Multiple Mixing and Local Rank Group Actions

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Abstract

Let $r \geq 1$; the Følner sequences $\{F^{n,1}\}_{n=1}^{\infty}, \{F^{n,2}\}_{n=1}^{\infty}, \ldots, \{F^{n,r}\}_{n=1}^{\infty}$ satisfy the *bounded intersection property* if there is a constant p such that for any $n \in \mathbf{N}$ and $1 \leq i \leq r$ each $F^{n,i}$ can intersect no more than p disjoint translates of $F^{n,1}, F^{n,2}, \ldots F^{n,r}$. They have *comparable magnitudes* if $0 < \underline{\lim}_n \frac{|F^{n,i}|}{|F^{n,j}|} < \infty$ for $1 \leq i, j \leq r$. Suppose G is a countable, Abelian group with an element of infinite order and let $\mathcal{X}$ be a mixing finite (or β-local, with $\beta > 1/2$) rank action of G on a probability space. Suppose further that the Følner sequences $\{F^{n,1}\}_{n=1}^{\infty}, \{F^{n,2}\}_{n=1}^{\infty}, \ldots, \{F^{n,r}\}_{n=1}^{\infty}$ indexing the r towers of the finite rank $\mathcal{X}$ (or the Følner sequence $\{F^{n}\}_{n=1}^{\infty}$ indexing the tower of the β-local rank $\mathcal{X}$) satisfy the bounded intersection property, and have comparable heights. Then $\mathcal{X}$ is mixing of all orders. We follow Ryzhikov's joining technique in our proof: the main theorem follows from showing that any pairwise independent joining of k copies of $\mathcal{X}$ is necessarily product measure.

1 Introduction

A mixing group action $(X, \mathcal{B}, \mu, G)$ is not generally multiply mixing ([?]); but if any pairwise independent joining of $(X, \mathcal{B}, \mu, G)$ is actually independent, then multiple mixing follows. In this paper we discuss this problem for certain mixing finite rank and local rank group actions. Kalikow [?] showed that rank one mixing transformations were 3-mixing, and Host [?] proved that mixing transformations with singular spectrum are mixing of all orders. Ryzhikov [?]

and [?] shows that finite rank and β-local ($\beta > 1/2$) mixing transformations are mixing of all orders by showing that pairwise independent self-joinings of the given system are necessarily product measure. Here we generalize Ryzhikov's result to certain finite rank and β-local rank group actions. The author would like to thank J. Choksi and A. del Junco, and especially I. Klemes, for several discussions.

2 Preliminaries

Throughout this paper G will denote a countable Abelian group. Let $\mathcal{X} = (X, \mathcal{B}, \mu, G)$ and $\mathcal{Y} = (Y, \mathcal{F}, \nu, G)$ be finite measure-preserving G–actions. To each element $g \in G$ there corresponds a measure-preserving transformation $T_g : X \to X$; however we will mostly use g to denote both the element of the group, the measure-preserving transformation it represents, and the unitary operator induced by g on $\mathcal{L}_2(X)$. A sequence $\{F^n\}_{n=1}^{\infty}$ of finite subsets of G is *Følner* if $\forall g \in G$,

$$\lim_{n\to\infty} \frac{|gF^n \Delta F^n|}{|F^n|} = 0\,.$$

For $i = 1, \ldots, r$, let $\{F^{n,i}\}_{n=1}^{\infty} = F^{1,i}, F^{2,i}, F^{3,i}, \ldots$ be a Følner sequence; we will write $\{F^{n,i}\}$ instead of $\{F^{n,i}\}_{n=1}^{\infty}$. We say that the r Følner sequences $\{F^{n,1}\}, \{F^{n,2}\}, \ldots, \{F^{n,r}\}$ have the bounded intersection property if there exists $p \in \mathbf{N}$ such that for each $n \in \mathbf{N}$ and $1 \le j \le r$, at most p disjoint translates of $F^{n,1}, F^{n,2}, \ldots, F^{n,r}$ can nontrivially intersect $F^{n,j}$. We will assume that the Følner sequence(s) defining our action $(X, \mathcal{B}, \mu, G)$ have the bounded intersection property. We will call such a p an *intersection bound* for the Følner sequences $\{F^{n,1}\}, \{F^{n,2}\}, \ldots, \{F^{n,r}\}$. Examples of groups which have natural Følner sequences satisfying this condition are $\mathbf{Z}^n$, countable direct sums of finite cyclic groups and direct sums of these two cases.

Furthermore, we will have to assume that the Følner sequences have comparable magnitude: i.e. that

$$0 < \underline{\lim}_{n} \frac{|F^{n,i}|}{|F^{n,j}|} < \infty\,. \tag{1}$$

for $1 \le i, j \le r$. Henceforth we will assume that the Følner sequences we are dealing with have the bounded intersection property, and also that (**??**) is true.

The Følner sequence $\{F^n\}$ satisfies the *Tempel'man condition* if $F^n \subset F^{n+1}$ for each n and there exists $K \in \mathbf{N}$ such that $|F^n(F^n)^{-1}| \le K|F^n|$ for all $n \in \mathbf{N}$. We mention the Tempel'man condition ([?]) only because it seems to be closely related to the bounded intersection property, in that, in all the standard examples we know of, they stand or fall together. However we do not know if it is possible to derive one from the other.

It is not difficult to find Følner sequences $\{F^n\}$ in $\mathbf{Z}$ which do not have the bounded intersection property. We give an example of a sequence $\{F^n\}$ so that n disjoint translates of F^n, say $F^n + t_1, \ldots, F^n + t_n$, intersect F^n nontrivially. F^n will consist of $n+1$ intervals, all but the last having cardinality n. These intervals will be separated by gaps of length β_i where

$$\beta_1 >> n^2 \tag{2}$$

and

$$\beta_k > \sum_{i=1}^{k-1} \beta_i \,. \tag{3}$$

Explicitly, we set $\beta_0 = 0$ and define

$$F^n := \bigcup_{k=0}^{n-1} \left[kn + \sum_{i=0}^{k} \beta_i, \;\; kn + \sum_{i=0}^{k} \beta_i + (n-1) \right] \;\bigcup\; \left[n^2 + \sum_{i=0}^{n} \beta_i, \;\; 2n^2 + \sum_{i=0}^{n} \beta_i \right] .$$

Finally for $k = 1, \ldots, n$ define

$$t_k := n^2 + \sum_{i=k}^{n} \beta_i \,.$$

It can be verified that

$$(F^n + t_k) \cap F^n = \left[(n+k)n + \sum_{i=0}^{n} \beta_i, \;\; (n+k)n + \sum_{i=0}^{n} \beta_i + (n-1) \right] .$$

Using (**??**) and (**??**), one can show that $(F^n + t_j) \cap (F^n + t_k) = \emptyset$ for $1 \leq j \neq k \leq n$.

A group action $(X, \mathcal{B}, \mu, G)$ is *rank* $\leq r$ if there exist r Følner sequences $\{F^{n,\,i}\}$ where $1 \leq i \leq r$, each indexing a measurable tower

$$X^{n,\,i} := \bigcup_{g \in F^{n,\,i}} X_g^{n,i} \,,$$

– if $g \in F^{n,\,i}$, then $X_g^{n,i}$ is a level of the tower $X^{n,\,i}$ – satisfying: for the sequence of measurable partitions

$$\mathcal{P}_n := \left\{ X_g^{n,i} : g \in F^{n,\,i}, i = 1, \ldots, r; \;\; X \backslash X^n \right\}$$

of X, where $X^n := \bigcup_{i=1}^{r} X^{n,\,i}$, we have

1. $\mu\,(X^n) \to 1$,

2. $hX_g^{n,i} = X_{hg}^{n,i}$ whenever $g \in F^{n,i} \cap h^{-1}F^{n,i}$,

3. For each measurable set A, and each $n \in \mathbf{N}$ and $1 \leq i \leq r$, there exists a set $A^{n,i}$ which is a union of elements of $\{X_g^{n,i}\}_{g \in F^{n,i}}$ and $\mu((A \cap X^{n,i})\Delta A^{n,i}) \rightarrow_n 0$ for each $i = 1, 2, \ldots, r$.

We let $a^{n,i}$ denote the (common) mass of a level in the i-th tower, i.e. $\mu(X_g^{n,i}) = a^{n,i}$ for all $g \in F^{n,i}$.

Property 3 says that the partitions $\mathcal{P}_n$ converge to $\mathcal{B}$ and to denote this we write $\mathcal{P}_n \rightarrow_n \epsilon$. We will call $\{F^{n,i}\}_{i=1}^r$ the Følner sequences associated with the rank (at most) r group action $\mathcal{X}$ and write $(\mathcal{X}, \{F^{n,i}\}_{i=1}^r)$ for $\mathcal{X}$ when we wish to specify $\{F^{n,i}\}_{i=1}^r$. We will say that $(\mathcal{X}, \{F^{n,i}\}_{i=1}^r)$ has the bounded intersection property if $\{F^{n,i}\}_{i=1}^r$ have the bounded intersection property. For a general scheme for constructing finite rank group actions, one can generalise the method used in [**?**] to construct rank one group actions. When $G = \mathbf{Z}$, what we call a finite rank action is in fact a generalisation of the usual finite rank transformation (which requires that the $F^{n,i}$ be intervals in $\mathbf{Z}$). Ferenczi [**?**] constructs a rank one $\mathbf{Z}$ action which is not loosely Bernoulli, and therefore not of finite rank. He calls this a **funny** rank one transformation. Ferenczi's example in fact has the bounded intersection property with $p = 4$. We say that the action $(X, \mathcal{B}, \mu, G)$ is *rank* r if the action is rank $\leq r$ and the action is not rank j for $1 \leq j < r$.

Let $0 < \beta \leq 1$. A group action $(X, \mathcal{B}, \mu, G)$ has *local rank* β (or is β-*local*) if there exists a Følner sequence $\{F^n\}$ and a sequence of measurable partitions

$$\mathcal{P}_n := \{X_g^n : g \in F^n; \; X \backslash X^n\}$$

of X, (where $X^n := \cup_{g \in F^n} X_g^n$), such that

1. $\mu(X^n) \rightarrow \beta$,

2. $hX_g^n = X_{hg}^n$ whenever $g \in F^n \cap h^{-1}F^n$,

3. For each measurable set A, and each $n \in \mathbf{N}$, there exists a set A^n which is a union of elements of $\{X_g^n\}_{g \in F^n}$ and $\mu((A \cap X^n)\Delta A^n) \rightarrow 0$.

In general, if $\{X^n\}$ is a sequence of towers satisfying Property 3, we will say it is an *approximating* sequence of towers. We will call the Følner sequence used above in the definition of a β-local group action the Følner sequence associated with the β-local group action $\mathcal{X}$ (and sometimes say that $\{F^n\}$ *generates* $X^n = \cup_{f \in F^n} X_f^n$) and write $(\mathcal{X}, \{F^n\})$ for the β-local group action. The measure of a level in the n-th tower will be denoted by a_n. Ergodic finite rank actions are examples of β-local actions - this can be seen by applying Lemma **??**.

If $\{g_j\}$ is a sequence in G we write $g_j \to \infty$, if whenever $V \subset G$ is finite, then only finitely many of the g_j's belong to V. A group action $\mathcal{X}$ is *2-mixing* if

$$\lim_{j\to\infty} \mu\,(A_1 \cap g_j A_2) = \mu\,(A_1)\mu\,(A_2)$$

$\forall A_1\,, A_2\, \in \mathcal{B}$ and for each sequence $g_j \to \infty$. $\mathcal{X}$ is *k-mixing* if

$$\lim_{j\to\infty} \mu\,(A_1 \cap g_j^1 A_1 \cap g_j^2 A_2 \cap \ldots \cap g_j^{k-1} A_k) = \mu\,(A_1)\ldots\mu\,(A_k)$$

where $\lim_{j\to\infty} g_j^i = \infty$ for $i = 1, 2\ldots, k-1$ and also $\lim_{j\to\infty}(g_j^i)^{-1} g_j^l = \infty$ whenever $i \neq l$. $\mathcal{X}$ is *mixing of all orders* if it is k-mixing for each $k \geq 2$.

Our main theorem is

Theorem 1 *Suppose that G is Abelian, countable and has an element of infinite order. Let $(X\,,\mathcal{B}\,,\mu\,,G)$ be a mixing finite rank action, or a mixing β-local action, with $\beta > 1/2$. In either case, assume that the Følner sequence(s) indexing the tower(s) of the action have the bounded intersection property, and have comparable magnitudes. Then $(X\,,\mathcal{B}\,,\mu\,,G)$ is mixing of all orders.*

To prove Theorem **??**, we use the method of joinings, generalizing [**?**].

