BRATTELI DIAGRAMS WHERE RANDOM ORDERS ARE IMPERFECT

J. JANSSEN, A. QUAS, AND R. YASSAWI

Abstract. For the simple Bratteli diagrams B where there is a single edge connecting any two vertices in consecutive levels, we show that a random order has uncountably many infinite paths if and only if the growth rate of the level-n vertex sets is super-linear. This gives us the dichotomy: a random order on a slowly growing Bratteli diagram admits a homeomorphism, while a random order on a quickly growing Bratteli diagram does not. We also show that for a large family of infinite rank Bratteli diagrams B, a random order on B does not admit a continuous Vershik map.

1. Introduction

Consider the following random process. For each natural number n, we have a collection of finitely many individuals. Each individual in the $n+1$-st collection randomly picks a parent from the n-th collection, and this is done for all n. If we know how many individuals there are in each generation, the question "How many infinite ancestral lines are there?" almost always has a common answer j: what is it? We can also make this game more general, by for each individual, changing the odds that he choose a certain parent, and ask the same question.

The information that we are given will come as a *Bratteli diagram* B (Definition 2.1), where each "individual" in generation n is represented by a vertex in the n-th vertex set V_n, and the chances that an individual $v \in V_{n+1}$ chooses $v' \in V_n$ as a parent is the ratio of the number of edges incoming to v with source v' to the total number of edges incoming to v. We consider the space $\mathcal{O}_B$ of *orders* on B (Definition 2.4) as a measure space equipped with the completion of the uniform product measure $\mathbb{P}$. A result in [BKY14] (stated as Theorem 3.1 here) tells us that there is some j, either a positive integer or infinite, such that a $\mathbb{P}$-random order ω possesses j maximal paths.

Bratteli diagrams, which were first studied in operator algebras, appeared explicitly in the measurable dynamical setting in [Ver81, Ver85], where it was shown that any ergodic invertible transformation of a Lebesgue space can be represented as a measurable "successor" (or *Vershik*) map on the space of infinite paths X_B in some Bratteli diagram B (Definition 2.9). The successor map, which is defined using an order on B, is not defined on the set of maximal paths in X_B, but as this set is typically a null set, it poses no problem in the measurable framework. Similar results were discovered in the topological setting in [HPS92]: any minimal homeomorphism on a Cantor Space has a representation as a (continuous, invertible) Vershik map which is defined on all of X_B for some Bratteli diagram B. To achieve this, the technique used in [HPS92] was to construct the order so that it had a

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unique minimal and maximal path, in which case the successor map extends uniquely to a homeomorphism of X_B. For such an order our quantity j takes the value 1. We were curious to see whether such an order is typical, and whether a typical order defined a continuous Vershik map. Note that the value j is not an invariant of topological dynamical properties that are determined by the Bratteli diagram's *dimension group* [Eff81], such as *strong orbit equivalence* [GPS95].

In this article we compute j for a large family of *infinite rank* Bratteli diagrams (Definition 2.3). Namely, in Theorem 4.2, we show that j is uncountable for the situation where any individual at stage n is equally likely to be chosen as a parent by any individual at stage $n+1$, whenever the generation growth rate is super-linear. If the generations grow at a slower rate than this, $j = 1$. We note that this latter situation has been studied in the context of gene survival in a variable size population, as in the Fisher-Wright model (e.g. [Sen74], [Don86]). We describe this connection in Section 4.

In Theorem 4.10 we generalise part of Theorem 4.2 to a large family of Bratteli diagrams. We can draw the following conclusion from these results. An order ω is called *perfect* if it admits a continuous Vershik map. Researchers working with Bratteli-Vershik representations of dynamical systems usually work with a subfamily of perfect orders called *proper* orders: those that have only one maximal and one minimal path. For a large class of simple Bratteli diagrams (including the ones we identify in Theorems 4.1 and 4.10), if $j > 1$, then a $\mathbb{P}$-random order is almost surely not perfect (Theorem 3.3), hence not proper. We note that this is in contrast to the case for finite rank diagrams, where almost any order put on any reasonable finite rank Bratteli diagram is perfect (Section 5, [BKY14]).

2. Bratteli diagrams and Vershik maps

In this section, we collect the notation and basic definitions that are used throughout the paper.

2.1. **Bratteli diagrams.**

Definition 2.1. A *Bratteli diagram* is an infinite graph $B = (V, E)$ such that the vertex set $V = \bigcup_{i\geq 0} V_i$ and the edge set $E = \bigcup_{i\geq 1} E_i$ are partitioned into disjoint subsets V_i and E_i where

(i) $V_0 = \{v_0\}$ is a single point;
(ii) V_i and E_i are finite sets;
(iii) there exists a range map r and a source map s, both from E to V, such that $r(E_i) = V_i$, $s(E_i) = V_{i-1}$.

Note that E may contain multiple edges between a pair of vertices. The pair (V_i, E_i) or just V_i is called the *i-th level* of the diagram B. A finite or infinite sequence of edges $(e_i : e_i \in E_i)$ such that $r(e_i) = s(e_{i+1})$ is called a *finite* or *infinite path*, respectively.

For $m < n$, $v \in V_m$ and $w \in V_n$, let $E(v, w)$ denote the set of all paths $\overline{e} = (e_1, \ldots, e_p)$ with $s(e_1) = v$ and $r(e_p) = w$. For a Bratteli diagram B, let X_B be the set of infinite paths starting at the top vertex v_0. We endow X_B with the topology generated by cylinder sets $\{U(e_j, \ldots, e_n) : j,\ n \in \mathbb{N}, \text{ and } (e_j, \ldots, e_n) \in E(v, w), v \in V_{j-1}, w \in V_n\}$, where $U(e_j, \ldots, e_n) := \{x \in X_B : x_i = e_i,\ i = j, \ldots, n\}$. With this topology, X_B is a 0-dimensional compact metrizable space.

Definition 2.2. Given a Bratteli diagram B, the n-th *incidence matrix* $F_n = (f^{(n)}_{v,w}),\ n \geq 0$, is a $|V_{n+1}| \times |V_n|$ matrix whose entries $f^{(n)}_{v,w}$ are equal to the number of edges between the vertices $v \in V_{n+1}$ and $w \in V_n$, i.e.

$$f^{(n)}_{v,w} = |\{e \in E_{n+1} : r(e) = v, s(e) = w\}|.$$

Next we define some families of Bratteli diagrams that we work with in this article.

