# How to prove that a sequence is not automatic

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**Abstract**

Automatic sequences have many properties that other sequences (in particular, non-uniformly morphic sequences) do not necessarily share. In this paper we survey a number of different methods that can be used to prove that a given sequence is not automatic. When the sequences take their values in a finite field $\mathbb{F}_q$, this also permits proving that the associated formal power series are transcendental over $\mathbb{F}_q(X)$.

## 1 Introduction

Automatic sequences can be found in several fields, particularly in view of their nature being "deterministic but possibly chaotic-like". They are more "regular" than other sequences; in particular, than general non-uniformly morphic sequences. Several papers prove that given sequences or families of sequences are not automatic, using a variety of methods. The purpose of this survey is to give a manual for proving (or trying to prove) that a given sequence is not automatic. The method essentially consists of finding, for each considered sequence, a relatively "easy-to-check" criterion for being automatic that is not satisfied by

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the sequence. In the case where the sequence takes its values in a finite field $\mathbb{F}_q$, proving that it is not $q$-automatic gives a proof of the transcendence of the associated formal power series over the field $\mathbb{F}_q(X)$, by means of a celebrated theorem of Christol (see [46, 47]).

We recall some definitions here.

- If $A$ is an *alphabet* (i.e., a finite set), we let $A^*$ denote the set of *words* over $A$, including the empty word (i.e., the set of finite sequences on $A$, including the empty sequence).
- The set $A^*$ can be equipped with a structure of a (free) *monoid*, with multiplication being *concatenation* of words.
- Let $A$ and $B$ be two alphabets. A *morphism* $\varphi$ from $A^*$ to $B^*$ is a map $A^* \to B^*$ such that for all $u, v \in A^*$ one has $\varphi(uv) = \varphi(u)\varphi(v)$. Clearly, a morphism is completely determined by its values on $A$ alone. If $A = B$, the morphism is called a *morphism on* $A$. If $A = \{a_1, \dots, a_d\}$, the *transition matrix* (or *adjacency matrix*) of $\varphi$ is the matrix $M = (m_{i,j})$, where $m_{i,j}$ is the number of occurrences of the letter $a_i$ in $\varphi(a_j)$.
- The morphism $\varphi$ is called *uniform* if the lengths of the images of each letter in $A$ by $\varphi$ are the same. If this length is equal to $q$, the morphism is called *$q$-uniform* or a *$q$-morphism*.
- Let $\varphi$ be a morphism on the alphabet $A$. If there exist a letter $a \in A$ and a non-empty word $v \in A^*$ such that $\varphi(a) = av$ and no $\varphi^k(v)$ is empty, then the words $\varphi^k(a) = av\varphi(v) \cdots \varphi^{k-1}(v)$ form a sequence of words of increasing length such that $\varphi^k(a)$ is a prefix of $\varphi^{k+1}(a)$ for all $k$. Thus there exists a unique infinite sequence that admits all the words $\varphi^k(a)$ as prefixes, namely the sequence

  $$a\, v\, \varphi(v)\, \varphi^2(v) \cdots \varphi^{k-1}(v) \cdots .$$

  This infinite sequence is a fixed point of $\varphi$ extended to infinite sequences on $A$. It is thus called a *fixed point* or an *iterative fixed point* of the morphism $\varphi$ (more precisely here *the* fixed point of the morphism $\varphi$ beginning with $a$). An iterative fixed point of a morphism is also called a *purely morphic* sequence.

  The extension of $\varphi$ to infinite sequences can be defined by

  $$\varphi(a_0 a_1 \cdots a_n \cdots) := \varphi(a_0)\varphi(a_1) \cdots \varphi(a_n) \cdots .$$

  What precedes can be rephrased in terms of *topological* notions. Let $\diamond$ be an element not belonging to the alphabet $A$. Let $A' := A \cup \{\diamond\}$. Every finite word $a_0 a_1 \cdots a_d$ over $A$ can be identified with the infinite sequence obtained by concatenating an infinite sequence of $\diamond$'s to this word, i.e., the sequence $a_0 a_1 \cdots a_d \diamond\diamond\diamond \cdots$, so that $A^* \cup A^{\mathbb{N}} \subset (A')^{\mathbb{N}}$. Now the set $(A')^{\mathbb{N}}$ is endowed with the product topology induced by the discrete topology on each copy of $A'$. Alternatively, this topology can be defined by a metric, where two sequences are "close" if they coincide on long prefixes. In this framework, extending the morphism $\varphi$ on $A^*$ by $\varphi(\diamond) := \diamond$, shows that $\varphi^k(a)$ *tends to* the iterative fixed point of $\varphi$, and that $\varphi$ is extended from $A^*$ to $(A')^{\mathbb{N}}$ by continuity.

- If a sequence is the image of the fixed point of a morphism by a 1-morphism, it is called *morphic*.

- If a sequence is morphic for a $q$-uniform morphism, it is called *$q$-automatic*. A sequence that is $q$-automatic for some $q \geq 2$ is called *automatic*.

- For $x$ a finite or infinite word by $x[i..j]$ we mean $x[i] \cdots x[j]$, where $x[i]$ is the $i$th letter of $x$.

The most famous example of an automatic sequence (more specifically, a 2-automatic sequence) is the (Prouhet-)Thue-Morse sequence $\mathbf{u} = (u_n)_{n \geq 0}$ where $u_n$ is the sum, reduced modulo 2, of the binary digits of $n$. It is not difficult to see that this sequence is the iterative fixed point, starting with 0, of the morphism defined on $\{0, 1\}$ by $0 \to 01$, $1 \to 10$.

One of the most famous examples of a (purely) morphic sequence is the binary Fibonacci sequence $\mathbf{f} = 01001010 \cdots$, defined as the iterative fixed point, starting with 0, of the morphism defined on $\{0, 1\}$ by $0 \to 01$, $1 \to 0$.

A famous 2-automatic sequence (that is not purely morphic) is the Golay–Shapiro sequence (also called the Rudin-Shapiro sequence). It is defined as the image by the 1-morphism $f$ of the iterative fixed point (beginning with $a$) of the morphism $\psi$ where $f$ and $\psi$ are defined as follows: $\psi$ is the 2-morphism defined on the alphabet $\{a, b, c, d\}$ by $a \to ab$, $b \to ac$, $c \to db$, $d \to dc$, and $f$ is defined by $a \to +1$, $b \to +1$, $c \to -1$, $d \to -1$.

One of the pioneering papers about automatic and morphic sequences is a famous paper of Cobham [49]. An early survey on the topic [6] was published in this journal in 1987. For a general approach to automatic and morphic sequences, the reader can consult, e.g., [15, 57, 62, 85].

## 2 Infinite $q$-kernels

The $q$-kernel of a sequence $\mathbf{a} = (a_n)_{n \geq 0}$ is the set of linearly-indexed subsequences

$$\mathrm{Ker}_q\ \mathbf{a} = \{(a_{q^k n + r})_{n \geq 0} :\ k \geq 0,\ r \in [0, q^k - 1]\}.$$

The set $\mathrm{Ker}_q\ \mathbf{a}$ is a subset of $A^{\mathbb{N}}$. A necessary and sufficient condition for a sequence to be $q$-automatic is that its $q$-kernel be finite (see, e.g., [15]). Thus, to prove that a sequence is not $q$-automatic, it suffices to exhibit some subset of its $q$-kernel of infinite cardinality. In particular, the following result often proves useful.

