K-THEORY FOR ÉTALE GROUPOID C*-ALGEBRAS VIA GROUPOID CORRESPONDENCES AND SPECTRAL SEQUENCES

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Statement of originality

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Abstract

We develop techniques that allow us to convert isomorphisms in the homology of ample groupoids into isomorphisms in the K-theory of their associated C*-algebras. We apply this to prove an “orbit-stabiliser” K-theory formula for certain inverse semigroup dynamical systems, allowing us to compute the K-theory of reduced C*-algebras of inverse semigroups and left regular algebras of finitely aligned left cancellative small categories.

To apply our techniques the isomorphism in homology must be induced by a correspondence of groupoids. Correspondences are a type of morphism of groupoids which can induce maps in the K-theory of the associated C*-algebras, whose isomorphisms are Morita equivalences. We extend this induced map in operator K-theory in a categorical fashion by constructing functors between the associated equivariant Kasparov categories and natural transformations between the associated K-theory functors. We also construct a map in homology from a proper correspondence of ample groupoids. This groupoid homology is related to the operator K-theory by Proietti and Yamashita’s implementation of the ABC spectral sequence. We develop general functoriality for the ABC spectral sequence, which is part of a framework developed by Meyer and Nest for a categorical approach to the Baum-Connes conjecture. From a proper correspondence of étale groupoids we obtain a morphism of associated ABC spectral sequences, allowing us to construct isomorphisms in K-theory from more tractable kinds of isomorphism.

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Contents

Introduction

"Groupoid" has been a key word in the vocabulary of the common C*-algebraist since Renault's seminal work on C*-algebras constructed from topological groupoids [72], which capture the essence of topological dynamics in a very broad sense. Renault's groupoid C*-algebras provide a common language for C*-algebras associated to a great variety of mathematical objects, each capturing some aspect of topological dynamics. This ranges from algebraic objects like groups and semigroups to combinatorial objects such as (higher rank) graphs, from coarse geometry to the classical dynamical systems of a homeomorphism on a compact metric space. Étale groupoids are topological groupoids which more closely resemble discrete-time dynamics, and provide models for a huge range of C*-algebras. Étale groupoids can be constructed from Cartan inclusions of C*-algebras [74], and there is a (twisted) étale groupoid model [47] for every C*-algebra which fits into the celebrated classification programme [25, 33, 42, 69, 83].

K-theory for C*-algebras extends topological K-theory of spaces and plays a crucial role in noncommutative geometry and index theory. It is an important tool for extracting useful information from a C*-algebra, and is one of the primary invariants in the classification of C*-algebras. It is therefore no surprise that the task of computing the K-theory of C*-algebras associated to étale groupoids is in general very difficult. Some of the most wide-reaching techniques developed to attack this problem involve connecting the operator-algebraic K-theory of the groupoid C*-algebra with invariants of a more topological nature, embracing the philosophy of noncommutative topology. The Baum-Connes conjecture [4, 5, 86] is an example of this, claiming an isomorphism between the topological K-theory of a groupoid and the operator K-theory. Matui's investigation [54–56] into Crainic and Moerdijk's homology theory for étale groupoids [19] reveals close links with K-theory, inspiring his HK conjecture and a flurry of activity [10, 24, 29, 65, 71, 75]. In particular, Proietti and Yamashita use a spectral sequence to connect the homology of an ample groupoid with torsion-free isotropy groups to the K-theory of its C*-algebra in [71].

In this document we investigate how correspondences $\Omega\colon G \to H$ of étale groupoids can help us to understand the relationship between the operator K-theory groups $K_*(C_r^*(G))$ and $K_*(C_r^*(H))$. Holkar introduced the notion of a groupoid correspondence $\Omega\colon G \to H$ in [38] from which he constructed a C*-correspondence $C^*(\Omega)\colon C^*(G) \to C^*(H)$. When Ω is proper this induces a map in K-theory. We develop many more functoriality results for constructions built from étale groupoids with respect to correspondences of those groupoids. As a result, we prove the following link between the operator K-theory and the homology of an ample groupoid:

Theorem A (See Corollary 7.29)**.** *Let G and H be Hausdorff ample groupoids. Then any proper correspondence $\Omega\colon G \to H$ induces a map in homology*

$$H_*(\Omega)\colon H_*(G) \to H_*(H).$$

Suppose further that G and H are second countable, have torsion-free isotropy groups and satisfy the Baum-Connes conjecture. If the induced map in homology $H_*(\Omega)$ is an isomorphism, then there is an isomorphism $K_*(C_r^*(G)) \cong K_*(C_r^*(H))$ in K-theory.

This is useful in situations where groupoid homology is easier to work with than operator K-theory. We apply Theorem A to compute the K-theory of the reduced C*-algebra of many inverse semigroups. Inverse semigroups are semigroups of partial symmetries that are intertwined with the theory of étale groupoids. They often arise in the theory of C*-algebras as systems of partial isometries.

Theorem B (Orbit-stabiliser K-theory formula for an inverse semigroup, Corollary 9.1)**.** *Let S be a countable inverse semigroup with stabiliser subgroups $\mathrm{Stab}_e(S) := \{s \in S \mid s^*s = e = ss^*\}$ for each idempotent $e \in E$. Suppose that each stabiliser subgroup is torsion-free and that the universal groupoid $G(S)$ is Hausdorff and satisfies the Baum-Connes conjecture. Then we may compute the K-theory of $C_r^*(S)$:*

$$K_*(C_r^*(S)) \cong \bigoplus_{\mathrm{orb}(e)\in S\backslash E^\times} K_*(C_r^*(\mathrm{Stab}_e(S))). \tag{0.1}$$

The direct sum in this formula is taken over the orbits $\mathrm{orb}(e)$ of the canonical action $S \curvearrowright E^\times$ of S on its non-zero idempotents. This reduces the problem of understanding the K-theory $K_*(C_r^*(S))$ to a problem for group C*-algebras. These groups are often considerably less complicated than S, and in many applications are trivial. The origins of this formula lie in Cuntz, Echterhoff and Li's computation of the K-theory associated to certain left cancellative monoids in [20, 21]. In [64] Norling applied these results to the reduced C*-algebra of an inverse semigroup, with more general inverse semigroups covered in [48]. We further extend the class of inverse semigroups covered at the price of requiring torsion-free stabilisers. The main condition we remove from [48] is the strong 0-E-unitarity of S, which enables us to make applications to dynamical systems which can't be described in terms of the partial action of a group. Without access to a useful group, we turn to groupoid methods. The universal groupoid $G(S)$ of S is a groupoid model for $C_r^*(S)$. We probe $G(S)$ with a proper correspondence from the discrete groupoid $S \ltimes E^\times$. Using Theorem A, this correspondence induces an isomorphism in K-theory, yielding the desired formula (0.1).