We say that $(X, \mathcal{B})$ is *regular* if X is compact and $\mathcal{B}$ is the Borel σ-algebra. This implies that $C(X) = \{f : X \to \mathbf{R}, f \text{ is continuous}\}$ is separable and also that any probability measure on $(X, \mathcal{B})$ is regular. In this section all spaces are regular. A probability measure λ is a *2-joining* of $\mathcal{X}$ and $\mathcal{Y}$ if $(\,X\times Y,\ \mathcal{B}\otimes\mathcal{F},\ \lambda,\ G\,)$ is a measure-preserving group action with the additional condition that

$$\lambda(\,A\,\times\,Y\,) = \mu\,(\,A\,)$$

and

$$\lambda(\,X\,\times\,B) = \nu\,(B)$$

for $A\, \in\, \mathcal{B}\,,\ B\, \in\, \mathcal{F}$ respectively. We will use the word *coupling* when the measure λ projects onto μ and ν, without necessarily preserving the group action. For a detailed account of joinings, see [**?**]. Note that by regularity, λ is a joining if and only if $\lambda(f \otimes 1) = \mu(f)$ and $\lambda(1 \otimes g) = \nu(g)$ for $f, g \in C(X), C(Y)$ respectively (where $f \otimes g(x, y) := f(x)g(y)$). If $\{(X_i,\, \mathcal{B}_i\,, \mu_i)\}_{i=1}^n$ are probability spaces and λ is a probability measure on $(\Pi_{i=1}^n X_i\,, \otimes_{i=1}^n \mathcal{B}_i)$, then we define $\pi_{i_1,i_2,\ldots,i_k}\lambda$ to be the projection of λ on $(\Pi_{j=1}^k X_{i_j}\,, \otimes_{j=1}^k \mathcal{B}_{i_j})$. λ is an *n-joining* of $\{(X_i,\, \mathcal{B}_i\,, \mu_i\,, G)\}_{i=1}^n$ if $(\Pi_{i=1}^n X_i,\ \otimes_{i=1}^n \mathcal{B}_i,\ G, \lambda)$ is a measure-preserving action and $\pi_k\lambda = \mu_k$ for $k = 1, \ldots, n$. For $n > 2$ it is natural to impose stronger conditions on λ : in particular, if λ is an n- joining of $\{X_i\}_{i=1}^n$, then we can require that

$$\pi_{i_1,i_2,\ldots,i_k}\lambda = \Pi_{j=1}^k \mu_{i_j}$$

whenever $i_1, i_2, \ldots i_k$ are k distinct elements of $\{1, 2, \ldots, n\}$. In this case we write $\lambda \in M(k, n)$ and say λ is *k-fold independent*.

For our purposes $(X_i, \mathcal{B}_i, \mu_i) = (X, \mathcal{B}, \mu)$ for $i = 1, 2 \ldots n$, and we use the notation $X^{(n)} := X \times X \times \ldots \times X$. Note that $M(k, n) \subset M(1, n)\ \forall\, k \geq 1$, and that $M(1, n) \subset M(X^{(n)})$, the set of probability measures on $X^{(n)}$. We will work with the weak-$*$ topology on $M(X^{(n)})$. With this (metrisable) topology, $\mu_k \to \mu$ iff $\mu_k(g) \to \mu(g)$ for $g \in C(X^{(n)})$. As the linear span of the family $\{\otimes_{i=1}^n f_i : f_i \in C(X)\}$ is dense in $C(X^{(n)})$, it is sufficient to check convergence on this family. Therefore if $\{\lambda_k\} \subset M(1, n)$, then $\lambda_k \to \lambda$ iff $\lambda_k(\Pi_{i=1}^n A_i) \to_k \lambda(\Pi_{i=1}^n A_i)$ for all measurable rectangles $(\Pi_{i=1}^n A_i)$. For, suppose that the latter is true; given $\otimes_{i=1}^n f_i \in C(X^{(n)})$, let $\{\phi_j^1\}, \{\phi_j^2\}, \ldots, \{\phi_j^n\}$ be sequences of simple functions tending uniformly to $f_1, f_2, \ldots f_n$ respectively. Then $|\lambda_k(\bigotimes_{i=1}^n f_i) - \lambda(\bigotimes_{i=1}^n f_i)| \leq \left|\lambda_k(\bigotimes_{i=1}^n f_i) - \lambda_k\left(\bigotimes_{i=1}^n \phi_j^i\right)\right| + \left|\lambda_k\left(\bigotimes_{i=1}^n \phi_j^i\right) - \lambda\left(\bigotimes_{i=1}^n \phi_j^i\right)\right| + \left|\lambda\left(\bigotimes_{i=1}^n \phi_j^i\right) - \lambda(\bigotimes_{i=1}^n f_i)\right|$, and first choosing j big enough so that the first and third summands are small, and then k big enough so that the second summand is small, the result follows. Conversely, suppose that $\lambda_n \to \lambda$, and let $(\Pi_{i=1}^n A_i)$ be measurable. Note that λ is a coupling. By regularity there exist compact K_i and open U_i where $K_i \subset A_i \subset U_i$ and $\mu_i(U_i \backslash K_i) < \epsilon/n$ for $i = 1, \ldots, n$. We can then find a continuous function f which is identically 1 on $\Pi_{i=1}^n K_i$ and 0 outside $\Pi_{i=1}^n U_i$. Thus for each k

$$|\lambda_k(\Pi_{i=1}^n A_i) - \lambda_k(f)| < \epsilon,$$

and the same is true for λ. A triangle inequality gives the result.

It now follows that although the maps T_g are not continuous, the set $M(1, n)$ is a closed subset of $M(X^{(n)})$ in the weak-$*$ topology, and hence it is compact.

Theorem **??** will follow from

Theorem 2 *Suppose that G is Abelian, countable and has an element of infinite order. Let $(X, \mathcal{B}, \mu, G)$ be a regular mixing finite rank action, or a regular mixing β-local action, with $\beta > 1/2$. In either case, assume that the Følner sequence(s) indexing the tower(s) of the action have the bounded intersection property, and have comparable magnitudes. Let $\nu \in M(2, k)$. Then $\nu = \mu^k$.*

We will only prove that $M(2, 3) = \{\mu^3\}$, the proof for $k > 3$ follows by the same method.

Note that regularity is required in the statement of Theorem **??**, but not Theorem **??**. The proof of Theorem **??** clarifies this.

Proof of Theorem ??: The finite rank hypothesis on $\mathcal{X}$ ensures that the measure algebra $\bar{\mathcal{B}}$ of μ-equivalence classes of sets is separable. Standard arguments then show that $\bar{\mathcal{B}}$ is isomorphic to the measure algebra of a regular space and the action of G on $\bar{\mathcal{B}}$ transfers to an action on this regular measure algebra, which can then be realized as a point action. This shows that there is no harm in assuming that $\mathcal{X}$ itself is regular. (Like most dynamical notions, the concepts of mixing and finite rank are obviously invariant under isomorphism of actions at the level of measure algebras. The concept of a joining, however, is not.)

We give the argument for $k = 3$, for simplicity. If $\mathcal{X}$ is not 3-mixing then there exist measurable sets A, B, C and sequences $\{k_j\}$ and $\{l_j\}$ of group elements, both tending to infinity, and $\epsilon > 0$ such that $k_j^{-1} l_j \to_j \infty$ satisfying

$$|\mu(A \cap l_j B \cap k_j C) - \mu(A)\mu(B)\mu(C)| \geq \epsilon$$

for all j. Consider the joining $\Delta_{k_j,l_j}(E_1 \times E_2 \times E_3) := \mu(E_1 \cap l_j E_2 \cap k_j E_3)$. If Δ^* is a limit point of the sequence $\{\Delta_{k_j,l_j}\}$ then $\Delta^* \neq \mu^3$. On the other hand 2-fold mixing implies that $\Delta^* \in M(2,3)$. This contradicts Theorem **??**. ■

In proving Theorem **??** we will restrict ourselves to the case $k = 3$. A similar argument shows that $M(k-1,k) = \{\mu^k\}$ for $k > 2$, so Theorem **??** follows by induction. Throughout the remainder of the paper we will assume that the underlying measure space is regular and convergence of measures will always mean weak-* convergence.

Let $(X, \mathcal{B}, \mu, G)$ be mixing and finite rank, and let $\nu \in M(2,3)$ be ergodic. A sketch of the proof that $\nu = \mu^3$ is as follows. It would be ideal if two things were to occur simultaneously: first, there is a sequence of products of some approximating towers

$$R^n = T^{n,1} \times T^{n,2} \times T^{n,3} \tag{4}$$

with $\underline{\lim}\,\nu(R^n) > 0$; and second that the "cubes" in R^n do not individually contain too much ν-mass. Ryzhikov would call such cubes ν-light. This ensures that ν is to some extent "smeared out" over substantially many of the cubes in R^n. The first property would mean that the measure ν_{R^n} - " the measure ν conditioned on R^n", would converge weak-* to ν (Lemma **??**). The second property is needed to show that $\nu_{R^n} \to \mu^3$. In the rank one case, (see [**?**] or [**?**]) finding R^n is not so hard. The $T^{n,i}$'s will not be the actual rank one towers, rather they will be subsets of the rank one towers. For example if $G = \mathbf{Z}$, and the Følner sequence is a sequence of intervals: $F^n = [0, h_n)$, then R^n could be some subset of

$$\cup_{i=0}^{(1-\delta)h_n} X_i^n \times \cup_{i=0}^{(1-\delta)h_n} X_i^n \times \cup_{(1-\delta)h_n}^{h_n} X_i^n,$$

(or its image under $T^{\delta h_n}$). The fact that ν is pairwise independent, and that the rank one tower X^n is most of the space, easily yields that

$$\underline{\lim}_n\, \nu(\cup_{i=0}^{(1-\delta)h_n} X_i^n \times \cup_{i=0}^{(1-\delta)h_n} X_i^n \times \cup_{(1-\delta)h_n}^{h_n} X_i^n) > 0.$$

Of course we then have to do a bit of work to show that our chosen R^n has positive ν-mass. The fact that $(X, \mathcal{B}, \mu, G)$ is mixing helps insure the second requirement, (mixing means that no level in X^n can go "heavily" into any other - Lemma **??**), which in turn is needed to guarantee that a certain sequence of probability weights we end up averaging against is uniformly small. Mixing is used a second time to invoke the Blum-Hanson theorem ([**?**]) which is where the uniform smallness of the afore-mentioned weights are needed.

If $(X, \mathcal{B}, \mu, G)$ is finite rank, the difficulty is to get a sequence of R^n's which have both positive ν mass *and* whose "cubes" are ν-light at the same time. If $(X, \mathcal{B}, \mu, G)$ has local rank, with towers X^n it is not even clear that $\overline{\lim}_n\ \nu(X^n \times X^n \times X^n) > 0$. In [**?**], Ryzhikov finds two sequences of sets $\{R^n\}$ and $\{S^n\}$ (as in the example above, $S^n \approx h_n R^n$, where $h_n \to \infty$ in G.) He then finds a sequence of *induced joinings* $\in M(1,3)$ satisfying both of the required properties for either R^n or S^n. These joinings are certain projections of relative products built using the disintegration of ν over $(X, \mathcal{B}, \mu, G)$. The sequence of induced joinings $\{\eta_j\}$ are chosen so that as $j \to \infty$, η_j has an increasingly large pairwise independent component and we show that this component is mostly product measure; so that $\eta_j = c_j \mu^3$, where $c_j \uparrow 1$. Lemma **??** then shows that $\nu = \mu^3$.

The rest of the paper will go in the following order: first we define induced joinings and show how an appropriate sequence of these joinings will give information about ν (Lemma **??**). Next we construct the sets $\{R^n\}$ and $\{S^n\}$, and then find the sequence of induced joinings (Proposition **??**) which give $\{R^n\}$ and $\{S^n\}$ the "right amount" of mass (Propositions **??** and **??**). Finally we will show that this sequence of measures have a large pairwise independent component (Lemma **??**) , and work with these components, and show that they are mostly product measure. This will allow us (Lemma **??**) to conclude that $\nu = \mu^3$.

3 Auxiliary Results

If A is a measurable subset of X and μ is a probability measure on X, we define the measure μ_A, the measure μ conditioned on the set A, as

$$\mu_A(F) = \frac{\mu(A \cap F)}{\mu(A)}.$$

If μ, μ_1 are probability measures and $0 < c \leq 1$, then the expression " $\mu \geq c\mu_1$" means that $\mu = c\mu_1 + (1-c)\mu_2$ for some probability measure μ_2.

The next simple lemma is used repeatedly in [**?**].

Lemma 1 *Let $(X, \mathcal{B}, \nu, G)$ be an ergodic measure-preserving group action, and let I_n be a sequence of measurable sets such that*

1 $\lim_n \nu(I_n) = c \neq 0$,

2 $\lim_n \nu(gI_n \Delta I_n) = 0$ *for each* $g \in G$.

Then $\lim_n \nu_{I_n} = \nu$.

Proof: Let $\nu_n := \nu_{I_n}$. We know that (by passing to a subsequence if necessary) that the sequence $\{\nu_n\}$ converges weak-$*$ to some measure λ. Now $\nu_n = f_n\, d\nu$ where $f_n = \frac{\chi_{I_n}}{\nu(I_n)}$, and by property 1, there is a K such that $|f_n| \leq K$ for all large n. We claim that if $\nu_n = f_n\, d\nu \to \lambda$ and $|f_n| \leq K$, then $\lambda = f\, d\nu$ for some f.

If not, then there is some L with $\lambda(L) \geq \epsilon$ and $\nu(L) = 0$. Find U open containing L such that $\nu(U) < \epsilon/2K$. Then $\nu_n(U) < \epsilon/2$ for all large n, and this is a contradiction since $\liminf_n \nu_n(U) \geq \lambda(U)$ for U open.

Thus $\nu_n = f_n\, d\nu \to f\, d\nu$. Note that this implies that $||f_n - f||_1 \to 0$. Next we claim that $\lambda \circ g^{-1} = \lambda$ for $g \in G$. To see this, note that

$$\left| \int_A f\, d\nu - \int_{g^{-1}A} f\, d\nu \right| = \left| \int_A (f - f \circ g)\, d\nu \right| \leq 2 \int |f - f_n|\, d\nu + \int_A |f_n - f_n \circ g|\, d\nu\,.$$

The first summand is small for large n. The second summand is also small since

$$\left| \int_A (f_n - f_n \circ g)\, d\nu \right| \leq \int |f_n - f_n \circ g|\; d\nu = \nu(I_n \Delta g^{-1} I_n) \to 0$$

as $n \to \infty$. Thus $\lambda << \nu$ and is G-invariant, and $\nu \geq c\lambda$. The ergodicity of ν forces $\lambda = \nu$. ∎

We will also need a version of the Blum-Hanson Theorem for group actions. ([?]).