**Theorem 1** *Let $\mathbf{a} = (a_n)_{n \geq 0}$ be a sequence and $q$ be an integer $\geq 2$. If there exists a sequence of integers $(r_k)_{k \geq 0}$ such that $r_k \in [0, q^k)$ and the subsequences $(a_{q^k n + r_k})_{n \geq 0}$ are all distinct, then the sequence $\mathbf{a}$ is not $q$-automatic.*

*Proof.* Immediate from the finite kernel property for automatic sequences. □

**Example 2** This last result has been used several times in the literature. In particular, a theorem of Christol [46, 47] asserts that a formal power series $\sum a_n X^n$ with coefficients in a finite field $\mathbb{F}_q$ is algebraic over the field of rational functions $\mathbb{F}_q(X)$ if and only if the sequence $(a_n)$ is $q$-automatic. Hence, proving that the series $\sum a_n X^n$ is transcendental over $\mathbb{F}_q(X)$ is equivalent to proving that the $q$-kernel of the sequence $(a_n)$ is not finite. Here are some examples of results on the non-finiteness of $q$-kernels of sequences.

- A variation on this method was used [5, Lemme fondamental, p. 281] to prove that the sequence of the sums of the base-$p$-digits of $R(n)$, reduced modulo $p$, is not $p$-automatic (where $R$ is a polynomial of degree at least 2 that sends the integers to the integers). Also see the generalization [14], where the sequence of the sums of digits is replaced with any *quasi-strongly-$B$-additive* sequence.

- (Non-)finiteness of the kernel is used to prove transcendence/algebraicity results for formal Drinfeld modules in [40, 41], where the author generalizes an unpublished proof by the first author of a conjecture of Laubie (see [40, Proposition 3.3.1] or [41, Proposition 1]): Let $(a_n)_{n\geq 0}$ be a sequence with values in a finite field $\mathbb{F}_q$. Then the formal power series $\sum a_n X^{q^n}$ is algebraic over $\mathbb{F}_q(X)$ if and only if the formal power series $\sum a_n X^n$ is rational (i.e., if and only if the sequence $(a_n)_{n\geq 0}$ is eventually periodic).

- The infinite fixed point $\mathbf{v} = (v_n)_{n\geq 0}$ of the morphism $a \to aab$, $b \to b$ is not 2-automatic, as proved in [17]. The study of the sequences $(v_{2^k n+2^k-k})_{n\geq 0}$ for $k \geq 1$ reveals that they are all distinct. This fixed point occurs in von Neumann's recursive definition of the integers (see, e.g., [61, Section 3.2]).

- Let $p$ be a prime number and $q$ a power of $p$. Let $\mathbb{F}_q$ denote a finite field with $q$ elements. The formal power series $\Pi_q$ defined by

$$\Pi_q := \prod_{k\geq 1} \left(1 - \frac{X^{q^k} - X}{X^{q^{k+1}} - X}\right)$$

  is an element of the field $\mathbb{F}_q((X^{-1}))$ (the field of Laurent series in $1/X$ with coefficients in $\mathbb{F}_q$) that is essentially an analog of $\pi$. A proof that $\Pi_q$ is transcendental over $\mathbb{F}_q(x)$ uses Christol's theorem and the fact that $(b_n)_{n\geq 0}$ is not $q$-automatic, where $(b_n)_{n\geq 0}$ is the characteristic function of the integers that can be written as a finite sum $n = \sum(q^j - 1)$. It can be proved that the sequences $(b_{q^k n+q^k-k})_{n\geq 0}$ are all distinct [7].

- The transcendence of the series $\Pi_q$ in the previous item, as well as the transcendence of the so-called *bracket series*, were proved in [8] by showing that a certain sequence $(v_n)_{n\geq 0}$ is not $q$-automatic, thanks to a corollary of Theorem 1 above: *if a sequence is $q$-automatic, then there are finitely many sequences distinct of the form $(v_{q^k(n+1)-1})_{n\geq 0}$, and hence the sequence $(v_{q^k-1})_{k\geq 0}$ is ultimately periodic.* A result of the same kind in the case of $2D$-automatic sequences was used in [22].

- Other transcendence results for values of Carlitz functions analogous to the Riemann zeta function, to the logarithm, etc. have been obtained in approaches similar to the previous two items, see, e.g., [28, 29, 30, 10, 76, 94, 92, 67].

- A very sophisticated criterion of transcendence based on the non-finiteness of the $q$-kernel of a sequence is given in [93].

- The fixed point $(d_n)_{n\geq 0}$ of the morphism $1 \to 121$, $2 \to 12221$ is not 2-automatic, which results from the (non-trivial) property that the subsequences $(d_{2^{2^k}n})_{n\geq 0}$ are all distinct (see [4], where the morphism above occurs in the drawing of a classical *kolam*). Note that the sequence $(e_n)_{n\geq 0}$ defined by $e_n = d_{n+1}$ for all $n \geq 0$ is the fixed point of the morphism $2 \to 211$, $1 \to 2$.

- Generalizing the result for the morphism $2 \to 211$, $1 \to 2$ in the previous item, it can be proved similarly that, for $k \geq 1$, the fixed point of the morphism $1 \to 1^{k-1}2$, $2 \to 1^{k-1}21^{k+1}$ is not $(k+1)$-automatic (see [23] where this family of morphisms occurs in the study of certain sum-free sets).

- It is proved in [80] that, if a sequence has arbitrarily long blocks in common with a Sturmian sequence, then it cannot be $q$-automatic for any $q \geq 2$. This is generalized from Sturmian sequences to generalized polynomials in [37].

- It is proved in [90] that the sequence of gaps between consecutive occurrences of a block $w$ with $|w| > 1$ in the Thue-Morse sequence is substitutive but not $k$-automatic for any $k \geq 2$.

Other kinds of proofs of non-finiteness of $q$-kernels have been used for "arithmetic" sequences, in particular for *multiplicative sequences*. Recall that a sequence $(a_n)_{n\geq 1}$ is called *multiplicative* if, for all $m, n \geq 1$ such that $\gcd(m, n) = 1$, one has $a_{mn} = a_m a_n$. Also recall that the sequence $(a_n)_{n\geq 1}$ is called *completely multiplicative* if, for all $m, n \geq 1$, one has $a_{mn} = a_m a_n$.

The following theorem is [95, Theorem 2].

**Theorem 3** ([95]) *Let $v > 1$ be an integer and $f$ a multiplicative function. Assume that for some integer $h \geq 1$ there exist infinitely many primes $q_1$ such that $f(q_1^h) \equiv 0 \pmod{v}$. Furthermore assume that there exist relatively prime integers $b$ and $c$ such that for all primes $q_2 \equiv c \pmod{b}$ we have $f(q_2) \not\equiv 0 \pmod{v}$. Then the sequence $(f(n))_{n\geq 1} \pmod{v}$ is not $q$-automatic for any $q \geq 2$.*

**Example 4** There are several papers about (completely) multiplicative functions and their (non-)automaticity in the literature. Here we give one theorem and a few recent references.

- Recall that the multiplicative sequences $\sigma_m$, $\varphi$ and $\mu$, are defined as follows: for $m \geq 1$ $\sigma_m(n) := \sum_{d|n} d^m$, $\varphi(n)$ is the Euler totient function, and $\mu(n)$ is the Möbius function. Theorem 3 was used in [95] to prove that $(\sigma_m(n) \pmod{v})_{n\geq 1}$ and $(\varphi(n) \pmod{v})_{n\geq 1}$ are not $q$-automatic for any $q \geq 2$ and $v \geq 3$. The case $v = 2$ is also addressed in [95, Thm. 7], where the author uses a theorem of Minsky and Papert (Theorem 23 below), and the fact that $(\mu(n) \pmod v)_{n\geq 1}$ is not $q$-automatic for any $q \geq 2$ and $v \geq 2$.

