{h}\in Q$. Therefore, we have

$$\Xi(\vec{g},\vec{h})\Xi(\vec{g},\vec{k}) = R(\vec{g},\vec{h})R(\vec{g},\vec{k})\frac{\sigma(\vec{g},\vec{h})}{\sigma(\vec{h},\vec{g})}\frac{\sigma(\vec{g},\vec{k})}{\sigma(\vec{k},\vec{g})} = R(\vec{g},\vec{h}+\vec{k})\frac{\sigma(\vec{g},\vec{h}+\vec{k})}{\sigma(\vec{h}+\vec{k},\vec{g})} = \Xi(\vec{g},\vec{h}+\vec{k}) \tag{C.5}$$

where in the second equality above we used the property that for any 2-cocycle σ, $\frac{\sigma(\vec{g},\vec{h})}{\sigma(\vec{h},\vec{g})}$ is a bicharacter. A similar argument can be used to show that $\Xi(\vec{g},\vec{k})\Xi(\vec{h},\vec{k}) = \Xi(\vec{g}+\vec{h},\vec{k})$. This shows that Ξ is a bicharacter.

C.2 Properties of $Z_{\mathcal{T}/Q,[\sigma]}$

Let us discuss some properties of $Z_{\mathcal{T}/Q,[\sigma]}$ which will be useful for our arguments. To that end, consider the general expression for $Z_{\mathcal{T}/Q,[\sigma]}$.

$$Z_{\mathcal{T}/Q,[\sigma]} = \sum_{\vec{g}\in Q}\sum_{\vec{p}\in B_{\vec{g}}}\chi_{\vec{p}}(q)\bar{\chi}_{\overline{\vec{p}+\vec{g}}}(q)\ , \tag{C.6}$$

where

$$B_{\vec{g}} := \left\{\vec{p}\ \middle|\ S_{\vec{h},\vec{p}}\,\Xi(\vec{h},\vec{g}) = 1\ ,\ \forall\vec{h}\in Q\right\}\ . \tag{C.7}$$

A basic observation is that these partition functions are of the form

$$Z_{\mathcal{T}/Q,[\sigma]} = \sum_{\vec{p},\vec{q}}\mathcal{M}_{\vec{p}\vec{q}}\chi_{\vec{p}}(q)\bar{\chi}_{\vec{q}}(\bar{q})\ , \tag{C.8}$$

where $\mathcal{M}_{\vec{p}\vec{q}}$ is a modular invariant matrix with entries consisting of 0's and 1's. Indeed, if

$$\chi_{\vec{p}}\bar{\chi}_{\overline{\vec{p}+\vec{g}}} = \chi_{\vec{q}}\bar{\chi}_{\overline{\vec{q}+\vec{h}}}\ , \tag{C.9}$$

then we should have $\vec{p}=\vec{q}$ and $\vec{p}+\vec{g}=\vec{q}+\vec{h}$ which implies that $\vec{g}=\vec{h}$. Therefore, the non-trivial terms contribute to the partition function without multiplicity.

Now let us discuss some properties of the set $B_{\vec{g}}$. For any $\vec{g}$, the set $B_{\vec{g}}$ is non-empty. To see this, let K be the group defined in equation (6.8). Let $\{e_i\}$ be a set of generators of this group. Let $\vec{h}\in Q\lhd K$ be the vector denoting an element of K in the basis $\{e_i\}$. Let $\{f_i\}$ be a basis of Q. Then we have

$$f_i = \sum_j L_{ij}e_j\ , \tag{C.10}$$

for some integer matrix L with non-negative entries. We will focus on $Q=\mathbb{Z}_2^k$. Therefore, the non-trivial entries of L_{ij} have the form 2^{r_j-1}. Let $\vec{h}_Q$ be the vector $\vec{h}$ written in the basis $\{f_i\}$. Then we have

$$\vec{h} = L^T\vec{h}_Q\ . \tag{C.11}$$

We introduced the basis $\{f_i\}$ because Ξ has a simple description in this basis. It can always be written as