Theorem 3 *Let $\mathcal{X}$ be a mixing action of a countable Abelian group G. Suppose that $\{a^n\}_{n \in \mathbf{N}}$ is a sequence of functions $a^n : G \to [0, \infty)$ satisfying*

1 $\sum_{g \in G} a^n(g) = 1, \ \forall\, n \in \mathbf{N}$

2 $\lim_{n \to \infty} \sup_{g \in G} a^n(g) = 0\,.$

Then for any $\phi \in \mathcal{L}_2(\mu)$,

$$\|\ A_n(\phi) - \langle \phi, 1 \rangle\ \|_2 \to 0$$

where $A_n(\phi) := \sum_{g \in G} a^n(g)(\phi \circ g)\,.$

Proof: We may assume that $\phi \in \mathcal{L}_2(X)$ has 0 mean and unit norm. Since $\mathcal{X}$ is mixing, then given $\epsilon > 0$, we may choose a finite subset $\mathcal{O}_\epsilon \subset G$ such that $| \langle \phi \circ g, \phi \rangle | < \epsilon/2$ whenever $g \in G \backslash \mathcal{O}_\epsilon$. Next choose N large enough such that

$$\sup_{g \in G} a^n(g) < \frac{\epsilon}{2|\mathcal{O}_\epsilon|}$$

for all $n > N$. Then $\left\langle \sum_{g\in G} a^n(g)(\phi \circ g), \sum_{g \in G} a^n(g)\,(\phi \circ g) \right\rangle$ can be split up into two summands,

$$\sum_{g\in G} a^n(g) \sum_{\{h: gh^{-1} \in \mathcal{O}_\epsilon\}} a^n(h) \left\langle \phi \circ (gh^{-1}), \phi \right\rangle$$

and

$$\sum_{g\in G} a^n(g) \sum_{\{h: gh^{-1} \in G \backslash \mathcal{O}_\epsilon\}} a^n(h) \left\langle \phi \circ (gh^{-1}), \phi \right\rangle \, .$$

The first summand can be bounded by $\epsilon/2|\mathcal{O}_\epsilon| \sum_{g\in G} a^n(g)|\mathcal{O}_\epsilon|$ and the second by $\epsilon/2 \sum_{g \in G} a^n(g) \sum_{h \in G} a^n(h)$, and using the fact that $\sum_{g\in G} a^n(g) = 1$, the result follows. ∎

3.1 Induced Joinings

Let $\nu \in M(2,3)$ be ergodic for $(Y, \mathcal{F}, G) := (X^3, \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B}, G)$. Using the disintegration

$$\nu(A \times B \times C) = \int_C \int_B \nu_{x,y}(A)\, d\mu\,(y)\, d\mu\,(x) \tag{5}$$

of ν with respect to the factor (X^2, μ^2) (see [**?**]), we can define the family of operators $\{J_x\}_{x \in X}$, where $J_x : \mathcal{L}_2(X) \to \mathcal{L}_2(X)$ is defined as $J_x f\,(y) := \nu_{x,y}(f)$ so that equation (**??**) can also be written as

$$\nu(A \times B \times C) = \int_C \langle J_x \chi_A,\, \chi_B \rangle \; d\mu\,(x) \tag{6}$$

Note that

$$J_x(\chi_X) = J_x^*(\chi_X) = 1 \tag{7}$$

and

$$J_x(f) \geq 0 \text{ and } J_x^*(f) \geq 0 \text{ whenever } f \geq 0\,. \tag{8}$$

We have

$$\begin{aligned}\nu\left(gA\times gB\times gC\right) &= \int_{gC}\left\langle J_x\chi_{gA},\,\chi_{gB}\right\rangle\,d\mu\left(x\right)\\ = \int_{gC}\left\langle J_x\chi_A\circ g^{-1},\,\chi_B\circ g^{-1}\right\rangle\,d\mu\left(x\right) &= \int_{gC}\left\langle gJ_xg^{-1}\chi_A,\,\chi_B\right\rangle\,d\mu\left(x\right)\\ = \int_{C}\left\langle gJ_{g^{-1}x}g^{-1}\chi_A,\,\chi_B\right\rangle\,d\mu\left(x\right) &\quad (\text{where } g \text{ and } g^{-1} \text{ act unitarily on } \mathcal{L}_2(X))\end{aligned}$$

and since ν is preserved by the action of G, then

$$\int_C\left\langle gJ_{g^{-1}x}g^{-1}\chi_A,\,\chi_B\right\rangle\,d\mu\left(x\right)=\int_C\left\langle J_x\chi_A,\,\chi_B\right\rangle\,d\mu\left(x\right)$$

and this is true for all measurable sets $A,\,B,\,C\,\in\mathcal{B}$. Hence

$$gJ_{g^{-1}x}g^{-1}\equiv J_x \tag{9}$$

or

$$gJ_xg^{-1}\equiv J_{gx}\,, \tag{10}$$

and this relation is clearly equivalent to the G-invariance of ν. Since $\nu\in M(2,3)$, in particular $\pi_{1,2}\nu=\mu$ so

$$\int_X\left\langle J_x\chi_A,\,\chi_B\right\rangle\,d\mu\left(x\right)=\nu(A\times B\times X)=\mu\left(A\right)\mu\left(B\right)=\left\langle\int\chi_A,\chi_B\right\rangle\,,$$

or, in the language of operators,

$$\int_X J_xd\mu\left(x\right)\equiv\int d\mu^2. \tag{11}$$

Conversely, if $\{J_x\}_{x\in X}$ is a measurable family of operators that satisfies (**??**), (**??**), (**??**) and (**??**) then the measure ν defined by (**??**) is a pairwise independent joining. If equation (**??**) is not satisfied then (**??**) still gives $\pi_{1,3}\nu=\pi_{2,3}\nu=\mu^2$.

Given $\nu\in M(2,3)$, let $\{J_x\}_{x\in X}$ be the family of operators defined by (**??**). If $k,\,l\,\in G$, then the *induced joining* $\eta_{k,l}$ is the measure defined by

$$\begin{aligned}\eta_{k,l}(A\times B\times C) &:= \int_C\left\langle J_{kx}\chi_A,\,J_{lx}\chi_B\right\rangle\,d\mu\left(x\right)\\ &= \int_C\int_X\nu_{(kx,y)}(A)\nu_{(lx,y)}(B)d\mu\left(y\right)d\mu\left(x\right).\end{aligned} \tag{12}$$

$\eta_{k,l}$ is the projection of a relative product: we will describe this when $l=e$, i.e. for the induced joinings $\eta_k:=\eta_{k,e}$. If we label the i-th copy of X in Y

as X_i for $i = 1, 2, 3$, then we have two factors of Y, namely $\pi_{2,3}Y$ and $\pi_{1,3}Y$, and we identify these two factors using the transformation $\phi : \pi_{2,3}Y \to \pi_{1,3}Y$ which is defined as:

$$\phi(x, y) := (kx, y) .$$

ϕ induces a joining $\eta_k^{'}$ on X^4, the relative product over ϕ:

$$\eta_k^{'}(A \times B \times C \times D) := \int_C \int_D \nu_{(kx,y)}(A)\nu_{(x,y)}(B) d\mu\,(y)\, d\,\mu\,(x)$$

and from this we see that $\eta_k = \pi_{1,2,3}\eta_k^{'}$.

Note that $\eta_{k,l}$ is G invariant; this follows using (**??**). Thus (Y, $\mathcal{F}$, $\eta_{k,l}$, G) is a measure preserving group action. We will only be using induced joinings $\eta_{k,g}$ where $k^{-1}g$ has infinite order.

As an example, we can look at the (well known) case when ν is a nontrivial 3-fold joining of the topological group $X = \{0, 1\}^{\mathbf{Z}}$, with the Borel σ-algebra and Haar measure, μ. Let $\phi : X^2 \to X^3$ be the map defined by $\phi((x, y)) = (x + y, x, y)$ and let $\nu := \mu^2 \circ \phi^{-1}$. Then ν is supported on the shift invariant set

$$E \;=\; \{(x + y, y, x) : x, y\, \in X\} .$$

Here

$$J_x f(y) \;= \nu_{x,y}(f) \;=\; f(x + y) .$$

Then $\eta_{k,l}$ is the measure

$$\eta_{k,l}(A \times B \times C) = \int_C \int_X \chi_A(\tau^l(x) + y)\, \chi_B(\tau^k(x) + y)\, d\mu\,(y)\, d\mu\,(x) ,$$

where τ is the shift. $\eta_{k,l}$ is supported on the set

$$\{(\tau^l(x) + y, \tau^k(x) + y, x, y) : x, y\, \in X\} .$$

In this case $\eta_{k,l}$ is ergodic, but not pairwise independent. So we cannot assume that $\eta_{k,l} \in M(2, 3)$ ($\pi_{1,3}\eta_{k,l}$ and $\pi_{2,3}\eta_{k,l}$ are product measure, but it is not clear that (**??**) holds); nor in general can we assume that $\eta_{k,l}$ is ergodic.

Let $\{c_j\}$ be a sequence of real numbers increasing to 1. While it is not clear that we can find an appropriate sequence of pairs of towers $\{R^n, S^n\}$ which are given positive mass by ν, we will be able to do this for a sequence of induced joinings $\{\eta_j\}$. Next we will show that this sequence has a large component which is pairwise independent, and this will set the stage to prove that $\eta_j \geq c_j\mu^3$. This will imply that $\nu = \mu^3$: we prove this in the next lemma.

Lemma 2 *Let* $c_j \uparrow_j 1$. *Let* $(X\,,\mathcal{B}\,,\mu\,,G)$ *be a mixing group action, and* ν *be a pairwise independent 3-joining of this action. Suppose that for some sequences* $\{g_j\}$ *and* $\{k_j\}$ *of group elements where* $g_j k_j^{-1} \to \infty$, *we have that*

$$\eta_{g_j,k_j} \geq c_j \mu^3$$

for each $j \in \mathbf{N}$, *where* η_{g_j,k_j} *are the induced joinings defined by (??). Then* $\nu = \mu^3$.

Proof: Let $\{J_x\}_{x\in X}$ be the family of operators defined by ν. Note that if $f,\ g,\ h,$ are bounded measurable functions of $(X,\ \mathcal{B},\ \mu)$, then

$$\mu^3(f\otimes g\otimes h) = \int f \left\langle \int g,\ h \right\rangle d\mu\ .$$

To show that $\nu = \mu^3$, it is then sufficient to show that $J_x = \int$ for μ-almost all x.

Let $\mathcal{F}$ be a countable basis of (real valued) bounded functions for $\mathcal{L}_2^0(X,\ \mu)$ such that $||f||_2 = 1$ for each $f \in \mathcal{F}$. Fix an f in $\mathcal{F}$, we shall show that $J_x f = 0$ for μ-almost all $x \in X$. For $\epsilon > 0$ and $g \in \mathcal{L}_\infty(X)$, define

$$E_{\epsilon,g} := \left\{ x \in X : ||J_x f - g||_2 < \frac{\epsilon}{1 + ||f||_\infty + ||g||_\infty} \right\}.$$

We claim that if $\mu\left(E_{\epsilon,g}\right) > 0$, then $||g||_2 < \sqrt{\epsilon}$. By assumption on η_{g_j,k_j},

$$\eta_{g_j,k_j} - c_j\mu^3 = (1-c_j)\rho,$$

where $\rho \in M(1,3)$. Hence for each $h \in \mathcal{L}_\infty(X)$

$$\left|\eta_{k_j,g_j}(f\otimes f\otimes h)\ -\ c_j\,\mu^3(f\otimes f\otimes h)\right| \leq (1-c_j)||f||_\infty^2||h||_\infty\,.$$

Therefore

$$\left|\int h(x)\langle J_{g_jx}^* J_{k_jx} f, f\rangle\, d\mu\,(x)\right| =$$
$$\left|\int h(x)\left(\langle J_{g_jx}^* J_{k_jx} f, f\rangle - c_j\langle \int f, f\rangle\right) d\mu\,(x)\right| \leq (1-c_j)||f||_\infty^2||h||_\infty.$$

By taking now $h = \ldots$,

where
$$A_j^{+\cdots}(\delta) := \left\{ x : \left\langle J_{g_jx}^* J_{k_jx} f, f \right\rangle \geq \ldots \delta \right\}$$

we obtain that for $\delta > 0$

$$\mu\left(\left\{ x : \left|\left\langle J_{g_jx}^* J_{k_jx} f, f\right\rangle\right| > \delta \right\}\right) < \frac{2(1-c_j)||f||_\infty^2}{\delta}\,.$$

Let $\delta = 1+||f||_\infty+||g||_\infty$. Assuming that $\mu\,(E_{\epsilon,g}) > 0$, and that $(X\,,\mathcal{B}\,,\mu\,,G)$ is mixing, we can find a j large enough such that $\mu\,(g_j^{-1}E_{\epsilon,g} \cap k_j^{-1}E_{\epsilon,g}) = \mu\,(k_j g_j^{-1}E_{\epsilon,g} \cap E_{\epsilon,g}) > 0$, and also so that

$$\left\{x : \left|\left\langle J^*_{g_j x} J_{k_j x} f, f\right\rangle\right| < \delta\right\} \cap g_j^{-1}E_{\epsilon,g} \cap k_j^{-1}E_{\epsilon,g} \neq \emptyset\,.$$

Thus for an x in this non-empty intersection, and for large j,

$$\begin{aligned}\langle g, g\rangle &= \left\langle g, g - J_{g_j x} f\right\rangle + \left\langle g - J_{k_j x} f, J_{g_j x} f\right\rangle + \left\langle J_{k_j x} f, J_{g_j x} f\right\rangle \\ &\leq ||g||_\infty ||g - J_{k_j x} f||_2 + ||J_{k_j x} f||_2 ||g - J_{g_j x} f||_2 + \left|\left\langle J_{k_j x} f, J_{g_j x} f\right\rangle\right| \\ &\leq (||g||_\infty + ||f||_\infty + 1)\,\frac{\epsilon}{(1+||f||_\infty+||g||_\infty)} = \epsilon\,,\end{aligned}$$

or $||g||_2 < \sqrt{\epsilon}$. This proves our claim. Now let $\mathcal{G}$ be a countable subset of bounded functions dense in $\{g \in \mathcal{L}_2(X,\,\mu) : ||g||_2 > \sqrt{\epsilon}\}$. What we've proved above implies that if $g \in \mathcal{G}$, then $\mu\,(E_{\epsilon,g}) = 0$. Let

$$F_\epsilon = \{x : ||J_x f||_2 > \sqrt{\epsilon}\}\,,$$

By density of $\mathcal{G}$, $\exists g \in \mathcal{G}$ such that $||J_x f - g||_2 < \frac{\epsilon}{(1+||f||_\infty+||g||_\infty)}$, so that

$$F_\epsilon \subset \bigcup_{g\in\mathcal{G}} E_{\epsilon,g}\,,$$

which means that $\mu\,(F_\epsilon) \leq \sum_{g\in\mathcal{G}} \mu\,(E_{\epsilon,g}) = 0$.

