Whittaker coefficients of automorphic forms and applications to analytic Number Theory

by

Félicien Comtat

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School of Mathematical Sciences

Queen Mary University of London

United Kingdom

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Chapter 1 of this thesis is based on my article *Optimal sup norm bounds for newforms on* GL_2 *with maximally ramified central character*, which appeared in Forum Mathematicum [**Com21**].

Abstract

The formalism of automorphic representations makes the study of automorphic forms amenable to representation-theoretic methods. In particular the Whittaker model, when it exists, permits to extract interesting arithmetic and analytic information. In this thesis, we give two instances of this principle, in which we are concerned respectively with a) bounding the values taken by and b) the distribution of the Satake parameters of certain automorphic forms.

In the first part of this thesis, carried out in Chapter 1, we study the problem of bounding the sup norms of L^2-normalized cuspidal automorphic newforms ϕ on GL_2 in the level aspect. Prior to this work, strong upper bounds were only available if the central character χ of ϕ is not too highly ramified. We establish a uniform upper bound in the level aspect for general χ. If the level N is a square, our result reduces to

$$\|\phi\|_\infty \ll N^{\frac{1}{4}+\epsilon},$$

at least under the Ramanujan Conjecture. In particular, when χ has conductor N, this improves upon the previous best known bound $\|\phi\|_\infty \ll N^{\frac{1}{2}+\epsilon}$ in this setup (due to Saha) and matches a lower bound due to Templier, thus our result is essentially optimal in this case.

In the second and more substantial part, carried out in Chapter 2, we develop a Kuznetsov type formula for the group GSp_4. To this end, we follow a relative trace formula approach, and we focus on giving a final formula that is as explicit as possible. In particular, our formula is valid for arbitrary level, arbitrary central character, and includes the Hecke eigenvalues. We then use this Kuznetsov formula in Chapter 3 to show that, as the level tends to infinity, the Satake parameters of automorphic forms on GSp_4, suitably weighted, equidistribute with respect to the Sato-Tate measure.

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Nicolas Bouvier

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Contents

General Introduction

Historically, Fourier's concern in studying trigonometric series was analytical in nature. One of his main discoveries is that an arbitrary piecewise continuous, periodic function on the reals can be expressed as a *Fourier series*. Since then, his results were vastly extended and generalised in numerous frameworks. Other areas of mathematics, such as algebra, topology and representation theory, were brought in, providing reinterpretations of Fourier theory from different perspectives and bearing with them new interests. While recovering a function via its "Fourier coefficients" is still an important motivation, Fourier theory has been generalised to new situations where this is not always possible.

One such generalisation is the notion of *Whittaker coefficients* of automorphic forms. Thanks to the close relationship between automorphic forms and the theory of representations, the Whittaker coefficients can be interpreted from a representation theoretic point of view, which in turn has consequences for number theory.

Let us give an overview of this principle. Let ϕ be an automorphic form for a connected reductive algebraic group G over a global field F. The right translates of ϕ by $G(\mathbb{A}_F)$ (the group of adelic points of G) generate a certain representation π. It is sufficient to consider the case when this representation π is irreducible. Now π may (or may not) have a (global) *Whittaker model*. On the other hand, the Whittaker coefficient $\mathscr{W}(\phi)$ of ϕ is a function on $G(\mathbb{A}_F)$ which is given by definition by a certain period integral of ϕ generalising the usual definition of Fourier coefficients. From the definition, it satisfies some invariance properties that guarantee that the map $\phi \mapsto \mathscr{W}(\phi)$, if not identically zero, takes value in a Whittaker model of π. We say that π is *globally generic* if $\{\mathscr{W}(\phi) : \phi \in \pi\}$ is non-zero. We henceforth assume this is the case.

It is known by the Flath tensor product theorem that π is isomorphic to a restricted tensor product over all places v of F of local representations π_v of $G(F_v)$: $\pi \simeq \bigotimes_v \pi_v$. The fact that π has a global Whittaker model immediately implies that each representation π_v has a *local Whittaker model*. The key point is that Whittaker models are (usually) well-behaved. Namely, assume that each local representation π_v has a *unique* Whittaker model. Then this implies that the global Whittaker model is itself unique, and moreover if $\phi \in \pi$ corresponds to a pure tensor $\bigotimes_v \phi_v$, then we have $\mathscr{W}(\phi)(g) = \prod_v \mathscr{W}_v(\phi_v)(g_v)$ for all $g \in G(\mathbb{A}_F)$, where $\mathscr{W}_v$ is an isomorphism between π_v and its local Whittaker model. This factorization property is important for number theory. Indeed, it provides an instance of a *local-global principle* as well as a connection with L-functions, as we shall see below in more details.

Now let us sketch how understanding the Whittaker coefficients can shed light on different aspects of automorphic forms. Firstly, when a *Whittaker expansion* is available, knowledge of the Whittaker coefficients of ϕ enables one to access information on ϕ itself. While the Whittaker coefficients of ϕ are a rather mysterious global object, one can study the local Whittaker coefficients, which “only” depend on the corresponding local representation theory. Using the factorisation property, one may be able to derive some useful information on the global Whittaker coefficients. This is an instance of a local to global principle. An important question which can be tackled this way is that of bounding the sup norm of ϕ. This question is addressed in the case of automorphic newforms for $\mathrm{GL}_2(\mathbb{Q})$ in Chapter 1 of this thesis. Note that in general the non-vanishing of Whittaker coefficients of ϕ is not automatic, and even if this is the case, it also does not guarantee that a Whittaker expansion is available for ϕ. For instance a Whittaker expansion in the traditional sense is not available for automorphic forms on GSp_4 even if they are generic.

Secondly, the factorization property provides some insight that the Whittaker coefficients "should" be related to L-functions. This is well known in the case of GL_2, where the local Whittaker coefficients of a Hecke newform ϕ coincide at unramified place with its Hecke eigenvalues, which are themselves the Dirichlet coefficients of the L-function attached to ϕ. In the situation of GSp_4, the Whittaker coefficients no longer coincide with the Hecke eigenvalues though they are closely related. If the Hecke eigenvalues can't be "directly" accessed through the Whittaker coefficients, they both are dictated by the *Satake parameters*. It is thus interesting to use the Whittaker coefficients to study some questions concerning the Satake parameters themselves. Two important questions concerning the Satake parameters of automorphic forms are their size and their distribution. In Chapter 3, we investigate the latter question for $\mathrm{GSp}_4(\mathbb{Q})$.

Instead of studying the Whittaker coefficients of a single automorphic form, one can rather study the Whittaker coefficients of automorphic forms in a given collection. When this collection consists of automorphic forms occurring in the spectral decomposition of a suitable space, a classical tool is the theory of relative trace formulae. One may consider the spectral expansion of a certain automorphic kernel, and take the Whittaker coefficients thereof. Using the Bruhat decomposition, one can equate this spectral term to a "geometric term". In Chapter 2, we follow this approach to develop a Kuznetsov formula for the group $\mathrm{GSp}_4(\mathbb{Q})$, on which our study of Satake parameters in Chapter 3 is based. Because the Kuznetsov formula is a central tool in analytic number theory, we believe Chapter 2 has its interest of its own, and we expect it can be used to tackle other applications in the future.

In addition to being a *tool* for studying the Whittaker coefficients of automorphic forms, relative trace formulae also provide a new *motivation* for studying them. Indeed, the geometric side involves some (generalised) Kloosterman sums, whose definition comes from the Bruhat decomposition, and this connection between Whittaker coefficients and Kloosterman sums furnishes new ways of analysing the latter (this line of investigation is not tackled in this thesis). The GL_2 Kloosterman sum are classical and naturally arise in other problems of number theory, and more generally for GL_n, the Kloosterman sums attached to the Weyl element $\left[\begin{smallmatrix} & 1 \\ I_{n-1} & \end{smallmatrix}\right]$ are hyper-Kloosterman sums, but it seems that in more general situations the Kloosterman sums do not occur "naturally" outside of the setting of relative trace formulae.

CHAPTER 1

Sup norm bounds for newforms on GL_2

1. Introduction

Let ϕ be a cuspidal automorphic form on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with conductor $N = \prod_p p^{n_p}$ and central character χ. Assume in addition ϕ is a newform, in the sense that there exists either a Maaß or holomorphic cuspidal newform f of weight k for $\Gamma_1(N)$ such that for all $g \in \mathrm{SL}_2(\mathbb{R})$ we have $\phi(g) = j(g,i)^{-k} f(g \cdot i)$, where as usual $j(g,z) = cz + d$ for $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and $z \in \mathbb{H}$. In particular, ϕ is bounded and has a finite L^2 norm, hence one may be interested in asking how its L^∞ and its L^2 norm relate. In the level aspect, one traditionally asks for bounds for $\|\phi\|_\infty = \sup_g |\phi(g)| = \sup_z |y^{\frac{k}{2}} f(z)|$ depending on N as $\|\phi\|_2$ is fixed. Subsequent investigations have shown that it is relevant for this problem to also take into account the conductor $C = \prod_p p^{c_p}$ of χ. Assuming that ϕ is L^2-normalized, the "trivial bound" is

$$1 \ll \|\phi\|_\infty \ll N^{\frac{1}{2}+\epsilon} \tag{1.1}$$

for any $\epsilon > 0$. Here and below, the implied constant may depend on ϵ and on the archimedean parameters of ϕ. The upper bound in (1.1) does not appear to have been written down previously for general N and C, but it can be deduced from the main result of [**Sah17**] for instance.

For squarefree N, the first non-trivial upper bound is due to Blomer and Holowinsky [**BH10**], and has been subject to several improvements by Harcos and Templier (and some unpublished work of Helfgott and Ricotta) culminating with the result of [**HT13**] which achieves the upper bound $N^{\frac{1}{3}+\epsilon}$. For non-squarefree N, the best result to date is due to Saha [**Sah17**], but it significantly improves on the trivial bound only when χ is not highly ramified (here and elsewhere we say χ is highly ramified if $c_p > \lceil \frac{n_p}{2} \rceil$ for some prime p). Indeed, if χ is not highly ramified and N is a perfect square, then Saha's result [**Sah17**] gives an upper bound of $N^{\frac{1}{4}+\epsilon}$. Recent work of Hu and Saha (see [**HS20**], especially the last paragraph of their introduction) suggests that this bound may be further improved in the compact case. On the other hand, if $N = C$ and if N is a perfect square, then Saha's result [**Sah17**] reduces to the trivial bound (1.1).

Templier was the first to provide evidence that the actual size of $\|\phi\|_\infty$ may depend on how ramified χ is. Namely, he proved in [**Tem14**] that whenever $N = C$ we have

$$\|\phi\|_\infty \gg N^{-\epsilon} \prod_{p^{n_p}\|N} p^{\frac{1}{2}\lfloor \frac{n_p}{2} \rfloor}. \tag{1.2}$$

In particular, if N is a square, then

$$\|\phi\|_\infty \gg N^{\frac{1}{4}-\epsilon}.$$

We shall prove the following comparable upper bound, which improves on [**Sah17**] when χ is highly ramified.

Theorem 1.1.1. *Let π be an unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ with central character ω_π. Let $N = \prod_p p^{n_p}$ be the conductor of π. Let $\phi \in \pi$*

be an L^2-normalized newform. Then

$$\|\phi\|_\infty \ll_{\epsilon,\pi_\infty} N^{\delta+\epsilon} \prod_{p|N} p^{\frac{1}{2}\lceil \frac{n_p}{2} \rceil},$$

where δ is any bound towards the Ramanujan Conjecture for π.

Theorem 1.1.1 provides for the first time non-trivial upper bounds for general N that do not get worse when the conductor C varies. As a point of comparison, the main result of [**Sah17**] had an additional factor of $\prod_p p^{\max\{0,c_p-\lceil \frac{n_p}{2} \rceil\}}$, which is larger than one precisely when χ is highly ramified. Furthermore, for $C = N$, in view of the lower bound (1.2) and assuming the Ramanujan Conjecture, our result is essentially optimal when N is a square. Note that the Ramanujan Conjecture is known by work of Deligne and Serre for ϕ arising from a holomorphic cusp form, and otherwise $\delta = \frac{7}{64}$ is admissible [**Kim03**].

Remark 1.1.1. *In* [**Sah17**]*, the appeal to a bound towards the Ramanujan Conjecture is avoided by using Hölder inequality to estimate separately L^2 averages of the Whittaker newforms at primes at which the central character is ramified and moments of the coefficients λ_π of the L-function attached to π. However, in our situation, we want to exploit the fact that the Whittaker coefficients are supported on arithmetic progressions of modulus L, say, as explained later. A similar technique as in* [**Sah17**] *would thus lead us to estimate moments of λ_π on these arithmetic progressions. One might expect that these moments are approximately L times smaller than the full moments, but such a result does not seem to be available. Hence, if we were to bound them by positivity by the full moments, we would expect an over-estimate of same order as L. Since estimates are known by Rankin-Selberg theory up* *to the eighth moments, and, as we shall see, $L \leq \prod_{c_p > \frac{n_p}{2}} p^{\lfloor \frac{n_p}{2} \rfloor}$, one should be able to replace N^δ in Theorem 1.1.1 with $\prod_{c_p > \frac{n_p}{2}} p^{\frac{1}{8}\lfloor \frac{n_p}{2} \rfloor}$, similarly as in Theorem 1.1 of* [**HNS19**]*. As pointed out by Andy Booker, with more work one can also interpolate between N^δ and the eighth moments estimate, which leads to a better bound. However, for the sake of brevity, we do not carry out these arguments.*

The lower bound (1.2) has been generalized by Saha in [**Sah16**] and subsequently by Assing in [**Ass19b**]. When χ is not maximally ramified, there is still a gap between the best known lower bound and the upper bound from Theorem 1.1.1. Finally, let us mention that the hybrid bounds over $\mathbb{Q}$ in [**Sah17**], which combines the Whittaker expansion with some amplification, still beats our result when χ is not highly ramified. For hybrid bounds over general number fields, we refer to the work of Assing [**Ass17**, **Ass19a**].

The proof proceeds by using Whittaker expansion to reduce the problem of bounding ϕ to that of understanding the local newforms attached to ϕ. By making use of the invariances of ϕ, we can restrict ourselves to evaluate these local newforms in the Whittaker model on some convenient cosets. The values of these local newforms have been computed [**Ass19b**, **Ass19a**] by using a "basic identity" derived from the Jacquet-Langlands local functional equations which was first expressed in this form in [**Sah16**]. In the non maximally ramified case, local bounds are slightly weaker than needed to obtain our result, and we take advantage of strong L^2-bounds due to Saha [**Sah17**] instead.

Actually, we are using the Whittaker expansion of a certain translate of ϕ, the "balanced newform". The main feature is that it is supported on arithmetic progressions, which enables us to get some savings. Though we are working adelically, this fact can also be seen classically by computing the Fourier expansion of the corresponding cusp form at cusps of large width. The situation is somewhat analogous to [**HNS19**], where the authors also get Whittaker expansions supported on arithmetic progressions.

Let us explain this analogy in the maximally ramified case – in which we get optimal upper bounds. As we shall see, in this case each local representation with ramified central character is of the form $\chi_1 \boxplus \chi_2$, where χ_1 has exponent of conductor n_p and χ_2 is unramified. Then the local balanced newform for π is a twist of the local balanced newform for $\chi_1\chi_2^{-1} \boxplus 1$. For representations of this type, the local balanced newform coincides with the p-adic microlocal lift as defined in [**Nel18**]. Now as explained in [**HNS19**], the microlocal lift is the split analogue of the minimal vectors used there. Therefore the fact that we get optimal sup norm bounds in this case is the direct analogue of Theorem 1.1 of [**HNS19**] which gives an optimal sup norm bound for automorphic forms of minimal type.

It is worth noticing that [**HNS19**], [**Sah20**] as well as the present work provide instances of the seemingly general principle according to which when considering very localized vectors, one is able to establish very good and sometimes optimal upper bounds. This is even the case when a Whittaker expansion is not available, as in [**Sah20**].

The analysis of local newforms is given in Section 2. The proof of Theorem 1.1.1 is given in Section 3.

2. Local bounds

In this section, F will denote a non-archimedean local field of characteristic zero with residue field $\mathbb{F}_q$. Let $\mathfrak{o}$ denote the ring of integers of F and $\mathfrak{p}$ its maximal ideal with uniformizer $t_\mathfrak{p}$. The discrete valuation associated to F will be denoted by $v_\mathfrak{p}$. We define $U(0) = \mathfrak{o}^\times$, and for $k \geq 1$, $U(k) = 1 + \mathfrak{p}^k$. We fix an additive unitary character ψ of F with conductor $\mathfrak{o}$. In the sequel, the Whittaker models given will be those with respect to ψ.

2.1. Generalities.

2.1.1. *Double coset decomposition.* Let $G = \mathrm{GL}_2(F)$, $K = \mathrm{GL}_2(\mathfrak{o})$. For $x \in F$ and $y \in F^\times$, consider the following elements

$$w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, a(y) = \begin{bmatrix} y & 0 \\ 0 & 1 \end{bmatrix}, n(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, z(y) = \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix}.$$

Then define the following subgroups

$$N = n(F),\ A = a(F^\times),\ Z = z(F^\times),$$

and, for $\mathfrak{a}$ an ideal of $\mathfrak{o}$,

$$K^{(1)}(\mathfrak{a}) = K \cap \begin{bmatrix} 1+\mathfrak{a} & \mathfrak{o} \\ \mathfrak{a} & \mathfrak{o} \end{bmatrix},\ K^{(2)}(\mathfrak{a}) = K \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{a} & 1+\mathfrak{a} \end{bmatrix}. \tag{1.3}$$

Note that for $\mathfrak{a} = \mathfrak{p}^n$, with n a non-negative integer, we have

$$K^{(2)}(\mathfrak{p}^n) = \begin{bmatrix} & 1 \\ t_{\mathfrak{p}}^n & \end{bmatrix} K^{(1)}(\mathfrak{p}^n) \begin{bmatrix} & 1 \\ t_{\mathfrak{p}}^n & \end{bmatrix}^{-1}. \tag{1.4}$$

From [**Sah16**, Lemma 2.13], for any integer $n \geq 0$ we have the following double coset decomposition

$$G = \coprod_{m \in \mathbb{Z}} \coprod_{\ell=0}^{n} \coprod_{\nu \in \mathfrak{o}^\times/(1+\mathfrak{p}^{\ell_n})} ZNg_{m,\ell,\nu}K^{(1)}(\mathfrak{p}^n), \tag{1.5}$$

where $\ell_n = \min\{\ell, n-\ell\}$, and

$$\begin{aligned} g_{m,\ell,\nu} &= a(t_{\mathfrak{p}}^m)wn(t_{\mathfrak{p}}^{-\ell}\nu) \\ &= \begin{bmatrix} 0 & t_{\mathfrak{p}}^m \\ -1 & -t_{\mathfrak{p}}^{-\ell}\nu \end{bmatrix}. \end{aligned}$$

Definition 1.2.1. Assume $n \geq 0$ is a fixed integer. Then for any $g \in G$ we define

$$(m(g), \ell(g), \nu(g)) \in \mathbb{Z} \times \{0, \cdots, n\} \times \mathfrak{o}^\times/(1 + \mathfrak{p}^{\ell(g)_n})$$

as the unique triple such that

$$g \in ZNg_{m(g),\ell(g),\nu(g)}K^{(1)}(\mathfrak{p}^n).$$

Remark 1.2.1. *Any $g \in \mathrm{GL}_2(F)$ belongs to some $ZNa(y)\kappa$ where $\kappa = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathfrak{o})$. Then by Remark 2.1 of* [**Sah17**]*, we have $\ell(g) = \min\{v_{\mathfrak{p}}(c), n\}$ and $m(g) = v_{\mathfrak{p}}(y) - 2\ell(g)$. In particular, if g is already an element of $\mathrm{GL}_2(\mathfrak{o})$, then g is in a coset of the form $g_{-2j,j,*}$.*

Now we determine the double cosets corresponding to certain elements of interest for the global application.

LEMMA 1.2.1. *Consider two integers $0 \le e \le n$. Let $g \in \mathrm{GL}_2(\mathfrak{o})a(t_{\mathfrak{p}}^{e})$. Then there exist a non-negative integer $\ell \le n$ and $\nu \in \mathfrak{o}^\times$ such that one of the following holds*

(1) either $\ell \le e$ and $g \in ZNg_{-e,\ell,\nu}K^{(1)}(\mathfrak{p}^n)$,

(2) or $e < \ell \le n$ and $g \in ZNg_{-2\ell+e,\ell,\nu}K^{(1)}(\mathfrak{p}^n)$,

where the subgroup $K^{(1)}(\mathfrak{p}^n)$ is defined in (1.3).

PROOF. We know by (1.5) that $g \in ZNg_{m,\ell,\nu}k_1$ for some $k_1 \in K^{(1)}(\mathfrak{p}^n)$ hence

$$gk_1^{-1}a(t_{\mathfrak{p}}^{-e}) \in ZNg_{m,\ell,\nu}a(t_{\mathfrak{p}}^{-e}).$$

Since $g \in Ka(t_{\mathfrak{p}}^{e})$, it follows that $gk_1^{-1}a(t_{\mathfrak{p}}^{-e}) \in K$. By Remark 1.2.1, it is then in the coset of some $g_{-2j,j,*}$ with $0 \le j \le n$. On the other hand,

$$\begin{aligned} g_{m,\ell,\nu}a(t_{\mathfrak{p}}^{-e}) &= a(t_{\mathfrak{p}}^{m})wn(t_{\mathfrak{p}}^{-\ell}\nu)a(t_{\mathfrak{p}}^{-e}) \\ &= a(t_{\mathfrak{p}}^{m})wa(t_{\mathfrak{p}}^{-e})n(t_{\mathfrak{p}}^{e-\ell}\nu) \\ &= t_{\mathfrak{p}}^{-e}a(t_{\mathfrak{p}}^{m+e})wn(t_{\mathfrak{p}}^{e-\ell}\nu). \end{aligned}$$

If $\ell \le e$ then

$$wn(t_{\mathfrak{p}}^{e-\ell}\nu) = \begin{bmatrix} & 1 \\ -1 & -t_{\mathfrak{p}}^{e-\ell}\nu \end{bmatrix} \in \mathrm{GL}_2(\mathfrak{o})$$

so by Remark 1.2.1 $a(t_{\mathfrak{p}}^{m+e})wn(t_{\mathfrak{p}}^{e-\ell}\nu)$ is in the coset of $g_{m+e,0,*}$. So in this case, $g_{-2j,j,*} = g_{m+e,0,*}$ thus $m = -e$ and we find that

$$g \in ZNg_{-e,\ell,\nu}K^{(1)}(\mathfrak{p}^n).$$

Otherwise $a(t_{\mathfrak{p}}^{m+e})wn(t_{\mathfrak{p}}^{e-\ell}\nu) = g_{m+e,\ell-e,\nu}$, therefore $g_{-2j,j,*} = g_{m+e,\ell-e,\nu}$ and we get $m + e = -2(\ell - e)$, so

$$g \in ZNg_{-2\ell+e,\ell,\nu}K^{(1)}(\mathfrak{p}^n).$$

□

2.1.2. *Characters and representations.* For χ a character of $F^\times$, we denote by $a(\chi)$ the exponent of the conductor of χ, that is the least non-negative integer n such that χ is trivial on $U(n)$. For π an irreducible admissible representation of G, we also denote by $a(\pi)$ the exponent of the conductor of π, that is the least non-negative integer n such that π has a $K^{(1)}(\mathfrak{p}^n)$-fixed vector. The central character of π will be denoted by ω_π.

2.1.3. *The local Whittaker newform.* Fix π a generic irreducible admissible unitarizable representation of G. From now on, we fix $n = a(\pi)$, and we shall assume that π is realized on its Whittaker model.

Definition 1.2.2. The normalized newform W_π attached to π is the unique $K^{(1)}(\mathfrak{p}^n)$-fixed vector such that $W_\pi(1) = 1$.

The normalized conjugate-newform W_π^* attached to π is the unique $K^{(2)}(\mathfrak{p}^n)$-fixed vector such that $W_\pi^*(1) = 1$.

REMARK 1.2.2. *By (1.4), the function* $\mathrm{GL}_2(F) \to \mathbb{C}$:

$$g \mapsto W_\pi \left(g \begin{bmatrix} & 1 \\ t_{\mathfrak{p}}^n & \end{bmatrix} \right)$$

is $K^{(2)}(\mathfrak{p}^n)$*-invariant. Thus there exists a complex number* α_π *such that*

$$W_\pi \left(\cdot \begin{bmatrix} & 1 \\ t_{\mathfrak{p}}^n & \end{bmatrix} \right) = \alpha_\pi W_\pi^*.$$

In addition, we have $W_\pi^*(g) = \omega_\pi(\det(g)) W_{\tilde{\pi}}(g)$, *where* $\tilde{\pi}$ *is the contragradient representation to* π. *Altogether, we get that*

$$W_\pi \left(\cdot \begin{bmatrix} & 1 \\ t_{\mathfrak{p}}^n & \end{bmatrix} \right) = \alpha_\pi \omega_\pi(\det(g)) W_{\tilde{\pi}}(g).$$

One can even show that $|\alpha_\pi| = 1$ *(see* [**Sah16**, Lemma 2.17], *or* [**Sah16**, Proposition 2.28] *for an exact formula in terms of* ϵ*-factors). Also note the following identity*

$$n(t_{\mathfrak{p}}^{\ell+m}\nu^{-1}) z(t_{\mathfrak{p}}^{\ell-n}\nu^{-1}) g_{m,\ell,\nu} \begin{bmatrix} & 1 \\ t_{\mathfrak{p}}^n & \end{bmatrix} = g_{m+2\ell-n,n-\ell,-\nu} \begin{bmatrix} 1 & \\ & -\nu^{-2} \end{bmatrix}, \tag{1.6}$$

which, combined with the above, enables one to restrict attention to those cosets satisfying $\ell \leq \frac{n}{2}$, *at the price of changing* π *to* $\tilde{\pi}$.

Assing has computed the local Whittaker newforms in great generality, and estimated them using the p-adic stationary phase method [**Ass19b**, **Ass19a**]. Let us briefly explain the basic ideas of his method. For any fixed $m \in \mathbb{Z}$ and $0 \leq \ell \leq n$ the function on $\mathfrak{o}^\times$ given by $\nu \mapsto W_\pi(g_{m,\ell,\nu})$ only depends on $\nu \mod (1+\mathfrak{p}^\ell)$. Thus, by Fourier inversion, there exist complex numbers $c_{m,\ell}(\mu)$ such that

$$W_\pi(g_{m,\ell,\nu}) = \sum_{\mu \in \tilde{X}(\ell)} c_{m,\ell}(\mu)\mu(\nu),$$

where $\tilde{X}(\ell)$ is the set of characters μ satisfying $\mu(t_\mathfrak{p}) = 1$ and $a(\mu) \leq \ell$.

Then, one may reformulate the Jacquet-Langlands local functional equation as an equality of power series in the variable q^s whose coefficients involve on one side the Fourier coefficients $c_{m,\ell}(\mu)$ one is interested in, and on the other side Gauss sums and values of the local newform at some diagonal matrices, both of which are known [**Sch02**]. This is the content of [**Sah16**, Proposition 2.23]. By identifying the coefficients of the power series appearing in both side, one is then able to compute inductively the coefficients $c_{m,\ell}(\mu)$, and, from there, the values of the local newform on each double coset.

This can be done for each local representation π, however Lemma 1.2.2 below (same as [**Sah16**, Lemma 2.36]) will enable us to restrict ourselves to principal series representations. By Remark 1.2.2, we can further restrict ourselves to the situation $\ell \leq \frac{n}{2}$. Finally, as we mentioned earlier, in our global application we shall use Saha's strong L^2-bound [**Sah17**], so what we are really interested in this section is only the support of the local newforms. Recall that we have fixed $n = a(\pi)$.

LEMMA 1.2.2. *Assume $a(\omega_\pi) > \frac{a(\pi)}{2}$. Then $\pi = \chi_1 \boxplus \chi_2$, where χ_1 and χ_2 are unitary characters with respective exponents of conductors $a_1 = a(\omega_\pi)$ and $a_2 = n - a(\omega_\pi)$.*

In the rest of this section, we shall only consider the case $a(\omega_\pi) > \frac{a(\pi)}{2}$, as the main point of our global application is to take advantage of primes at which the central character is highly ramified. Thus for our purpose, we only have to consider $\pi = \chi_1 \boxplus \chi_2$ with $a_2 < \frac{n}{2} < a_1$, where from now on we denote $a_1 = a(\chi_1)$ and $a_2 = a(\chi_2)$. We first state the case of maximally ramified principal series.

LEMMA 1.2.3. *Let π be a generic irreducible admissible unitarizable representation of G with exponent of conductor $a(\pi) = n > 1$. Assume $a(\omega_\pi) = a(\pi)$. Then there exists $\nu_1 \in \mathfrak{o}^\times$ such that for all $m \in \mathbb{Z}$ and for $0 \leq \ell \leq \frac{n}{2}$, we have*

$$|W_\pi(g_{m,0,\nu})| = \mathbb{1}_{m \geq -n} q^{-\frac{m+n}{2}},$$

$$|W_\pi(g_{-n-\ell,\ell,\nu})| = \begin{cases} q^{\frac{\ell}{2}} & \text{if } \nu \in \nu_1 + \mathfrak{p}^\ell, \\ 0 & \text{if } \nu \notin \nu_1 + \mathfrak{p}^\ell, \end{cases}$$

and if $0 < \ell < n$ and $m + \ell \neq -n$ then $W_\pi(g_{m,\ell,\nu}) = 0$.

PROOF. This follows from Lemma 3.4 and proof of Lemma 5.8 in [**Ass19b**]. $\square$

In particular, one sees that in this case the local Whittaker newform is essentially supported on an arithmetic progressions. The case $1 \leq a_2 < \frac{n}{2} < a_1$ is a bit more complicated, but one may obtain a result similar in flavour. Work of Assing [**Ass19a**] gives precise bounds for the local newform, however these local bounds are slightly weaker than what we need for our global application. Consequently, we only give here statements regarding the support of the local newform, and we shall rely on strong bounds for the L^2 mass [**Sah17**].

LEMMA 1.2.4. *Let π be a generic irreducible admissible unitarizable representation of G with exponent of conductor $n > 1$. Assume $\frac{n}{2} < a(\omega_\pi) < n$. Set $a_1 = a(\omega_\pi)$ and $a_2 = n - a_1$. Assume moreover $F = \mathbb{Q}_p$. There exists $\nu_1 \in \mathfrak{o}^\times$ such that if $m \in \mathbb{Z}$ and $0 \leq \ell \leq \frac{n}{2}$, then have $W_\pi(g_{m,\ell,\nu}) = 0$ unless one of the following holds:*

(1) $\ell < a_2$ and $m = -n$,
(2) $\ell = a_2$ and $m \geq -n$,
(3) $\ell > a_2$, $m = -a_1 - \ell$ and $\nu \in {\nu_1}^{-1} + {t_\mathfrak{p}}^{\ell - a_2}\mathfrak{o}^\times$

PROOF. This follows almost directly from inspection of the cases in Lemma 3.4.12 in [**Ass19a**]. Since we are taking $F = \mathbb{Q}_p$, the quantity κ_F defined in [**Ass19a**] equals one, so the only bothersome case is $a_2 < \ell \leq \frac{a_1+a_2}{2}$ when $a_2 = 1$. By [**Ass19a**, Lemma 3.3.9], for $a_2 < \ell < a_1$ we must have $m = -a_1 - \ell$, so it only remains to see that the congruence condition also holds. If $\ell \leq \frac{a_1}{2}$, this follows from Case I of the proof of Lemma [**Ass19a**, Lemma 3.4.12]. The only remaining case is thus $\ell = \frac{1+a_1}{2}$, which only occurs for a_1 odd, hence $a_1 \geq 3$, so $a_1 - a_2 \geq 2\kappa_F$. As seen from Case VI.2 of the proof, this last condition is enough to get the congruence condition. $\square$

2.2. Archimedean case. The local representation at the infinite place is a generic irreducible admissible unitary representation π of $\mathrm{GL}_2(\mathbb{R})$. Let ψ be the additive character of $\mathbb{R}$ given by $\psi(x) = e^{2i\pi x}$. The lowest weight vector in the Whittaker model with respect to ψ is given by

$$W_\pi(n(x)a(y)) = e^{2i\pi x}\kappa(y), \tag{1.7}$$

where κ is determined by the form of the representation π. We shall use that for $y \in \mathbb{R}^\times$

$$\kappa(y) \ll |y|^{-\epsilon} e^{(-2\pi+\epsilon)|y|}. \tag{1.8}$$

uniformly in y. To see this, let us examine the possibilities for π.

2.2.1. *Principal series representations.* If $\pi = \chi_1 \boxplus \chi_2$, where $\chi_i = \mathrm{sgn}^{m_i}|.|^{s_i}$ with $0 \le m_2 \le m_1 \le 1$ integers and $s_1 + s_2 \in i\mathbb{R}$ and $s_1 - s_2 \in i\mathbb{R} \cup (-1,1)$ then the lowest weight vector is given by

$$\kappa(y) = \begin{cases} \mathrm{sgn}(y)^{m_1} |y|^{\frac{s_1+s_2}{2}} |y|^{\frac{1}{2}} K_{\frac{s_1-s_2}{2}}(2\pi|y|) & \text{if} \quad m_1 = m_2 \\ |y|^{\frac{s_1+s_2}{2}} |y| \left(K_{\frac{s_1-s_2-1}{2}}(2\pi|y|) + \mathrm{sgn}(y) K_{\frac{s_1-s_2+1}{2}}(2\pi|y|) \right) & \text{if} \quad m_1 \neq m_2, \end{cases}$$

where K_ν is the K-Bessel function of index ν. By [**HM06**, Proposition 7.2], we have the following estimate.

LEMMA 1.2.5. *Let $\sigma > 0$. For $\Re(\nu) \in (-\sigma, \sigma)$ we have*

$$K_\nu(u) \ll_{\nu,\epsilon} \begin{cases} u^{-\sigma-\epsilon} & \text{if} \quad 0 < u \le 1 + \frac{\pi}{2}|\Im(\nu)|, \\ u^{-\frac{1}{2}} e^{-u} & \text{if} \quad u > 1 + \frac{\pi}{2}|\Im(\nu)|. \end{cases}$$

In particular, taking $\sigma = \frac{1}{2}$ if $m_1 = m_2$ and $\sigma = 1$ otherwise, (1.8) follows in this case.

2.2.2. *Discrete series representations.* If π is the unique irreducible subrepresentation of $\chi_1 \boxplus \chi_2$, where $\chi_i = \mathrm{sgn}^{m_i}|.|^{s_i}$ with $0 \le m_2 \le m_1 \le 1$ integers and $s_1 + s_2 \in i\mathbb{R}$ and $s_1 - s_2 \in \mathbb{Z}_{>0}$, $s_1 - s_2 \equiv m_1 - m_2 + 1 \mod 2$, then the lowest weight vector is given by

$$\kappa(y) = |y|^{\frac{s_1+s_2}{2}} y^{\frac{s_1-s_2+1}{2}} (1 + \operatorname{sgn}(y)) e^{-2\pi y},$$

and we see that it satisfies again the estimate (1.8).

3. Global computations

3.1. Notations. Let $\mathbb{A}_{\mathbb{Q}}$ denote the ring of adèles of $\mathbb{Q}$ and let ψ be the unique additive character of $\mathbb{A}_{\mathbb{Q}}$ that is unramified at each finite place and equals $x \mapsto e^{2i\pi x}$ at $\mathbb{R}$. For any local object defined in Section 2, we use the subscript $_p$ to denote this object defined over $\mathbb{Q}_p$. We also fix in all the sequel

$$\Gamma_\infty = \mathrm{SO}_2(\mathbb{R}). \tag{1.9}$$

Let $\pi = \otimes_{p \leq \infty} \pi_p$ be a unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ with central character ω_π. Let $N = \prod_p p^{n_p}$ be the conductor of π and let $C = \prod_p p^{c_p}$ be the conductor of ω_π. In particular $C \mid N$. Let us introduce some notation to denote respectively the set of primes for which Lemma 1.2.2 do or do not apply, namely

$$\mathscr{H} = \left\{ p \mid N : c_p > \frac{n_p}{2} \right\} \text{ and } \mathscr{L} = \left\{ p \mid N : c_p \leq \frac{n_p}{2} \right\}. \tag{1.10}$$

We also denote by S_N the set of prime numbers dividing N, so that

$$S_N = \mathscr{H} \cup \mathscr{L}.$$

Then according to Lemma 1.2.2, π_p is an irreducible principal series representation for each prime $p \in \mathscr{H}$, and we have corresponding local exponents of conductors $a_1(p) = c_p$ and $a_2(p) = n_p - c_p$. Finally, for any set of primes $\mathscr{P}$, define $\Psi(\mathscr{P})$ to be the set of positive integers having all their prime divisors among $\mathscr{P}$. We shall use the following obvious result.

LEMMA 1.3.1. *Let $\mathscr{P}$ be a finite set of primes. Then for all $0 < \alpha \leq \frac{1}{\log(2)}$ we have*

$$\sum_{s\in\Psi(\mathscr{P})} s^{-\alpha} = \prod_{p\in\mathscr{P}} \frac{1}{1-p^{-\alpha}} \leq \left(\frac{e}{\alpha(e-1)}\right)^{\#\mathscr{P}}.$$

PROOF. Since 2 is the smallest prime we have $\frac{1}{1-2^{-\alpha}} \leq \frac{1}{1-p^{-\alpha}}$ and since the function $\mathbb{R}_{>0} \to \mathbb{R}, \alpha \mapsto \frac{\alpha}{1-2^{-\alpha}}$ is increasing, for α in the said range we have $\frac{1}{1-2^{-\alpha}} \leq \frac{e}{\alpha(e-1)}$. $\square$

3.2. The Whittaker expansion. Let $\phi \in \pi$ be an L^2-normalized newform. Define the global Whittaker newform on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ by

$$W_\phi(g) = \int_{\mathbb{Q}\backslash\mathbb{A}_\mathbb{Q}} \phi(n(x)g)\psi(-x)\,dx.$$

It factors as

$$W_\phi(g) = c_\phi \prod_{p\leq\infty} W_p(g_p),$$

where W_p are as defined in the first two sections, and c_ϕ is a constant that satisfies

$$2\xi(2)c_\phi^2 \| \prod_{p\leq\infty} W_p\|_{reg}^2 = 1,$$

with

$$\| \prod_{p\leq\infty} W_p\|_{reg} = L(\pi, \mathrm{Ad}, 1) \prod_{p\leq\infty} \frac{\zeta_p(2)\|W_p\|_2}{\zeta_p(1)L_p(\pi, \mathrm{Ad}, 1)},$$

see [**MV10**, Lemma 2.2.3]. In turn, we have the Whittaker expansion

$$\phi(g) = \sum_{q\in\mathbb{Q}^\times} W_\phi(a(q)g) = c_\phi \sum_{q\in\mathbb{Q}^\times} \prod_{p\leq\infty} W_p(a(q)g_p) \tag{1.11}$$

for any $g \in \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$. Our strategy to bound $\|\phi\|_\infty$ will be to bound for all g

$$|c_\phi| \sum_{q \in \mathbb{Q}^\times} \prod_{p \leq \infty} |W_p(a(q)g_p)| \geq |\phi(g)|,$$

that is, we do not take advantage of the potential oscillations in the Whittaker expansion. First, we give a bound for the constant c_ϕ appearing here. By [**HL94**] we have

$$L(\pi, Ad, 1) \gg N^{-\epsilon}.$$

For p unramified,

$$\frac{\zeta_p(2)\|W_p\|_2}{\zeta_p(1)L_p(\pi, \mathrm{Ad}, 1)} = 1.$$

For p ramified, we have

$$L_p(\pi, \mathrm{Ad}, 1) \asymp 1 \text{ and } 1 \leq \|W_p\|_2 \leq 2$$

(see [**Sah16**, Lemma 2.16]). Consequently, $|c_\phi| \ll N^\epsilon$. We shall also use that for any integer n coprime to N, we have

$$\prod_{p \nmid N} W_p(a(n)) = n^{-\frac{1}{2}} \lambda_\pi(n), \tag{1.12}$$

where $\lambda_\pi(n)$ is the n-th coefficient of the finite part of the L-function attached to π.

3.3. Generating domains. Using invariances of automorphic forms, we can restrict their argument to lie in some convenient set of representatives. We first describe such generating domains.

DEFINITION 1.3.1. We denote by $\mathscr{D}_N$ be the set of $g \in \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ such that

- $g_\infty = n(x)a(y)$ for some $x \in \mathbb{R}$ and $y \geq \frac{\sqrt{3}}{2}$,
- $g_p = 1$ for all $p \nmid N$,
- $g_p \in \mathrm{GL}_2(\mathbb{Z}_p)$ for all p.

LEMMA 1.3.2. *Let $\Gamma = \prod_{p \leq \infty} \Gamma_p$ be a subgroup of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ such that $\Gamma_\infty = \mathrm{SO}_2(\mathbb{R})$, for all finite p the group Γ_p is an open subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$ whose image by the determinant map is $\mathbb{Z}_p^\times$, and $\Gamma_p = \mathrm{GL}_2(\mathbb{Z}_p)$ for $p \nmid N$. Then the subset $\mathscr{D}_N$ of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ given by Definition 1.3.1 contains representatives of each double coset of $Z(\mathbb{A}_\mathbb{Q})\,\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})/\Gamma$.*

PROOF. By the strong approximation theorem, any $g \in \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ can be written as $g_\infty \gamma k$ with $g_\infty \in \mathrm{GL}_2^+(\mathbb{R})$, $\gamma \in \mathrm{GL}_2(\mathbb{Q})$, and $k \in \Gamma$. Multiplying on the left by γ^{-1} and on the right by k^{-1}, we can first assume that $g_p = 1$ for all finite p. Next, let $z = g_\infty \cdot i$. Then there is $\sigma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\Im(\sigma \cdot z) \geq \frac{\sqrt{3}}{2}$. After multiplying on the left by σ and on the right by $\prod_{p \nmid N} \sigma^{-1}$, we can instead assume that $g_p = 1$ for all $p \nmid N$, $g_p \in \mathrm{GL}_2(\mathbb{Z}_p)$ for $p \mid N$, and $\Im(g_\infty z) \geq \frac{\sqrt{3}}{2}$. Finally, multiplying by an element of $\mathrm{SO}_2(\mathbb{R})$, we can assume that g_∞ is of the form $n(x)a(y)$ with $y \geq \frac{\sqrt{3}}{2}$. □

Instead of evaluating our newform ϕ on elements of our generating domain $\mathscr{D}_N$, we shall rather use it with a certain translate of ϕ, the "balanced newform".

LEMMA 1.3.3. *Consider the subgroup $K^{(1)}$ of $\mathrm{GL}_2(\mathbb{A}_Q)$ defined by*

$$K^{(1)} = \Gamma_\infty \prod_{p<\infty} K^{(1)}(p^{n_p}\mathbb{Z}_p),$$

where the local subgroups Γ_∞ and $K^{(1)}(p^{n_p}\mathbb{Z}_p)$ are defined in (1.9) and (1.3) respectively. For each prime p dividing N, let e_p be an integer with $0 \leq e_p \leq n_p$. Let $\mathscr{D}_N$ be *the subset of* $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ *given by Definition 1.3.1. Then the set*

$$\mathscr{D}_N \prod_{p|N} a(p^{e_p}) \subset \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$$

contains representatives of each double coset of $Z(\mathbb{A}_\mathbb{Q})\,\mathrm{GL}_2(\mathbb{Q})\backslash\,\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})/K^{(1)}$.

Proof. Let

$$\Gamma_p = \mathrm{GL}_2(\mathbb{Z}_p) \cap \begin{bmatrix} 1 + p^{n_p}\mathbb{Z}_p & p^{e_p}\mathbb{Z}_p \\ p^{n_p - e_p}\mathbb{Z}_p & \mathbb{Z}_p \end{bmatrix},$$

and $\Gamma = \prod_{p \le \infty} \Gamma_p$. Let $g \in \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. By Lemma 1.3.2 there exists $g_d \in \mathscr{D}_N$ such that we have the following equality of double cosets

$$Z(\mathbb{A})\,\mathrm{GL}_2(\mathbb{Q}) g \prod_p a(p^{-e_p})\Gamma = Z(\mathbb{A})\,\mathrm{GL}_2(\mathbb{Q}) g_d \Gamma.$$

In particular, for each $p \mid N$ there exists $k_p \in \Gamma_p$ such that

$$Z(\mathbb{A})\,\mathrm{GL}_2(\mathbb{Q}) g \prod_p a(p^{-e_p}) = Z(\mathbb{A})\,\mathrm{GL}_2(\mathbb{Q}) g_d k_p.$$

Now if

$$k_p = \begin{bmatrix} 1 + ap^{n_p} & bp^{e_p} \\ cp^{n_p - e_p} & d \end{bmatrix},$$

then

$$a(p^{-e_p}) k_p a(p^{e_p}) = \begin{bmatrix} 1 + ap^{n_p} & b \\ cp^{n_p} & d \end{bmatrix} \in K^{(1)}(p^{n_p}\mathbb{Z}_p).$$

Hence writing

$$\begin{aligned} Z(\mathbb{A})\,\mathrm{GL}_2(\mathbb{Q})g &= Z(\mathbb{A})\,\mathrm{GL}_2(\mathbb{Q})g_d \prod_{p\mid N} k_p a(p^{e_p}) \\ &= Z(\mathbb{A})\,\mathrm{GL}_2(\mathbb{Q})(g_d \prod_{p\mid N} a(p^{e_p})) \prod_{p\mid N} a(p^{-e_p}) k_p a(p^{e_p}), \end{aligned}$$

we find that the double coset $Z(\mathbb{A})\,\mathrm{GL}_2(\mathbb{Q})gK^{(1)}$ contains the element $g_d \prod_{p\mid N} a(p^{e_p})$, which belongs to $\mathscr{D}_N \prod_{p\mid N} a(p^{e_p})$. $\square$

By Lemma 1.3.3, we can restrict ourselves to evaluate $|\phi|$ on $\mathscr{D}_N \prod_p a(p^{e_p})$, where the exponents e_p may be conveniently chosen. Of course, this is equivalent to evaluate its right translate by $\prod_p a(p^{e_p})$ on $\mathscr{D}_N$. Now, by Lemma 1.2.1 of Section 2, we can describe this generating domain in terms of the explicit representatives corresponding to each local double coset decomposition.

LEMMA 1.3.4. *Let $\mathscr{D}_N$ be the subset of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ given by Definition 1.3.1. Let $g \in \mathscr{D}_N \prod_{p\mid N} a(p^{e_p}) \subset \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. Then g satisfies the following.*

- *$g_\infty = n(x)a(y)$ for some $x \in \mathbb{R}$ and $y \geq \frac{\sqrt{3}}{2}$,*
- *$g_p = 1$ for all $p \nmid N$,*
- *Let $p \mid N$. If $\ell(g_p) \leq e_p$ then $m(g_p) = -e_p$, and if $\ell(g_p) > e_p$ then $m(g_p) = -2\ell(g_p) + e_p$, where we have used notations of Definition 1.2.1.*

PROOF. This follows immediately from Definition 1.3.1 and Lemma 1.2.1. $\square$

In particular the (optimal) choice $e_p = \lfloor \frac{n_p}{2} \rfloor$ for all $p \mid N$, together with Remark 1.2.2, motivates the following definition.

DEFINITION 1.3.2. Let $\mathscr{I}_N$ be the set of $g \in \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ such that

- $g_\infty = n(x)a(y)$ for some $x \in \mathbb{R}$ and $y \geq \frac{\sqrt{3}}{2}$,
- $g_p = 1$ for all $p \nmid N$,
- for all $p \mid N$ we have $\ell(g_p) \leq \frac{n_p}{2}$ and $m(g_p) \in \left\{-\lfloor \frac{n_p}{2} \rfloor, -\lceil \frac{n_p}{2} \rceil\right\}$.

REMARK 1.3.1. *Note that for $p \mid N$ we do not require $g_p \in \mathrm{GL}_2(\mathbb{Z}_p)a(p^{e_p})$, but only the stated conditions about $\ell(g_p)$ and $m(g_p)$.*

Finally, let us state the quantity we shall actually bound.

LEMMA 1.3.5. *Recall notations from § 3.1. For each $S \subset S_N$, define*

$$\phi^S(g) = \phi\left(g \prod_{p \in S} \begin{bmatrix} & 1 \\ p^{n_p} & \end{bmatrix}\right).$$

Then

$$\|\phi\|_\infty = \max_{S \subset S_N} \sup_{g \in \mathscr{I}_N} |\phi^S(g)|. \tag{1.13}$$

Moreover, for each subset $S \subset S_N$ and for every $g \in \mathscr{I}_N$ we have

$$|\phi^S(g)| \leq |c_\phi| \sum_{q \in \mathbb{Q}^\times} \left| \prod_{p \mid N} W_p^S(a(q) g_{m(g_p), \ell(g_p), \nu(g_p)}) \prod_{p \nmid N} W_p(a(q)) W_\infty(a(q)n(x)a(y)) \right|, \tag{1.14}$$

where $W_p^S = W_p$ if $p \notin S$, and W_p^S is the normalized local newform attached to the contragradient $\tilde{\pi}_p$ if $p \in S$.

PROOF. For each $p \mid N$, set $e_p = \lfloor \frac{n_p}{2} \rfloor$. For convenience, also set $e'_p = \lceil \frac{n_p}{2} \rceil$. Then by Lemma 1.3.3, we have

$$\|\phi\|_\infty = \sup_{g \in \mathscr{D}_N \prod_{p|N} a(p^{e_p})} |\phi(g)|.$$

We now prove that

$$\sup_{g \in \mathscr{D}_N \prod_{p|N} a(p^{e_p})} |\phi(g)| \leq \max_S \sup_{g \in \mathscr{I}_N} |\phi^S(g)|. \tag{1.15}$$

By Lemma 1.3.4 we have

$$\sup_{g \in \mathscr{D}_N \prod_{p|N} a(p^{e_p})} |\phi(g)| \leq \sup_{\substack{x \in \mathbb{R}, y \geq \frac{\sqrt{3}}{2} \\ m(g_p) = -e_p \text{ if } \ell(g_p) \leq \frac{n_p}{2} \\ m(g_p) = -2\ell(g_p) + e_p \text{ otherwise}}} \left| \phi\left(n(x)a(y) \prod_{p|N} g_p \right) \right|. \tag{1.16}$$

By (1.6) we have

$$g_{-2\ell_p+e_p,\ell_p,\nu_p} \begin{bmatrix} 1 & \\ & -\nu_p^{-2} \end{bmatrix} = n(-p^{e_p-\ell_p}\nu^{-1}) z(-p^{-\ell_p}\nu^{-1}) g_{-e'_p,n_p-\ell_p,-\nu_p} \begin{bmatrix} & 1 \\ p^{n_p} & \end{bmatrix}.$$

Using this identity at each prime belonging to the set S of primes p satisfying $\ell(g_p) > \frac{n_p}{2}$ in the right hand side of (1.16) we obtain by right-$K^{(1)}$ invariance of ϕ

$$\sup_{\substack{x \in \mathbb{R}, y \geq \frac{\sqrt{3}}{2} \\ m(g_p) = -e_p \text{ if } \ell(g_p) \leq \frac{n_p}{2} \\ m(g_p) = -2\ell(g_p) + e_p \text{ otherwise}}} \left| \phi\left(n(x)a(y) \prod_{p|N} g_p \right) \right| \leq M(\phi) \tag{1.17}$$

where we have set

$$M(\phi) = \max_{S \subset S_N} \sup_{\substack{x \in \mathbb{R}, y \geq \frac{\sqrt{3}}{2} \\ m(g_p) = -e_p \text{ if } p \in S \\ m(g_p) = -e'_p \text{ otherwise} \\ \ell(g_p) \leq \frac{n_p}{2}}} \left| \phi^S \left(n(x)a(y) \prod_{p|N} g_p \right) \right|.$$

Combining (1.16), (1.17) and the definition of $\mathscr{I}_N$, we obtain the bound (1.15). From definition, it is clear that

$$\|\phi\|_\infty \geq \max_{S \subset S_N} \sup_{g \in \mathscr{I}_N} |\phi^S(g)|$$

so (1.13) follows.