$$\Xi(\vec{h}_Q, \vec{g}_Q) = e^{\pi i \vec{h}_Q^T X g_Q} \ , \tag{C.12}$$

where X is a symmetric integer matrix with diagonal entries equal to 1 [190]. Now, we have

$$S_{\vec{h},\vec{p}}\ \Xi(\vec{h}, \vec{g}) = e^{\pi i \vec{h} M A \vec{p}} e^{\pi i \vec{h}_Q^T X g_Q} = e^{\pi i \vec{h}_Q^T L M A \vec{p}} e^{\pi i \vec{h}_Q^T X g_Q} \ . \tag{C.13}$$

Therefore, the constraint (C.7) can be simplified to get

$$h_Q^T (LMA\vec{p} + X\vec{g}_Q) = 0 \text{ mod } 2\ \forall \vec{h}_Q \in \mathbb{Z}_2^k \ . \tag{C.14}$$

We get

$$LMA\vec{p} = \vec{\alpha} - X\vec{g}_Q \ . \tag{C.15}$$

where $\vec{\alpha}$ satisfies $\vec{h}_Q \cdot \vec{\alpha} = 0$ mod 2 $\forall \vec{h}_Q \in \mathbb{Z}_2^k$. This equation always has a solution since LMA is a full rank matrix. Therefore, we find that $B_{\vec{g}}$ is a non-empty set for all $\vec{g}$.

Let us look at how $B_{\vec{g}}$ are related to $B_{\vec{0}}$. For $\vec{g} = \vec{0}$, the constraint (C.7) reduces to

$$S_{\vec{h},\vec{p}} = 1\ \forall \vec{h} \in \mathbb{Z}_2^k \ . \tag{C.16}$$

$B_{\vec{0}}$ is the the set of solutions to this constraint. In the bulk TQFT, solutions to this constraint are the Wilson lines which braid trivially with all $\vec{h} \in \mathbb{Z}_2^k$. Using Theorem 3.2 in [49], we have

$$|B_{\vec{0}}| = \frac{|K|}{2^k} \ . \tag{C.17}$$

Now, given some solution $\vec{p} \in B_{\vec{g}}$, $\vec{p}+\vec{q} \in B_{\vec{g}}$ where $\vec{q} \in B_{\vec{0}}$. Moreover, given $\vec{p}_1, \vec{p}_2 \in B_g$, we have

$$\begin{aligned} S_{\vec{h},\vec{p}_1}\ \Xi(\vec{h}, \vec{g}) = 1 = S_{\vec{h},\vec{p}_2}\ \Xi(\vec{h}, \vec{g})\ \forall \vec{h} \in \mathbb{Z}_2^k \\ \implies S_{\vec{h},\vec{p}_1 - \vec{p}_2} = 1\ \forall \vec{h} \in \mathbb{Z}_2^k \ . \end{aligned} \tag{C.18}$$

Therefore, $\vec{p}_1 - \vec{p}_2$ belongs to $B_{\vec{0}}$. This shows that given some $\vec{p} \in B_{\vec{g}}$, all other elements of $B_{\vec{g}}$ are of the form $\vec{p} + \vec{q}$ where $\vec{q} \in B_{\vec{0}}$. Therefore, we have

$$|B_{\vec{g}}| = |B_{\vec{0}}| = \frac{|K|}{2^k} \ . \tag{C.19}$$

This argument implies that the total number of terms in the partition function $Z_{\mathcal{T}/Q,[\sigma]}$ is always $|\mathbb{Z}_2^k| \otimes \frac{|K|}{2^k} = |K|$. Therefore, the map (6.21) gives us a code with 2^n elements. Hence, the stabilizer code corresponding to the partition function $Z_{\mathcal{T}/Q,[\sigma]}$ is self-dual.