Definition 2.3. Let B be a Bratteli diagram.

(1) We say B has *finite rank* if for some k, $|V_n| \leq k$ for all $n \geq 1$.
(2) We say that B is *simple* if for any level m there is $n > m$ such that $E(v, w) \neq \emptyset$ for all $v \in V_m$ and $w \in V_n$.
(3) We say that a Bratteli diagram is *completely connected* if all entries of its incidence matrices are positive.

In this article we work only with completely connected Bratteli diagrams.

2.2. **Orderings on a Bratteli diagram.**

Definition 2.4. A Bratteli diagram $B = (V, E)$ is called *ordered* if a linear order '$>$' is defined on every set $r^{-1}(v)$, $v \in \bigcup_{n\geq 1} V_n$. We use ω to denote the corresponding partial order on E and write (B, ω) when we consider B with the ordering ω. Denote by $\mathcal{O}_B$ the set of all orderings on B.

Every $\omega \in \mathcal{O}_B$ defines a *lexicographic* partial ordering on the set of finite paths between vertices of levels V_k and V_l: $(e_{k+1}, \ldots, e_l) > (f_{k+1}, \ldots, f_l)$ if and only if there is an i with $k + 1 \leq i \leq l$, $e_j = f_j$ for $i < j \leq l$ and $e_i > f_i$. It follows that, given $\omega \in \mathcal{O}_B$, any two paths from $E(v_0, v)$ are comparable with respect to the lexicographic ordering generated by ω. If two infinite paths are *tail equivalent*, i.e. agree from some vertex v onwards, then we can compare them by comparing their initial segments in $E(v_0, v)$. Thus ω defines a partial order on X_B, where two infinite paths are comparable if and only if they are tail equivalent.

Definition 2.5. We call a finite or infinite path $e = (e_i)$ *maximal (minimal)* if every e_i is maximal (minimal) amongst the edges from $r^{-1}(r(e_i))$.

Notice that, for $v \in V_i$, $i \geq 1$, the minimal and maximal (finite) paths in $E(v_0, v)$ are unique. Denote by $X_{\max}(\omega)$ and $X_{\min}(\omega)$ the sets of all maximal and minimal infinite paths in X_B, respectively. It is not hard to show that $X_{\max}(\omega)$ and $X_{\min}(\omega)$ are non-empty closed subsets of X_B. If B is completely connected, then $X_{\max}(\omega)$ and $X_{\min}(\omega)$ have no interior points.

Given a Bratteli diagram B, we can describe the set of all orderings $\mathcal{O}_B$ in the following way. Given a vertex $v \in V \backslash V_0$, let P_v denote the set of all orders on $r^{-1}(v)$; an element in P_v is denoted by ω_v. Then $\mathcal{O}_B$ can be represented as

$$\mathcal{O}_B = \prod_{v \in V\backslash V_0} P_v. \tag{2.1}$$

We write an element of $\mathcal{O}_B$ as $(\omega_v)_{v\in V\backslash V_0}$. An Nth level *cylinder set* is a set of the form $\bigcap_{v\in\bigcup_{i=1}^N V_i}[w^*_v]$, where $[w^*_v] = \{\omega : \omega_v = \omega^*_v\}$. The collection of Nth level cylinder sets forms a finite σ-algebra, $\mathcal{F}_N$. We let $\mathcal{B}$ denote the σ-algebra generated by $\bigcup_N \mathcal{F}_N$ and equip $(\mathcal{O}_B, \mathcal{B})$ with the product measure, $\mathbb{P}' = \prod_{v\in V\backslash V_0} \mathbb{P}_v$ where $\mathbb{P}_v$ is the uniform measure on

P_v: $\mathbb{P}_v(\{i\}) = (|r^{-1}(v)|!)^{-1}$ for every $i \in P_v$ and $v \in V \backslash V_0$. Finally, it will be convenient to extend the measure space $(\mathcal{O}_B, \mathcal{B}, \mathbb{P}')$ to its completion, $(\mathcal{O}_B, \mathcal{F}, \mathbb{P})$. (The reason for the use of the completion is that the subset of $\mathcal{O}_B$ consisting of orders with uncountably many maximal paths may not be $\mathcal{B}$-measurable, but will shown to be $\mathcal{F}$-measurable.)

Definition 2.6. Let B be a Bratteli diagram, and $n_0 = 0 < n_1 < n_2 < \ldots$ be a strictly increasing sequence of integers. The *telescoping of B to (n_k)* is the Bratteli diagram B', whose k-level vertex set $V'_k = V_{n_k}$ and whose incidence matrices (F'_k) are defined by

$$F'_k = F_{n_{k+1}-1} \circ \ldots \circ F_{n_k},$$

where (F_n) are the incidence matrices for B.

If B' is a telescoping of B, then there is a natural injection $L : \mathcal{O}_B \to \mathcal{O}_{B'}$. Note that unless $|V_n| = 1$ for all but finitely many n, $L(\mathcal{O}_B)$ is a set of zero measure in $\mathcal{O}_{B'}$.

2.3. **Vershik maps.**

Definition 2.7. Let (B, ω) be an ordered Bratteli diagram. The *successor map*, s_ω is the map from $X_B \setminus X_{\max}(\omega)$ to $X_B \setminus X_{\min}(\omega)$ defined by $s_\omega(x_1, x_2, \ldots) = (x_1^0, \ldots, x_{k-1}^0, \overline{x_k}, x_{k+1}, x_{k+2}, \ldots)$, where $k = \min\{n \geq 1 : x_n \text{ is not maximal}\}$, $\overline{x_k}$ is the successor of x_k in $r^{-1}(r(x_k))$, and $(x_1^0, \ldots, x_{k-1}^0)$ is the minimal path in $E(v_0, s(\overline{x_k}))$.

Definition 2.8. Let (B, ω) be an ordered Bratteli diagram. We say that $\varphi = \varphi_\omega : X_B \to X_B$ is a *continuous Vershik map* if it satisfies the following conditions:

- i φ is a homeomorphism of the Cantor set X_B;
- ii $\varphi(X_{\max}(\omega)) = X_{\min}(\omega)$;
- iii $\varphi(x) = s_\omega(x)$ for all $x \in X_B \setminus X_{\max}(\omega)$.