- Recent papers on (completely) multiplicative sequences that are (non-)automatic use a variety of methods: see [13, 66, 74, 70, 71, 72, 73]. In particular Konieczny, Lemańczyk, and Müllner [73] give a complete characterization of automatic multiplicative sequences.

**Remark 5** A classical property of regular languages is that they satisfy the "pumping lemma"[1] (see, e.g., [15, Lemma 4.2.1]). Let $C_q$ be the set of all canonical base-$q$ representations of natural numbers (with no leading zeroes). Since a sequence $(a_n)_{n\geq 0}$ is $q$-automatic if and only if all the languages

$$L_b = \{n_q : \ n_q \in C_q \text{ is the } q\text{-ary expansion of } n \text{ and } a_n = b\}$$

are regular, a way to prove that a sequence $(a_n)_{n\geq 0}$ is not $q$-automatic is to prove that for some value $b$, the language $L_b$ is not regular —which can be done, e.g., by using the pumping lemma.

As an example of this approach, consider the characteristic sequence of the set $S = \{2^n(2^n - 1) : n \geq 0\}$. Here the language of base-2 expansions of $S$ is $\{1^n 0^n : n \geq 0\}$, a classical example of a non-regular language. Thus the characteristic sequence of $S$ is not 2-automatic. Another example is given in [87]: the fixed point beginning with $c$ of the morphism $c \to cba$, $a \to aa$, and $b \to b$, which is proved to be non-2-automatic by showing that the language $L := \{10^{n-\lfloor \log_2 n\rfloor -1}(n)_2 : \ n \geq 1\} \cup \{1\}$ is not regular.

**Remark 6** We have not explicitly spoken of $q$-automata. But, actually, using the finiteness of the $q$-kernel or the pumping lemma is essentially possible because the "$q$-automaton" behind an automatic sequence has a *finite* number of states. A nice recent paper [75] invokes this property to prove that $(\ell_b(n))_{n\geq 1}$, is not a $q$-automatic sequence for any $q \geq 2$, where $\ell_b(n)$ is the last nonzero digit of $n!$ and $b \geq 2$ is a fixed base such that, if $b = p_1^{a_1} p_2^{a_2} \cdots$, then there exist at least two $p_i$'s such that $a_i(p_i - 1) = \max\{a_j(p_j - 1) : \ j = 1, 2, \ldots\}$. These integers form the sequence A135710 in [89] (A135710 $= 12, 45, 80, 90, 144, 180, 189, 240, 360, \ldots$).

**Remark 7** Similarly, the Myhill-Nerode theorem (see, e.g., [64, Thm. 3.9]) can be used to prove that a language is not regular. In fact, the $k$-kernel of a binary sequence $(s_n)_{n\geq 0}$ is essentially the same size as the number of Myhill-Nerode equivalence classes of the language of reversed canonical base-$k$ representations of $\{n : s_n = 1\}$.

# 3 Irrational frequencies

Suppose a sequence $(a_n)_{n\geq 0}$ is $q$-automatic for some $q \geq 2$. Then, if the *letter frequencies*

$$\lim_{n\to\infty} \frac{1}{N} \#\{n \leq N : \ a_n = b\}$$

[1] Though some people try to translate it literally, the correct name of this lemma in French is "le lemme de l'étoile".

exist, they must be rational [49]. Hence if the frequency of occurrence of some letter in a sequence $(a_n)_{n\geq 0}$ taking its values in a finite alphabet exists and is irrational, the sequence cannot be $q$-automatic for any $q \geq 2$.

**Example 8** An example of the claim above is given by the Sturmian sequences (recall that a sequence is Sturmian if, for all $k \geq 0$, it contains exactly $k+1$ distinct blocks of length $k$). A Sturmian sequence necessary takes only two values, and the frequencies of occurrence of these two values are irrational. Note that this remark makes the title of [91] redundant, and underlines a mistake in [34] (about that paper, also see the review MR 2006a:68140 by P. Séébold). Speaking of non-automaticity of Sturmian sequences, it is worth noting that, more generally, *dendric sequences* cannot be $q$-automatic for any $q \geq 2$: this is proved in [32, Corollary 15], by first proving that these sequences admit no rational topological dynamical eigenvalue (see Section 10). Note that, as indicated in that paper, Sturmian sequences, episturmian sequences, and codings of interval exchanges are particular examples of dendric sequences.

It may happen that all frequencies exist and are rational, but, in this case, a variation on the observation above can be useful:

**Theorem 9** *If a sequence $(a_n)_{n\geq 0}$ with values in a finite alphabet is such that the frequency of some block occurring in this sequence exists and is irrational, then the sequence cannot be $q$-automatic for any $q \geq 2$.*

*Proof.* The sequence of consecutive overlapping blocks of length $k$ occurring in a $q$-automatic sequence is also $q$-automatic. □

**Example 10** Theorem 9 was used in the following examples.

- To show that the sequence of moves in the cyclic tower of Hanoi algorithm is not $q$-automatic, for any $q \geq 2$, it was proved in [9] that the frequency of some three-letter word on the alphabet of moves exists and is not rational.
- It is shown in [11] that the language of all primitive words over a finite alphabet is not unambiguously context-free by proving that the square of the Möbius function, $(\mu^2(n))_{n\geq 1}$, is not automatic: the frequencies of the values taken by $\mu^2$ are $6/\pi^2$ and $1 - 6/\pi^2$, which are irrational.

**Remark 11**

- Unfortunately, this method does not work all the time: there exist morphic sequences that are non-automatic, "although" the frequencies of all words occurring in the sequence exist and are rational. For example, consider the fixed point of the morphism $2 \to 211$, $1 \to 2$. The dominant eigenvalue of the matrix of this morphism is equal to 2. Furthermore, the morphism $f$ is *primitive*: this means that there is a constant $d$ such that $a$ appears in $f^d(b)$ for all letters $a, b$ in the alphabet. For each $\ell \geq 1$ the associated morphism generating the sequence of overlapping blocks of length $\ell$ is also primitive and has 2 as dominant eigenvalue (see, e.g., [85, Section 5.4.1]). Hence the frequency of occurrence of every block is a rational number (recall that the vector of frequencies of a fixed point of a primitive morphism is the normalized eigenvector associated with the dominant eigenvalue, and that linear equations with rational coefficients have rational solutions). But the fixed point of $2 \to 211$, $1 \to 2$ is not $q$-automatic for any $q \geq 2$ (see [4]).

- If the frequency for a given letter occurring in a sequence does not exist, one can replace the limit in the definition of the frequency with limsup or liminf: a result in [26] asserts that, if the sequence is automatic, then both quantities limsup and liminf are rational.

Another result dealing with frequencies, stated in [18], is worth noting.

**Theorem 12** **(**[18]) *If the adjacency matrix of a primitive non-uniform morphism has an irrational dominant eigenvalue, then an iterative fixed point of this morphism cannot be automatic.*

**Example 13** Theorem 12 was used in [18] to prove the non-automaticity of fixed points of morphisms related to Grigorchuk-like groups. For example, the fixed point of the morphism defined on $\{a, b, c, d\}$ by $a \to aca$, $b \to d$, $c \to aba$, $d \to c$ (see [25, Theorem 4.1]) is not automatic. Namely, as noted in [18], the matrix of this morphism is primitive and its characteristic polynomial, which is equal to $x^4 - 2x^3 - 2x^2 - x + 2$, clearly has no rational root.