The second claim follows from the Whittaker expansion (1.11). Observe that by Remark 1.2.2,

$$\left| W_p \left(g_p \begin{bmatrix} & 1 \\ p^{n_p} & \end{bmatrix} \right) \right| = |\tilde{W}_p(g_p)|,$$

where $\tilde{W}_p$ is the normalized local newform attached to the contragradient $\tilde{\pi}_p$. The identity

$$a(q)n(x) = n(qx)a(q)$$

and the left invariance of the modulus of the local Whittaker newforms by NZ give (1.14). $\square$

As we shall be interested in the support of the Whittaker expansion, we make now the following definition.

DEFINITION 1.3.3. Keep notation as in Lemma 1.3.5. For every $S \subset S_N$ and $g \in \mathscr{I}_N$ we define $\mathrm{Supp}(g; S)$ as the set of rational numbers $q \in \mathbb{Q}^\times$ such that

$$\prod_{p|N} W_p^S(a(q)g_{m(g_p),\ell(g_p),\nu(g_p)}) \prod_{p\nmid N} W_p(a(q))W_\infty(a(q)n(x)a(y)) \neq 0.$$

NOTATION 1.3.1. From now on we fix $g \in \mathscr{I}_N$ and $S \subset S_N$ (in the notation of § 3.1 and Definition 1.3.2), and we define for each $p \mid N$, $\ell_p = \ell(g_p)$, $\epsilon_p = -m(g_p)$, $\epsilon'_p = n_p - \epsilon_p$, and $\nu_p = \nu(g_p)$. We then define the following integers

$$L = \textstyle\prod_{p|N} p^{\ell_p}, \quad N_1 = \prod_p p^{\epsilon_p}, \quad N_2 = \prod_p p^{\epsilon'_p},$$

as well as the sets of primes

$$\begin{aligned} \mathscr{H}_- &= \{p \in \mathscr{H} : \ell_p < a_2(p)\}, \\ \mathscr{H}_= &= \{p \in \mathscr{H} : \ell_p = a_2(p)\}, \\ \mathscr{H}_+ &= \{p \in \mathscr{H} : \ell_p > a_2(p)\}, \end{aligned} \tag{1.18}$$

where $a_2(p) = n_p - c_p$ is the exponent of the conductor of the local character χ_2 (note that in the case where $N = C$, we have $\mathscr{H}_- = \varnothing$ and $\mathscr{H}_=$ coincides with the set of primes dividing N and not dividing L). If $M = \prod_p p^{m_p}$ is any integer, we may use the notation

$$M^\star = \prod_{p \in \mathscr{H}_\star} p^{m_p}$$

for $\star \in \{+, -, =\}$.

3.4. Sup norms: maximally ramified case. In this subsection, we are assuming $N = C$ and we prove Theorem 1.1.1 in this special case, as the proof becomes simpler. We first determine the support of the "Whittaker expansion" (1.14).

LEMMA 1.3.6. *Recall Notation 1.3.1. There is a map*

$$\Psi(\mathscr{H}_=) \to \{1, \cdots, L\}$$

$$s \mapsto t_s$$

such that

$$\mathrm{Supp}(g; S) \subseteq \left\{ \frac{s}{N_2 L}(t_s + jL) : s \in \Psi(\mathscr{H}_=), j \in \mathbb{Z} \text{ with } t_s + jL \text{ coprime to } N \right\}.$$

PROOF. Let $q = \prod_p p^{q_p} \in \mathbb{Q}^\times$. Assume $q \in \mathrm{Supp}(g; S)$. First, if $p \nmid N$ then we must have $q_p \geq 0$. So $\mathrm{sgn}(q) \prod_{p \nmid N} p^{q_p}$ is an integer. We shall see that it satisfies a certain congruence condition. Consider now a prime $p \mid N$, if $q = p^{q_p} u \in \mathbb{Q}^\times$ with $u \in \mathbb{Z}_p^\times$, we have

$$a(q) g_{-\epsilon_p, \ell_p, \nu_p} = g_{q_p - \epsilon_p, \ell_p, \nu_p u^{-1}} \begin{bmatrix} 1 & \\ & u \end{bmatrix} = g_{q_p - \epsilon_p, \ell_p, \nu_p p^{q_p} q^{-1}} \begin{bmatrix} 1 & \\ & q p^{-q_p} \end{bmatrix}. \tag{1.19}$$

By Lemma 1.2.3 (applied either to π_p if $p \notin S$ or to $\tilde{\pi}_p$ if $p \in S$), if $\ell_p = 0$ then $q_p - \epsilon_p \geq -n_p$, so $q_p \geq -\epsilon'_p$. It follows that

$$s \doteq \prod_{\substack{p \mid N \\ p \nmid L}} p^{q_p + \epsilon'_p} \in \Psi(\mathscr{H}_=).$$

On the other hand, if $\ell_p > 0$ then $q_p - \epsilon_p = -n_p - \ell_p$, so $q_p = -\epsilon'_p - \ell_p$. Now fix a prime $p_0 \mid L$ (so $\ell_{p_0} > 0$), and write

$$\begin{aligned}
\operatorname{sgn}(q) s \prod_{p \nmid N} p^{q_p} &= \operatorname{sgn}(q) \prod_{p \nmid N} p^{q_p} \prod_{\substack{p \mid N \\ p \nmid L}} p^{q_p + \epsilon'_p} \\
&= \operatorname{sgn}(q) \prod_{p \nmid N} p^{q_p} \prod_{\substack{p \mid N \\ p \nmid L}} p^{q_p + \epsilon'_p} \prod_{\substack{p \mid L \\ p \neq p_0}} p^{q_p + \epsilon'_p + \ell_p} \\
&= \operatorname{sgn}(q) \prod_{p \neq p_0} p^{q_p} \prod_{\substack{p \mid N \\ p \nmid L}} p^{\epsilon'_p} \prod_{\substack{p \mid L \\ p \neq p_0}} p^{\epsilon'_p + \ell_p} \\
&= \left(p_0^{-q_{p_0}} q \right) \left(\prod_{\substack{p \mid N \\ p \nmid L}} p^{\epsilon'_p} \prod_{\substack{p \mid L \\ p \neq p_0}} p^{\epsilon'_p + \ell_p} \right).
\end{aligned}$$

By Lemma 1.2.3 and equality (1.19), $p_0^{-q_{p_0}} q$ satisfies a certain congruence condition modulo $p^{\ell_{p_0}} \mathbb{Z}_{p_0}$. In addition $\prod_{\substack{p \mid N \\ p \nmid L}} p^{\epsilon'_p} \prod_{\substack{p \mid L \\ p \neq p_0}} p^{\epsilon'_p + \ell_p}$ is clearly in $\mathbb{Z}_{p_0}^{\times}$. So we just showed that the integer $\operatorname{sgn}(q) s \prod_{p \nmid N} p^{q_p}$ satisfies a certain congruence condition modulo $p_0^{\ell_{p_0}}$. Applying the same reasoning with each prime dividing L, we obtain by the Chinese remainder theorem a condition of the type

$$\operatorname{sgn}(q) s \prod_{p \nmid N} p^{q_p} \equiv r_0 \mod L.$$

Since in addition L and s are coprime, we can write

$$\operatorname{sgn}(q) \prod_{p \nmid N} p^{q_p} = t_s + jL \tag{1.20}$$

for some integer $t_s \equiv r_0 s^{-1} \mod L$, and j ranging over $\mathbb{Z}$. Finally,

$$q = \operatorname{sgn}(q) \prod_{p|L} p^{-\epsilon'_p - \ell_p} \prod_{\substack{p|N \\ p\nmid L}} p^{q_p} \prod_{p\nmid N} p^{q_p} = \frac{s}{N_2 L}(t_s + jL).$$

□

We now compute the size of each term in "the Whittaker expansion" (1.14).

LEMMA 1.3.7. *Keep notations from Notation 1.3.1 and Lemma 1.3.5. Let* $q = \frac{s}{N_2 L}(t_s + jL)$ *as in Lemma 1.3.6. Then we have*

$$\left| \prod_{p|N} W_p^S(a(q) g_{-\epsilon_p, \ell_p, \nu_p}) \prod_{p\nmid N} W_p(a(q)) \right| = L^{\frac{1}{2}} s^{-\frac{1}{2}} |t_s + jL|^{-\frac{1}{2}} \left| \lambda_\pi(|t_s + jL|) \right|.$$

PROOF. For q of this form, using (1.20) and (1.12), we have

$$\begin{aligned} \prod_{p\nmid N} W_p(a(q)) &= \prod_{p\nmid N} W_p(t_s + jL) \\ &= (|t_s + jL|)^{-\frac{1}{2}} \lambda_\pi(|t_s + jL|), \end{aligned}$$

and Lemma 1.2.3 (observe that the contragradient representation $\tilde{\pi}_p$ satisfies the same hypothesis as π_p) together with equality (1.19) give

$$\begin{aligned} \left| \prod_{p|N} W_p^S(a(q) g_{-\epsilon_p, \ell_p, \nu_p}) \right| &= \left| \prod_{p|N} W_p^S(g_{q_p - \epsilon_p, \ell_p, \nu_p p^{q_p} q^{-1}}) \right| \\ &= L^{\frac{1}{2}} \prod_{\ell_p = 0} p^{-\frac{q_p - \epsilon_p + n_p}{2}} = L^{\frac{1}{2}} s^{-\frac{1}{2}}. \end{aligned}$$

$\square$

By Combining Lemmas 1.3.6 and 1.3.7 the "Whittaker expansion" (1.14) is thus bounded above by

$$c_\phi L^{\frac{1}{2}} \sum_{s \in \Psi(\mathscr{H}_=)} s^{-\frac{1}{2}} \sum_{j \in \mathbb{Z}} |t_s + jL|^{-\frac{1}{2}+\delta+\epsilon} \kappa\left(\frac{t_s + jL}{N_2 L} sy\right).$$

Using estimate (1.8), we first evaluate the j-sum as follows:

$$\begin{aligned}
&\sum_{j \in \mathbb{Z}} |t_s + jL|^{-\frac{1}{2}+\delta+\epsilon} \kappa\left(\frac{t_s + jL}{N_2 L} sy\right) \\
&\qquad \ll \left(\frac{sy}{N_2 L}\right)^{-\epsilon} \sum_{j \in \mathbb{Z}} |t_s + jL|^{-\frac{1}{2}+\delta+\epsilon} \exp\left((-2\pi + \epsilon)\frac{|t_s + jL|}{N_2 L} sy\right) \\
&\qquad \ll \left(\frac{sy}{N_2 L}\right)^{-\epsilon} \left(1 + \int_{\mathbb{R}} |tL|^{-\frac{1}{2}+\delta+\epsilon} \exp\left((-2\pi + \epsilon)\frac{|t|}{N_2} sy\right) dt\right) \\
&\qquad \ll \left(\frac{N_2 L}{sy}\right)^{2\epsilon} \left(1 + \left(\frac{N_2}{Lsy}\right)^{\frac{1}{2}} \left(\frac{N_2 L}{sy}\right)^{\delta}\right).
\end{aligned}$$

Altogether, using Lemma 1.3.1 and the fact that $A^{\omega(N)} \ll_\epsilon N^\epsilon$ for any fixed $A > 0$ we get

$$|\phi(g)| \ll c_\phi \left(\frac{N_2 L}{y}\right)^{\epsilon} \left(L^{\frac{1}{2}} + L^{\delta} \left(\frac{N_2}{y}\right)^{\frac{1}{2}+\delta}\right) \ll N^{2\epsilon}\left(L^{\frac{1}{2}} + N_2^{\frac{1}{2}} N^{\delta}\right)$$

since $c_\phi \ll N^\epsilon$, $y \geq \frac{\sqrt{3}}{2}$ and $N_2 L \leq N$. This establishes Theorem 1.1.1 when $N = C$ because we have $L \leq N^{\frac{1}{2}}$ and $N_2 \leq \prod_{p|N} p^{\lceil \frac{n_p}{2} \rceil}$.

3.5. Sup norms: general ramification. Finally, let us address the necessary modifications when we do not make any assumption about the conductor of χ. The analysis of the local Whittaker newform W_p is similar, but with more cases to take into account, depending on which of the sets (1.10) the prime p belongs. In particular, it still holds that for all $p \in \mathscr{H}$ we have $\pi_p = \chi_1 \boxplus \chi_2$, but the exponents $a_2(p) = n_p - c_p$ of the conductor of the local characters χ_2 may not all equal zero. We thus also get a Whittaker expansion supported on arithmetic progressions dictated by the primes at which the central character is highly ramified. The rest of our argument differs from the maximally ramified case, as we rather use strong L^2-averages of the local newforms, in the spirit of [**Sah17**], instead of the local bounds. Of course, in the maximally ramified case, these L^2-averages follow immediately from the computation of the support of the local newform W_p and the local bound, so the difference on the argument is mainly expository.

We first determine the support of the "Whittaker expansion" (1.14) in this more general case.

LEMMA 1.3.8. *Recall Notation 1.3.1. There is a map*

$$\Psi(\mathscr{H}_=) \times \Psi(\mathscr{L}) \to \left\{1, \cdots, \frac{L^+C^+}{N^+}\right\}$$

$$(s,u) \mapsto t_{su}$$

such that

$$\mathrm{Supp}(g;S) \subseteq \left\{ su\frac{N^+}{N_2L^+C^+}\left(t_{su} + j\frac{L^+C^+}{N^+}\right), s \in \Psi(\mathscr{H}_=), u \in \Psi(\mathscr{L}), j \in \mathbb{Z} \right.$$
$$\left. \text{with } t_{su} + j\frac{L^+C^+}{N^+} \text{ coprime to } N \right\}.$$

REMARK 1.3.2. *It is immediate by unravelling the definitions that $\frac{L^+C^+}{N^+}$ is an integer.*

PROOF. The reasoning is quite similar to the proof of Lemma 1.3.6, but we use [**Sah17**, Proposition 2.10] for the primes in $\mathscr{L}$ and Lemma 1.2.4 instead of Lemma 1.2.3 for those primes in $\mathscr{H}$. Fix $q = \prod_p p^{q_p} \in \mathrm{Supp}(g; S)$. As before, $\mathrm{sgn}(q) \prod_{p \nmid N} p^{q_p}$ is an integer and we shall see it satisfies some congruence condition. If $p \in \mathscr{H}_=$ or $p \in \mathscr{L}$ then examination of either Lemma 1.2.4 or [**Sah17**, Proposition 2.10] gives $q_p \geq -\epsilon_p'$. So

$$su \doteq \prod_{p \in \mathscr{L} \cup \mathscr{H}_=} p^{q_p + \epsilon_p'} \in \Psi(\mathscr{L} \cup \mathscr{H}_=).$$

In addition Lemma 1.2.4 gives that for $p \in \mathscr{H}_-$ we have $q_p = -\epsilon_p'$, and for $p \in \mathscr{H}_+$ we have $q_p = \epsilon_p - \ell_p - a_1(p)$. Fix $p_0 \in \mathscr{H}_+$ and write

$$\begin{aligned}
\mathrm{sgn}(q) su \prod_{p \nmid N} p^{q_p} &= \mathrm{sgn}(q) \prod_{p \nmid N} p^{q_p} \prod_{p \in \mathscr{L} \cup \mathscr{H}_=} p^{q_p + \epsilon_p'} \\
&= \mathrm{sgn}(q) \prod_{p \nmid N} p^{q_p} \prod_{p \in \mathscr{L} \cup \mathscr{H}_=} p^{q_p + \epsilon_p'} \prod_{p \in \mathscr{H}_-} p^{q_p + \epsilon_p'} \prod_{\substack{p \in \mathscr{H}_+ \\ p \neq p_0}} p^{q_p - (\epsilon_p - \ell_p - a_1(p))} \\
&= \mathrm{sgn}(q) \prod_{p \neq p_0} p^{q_p} \prod_{p \in \mathscr{L} \cup \mathscr{H}_= \cup \mathscr{H}_-} p^{\epsilon_p'} \prod_{\substack{p \in \mathscr{H}_+ \\ p \neq p_0}} p^{\ell_p + a_1(p) - \epsilon_p}.
\end{aligned}$$

By Lemma 1.2.4 the rational number $\mathrm{sgn}(q) \prod_{p \neq p_0} p^{q_p}$ satisfies a congruence condition modulo $p^{\ell_{p_0} - a_2(p_0)} \mathbb{Z}_{p_0}$. Then using the Chinese remainder theorem we see that $\mathrm{sgn}(q) su \prod_{p \nmid N} p^{q_p}$ is an integer satisfying a congruence condition modulo $\frac{L^+C^+}{N^+}$. It follows that we can write

$$\operatorname{sgn}(q)\prod_{p\nmid N}p^{q_p}=t_{su}+j\frac{L^+C^+}{N^+}.$$

Finally,

$$\begin{aligned}q&=\operatorname{sgn}(q)\prod_{p\nmid N}p^{q_p}\prod_{p\in\mathscr{L}\cup\mathscr{H}_=}p^{q_p}\prod_{p\in\mathscr{H}_-}p^{-\epsilon'_p}\prod_{p\in\mathscr{H}_+}p^{\epsilon_p-\ell_p-a_1(p)}\\&=\left(t_{su}+j\frac{L^+C^+}{N^+}\right)\frac{su}{\prod_{p\in\mathscr{L}}p^{\epsilon'_p}N_2^=}\frac{1}{N_2^-}\frac{N_1^+}{L^+C^+}.\end{aligned}$$

□

If we were now to proceed following the exact same strategy as in the maximally ramified case, then we would get a worse estimate because of weaker local bounds for the local newform in the case $\ell_p=\frac{n_p}{2}$ (see [**Ass19b**, Lemma 5.10]). Instead, we rely on L^2-averages of the local newvectors established by Saha [**Sah17**]. To this end, we make first the following trivial lemma.

LEMMA 1.3.9. *Suppose $(a_n)_{n\in\mathbb{Z}}$, $(b_n)_{n\in\mathbb{Z}}$ are two families of positive real numbers such that $\sum_{n\in\mathbb{Z}}a_nb_n$ converges absolutely*[1]*, and a_n is periodic with period T. Let M be such that*

$$\sum_{n=0}^{T-1}a_n^2\le M.$$

Then we have

$$\sum_{n\in\mathbb{Z}}a_nb_n\le M^{\frac12}\sum_{k\in\mathbb{Z}}\left(\sum_{j=0}^{T-1}b_{Tk+j}^2\right)^{\frac12}.$$

[1]meaning the partial sums $\sum_{n\in J}|a_nb_n|$ indexed by finite sets $J\subset\mathbb{Z}$ are uniformly bounded.

PROOF. We regroup the series in sums of length T and we apply Cauchy-Schwarz for each of these. □

Next, we express the "Whittaker expansion" (1.14) so as to be tackled by previous lemma.

LEMMA 1.3.10. *Recall Notation 1.3.1. Then*

$$|\phi^S(g)| \leq |c_\phi| \sum_{s\in\Psi(\mathscr{H}_=)} \sum_{u\in\Psi(\mathscr{L})} \sum_{n\in\mathbb{Z}} a_n b_n,$$

where a_n is periodic with period L and satisfies

$$\sum_{n=0}^{L-1} a_n^2 \ll N^\epsilon L(su)^{-\frac{1}{2}} \tag{1.21}$$

and

$$b_n = |n|^{-\frac{1}{2}} \left|\lambda_\pi(n)\,\kappa\left(\frac{N^+ suny}{N_2L^+C^+}\right)\right| \mathbb{1}_{n\equiv t_{su} \mod \frac{L^+C^+}{N^+}}.$$

PROOF. The claim will follow from the "Whittaker expansion" (1.14)

$$|\phi^S(g)| \leq |c_\phi| \sum_{q\in\mathbb{Q}^\times} \left| \prod_{p|N} W_p^S(g_{q_p-\epsilon_p,\ell_p,\nu_p p^{q_p} q^{-1}}) \prod_{p\nmid N} W_p(a(q)) W_\infty(a(q)n(x)a(y))) \right|,$$

together with (1.7), (1.12) and Lemma 1.3.8 once we have shown that the sequence defined by

$$a_n = \left| \prod_{p|N} W_p^S\left(a\left(\frac{N^+ sun}{N_2L^+C^+}\right) g_{-\epsilon_p,\ell_p,\nu_p}\right)\right|$$

satisfies the desired properties. For each $p \mid N$, let us distinguish cases depending on which of the sets defined in (1.10) and (1.18) contains p. For all $v \in \mathbb{Z}_p^\times$ we have

$$W_p^S\left(a\left(\frac{N^+suv}{N_2L^+C^+}\right)g_{-\epsilon_p,\ell_p,\nu_p}\right) = \begin{cases} W_p^S\left(a\left(v\right)g_{u_p-n_p,\ell_p,*}\right) \text{ if } p \in \mathscr{L} \\ W_p^S\left(a\left(v\right)g_{-n_p,\ell_p,*}\right) \text{ if } p \in \mathscr{H}_- \\ W_p^S\left(a\left(v\right)g_{s_p-n_p,\ell_p,*}\right) \text{ if } p \in \mathscr{H}_= \\ W_p^S\left(a\left(v\right)g_{-\ell_p-a_1(p),\ell_p,*}\right) \text{ if } p \in \mathscr{H}_+, \end{cases}$$

where each $*$ is independent of v. By [**Sah17**, Proposition 2.10], we then get

$$\int_{v\in\mathbb{Z}_p^\times}\left|W_p^S\left(a\left(\frac{N^+suv}{N_2L^+C^+}\right)g_{-\epsilon_p,\ell_p,\nu_p}\right)\right|^2 d^\times v \ll \begin{cases} p^{-\frac{u_p}{2}} \text{ if } p \in \mathscr{L} \\ 1 \text{ if } p \in \mathscr{H}_- \\ p^{-\frac{s_p}{2}} \text{ if } p \in \mathscr{H}_= \\ 1 \text{ if } p \in \mathscr{H}_+. \end{cases}$$

Now by [**Sah17**, Remark 2.12], for each $p \mid N$ and each fixed $s \in \Psi(\mathscr{H}_=)$ and $u \in \Psi(\mathscr{L})$, the map on $\mathbb{Z}_p^\times$ given by

$$v \mapsto \left|W_p^S\left(a\left(\frac{N^+suv}{N_2L^+C^+}\right)g_{-\epsilon_p,\ell_p,\nu_p}\right)\right|$$

is $U_p(\ell_p)$-invariant. Hence by the Chinese remainder theorem, these give rise to a map on $(\mathbb{Z}/L\mathbb{Z})^\times$ given by

$$(r \mod L) \mapsto \prod_{p\mid N}\left|W_p^S\left(a\left(\frac{N^+sur}{N_2L^+C^+}\right)g_{-\epsilon_p,\ell_p,\nu_p}\right)\right|,$$

and by Lemma 1.3.8, if $a_n \neq 0$ then n is coprime to N, thus the sum (1.21) is just

$$\sum_{r\in(\mathbb{Z}/L\mathbb{Z})^\times} \prod_{p|N} \left| W_p^S \left(a\left(\frac{N^+ sur}{N_2 L^+ C^+}\right) g_{-\epsilon_p,\ell_p,\nu_p}\right)\right|^2$$
$$= \varphi(L) \prod_{p|N} \int_{v\in\mathbb{Z}_p^\times} \left| W_p^S \left(a\left(\frac{N^+ suv}{N_2 L^+ C^+}\right) g_{-\epsilon_p,\ell_p,\nu_p}\right)\right|^2 d^\times v$$
$$\ll N^\epsilon L (su)^{-\frac{1}{2}},$$

where φ is Euler's totient. □

By combining Lemmas 1.3.9 and 1.3.10 it follows

$$|\phi(g)| \ll N^\epsilon L^{\frac{1}{2}} \sum_{s\in\Psi(\mathscr{H}_=)} s^{-\frac{1}{4}} \sum_{u\in\Psi(\mathscr{L})} u^{-\frac{1}{4}} \sum_{k\in\mathbb{Z}} S_k^{\frac{1}{2}}, \tag{1.22}$$

where

$$S_k = \sum_{j=0}^{L-1} b_{Lk+j}^2, \tag{1.23}$$

and b_n is defined in Lemma 1.3.10.

LEMMA 1.3.11. *For all $k \geq 1$ the sum (1.23) satisfies*

$$S_k \ll \frac{N^+}{L^+C^+} L^{2\delta} \left(\frac{N}{suy}\right)^\epsilon k^{-1+2\delta+\epsilon} \exp\left(-\pi \frac{N^+ suyL}{N_2 L^+ C^+} k\right),$$

and the same estimate holds for $k \leq -2$ upon replacing k with $-k-1$ in the right hand side. Finally,

$$S_0, S_{-1} \ll \left(\frac{N_2}{suy}\right)^\epsilon \left(1 + \frac{N^+}{L^+C^+}\left(\frac{N_2 L^+ C^+}{N^+ suy}\right)^{2\delta+\epsilon}\right).$$

PROOF. For those intervals $[kL, (k+1)L]$ not containing zero we use estimate (1.8) then we bound S_k by the number of terms multiplied by the largest term. Since $\frac{L^+C^+}{N^+}$ divides L, the congruence condition on $n = Lk + j$ modulo $\frac{L^+C^+}{N^+}$ is equivalent to the same congruence condition on j. We thus get, for $k \geq 1$

$$S_k \ll \frac{N^+}{L^+C^+} L^{2\delta} \left(\frac{N}{suy}\right)^{\epsilon} k^{-1+2\delta+\epsilon} \exp\left(-\pi \frac{N^+ suyL}{N_2 L^+ C^+} k\right).$$

For $k = 0$ we have

$$\begin{aligned} S_0 \ll \left(\frac{N}{suy}\right)^{\epsilon} \Bigg(1+\int_0^{\infty} \left(t_{su} + t\frac{L^+C^+}{N^+}\right)^{-1+2\delta+\epsilon} \\ \times \exp\left(-\pi\frac{N^+suy}{N_2L^+C^+}\left(t_{su}+t\frac{L^+C^+}{N^+}\right)\right) dt\Bigg) \\ \ll \left(\frac{N}{suy}\right)^{\epsilon}\left(1+\frac{N^+}{L^+C^+}\left(\frac{N_2L^+C^+}{N^+suy}\right)^{2\delta+2\epsilon}\right). \end{aligned}$$

The analogous results for $k < 0$ follow by changing k to $-k-1$ and changing $t_{s,u}$ to $\frac{L^+C^+}{N^+} - t_{s,u}$. $\square$

By a similar argument as in § 3.4, Lemma 1.3.11 implies

$$\sum_{k\in\mathbb{Z}} S_k^{\frac{1}{2}} \ll \left(\frac{N}{suy}\right)^{\epsilon}\left(1+\left(\frac{N^+}{L^+C^+}\right)^{\frac{1}{2}}\left(\frac{N_2L^+C^+}{N^+suy}\right)^{\delta+\epsilon}+\left(\frac{N_2}{Lsuy}\right)^{\frac{1}{2}}\left(\frac{N_2L^+C^+}{N^+suy}\right)^{\delta+\epsilon}\right).$$

Substituting this into (1.22) and using Lemma 1.3.1 we obtain

$$|\phi(g)| \ll N^{\delta+\epsilon}\left(L^{\frac{1}{2}} + N_2^{\frac{1}{2}}\right).$$

Lemma 1.3.5 together with the results from Section 3.4 and 3.5 finishes the proof of Theorem 1.1.1.

CHAPTER 2

A relative trace formula approach to the Kuznetsov formula for GSp_4

1. Introduction

In this chapter we develop a Kuznetsov formula for the group GSp_4. To motivate our results, we first recall the Kuznetsov formula for GL_2, an identity relating spectral information about the quotient space $\Gamma\backslash\mathbb{H}$ (where Γ is a congruence subgroup) to some arithmetic input.

For arbitrarily chosen nonzero integers n and m and any reasonable test function h, the spectral side involves the quantity

$$h(t_u)a_m(u)\overline{a_n(u)}, \tag{2.1}$$

where u ranges over eigenfunctions of the Laplace operator involved in the spectral decomposition of $L^2(\Gamma\backslash\mathbb{H})$, $a_m(u)$ is the m-th Fourier coefficient of u, and t_u is the corresponding spectral parameter. More precisely, the spectrum of $L^2(\Gamma\backslash\mathbb{H})$ can be described as the direct sum of the *discrete spectrum* and the *continuous spectrum*. The discrete spectrum is the direct sum of 1-dimensional subspaces spanned by cuspidal Maaß forms (the *cuspidal spectrum*) plus the constant function (the *residual spectrum*). The continuous spectrum is a direct integral of 1-dimensional subspaces spanned by the Eisenstein series. The spectral side of the Kuznetsov formula correspondingly splits as a discrete sum over Maaß forms plus a continuous integral over Eisenstein series.

The arithmetic-geometric side is a sum of two contributions, that may be seen as the contributions from the two elements of the Weyl group of GL_2. The identity contribution is given by the delta symbol $\delta_{n,m}$ times the integral of the spectral test function h against the spectral measure $\frac{t}{\pi^2}\tanh(\pi t)dt$. For this reason, the Kuznetsov formula may be viewed as a result of quasi-orthogonality for the Fourier coefficients $a_m(\cdot)$ and $a_n(\cdot)$, provided the remaining contribution can be controlled. The latter consists of a sum of Kloosterman sums weighted by some integral transform of the test function h, involving Bessel functions.

Applications of the Kuznetsov formula involve using known results about any of the two sides to derive information about the other side. On one hand, the flexibility allowed by the choice of the test function h enables one to use known bounds about the Kloosterman sums to study the distribution of the discrete spectrum and the size of the Fourier coefficients of Maaß forms. On the other hand, understanding the Fourier coefficients of Maaß forms as well as the integral transform appearing on the geometric side yields strong bounds for sums of Kloosterman sums.

Recently, Kuznetsov formulae have been developed by Blomer and Buttcane for GL_3 (see [**Blo13**, **But13**, **But16**, **But19**, **But20b**, **But21**, **But22**, **BB19**, **BBM17**]), with similar applications as described above. It would thus be interesting to establish the corresponding formulae for other groups.

In the classical proof of the GL_2 Kuznetsov formula, one computes the inner product of two Poincaré series in two different ways, one involving the spectral decomposition of $L^2(\Gamma\backslash\mathbb{H})$, and the other one by computing the Fourier coefficients of the Poincaré series and unfolding. This gives a "pre-Kuznetsov formula", that one then proceeds to integrate against the test function h, obtaining on the geometric side the integral transforms of h mentioned above.

Another approach, that one may call the relative trace formula approach to the Kuznetsov formula, builds upon the relative trace formula that was introduced by Jacquet [**JL85**]. In the case of GL_2, the relative trace formula approach to the Kuznetsov formula is apparently based on unpublished work of Zagier, detailed in [**Joy90**]. This approach is developed in the adelic framework in [**KL13**] for the congruence subgroup $\Gamma = \Gamma_1(N)$. It proceeds by integrating an automorphic kernel

$$K_f(x,y) = \sum_{\gamma \in \mathrm{PGL}_2(\mathbb{Q})} f(x^{-1}\gamma y),$$

where $x, y \in \mathrm{GL}_2(\mathbb{A})$ and $f : \mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$ is a suitable test function. The spectral expansion of the kernel will then involve the quantity $\tilde{f}(t_u)u(x)\overline{u(y)}$, where u ranges over the eigenfunctions involved in the spectral decomposition of $L^2(\Gamma(N)\backslash\mathbb{H})$, t_u is the spectral parameter of u, and $\tilde{f}$ is the *spherical transform* of f. Thus integrating $K_f(x,y)$ against a suitable character on $U \times U$, where $U = \left[\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right]$, one gets the quantity (2.1) with $h = \tilde{f}$. On the other hand, using the Bruhat decomposition for $\mathrm{PGL}_2(\mathbb{Q})$, one can decompose the integral over $U \times U$ as a sum over elements of the Weyl group and over diagonal matrices in $\mathrm{PGL}_2(\mathbb{Q})$ of some orbital integrals. In the case of the identity element, at most one diagonal matrix will have a non-zero contribution, which will turn out to be a delta symbol times some integral transform of the function f. In the case of the longest element in the Weyl group, each positive integer in $N\mathbb{Z}$ will have a nonzero contribution, given by a Kloosterman sum times a second kind of integral transform of f. A more refined version is then obtained by taking the Mellin transform of the primitive formula obtained. Note that in this approach, one gets on the geometric side some integral transforms of the function f, hence one has to do some extra work to relate these to the test function $h = \tilde{f}$ appearing in the spectral side.

A couple of remarks are in order about the choice of f. Firstly, the spectral expansion of the kernel involves the spectral decomposition of $L^2(\mathbb{R}_{>0}\operatorname{GL}_2(\mathbb{Q})\backslash\operatorname{GL}_2(\mathbb{A}))$ rather than $L^2(\Gamma\backslash\mathbb{H})$. By restricting f to be left and right K_∞-invariant (where $K_\infty = \mathrm{SO}_2$), only right-K_∞-invariant automorphic forms ϕ (thus corresponding to adelization of functions on the homogeneous space $\mathbb{H} = SL_2(\mathbb{R})/K_\infty$) will show up in the spectral expansion of the kernel, but other choices are possible. Also one may choose the test function f at unramified places so as to get a final formula that include the Hecke eigenvalues of a fixed Hecke operator of index coprime to the level N.

Our plan is to implement the relative trace formula approach in the case of GSp_4. In contrast to the case of GL_2, there is more than one non-conjugate unipotent subgroups U. Choosing U to be the unipotent radical of the Borel subgroup (that is the minimal parabolic subgroup) will yield Whittaker coefficients of the automorphic forms involved (instead of the Fourier coefficients). The Whittaker coefficients have a "multiplicity one" property, which ensures that the global Whittaker coefficients factor into a product of local coefficients. These local Whittaker coefficients can be written down in terms of local Satake parameters, which is important for applications. Also in contrast to the case of GL_2, not every automorphic form has non-identically zero Whittaker coefficients. For instance, Siegel modular forms give rise to automorphic forms whose Whittaker coefficients vanish identically. Thus, only *generic* automorphic forms (*i.e,* with non-identically zero Whittaker coefficients) will survive the integration on $U \times U$ and contribute to the final formula.

In Section 2 below we introduce the group GSp_4 and the structure theory that shall be needed. In Section 3, we introduce the basic representation-theoretic notions and tools: in subsection 3.1 we introduce the Whittaker coefficients of automorphic forms, that is the basic object which will appear in our relative trace formula. In subsection 3.2 we introduce the automorphic kernel associated to a test function f, on which the relative trace formula approach is based. This kernel induces a certain global operator $R(f)$, that factors as a tensor product of local operators. The automorphic forms appearing in the spectral side of the relative trace formula range over an orthonormal basis of eigenfunctions of $R(f)$, plus an analogous continuous contribution. The construction of this eigenbasis is done by studying the local operators corresponding to $R(f)$. This is the object of the following next two subsections, where we discuss these local operators at the finite places and at the Archimedean place respectively. As we explain there, the former are given by the *Hecke algebra* while the latter amounts to the *spherical transform.* We also include a discussion of the Whittaker function at the Archimedean place, and an integral transform related to it, that will eventually appear in the geometric side of the relative trace formula. In Section 4, we introduce the Eisenstein series involved in the spectral expansion of the automorphic kernel, and we derive the spectral side of the relative trace formula in an explicit fashion. In Section 5, we deal with the geometric side of the relative trace formula. In the first two subsections we introduce the relevant orbital integrals and we study them globally. We then switch to a local analysis. In subsection 5.3 we study the orbital integrals at the Archimedean place. This involves the integral transform that was mentioned earlier, as well as a certain interchange of integrals conjecture (that we will not need for our final application in Chapter 3). The finite part of the orbital integrals – which gives rises to generalised Kloosterman sums – is studied in subsection 5.4. While until that point we work with a general congruence subgroup, in subsection 5.4 we fix a choice of congruence subgroup (the *Borel congruence subgroup*) in order to write down explicitly the corresponding Kloosterman sums. Other choices of congruence subgroups would be possible, but we do not pursue this here. Finally, in Section 6 we assemble the material from previous sections and we write down the Kuznetsov formula explicitly (in Theorem 2.6.1) by equating the spectral side of the relative trace formula to the geometric side.

Let us briefly sketch some similarities and differences with the Kuznetsov formula for GL_3. These groups both have rank 2, but GL_3 has root system of type A_2 and GSp_4 has root system of type C_2. On the spectral side, the continuous contribution is in both cases given on the one hand by *minimal Eisenstein series*, (that is, attached to the minimal parabolic subgroup), and on the other hand by Eisenstein series induced from non-minimal parabolic subgroups by Maaß forms on GL_2. However, in the case of GL_3, the two non-minimal proper standard parabolic subgroup are *associated*, hence by Langlands theory their Eisenstein series are essentially the same. On the other hand, for GSp_4, we have two distinct non-associated such parabolic subgroups, giving rise to two distinct kinds of Eisenstein series. As for as the geometric side, the Weyl group of GL_3 has six elements, while the Weyl group of GSp_4 has eight. However, it seems interesting to notice that in both case, only the identity element and the longest three elements in the Weyl group have a non-zero contribution, thus eventually giving in total four distinct terms.

Finally, let us mention that Siu Hang Man has independently derived a Kuznetsov formula for Sp_4 using the more classical technique of computing the inner product of Poincaré series, and has derived some applications towards the Ramanujan Conjecture [**SHM21**]. However, because the techniques employed and the final formulae differ, the author believes that our works are complementary rather than redundant. Indeed, the flexibility offered by the adelic framework enables us to treat the test function differently at each place. As a result, by choosing an appropriate test function at finite places, our formula might incorporate the eigenvalues of an arbitrary Hecke operator. Furthermore, at the Archimedean place, we make use of two deep theorems of functional analysis on real reductive groups (namely Harish-Chandra inversion theorem and Wallach's Whittaker inversion theorem) in order to produce an arbitrary Paley-Wiener test function on the spectral side, and relate it explicitly to its transform appearing on the arithmetic side. As a last point, working with GSp_4 instead of Sp_4 enables us to work with a central character.

2. Generalities

Definition 2.2.1. The general symplectic group of degree 2 over a field $\mathbb{F}$ is the group

$$\mathrm{GSp}_4(\mathbb{F}) = \{\mathsf{g} \in \mathrm{Mat}_4(\mathbb{F}) : \exists \mu \in \mathbb{F}^\times, {}^\top\mathsf{g}\mathsf{J}\mathsf{g} = \mu \mathsf{J}\},$$

where $\mathsf{J} = \left[\begin{smallmatrix} & \mathrm{I}_2 \\ -\mathrm{I}_2 & \end{smallmatrix}\right]$ and ${}^\top\mathsf{g}$ denotes the transpose matrix of g.

Note that some authors use different realizations of GSp_4, for instance the realization used in [**RS07**] (to which we refer, along with [**RS16**], for expository details) is conjugated in GL_4 to ours by the matrix $\left[\begin{smallmatrix} & 1 & & \\ 1 & & & \\ & & 1 & \\ & & & 1 \end{smallmatrix}\right]$. From now on we denote $G = \mathrm{GSp}_4$. The scalar $\mu = \mu(\mathsf{g})$ in the definition is called the **multiplier system**. The **Cartan involution** of G is given by $\theta(\mathsf{g}) = {}^\top\mathsf{g}^{-1} = \mu(\mathsf{g})^{-1}\mathsf{J}\mathsf{g}\mathsf{J}^{-1}$. The centre of G consists of all the invertible scalar matrices. We fix a maximal torus in $G(\mathbb{F})$

$$T(\mathbb{F}) = \left\{ \left[\begin{smallmatrix} x & & & \\ & y & & \\ & & tx^{-1} & \\ & & & ty^{-1} \end{smallmatrix}\right] : x, y, t \in \mathbb{F}^\times \right\}.$$

Definition 2.2.2. The symplectic group of degree 2 over a field $\mathbb{F}$ is the group

$$\mathrm{Sp}_4(\mathbb{F}) = \{\mathsf{g} \in G(\mathbb{F}) : \mu(\mathsf{g}) = 1\}.$$

The centre of Sp_4 is $\{\pm 1\}$, and a maximal torus in $\mathrm{Sp}_4(\mathbb{F})$ is given by

$$A(\mathbb{F}) = T(\mathbb{F}) \cap \mathrm{Sp}_4(\mathbb{F}) = \left\{ \left[\begin{smallmatrix} x & & & \\ & y & & \\ & & x^{-1} & \\ & & & y^{-1} \end{smallmatrix}\right] : x, y \in \mathbb{F}^\times \right\}.$$

We denote by $\mathfrak{a}$ the Lie algebra of $A(\mathbb{R})$.

2.1. Weyl group. Let $N(T)$ be the normalizer of T. The Weyl group $\Omega = N(T)/T$ is generated by (the images of) $s_1 = \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix}$ and $s_2 = \begin{bmatrix} & & 1 & \\ & 1 & & \\ -1 & & & \\ & & & 1 \end{bmatrix}$, and consists of the (images of the) eight elements

$$1, s_1, s_2, s_1 s_2 = \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ -1 & & & \end{bmatrix}, s_2 s_1 = \begin{bmatrix} & & & 1 \\ 1 & & & \\ & -1 & & \\ & & 1 & \end{bmatrix},$$

$$s_1 s_2 s_1 = \begin{bmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & -1 & & \end{bmatrix}, s_2 s_1 s_2 = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix}, (s_1 s_2)^2 = \mathsf{J}.$$

2.2. Compact subgroups. A choice of maximal compact subgroup of $G(\mathbb{R})$ is given by the set K_0 of fixed points of the Cartan involution θ. An easy computation shows

$$K_0 = K_\infty \sqcup \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} K_\infty,$$

where

$$K_\infty = \left\{ \left[\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix} \right] : A^\top A + B^\top B = \mathrm{I}_2, A^\top B = B^\top A \right\}.$$

The condition

$$\begin{cases} A^\top A + B^\top B = \mathrm{I}_2 \\ A^\top B = B^\top A \end{cases}$$

is equivalent to $A + iB \in U(2)$, hence K_∞ is isomorphic to $U(2)$.

For each prime p we also consider a (compact open) congruence subgroup $\Gamma_p \subset G(\mathbb{Z}_p)$, with the properties that $\Gamma_p = G(\mathbb{Z}_p)$ for all but finitely many p and the multiplier system μ is surjective from Γ_p to $\mathbb{Z}_p^\times$ for all p. This implies we have the **strong approximation**: setting $\Gamma = K_\infty \prod_p \Gamma_p$, we have

$$G(\mathbb{A}) = G(\mathbb{R})^\circ G(\mathbb{Q}) \Gamma,$$

where $G(\mathbb{R})^\circ$ is the connected component of the identity and $\mathbb{A}$ is the ring of adèles of $\mathbb{Q}$. Moreover we have the **Iwasawa decomposition** $G(\mathbb{A}) = P(\mathbb{A})K$ for all standard parabolic subgroups P, where $K = K_\infty \prod_p G(\mathbb{Z}_p)$.

2.3. Parabolic subgroups. Parabolic subgroups are subgroups such that G/P is a projective variety. Given a minimal parabolic subgroup P_0, **standard parabolic subgroups** (with respect to P_0) are those parabolic subgroups that contain P_0. If P is a standard parabolic subgroup defined over $\mathbb{Q}$, the **Levi decomposition** of P is a semidirect product $P = N_P M_P$ where M_P is a reductive subgroup and N_P is a normal unipotent subgroup. We give here the three non-trivial standard (with respect to our choice of $P_0 = B$) parabolic subgroups and their Levi decompositions.

2.3.1. *Borel subgroup.* The **Borel subgroup** is the minimal standard parabolic subgroup. It is given by

$$B = \begin{bmatrix} * & & * & * \\ * & * & * & * \\ & & * & * \\ & & & * \end{bmatrix} \cap \mathrm{GSp}_4$$

and has Levi decomposition $B = UT = TU$, where

$$U = \left\{ \mathsf{u}(x, a, b, c) : a, b, c, x \in \mathbb{F} \right\},$$

where

$$\mathsf{u}(x, a, b, c) = \begin{bmatrix} 1 & & c & a-cx \\ x & 1 & a & b \\ & & 1 & -x \\ & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ x & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & c & a-cx \\ x & 1 & a-cx & b-x(a-cx) \\ & & 1 & -x \\ & & & 1 \end{bmatrix}.$$

We have the **Bruhat decomposition**

$$G = \coprod_{\sigma \in \Omega} B\sigma B = \coprod_{\sigma \in \Omega} UT\sigma U.$$

For each element σ of the Weyl group, define $U_\sigma = U \cap \sigma U \sigma^{-1}$, and $\overline{U}_\sigma = U \cap \sigma^\top U \sigma^{-1}$. Then we have $U = U_\sigma \overline{U}_\sigma = \overline{U}_\sigma U_\sigma$ and $U_\sigma \cap \overline{U}_\sigma = \{1\}$, and the Bruhat decomposition can be written

$$G = \coprod_{\sigma \in \Omega} \overline{U}_\sigma T \sigma U = \coprod_{\sigma\in\Omega} UT\sigma\overline{U}_{\sigma^{-1}}.$$

We write the Iwasawa decomposition for $\mathrm{Sp}_4(\mathbb{R})$ as follows.

Definition 2.2.3. For every $\mathsf{g} \in \mathrm{Sp}_4(\mathbb{R})$ there is a unique element $A(g) \in \mathfrak{a}$, such that

$$\mathsf{g} \in U \exp(A(g)) K_\infty.$$

2.3.2. *Klingen subgroup.* The **Klingen subgroup** is

$$\mathrm{P_K} = \begin{bmatrix} * & & * & * \\ * & * & * & * \\ * & & * & * \\ & & & * \end{bmatrix} \cap \mathrm{GSp}_4\,.$$

It has Levi decomposition $\mathrm{P_K} = \mathrm{N_K}\,\mathrm{M_K}$, where

$$\mathrm{N_K} = U_{s_2} = \left\{ \begin{bmatrix} 1 & & & y \\ x & 1 & y & z \\ & & 1 & -x \\ & & & 1 \end{bmatrix}, x, y, z \in \mathbb{F} \right\}$$

and

$$\mathrm{M_K} = \left\{ \begin{bmatrix} a & & b & \\ & t & & \\ c & & d & \\ & & & t^{-1}\delta \end{bmatrix}, t \in \mathbb{F}^\times, \delta = \det\left(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\right) \neq 0 \right\}.$$

We have $\mathrm{M_K} \simeq \mathrm{GL}_2 \times \mathrm{GL}_1$, and if $\mathsf{m} = \begin{bmatrix} a & & b & \\ & t & & \\ c & & d & \\ & & & t^{-1}\delta \end{bmatrix} \in \mathrm{M_K}$ and $\mathsf{n} \in \mathrm{N_K}$ we define

$$\mathrm{Proj}_{\mathrm{P_K}}^{\mathrm{GL}_2}(\mathsf{nm}) = \mathrm{Proj}_{\mathrm{P_K}}^{\mathrm{GL}_2}(\mathsf{mn}) = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$$

$$\mathrm{Proj}_{\mathrm{P_K}}^{\mathrm{GL}_1}(\mathsf{nm}) = \mathrm{Proj}_{\mathrm{P_K}}^{\mathrm{GL}_1}(\mathsf{mn}) = t.$$

The centre of $\mathrm{M_K}$ is $\mathrm{A_K} = \left\{ \begin{bmatrix} u & & & \\ & t & & \\ & & u & \\ & & & t^{-1}u^2 \end{bmatrix}, t, u \in \mathbb{F}^\times \right\}$.

2.3.3. *Siegel subgroup.* The **Siegel subgroup** is

$$\mathrm{P_S} = \left[\begin{smallmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{smallmatrix}\right] \cap \mathrm{GSp}_4 .$$

It has Levi decomposition $\mathrm{P_S} = \mathrm{N_S}\,\mathrm{M_S}$, where

$$\mathrm{N_S} = U_{s_1} = \left\{ \left[\begin{smallmatrix} 1 & & x & y \\ & 1 & y & z \\ & & 1 & \\ & & & 1 \end{smallmatrix}\right], x, y, z \in \mathbb{F} \right\}$$

and

$$\mathrm{M_S} = \left\{ \left[\begin{smallmatrix} A & \\ & t^\top A^{-1} \end{smallmatrix}\right], A \in \mathrm{GL}_2(\mathbb{F}), t \in \mathbb{F}^\times \right\} .$$

We have $\mathrm{M_S} \simeq \mathrm{GL}_2 \times \mathrm{GL}_1$, and if $\mathsf{m} = \left[\begin{smallmatrix} A & \\ & t^\top A^{-1} \end{smallmatrix}\right] \in \mathrm{M_K}$ and $\mathsf{n} \in \mathrm{N_K}$ we define

$$\mathrm{Proj}_{\mathrm{P_S}}^{\mathrm{GL}_2}(\mathsf{nm}) = \mathrm{Proj}_{\mathrm{P_S}}^{\mathrm{GL}_2}(\mathsf{mn}) = A$$

$$\mathrm{Proj}_{\mathrm{P_S}}^{\mathrm{GL}_1}(\mathsf{nm}) = \mathrm{Proj}_{\mathrm{P_S}}^{\mathrm{GL}_1}(\mathsf{mn}) = \mu(\mathsf{m}) = t.$$

The centre of $\mathrm{M_S}$ is $\mathrm{A_S} = \left\{ \left[\begin{smallmatrix} u & & & \\ & u & & \\ & & tu^{-1} & \\ & & & tu^{-1} \end{smallmatrix}\right], t, u \in \mathbb{F}^\times \right\}$.