■

4 D-approximations.

If $(X\,,\mathcal{B}\,,\mu\,,G)$ is a rank one mixing group action, then any level of the nth tower X^n cannot get mapped "heavily" into itself by increasing elements of G. In other words, the situation

$$\overline{\lim_n}\,\frac{\mu\,(h_n X^n_{i_n} \cap X^n_{i_n})}{\mu\,(X^n_{i_n})} > 0$$

is not possible, for any sequence $i_n \in F^n$, and $h_n \to \infty$. This is not necessarily true for rank r actions; what is true though is that one can find two nontrivial (in measure) "sub"-towers, Y^n and $Y^{n,2}$, such that the levels of Y^n do not get mapped heavily into the levels of $Y^{n,2}$. This is one of the things that we show in this section.

We first generalise Ryzhikov's definition of a D-approximation (see [**?**]). We say that the group action $(X\,,\mathcal{B}\,,\mu\,,G)$ has a *D-approximation* if there exist 2 Følner sequences $\{F^n\}$ and $\{G^n\}$ which generate the approximating towers $X^{n,1} := \cup_{f\in F^n} X_f^{n,1}$ and $X^{n,2} := \cup_{g\in G^n} X_g^{n,2}$ respectively, a sequence of group elements $h_n \to \infty$, and sequences of subsets of group elements $\{A^n\}$, $\{B^n\}$, satisfying:

1. $\{A^n\}$ is Følner, $A^n \subset F^n$, $B^n \subset F^n$ and $|A^n| \geq \alpha_1 |F^n|$, $|B^n| \geq \alpha_2 |F^n|$, for some positive constants $\alpha_1,\ \alpha_2$;
2. $h_n A^n \subset F^n$, and $h_n B^n \cap F^n = \emptyset$;
3. For some $\delta > 0$, for each $f \in B^n$ there exists $Y_f^n \subset X_f^{n,1}$ such that $\mu\left(Y_f^n\right) \geq \delta\,\mu\left(X_f^{n,1}\right)$, $h_n\left(\cup_{f\in B^n} Y_f^n\right) \subset X^{n,2}$, and
$$\underline{\lim}_n\, \mu\left(h_n Y_{f_n}^n | X_{g_n}^{n,2}\right) = 0$$
whenever $f_n \in B^n$ and $g_n \in G^n$.

(Note that the two Følner sequences $\{F^n\}$ and $\{G^n\}$ need not necessarily be distinct.)

If $(X\,,\mathcal{B}\,,\mu\,,G)$ is a local rank ergodic action (so the following remark also applies for finite rank actions) with local tower $X^n = \cup_{f\in F^n} X_f^n$ and (passing to a subsequence if necessary) $\lim_n \mu\left(X^n\right) = c > 0$ then as X^n is approximately invariant, by Lemma **??**, we have for any $A \in \mathcal{B}$,

$$\mu\left(A|X^n\right) \to_n \mu\left(A\right)\,.$$

We use this in:

Lemma 3 *Let $(X\,,\mathcal{B}\,,\mu\,,G)$ be a local rank mixing G-action, with associated Følner sequence $\{F^n\}$. Then*

$$\lim_{n\to\infty} \sup_{g\neq e} \frac{\mu\left(gX_{f_n}^n \cap X_{f_n}^n\right)}{\mu\left(X_{f_n}^n\right)} = 0 \tag{13}$$

whenever $f_n \in F^n$.

Proof: We'll assume that $e \in F^n$ for each n and show that equation (**??**) is true for $f_n = e$; the case for an arbitrary sequence $\{f_n\}$ is similar. Suppose the lemma is false, i.e. there exist sequences $\{n_k\} \subset \mathbf{N}$ and $\{g_k\} \subset G$ such that $g_k \neq e$ and

$$\frac{\mu\left(g_k X_e^{n_k} \cap X_e^{n_k}\right)}{\mu\left(X_e^{n_k}\right)} \to c \neq 0\,.$$

Passing to a subsequence if necessary, we can assume that $g_k \to \infty$.
(For otherwise we can pass to a subsequence along which $g_k = g$ with $g \neq 0$. But since $\{F^{n_k}\}$ is Følner, for large k there exist distinct f and f', both in F^{n_k} such that $fg = f'$. Therefore, $\mu\left(gX_e^{n_k} \cap X_e^{n_k}\right) = \mu\left(f^{-1}f'X_e^{n_k} \cap X_e^{n_k}\right) = \mu\left(X_{f'}^{n_k} \cap X_f^{n_k}\right) = 0$, a contradiction.)
Let $\beta = \lim_n \mu\left(X^n\right)$. Choose a set A with $0 < \mu\left(A\right) < c\,\beta$, and for each k choose a set $F^{n_k}(A) \subset F^{n_k}$ so that the sets

$$A^{n_k} = \bigcup_{f \in F^{n_k}(A)} X_f^{n_k}$$

satisfy $\mu\left((A \cap X^{n_k}) \triangle A^{n_k}\right) \to 0$.

We have

$$\mu\left(g_k A^{n_k} \cap A^{n_k}\right) \geq |F^{n_k}(A)| \mu\left(g_k X_e^{n_k} \cap X_e^{n_k}\right) \tag{14}$$

$$= \mu\left(X_e^{n_k}\right) |F^{n_k}(A)| \frac{\mu\left(gX_e^{n_k} \cap X_e^{n_k}\right)}{\mu\left(X_e^{n_k}\right)} \tag{15}$$

$$= \mu\left(A^{n_k}\right) \frac{\mu\left(gX_e^{n_k} \cap X_e^{n_k}\right)}{\mu\left(X_e^{n_k}\right)} \to \beta\,\mu\left(A\right) c\,. \tag{16}$$

On the other hand,

$$\lim_{k\to\infty} \mu\left(g_k A^{n_k} \cap A^{n_k}\right) \leq \lim_{k\to\infty} \mu\left(g_k A \cap A\right) = \left(\mu\left(A\right)\right)^2$$

which together with (**??**) leads to a contradiction.

■

We use Lemma **??** to prove Proposition **??** below. We remark first that given $\epsilon > 0$, we can always find a sequence of group elements $h_n \to \infty$ such that

$$\epsilon \leq \frac{|h_n^{-1}F^n \cap F^n|}{|F^n|} \leq 2\epsilon\,.$$

For, if l is an element of infinite order, choose N so that

$$\frac{|l^{-1}F^n \cap F^n|}{|F^n|} = \frac{|lF^n \cap F^n|}{|F^n|} > 1 - \epsilon/2$$

for $n > N$. Letting

$$a_k := \frac{|l^k F^n \cap F^n|}{|F^n|},$$

we have, for $n > N$:

$$a_0 = 1,\ \ |a_{k+1} - a_k| < \epsilon \text{ and } a_k \neq 0 \text{ iff } l^k \in F^n\, F^{n\,-1},\ \text{a finite set.} \tag{17}$$

Choosing h_n to be the appropriate sequence of powers of l establishes the remark.

Proposition 1 *If* $(X, \mathcal{B}, \mu, G)$ *is* β*-local with* $\beta > 1/2$*, then* $(X, \mathcal{B}, \mu, G)$ *has a D-approximation.*

Proof: Let $\alpha > 0$ be chosen so that $\beta(1-2\alpha) > 1/2$. By the remark above we can find a sequence $\{h_n^{-1}\}$ of group elements such that $h_n^{-1} \to \infty$ and

$$\alpha \leq \frac{|h_n^{-1} F^n \cap F^n|}{|F^n|} \leq 2\alpha\,.$$

Define $A^n := h_n^{-1} F^n \cap F^n$, $B^n := F^n \backslash A^n$. Then A^n, B^n are Følner sequences, $h_n A^n \subset F^n$, and $h_n B^n \cap F^n = \emptyset$. For $g \in B^n$, let $Y_g^n := X_g^n$. Recalling that a_n is the measure of any level of the local tower, we have:

$$\underline{\lim}_n\, \mu \left(\cup_{g \in B^n} X_g^n\right) = \underline{\lim}_n\, a_n |B^n| > \underline{\lim}_n\, a_n |F^n|(1-2\alpha) = \beta(1-2\alpha) > \frac{1}{2}$$

by choice of α. It follows that

$$\underline{\lim}_n\, \mu \left(h_n \cup_{g \in B^n} X_g^n \cap \cup_{f \in F^n} X_f^n\right) > 0\,.$$

Suppose now that there exist sequences $\{f_n\}$ and $\{g_n\}$ in B^n and F^n respectively such that

$$\overline{\lim}_n\, \mu \left(h_n X_{f_n}^n | X_{g_n}^n\right) = c > 0\,. \tag{18}$$

Then

$$\overline{\lim}_n\, \frac{\mu \left(f_n g_n^{-1} h_n X_{f_n}^n \cap X_{f_n}^n\right)}{\mu \left(X_{f_n}\right)} = c > 0\,,$$

and since $f_n \in B^n = F^n \backslash h_n^{-1} F^n$ and $g_n \in F^n$, then $f_n g_n^{-1} h_n \neq e$. This would contradict Lemma **??**. Thus equation (**??**) is false, and Property 3 of the definition of D-approximation is also satisfied. ■

Let $(X, \mathcal{B}, \mu, G)$ be a rank r group action, generated by $\{F^{n,i}\}$, for $i = 1, 2, \ldots, r$. Let h be an element of infinite order in G, and let $\alpha > 0$ be small and fixed. Choose sequences of powers of h, say $h_n^{(i)} := h^{k_n(i)}$, for $i = 1, 2, \ldots, r$ such that

$$\frac{\alpha}{2^{r-1}} < \frac{|(h_n^{(i)})^{-1} F^{n,i} \cap F^{n,i}|}{|F^{n,i}|} < \frac{2\alpha}{2^{r-1}}\,. \tag{19}$$

If $f \in F^{n,i}$, define

$$X_f^{n, i \to j} = \{x \in X_f^{n,i} : h_n^{(i)} x \in X^{n,j}\}\,.$$

Also define

$$X^{n,\,i\to j} = \cup_{\{f\in F^{n,\,i}\setminus (h_n^{(i)})^{-1}F^{n,\,i}\}} X_f^{n,\,i\to j}\,.$$

If

$$\overline{\lim_n}\,\mu\left(X^{n,\,i\to i}\right) > 0$$

for some $i\in\{1,2,\ldots,r\}$, then a D-approximation exists much in the same way that it existed for local rank actions: namely for this i, we let $A^n := (h_n^{(i)})^{-1}F^{n,\,i}\cap F^{n,\,i}$, $B^n := F^{n,\,i}\setminus A^n$, these sequences will do the trick (letting $F^n = G^n = F^{n,\,i}$).

Thus we will make the assumption that

$$\overline{\lim_n}\,\mu\left(X^{n,\,i\to i}\right) = 0$$

for $i = 1,2,\ldots,r$, since this is the only case that remains. In this case we will need some extra restrictions on our choices of $h_n^{(i)}$. We can choose the elements $h_n^{(i)}$ to satisfy both equation (**??**), and also

$$\frac{|\left(h_n^{(i)}\cdot\prod_{j\in J} h_n^{(j)}\right)^{-1} F^{n,\,i}\cap F^{n,\,i}|}{|F^{n,\,i}|} < \frac{2\alpha}{2^{r-1}} \tag{20}$$

where $J\subset\{1,\ldots,r\}$ and $i\notin J$. To see this, when choosing the $h_n^{(i)}$'s, simply let them be the highest power of l (the element of infinite order) such that

$$\frac{\alpha}{2^{r-1}} \le \frac{|(h_n^{(i)})^{-1}F^{n,\,i}\cap F^{n,\,i}|}{|F^{n,\,i}|} \le \frac{2\alpha}{2^{r-1}}\,.$$

Then using the properties listed in (**??**), with $\epsilon = \frac{\alpha}{2^{r-1}}$, it can be seen that (**??**) will hold. We will also have

$$|F_0^{n,i}| < 2\alpha \tag{21}$$

where

$$F_0^{n,i} \;=\; F^{n,\,i}\cap \bigcup_{\substack{J\subset\{1\ldots r\}\\ i\notin J}} \left(h_n^{(i)}\cdot\prod_{j\in J} h_n^{(j)}\right)^{-1} F^{n,\,i}.$$

For $f\in F^{n,\,i}$, $g\in F^{n,\,j}$, define

$$X_{f\to g}^{n,i\to j} := \{x\in X_f^{n,\,i} : h_n^{(i)}x\in X_g^{n,\,j}\}\,;$$

so $X_{f\to g}^{n,i\to j}\subset X_f^{n,i}$ for all $g\in F^{n,\,j}$. We say that $X_f^{n,i}$ goes ϵ-*heavily* into $X_g^{n,j}$ under $h_n^{(i)}$ if

$$\mu\left(X_{f\to g}^{n,i\to j}|X_f^{n,i}\right)\ge\epsilon\,,$$

and if $\mu\left(X^{n,i\to j}_{f\to g}|X^{n,i}_f\right)<\epsilon$, then we say that $X^{n,i}_f$ goes ϵ-*lightly* into $X^{n,j}_g$ under $h^(_n i)$. For $\epsilon>0$ and $f\in F^{n,\,i}$, define

$$X^{n,\,i\to j}_f(\epsilon):=\bigcup_{\{g\in F^{n,\,j}:\,\mu\left(X^{n,i\to j}_{f\to g}|X^{n,i}_f\right)<\epsilon\}} X^{n,i\to j}_{f\to g}$$

for $j=1,\dots r$. Thus $X^{n,\,i\to j}_f(\epsilon)$ is the part of $X^{n,i}_f$ which moves lightly into $X^{n,j}$.