We develop a further extension of the scope of the orbit-stabiliser K-theory formula (see Theorem 8.1) which is designed to also handle the setting of a left cancellative small category Λ. As for left cancellative monoids, we have a left regular representation $\Lambda \curvearrowright \ell^2(\Lambda)$. The associated C*-algebra is studied by Spielberg in [79, 80] using groupoid methods, and by Li in [49] with an inverse semigroup approach to the groupoid. We say a left cancellative small category Λ is *finitely aligned* if the intersection of principal ideals $\lambda\Lambda \cap \mu\Lambda$ can be written as a finite union $\bigcup_{\nu \in F} \nu\Lambda$ of principal ideals, which can be viewed as a weakening of the condition on a left cancellative monoid of having right LCMs. For this class, the groupoid G_Λ assigned by Spielberg and Li to Λ models the left regular algebra $C^*_\lambda(\Lambda)$. One of the motivating examples covered is that of finitely aligned higher rank graphs, where $C^*_\lambda(\Lambda)$ can be identified with the Toeplitz algebra $\mathcal{T}C^*(\Lambda)$ (see [79, Remark 8.4]). The special case of single alignment relates to the existence of right LCMs and allows for a description of the left regular algebra as the reduced C*-algebra of an inverse semigroup, in which case Theorem B directly applies. The more general setting of finitely aligned left cancellative small categories calls for a more general K-theory formula[1] (see Theorem 8.1). Applying this, we can compute the K-theory of the left regular algebra of a finitely aligned left cancellative small category in terms of its (discrete) groupoid Λ^* of invertible elements.

Theorem C (K-theory formula for a finitely aligned left cancellative small category, Corollary 9.2)**.** *Let Λ be a countable manageably finitely aligned left cancellative small category such that G_Λ has torsion-free isotropy groups, is Hausdorff and satisfies the Baum-Connes conjecture. Then*

$$K_*(C^*_\lambda(\Lambda)) \cong K_*(C^*_r(\Lambda^*)).$$

For each of these conditions on the groupoid G_Λ we discuss more easily verifiable sufficient (or equivalent) conditions in Chapter 9. *Manageability* is a technical condition which allows us to contain the finite alignment of Λ, but we do need it (see Example 9.4). As a special case of this formula we recover Fletcher's computation in [32] of the K-theory of the Toeplitz algebra $\mathcal{T}C^*(\Lambda)$ of a finitely aligned higher rank graph Λ:

$$K_0(\mathcal{T}C^*(\Lambda)) \cong \bigoplus_{v \in \Lambda^0} \mathbb{Z}, \qquad K_1(\mathcal{T}C^*(\Lambda)) = 0.$$

The proofs of the previous iterations of the K-theory formula [20–22, 48] use the Going Down principle for a group Γ [17], which gives conditions on an equivariant Kasparov cycle to induce an isomorphism in K-theory. The theory of these cycles is called equivariant KK-theory, where they provide the morphisms in the equivariant Kasparov category KK^Γ of Γ-C*-algebras, which is an invaluable tool for studying K-theoretic questions surrounding Γ. Le Gall extended equivariant KK-theory to

[1] After this thesis was submitted, the author spoke to Victor Wu who has independently worked on extending the Cuntz Echterhoff Li formula to the finitely aligned setting with Nathan Brownlowe, Jack Spielberg and Anne Thomas, but with the dynamics given by the action of a group.

the setting of groupoids in [46]. We might then hope to apply the Going Down principle for groupoids [8,9,11] to prove the K-theory formula. The problem is that there is no clear Kasparov cycle which is equivariant with respect to a single groupoid to make use of. Instead what we can do is find a topological correspondence which is equivariant with respect to an inverse semigroup, inducing a correspondence of groupoids. We cannot work entirely within the equivariant Kasparov category of a single groupoid, so we relate the two categories using this correspondence. This is the reason that we started studying groupoid correspondences, and they turn out to be an extremely useful tool for transferring information between groupoids.

Our approach combines groupoid correspondences with the categorical approach to the Baum-Connes conjecture. The Baum-Connes conjecture for a group Γ asserts that a particular map μ_Γ called the *Baum-Connes assembly map*

$$\mu_\Gamma \colon K_*^{\mathrm{top}}(\Gamma) \to K_*(C_r^*(\Gamma))$$

is an isomorphism. The left hand side $K_*^{\mathrm{top}}(\Gamma)$ is the topological K-theory which has a more topological flavour and is in principle easier to compute than the operator K-theory $K_*(C_r^*(\Gamma))$. Indeed, a spectral sequence converging to $K_*^{\mathrm{top}}(\Gamma)$ is covered in [60]. In the categorical approach to the Baum-Connes conjecture the equivariant Kasparov category KK^Γ of Γ-C*-algebras takes centre stage, viewed as a triangulated category. This perspective was introduced by Meyer and Nest in [58], who reformulate the topological K-theory $K_*^{\mathrm{top}}(\Gamma; A)$ with coefficients in a Γ-C*-algebra A as a *localisation* $\mathbb{L}F_*(A)$ of the operator K-theory functor $F_* = K_*(\Gamma \ltimes_r -)\colon \mathrm{KK}^\Gamma \to \mathsf{Ab}_*$. The assembly map μ_Γ drops out as a feature of the process of localisation. Meyer and Nest develop a general framework for doing homological algebra in triangulated categories [57–59], introducing the concepts of localisations $\mathbb{L}F_*(A)$ and derived functors $\mathbb{L}_n F_*(A)$. Meyer relates these with the ABC spectral sequence, which converges to the localisation $\mathbb{L}F_*(A)$ with second sheet given by the derived functors $\mathbb{L}_n F_*(A)$:

$$\mathbb{L}_p F_q(A) \Rightarrow \mathbb{L}F_{p+q}(A)$$

The categorical approach to the Baum-Connes conjecture of an étale groupoid G is developed in [11, 71], using the groupoid equivariant Kasparov category KK^G. Bönicke and Proietti discuss in [11] how to localise in this setting to frame the Baum-Connes assembly map

$$\mu_{G,A} \colon K_*^{\mathrm{top}}(G; A) \to K_*(G \ltimes_r A)$$

in the Meyer-Nest theory. Proietti and Yamashita apply the ABC spectral sequence to an ample groupoid G with torsion-free isotropy groups in [71], identifying the derived functors $\mathbb{L}_n F_*(A)$ with groupoid homology groups $H_n(G; K_*(A))$. In the presence of the Baum-Connes conjecture they obtain a spectral sequence which converges to the operator K-theory $K_*(G \ltimes_r A)$:

$$H_p(G; K_q(A)) \Rightarrow K_{p+q}(G \ltimes_r A)$$

Morally this says that $K_*(G \ltimes_r A)$ is built out of the homology groups $H_p(G; K_q(A))$. This is similar to the sense in which the middle object of a short exact sequence is built out of the outer objects, where the building blocks alone do not fully determine the built object. Determining the middle object of a short exact sequence from its outer objects is known as an extension problem, and in general we have to solve many extension problems to determine $K_*(G \ltimes_r A)$ from $H_p(G; K_q(A))$. In low dimensions this is a feasible task, which has been carried out in [10] to verify the HK conjecture for principal ample groupoids with dynamic asymptotic dimension at most 2. We instead use the comparison theorem for spectral sequences [88, Theorem 5.2.12], culminating in Theorem A. A morphism of short exact sequences which is an isomorphism on the outer objects must be an isomorphism on the middle objects, and similarly a morphism of Proietti and Yamashita's spectral sequences that is an isomorphism $H_p(G; K_q(A)) \cong H_p(H; K_q(B))$ on the homology groups must induce an isomorphism $K_*(G \ltimes_r A) \cong K_*(H \ltimes_r B)$ in K-theory.