2.4. Lie algebras and characters. Following Arthur [**Art05**], we parametrize the characters of the Levi components of the parabolic subgroups by the duals of the Lie algebras of their centres. We fix $|\cdot|_{\mathbb{A}} = \prod_v |\cdot|_v$ the **standard adelic absolute value**. Let $P = M_P N_P$ be a standard parabolic subgroup, and A_P be the centre of M_P. Then there is a surjective homomorphism

$$H_P : M_P(\mathbb{A}) \to \mathrm{Hom}_{\mathbb{Z}}(X(M_P), \mathbb{R})$$

defined by

$$(H_P(\mathsf{m}))\,(\chi) = \log(|\chi(\mathsf{m})|_{\mathbb{A}}), \tag{2.2}$$

where we write $X(H)$ for the group of homomorphisms (of *algebraic groups*) $H \to \mathrm{GL}_1$ that are defined over $\mathbb{Q}$. On the other hand, we may identify the vector space $\mathrm{Hom}_{\mathbb{Z}}(X(M_P), \mathbb{R})$ with the Lie algebra $\mathfrak{a}_P \oplus \mathfrak{z}$ of $A_P(\mathbb{R})$ (where $\mathfrak{a}_P$ is the Lie algebra of $A_P(\mathbb{R}) \cap \mathrm{Sp}_4(\mathbb{R})$ and $\mathfrak{z}$ is the Lie algebra of the centre). Denote by $\mathfrak{a}_P^*$ the dual of $\mathfrak{a}_P$, by $\mathfrak{a}_P^*(\mathbb{C}) = \mathfrak{a}_P^* \otimes \mathbb{C}$ its complexification, and similarly for $\mathfrak{z}$. If $\nu \in \mathfrak{a}_P^*(\mathbb{C}) \oplus \mathfrak{z}^*(\mathbb{C})$, then the map $M_P(\mathbb{A}) \to \mathbb{C}$:

$$\mathsf{m} \mapsto \exp(\langle \nu, H_P(\mathsf{m}) \rangle), \tag{2.3}$$

where $\langle , \rangle$ is the pairing between $\mathfrak{a}_P^*(\mathbb{C}) \oplus \mathfrak{z}^*(\mathbb{C})$ and $\mathfrak{a}_P(\mathbb{C}) \oplus \mathfrak{z}(\mathbb{C})$, defines a character of $M_P(\mathbb{A})$. Moreover characters of $Z(\mathbb{A})$ correspond to $\mathfrak{z}^*(\mathbb{C})$ while characters that are trivial on $Z(\mathbb{A})$ correspond to $\mathfrak{a}_P^*(\mathbb{C})$. For convenience, when $P = B$ we shall use the notation $\mathfrak{a}_{\mathbb{C}}^*$ for $\mathfrak{a}_P^*(\mathbb{C})$.

3. Representations

The object of this section is to introduce the basic notions and tools of representation theory that shall be needed for the relative trace formula – essentially for the spectral side. The first two subsections are global. In these, we introduce the objects that are the central topic of this work: the notion of Whittaker coefficients, which is what appear in the final formula, and the automorphic kernel, on which the whole relative trace formula approach is built. As explained in the introduction, and as we shall see in more details below, the automorphic kernel K_f induces a certain operator $R(f)$. This corresponds to turning the regular right representation of the group $G(\mathbb{A})$ into a representation of the algebra of "nice" functions f on $G(\mathbb{A})$. By choosing f

appropriately, we can ensure that the spectral expansion of the kernel K_f – and hence, *in fine*, the spectral side of the relative trace formula – can be expressed in an orthonormal system consisting of functions that are fixed by our choice of compact subgroup Γ, and that are moreover eigenfunctions of the operator $R(f)$. This is the object of the last two subsections, where we work locally, at the finite places and at the Archimedean place respectively.

The test function $f : G(\mathbb{A}) \to \mathbb{C}$ that we will eventually choose has the property that it factors as a product over all places $f(g) = \prod_v f_v(g_v)$ where each f_v is a function on $G(\mathbb{Q}_v)$. At unramified primes p, we can choose f_p to correspond to an arbitrary Hecke operator. As a result, our relative trace formula incorporates the corresponding Hecke eigenvalues. As we explain in the last section, at $v = \infty$, the local eigenvalue is given by the so-called spherical transform $\tilde{f}_\infty$. Thus, the spherical transform $\tilde{f}_\infty$ eventually plays the role of the test function on the spectral side. In particular, we discuss what class of test functions $\tilde{f}_\infty$ we can generate subject to our assumptions of f. We also include a discussion of the Whittaker function and of a certain integral transform related to it, that eventually appears in the geometric side of the trace formula. Our objective is to relate it as explicitly as possible to the test function $\tilde{f}_\infty$ appearing on the spectral side.

3.1. Generic representations. In this subsection, we introduce the notion of generic representations and of Whittaker coefficients. We briefly discuss the factorisation property of the Whittaker coefficients. We also relate the genericity of a representation of GSp_4 to that of its restriction to Sp_4 (and similarly with "genericity" replaced with "cuspidality"). This is for the later purpose of using a result of Kim dealing with automorphic representations on Sp_4.

3.1.1. *Generic characters.* A character ψ of $U(\mathbb{Q})\backslash U(\mathbb{A})$ is said to be **generic** if its differential is non-trivial on each of the eigenspaces $\mathfrak{n}_\alpha$ corresponding to simple roots α (where $\mathfrak{n}$ is the Lie algebra of $N_B = U$). Explicitly, if θ is the standard additive character of $\mathbb{A}/\mathbb{Q}$ and $\mathbf{m} = (m_1, m_2) \in (\mathbb{Q}^\times)^2$, generic characters of $U(\mathbb{A})$ are given by

$$\psi_{\mathbf{m}}\left(\begin{bmatrix} 1 & & c & a-cx \\ x & 1 & a & b \\ & & 1 & -x \\ & & & 1 \end{bmatrix}\right) = \theta(m_1 x + m_2 c). \tag{2.4}$$

Note that all generic characters may be obtained from each other by conjugation by an element of T/Z, as we have for all $\mathsf{u} \in U(\mathbb{A})$

$$\psi_{\mathbf{m}}(\mathsf{u}) = \psi_{\mathbf{1}}\left(\mathsf{t}_{\mathbf{m}}^{-1}\mathsf{u}\mathsf{t}_{\mathbf{m}}\right), \tag{2.5}$$

where

$$\mathsf{t}_{\mathbf{m}} = \begin{bmatrix} m_1 & & & \\ & 1 & & \\ & & m_1 m_2 & \\ & & & m_1^2 m_2 \end{bmatrix}. \tag{2.6}$$

In the sequel we may occasionally just write ψ for $\psi_{\mathbf{1}}$.

3.1.2. *Whittaker coefficients and generic representations.* If ϕ is any automorphic form on $G(\mathbb{A})$ and ψ a generic character, the **ψ-Whittaker coefficient** of ϕ is by definition the function $\mathscr{W}_\psi(\phi) : G(\mathbb{A}) \to \mathbb{C}$ given by

$$\mathscr{W}_\psi(\phi)(\mathsf{g}) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \phi(\mathsf{u}\mathsf{g})\psi(\mathsf{u})^{-1}\, d\mathsf{u}. \tag{2.7}$$

ϕ is called ψ-generic if $\mathscr{W}_\psi(\phi)$ is not identically zero as a function of g. Changing variable and using the left-$G(\mathbb{Q})$-invariance of ϕ, note that we have

$$\mathscr{W}_{\psi_{\mathbf{m}}}(\phi)(\mathsf{g}) = \frac{1}{|m_1^4 m_2^3|}\mathscr{W}_{\psi_{\mathbf{1}}}(\phi)(\mathsf{t}_{\mathbf{m}}^{-1}\mathsf{g}).$$

In particular, ϕ is ψ-generic for some generic character ψ if and only if it is ψ-generic for any generic character ψ, henceforth we shall just say ϕ is **generic**. An irreducible automorphic representation (π, V_π) is called generic if V_π contains a generic automorphic form ϕ. Equivalently, every automorphic form in the space of a generic irreducible automorphic representation π is generic, since otherwise the kernel of the map $\phi \mapsto \mathscr{W}_\psi(\phi)$ would be lead to a non-trivial invariant subspace of π, contradicting the irreducibility of π.

If π is an irreducible generic automorphic representation, then the space of Whittaker coefficients $\mathscr{W}_\psi(\phi)$ of elements $\phi \in \pi$ provides a **ψ-global Whittaker model** of π, which is by definition a space W_π of functions $w : G(\mathbb{A}) \to \mathbb{C}$ of moderate growth and satisfying $w(\mathsf{ug}) = \psi(\mathsf{u})w(\mathsf{g})$ for all $\mathsf{u} \in U(\mathbb{A})$, with the property that W_π is stable by right translation by $G(\mathbb{A})$ and moreover the resulting representation is isomorphic to π. Now by the Flath tensor product theorem, the representation π factors as a restricted tensor product $\pi \simeq \bigotimes_v \pi_v$ of local representations of π_v. The existence of a global Whittaker model for π then ensures that each local representation π_v also has a **local Whittaker model** W_{π_v} (whose definition is a local analogue of the global Whittaker model). Moreover, it is known that a local Whittaker model of π_v, if it exists, is unique. It can be seen that this implies that if $\phi \in \pi$ is a factorizable vector then we have for all $g \in G(\mathbb{A})$

$$\mathscr{W}_\psi(\phi)(g) = \prod_v \mathscr{W}_v(\phi_v)(g_v) \tag{2.8}$$

where $\mathscr{W}_v$ is an isomorphism between π_v and W_{π_v}. Now if H_v is a certain subgroup of $G(\mathbb{Q}_v)$ such that $\dim(\pi_v^{H_v}) = 1$ then since, by definition the action of $G(\mathbb{Q}_v)$ on W_{π_v} is isomorphic to π_v, it follows that W_{π_v} contains a unique (up to scalar multiplication) function that is right-invariant by H_v. When the subgroup H_v is implicit, we shall loosely refer to this function as the **local Whittaker function**. In some cases, the local Whittaker function can be determined by purely local methods (*e.g.* by calculating the *Jacquet integral* at the Archimedean place, or by the Casselman-Shalika at finite primes). The point is that if we assume moreover that ϕ is fixed by the subgroup H_v, then the component $\mathscr{W}_v(\phi_v)$ in (2.8) is then given (up to scalar multiplication) by the corresponding Whittaker function. In particular, taking $v = \infty$, we can factor

$$\mathscr{W}_\psi(\phi)(g) = \mathscr{W}_{\pi_\infty}(g_\infty)\mathscr{W}_{\mathrm{fin}}(\phi)(g_{\mathrm{fin}}),$$

where $\mathscr{W}_{\pi_\infty}$ is the Archimedean Whittaker function associated to the representation π_∞ (for which we have explicit formulae by work of Niwa [**Niw95**] and Ishii [**Ish05**], see below) and the product over finite places $\mathscr{W}_{\mathrm{fin}}(\phi)$ contains the arithmetic information. Thus, in the final relative trace formula, we might view $\mathscr{W}_{\pi_\infty}$ as being part of the spectral test function. In fact, as it turns out, the test function arising in our final formula in front of the "arithmetic part" of the Whittaker coefficients, as well as in various integral transforms in the geometric side (at least under Conjecture 2.5.1), is

$$\tilde{f}_\infty(\nu_\pi)\mathscr{W}_{\pi_\infty}(\mathsf{t}_1)\overline{\mathscr{W}_{\pi_\infty}}(\mathsf{t}_2),$$

where ν_π is the spectral parameter of π_∞ and $\mathsf{t}_1, \mathsf{t}_2$ are two fixed diagonal matrices. We refer to subsections 3.2 and 3.4 below for more details.

Let us now return to the definition of the Whittaker coefficients. Since U may as well be viewed as the unipotent part of the minimal parabolic subgroup of Sp_4, we can define the Whittaker coefficients of automorphic forms ϕ on Sp_4 in the exact same way as (2.7), except the argument is restricted to $\mathrm{Sp}_4(\mathbb{A})$. This gives a similar notion of generic automorphic forms and generic representations for Sp_4. Later on, we shall restrict automorphic forms on GSp_4 to Sp_4. Let us briefly explain the corresponding operations on automorphic representations.

Definition 2.3.1. Let (π, V_π) be an automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$ realized by right translation on a subspace of $L^2(G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A}))$. We define a representation $\operatorname{res}\pi$ of $\mathrm{Sp}_4(\mathbb{A})$ as the action of $\mathrm{Sp}_4(\mathbb{A})$ on $\left\{\phi|_{\mathrm{Sp}_4(\mathbb{A})} : \phi \in V_\pi\right\}$. It is a quotient of the restriction $\operatorname{Res}\pi = \pi|_{\mathrm{Sp}_4(\mathbb{A})}$.

The representation $\operatorname{res}\pi$ does not have finite length in general. However, the following shall be useful later on.

Lemma 2.3.1. *Let π be an irreducible automorphic representation π of $G(\mathbb{A})$ that occurs discretely in $L^2(G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A}))$. Then π is generic if and only if* $\operatorname{res}\pi$ *has a generic constituent.*

Proof. Fix a generic character ψ. Note that for any automorphic form ϕ on $G(\mathbb{A})$ we have $\mathscr{W}_\psi(\phi|_{\mathrm{Sp}_4}) = (\mathscr{W}_\psi(\phi))\,|_{\mathrm{Sp}_4}$. From this, it is clear that if $\operatorname{res}\pi$ has a generic constituent then π is generic. Let us show the converse. Assume no constituent of

$\operatorname{res}\pi$ is generic, so for all $\phi \in V_\pi$,

$$\mathscr{W}_\psi(\phi)|_{\mathrm{Sp}_4(\mathbb{A})} = 0.$$

Let $\phi \in \pi$ and $\mathsf{g} \in G(\mathbb{A})$. Then $\pi(\mathsf{g})\phi \in V_\pi$ hence

$$\mathscr{W}_\psi(\phi)(\mathsf{g}) = \mathscr{W}_\psi(\pi(\mathsf{g})\phi)(1) = 0.$$

Thus π is not generic. $\square$

We now prove a similar lemma for the restriction of non-cuspidal representations.

LEMMA 2.3.2. *Let π be an irreducible automorphic representation π of $G(\mathbb{A})$ that occurs discretely in $L^2(G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A}))$. Then π is non-cuspidal if and only if* $\operatorname{res}\pi$ *has no cuspidal constituent.*

PROOF. Recall π is cuspidal if the constant term

$$C_P(\phi)(\mathsf{g}) = \int_{N_P(\mathbb{Q})\backslash N_P(\mathbb{A})} \phi(\mathsf{ug})\, d\mathsf{u}$$

of some (equivalently, any, since π is irreducible) function ϕ in the space of π vanishes identically for all parabolic subgroup P. The exact same proof as Lemma 2.3.1, replacing the generic character ψ by 1 (and U by N_P), shows that π is non-cuspidal if and only if $\operatorname{res}\pi$ has a non-cuspidal component. However, we want to show that if π is non-cuspidal, then $\operatorname{res}\pi$ has no cuspidal component. So suppose that $\operatorname{res}\pi$ has a cuspidal component. This means there is $\phi \in V_\pi$ such that $(C_P(\phi))|_{\mathrm{Sp}_4(\mathbb{A})} = 0$ for all parabolic P. We want to show that $C_P(\phi)$ is identically zero on $\mathrm{GSp}_4(\mathbb{A})$. Now changing variables and using the left-invariance of ϕ under $\mathrm{GSp}_4(\mathbb{Q})$, if $\mathsf{t} \in T(\mathbb{Q})$

then we have $C_P(\phi)(\mathsf{tg}) = C_\phi(\mathsf{g})$. In addition, if $\mathsf{z} \in Z(\mathbb{A})$ then $C_\phi(\mathsf{zg}) = \omega_\pi(\mathsf{z})C_\phi(\mathsf{g})$. Moreover, since π is an admissible representation, ϕ is right-invariant by $\mathrm{GSp}_4(\mathbb{Z}_p)$ for almost all prime p. It follows that there exists a finite set of places S such that for any $\mathsf{g} \in \mathrm{GSp}_4(\mathbb{A})$, if $\mu(\mathsf{g}) \in \mathbb{Q}^\times(\mathbb{A}^\times)^2 \prod_{p \notin S} \mathbb{Z}_p^\times$ then $C_P(\phi)(\mathsf{g}) = 0$. The following lemma concludes the proof. $\square$

LEMMA 2.3.3. *Let S be any finite set of places containing ∞. We have*

$$\mathbb{Q}^\times(\mathbb{A}^\times)^2 \prod_{p \notin S} \mathbb{Z}_p^\times = \mathbb{A}^\times.$$

PROOF. Let $x \in \mathbb{A}^\times$. By strong approximation, we have $x = qu$, with $q \in \mathbb{Q}^\times$ and $u \in \mathbb{R}_{>0} \prod_{p<\infty} \mathbb{Z}_p^\times$. Now by the Chinese Remainders Theorem, there exists an integer $n > 0$ such that for all finite $p \in S$, we have $nu_p \in (\mathbb{Z}_p^\times)^2$. For all $p \notin S$, let $\epsilon_p \in \mathbb{Z}_p^\times$ such that $\epsilon_p n u_p \in (\mathbb{Z}_p^\times)^2$. Define $\epsilon_p = 1$ for $p \in S$. Then $n\epsilon u \in (\mathbb{A}^\times)^2$, and $x = (qn^{-1})(n\epsilon u) \prod_{p \notin S} \epsilon_p^{-1}$. $\square$

3.2. The basic kernel. In this subsection we introduce the basic kernel, and we sketch how to use it to obtain a relative trace formula involving the Whittaker coefficients of an orthonormal basis of Γ-invariant automorphic forms. We first need to introduce the space on which this kernel operates. Recall that we have fixed $G = \mathrm{GSp}_4$, though most of the discussion is valid for arbitrary reductive algebraic groups G over $\mathbb{Q}$.

The group $Z(\mathbb{Q})Z(\mathbb{R})\backslash Z(\mathbb{A})$ is compact and it acts by right translation on the Hilbert space $L^2(G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A}))$. Since $Z(\mathbb{Q})Z(\mathbb{R})\backslash Z(\mathbb{A})$ is abelian, its irreducible representations are characters, thus by Peter-Weyl theorem we have

$$L^2(G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A})) = \bigoplus_{\omega} L^2(G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A}), \omega),$$

where the orthogonal direct sum ranges all characters of $Z(\mathbb{A})$ that are trivial on $Z(\mathbb{Q})Z(\mathbb{R})$, and $L^2(G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A}), \omega)$ is the subspace of $L^2(G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A}))$ of functions $\phi : G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A}) \to \mathbb{C}$ satisfying

$$\phi(\mathsf{gz}) = \omega(\mathsf{z})\phi(\mathsf{g})$$

for all $\mathsf{z} \in Z(\mathbb{A})$. Fix such a character ω. If $f : G(\mathbb{A}) \to \mathbb{C}$ is a measurable function that satisfies

- $f(\mathsf{gz}) = \overline{\omega}(\mathsf{z})f(\mathsf{g})$ for all $\mathsf{z} \in Z(\mathbb{A})$,
- f is compactly supported modulo $Z(\mathbb{A})$,

then we define an operator $R(f)$ on $L^2(G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A}), \omega)$ by

$$R(f)\phi(\mathsf{x}) = \int_{\overline{G}(\mathbb{A})} f(\mathsf{y})\phi(\mathsf{xy})\, d\mathsf{y},$$

where $\overline{G}$ denotes G/Z. By $G(\mathbb{Q})$-invariance of ϕ, we have

$$R(f)\phi(\mathsf{x}) = \int_{\overline{G}(\mathbb{A})} f(\mathsf{x}^{-1}\mathsf{y})\phi(\mathsf{y})d\mathsf{y} = \sum_{\gamma\in\overline{G}(\mathbb{Q})} \int_{\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})} f(\mathsf{x}^{-1}\gamma\mathsf{y})\phi(\mathsf{y})\, d\mathsf{y}.$$

Hence, setting for $\mathsf{x}, \mathsf{y} \in G(\mathbb{A})$

$$K_f(\mathsf{x}, \mathsf{y}) = \sum_{\gamma\in\overline{G}(\mathbb{Q})} f(\mathsf{x}^{-1}\gamma\mathsf{y}), \tag{2.9}$$

we have

$$R(f)\phi(\mathsf{x}) = \int_{\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})} K_f(\mathsf{x},\mathsf{y})\phi(\mathsf{y})\,d\mathsf{y}. \tag{2.10}$$

Now let us argue informally to motivate the more technical actual reasoning which will be done later. Let us pretend that $K_f(\mathsf{x},.)$ belongs to $L^2(G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A}),\omega)$, and that $L^2(G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A}),\omega)$ has a Hilbert orthonormal basis $\mathscr{B}$. Then we would have

$$K_f(\mathsf{x},.) = \sum_{\phi\in\mathscr{B}} \langle K(\mathsf{x},.)|\overline{\phi}\rangle\overline{\phi}.$$

But equation (2.10) says that $\langle K_f(\mathsf{x},.)|\overline{\phi}\rangle = R(f)\phi(\mathsf{x})$. Thus we might expect a spectral expansion of the kernel of the form

$$K_f(\mathsf{x},\mathsf{y}) = \sum_{\phi\in\mathscr{B}} R(f)\phi(\mathsf{x})\overline{\phi(\mathsf{y})}. \tag{2.11}$$

If moreover each element ϕ of our basis $\mathscr{B}$ is an eigenfunction of the operator $R(f)$, say

$$R(f)\phi = \lambda_f(\phi) \tag{2.12}$$

then the above expansion becomes

$$K_f(\mathsf{x},\mathsf{y}) = \sum_{\phi\in\mathscr{B}} \lambda_f(\phi)\phi(\mathsf{x})\overline{\phi(\mathsf{y})}.$$

Finally, integrating $K_f(\mathsf{x}\mathsf{t}_1,\mathsf{y}\mathsf{t}_2)$ on $U\times U$ against a character $\overline{\psi_1(\mathsf{x})}\psi_2(\mathsf{y})$ would then yield a spectral equality involving the Whittaker coefficients and the eigenvalues

$\lambda_f(\phi)$, of the form

$$(2.13)\quad \int_{(U(\mathbb{Q})\backslash U(\mathbb{A}))^2} K_f(\mathsf{x}\mathsf{t}_1,\mathsf{y}\mathsf{t}_2)\overline{\psi_1(\mathsf{x})}\psi_2(\mathsf{y})\,d\mathsf{x}\,d\mathsf{y} = \sum_{\phi\in\mathscr{B}} \lambda_f(\phi)\mathscr{W}_{\psi_1}(\phi)(\mathsf{t}_1)\overline{\mathscr{W}_{\psi_2}(\phi)(\mathsf{t}_2)}.$$

Note that in the last step we need (2.11) to hold not only in the L^2 sense, but pointwise, as $(U(\mathbb{Q})\backslash U(\mathbb{A}))^2$ has measure zero.

Of course, $L^2(G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A}),\omega)$ *does not* have a Hilbert orthonormal basis, due to the presence of continuous spectrum. However, after adding the proper continuous contribution, a spectral expansion of the form (2.11) has been proved by Arthur [**Art78**, pages 928-934], building on the spectral decomposition of the space $L^2(G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A}),\omega)$ by Langlands. We may then reduce from global to local as follows. By general theory, we may choose automorphic forms ϕ appearing in the spectral expansion of the kernel to be factorizable vectors $\phi_\infty \otimes \bigotimes_p \phi_p$. If moreover we take f factorizable, say $f = f_\infty \prod_p f_p$, then the computation of $R(f)\phi$ reduces to the computation of the action of each local component f_v on ϕ_v. By choosing the local components f_v appropriately, we can ensure that each ϕ_v is an eigenvector of the operator corresponding to f_v, so that (2.12) holds. The determination of $\lambda_f(\phi)$ then amounts, at the infinite place, to the study of the spherical transform of f_∞, and at finite places p, of the action of the local Hecke algebra. Specifically, from now on we assume f is as follows.

Assumption 2.1. From now on we assume $f = f_\infty \prod_p f_p$ where

- f_∞ is a smooth, left and right K_∞-invariant and $Z(\mathbb{R})$-invariant function on $G(\mathbb{R})$, whose support is compact modulo the centre and contained in $G^\circ(\mathbb{R}) = \{\mathsf{g} \in G(\mathbb{R}) : \mu(\mathsf{g}) > 0\}$.
- for all prime p, f_p is a left and right Γ_p-invariant function on $G(\mathbb{Q}_p)$, satisfying $f_p(\mathsf{gz}) = \overline{\omega_p}(\mathsf{z}) f(\mathsf{g})$ for all $\mathsf{z} \in Z(\mathbb{Q}_p)$, and compactly supported modulo the centre,
- whenever $\Gamma_p \neq G(\mathbb{Z}_p)$, we have

$$f_p(\mathsf{g}) = \begin{cases} \frac{\overline{\omega_p}(\mathsf{z})}{\mathrm{Vol}(\overline{\Gamma_p})} \text{ if there exists } \mathsf{z} \in Z(\mathbb{Q}_p) \text{ such that } \mathsf{g} \in \mathsf{z}\Gamma_p \\ 0 \text{ otherwise.} \end{cases}$$

Note that this assumption can be fulfilled if and only if we have the following compatibility condition

Assumption 2.2. For each prime p, the restriction of ω_p to $\Gamma_p \cap Z(\mathbb{Q}_p)$ is trivial.

Let us recall the following result [**KL13**, Lemma 3.10].

Proposition 2.3.1. *Let G be a locally compact group, let $K \subset G$ be a closed subgroup, and let π be a unitary representation of G on a Hilbert space V with central character ω. Let $f : G \to \mathbb{C}$ be any left and right K-invariant function satisfying*

- $f(\mathsf{gz}) = \overline{\omega}(\mathsf{z}) f(\mathsf{g})$ *for all* z *in the centre* Z *of* G*,*
- $|f|$ *is integrable on* G/Z*.*

Then the operator $\overline{\pi}(f)$ on V defined by

$$\overline{\pi}(f)v = \int_{G/Z} f(\mathsf{g})\pi(\mathsf{g})v\, d\mathsf{g}$$

has its image in the K-fixed subspace V^K and annihilates the orthogonal complement of this subspace.

Because of Assumption 2.1, this result implies only Γ-fixed automorphic forms having central character ω will appear in the spectral decomposition of K_f. These automorphic forms come from admissible irreducible representations with central character ω and having a Γ-fixed vector. In turn, these representations factor as restricted tensor products of local representations having similar local properties. Furthermore, only those automorphic forms ϕ that are generic will survive the integration against a generic character on U, hence we may restrict attention to local representations that are generic. We now switch to a local set-up and we treat the finite places and the Archimedean place in the next two subsections respectively.

3.3. Non-Archimedean Hecke algebras. Let p be a prime number, and let $f_p : G(\mathbb{Q}_p) \to \mathbb{C}$ be the local component of the function f in Assumption 2.1. Let (π, V) be a unitary representation of $G(\mathbb{Q}_p)$ with central character ω_p. Throughout this section the Haar measure on $G(\mathbb{Q}_p)$ is normalised so that $K_p = G(\mathbb{Z}_p)$ has volume one. By Proposition 2.3.1 we have an operator

$$\overline{\pi}(f_p)v = \int_{\overline{G}(\mathbb{Q}_p)} f(\mathsf{g})\pi(\mathsf{g})v\, d\mathsf{g}. \tag{2.14}$$

acting on the Γ_p-fixed subspace V^{Γ_p} and annihilating the orthogonal complement of this subspace.

First, let us consider the case $\Gamma_p \neq G(\mathbb{Z}_p)$. Then any Γ_p-fixed vector $v \in V$ is also fixed by $\overline{\pi}(f_p)$, since in this case by Assumption 2.1 we have

$$\overline{\pi}(f_p)v = \frac{1}{\mathrm{Vol}(\overline{\Gamma_p})} \int_{\overline{\Gamma}_p} \pi(\mathsf{g})v \, d\mathsf{g} = v.$$

We now turn to the situation $\Gamma_p = K_p = G(\mathbb{Z}_p)$ (in particular, the character ω_p must be unramified). We have have the following [**RS07**, Theorem 7.5.1].

Proposition 2.3.2. *Let* (π, V) *be an irreducible, admissible, representation of* $G(\mathbb{Q}_p)$*. Assume* π *has a non-zero* K_p*-fixed vector. Then* V^{K_p} *has dimension* 1*.*

Remark 2.3.1. *In* [**RS07**, Theorem 7.5.1] *it is assumed* π *has trivial central character. However, in our situation, the fact that* π *has a non-zero* K_p*-fixed vector forces the central character to be unramified. We can thus twist our representation by an unramified character to reduce to the hypothesis of* [**RS07**].

By definition, any non-zero vector ϕ in V^{K_p} is then called the **spherical vector**. Since $\overline{\pi}(f_p)$ acts on V^{K_p} which is one-dimensional, the spherical vector is an eigenvector of $\overline{\pi}(f_p)$. Finally, let us relate the operator $\overline{\pi}(f_p)$ to the action of the unramified Hecke algebra. The **local Hecke algebra** $\mathscr{H}(K_p)$ is the vector space of left and right K_p-invariant compactly supported functions $f : G(\mathbb{Q}_p) \to \mathbb{C}$, endowed with the convolution product

$$(f * h)(\mathsf{g}) = \int_{G(\mathbb{Q}_p)} f(\mathsf{g}\mathsf{x}^{-1})h(\mathsf{x}) \, d\mathsf{x}.$$

If (π, V) is a smooth representation of $G(\mathbb{Q}_p)$, then the Hecke algebra $\mathscr{H}(K_p)$ acts on the K_p-invariant subspace V^{K_p} by

$$\pi(f)v = \int_{G(\mathbb{Q}_p)} f(\mathsf{g})\pi(\mathsf{g})v \, d\mathsf{g}.$$

LEMMA 2.3.4. *Let f be a bi-K_p-invariant function on $G(\mathbb{Q}_p)$, with a (unramified) central character, and compactly supported modulo the centre. There exists a compactly supported bi-K_p-invariant function $\tilde{f}$ on $G(\mathbb{Q}_p)$ and a complete set of representatives $\overline{G}$ of $G(\mathbb{Q}_p)/\mathbb{Q}_p^\times$ satisfying $\tilde{f}(\mathsf{gz}) = f(\mathsf{g})\mathbb{1}_{\mathbb{Z}_p^\times}(\mathsf{z})$ for all $\mathsf{g} \in \overline{G}$ and $\mathsf{z} \in \mathbb{Q}_p^\times$.*

PROOF. By the Cartan decomposition we have

$$G(\mathbb{Q}_p) = \coprod_{\substack{i,j,t \in \mathbb{Z} \\ i \leq j \leq t-j}} K_p \begin{bmatrix} p^i & & & \\ & p^j & & \\ & & p^{t-i} & \\ & & & p^{t-j} \end{bmatrix} K_p.$$

Thus we have

$$G(\mathbb{Q}_p)/\mathbb{Q}_p^\times = \left(\coprod_{\substack{j \geq 0 \\ t \geq 2j}} K_p \begin{bmatrix} 1 & & & \\ & p^j & & \\ & & p^t & \\ & & & p^{t-j} \end{bmatrix} K_p \right) \Big/ \mathbb{Z}_p^\times.$$

Fix a complete set of representatives $\overline{K}_p$ of $K_p/\mathbb{Z}_p^\times$. Then

$$\overline{G} = \coprod_{\substack{j \geq 0 \\ t \geq 2j}} \overline{K}_p \begin{bmatrix} 1 & & & \\ & p^j & & \\ & & p^t & \\ & & & p^{t-j} \end{bmatrix} \overline{K}_p$$

is a complete set of representatives of $G(\mathbb{Q}_p)/\mathbb{Q}_p^\times$. Moreover, defining

$$S = \coprod_{\substack{j \geq 0 \\ t \in \mathbb{Z}}} K_p \begin{bmatrix} 1 & & & \\ & p^j & & \\ & & p^t & \\ & & & p^{t-j} \end{bmatrix} K_p \cap \operatorname{Supp}(f) = (\mathbb{Z}_p^\times \overline{G}) \cap \operatorname{Supp}(f),$$

the function $\tilde{f} = \mathbb{1}_S \times f$ has the desired properties. $\square$

Now the function $\tilde{f}_p$ attached to f_p by Lemma (2.3.4) is an element of the Hecke algebra, and we have $\pi(\tilde{f}_p) = \overline{\pi}(f_p)$, as

$$\pi(\tilde{f}_p)v = \int_{G(\mathbb{Q}_p)} \tilde{f}(\mathsf{g})\pi(g)v\,d\mathsf{g} = \int_{G(\mathbb{Q}_p)/\mathbb{Q}_p^\times} \int_{\mathbb{Q}_p^\times} f_p(\mathsf{g})\mathbb{1}_{\mathbb{Z}_p^\times}(\mathsf{z})\pi(\mathsf{g})v\,d\mathsf{z}\,d\mathsf{g} = \overline{\pi}(f_p)v.$$

We summarize the above discussion in the following proposition.

PROPOSITION 2.3.3. *Let p be a prime number, and f_p be the local component of the function f in Assumption 2.1. Let (π, V) be an irreducible unitary representation of $G(\mathbb{Q}_p)$ with central character ω_p. Then the operator $\overline{\pi}(f_p)$ from Proposition 2.3.1 acts by a scalar $\lambda_\pi(f_p)$ on the Γ_p fixed subspace V^{Γ_p} and annihilates the orthogonal complement of this subspace. Moreover, if $\Gamma_p \neq G(\mathbb{Z}_p)$ then $\lambda_\pi(f_p) = 1$, and if $\Gamma_p = G(\mathbb{Z}_p)$ then $\overline{\pi}(f_p)$ equals the Hecke operator $\pi(\tilde{f}_p)$, where $\tilde{f}_p$ is given by Lemma 2.3.4.*

3.4. The Archimedean representation. In this section we discuss various aspects of the Archimedean component of the automorphic representations involved in the spectral expansion of the automorphic kernel K_f. We first show that in our situation this representation must be an irreducible principal series representation, that is full induced from the Borel subgroup. A representation of $G(\mathbb{R})$ which has a non-zero K_∞-fixed vector is called **spherical**.

PROPOSITION 2.3.4. *Any generic irreducible spherical representation (π, V) of $G(\mathbb{R})$ is a principal series representation.*

The author wishes to thank Ralf Schmidt for communicating the following argument.

Proof. As explained at the end of [**Vog78**], the generic representations are exactly the "large" ones, *i.e.*, those with maximal Gelfand-Kirillov dimension. The Gelfand-Kirillov dimension of all irreducible representations of $\mathrm{GSp}_4(\mathbb{R})$ have been calculated in [**Ver19**, Appendix A]. In particular the maximal Gelfand-Kirillov dimension is 4, and the irreducible large representations are either discrete series or limit of discrete series, induced from the Siegel parabolic subgroup, Langlands quotient of representation induced from the Klingen subgroup, or principal series representations. Now the multiplicity of each possible K_∞-type are described in [**Ver19**, Chapter 4], and among large representations of $\mathrm{GSp}_4(\mathbb{R})$ only principal series representations contain the trivial K_∞-type. $\square$

It is then known by [**Ver19**, Chapter 4] that the trivial K_∞-type occurs in π with multiplicity one, that is to say there is a unique (up to scalar multiplication) K_∞-fixed vector in the space V. Moreover, π has a unique **Whittaker model**, and the image of a non-zero K_∞-fixed vector is by definition given by the **Whittaker function**. The Whittaker function is an eigenfunction of the centre of the universal enveloping algebra, which acts as an algebra of differential operators. One may then obtain a system of partial differential equations characterizing the Whittaker function, and compute it explicitly. The Whittaker function may also be computed by the mean of the Jacquet integral. This has been done by Niwa [**Niw95**] and Ishii [**Ish05**]. We shall return to this in § 3.4.3 below.

3.4.1. *The spherical transform.* In this section we discuss the spherical transform for $\mathrm{Sp}_4(\mathbb{R})$, which in some sense is the Archimedean analogue of the Hecke operators studied in subsection 3.3. Note that the arguments of this section work, with the required modifications, for an arbitrary real connected semisimple Lie group with finite centre (see [**Hel84**]). We normalize the Haar measure on $\mathrm{Sp}_4(\mathbb{R})$ so that K_∞ has measure 1. If h is any bi-K_∞-invariant compactly supported function on $\mathrm{Sp}_4(\mathbb{R})$, its **spherical transform** is the function $\tilde{h}$ defined on $\mathfrak{a}^*(\mathbb{C})$ by

$$\tilde{h}(\nu) = \int_{\mathrm{Sp}_4(\mathbb{R})} h(\mathsf{g})\phi_{-\nu}(\mathsf{g})\, d\mathsf{g}, \tag{2.15}$$

where

$$\phi_{-\nu}(\mathsf{g}) = \int_{K_\infty} e^{\langle \rho-\nu, A(\mathsf{kg})\rangle}\, d\mathsf{k} \tag{2.16}$$

is the **spherical function** with parameter $-\nu$ (here ρ is the half-sum of positive roots).

Proposition 2.3.5. *Let f_∞ be the Archimedean component of the function f in Assumption 2.1. Let (π, V) be a generic irreducible unitary representation representation of $G(\mathbb{R})$ with trivial central character. Then the operator $\overline{\pi}(f_\infty)$ from Proposition 2.3.1 acts by a scalar $\lambda_\pi(f_\infty)$ on the K_∞ fixed subspace V^{K_∞} and annihilates the orthogonal complement of this subspace. Moreover, provided this subspace V^{K_∞} is non zero, then π is a principal series representation, and $\lambda_\pi(f_\infty) = \tilde{f}_\infty(-\nu)$, where $\tilde{f}_\infty$ is the spherical transform of f_∞ and ν is the spectral parameter of π.*

Proof. If V^{K_∞} is zero then by Proposition 2.3.1 the statement is vacuous. Assume now π has a non-zero fixed vector. By Proposition 2.3.4, π is then a principal series. Then V^{K_∞} is one-dimensional, so if v is any K_∞-fixed vector in V then we have

$$\pi(f_\infty)v = \lambda_\pi(f_\infty)v \tag{2.17}$$

for some complex number $\lambda_\pi(f)$. Since π is induced by a character of the Borel subgroup, to compute the eigenvalue $\lambda_\pi(f)$, we may realize π as acting by right translation on a space of functions ϕ satisfying for all $\mathsf{g} \in G(\mathbb{R})$, $\mathsf{u} \in U(\mathbb{R})$ and $\mathsf{a} \in T^+(\mathbb{R})$

$$\phi(\mathsf{uag}) = e^{\langle \rho+\nu, \log(\mathsf{a})\rangle}\phi(\mathsf{g}), \tag{2.18}$$

where $\nu \in \mathfrak{a}^*(\mathbb{C})$ is the **spectral parameter** of π. We may view a $Z(\mathbb{R})$-invariant function supported on $G(\mathbb{R})^+$ as a function on $\mathrm{Sp}_4(\mathbb{R})$, so the operator $\overline{\pi}(f)$ of Proposition 2.3.1 is given by

$$\overline{\pi}(f_\infty)v = \int_{\overline{G}(\mathbb{R})} f_\infty(\mathsf{g})\pi(\mathsf{g})v \, d\mathsf{g} = \int_{\mathrm{Sp}_4(\mathbb{R})} f_\infty(\mathsf{g})\pi(\mathsf{g})v \, d\mathsf{g}. \tag{2.19}$$

If ϕ is a non-zero K_∞-fixed function satisfying (2.18) then because of the Iwasawa decomposition we must have $\phi(1) \neq 0$. Using the integration formula [**Hel84**, Ch. I Corollary 5.3] and right-K_∞ invariance we may compute

$$\begin{aligned} \pi(f_\infty)\phi(1) &= \int_{\mathrm{Sp}_4(\mathbb{R})} f_\infty(\mathsf{g})\pi(\mathsf{g})\phi(1)\, d\mathsf{g} \\ &= \int_{K_\infty}\int_{UA^+} f_\infty(\mathsf{au})\phi(\mathsf{au})\, d\mathsf{a}\, d\mathsf{u}\, d\mathsf{k} = \int_{UA^+} f_\infty(\mathsf{au})e^{\langle \rho_{\mathrm{B}}+\nu, \log(\mathsf{a})\rangle}\, d\mathsf{a}\, d\mathsf{u}\, \phi(1), \end{aligned}$$

where A^+ is the subgroup of $A(\mathbb{R})$ with positive diagonal entries. Therefore, using the Iwasawa decomposition and left-K_∞ invariance of f_∞, the eigenvalue $\lambda_\pi(f)$ is given by

$$\begin{aligned}\lambda_\pi(f) &= \int_{\mathrm{Sp}_4(\mathbb{R})} f_\infty(\mathsf{g}) e^{\langle \rho+\nu, A(\mathsf{g})\rangle}\, d\mathsf{g} \\ &= \int_{K_\infty}\int_{\mathrm{Sp}_4(\mathbb{R})} f_\infty(\mathsf{g}) e^{\langle \rho+\nu, A(\mathsf{kg})\rangle}\, d\mathsf{g}\, d\mathsf{k} = \int_{\mathrm{Sp}_4(\mathbb{R})} f_\infty(\mathsf{g})\phi_\nu(\mathsf{g})\, d\mathsf{g} = \tilde{f}_\infty(-\nu).\end{aligned}$$

□

The spherical transform $\tilde{f}_\infty$ will thus play the role of the test function on the spectral side of our formula. On the other hand, the geometric side will involve some different integral transforms of our test function f_∞. It is therefore natural to investigate the analytic properties of $\tilde{f}_\infty$, and to seek to recover f_∞ from $\tilde{f}_\infty$. This can be achieved by the Paley-Wiener theorem and Harish-Chandra inversion theorem.

3.4.2. *The Paley-Wiener theorem and Harish-Chandra inversion theorem.* The material in this section is taken from [**Hel84**]. As in 3.4.1, the arguments are valid for arbitrary real connected semisimple Lie groups with finite centre. Let us introduce a bit of notation. We denote by $\langle , \rangle$ the Killing form on the Lie algebra of $\mathrm{Sp}_4(\mathbb{R})$, and we define for each $\nu \in \mathfrak{a}^*$ a vector $A_\nu \in \mathfrak{a}$ by $\nu(H) = \langle A_\nu, H\rangle$ for all $H \in \mathfrak{a}$. We then define $\langle \lambda, \nu\rangle = \langle A_\lambda, A_\nu\rangle$. We define $\mathfrak{a}_+$ as the subset of elements $H \in \mathfrak{a}$ satisfying $\alpha(H) > 0$ for all $\alpha \in \Phi_{\mathrm{B}}$, and $\mathfrak{a}_+^* = \{\nu \in \mathfrak{a} : A_\nu \in \mathfrak{a}_+\}$. Explicitly the Killing form is given by $\langle X, Y\rangle = 6Tr(XY)$ and $\mathfrak{a}_+ = \left\{ \begin{bmatrix} x & & & \\ & y & & \\ & & -x & \\ & & & -y \end{bmatrix} : 0 < x < y \right\}$.

Harish-Chandra's c-**function** captures the asymptotic behaviour of the spherical function and it gives the Plancherel measure. More precisely, by Theorem 6.14

of [**Hel84**, Chap. IV], if $H \in \mathfrak{a}^+$ and $\nu \in \mathfrak{a}_+^*$ then we have

$$\lim_{t\to+\infty} e^{\langle -\nu+\rho, tH\rangle}\phi_{-i\nu}(\exp(tH)) = c(-i\nu).$$

Moreover, $c(\nu)$ is given, for $\nu \in \mathfrak{a}_+^*$, by the absolutely convergent integral

$$c(\nu) = \int_{U(\mathbb{R})} e^{\langle \nu+\rho, A(\mathsf{Ju})\rangle}\, d\mathsf{u}, \tag{2.20}$$

where the measure $d\mathsf{u}$ is normalized so that $c(\rho) = 1$, and has meromorphic continuation to $\mathfrak{a}^*(\mathbb{C})$ given in our situation by the expression

$$c(-i\nu) = c_0 \prod_{\alpha\in\Phi} \frac{2^{-\langle i\nu,\alpha_0\rangle}\Gamma(\langle i\nu, \alpha_0\rangle)}{\Gamma\left(\frac{\frac{3}{2}+\langle i\nu,\alpha_0\rangle}{2}\right)\Gamma\left(\frac{\frac{1}{2}+\langle i\nu,\alpha_0\rangle}{2}\right)},$$

where Φ is the set of roots, $\alpha_0 = \frac{\alpha}{\langle\alpha,\alpha\rangle}$ and the constant c_0 is such that $c(\rho) = 1$. Using the duplication formula $\Gamma(z)\Gamma(z+\frac{1}{2}) = \pi^{\frac{1}{2}}2^{1-2z}\Gamma(2z)$, we can rewrite this as

$$c(-i\nu) = \frac{c_0}{4\pi^2}\prod_{\alpha\in\Phi}\frac{\Gamma(\langle i\nu,\alpha_0\rangle)}{\Gamma(\frac{1}{2}+\langle i\nu,\alpha_0\rangle)}.$$

We then have the following theorems

THEOREM 2.3.1 (Paley-Wiener theorem). *Let $\mathscr{H}^R(\mathfrak{a}_{\mathbb{C}}^*)$ the set of Ω-invariant entire functions h on $\mathfrak{a}_{\mathbb{C}}^*$ such that for all $N \geq 0$ we have*

$$h(\nu) \ll_N (1+|\nu|)^{-N} e^{R|\Re(\nu)|}.$$

Let

$$\mathscr{H}(\mathfrak{a}_{\mathbb{C}}^*) = \bigcup_{R>0} \mathscr{H}^R(\mathfrak{a}_{\mathbb{C}}^*).$$

Then the spherical transform $f \mapsto \tilde{f}$ *is a bijection from* $C_c^\infty(K_\infty \backslash \operatorname{Sp}_4(\mathbb{R})/K_\infty)$ *to* $\mathscr{H}(\mathfrak{a}_\mathbb{C}^*)$.

Theorem 2.3.2 (Inversion theorem). *There is a constant* c *such that for every function* $f \in C_c^\infty(K\backslash \operatorname{Sp}_4(\mathbb{R})/K)$ *we have for all* $\mathsf{g} \in \operatorname{Sp}_4(\mathbb{R})$

$$cf(\mathsf{g}) = \int_{\mathfrak{a}^*} \tilde{f}(-i\nu)\phi_{-i\nu}(\mathsf{g}) \frac{d\nu}{c(i\nu)c(-i\nu)}. \tag{2.21}$$

Remark 2.3.2. *The constant* c *may be worked out by Exercise C.4 of* [**Hel84**, Chap. IV].

Remark 2.3.3. *Using formulae* $\Gamma(iz)\Gamma(-iz) = \frac{\pi}{z \sinh \pi z}$ *and* $\Gamma(\frac{1}{2} - iz)\Gamma(\frac{1}{2} + iz) = \frac{\pi}{\cosh \pi z}$, *the Plancherel measure is given by*

$$\frac{d\nu}{c(i\nu)c(-i\nu)} = \frac{16\pi^4}{c_0^2} \prod_{\alpha \in \Phi} \langle \nu, \alpha_0 \rangle \tanh(\pi \langle \nu, \alpha_0 \rangle) d\nu. \tag{2.22}$$

3.4.3. *The Whittaker function and the Jacquet integral.* As mentioned above, the Whittaker function is a non-zero K_∞-fixed vector in the Whittaker model, and it is unique up to scaling. It is given by (meromorphic continuation of) the Jacquet integral. Namely, if ψ is a generic character of $U(\mathbb{R})$, we have the **Jacquet integral**

$$W(\nu, \mathsf{g}, \psi) = \int_{U(\mathbb{R})} e^{\langle \rho + \nu, A(\mathsf{Jug}) \rangle} \overline{\psi(\mathsf{u})}\, d\mathsf{u}. \tag{2.23}$$

The Jacquet integral converges absolutely for $\Re(\nu) \in \mathfrak{a}_+^*$, as may be seen by using the absolute convergence of (2.20) and computing

$$|W(\nu, \mathsf{g}, \psi)| \leq \int_{U(\mathbb{R})} |e^{\langle \nu + \rho, A(\mathsf{Jug}) \rangle}|\, d\mathsf{u} = e^{\langle \rho - \Re(\nu), A(\mathsf{g}) \rangle} c(\Re(\nu)). \tag{2.24}$$

Moreover, it has meromorphic continuation to all $\nu \in \mathfrak{a}_{\mathbb{C}}^*$. Ishii [**Ish05**] computed explicit integral representations for the normalized Jacquet integral

$$\mathscr{W}(\nu, \mathsf{g}, \psi) = \frac{1}{4\pi^2} \prod_{\alpha \in \Phi} \Gamma\left(\frac{1}{2} + \langle \nu, \alpha_0 \rangle\right) W(\nu, \mathsf{g}, \psi), \tag{2.25}$$

namely if $\mathsf{a} = \begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & a_1^{-1} & \\ & & & a_2^{-1} \end{bmatrix} \in A^+$ then for any $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ we have (note the different choice of minimal parabolic subgroup)

$$\begin{aligned} \mathscr{W}(\nu, \mathsf{a}, \psi) = 2a_1 a_2^2 \int_0^\infty \int_0^\infty & K_{\frac{\nu_2-\nu_1}{2}}(2\pi v_1) K_{\frac{\nu_1+\nu_2}{2}}(2\pi v_2) \\ & \times \exp\left(-\pi\left(\frac{a_2^2}{v_1 v_2} + \frac{v_1 v_2}{a_1^2} + a_1^2\left(\frac{v_1}{v_2} + \frac{v_2}{v_1}\right)\right)\right) \frac{dv_1 dv_2}{v_1 v_2}. \end{aligned} \tag{2.26}$$

This implies in particular that the normalized Jacquet integral satisfies the functional equations

$$\mathscr{W}(\sigma \cdot \nu, \mathsf{g}, \psi) = \mathscr{W}(\nu, \mathsf{g}, \psi) \tag{2.27}$$

for all $\sigma \in \Omega$. If $\mathsf{t} \in A^+$ and if we denote by ψ_{t} the character $\psi_{\mathsf{t}}(\mathsf{u}) = \psi(\mathsf{t}^{-1}\mathsf{u}\mathsf{t})$, then it is easy to see (first by a change of variable in the domain where the Jacquet integral is absolutely convergent, then by meromorphic continuation) that

$$W(\nu, \mathsf{g}, \psi_{\mathsf{t}}) = e^{\langle \rho - \nu, \log(\mathsf{t}) \rangle} W(\nu, \mathsf{t}^{-1}\mathsf{g}, \psi). \tag{2.28}$$

3.4.4. *Wallach's Whittaker transform.* Theorem 2.3.3 below is a consequence of [**Wal92**, Ch. 15], which is valid for arbitrary real reductive groups. However in order to avoid introducing additional notation, we stick to the case of $\mathrm{Sp}_4(\mathbb{R})$. Let $C_c^\infty(U \backslash \mathrm{Sp}_4(\mathbb{R})/K, \psi)$ be the space of functions f on $\mathrm{Sp}_4(\mathbb{R})$ satisfying $f(\mathsf{ugk}) =$

$\psi(\mathsf{u})f(\mathsf{g})$ for all $\mathsf{u} \in U(\mathbb{R})$, for all $\mathsf{k} \in K_\infty$ and for all $\mathsf{g} \in \mathrm{Sp}_4(\mathbb{R})$, and such that f is smooth and has a compact support modulo $U(\mathbb{R})$.