Proposition 2 *Let (X,G) be a rank r mixing action, generated by the Følner sequences $\{F^{n,\,i}\}$, $i=1,2,\dots,r$, and let $h^{(i)}_n$ be as in formula (**??**). If*

$$\overline{\lim_n}\,\mu\left(X^{n,\,i\to i}\right)=0$$

for $i=1,2,\dots,r$, then there exist $i_0\neq j_0\in\{1,\dots,r\}$, and $\exists\,\delta>0\;\forall\,\epsilon>0\;\exists\,n=n(\epsilon)$ such that for $n\geq n(\epsilon)$, for all $f\in B^n\subset F^{n,\,i_0}\backslash F^{n,\,i_0}_0$ where $|B^n|\geq\delta\,|F^{n,\,i_0}|$, we have

$$\mu\left(X^{n,\,i_0\to j_0}_f(\epsilon)|\,X^{n,i_0}_f\right)\geq\delta\,. \tag{22}$$

Setting $A^n:=(h^{(i_0)}_n)^{-1}F^{n,\,i_0}\cap F^{n,\,i_0}$, and B^n as above, we obtain a D-approximation.

Proof: Suppose not. Then for all $\delta>0$, there exists some ϵ and a sequence $n_k\to\infty$ such that equation (**??**) with $n=n_k$ is true for less than $\delta\,|F^{n_k,\,i}|$ values of f in $F^{n_k,\,i}$, for all $i\in\{1,\dots,r\}$.

In Figure **??**, in the leftmost tower, region C represents the levels which are indexed by elements in $F^{n,1}_0$. Region C is fixed but small (it depends on the choice of α in formula (**??**)). Region B represents the levels in $X^{n_k,1}$ which do satisfy inequality (**??**), for some i_0, and region A represents the part of $X^{n_k,\,1}$ which moves ϵ-lightly into other towers. Both regions A and B can be made small by chosing δ small; and regions A, B and C are present and small in all towers.

We choose an $f\in F^{n_k,\,1}$ such that

- f is not indexing a level in regions B or C;
- $\exists j_1,\exists f_1\in F^{n_k,j_1}$ with f_1 not indexing a level in regions B or C, and such that $\mu\left(X^{n_k,\,1\to j_1}_{f\to f_1}|X^{n,1}_f\right)\geq\epsilon$,
- $h^{(1)}_{n_k}X^{n_k,\,1}_f\cap X^{n_k,\,j_1}_{f_1}$ is in the part of $X^{n_k,j_1}_{f_1}$ which moves ϵ-heavily out of $X^{n_k,\,j_1}$ under $h^{(j_1)}_{n_k}$, into $X^{n_k,\,j_2}_{f_2}$ where $f_2\in F^{n_k,j_2}\backslash F^{n_k,j_2}_0$, so that

$$\mu\left(h^{(j_1)}_{n_k}h^{(1)}_{n_k}(X^{n_k,1}_f)\cap X^{n_k,j_2}_{f_2}|X^{n_k,1}_f\right)\geq\epsilon^2\,.$$

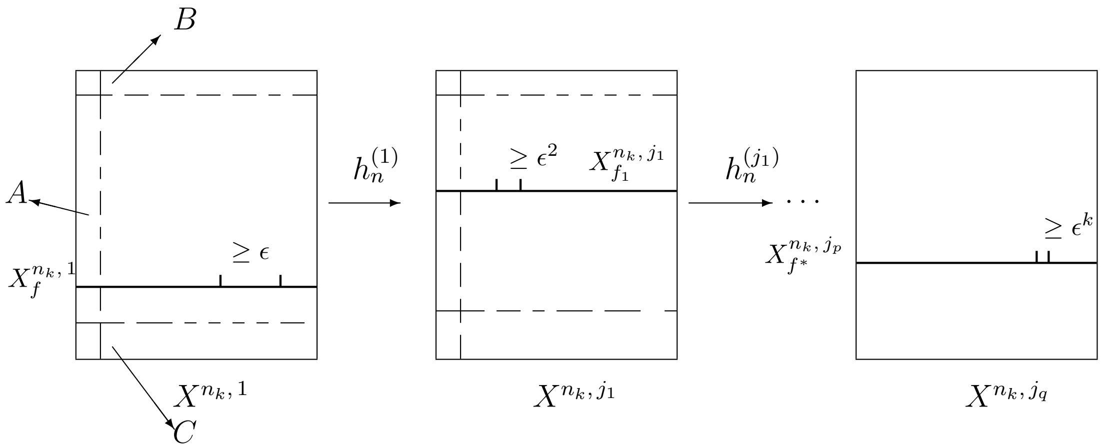

Figure 1: How one level moves heavily into following towers

- Repeating this process at most r times, until we see the same tower twice in our picture, we will find a $j_p \in \{1, \dots, r\}$, and $f_p \in F^{n_k,j_p} \backslash F_0^{n_k,j_p}$, and $f^* \in F^{n,j_p}$ such that

$$\mu\left(h_{n_k}^{(j_q)} \dots h_{n_k}^{(j_p)} X_{f_p}^{n_k,j_p} \cap X_{f^*}^{n_k,j_p} | X_{f_p}^{n_k,j_p}\right) \geq \epsilon^r .$$

Since $f_p \in F^{n_k,j_p} \backslash F_0^{n_k,j_p}$, then

$$h_{n_k}^{(j_q)} \dots h_{n_k}^{(j_p)} (f^*)^{-1} f_p \neq e .$$

Now it is possible to obtain a contradiction to mixing, using Lemma **??**.

We now construct the D-approximation. Let $\epsilon_n \to 0$, and, by passing to a subsequence if necessary, find a sequence $\{B^n\}$ such that elements in B^n satisfy equation (**??**) for $\epsilon = \epsilon_n$. Suppose wlog that we found $i_0 = 1,\ j_0 = 2$ in Proposition **??**. Then let $h_n := (h_n^{(1)})^{-1},\ \ A^n := h_n^{(1)} F^{n,1} \cap F^{n,1}$. Furthermore, for $g \in B^n$, let

$$Y_g^n := X_g^{n,1\to 2}(\epsilon_n) = \bigcup_{\{f \in F^{n,2} :\, \mu(X_{g\to f}^{n,1\to 2} | X_g^{n,1}) < \epsilon\}} X_{g\to f}^{n,1\to 2} .$$

Then

$$\varliminf_n \mu\left(\bigcup_{g \in B^n} Y_g^n\right) \geq \varliminf_n \delta\, |F^{n,1}| \delta a^{n,1} = \delta^2/2 .$$

Also,

$$h_n\left(\bigcup_{g \in B^n} Y_g^n\right) \subset \bigcup_{f \in F^{n,2}} X_f^{n,2} ,$$

so that

$$\varliminf_n \mu\left(h_n\left(\bigcup_{g\in B^n} Y_g^n\right)\bigcap\left(\bigcup_{f\in F^{n,2}} X_f^{n,2}\right)\right)=\delta^2/2\,.$$

Finally, note that by construction, for any $g \in B^n$, $f \in F^{n,2}$, we have

$$\mu\left(h_n Y_g^n | X_f^{n,2}\right) < \epsilon_n \to_n 0\,,$$

i.e. condition 3 of the definition of D-approximation is also satisfied. ∎

We have therefore shown that a D-approximation exists for all rank r mixing group actions (X, G) where G is countable, Abelian, and with an element of infinite order.

Henceforth we will assume that A^n, $B^n \subset F^{n,1}$ and that

$$h_n\left(\bigcup_{g\in B^n} Y_g^n\right)\subset X^{n,2}$$

- the other cases are similar. If $(k,l,m) \in A^n \times A^n \times B^n$, where A^n, B^n are the sets coming out of the D-approximation, then we will denote the cube (or rather cuboid) $X_k^{n,1} \times X_l^{n,1} \times Y_m^n$ by $Y_{k,l,m}^n$. We can then form the sets

$$R^n := \bigcup_{(k,l,m)\in A^n\times A^n\times B^n} Y_{k,l,m}^n \tag{23}$$

and

$$S^n := \bigcup_{(k,l,m)\in h_nA^n\times h_nA^n\times F^{n,2}} X_{k,l,m}^n\,. \tag{24}$$

If a pairwise independent ergodic measure gave these sets nontrivial measure, then we would be set to prove the mixing theorem for finite rank actions. This need not be the case, but Proposition **??** shows that we can find induced joinings which give the towers R^n (and therefore S^n) measure almost as large as the product measure of these towers; and furthermore these same induced joinings will also give the set $\left(\bigcup_{(k,l)\in A^n\times A^n} X_{k,l}^n\right)\times X$ measure comparable to its product mass. The latter will insure that the induced joinings have a large pairwise independent component, as we shall see in Lemma **??**.

In what follows, the sequence of sets $\{U_n\}$ is the sequence $\{\cup_{f\in A^n} X_f^{n,1}\}$ and $\{U_n'\}$ is the sequence $\{\cup_{g\in B^n} Y_g^n\}$. We recall Ramsey's theorem (see for example [**?**]): given $l \in \mathbf{N}$, there exists $r(l)$ such that if A is any subset of a countable set H of cardinality $r(l)$, with $*$ a 2-colouring (a binary relation) of A, then A contains a set B of cardinality l such that either $b * b'$ for all b, $b' \in B$ or $\neg(b * b')$ for all b, $b' \in B$.

The following Lemma will be used in the proof of Proposition **??**.

Lemma 4 *Suppose* $F_1,\ F_2,\ \ldots, F_l\ \in \mathbf{L}_2\,(\mu)$, $0 \le F_i \le 1$ *and* $\int F_i\, d\mu\ = u$ *for* $i = 1, 2, \ldots, l$. *Then there exist distinct* $i,\ k$ *such that*

$$\langle F_i, F_k\rangle\ \ge\ \frac{u^2 - l^{-1}u}{1 - l^{-1}}\ \ge\ u^2 - \frac{1}{l-1}\ .$$

Proof: Note that since $0 \le F_i \le 1$, then

$$\langle F_i, F_i\rangle\ \le\ \langle F_i, 1\rangle\ .$$

If $\phi := \sum_{i=1}^{l} F_i$, then by the Cauchy-Schwarz inequality,

$$< \phi, \phi\ \ > \ge\ < \phi, 1 >^2 = (lu)^2\ .$$

Hence

$$\begin{aligned} &\max_{i \ne k}\ \langle F_i, F_k\rangle \\ \ge\ & \frac{< \phi, \phi > - \sum_{i=1}^{l} \langle F_i, F_i\rangle}{l^2 - l} \\ \ge\ & \frac{l^2u^2 - l\,u}{l^2 - l} = \frac{u^2 - l^{-1}u}{1 - l^{-1}}\ \ge\ u^2 - \frac{1}{l-1}\,. \end{aligned}$$

■

Proposition 3 *Let* $(X,\ \mathcal{B},\ G,\ \mu)$ *be a group action,* ν *be pairwise independent and* $\{\eta_{g,k}\}_{g,k\,\in G}$ *the family of induced measures generated by* ν. *Let* $c_j \uparrow 1$. *If* $\{U_n\}$, $\{U_n^{'}\}$ *are sequences of measurable sets such that*

$$\lim_n \mu\,(U_n) = d > 0\,,\quad \lim_n \mu\,(U_n^{'}) = d' > 0,$$

then there exist sequences $\{g_j\}$, $\{k_j\}$ *of group elements of infinite order such that*

$$\lim_n g_j k_j^{-1} = \infty,$$

and

$$\overline{\lim_n}\ \eta_{g_j,k_j}(U_n \times U_n \times X) \ge c_j d^2 \tag{25}$$

and

$$\overline{\lim_n}\ \eta_{g_j,k_j}(U_n \times U_n \times U_n^{'}) \ge c_j d^2 d' \tag{26}$$

for each $j\ \in \mathbf{N}$.

Proof:

Fix $n\ \in \mathbf{N}$ for the time being. Choose a natural number l large enough so that

$$\frac{\mu\,(U_n)^2 - l^{-1}\mu\,(U_n)}{1 - l^{-1}} > \mu\,(U_n)^2 c_1\ .$$

Note that l can be chosen independently of large n. Let

$$H := \{h \in G : h \text{ is of infinite order}\}.$$

Choose and fix a set $A \subset H$ such that $|A| = r(l)$, where $r(l)$ is given by Ramsey's Theorem. Let $g \in G$. Define $F_g : X^2 \to [0,1]$ as

$$F_g(x,y) = \nu_{(gx,y)}(U_n).$$

Then if g, $k \in G$, and $\mu_n := \mu_{U'_n}$, we have

$$\begin{aligned} \eta_{g,k}(U_n \times U_n \times U'_n) &= \mu(U'_n) \int_X \int_X F_g(x,y) F_k(x,y) d\mu(y) d\mu_n(x) \\ &= \mu(U'_n) \langle F_g, F_k \rangle_{\mu \times \mu_n}. \end{aligned}$$

Assumption: Suppose that there is no pair of group elements g, k in A such that both

$$\mu(U'_n) \langle F_g, F_k \rangle_{\mu \times \mu_n} = \eta_{g,k}(U_n \times U_n \times U'_n) \geq c_1 \mu(U_n)^2 \mu(U'_n) \tag{27}$$

and

$$\langle F_g, F_k \rangle = \eta_{g,k}(U_n \times U_n \times X) \geq c_1 (\mu(U_n))^2. \tag{28}$$

hold. Define the 2-colouring $*$ ($*$ depends on n) on A as: $g * k$ iff equation (**??**) does hold for the pair g, k. If $g * k$, then because of our assumption, equation (**??**) does not hold for the pair g, k.