The main technical achievement in this work is the construction of morphisms of ABC spectral sequences associated to étale groupoids from a proper correspondence of these groupoids. As the focus of the categorical approach to Baum-Connes is the equivariant Kasparov category, the first ingredient we introduce is an induction functor

$$\mathrm{Ind}_\Omega \colon \mathrm{KK}^H \to \mathrm{KK}^G$$

associated to a correspondence $\Omega \colon G \to H$. This is based on the subgroupoid induction functor constructed in [8]. In order to transfer information between the operator K-theory functors, we construct a natural transformation

$$\alpha_\Omega \colon K_*(G \ltimes \mathrm{Ind}_\Omega -) \Rightarrow K_*(H \ltimes -) \colon \mathrm{KK}^H \rightrightarrows \mathsf{Ab}_*.$$

We use the universal crossed product because it has better functoriality properties, and the resulting ABC spectral sequences will not see the difference between the reduced and universal crossed products. We construct α_Ω by building a proper correspondence $G \ltimes \mathrm{Ind}_\Omega B \to H \ltimes B$ for each H-C*-algebra $B \in \mathrm{KK}^H$. This in turn is made by equipping $\Omega \colon G \to H$ with C*-coefficients and forming a crossed product correspondence construction[2]. This combines Holkar's construction of $C^*(\Omega)$ with the equivalences of groupoid crossed products in [61]. The third and final ingredient is a morphism $f_\Omega \colon C_0(G^0) \to \mathrm{Ind}_\Omega C_0(H^0)$ when Ω is proper. We may also use the notion of a proper groupoid correspondence with C*-coefficients $(E, \Omega) \colon (A, G) \to (B, H)$ as our starting point, in which case this third ingredient is a morphism $f_E \colon A \to \mathrm{Ind}_\Omega B$ which we build using a universal property of the correspondence $(\mathrm{Ind}_\Omega B, G) \to (B, H)$. The triple $(\mathrm{Ind}_\Omega, \alpha_\Omega, f_E)$ is exactly the information we need to construct a morphism of the ABC spectral sequences associated to (A, G) and (B, H).

[2]Groupoid correspondences with C*-coefficients and their crossed products may be known to experts, but we are not aware of their presence in the literature.

The input for the ABC spectral sequence is a quadruple $(\mathfrak{T}, \mathfrak{I}, F, A)$, consisting of a triangulated category $\mathfrak{T}$, a homological ideal $\mathfrak{I} \triangleleft \mathfrak{T}$, a functor $F\colon \mathfrak{T} \to \mathsf{Ab}_*$ and an object $A \in \mathfrak{T}$. In the setting of an étale groupoid G and a G-C*-algebra A, we take KK^G as our triangulated category and $A \in \mathrm{KK}^G$ as our object. The ideal $\mathfrak{I}$ is typically constructed from a family of subgroupoids of G, and the functor F is usually $K_*(G \ltimes -)$ or $K_*(G \ltimes_r -)$. We define a category of these quadruples which we call the ABC category, and develop functoriality of the localisation $\mathbb{L}F_*(A)$, the derived functors $\mathbb{L}_n F_*(A)$ and the ABC spectral sequence with respect to this category:

Theorem D (See Theorem 7.14)**.** *Morphisms $\mathfrak{m}\colon \mathfrak{M} \to \mathfrak{M}'$ in the ABC category functorially induce morphisms of ABC spectral sequences $\mathrm{ABC}(\mathfrak{m})\colon \mathrm{ABC}(\mathfrak{M}) \to \mathrm{ABC}(\mathfrak{M}')$. Moreover,*

(i) the map on the second sheet is given by the derived functor maps $\mathbb{L}_n(\mathfrak{m})$,

(ii) the map on the limit sheet agrees with the localisation map $\mathbb{L}(\mathfrak{m})$.

The morphisms in the ABC category are triples just like $(\mathrm{Ind}_\Omega, \alpha_\Omega, f_E)$ which we constructed from a proper correspondence $(E, \Omega)\colon (A, G) \to (B, H)$ with C*-coefficients. As a result, $(\mathrm{Ind}_\Omega, \alpha_\Omega, f_E)$ induces a morphism of spectral sequences. Theorem A drops out as a corollary of this.

In Chapter 1, we cover preliminaries for studying étale groupoid equivariant KK-theory. We introduce étale groupoids and their correspondences, giving examples of these correspondences. Equivariant KK-theory is built from C*-algebras and their Hilbert modules each equipped with actions of a groupoid G. This requires us to understand how these C*-algebras and their Hilbert modules can fibre over the unit space G^0 of the groupoid. This leads us to cover a significant amount of Banach bundle theory before we can discuss the equivariant correspondence category Corr^G which forms the basis of the equivariant Kasparov category KK^G.

Chapter 2 is about the interaction between a groupoid correspondence $\Omega\colon G \to H$ and the equivariant correspondence categories Corr^G and Corr^H. We introduce the induction functor $\mathrm{Ind}_\Omega\colon \mathsf{Corr}^H \to \mathsf{Corr}^G$ and then the notion of a groupoid correspondence with C*-coefficients. We construct the associated crossed products and prove their fundamental properties. We then build the correspondences that underlie the induction natural transformation $\alpha_\Omega\colon K_*(G \ltimes \mathrm{Ind}_\Omega -) \Rightarrow K_*(H \ltimes -)$ and the morphism $f_E\colon A \to \mathrm{Ind}_\Omega B$. At each stage we show how our constructions are compatible with composition of correspondences.

Chapter 3 builds on the previous chapter, now working in groupoid equivariant KK-theory. We give a quick overview of groupoid equivariant KK-theory, and then construct the KK-theoretic induction functor $\mathrm{Ind}_\Omega\colon \mathrm{KK}^H \to \mathrm{KK}^G$ and prove naturality of α_Ω. Again, we check compatibility with composition of correspondences.

In Chapter 4 we deal with the groupoid homology side of things. We give a module-theoretic approach to ample groupoid homology and use it to construct a map in homology $H_*(\Omega)\colon H_*(G) \to H_*(H)$ from a proper correspondence $\Omega\colon G \to H$.

Chapter 5 is a quick overview of spectral sequences.

Chapter 6 reviews the framework of homological algebra in triangulated categories as developed in [57–59]. This first covers the basics of triangulated categories, homological ideals and doing homological algebra relative to these ideals. We then discuss localisation with respect to complementary subcategories and how this relates to the Baum-Connes conjecture, highlighting some contributions from [11,71].

In Chapter 7 we define the ABC category and develop functoriality of constructions from the previous chapter. This includes the derived functors, localisations and ultimately the ABC spectral sequence. We prove Theorem D and deduce Theorem A.

Chapter 8 uses Theorem A to prove a general version of the orbit-stabiliser K-theory formula (Theorem 8.1). We also sketch how to approach the removal of the torsion-free condition.

Chapter 9 explains how to recover Theorems B and C from the general orbit-stabiliser K-theory formula, with a discussion of in what situations the conditions for the formulae are met.

Finally, we look to the future in Chapter 10 with a discussion of the questions which leap out at us from this work.

1. Étale groupoids and their interactions with C*-algebras and Hilbert modules

The goal of this document is to explore the K-theory $K_*(C^*_r(G))$ of the reduced C*-algebra of an étale groupoid G. We need to work in the setting of groupoid equivariant KK-theory, for which we will need a solid grasp of how étale groupoids interact with C*-algebras and their Hilbert modules. The purpose of this chapter is to serve as an introduction to these matters.

The material in this chapter is largely well-known, and is collected and repackaged here to be applied later in this document. Consequently, we will not provide proofs for everything in this section. Much of the material in this chapter can be found in [8], which also deals with groupoid equivariant KK-theory. For more details on correspondences of étale groupoids, see [3]. Another good source on $C_0(X)$-algebras, Banach bundles and groupoid actions on them is [61]. For $C_0(X)$-algebras and their bundles, see Appendix C in [89]. For a thorough "section-forward" treatment of Banach bundles, see [67]. We will assume that the reader is familiar with Hilbert modules, for which [44] is a great introduction.