Theorem 2.3.3 (Wallach's Whittaker inversion). *For $f \in C_c^\infty(U\backslash \mathrm{Sp}_4(\mathbb{R})/K, \psi)$ define the* **Whittaker transform**

$$W(f)(\nu) = c\int_{A^+} f(\mathsf{a})W(i\nu, \mathsf{a}, \overline{\psi})e^{-2\langle\rho,\log\mathsf{a}\rangle}\,d\mathsf{a},$$

where the constant c is the same as in Theorem 2.3.2. Then we have

$$f = \mathscr{T}(W(f)),$$

where

$$\mathscr{T}(\alpha)(\mathsf{a}) = \int_{\mathfrak{a}^*} \alpha(\nu)W(-i\nu, \mathsf{a}, \psi)\frac{d\nu}{c(i\nu)c(-i\nu)}.$$

3.4.5. *An integral transform.* Let $\mathsf{g} \in G(\mathbb{R})$, $\mathsf{t} \in A^+$ and let ψ be a generic character of $U(\mathbb{R})$. When dealing with the geometric side of the relative trace formula, we shall be interested in the integral

$$I(f_\infty) = \int_{U(\mathbb{R})} f_\infty(\mathsf{tug})\overline{\psi}(\mathsf{u})\,d\mathsf{u}.$$

Using expression (2.16) and applying Theorem 5.20 of [**Hel84**, Ch.I] that relates integration on K_∞ to integration on $U(\mathbb{R})$, one may establish the following identity for all $\nu \in \mathfrak{a}_{\mathbb{C}}^*$

$$\phi_\nu(\mathsf{g}) = \int_{U(\mathbb{R})} e^{\langle\rho+\nu, A(\mathsf{Jug})\rangle}e^{\langle\rho-\nu, A(\mathsf{Ju})\rangle}\,d\mathsf{u}. \tag{2.29}$$

From this identity, ignoring convergence issues and treating integrals as if they were absolutely convergent, one may heuristically expect the following

$$\int_{U(\mathbb{R})} \phi_\nu(\mathsf{tug})\overline{\psi(\mathsf{u})}\, d\mathsf{u} = W(\nu, \mathsf{g}, \psi)W(-\nu, \mathsf{t}^{-1}, \overline{\psi}). \tag{2.30}$$

However, the domain of absolute convergence of the two Jacquet integral in the right hand side are complementary from each other, and the integral in the left hand side is likely not absolutely convergent, making such a result, where the left hand side is (optimistically) a semi-convergent integral and the right-hand side is defined by meromorphic continuation, likely difficult to prove.

Carrying on with this heuristic and using Theorem 2.3.2, let us write

$$\begin{aligned} cI(f_\infty) &= \int_{U(\mathbb{R})}\int_{\mathfrak{a}^*} \tilde{f}_\infty(-i\nu)\phi_{-i\nu}(\mathsf{tug})\frac{d\nu}{c(i\nu)c(-i\nu)}\overline{\psi}(\mathsf{u})\, d\mathsf{u} \\ &= \int_{\mathfrak{a}^*} \tilde{f}_\infty(-i\nu)\int_{U(\mathbb{R})} \phi_{-i\nu}(\mathsf{tug})\overline{\psi}(\mathsf{u})d\mathsf{u}\frac{d\nu}{c(i\nu)c(-i\nu)} \\ &= \int_{\mathfrak{a}^*} \tilde{f}_\infty(-i\nu)W(-i\nu, \mathsf{g}, \psi)W(i\nu, \mathsf{t}^{-1}, \overline{\psi})\frac{d\nu}{c(i\nu)c(-i\nu)}. \end{aligned}$$

Unlike (2.30), this equality seems more reasonable. Indeed, the left hand side is absolutely convergent because f_∞ is compactly supported, and in the right hand side $\tilde{f}_\infty$ has rapid decay. We now give a rigorous proof of the following theorem.

Theorem 2.3.4. *Let f_∞ be a smooth, bi-K_∞-invariant, compactly supported function on $\mathrm{Sp}_4(\mathbb{R})$. Let $\mathsf{g}_1, \mathsf{g}_2 \in G(\mathbb{R})$, and let ψ be a generic character of $U(\mathbb{R})$. Then we have*

$$c\int_{U(\mathbb{R})} f_\infty(\mathsf{g}_2\mathsf{u}\mathsf{g}_1)\overline{\psi}(\mathsf{u})\, d\mathsf{u} = \int_{\mathfrak{a}^*} \tilde{f}_\infty(-i\nu)W(-i\nu, \mathsf{g}_1, \psi)W(i\nu, \mathsf{g}_2^{-1}, \overline{\psi})\frac{d\nu}{c(i\nu)c(-i\nu)},$$

where $W(\nu,\cdot,\psi)$ is the ψ-Whittaker function of the principal series with spectral parameter ν.

REMARK 2.3.4. *Note that Wallach's Whittaker inversion theorem holds true for arbitrary real reductive groups. Thus Theorem 2.3.4 also holds true for general real reductive groups (with the relevant notations).*

PROOF. In the variable g_1, both sides transform on the left by $U(\mathbb{R})$ according to ψ, and are K_∞-invariant on the right. Thus by the Iwasawa decomposition, it suffices to prove it for $\mathsf{g}_1 = \mathsf{a} \in A^+$. Similarly, in the variable g_2, both sides transform on the right by $U(\mathbb{R})$ according to ψ, and are K_∞-invariant on the left, thus it suffices to prove it for $\mathsf{g}_2 = \mathsf{t} \in A^+$. Also, by (2.28), we may restrict ourselves to $\mathsf{t} = 1$. With notations of Theorem 2.3.3, we have

$$\int_{\mathfrak{a}^*} \tilde{f}_\infty(-i\nu) W(-i\nu, \mathsf{a}, \psi) W(i\nu, 1, \overline{\psi}) \frac{d\nu}{c(i\nu)c(-i\nu)} = \mathscr{T}(\alpha)(\mathsf{a}),$$

where

$$\alpha(\nu) = \tilde{f}_\infty(-i\nu) W(i\nu, 1, \overline{\psi}).$$

Moreover the map $F : \mathsf{g} \mapsto \int_{U(\mathbb{R})} f_\infty(\mathsf{ug})\overline{\psi}(\mathsf{u})\, d\mathsf{u}$ belongs to $C_c^\infty(U \backslash \mathrm{Sp}_4(\mathbb{R})/K, \psi)$ since f_∞ is smooth and compactly supported. Hence by Wallach's Whittaker inversion it suffices to show that $\alpha = W(F)$, that is for all $\nu \in \mathfrak{a}^*$ we have

$$\alpha(\nu) = \int_{A^+} e^{-2\langle \rho, \log \mathsf{a}\rangle} \int_{U(\mathbb{R})} f_\infty(\mathsf{ua})\overline{\psi}(\mathsf{u})\, d\mathsf{u}\, W(i\nu, \mathsf{a}, \overline{\psi})\, d\mathsf{a}. \tag{2.31}$$

Since both sides are meromorphic in ν, it suffices to show this for $\Re(i\nu) \in \mathfrak{a}_+^*$. In this region, the Jacquet integral $W(i\nu, \mathsf{a}, \overline{\psi}) = \int_{U(\mathbb{R})} e^{\langle \rho + i\nu, A(\mathsf{J}\mathsf{u}_1\mathsf{a})\rangle} \psi(\mathsf{u}_1) d\mathsf{u}_1$ converges absolutely. Hence the integral in (2.31) may then be written as

$$\int_{A^+}\int_{U(\mathbb{R})} f_\infty(\mathsf{au})\overline{\psi}(\mathsf{aua}^{-1})\,d\mathsf{u}\,W(i\nu,\mathsf{a},\overline{\psi})\,d\mathsf{a} = \int_{A^+}\int_{U(\mathbb{R})} f_\infty(\mathsf{au})W(i\nu,\mathsf{au},\overline{\psi})\,d\mathsf{u}\,d\mathsf{a}$$
$$= \int_{A^+}\int_{U(\mathbb{R})} f_\infty(\mathsf{au})\int_{U(\mathbb{R})} e^{\langle\rho+i\nu,A(\mathsf{Ju}_1\mathsf{au})\rangle}\psi(\mathsf{u}_1)\,d\mathsf{u}_1\,d\mathsf{u}\,d\mathsf{a}.$$

Write $\mathsf{Ju}_1 = \mathsf{u}_2\exp(A(\mathsf{Ju}_1))\mathsf{k}_0(\mathsf{Ju}_1)$ with $\mathsf{u}_2 \in U(\mathbb{R})$ and $\mathsf{k}_0(\mathsf{Ju}_1) \in K_\infty$. Then $A(\mathsf{Ju}_1\mathsf{au}) = A(\mathsf{Ju}_1) + A(\mathsf{k}_0\mathsf{au})$. So the integral we have to evaluate becomes

$$\int_{A^+}\int_{U(\mathbb{R})} e^{\langle\rho+i\nu,A(\mathsf{Ju}_1)\rangle}\psi(\mathsf{u}_1)\int_{U(\mathbb{R})} f_\infty(\mathsf{au})e^{\langle\rho+i\nu,A(\mathsf{k}_0(\mathsf{Ju}_1)\mathsf{au})\rangle}\,d\mathsf{u}\,d\mathsf{u}_1\,d\mathsf{a}$$
$$= \int_{A^+}\int_{U(\mathbb{R})} e^{\langle\rho+i\nu,A(\mathsf{Ju}_1)\rangle}\psi(\mathsf{u}_1)\int_{\mathrm{Sp}_4(\mathbb{R})} f_\infty(\mathsf{g})e^{\langle\rho+i\nu,A(\mathsf{k}_0(\mathsf{Ju}_1)\mathsf{g})\rangle}\,d\mathsf{g}\,d\mathsf{u}_1\,d\mathsf{a}$$
$$= \int_{A^+}\int_{U(\mathbb{R})} e^{\langle\rho+i\nu,A(\mathsf{Ju}_1)\rangle}\psi(\mathsf{u}_1)\int_{\mathrm{Sp}_4(\mathbb{R})} f_\infty(\mathsf{g})e^{\langle\rho+i\nu,A(\mathsf{g})\rangle}\,d\mathsf{g}\,d\mathsf{u}_1\,d\mathsf{a}$$
$$= W(i\nu,1,\overline{\psi})\tilde{f}(-i\nu).$$

$\square$

3.4.6. *Estimates for the Whittaker function.* We close this section with some estimates for the Whittaker function to be used later on. We begin with recalling the following estimate for Bessel K functions from [**HM06**, Proposition 7.2].

LEMMA 2.3.5. *Let $\sigma > 0$. For $\Re(\nu) \in [-\sigma,\sigma]$ we have for all $\epsilon > 0$*

$$K_\nu(u) \ll_{\sigma,\epsilon} \begin{cases} (1+|\Im(\nu)|)^{\sigma+\epsilon}u^{-\sigma-\epsilon} & \text{if } \ 0 < u \le 1+\frac{\pi}{2}|\Im(\nu)|, \\ u^{-\frac{1}{2}}e^{-u} & \text{if } \ u > 1+\frac{\pi}{2}|\Im(\nu)|. \end{cases}$$

In the following lemma, we have only used trivial bounds and haven't sought for optimality.

LEMMA 2.3.6. *Let* $\sigma > 0$. *Let* $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ *with* $-\sigma < \frac{\Re(\nu_1-\nu_2)}{2}, \frac{\Re(\nu_1+\nu_2)}{2} < \sigma$ *and* $\mathsf{a} \in A^+$. *For simplicity, set* $r_1 = \frac{|\Im(\nu_1-\nu_2)|}{2}$ *and* $r_2 = \frac{|\Im(\nu_1+\nu_2)|}{2}$. *Then for all* $\epsilon > 0$ *we have*

$$\begin{aligned}\mathscr{W}(\nu, \mathsf{a}, \psi) &\ll (1+r_1)^{\sigma+1+\epsilon}(1+r_2)^{\sigma+1+\epsilon}a_1a_2^{-2\sigma-\epsilon}\\ &+ (1+r_1)^{-\frac{3}{2}}(1+r_2)^{-\frac{3}{2}}a_1a_2^2\\ &+ (1+r_1)^{\sigma+\epsilon}(1+r_2)^{-(\sigma+\frac{5}{2}+\epsilon)}a_1^{-2\sigma-1-\epsilon}a_2^2\\ &+ (1+r_1)^{-(\sigma+\frac{5}{2}+\epsilon)}(1+r_2)^{\sigma+\epsilon}a_1^{-2\sigma-1-\epsilon}a_2^2.\end{aligned}$$

PROOF. This follows from the explicit integral representation (2.26) together with Lemma 2.3.5. □

PROPOSITION 2.3.6. *Let* $\mathsf{a} \in A^+$. *Then, for* $\Re(\nu)$ *small enough we have for all* $\epsilon > 0$

$$W(\nu, \mathsf{a}, \psi) \ll_{\Re(\nu),\mathsf{a}} \prod_{\alpha\in\Phi} |\langle \Im(\nu), \alpha_0\rangle|^{2|\langle \Re(\nu),\alpha_0\rangle|+\epsilon}.$$

PROOF. First observe that, if $\Re(\nu) \in \mathfrak{a}_+^*$, then the claim follows from the trivial bound (2.24). Next, if $\Re(\nu)$ belongs to any open Weyl chamber, there is $\sigma \in \Omega$ such that $Re(\sigma \cdot \nu) \in \mathfrak{a}_+^*$. The functional equation (2.27) gives

$$W(\nu, \mathsf{a}, \psi) = \prod_{\alpha\in\Phi} \frac{\Gamma\left(\frac{1}{2} + \langle \sigma\cdot\nu, \alpha_0\rangle\right)}{\Gamma\left(\frac{1}{2} + \langle \nu, \alpha_0\rangle\right)} W(\sigma\cdot\nu, \mathsf{a}, \psi).$$

Since the Weyl group acts by permutation on the set of (positive and negative) roots, the product can be written as

$$\prod_{\alpha\in\Phi_\sigma}\frac{\Gamma\left(\frac{1}{2}-\langle\nu,\alpha_0\rangle\right)}{\Gamma\left(\frac{1}{2}+\langle\nu,\alpha_0\rangle\right)}\ll\prod_{\alpha\in\Phi_\sigma}|\langle\Im(\nu),\alpha_0\rangle|^{-2\langle\Re(\nu),\alpha_0\rangle},$$

where Φ_σ is the set of positive roots whose image by σ is a negative root and we have used that $|\Gamma(x+iy)|\sim\sqrt{2\pi}e^{-\frac{\pi}{2}|y|}|y|^{x-\frac{1}{2}}$ as $|y|\to\infty$ and that the numerator has no poles because $\Re(\nu)$ is small enough. But if $\sigma\cdot\alpha$ is a negative root then we have $\langle\Re(\nu),\alpha_0\rangle<0$ and so we are done in this case again. Finally, if $\Re(\nu)$ belongs to a wall of a Weyl chamber, by Lemma 2.3.6 we may apply the Phragmén-Lindelöf principle to deduce the result. $\square$

4. Eisenstein series and the spectral decomposition

The goal of Eisenstein series is to describe the continuous spectrum. The latter is an orthogonal direct sum over standard parabolic subgroups P, each summand of which is a direct integral parametrized by $i\mathfrak{a}_P^*$. Eisenstein series will give intertwining operators from some representation induced from M_P to the corresponding part of the continuous spectrum. One thus wants to define $E(\cdot,\phi,\nu)$ for ϕ in the space $\mathscr{H}_P$ of the aforementioned induced representation, and for $\nu\in i\mathfrak{a}_P^*$. Because of convergence issues, one originally defines $E(\cdot,\phi,\nu)$ for ϕ lying a certain dense space of automorphic forms $\mathscr{H}_P^0\subset\mathscr{H}_P$ and for $\nu\in\mathfrak{a}_P^*(\mathbb{C})$ with large enough real part. The definition is then extended to all ϕ in the completion of $\mathscr{H}_P^0$ and to all $\nu\in\mathfrak{a}_P^*(\mathbb{C})$. Our exposition follows Arthur, and in particular [**Art05**]. As before, we are setting $G=\mathrm{GSp}_4$, but the results discussed here hold (with the necessary modifications) in more generality, namely for any connected reductive group over $\mathbb{Q}$.

4.1. Definition of Eisenstein series. Fix a standard parabolic subgroup $P = N_P M_P$ throughout this section, and let A_P be the centre of M_P, and $A_P^+(\mathbb{R})$ be the connected component of 1 in $A_P(\mathbb{R})$. Let $R_{M_P,\mathrm{disc}}$ be the restriction of the right regular representation of $M_P(\mathbb{A})$ on the subspace of $L^2(M_P(\mathbb{Q})A_P^+(\mathbb{R})\backslash M_P(\mathbb{A}))$ that decompose discretely. For $\nu \in \mathfrak{a}_P^*(\mathbb{C})$, consider the tensor product given by $R_{M_P,\mathrm{disc},\nu}(\mathsf{x}) = R_{M_P,\mathrm{disc}}(\mathsf{x})e^{\langle \nu, H_{M_P}(\mathsf{x})\rangle}$ for $\mathsf{x} \in M_P(\mathbb{A})$. The continuous spectrum is described via the Eisenstein series in terms of the induced representation

$$\mathscr{I}_P(\nu) = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\mathrm{I}_{N_P(\mathbb{A})} \otimes R_{M_P,\mathrm{disc},\nu}).$$

The space of this induced representation is independent of ν and is given in the following definition.

Definition 2.4.1. With notations as above, define $\mathscr{H}_P$ to be the Hilbert space obtained by completing the space $\mathscr{H}_P^0$ of functions

$$\phi : N_P(\mathbb{A})M_P(\mathbb{Q})A_P^+(\mathbb{R})\backslash G(\mathbb{A}) \to \mathbb{C} \tag{2.32}$$

such that

(1) for any $\mathsf{x} \in G(\mathbb{A})$, the function $M_P(\mathbb{A}) \to \mathbb{C}, \mathsf{m} \mapsto \phi(\mathsf{m}\mathsf{x})$ is $\mathscr{Z}_{M_P}$-finite, where $\mathscr{Z}_{M_P}$ is the centre of the universal enveloping algebra of $\mathfrak{M}_P(\mathbb{C})$,
(2) ϕ is right K-finite,
(3) $\|\phi\|^2 = \int_K \int_{A_P(\mathbb{R})^+ M_P(\mathbb{Q})\backslash M_P(\mathbb{A})} |\phi(\mathsf{m}\mathsf{k})|^2 \, d\mathsf{m} \, d\mathsf{k} < \infty$.

Then the representation $\mathscr{I}_P(\nu)$ acts on $\mathscr{H}_P$ via

$$(\mathscr{I}_P(\nu,\mathsf{y})\phi)(\mathsf{x}) = \phi(\mathsf{xy}) \exp(\langle \nu + \rho_P, H_P(\mathsf{xy})\rangle) \exp(-\langle \nu + \rho_P, H_P(\mathsf{x})\rangle)$$

for $\mathsf{x},\mathsf{y} \in G(\mathbb{A})$, and is unitary for $\nu \in i\mathfrak{a}_P(\mathbb{C})$.

We now define the Eisenstein series attached to P. We extend H_P to $P(\mathbb{Q})\backslash G(\mathbb{A})$ by setting $H_P(\mathsf{nmk}) = H_P(\mathsf{m})$ ($\mathsf{n} \in N_P, \mathsf{m} \in M_P, \mathsf{k} \in K$), therefore the expression in the following proposition is well defined.

Proposition 2.4.1. *For $\nu \in \mathfrak{a}_P^*(\mathbb{C})$ with large enough real part, if $\mathsf{x} \in G(\mathbb{A})$ and $\phi \in \mathscr{H}_P^0$, the* **Eisenstein series**

$$E(\mathsf{x},\phi,\nu) = \sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \phi(\delta\mathsf{x}) \exp(\langle \nu + \rho_P, H_P(\delta\mathsf{x})\rangle)$$

converges absolutely.

The relation between the induced representation $\mathscr{I}_P(\nu,\mathsf{y})$ on $\mathscr{H}_P$ defined above and the regular right representation on the corresponding space of Eisenstein series is given formally by

$$E(\mathsf{x},\mathscr{I}_P(\nu,\mathsf{y})\phi,\nu) = E(\mathsf{xy},\phi\,\nu).$$

Langlands provided analytic continuation of Eisenstein series, as well as the spectral decomposition of $L^2(Z(\mathbb{R})G(\mathbb{Q})\backslash G(\mathbb{A}))$. The latter gives a decomposition of the right regular representation R as direct sum over association classes of parabolic subgroups. The class of G, viewed as a parabolic subgroup itself, gives the **discrete spectrum**. It consists on one hand of cuspidal functions on $Z(\mathbb{R})G(\mathbb{Q})\backslash G(\mathbb{A})$ and on the other hand of residues of Eisenstein series attached to proper parabolic subgroups. The contribution of the other classes is given by direct integrals of corresponding induced representations and gives the **continuous spectrum**. For a nice survey, we refer the reader to [**Art05**]. We now describe explicitly the Eisenstein series that are relevant for us.

4.2. Action of the centre and of the compact Γ. Since our test function f is bi-Γ-invariant and has central character ω, Eisenstein series occurring in the spectral expansion of its kernel K_f are only from the subspaces of $\mathscr{H}_P$ satisfying similar properties (see Lemma 2.4.5 below for a formal justification). Using the Peter-Weyl Theorem, we can further deduce:

LEMMA 2.4.1. *Let P be a standard parabolic subgroup and A_P it centre. Let $\mathscr{H}_P^\Gamma(\omega)$ be the closed subspace of $\mathscr{H}_P$ consisting in functions ϕ such that for all $\mathsf{z} \in Z(\mathbb{A})$ and $\mathsf{k} \in \Gamma$, we have $\phi(\mathsf{zgk}) = \omega(\mathsf{z})\phi(\mathsf{g})$. Then*

$$\mathscr{H}_P^\Gamma(\omega) = \bigoplus_\chi \mathscr{H}_P^\Gamma(\chi) \tag{2.33}$$

where the χ-orthogonal direct sum ranges over characters of $A_P^+(\mathbb{R})A_P(\mathbb{Q})\backslash A_P(\mathbb{A})$ that coincide with ω on $Z(\mathbb{A})$, and $\mathscr{H}_P^\Gamma(\chi)$ is the subspace of $\mathscr{H}_P^\Gamma(\omega)$ consisting in functions ϕ such that for all $\mathsf{z} \in A_P(\mathbb{A})$, $\phi(\mathsf{zg}) = \chi(\mathsf{z})\phi(\mathsf{g})$.

4.3. Explicit description of Eisenstein series. Write the decomposition of $R_{M_P,\mathrm{disc}}$ into irreducible representations $\pi = \bigotimes_v \pi_v$ of $M_P(\mathbb{A})/A_P(\mathbb{R})^+$ as $R_{M_P,\mathrm{disc}} = \bigoplus_\pi \pi = \bigoplus_\pi (\bigotimes_v \pi_v)$. Then we have

$$\mathscr{I}_P(\nu) = \bigoplus_\pi \mathscr{I}_P(\pi_\nu) = \bigoplus_\pi \left(\bigotimes_v \mathscr{I}_P(\pi_{v,\nu}) \right).$$

Moreover the representation space of each $\mathscr{I}_P(\pi_\nu)$ does not depend on ν. Hence, to describe the spaces $\mathscr{H}_P^\Gamma(\chi)$ it suffices to describe

- the irreducible representations π with central character χ occurring $R_{M_P,\mathrm{disc}}$,
- the Γ-fixed subspace of each representation $\mathscr{I}_P(\pi_\nu)$.

By the Iwasawa decomposition, elements of this space may be viewed as families of functions indexed by K/Γ satisfying some compatibility condition that we proceed to make explicit now. We also prove that the Archimedean part of $\mathscr{I}_P(\pi_\nu)$ is a principal series representation, and we provide its spectral parameter.

4.3.1. *Borel Eisenstein series.*

LEMMA 2.4.2. *The irreducible representations occurring in $R_{T,\mathrm{disc}}$ are precisely characters χ of $T^+(\mathbb{R})T(\mathbb{Q})\backslash T(\mathbb{A})$. Let χ be such character and $\nu \in i\mathfrak{a}^*$. The Archimedean part of $\mathscr{I}_B(\chi_\nu)$ is an irreducible principal series representation with spectral parameter ν.*

PROOF. The first part is because $T^+(\mathbb{R})T(\mathbb{Q})\backslash T(\mathbb{A})$ is abelian. For the second part, since $\chi_\infty = 1$ we have $\mathscr{I}_B(\chi_\nu)_\infty = \mathscr{I}_B(e^\nu)$, which is irreducible because $\nu \in i\mathfrak{a}^*$ (see [**Mui09**, Lemma 5.1]). □

Characters χ of $T^+(\mathbb{R})T(\mathbb{Q})\backslash T(\mathbb{A})$ that coincide with ω on $Z(\mathbb{A})$ are in one-to-one correspondence with triplets $(\omega_1, \omega_2, \omega_3)$ of characters of $\mathbb{R}_{>0}\mathbb{Q}^\times\backslash\mathbb{A}^\times$ satisfying $\omega_1\omega_2\omega_3^2 = \omega$, via

$$\chi\left(\begin{bmatrix} x &&& \\ & y && \\ && tx^{-1} & \\ &&& ty^{-1}\end{bmatrix}\right) = \omega_1(x)\omega_2(y)\omega_3(t).$$

Define a character of B by

$$\omega\left(\begin{bmatrix} x & * & * & * \\ & y & * & * \\ & & tx^{-1} & * \\ & & & ty^{-1} \end{bmatrix}\right) = \omega_1(x)\omega_2(y)\omega_3(t)$$

(note that this notation is sound, as it coincides with our original ω on scalar matrices).

Proposition 2.4.2. *Let $\chi = (\omega_1, \omega_2, \omega_3)$ with $\omega_1\omega_2\omega_3^2 = \omega$. Consider $(\phi_k)_{\mathsf{k}\in K/\Gamma}$ such that*

(1) for all k, $\phi_{\mathsf{k}} \in \mathbb{C}$,

(2) if $\gamma \in K \cap B(\mathbb{A})$ then for all k, $\phi_{\mathsf{k}} = \chi(\gamma^{-1})\phi_{\gamma\mathsf{k}}$.

Then the function on $G(\mathbb{A})$ given for $\mathsf{u} \in U(\mathbb{A}), \mathsf{t} \in T(\mathbb{A}), \mathsf{k} \in K$ by

$$\phi(\mathsf{utk}) = \chi(\mathsf{t})\phi_{\mathsf{k}}, \tag{2.34}$$

is well-defined and belongs to ${\mathscr{H}_B}^{\Gamma}(\chi)$. Moreover, every function in ${\mathscr{H}_B}^{\Gamma}(\chi)$ has this shape.

Proof. We first prove that ϕ is well-defined. Suppose $\mathsf{u}_1\mathsf{t}_1\mathsf{k}_1 = \mathsf{u}_2\mathsf{t}_2\mathsf{k}_2$. In particular $\mathsf{k}_1\mathsf{k}_2^{-1} = (\mathsf{u}_1\mathsf{t}_1)^{-1}(\mathsf{u}_2\mathsf{t}_2) \in B(\mathbb{A}) \cap K$. Therefore

$$\chi(\mathsf{t}_1)\phi_{\mathsf{k}_1} = \chi(\mathsf{t}_1)\chi(\mathsf{k}_1\mathsf{k}_2^{-1})\phi_{\mathsf{k}_2} = \chi(\mathsf{t}_2)\phi_{\mathsf{k}_2}.$$

Next we show that ϕ belongs indeed to ${\mathscr{H}_B}^{\Gamma}(\omega_1, \omega_2, \omega_3)$. The fact that ϕ is invariant on the left by $U(\mathbb{A})T(\mathbb{Q})T(\mathbb{R})^+$, the right invariance by Γ and the fact that ϕ transforms under $T(\mathbb{A})$ according to χ are obvious from the definition. Finally,

$$\begin{aligned}\int_K \int_{T(\mathbb{R})^+T(\mathbb{Q})\backslash T(\mathbb{A})} |\phi(\mathsf{mk})|^2 \, d\mathsf{m}\, d\mathsf{k} &= \int_K \int_{(\mathbb{R}_{>0}\mathbb{Q}^\times\backslash\mathbb{A}^\times)^3} |\phi_{\mathsf{k}}|^2 \, d\mathsf{m}\, d\mathsf{k} \\ &= \mathrm{Vol}(\mathbb{R}_{>0}\mathbb{Q}^\times\backslash\mathbb{A}^\times)^3 \,\mathrm{Vol}(\Gamma) \sum_{\mathsf{k}\in K/\Gamma} |\phi_{\mathsf{k}}|^2 < \infty\end{aligned}$$

since $\mathbb{R}_{>0}\mathbb{Q}^\times\backslash\mathbb{A}^\times$ is compact and K/Γ is finite. As a last point, we show that we thus exhaust all of ${\mathscr{H}_B}^\Gamma(\omega_1,\omega_2,\omega_3)$. Let $\phi \in {\mathscr{H}_B}^\Gamma(\omega_1,\omega_2,\omega_3)$. Define

$$\phi_{\mathsf{k}} = \phi(\mathsf{k}).$$

Then it is clear that equation (2.34) holds. As for condition 2, note that if $\gamma = \mathsf{t}_\gamma \mathsf{u}_\gamma \in K \cap B(\mathbb{A})$ with $\mathsf{t}_\gamma \in T(\mathbb{A})$ and $\mathsf{u}_\gamma \in U(\mathbb{A})$ then

$$\begin{aligned}\phi_{\gamma k} &= \phi(\gamma\mathsf{k}) \\ &= \phi(\mathsf{t}_\gamma\mathsf{u}_\gamma\mathsf{k}) = \chi(\mathsf{t}_\gamma)\phi(\mathsf{k}) \\ &= \chi(\gamma)\phi_{\mathsf{k}}.\end{aligned}$$

$\square$

REMARK 2.4.1. *Consider the action of $K \cap B(\mathbb{A})$ on K/Γ by multiplication on the left. Then the compatibility condition 2. can only be met if χ is trivial on the stabilizer of each element $\mathsf{k} \in K/\Gamma$ such that $\phi_{\mathsf{k}} \neq 0$. Thus the dimension of ${\mathscr{H}_B}^\Gamma(\chi)$ is the number of distinct orbits of such elements.*

4.3.2. *Klingen Eisenstein series.* The set of characters χ of $\mathrm{A_K}^+(\mathbb{R})\,\mathrm{A_K}(\mathbb{Q})\backslash\,\mathrm{A_K}(\mathbb{A})$ that coincide with ω on $Z(\mathbb{A})$ is in one-to-one correspondence with pairs (ω_1,ω_2) of characters of $\mathbb{R}_{>0}\mathbb{Q}^\times\backslash\mathbb{A}^\times$ satisfying $\omega_1\omega_2 = \omega$, via

$$\chi\left(\begin{bmatrix} u & & & \\ & t & & \\ & & u & \\ & & & t^{-1}u^2 \end{bmatrix}\right) = \omega_1(u)\omega_2(t).$$

For convenience, if $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}_2$, define $\iota_A = \begin{bmatrix} a & & b & \\ & 1 & & \\ c & & d & \\ & & & \det(A) \end{bmatrix} \in \mathrm{M_K}$.

Lemma 2.4.3. *Let $\chi = (\omega_1, \omega_2)$ with $\omega_1\omega_2 = \omega$. The irreducible representations with central character χ occurring in $R_{\mathrm{M_K},\mathrm{disc}}$ are twists $\omega_2 \otimes \pi$, where π occurs in the discrete spectrum of $L^2(\mathbb{R}_{>0}\,\mathrm{GL}_2(\mathbb{Q})\backslash\,\mathrm{GL}_2(\mathbb{A}))$ and has central character ω_1.*

Proof. Let π be an irreducible representations with central character χ occurring in $R_{\mathrm{M_K},\mathrm{disc}}$. By definition, the space of π is contained in the subspace of $L^2(\mathrm{M_K}(\mathbb{Q})\,\mathrm{A_K}^+(\mathbb{R})\backslash\,\mathrm{M_K}(\mathbb{A}))$ consisting of functions with central character χ. This subspace identifies with the space $L^2(\mathbb{R}_{>0}\,\mathrm{GL}_2(\mathbb{Q})\backslash\,\mathrm{GL}_2(\mathbb{A}), \omega_1)$ via

$$\phi \mapsto \left(\begin{bmatrix} a & & b & \\ & t & & \\ c & & d & \\ & & & t^{-1}\det(A) \end{bmatrix} \mapsto \omega_2(t)\phi(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right])\right).$$

$\square$

Proposition 2.4.3. *Let $\chi = (\omega_1, \omega_2)$ with $\omega_1\omega_2 = \omega$. Let (π, V_π) occur in the discrete spectrum of $L^2(\mathbb{R}_{>0}\,\mathrm{GL}_2(\mathbb{Q})\backslash\,\mathrm{GL}_2(\mathbb{A}))$ with central character ω_1. Consider $(\phi_k)_{\mathsf{k}\in K/\Gamma}$ such that*

1. *for all k, $\phi_{\mathsf{k}} \in V_\pi$,*
2. *if $\gamma \in K \cap \mathrm{P_K}(\mathbb{A})$ then for all k, $\phi_{\mathsf{k}}(\cdot\,\mathrm{Proj}^{\mathrm{GL}_2}_{\mathrm{P_K}}(\gamma)) = \omega_2 \circ \mathrm{Proj}^{\mathrm{GL}_1}_{\mathrm{P_K}}(\gamma^{-1})\phi_{\gamma k}$.*

Then the function on $G(\mathbb{A})$ given for $\mathsf{n} \in \mathrm{N_K}(\mathbb{A}), \mathsf{m} \in \mathrm{M_K}(\mathbb{A}), \mathsf{k} \in K$ by

$$\phi(\mathsf{nmk}) = \omega_2 \circ \mathrm{Proj}^{\mathrm{GL}_1}_{\mathrm{P_K}}(\mathsf{m})\phi_{\mathsf{k}}(\mathrm{Proj}^{\mathrm{GL}_2}_{\mathrm{P_K}}(\mathsf{m})) \tag{2.35}$$

is well-defined and belongs to $\mathscr{I}_{\mathrm{P_K}}((\omega_2\otimes\pi)_\nu)^\Gamma$. *Moreover, every function belonging to* $\mathscr{I}_{\mathrm{P_K}}((\omega_2\otimes\pi)_\nu)^\Gamma$ *has this shape.*

REMARK 2.4.2. *Condition (2) implies that each* ϕ_{k} *is right-*$\mathrm{SO}_2(\mathbb{R})$*-invariant (and hence must be an adelic Maaß form or a character). Indeed, let* $v\le\infty$ *and let* k_v *be a compact subgroup of* $\mathrm{GL}_2(\mathbb{Q}_v)$ *such that*

$$\{\iota_A : A\in k_v\}\subset K_v.$$

Assume moreover that $K_v=\Gamma_v$. *Then* K/Γ *is left invariant by* Γ_v, *hence for all* $A\in k_v$ *we have* $\phi_{\mathsf{k}}(\cdot A)=\phi_{\iota_A\mathsf{k}}=\phi_{\mathsf{k}}$. *In particular, for* $v=\infty$, *we may take* $k_v=\mathrm{O}_2(\mathbb{R})$, *hence the claim.*

PROOF. We first prove that ϕ is well-defined. Suppose $\mathsf{n}_1\mathsf{m}_1\mathsf{k}_1=\mathsf{n}_2\mathsf{m}_2\mathsf{k}_2$. In particular $\mathsf{k}_2\mathsf{k}_1^{-1}=(\mathsf{n}_2\mathsf{m}_2)^{-1}(\mathsf{n}_1\mathsf{m}_1)\in\mathrm{P_K}(\mathbb{A})\cap K$. Therefore we have

$$\mathrm{Proj}_{\mathrm{P_K}}^{\mathrm{GL}_2}(\mathsf{m}_1)=\mathrm{Proj}_{\mathrm{P_K}}^{\mathrm{GL}_2}(\mathsf{n}_1\mathsf{m}_1)=\mathrm{Proj}_{\mathrm{P_K}}^{\mathrm{GL}_2}(\mathsf{n}_2\mathsf{m}_2\mathsf{k}_2\mathsf{k}_1^{-1})=\mathrm{Proj}_{\mathrm{P_K}}^{\mathrm{GL}_2}(\mathsf{m}_2)\,\mathrm{Proj}_{\mathrm{P_K}}^{\mathrm{GL}_2}(\mathsf{k}_2\mathsf{k}_1^{-1}).$$

Then

$$\begin{aligned}\omega_2\circ\mathrm{Proj}_{\mathrm{P_K}}^{\mathrm{GL}_1}(\mathsf{m}_1)\phi_{\mathsf{k}_1}(\mathrm{Proj}_{\mathrm{P_K}}^{\mathrm{GL}_2}(\mathsf{m}_1)) &= \omega_2\circ\mathrm{Proj}_{\mathrm{P_K}}^{\mathrm{GL}_1}(\mathsf{m}_1)\phi_{\mathsf{k}_1}(\mathrm{Proj}_{\mathrm{P_K}}^{\mathrm{GL}_2}(\mathsf{m}_2)\,\mathrm{Proj}_{\mathrm{P_K}}^{\mathrm{GL}_2}(\mathsf{k}_2\mathsf{k}_1^{-1}))\\ &= \omega_2\circ\mathrm{Proj}_{\mathrm{P_K}}^{\mathrm{GL}_1}(\mathsf{m}_1)\omega_2\circ t(\mathsf{k}_1\mathsf{k}_2^{-1})\phi_{\mathsf{k}_2}(\mathrm{Proj}_{\mathrm{P_K}}^{\mathrm{GL}_2}(\mathsf{m}_2))\\ &= \omega_2\circ\mathrm{Proj}_{\mathrm{P_K}}^{\mathrm{GL}_1}(\mathsf{m}_2)\phi_{\mathsf{k}_2}(\mathrm{Proj}_{\mathrm{P_K}}^{\mathrm{GL}_2}(\mathsf{m}_2)).\end{aligned}$$

Next we show that ϕ belongs indeed to $\mathscr{I}_{\mathrm{P_K}}((\omega_2\otimes\pi)_\nu)^\Gamma$. The fact that ϕ is invariant on the left by $\mathrm{N_K}(\mathbb{A})\,\mathrm{M_K}(\mathbb{Q})\,\mathrm{A_K}^+(\mathbb{R})$ and on the right by Γ are obvious from the definition. The fact that ϕ is square integrable follows from

$$\begin{aligned}\int_K \int_{\mathrm{A_K}^+(\mathbb{R})\,\mathrm{M_K}(\mathbb{Q})\backslash\,\mathrm{M_K}(\mathbb{A})} |\phi(\mathsf{mk})|^2\, d\mathsf{m}\, d\mathsf{k} &= \int_K \int_{\mathrm{A_K}^+(\mathbb{R})\,\mathrm{M_K}(\mathbb{Q})\backslash\,\mathrm{M_K}(\mathbb{A})} |\phi_{\mathsf{k}}(\mathrm{Proj}_{\mathrm{P_K}}^{\mathrm{GL}_2}(\mathsf{m}))|^2\, d\mathsf{m}\, d\mathsf{k} \\ &= \sum_{\mathsf{k}\in K/\Gamma} \mathrm{Vol}(\Gamma) \int_{\mathbb{R}_{>0}\,\mathrm{GL}_2(\mathbb{Q})\backslash\,\mathrm{GL}_2(\mathbb{A})} \int_{\mathbb{R}_{>0}\mathbb{Q}^\times\backslash\mathbb{A}^\times} |\phi_{\mathsf{k}}(\mathsf{x})|^2\, d\mathsf{t}\, d\mathsf{x} < \infty\end{aligned}$$

since ϕ_{k} is square integrable, $\mathbb{R}_{>0}\mathbb{Q}^\times\backslash\mathbb{A}^\times$ is compact and K/Γ is finite. Finally, we need to show that for all $\mathsf{g} = \mathsf{nmk}$, the function $\phi_{\mathsf{g}} : \mathrm{M_K}(\mathbb{A}) \to \mathbb{C}, \mathsf{m}_1 \mapsto \phi(\mathsf{m}_1\mathsf{g})$ transform under $\mathrm{M_K}(\mathbb{A})$ on the right according to $\omega_2 \otimes \pi$. Indeed, for $\mathsf{m}_1 \in \mathrm{M_K}(\mathbb{A})$ we have

$$\phi_{\mathsf{g}}(\mathsf{m}_1) = \phi(\mathsf{m}_1\mathsf{nmk}) = \phi(\underbrace{\mathsf{m}_1\mathsf{n}\mathsf{m}_1^{-1}}_{\in \mathrm{N_K}(\mathbb{A})}\mathsf{m}_1\mathsf{mk}) = \omega_2 \circ \mathrm{Proj}_{\mathrm{P_K}}^{\mathrm{GL}_1}(\mathsf{m})\omega_2 \circ \mathrm{Proj}_{\mathrm{P_K}}^{\mathrm{GL}_1}(\mathsf{m}_1)\phi_{\mathsf{k}}(\mathsf{m}_1)$$

hence the claim since $\phi_{\mathsf{k}} \in V_\pi$.

As a last point, we show that $\mathscr{I}_{\mathrm{P_K}}((\omega_2 \otimes \pi)_\nu)^\Gamma$ consists exactly in such functions. Let $\phi \in \mathscr{I}_{\mathrm{P_K}}((\omega_2 \otimes \pi)_\nu)^\Gamma$. Define

$$\phi_{\mathsf{k}}(A) = \phi(\iota_A \mathsf{k}).$$

Then it is clear that equation (2.35) holds. As for condition (2), note that if $\gamma = \mathsf{n}_\gamma \mathsf{m}_\gamma \in K \cap \mathrm{P}_{\mathrm{K}}(\mathbb{A})$ then

$$\begin{aligned}
\phi_{\mathsf{k}}(A \operatorname{Proj}_{\mathrm{P}_{\mathrm{K}}}^{\mathrm{GL}_2}(\gamma)) &= \phi(\iota_A \iota_{\operatorname{Proj}_{\mathrm{P}_{\mathrm{K}}}^{\mathrm{GL}_2}(\gamma)} \mathsf{k}) \\
&= \phi\left(\iota_A \begin{bmatrix} 1 & & & \\ & \operatorname{Proj}_{\mathrm{P}_{\mathrm{K}}}^{\mathrm{GL}_1}(\gamma)^{-1} & & \\ & & 1 & \\ & & & \operatorname{Proj}_{\mathrm{P}_{\mathrm{K}}}^{\mathrm{GL}_1}(\gamma) \end{bmatrix} \mathsf{m}_\gamma \mathsf{k}\right) \\
&= \omega_2 \circ \operatorname{Proj}_{\mathrm{P}_{\mathrm{K}}}^{\mathrm{GL}_1}(\gamma^{-1}) \phi(\iota_A \mathsf{n}_\gamma^{-1} \gamma \mathsf{k}) \\
&= \omega_2 \circ \operatorname{Proj}_{\mathrm{P}_{\mathrm{K}}}^{\mathrm{GL}_1}(\gamma^{-1}) \phi(\underbrace{\iota_A \mathsf{n}_\gamma^{-1} \iota_A^{-1}}_{\in \mathrm{N}_{\mathrm{K}}} \iota_A \gamma \mathsf{k}) \\
&= \omega_2 \circ \operatorname{Proj}_{\mathrm{P}_{\mathrm{K}}}^{\mathrm{GL}_1}(\gamma^{-1}) \phi_{\gamma \mathsf{k}}(A).
\end{aligned}$$

Finally, by definition of $\mathscr{I}_{\mathrm{P}_{\mathrm{K}}}((\omega_2 \otimes \pi)_\nu)$ the function $\mathsf{m} \mapsto \phi(\mathsf{mk})$ transforms under $\mathrm{M}_{\mathrm{K}}(\mathbb{A})$ on the right according to $\omega_2 \otimes \pi$, from which follows ϕ_{k} transforms according to π. $\square$

Finally we prove the following

PROPOSITION 2.4.4. *Let (π, V_π) be a representation occuring in the discrete spectrum of $L^2(\mathbb{R}_{>0} \mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}))$ with central character ω_1. $\mathscr{I}_{\mathrm{P}_{\mathrm{K}}}((\omega_2 \otimes \pi)_\nu)$ has a K_∞-fixed vector if and only if π has a $\mathrm{O}_2(\mathbb{R})$-fixed vector. In this case, $\mathscr{I}_{\mathrm{P}_{\mathrm{K}}}((\omega_2 \otimes \pi)_\nu)_\infty$ is generic if and only if π_∞ is a principal series. Finally if π_∞ is a spherical principal series with spectral parameter s and $\nu \in i\mathfrak{a}_{\mathrm{P}_{\mathrm{K}}}^*$ then $\mathscr{I}_{\mathrm{P}_{\mathrm{K}}}((\omega_2 \otimes \pi)_\nu)_\infty$ is a principal series representation with spectral parameter $\nu + \nu_K(s)$, where $\nu_K(s)$ is the element of $\mathfrak{a}^*(\mathbb{C})$ corresponding to the character $\begin{bmatrix} y^{\frac{1}{2}} & & & \\ & u & & \\ & & y^{-\frac{1}{2}} & \\ & & & u^{-1} \end{bmatrix} \mapsto |y|^s$.*

Proof. The first claim follows immediately from Proposition 2.4.3. By the spectral decomposition for GL_2, if π has a $\mathrm{O}_2(\mathbb{R})$-fixed vector then π_∞ is either a character or a principal series. But representations induced from a character of the Klingen subgroup are not generic, as seen in row IIb of Table A.1 in [**RS07**]. This shows the second claim. Finally assume π_∞ is a spherical principal series on GL_2 with spectral parameter s. Then we might see π_∞ as the representation of $\mathrm{PGL}_2(\mathbb{R})$ induced from the character $\chi_s : \begin{bmatrix} y^{\frac{1}{2}} & x \\ & \pm y^{-\frac{1}{2}} \end{bmatrix} \mapsto |y|^s$, where s is either an imaginary number or a real number with $0 < |s| < \frac{1}{2}$. Define the following subgroups: $N_1 = \overline{U}_{s_2} = \begin{bmatrix} 1 & & & \\ & 1 & * & \\ & & 1 & \\ & & & 1 \end{bmatrix}$, $A_1 = \left\{ \begin{bmatrix} y^{\frac{1}{2}} & & & \\ & 1 & & \\ & & \pm y^{-\frac{1}{2}} & \\ & & & 1 \end{bmatrix} : y \neq 0 \right\}$, $M_1 = \{\iota_A : A \in \mathrm{PGL}_2(\mathbb{R})\}$. Note that $N_1 \, \mathrm{N_K} = U$, $A_1 \, \mathrm{A_K}(\mathbb{R}) = T(\mathbb{R})$ and $M_1 \, \mathrm{A_K} = \mathrm{M_K}$. We might view χ_s as a character of $A_1 N_1$. Since ω_2 is trivial on $\mathrm{A_K}(\mathbb{R})$, inducing in stages, we get

$$\begin{aligned}
\mathscr{I}_{\mathrm{P_K}}((\omega_2 \otimes \pi)_\nu)_\infty &= \mathrm{Ind}_{\mathrm{P_K}(\mathbb{R})}^{G(\mathbb{R})} \left(\mathrm{I}_{\mathrm{N_K}(\mathbb{R})} \otimes e^\nu \otimes \pi_\infty \right) \\
&= \mathrm{Ind}_{\mathrm{P_K}(\mathbb{R})}^{G(\mathbb{R})} \left(\mathrm{I}_{\mathrm{N_K}(\mathbb{R})} \otimes e^\nu \otimes \mathrm{Ind}_{A_1 N_1}^{M_1}(\chi_s) \right) \\
&= \mathrm{Ind}_{\mathrm{P_K}(\mathbb{R})}^{G(\mathbb{R})} \mathrm{Ind}_{B(\mathbb{R})}^{\mathrm{P_K}(\mathbb{R})} \left(\mathrm{I}_{\mathrm{N_K}(\mathbb{R})} \otimes \mathrm{I}_{N_1} \otimes e^{\nu + \nu_{\mathrm{K}}(s)} \right) \\
&= \mathrm{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})} (\mathrm{I}_U \otimes e^{\nu + \nu_{\mathrm{K}}(s)}).
\end{aligned}$$

Since $\nu \in i\mathfrak{a}^*$, by Lemma 5.1 of [**Mui09**] this representation is irreducible. $\square$

4.3.3. *Siegel Eisenstein series.*

Characters χ of $\mathrm{A_S}^+(\mathbb{R}) \, \mathrm{A_S}(\mathbb{Q}) \backslash \mathrm{A_S}(\mathbb{A})$ that coincide with ω on $Z(\mathbb{A})$ are in one-to-one correspondence with pairs (ω_1, ω_2) of characters of $\mathbb{R}_{>0}\mathbb{Q}^\times \backslash \mathbb{A}^\times$ satisfying $\omega_1 \omega_2^2 = \omega$, via

$$\chi\left(\begin{bmatrix} u & & & \\ & u & & \\ & & tu^{-1} & \\ & & & tu^{-1} \end{bmatrix} \right) = \omega_1(u)\omega_2(t).$$

For convenience, if $A \in \mathrm{GL}_2$, define $\iota_A = \left[\begin{smallmatrix} A & \\ & {}^{\top}A^{-1} \end{smallmatrix}\right] \in \mathrm{M}_{\mathrm{S}}$.

LEMMA 2.4.4. *Let $\chi = (\omega_1, \omega_2)$ with $\omega_1\omega_2^2 = \omega$. The irreducible representations with central character χ occurring in $R_{\mathrm{M_S},\mathrm{disc}}$ are twists $\omega_2 \otimes \pi$, where π occurs in the discrete spectrum of $L^2(\mathbb{R}_{>0}\,\mathrm{GL}_2(\mathbb{Q})\backslash\,\mathrm{GL}_2(\mathbb{A}))$ and has central character ω_1.*

PROOF. Similar as Lemma 2.4.3 with trivial modifications where required. □

PROPOSITION 2.4.5. *Let $\chi = (\omega_1, \omega_2)$ with $\omega_1\omega_2^2 = \omega$. Let (π, V_π) occur in the discrete spectrum of $L^2(\mathbb{R}_{>0}\,\mathrm{GL}_2(\mathbb{Q})\backslash\,\mathrm{GL}_2(\mathbb{A}))$ with central character ω_1. Consider $(\phi_{\mathsf{k}})_{\mathsf{k}\in K/\Gamma}$ such that*

(1) for all k, $\phi_{\mathsf{k}} \in V_\pi$,
(2) if $\gamma \in K \cap \mathrm{P_S}(\mathbb{A})$ then for all k, $\phi_{\mathsf{k}}(\cdot\,\mathrm{Proj}_{\mathrm{P_S}}^{\mathrm{GL}_2}(\gamma)) = \omega_2 \circ \mu(\gamma^{-1})\phi_{\gamma\mathsf{k}}$.

Then the function on $G(\mathbb{A})$ given for $\mathsf{n} \in \mathrm{N_S}(\mathbb{A}), \mathsf{m} \in \mathrm{M_S}(\mathbb{A}), \mathsf{k} \in K$ by

$$\phi(\mathsf{nmk}) = \omega_2 \circ \mu(\mathsf{m})\phi_{\mathsf{k}}(\mathrm{Proj}_{\mathrm{P_S}}^{\mathrm{GL}_2}(\mathsf{m})) \tag{2.36}$$

is well-defined and belongs to $\mathscr{I}_{\mathrm{P_S}}((\omega_2 \otimes \pi)_\nu)^\Gamma$. Moreover, every function belonging to $\mathscr{I}_{\mathrm{P_S}}((\omega_2 \otimes \pi)_\nu)^\Gamma$ has this shape.

REMARK 2.4.3. *Similarly as Remark 2.4.2, condition (2) implies that each ϕ_{k} is right-$\mathrm{O}_2(\mathbb{R})$-invariant (and hence must be an adelic Maaß form or a character).*

PROOF. Same proof as Proposition 2.4.3, with trivial modifications where required.

□

PROPOSITION 2.4.6. *Let (π, V_π) be a representation occuring in the discrete spectrum of $L^2(\mathbb{R}_{>0}\,\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}))$ with central character ω_1. $\mathscr{I}_{\mathrm{P_S}}((\omega_2 \otimes \pi)_\nu)$ has a K_∞-fixed vector if and only if π has a $\mathrm{O}_2(\mathbb{R})$-fixed vector. In this case, $\mathscr{I}_{\mathrm{P_S}}((\omega_2\otimes\pi)_\nu)_\infty$ is generic if and only if π_∞ is a principal series. Finally if π_∞ is a spherical principal series with spectral parameter s and $\nu \in i\mathfrak{a}^*_{\mathrm{P_S}}$ then $\mathscr{I}_{\mathrm{P_K}}((\omega_2 \otimes \pi)_\nu)_\infty$ is a principal series representation with spectral parameter $\nu + \nu_S(s)$, where $\nu_S(s)$ is the element of $\mathfrak{a}^*(\mathbb{C})$ corresponding to the character* $\begin{bmatrix} y^{\frac12}u &&& \\ & y^{-\frac12}u && \\ && y^{-\frac12}u^{-1} & \\ &&& y^{\frac12}u^{-1} \end{bmatrix} \mapsto |y|^s$.