By Ramsey's Theorem, A contains a set B of cardinality l where either $g * k$ for all g, $k \in B$, or $\neg(g * k)$ for all g, $k \in B$. (B is really B_n and $* = *_n$.) If $g * k$ for all g, $k \in B$, then equation (**??**) does not hold for all g, $k \in B$, i.e.

$$\langle F_g, F_k \rangle_{\mu \times \mu_n} < c_1 (\mu(U_n))^2.$$

Similarly if $\neg(g * k)$ for g, $k \in B$, then

$$\langle F_g, F_k \rangle < c_1 (\mu(U_n))^2$$

for g, $k \in B$. Lemma **??** tells us that neither possibility can happen. So our initial assumption was false, i.e. there exist elements g, $k \in A$ such that both equations (**??**) and (**??**) hold. Repeating this argument for all large n, (with the same set A) there exists a sequence of pairs of elements $g_1(n)$, $k_1(n)$ in A such that both equations (**??**) and (**??**) hold. As A is finite, then there exists a pair g_1, k_1 which work for infinitely many n's, i.e. such that both equations (**??**) and (**??**) hold for $j = 1$. The proof can be repeated for any $j \in \mathbf{N}$, except that we vary the set $A = A(j)$, taking it to consist of sufficiently spaced integers so that $g_j k_j^{-1} \to \infty$; and only working with the subsequence of n's that one obtained at stage $j-1$. Taking a diagonal subsequence, we have equations (**??**) and (**??**) for all $j \in \mathbf{N}$.

■

5 Long Orbits and Almost Pairwise Independence

In this section $(\mathcal{X}, G)$ is a rank r or local rank ($\beta > 1/2$) mixing G-action as in Theorem **??**, and $\eta \in M(1,3)$. For $(f,\, g,\, h) \in F^{n,\,i} \times F^{n,\,j} \times F^{n,\,k}$ where $1 \le i,\, j,\, k \le r$, we denote $X_f^{n,i} \times X_g^{n,j} \times X_h^{n,\,k}$ by $X_{f,g,h}^{n,\,i,j,k}$ and call this object a cube. For $1 \le i,\, j,\, k \le r$, we let

$$X^{n,\,i,j,k} := X^{n,\,i} \times X^{n,\,j} \times X^{n,\,k}\,;$$

i.e. $X^{n,\,i,j,k}$ stands for the product of the towers i, j, and k at the n-th stage. By an orbit in G^3 we mean any

$$G^3_{g_0,g_1} := \{(l, lg_0, lg_1) : l \in G\},$$

that is, an orbit of the diagonal action of G on G^3 by translation. By an *n-orbit* in $X^{n,\,i,j,k}$ we mean

$$\mathcal{O}^{n,\,i,j,k}_{g_0,g_1} := \bigcup_{\{l \in F^{n,\,i} \cap F^{n,\,j} g_0^{-1} \cap F^{n,\,k} g_1^{-1}\}} X^{n,\,i,j,k}_{(l,lg_0,lg_1)}$$

Thus each $X^{n,\,i,j,k}$ is partitioned into n-orbits and each n-orbit is a union of cubes of equal η-measure. The length of an n-orbit is the number of cubes in it. For $(f,g,h) \in F^{n,\,i} \times F^{n,\,j} \times F^{n,\,k}$, $\mathcal{O}(X^{n,\,i,j,k}_{f,g,h})$ will denote the n-orbit containing the cube $X^{n,\,i,j,k}_{f,g,h}$.

Let us say an n-orbit in $X^{n,\,i,j,k}$ is δ-long if its length is at least $\delta \min_{1 \le i \le r} |F^{n,\,i}|$ (otherwise we call it δ-short) and let $\mathcal{O}^n(\delta^c)$ denote the union of the δ-short orbits, and $\mathcal{O}^{n,\,i,j,k}(\delta^c)$ the union of the δ-short orbits in $X^{n,\,i,j,k}$. Similarly, we call $F^{n,\,i} \cap gF^{n,\,j} \cap g^* F^{n,\,k}$ a δ-short intersection if $|F^{n,\,i} \cap gF^{n,\,j} \cap g^* F^{n,\,k}| \le \delta \min_{1 \le i \le r} |F^{n,\,i}|$. In the next lemma we will show that the η-mass of $\mathcal{O}^{n,\,1,1,1}(\delta^c)$ is small in the limit. The argument can be copied to give the smallness of $\eta(\mathcal{O}^{n,\,i,j,k}(\delta^c))$ for any $1 \le i,\, j,\, k \le r$. Summing over these r^3 possibilities will give that $\underline{\lim}_n\, \eta(\mathcal{O}^n(\delta^c)) \le r^3 p^2\, \delta$.

Lemma 5 *If p is an intersection bound for $\{F^{n,\,1}, F^{n,\,2}, \ldots, F^{n,\,r}\}$ then*

$$\overline{\lim_n}\, \eta(\mathcal{O}^{n,\,1,1,1}(\delta^c) \le p^2 \delta\,.$$

Proof: We fix n; to ease notation, we assume that $e \in F^{n,\,1}$, so that $X_e^{n,\,1}$ will be considered as the "base" of the tower $X^{n,\,1}$. We will consider all finite subsets $\gamma = \{g_1,\, g_2,\, \ldots, g_q\}$ of G with the properties that the $\{g_k F^{n,\,1}\}_{k=1}^q$ are pairwise disjoint and all intersect $F^{n,\,1}$ non-trivially , so $|\gamma| \le p$. We call such a γ a *configuration* and let Γ denote the space of all configurations. Clearly Γ

is finite. If $\gamma = \{g_1, g_2, \ldots, g_q\}$ is a configuration and $k \in G$, then $k\,\gamma$ is the set $\{kg_1, kg_2, \ldots kg_q\}$.

For $x \in X$ we let $R^{n,1}(x) = \{g \in G : gx \in X_e^{n,1}\}$, the set of "return times" to the base of the n-th tower $X^{n,1}$. By an n-*block* in x we mean any $gF^{n,1}$ with $g \in R^{n,1}(x)$, and g is called the *base time* of this n-block. The n-blocks in x are disjoint translates of $F^{n,1}$. For $x \in X^{n,1}$ we denote by $B^n(x)$ the n-block in x containing $e \in G$, namely $B^n(x) = k^{-1}F^{n,1}$ if $x \in X_k^{n,1}$.

For $(x, y) \in X_k^{n,1} \times X$ we create a configuration $\gamma(x, y)$ by letting $g_1, \ldots, g_q$ denote the base times of the n-blocks in y which intersect $B^n(x)$ and defining

$$\gamma(x, y) = k\,\{g_1, g_2, \ldots, g_q\} = \{kg_1, \ldots, kg_q\}\,. \tag{29}$$

For $(x, y, z) \in X_k^{n,1} \times X^2$, if

$$\gamma(x, z) = k\{h_1, h_2, \ldots, h_s\}, \tag{30}$$

then let

$$\gamma_1(x, y, z) := \gamma(x, y) \text{ and } \gamma_2(x, y, z) := \gamma(x, z).$$

We view the map $Q : (x, y, z) \to (\gamma_1(x, y, z), \gamma_2(x, y, z))$ as a partition of $X^{n,1} \times X^2$ indexed by Γ^2, so we will write $Q_{(\gamma_1,\gamma_2)} = Q^{-1}((\gamma_1, \gamma_2))$. Thus we are partitioning $X^{n,1} \times X^2$ according to the pattern, up to a shift, formed by $B^n(x)$ and the $F^{n,1}$-blocks in y and z which intersect $B^n(x)$.

For $(\gamma_1, \gamma_2) \in \Gamma^2$, and $k \in F^{n,1}$ let

$$Q_{(\gamma_1,\gamma_2),k} := \{(x, y, z) \in Q_{(\gamma_1,\gamma_2)} : x \in X_k^{n,1}\} \subset X_k^{n,1} \times X^2\,.$$

We call $Q_{(\gamma_1,\gamma_2),k}$ a *cell* of $Q_{(\gamma_1,\gamma_2),}$. We claim that

$$\eta\,(Q_{(\gamma_1,\gamma_2),k}) = \eta(Q_{(\gamma_1,\gamma_2),e})$$

for each $k \in F^{n,1}$. This is true as $Q_{(\gamma_1,\gamma_2),e}$ is mapped to $Q_{(\gamma_1,\gamma_2),k}$ by the transformation k.

Define

$$S((\gamma_1, \gamma_2), 1, 1, 1) := \bigcup_{\{(g,h): g\in\gamma_1,\ h\in\gamma_2\}} F^{n,1} \cap gF^{n,1} \cap hF^{n,1}\,.$$

It can be verified that $k^* \in S((\gamma_1, \gamma_2), 1, 1, 1)$ iff $Q_{(\gamma_1,\gamma_2),k^*} \subset X^{n,1,1,1}$.

Now suppose that $k \in F^{n,1} \cap gF^{n,1} \cap hF^{n,1}$, and suppose that $\mathcal{O}(X_{k,l,m}^{n,1,1,1}) \cap Q_{(\gamma_1,\gamma_2),k}$ is nontrivial. Then $\mathcal{O}(X_{k,l,m}^{n,1,1,1})$ intersects $Q_{(\gamma_1,\gamma_2),k^*}$ nontrivially, whenever $k^* \in F^{n,1} \cap gF^{n,1} \cap hF^{n,1}$. Thus δ- short orbits can only intersect cells which belong to a δ-short intersection. There are at most p^2 intersections. Noting that the cells of $Q_{(\gamma_1,\gamma_2)}$ all have the same η-mass, we have

$$\eta(\mathcal{O}^{n,1,1,1}(\delta^c)) \leq \sum_{\gamma_1,\gamma_2} p^2\,\delta|F^{n,1}|\eta(Q_{(\gamma_1,\gamma_2),e}) \leq p^2\delta|F^{n,1}|\mu\,(X_e^{n,1})$$

since the $Q_{(\gamma_1,\gamma_2),k}$ form a partition of $X_k^{n,1} \times X^2$ and $\eta \in M(1,3)$. Letting $n \to \infty$, we get the result.

■

Note that as long as we have this bounded intersection property for Følner sequences, the proof of Lemma **??** can be copied to give that the δ-short orbits in $X^{n,i} \times X^{n,j}$ are of small η-mass, for *any* joining $\eta \in M(1,2)$ (ergodic or not) which preserves the action of G. This is used in Lemma **??**. The same is true when looking at a local rank transformation - that the δ-short orbits in $X^n \times X^n \times X^n$ are of small η-mass, whenever $\eta \in M(1,3)$.

Lemma 6 *Suppose* $(X, \mathcal{B}, \mu, G, \{A^n\})$ *is a* β*-local mixing action. Let* η *be a (not necessarily ergodic) 2-joining of* $(X, \mathcal{B}, \mu, G)$ *with itself and suppose that*

$$\lim_n \eta\left(\bigcup_{(k,l)\in A^n\times A^n} X_{k,l}^n\right) = c > 0\,. \tag{31}$$

Then η *either has an off-diagonal measure as a (nontrivial) component, or* μ^2 *as a nontrivial component. In the case when* $\eta = \pi_{1,2}\eta_{g,k}$ *is the projection of an induced joining, then* $\eta_{g,k} \geq c\beta^{-2}\mu^2$*, where*

$$\beta = \lim_n \mu\left(\bigcup_{k\in A^n} X_k^n\right)\,.$$

Proof:

Let

$$\mathcal{O}^n(\delta) := \bigcup_{\{(k,l)\in A^n\times A^n : |A^n k^{-1}\cap A^n l^{-1}|\geq \delta |A^n|\}} \mathcal{O}\left(X_{k,l}^n\right).$$

Fix some small δ, such that

$$\varliminf_n \eta\left(\mathcal{O}^n(\delta)\right) > 0\,,$$

By passing to a subsequence if necessary, assume that this liminf is a limit. A limit point of the measures η^n will be a nontrivial component of η, by Lemma **??**. We can write η^n as

$$\eta^n = \sum_{\{(k,l):|A^n k^{-1}\cap A^n l^{-1}|\geq\delta|A^n|\}} a_{k,l}^n \eta_{k,l}^n$$

where $\eta_{k,l}^n := \eta_{\mathcal{O}(X_{k,l}^n)}$ and $a_{k,l}^n := \frac{\eta(\mathcal{O}\left(X_{k,l}^n\right))}{\eta(\mathcal{O}^n(\delta))}$. Define

$$\lambda^n := \sum_{\{(k,l):|A^n k^{-1}\cap A^n l^{-1}|\geq\delta|A^n|\}} a_{k,l}^n \Delta_{k,l}^n$$

where $\Delta^n_{k,l}$ is the measure $\Delta_{k,l}$ conditioned on the set $\mathcal{O}(X^n_{k,l})$. (Recall that $\Delta_{k,l}$ is the graph joining: $\Delta_{k,l}(A \times B) := \mu\ (lk^{-1}A \cap B)$). Since $(X, G, \{A^n\})$ is β-local, then $\{\lambda^n\}$ and $\{\eta^n\}$ have the same limit points. To see this, note that $\lambda^n\ (A_1 \times A_2) = \eta^n\ (A_1 \times A_2)$ whenever A_1, A_2 are unions of levels in X^j, for $n \geq j$. If A_1, A_2 are arbitrary measurable, find sequences $\{A^j_i\}_{j\in\mathbf{N}}$ for $i = 1, 2$ such that A^j_i is a union of levels in X^j and

$$\mu\ \left((A_i \cap X^j)\Delta A^j_i\right) \to_j 0\,.$$

Since

$$\lambda^n\left((A_1 \times A_2 \cap X^j \times X^j)\Delta(A^j_1 \times A^j_2)\right) \leq \sum_{i=1}^{2} \lambda^n\left(((A_i \cap X^j)\Delta A^j_i) \times X\right), \quad (32)$$

then as $j \to \infty$, this quantity tends to 0 for each n. Inequality (**??**) also holds for η^n, so for a fixed j, the quantity on the right hand side of equation (**??**) will also be small, Thus λ^n, η^n have the same limit points.