If there is an s_ω-invariant measure μ on X_B such that $\mu(X_{\max}(\omega)) = \mu(X_{\min}(\omega)) = 0$, then we may extend s_ω to a measure-preserving transformation ϕ_ω of X_B. In this case, we call ϕ_ω a *measurable Vershik map* of (X_B, μ). Note that in our case, $X_{\max}(\omega)$ and $X_{\min}(\omega)$ have empty interiors, so that there is at most one continuous extension of the successor map to the whole space.

Definition 2.9. Let B be a Bratteli diagram. We say that an ordering $\omega \in \mathcal{O}_B$ is *perfect* if ω admits a continuous Vershik map φ_ω on X_B. If ω is not perfect, we call it *imperfect*.

Let $\mathcal{P}_B \subset \mathcal{O}_B$ denote the set of perfect orders on B.

3. The size of certain sets in $\mathcal{O}_B$.

A finite rank version of the following result was shown in [BKY14] as a corollary of the Kolmogorov 0–1 law; the proof for non-finite rank diagrams is the same.

Theorem 3.1. *Let B be a simple Bratteli diagram. Then there exists $j \in \mathbb{N} \cup \{\infty\}$ such that for $\mathbb{P}$-almost all orderings, $|X_{\max}(\omega)| = j$.*

By symmetry (since the sets $X_{\max}(\omega)$ and $X_{\min}(\omega)$ have the same distribution), it follows that $|X_{\min}(\omega)| = j$ almost surely as well.

Example 3.2. It is not difficult, though contrived, to find a simple finite rank Bratteli diagram B where almost all orderings are not perfect. Let $V_n = V = \{v_1, v_2\}$ for $n \geq 1$, and define $m^{(n)}_{v,w} := \frac{f^{(n)}_{v,w}}{\sum_w f^{(n)}_{v,w}}$: i.e. $m^{(n)}_{v,w}$ is the proportion of edges with range $v \in V_{n+1}$ that have source $w \in V_n$. Suppose that $\sum_{n=1}^{\infty} m^{(n)}_{v_i,v_j} < \infty$ for $i \neq j$. Then for almost all orderings, there is some K such that for $n > K$, the sources of the two maximal/minimal edges at level n are distinct, i.e. $j = 2$. The assertion follows from [BKY14, Theorem 5.4].

We point out that given an unordered Bratteli diagram, B, if B is equipped with two proper orderings ω and ω', then the resulting topological Vershik dynamical systems s and s' are strong orbit equivalent [GPS95]. Likewise, if B equipped with a perfect ordering is telescoped, the topological Vershik systems are conjugate. The number of maximal paths that a random order on B possesses is not invariant under telescoping. Take for example a Bratteli diagram B where odd levels consist of a unique vertex and even levels have n^2 vertices. let all incidence matrices have all entries equal to 1. By Theorem 4.2, a random order on B has infinitely many maximal paths. On the other hand, B can be telescoped to a diagram B' with only one vertex at each level, so that B' has a unique maximal path.

A finite rank version of the following result is proved in Theorem 5.4 of [BKY14].

Theorem 3.3. *Suppose that B is a completely connected Bratteli diagram of infinite rank so that $\mathbb{P}$-almost all orderings have j maximal and minimal elements. If $j > 1$, then $\mathbb{P}$-almost all orderings are imperfect.*

For an $\omega \in \Omega$, in order that s_ω can be extended continuously to X, it is necessary that for each n, there exists an N such that knowledge of any path, x, in $X \setminus X_{\max}$ up to level N determines $s_\omega(x)$ up to level n. Conversely to show ω is imperfect, one shows that there exists an n such that for each N, the first N terms of x do not determine the first n terms of $s(x)$. In fact, we show that for any n and N, there exists a sequence of values, K, such that if one considers the collection, $\mathcal{M}_K$ of finite paths from the Kth level to the root that are non-maximal in the Kth edge, but maximal in all prior edges, the set of ω such that the first N edges of x determine the first n edges of $s_\omega(x)$ is of measure 0. The following lemma provides a key combinatorial estimate that we use in the proof of the theorem. We make use of the obvious correspondence between the collection of orderings on a set S with n elements, and the collection of bijections from $\{1, 2, \ldots, n\}$ to S.

Lemma 3.4. *Let S be a finite set of size n and let F and G be maps from S into a set R with G non-constant. Let the set Σ of total orderings on S be equipped with the uniform probability measure.*

Then

$$\mathbb{P}(O) \leq \tfrac{1}{n-1}, \text{ where } O = \big\{\sigma \in \Sigma\colon F(\sigma(i)) = G(\sigma(i+1)) \text{ for all } 1 \leq i < n\big\}. \tag{3.1}$$

Proof. Let V be the union of the range of F and the range of G. Form a directed multigraph $\mathcal{G} = (V, E)$ as follows. For $1 \leq i \leq n$, define the ordered pair $e_i = (G(i), F(i))$. Let $E = \{e_1, e_2, \ldots, e_n\}$. Now let $\sigma \in O$. Then for $1 \leq i < n$, the range of $e_{\sigma(i)}$ equals the source of $e_{\sigma(i+1)}$. Therefore, $e_{\sigma(1)} e_{\sigma(2)} \ldots e_{\sigma(n)}$ is an Eulerian trail in $\mathcal{G}$.

It is straightforward to check that the map from O to Eulerian trails is bijective, and thus we need to bound the number of Eulerian trails in $\mathcal{G}$. To do this, note that each Eulerian trail induces an ordering on the out-edges of each vertex. Let $V = \{v_1, \dots, v_k\}$, and let n_i be the number of out-edges of v_i. Since G is non-constant, there are at least two directed edges with different sources, and thus $n_i \le n-1$ for $1 \le i \le k$. The number of orderings of out-edges is $n_1!n_2!\dots n_k!$.

We distinguish two cases. If all vertices have out-degree equal to in-degree, then each Eulerian trail is in fact an Eulerian circuit. An Eulerian circuit corresponds to n different Eulerian trails, distinguished by their starting edge. To count the number of circuits, we may fix a starting edge e^*, and then note that each circuit induces exactly one out-edge ordering if we start following the circuit at this edge. Note that in each such ordering, the edge e^* must be the first in the ordering of the out-edges of its source. We may choose e^* such that its source, say v_1, has maximum out-degree. Thus the number of compatible out-edge orderings is at most $(n_1-1)!n_2!\dots n_k!$ This expression is maximized, subject to the conditions $n_1 + n_2 + \dots + n_k \le n$ and $n_i \le n_1 \le n-1$ for $1 \le i \le k$, when $k = 2$ and $n_1 = n-1$, $n_2 = 1$. Therefore, there are at most $(n-2)!$ Eulerian circuits, so at most $n(n-2)!$ Eulerian trails and elements of O.