# 4 Synchronization

Suppose $f$ is a function from $\mathbb{N}$ to $\mathbb{N}$. If there is a deterministic finite automaton that recognizes, in parallel, the base-$k$ representations of $n$ and $f(n)$, then we say that $f$ is *$k$-synchronized*. The following result is very useful for proving sequences not automatic.

**Theorem 14** *If $f$ is $k$-synchronized, then*

*(a)* $f = \mathcal{O}(n)$*;*

*(b) If* $f = o(n)$ *then* $f(n) = \mathcal{O}(1)$*.*

*(c) If there is an increasing subsequence* $n_1 < n_2 < \cdots$ *such that* $\lim_{i\to\infty} f(n_i)/n_i = 0$*, then there is a constant $C$ such that $f(n) = C$ for infinitely many $n$.*

For proofs, see [42, 88].

Many functions dealing with $k$-automatic sequences are $k$-synchronized. A fairly detailed list is contained in [88] and includes such quantities as

- appearance (length of shortest prefix containing all length-$n$ blocks); [45]

- repetitivity index (minimum distance between two consecutive occurrences of a length-$n$ block); [43]
- the uniform recurrence function (maximum distance between two consecutive occurrences of a length-$n$ block); [45]
- condensation (length of the shortest block containing all length-$n$ blocks); [59]
- separator length (length of the shortest word beginning at position $n$ not appearing previously in the sequence); [58, 42]
- palindrome separation (longest distance between two consecutive length-$n$ blocks, both of which are palindromes); [88]
- repetition word length (for each $n$, the length of the shortest prefix $w$ of $\mathbf{x}[n..\infty]$ for which either $w$ is a suffix of $\mathbf{x}[0..n-1]$ or vice versa); [77]
- largest square centered at a given position; [88]
- shortest square beginning at a given position; [88]
- longest palindromic suffix of a length-$n$ prefix; [33]
- the Bugeaud-Kim function (the length of the shortest prefix of $\mathbf{x}$ containing two possibly overlapping occurrences of some length-$n$ block); [36]
- block complexity; [60] and
- first occurrence of a run of length $\geq n$. [88]

As a consequence, we immediately get that if $\mathbf{x}$ is a $k$-automatic sequence, then the bounds in Theorem 14 hold. This gives a method for proving some sequences non-automatic. Let us consider some examples.

**Example 15** A *run* in a sequence is a block of consecutive identical values. Schlage-Puchta proved the following lemma about runs in automatic sequences: if an automatic sequence $(a_n)_{n\geq 0}$ has arbitrarily long runs, then there exists a constant $c > 0$ such that $a_n = a$ for $n \in [x, (1+c)x]$ and infinitely many $x$. See [86]. With this lemma he was able to prove that the sequence $(\mu(n) \bmod p)_{n\geq 1}$ is not automatic, for all primes $p$. Schlage-Puchta's lemma immediately follows from Theorem 14 and the observation that the function $f$ mapping $n$ to the first position $m$ where $a_m = a_{m+1} = \cdots = a_{m+n-1}$ is $k$-synchronized.

Even further, the $O(n)$ upper bound on the growth rate of automatic sequences applies to sequences defined over many other kinds of numeration systems, such as Fibonacci numeration, Tribonacci numeration, and so forth.

**Example 16** Consider the fixed point $\mathbf{v}$ of the morphism $a \rightarrow aab$, $b \rightarrow b$, which we discussed previously. The function mapping $n$ to the starting position of the first occurrence of a run of length $n$ in a $k$-automatic sequence is $k$-synchronized and hence must be in $\mathcal{O}(n)$.

But the first occurrence of run of length $n$ in $\mathbf{v}$ begins at position $2^{n+1} - n - 1$, which is clearly not in $\mathcal{O}(n)$. Hence $\mathbf{v}$ is not $k$-automatic for any $k$. By the remark above, $\mathbf{v}$ cannot be 'automatic' in any numeration system at all (e.g., Fibonacci, Ostrowski, etc.), provided the numeration system has certain properties.

# 5 Block complexity

As we saw in the last section, the (block-)complexity (aka factor complexity, aka subword complexity) of a sequence $\mathbf{a}$ is the function $p_{\mathbf{a}}(n)$ counting the number of distinct blocks of length $n$ that occur in $\mathbf{a}$. Hence the block complexity of an automatic sequence is in $\mathcal{O}(n)$ (see, e.g., [49, Thm. 2]). Thus we have

**Theorem 17** *If the (block-)complexity of a sequence taking its values in a finite alphabet is not in $\mathcal{O}(n)$, then the sequence cannot be automatic. If the appearance function of a sequence taking its values in a finite alphabet is not in $\mathcal{O}(n)$, then the sequence cannot be automatic.*

This is one of the most useful criteria for proving non-automaticity.

**Example 18** Theorem 17 was used in various contexts.

- A first example, in the more general context of $2D$-sequences, is the study of complexity of the Pascal triangle modulo $d$: in this case the complexity is the rectangle-complexity; $p(u, v)$ is the number of different rectangles of size $(u \times v)$. It was proved in [12] that, for $d \geq 2$, the complexity of the sequence $(\binom{m}{n} \pmod{d})_{m,n\geq 0}$ is $\Theta(\max(u, v)^{2\omega(d)})$, where $\omega(d)$ is the number of distinct prime divisors of $d$. In particular this double sequence is not $q$-automatic for any $q \geq 2$ if $q$ is not a prime power. (This result was generalized to linear cellular automata by Berthé [31].)

- If $\mathbf{t} = (t_n)_{n\geq 0}$ is the Thue-Morse sequence (defined, e.g., as the fixed point of the morphism $0 \to 01$, $1 \to 10$) and $H$ is a polynomial with rational coefficients sending the integers to the integers and such that $\deg H \geq 2$, then the block complexity of the sequence $(t_{H(n)})_{n\geq 0}$ grows exponentially [81, Corollary 3], which proves that this sequence is not $q$-automatic for any $q \geq 2$ (this was proved only for $q = 2$ in [5]). Furthermore it is proved in [52] that the sequence $(t_{n^2})_{n\geq 0}$ is normal, and, more generally, in [83] that the sequences $(u_{n^2} \bmod d)_{n\geq 0}$ are normal, where $(u_n)_{n\geq 0}$ is a digital sequence in base $q$ in the sense of [44] and $d$ an integer prime to $q - 1$ and to $\gcd\{u_n,\ n \in \mathbb{N}\}$.

- Another example is [55, 56], where it is proved that the fixed point $\mathbf{v}$ of the morphism $a \to aab$, $b \to b$ is not $q$-automatic for any $q \geq 2$, by proving that its block-complexity is in $\Theta(n^2)$.

- The following result was proved in [66]: if $\mathbf{u} = (u_n)_{n\geq 1}$ is a completely multiplicative sequence (i.e., $u_{mn} = u_m u_n$ for all $m, n \geq 1$), taking finitely many values in a field $K$, and if the number of primes $p$ such that $u(p) \neq 1_K$ is finite, then the subword complexity of $\mathbf{u}$ is $\Theta(n^t)$ where $t$ is the number of primes $p$ such that $u(p) \notin \{0_K, 1_K\}$.

An example of application is that the sequence $((-1)^{\nu_2(n)+\nu_3(n)})_{n\geq 1}$ is not $q$-automatic for any $q$, where $\nu_p(n)$ is the highest exponent $j$ such that $p^j \mid n$: its complexity is in $\Theta(n^2)$.