There may exist a pair of group elements (k, l), and a subsequence of n's such that $\lim_n a^n_{(k,l)} > 0$, in which case by ergodicity of μ, and therefore of $\Delta_{k,l}$,

$$\begin{aligned} \eta &\geq \underline{\lim_n}\, \eta\left(\mathcal{O}^n(\delta)\right) \lim_n \eta^n = \underline{\lim_n}\, \eta\left(\mathcal{O}^n(\delta)\right) \lim_n \lambda^n \\ &\geq \underline{\lim_n}\, \eta\left(\mathcal{O}^n(\delta)\right) \lim_n a^n_{(k,l)} \lim_n \Delta^n_{k,l} = \underline{\lim_n}\, \eta\left(\mathcal{O}^n(\delta)\right) \lim_n a^n_{(k,l)} \Delta_{k,l}, \end{aligned}$$

so that η has an off-diagonal measure as a component. If no such (k, l) exists then $\lim_n a^n_{k,l} = 0$ for each pair (k, l), in which case by mixing,

$$\sum_{\{(k,l):|A^n k^{-1} \cap A^n l^{-1}| \geq \delta |A^n|\}} a^n_{k,l} \Delta_{k,l} \to_n \mu^2\,,$$

and by the ergodicity of μ^2 and the fact that λ^n is a nontrivial component of the above measure, it follows that the only limit point of λ^n is μ^2. Thus μ^2 is a component of η.

We now look at what happens when $\eta = \pi_{1,2}\eta_{g,k}$ is the projection of an induced joining. In this case we discuss why the first possibility, that $\pi_{1,2}\eta_{g,k}$ has a diagonal measure as a nontrivial component, leads to a contradiction. Suppose that

$$\pi_{1,2}\eta_{g,k} \geq a\Delta_{(p,q)}$$

for some $p,\ q$, in G. Thus if A, B are measurable sets, we have

$$\int_X < J^*_{gx} J_{kx} \chi_{q^{-1}pA},\ \chi_B > d\mu\,(x) \geq a\Delta\,(A \times B)\ ,$$

where Δ is diagonal measure. Let $D(X) := \{(x,x) : x \in X\}$. Expanding the measure on the left hand side, we have

$$\int_X \int_X \nu_{kx,y} \circ q^{-1}p \times \nu_{gx,y} \, d\mu\,(y) \, d\mu\,(x) \geq a\Delta$$

so that

$$\mu^2 \left(\{(x,y) : \nu_{kx,y} \circ q^{-1}p \times \nu_{gx,y}(D(X)) > 0\}\right) > 0\,.$$

Now the only way that the measure $\nu_{kx,y} \circ q^{-1}p \times \nu_{gx,y}$ can give positive mass to $D(X)$ is if $\nu_{kx,y} \circ q^{-1}p$ and $\nu_{gx,y}$ have common discrete components: say

$$\nu_{gx,y} \;\geq\; a_{(x,y)}\, \delta_{t(x,y)}$$

and

$$\nu_{kx,y} \circ q^{-1}p \;\geq\; a_{(x,y)}\, \delta_{t(x,y)}$$

where $\delta_{t(x,y)}$ is the point mass at $t(x,y) \in X$ and $0 < a_{(x,y)} \leq 1$. Thus if $\nu' := \nu \circ (q^{-1}p \times e \times k^{-1})$ and $\nu'' := \nu \circ (e \times e \times g^{-1})$, i.e.

$$\nu''\,(A \times B \times C) = \int_{g^{-1}C} \nu_x\,(A \times B)\, d\mu\;(x) = \int_C \int_B \nu_{gx,y}(A) d\mu\,(y) d\mu\,(x)\,,$$

and

$$\nu'\,(A \times B \times C) = \int_C \int_B \nu_{kx,y} \circ q^{-1}p(A) d\mu\,(y) d\mu\,(x)$$

then ν', ν'' have a common nontrivial component. By assumption, ν is ergodic, and hence so are ν', ν'', both being isomorphic to $(X^3,\, \mathcal{B} \times \mathcal{B} \times \mathcal{B},\, G,\, \nu)$. Hence $\nu' = \nu''$. Thus ν' is a (ergodic) joining of the ergodic system $(X^2,\, \mathcal{B} \times \mathcal{B},\, q^{-1}p \times k^{-1}g,\, \mu^2)$ with $(X,\, \mathcal{B},\, e,\, \mu)$. Recall that $k^{-1}g$ was chosen to have infinite order, and if $q^{-1}p$ has infinite order, then $(X^2,\, \mathcal{B} \times \mathcal{B},\, q^{-1}p \times k^{-1}g,\, \mu^2)$ is ergodic and so ν' must be product measure, which is only possible if ν itself were. If $q^{-1}p$ has finite order, then ν' joins a mixing system $((X,\, \mathcal{B},\, \mu, k^{-1}g))$ with a periodic one, and this is also not possible (since then ν' joins the power of a mixing system, which is ergodic, with the identity) unless ν' (and hence ν) were product measure.

It remains to be shown that the "size" of this component is $c\beta^{-2}$ when $\eta = \pi_{1,2}\eta_{g,k}$. Let

$$\eta = kd\,\mu^2 + \theta$$

be the Radon-Nikodym decomposition of η with respect to μ^2, where θ is a measure which is singular with respect to μ^2 and $k > 0$. Now

$$\overline{\lim_n}\, \theta \left(\bigcup_{(p,q) \in A^n \times A^n} X^n_{p,q} \right) = 0\,;$$

or else we could repeat the whole above argument to conclude that θ has μ^2 as a component (by the above argument, η, and therefore θ can't have an off diagonal as a component), and this contradicts the fact that $\theta \perp \mu^2$. Hence

$$c = \lim_n \eta \left(\bigcup_{(p,q)\in A^n \times A^n} X^n_{p,q} \right) = k \lim_n \mu^2 \left(\bigcup_{(p,q)\in A^n \times A^n} X^n_{p,q} \right) = k\beta^2$$

or $k = c\beta^{-2}$. ■

We have shown therefore that if $\eta_{g,k}$ is an induced joining satisfying equation (??), then

$$\pi_{1,2}\eta_{g,k} = \kappa\mu^2 + \theta\,, \tag{33}$$

where $\kappa \geq c\beta^{-2}$, and $\theta \perp \mu^2$. Note that (??) implies that there exists $\eta^*_{g,k}$ pairwise independent with

$$\eta^*_{g,k} \geq \kappa\eta_{g,k} \geq c\,\beta^{-2}\,\eta_{g,k}\,.$$

6 Proof of the Main Theorem

Recall (formulas (??) and (??)) that

$$R^n := \bigcup_{(k,l,m)\in A^n \times A^n \times B^n} Y^n_{k,l,m}$$

and

$$S^n := \bigcup_{(k,l,m)\in h_n A^n \times h_n A^n \times F^{n,\,2}} X^n_{k,l,m}$$

were the products of towers constructed using the sets A^n and B^n coming from the D-approximation. R^n and S^n were constructed so that the third component of R^n moved μ-lightly into the third component of S^n. In the next lemma we move this up to the level of joinings. Namely we shall show that if η is pairwise independent such that

$$\overline{\lim_n}\, \eta(h_n R^n \cap S^n) > 0$$

then "one of R^n or S^n will have substantially many $\eta-$light cubes." This is made rigorous in Lemma **??**.

Let $c_j \uparrow 1$. Proposition **??** gives us a sequence of induced joinings η_{g_j,k_j} which satisfy

$$\overline{\lim_n}\, \eta_{g_j,k_j}\,(R^n) \geq c_j d^2 d'\,,$$

and

$$\overline{\lim_n}\, \eta_{g_j,k_j} \left(\left(\bigcup_{k,l\,\in A^n \times A^n} X^n_{k,l} \right) \times X \right) \geq c_j d^2\,,$$

where $\lim_n \mu\left(\cup_{k \in A^n} X_k^n\right) = d > 0$, and $\lim_n \mu\left(\cup_{k \in B^n} Y_k^n\right) = d' > 0$,

Lemma **??** tells us that

$$\eta_{g_j,k_j} \geq c_j \eta^*_{g_j,k_j}$$

where $\eta^*_{g_j,k_j}$ is pairwise independent. Thus for a large enough j,

$$\overline{\lim_n}\, \eta^*_{g_j,k_j}(R^n) > c_j^* d^2 d^{'} ,$$

where $c_j^* \to_j 1$. We will fix such a large enough j, and for ease of notation write this measure $\eta^*_{g_j,k_j} = \eta^j$. We define a cube $Y^n_{k,l,m} \in R^n$ to be ϵ-light (for η) if $\eta(Y^n_{k,l,m}) \leq \epsilon (a^{n,1})^2$; similarly for a cube in S_n. Let

$$L_\epsilon^n = \bigcup \{Y^n_{k,l,m} \,:\, (k,l,m) \in A^n \times A^n \times B^n \text{ and } Y^n_{k,l,m} \text{ is } \epsilon - light\} ,$$

and

$$\underline{L}_\epsilon^n = \bigcup \{X^n_{k,l,m} : (k,l,m) \in h_n A^n \times h_n A^n \times F^{n,2} \text{ and } X^n_{k,l,m} \text{ is } \epsilon - light\} .$$

Let $Di(\eta) = \lim_{\epsilon \to 0} \underline{\lim}_n\, \eta(L^n_\epsilon)$, and $\underline{D}i(\eta) = \lim_{\epsilon \to 0} \underline{\lim}_n\, \eta(\underline{L}^n_\epsilon)$.

Lemma 7 *Let $\eta \in M(2,3)$ where $(X, \mathcal{B}, \mu, G)$ is finite rank or β-local rank, $\beta > 1/2$, and G is countable, Abelian, and has an element of infinite order. If*

$$\overline{\lim_n}\, \eta(R^n) \,>\, 0 ,$$

then $Di(\eta) > 0$ or $\underline{D}i(\eta) > 0$.

Proof: Suppose that both $Di(\eta) = 0$ and $\underline{D}i(\eta) = 0$. Given any ι small, there exists a (fixed) ϵ such that $\underline{\lim}_n\, \eta\left(L^n_\epsilon\right) < \iota$ and $\underline{\lim}_n\, \eta\left(\underline{L}^n_\epsilon\right) < \iota$. If $C^n_{k,l} := \cup_{m \in B^n} Y^n_{k,l,m}$, then $h_n C^n_{k,l} \subset \underline{C}^n_{h_n k, h_n l} := \cup_{m \in F^{n,2}} X^n_{h_n k, h_n l, m}$. Since η is pairwise independent, then

$$\eta\left(C^n_{k,l}\right) \,\leq\, \eta\left(X^n_{k,l} \times X\right) = (a^{n,1})^2 ;$$

so there exist at most $1/\epsilon$ ϵ-heavy cubes in $C^n_{k,l}$, and similarly for $\underline{C}^n_{h_n k, h_n l}$. If H^n_ϵ, $\underline{H}^n_\epsilon$ are the complements of L^n_ϵ, $\underline{L}^n_\epsilon$ in R^n, S^n respectively, then if we have chosen ι small enough, we can state that

$$\overline{\lim_n}\, \eta(h_n H^n_\epsilon \,\cap\, \underline{H}^n_\epsilon) > 0 . \tag{34}$$

Since η is pairwise independent then for any $m \in B^n$, $m' \in F^{n,2}$,

$$\eta\left(h_n(Y^n_{k,l,m}) \cap X^n_{h_n k, h_n l, m'}\right) \leq a^{n,1} a^{n,2} \mu\left(h_n Y^n_m | X^n_{m'}\right) .$$

Now the image of an ϵ-heavy cube in $C^n_{k,l}$ under h_n can intersect at most $1/\epsilon$ heavy cubes in $\underline{C}^n_{h_nk,h_nl}$. Thus

$$\begin{aligned}\mu\left(h_nH^n_\epsilon\,\cap\,\underline{H}^n_\epsilon\right) &= \sum_{k,l\,\in A^n}\mu\,\left(h_n(C^n_{k,l}\cap H^n_\epsilon)\cap(\underline{C}^n_{h_nk,h_nl}\cap\underline{H}^n_\epsilon)\right)\\ &\le \sum_{k,l\,\in A^n}\frac{1}{\epsilon^2}a^{n,1}a^{n,2}\sup_{\{m\,\in B^n,\,m'\in F^{n,\,2}\}}\mu\,\left(h_nY^n_m|X^n_{m'}\right)\\ &\le |F^{n,\,1}|^2\frac{1}{\epsilon^2}a^{n,1}a^{n,2}\sup_{\{m\,\in B^n,\,m'\in F^{n,\,2}\}}\mu\left(h_nY^n_m|X^n_{m'}\right)\end{aligned}$$

The sets R^n and S^n were built in such a way (Property 3 in the definition of D-approximation) that

$$\overline{\lim_n}\sup_{\{m\,\in B^n,\,m'\in F^{n,\,2}\}}\mu\left(h_nY^n_m|X^n_{m'}\right)=0\,.$$

Since ϵ is fixed, and we are assuming that $0<\underline{\lim}\,|F^{n,\,1}|/|F^{n,\,2}|<\infty$, then

$$\mu\left(h_nH^n_\epsilon\,\cap\,\underline{H}^n_\epsilon\right)\to 0\,.$$

This contradicts equation (**??**) .