1.1. **Étale groupoids and groupoid correspondences.** Let us first fix some conventions for étale groupoids. A *topological groupoid* G is a topological space G with a distinguished subspace G^0 called the *unit space*, with continuous maps $r, s\colon G \rightrightarrows G^0$ that assign a *range* and a *source* to each groupoid element. Elements g and h with matching range and source $r(h) = s(g)$ can be *multiplied* or *composed* to form an element gh with range $r(g)$ and source $s(h)$. Multiplication is a continuous associative map $(g, h) \mapsto gh\colon G^2 \to G$, where $G^2 := \{(g, h) \in G \times G \mid s(g) = r(h)\}$ is the space of composable pairs. Elements of G^0 act as identities in that each element is its own range and source, and $r(g)g = g = gs(g)$ for each $g \in G$. There is a continuous *inversion* map $g \mapsto g^{-1}\colon G \to G$ that swaps the range and source of each element such that $g^{-1}g = s(g)$ and $gg^{-1} = r(g)$ for each $g \in G$. An *étale groupoid* G is a topological groupoid such that the range map $r\colon G \to G^0$ is a local homeomorphism and the unit space G^0 is open in G.

Standing assumption. Unless otherwise stated we assume our étale groupoids to be locally compact and Hausdorff. Local compactness is essential to study associated C*-algebras. There are still reasonable C*-algebras to study if we weaken the Hausdorff condition to Hausdorffness of the unit space G^0, in which case G is locally Hausdorff. If we say non-Hausdorff étale groupoid, this is what we mean. However, this complicates many of the details and tools that we need are no longer available. In particular, our (current) approach to the Baum-Connes conjecture requires G to be Hausdorff. By default topological spaces are locally compact and Hausdorff (LCH) unless otherwise stated or built from another space (e.g. a quotient space may not be Hausdorff).

We write $G^x := r^{-1}(x)$ and $G_x := s^{-1}(x)$ for the range and source fibres of an étale groupoid G at $x \in G^0$. These are discrete spaces because G is étale. In general, if a continuous map $f\colon Y \to X$ of topological spaces is understood, we may write Y_x or Y^x for the fibre $f^{-1}(x)$ at $x \in X$. The *isotropy group* G_x^x of G at $x \in G^0$ is the group of arrows with range and source x. If $f\colon X \to Z$ and $g\colon Y \to Z$ are continuous maps, the *fibre product* $X \times_{f,Z,g} Y$ is the space $\{(x, y) \in X \times Y \mid f(x) = g(y)\}$. If the maps f and g are understood, we may write $X \times_Z Y$. For example, the space of composable elements G^2 of G is the fibre product $G \times_{s,G^0,r} G$. A subspace $U \subseteq G$ on which r and s are injective is called a *bisection*. Particularly important are open bisections, on which r and s are homeomorphisms onto their (open) images. The set of open bisections forms a basis for the topology of an étale groupoid.

The basic notion of a morphism of topological groupoids is a continuous groupoid homomorphism, which is a continuous structure preserving map. In the setting of étale groupoids, we are especially interested in a special class of these called étale homomorphisms:

Definition 1.1 (Étale homomorphism)**.** Let G and H be étale groupoids. An *étale homomorphism* $\varphi\colon G \to H$ is a groupoid homomorphism that is also a local homeomorphism.

Inverse semigroups are an important source of étale groupoids. An *inverse semigroup* S is a semigroup such that for each $s \in S$ there is a unique element $s^* \in S$ called the *adjoint* of s satisfying $s^*ss^* = s^*$ and $ss^*s = s$. We will always take our inverse semigroups to have a distinguished 0 element. The set $E = E(S)$ of idempotents forms a semilattice under multiplication. We think of the elements of an inverse semigroup as partial symmetries, and this is reflected in how they act. An inverse semigroup acts on a space X by *partial homeomorphisms*, which are homeomorphisms $\alpha\colon \operatorname{dom}\alpha \to \operatorname{ran}\alpha$ between open subsets of X. The composition $\alpha \circ \beta$ of two partial homeomorphisms is the partial homeomorphism $x \mapsto \alpha(\beta(x))$ whose domain consists of the x in $\operatorname{dom}\beta$ with $\beta(x) \in \operatorname{dom}\alpha$. The set of partial homeomorphisms of X forms an inverse semigroup $\mathrm{PHom}(X)$, with the adjoint given by the inverse.

Definition 1.2 (Inverse semigroup action)**.** A *(left) action* $S \curvearrowright X$ of an inverse semigroup S on a space X is a homomorphism $\alpha\colon S \to \mathrm{PHom}(X)$ to the inverse semigroup of partial homeomorphisms of X. The domain and range of $\alpha(s)$ for $s \in S$ may be written $\operatorname{dom}_X s$ and $\operatorname{ran}_X s$ if we want to emphasise the space. We usually write $s \cdot x$ for $\alpha(s)(x)$.

A continuous map $f\colon X \to Y$ between S-spaces is *S-equivariant* if $x \in \operatorname{dom}_X s$ implies that $f(x) \in \operatorname{dom}_Y s$ and $f(s \cdot x) = s \cdot f(x)$. From an inverse semigroup action we may build an étale groupoid:

Definition 1.3 (Transformation groupoid of an inverse semigroup action)**.** Let $S \curvearrowright X$ be an inverse semigroup action. Consider the space

$$S \ltimes X := \{(s,x) \in S \times X \mid x \in \operatorname{dom} s\}/\sim,$$

where $(s,x) \sim (t,x)$ if there is an idempotent $e \in E$ with $x \in \operatorname{dom} e$ and $se = te$. We take the quotient topology of the subspace topology of the product topology on $S \times X$, where S is given the discrete topology, and we write $[s,x]$ for the equivalence class of (s,x). We define range and source maps by $r([t,x]) = t \cdot x$, $s([t,x]) = x$ for $[t,x] \in S \ltimes X$ and a composition $[t, s \cdot x][s,x] = [ts, x]$. This turns $S \ltimes X$ into a (possibly non-Hausdorff) étale groupoid called the *transformation groupoid* of $S \curvearrowright X$ whose unit space is an open subset of X.

Each inverse semigroup element s defines an open bisection $\{[s,x] \mid x \in \operatorname{dom} s\}$ of the groupoid $S \ltimes X$, and in fact every étale groupoid can be viewed as a transformation groupoid of its inverse semigroup of open bisections acting on its unit space.

It's important for us to understand how a groupoid itself can act. As there is a space component of a groupoid, this must be respected in an action, and so it only acts on objects that are fibred over its unit space. The symmetries induced by elements of the groupoid are between fibres of the object. A fundamental example of how groupoids act is on topological spaces:

Definition 1.4 (Groupoid action)**.** Let G be an étale groupoid and let X be a topological space. A *(left) groupoid action* $G \curvearrowright X$ consists of:

- a continuous map $\tau\colon X \to G^0$ called the *anchor map*,
- a continuous map $\alpha\colon G \times_{s,G^0,\tau} X \to X$ called the *action map*. We usually write $g \cdot x$ for $\alpha(g,x)$.

The pair (τ, α) is an action if whenever $g, h \in G$ and $x \in X$ satisfy $s(g) = r(h)$ and $s(h) = \tau(x)$, we have $\tau(h \cdot x) = r(h)$ and $gh \cdot x = g \cdot (h \cdot x)$. Each groupoid element $g \in G$ induces a homeomorphism $\alpha_g\colon X_{s(g)} \to X_{r(g)}$ that sends x to $g \cdot x$, where we write $X_z = \tau^{-1}(z)$ for the fibre of X at $z \in G^0$. The action map $\alpha\colon G \times_{s,G^0,\tau} X \to X$ is automatically a surjective local homeomorphism [3, Lemma 2.11]. A *right action* $X \curvearrowleft G$ is defined similarly, with an action map $X \times_{\tau,G^0,r} G \to X$ satisfying mirrored conditions.