PROOF. Same proof as Proposition 2.4.4, with trivial modifications where required. $\square$

4.4. Spectral expansion of the kernel. We now give (in Corollary 2.4.1) the spectral expansion of the kernel. This follows directly from work of Arthur combined with the discussion of Sections 3.3, 3.4 and 4.3. For technical reasons, we need the absolute convergence of the spectral expansion. This is the content of Proposition 2.4.8. The automorphic forms involved in the spectral expansion of the automorphic kernel are precisely those whose Whittaker coefficients will appear in the relative trace formula. In particular, the non-generic ones will by definition have a zero contribution to the spectral side of the relative trace formula, even though they do appear in the spectral expansion of the kernel.

DEFINITION 2.4.2. For each standard parabolic P we choose an orthonormal basis $\mathscr{B}_P$ of $\mathscr{H}_P(\omega)$ such that

(1) if $R_{M_P,\mathrm{disc}} = \bigoplus_\pi \pi$ is the decomposition of the restriction of the right regular representation of $M_P(\mathbb{A})$ on the subspace of $L^2(M_P(\mathbb{Q})A_P^+(\mathbb{R})\backslash G(\mathbb{A}))$ that decompose discretely, then $\mathscr{B}_P = \bigcup_\pi \mathscr{B}_\pi$, where each $\mathscr{B}_\pi$ is a basis of the space of the corresponding induced representation $\mathscr{I}_P(\pi_\nu)$, (note that this space does not depend on ν).

(2) for each representation $\pi = \bigotimes_v \pi_v$ as above, for each place v there is an orthonormal basis $\mathscr{B}_{\pi,v}$ of the local representation π_v such that $\mathscr{B}_\pi$ consists of factorizable vectors $\phi = \bigotimes_{v\leq\infty} \phi_v$ where each ϕ_v belongs to the corresponding $\mathscr{B}_{\pi,v}$.

(3) for each representation π_v, we have $\mathscr{B}_{\pi,v} = \bigcup_\tau \mathscr{B}_{\pi,v,\tau}$, where the union is over the irreducible representations τ of Γ_v, and $\mathscr{B}_{\pi,v,\tau}$ is a basis of the space of π_v consisting of vectors ϕ satisfying $\pi_v(\gamma)\phi = \tau(\gamma)\phi$ for all $\gamma \in \Gamma_v$.

Note that conditions (2) and (3) imply in particular that elements of $\mathscr{B}_P$ are in $\mathscr{H}_P^0$.

Definition 2.4.3. For each standard parabolic P and for each irreducible representation π occurring in $R_{M_P,\mathrm{disc}}$, define $\mathscr{B}_{\pi,1}$ to be the subset of $\mathscr{B}_\pi$ consisting in vectors ϕ whose each local component ϕ_v belongs to $\mathscr{B}_{\pi,v,1}$, and set $\mathscr{B}_P^\Gamma = \bigcup_\pi \mathscr{B}_{\pi,1}$. If χ is a character of $A_P(\mathbb{A})$, define

$$\mathscr{G}_P(\Gamma,\chi) = \bigcup_\pi \mathscr{B}_{\pi,1},$$

where the union runs over representations π with central character χ and such that the induced representations $\mathscr{I}_P(\pi_\nu)$ are generic.

If $u \in \mathscr{H}_P(\omega)$, define

$$\mathscr{I}_P(\nu, f)u = \int_{\overline{G(\mathbb{A})}} f(\mathsf{y})\mathscr{I}_P(\nu, \mathsf{y})u \, d\mathsf{y}.$$

PROPOSITION 2.4.7. *Let $\nu \in i\mathfrak{a}_P^*$. Let $u \in \mathscr{B}_P$. Then either $\mathscr{I}_P(\nu, f)u = 0$ or $u \in \mathscr{B}_P^{\Gamma}$. In the latter case, say $u \in \mathscr{B}_\pi$. Then if π is generic we have*

$$\mathscr{I}_P(\nu, f)u = \lambda_f(u, \nu)u,$$

where $\lambda_f(u, \nu) = \lambda_{f_\infty}(u, \nu)\lambda_{f_{\mathrm{fin}}}(u, \nu)$, and

$$\lambda_{f_\infty}(u, \nu) = \begin{cases} \tilde{f}_\infty(\nu) \text{ if } P = B, \\ \tilde{f}_\infty(\nu + \nu_K(s_u)) \text{ if } P = \mathrm{P_K} \text{ and } \pi_\infty \text{ has spectral parameter } s_u, \\ \tilde{f}_\infty(\nu + \nu_S(s_u)) \text{ if } P = \mathrm{P_S} \text{ and } \pi_\infty \text{ has spectral parameter } s_u, \\ \tilde{f}_\infty(\nu_u) \text{ if } P = G \text{ and } \pi_\infty \text{ has spectral parameter } \nu_u, \end{cases}$$

and, following notations of Proposition 2.3.3, $\lambda_{f_{\mathrm{fin}}}(u, \nu)$ is the eigenvalue of the Hecke operator

$$\bigotimes_{\Gamma_p = G(\mathbb{Z}_p)} \overline{\pi_{p,\nu}}(\tilde{f}_p).$$

REMARK 2.4.4. *If $P = G$ then $\mathfrak{a}_P = \{0\}$ and $\mathscr{I}_P(\nu, f) = R(f)$.*

PROOF. This is a combination of Propositions 2.3.3, 2.3.5, Lemma 2.4.2 and Propositions 2.4.4 and 2.4.6. □

The following statement [**Art78**, pages 928-935] may be viewed as a rigorous version of the informal discussion in Section 3.2.

LEMMA 2.4.5. *Let f as in Assumption 2.1. Then for $\mathsf{x}, \mathsf{y} \in G(\mathbb{A})$ we have a pointwise equality*

$$K_f(\mathsf{x}, \mathsf{y}) = \sum_P n_P^{-1} \int_{i\mathfrak{a}_P^*} \sum_{u \in \mathscr{B}_P} E(\mathsf{x}, \mathscr{I}_P(\nu, f)u, \nu)\overline{E(\mathsf{y}, u, \nu)}\, d\nu.$$

Here, $n_G = 1$, $n_B = 8$, $n_{\mathrm{P_K}} = 2$ and $n_{\mathrm{P_S}} = 2$.

However, for the later purpose of interchanging integration order, we want to show that the above expressions for the kernel converge absolutely. To this end, we need the following stronger statement.

PROPOSITION 2.4.8. *Let f as in Assumption 2.1. Then the following expression defines a continuous function in the variables $\mathsf{x} \in G(\mathbb{A})$ and $\mathsf{y} \in G(\mathbb{A})$, which is moreover bounded on any compact subset of $(G(\mathbb{Q})\backslash G(\mathbb{A}))^2$:*

$$K_{abs}(\mathsf{x}, \mathsf{y}) = \sum_P n_P^{-1} \int_{i\mathfrak{a}_P^*} \sum_{u \in \mathscr{B}_P} |E(\mathsf{x}, \mathscr{I}_P(\nu, f)u, \nu)\overline{E(\mathsf{y}, u, \nu)}|\, d\nu.$$

We do not give a proof of this proposition here, as a similar statement was proven in the setting of GL_2 in § 6 of [**KL13**], the proof thereof can be directly adapted. For completeness, we include a proof in Appendix A. By combining it with Lemmas 2.4.2, 2.4.3, 2.4.5 and Proposition 2.4.7, we obtain the following corollary.

COROLLARY 2.4.1. *Let f as in Assumption 2.1. Then for $\mathsf{x}, \mathsf{y} \in G(\mathbb{A})$ we have a pointwise equality*

$$K_f(\mathsf{x}, \mathsf{y}) = K_{\mathrm{disc}}(\mathsf{x}, \mathsf{y}) + K_B(\mathsf{x}, \mathsf{y}) + K_K(\mathsf{x}, \mathsf{y}) + K_S(\mathsf{x}, \mathsf{y}) + K_{ng}(\mathsf{x}, \mathsf{y}),$$

where

$$K_{\mathrm{disc}}(\mathsf{x},\mathsf{y}) = \sum_{u\in\mathscr{G}(\Gamma,\omega)} \tilde{f}_\infty(\nu_u)\lambda_{f_{\mathrm{fin}}}(u)u(\mathsf{x})\overline{u(\mathsf{y})},$$

$$K_B(\mathsf{x},\mathsf{y}) = \frac{1}{8}\sum_{\omega_1\omega_2\omega_3^2=\omega}\ \sum_{u\in\mathscr{G}_B(\Gamma,\omega_1,\omega_2,\omega_3)} \int_{i\mathfrak{a}^*} \tilde{f}_\infty(\nu)\lambda_{f_{\mathrm{fin}}}(u,\nu)E(\mathsf{x},u,\nu)\overline{E(\mathsf{y},u,\nu)}\,d\nu,$$

$$K_K(\mathsf{x},\mathsf{y}) = \frac{1}{2}\sum_{\omega_1\omega_2=\omega}\ \sum_{u\in\mathscr{G}_{\mathrm{P_K}}(\Gamma,\omega_1,\omega_2)} \int_{i\mathfrak{a}_K^*} \tilde{f}_\infty(\nu+\nu_K(s_u))\lambda_{f_{\mathrm{fin}}}(u,\nu)E(\mathsf{x},u,\nu)\overline{E(\mathsf{y},u,\nu)}\,d\nu,$$

$$K_S(x,y) = \frac{1}{2}\sum_{\omega_1\omega_2^2=\omega}\ \sum_{u\in\mathscr{G}_{\mathrm{P_S}}(\Gamma,\omega_1,\omega_2)} \int_{i\mathfrak{a}_S^*} \tilde{f}_\infty(\nu+\nu_S(s_u))\lambda_{f_{\mathrm{fin}}}(u,\nu)E(\mathsf{x},u,\nu)\overline{E(\mathsf{y},u,\nu)}\,d\nu,$$

and all the automorphic forms involved in K_{ng} are not generic.

Actually, no automorphic form from the residual spectrum is generic, as shown by the following lemma. Thus K_{disc} consists only in elements from the cuspidal spectrum.

LEMMA 2.4.6. *Let (π, V_π) be any irreducible representation occurring in the residual spectrum of $L^2(Z(\mathbb{R})G(\mathbb{Q})\backslash G(\mathbb{A}),\omega)$. Then π is non-generic.*

PROOF. We will rely on results of Kim that describe the residual spectrum of Sp_4. Thus we first need to show that the representation $\operatorname{res}\pi$ given by Definition 2.3.1 belongs to the residual spectrum of $\mathrm{Sp}_4(\mathbb{A})$. First, $\operatorname{res}\pi$ occurs in the discrete spectrum of $L^2(\mathrm{Sp}_4(\mathbb{Q})\backslash \mathrm{Sp}_4(\mathbb{A}))$, because there are only finitely many possibilities for the Archimedean component of any irreducible representation occurring in $\operatorname{res}\pi$. Moreover $\operatorname{res}\pi$ is not cuspidal by Lemma 2.3.2. Hence $\operatorname{res}\pi$ belongs to the residual spectrum of $\mathrm{Sp}_4(\mathbb{A})$, as claimed. In view of Lemma 2.3.1, it suffices to prove that the residual spectrum of $\mathrm{Sp}_4(\mathbb{A})$ is not generic. By Theorem 3.3 and Remark 3.2 of [**Kim95**], the representations occurring from poles of Siegel Eisenstein series are non-generic.

Similarly, by Theorem 4.1 and Remark 4.2 of [**Kim95**], the representations occurring from poles of Klingen Eisenstein series are non-generic. Finally, by [**Kim95**] § 5.3, irreducible representations π occurring from the poles of Borel Eisenstein series are described as follows. On the one hand, we have the space of constant functions, which is clearly not generic. On the other hand, for every non-trivial quadratic grössencharacter μ of $\mathbb{Q}$ we have a representation $B(\mu)$ whose local components are irreducible subquotients of the induced representation $\mathrm{Ind}_B^{\mathrm{Sp}_4}(|\cdot|_v\mu_v \times \mu_v)$. Therefore, in the terminology of [**RS07**, § 2.2], for all prime p, π_p belongs to Group V if $\mu_p \neq 1$, and to Group VI if $\mu_p = 1$. Now by Table A.2 of [**RS07**], we see that the only generic representations in Group V and VI are those from Va and VIa. But Table A.12 shows that neither of these have a K_p-fixed vector. Since almost all π_p contain a K_p-fixed vector, at least one local component of π must be non-generic, and thus π is not globally generic. $\square$

4.5. The spectral side of the trace formula. Let $\psi_1 = \psi_{\mathbf{m}_1}, \psi_2 = \psi_{\mathbf{m}_2}$ be generic characters of $U(\mathbb{A})/U(\mathbb{Q})$. Fix $\mathsf{t}_1, \mathsf{t}_2 \in A^+$ and consider the basic integral

$$I = \int_{(U(\mathbb{Q})\backslash U(\mathbb{A}))^2} K_f(\mathsf{x}\mathsf{t}_1, \mathsf{y}\mathsf{t}_2)\overline{\psi_{\mathbf{m}_1}(\mathsf{x})}\psi_{\mathbf{m}_2}(\mathsf{y})\, d\mathsf{x}\, d\mathsf{y}. \tag{2.37}$$

Our goal is to compute it in two different ways – using the spectral decomposition of the kernel K_f on the one hand, and its expression as a series together with the Bruhat decomposition on the other hand. The latter will constitute the geometric side and will be addressed in Section 5. We now focus on the former. Using the spectral expansion of the kernel K_f given by Lemma (2.4.5), we can evaluate the basic integral (2.37) as

$$\int_{(U(\mathbb{Q})\backslash U(\mathbb{A}))^2} \sum_P n_P^{-1} \int_{i\mathfrak{a}_P^*} \sum_{u\in\mathscr{B}_P} E(\mathsf{x}\mathsf{t}_1, \mathscr{I}_P(\nu, f)u, \nu)\overline{E(\mathsf{y}\mathsf{t}_2, u, \nu)}\, d\nu\, \overline{\psi_{\mathbf{m}_1}(x)}\psi_{\mathbf{m}_2}(\mathsf{y})\, d\mathsf{x}\, d\mathsf{y}.$$

By Proposition 2.4.8, this expression is absolutely integrable since $(U(\mathbb{Q})\backslash U(\mathbb{A}))^2$ is compact. Thus we may interchange integration order, thus obtaining the Whittaker coefficients of the automorphic forms involved here. By Corollary 2.4.1, we get a discrete contribution and a residual contribution, and a continuous contribution – which itself splits into the contribution of the various parabolic classes. Thus the spectral side of the Kuznetsov formula is given as follows.

PROPOSITION 2.4.9. *We have* $I = \frac{1}{(\mathbf{m}_{11}\mathbf{m}_{21})^4|\mathbf{m}_{12}\mathbf{m}_{21}|^3}\left(\Sigma_{\mathrm{disc}} + \Sigma_B + \Sigma_K + \Sigma_S\right)$, *where*

$$\Sigma_{\mathrm{disc}} = \sum_{u\in\mathscr{G}(\Gamma,\omega)} \tilde{f}_\infty(\nu_u)\lambda_{f_{\mathrm{fin}}}(u)\mathscr{W}_\psi(u)(\mathsf{t}_1\mathsf{t}_{\mathbf{m}_1}^{-1})\overline{\mathscr{W}_\psi(u)}(\mathsf{t}_2\mathsf{t}_{\mathbf{m}_2}^{-1}),$$

$$\Sigma_B = \frac{1}{8}\sum_{\omega_1\omega_2\omega_3^2=\omega}\ \sum_{u\in\mathscr{G}_B(\Gamma,\omega_1,\omega_2,\omega_3)} \int_{i\mathfrak{a}^*} \tilde{f}_\infty(\nu)\lambda_{f_{\mathrm{fin}}}(u,\nu)$$
$$\times\, \mathscr{W}_\psi(E(\cdot,u,\nu))(\mathsf{t}_1\mathsf{t}_{\mathbf{m}_1}^{-1})\overline{\mathscr{W}_\psi(E(\cdot,u,\nu))}(\mathsf{t}_2\mathsf{t}_{\mathbf{m}_2}^{-1})\, d\nu,$$

$$\Sigma_K = \frac{1}{2}\sum_{\omega_1\omega_2=\omega}\ \sum_{u\in\mathscr{G}_{\mathrm{P_K}}(\Gamma,\omega_1,\omega_2)} \int_{i\mathfrak{a}_K^*} \tilde{f}_\infty(\nu+\nu_K(s_u))\lambda_{f_{\mathrm{fin}}}(u,\nu)$$
$$\times\, \mathscr{W}_\psi(E(\cdot,u,\nu))(\mathsf{t}_1\mathsf{t}_{\mathbf{m}_1}^{-1})\overline{\mathscr{W}_\psi(E(\cdot,u,\nu))}(\mathsf{t}_2\mathsf{t}_{\mathbf{m}_2}^{-1})\, d\nu,$$

$$\Sigma_S = \frac{1}{2} \sum_{\omega_1 \omega_2^2 = \omega} \sum_{u \in \mathscr{G}_{\mathrm{P_S}}(\Gamma, \omega_1, \omega_2)} \int_{i\mathfrak{a}_S^*} \tilde{f}_\infty(\nu + \nu_S(s_u)) \lambda_{f_{\mathrm{fin}}}(u, \nu)$$
$$\times \mathscr{W}_\psi(E(\cdot, u, \nu))(\mathsf{t}_1 \mathsf{t}_{\mathbf{m}_1}^{-1}) \overline{\mathscr{W}_\psi(E(\cdot, u, \nu))}(\mathsf{t}_2 \mathsf{t}_{\mathbf{m}_2}^{-1})\, d\nu.$$

5. The geometric side of the trace formula

We now return to computing the basic integral (2.37) using the Bruhat decomposition. The resulting expression will constitute the geometric side of the relative trace formula. Similarly as in Section 3, we first work globally before switching to a local framework. The Weyl group of GSp_4 has eight elements. However, as we show in subsection 5.1 below, only the identity element and the longest three elements have a non-zero contribution. In subsection 5.2 we then obtain a "uniform" expression for the (global) *relevant orbital integrals.* In subsection 5.3, we use the integral transform that was discussed in § 3.4.5 to express the Archimedean part of the relevant orbital integrals in terms of on integral over $\mathfrak{a}^*$ of the test function occurring on the spectral side of the relative trace formula. However, this integral will appear "inside" another integral over a certain subgroup $\overline{U}_\sigma(\mathbb{R})$. Conjecturally, we can interchange integration order and replace the integral over $\overline{U}_\sigma(\mathbb{R})$ with some *generalised Bessel functions.* However, we will not need this conjecture in our application in Chapter 3. In subsection 5.4, we eventually specialise the congruence subgroup Γ to be the *Borel congruence subgroup*, which allows us to give an explicit expression for the finite part of the relevant orbital integrals. As expected, the identity contribution is (at least under some simplifying hypothesis) a delta symbol, while the other three contributions give sums of various kinds of generalised Kloosterman sums for which we give explicit expressions.

Breaking the sum (2.9) over $U(\mathbb{Q})\times U(\mathbb{Q})$ orbits leads to a sum over representatives of the double cosets of $U\backslash G/U$ of orbital integrals. Specifically, set $H = U \times U$, acting on G by

$$(\mathsf{x},\mathsf{y})\cdot\delta = \mathsf{x}^{-1}\delta\mathsf{y},$$

and denote by H_δ the stabilizer of δ. Since f has compact support, the infinite sum $\sum_{\delta\in\overline{G}(\mathbb{Q})}|f(\mathsf{t}_1^{-1}\mathsf{x}^{-1}\delta\mathsf{y}\mathsf{t}_2)|$ is in fact locally finite and hence defines a continuous function in x and y on the compact set $H(\mathbb{Q})\backslash H(\mathbb{A})$. Thus we may interchange summation and integration order, getting

$$\begin{aligned} I &= \int_{H(\mathbb{Q})\backslash H(\mathbb{A})}\sum_{\delta\in\overline{G}(\mathbb{Q})} f(\mathsf{t}_1^{-1}\mathsf{x}^{-1}\delta\mathsf{y}\mathsf{t}_2)\overline{\psi_{\mathbf{m}_1}(\mathsf{x})}\psi_{\mathbf{m}_2}(\mathsf{y})\,d\mathsf{x}\,d\mathsf{y} \\ &= \sum_{\delta\in\overline{G}(\mathbb{Q})}\int_{H(\mathbb{Q})\backslash H(\mathbb{A})} f(\mathsf{t}_1^{-1}\mathsf{x}^{-1}\delta\mathsf{y}\mathsf{t}_2)\overline{\psi_{\mathbf{m}_1}(\mathsf{x})}\psi_{\mathbf{m}_2}(\mathsf{y})\,d\mathsf{x}\,d\mathsf{y} \\ &= \sum_{\delta\in U(\mathbb{Q})\backslash\overline{G}(\mathbb{Q})/U(\mathbb{Q})} I_\delta(f), \end{aligned}$$

where

$$I_\delta(f) = \int_{H_\delta(\mathbb{Q})\backslash H(\mathbb{A})} f(\mathsf{t}_1^{-1}\mathsf{x}^{-1}\delta\mathsf{y}\mathsf{t}_2)\overline{\psi_{\mathbf{m}_1}(\mathsf{x})}\psi_{\mathbf{m}_2}(\mathsf{y})\,d(\mathsf{x},\mathsf{y}), \tag{2.38}$$

and $d(\mathsf{x},\mathsf{y})$ is the quotient measure on $H_\delta(\mathbb{Q})\backslash H(\mathbb{A})$. Using the Bruhat decomposition $G = B\Omega B = \coprod_{\sigma\in\Omega} U\sigma TU$, we have

$$U\backslash\overline{G}/U = \coprod_{\sigma\in\Omega}\sigma\overline{T}, \tag{2.39}$$

where $\overline{T} = T/Z$. We can then compute separately the contribution from each element from the Weyl group. Writing $H(\mathbb{A}) = H_\delta(\mathbb{A}) \times (H_\delta(\mathbb{A})\backslash H(\mathbb{A}))$, we can factor out the integral of $\overline{\psi_{\mathbf{m}_1}} \otimes \psi_{\mathbf{m}_2}$ over the compact group $H_\delta(\mathbb{Q})\backslash H_\delta(\mathbb{A})$ in (2.38). Therefore, $I_\delta(f)$ vanishes unless the character $\overline{\psi_{\mathbf{m}_1}} \otimes \psi_{\mathbf{m}_2}$ is trivial on $H_\delta(\mathbb{A})$. Following Knightly and Li (and Jacquet), we shall call the orbits $H \cdot \delta$ such that $\overline{\psi_{\mathbf{m}_1}} \otimes \psi_{\mathbf{m}_2}$ is trivial on $H_\delta(\mathbb{A})$ **relevant**.

5.1. Relevant orbits. In order to characterize the relevant orbits, let us introduce a bit of notation. A set of representatives of $T(\mathbb{Q})/Z(\mathbb{Q})$ is given by the elements

$$\delta_1 \doteq \begin{bmatrix} d_1 & & & \\ & 1 & & \\ & & d_2 & \\ & & & d_1 d_2 \end{bmatrix}, d_1, d_2 \in \mathbb{Q}^\times. \tag{2.40}$$

For each $\sigma \in \Omega$, the corresponding set of representatives of $\sigma\overline{T}(\mathbb{Q})$ in (2.39) is given by elements of the form

$$\delta_\sigma = \sigma\delta_1, \tag{2.41}$$

and $H_{\delta_\sigma}(\mathbb{A})$ consists of pairs $(\mathsf{u}, \delta_\sigma^{-1}\mathsf{u}\delta_\sigma) = (\mathsf{u}, \delta_1^{-1}\sigma^{-1}\mathsf{u}\delta_1\sigma)$ such that both component lie in $U(\mathbb{A})$. Since conjugation by δ_1 preserves $U(\mathbb{A})$, the condition that the second component lies in $U(\mathbb{A})$ is equivalent to $\mathsf{u} \in U(\mathbb{A})$ and $\sigma^{-1}\mathsf{u}\sigma \in U(\mathbb{A})$, and hence $\mathsf{u} \in U_\sigma(\mathbb{A})$. We accordingly make the following definition.

Definition 2.5.1. For $\sigma \in \Omega$, define

$$D_\sigma(\mathbb{A}) = U_\sigma(\mathbb{A}) \times \sigma^{-1}U_\sigma(\mathbb{A})\sigma. \tag{2.42}$$

Then we have

$$H_{\delta_\sigma}(\mathbb{A}) = \{(\mathsf{u}, \delta_\sigma^{-1}\mathsf{u}\delta_\sigma) : \mathsf{u} \in U_\sigma(\mathbb{A})\} \subset D_\sigma(\mathbb{A}). \tag{2.43}$$

Lemma 2.5.1. *The relevant orbits are the ones corresponding to the following elements:*

- $\sigma = 1$ *with* $\delta_1 = {t_{\mathbf{m}_2}}^{-1} t_{\mathbf{m}_1} = \begin{bmatrix} \frac{\mathbf{m}_{11}}{\mathbf{m}_{21}} & & & \\ & 1 & & \\ & & \frac{\mathbf{m}_{11}\mathbf{m}_{12}}{\mathbf{m}_{21}\mathbf{m}_{22}} & \\ & & & \frac{\mathbf{m}_{11}^2\mathbf{m}_{12}}{\mathbf{m}_{21}^2\mathbf{m}_{22}} \end{bmatrix}$,
- $\sigma = s_1s_2s_1$ *with* δ_1 *satisfying* $d_1\mathbf{m}_{12} = d_2\mathbf{m}_{22}$,
- $\sigma = s_2s_1s_2$ *with* δ_1 *satisfying* $\mathbf{m}_{11} = -d_1\mathbf{m}_{21}$,
- $\sigma = s_1s_2s_1s_2 = \mathsf{J}$ *with no condition on* δ_1.

Proof. For each representative δ_σ as in (2.41), let us fix $\mathsf{u}_1 \in U_\sigma(\mathbb{A})$, and compute $\delta_\sigma^{-1}\mathsf{u}_1\delta_\sigma$ in order to determine under which condition $\psi_{\mathbf{m}_1} \otimes \overline{\psi_{\mathbf{m}_2}}$ is trivial on $H_{\delta_\sigma}(\mathbb{A})$. For $\sigma = 1$, we have $U_\sigma = U$, hence we may take $\mathsf{u}_1 = \begin{bmatrix} 1 & & c & a-cx \\ x & 1 & a & b \\ & & 1 & -x \\ & & & 1 \end{bmatrix}$. Then we have $\delta^{-1}\mathsf{u}_1\delta = \begin{bmatrix} 1 & & c\frac{d_2}{d_1} & (a-cx)d_2 \\ xd_1 & 1 & ad_2 & bd_1d_2 \\ & & 1 & -xd_1 \\ & & & 1 \end{bmatrix}$. Thus, by (2.5), the condition that $\psi_{\mathbf{m}_1} \otimes \overline{\psi_{\mathbf{m}_2}}$ be trivial on $H_{\delta_1}(\mathbb{A})$ is equivalent to $\delta_1 = {t_{\mathbf{m}_2}}^{-1}t_{\mathbf{m}_1}$.

For $\sigma = s_1$, we have $U_\sigma(\mathbb{A}) = \left\{ \begin{bmatrix} 1 & & c & a \\ & 1 & a & b \\ & & 1 & \\ & & & 1 \end{bmatrix} : a, b, c \in \mathbb{A} \right\}$, and if $\mathsf{u}_1 = \begin{bmatrix} 1 & & c & a \\ & 1 & a & b \\ & & 1 & \\ & & & 1 \end{bmatrix}$, then $\delta^{-1}\mathsf{u}_1\delta = \begin{bmatrix} 1 & & b\frac{d_2}{d_1} & ad_2 \\ & 1 & ad_2 & cd_1d_2 \\ & & 1 & \\ & & & 1 \end{bmatrix}$, hence the condition that $\psi_{\mathbf{m}_1} \otimes \overline{\psi_{\mathbf{m}_2}}$ be trivial on $H_{\delta_{s_1}}(\mathbb{A})$ is equivalent to $\theta\left(\mathbf{m}_{12}c - \mathbf{m}_{22}\frac{d_2}{d_1}b\right) = 1$ for all $b, c \in \mathbb{A}$, which is equivalent to $\mathbf{m}_{12} = \mathbf{m}_{22} = 0$ and thus contradicts the fact that $\psi_{\mathbf{m}_1}$ and $\psi_{\mathbf{m}_2}$ are generic.

Similar calculations show that $\sigma = s_2, s_1s_2$ and s_2s_1 yield no relevant orbit.

For $\sigma = s_1 s_2 s_1$ we have $U_\sigma(\mathbb{A}) = \left\{ \begin{bmatrix} 1 & & c & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} : c \in \mathbb{A} \right\}$, and if $\mathsf{u}_1 = \begin{bmatrix} 1 & & c & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$ then we have $\delta^{-1}\mathsf{u}_1\delta = \begin{bmatrix} 1 & & c\frac{d_2}{d_1} & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$, hence the condition that $\psi_{\mathbf{m}_1} \otimes \overline{\psi_{\mathbf{m}_2}}$ be trivial on $H_{\delta_{s_{121}}}(\mathbb{A})$ is equivalent to $\theta\left(\left(\mathbf{m}_{12} - \mathbf{m}_{22}\frac{d_2}{d_1}\right) c\right) = 1$ for all $c \in \mathbb{A}$. This is equivalent to $d_1\mathbf{m}_{12} = d_2\mathbf{m}_{22}$.

The calculation for $\sigma = s_2 s_1 s_2$ is similar. Finally, for $\sigma = s_1 s_2 s_1 s_2 = \mathsf{J}$ the long Weyl element, $H_{\delta_{s_{1212}}}(\mathbb{A})$ is trivial. $\square$

A case by case calculation also shows the following.

LEMMA 2.5.2. *Let $\sigma \in \Omega$. Then there exists $\delta \in \overline{T}(\mathbb{Q})$ such that the orbit of $\delta_\sigma = \sigma\delta$ is relevant if and only if $U_\sigma = \{\mathsf{u} \in U : \sigma^{-1}\mathsf{u}\sigma = \mathsf{u}\}$.*

In the sequel, we shall call such elements of the Weyl group **relevant** as well. In particular, by definition of the relevant orbits, and by (2.43), we have the following.

COROLLARY 2.5.1. *Suppose that the orbit of $\delta_\sigma = \sigma\delta$ is relevant. Then for all $\mathsf{u} \in U_\sigma$ we have $\psi_{\mathbf{m}_2}(\delta^{-1}\mathsf{u}\delta) = \psi_{\mathbf{m}_1}(\mathsf{u})$.*

5.2. General shape of the relevant orbital integrals. In this subsection we a uniform parameterization of the relevant orbits. We thus obtain a "uniform expression" for the relevant orbital integrals. In particular, this expression shows that the (global) relevant orbital integrals factor as a product of local orbital integrals, and thus can be analysed locally, which will be the object of the subsequent subsections.

LEMMA 2.5.3. *For each* $\delta_1 \in \overline{T}(\mathbb{Q})$ *and* $\sigma \in \Omega$, *the map*

$$\varphi : D_\sigma(\mathbb{A}) \to D_\sigma(\mathbb{A})$$
$$(\mathsf{u}_1, \mathsf{u}_2) \mapsto (\mathsf{u}_1, \delta_\sigma^{-1}\mathsf{u}_1^{-1}\delta_\sigma\mathsf{u}_2)$$

induces a bijective map

$$\begin{aligned} H_{\delta_\sigma}(\mathbb{Q})\backslash D_\sigma(\mathbb{A}) &\to (U_\sigma(\mathbb{Q}) \times \{1\})\backslash D_\sigma(\mathbb{A}) \\ &\cong (U_\sigma(\mathbb{Q})\backslash U_\sigma(\mathbb{A})) \times \left(\sigma^{-1}U_\sigma(\mathbb{A})\sigma\right) \end{aligned}$$

preserving the quotient measures.

PROOF. To prove φ is well defined it is sufficient to prove that for any $(\mathsf{u}_1, \mathsf{u}_2) \in U_\sigma(\mathbb{A}) \times \sigma^{-1}U_\sigma(\mathbb{A})$ we have $\varphi_2(\mathsf{u}_1, \mathsf{u}_2) = \delta_\sigma^{-1}\mathsf{u}_1^{-1}\delta_\sigma\mathsf{u}_2 \in \sigma^{-1}U_\sigma(\mathbb{A})$. This is equivalent to the condition $\sigma\varphi_2(\mathsf{u}_1, \mathsf{u}_2)\sigma^{-1} \in U_\sigma(\mathbb{A})$, which in turn is equivalent to

$$\begin{cases} \sigma\varphi_2(\mathsf{u}_1, \mathsf{u}_2)\sigma^{-1} \in U(\mathbb{A}) \\ \varphi_2(\mathsf{u}_1, \mathsf{u}_2) \in U(\mathbb{A}). \end{cases}$$

But $\varphi_2(\mathsf{u}_1, \mathsf{u}_2) = \delta_1^{-1}\sigma^{-1}\mathsf{u}_1^{-1}\sigma\delta_1\mathsf{u}_2$, and since $\mathsf{u}_1 \in U_\sigma(\mathbb{A})$, we have $\sigma^{-1}\mathsf{u}_1^{-1}\sigma \in U(\mathbb{A})$ and it follows $\varphi_2(\mathsf{u}_1, \mathsf{u}_2) \in U(\mathbb{A})$ as desired. On the other hand,

$$\begin{aligned} \sigma\varphi_2(\mathsf{u}_1, \mathsf{u}_2)\sigma^{-1} &= \sigma\delta_1^{-1}\sigma^{-1}\mathsf{u}_1^{-1}\sigma\delta_1\mathsf{u}_2\sigma^{-1} \\ &= (\sigma\delta_1\sigma^{-1})^{-1}\mathsf{u}_1^{-1}(\sigma\delta_1\sigma^{-1})\sigma\mathsf{u}_2\sigma^{-1}. \end{aligned}$$

By definition of the Weyl group, $\sigma\delta_1\sigma^{-1} \in T(\mathbb{A})$ so $(\sigma\delta_1\sigma^{-1})^{-1}\mathsf{u}_1^{-1}(\sigma\delta_1\sigma^{-1}) \in U(\mathbb{A})$. Furthermore, $\sigma u_2\sigma^{-1} \in U_\sigma(\mathbb{A}) \subset U(\mathbb{A})$ and it also follows that $\sigma\varphi_2(\mathsf{u}_1, \mathsf{u}_2)\sigma^{-1} \in U(\mathbb{A})$.

Next, for $\mathsf{h} = (\mathsf{h}_1, \mathsf{h}_2) \in H_{\delta_\sigma}(\mathbb{Q})$, we clearly have $\varphi(\mathsf{h}) = (\mathsf{h}_1, 1)$, and

$$\begin{aligned}\varphi(\mathsf{h}(\mathsf{u}_1, \mathsf{u}_2)) &= (\mathsf{h}_1\mathsf{u}_1, \delta_\sigma^{-1}\mathsf{u}_1^{-1}\underbrace{\mathsf{h}_1^{-1}\delta_\sigma\mathsf{h}_2}_{=\delta_\sigma}\mathsf{u}_2)\\ &= \varphi(\mathsf{h})\varphi(\mathsf{u}_1, \mathsf{u}_2).\end{aligned}$$

Finally if we define $\psi(\mathsf{u}_1, \mathsf{u}_2) = (\mathsf{u}_1^{-1}, \mathsf{u}_2)$, then $\psi \circ \varphi$ is an involution, and in particular φ is bijective, which establishes the lemma. □

COROLLARY 2.5.2. *Let $\delta_1 \in \overline{T}(\mathbb{Q})$ and σ be a relevant element of the Weyl group. We have a measure preserving map*

$$\begin{aligned}\varphi : H_{\delta_\sigma}(\mathbb{Q})\backslash H(\mathbb{A}) &\to (U_\sigma(\mathbb{Q})\backslash U_\sigma(\mathbb{A})) \times (U_\sigma(\mathbb{A})\backslash U(\mathbb{A})) \times U(\mathbb{A})\\ (\mathsf{x}, \mathsf{y}) &\mapsto (U_\sigma(\mathbb{Q})\mathsf{u}_1, U_\sigma(\mathbb{A})\mathsf{u}_2, \mathsf{u}_3)\end{aligned}$$

with $\mathsf{u}_1\mathsf{u}_2 = \mathsf{x}$ and $\mathsf{u}_3 = \delta_\sigma^{-1}\mathsf{u}_1^{-1}\delta_\sigma\mathsf{y}$.

REMARK 2.5.1. *The assumption that σ is relevant is not really needed here, but it simplifies slightly the proof.*

PROOF. The quotient $U_\sigma\backslash U$ may be identified with $\overline{U}_\sigma$, and the map $U_\sigma \times \overline{U}_\sigma, (\mathsf{u}_\sigma, \mathsf{u}_1) \mapsto \mathsf{u}_\sigma\mathsf{u}_1$ preserves the Haar measures. Define $\overline{D}_\sigma = \overline{U}_\sigma \times \overline{U}_\sigma$. Using that σ is relevant and hence, by Lemma 2.5.2, that $D_\sigma(\mathbb{A}) = U_\sigma(\mathbb{A}) \times U_\sigma(\mathbb{A})$, we obtain a measure preserving map

$$\begin{aligned}H_{\delta_\sigma}(\mathbb{Q})\backslash H(\mathbb{A}) &\to (H_{\delta_\sigma}(\mathbb{Q})\backslash D_\sigma(\mathbb{A})) \times \overline{D}_\sigma\\ H_{\delta_\sigma}(\mathbb{Q})(\mathsf{x}, \mathsf{y}) &\mapsto (H_{\delta_\sigma}(\mathbb{Q})(\mathsf{x}_\sigma, \mathsf{y}_\sigma), (\mathsf{x}_1, \mathsf{y}_1)).\end{aligned}$$

Composing the first coordinate with the map obtained in Lemma 2.5.3, we get a measure preserving map

$$H_{\delta_\sigma}(\mathbb{Q})\backslash H(\mathbb{A}) \to ((U_\sigma(\mathbb{Q}) \times \{1\})\backslash D_\sigma(\mathbb{A})) \times \overline{D}_\sigma$$
$$H_{\delta_\sigma}(\mathbb{Q})(\mathsf{x},\mathsf{y}) \mapsto \left((U_\sigma(\mathbb{Q}) \times \{1\})\,(\mathsf{x}_\sigma, \delta_\sigma^{-1}\mathsf{x}_\sigma^{-1}\delta_\sigma\mathsf{y}_\sigma), (\mathsf{x}_1,\mathsf{y}_1)\right).$$

Finally, composing with $U_\sigma(\mathbb{A}) \times \overline{U}_\sigma \to U(\mathbb{A}), (\mathsf{y}_\sigma, \mathsf{y}_1) \mapsto \mathsf{y}_\sigma\mathsf{y}_1$ we obtain

$$H_{\delta_\sigma}(\mathbb{Q})\backslash H(\mathbb{A}) \to (U_\sigma(\mathbb{Q})\backslash U_\sigma(\mathbb{A})) \times \overline{U}_\sigma(\mathbb{A}) \times U(\mathbb{A})$$
$$H_{\delta_\sigma}(\mathbb{Q})(\mathsf{x},\mathsf{y}) \mapsto \left(U_\sigma(\mathbb{Q})\mathsf{x}_\sigma, \mathsf{x}_1, \delta_\sigma^{-1}\mathsf{x}_\sigma^{-1}\delta_\sigma\mathsf{y}\right).$$

□

PROPOSITION 2.5.1. *Let $H \cdot \delta_\sigma$ be a relevant orbit. Then the integral (2.38) can be expressed as*

$$I_{\delta_\sigma}(f) = \int_{U_\sigma(\mathbb{A})\backslash U(\mathbb{A})} \int_{U(\mathbb{A})} f(\mathsf{t}_1^{-1}\mathsf{u}\delta_\sigma\mathsf{u}_1\mathsf{t}_2)\psi_{\mathbf{m}_1}(\mathsf{u})\psi_{\mathbf{m}_2}(\mathsf{u}_1)\,d\mathsf{u}\,d\mathsf{u}_1. \tag{2.44}$$

Moreover, it factors as $I_{\delta_\sigma}(f) = I_{\delta_\sigma}(f_\infty)I_{\delta_\sigma}(f_{\mathrm{fin}})$, where we have set $f_{\mathrm{fin}} = \prod_p f_p$.

REMARK 2.5.2. *Note that the integral is well-defined by Corollary 2.5.1.*

REMARK 2.5.3. *By Assumption 2.1, the support of f_∞ is included in $G^\circ(\mathbb{R}) = \{\mathsf{g} \in G(\mathbb{R}), \mu(\mathsf{g}) > 0\}$. Therefore, if $\delta_1 = \begin{bmatrix} d_1 & & & \\ & 1 & & \\ & & d_2 & \\ & & & d_1d_2 \end{bmatrix}$, we have $I_{\delta_\sigma}(f_\infty) \neq 0$ only if $d_1d_2 > 0$.*

PROOF. By Corollary 2.5.2 we can make the change of variable $(\mathsf{u}_1,\mathsf{u}_2,\mathsf{u}_3)=\varphi(\mathsf{x},\mathsf{y})$ in (2.38). So we get

$$I_\delta(f)=\int_{U_\sigma(\mathbb{Q})\backslash U_\sigma(\mathbb{A})}\int_{U_\sigma(\mathbb{A})\backslash U(\mathbb{A})}\int_{U(\mathbb{A})} f(\mathsf{t}_1^{-1}\mathsf{u}_2^{-1}\delta_\sigma\mathsf{u}_3\mathsf{t}_2)) \times\overline{\psi_{\mathbf{m}_1}(\mathsf{u}_1\mathsf{u}_2)}\psi_{\mathbf{m}_2}(\delta_\sigma^{-1}\mathsf{u}_1\delta_\sigma\mathsf{u}_3)\,d\mathsf{u}_3\,d\mathsf{u}_2\,d\mathsf{u}_1. \tag{2.45}$$

We have

$$\begin{aligned}\overline{\psi_{\mathbf{m}_1}(\mathsf{u}_1\mathsf{u}_2)}\psi_{\mathbf{m}_2}(\delta_\sigma^{-1}\mathsf{u}_1\delta_\sigma\mathsf{u}_3)&=\overline{\psi_{\mathbf{m}_1}(\mathsf{u}_2)\psi_{\mathbf{m}_1}(\mathsf{u}_1)}\psi_{\mathbf{m}_2}(\delta_\sigma^{-1}\mathsf{u}_1\delta_\sigma)\psi_{\mathbf{m}_2}(\mathsf{u}_3)\\&=\overline{\psi_{\mathbf{m}_1}(\mathsf{u}_2)}\psi_{\mathbf{m}_2}(\mathsf{u}_3)\end{aligned}$$

since $(\mathsf{u}_1,\delta_\sigma^{-1}\mathsf{u}_1\delta_\sigma)\in H_\delta(\mathbb{A})$ and we assume $H\cdot\delta_\sigma$ is relevant orbit. Reporting this equality in (2.45), we get

$$I_{\delta_\sigma}(f)=\int_{U_\sigma(\mathbb{A})\backslash U(\mathbb{A})}\int_{U(\mathbb{A})} f(\mathsf{t}_1^{-1}\mathsf{u}_2^{-1}\delta_\sigma\mathsf{u}_3\mathsf{t}_2)\overline{\psi_{\mathbf{m}_1}(\mathsf{u}_2)}\psi_{\mathbf{m}_2}(\mathsf{u}_3)\,d\mathsf{u}_3\,d\mathsf{u}_2.$$

Write $\mathsf{u}_3=\mathsf{u}_\sigma\mathsf{u}_1$ with $\mathsf{u}_\sigma\in U_\sigma$ and $\mathsf{u}_1\in U_\sigma\backslash U$. Then by Lemma 2.5.2 we have

$$\mathsf{u}_2^{-1}\delta_\sigma\mathsf{u}_3=\mathsf{u}_2^{-1}\sigma\delta\mathsf{u}_\sigma\mathsf{u}_1=\mathsf{u}_2^{-1}\sigma\delta\mathsf{u}_\sigma\delta^{-1}\delta\mathsf{u}_1=\mathsf{u}_2^{-1}\delta\mathsf{u}_\sigma\delta^{-1}\sigma\delta\mathsf{u}_1,$$

and by Corollary 2.5.1 we have

$$\begin{aligned}\overline{\psi_{\mathbf{m}_1}(\mathsf{u}_2)}\psi_{\mathbf{m}_2}(\mathsf{u}_3)&=\overline{\psi_{\mathbf{m}_1}(\mathsf{u}_2)}\psi_{\mathbf{m}_2}(\mathsf{u}_\sigma\mathsf{u}_1)\\&=\overline{\psi_{\mathbf{m}_1}(\mathsf{u}_2)}\psi_{\mathbf{m}_1}(\delta\mathsf{u}_\sigma\delta^{-1})\psi_{\mathbf{m}_2}(\mathsf{u}_1)=\overline{\psi_{\mathbf{m}_1}(\delta\mathsf{u}_\sigma^{-1}\delta^{-1}\mathsf{u}_2)}\psi_{\mathbf{m}_2}(\mathsf{u}_1).\end{aligned}$$

Setting $\mathsf{u}=\delta\mathsf{u}_\sigma^{-1}\delta^{-1}\mathsf{u}_2$ we get the result. □

5.3. The Archimedean orbital integrals. In this subsection we use the integral transform that was discussed in § 3.4.5 to express the Archimedean part of the relevant orbital integrals in terms of spherical transform $\tilde{f}_\infty$ occurring on the spectral side of the relative trace formula. Moreover, under a conjectural interchange of integral (due to Buttcane), we obtain a neater expression involving the quantity $\tilde{f}_\infty(-i\nu)W(i\nu, \mathsf{t}_1\mathsf{t}_{\mathbf{m}_1}^{-1}, \psi_{\mathbf{1}})W(-i\nu, \mathsf{t}_2\mathsf{t}_{\mathbf{m}_2}^{-1}, \overline{\psi_{\mathbf{1}}})$ occurring in the spectral side of the relative trace formula, together with some *generalised Bessel functions* $K_\sigma(-i\nu, \cdot, \psi_{\mathbf{1}})$.

By Theorem 2.3.4 and using (2.28) we have the following

LEMMA 2.5.4. *Let $H \cdot \delta_\sigma$ be a relevant orbit. Then the corresponding Archimedean orbital integral $I_{\delta_\sigma}(f_\infty)$ is given by the following expression*

$$\frac{1}{c}\frac{\Delta_\sigma(\mathsf{t}_{\mathbf{m}_2})}{|\mathbf{m}_{11}^4\mathbf{m}_{12}^3|}\int_{U_\sigma(\mathbb{R})\backslash U(\mathbb{R})}\int_{\mathfrak{a}^*}\tilde{f}_\infty(-i\nu)W(i\nu, \mathsf{t}_{\mathbf{m}_1}^{-1}\mathsf{t}_1, \psi_{\mathbf{1}})$$
$$\times W(-i\nu, \mathsf{t}_{\mathbf{m}_1}^{-1}\delta_\sigma\mathsf{t}_{\mathbf{m}_2}\mathsf{u}_1\mathsf{t}_{\mathbf{m}_2}^{-1}\mathsf{t}_2, \overline{\psi_{\mathbf{1}}})\frac{d\nu}{c(i\nu)c(-i\nu)}\psi_{\mathbf{1}}(\mathsf{u}_1)\,d\mathsf{u}_1,$$

where the constant c is the one appearing in the spherical inversion theorem and Δ_σ is the modulus character of the group $U_\sigma(\mathbb{R})\backslash U(\mathbb{R})$.

Note that the above integral is well-defined. More generally, let ψ be a generic character, let σ be a relevant element of the Weyl group and let $g : i\mathfrak{a}^* \to \mathbb{C}$ be a measurable function. Then by Lemma 2.5.2, for all $\mathsf{t} \in G(\mathbb{R})$ the integral

$$\int_{U_\sigma(\mathbb{R})\backslash U(\mathbb{R})}\int_{\mathfrak{a}^*} g(-i\nu)W(-i\nu, \mathsf{y}\sigma\mathsf{u}_1\mathsf{t}, \psi)\,d\nu\,\psi(\mathsf{u}_1)\,d\mathsf{u}_1$$

is well defined as long as for all $\mathsf{u} \in U_\sigma(\mathbb{R})$ the commutator $\mathsf{y}\mathsf{u}\mathsf{y}^{-1}\mathsf{u}^{-1}$ belongs to $U_0(\mathbb{R}) = \left\{ \begin{bmatrix} 1 & & & a \\ & 1 & a & b \\ & & 1 & \\ & & & 1 \end{bmatrix}, a, b \in \mathbb{R} \right\}$. The following conjecture due to Buttcane [**But20a**], if true, will enable us to take the $\mathfrak{a}^*$ integral out in Lemma 2.5.4.

Conjecture 2.5.1 (Interchange of integral). *Let g be holomorphic with rapid decay on an open tube domain of $\mathfrak{a}^*_{\mathbb{C}}$ containing $\mathfrak{a}^*$, and let $\mathsf{t} \in G(\mathbb{R})$. Let ψ be a generic character, and let σ be a relevant element of the Weyl group. Then for almost all $\mathsf{y} \in \mathrm{Sp}_4(\mathbb{R})$ satisfying $\mathsf{y}\mathsf{u}\mathsf{y}^{-1}\mathsf{u}^{-1} \in U_0(\mathbb{R})$ for all $\mathsf{u} \in U_\sigma(\mathbb{R})$ we have*

$$\int_{U_\sigma(\mathbb{R})\backslash U(\mathbb{R})} \int_{\mathfrak{a}^*} g(-i\nu) W(-i\nu, \mathsf{y}\sigma\mathsf{u}_1\mathsf{t}, \psi)\, d\nu\, \overline{\psi(\mathsf{u}_1)}\, d\mathsf{u}_1 = \int_{\mathfrak{a}^*} g(-i\nu) \tilde{K}_\sigma(-i\nu, \mathsf{y}, \mathsf{t})\, d\nu$$

where

$$\tilde{K}_\sigma(-i\nu, \mathsf{y}, \mathsf{t}) = \lim_{R \to 0} \int_{U_\sigma(\mathbb{R})\backslash U(\mathbb{R})} h\left(\frac{\|\mathsf{u}_1\|}{R}\right) W(-i\nu, \mathsf{y}\sigma\mathsf{u}_1\mathsf{t}, \psi)\overline{\psi(\mathsf{u}_1)}\, d\mathsf{u}_1,$$

for some fixed, smooth, compactly supported h with $h(0) = 1$. Moreover $\tilde{K}_\sigma$ is entire in ν and smooth and polynomially bounded in t and y for $\Re(-i\nu)$ in some fixed compact set.