If not all vertices have out-degree equal to in-degree, then either no Eulerian trail exists and the lemma trivially holds, or exactly one vertex, say v_1, has out-degree greater than in-degree, and this vertex must be the starting vertex of every trail. In this case, an ordering of out-edges from all vertices precisely determines the trail. The number of out-edge orderings (and good bijections) in this case is bounded above by $(n-1)!$.

Therefore, O consists of at most $n(n-2)!$ orderings (out of $n!$), and the lemma follows. □

Proof of Theorem 3.3. Note that if $|V_n| = 1$ for infinitely many n, then any order on B has exactly one maximal and one minimal path. So we assume that $|V_n| \ge 2$ for all large n.

We first define some terminology. Recall that $s(e)$ and $r(e)$ denote the source and range of the edge e respectively. Given an order $\omega \in \mathcal{O}_B$, we let $e_{\alpha,\omega}(v)$ be the αth edge in $r^{-1}(v)$ If $v \in V_{N'}$ for some $N' > n$, we let $t_{n,\omega}(v)$ be the element of V_n that the maximal incoming path to v goes through. We call $t_{n,\omega}(v)$ the *n-tribe* of v. Similarly the *n-clan* of v, $c_{n,\omega}(v)$ is the element of V_n through which the minimal incoming path to v passes. If n is such that for any $N > n$, the elements of V_N belong to at least two n-clans (or n-tribes), we shall say that ω has at least two infinite n-clans (or n-tribes.)

Let $N > n$ and define $C_{n,N}$ to be the set of orders ω such that if the non-maximal paths x and y agree to level N, then their successors $s_\omega(x)$ and $s_\omega(y)$ agree to level n. Note that $\mathcal{P}_B \subset \bigcap_{n=1}^\infty \bigcup_{N=n}^\infty C_{n,N}$. In what follows we show that this last set has zero mass.

Fix n and N with $N > n$, and take any $N' > N$. Any order $\omega \in C_{n,N}$ must satisfy the following constraints: given any two non-maximal edges whose sources in $V_{N'}$ belong to the same N-tribe, their successors must belong to the same n-clan. In particular, if v and v' are vertices in $V_{N'}$ such that the sources of $e_{\alpha,\omega}(v)$ and $e_{\beta,\omega}(v)$ belong to the same N-tribe, where α and β are both non-maximal, then the sources of $e_{\alpha+1,\omega}(v)$ and $e_{\beta+1,\omega}(v')$ must belong to the same n-clan. That is, there is a map $f \colon V_N \to V_n$ such that for any $v \in V_{N'}$ and any non-maximal α, $f(t_{N,\omega}(s(e_{\alpha,\omega}(v)))) = c_{n,\omega}(s(e_{\alpha+1,\omega}(v)))$. We think of this f as mapping N-tribes to n-clans. This is illustrated in Figure 1.

Motivated by the preceding remark, if $N' > N > n$, we define two subsets of $\mathcal{O}_B$. We let $D_{n,N'}$ be the set of orders such that $V_{N'}$ contains members of at least two n-clans; and $E_{n,N,N'}$ to be the subset of orders in $\mathcal{D}_{n,N'-1}$ which additionally satisfy the condition (*):

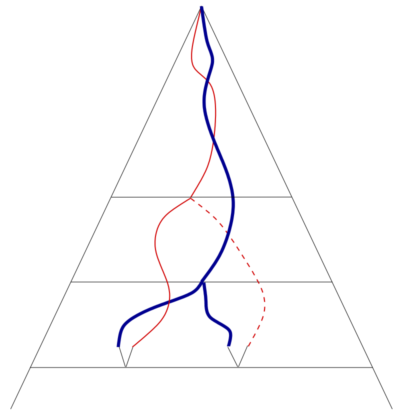

FIGURE 1. The maximal upward paths from $s(e_{\alpha,\omega}(v))$ and $s(e_{\beta,\omega}(v'))$ (blue and bold) agree above level N, so the minimal upward path from path with range $s(e_{\beta+1,\omega}(v'))$ (red, dashed, not bold) must hit the same vertex in the nth level as the minimal upward path from $s(e_{\alpha+1,\omega}(v))$ (red, solid, not bold).

> There is a function $f\colon V_N \to V_n$ such that for all $v \in V_{N'}$, if α is a non-maximal edge entering v then $f(t_{N,\omega}(s(e_{\alpha,\omega}(v)))) = c_{n,\omega}(s(e_{\alpha+1,\omega}(v)))$.

We observe that $D_{n,N'}$ and $E_{n,N,N'}$ are $\mathcal{F}_{N'}$-measurable. We compute $\mathbb{P}(E_{n,N,N'}|\mathcal{F}_{N'-1})$. Since $D_{n,N'-1}$ is $\mathcal{F}_{N'-1}$ measurable, we have $\mathbb{P}(E_{n,N,N'}|\mathcal{F}_{N'-1})(\omega)$ is 0 for $\omega \notin D_{n,N'-1}$. For a fixed map $f\colon V_N \to V_n$, and a fixed vertex $v \in V_{N'}$, and $\omega \in D_{n,N'-1}$, the conditional probability given $\mathcal{F}_{N'-1}$ that (*) with the specific function f is satisfied at v is at most $1/(|V_{N'-1}| - 1)$. To see this, notice that for $\omega \in D_{n,N'-1}$, the n-clan is a non-constant function of $V_{N'-1}$, so that the hypothesis of Lemma 3.4 is satisfied, with $F = f \circ t_{N,\omega} \circ s$ and $G = c_{n,\omega} \circ s$, both applied to the set of incoming edges to v. Also, since B is completely connected, there are at least $|V_{N'-1}|$ edges coming into v.

Since these are independent events conditioned on $\mathcal{F}_{N'-1}$, the conditional probability that (*) is satisfied for the fixed function f over all $v \in V_{N'}$ is at most $1/(|V_{N'-1}| - 1)^{|V'_N|}$. There are $|V_n|^{|V_N|}$ possible functions f that might satisfy (*). Hence we obtain

$$\mathbb{P}(E_{n,N,N'}) \leq \frac{|V_n|^{|V_N|}\mathbb{P}(D_{n,N'-1})}{(|V_{N'-1}| - 1)^{|V_{N'}|}},$$

so that for fixed n and N with $n < N$, one has $\liminf_{N'\to\infty} \mathbb{P}(E_{n,N,N'}) = 0$. By the hypothesis, for any $\epsilon > 0$, there exists $m(\epsilon)$ such that $\mathbb{P}(R_n) > 1 - \epsilon$ for all $n > m(\epsilon)$, where $R_n = \{\omega \in \mathcal{O}_B : \omega \text{ has at least 2 infinite } n\text{-clans}\}$.