Example 1.5. Every groupoid G acts on its unit space G^0 in the following way. The anchor map $G^0 \to G^0$ is the identity and for $g \in G$, we define $g \cdot s(g) = r(g)$.

Definition 1.6 (G-spaces)**.** A *(left) G-space* X is a locally compact Hausdorff space equipped with a left action of G. We write LCH^G for the category of G-spaces with morphisms given by G-equivariant continuous maps, and $\mathsf{LCH}^G_{\mathrm{loc}}$ for the subcategory of G-equivariant local homeomorphisms.

This differs from the definition of G-space in [10] as we do not require the anchor map to be étale.

Definition 1.7 (Orbits)**.** Let $G \curvearrowright X$ be an action of an étale groupoid with anchor map $\tau\colon X \to G^0$ and let $x \in X$. The *orbit* $[x]_G$ or $\operatorname{orb}(x)$ of x is the subspace $\{g \cdot x \mid g \in G_{\tau(x)}\}$ of X. The *quotient of X by G* or *orbit space* X/G carries the quotient topology with quotient map $x \mapsto [x]_G\colon X \to X/G$.

Actions of discrete groups on topological spaces give rise to an important class of étale groupoids. These transformation groupoids can also be constructed for the action of an étale groupoid.

Definition 1.8 (Transformation groupoid of a groupoid action)**.** Let $G \curvearrowright X$ be an action of an étale groupoid on a locally compact Hausdorff space X with anchor map $\tau\colon X \to G^0$. The *transformation groupoid* is the fibre product

$$G \ltimes X := G \times_{s,G^0,\tau} X = \{(g,x) \in G \times X \mid s(g) = \tau(x)\}.$$

The unit space is given by $\{(\tau(x), x) \mid x \in X\}$ which we identify with X. The range and source maps are given by $r(g,x) = g \cdot x$, $s(g,x) = x$, and composition is given by $(g, h \cdot x)(h,x) = (gh, x)$.

As with group actions, we are interested in special properties of our actions, such as freeness and properness.

Definition 1.9 (Free, proper and étale actions)**.** Let G be an étale groupoid and let $G \curvearrowright X$ be a (left) action with anchor map $\tau\colon X \to G^0$. We say that the action is:

- *free* if for $g \in G$ and $x \in X$, $g \cdot x = x$ implies that $g \in G^0$.
- *proper* if the map $(g,x) \mapsto (g, g \cdot x)\colon G \times_{s,G^0,\tau} X \to X \times X$ is proper.
- *étale* if $\tau\colon X \to G^0$ is a local homeomorphism.

We say that G is *proper* if the action $G \curvearrowright G^0$ of G on its unit space is proper, and we say that G is *principal* if the action $G \curvearrowright G^0$ is free.

Example 1.10. Every étale groupoid G acts on itself by left multiplication. The anchor map is $r\colon G \to G^0$, and the action map is given by $g \cdot h := gh$ for $(g,h) \in G^2$. This action is free because if $g \cdot h = h$ we can apply h^{-1} on the right to see that $g \in G^0$. It is étale because the range map is a local homeomorphism. Finally, it is proper because the map $(g,h) \mapsto (g, gh)\colon G \times_{s,G^0,r} G \to G \times G$ is a homeomorphism onto the closed subspace $G \times_{r,G^0,r} G$.

One of the main advantages of proper actions is that they have well-behaved quotients.

Proposition 1.11 (Lemma 1.2.11 in [7], Lemma 2.12 and Proposition 2.19 in [3])**.** *Let G be an étale groupoid and let $G \curvearrowright X$ be a proper action. Then the orbit space $G\backslash X$ is Hausdorff. If the action is also free, then the quotient map $q\colon X \to G\backslash X$ is a local homeomorphism.*

Proposition 1.12. *Let $G \curvearrowright X$ be a free proper étale action of an étale groupoid G with anchor map $\tau\colon X \to G^0$. Then the open sets $U \subseteq X$ such that $g \cdot u \mapsto g\colon G \cdot U \to G_{\tau(U)}$ is well-defined and a homeomorphism form a basis for the topology on X.*

Proof. Let $x \in X$ and let V_0 be an open neighbourhood of x. By Proposition 1.11 we may find an open neighbourhood V_1 of x such that $q\colon V_1 \to G\backslash X$ is a homeomorphism onto its image. By étaleness, we may find an open neighbourhood V_2 of x such that $\tau\colon V_2 \to G^0$ is a homeomorphism onto its image. Taking $U = V_0 \cap V_1 \cap V_2$, we obtain homeomorphisms $(g, u) \mapsto g \cdot u\colon G \times_X U \to G \cdot U$ and $(g, u) \mapsto g\colon G \times_X U \to G_{\tau(U)}$. □

Proper actions allow us to construct further proper actions.

Proposition 1.13 (Proposition 2.20 in [87])**.** *Let G be an étale groupoid, let $X \curvearrowleft G$ be an action and let $G \curvearrowright Y$ be a proper action. Then the diagonal action $X \times_{G^0} Y \curvearrowleft G$ given by $(x, y) \cdot g = (x \cdot g, g^{-1} \cdot y)$ is proper.*

The main focus of our study will be groupoid correspondences. Building on work of Holkar [37–39], we view these as morphisms of groupoids. One motivation to study them is that the groupoid C*-algebra construction is functorial with respect to correspondences; from a groupoid correspondence we can construct a C*-correspondence. As we are working with étale groupoids, we use a restricted notion that might best be called an étale correspondence, as in [3, Definition 3.1].

Definition 1.14 (Groupoid correspondence)**.** Let G and H be étale groupoids with unit spaces X and Y. A *groupoid correspondence* $\Omega\colon G \to H$ is a topological space Ω with a left G-action and a right H-action with anchor maps $\rho\colon \Omega \to X$ and $\sigma\colon \Omega \to Y$ called the *range* and *source* such that:

- The G-action commutes with the H-action - Ω is a *G-H-bispace.*
- The right action $\Omega \curvearrowleft H$ is free, proper and étale.

We may also call this a G-H correspondence and we may refer to it by $G \curvearrowright \Omega \curvearrowleft H$ or simply by Ω. If we want to highlight the correspondence Ω, we may write σ_Ω and ρ_Ω instead of σ and ρ. For $x \in X$ and $y \in Y$, we write Ω^x and Ω_y for the range and source fibres $\rho^{-1}(x)$ and $\sigma^{-1}(y)$.

Example 1.15 (Identity correspondence)**.** For any étale groupoid G, the actions of left and right multiplication form a groupoid correspondence $G \curvearrowright G \curvearrowleft G$ which we call the *identity correspondence*.

Definition 1.16 (Proper groupoid correspondence)**.** We say that a groupoid correspondence $G \curvearrowright \Omega \curvearrowleft H$ is *proper* if the map $\overline{\rho}\colon \Omega/H \to X$ induced by the left anchor map $\rho\colon \Omega \to X$ is proper.