Note that Conjecture 2.5.1 is not needed for $\sigma = 1$ since in this case we have $U_\sigma = U$ and hence $\tilde{K}_\sigma(-i\nu, \mathsf{y}, \mathsf{t}) = W(-i\nu, \mathsf{y}\sigma\mathsf{t}, \psi)$. Consider now the case of $\sigma = \mathsf{J}$ the long Weyl element. In this case U_σ is trivial. Let $\mathsf{u} \in U(\mathbb{R})$ and $\mathsf{k} \in K_\infty$. Then changing variables and using the fact that the map $\mathrm{GSp}_4(\mathbb{R}) \to \mathbb{C} : \mathsf{g} \mapsto W(-i\nu, \mathsf{y}\mathsf{g}, \psi)$ is right-K_∞ invariant we have for all $\mathsf{u} \in U(\mathbb{R}), \mathsf{t} \in T(\mathbb{R})$ and $\mathsf{k} \in K_\infty$

$$\tilde{K}_\sigma(-i\nu, \mathsf{y}, \mathsf{utk}) = \psi(\mathsf{u})\tilde{K}_\sigma(-i\nu, \mathsf{y}, \mathsf{t}). \tag{2.46}$$

Moreover $\tilde{K_\sigma}(-i\nu, \mathsf{y}, \cdot)$ is an eigenfunction of the centre of the universal enveloping algebra in each variable, with eigenvalue matching those of $W(-i\nu, \cdot, \psi)$. It follows from the uniqueness of the Whittaker model that

$$\tilde{K_\sigma}(-i\nu, \mathsf{y}, \mathsf{t}) = K_\sigma(-i\nu, \mathsf{y}, \psi) W(-i\nu, \mathsf{t}, \psi)$$

for some function $K_\sigma(-i\nu, \mathsf{y}, \psi)$ that we call the long Weyl element **Bessel function**. $K_\sigma(-i\nu, \cdot, \psi)$ is itself an eigenfunction of the centre of the universal enveloping algebra with eigenvalue matching those of $W(-i\nu, \cdot, \psi)$, and satisfies for all $\mathsf{u} \in U(\mathbb{R})$ the transformation rule

$$K_\sigma(-i\nu, \mathsf{u}\mathsf{y}, \psi) = \psi(\mathsf{u}) K_\sigma(-i\nu, \mathsf{y}, \psi) = K_\sigma(-i\nu, \mathsf{y}\sigma\mathsf{u}\sigma^{-1}, \psi). \tag{2.47}$$

For the remaining two relevant elements of the Weyl group $\tilde{K_\sigma}(-i\nu, \mathsf{y}, \cdot)$ still satisfies relation (2.46) for all $\mathsf{u} \in U(\mathbb{R})$. Thus there is still a factorisation $\tilde{K_\sigma}(-i\nu, \mathsf{y}, \mathsf{t}) = K_\sigma(-i\nu, \mathsf{y}, \psi) W(-i\nu, \mathsf{t}, \psi)$ for appropriate y, where $K_\sigma(-i\nu, \mathsf{y}, \psi)$ is defined to be the σ-**Bessel function**. However because of the restriction on y, the conditions satisfied by $K_\sigma(-i\nu, \mathsf{y}, \psi)$ are more complicated.

Buttcane has announced a proof for Conjecture 2.5.1 in a more general context, but as far as we are aware the proof is not publicly available yet. Assuming Conjecture 2.5.1 yields a uniform expression for the Archimedean integrals attached to the various elements of the Weyl group.

Proposition 2.5.2. *Assume Conjecture 2.5.1. Let $H \cdot \delta_\sigma$ be a relevant orbit. Then the corresponding Archimedean orbital integral is given by*

$$I_{\delta_\sigma}(f_\infty) = \frac{1}{c}\frac{\Delta_\sigma(\mathtt{t}_{\mathbf{m}_2})}{|\mathbf{m}_{11}^4\mathbf{m}_{12}^3|}\int_{\mathfrak{a}^*}\tilde{f}_\infty(-i\nu)K_\sigma(-i\nu,\mathtt{t}_{\mathbf{m}_1}^{-1}\sigma\delta\mathtt{t}_{\mathbf{m}_2}\sigma^{-1},\overline{\psi_{\mathbf{1}}})$$
$$\times W(i\nu,\mathtt{t}_{\mathbf{m}_1}^{-1}\mathtt{t}_1,\psi_{\mathbf{1}})W(-i\nu,\mathtt{t}_{\mathbf{m}_2}^{-1}\mathtt{t}_2,\overline{\psi_{\mathbf{1}}})\frac{d\nu}{c(i\nu)c(-i\nu)}$$

where the constant c is the one appearing in the spherical inversion theorem and Δ_σ is the modular character of the group $U_\sigma(\mathbb{R})\backslash U(\mathbb{R})$.

Proof. We apply the statement of Conjecture 2.5.1 to the integral in Lemma 2.5.4 for the function defined by

$$g(-i\nu) = \frac{1}{c(i\nu)c(-i\nu)}\tilde{f}_\infty(-i\nu)W(i\nu,\mathtt{t}_{\mathbf{m}_1}^{-1}\mathtt{t}_1,\psi_{\mathbf{m}_1}),$$

which has rapid decay by the rapid decay of $\tilde{f}_\infty$ (Theorem 2.3.1), the explicit expression (2.22) for the spectral measure, and the estimate for the Whittaker function in the spectral aspect given by Proposition 2.3.6. □

5.4. Symplectic Kloosterman sums. In this subsection, the non-Archimedean part of the orbital integrals is computed when the finite part of the test function satisfies the following.

Assumption 2.3. Recall from Assumption 2.1 that we assume $f = f_\infty \prod_p f_p$ has central character ω. We now further assume that there are two coprime positive integers N and $\mathtt{n}$ such that ω is trivial on $(1+N\hat{\mathbb{Z}}) \cap \hat{\mathbb{Z}}^\times$, and the function f_{fin} is supported on $Z(\mathbb{A}_{\text{fin}})M(\mathtt{n}, N)$ and satisfies

$$f_{\text{fin}}(\mathsf{zm}) = \frac{1}{\text{Vol}(\overline{B_1(N)})}\overline{\omega}(\mathsf{z})$$

for $\mathsf{z} \in Z(\mathbb{A}_{\text{fin}})$ and $\mathsf{m} \in M(\mathtt{n}, N)$, where

$$M(\mathtt{n}, N) = \left\{\mathsf{g} \in G(\mathbb{A}_{\text{fin}}) \cap \text{Mat}_4(\hat{\mathbb{Z}}) : \mathsf{g} \equiv \begin{bmatrix} * & & * & * \\ * & 1 & * & * \\ & & * & * \\ & & & * \end{bmatrix} \mod N, \mu(\mathsf{g}) \in \mathtt{n}\,\hat{\mathbb{Z}}^\times\right\},$$

and

$$B_1(N) = \left\{\mathsf{g} \in G(\hat{\mathbb{Z}}) : \mathsf{g} \equiv \begin{bmatrix} * & & * & * \\ * & 1 & * & * \\ & & * & * \\ & & & * \end{bmatrix} \mod N\right\}.$$

Remark 2.5.4. *With this choice, $f = \bigotimes_p f_p$, and each f_p is left and right Γ_p-invariant, where*

$$\Gamma_p = \Gamma_p(N) = \{\mathsf{g} \in G(\mathbb{Z}_p) : \mathsf{g} \equiv \begin{bmatrix} * & & * & * \\ * & 1 & * & * \\ & & * & * \\ & & & * \end{bmatrix} \mod N\}.$$

In particular, if $x, c \in \mathbb{Z}_p$ then Γ_p contains the matrix $\begin{bmatrix} 1 & & c & -cx \\ x & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix}$. Thus if ϕ is right-Γ_p-invariant for all prime p, changing variables $\mathsf{u} \mapsto \mathsf{u}\begin{bmatrix} 1 & & c & -cx \\ x & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix}$ in the integral expression of the $\psi_{\mathbf{m}}$-Whittaker coefficient of ϕ, we get

$$\mathscr{W}_{\psi_{\mathbf{m}}}(\phi)(\mathsf{g}) = \overline{\theta(m_1x + m_2c)}\mathscr{W}_{\psi_{\mathbf{m}}}(\phi)(\mathsf{g})$$

for all g. Therefore $\mathscr{W}_{\psi_{\mathbf{m}}}(\phi) = 0$ unless m_1 and m_2 are integers. Henceforth, we shall assume $\mathbf{m}_1$ and $\mathbf{m}_2$ are two pairs of integers.

Remark 2.5.5. *Note that $\Gamma = K_\infty \prod_p \Gamma_p(N)$ is contained both in the Borel, Klingen, Siegel, and paramodular congruence subgroup of level N, thus any automorphic form that is fixed by one of these groups is also fixed by Γ, and hence will appear in* *our formula. One could fix a different choice of congruence subgroup, and accordingly define different types of Kloosterman sums.*

Under Assumption 2.3, the element δ_1 in Lemma 2.5.1 must satisfy more conditions in order for the corresponding orbital integral to be non-zero.

LEMMA 2.5.5. *Let $\sigma \in \Omega$, $\delta_1 = \begin{bmatrix} d_1 & & & \\ & 1 & & \\ & & d_2 & \\ & & & d_1 d_2 \end{bmatrix}$ such that the orbit of $\delta_\sigma = \sigma\delta_1$ is relevant. Assume $I_{\delta_\sigma}(f_{\text{fin}}) \neq 0$. Then there is an integer s such that $d_1 d_2 = \pm \frac{\mathtt{n}}{s^2}$.*

PROOF. For all $\mathsf{u} \in U(\mathbb{A})$ and $\mathsf{u}_1 \in U_\sigma(\mathbb{A})\backslash U(\mathbb{A})$ we have $\mu(\mathsf{u}\delta_\sigma \mathsf{u}_1) = d_1 d_2$. So by Assumption 2.3, $\mathsf{u}\delta_\sigma\mathsf{u}_1$ belongs to the support of f only if $d_1 d_2 \in \mathbb{A}_{\text{fin}}^2 \hat{\mathbb{Z}}^\times\, \mathtt{n}$. Since $d_1 d_2$ is a rational number, there must be a rational number s such that $d_1 d_2 = \pm\frac{\mathtt{n}}{s^2}$. But the second diagonal entry of $s\sigma^{-1}\mathsf{u}\delta_\sigma\mathsf{u}_1$ is s therefore s must belong to $\hat{\mathbb{Z}}$, hence s is an integer. □

Henceforth, we shall assume δ is as in Lemma 2.5.5. By Remark 2.5.3, we could also assume that $d_1 d_2 > 0$ (which would then fix the sign in the equality $d_1 d_2 = \pm\frac{\mathtt{n}}{s^2}$ above). However, we do not need doing so for now, and we shall not, in view of possible applications with a different choice of test function at the Archimedean place.

REMARK 2.5.6. *Consider the case $N = \mathtt{n} = 1$. Then*

$$M(\mathtt{n}, N) = \mathrm{GSp}_4(\hat{\mathbb{Z}}) = \prod_p \mathrm{GSp}_4(\mathbb{Z}_p).$$

For simplicity, set $\eta = \delta_\sigma$, and if p is a prime and $x \in G(\mathbb{A})$, write x_p for the p-th component of x. Also write $\psi_{p,1}$ and $\psi_{p,2}$ for the local p-th components of the characters $\psi_{\mathbf{m}_1}$ and $\psi_{\mathbf{m}_2}$, respectively. In particular, these characters are trivial on

$U(\mathbb{Z}_p)$. Then we have

$$\begin{aligned}I_{\delta_\sigma}(f_{\mathrm{fin}}) &= \frac{1}{\mathrm{Vol}(\overline{B_1(N)})}\int_{U_\sigma(\mathbb{A}_{\mathrm{fin}})\backslash U(\mathbb{A}_{\mathrm{fin}})}\int_{U(\mathbb{A}_{\mathrm{fin}})} \mathbb{1}_{\mathrm{GSp}_4(\hat{\mathbb{Z}})}(s\mathsf{u}\eta\mathsf{v})\psi_{\mathbf{m}_1}(\mathsf{u})\psi_{\mathbf{m}_2}(\mathsf{v})\,d\mathsf{u}\,d\mathsf{v}\\ &= \prod_p \frac{1}{\mathrm{Vol}(\overline{\Gamma_p(N)})}\int_{U_\sigma(\mathbb{Q}_p)\backslash U(\mathbb{Q}_p)}\int_{U(\mathbb{Q}_p)} \mathbb{1}_{\mathrm{GSp}_4(\mathbb{Z}_p)}(s_p\mathsf{u}_p\eta_p\mathsf{v}_p)\psi_{p,1}(\mathsf{u}_p)\psi_{p,2}(\mathsf{v}_p)\,d\mathsf{u}_p\,d\mathsf{v}_p.\end{aligned}$$

For all but finitely many primes p, the entries of $s_p\eta_p$ are in $\mathbb{Z}_p^\times$. For those primes, by the explicit Bruhat decomposition (see Lemma 2.5.7 below), the condition $s_p\mathsf{u}_p\eta_p\mathsf{v}_p \in \mathrm{GSp}_4(\mathbb{Z}_p)$ is equivalent to $\mathsf{u}_p \in U(\mathbb{Z}_p)$ and $\mathsf{v}_p \in U_\sigma(\mathbb{Z}_p)\backslash U(\mathbb{Z}_p)$, and hence

$$\int_{U_\sigma(\mathbb{Q}_p)\backslash U(\mathbb{Q}_p)}\int_{U(\mathbb{Q}_p)} \mathbb{1}_{\mathrm{GSp}_4(\mathbb{Z}_p)}(s_p\mathsf{u}_p\eta_p\mathsf{v}_p)\psi_{p,1}(\mathsf{u}_p)\psi_{p,2}(\mathsf{v}_p)\,d\mathsf{u}_p\,d\mathsf{v}_p = 1.$$

*For the remaining primes p, noticing that $U_\sigma(\mathbb{Q}_p)\backslash U(\mathbb{Q}_p)$ may be identified with the subgroup $\overline{U}_\sigma(\mathbb{Q}_p) = U(\mathbb{Q}_p) \cap \sigma^\top U(\mathbb{Q}_p)\sigma^{-1}$, the local integral equals the Kloosterman sum $\mathrm{Kl}(\eta, \psi_{p,1}, \psi_{p,2})$ as defined in [**SHM20**] when $\eta \in \mathrm{Sp}_4(\mathbb{Q}_p)$ (note that we denote here by U_σ what is denoted there by $\overline{U}_{\sigma^{-1}}$, and conversely).*

We now treat separately the contribution from each relevant element of the Weyl group from a global point of view. To alleviate notations, we shall not include N and ω in the argument of the Kloosterman sums we proceed to define.

5.4.1. *The identity contribution.*

Definition 2.5.2. Let a, b, d, N be integers such that $d \mid N$. Then the following sum is well-defined

$$S(a, b, d, N) = \sum_{\substack{x,y \in \mathbb{Z}/N\mathbb{Z} \\ d \mid xy}} e\left(\frac{ax + by}{N}\right).$$

Lemma 2.5.6. *Let a, b, d, N be integers such that $d \mid N$. Write $a = \prod_i p_i^{a_i}$, where p_i are distinct primes and a_i are integers, and similarly for b, d, N. Then we have*

$$S(a, b, d, N) = \prod_i S(p_i^{a_i}, p_i^{b_i}, p_i^{d_i}, p_i^{N_i}).$$

Moreover if n is a positive integer, i, j, k are non-negative integers with $k \leq n$ and p is a prime, then we have

$$\begin{aligned} S(p^i, p^j, p^k, p^n) &= p^{2n-k}(1 - p^{-1}) \max(0, k + 1 - \max(0, n - i) - \max(0, n - j)) \\ &\quad + p^{2n-k-1} \left(\mathbb{1}_{\substack{i \geq n \\ j \geq n}} - \mathbb{1}_{\substack{i<n \\ j<n \\ i+j \geq 2n-k-1}} \right). \end{aligned}$$

In particular, it follows that $S(p^i, p^j, p^k, p^n)$ is non-zero only if

$$(n - i) + (n - j) \leq k + 1. \tag{2.48}$$

Proof. The factorization is immediate from the Chinese remainder theorem. Now let us evaluate $S = S(p^i, p^j, p^k, p^n)$. We have (here, abusing notation slightly, we set $v_p(0) = n$)

$$S = \sum_{h=0}^{k} \sum_{\substack{x \in \mathbb{Z}/p^n\mathbb{Z} \\ v_p(x) = h}} e\left(\frac{p^i x}{p^n}\right) \sum_{\substack{y \in \mathbb{Z}/p^n\mathbb{Z} \\ v_p(y) \geq k-h}} e\left(\frac{p^j y}{p^n}\right) + \sum_{\substack{x \in \mathbb{Z}/p^n\mathbb{Z} \\ v_p(x) \geq k+1}} e\left(\frac{p^i x}{p^n}\right) \sum_{y \in \mathbb{Z}/p^n\mathbb{Z}} e\left(\frac{p^j y}{p^n}\right).$$

Now if ℓ is any non-negative integer, we have

$$\sum_{\substack{y\in\mathbb{Z}/p^n\mathbb{Z}\\ v_p(y)\geq\ell}} e\left(\frac{p^j y}{p^n}\right) = \begin{cases} p^{n-\ell} \text{ if } j+\ell\geq n \text{ and } \ell\leq n \\ 0 \text{ otherwise.}\end{cases}$$

Hence

$$S = \sum_{h=0}^{k-\max(0,n-j)} p^{n-k+h} \sum_{\substack{x\in\mathbb{Z}/p^n\mathbb{Z}\\ v_p(x)=h}} e\left(\frac{p^i x}{p^n}\right) + p^{2n-k-1}\mathbb{1}_{\substack{j\geq n\\ i+k+1\geq n\\ k<n}}.$$

Now

$$\sum_{\substack{x\in\mathbb{Z}/p^n\mathbb{Z}\\ v_p(x)=h}} e\left(\frac{p^i x}{p^n}\right) = \sum_{\substack{x\in\mathbb{Z}/p^n\mathbb{Z}\\ v_p(x)\geq h}} e\left(\frac{p^i x}{p^n}\right) - \sum_{\substack{x\in\mathbb{Z}/p^n\mathbb{Z}\\ v_p(x)\geq h+1}} e\left(\frac{p^i x}{p^n}\right),$$

hence the h-sum becomes

$$\sum_{h=\max(0,n-i)}^{k-\max(0,n-j)} p^{2n-k}(1-p^{-1})$$
$$- p^{2n-k-1}\mathbb{1}_{0\leq n-i-1\leq k-\max(0,n-j)} + p^{2n-k-1}\mathbb{1}_{k-\max(0,n-j)=n},$$

so

$$S = p^{2n-k}(1-p^{-1})\max(0,k+1-\max(0,n-i)-\max(0,n-j))$$
$$+p^{2n-k-1}(\mathbb{1}_{\substack{j\geq n\\ i+k+1\geq n\\ k<n}} - \mathbb{1}_{0\leq n-i-1\leq k-\max(0,n-j)} + \mathbb{1}_{k-\max(0,n-j)=n}).$$

Finally, it can be checked by inspection of cases that

$$\mathbb{1}_{\substack{j\geq n\\ i+k+1\geq n\\ k<n}} - \mathbb{1}_{0\leq n-i-1\leq k-\max(0,n-j)} + \mathbb{1}_{k-\max(0,n-j)=n} = \mathbb{1}_{\substack{i\geq n\\ j\geq n}} - \mathbb{1}_{\substack{i<n\\ j<n\\ i+j\geq 2n-k-1}}.$$

$\square$

PROPOSITION 2.5.3. *Let* $\sigma = 1$, $\delta_1 = \begin{bmatrix} d_1 & & & \\ & 1 & & \\ & & d_2 & \\ & & & d_1 d_2 \end{bmatrix}$ *with* $d_1 d_2 = \pm\frac{\mathtt{n}}{s^2}$ *for some integer* s. *Then* $I_{\delta_\sigma}(f_{\text{fin}}) = 0$ *unless all of the following hold:*

(1) s *divides* $\mathtt{n}$*,*

(2) $d_1 = \frac{\mathbf{m}_{11}}{\mathbf{m}_{21}}$ *and* sd_1 *is an integer dividing* $\mathtt{n}$*,*

(3) $d_2 = \frac{\mathbf{m}_{11}\mathbf{m}_{12}}{\mathbf{m}_{21}\mathbf{m}_{22}}$.

If all these conditions are met, let $d = \gcd(s, sd_1 s, d_2, \frac{\mathtt{n}}{s})$*, and* $D = \gcd(sd_1, \frac{n}{s})$*. Then*

$$I_{\delta_\sigma}(f_{\text{fin}}) = \frac{\overline{\omega_N(s)}}{\mathrm{Vol}(\overline{B_1(N)})} \frac{\mathtt{n}\, d}{|s^3 d_1|} S\left(\mathbf{m}_{11}\frac{\mathtt{n}}{D}, \mathbf{m}_{12} sd_1, d, \mathtt{n}\right), \tag{2.49}$$

where $\omega_N(s) = \prod_{p|N} \omega_p(s)$.

REMARK 2.5.7. *The integer* s *is only determined up to sign. However, expression (2.49) does not depend on the sign of* s*, since* $S(a, b, d, \mathtt{n}) = S(a, -b, d, \mathtt{n})$ *and* $\omega_N(-1) = \omega(-1) = 1$ *as* $\omega_p(-1) = 1$ *for all* $p \nmid N$.

REMARK 2.5.8. *The two pairs of integers* $\mathbf{m}_1$ *and* $\mathbf{m}_2$ *essentially play symmetric roles in our formula. More precisely, for our choice of test function* f*, the operator* $\omega_N(\mathtt{n})^{\frac{1}{2}} R(f)$ *is self-adjoint. Thus exchanging* $\mathbf{m}_1$ *and* $\mathbf{m}_2$ *amounts to take the complex conjugate of the spectral side and multiply it by* $\omega_N(\mathtt{n})$*. Hence the geometric side, and in particular the identity contribution, should enjoy the same symmetries. Proposition 2.5.3 says that the identity element has a non-zero contribution only if there is an integer* t *dividing* $\mathtt{n}$ *with* $\frac{\mathtt{n}}{t} = \pm\frac{\mathbf{m}_{12}}{\mathbf{m}_{22}} t$ *and such that* $s = \frac{\mathbf{m}_{21}}{\mathbf{m}_{11}} t$ *is also an integer dividing* $\mathtt{n}$*. This condition is indeed symmetric, as interchanging* $\mathbf{m}_1$ *and* $\mathbf{m}_2$ *amounts to replace* t *with* $\frac{\mathtt{n}}{t}$ *and* s *with* $\frac{\mathtt{n}}{s}$*. In addition, we have* $S\left(\mathbf{m}_{11}\frac{\mathtt{n}}{\gcd(t, \frac{\mathtt{n}}{s})}, \mathbf{m}_{12} t, d, \mathtt{n}\right) = S\left(\mathbf{m}_{21}\frac{\mathtt{n}}{\gcd(s, \frac{\mathtt{n}}{t})}, \mathbf{m}_{22}\frac{\mathtt{n}}{t}, d, \mathtt{n}\right)$*. Finally, using that*

$|s^3 d_1| = \left|\mathtt{n}^3 \frac{\mathbf{m}_{21}^4 \mathbf{m}_{22}^3}{\mathbf{m}_{11}^4 \mathbf{m}_{12}^3}\right|^{\frac{1}{2}}$, *multiplying* $\frac{\mathtt{n}}{|s^3 d_1|}$ *by the factor* $\frac{1}{|\mathbf{m}_{11}^4 \mathbf{m}_{12}^3|}$ *that comes from the Archimedean part in Proposition 2.5.2 gives* $\mathtt{n}^{-\frac{1}{2}}(\mathbf{m}_{11}\mathbf{m}_{21})^{-2}|\mathbf{m}_{12}\mathbf{m}_{22}|^{-\frac{3}{2}}$.

REMARK 2.5.9. *In the case* $\mathtt{n} = 1$ *we must have* $s = \pm 1$ *and hence* $\mathbf{m}_{11} = \pm \mathbf{m}_{21}$. *Together with the condition* $d_1 d_2 = \pm \frac{\mathbf{m}_{11}^2 \mathbf{m}_{12}}{\mathbf{m}_{21}^2 \mathbf{m}_{22}} = \frac{\mathtt{n}}{s^2}$ *this also gives* $\mathbf{m}_{12} = \pm \mathbf{m}_{22}$.

REMARK 2.5.10. *Using condition (2.48) we find that the contribution from the identity element is non-zero only if for all prime* $p \mid \mathtt{n}$ *we have*

$$v_p(s) \le v_p(\mathbf{m}_{21}) + v_p(\mathbf{m}_{21}) + \min(0, v_p(\mathbf{m}_{21}) - v_p(\mathbf{m}_{11})) + 1,$$

which in turn implies that for all prime p *we have*

$$v_p(\mathtt{n}) \le 2\min(v_p(\mathbf{m}_{11}), v_p(\mathbf{m}_{21})) + v_p(\mathbf{m}_{12}) + v_p(\mathbf{m}_{22}) + 1.$$

PROOF. The finite part of the orbital integral corresponding to the identity element reduces to

$$I_{\delta_\sigma}(f_{\mathrm{fin}}) = \int_{U(\mathbb{A}_{\mathrm{fin}})} f(\mathsf{u}\delta)\psi_{\mathbf{m}_1}(\mathsf{u})d\mathsf{u} = \int_{U(\mathbb{A}_{\mathrm{fin}})} f(s\mathsf{u}\delta)\psi_{\mathbf{m}_1}(\mathsf{u})\,d\mathsf{u}.$$

Assume it is non-zero. Note that by Lemma 2.5.5 we have $\mu(s\mathsf{u}\delta) = \mathtt{n}$. Then by Assumption 2.3, $s\mathsf{u}\delta \in \mathrm{Supp}(f)$ if and only if $s\mathsf{u}\delta \in \mathrm{Mat}_4(\hat{\mathbb{Z}})$. In particular, each entry of $s\delta$ must be an integer. Furthermore by Lemma 2.5.1 we must have $\delta = \begin{bmatrix} d_1 & & & \\ & 1 & & \\ & & d_2 & \\ & & & d_1 d_2 \end{bmatrix}$ with $d_1 = \frac{\mathbf{m}_{11}}{\mathbf{m}_{21}}$, $d_2 = \frac{\mathbf{m}_{11}\mathbf{m}_{12}}{\mathbf{m}_{21}\mathbf{m}_{22}}$. So we learn that $sd_1 = s\frac{\mathbf{m}_{11}}{\mathbf{m}_{21}} \in \mathbb{Z}$, $s \mid \mathtt{n}$, and $sd_1 \mid \mathtt{n}$. Now let us examine the non-diagonal entries of $s\mathsf{u}\delta$. Write $u = \begin{bmatrix} 1 & & c & a-cx \\ x & 1 & a & b \\ & & 1 & -x \\ & & & 1 \end{bmatrix}$. Then the following conditions must hold:

(1) $sd_1 x \in \hat{\mathbb{Z}}$ and $\frac{\mathtt{n}}{s}x \in \hat{\mathbb{Z}}$,

(2) $c' \doteq \frac{\mathtt{n}}{sd_1}c \in \hat{\mathbb{Z}}$,

(3) $a' \doteq \frac{\mathtt{n}}{sd_1}a \in \hat{\mathbb{Z}}$,

(4) $\frac{\mathtt{n}}{s}(a - cx) \in \hat{\mathbb{Z}}$,

(5) $b' \doteq \frac{\mathtt{n}}{s}b \in \hat{\mathbb{Z}}$.

Condition (1) is equivalent to $x \in \frac{1}{D}\hat{\mathbb{Z}}$, where $D = \gcd(sd_1, \frac{\mathtt{n}}{s})$ (note that $sd_1 \mid sD$). Set $x' = Dx$. Then condition (4) gives $d_1 a' - \frac{d_1}{D}c'x' \in \hat{\mathbb{Z}}$. Combined with conditions (1), (2) and (3), this is equivalent to $c'x' \equiv Da' \mod \frac{D}{d_1}$. Now, $\psi_{\mathbf{m}_1}(u) = \theta_{\text{fin}}(\mathbf{m}_{11}x + \mathbf{m}_{12}c)$ and $f(su\delta) = \frac{\overline{\omega_N(s)}}{\operatorname{Vol}(\overline{B_1(N)})}$. Therefore integration with respect to b gives $\operatorname{Vol}\left(\frac{s}{\mathtt{n}}\hat{\mathbb{Z}}\right)\frac{\overline{\omega_N(s)}}{\operatorname{Vol}(\overline{B_1(N)})} = \frac{\mathtt{n}}{s}\frac{\overline{\omega_N(s)}}{\operatorname{Vol}(\overline{B_1(N)})}$. Next, changing variables $x = \frac{1}{D}x'$ and $c = \frac{sd_1}{\mathtt{n}}c'$, for fixed a the x,c-integral is

$$I(a) = \frac{\overline{\omega_N(s)}}{\operatorname{Vol}(\overline{B_1(N)})}\frac{\mathtt{n}^2\, D}{s^2 d_1}\int\int_{c'x' \equiv Da' \mod \frac{D}{d_1}} \theta_{\text{fin}}\left(\mathbf{m}_{11}\frac{x'}{D} + \mathbf{m}_{12}\frac{sd_1}{\mathtt{n}}c'\right) dx'\, dc'.$$

Since $D \mid sd_1$ and $\mathbf{m}_{12}\frac{s^2 d_1^2}{\mathtt{n}} = \mathbf{m}_{22}$, and since θ_{fin} is trivial on $\hat{\mathbb{Z}}$ the integrand is constant on cosets $x' + sd_1\hat{\mathbb{Z}}$ and $c' + sd_1\hat{\mathbb{Z}}$. As $sd_1 \mid sD$, it is also constant on cosets $x' + sD\hat{\mathbb{Z}}$ and $c' + sD\hat{\mathbb{Z}}$. Therefore we get

$$I(a) = \frac{\overline{\omega_N(s)}}{\operatorname{Vol}(\overline{B_1(N)})}\frac{\mathtt{n}^2}{|Dd_1 s^4|}\sum_{\substack{x,y \in \mathbb{Z}/sD\mathbb{Z} \\ xy \in Da' + \frac{D}{d_1}\hat{\mathbb{Z}}}} e\left(\frac{\mathbf{m}_{11}x}{D} + \frac{\mathbf{m}_{12}sd_1 y}{\mathtt{n}}\right).$$

Finally the a integrand depends only on $a' \mod \frac{D}{d_1}\hat{\mathbb{Z}}$, thus, setting $d = \gcd\left(D, \frac{D}{d_1}\right) = \gcd(s, sd_1sd_2, \frac{\mathtt{n}}{s})$ we get

$$\begin{aligned} I_{\delta_\sigma}(f_{\text{fin}}) &= \frac{\overline{\omega_N(s)}}{\mathrm{Vol}(\overline{B_1(N)})} \frac{\mathtt{n}^3}{|s^5D^2d_1|} \sum_{a \in \mathbb{Z}/\frac{D}{d_1}\mathbb{Z}} \sum_{\substack{x,y \in \mathbb{Z}/sD\mathbb{Z} \\ xy \in Da + \frac{D}{d_1}\hat{\mathbb{Z}}}} e\left(\frac{\mathbf{m}_{11}x}{D} + \frac{\mathbf{m}_{12}sd_1y}{\mathtt{n}}\right) \\ &= \frac{\overline{\omega_N(s)}}{\mathrm{Vol}(\overline{B_1(N)})} \frac{\mathtt{n}^3}{|s^5D^2d_1|} d \sum_{\substack{x,y \in \mathbb{Z}/sD\mathbb{Z} \\ xy \in d\mathbb{Z}}} e\left(\frac{\mathbf{m}_{11}x}{D} + \frac{\mathbf{m}_{12}sd_1y}{\mathtt{n}}\right) \\ &= \frac{\overline{\omega_N(s)}}{\mathrm{Vol}(\overline{B_1(N)})} \frac{\mathtt{n}}{|s^3d_1|} d \sum_{\substack{x,y \in \mathbb{Z}/\mathtt{n}\mathbb{Z} \\ xy \in d\mathbb{Z}}} e\left(\frac{\mathbf{m}_{11}x}{D} + \frac{\mathbf{m}_{12}sd_1y}{\mathtt{n}}\right). \qquad \square \end{aligned}$$

5.4.2. *The contribution from the longest Weyl element.* The following lemma makes it explicit how to compute the Bruhat decomposition for elements in the cell of the long Weyl element. One could do the same for each element of the Weyl group, but, as it is straightforward calculations, we only include this case for the sake of clarity in latter arguments.

LEMMA 2.5.7. *Let $\mathbb{F}$ be a field, and let $\mathsf{g} \in \mathrm{GSp}_4(\mathbb{F})$. Assume*

$$\mathsf{g} = \begin{bmatrix} 1 & & c_1 & a_1 \\ x_1 & 1 & a_1+c_1x_1 & b_1 \\ & & 1 & -x_1 \\ & & & 1 \end{bmatrix} \mathsf{J} \begin{bmatrix} t_1 & & & \\ & t_2 & & \\ & & t_3t_1^{-1} & \\ & & & t_3t_2^{-1} \end{bmatrix} \begin{bmatrix} 1 & & c_2 & a_2-c_2x_2 \\ x_2 & 1 & a_2 & b_2 \\ & & 1 & -x_2 \\ & & & 1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \left[\begin{array}{cc|cc} a_{11} & a_{12} & \multicolumn{2}{c}{B} \\ a_{21} & a_{22} & & \\ \hline c_{11} & c_{12} & d_{11} & d_{12} \\ c_{21} & c_{22} & d_{21} & d_{22} \end{array}\right].$$

Set

$$\Delta_1 = \begin{bmatrix} a_{11} & a_{12} \\ c_{21} & c_{22} \end{bmatrix}, \quad \Delta_2 = \begin{bmatrix} c_{12} & d_{11} \\ c_{22} & d_{21} \end{bmatrix}.$$

Then

$$t_3 = \mu(g), \quad t_2 = -c_{22}, \quad t_1 t_2 = \det(C),$$

$$x_1 = -\frac{c_{12}}{c_{22}}, \quad x_2 = \frac{c_{21}}{c_{22}}, \quad c_1 = \frac{\det(\Delta_1)}{\det(C)}, \quad c_2 = -\frac{\det(\Delta_2)}{\det(C)},$$

$$a_1 = \frac{a_{12}}{c_{22}} \quad a_2 = \frac{d_{21}}{c_{22}} \quad b_1 = \frac{a_{22}}{c_{22}} \quad b_2 = \frac{d_{22}}{c_{22}}.$$

Moreover, if $\mathsf{g} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathrm{GSp}_4(\mathbb{F})$ *with* $C = \left[\begin{smallmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{smallmatrix}\right]$ *satisfying* $\det(C) \neq 0$ *and* $c_{22} \neq 0$ *then* $\mathsf{g} \in U\mathsf{J}TU$.

Proof. The first claims follow by computing explicitly

$$C = \left[\begin{smallmatrix} 1 & -x_1 \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} -t_1 & \\ & -t_2 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ x_2 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} -t_1+t_2x_1x_2 & t_2x_1 \\ -t_2x_2 & -t_2 \end{smallmatrix}\right],$$

$$\Delta_1 = \left[\begin{smallmatrix} c_1 & a_1 \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} -t_1 & \\ & -t_2 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ x_2 & 1 \end{smallmatrix}\right], \qquad \Delta_2 = \left[\begin{smallmatrix} 1 & -x_1 \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} -t_1 & \\ & -t_2 \end{smallmatrix}\right]\left[\begin{smallmatrix} & c_2 \\ 1 & a_2 \end{smallmatrix}\right],$$

$$D = \left[\begin{smallmatrix} 1 & -x_1 \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} -t_1 & \\ & -t_2 \end{smallmatrix}\right]\left[\begin{smallmatrix} c_2 & a_2-c_2x_2 \\ a_2 & b_2 \end{smallmatrix}\right], \qquad A = \left[\begin{smallmatrix} c_1 & a_1 \\ a_1+c_1x_1 & b_1 \end{smallmatrix}\right]\left[\begin{smallmatrix} -t_1 & \\ & -t_2 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ x_2 & 1 \end{smallmatrix}\right].$$

To prove the last claim, it suffices to show that provided $\det C \neq 0$ and $c_{22} \neq 0$, there exist at most one $\mathsf{g} \in \mathrm{GSp}_4(\mathbb{F})$ with the specified values for $\mu(\mathsf{g})$, C, a_{12}, a_{22}, d_{21}, d_{22}, $\det(\Delta_1)$ and $\det(\Delta_2)$. Since $c_{22} \neq 0$, the values of a_{12}, c_{21} and $\det(\Delta_1) = a_{11}c_{22} - c_{21}a_{12}$ determine the value of a_{11}. The equation ${}^{\top}AC = {}^{\top}CA$ then gives $a_{12}c_{11} + a_{22}c_{21} = a_{11}c_{12} + a_{21}c_{22}$, which determines the value of a_{21} hence of A. The same reasoning using $\det(\Delta_1)$ and $C^{\top}D = D^{\top}C$ instead similarly fixes D. Finally the equation ${}^{\top}AD - {}^{\top}CB = \mu(\mathsf{g})$ fixes B since we are assuming C is invertible. $\square$

Definition 2.5.3. Let s, d, m be three non-zero integers and define

$$C_{\mathsf{J}}(s,d,m) = \{\mathsf{g} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \det(C) = d, c_{22} = -s, \mu(\mathsf{g}) = m\}$$

and

$$\Gamma_{\mathsf{J}}(N,s,d,m) = B_1(N) \cap \mathrm{Mat}_4(\mathbb{Z}) \cap C_{\mathsf{J}}(s,d,m).$$

For $\mathsf{g} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathrm{GSp}_4$, let $\Delta_1 = \left[\begin{smallmatrix} a_{11} & a_{12} \\ c_{21} & c_{22} \end{smallmatrix}\right]$ and $\Delta_2 = \left[\begin{smallmatrix} c_{12} & d_{11} \\ c_{22} & d_{21} \end{smallmatrix}\right]$. Then, for $\mathbf{m}_1, \mathbf{m}_2$ two pair of non-zero integers, we define the following generalized twisted Kloosterman sum

$$\mathrm{Kl}_{\mathsf{J}}(\mathbf{m}_1, \mathbf{m}_2, s, d, m) = \sum_{\mathsf{g} \in U(\mathbb{Z}) \backslash \Gamma_{\mathsf{J}}(N,s,d,m)/U(\mathbb{Z})} \overline{\omega_N}(a_{22}) e\left(\frac{\mathbf{m}_{11}c_{12} - \mathbf{m}_{21}c_{21}}{s} + \frac{\mathbf{m}_{12}\det(\Delta_1) - \mathbf{m}_{22}\det(\Delta_2)}{d}\right).$$

Remark 2.5.11. *Using Lemma 2.5.7, we can see that* $\mathrm{Kl}_{\mathsf{J}}(\mathbf{m}_1, \mathbf{m}_2, s, d, m)$ *is well defined. Indeed, matrices in* $\Gamma_{\mathsf{J}}(\mathtt{n}, N, d, s)$ *are of the form*

$$\mathsf{g} = \mathsf{u}(x_1, a_1, b_1, c_1)\mathsf{J}\begin{bmatrix} \frac{d}{s} & & & \\ & s & & \\ & & m\frac{s}{d} & \\ & & & \frac{m}{s} \end{bmatrix}\mathsf{u}(x_2, a_2, b_2, c_2).$$

Then $\frac{c_{12}}{s} = x_1, \frac{c_{21}}{s} = -x_2$, $\frac{\det(\Delta_1)}{d} = c_1$ *and* $\frac{\det(\Delta_2)}{d} = -c_2$. *Now multiplying* g *on the left (resp. on the right) by an element of* $U(\mathbb{Z})$ *does not change the classes of* x_1 *and* c_1 *(resp.* x_2 *and* c_2*) in* $\mathbb{R}/\mathbb{Z}$.

Proposition 2.5.4. *Let* $\sigma = \mathsf{J}$, $\delta_1 = \begin{bmatrix} d_1 & & & \\ & 1 & & \\ & & d_2 & \\ & & & d_1d_2 \end{bmatrix}$ *with* $d_1d_2 = \pm\frac{\mathtt{n}}{s^2}$ *for some integer* s. *Then we have* $I_{\delta_\sigma}(f_{\mathrm{fin}}) = \frac{1}{\mathrm{Vol}(B_1(N))}\mathrm{Kl}_J(\mathbf{m}_1, \mathbf{m}_2, s, d_1s^2, s^2d_1d_2)$.

Remark 2.5.12. *The set* $\Gamma_{\mathsf{J}}(N, s, d_1s^2, s^2d_1d_2)$ *is non-empty only if* N *divides* s *and* N^2 *divides* d_1s^2.

PROOF. The finite part of the orbital integral corresponding to the longest Weyl element reduces to

$$\begin{aligned} I_{\delta_\sigma}(f_{\mathrm{fin}}) &= \int_{U(\mathbb{A}_{\mathrm{fin}})}\int_{U(\mathbb{A}_{\mathrm{fin}})} f(\mathsf{u}_1\mathsf{J}\delta\mathsf{u}_2)\psi_{\mathbf{m}_1}(\mathsf{u}_1)\overline{\psi_{\mathbf{m}_2}(\mathsf{u}_2)}\,d\mathsf{u}_1\,d\mathsf{u}_2 \\ &= \int_{U(\mathbb{A}_{\mathrm{fin}})}\int_{U(\mathbb{A}_{\mathrm{fin}})} f(s\mathsf{u}_1\mathsf{J}\delta\mathsf{u}_2)\psi_{\mathbf{m}_1}(\mathsf{u}_1)\overline{\psi_{\mathbf{m}_2}(\mathsf{u}_2)}\,d\mathsf{u}_1\,d\mathsf{u}_2. \end{aligned}$$

By Assumption 2.3 we have $s\mathsf{u}_1\mathsf{J}\delta\mathsf{u}_2 \in \mathrm{Supp}(f)$ if and only if $s\mathsf{u}_1\mathsf{J}\delta\mathsf{u}_2 = \left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right] \in Z(\mathbb{A}_{\mathrm{fin}})M(\mathtt{n},N)$. In this case, we have $f(s\mathsf{u}_1\mathsf{J}\delta\mathsf{u}_2) = \frac{\overline{\omega_N}(a_{22})}{\mathrm{Vol}(\overline{B_1(N)})}$, and Lemma 2.5.7 shows that

$$\psi_{\mathbf{m}_1}(\mathsf{u}_1)\psi_{\mathbf{m}_2}(\mathsf{u}_2) = e\left(\frac{-\mathbf{m}_{11}c_{12}+\mathbf{m}_{21}c_{21}}{c_{22}} + \frac{\mathbf{m}_{12}\det(\Delta_1)-\mathbf{m}_{22}\det(\Delta_2)}{\det(C)}\right).$$

Moreover, f is left and right $U(\hat{\mathbb{Z}})$-invariant, and the characters $\psi_{\mathbf{m}_1}$ and $\psi_{\mathbf{m}_2}$ are trivial on $\hat{\mathbb{Z}}$. Therefore, if we consider the map $\varphi : U(\mathbb{A}_{\mathrm{fin}})\times U(\mathbb{A}_{\mathrm{fin}}) \to G(\mathbb{A}), (\mathsf{u}_1,\mathsf{u}_2)\mapsto s\mathsf{u}_1\mathsf{J}\delta\mathsf{u}_2$, we have

$$\begin{aligned} I_{\delta_\sigma}(f_{\mathrm{fin}}) = \sum_{U(\hat{\mathbb{Z}})\backslash(M(\mathtt{n},N)\cap Im(\varphi))/U(\hat{\mathbb{Z}})} & \frac{\overline{\omega_N}(a_{22})}{\mathrm{Vol}(\overline{B_1(N)})} \\ & \times e\left(\frac{-\mathbf{m}_{11}c_{12}+\mathbf{m}_{21}c_{21}}{c_{22}} + \frac{\mathbf{m}_{12}\det(\Delta_1)-\mathbf{m}_{22}\det(\Delta_2)}{\det(C)}\right). \end{aligned}$$

Now by Lemma 2.5.7, $Im(\varphi) = C_{\mathsf{J}}(s, d_1s^2, s^2d_1d_2)$. Therefore,

$$U(\hat{\mathbb{Z}})\backslash(M(\mathtt{n},N)\cap Im(\varphi))/U(\hat{\mathbb{Z}})$$

may be identified to $U(\mathbb{Z})\backslash\Gamma_{\mathsf{J}}(N,s,d_1s^2,s^2d_1d_2)/U(\mathbb{Z})$. $\square$

5.4.3. *Contribution from $\sigma = s_1s_2s_1$.*

Definition 2.5.4. Let s, d, m be three non-zero integers and define

$$C_{121}(s,d,m) = \{\mathtt{g} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \det(C) = 0, c_{22} = -s, \det(\Delta_2) = d, \mu(\mathtt{g}) = m\}$$

and

$$\Gamma_{121}(N,s,d,m) = B_1(N) \cap \mathrm{Mat}_4(\mathbb{Z}) \cap C_{121}(s,d,m).$$

For $\mathtt{g} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathrm{GSp}_4$, let $\Delta_3 = \left[\begin{smallmatrix} a_{12} & b_{11} \\ c_{22} & d_{21} \end{smallmatrix}\right]$. Then we define the following generalized twisted Kloosterman sum

$$\mathrm{Kl}_{121}(\mathbf{m}_1, \mathbf{m}_2, s, d, m) = \\ \sum_{\mathtt{g} \in U(\mathbb{Z}) \backslash \Gamma_{121}(N,s,d,m) / \overline{U}_\sigma(\mathbb{Z})} \overline{\omega_N}(a_{22}) e\left(\frac{\mathbf{m}_{11} c_{12} - \mathbf{m}_{21} c_{21}}{s} + \frac{\mathbf{m}_{12}\det(\Delta_3)}{d}\right).$$

By a similar argument as in the case of the long Weyl element, $\mathrm{Kl}_{121}(\mathbf{m}_1, \mathbf{m}_2, s, d, m)$ is well-defined, and together with the condition on δ from Lemma 2.5.1 we get the following.

Proposition 2.5.5. *Let* $\sigma = s_1 s_2 s_1$, $\delta_1 = \left[\begin{smallmatrix} d_1 & & & \\ & 1 & & \\ & & d_2 & \\ & & & d_1 d_2 \end{smallmatrix}\right]$ *with* $d_1 d_2 = \pm \frac{\mathtt{n}}{s^2}$ *for some integer* s *and* $d_1 \mathbf{m}_{12} = d_2 \mathbf{m}_{22}$. *Then we have*

$$I_{\delta_\sigma}(f_{\mathrm{fin}}) = \frac{1}{\mathrm{Vol}(\overline{B_1(N)})} \mathrm{Kl}_{121}(\mathbf{m}_1, \mathbf{m}_2, s, d_2 s^2, s^2 d_1 d_2).$$

Remark 2.5.13. *The set* $\Gamma_{121}(\mathtt{n}, N, s, d_2 s^2, s^2 d_1 d_2)$ *is non-empty only if* N *divides* s *and* N^2 *divides* $d_2 s^2$.

5.4.4. *Contribution from* $\sigma = s_2 s_1 s_2$.

DEFINITION 2.5.5. Let s, d be three non-zero integers and define

$$C_{212}(s,d,m) = \{\mathtt{g} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \det(C) = -d, c_{22} = 0, c_{21} = -s, \mu(\mathtt{g}) = m\}$$

and

$$\Gamma_{212}(N,s,d,m) = \mathrm{Mat}_4(\mathbb{Z}) \cap B_1(N) \cap C_{212}.$$

We define the following generalized twisted Kloosterman sum

$$\mathrm{Kl}_{212}(\mathbf{m}_1, \mathbf{m}_2, s, d, m) = \sum_{g \in U(\mathbb{Z}) \backslash \Gamma_{212}(N,s,d,m)/\overline{U}_\sigma(\mathbb{Z})} \overline{\omega_N}(a_{22}) e\left(\frac{\mathbf{m}_{11} c_{11} - \mathbf{m}_{22} d_{21}}{s} - \frac{\mathbf{m}_{12} \det(\Delta_1)}{d}\right).$$

By a similar argument as above, $\mathrm{Kl}_{212}(\mathbf{m}_1, \mathbf{m}_2, d, s)$ is well defined, and we have the following.

PROPOSITION 2.5.6. *Let* $\sigma = s_1 s_2 s_1$, $\delta_1 = \begin{bmatrix} d_1 & & & \\ & 1 & & \\ & & d_2 & \\ & & & d_1 d_2 \end{bmatrix}$ *with* $d_1 d_2 = \pm \frac{\mathtt{n}}{s^2}$ *for some integer* s *and* $\mathbf{m}_{11} = -d_1 \mathbf{m}_{21}$. *Then we have*

$$I_{\delta_\sigma}(f_{\mathrm{fin}}) = \frac{1}{\mathrm{Vol}(\overline{B_1(N)})} \mathrm{Kl}_{212}(\mathbf{m}_1, \mathbf{m}_2, sd_1, d_1 s^2, sd_1 d_2).$$

REMARK 2.5.14. *The set* $\Gamma_{212}(\mathtt{n}, N, d_1 s^2, d_1 s)$ *is non-empty only if* N *divides* $d_1 s$ *and* N^2 *divides* $d_1 s^2$.

6. The final formula

We now assemble the material from previous sections to obtain our relative trace formula. Let $N \geq 1$ be an integer. We define the **adelic congruence subgroup**

$B_1(N)$ to be matrices of the form $\mathsf{g}_\infty \mathsf{g}_{\mathrm{fin}}$ where $\mathsf{g}_\infty \in K_\infty$ and $\mathsf{g}_{\mathrm{fin}} \in \{\mathsf{g} \in G(\hat{\mathbb{Z}}) : \mathsf{g} \equiv \left[\begin{smallmatrix} * & & * & * \\ * & 1 & * & * \\ & & * & * \\ & & & * \end{smallmatrix}\right] \mod N\}$. Fix a character $\omega : \mathbb{Q}^\times \mathbb{R}^\times \backslash \mathbb{A}^\times \to \mathbb{C}$, that we may see as a character of the centre of $G(\mathbb{A})$. Assume that ω is trivial on $(1+N\hat{\mathbb{Z}}) \cap \hat{\mathbb{Z}}^\times$, and define $\omega_N(t) = \prod_{p|N} \omega(t_p)$. For each standard parabolic subgroup $P = N_P M_P$ (including G itself), consider the space $\mathscr{H}_P$ defined in Section 4.1. For each character χ of the centre of M_P whose restriction to the centre of G coincides with ω, let $\mathscr{G}_P(N, \chi)$ be an orthonormal basis consisting of factorizable vectors of the subspaces of functions ϕ in $\mathscr{H}_P$ that are generic, $B_1(N)$-fixed, and have central character χ. Specifically,

- If $P = G$ then $\mathscr{G}(N, \omega) = \mathscr{G}_P(N, \omega)$ consists of cuspidal eigenfunctions of the centre of the universal enveloping algebra in $L^2(Z(\mathbb{R})G(\mathbb{Q}) \backslash G(\mathbb{A}), \omega)^{B_1(N)}$,
- If $P = B$, each such character χ may be identified with a triplet of characters $(\omega_1, \omega_2, \omega_3)$ satisfying $\omega_1 \omega_2 \omega_3^2 = \omega$. Choose a set of representatives $S = \{\mathsf{k}_1, \cdots, \mathsf{k}_d\}$ of $(K \cap B(\mathbb{A})) \backslash K / B_1(N)$. Then there is a basis $(e_i)_{1 \leq i \leq d}$ of $\mathbb{C}^S$ such that functions in $\mathscr{G}_P(N, \omega_1, \omega_2, \omega_3)$ are of the form

$$\phi_j^{\mathrm{B}}(\mathsf{b}\mathsf{k}_i\gamma) = \chi(\mathsf{b}) e_j(\mathsf{k}_i)$$

  for $\mathsf{b} \in B(\mathbb{A})$, $\gamma \in B_1(N)$.
- If $P = \mathrm{P_K}$, each such character χ may be identified with a pair of characters (ω_1, ω_2) satisfying $\omega_1 \omega_2 = \omega$. Choose a set of representatives $S = \{\mathsf{k}_1, \cdots, \mathsf{k}_d\}$ of $(K \cap \mathrm{P_K}(\mathbb{A})) \backslash K / B_1(N)$. For $1 \leq i \leq d$, consider the compact subgroup of GL_2 given by $C_i = \mathrm{Proj}_{\mathrm{P_K}}^{\mathrm{GL}_2}\left(\mathrm{Stab}_{K \cap \mathrm{P_K}(\mathbb{A})}(\mathsf{k}_i)\right)$. Let $d_i = \dim(\pi^{C_i})$ and $d_\pi = \sum_1^d d_i$. Then, for each cuspidal automorphic representation π of GL_2 with central character ω_1 and whose Archimedean component is a principal

series there is a basis $(u_j)_{1\le j\le d_\pi} = ((u_{j,i})_{1\le i\le d})_{1\le j\le d_\pi}$ of $\prod_i \pi^{C_i}$ such that functions in $\mathscr{G}_P(N,\omega_1,\omega_2)$ are of the form

$$\phi^{\mathrm{K}}_{\pi,j}(\mathsf{pk}_i\gamma) = \omega_2(\mathsf{p})u_{j,i}(\mathrm{Proj}^{\mathrm{GL}_2}_{\mathrm{P_K}}(\mathsf{p}))$$

for $\mathsf{p}\in \mathrm{P_K}(\mathbb{A})$, $\gamma\in B_1(N)$. In particular each $u_{i,j}$ is a GL_2 adelic Maaß form.