Since $C_{n,N} \cap R_n \subset E_{n,N,N'}$ for all $N' > N > n$, we conclude that $\mathbb{P}(C_{n,N} \cap R_n) = 0$ for $N > n$, so that $\mathbb{P}(C_{n,N}) \leq \epsilon$ for $N > n > m(\epsilon)$. Now since $\mathcal{P}_B \subset \bigcap_{n=1}^{\infty} \bigcup_{N=n}^{\infty} C_{n,N}$ and $C_{n,N} \subset C_{n,N+1}$ for each $N \geq n$, we conclude that $\mathbb{P}(\mathcal{P}_B) = 0$. □

4. Diagrams whose orders are almost always imperfect

4.1. **Bratteli diagrams and the Wright-Fisher model.** Let $\mathbf{1}_{a\times b}$ denote an $a \times b$ matrix, all of whose entries are 1. If V_n is the n-th vertex set in B, define $M_n = |V_n|$. In this subsection, all Bratteli diagrams that we consider have incidence matrices $F_n = \mathbf{1}_{M_{n+1}\times M_n}$ for each n.

We wish to give conditions on (M_n) so that a $\mathbb{P}$-random order has infinitely many maximal paths. We first comment on the relation between our question and the Wright-Fisher model in population genetics. Given a subset $A \subset V_k$, and an ordering $\omega \in \mathcal{O}_B$, we let $S^{\omega}_{k,n}(A)$ for $n > k$ be the collection of vertices v in V_n such that the unique upward maximal path in the ω ordering through v passes through A. Informally, when we consider the tree formed by all maximal edges then $S^{\omega}_{k,n}(A)$ is the set of vertices in V_n that have "ancestors" in A.

Let $Y_n = |S^{\omega}_{k,n}(A)|/M_n$. We observe that conditional on Y_n, each vertex in V_{n+1} has probability Y_n of belonging to $S^{\omega}_{k,n+1}(A)$ (since each vertex in V_{n+1} chooses an independent ordering of V_n from the uniform distribution), so that the distribution of $|S^{\omega}_{k,n+1}(A)|$ conditional on Y_n is binomial with parameters M_{n+1} and Y_n. In particular, (Y_n) is a martingale with respect to the natural filtration $(\mathcal{F}_n)$, where $\mathcal{F}_n$ is the σ-algebra generated by the nth level cylinder sets. Since (Y_n) is a bounded martingale, it follows from the martingale convergence theorem that (Y_n) almost surely converges to some limit Y_∞ where $0 \leq Y_\infty \leq 1$.

It turns out that the study of maximal paths is equivalent to the Wright-Fisher model in population genetics. Here one studies populations where there are disjoint generations; each population member inherits an allele (gene type) from a uniformly randomly chosen member of the previous generation.

To compare the randomly ordered Bratteli diagram and Wright-Fisher models, the vertices in V_n represent the nth generation and a vertex $v \in V_{n+1}$ 'inherits an allele' from $w \in V_n$ if the edge from w to v is the maximal incoming edge to v. Since for each v, one of the $M_n!$ orderings of V_n is chosen uniformly at random, the probability that any element of V_n is the source of the maximal incoming edge to v is $1/M_n$. Since the orderings are chosen independently, the ancestor of $v \in V_{n+1}$ is independent of the ancestor of any other $v' \in V_{n+1}$.

Analogous to Y_n, in the Wright-Fisher context, one studies the proportion of the population that have various alleles. If one declares the vertices in $A \subset V_k$ to have allele type $\mathbf{A}$ and the other vertices in that level to have allele type $\mathbf{a}$, then there is a maximal path through A if and only if in the Wright-Fisher model, the allele $\mathbf{A}$ persists - that is there exist individuals in all levels beyond the nth with type $\mathbf{A}$ alleles.

In a realization of the Wright-Fisher model, an allele type is said to *fixate* if the proportion Y_n of individuals with that allele type in the nth level converges to 0 or 1 as $n \to \infty$. An allele type is said to become *extinct* if $Y_n = 0$ for some finite level, or to *dominate* if $Y_n = 1$ for some finite level.

Theorem 4.1. [Don86, Theorem 3.2] *Consider a Wright-Fisher model with population structure $(M_n)_{n\geq 0}$. Then domination of one of the alleles occurs almost surely if and only if $\sum_{n\geq 0} 1/M_n = \infty$.*

Theorem 4.1 also holds if in the Wright-Fisher model, individuals can inherit one of r alleles with $r \geq 2$. We exploit this below by letting each member of a chosen generation have a distinct allele type.

To indicate the flavour of the arguments, we give a proof of the simpler fact that if $\sum_{n\geq 0} 1/M_n = \infty$ then each allele type fixates. To see this, let $Q_n = Y_n(1 - Y_n)$. Now we have

$$\mathbb{E}(Q_n|\mathcal{F}_{n-1}) = Y_{n-1} - Y_{n-1}^2 - (\mathbb{E}(Y_n^2|Y_{n-1}) - \mathbb{E}(Y_n|Y_{n-1})^2) = Q_{n-1} - \mathrm{Var}(Y_n|Y_{n-1}).$$

Since $M_n Y_n$ is binomial with parameters M_n and Y_{n-1},

$$\mathrm{Var}(Y_n|Y_{n-1}) = (1/M_n^2)(M_n Y_{n-1}(1-Y_{n-1})) = Q_{n-1}/M_n.$$

This gives $\mathbb{E}(Q_n|\mathcal{F}_{n-1}) = (1-1/M_n)Q_{n-1}$. Now using the tower property of conditional expectations, we have $\mathbb{E}Q_n = \mathbb{E}(Q_n|\mathcal{F}_0) = \prod_{j=1}^n (1-1/M_j)\mathbb{E}Q_0$, which converges to 0. As noted above, the sequence $(Y_n(\omega))$ is convergent for almost all ω to $Y_\infty(\omega)$ say. It follows that $Q_n(\omega)$ converges pointwise to $Y_\infty(1-Y_\infty)$. By the bounded convergence theorem, we deduce that $\mathbb{E}Y_\infty(1-Y_\infty)=0$, so that Y_∞ is equal to 0 or 1 almost everywhere.