Example 1.17 (Étale homomorphism correspondence)**.** Let $\varphi\colon G \to H$ be an étale homomorphism of étale groupoids, and let $\varphi^0\colon G^0 \to H^0$ be its restriction to the unit space. Consider the space

$$\Omega_\varphi := G^0 \times_{\varphi^0, H^0, r} H = \{(x,h) \in G^0 \times H \mid \varphi(x) = r(h)\}.$$

We define a left action $G \curvearrowright \Omega$ by $g \cdot (s(g), h) = (r(g), \varphi(g)h)$ and a right action $\Omega \curvearrowleft H$ by $(x,h) \cdot h' = (x, hh')$. The correspondence Ω_φ is proper, because $\overline{\rho}\colon \Omega_\varphi/H \to G^0$ is a homeomorphism with inverse given by $x \mapsto [x, \varphi(x)]_H$.

Every homomorphism $\varphi\colon G \to H$ of discrete groups is an étale homomorphism viewing the groups as étale groupoids. The associated correspondence Ω_φ is given by actions $G \curvearrowright H \curvearrowleft H$, with G acting through φ on the left and H acting on the right by multiplication.

Example 1.18 (Algebraic morphisms)**.** Let G and H be étale groupoids. An *algebraic morphism* $G \to H$ is an action $G \curvearrowright H$ that commutes with the right multiplication action $H \curvearrowleft H$. These are studied in [14] and under the name "translation action" in [53]. Like étale homomorphisms, these also generalise homomorphisms of discrete groups. Unlike étale homomorphisms, an algebraic morphism induces a $*$-homomorphism of the groupoid C*-algebras.

Example 1.19 (Topological correspondences)**.** Let X and Y be locally compact Hausdorff spaces considered as groupoids with only identity arrows. A *topological correspondence* $\Omega\colon X \to Y$ is a locally compact Hausdorff space Ω with a continuous map $\rho\colon \Omega \to X$ and a local homeomorphism $\sigma\colon \Omega \to Y$. The correspondence is proper if and only if $\rho\colon \Omega \to X$ is proper. As a special case, consider a compact open cover $\mathcal{U}$ of a totally disconnected space Y. Setting $\Omega = \sqcup_{U \in \mathcal{U}} U$, the canonical map $\Omega \to Y$ is a local homeomorphism, and the indexing map $\Omega \to \mathcal{U}$ is proper. We obtain a proper correspondence $\Omega\colon \mathcal{U} \to Y$.

Our new examples of groupoid correspondences come from situations with an ambient inverse semigroup. Given a topological correspondence which is equivariant with respect to an inverse semigroup, we obtain a correspondence of the associated transformation groupoids.

Example 1.20 (Inverse semigroup equivariant topological correspondence)**.** Let $\Omega\colon X \to Y$ be a topological correspondence, and suppose that we have actions $S \curvearrowright X$, $S \curvearrowright Y$ and $S \curvearrowright \Omega$ of an inverse semigroup S such that $\rho\colon \Omega \to X$ and $\sigma\colon \Omega \to Y$ are S-equivariant. We call $\Omega\colon X \to Y$ an *S-equivariant topological correspondence*, and it induces a correspondence $S \ltimes \Omega\colon S \ltimes X \to S \ltimes Y$. We have anchor maps $\rho \circ r_{S\ltimes\Omega}\colon S \ltimes \Omega \to X$ and $\sigma \circ s_{S\ltimes\Omega}\colon S \ltimes \Omega \to Y$. The actions $S \ltimes X \curvearrowright S \ltimes \Omega$ and $S \ltimes \Omega \curvearrowleft S \ltimes Y$ are given for $s, t \in S$, $\omega \in \Omega$ with $\omega \in \mathrm{dom}_\Omega\, st$ by $[s, \rho(t \cdot \omega)] \cdot [t, \omega] = [st, \omega]$ and $[s, t \cdot \omega] \cdot [t, \sigma(\omega)] = [st, \omega]$. If $\Omega\colon X \to Y$ is proper, then so is $S \ltimes \Omega\colon S \ltimes X \to S \ltimes Y$.

Example 1.21 (Action correspondences)**.** Let G be an étale groupoid and let $G \curvearrowright X$ be an action with anchor map $\tau\colon X \to G^0$. Then there is a correspondence $G \curvearrowright G \ltimes X \curvearrowleft G \ltimes X$ with bispace $G \ltimes X$ called the *action correspondence*. The right action is given by right multiplication in $G \ltimes X$. The left action has anchor map given by $(g, x) \mapsto r(g)\colon G \ltimes X \to G^0$ and action map given by $h \cdot (g, x) = (hg, x)$ when $r(g) = s(h)$. This is a special case of an algebraic morphism. The action correspondence is proper if and only if τ is a proper map.

We can also build a correspondence of groupoids using only the "right hand side" data.

Example 1.22. Let H be an étale groupoid and let Ω be a free, proper, étale right H-space. Then Ω is a proper correspondence from Ω/H to H.

Definition 1.23 (Bisections in groupoid correspondences)**.** Let $\Omega\colon G \to H$ be a correspondence of étale groupoids. A *bisection* of Ω is a subspace $U \subseteq \Omega$ such that $\sigma\colon \Omega \to H^0$ and $q\colon \Omega \to \Omega/H$ are injective on U.

These are analogous to bisections in étale groupoids. The set of open bisections (sometimes called *slices*, see [3, Definition 7.2]) forms a basis for the topology of Ω, so we often restrict attention to open bisections in order to work locally with correspondences. The open bisections in G and H act on the open bisections of Ω: given open bisections $U \subseteq \Omega$, $V \subseteq G$ and $W \subseteq H$, the sets

$$V \cdot U = \{v \cdot u \in \Omega \mid (v, u) \in V \times_{G^0} U\}$$
$$U \cdot W = \{u \cdot w \in \Omega \mid (u, w) \in U \times_{H^0} W\}$$

are also open bisections of Ω. For each open bisection $U \subseteq \Omega$, the space $UH = \{u \cdot h \mid u \in U,\ h \in H^{\sigma(u)}\}$ can be viewed as a (proper) correspondence from $q(U)$ to H.

To view groupoid correspondences as morphisms, we need to know how to compose them. While it may be clear how to compose correspondences that arise from étale homomorphisms or from actions, the general construction for composing arbitrary correspondences is more involved.

Definition 1.24 (Composition of groupoid correspondences)**.** Let $\Omega\colon G \to H$ and $\Lambda\colon H \to K$ be groupoid correspondences, and let the unit spaces of G, H and K be X, Y and Z respectively. Consider the diagonal action of H on $\Omega \times_Y \Lambda$ given by $(\omega, \lambda) \cdot h := (\omega \cdot h, h^{-1} \cdot \lambda)$ for $(\omega, \lambda) \in \Omega \times_Y \Lambda$ and $h \in H^{\sigma(\omega)}$. The *composition* $\Lambda \circ \Omega\colon G \to K$ is given by the quotient space

$$\Lambda \circ \Omega := \Omega \times_H \Lambda := (\Omega \times_Y \Lambda)/H$$

which is naturally a G-K-bispace. The left G-action is induced by the action on Ω and the right K-action is induced by the action on Λ. Explicitly, the actions are given by the following formulae.

$$\begin{aligned} &G \curvearrowright \Lambda \circ \Omega & g \cdot [\omega, \lambda]_H &:= [g \cdot \omega, \lambda]_H & (\omega, \lambda) \in \Omega \times_Y \Lambda,\ g \in G_{\rho(\omega)} \\ &\Lambda \circ \Omega \curvearrowleft K & [\omega, \lambda]_H \cdot k &:= [\omega, \lambda \cdot k]_H & (\omega, \lambda) \in \Omega \times_Y \Lambda,\ k \in K^{\sigma(\lambda)} \end{aligned}$$

This also induces a composition of open bisections. Given open bisections $U \subseteq \Omega$ and $V \subseteq \Lambda$, the *composition* $V \circ U$ is given by

$$V \circ U := \{[u, v]_H \in \Lambda \circ \Omega \mid (u, v) \in U \times_Y V\}$$

which is an open bisection in $\Lambda \circ \Omega$ [3, Lemma 7.14].