C.3 Permutation modular invariants and Non-degenerate Ξ

In this appendix, we prove the following claim:

Claim C.3.1 *A code CFT is an MPMI if and only if $\Xi(\vec{g},\vec{h})$, defined in (6.19), is non-degenerate.*

To understand this claim, let us consider the general expression for the partition function $Z_{\mathcal{T}/Q,[\sigma]}$.

$$Z_{\mathcal{T}/Q,[\sigma]} = \sum_{\vec{g}\in Q}\sum_{\vec{p}\in B_{\vec{g}}} \chi_{\vec{p}}(q)\bar{\chi}_{\overline{\vec{p}+\vec{g}}}(q) \ , \tag{C.20}$$

where

$$B_{\vec{g}} := \left\{\vec{p}\ \middle|\ S_{\vec{h},\vec{p}}\,\Xi(\vec{h},\vec{g}) = 1 \ , \ \forall\vec{h}\in Q\right\} \ . \tag{C.21}$$

For the partition function to be given by a permutation modular invariant, we know that $\vec{p}$ as well as $\vec{g}+\vec{p}$ should not repeat in the terms of the partition function. Moreover, $\vec{p}$ should take values in all representations of the chiral algebra. Therefore, it is clear that we need to satisfy the constraint $B_{\vec{g}}\cap B_{\vec{h}} = \emptyset$ for $\vec{h}\neq\vec{g}\in\mathbb{Z}_2^k$.

Let us restrict our attention to the case of partition functions which admit a qubit quantum code description. Then we know that the 1-form symmetry $Q=\mathbb{Z}_2^k$ of the bulk TQFT should be anomaly free. Therefore, if $\vec{p}\in B_g$, then $\vec{p}+\vec{g}\in B_{\vec{g}}$. This follows from

$$S_{\vec{h},\vec{p}+\vec{g}} = S_{\vec{h},\vec{p}}S_{\vec{h},\vec{g}} = S_{\vec{h},\vec{p}} \ , \tag{C.22}$$

where we have used the fact that S is a bicharacter and $S_{\vec{h},\vec{g}} = 1\ \forall\vec{h}$ since $\mathbb{Z}_2^k$ is anomaly free. Therefore, if $B_{\vec{g}}\cap B_{\vec{h}} = \emptyset$ for $\vec{h}\neq\vec{g}\in\mathbb{Z}_2^k$, then $\vec{g}+\vec{p}$ cannot be the solution to (C.21) for some $\vec{h}\neq g$. Therefore, $\vec{g}+\vec{p}$ also does not repeat for different terms in the partition function. This fact, along with (C.19), then also guarantees that $\vec{p}$ takes values in all representations.

Therefore, we find that it is necessary and sufficient to satisfy the constraint

$$B_{\vec{g}}\cap B_{\vec{h}} = \emptyset \text{ for } \vec{h}\neq\vec{g}\in\mathbb{Z}_2^k \tag{C.23}$$

to have a permutation modular invariant. It is clear from (C.21) that if $\Xi(\vec{h},\vec{g}) = \Xi(\vec{h},\vec{l})\ \forall\vec{h}\in\mathbb{Z}_2^k$, then $B_{\vec{g}} = \vec{B}_{\vec{l}}$. Also, suppose $\vec{p}$ belongs to both $B_{\vec{g}}$ and $B_{\vec{l}}$. Then using (C.21), we find that $\Xi(\vec{h},\vec{g}) = \Xi(\vec{h},\vec{l})\ \forall\ \vec{h}\in\mathbb{Z}_2^k$. Therefore, satisfying (C.23) is the same as having a non-degenerate $\Xi(\vec{g},\vec{h})$.

C.4 The Verlinde subgroup

Using the results in Appendix C.3, we know that a non-permutation modular invariant necessarily leads to states of the form $(\vec{0}, \bar{\vec{g}})$ where $\vec{g} \neq \vec{0}$. The states $(\vec{0}, \bar{\vec{g}})$ form a group under fusion we call $E \simeq Z_2^t$. In this appendix, we will discuss how we can extend the chiral algebra using E to get a permutation modular invariant. Then we will discuss how this gives symmetries generated by Verlinde lines which are used to construct the Verlinde subgroup.