- The last two, spectacular, examples that we give in this section are results from [3, 1] and [35]:
  - [3, 1] The sequence of digits of an algebraic irrational real in base $b \geq 2$ is not $q$-automatic for any $q \geq 2$. More generally the main result of [3, 1] reads: *The complexity of the $b$-ary expansion of every irrational algebraic number satisfies the property* $\liminf_{n\to\infty} p(n)/n = +\infty$.
  - [35] (also see [2]) Let $\alpha$ be a positive real which is algebraic over $\mathbb{Q}$ of degree at least 3, and such that its continued fraction expansion has finitely many distinct partial quotients. Then this sequence of partial quotients is not $q$-automatic for any $q \geq 2$. More generally the main result of [35] reads: *If the sequence of partial quotients of a positive algebraic real of degree $\geq 3$ takes finitely many distinct values, then the (block-)complexity of this sequence of partial quotients satisfies the property* $\liminf_{n\to\infty} p(n)/n = +\infty$.

# 6 Gaps and runs

Cobham [49] proved the following useful result about gaps in automatic sequences. (A similar result was found independently by Minsky and Papert [78].)

**Theorem 19** *Let $\mathbf{x} = (x(n))_{n\geq 0}$ be a $k$-automatic sequence over $\Delta$. Let $d \in \Delta$. Define $\alpha_j$ to be the position of the $j$'th occurrence of $d$ in $\mathbf{x}$. (More formally, if $|\mathbf{x}[0..t-1]|_d = j-1$ and $\mathbf{x}[t] = d$, then $\alpha_j = t$.) Then either*

$$\limsup_{n\to\infty} \frac{|\mathbf{x}[0..n-1]|_d}{\log n} < \infty \quad \textit{or} \quad \liminf_{j\to\infty} (\alpha_{j+1} - \alpha_j) < \infty \quad \textit{(or both).}$$

**Remark 20** It is possible for both alternatives to hold. For example, consider the characteristic sequence of the set $\{2^n : n \geq 1\} \cup \{2^n - 1 : n \geq 1\}$, and $d = 1$.

As an application let us prove the following:

**Corollary 21** *Let $p$ be a polynomial with rational coefficients such that $p(\mathbb{N}) \subseteq \mathbb{N}$. Then the characteristic sequence $\mathbf{c}$ of the set $\{p(i) : i \geq 0\}$ is $k$-automatic if and only if $\deg p < 2$.*

*Proof.* If $\deg p < 2$, then this characteristic sequence is ultimately periodic, and hence $k$-automatic. Otherwise assume $\deg p \geq 2$, and $\mathbf{c}$ is $k$-automatic. Take $d = 1$ in Theorem 19. We have $\alpha_{j+1} - \alpha_j = p(j'+1) - p(j')$ for $j' = j + c$, and $j$ sufficiently large and $c$ a constant.

But this difference is a polynomial of degree $(\deg p) - 1$ and hence goes to $\infty$ as $j$ gets large, so the theorem tells us that

$$\limsup_{n\to\infty} \frac{|\mathbf{x}[0..n-1]|_d}{\log n} < \infty.$$

But $|\mathbf{x}[0..n-1]|_d = \Theta(n^{1/s})$, where $s = \deg p$, a contradiction. Hence $\mathbf{c}$ cannot be $k$-automatic.

Let us give another application found in [69], which in particular answers a question in [21].

**Corollary 22 (Kärki–Lacroix–Rigo)** *Let $r$ be an integer $\geq 1$. Let $F$ be the set of maps $\{\varphi_0, \varphi_1, \ldots, \varphi_r\}$ with $\varphi_0(x) = x$ and there exist $k_i, \ell_i \in \mathbb{Z}$, with $2 \leq k_1 \leq k_2 \leq \cdots k_r$ such that $\varphi_i(x) = k_i x + \ell_i$ for each $i \in [1, r]$. Let $F(S)$ be defined for any set of integers $S$ by $F(s) := \{\varphi(s) : s \in S, \varphi \in F\}$. Let $I$ be any finite set of integers. Define $F^0(I) = I$, and $F^{m+1}(I) := F(F^m(I))$ for $m \geq 0$. Let $X := \cup_{m\geq 0} F^m(I)$ and suppose that $X \subset \mathbb{N}$. If $\sum_{1\leq t\leq r} k_t^{-1} < 1$ and if there exist $i, j$ such that $k_i$ and $k_j$ are multiplicatively independent, then the characteristic sequence of $X$ is not $k$-automatic for any $k \geq 2$.*

Another result about gaps is the following theorem [78].

**Theorem 23 (Minsky-Papert)** *Let $\mathbf{a}$ be a $k$-automatic sequence. Let $a$ be a value occurring in $\mathbf{a}$ infinitely often. Suppose that the frequency of occurrences of $a$ in $\mathbf{a}$ is zero. Then, letting $\alpha_j$ denote the index of $j$-th occurrence of $a$, one has $\limsup \alpha_{j+1}/\alpha_j > 1$.*

As indicated above (second item in Example 4) an application of this result can be found, e.g., in [95]. Note that a more precise version of Theorem 23 was given by Cobham in [49, Theorem 12]: it was used to prove results of non-automaticity, e.g., in [16].

**Remark 24** Note that several other results of Cobham can be used to prove the non-automaticity of morphic sequences. For example, [49, Theorem 9] was used by Mkaouar [79] to prove that the sequence of partial quotients of the Baum-Sweet power series is not $k$-automatic, if $k \geq 2$ is any integer multiplicatively independent of 2 (the case where $k$ is a power of 2 is addressed earlier in that paper); also see a generalization by Yao in [93, Théorème 3]. To conclude this remark, we note that several results in [93] state that if the positions of the runs in a sequence satisfy certain growth properties, then the sequence is not automatic.

# 7 Dirichlet series

Given an automatic sequence $(a_n)_{n\geq 0}$ with values in the complex numbers, it is proved in [19] that the Dirichlet series $\sum a_n/(n+1)^s$ possesses a meromorphic continuation to the entire complex plane, and that its poles, if any, belong to a finite number of left semi-lattices. This result was used in [50] to prove that certain arithmetic sequences are not automatic: their Dirichlet series cannot be meromorphically continued to the whole complex plane, or the continuation violates the condition on the poles given above.

**Example 25** Several arithmetic sequences were proved to be non-automatic [50] by using the properties of their Dirichlet series. In particular, we mention the following:

- Let $\Omega(n)$ be the number of primes (counted with multiplicity) that divide $n$. The Liouville function $\lambda$ is defined by $\lambda(n) = (-1)^{\Omega(n)}$. Then the sequence $(\lambda(n))_{n\geq 1}$ is not $q$-automatic for any $q \geq 2$.

- The characteristic function of the prime numbers is not $q$-automatic for any $q \geq 2$. (Note that this was proved in a different way in [63].)

- The characteristic function of the prime powers is not $q$-automatic for any $q \geq 2$. (Note that this was proved in a different way in [78].)

- The sequences $(q_m(n))_{n\geq 1}$, for $m \geq 2$, are not $q$-automatic for any $q \geq 2$, where $q_m(n)$ is defined by
$$q_m(n) = \begin{cases} 0, & \text{if } p^m \mid n \text{ for some prime } p; \\ 1, & \text{otherwise.} \end{cases}$$

**Remark 26** Also note that Dirichlet series were used in [66] to give an alternative proof of the non-automaticity of certain completely multiplicative sequences.

# 8 Orbit properties

We define the *shift map* $T$ on sequences taking their values in a finite set as follows.