We introduced the notation $X \times_H Y := (X \times_{H^0} Y)/H$ here. This makes sense whenever we have actions $X \curvearrowleft H$ and $H \curvearrowright Y$, and will be a locally compact Hausdorff space whenever one of the actions is free and proper.

Proposition 1.25 (Propositions 5.7 and 6.5 in [3])**.** *The composition of groupoid correspondences is a groupoid correspondence, and the composition of proper correspondences is proper. Furthermore, this composition is associative up to canonical isomorphisms of correspondences, and the identity correspondence forms an identity up to canonical isomorphisms.*

This allows us to define the correspondence category of groupoids and its subcategory of proper correspondences. Many constructions built from étale groupoids can be turned into functors from these categories. The chief technical results of this document will be of this nature.

Definition 1.26 (Categories of groupoid correspondences)**.** The correspondence category $\mathsf{GpdCorr}$ of groupoids is the category whose objects are étale groupoids and whose morphisms are isomorphism classes of groupoid correspondences. The subcategory of proper correspondences is denoted $\mathsf{GpdCorr}_p$.

Example 1.27 (Morita equivalences)**.** Let G and H be étale groupoids. A Morita equivalence from G to H is a G-H-bispace Ω such that

- the left action $G \curvearrowright \Omega$ and the right action $\Omega \curvearrowleft H$ are free and proper.

- the maps $\overline{\rho}\colon \Omega/H \to G^0$ and $\overline{\sigma}\colon G\backslash\Omega \to H^0$ induced by the anchor maps are homeomorphisms.

The second condition implies that the actions are étale, so this is a (proper) correspondence. Furthermore, Morita equivalences are exactly the invertible groupoid correspondences [1, Theorem 2.30].

Many properties are preserved by Morita equivalences, such as properness [87, Proposition 2.36] and the K-theory of the associated C*-algebra.

The philosophy behind the work in this document is to use correspondences $\Omega\colon G \to H$ to transfer information between the étale groupoids G and H. A basic example of this, which will be echoed throughout the rest of the document, is the construction of a G-space from an H-space.

Definition 1.28 (Induced space)**.** Let $\Omega\colon G \to H$ be a correspondence of étale groupoids and let Y be an H-space. Consider the diagonal action of H on $\Omega \times_{H^0} Y$. The *induced G-space* $\operatorname{Ind}_\Omega Y$ is the space

$$\operatorname{Ind}_\Omega Y := \Omega \times_H Y,$$

The action $G \curvearrowright \operatorname{Ind}_\Omega Y$ is given by $g \cdot [\omega, y]_H = [g \cdot \omega, y]_H$ for $g \in G$ and $(\omega, y) \in \Omega \times_{H^0} Y$ with $s(g) = \rho(\omega)$. We obtain a functor $\operatorname{Ind}_\Omega\colon \mathsf{LCH}^H \to \mathsf{LCH}^G$.

Given another étale groupoid K, a correspondence $\Lambda\colon H \to K$ and a K-space Z, the G-spaces $\operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda Z$ and $\operatorname{Ind}_{\Lambda\circ\Omega} Z$ are naturally isomorphic. We obtain a natural isomorphism $\operatorname{Ind}_\Omega \circ \operatorname{Ind}_\Lambda \cong \operatorname{Ind}_{\Lambda\circ\Omega}$. Furthermore, given a G-space X, the G-space $\operatorname{Ind}_G X$ induced by the identity correspondence $G\colon G \to G$ is naturally isomorphic to X. All of this foreshadows what can be said for C*-algebras and their Hilbert modules equipped with actions of K.

1.2. **C*-algebras and Hilbert modules fibred over topological spaces.** In order to describe how a groupoid can act on a C*-algebra or a Hilbert module, we need to first understand how a C*-algebra can be fibred over the unit space of the groupoid. A continuous map $f\colon Y \to X$ of locally compact Hausdorff spaces is equivalent via Gelfand duality to a non-degenerate $*$-homomorphism $f^*\colon C_0(X) \to C_b(Y) \cong M(C_0(Y))$. This turns $C_0(Y)$ into a $C_0(X)$-*algebra*, which is our fundamental notion of a C*-algebra fibred over X.

Definition 1.29 ($C_0(X)$-algebra and its fibres)**.** A $C_0(X)$-*algebra* is a C*-algebra A equipped with a non-degenerate $*$-homomorphism from $C_0(X)$ to the centre of its multiplier algebra (the *structure map*).

$$\varphi\colon C_0(X) \to ZM(A)$$

We often omit the structure map from the notation and write ξa instead of $\varphi(\xi)(a)$ for $\xi \in C_0(X)$ and $a \in A$. *Non-degeneracy* is the condition that $C_0(X)A = \operatorname{span}\{\xi a \mid \xi \in C_0(X),\ a \in A\}$ is dense in A.

For each $x \in X$, we define the *fibre* A_x of A at x to be the quotient C*-algebra $A/\overline{C_0(X \setminus \{x\})A}$. For an element $a \in A$, its fibre a_x at x is its image in the quotient algebra A_x. For each open subset $U \subseteq X$, we write UA for the subalgebra $\overline{C_0(U)A}$, which is a $C_0(U)$-algebra.

Example 1.30. Let $f\colon Y \to X$ be a continuous map of locally compact Hausdorff spaces. Then $f^*\colon C_0(X) \to C_b(Y)$ turns $C_0(Y)$ into a $C_0(X)$-algebra whose fibre at $x \in X$ is $C_0(Y_x)$, where $Y_x = f^{-1}(x)$ is the fibre of Y at x.

We are interested in the $*$-homomorphisms between $C_0(X)$-algebras that respect the $C_0(X)$-structures. These are called $C_0(X)$-linear.

Definition 1.31 ($C_0(X)$-linear)**.** A $*$-homomorphism $\varphi\colon A \to B$ between $C_0(X)$-algebras is $C_0(X)$*-linear* if $\varphi(\xi a) = \xi\varphi(a)$ for each $a \in A$ and $\xi \in C_0(X)$. This induces a $*$-homomorphism $\varphi_x\colon A_x \to B_x$ for each $x \in X$, called the *fibre* of φ at x.

Definition 1.32 (Category of $C_0(X)$-algebras)**.** The category $\mathsf{C^*\text{-}alg}^X$ of $C_0(X)$-algebras consists of all the $C_0(X)$-algebras, with morphisms given by the $C_0(X)$-linear $*$-homomorphisms.

The direct sum $A \oplus B$ of $C_0(X)$-algebras is a $C_0(X)$-algebra. If A is a $C_0(X)$-algebra and B is a C*-subalgebra of A invariant under the $C_0(X)$ action, then B is a $C_0(X)$-algebra. If B is furthermore an ideal, the quotient algebra A/B is a $C_0(X)$-algebra.

We can learn a lot about a $C_0(X)$-algebra A through its fibres $(A_x)_{x\in X}$.