To that end, let $\vec{\gamma}$ denote a representative of the orbit $\{\vec{\gamma} + \vec{b} | \vec{b} \in E\}$ and $\vec{\gamma} \in Q$. Now, since $\Xi(\vec{h}, \vec{a}) = 1$ for any $\vec{a} \in E$ and $\vec{h} \in \mathbb{Z}_2^k$, $B_{\vec{g}} = B_{\vec{a}+\vec{g}}$. That is, $B_{\vec{g}}$ only depends on the E-orbit of $\vec{g}$. Therefore

$$Z_{\mathcal{T}/Q,[\sigma]} = \sum_{\vec{\gamma}} \sum_{\vec{p} \in B_{\vec{\gamma}}} \sum_{\vec{b} \in E} \chi_{\vec{p}}(q) \overline{\chi}_{\overline{\vec{p}+\vec{\gamma}+\vec{b}}}(\bar{q}) \ , \tag{C.24}$$

where the subscript on $B_{\vec{\gamma}}$ indicates that the set of elements in $B_{\vec{g}}$ only depends on the E-orbit of $\vec{g}$.

For a given $\vec{g}$ and $\vec{p} \in B_{\vec{g}}$, $\vec{p} + \vec{a}$, for any $\vec{a} \in E$, also belongs to $B_{\vec{g}}$. This statement follows from that fact that $\vec{a}, \vec{g} \in Q$ braid trivially with each other. Therefore, we can put the elements of B_g in orbits under the action of E. Let $\vec{\rho}$ denote the representative of an orbit $\{\vec{p} + \vec{a} | \vec{a} \in E\}$ and $\vec{p} \in B_{\vec{g}}$. Then the partition function becomes

$$Z_{\mathcal{T}/Q,[\sigma]} = \sum_{\vec{\gamma}} \sum_{\vec{\rho} \in B_{\vec{\gamma}}} \sum_{\vec{b} \in E} \sum_{\vec{a} \in E} \chi_{\vec{\rho}+\vec{a}} \overline{\chi}_{\overline{\vec{\rho}+\vec{a}+\vec{\gamma}+\vec{b}}} \ . \tag{C.25}$$

In writing this, we have split the sum over $\vec{p}$ for a given $\vec{g}$ into a sum over E orbits. We know that $\vec{a} + \vec{b}$ is also an element of E. Since we are summing over all elements in the group E, we can change variables and obtain

$$Z_{\mathcal{T}/Q,[\sigma]} = \sum_{\vec{\gamma}} \sum_{\vec{\rho} \in B_{\vec{\gamma}}} \sum_{\vec{b} \in E} \sum_{\vec{a} \in E} \chi_{\vec{\rho}+\vec{a}} \overline{\chi}_{\overline{\vec{\rho}+\vec{\gamma}+\vec{b}}} = \sum_{\vec{\gamma}} \sum_{\vec{\rho} \in B_{\vec{\gamma}}} \left(\sum_{\vec{a} \in E} \chi_{\vec{\rho}+\vec{a}} \right) \left(\sum_{\vec{b} \in E} \overline{\chi}_{\overline{\vec{\rho}+\vec{\gamma}+\vec{b}}} \right) \ . \tag{C.26}$$

Therefore, we can enlarge the chiral algebra where the vaccum character of the new chiral algebra is given by $\tilde{\chi}_{\vec{0}} := \sum_{\vec{a} \in E} \chi_{\vec{a}}$ and, more generally

$$\tilde{\chi}_{\vec{\rho}} := \sum_{\vec{a} \in E} \chi_{\vec{\rho}+\vec{a}} \ . \tag{C.27}$$

Then the partition function becomes

$$Z_{\mathcal{T}/Q,[\sigma]} = \sum_{\vec{\gamma}} \sum_{\vec{\rho} \in B_{\vec{\gamma}}} \tilde{\chi}_{\vec{\rho}} \overline{\tilde{\chi}}_{\overline{\vec{\rho}+\vec{\gamma}}} \ . \tag{C.28}$$