We shall use Theorem 4.1 to prove the first part of the following theorem.

Theorem 4.2. *Consider a Bratteli diagram with $M_n \geq 1$ vertices in the nth level and whose incidence matrices are all of the form $\mathbf{1}_{M_{n+1}\times M_n}$. We have the following dichotomy:*

If $\sum_n 1/M_n = \infty$, then there is $\mathbb{P}$-almost surely a unique maximal path.

If $\sum_n 1/M_n < \infty$, then there are $\mathbb{P}$-almost surely uncountably many maximal paths.

To prove the second part of this result we will need the following tool. Recall the definition of $S^\omega_{k,n}(A)$, from the beginning of Section 4.1.

Proposition 4.3. *Consider a Wright-Fisher model with population structure $(M_n)_{n\geq 0}$. Suppose that $\sum_{n\geq 0} 1/M_n < \infty$. Then for each $\epsilon > 0$ and $\eta > 0$, there exists an $l > 0$ such that for any $\mathcal{F}_l$-measurable random subset, $A(\omega)$, of V_l (that is an $\mathcal{F}_l$-measurable map $\Omega \to \mathcal{P}(V_l)$) and any $L > l$,*

$$\mathbb{P}\left(\left|\frac{|A(\omega)|}{|V_l|} - \frac{|S^\omega_{l,L}(A(\omega))|}{|V_L|}\right| \geq \eta\right) < \epsilon.$$

Proof. Let l be chosen so that $\sum_{n=l+1}^\infty 1/M_n < 4\epsilon\eta^2$ and let $L > l$. For $n > l$, let $Y_n = \big|S^\omega_{l,n}\big(A(\omega)\big)\big|/|V_n|$. Recall that (Y_n) is a martingale with respect to the filtration $(\mathcal{F}_n)$. Set $Z_n = (Y_n - Y_l)^2$ and notice that $(Z_n)_{n\geq l}$ is a bounded sub-martingale by the conditional expectation version of Jensen's inequality.

We have

$$\begin{aligned}
\mathbb{E}Z_L &= \mathbb{E}\big(\mathbb{E}((Y_L - Y_l)^2|\mathcal{F}_l)\big)\\
&= \mathbb{E}\big(\mathbb{E}(Y_L^2 - Y_l^2|\mathcal{F}_l)\big)\\
&= \mathbb{E}(Y_L^2 - Y_l^2)\\
&= \sum_{j=l}^{L-1} \mathbb{E}(Y_{j+1}^2 - Y_j^2).
\end{aligned}$$

A calculation shows that

$$\begin{aligned}
\mathbb{E}(Y_{j+1}^2 - Y_j^2|\mathcal{F}_j) &= \mathbb{E}(Y_{j+1}^2|\mathcal{F}_j) - \mathbb{E}(Y_{j+1}|\mathcal{F}_j)^2\\
&= \mathrm{Var}(Y_{j+1}|\mathcal{F}_j)\\
&= \frac{Y_j(1-Y_j)}{M_{j+1}}
\end{aligned}$$

so that $\mathbb{E}(Y_{j+1}^2 - Y_j^2) \leq 1/(4M_{j+1})$ and we obtain $\mathbb{E}Z_L \leq \sum_{j=l+1}^L 1/(4M_j)$. In particular we have $\mathbb{E}Z_L \leq \epsilon\eta^2$. The claim follows from Markov's inequality.

□

Proof of Theorem 4.2. Suppose first that $\sum_n 1/M_n = \infty$. We show for all k, with probability 1, there exists $n > k$ such that all maximal paths from each level n vertex to the root vertex pass through a single vertex at level k.

To do this, we consider the M_k vertices at level k to each have a distinct allele type. By Theorem 4.1, there is for almost every ω, a level n such that by level n one of the M_k allele types has dominated all the others. This is a direct translation of the statement that we need, which is that every maximal finite path with range in V_n passes through the same vertex in V_k.

Now we consider the case $\sum_n 1/M_n < \infty$. In this case, we identify a sequence (n_k) of levels. We start with a single allele at v_0; and evolve it to level n_1, where it is split into two almost equal sub-alleles. This *evolve-and-split* operation is repeated inductively, evolving the two alleles at level n_1 to level n_2 and splitting each one giving four sub-alleles and so on, so that there are 2^k alleles in the generations of the Bratteli diagram between the n_kth and n_{k+1}st. We show that with very high probability, they all persist and maintain a roughly even share of the population. This splitting allows us to find, with probability arbitrarily close to one, a surjective map from the set of maximal paths to all possible sequences of 0s and 1s.

Fix a small $\kappa > 0$. Using Proposition 4.3, choose an increasing sequence of levels $(n_k)_{k\geq 1}$ with the property that $M_{n_k} > 4^k$ and that for any random $\mathcal{F}_{n_k}$-measurable subset, $A(\omega)$ of V_{n_k}, one has with probability at least $1 - \kappa 4^{-k}$,

$$\left| \frac{|S^\omega_{n_k,n_{k+1}}(A(\omega))|}{|V_{n_{k+1}}|} - \frac{|A(\omega)|}{|V_{n_k}|} \right| < 4^{-(k+1)}. \tag{4.1}$$

Let $n_0 = 0$ and let $A_\epsilon(\omega) = V_0$ (here ϵ stands for the empty string). We inductively define a collection of $\mathcal{F}_{n_k}$-measurable subsets of V_{n_k} indexed by strings of 0's and 1's of length k. Suppose that for each string s of length k, $A_s(\omega)$ is a random $\mathcal{F}_{n_k}$-measurable subset of V_{n_k}. Then we let $A_{s0}(\omega)$ be the first half of $S^\omega_{n_k,n_{k+1}}(A_s(\omega))$ and $A_{s1}(\omega)$ be the second half (by the first half of a subset A of V_n, we mean the subset consisting of the first $\lceil \frac{|A|}{2} \rceil$ elements of A with respect to the fixed indexing of V_n and the second half is the subset consisting of the last $\lfloor \frac{|A|}{2} \rfloor$ elements of A). By the union bound, we see that with probability at least $1 - (\sum_{k=1}^\infty 2^k \kappa 4^{-k}) = 1 - \kappa$, the sets satisfy for each $s \in \{0,1\}^k$,

$$\left| \frac{|S^\omega_{n_k,n_{k+1}}(A_s(\omega))|}{|V_{n_{k+1}}|} - \frac{|A_s(\omega)|}{|V_{n_k}|} \right| < 4^{-(k+1)}. \tag{4.2}$$