- If $P=\mathrm{P_S}$, each such character χ may be identified with a pair of characters (ω_1,ω_2) satisfying $\omega_1\omega_2^2=\omega$. Choose a set of representatives $S=\{\mathsf{k}_1,\cdots,\mathsf{k}_d\}$ of $(K\cap \mathrm{P_S}(\mathbb{A}))\backslash K/B_1(N)$. Keeping notations of § 4.3.3, for $1\le i\le d$, consider the compact subgroup of GL_2 given by $C_i = \mathrm{Proj}^{\mathrm{GL}_2}_{\mathrm{P_S}}\left(\mathrm{Stab}_{K\cap \mathrm{P_S}(\mathbb{A})}(\mathsf{k}_i)\right)$. Let $d_i=\dim(\pi^{C_i})$ and $d_\pi=\sum_1^d d_i$. Then, for each cuspidal automorphic representation π of GL_2 with central character ω_1 and whose Archimedean component is a principal series there is a basis $(u_j)_{1\le j\le d_\pi} = ((u_{j,i})_{1\le i\le d})_{1\le j\le d_\pi}$ of $\prod_i \pi^{C_i}$ such that functions in $\mathscr{G}_P(N,\omega_1,\omega_2)$ are of the form

$$\phi^{\mathrm{S}}_{\pi,j}(\mathsf{pk}_i\gamma) = \omega_2\circ\mu(\mathsf{p})u_{j,i}(\mathrm{Proj}^{\mathrm{GL}_2}_{\mathrm{P_S}}(\mathsf{p}))$$

for $\mathsf{p}\in \mathrm{P_S}(\mathbb{A})$, $\gamma\in B_1(N)$. In particular each $u_{i,j}$ is a GL_2 adelic Maaß form.

Now fix an integer $\mathtt{n}>0$ coprime to N. Consider

$$M(\mathtt{n},N) = \left\{\mathsf{g}\in G(\mathbb{A}_{\mathrm{fin}})\cap \mathrm{Mat}_4(\hat{\mathbb{Z}}) : \mathsf{g}\equiv \begin{bmatrix} * & & * & * \\ * & 1 & * & * \\ & & * & * \\ & & & * \end{bmatrix} \mod N, \mu(g)\in \mathtt{n}\,\hat{\mathbb{Z}}^\times\right\}.$$

Define the $\mathtt{n}$**-th Hecke operator of level** $B_1(N)$ by

$$T_{\mathtt{n}}\phi(\mathsf{g}) = \int_{M(\mathtt{n},N)} \phi(\mathsf{gx})\,d\mathsf{x}.$$

Then for every standard parabolic subgroup P, for every element $u \in \mathscr{G}_P$ and for every $\nu \in i\mathfrak{a}_P^*$, the Eisenstein series $E(\cdot, u, \nu)$ is an eigenfunction of $T_{\mathtt{n}}$. We shall denote the corresponding eigenvalue by $\lambda_{\mathtt{n}}(u, \nu)$. Then we have the following.

THEOREM 2.6.1. *Let $\mathbf{m}_1$, $\mathbf{m}_2$ be two pairs of non-zero integers, $\mathsf{t}_1, \mathsf{t}_2 \in A^+$. Let h be a Paley-Wiener function on $\mathfrak{a}_{\mathbb{C}}$ and let c be the constant appearing in Theorem 2.3.2. Then we have*

$$c(\Sigma_{cusp} + \Sigma_B + \Sigma_K + \Sigma_S) = \frac{1}{\mathrm{Vol}(\overline{B_1(N)})}(K_1 + K_{121} + K_{212} + K_J).$$

The expression $\Sigma_{cusp} + \Sigma_B + \Sigma_K + \Sigma_S$ is given by

$$\Sigma_{cusp} = \sum_{u \in \mathscr{G}(N,\omega)} h(\nu_u)\lambda_{\mathtt{n}}(u)\mathscr{W}_\psi(u)(\mathsf{t}_1\mathsf{t}_{\mathbf{m}_1}^{-1})\overline{\mathscr{W}_\psi(u)}(\mathsf{t}_2\mathsf{t}_{\mathbf{m}_2}^{-1}),$$

$$\Sigma_B = \frac{1}{8}\sum_{\omega_1\omega_2\omega_3^2=\omega}\ \sum_{u \in \mathscr{G}_B(N,\omega_1,\omega_2,\omega_3)} \int_{i\mathfrak{a}^*} h(\nu)\lambda_{\mathtt{n}}(u,\nu)$$
$$\times \mathscr{W}_\psi(E(\cdot,u,\nu))(\mathsf{t}_1\mathsf{t}_{\mathbf{m}_1}^{-1})\overline{\mathscr{W}_\psi(E(\cdot,u,\nu))}(\mathsf{t}_2\mathsf{t}_{\mathbf{m}_2}^{-1})\,d\nu,$$

$$\Sigma_K = \frac{1}{2}\sum_{\omega_1\omega_2=\omega}\ \sum_{u \in \mathscr{G}_{\mathrm{P_K}}(N,\omega_1,\omega_2)} \int_{i\mathfrak{a}_K^*} h(\nu + \nu_K(s_u))\lambda_{\mathtt{n}}(u,\nu)$$
$$\times \mathscr{W}_\psi(E(\cdot,u,\nu))(\mathsf{t}_1\mathsf{t}_{\mathbf{m}_1}^{-1})\overline{\mathscr{W}_\psi(E(\cdot,u,\nu))}(\mathsf{t}_2\mathsf{t}_{\mathbf{m}_2}^{-1})\,d\nu,$$

$$\Sigma_S = \frac{1}{2}\sum_{\omega_1\omega_2^2=\omega}\ \sum_{u \in \mathscr{G}_{\mathrm{P_S}}(N,\omega_1,\omega_2)} \int_{i\mathfrak{a}_S^*} h(\nu + \nu_S(s_u))\lambda_{\mathtt{n}}(u,\nu)$$
$$\times \mathscr{W}_\psi(E(\cdot,u,\nu))(\mathsf{t}_1\mathsf{t}_{\mathbf{m}_1}^{-1})\overline{\mathscr{W}_\psi(E(\cdot,u,\nu))}(\mathsf{t}_2\mathsf{t}_{\mathbf{m}_2}^{-1})\,d\nu,$$

where ν_u *(resp.* s_u*) is the spectral parameter of the representation of* $GSp_4(\mathbb{R})$ *(resp.* $GL_2(\mathbb{R})$*) attached to* u*,* ν_K *and* ν_S *are given by Propositions 2.4.4 and 2.4.6. On the right hand side,*

- K_1 *is non-zero only if there is an integer* t *dividing* $\mathtt{n}$ *with* $\frac{\mathtt{n}}{t} = \frac{\mathbf{m}_{12}}{\mathbf{m}_{22}} t$ *and such that* $s = \frac{\mathbf{m}_{21}}{\mathbf{m}_{11}} t$ *is also an integer dividing* $\mathtt{n}$*, in which case, setting* $d = \gcd(s, \frac{\mathtt{n}}{s}, t, \frac{\mathtt{n}}{t})$ *and*

$$T(\mathtt{n}, \mathbf{m}_1, \mathbf{m}_2) = d \times \overline{\omega_N(s)}\, \mathtt{n}^{-\frac{1}{2}} (\mathbf{m}_{11}\mathbf{m}_{21})^{-2} |\mathbf{m}_{12}\mathbf{m}_{22}|^{-\frac{3}{2}}$$
$$\times S\left(\mathbf{m}_{11} \frac{\mathtt{n}}{\gcd(t, \frac{\mathtt{n}}{s})}, \mathbf{m}_{12} t, d, \mathtt{n}\right)$$

we have

$$K_1 = T(\mathtt{n}, \mathbf{m}_1, \mathbf{m}_2) \int_{\mathfrak{a}^*} h(-i\nu) W(i\nu, \mathtt{t}_{\mathbf{m}_1}^{-1} \mathtt{t}_1, \psi) W(-i\nu, \mathtt{t}_{\mathbf{m}_2}^{-1} \mathtt{t}_2, \overline{\psi}) \frac{d\nu}{c(i\nu) c(-i\nu)}.$$

- *The contribution of the long Weyl element is*

$$K_J = \sum_{\substack{N|s \\ N^2|k}} \mathrm{Kl}_J(\mathbf{m}_1, \mathbf{m}_2, s, k, \mathtt{n}) I_J(h) \left(\frac{k}{s^2}, \frac{\mathtt{n}}{k}\right),$$

- *The contribution of* $s_1 s_2 s_1$ *is non-zero only if* $\mathtt{n}\frac{\mathbf{m}_{12}}{\mathbf{m}_{22}} = b^2$ *for some rational number* b*, in which case it is given by*

$$K_{121} = \mathbf{m}_{22} \sum_{N|kb} \mathrm{Kl}_{121}(\mathbf{m}_1, \mathbf{m}_2, Nk, bNk, \mathtt{n}) I_{121}(h) \left(\frac{\mathtt{n}}{Nkb}, \frac{b}{Nk}\right),$$

- *The contribution K_{212} of $s_2 s_1 s_2$ is given by*

$$\mathbf{m}_{21} \sum_{\substack{\mathbf{m}_{21} N | s\mathbf{m}_{11} \\ \mathbf{m}_{21} N^2 | s^2 \mathbf{m}_{11}}} \mathrm{Kl}_{212}\left(\mathbf{m}_1, \mathbf{m}_2, -s\frac{\mathbf{m}_{11}}{\mathbf{m}_{21}}, -s^2\frac{\mathbf{m}_{11}}{\mathbf{m}_{21}}, \mathtt{n}\right) I_{212}(h)\left(-\frac{\mathbf{m}_{11}}{\mathbf{m}_{21}}, -\frac{\mathtt{n}}{s^2}\frac{\mathbf{m}_{21}}{\mathbf{m}_{11}}\right),$$

where we have defined $I_\sigma(h)(d_1, d_2)$ as the integral

$$\begin{aligned} &\int_{\mathfrak{a}^*} h(-i\nu) W(i\nu, \mathsf{t}_{\mathbf{m}_1}^{-1}\mathsf{t}_1, \psi) \\ &\times \int_{U_\sigma(\mathbb{R})\backslash U(\mathbb{R})} W\left(-i\nu, \mathsf{t}_{\mathbf{m}_1}^{-1}\sigma \begin{bmatrix} d_1 & & \\ & 1 & \\ & & d_2 & \\ & & & d_1 d_2 \end{bmatrix} \mathsf{t}_{\mathbf{m}_2}\mathsf{u}_1\mathsf{t}_{\mathbf{m}_2}^{-1}\mathsf{t}_2, \overline{\psi}\right) \psi(\mathsf{u}_1)\, d\mathsf{u}_1 \frac{d\nu}{c(i\nu)c(-i\nu)}. \end{aligned}$$

Moreover, if Conjecture 2.5.1 is true then we have

$$\begin{aligned} I_\sigma(h)(d_1, d_2) = \int_{\mathfrak{a}^*} h(-i\nu) K_\sigma &\left(-i\nu, \mathsf{t}_{\mathbf{m}_1}^{-1}\sigma \begin{bmatrix} d_1 & & \\ & 1 & \\ & & d_2 & \\ & & & d_1 d_2 \end{bmatrix} \mathsf{t}_{\mathbf{m}_2}\sigma^{-1}, \overline{\psi}\right) \\ &\times W(i\nu, \mathsf{t}_1\mathsf{t}_{\mathbf{m}_1}^{-1}, \psi) W(-i\nu, \mathsf{t}_2\mathsf{t}_{\mathbf{m}_2}^{-1}, \overline{\psi}) \frac{d\nu}{c(i\nu)c(-i\nu)}, \end{aligned}$$

where the generalised Bessel functions K_σ have been defined in § 5.3.

CHAPTER 3

Equidistribution of Satake parameters of automorphic forms for GSp_4

1. Introduction

The distribution of Satake parameters of automorphic forms is a classical problem in number theory. In the case of GL_2, the Sato-Tate conjecture states that, for a fixed typical newform u of trivial central character, the Hecke eigenvalues $\lambda_p(u)$ (which in this case are $\alpha_p(u) + \alpha_p(u)^{-1}$, where $\alpha_p(u)$ are the corresponding Satake parameters) equidistribute with respect to the Sato-Tate measure $d\mu_{\mathrm{ST}}$ as p varies among primes not dividing the level. The Sato-Tate conjecture is known for holomorphic forms of weight $k \geq 2$ [**BLGHT11**]. This is usually referred as the "*horizontal*" distribution problem.

On the other hand, one can fix the prime p and allow u to vary, making the problem amenable to the (Selberg or Arthur) trace formula. This easier problem is known as the "*vertical*" distribution problem and it asks for the distribution of the Satake parameters $\alpha_p(u)$ as u varies and as the weight or level tends to infinity. This problem has been addressed for GL_2 independently by Bruggeman [**Bru78**] and Sarnak [**Sar87**] for Maaß forms and by Serre [**Ser97**] and Duke-Conrey-Farmer [**CDF97**] for holomorphic forms. The relevant measure in this case is not the Sato-Tate measure, but the p-adic Plancherel measure, given by $d\mu_p^{\mathrm{Pl}}(x) = \frac{p+1}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}}) - x^2} d\mu_{\mathrm{ST}}(x)$.

A similar problem, the **weighted** vertical equidistribution problem is obtained by replacing the use of the trace formula by a relative trace formula. This corresponds to count every automorphic form with a certain harmonic weight coming from the relative trace formula, and to ask for the weighted vertical distribution of the Satake parameters. This has been done by Knightly and Li for holomorphic forms using the Petersson formula [**Li04**, **KL08**] and for Maaß forms using the Kuznetsov formula [**KL13**]. Interestingly, in the weighted vertical equidistribution problem, the limiting measure is the Sato-Tate measure μ_{ST}, independently of the choice of the prime p not dividing the level.

Moving away from the case of GL_2, the unweighted vertical equidistribution problem has been tackled for groups admitting discrete series at the infinite place by the work of Shin [**Shi12**] and Shin-Templier [**ST16**], and by Matz-Templier [**MT21**] for Maaß forms on $\mathrm{SL}_n \,/\, \mathrm{SO}_n$. Kim, Wakatsuki and Yamauchi [**KWY20**] have also investigated the situation of Siegel modular forms on GSp_4. As in the case of GL_2, in this type of problem,the limiting distribution is the p-adic *Plancherel measure* μ_p^{Pl}, which converges to the Sato-Tate measure μ_{ST} as the prime p tends to infinity.

On the other hand, the weighted vertical equidistribution problem has been treated for Siegel modular forms by Kowalski, Saha and Tsimerman [**KST12**] and Dickson [**Dic15**] in the case of GSp_4, and by Knightly and Li [**KL19**] for GSp_{2n}. The situation of Maaß forms is known by [**BBR14**] for GL_3, and conjecturally for

PGL_n by [**Zho14**]. Here again, the situation is analogous to GL_2 in that the limiting measure is the *Sato-Tate measure* μ_{ST}.

In this chapter, we apply the Kuznetsov formula to treat the weighted vertical equidistribution problem for the whole generic, spherical at infinity spectrum of GSp_4 for the group $B_1(N)$. In order to obtain a weighted vertical equidistribution for Maaß forms on GSp_4, we would still need to bound the contribution from the continuous spectrum. This is work in progress. We access the Satake parameters via the local Whittaker function, using the Casselman-Shalika formula. To derive the equidistribution result, we need to show that the set of test functions we can generate this way spans the relevant space of test functions. More precisely, our approach allows us to choose the test function to be an arbitrary Ω-invariant Laurent polynomial, which is shown to be sufficient by a Stone-Weierstraß density argument. This is done in Section 3 below. Finally, for a fixed Laurent polynomial $\mathscr{g}$, the non-diagonal contribution on the geometric side is shown to vanish identically for N large enough (in terms of $\mathscr{g}$). In other words, the first "moments" of the weighted distribution of the Satake parameters coincide exactly with the corresponding moments of the Sato-Tate distribution, until a certain point that depends on N and that goes to infinity with N.

2. Satake parameters

We follow the exposition of [**Pit19**] and [**KST12**]. Let p be a prime number. Let σ, χ_1, χ_2 be unramified characters of $\mathbb{Q}_p^\times$. They determine an unramified character of the Borel subgroup B of $\mathrm{GSp}_4(\mathbb{Q}_p)$ that is trivial on the unipotent radical, and whose values on the diagonal are given by $\chi\left(\begin{bmatrix} x & & & \\ & y & & \\ & & tx^{-1} & \\ & & & ty^{-1}\end{bmatrix}\right) = \sigma(t)\chi_1(x)\chi_2(y)$. The representation $\chi_1 \times \chi_2 \rtimes \sigma$ of $\mathrm{GSp}_4(\mathbb{Q}_p)$ obtained by normalized induction from χ has a unique subquotient $\pi(\sigma, \chi_1, \chi_2)$ that is **spherical**, meaning it contains a non-zero vector fixed by $K_p = \mathrm{GSp}_4(\mathbb{Z}_p)$. One can check that the central character of $\pi(\sigma, \chi_1, \chi_2)$ is $\sigma^2\chi_1\chi_2$. Moreover, two such representations are isomorphic to each other if and only if their inducing characters are equal modulo the action of the Weyl group.

It is known that any irreducible admissible representation π of $\mathrm{GSp}_4(\mathbb{Q}_p)$ that is spherical is of the form $\pi(\sigma, \chi_1, \chi_2)$ for some unramified characters σ, χ_1, χ_2 of $\mathbb{Q}_p^\times$. Now any unramified character of $\mathbb{Q}_p$ is determined by its value at p. Hence π is determined by the tuple of non-zero complex numbers $(\sigma(p), \chi_1(p), \chi_2(p))$ modulo the action of the Weyl group. If moreover π has trivial central character, it holds that $\sigma^2(p)\chi_1(p)\chi_2(p) = 1$, hence π is completely determined by the pair $(x, y) = (\sigma(p), \sigma(p)\chi_1(p)) \in \mathbb{C}^\times \times \mathbb{C}^\times$, modulo the action of the Weyl group. These are the **Satake parameters** of π. The action of the Weyl group is generated by the two transformations

$$(x, y) \mapsto (x, y^{-1}) \text{ and } (x, y) \mapsto (y^{-1}, x^{-1}). \tag{3.1}$$

Let π be a spherical irreducible admissible representation of $\mathrm{GSp}_4(\mathbb{Q}_p)$ with trivial central character and assume moreover that π is unitary and generic. By [**PS09**, Proposition 3.1], its Satake parameters (x, y) must satisfy $p^{-\frac{1}{2}} < |x|, |y| < p^{\frac{1}{2}}$ together with one of the following conditions:

(1) $(x, y) \in \mathbb{S}^1 \times \mathbb{S}^1$, the tempered case,

(2) $(x, y) \in \mathbb{S}^1 \times \mathbb{R}$ or $(x, y) \in \mathbb{R} \times \mathbb{S}^1$,

(3) $(x, y) \in R_1 = \{(\lambda z, \lambda^{-1} z), \lambda > 0, z \in \mathbb{S}^1\}$ or $(x, y) \in R_2 = \{(\lambda z, \lambda z^{-1}), \lambda > 0, z \in \mathbb{S}^1\}$,

(4) $(x, y) \in \mathbb{R} \times \mathbb{R}$.

Accordingly, we define the space $\mathscr{Y}$ of putative Satake parameters to be the quotient of

$$X = C_p \cap \big((\mathbb{S}_1 \times \mathbb{S}_1) \cup (\mathbb{S}^1 \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{S}^1) \cup R_1 \cup R_2 \cup (\mathbb{R} \times \mathbb{R})\big)$$

by the action of the Weyl group described in (3.1), where C_p is the compact set $\{(z_1, z_2) \in \mathbb{C}^2, p^{-\frac{1}{2}} \leq |z_1|, |z_2| \leq p^{\frac{1}{2}}\}$. The following remark is trivial but turns out to be important for later use of the Stone-Weierstraß theorem.

REMARK 3.2.1. *If $(x, y) \in \mathscr{Y}$ then $\overline{x}$ is equal to either x, $-x$, x^{-1}, y or y^{-1}.*

We parametrize the subset of $\mathscr{Y}$ corresponding to tempered representations $(\mathbb{S}^1 \times \mathbb{S}^1)/\Omega \subset \mathscr{Y}$ by $(\mathbb{S}^1 \times \mathbb{S}^1)/\Omega = \{(e^{i\theta_1}, e^{i\theta_2}) : 0 \leq \theta_1 \leq \theta_2 \leq \pi\}$. The **Sato-Tate measure** is supported on the tempered spectrum, and, in these coordinates, it is given by

$$d\mu_{ST}(\theta_1, \theta_2) = \frac{4}{\pi^2}(\cos\theta_1 - \cos\theta_2)^2 \sin^2\theta_1 \sin^2\theta_2 d\theta_1 d\theta_2. \tag{3.2}$$

3. The Whittaker function

Let π be an irreducible spherical admissible generic representation of $\mathrm{GSp}_4(\mathbb{Q}_p)$ with trivial central character. Let (x, y) be the Satake parameters of π, and let the **normalized Whittaker function** $\mathscr{W}(x,y)$ be the unique right-K_p-invariant function in the Whittaker model of π that satisfies $\mathscr{W}(x,y)(1) = 1$ (the fact that a non-zero K_p-fixed vector in the Whittaker model of π doesn't vanish at 1 follows from the proof of [**RS07**, Corollary 7.1.5]). By uniqueness of the Whittaker model, the map $(x,y) \mapsto \mathscr{W}(x,y)$ is invariant by the Weyl group. By the Casselman-Shalika formula [**CS80**], see also [**RS07**, Formula (7.3)], the value of $\mathscr{W}(x,y)$ on diagonal matrices is given by

$$\mathscr{W}_{a,b,c}(x,y) \doteq \mathscr{W}(x,y)\left(\begin{bmatrix} p^a & & & \\ & p^b & & \\ & & p^{c-a} & \\ & & & p^{c-b} \end{bmatrix}\right) = \begin{cases} \frac{W_{a,b,c}(x,y)}{W_{0,0,0}(x,y)} & \text{if } b \geq a \text{ and } 2a \geq c \\ 0 & \text{otherwise,} \end{cases} \tag{3.3}$$

where for all integers a, b, c we define

$$\begin{aligned} W_{a,b,c}(x,y) = p^{-2b-a+3c/2}x^{-3}((x^{b-a+1} - x^{a-b-1})(y^{a+b+2-c} - y^{c-a-b-2}) \\ -(y^{b-a+1} - y^{a-b-1})(x^{a+b-c+2} - x^{c-a-b-2})). \end{aligned} \tag{3.4}$$

We now prove the relevant results of functional analysis we need about the Whittaker function for our equidistribution result.

LEMMA 3.3.1. *The functions $(W_{a,b,c})_{\substack{c \leq a \leq b \\ c \in \{0,1\}}}$ span the space $V \subset \mathbb{C}[x, x^{-1}, y, y^{-1}]$ of Laurent polynomials $P(x,y)$ that satisfy $P(x, y^{-1}) = -P(x,y)$ and $P(y^{-1}, x^{-1}) = -(xy)^3 P(x,y)$.*

PROOF. The action (3.1) of the Weyl group on Satake parameters extends to an action on Laurent polynomials, and it is clear that any Laurent polynomial in V is a linear combination of Laurent polynomials of the form $L_{n,m}(x,y) =$

$x^{-3}\sum_{\sigma\in\Omega}\operatorname{sgn}(\sigma)\sigma(x^n y^m)$ with $0<n<m$. But $L_{n,m}=p^{2b+a-3c/2}W_{a,b,c}$ where $a=\frac{m-n+c-1}{2}$, $b=\frac{n+m+c-3}{2}$ and $c=(n+m)\mod 2$. $\square$

LEMMA 3.3.2. *The space of functions spanned by* $(\mathscr{W}_{a,b,c})_{\substack{c\le a\le b\\ c\in\{0,1\}}}$ *is dense in the set* $\mathscr{C}(\mathscr{Y})$ *of continuous functions on* $\mathscr{Y}$.

PROOF. Using notations of Lemma 3.3.1, we clearly have $W_{0,0,0}(x,y)\mathbb{C}[X+Y,XY]\subset V$, where $X=x+x^{-1}$ and $Y=y+y^{-1}$. Hence every element of $\mathbb{C}[X+Y,XY]$ can be written as a linear combination of functions of the form $\mathscr{W}_{a,b,c}$ with $0<b-a+1<a+b+2-c$ and $c\in\{0,1\}$. So it suffices to show that $\mathbb{C}[X+Y,XY]$ is dense in $\mathscr{C}(\mathscr{Y})$. By Remark 3.2.1, the algebra $\mathbb{C}[X+Y,XY]$ is stable under complex conjugation, hence by the Stone-Weierstraß theorem, it suffices to show that the two functions $X+Y$ and XY separate the points on $\mathscr{Y}$. But, for fixed (u,v), the (at most) two solutions for (X,Y) of the system $\begin{cases}X+Y=u\\ XY=v\end{cases}$ are symmetric to each other, and each value for X (resp. for Y) gives two possible solutions for x (resp. for y), that are inverse from each other. Thus the (at most) eight solutions for (x,y) are equal modulo the action of the Weyl group, and hence represent the same point in $\mathscr{Y}$. $\square$

LEMMA 3.3.3. *Let* $a_1,b_1,c_1,\ a_2,b_2,c_2$ *be integers and, for ease of notation, set* $n_j=b_j-a_j+1$ *and* $m_j=a_j+b_j-c_j+2$, *and* $h=2(b_1+b_2)+(a_1+a_2)-3(c_1+c_2)/2$. *Assume* $0<n_j<m_j$ *and* $c_j\in\{0,1\}$ *for* $j=1,2$. *Then*

$$p^h\int_{\mathscr{Y}}\mathscr{W}_{a_1,b_1,c_1}\overline{\mathscr{W}_{a_2,b_2,c_2}}\,d\mu_{ST}=\begin{cases}1 \text{ if } a_1=a_2, b_1=b_2 \text{ and } c_1=c_2\\ 0 \text{ otherwise.}\end{cases}$$

Proof. Note that $W_{0,0,0}(e^{i\theta_1}, e^{i\theta_2}) = -8e^{-3i\theta_1}\sin\theta_1\sin\theta_2(\cos\theta_1 - \cos\theta_2)$. Hence combining the definition of the Sato-Tate measure (3.2) with formulae (3.3), (3.4) the integral we have to evaluate becomes

$$\frac{4}{\pi^2}\int_0^{\pi}\int_0^{\theta_2}(\sin(n_1\theta_1)\sin(m_1\theta_2) - \sin(m_1\theta_1)\sin(n_1\theta_2)) \times \cdots$$

$$\cdots \times (\sin(n_2\theta_1)\sin(m_2\theta_2) - \sin(m_2\theta_1)\sin(n_2\theta_2))\, d\theta_1\, d\theta_2$$

$$= \frac{1}{2\pi^2}[2I(n_1,n_2)I(m_1,m_2) - 2I(n_1,m_2)I(m_1,n_2)],$$

where

$$I(n,m) = \int_0^{2\pi}\sin(n\theta)\sin(m\theta)\, d\theta = \begin{cases}\operatorname{sgn}(mn)\pi \text{ if } |n| = |m| \\ 0 \text{ otherwise.}\end{cases}$$

Since $n_j, m_j > 0$, the term $I(n_1,n_2)I(m_1,m_2)$ is non-zero if and only if $n_1 = n_2$ and $m_1 = m_2$. The term $I(n_1,m_2)I(m_1,n_2)$ is non-zero if and only if $n_1 = m_2$ and $m_1 = n_2$, but this contradicts the assumption that $n_2 < m_2$ and $n_1 < m_1$. □

4. Vanishing of the geometric side

The contribution from the non-identity elements in the geometric side is given by sums over diagonal matrices δ whose entries satisfy various divisibility conditions modulo N. On the other hand, the test function f_∞ is compactly supported modulo the centre, and it so is its integral transform appearing on the geometric side. The upshot is for N large enough, every δ subject to the relevant divisibility conditions lies outside of the support of the corresponding integral transform, and thus the corresponding geometric terms vanishes. We now make this argument more precise.

LEMMA 3.4.1. *Let $K \subset \mathrm{GSp}_4(\mathbb{R})$ be compact. Let $X = \mathbb{R}^\times K$. There exists a constant C_K such that every element $\mathsf{x} \in X$ has a Bruhat decomposition $\mathsf{x} = \mathsf{u}_1\sigma\delta\mathsf{u}_2$ where the entries of $\frac{1}{\sqrt{|\mu(\delta)|}}\delta$ satisfy*

- *the second diagonal entry lies in $[-C_K, C_K]$,*
- *if j_0 is the unique index such that $\sigma^{-1}_{4j_0} = 1$, then the j_0-th diagonal entry lies in $[-C_K, C_K]$.*

PROOF. Let $K' = \{\pm\frac{1}{\sqrt{|\mu(\mathsf{k})|}}\mathsf{k} : \mathsf{k} \in K\}$. Then K' is itself compact, and in particular the elements of K' have bounded entries. Now let $\mathsf{x} = \mathsf{z}\mathsf{k}$ with $\mathsf{z} \in \mathbb{R}$ and $\mathsf{k} \in K$. Let $\mathsf{k}' = \pm\frac{1}{\sqrt{|\mu(\mathsf{k})|}}\mathsf{k} \in K'$. Then

$$\begin{aligned}\frac{1}{\sqrt{|\mu(\delta)|}}\delta &= \frac{1}{\sqrt{|\mu(x)|}}\sigma^{-1}\mathsf{u}_1^{-1}\mathsf{x}\mathsf{u}_2^{-1} \\ &= \sigma^{-1}\mathsf{u}_1^{-1}\mathsf{k}'\begin{bmatrix} 1 & & * & * \\ * & 1 & * & * \\ & & 1 & * \\ & & & 1\end{bmatrix} \\ &= \sigma^{-1}\mathsf{u}_1^{-1}\begin{bmatrix} * & k'_{12} & * & * \\ * & k'_{22} & * & * \\ * & k'_{32} & * & * \\ * & k'_{42} & * & *\end{bmatrix}.\end{aligned}$$

Without loss of generality, we may assume $\mathsf{u}_1 \in \overline{U}_\sigma$. But one may easily check that if i_0 is the unique index such that $\sigma^{-1}_{2i_0} = 1$ then for all $\mathsf{u} \in \overline{U}_\sigma$ we have $\mathsf{u}_{i_0j} = 0$ for $i_0 \neq j$, and $\mathsf{u}_{i_0i_0} = 1$. It follows $\delta_{22} = k'_{i2}$ is bounded. Now we may instead assume that $\mathsf{u}_2 \in \overline{U}_{\sigma^{-1}}$. Checking that elements $\mathsf{u} \in \overline{U}_{\sigma^{-1}}$ satisfy $\mathsf{u}_{ij_0} = 0$ for $i \neq j_0$ and $\mathsf{u}_{j_0j_0} = 1$, a similar calculation establishes the second claim. $\square$

COROLLARY 3.4.1. *Assume $\sigma \neq 1$. Then for N large enough (in terms of h, t_1 and t_2) the term K_σ in the geometric side of the Kuznetsov formula vanishes.*

Proof. We use expression (2.44) for the Archimedean part of the orbital integrals. Let $K = \mathsf{t}_1 \operatorname{Supp}(f_\infty)\mathsf{t}_2^{-1}$. Then $K_\sigma \neq 0$ only if there exists $\mathsf{u}, \mathsf{u}_1 \in U(\mathbb{R})$ such that $\mathsf{u}\sigma\delta\mathsf{u}_1 \in K$. Now if $\sigma = \mathsf{J}$ then by Remark 2.5.12 the only elements δ contributing to K_J satisfy

$$\frac{1}{\sqrt{|\mu(\delta)|}}\delta = \begin{bmatrix} \frac{Nm}{k} & & & \\ & Nk & & \\ & & \pm\frac{k}{Nm} & \\ & & & \pm\frac{1}{Nk} \end{bmatrix}$$

for some non-zero integers k, m. In particular, for $N \geq C_K$, by Lemma 3.4.1, we get $K_\mathsf{J} = 0$. If $\sigma = s_1 s_2 s_1$ then by Remark 2.5.13 the only elements δ contributing to K_{121} satisfy

$$\frac{1}{\sqrt{|\mu(\delta)|}}\delta = \begin{bmatrix} \frac{k}{Nm} & & & \\ & Nk & & \\ & & \pm\frac{Nm}{k} & \\ & & & \pm\frac{1}{Nk} \end{bmatrix}$$

for some non-zero integers k, m. In particular, for $N \geq C_K$, by Lemma 3.4.1, we get $K_{121} = 0$. Finally, if $\sigma = s_1 s_2 s_1$ then by Remark 2.5.14 the only elements δ contributing to K_{212} satisfy

$$\frac{1}{\sqrt{|\mu(\delta)|}}\delta = \begin{bmatrix} Nk & & & \\ & \frac{Nm}{k} & & \\ & & \pm\frac{1}{Nk} & \\ & & & \pm\frac{k}{Nm} \end{bmatrix}$$

for some non-zero integers k, m. But in this case we have $j_0 = 1$ in Lemma 3.4.1, and thus we see that again for $N \geq C_K$, we have $K_{212} = 0$. $\square$

5. The equidistribution result

We are now ready to prove our equidistribution result. We start with a technical functional analysis lemma.

LEMMA 3.5.1. *Let $(\Pi, d\varpi)$ be a measured space. Let K be a compact space endowed with a Borel probability measure μ. Consider a non-negative measurable function $w : \Pi \to \mathbb{R}_{\geq 0}$ and a sequence $(X(N))_{N>0}$ of measurable sets $X(N) \subseteq \Pi$ such that for all N we have*

$$0 < \int_{X(N)} w(\varpi)\, d\varpi < \infty.$$

Assume that there is a measurable function $\mathcal{S} : \Pi \to K$ and a dense subspace W of the space $\mathscr{C}(K)$ of complex-valued continuous functions on K endowed with the sup-norm topology, such that for all $g \in W$ we have

$$\lim_{N\to\infty} \frac{\int_{X(N)} w(\varpi) g \circ \mathcal{S}(\varpi)\, d\varpi}{\int_{X(N)} w(\varpi)\, d\varpi} = \int_K g\, d\mu. \tag{3.5}$$

Then the same holds for all $g \in \mathscr{C}(K)$.

PROOF. Observe that since w is non-negative we have

$$\left| \frac{\int_{X(N)} w(\varpi) g \circ \mathcal{S}(\varpi)\, d\varpi}{\int_{X(N)} w(\varpi)\, d\varpi} \right| \leq \|g\|_\infty. \tag{3.6}$$

It suffices to show that the left hand side of (3.5) is defined for all $g \in \mathscr{C}(K)$. Indeed, both side will then define continuous linear functionals on $\mathscr{C}(K)$, that coincide on the dense subspace W, hence are equal. Now if $g = w + h$ with $w \in W$ and $\|h\|_\infty \leq \epsilon$, it follows by linearity from (3.6) that the limits of all the converging subsequences of $\left(\frac{\int_{X(N)} w(\varpi) g \circ \mathcal{S}(\varpi)\, d\varpi}{\int_{X(N)} w(\varpi)\, d\varpi} \right)_{N>0}$ are in an ϵ-neighborhood of $\int_K w\, d\mu$, and in particular they are at a distance at most 2ϵ from each other. But by density of W, this holds for arbitrary $\epsilon > 0$, and hence all the converging subsequences have the same limit, which implies that the left hand side of (3.5) is well defined. $\square$

To state the equidistribution result, let us set up some notations.

Definition 3.5.1. Define Π to be the set of pairs $\varpi = (\pi, u)$ where π is a unitary automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ and $u \in \pi$. We endow Π with a measure $d\varpi$ in the following way. Recall from § 4.3 that $\pi \simeq \mathscr{I}(\sigma_\nu)$ for some parabolic subgroup P (with possibly P=G), $\nu \in i\mathfrak{a}_P^*$ (in case $P = G$, this space is $\{0\}$) and σ a representation occurring discretely in the spectrum of M_P. We then put $d\varpi = d\nu \otimes |\cdot|_{R_{M_P,disc}} \otimes |\cdot|_\pi$, where $d\nu$ is the Haar measure on $i\mathfrak{a}_P^*$, $|\cdot|_{R_{M_P,disc}}$ is the counting measure on $R_{M_P,disc}$ and $|\cdot|_\pi$ is the counting measure on the space of π.

Definition 3.5.2. For any integer N and each Dirichlet character ω, we define $X(N,\omega) \subset \Pi$ to be the set of pairs[1] $\varpi = (\pi, E(\cdot, u, \nu))$ where $\pi \simeq \mathscr{I}(\sigma_\nu)$ has central character ω and $u \in \mathscr{B}_{\sigma,1}$, the basis of the $B_1(N)$-invariant subspace of $\mathscr{I}(\sigma_\nu)$ described in § 4.4.

Definition 3.5.3. Fix a matrix $\mathsf{t} \in A^+$, and a Paley-Wiener function h on $\mathfrak{a}_\mathbb{C}$ such that $h(\nu_\pi) \geq 0$ for all spectral parameters ν_π. Given $\varpi = (\pi, u) \in \Pi$, define a spectral weight

$$w(\varpi) \doteq |W_\psi(u)(\mathsf{t})|^2 h(\nu_\pi).$$

Remark 3.5.1. *The existence of a Paley-Wiener function h on $\mathfrak{a}_\mathbb{C}$ such that $h(\nu_\pi) \geq 0$ for all spectral parameters ν_π and $\int_{X(N,\omega)} w(\pi)\, d\pi > 0$ for N large enough is proved in Corollary 3.5.1 below.*

Remark 3.5.2. *According to the Lapid-Mao Conjecture, when $\mathsf{t} = 1$ and π is cuspidal, the weight $w(\varpi)$ should be related to $L(1, \pi, Ad)$, the value at 1 of the adjoint*

[1] recall that the discrete spectrum corresponds to $P = G$, in which case we have $\nu = 0$, $\pi = \mathscr{I}_P(\pi_\nu)$ and $E(\cdot, u, \nu) = u$.

L-function of π. Chen and Ichino have proved that the Lapid-Mao conjecture holds for automorphic representations π of GSp_4 *such that π_∞ is a principal series and π has squarefree paramodular conductor (see* [**CI19**, Theorem 2.1]*). In particular, since $B(N)$ is contained in the paramodular subgroup of level N, when N is squarefree and $\omega = 1$ we have*

$$|W_\psi(u)(1)|^2 = 2^{-c}\zeta(2)\zeta(4)\frac{|\mathscr{W}(\nu_u, 1, \psi)|^2}{L(1, \pi, Ad)}\prod_v \frac{1}{C(\pi_v)},$$

for all $u \in \mathscr{G}(N, \omega)$, where

- *$\mathscr{W}(\nu, 1, \psi)$ is the normalised Jacquet integral defined in (2.25),*
- $c = \begin{cases} 1 \text{ if } \pi \text{ is stable,} \\ 2 \text{ if } \pi \text{is endoscopic,} \end{cases}$
- $C(\pi_v) = \begin{cases} 1 \text{ if } v \nmid N\infty, \\ \frac{\zeta_p(4)}{p\zeta_p(2)} \text{ if } v = p \mid N, \\ 2^{-4} \text{ if } v = \infty. \end{cases}$

Definition 3.5.4. Let p be a prime. Fix an integer N coprime to p and a Dirichlet character ω modulo N such that $\omega(p) = 1$. For $\pi \simeq \bigotimes_v \pi_v$ an automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ and $u \in \pi$, define $\mathscr{S}_p(\pi) = (\alpha_p(u), \beta_p(u)) \in \mathscr{Y}$ as the Satake parameters of the local representation π_p. Finally, define a measure μ_N on $\mathscr{Y}$ as the push-forward of the measure $w(\varpi)d\varpi$ on $X(N, \omega)$ along $\mathscr{S}_p$.

Our main results say that, as N gets large, the Satake parameters at p of the whole $B_1(N)$-invariant generic spectrum, suitably weighted, equidistribute with respect to the Sato-Tate measure.

THEOREM 3.5.1. *Fix a prime number p. For each integer N coprime to p, pick a Dirichlet character ω_N modulo N such that $\omega_N(p) = 1$. For brevity, set $X(N) = X(N, \omega_N)$. Then the probability measure $\frac{1}{\mu_N(\mathscr{Y})}\mu_N$ converges weakly to the Sato-Tate measure (3.2) as N tends to infinity. This means that for any continuous Ω-invariant function $\mathscr{g}$ on $\mathbb{C}^2$ we have*

$$\lim_{N\to\infty} \frac{\int_{X(N)} w(\varpi)\mathscr{g}(\alpha_p(u), \beta_p(u))\, d\varpi}{\int_{X(N)} w(\varpi)\, d\varpi} = \int_{\mathbb{C}^2/\Omega} \mathscr{g}(x, y)\, d\mu_{ST}. \tag{3.7}$$

REMARK 3.5.3. *The proof shows that when $\mathscr{g}$ is a fixed Laurent polynomial, identity (3.7) is actually an equality for N large enough.*

REMARK 3.5.4. *A more interesting result would be that the Satake parameters of Maaß forms, weighted with the same weight, equidistribute with respect to the Sato-Tate measure. This is equivalent to showing that the part of the measure $\frac{1}{\mu_N(\mathscr{Y})}\mu_N$ that is supported on the continuous spectrum converges weakly to zero. This is work in progress*

PROOF. We apply the Kuznetsov formula with $\mathtt{n} = 1$, $\mathbf{m}_1 = (1, 1)$, $\mathbf{m}_2 = (p^i, p^j)$, $\mathtt{t}_1 = \mathtt{t}$ and $\mathtt{t}_2 = \mathtt{t}\mathtt{t}_{\mathbf{m}_2}$. Let $c = 2i + j \mod 2$ and $b = \frac{2i+j+c}{2}$ and $a = b - i$, so that $c \le a \le b$ and

$$\mathtt{t}_{\mathbf{m}_2}^{-1} = \begin{bmatrix} p^{-i} & & & \\ & 1 & & \\ & & p^{-i-j} & \\ & & & p^{-2i-j} \end{bmatrix} = p^{-b} \begin{bmatrix} p^{a} & & & \\ & p^{b} & & \\ & & p^{c-a} & \\ & & & p^{c-b} \end{bmatrix}.$$

Let $(\pi, u) \in X(N)$. Then $\mathscr{W}_\psi(u)$ defines a factorizable vector in the global Whittaker model of π, hence for all $\mathsf{g} \in G(\mathbb{A})$ we have

$$\mathscr{W}_\psi(u)(\mathsf{g}) = \prod_v W_v(\mathsf{g}_v),$$

where for each place v, W_p is a certain vector in the Whittaker model of π_v that is fixed by the corresponding local component of $B_1(N)$. For v prime, $v \neq p$, we have $\mathsf{t}_{\mathbf{m}_2}^{-1} \in B_1(p^{n_v})$ and hence $W_v(\mathsf{t}_{\mathbf{m}_2}^{-1}) = W_v(1)$. For $v = p$, we have $B_1(p^{n_p}) = K_p$ hence by uniqueness of the K_p-fixed vector we must have $W_p(\mathsf{t}_{\mathbf{m}_2}^{-1}) = W_p(1)\mathscr{W}_{a,b,c}(\alpha_p, \beta_p)$. So we get

$$W_\psi(u)(\mathsf{t}_2\mathsf{t}_{\mathbf{m}_2}^{-1}) = W_\psi(u)(\mathsf{t})\mathscr{W}_{a,b,c}(\alpha_p, \beta_p),$$

and the spectral side of the Kuznetsov formula is

$$c\Sigma = \int_{X(N)} w(\varpi)\mathscr{W}_{a,b,c}(\alpha_p(u), \beta_p(u))\, d\varpi.$$

The identity contribution in the geometric side is

$$K_1 = \delta_{(i,j)=(0,0)} \int_{\mathfrak{a}^*} h(-i\nu)W(i\nu, \mathsf{t}, \psi)W(-i\nu, \mathsf{t}, \overline{\psi})\frac{d\nu}{c(i\nu)c(-i\nu)}.$$

In particular, taking $(i, j) = (0, 0)$, since $\mathrm{Vol}(\overline{B_1(N)}) = \frac{1}{[K:B(N)]}$, by Corollary 3.4.1 we have that for N large enough

$$\text{(3.8)}\quad c\int_{X(N)} w(\varpi)\, d\varpi = [K : B(N)] \int_{\mathfrak{a}^*} h(-i\nu)W(i\nu, \mathsf{t}, \psi)W(-i\nu, \mathsf{t}, \overline{\psi})\frac{d\nu}{c(i\nu)c(-i\nu)}.$$

It follows that for all $c \leq a \leq b$ with $c \in \{0,1\}$ we have for N large enough (in terms of a, b, c)

$$\frac{\int_{X(N)} w(\varpi)\mathscr{W}_{a,b,c}(\alpha_p(u), \beta_p(u))\,d\varpi}{\int_{X(N)} w(\varpi)\,d\varpi} = \begin{cases} 1 \text{ if } a = b = c = 0 \\ 0 \text{ otherwise.} \end{cases}$$

In view of Lemma 3.3.3, the equidistribution statement (3.7) holds when $\mathscr{g} = \mathscr{W}_{a,b,c}$, and by linearity it still holds when $\mathscr{g}$ belongs to the subspace W of $\mathscr{C}(\mathscr{Y})$ that is spanned by $(\mathscr{W}_{a,b,c})_{\substack{c \leq a \leq b \\ c \in \{0,1\}}}$. The result follows from Lemma 3.5.1 and Lemma 3.3.2. $\square$

Lemma 3.5.2. *For $\epsilon \geq 0$ define $B_\epsilon = \{\mathsf{g} \in \mathrm{Mat}_4(\mathbb{R}) : \|\mathsf{g}\| \leq \epsilon\}$, and $B_\epsilon^K = \{\mathsf{k}\mathsf{g}^\top\mathsf{k} : \|\mathsf{g}\| \leq \epsilon, \mathsf{k} \in K_\infty\}$ where $\|\mathsf{g}\| = \max_{i,j} |\mathsf{g}_{i,j}|$ and $S_\epsilon = \{\mathsf{g} \in \mathrm{Sp}_4(\mathbb{R}), \mathsf{g}^\top\mathsf{g} \in 1 + B_\epsilon^K\}$. Then we have*

$$K_\infty S_\epsilon K_\infty = S_\epsilon,$$

and for $\epsilon, \delta > 0$ small enough we have

$$S_\epsilon^{-1} \subseteq S_{5\epsilon}$$

and

$$S_\epsilon \cdot S_\delta \subseteq 1 + B_{4\delta(5+16\epsilon)}.$$

Proof. The first claim is obvious from the fact that $K_\infty = \{\mathsf{g} \in \mathrm{Sp}_4(\mathbb{R}) : \mathsf{g}^\top\mathsf{g} = 1\}$. Now observe that if $\mathsf{g}^\top\mathsf{g} \in 1 + B_\epsilon$ then we have

$$\|\mathsf{g}\mathsf{a}^\top\mathsf{g}\| \leq 4(1+\epsilon)\|a\|. \tag{3.9}$$

for all $\mathsf{a} \in \mathrm{Mat}_4(\mathbb{R})$. In particular, taking $\epsilon = 0$, we obtain

$$B_\epsilon \subseteq B_\epsilon^K \subseteq B_{4\epsilon}. \tag{3.10}$$

The second claims then follows from the Taylor expansion of the map $\mathsf{g} \mapsto \mathsf{g}^{-1}$ at $\mathsf{g} = 1$. Now let $\mathsf{g}_1 \in S_\epsilon$, $\mathsf{g}_2 \in S_\delta$. Then by (3.10) we have

$$(\mathsf{g}_1\mathsf{g}_2)^\top(\mathsf{g}_1\mathsf{g}_2) \in \mathsf{g}_1(1 + B_{4\delta})^\top \mathsf{g}_1 \subseteq 1 + B_{4\delta} + \mathsf{g}_1 {B_{4\delta}}^\top \mathsf{g}_1$$

and the result follows using (3.9). □

COROLLARY 3.5.1. *Fix* $\mathsf{t} \in A+$. *Let* $F : \mathrm{Sp}_4(\mathbb{R}) \to \mathbb{R}$ *satisfying the following hypothesis*

- F *is smooth and bi-*K_∞*-invariant,*
- $\mathrm{Supp}(F) = S_\epsilon$ *as defined in Lemma 3.5.2.*
- F *only assumes non-negative values.*

Let $f_\infty = F^* {*} F$, *where* $F^*(\mathsf{g}) = \overline{F}(\mathsf{g}^{-1})$. *Then we have* $\tilde{f}_\infty(\nu_\pi) \geq 0$ *for all automorphic representation* π *of* GSp_4 *and if* $\epsilon > 0$ *is small enough then for all* N *large enough and for all Dirichlet character* ω *modulo* N *we have* $\int_{X(N)} w(\varpi)\, d\varpi > 0$.