In fact, this is again a permutation modular invariant. To see this, let $\vec{\delta}$ and $\vec{\epsilon}$ lie in two distinct E-orbits. Then the sets $B_{\vec{\delta}}$ and $B_{\vec{\epsilon}}$ have no common elements since otherwise (using the bi-character nature of Ξ)

$$\Xi(\vec{h},\vec{\delta}) = \Xi(\vec{h},\vec{\epsilon}) \ , \forall \vec{h} \in \mathbb{Z}_2^k \ \Rightarrow \ \Xi(\vec{h},\vec{\delta}+\vec{\epsilon}) = 1 \ . \tag{C.29}$$

Therefore, $\vec{\delta}+\vec{\epsilon}$ would be an element of E which would imply that $\vec{\delta}$ and $\vec{\epsilon}$ are in the same E-orbit (a contradiction). Therefore, for every $\vec{\gamma}$, the sum over $\vec{\rho}$ is over elements which do not repeat for any $\vec{\eta} \neq \vec{\gamma}$. Also, we know that $\vec{\rho}+\vec{\gamma} \in B_{\vec{\gamma}}$ if $\vec{\rho} \in B_{\vec{\gamma}}$. As a result, in (C.28), the values of $\overline{\vec{\rho}+\vec{\gamma}}$ do no repeat either. In other words, after enlarging the chiral algebra, we end up with a permutation modular invariant theory with respect to this new chiral algebra.

It is now clear that we have Verlinde lines labelled by primaries with respect to the enlarged chiral algebra. Then, consider the following defect partition function

$$Z^{\vec{\zeta}}_{\mathcal{T}/Q,[\sigma]} = \sum_{\vec{\gamma}} \sum_{\vec{\rho}\in B_{\vec{\gamma}}} \tilde{\chi}_{\vec{\rho}+\vec{\zeta}} \bar{\tilde{\chi}}_{\overline{\vec{\rho}+\vec{\gamma}}} \ . \tag{C.30}$$

To get a map to the corresponding code elements, it is easier to use (C.27) and substitute

$$\begin{aligned} Z^{\vec{\zeta}}_{\mathcal{T}/Q,[\sigma]} &= \sum_{\vec{\gamma}} \sum_{\vec{\rho}\in B_{\vec{\gamma}}} \left(\sum_{\vec{a}\in E} \chi_{\vec{\rho}+\vec{\zeta}+\vec{a}}\right)\left(\sum_{\vec{b}\in E} \chi_{\vec{\rho}+\vec{\gamma}+\vec{b}}\right) \\ &= \sum_{\vec{\gamma}} \sum_{\vec{\rho}\in B_{\vec{\gamma}}} \sum_{\vec{a},\vec{b}\in E} \chi_{(\vec{\rho}+\vec{a})+\vec{\zeta}} \chi_{(\vec{\rho}+\vec{a})+\vec{\gamma}+\vec{a}+\vec{b}} \ . \end{aligned} \tag{C.31}$$

When we sum over $\vec{a} \in E$, the term $\vec{\rho}+\vec{a}$ runs over the E-orbit of $\vec{\rho} \in B_{\vec{\gamma}}$. Also, the term $\vec{a}+\vec{b}$ is just a permutation of $\vec{b}$. Since we are summing over all $\vec{b} \in E$ as well, we can simplify the expression above to get

$$Z^{\vec{\zeta}}_{\mathcal{T}/Q,[\sigma]} = \sum_{\vec{\gamma}} \sum_{\vec{p}\in B_{\vec{\gamma}}} \sum_{\vec{b}\in E} \chi_{\vec{p}+\vec{\zeta}} \chi_{\vec{p}+\vec{\gamma}+\vec{b}} = \sum_{\vec{g}} \sum_{\vec{p}\in B_{\vec{g}}} \chi_{\vec{p}+\vec{\zeta}} \chi_{\vec{p}+\vec{g}} \ . \tag{C.32}$$