In particular, this suffices to ensure that the sets $A_s(\omega)$ are non-empty for each finite string of 0's and 1's. Now we define a map from the collection of maximal paths to $\{0,1\}^{\mathbb{N}}$: for each k, the $A_s(\omega)$ for $s \in \{0,1\}^k$ partition V_{n_k}. Given $x \in X_{\max}(\omega)$, there is a unique sequence $\iota(x) = i_1 i_2 \ldots \in \{0,1\}^{\mathbb{N}}$ such that the k_nth edge, $r(x_{k_n}) \in A_{i_1 \ldots i_n}(\omega)$. The map $\iota\colon X_{\max}(\omega) \to \{0,1\}^{\mathbb{N}}$ is then continuous. For any ω satisfying (4.2), the map $\iota\colon X_{\max}(\omega) \to \{0,1\}^{\mathbb{N}}$ is surjective. Hence for each $\kappa > 0$, we have exhibited a measurable subset of Ω with measure $1 - \kappa$ for which there are uncountably many maximal paths. By completeness of the measure, it follows that almost every ω has uncountably many maximal paths.

□

4.2. **Other Bratteli diagrams whose orders support many maximal paths.** Next we partially extend the results in Section 4.1 to a larger family of Bratteli diagrams.

Definition 4.4. Let B be a Bratteli diagram.

- We say that B is *superquadratic* if there exists $\delta > 0$ so that $M_n \geq n^{2+\delta}$ for all large n.
- Let B be superquadratic with constant δ. We say that B is *exponentially bounded* if $\sum_{n=1}^{\infty} |V_{n+1}| \exp(-|V_n|/n^{2+2\delta/3})$ converges.

We remark that the condition that B is exponentially bounded is very mild.

In Theorem 4.10 below we show that Bratteli diagrams satisfying these conditions have infinitely many maximal paths. Given $v \in V_{n+1}$, define

$$V_n^{v,i} := \{w \in V_n : f_{v,w}^{(n)} = i\},$$

so that if the incidence matrix entries for B are all positive and bounded above by r, then $V_n = \bigcup_{i=1}^{r} V_n^{v,i}$ for each $v \in V_{n+1}$.

Definition 4.5. Let B be a Bratteli diagram with positive incidence matrices. We say that B is *impartial* if there exists an integer r so that all of B's incidence matrix entries are bounded above by r, and if there exists some $\alpha \in (0,1)$ such that for any n, any $i \in \{1, \ldots, r\}$ and any $v \in V_{n+1}$, $|V_n^{v,i}| \geq \alpha |V_n|$.

In other words, B is impartial if for any row of any incidence matrix, no entry occurs disproportionately rarely or often with respect to the others. For example, fixing r, if we let $|V_n| = r(n+1)$, and let each row of F_n consist of any vector with entries equidistributed from $\{1, \ldots, r\}$, the resulting Bratteli diagram is impartial. Note that our diagrams in Theorem 4.2 are impartial. However the vertex sets can grow as fast as we want, so the diagrams are not necessarily exponentially bounded. We remark also that if a Bratteli diagram is impartial, then it is completely connected, which means that we can apply Theorem 3.3 if $j > 1$.

Definition 4.6. Suppose that B is a Bratteli diagram each of whose incidence matrices has entries with a maximum value of r. We say that $A \subset V_n$ is (β, ϵ)*-equitable* for B if for each $v \in V_{n+1}$ and for each $i = 1, \ldots, r$,

$$\left| \frac{|V_n^{v,i} \cap A|}{|V_n^{v,i}|} - \beta \right| \leq \epsilon.$$

In the case $\beta = \frac{1}{2}$, we shall speak simply of ϵ-equitability.

Given $v \in V \backslash V_0$ and an order $\omega \in \mathcal{O}_B$, recall that we use $\widetilde{e}_v = \widetilde{e}_v(\omega)$ to denote the maximal edge with range v.

Lemma 4.7. *Suppose that B is impartial. Let $A \subset V_n$ be (β, ϵ)-equitable, and $v \in V_{n+1}$. Let the random variable X_v be defined as*

$$X_v(\omega) = \begin{cases} 1 & \text{if } s(\widetilde{e}_v) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\beta - \epsilon \leq \mathbb{E}(X_v) \leq \beta + \epsilon$.

Proof. We have

$$\begin{aligned}\mathbb{E}(X_v) &= \frac{\sum_{j=1}^r j|A\cap V_n^{v,j}|}{\sum_{j=1}^r j|V_n^{v,j}|}\\ &\le \frac{\sum_{j=1}^r j|V_n^{v,j}|(\beta+\epsilon)}{\sum_{j=1}^r j|V_n^{v,j}|} = \beta+\epsilon,\end{aligned}$$

the last inequality following since A is ϵ-equitable. Similarly, $\mathbb{E}(X_v) \ge \beta - \epsilon$. □

Lemma 4.8. *Let B be an impartial Bratteli diagram with impartiality constant α and the property that each entry of each incidence matrix is between 1 and r. Let β, δ and ϵ be positive, let $(p_v)_{v\in V_N}$ satisfy $|p_v - \beta| < \delta$ for each $v \in V_N$ and let $A \subset V_N$ be a randomly chosen subset, where each v is included with probability p_v independently of the inclusion of all other vertices. Then the probability that A fails to be $(\beta, \delta+\epsilon)$-equitable is at most $2r|V_{N+1}|e^{-\alpha|V_N|\epsilon^2}$.*

Proof. Let $(Z_v)_{v\in V_N}$ be $\mathbf{1}_{v\in A}$, so that these are independent Bernoulli random variables, where Z_v takes the value 1 with probability p_v

For $u \in V_{N+1}$ and $1 \le i \le r$, define

$$Y_{u,i} := \frac{1}{|V_N^{u,i}|}\sum_{v\in V_N^{u,i}} Z_v = \frac{|\{v\in V_N^{u,i} : v\in A\}|}{|V_N^{u,i}|} = \frac{|A\cap V_N^{u,i}|}{|V_N^{u,i}|}\,. \tag{4.3}$$