■

Now we can proceed to show that

Theorem 4 *If $\eta\in M(2,3)$ is ergodic and $Di(\eta)>0$ or $\underline{Di}(\eta)>0$, then $\eta=\mu^3$.*

Proof: Assume that $Di(\eta)>0$ (the case where $\underline{Di}(\eta)>0$ is similar). Let $\epsilon_n\to 0$, by passing to a subsequence if necessary, we can assume that $\lim_n\eta\left(L^n_{\epsilon_n}\right)>0$. Using Lemma **??**, we choose a δ small enough so that if

$$L^n:=L^n_{\epsilon_n}\cap\mathcal{O}^n(\delta),$$

then $\overline{\lim}\eta(L^n)>0$. We will now start slicing L^n into "2-dimensional" slices and then "1-dimensional" fibres. If for $m\in B^n$ we let

$$L^n(m):=\bigcup_{\{(k,l)\,\in A^n\times A^n:Y^n_{k,l,m}\,\subset L^n\}}Y^n_{k,l,m}$$

and

$$L^n(m,\,\delta):=\bigcup_{\{(k,l)\,\in A^n\times A^n:Y^n_{k,l,m}\,\subset L^n\}}\mathcal{O}(Y^n_{k,l,m})\,;$$

then

$$\eta(L^n)\;=\;\sum_{m\,\in B^n}\eta(L^n(m))\;\le\;\frac{1}{\delta\,|F^{n,1}|}\sum_{m\,\in B^n}\eta\left(L^n(m,\delta)\right)\,,$$

and since $|B^n| \geq \delta\,|F^{n,1}|$, then taking limits, there exists a sequence $\{m_n\}$ such that (passing to a subsequence if necessary)

$$\lim_n \eta\,(L^n(m_n,\,\delta)) > 0\,.$$

Now

$$L^n(m_n,\delta) = \bigcup_{l\,\in A^n} L^n(l,m_n,\delta);$$

- here $L^n(l,m_n,\delta)$ is the fibre $\bigcup_{\{k:Y^n_{k,l,m_n}\in L^n(m_n,\delta)\}} \mathcal{O}(Y^n_{k,l,m_n})$. Thus we can find a d small enough so that we can pass down to a subset of these fibres indexed by $A^{n,*} \subset A^n$ each satisfying $\eta(L^n(l,m_n,\delta)) > da^{n,1}$ and also so that

$$\lim_n \eta\left(\bigcup_{l\,\in A^{n,*}} L^n(l,m_n,\delta)\right) > 0.$$

Let $L^n(m_n,\delta) := \cup_{l\,\in A^{n,*}} L^n(l,m_n,\delta)$ be the union of these suitable fibres. This will ensure later on that the Blum-Hanson coefficients are uniformly small.

Note that $L^n(m_n,\,\delta)$ is approximately invariant: If $g\,\in G$ and $K\,\in \mathbf{N}$ is given, choose n so large that $|g^{-1}F^{n,1}\cap F^{n,1}| > (1-\delta/K)|F^{n,1}|$. If $(k,l,m_n)\in L^n(m_n)$, then $\mathcal{O}(Y^n_{k,l,m_n})$ is at least δ-long, so at least $\delta(1-3/K)|F^{n,1}|$ of the cubes in $\mathcal{O}(Y^n_{k,l,m_n})$ stay in $\mathcal{O}(Y^n_{k,l,m_n})$ after g acts on them. This is true for the orbit of any $Y^n_{k,l,m}$ in $L^n(m_n)$. By letting $K\to\infty$, $L^n(m_n,\,\delta)$ becomes increasingly invariant. Thus by Lemma **??**,

$$\eta_{L^n(m_n,\,\delta)} \rightarrow_n \eta\,.$$

We can write

$$\eta_{L^n(m_n,\,\delta)} = \sum_{l\,\in A^{n,*}} a^n_l \eta^n_{l,m_n}$$

where

$$\eta^n_{l,m_n} := \eta_{L^n(l,m_n,\delta)}$$

and

$$a^n_l := \frac{\eta\,(L^n(l,m_n,\delta))}{\eta\,(L^n(m_n,\delta))}\,.$$

We can write η^n_{l,m_n} as $\sum_{k\,\in A^n} b^n_k \eta^n_{k,l,m_n}$ where

$$\eta^n_{k,l,m_n} := \eta_{\mathcal{O}(Y^n_{k,l,m_n})}$$

and $b^n_k := \eta\left(\mathcal{O}(Y^n_{k,l,m_n})\right)/\eta\,(L^n(l,m_n,\delta))$. Note that

$$b^n_k \leq \frac{\epsilon_n (a^{n,1})^2\,\delta|F^{n,1}|}{da^{n,1}} < \text{ constant}\cdot\,\epsilon_n \rightarrow 0$$

uniformly for k as $n \to \infty$. This will justify the use of the Blum-Hanson Theorem shortly.

Now define the measures $\tau^n_{k,l,m} := (\mu^3)_{\mathcal{O}(Y^n_{k,l,m})}$, and also

$$\lambda^n := \sum_{l \in A^{n,*}} a^n_l \sum_{k \in A^n} b^n_k \tau^n_{k,l_n,m_n} .$$

We can show, in a way similar to what we did in Lemma **??** (by comparing λ^n to $\eta_{L^n(m_n,\delta)}$ on $A \times B \times C$ where $A,\ B,\ C$ are unions of levels of the towers, and then using approximation arguments), that

$$\lambda^n \to \eta .$$

Let

$$\tilde{\mathcal{O}}(X^n_{k,l,m}) := \bigcup_{\{g \in F^{n,1}k^{-1} \cap F^{n,1}l^{-1}\}} gX^n_{k,l,m} = \bigcup_{\{g \in F^{n,1}k^{-1} \cap F^{n,1}l^{-1}\}} X^n_{gk} \times X^n_{gl} \times gX^n_m$$

and

$$\theta^n_{k,l,m} := \mu^3_{\tilde{\mathcal{O}}(X^n_{k,l,m})} .$$

Note that $\theta^n_{k,l,m} \circ (k^{-1}e_1 \times i \times i) = \theta^n_{e_1,l,m}$ and

$$\frac{\mu^3\left(\mathcal{O}\ (X^n_{e_1,l,m})\right)}{\mu^3\left(\mathcal{O}(X^n_{k,l,m})\right)} \le \frac{|F^n|}{\delta |F^n|} = \frac{1}{\delta}\ ;$$

hence $f^n_m := \frac{d\tau^n_{k,l,m}}{d\theta^n_{l,l,m}} \le 1/\delta$. We have

$$\begin{aligned}
& \tau^n(\, A \times B \times C\,) \\
= \ & \sum_l a^n_l \sum_m b^n_{l,m} \left(\int \chi_{A\times B\times C}(x,y,z)\, d\tau^n_{e,l,m}\,(x,y,z) \right) \\
= \ & \sum_l a^n_l \sum_m b^n_{l,m} \left(\int \chi_{A\times B\times C}(x,y,z) f_{m,n}(x,y,z)\, d\theta^n_{e,l,m}\,(x,y,z) \right) \\
\le \ & \frac{1}{\delta} \sum_l a^n_l \sum_m b^n_{l,m} \left(\int \chi_{A\times B\times C}(x,y,z)\, d\theta^n_{e,l,m}\,(x,y,z) \right) \\
= \ & \frac{1}{\delta} \sum_l a^n_l \sum_m b^n_{l,m} \int \chi_{A\times B}(x,y)(\chi_C(z) - \mu\,(C))\, d\theta^n_{e,l,m}\,(x,y,z) \\
+ \ & \frac{1}{\delta} \sum_l a^n_l \sum_m b^n_{l,m} \int \mu\,(C) \chi_{A\times B}(x,y)\, d\theta^n_{e,l,m}\,(x,y,z) \\
\le \ & \frac{1}{\delta} \sum_l a^n_l \left| \sum_m b^n_{l,m} \int \chi_{A\times B}(x,y)(\chi_C(z) - \mu\,(C))\, d\theta^n_{e,l,m}\,(x,y,z) \right| \\
+ \ & \frac{1}{\delta} \sum_l a^n_l \sum_m b^n_{l,m} \int \mu\,(C) \chi_{A\times B}(x,y)\, d\theta^n_{e,l,m}(x,y,z).
\end{aligned}$$

Now the second term

$$\frac{1}{\delta}\sum_l a_l^n \sum_m b_{l,m}^n \int \mu\,(C)\chi_{A\times B}(x,y)\,d\theta_{e,l,m}^n(x,y,z)$$

is just $\frac{1}{\delta}\mu\,(C)\sum_l a_l^n \sum_m b_{l,m}^n \theta_{e,l,m}^n(A\times B\times X) \to_n \frac{1}{\delta}\mu\,(C)\mu^2(A\times B)$. To see this, note that the probability measures

$$\begin{aligned}
\theta^n(A\times B) &:= \sum_l a_l^n \sum_m b_{l,m}^n \theta_{e,l,m}^n(A\times B\times X)\\
&= \sum_l a_l^n \mu^2_{(\bigcup_{g\in F^{n,1}\cap F^{n,1}\,l^{-1}} X^n_{ge,gl})}(A\times B)\\
&= \sum_l \frac{a_l^n}{|F^n\,a\cap F^n\,al^{-1}|f_n^2}\ \mu^2\left(A\times B\bigcap\left(\bigcup_{g\in F^{n,1}\cap F^{n,1}\,l^{-1}} X^n_{ge,gl}\right)\right)
\end{aligned}$$

and

$$\frac{a_l^n}{|F^{n,1}\cap F^{n,1}\,l^{-1}|f_n^2} \le \frac{a_l^n}{\delta|F^{n,1}|f_n^2} \le \frac{|F^{n,1}|f_n^2}{\nu(\mathcal{O}^{n,*}(e,\delta))\,\delta\,|F^{n,1}|f_n^2} \le K$$

for all n, l. Thus a weak star limit of the measures θ^n has to be absolutely continuous with respect to μ^2, as well as being invariant. By ergodicity of μ^2, the limit in fact is μ^2.

As for the first term, we have

$$\begin{aligned}
&\frac{1}{\delta}\sum_l a_l^n \left|\sum_m b_{l,m}^n \int \chi_{A\times B}(x,y)(\chi_C(z)-\mu\,(C))\,d\theta_{e,l,m}^n\,(x,y,z)\right|\\
=\ &\frac{1}{\delta}\sum_l a_l^n \left|\sum_m b_{l,m}^n \int \chi_{A\times B}(x,y)(\chi_C(m^{-1}z)-\mu\,(C))\,d\theta_{e,l,e}^n\,(x,y,z)\right|\\
\le\ &\frac{1}{\delta}\sum_l a_l^n \int \left|\sum_m b_{l,m}^n(\chi_{mC}(z)-\mu\,(C))\chi_{A\times B}(x,y)\right|\,d\theta_{e,l,e}^n\,(x,y,z)\\
\le\ &\frac{1}{\delta}\sum_l a_l^n \int \left|\sum_m b_{l,m}^n(\chi_{mC}(z)-\mu\,(C))\,d\pi_3\theta_{e,l,e}^n\,(x,y,z)\right|\\
\le\ &\frac{1}{\delta^2}\sum_l a_l^n \int \left|\sum_m b_{l,m}^n\chi_{mC}(z)-\mu\,(C)\right|d\mu\\
\le\ &\frac{1}{\delta^2}\sum_l a_l^n \left\|\sum_m b_{l,m}^n\chi_{mC}-\mu\,(C)\right\|_{2,\mu}
\end{aligned}$$

and by the Blum Hanson Theorem, this last term tends to 0 as $n \to \infty$. This completes the proof when η is ergodic. ■

We apply Theorem **??** to the pairwise independent components η'_{g_j,k_j}, (which we agreed to call η^j) of the chosen induced joinings η_{g_j,k_j}. If η^j is not necessarily ergodic, we only have a component of η^j being product measure. For, suppose that $Di(\eta^j) > 0$, we proceed as in the proof of Theorem **??**, but this time we can only say that

$$\eta^j \; \geq \; \eta^j(L^n)\,(\eta^j)_{L^n} \,.$$

Theorem **??** shows that $\lim_n (\eta^j)_{L^n} = \mu^3$. But now suppose that

$$\eta^j = k_j \mu^3 + \theta^j$$

is the Radon-Nikodym decomposition of η^j with respect to product measure. As we argued in Lemma **??**, θ^j satisfies

$$\overline{\lim_n}\, \theta^j(L^n) = 0$$

or else we could repeat the procedure above to show that $Di(\theta_j) > 0$, and then that θ^j has a component which is product measure, a contradiction, since $\mu^3 \perp \theta^j$. Hence

$$k_j = \lim_n \frac{\eta^j(L^n)}{\mu^3(L^n)} \geq c'_j,$$

since the induced joinings were chosen so that $\lim_n \eta^j(L^n) \geq c'_j \lim_n \mu^3(L^n)$; and so $\eta^j \geq c'_j \mu^3$. Thus

$$\eta_{g_j,k_j} \geq c_j \eta'_{g_j,k_j} \geq c_j\, c'_j \mu^3 \,.$$

Now we can apply Lemma **??** to get the long awaited fact that $\nu = \mu^3$.

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