**Definition 27**

- If $\mathbf{u} = (u_n)_{n\geq 0}$ is a sequence, then $T\mathbf{u} := (v_n)_{n\geq 0}$, where $v_n := u_{n+1}$ for all $n \geq 0$. In other words, $T(u_0u_1u_2\cdots) := u_1u_2u_3\cdots$.

- The orbit of a sequence $\mathbf{u} = (u_n)_{n\geq 0}$ under the shift is the set of sequences obtained from $\mathbf{u}$ by iterating $T$, namely $\{\mathbf{u}, T(\mathbf{u}), T^2(\mathbf{u}), \ldots\} = \{T^k(\mathbf{u}) : \ k \geq 0\}$.

- The orbit closure of a sequence is the closure (for the product topology on the set of sequences) of the closure of this orbit.

As proved in [20, Theorem 6]: *Let $q \geq 2$. The lexicographically least sequence in the orbit closure of a $q$-automatic sequence is also $q$-automatic.* Thus we get the following result:

**Theorem 28** ([18]) *Let $\mathbf{x} = (x_n)_{n\geq 0}$ be a sequence over some alphabet $\mathcal{A}$. Let $\mathcal{A}'$ be a proper subset of $\mathcal{A}$. Suppose that there exists a sequence $\mathbf{y} = (y_n)_{n\geq 0}$ on $\mathcal{A}'$ with the property that each of its prefixes appears in $\mathbf{x}$. Let $q \geq 2$. If no sequence in the closed orbit of $\mathbf{y}$ under the shift is $q$-automatic, then $\mathbf{x}$ is not $q$-automatic. In particular, if $\mathbf{y}$ is Sturmian, or $\mathbf{y}$ is uniformly recurrent and its complexity is not in $\mathcal{O}(n)$, then $\mathbf{x}$ is not $q$-automatic for any $q \geq 2$.*

**Example 29** Theorem 28 was used in [18] to prove that the two fixed points of morphisms respectively given in [27, Theorem 2.9] and [24, Theorem 4.5] are not $q$-automatic for any $q \geq 2$, namely

- The fixed point beginning with $a$ of the morphism $a \to aca$, $b \to bc$, $c \to b$ is not automatic.
- The fixed point beginning with $a$ of the morphism $a \to aca$, $c \to cd$, $d \to c$ is not automatic.

**Remark 30** One of the referees of the paper [18] has just noted that Theorem 28 can also be deduced from Theorem A in [39]. Furthermore this approach does not require that $\mathcal{A}'$ be a proper subset of $\mathcal{A}$.

# 9 When non-$q$-automaticity implies non-automaticity

Recall that a sequence is called non-automatic if it is not $q$-automatic for any integer $q \geq 2$. A nice and deep theorem in [53, Theorem 1 and Corollary 6] implies the following result.

**Theorem 31** ([53]) *Let $A$ be a finite alphabet. Suppose that $(a_n)_{n\geq 0}$ is the image by a non-erasing morphism of an iterative fixed point beginning with some letter $b$ of a morphism $\sigma$, such that all letters in $A$ occur in $\sigma^\infty(b)$. Let $\alpha$ be the dominant eigenvalue of the adjacency matrix of $\sigma$. If $(a_n)_{n\geq 0}$ is $q$-automatic and not ultimately periodic, then $\alpha$ and $q$ must be multiplicatively dependent (i.e., there exist two integers $s$ and $t$ with $\alpha^s = q^t$). In particular, if $\alpha$ is an integer, and the sequence $(a_n)_{n\geq 0}$ is automatic and not ultimately periodic, then it must be $\alpha$-automatic.*

**Example 32** Let us consider the morphism $a \to aab$, $b \to b$ one more time. We know that its fixed point $\mathbf{v}$ is not 2-automatic (as proved in[17]), while the dominant eigenvalue of the adjacency matrix is equal to 2. This proves once more that it is not $q$-automatic for any $q \geq 2$.

**Remark 33** Theorem 31 above contains, as a particular case, the celebrated Cobham theorem [48] which asserts that, *if $q$ and $r$ are two integers $\geq 1$ that are multiplicatively independent, then a sequence that is both $q$-automatic and $r$-automatic must be eventually periodic.* (The dominant eigenvalue of the adjacency matrix of the uniform morphism behind a $d$-automatic sequence is $d$.) This theorem can be used to prove that a sequence is not $r$-automatic, if it is already known to be $q$-automatic and not eventually periodic (with $q$ and $r$ multiplicatively independent). A nice generalization can be found in [38], where the notion of an *almost everywhere $q$-automatic* sequence (suggested by J.-M. Deshouillers) is introduced (this is a sequence that coincides with a $q$-automatic sequence on a set of natural density 1): in particular, the authors of [38] use this notion to prove a result of non-automaticity.

# 10 A "dynamical" approach

In this section we will consider *(discrete) dynamical systems* associated with sequences. The papers on the subject usually stick to the language of dynamics, which can differ from the language used by combinatorists on words. We will expand the dynamical approach a bit: in particular, we will try, as far as possible, to use the language of combinatorics of words, e.g., by "translating" the dynamical notions. Our purpose is to give some details about how to use the concept of *dynamical eigenvalues* for morphic sequences to spot whether a fixed point of a morphism $\varphi$ is automatic.

If $\mathbf{a} = (a_n)_{n\geq 0}$ is a sequence on a finite alphabet $A$, it defines a *shift dynamical system* $(X_{\mathbf{a}}, T)$ as follows. Let $T$ be the left shift map defined in Section 8 and let $X_{\mathbf{a}}$ be the (topological) closure of $\{T^k(\mathbf{a}) : k \geq 0\}$, where the topology on sequences is the one where two sequences are close if they agree on a long enough initial block. The elements of $X_{\mathbf{a}}$ are exactly the sequences all of whose subwords appear in $\mathbf{a}$. If $\varphi$ is a primitive morphism, then any *ultimately $\varphi$-periodic* $\mathbf{a}$ i.e., any sequence $\mathbf{a}$ such that $\varphi^i(\mathbf{a}) = \varphi^j(\mathbf{a})$ for some $i, j \geq 1$ with $i \neq j$) defines the same shift dynamical system, so we can write $X_\varphi$ instead of $X_{\mathbf{a}}$. Also with the assumption of primitivity, $X_\varphi = X_{\varphi^j}$ for each $j \geq 1$, so we can assume, up to replacing $\varphi$ by an iterate, that an ultimately $\varphi$-periodic sequence is a $\varphi$-fixed point.

We say that $\lambda \in \mathbb{C}$ is a (topological) *dynamical eigenvalue* for $(X_{\mathbf{a}}, T)$ if there is a (continuous) function $f : X_{\mathbf{a}} \to \mathbb{C}$ such that $f \circ T = \lambda f$. An eigenvalue for a dynamical system captures notions of periodicity in it. For example, suppose that $\mathbf{a}$ is a fixed point of a primitive $q$-uniform morphism $\varphi$. Recognizability [82] implies that if $\mathbf{a}$ is not ultimately periodic, then any $x \in X_{\mathbf{a}}$ can be de-substituted in a unique way, i.e., $x = T^k\varphi^n(y)$ for a unique $y \in X_{\mathbf{a}}$ and a unique $k \in [0, q^n)$. Using this de-substitution for $x$, if we define $f(x) = e^{\frac{2\pi i k}{q^n}}$, then $f$ is an eigenfunction for the eigenvalue $e^{\frac{2\pi i}{q^n}}$. Therefore for any natural number $n$, the number $\lambda = e^{\frac{2\pi i}{q^n}}$ is an eigenvalue for $(X_{\mathbf{a}}, T)$.