PROOF. We have $R(F^* * F) = R(F^*) \circ R(F)$, and $R(F^*)$ is the adjoint of $R(F)$, hence the eigenvalues of $R(f_\infty)$ are non-negative. But those are precisely $h(\nu_\pi)$. Now by (3.8) it suffices to show that

$$\int_{\mathfrak{a}^*} h(-i\nu) W(i\nu, \mathsf{t}, \psi) W(-i\nu, \mathsf{t}, \overline{\psi}) \frac{d\nu}{c(i\nu)c(-i\nu)} \neq 0.$$

By Theorem 2.3.4, this is the same as showing that

$$\int_{U(\mathbb{R})} f_\infty(\mathsf{t}^{-1}\mathsf{u}\mathsf{t})\overline{\psi}(\mathsf{u})\,d\mathsf{u} \neq 0.$$

By definition of f_∞, this integral equals

$$\int_{U(\mathbb{R})}\int_{\mathrm{Sp}_4(\mathbb{R})} F(\mathsf{y}^{-1})F(\mathsf{y}^{-1}\mathsf{t}^{-1}\mathsf{u}\mathsf{t})\overline{\psi}(\mathsf{u})\,d\mathsf{y}\,d\mathsf{u}.$$

Now by Lemma 3.5.2, if both y^{-1} and $\mathsf{y}^{-1}\mathsf{t}^{-1}\mathsf{u}\mathsf{t} \in \mathrm{Supp}(F) = S_\epsilon$ then $\mathsf{t}^{-1}\mathsf{u}\mathsf{t} \in 1 + B_{21\epsilon}$ and hence $\psi(\mathsf{u}) = 1 + O(\epsilon\|\mathsf{t}^{-1}\| \cdot \|\mathsf{t}\|)$. So it suffices to show that

$$\int_{U(\mathbb{R})}\int_{\mathrm{Sp}_4(\mathbb{R})} F(\mathsf{y}^{-1})F(\mathsf{y}^{-1}\mathsf{t}^{-1}\mathsf{u}\mathsf{t})\,d\mathsf{y}\,d\mathsf{u} \neq 0.$$

But if both $\mathsf{y}^{-1} \in \mathrm{Supp}(F)$ and $\mathsf{t}^{-1}\mathsf{u}\mathsf{t} \in S_{\frac{\epsilon}{21}} \subset \mathrm{Supp}(F)$ then by Lemma 3.5.2 we have $\mathsf{y}^{-1}\mathsf{t}^{-1}\mathsf{u}\mathsf{t} \in S_\epsilon = \mathrm{Supp}(F)$. Since S_ϵ and $U(\mathbb{R}) \cap S_{\frac{\epsilon}{21}}$ have positive measure and since F is non-negative, this proves the claim. $\square$

APPENDIX A

Absolute convergence of the kernel

For completeness, we give a proof of Proposition 2.4.8. The proof is directly adapted from [**KL13**], where the case of GL_2 was treated. Here we give a proof for general connected reductive algebraic groups over $\mathbb{Q}$. We start with recalling some definition and facts from [**Art05**].

1. Langlands spectral decomposition

Fix P_0 a minimal parabolic subgroup. Let $K = \prod_p K_p$ be a compact subgroup of $G(\mathbb{A})$ such that K_∞ is a maximal compact subgroup of the connected component $G^\circ(\mathbb{R})$ of 1 in $G(\mathbb{R})$ and K_p is a maximal compact subgroup of $G(\mathbb{Q}_p)$ for all prime p, and such that we have $G = P_0 K$. Many of the definitions we gave in Chapter 2 can be directly adapted with this choice of K.

Definition A.1.1. We let $A_G(\mathbb{Q})$ be the largest central subgroup of G over $\mathbb{Q}$ that is a $\mathbb{Q}$-split torus. Let $A_G^+(\mathbb{R})$ be the connected component of identity in $A_G(\mathbb{R})$. Then we have $G(\mathbb{A}) = A_G^+(\mathbb{R})G^1(\mathbb{A})$, where $G^1(\mathbb{A}) = \{\mathsf{g} \in G(\mathbb{A}), H_G(\mathsf{g}) = 0\}$ and H_G was defined (in the particular case of $G = \mathrm{GSp}_4$) in Section 2.4 of Chapter 2.

Definition A.1.2. If P is a parabolic subgroup with Levi decomposition $P = N_P M_P$, we let A_P be the centre of M_P, and we let $\mathfrak{a}_P$ be the Lie algebra of $A_P(\mathbb{R}) \cap G^1(\mathbb{A})$.

Remark A.1.1. *Note that this definition differs from* [**Art05**]. *This is because we are interested in the spectral decomposition of* $L^2(A_G(\mathbb{R})G(\mathbb{Q})\backslash G(\mathbb{A}))$ *instead of* $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$.

Definition A.1.3. If P and P' are two standard parabolic subgroups, let $\Omega(\mathfrak{a}_P, \mathfrak{a}_{P'})$ be the set of distinct linear isomorphisms from $\mathfrak{a}_P \subset \mathfrak{a}_{P_0}$ onto $\mathfrak{a}_{P'} \subset \mathfrak{a}_{P_0}$ obtained by restriction of elements in the Weyl group Ω. If $\Omega(\mathfrak{a}_P, \mathfrak{a}_{P'})$ is non-empty, we say that P and P' are *associated*.

Remark A.1.2. *In the case of* GSp_4, *two standard parabolic subgroups are associated if and only if they are equal.*

For each pair of standard parabolic subgroups P and P' and for each $s \in \Omega(\mathfrak{a}_P, \mathfrak{a}_{P'})$, there is an intertwining operator between the representations $\mathscr{I}_P(\nu)$ and $\mathscr{I}_{P'}(s\nu)$ (whose definition is given in § 4.1).

Definition A.1.4. Let $s \in \Omega(\mathfrak{a}_P, \mathfrak{a}_{P'})$, let $\phi \in \mathscr{H}_P^0$, and let $\nu \in \mathfrak{a}_P^*(\mathbb{C})$ with large enough real part. Then for every $\mathsf{x} \in G(\mathbb{A})$ the following integral converges absolutely to an analytic function in ν

$$(M(s,\nu)\phi)(\mathsf{x}) = \exp(-\langle s\nu+\rho_{P'}, H_{P'}(\mathsf{x})\rangle) \int_{N_s(\mathbb{A})} \phi(s^{-1}\mathsf{n}\mathsf{x}) \exp(\langle \nu+\rho_P, H_P(s^{-1}\mathsf{n}\mathsf{x})\rangle)\, d\mathsf{n},$$

where $N_s = (N_{P'} \cap sN_P s^{-1})\backslash N_{P'}$ (here we identify $s \in \Omega(\mathfrak{a}_P, \mathfrak{a}_{P'})$ with a representative in $G(\mathbb{Q})$).

Now fix a finite index subgroup $\Gamma = \prod_p \Gamma_p$ of K with $\Gamma_\infty = K_\infty$, and a character of $A_G(\mathbb{A})$ that is trivial on $A_G(\mathbb{Q})A_G(\mathbb{R})$ and on $A_G(\mathbb{A}) \cap \Gamma$. We denote by $L^2(\omega)$ the subspace of $L^2(A_G(\mathbb{R})G(\mathbb{Q})/G(\mathbb{A}))$ consisting of functions that are right-Γ-invariant and that have central character ω. The spectral decomposition of $L^2(\omega)$ is due to Langlands.

THEOREM A.1.1. *(1) Suppose $\phi \in \mathscr{H}_P^0$. Then $E(x, \phi, \nu)$ and $M(s, \nu)\phi$ can be analytically continued to meromorphic functions of $\nu \in \mathfrak{a}_P^*(\mathbb{C})$ that satisfy the functional equations*

$$E(\mathsf{x}, M(s, \nu)\phi, s\nu) = E(\mathsf{x}, \phi\, nu)$$

and

$$M(s_1 s_2, \nu) = M(s_1, s_2\nu)M(s_2, \nu).$$

Moreover both $E(\mathsf{x}, \phi, \nu)$ and $M(s, \nu)\phi$ are analytic in $\nu \in i\mathfrak{a}_P^*$, and $M(s, \nu)$ extends to a unitary operator from $\mathscr{H}_P$ to $\mathscr{H}_{P'}$.

(2) For each association class $\mathscr{P}$ of standard parabolic subgroups, let $\mathscr{L}_{\mathscr{P}}$ be the Hilbert space of families of measurable functions $F = (F_P)_{P \in \mathscr{P}}$ with

$$F_P : i\mathfrak{a}_P^* \to \mathscr{H}_P^\Gamma(\omega)$$

satisfying

$$F_{P'}(s\nu) = M(s, \nu)F_P(\nu)$$

and

$$\|F\|^2 \doteq \sum_{P \in \mathscr{P}} \frac{1}{n_P} \int_{i\mathfrak{a}_P^*} \|F_P(\nu)\|^2 \, d\nu < \infty,$$

where

$$n_P = \sum_{P' \in \mathscr{P}} \#\Omega(\mathfrak{a}_P, \mathfrak{a}_{P'}).$$

Then the mapping that sends F to the function $S(F)$ defined for $\mathsf{x} \in G(\mathbb{A})$ by

$$S(F)(x) = \sum_{P \in \mathscr{P}} \frac{1}{n_P} \int_{i\mathfrak{a}_P^*} E(\mathsf{x}, F_P(\nu), \nu)\, d\nu,$$

defined for when F_P is a smooth compactly supported function on $i\mathfrak{a}_P^*$ with values in a finite dimensional subspace of $\mathscr{H}_P^0$, extends to a unitary mapping from $\mathscr{L}_{\mathscr{P}}$ onto a closed $G(\mathbb{A})$-invariant subspace $L^2_{\mathscr{P}}(\omega)$ of $L^2(\omega)$. Moreover, we have an orthogonal direct sum decomposition

$$L^2(\omega) = \bigoplus_{\mathscr{P}} L^2_{\mathscr{P}}(\omega).$$

2. The geometric kernel and the spectral kernel

In this section we prove the absolute convergence of the spectral expression of the kernel associated to a function f satisfying the following.

Assumption A.1. Consider a measurable function $f : G(\mathbb{A}) \to \mathbb{C}$ with the following properties.

- $f(\mathsf{gz}) = \overline{\omega}(\mathsf{z}) f(\mathsf{g})$ for all $\mathsf{z} \in A_G(\mathbb{A})$ and $\mathsf{g} \in G(\mathbb{A})$,
- f is compactly supported modulo A_G,
- f is left and right Γ-invariant,
- $f = f_\infty f_{\mathrm{fin}}$ where f_∞ is smooth and has its support contained in $G^\circ(\mathbb{R})$.

Associated to f we have the operator $R(f)$ acting on $L^2(\omega)$ by

$$R(f)\phi(\mathsf{x}) = \int_{\overline{G}(\mathbb{A})} f(\mathsf{y})\phi(\mathsf{xy})\, d\mathsf{y} = \int_{\overline{G}(\mathbb{Q})\backslash \overline{G}(\mathbb{A})} K_f(\mathsf{x}, \mathsf{y})\psi(\mathsf{y})\, d\mathsf{y},$$

where

$$K_f(\mathsf{x},\mathsf{y}) = \sum_{\gamma\in\overline{G}(\mathbb{Q})} f(\mathsf{x}^{-1}\gamma\mathsf{y})$$

and $\overline{G} = A_G\backslash G$. Since f is continuous and compactly supported modulo $A_G(\mathbb{A})$ and since $\overline{G}(\mathbb{Q})$ is a discrete subset of $\overline{G}(\mathbb{A})$, the series defining $K_f(\mathsf{x},\mathsf{y})$ is locally finite and hence the latter is a continuous function on $G(\mathbb{A})\times G(\mathbb{A})$. By Theorem A.1.1 the corresponding operator $\mathscr{I}_P(f,\nu)$ on $\mathscr{H}_P^\Gamma(\omega)$ given by

$$\mathscr{I}_P(f,\nu)\phi = \int_{\overline{G}(\mathbb{A})} f(\mathsf{y})\mathscr{I}_P(\mathsf{y},\nu)\phi\,d\mathsf{y}$$

satisfies $R(f)\circ S = S\circ\mathscr{I}_P(f)$. In addition, we have a convolution product $f*g$ given by

$$(f*g)(\mathsf{x}) = \int_{\overline{G}(\mathbb{A})} f(\mathsf{y})g(\mathsf{y}^{-1}\mathsf{x})\,d\mathsf{y},$$

and we have $R(f*g) = R(f)\circ R(g)$. Finally, the adjoint of $R(f)$ is $R(f^*)$, where $f^*(\mathsf{g}) = \overline{f}(\mathsf{g}^{-1})$.

Lemma A.2.1. *Fix an association class $\mathscr{P}$ of parabolic subgroups. Let $J=(J_P)_{P\in\mathscr{P}}$ be a family of compact sets $J_P\subset i\mathfrak{a}_P^*$ satisfying the symmetry condition $sJ_P = J_{P'}$ for all $s\in\Omega(\mathfrak{a}_P,\mathfrak{a}_{P'})$. Moreover, for each $P\in\mathscr{P}$ let Q_P be a finite set of irreducible representations π with central character ω, occurring in $R_{M_P,\mathrm{disc}}$, and with the property that if $\pi\in Q_P$ then there exists $\pi'\in Q_{P'}$ such that $M(s,\nu)\mathscr{I}_P(\pi_\nu) = \mathscr{I}_P(\pi'_{s\nu})$ for all $s\in\Omega(\mathfrak{a}_P,\mathfrak{a}_{P'})$ and all $\nu\in i\mathfrak{a}_P$*[1]*. Finally, for each $\pi\in Q_P$, let $\mathscr{B}_\pi$ be an orthonormal*

[1] note that the space of $\mathscr{I}_P(\pi_\nu)$ does not depend on ν and hence this condition does not actually depend on ν either.

basis of the finite dimensional space $\mathscr{I}_P(\pi^\Gamma_\nu)$ consisting of elements of $\mathscr{H}^0_P$. Define

$$K^{Q,J}_{\mathscr{P}}(\mathsf{x},\mathsf{y})=\sum_{P\in\mathscr{P}}\frac{1}{n_P}\int_{J_P}\sum_{\pi\in Q_P}\sum_{u\in\mathscr{B}_\pi}E(\mathsf{x},\mathscr{I}_P(\nu,f)u,\nu)\overline{E(\mathsf{y},\mathscr{I}_P(\nu,f)u,\nu)}\,d\nu.$$

Then there exists a bounded linear operator $T_{\mathscr{P}}$ on $L^2_{\mathscr{P}}(\omega)$ such that for all $\psi\in L^2_{\mathscr{P}}(\omega)$ that is bounded and have compact support modulo $G(\mathbb{Q})A_G(\mathbb{A})$ we have

$$(T_{\mathscr{P}}\psi)(\mathsf{x})=\int_{\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})}K^{Q,J}_{\mathscr{P}}(\mathsf{x},\mathsf{y})\psi(\mathsf{y})\,d\mathsf{y}$$

for almost all $\mathsf{x}\in G(\mathbb{A})$.

PROOF. Let $\mathscr{L}^Q_{\mathscr{P}}$ be the subspace of $\mathscr{L}_{\mathscr{P}}$ consisting of those F such that F_P has it image contained in $\bigoplus_{\pi\in Q_P}\pi^\Gamma$ for all $P\in\mathscr{P}$. Let Let $\mathscr{L}^{^cQ}_{\mathscr{P}}$ be the subspace of $\mathscr{L}_{\mathscr{P}}$ consisting of those F such that F_P has it image contained in $\bigoplus_{\pi\notin Q_P}\pi^\Gamma$ for all $P\in\mathscr{P}$, so we have the orthogonal decomposition

$$\mathscr{L}_{\mathscr{P}}=\mathscr{L}^Q_{\mathscr{P}}\oplus\mathscr{L}^{^cQ}_{\mathscr{P}}.$$

Next, let $\mathscr{L}^{Q,J}_{\mathscr{P}}$ be the subspace of $\mathscr{L}^Q_{\mathscr{P}}$ consisting of those F such that F_P is supported on J_P for all $P\in\mathscr{P}$. Then we have the orthogonal decomposition

$$\mathscr{L}^Q_{\mathscr{P}}=\mathscr{L}^{Q,J}_{\mathscr{P}}\oplus\mathscr{L}^{Q,{}^cJ}_{\mathscr{P}},$$

where ${}^cJ_P=i\mathfrak{a}^*_P-J_P$. Taking the image of these decomposition by S, by Theorem A.1.1 we have

$$L_{\mathscr{P}}=L^Q_{\mathscr{P}}\oplus L^{^cQ}_{\mathscr{P}}$$

and

$$L_{\mathscr{P}}^{Q} = L_{\mathscr{P}}^{Q,J} \oplus L_{\mathscr{P}}^{Q,{}^cJ},$$

where $L_{\mathscr{P}}^{Q} = S(\mathscr{L}_{\mathscr{P}}^{Q}) \subset L_{\mathscr{P}}^{2}(\omega)$ and so on. Define $S_{Q,J}^{-1} : L_{\mathscr{P}}^{2}(\omega) \to \mathscr{L}_{\mathscr{P}}^{Q,J}$ by

$$S_{Q,J}^{-1}\phi = \begin{cases} S^{-1}\phi \text{ if } \phi \in L_{\mathscr{P}}^{Q,J} \\ 0 \text{ if } \phi \in L_{\mathscr{P}}^{Q,{}^cJ} \text{ or } \phi \in L_{\mathscr{P}}^{{}^cQ}. \end{cases}$$

Explicitly, $S_{Q,J}^{-1} = S_{J}^{-1} \circ P_Q$ where P_Q is the orthogonal projection of $L_{\mathscr{P}}^{2}(\omega)$ onto $L_{\mathscr{P}}^{Q}$, and for $\phi \in L_{\mathscr{P}}^{Q}$ we have

$$(S_J^{-1}\phi)_P(\nu) = \begin{cases} (S^{-1}\phi)_P(\nu) \text{ if } \nu \in J_P \\ 0 \text{ otherwise.} \end{cases}$$

The restriction of $S_{Q,J}^{-1}$ to $L_{\mathscr{P}}^{Q,J}$ is an isomorphism of Hilbert spaces. Moreover

$$P_{Q,J} \doteq (S_{Q,J}^{-1})^{*} \circ S_{Q,J}^{-1}$$

is the orthogonal projection of $L_{\mathscr{P}}^{2}(\omega)$ onto $L_{\mathscr{P}}^{Q,J}$, and hence we have

$$S_{Q,J}^{-1} = S^{-1} \circ P_{Q,J}.$$

Also note that we have $S^{-1} \circ P_Q = \hat{P}_Q \circ S^{-1}$, where $\hat{P}_Q$ is the orthogonal projection of $\mathscr{L}_{\mathscr{P}}$ onto $\mathscr{L}_{\mathscr{P}}^{Q}$. Define

$$T_{\mathscr{P}} = P_{Q,J} \circ R(f * f^{*}) \circ P_{Q,J},$$

Then $T_{\mathscr{P}}$ is bounded because $\|T_{\mathscr{P}}\| \leq \|R(f * f^*)\| \leq \|f * f^*\|_1$ by [**KL06**, p. 140]. Now for $\psi_1, \psi_2 \in L^2_{\mathscr{P}}(\omega)$ bounded compactly supported we have

$$\begin{aligned}
\langle T_{\mathscr{P}}\psi_1, \psi_2\rangle &= \langle P_{Q,J} \circ R(f * f^*) \circ P_{Q,J}\psi_1, \psi_2\rangle \\
&= \langle R(f^*) \circ P_{Q,J}\psi_1, R(f^*) \circ P_{Q,J}\psi_2\rangle \\
&= \langle P_{Q,J} \circ R(f^*)\psi_1, P_{Q,J} \circ R(f^*)\psi_2\rangle \\
&= \langle S^{-1} \circ P_{Q,J} \circ R(f^*)\psi_1, S^{-1} \circ P_{Q,J} \circ R(f^*)\psi_2\rangle \\
&= \langle S^{-1}_{Q,J} \circ R(f^*)\psi_1, S^{-1}_{Q,J} \circ R(f^*)\psi_2\rangle \\
&= \sum_{P\in\mathscr{P}} \frac{1}{n_P} \int_{J_P} \langle (S^{-1} \circ P_Q \circ R(f^*)\psi_1)_P, (S^{-1} \circ P_Q \circ R(f^*)\psi_2)_P\rangle \, d\nu \\
&= \sum_{P\in\mathscr{P}} \frac{1}{n_P} \int_{J_P} \langle (\hat{P}_Q \circ S^{-1} \circ R(f^*)\psi_1)_P, (\hat{P}_Q \circ S^{-1} \circ R(f^*)\psi_2)_P\rangle \, d\nu \\
&= \sum_{P\in\mathscr{P}} \frac{1}{n_P} \int_{J_P} \langle \hat{P}_Q \circ \mathscr{I}_P(f^*, \nu) \circ (S^{-1}\psi_1)_P, \hat{P}_Q \circ \mathscr{I}_P(f^*, \nu) \circ (S^{-1}\psi_1)_P\rangle \, d\nu \\
&= \sum_{P\in\mathscr{P}} \frac{1}{n_P} \int_{J_P} \sum_{\pi\in Q_P} \sum_{u\in\mathscr{B}_\pi} \langle (S^{-1}\psi_1)_P, \mathscr{I}_P(f, \nu)u\rangle \overline{\langle (S^{-1}\psi_2)_P, \mathscr{I}_P(f, \nu)u\rangle} \, d\nu \\
&= \sum_{P\in\mathscr{P}} \frac{1}{n_P} \int_{J_P} \sum_{\pi\in Q_P} \sum_{u\in\mathscr{B}_\pi} \langle (\psi_1, E(\cdot, \mathscr{I}_P(f, \nu)u, \nu)\rangle \overline{\langle (\psi_1, E(\cdot, \mathscr{I}_P(f, \nu)u, \nu)\rangle} \, d\nu \\
&= \int_{(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A}))^2} K^{Q,J}_{\mathscr{P}}(\mathsf{x}, \mathsf{y})\psi_1(\mathsf{y})\overline{\psi_2(\mathsf{x})} \, d\mathsf{y} \, d\mathsf{x}.
\end{aligned}$$

The interchange of summation and integration order is justified because Eisenstein series are continuous, J_P is compact, the u-sum is finite, and ψ_1, ψ_2 are bounded with compact support modulo $A_G(\mathbb{A})G(\mathbb{Q})$. □

Lemma A.2.2. *For each association class of parabolic subgroups $\mathscr{P}$, fix J and Q as in Lemma A.2.1, and let $T_{\mathscr{P}}$ be the corresponding bounded linear operator. Let*

$T = \sum_{\mathscr{P}} T_{\mathscr{P}}$. *Then for all* $\psi \in L^2(\omega)$ *bounded and compactly supported modulo* $G(\mathbb{Q})A_G(\mathbb{A})$ *we have*

$$\langle T\psi, \psi\rangle \le \langle R(f * f^*)\psi, \psi\rangle.$$

PROOF. For each class $\mathscr{P}$ let $P_{\mathscr{P}}$ be the orthogonal projection of $L^2(\omega)$ onto $L^2_{\mathscr{P}}(\omega)$. For each irreducible representation $\pi \notin Q_P$ with central character ω, fix an orthonormal basis $\mathscr{B}_\pi$ of $\mathscr{I}_p(\pi_\nu)^\Gamma$. Thus $\bigcup_\pi \mathscr{B}_\pi$ is an orthonormal basis of $\mathscr{H}_P^\Gamma(\omega)$. By the proof of Lemma A.2.1 above, we have

$$\begin{aligned}
\langle T_{\mathscr{P}}\psi, \psi\rangle &= \sum_{P\in\mathscr{P}} \frac{1}{n_P} \int_{J_P} \sum_{\pi\in Q_P} \sum_{u\in\mathscr{B}_\pi} \left|(\psi, E(\cdot, I_P(f,\nu)u, \nu)\right|^2 \, d\nu \\
&\le \sum_{P\in\mathscr{P}} \frac{1}{n_P} \int_{i\mathfrak{a}_P^*} \sum_{\pi\subset R_{M_p,\mathrm{disc}}} \sum_{u\in\mathscr{B}_\pi} \left|(\psi, E(\cdot, I_P(f,\nu)u, \nu)\right|^2 \, d\nu \\
&= \sum_{P\in\mathscr{P}} \frac{1}{n_P} \int_{i\mathfrak{a}_P^*} \sum_{\pi\subset R_{M_p,\mathrm{disc}}} \sum_{u\in\mathscr{B}_\pi} \left|(R(f^*)\psi, E(\cdot, u, \nu)\right|^2 \, d\nu \\
&= \langle P_{\mathscr{P}} \circ R(f^*)\psi, P_{\mathscr{P}} \circ R(f^*)\psi\rangle \\
&= \langle P_{\mathscr{P}} \circ R(f * f^*)\psi, P_{\mathscr{P}}\psi\rangle
\end{aligned}$$

since $P_{\mathscr{P}}$ commutes with $R(f)$. $\square$

We shall use the following result from [**GGK03**, Lemma 5.2.1].

LEMMA A.2.3. *Let* X *be a Radon measure space, and let* T *be an operator on* $L^2(X)$. *Suppose there is a continuous function* $K(\mathsf{x}, \mathsf{y})$ *on* $X \times X$ *such that for all* $\psi \in L^2(X)$ *that is bounded and compactly supported we have*

$$T\psi = \int_X K(\mathsf{x}, \mathsf{y})\psi(\mathsf{y}) \, d\mathsf{y}$$

and

$$\langle T\psi, \psi\rangle \geq 0.$$

Then $K(\mathsf{x},\mathsf{x}) \geq 0$ *for all* $\mathsf{x} \in X$.

PROPOSITION A.2.1. *For each association class of parabolic subgroups* $\mathscr{P}$*, and for each irreducible representation* π *with central character* ω *occurring in* $R_{M_P,\mathrm{disc}}$*, fix an orthonormal basis of* $\mathscr{I}_P(\pi_\nu)^\Gamma$ *consisting of elements of* $\mathscr{H}_P^0$*. For all* $\mathsf{x} \in G(\mathbb{A})$ *we have*

$$\sum_{\mathscr{P}} \sum_{P\in\mathscr{P}} \frac{1}{n_P} \int_{i\mathfrak{a}_P^*} \sum_{\pi} \sum_{u\in\mathscr{B}_\pi} |E(\mathsf{x}, \mathscr{I}_P(\nu, f)u, \nu)|^2 \, d\nu \leq K_{f*f^*}(\mathsf{x},\mathsf{x}).$$

PROOF. For each association class of parabolic subgroup $\mathscr{P}$, consider $J_{\mathscr{P}}$ and $Q_{\mathscr{P}}$ as in Lemma A.2.1, and let $K^{Q,J}(\mathsf{x},\mathsf{y}) = \sum_{\mathscr{P}} K_{\mathscr{P}}^{Q_{\mathscr{P}},J_{\mathscr{P}}}(\mathsf{x},\mathsf{y})$. Then $K^{Q,J}(\mathsf{x},\mathsf{y})$ is a continuous function on $G(\mathbb{A}) \times G(\mathbb{A})$ since each J_P is compact, each Q_P is finite, and the Eisenstein series are continuous. The geometric kernel $K_{f*f^*}(\mathsf{x},\mathsf{y})$ is also continuous. By Lemma A.2.2 we have $\langle (R(f * f^* - T))\psi, \psi\rangle \geq 0$ for all bounded $\psi \in L^2(\omega)$ with compact support. Hence by Lemma A.2.3 we get for all Q, J as above

$$K^{Q,J}(\mathsf{x},\mathsf{x}) \leq K_{f*f^*}(\mathsf{x},\mathsf{x}).$$

Hence

$$\sum_{\mathscr{P}} \sum_{P\in\mathscr{P}} \frac{1}{n_P} \int_{i\mathfrak{a}_P^*} \sum_{\pi} \sum_{u\in\mathscr{B}_\pi} |E(\mathsf{x}, \mathscr{I}_P(\nu, f)u, \nu)|^2 \, d\nu = \sup_{Q,J} K^{Q,J}(\mathsf{x},\mathsf{x}) \leq K_{f*f^*}(\mathsf{x},\mathsf{x}).$$

□

The following lemma is due to Duflo and Labesse [**DL71**] for GL_2, see also [**Art78**, Lemma 4.1] for the general case.

LEMMA A.2.4. *There exist* $h_{1,\infty}, h_{2,\infty}, h_{3,\infty}, h_{4,\infty}$ *smooth functions on* $G(\mathbb{R})/A_G(\mathbb{R})$ *that are bi-*K_∞*-invariant, and whose support is contained in* $G^0(\mathbb{R})$ *and is compact modulo* $A_G(\mathbb{R})$*, such that* $f_\infty = h_{1,\infty} * h_{2,\infty} + h_{3,\infty} * h_{4,\infty}$.

PROOF. Same as [**KL13**, Lemma 6.9] but we use [**Art78**, Lemma 4.1] instead of [**DL71**, I.1.11]. □

LEMMA A.2.5. *Fix a parabolic subgroup* P*. Let* $\pi = \bigotimes_{p\leq\infty}$ *be an irreducible representation occurring in* $R_{M_P,\mathrm{disc}}$*. Then the finite dimensional subspace* $\mathscr{I}_P(\pi_\infty)^{K_\infty}$ *has a basis* $\mathscr{B}_{\pi_\infty}$ *such that for every smooth functions bi-*K_∞*-invariant function* h *on* $G(\mathbb{R})/A_G(\mathbb{R})$ *whose support is contained in* $G^\circ(\mathbb{R})$ *and is compact modulo* $A_G(\mathbb{R})$*, and for all* $\nu \in i\mathfrak{a}_P^*$ *the elements of* $\mathscr{B}_{\pi_\infty}$ *are eigenfunctions of* $\mathscr{I}_P(h,\nu)$*.*

PROOF. Let V_π be the representation space of $\mathscr{I}_P(\pi_{\infty,\nu})$. We have an orthogonal decomposition

$$V_\pi = \bigoplus_\rho V_\rho \tag{A.1}$$

where V_ρ is a irreducible $G^\circ(\mathbb{R})$-invariant subspaces. By [**Kna86**, Theorem 8.1] the dimension of the K_∞-fixed subspace $V_\rho^{K_0}$ in each V_ρ is at most one. When $V_\rho^{K_0} \neq \{0\}$, write $V_\rho^{K_0} = \mathbb{C}e_\rho$ with $\|e_\rho\| = 1$. Since h is supported on $G^\circ(\mathbb{R})$, we may apply Proposition 2.3.1 to each representation ρ of $G^\circ(\mathbb{R})$, obtaining that e_ρ is an eigenvector of $\mathscr{I}_P(h,\nu)$. Note that $\mathscr{B}_{\pi_\infty} = (e_\rho)_\rho$ does not depend on ν because both V_π and the K_∞-fixed subspace $V_\pi^{K_\infty}$ are independent of ν. □

THEOREM A.2.1. *Let* f *satisfying Assumption A.1. Assume that for each parabolic subgroup* P *and for each irreducible representation* π *occurring in* $R_{M_P,\mathrm{disc}}$*, the space*

$\mathscr{I}_P(\pi)^\Gamma$ has an orthonormal basis $\mathscr{B}_\pi$ consisting of factorizable vectors $u_\infty \otimes u_{\mathrm{fin}}$ such that $u_\infty \in \mathscr{B}_{\pi_\infty}$ and u_{fin} is a common eigenfunction of all the operators $\mathscr{I}_P(f_{\mathrm{fin}}, \nu)$ for $\nu \in \mathfrak{a}_P^*$. Then the series

$$K_{abs}(\mathsf{x}, \mathsf{y}) = \sum_{\mathscr{P}} \sum_{P \in \mathscr{P}} \frac{1}{n_P} \int_{i\mathfrak{a}_P^*} \sum_{\pi} \sum_{u \in \mathscr{B}_\pi} |E(\mathsf{x}, \mathscr{I}_P(\nu, f)u, \nu) E(\mathsf{y}, u, \nu)| \, d\nu$$

converges absolutely and defines a function that is bounded on compact subsets of $G(\mathbb{A}) \times G(\mathbb{A})$ and continuous in x and y separately.

Proof. First, by Lemma A.2.4, we may assume $f_\infty = h_{1,\infty} * h_{2,\infty} + h_{3,\infty} * h_{4,\infty}$. Let T be the function on $G(\mathbb{A}_{\mathrm{fin}})$ defined by

$$T(\mathsf{g}) = \begin{cases} \frac{\omega(\mathsf{z})}{\mathrm{Vol}(\overline{\Gamma_{\mathrm{fin}}})} \text{ if there exists } \mathsf{z} \in A_G(\mathbb{A}_{\mathrm{fin}}) \text{ such that } \mathsf{g} \in \mathsf{z}\Gamma_{\mathrm{fin}} \\ 0 \text{ otherwise .} \end{cases}$$

Then $f = h_1 * h_2 + h_3 * h_4$, where $h_1 = h_{1,\infty} f_{\mathrm{fin}}$, $h_2 = h_{2,\infty} T$ and similarly for h_3 and h_4. Moreover if ϕ is right-Γ-invariant then we have $R(T)\phi = \phi$. Thus each function h_i (and a fortiori their convolution) satisfy the same conditions as f. Hence by the triangle inequality, it suffices to consider the case $f = h_1 * h_2$. By Lemma A.2.5 for $u \in \mathscr{B}_\pi$, we can write $\mathscr{I}_P(h_j, \nu)u = \lambda_j(\nu)u$ for all $\nu \in i\mathfrak{a}_P^*$. Then we have

$$E(\mathsf{x}, \mathscr{I}_P(\nu, f)u, \nu) = \lambda_1(\nu)\lambda_2(\nu)E(\mathsf{x}, u, \nu)$$

thus

$$\begin{aligned} E(\mathsf{x}, \mathscr{I}_P(\nu, f)u, \nu)\overline{E(\mathsf{y}, u, \nu)} &= \lambda_1(\nu)\lambda_2(\nu)E(\mathsf{x}, u, \nu)\overline{E(\mathsf{y}, u, \nu)} \\ &= E(\mathsf{x}, \mathscr{I}_P(\nu, h_1)u, \nu)\overline{E(\mathsf{y}, \mathscr{I}_P(\nu, h_2^*)u, \nu)}. \end{aligned}$$

Now consider any subset S_P of all the irreducible representations π occurring in $R_{M_P,\mathrm{disc}}$ and any measurable subset R_P of $i\mathfrak{a}_P^*$. Then by the Cauchy-Schwarz inequality we have

$$\sum_{\mathscr{P}}\sum_{P\in\mathscr{P}}\frac{1}{n_P}\int_{i\mathfrak{a}_P^*-R_P}\sum_{\pi\notin S_P}\sum_{u\in\mathscr{B}_\pi}|E(\mathsf{x},\mathscr{I}_P(\nu,f)u,\nu)E(\mathsf{y},u,\nu)|\ d\nu$$
$$=\sum_{\mathscr{P}}\sum_{P\in\mathscr{P}}\frac{1}{n_P}\int_{i\mathfrak{a}_P^*-R_P}\sum_{\pi\notin S_P}\sum_{u\in\mathscr{B}_\pi}|E(\mathsf{x},\mathscr{I}_P(\nu,h_1)u,\nu)E(\mathsf{y},\mathscr{I}_P(\nu,h_2^*)u,\nu)|\ d\nu$$
$$\leq\left(\sum_{\mathscr{P}}\sum_{P\in\mathscr{P}}\frac{1}{n_P}\int_{i\mathfrak{a}_P^*-R_P}\sum_{\pi\notin S_P}\sum_{u\in\mathscr{B}_\pi}|E(\mathsf{x},\mathscr{I}_P(\nu,h_1)u,\nu)|^2\ d\nu\right)^{\frac{1}{2}}$$
$$\times\left(\sum_{\mathscr{P}}\sum_{P\in\mathscr{P}}\frac{1}{n_P}\int_{i\mathfrak{a}_P^*-R_P}\sum_{\pi\notin S_P}\sum_{u\in\mathscr{B}_\pi}|E(\mathsf{y},\mathscr{I}_P(\nu,h_2^*)u,\nu)|^2\ d\nu\right)^{\frac{1}{2}}$$
$$\leq K_{h_1*h_1^*}(\mathsf{x},\mathsf{x})K_{h_2^**h_2}(\mathsf{y},\mathsf{y})$$

by Proposition A.2.1. Since both kernels are continuous, they are in particular bounded on compact sets, which proves the first part of the theorem. Now let us fix $\mathsf{x}\in G(\mathbb{A})$ and prove the continuity of $K_{abs}(\mathsf{x},\mathsf{y})$ in y. Fix an arbitrary compact set $U\subset G(\mathbb{A})$. It suffices to show that the series/integral defining $K_{abs}(\mathsf{x},\mathsf{y})$ converges uniformly in $\mathsf{y}\in U$. Let C an upper bound for $K_{h_2^**h_2}(\mathsf{y},\mathsf{y})$ on U. Fix $\epsilon>0$. Since

$$\sum_{\mathscr{P}}\sum_{P\in\mathscr{P}}\frac{1}{n_P}\int_{i\mathfrak{a}_P^*}\sum_{\pi}\sum_{u\in\mathscr{B}_\pi}|E(\mathsf{x},\mathscr{I}_P(\nu,h_1)u,\nu)|^2\ d\nu<\infty,$$

if S_P is a large enough finite set and R_P is a large enough compact set for all parabolic subgroup P then

$$\sum_{\mathscr{P}}\sum_{P\in\mathscr{P}}\frac{1}{n_P}\int_{i\mathfrak{a}_P^*-R_P}\sum_{\pi\notin S_P}\sum_{u\in\mathscr{B}_\pi}|E(\mathsf{x},\mathscr{I}_P(\nu,h_1)u,\nu)|^2\ d\nu<\frac{\epsilon^2}{C^2}.$$

Therefore by the above,

$$\sum_{\mathscr{P}}\sum_{P\in\mathscr{P}}\frac{1}{n_P}\int_{i\mathfrak{a}_P^*-R_P}\sum_{\pi\notin S_P}\sum_{u\in\mathscr{B}_\pi}\left|E(\mathsf{x},\mathscr{I}_P(\nu,f)u,\nu)E(\mathsf{y},u,\nu)\right|\,d\nu$$
$$\leq\left(\sum_{\mathscr{P}}\sum_{P\in\mathscr{P}}\frac{1}{n_P}\int_{i\mathfrak{a}_P^*-R_P}\sum_{\pi\notin S_P}\sum_{u\in\mathscr{B}_\pi}\left|E(\mathsf{x},\mathscr{I}_P(\nu,h_1)u,\nu)\right|^2\,d\nu\right)^{\frac{1}{2}}K_{h_2^* * h_2}(\mathsf{y},\mathsf{y})$$
$$\leq\epsilon,$$

which establishes the result. The same reasoning holds after exchanging x and y. $\square$

Bibliography

[Art05] James Arthur, *An introduction to the trace formula*, Harmonic analysis, the trace formula, and Shimura varieties, Clay Math. Proc., vol. 4, Amer. Math. Soc., Providence, RI, 2005, pp. 1–263.

[Art78] ———, *A trace formula for reductive groups. I. Terms associated to classes in $G(\mathbf{Q})$*, Duke Math. J. **45** (1978), no. 4, 911–952.

[Ass17] Edgar Assing, *On sup-norm bounds part I: ramified Maaß newforms over number fields.* (2017), preprint, available at https://arxiv.org/abs/1710.00362.

[Ass19a] ———, *Local Analysis of Whittaker New Vectors and Global Applications.* (2019), Ph.D. thesis, available at https://research-information.bris.ac.uk/en/theses/local-analysis-of-whittaker-new-vectors-and-global-applications(fd1d8115-513c-48db-94de-79abb60c5c89).html.

[Ass19b] ———, *On the size of p-adic Whittaker functions*, Trans. Amer. Math. Soc. **372** (2019), no. 8, 5287–5340.

[BLGHT11] Tom Barnet-Lamb, David Geraghty, Michael Harris, and Richard Taylor, *A family of Calabi-Yau varieties and potential automorphy II*, Publ. Res. Inst. Math. Sci. **47** (2011), no. 1, 29–98.

[Blo13] Valentin Blomer, *Applications of the Kuznetsov formula on $GL(3)$*, Invent. Math. **194** (2013), no. 3, 673–729.

[BB19] Valentin Blomer and Jack Buttcane, *Global decomposition of* GL(3) *Kloosterman sums and the spectral large sieve*, J. Reine Angew. Math. **757** (2019), 51–88.

[BBM17] Valentin Blomer, Jack Buttcane, and Péter Maga, *Applications of the Kuznetsov formula on* GL(3*) II: the level aspect*, Math. Ann. **369** (2017), no. 1-2, 723–759.

[BBR14] Valentin Blomer, Jack Buttcane, and Nicole Raulf, *A Sato-Tate law for* GL(3), Comment. Math. Helv. **89** (2014), no. 4, 895–919.

[BH10] Valentin Blomer and Roman Holowinsky, *Bounding sup-norms of cusp forms of large level*, Invent. Math. **179** (2010), no. 3, 645–681.

[Bru78] R. W. Bruggeman, *Fourier coefficients of cusp forms*, Invent. Math. **45** (1978), no. 1, 1–18.

[But13] Jack Buttcane, *On sums of $SL(3,\mathbb{Z})$ Kloosterman sums*, Ramanujan J. **32** (2013), no. 3, 371–419.

[But16] ———, *The spectral Kuznetsov formula on $SL(3)$*, Trans. Amer. Math. Soc. **368** (2016), no. 9, 6683–6714.

[But19] ———, *The arithmetic Kuznetsov formula on* GL(3)*, I: The Whittaker case*, Rev. Mat. Iberoam. **35** (2019), no. 7, 2093–2118.

[But20a] ———, *Bessel functions outside of $GL(2)$* (2020), conference talk, available at https://old.renyi.hu/seminars/ocaf202020200605_Buttcane.pdf.

[But20b] ———, *Kuznetsov, Petersson and Weyl on* GL(3)*, I: The principal series forms*, Amer. J. Math. **142** (2020), no. 2, 595–626.

[But21] ———, *Kuznetsov, Petersson and Weyl on $GL(3)$, II: The generalized principal series forms*, Math. Ann. **380** (2021), no. 1-2, 231–266.

[But22] ———, *The arithmetic Kuznetsov formula on GL(3), II: The general case*, Algebra Number Theory **16** (2022), no. 3, 567–646.

[CS80] W. Casselman and J. Shalika, *The unramified principal series of p-adic groups. II. The Whittaker function*, Compositio Math. **41** (1980), no. 2, 207–231.

[CI19] Shih-Yu Chen and Atsushi Ichino, *On Petersson norms of generic cusp forms and special values of adjoint L-functions for* GSp_4 (2019), available at https://arxiv.org/abs/1902.06429.

[Com21] Félicien Comtat, *Optimal sup norm bounds for newforms on* GL_2 *with maximally ramified central character*, Forum Math. **33** (2021), no. 1, 1–16.

[CDF97] J. B. Conrey, W. Duke, and D. W. Farmer, *The distribution of the eigenvalues of Hecke operators*, Acta Arith. **78** (1997), no. 4, 405–409.

[Dic15] Martin Dickson, *Local spectral equidistribution for degree two Siegel modular forms in level and weight aspects*, Int. J. Number Theory **11** (2015), no. 2, 341–396.

[DL71] Michel Duflo and Jean-Pierre Labesse, *Sur la formule des traces de Selberg*, Ann. Sci. École Norm. Sup. (4) **4** (1971), 193–284.

[GGK03] Israel Gohberg, Seymour Goldberg, and Marinus A. Kaashoek, *Basic classes of linear operators*, Birkhäuser Verlag, Basel, 2003.

[GK82] S. S. Gelbart and A. W. Knapp, *L-indistinguishability and R groups for the special linear group*, Adv. in Math. **43** (1982), no. 2, 101–121.

[HM06] Gergely Harcos and Philippe Michel, *The subconvexity problem for Rankin-Selberg L-functions and equidistribution of Heegner points. II*, Invent. Math. **163** (2006), no. 3, 581–655.

[HT13] Gergely Harcos and Nicolas Templier, *On the sup-norm of Maass cusp forms of large level. III*, Math. Ann. **356** (2013), no. 1, 209–216.

[Hel84] Sigurdur Helgason, *Groups and Geometric Analysis: Integral Geometry, Invariant Differential Operators, and Spherical Functions*, Mathematical Surveys and Monographs, Academic Press, 1984.

[HL94] Jeffrey Hoffstein and Paul Lockhart, *Coefficients of Maass forms and the Siegel zero*, Ann. of Math. (2) **140** (1994), no. 1, 161–181. With an appendix by Dorian Goldfeld, Hoffstein and Daniel Lieman.

[HNS19] Yueke Hu, Paul D. Nelson, and Abhishek Saha, *Some analytic aspects of automorphic forms on* $\mathrm{GL}(2)$ *of minimal type*, Comment. Math. Helv. **94** (2019), no. 4, 767–801.

[HS20] Yueke Hu and Abhishek Saha, *Sup-norms of eigenfunctions in the level aspect for compact arithmetic surfaces, II: newforms and subconvexity*, Compos. Math. **156** (2020), no. 11, 2368–2398.

[Ish05] Taku Ishii, *On principal series Whittaker functions on* $\mathrm{Sp}(2,\mathbf{R})$, J. Funct. Anal. **225** (2005), no. 1, 1–32.

[JL85] H. Jacquet and K. F. Lai, *A relative trace formula*, Compositio Math. **54** (1985), no. 2, 243–310.

[Joy90] David Joyner, *On the Kuznetsov-Bruggeman formula for a Hilbert modular surface having one cusp*, Math. Z. **203** (1990), no. 1, 59–104.

[Kim95] Henry H. Kim, *The residual spectrum of* Sp_4, Compositio Math. **99** (1995), no. 2, 129–151.

[Kim03] ______, *Functoriality for the exterior square of* GL_4 *and the symmetric fourth of* GL_2, J. Amer. Math. Soc. **16** (2003), no. 1, 139–183. With appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak.

[KWY20] Henry H. Kim, Satoshi Wakatsuki, and Takuya Yamauchi, *An equidistribution theorem for holomorphic Siegel modular forms for* GSp_4 *and its applications*, J. Inst. Math. Jussieu **19** (2020), no. 2, 351–419.

[Kna86] Anthony W. Knapp, *Representation theory of semisimple groups*, Princeton Mathematical Series, vol. 36, Princeton University Press, Princeton, NJ, 1986. An overview based on examples.

[KL08] Andrew Knightly and Charles Li, *Petersson's trace formula and the Hecke eigenvalues of Hilbert modular forms*, Modular forms on Schiermonnikoog, Cambridge Univ. Press, Cambridge, 2008, pp. 145–187.

[KL13] A. Knightly and C. Li, *Kuznetsov's trace formula and the Hecke eigenvalues of Maass forms*, Mem. Amer. Math. Soc. **224** (2013), no. 1055, vi+132.

[KL19] Andrew Knightly and Charles Li, *On the distribution of Satake parameters for Siegel modular forms*, Doc. Math. **24** (2019), 677–747.

[KL06] ______, *Traces of Hecke operators*, Mathematical Surveys and Monographs, vol. 133, American Mathematical Society, Providence, RI, 2006.

[KST12] Emmanuel Kowalski, Abhishek Saha, and Jacob Tsimerman, *Local spectral equidistribution for Siegel modular forms and applications*, Compos. Math. **148** (2012), no. 2, 335–384.

[Li04] Charles C. C. Li, *Kuznietsov trace formula and weighted distribution of Hecke eigenvalues*, J. Number Theory **104** (2004), no. 1, 177–192.

[MT21] Jasmin Matz and Nicolas Templier, *Sato-Tate equidistribution for families of Hecke-Maass forms on* $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$, Algebra Number Theory **15** (2021), no. 6, 1343–1428.

[MV10] Philippe Michel and Akshay Venkatesh, *The subconvexity problem for* GL_2, Publ. Math. Inst. Hautes Études Sci. **111** (2010), 171–271.

[Mui09] Goran Muić, *Intertwining operators and composition series of generalized and degenerate principal series for* $\mathrm{Sp}(4,\mathbb{R})$, Glas. Mat. Ser. III **44(64)** (2009), no. 2, 349–399.

[Nel18] Paul D. Nelson, *Microlocal lifts and quantum unique ergodicity on* $GL_2(\mathbb{Q}_p)$, Algebra Number Theory **12** (2018), no. 9, 2033–2064.

[Niw95] Shinji Niwa, *Commutation relations of differential operators and Whittaker functions on* $\mathrm{Sp}_2(\mathbf{R})$, Proc. Japan Acad. Ser. A Math. Sci. **71** (1995), no. 8, 189–191.

[Pit19] Ameya Pitale, *Siegel modular forms*, Lecture Notes in Mathematics, vol. 2240, Springer, Cham, 2019. A classical and representation-theoretic approach.

[PS09] Ameya Pitale and Ralf Schmidt, *Ramanujan-type results for Siegel cusp forms of degree 2*, J. Ramanujan Math. Soc. **24** (2009), no. 1, 87–111.

[RS16] Brooks Roberts and Ralf Schmidt, *Some results on Bessel functionals for* $\mathrm{GSp}(4)$, Doc. Math. **21** (2016), 467–553.

[RS07] ———, *Local newforms for GSp(4)*, Lecture Notes in Mathematics, vol. 1918, Springer, Berlin, 2007.

[Sah16] Abhishek Saha, *Large values of newforms on* $\mathrm{GL}(2)$ *with highly ramified central character*, Int. Math. Res. Not. IMRN **13** (2016), 4103–4131.

[Sah17] ———, *Hybrid sup-norm bounds for Maass newforms of powerful level*, Algebra Number Theory **11** (2017), no. 5, 1009–1045.

[Sah20] ———, *Sup-norms of eigenfunctions in the level aspect for compact arithmetic surfaces*, Math. Ann. **376** (2020), no. 1-2, 609–644.

[Sar87] Peter Sarnak, *Statistical properties of eigenvalues of the Hecke operators*, Analytic number theory and Diophantine problems (Stillwater, OK, 1984), Progr. Math., vol. 70, Birkhäuser Boston, Boston, MA, 1987, pp. 321–331.

[Sch02] Ralf Schmidt, *Some remarks on local newforms for* GL(2), J. Ramanujan Math. Soc. **17** (2002), no. 2, 115–147.

[Ser97] Jean-Pierre Serre, *Répartition asymptotique des valeurs propres de l'opérateur de Hecke* T_p, J. Amer. Math. Soc. **10** (1997), no. 1, 75–102 (French).

[Shi12] Sug Woo Shin, *Automorphic Plancherel density theorem*, Israel J. Math. **192** (2012), no. 1, 83–120.

[ST16] Sug Woo Shin and Nicolas Templier, *Sato-Tate theorem for families and low-lying zeros of automorphic L-functions*, Invent. Math. **203** (2016), no. 1, 1–177. Appendix A by Robert Kottwitz, and Appendix B by Raf Cluckers, Julia Gordon and Immanuel Halupczok.

[SHM20] Siu Hang Man, *Symplectic Kloosterman Sums and Poincaré Series* (2020), available at https://arxiv.org/abs/2006.03036.

[SHM21] ______, *A Density Theorem for $Sp(4)$* (2021), available at https://arxiv.org/abs/2101.09602.

[Tem14] Nicolas Templier, *Large values of modular forms*, Camb. J. Math. **2** (2014), no. 1, 91–116.

[Ver19] Colin VerNooy, *K-types and Invariants for the Representations of $GSp(4,\mathbb{R})$*, Ph.D. Thesis, 2019, https://hdl.handle.net/11244/321122.

[Vog78] David A. Vogan Jr., *Gel′fand-Kirillov dimension for Harish-Chandra modules*, Invent. Math. **48** (1978), no. 1, 75–98.

[Wal92] Nolan R. Wallach, *Real reductive groups. II*, Pure and Applied Mathematics, vol. 132, Academic Press, Inc., Boston, MA, 1992.

[Zho14] Fan Zhou, *Weighted Sato-Tate vertical distribution of the Satake parameter of Maass forms on* PGL(N), Ramanujan J. **35** (2014), no. 3, 405–425.