Using Hoeffding's inequality [Hoe63], since $\beta-\delta \le \mathbb{E}(Y_{u,i}) \le \beta+\delta$ we have that

$$\begin{aligned}\mathbb{P}(\{|Y_{u,i}-\beta| \ge (\delta+\epsilon)\}) &\le \mathbb{P}(\{|Y_{u,i}-\mathbb{E}(Y_{u,i})| \ge \epsilon\})\\ &\le 2e^{-2|V_N^{u,i}|\epsilon^2} \le 2e^{-2\alpha|V_N|\epsilon^2}.\end{aligned}$$

This implies that

$$\mathbb{P}\left(\bigcup_{i=1}^r \bigcup_{u\in V_{N+1}} \{|Y_{u,i}-\beta| \ge \delta+\epsilon\}\right) \le 2r|V_{N+1}|e^{-2|V_N|\alpha\epsilon^2}, \tag{4.4}$$

□

Lemma 4.9. *Suppose that B is impartial, superquadratic and exponentially bounded. Then for any ϵ small there exist n and $A \subset V_n$ such that A is $(\frac{1}{2}, \epsilon)$-equitable.*

Proof. We apply the probabilistic method. Let r and α be as in the statement of Lemma 4.8 and apply that lemma with $p_v = \frac{1}{2}$ for each $v \in V_n$. By the superquadratic and exponentially bounded properties, one has $2r|V_{n+1}|e^{-2\alpha|V_n|\epsilon^2} < 1$ for large n. Since the probability that a randomly chosen set is $(\frac{1}{2}, \epsilon)$-equitable is positive, the existence of such a set is guaranteed. □

Theorem 4.10. *Suppose that B is a Bratteli diagram that is impartial, superquadratic and exponentially bounded. Then $\mathbb{P}$-almost all orders on B have infinitely many maximal paths.*

We note that in the special case where B is defined as in Section 4.1, the following proof can be simplified and does not require the condition that B is exponentially bounded. Instead of beginning our procedure with an equitable set, which is what we do below, we can start with any set $A_N \subset V_N$ whose size relative to V_N is around 1/2.

Proof. Since B is superquadratic, we find a sequence (ϵ_j) such that

$$\sum_{j=1}^{\infty} \epsilon_j < \infty \text{ and} \tag{4.5}$$

$$M_j \epsilon_j^2 \geq j^{\gamma} \text{ for some } \gamma > 0 \text{ and large enough } j. \tag{4.6}$$

Fix N so that (4.6) holds for all $j \geq N$, and let N be large enough so that $\sum_{j=N}^{\infty} \epsilon_j < \frac{1}{2}$. Moreover, we can also choose our sequence (ϵ_j) and our N large enough so that there exists a set $A_N \subset V_N$ which is ϵ_N-equitable: by Lemma 4.9, this can be done. For all $k \geq 0$, define also

$$\delta_{N+k} = \sum_{i=0}^{k} \epsilon_{N+i}.$$

Finally, let r be so that all entries of all F_n are bounded above by r.

Define recursively, for all integers $k > 0$ and all $v \in V_{N+k}$, the Bernoulli random variables $\{X_v : \mathcal{O}_B \to \{0,1\} : v \in V_{N+k}\}$, and the random sets $\{A_{N+k} : \mathcal{O}_B \to 2^{V_{N+k}} : k \geq 1\}$, where $X_v(\omega) = 1$ if $s(\widetilde{e}_v) \in A_{N+k-1}$, and 0 otherwise, and $A_{N+k} = \{v \in V_{N+k} \ : \ X_v = 1\}$.

We shall show that for a large set of ω, each set A_{N+k} is δ_{N+k}-equitable. This implies that the size of A_{N+k} is not far from $\frac{1}{2}|V_{N+k}|$. For, if $k \geq 1$, define the event

$$D_{N+k} := \{\omega : A_{N+k} \text{ is } \delta_{N+k} - \text{equitable}\}.$$

We claim that

$$\mathbb{P}(D_{N+k+1}|D_{N+k}) \geq 1 - 2r|V_{N+k+2}|e^{-2\alpha|V_{N+k+1}|\epsilon_{N+k+1}^2}.$$

To see this, notice that if $\omega \in D_{N+k}$, then by Lemma 4.7, given $\mathcal{F}_{N+k}$, each vertex in V_{N+k+1} is independently present in A_{N+k+1} with probability in the range $[\beta - \delta_{N+k}, \beta + \delta_{N+k}]$. Hence by Lemma 4.8, A_{N+k+1} is δ_{N+k+1}-equitable with probability at least $1 - 2r|V_{N+k+2}|e^{-\alpha|V_{N+k+1}|\epsilon_{N+k+2}^2}$.

Next we show that our work implies that a random order has at least two maximal paths. Let $\gamma = \frac{1}{2} - \sum_{j=N}^{\infty} \epsilon_j$. Notice that if $A_n \neq V_n$ for all $n > N$, then there are at least two maximal paths. By our choice of N and $\gamma > 0$ we have that

$$\begin{aligned}
\mathbb{P}(\{\omega : |X_{\max}(\omega)| \geq 2\}) &\geq \mathbb{P}\left(\bigcap_{k=1}^{\infty}\left\{\omega : \gamma \leq \frac{|A_{N+k}|}{|V_{N+k}|} \leq 1-\gamma\right\}\right)\\
&\geq \mathbb{P}\left(\bigcap_{k=1}^{\infty} D_{N+k}\right)\\
&= \lim_{n\to\infty} \mathbb{P}(D_{N+1}) \prod_{k=1}^{n} \mathbb{P}(D_{N+k+1}|D_{N+k})\\
&\geq \lim_{n\to\infty} \mathbb{P}(D_{N+1}) \prod_{k=1}^{n} (1 - 2r|V_{N+k+2}|e^{-2|V_{N+k+1}|\alpha\epsilon_{N+k+1}^2}),
\end{aligned}$$

and the condition that B is superquadratic and exponentially bounded ensures that this last term converges to a non-zero value.

We can repeat this argument to show that for any natural k, a random order has at least k maximal paths. We remark also that the techniques of Section 4.1 could be generalized to show that a random order would have uncountably many maximal paths.

We now apply Theorem 3.3. □

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Department of Mathematics and Statistics, Dalhousie University, Canada
E-mail address: `Jeannette.Janssen@dal.ca`

Department of Mathematics and Statistics, University of Victoria, Canada
E-mail address: `aquas(a)uvic.ca`

Department of Mathematics, Trent University, Canada
E-mail address: `ryassawi@trentu.ca`