Note that $\vec{\zeta}$ need not be an order-two element of the MTC of the original chiral algebra, even though it may be an order-two element in the MTC of the extended chiral algebra. In fact, if $\vec{\zeta}$ is not an order-two element of the original MTC, then we cannot relate the defect operators $\{\mathcal{O}^{\vec{\zeta}}_{\vec{p}+\vec{\zeta},\vec{p}+\vec{g}}\}$ to a Pauli group element. If $\vec{\zeta}$ is order two, then from the terms in (C.32), we get the Pauli group elements

$$\{\mathcal{O}^{\vec{\zeta}}_{\vec{p}+\vec{\zeta},\vec{p}+\vec{g}}\} \leftrightarrow X^{M(\vec{g}+\vec{\zeta})} \circ Z^{A(\vec{p}+\vec{\zeta})} \ . \tag{C.33}$$

Note that here $\vec{p} \in B_{\vec{g}}$ is not independent of $\vec{g}$. In general, our RCFTs will have other sources of order-two lines that furnish the remainder of the Pauli group (as discussed in Section 6.3). In the extreme example of theories that are modular-invariant holomorphic RCFTs times modular-invariant anti-holomorphic RCFTs, all order-two lines are non-Verlinde lines.

Since the Verlinde subgroup $\mathcal{V}_{\mathcal{T}/Q}$ is formed by order two elements, it is isomorphic to $\mathbb{Z}_2^{N_v}$. Here N_v is the number of Pauli group elements obtained from the defect partition functions (C.32). In general $|\mathcal{V}_{\mathcal{T}/Q}|$ will depend on the choice of the group Q by which we orbifold the CFT with the charge-conjugation partition function to get $Z_{\mathcal{T}/Q,[\sigma]}$. But when the group K defined in (6.8) is such that $n_{A_2} = n_{B_2} = n_{C_2} = n_{D_2} = n_{E_2} = n_{F_2} = 0$, then we can find a general expression for $|\mathcal{V}_{\mathcal{T}/Q}|$. This constraint is the same as imposing that K does not have any $\mathbb{Z}_2$ factors. Note that we also ignore decoupled CFT factors described by A_{q^r} and B_{q^r}.

Consider the general expression of the S matrix $S_{\vec{p},\vec{q}} = e^{\frac{2\pi i}{2}\vec{p}^T MA\vec{q}}$. Consider an element $\vec{p} \in B_{\vec{0}}$ which satisfies

$$S_{\vec{h},\vec{p}} = 1 \forall \vec{h} \in Q = \mathbb{Z}_2^k \implies \vec{h}^T MA\vec{p} = 0 \bmod 2 \ . \tag{C.34}$$

Note that since $\vec{h}$ is an order two vector, $h^T M$ is an integer vector. Moreover, $\vec{h}$ has even components. A is also an integer matrix by definition. Let $\vec{p}$ be an order two vector. Then, it has even components. This follows from our assumption that K does not have any $\mathbb{Z}_2$ factors. Therefore, any order two vector satisfies the constraint (C.34). That is, all the 2^n distinct order two elements belong to $B_{\vec{0}}$, where n is the number of qubits in the corresponding quantum code or equivalently the length of the vector $\vec{p}$.

When we enlarge the chiral algebra to obtain a permutation modular invariant, these 2^n order two elements are put into orbits under the group E. Each such orbit defines a Verlinde line whose defect partition function gives 2^n Pauli group elements. This follows form the fact that the partition function itself gives 2^n distinct stabilizer elements, as we showed in Appendix C.2. Therefore, the size of the Verlinde subgroup is

$$2^{n-t} \times 2^n \ , \tag{C.35}$$

where $|E| = 2^t$. If the Schellekens algebra gives a permutation modular invariant, $t = 0$ and the Verlinde subgroup has size $2^n \times 2^n = 4^n$. Therefore, we get the full Pauli group.

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