The work of Kamae [68] and Dekking [51] shows that for primitive $q$-uniform morphisms, the only other possibility for an eigenvalue is some $e^{\frac{2\pi i}{h}}$, where $h \in \mathbb{N}$ is prime to $q$ (and in fact turns out to divide $q-1$). Here $h$ is the *height* of $\varphi$, or, equivalently, the height of one (any) of its fixed points $\mathbf{a}$. Define $w$ to be a *return word* to the letter $a$ for $\varphi$ if $w$ starts with $a$ and $wa$ occurs in $\varphi^n(b)$ for some $n$ and $b$. The height $h$ is defined to be

$$h = h(\varphi) = h(\mathbf{a}) := \gcd\{|w| : \ w \text{ is a return word to } a_0 \text{ in } \mathbf{a}\}.$$

As $\varphi$ is primitive, its height does not depend on the choice of the fixed point $\mathbf{a}$. The height of $\varphi$ entirely depends on the return word structure of $\varphi$.

Call a sequence $\boldsymbol{a}$ *aperiodic* if it is not eventually periodic, and call it *minimal* if for any factor $\boldsymbol{w}$ that appears in $\boldsymbol{a}$, there is a constant $k$ such that $\boldsymbol{w}$ appears in every factor of $\boldsymbol{a}$ of length at least $k$. The theorem of Dekking and Kamae was extended to shifts defined by aperiodic automatic sequences which are codings of a primitive $q$-morphism in [84]. Combining this result with the fact that any minimal automatic sequence can be realised as the pointwise image of a fixed point of a primitive $q$-uniform morphism [49], we obtain the following. This result can be seen as a dynamical version of Cobham's theorem.

**Theorem 34 (Müllner–Yassawi)** *Let $\mathbf{a}$ be a minimal aperiodic $q$-automatic sequence, and let $h$ be the height of $\mathbf{a}$. Then the eigenvalues of $X_{\mathbf{a}}$ are the $q^n$-th roots of unity, $\forall n \geq 1$, and $e^{\frac{2\pi i}{h}}$.*

Another way of capturing the notion of an eigenvalue $\lambda$ for $\varphi$ is as follows. Given $\mathbf{a} = (a_n)_{n\geq 0}$, a fixed point of a $q$-uniform morphism, we notice that if $a_j = a_k$, $j < k$, i.e., if $w = a[j..k-1]$ is a return word to $a_j$, then for each natural number $\ell$, $a[jq^\ell..(j+1)q^\ell - 1] = a[kq^\ell..(k+1)q^\ell - 1]$. If $n \in \mathbb{N}$, then writing $\lambda = e^{\frac{2\pi i}{q^n}}$, we capture the continuity of the eigenfunction $f$ by noting that

$$\lim_{\ell\to\infty} \lambda^{|\varphi^\ell(w)|} = \lim_{\ell\to\infty} \lambda^{|w|q^\ell} = 1. \tag{1}$$

In fact this limit is attained for $\ell \geq n$, but Host [65] showed that this notion extends to fixed points $\mathbf{a}$ for arbitrary primitive morphisms $\varphi$, and there in general $\lambda$ is not necessarily a root of unity[2].

**Theorem 35 (Host)** *Let $\varphi$ be a primitive morphism with an aperiodic fixed point and such that whenever $\varphi(a)$ starts with $b$, then $\varphi(b)$ starts with $b$. Then $\lambda$ is an eigenvalue for $(X_\varphi, T)$ if and only if*

$$\lim_{\ell\to\infty} \lambda^{|\varphi^\ell(w)|} = 1 \tag{2}$$

*whenever $w$ is a return word.*

One could extend these results to non-primitive morphisms, although care will need to be taken with letters that are not recurrent, the existence of eventually periodic $\varphi$-fixed points, etc.

The following example illustrates Host's result.

**Example 36** Consider the Fibonacci morphism $a \to ab$, $b \to a$. It can be checked that 1 is the gcd of the lengths of all return words to $a$ and also to $b$, so we can apply Lemma 37. The adjacency matrix has unit determinant, leading eigenvalue the golden ratio $\phi$, with an eigenvector whose entries are in $\mathbb{Z}[1, \frac{1}{\phi}]$. Lemma 37 tells us that the eigenvalues of $\mathbf{a}$ must belong to $\mathbb{Z}[1, \frac{1}{\phi}]$. In particular, $\mathbf{a}$ cannot be $q$-automatic for any $q$.

Note that if $\mathbf{a}$ is aperiodic, $q$-automatic, and also the fixed point of a primitive morphism $\varphi$, then, since $X_{\mathbf{a}} = X_\varphi$, the eigenvalues of $X_\varphi$ must equal the eigenvalues of $X_{\mathbf{a}}$, so Theorem 34 tells us that $\varphi$ must have all $q^n$-th roots of unity as eigenvalues. From (2) we immediately see that $q^n$ must divide $|\varphi^\ell(w)|$ whenever $w$ is a return word (for large enough

[2] In fact Host proved more, namely that any *measurable* eigenvalue, i.e., one which has a Borel-measurable eigenfunction, must be a topological eigenvalue, i.e., one which has a continuous eigenfunction, by showing that a measurable eigenvalue necessarily satisfies (2).

$\ell$). The following lemma, a mild modification of one in [65], tells us how to get information about the eigenvalues of $(X_\varphi, T)$ from the eigenvalues of the adjacency matrix $M$ of $\varphi$, in the case that it is invertible and the height of $\varphi$ equals 1. Let $\mathbf{t}$ denote the row vector all of whose entries equal $t$.

**Lemma 37** *Let $\varphi$ be a primitive morphism on the alphabet $A$ of cardinality $d$, with an aperiodic fixed point $\mathbf{a}$. If $\lambda = e^{2\pi i t}$ satisfies*

$$\lim_{\ell\to\infty} \lambda^{|\varphi^\ell(a)|} = 1 \tag{3}$$

*for each $a \in A$, then we can write $\mathbf{t} = \mathbf{t}_1 + \mathbf{t}_2$ where $\mathbf{t}_1 M^n \to 0$ and $\mathbf{t}_2 M^n \in \mathbb{Z}^d$ for all $n$ large. Furthermore if $M$ is invertible and $\psi$ is any eigenvalue for $M$ with $|\psi| \geq 1$ and eigenvector $\boldsymbol{\omega}$, the sum of whose entries is equal to 1, then $t = \frac{r}{s}$ where $r$ is an integer combination of the entries of $\boldsymbol{\omega}$ and $s$ divides $\det(M^k)$ for some $k$.*

*Proof.*

Let $A = \{a_1, \ldots, a_d\}$. Recalling that the entries of the $i$-th column of $M^n$ sum to $|\varphi^n(a_i)|$, Assumption (3) implies that

$$\lim_{n\to\infty} \mathbf{t}M^n \equiv \mathbf{0} \pmod 1.$$

We can therefore write

$$\mathbf{t}M^n = \mathbf{u}_n + \mathbf{v}_n$$

where $\mathbf{u}_n \in \mathbb{Z}^d$ and $\mathbf{v}_n \to \mathbf{0}$ as $n \to \infty$. For each $n$

$$\begin{aligned} \mathbf{u}_{n+1} + \mathbf{v}_{n+1} &= \mathbf{t}M^{n+1} = (\mathbf{u}_n + \mathbf{v}_n)\, M \\ &= \mathbf{u}_n M + \mathbf{v}_n M, \end{aligned}$$

so that

$$\mathbf{u}_{n+1} - \mathbf{u}_n M = \mathbf{v}_n M - \mathbf{v}_{n+1}.$$

As the right hand side of this last expression converges to the vector $\mathbf{0}$, so does the left. But $\mathbf{u}_{n+1} - \mathbf{u}_n M$ is an integer valued row vector. We conclude that there exists $n^*$ such that

$$\mathbf{u}_{n^*+m} - \mathbf{u}_{n^*} M^m = \mathbf{0} \text{ for all } m \geq 1.$$

We can find a vector $\mathbf{t}_2$ such that $\mathbf{t}_2 M^{n^*+d} = \mathbf{u}_{n^*} M^d$, and so

$$\mathbf{u}_n = \mathbf{t}_2 M^n \text{ for } n \geq n^* + d + 1.$$

Write $\mathbf{t}_1 := \mathbf{t} - \mathbf{t}_2$. Then

$$\mathbf{t}_1 M^n = \mathbf{t}M^n - \mathbf{t}_2 M^n = \mathbf{t}M^n - \mathbf{u}_n = \mathbf{v}_n$$

for $n \geq n^* + d + 1$. But $\mathbf{v}_n \to \mathbf{0}$ as $n \to \infty$, so $\mathbf{t}_1 M^n \to \mathbf{0}$ as $n \to \infty$. Now since $|\psi| \geq 1$, we conclude that $\mathbf{t_1}\boldsymbol{\omega} = 0$ (write $\mathbf{t_1} M^n \omega = \mathbf{t_1} \psi^n \boldsymbol{\omega}$ and use that $\mathbf{t_1} M^n \to 0$). We have

$$t = \mathbf{t}\boldsymbol{\omega} = (\mathbf{t_1} + \mathbf{t_2})\boldsymbol{\omega} = \mathbf{t_2}\boldsymbol{\omega}.$$

Finally, if $M$ is invertible, then $\mathbf{t}_2 = \mathbf{u}_{n^*+d+1} M^{-(n^*+d+1)}$. The result follows. $\square$

**Remark 38** There is an analogue of Lemma 37 in the case where the greatest common divisor of the lengths of all return words to some $a$ is larger than 1, i.e., where the limit (3) is not constantly equal to 1. Namely we apply the same proof as in the lemma, but starting with $\mathbf{t}(M - I)M^n$ instead of $\mathbf{t}M^n$. Also note that Ferenczi, Mauduit, and Nogueira [54] describe how to recover all dynamical eigenvalues from the adjacency matrix $M$.

**Example 39** Consider the morphism $1 \to 2$, $2 \to 211$ with fixed point $\mathbf{a} = 21122211211 \cdots$ Since 11 and 22 each appear in $\mathbf{a}$, then 1 is the gcd of the lengths of all return words to 1 and also to 2, so we can apply Lemma 37. The adjacency matrix has determinant $-2$ and eigenvalues $-1, 2$. Suppose that $\lambda = e^{2\pi i t}$ is a dynamical eigenvalue (for $(X_\varphi, T)$). Lemma 37 tells us that we can write $\mathbf{t} = \mathbf{t}_1 + \mathbf{t}_2$ where $\mathbf{t}_1 M^n \to 0$ and $\mathbf{t}_2 M^j$ belong to $\mathbb{Z}^d$ for some $j \in \mathbb{N}$. But as $M$ has no eigenvalues inside the unit circle, this means that $\mathbf{t}_1 = 0$, so that $\mathbf{t} = (\frac{a}{2^j}, \frac{a}{2^j})$ for some odd integer $a$. Hence the fixed point cannot be $q$-automatic, for $q > 2$. Now one verifies that if $n \geq 2$ and $j \geq 1$, then $\mathbf{t}M^n = (\frac{a_n}{2^j}, \frac{b_n}{2^j})$, where $a_n$ and $b_n$ are odd. But $\mathbf{t}M^n \to 0 \pmod{\mathbb{Z}^2}$. Therefore $j = 0$. Hence $\mathbf{a}$ does not have all (nor any, in fact) $2^j$-th roots of unity as dynamical eigenvalues, and therefore it is not 2-automatic.

**Example 40** Consider any morphism with adjacency matrix $\begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}$, and satisfying the conditions of Theorem 35. Here also, 1 is the gcd of the lengths of all return words to each of the two letters. We cannot apply Lemma 37 since the adjacency matrix is not invertible. But we have $|\varphi^n(a)| = 5^{n-1} \cdot 4$, $|\varphi^n(b)| = 5^{n-1} \cdot 8$. By Theorem 35, $e^{\frac{2\pi i}{5^n}}$ is an eigenvalue for each $n$. This suggests that any fixed point might be 5-automatic, and indeed [18, Theorem 1] gives this.

# 11 Conclusion. A strategy for proving that fixed points of non-uniform morphisms are not automatic

The previous sections addressed the question whether "general" sequences are automatic, but with an emphasis on sequences that are fixed points of non-uniform morphisms. For the latter, what precedes suggest a general strategy. Suppose that we are given the iterative fixed point $\mathbf{u} = (u_n)_{n \geq 0}$ of a non-uniform morphism $\varphi$, whose transition matrix has spectral radius $\rho$. The following steps can be followed.

First preliminary case: the morphism $\varphi$ is primitive.

- If the morphism $\varphi$ is primitive and if $\rho$ is not an integer, then $\mathbf{u}$ is not $q$-automatic for any $q \geq 2$. The case where $\rho$ is irrational is covered in Theorem 12 above. The case where $\rho$ is rational but not integer is addressed at the end of the first item below.

- Compute the dynamical eigenvalues of $X_{\mathbf{u}}$ and try to apply Theorem 34 in Section 10 above

General case: no assumption of primitivity for the morphism $\varphi$.

- If $\rho^k$ is never an integer for $k$ integer $\geq 1$, then the sequence $(u_n)_{n\geq 0}$ is not $q$-automatic for any $q \geq 2$ (Theorem 31 above). Note that in particular, this is the case if there exists some integer $t \geq 1$ such that $\rho^t$ is a rational number but not an integer.
- If $\rho^k = d$ for some integer $d \geq 2$, thus, using Theorem 31 again, $(u_n)_{n\geq 0}$ is either not $q$-automatic for any $q \geq 2$, or it is $d^\ell$-automatic for some $\ell \geq 1$ (hence $d$-automatic), or it is ultimately periodic (hence $d$-automatic). Thus, proving that the sequence $(u_n)_{n\geq 0}$ is not $q$-automatic for any $q \geq 2$ is the same as proving that it is not $d$-automatic.

Thus we see that, up to replacing the morphism $\varphi$ with some integer power $\varphi^k$ (note that this replaces $\rho$ with $\rho^k$), the case that is not "immediate" now is the case where $\rho$ is an integer. In this situation, we can (try to) use one of the properties previously described:

∗ exhibiting infinitely many distinct elements in the $d$-kernel of $(u_n)_{n\geq 0}$,

∗ proving that some block occurs in $(u_n)_{n\geq 0}$ with irrational frequency,

∗ proving that the complexity of $(u_n)_{n\geq 0}$ is not in $\mathcal{O}(n)$,

∗ finding gaps of "wrong" size in the sequence of integers $\{n : u_n = a\}$ for some value $a$,

∗ looking at the Dirichlet series associated with $(u_n)_{n\geq 0}$,

∗ studying the closed orbit of $(u_n)_{n\geq 0}$ under the shift, and so forth.

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