Proposition 1.33 (Proposition C.10 in [89])**.** *Let A be a $C_0(X)$-algebra. Then for each $a \in A$, the following hold.*

- *The map $x \mapsto \|a_x\|$ is upper-semicontinuous and vanishes at infinity, which means that for each $\epsilon > 0$, the subspace $\{x \mid \|a_x\| \geq \epsilon\}$ of X is compact.*
- *The norm of a is given by $\|a\| = \sup_{x\in X} \|a_x\|$.*
- *For each $\xi \in C_0(X)$, the fibre of ξa at $x \in X$ is given by $(\xi a)_x = \xi(x)a_x$.*

In fact, we can understand each element $a \in A$ of a $C_0(X)$-algebra through the associated function $x \mapsto a_x$. This is a function from X to the *total space* $\mathcal{A} = \bigsqcup_{x\in X} A_x$. There will turn out to be a natural topology on the total space $\mathcal{A}$ turning it into what we call a *Banach bundle* over X.

Definition 1.34 (Banach bundle)**.** A *Banach bundle* over a locally compact Hausdorff space X is a topological space $\mathcal{A}$ (not necessarily LCH) equipped with a continuous, open surjection $p\colon \mathcal{A} \to X$ and complex Banach space structures on each fibre $A_x := p^{-1}(\{x\})$ satisfying the following conditions.

- The map $a \mapsto \|a\|$ is upper-semicontinuous from $\mathcal{A}$ to $\mathbb{R}_{\geq 0}$ (that is, for all $\epsilon > 0$, $\{a \in \mathcal{A} \mid \|a\| \geq \epsilon\}$ is closed).
- The map $(a, b) \mapsto a + b$ is continuous from $\mathcal{A} \times_X \mathcal{A}$ to $\mathcal{A}$.
- For each $\lambda \in \mathbb{C}$, the map $a \mapsto \lambda a$ is continuous from $\mathcal{A}$ to $\mathcal{A}$.
- If $(a_i)_i$ is a net in $\mathcal{A}$ such that $p(a_i) \to x$ and $\|a_i\| \to 0$, then $a_i \to 0_x$ (where 0_x is the zero element in A_x).

This is sometimes called an upper-semicontinuous Banach bundle in the literature, to stress that the norm need only be upper-semicontinuous. We choose to take this as the default notion, and refer to a Banach bundle with the additional requirement of a continuous norm as a *continuous* Banach bundle.

Example 1.35 (Trivial Banach bundle)**.** Let X be a locally compact Hausdorff space. The *trivial Banach bundle* over X is the Banach bundle $X \times \mathbb{C} \to X$, with the product topology.

Definition 1.36 (Morphisms of Banach bundles)**.** A *map* of Banach bundles over X is a function $\varphi\colon \mathcal{A} \to \mathcal{B}$ that restricts to a bounded linear map $\varphi_x\colon A_x \to B_x$ on the fibres at each $x \in X$, such that $\{\varphi_x \mid x \in X\}$ is bounded. The linear map φ_x is called the fibre of φ at x. We do not require a map of Banach bundles to be continuous, and we refer to a continuous map of Banach bundles as a *morphism* of Banach bundles.

Definition 1.37 (C*-bundles)**.** A *C*-bundle* is a Banach bundle $\mathcal{A} \to X$ with the structure of a C*-algebra on each fibre A_x, such that the operations of multiplication $m\colon \mathcal{A} \times_X \mathcal{A} \to \mathcal{A}$ and involution ${}^*\colon \mathcal{A} \to \mathcal{A}$ are continuous. A *morphism* of C*-bundles over X is a morphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ of the underlying Banach bundles such that each fibre φ_x is a $*$-homomorphism.

The following result can be used to get a handle on the topology of a Banach bundle. In the setting of C*-bundles this is [89, Proposition C.20], whose proof works verbatim for general Banach bundles.

Proposition 1.38. *Let $p\colon \mathcal{A} \to X$ be a Banach bundle and let $(a_i)_i$ be a net in $\mathcal{A}$ such that $p(a_i) \to p(a)$ for some $a \in \mathcal{A}$. Suppose that for all $\epsilon > 0$ there is a net $(u_i)_i$ in $\mathcal{A}$ and $u \in \mathcal{A}$ with $p(u_i) = p(a_i)$ and $p(u) = p(a)$ such that*

- *$u_i \to u$ in $\mathcal{A}$,*

- *$\|a - u\| < \epsilon$, and*
- *$\|a_i - u_i\| < \epsilon$ for large i.*

Then $a_i \to a$.

Definition 1.39 (Sections of Banach bundles)**.** A *section* of a Banach bundle $p\colon \mathcal{A} \to X$ is a map $s\colon X \to \mathcal{A}$ such that $p \circ s = \mathrm{id}_X$. The set of continuous sections is denoted $\Gamma(X, \mathcal{A})$. We define the *support* $\mathrm{supp}(e)$ of a continuous section $e\colon X \to \mathcal{A}$ to be the closure of $\{x \in X \mid e(x) \neq 0\}$. The spaces $\Gamma_b(X, \mathcal{A})$, $\Gamma_0(X, \mathcal{A})$ and $\Gamma_c(X, \mathcal{A})$ consist of the continuous sections that are bounded, vanish at infinity and are compactly supported respectively. If Y is a closed subspace of X, we may write $\Gamma(Y, \mathcal{A})$ as shorthand for the space of continuous sections $\Gamma(Y, \mathcal{A}\restriction_Y)$, where $\mathcal{A}\restriction_Y = p^{-1}(Y) \to Y$ is the restriction of $\mathcal{A}$ to Y.

The topology on the total space $\mathcal{A}$ of a Banach bundle can sometimes be difficult to work with directly, it is usually not locally compact and often not Hausdorff [89, Example C.27]. It can instead be helpful to consider the space $\Gamma(X, \mathcal{A})$ of continuous sections of the bundle $\mathcal{A} \to X$. In fact, we can construct the topology of a Banach bundle if we have a large enough collection of sections which we want to be continuous.

Proposition 1.40. *Let $(A_x)_{x\in X}$ be a collection of Banach spaces and let Γ be a complex vector space of sections $X \to \mathcal{A} = \bigsqcup_{x\in X} A_x$ such that*

- *for each $a \in \Gamma$, $x \mapsto \|a(x)\|$ is upper-semicontinuous,*
- *for each $x \in X$, the set $\{a(x) \mid a \in \Gamma\}$ is dense in A_x.*

Then there is a unique topology on $\mathcal{A}$ turning $\mathcal{A} \to X$ into a Banach bundle such that all the elements of Γ are continuous.

The proof of this is technical but fairly straightforward, so we omit it. For continuous Banach bundles, see II.13.18 in [31]. For C*-bundles, see Theorem C.25 in [89]. Since a large part of our analysis of Banach bundles $\mathcal{A} \to X$ will be through their sections, it is good to know that continuous sections $X \to \mathcal{A}$ exist in abundance. We have access to the following result because we assume X to be locally compact and Hausdorff. For continuous Banach bundles this is due to Douady and Soglio-Hérault [31, Appendix C], and it has been noted true for general Banach bundles by Hofmann [36], although the details remain unpublished [35].

Proposition 1.41 (Banach bundles have enough sections)**.** *Let $\mathcal{A} \to X$ be a Banach bundle. Then $\mathcal{A} \to X$ has enough sections in the sense that for each $x \in X$ and $a_x \in A_x$ there is a continuous section $a \in \Gamma(X, \mathcal{A})$ such that $a(x) = a_x$.*

We can use Proposition 1.40 to associate a C*-bundle to a given $C_0(X)$-algebra A whose fibre at $x \in X$ is A_x. The topology on this bundle comes from the collection of sections given by $x \mapsto a_x$ for $a \in A$.

Definition 1.42 (Associated bundles of C*-algebras)**.** Let A be a $C_0(X)$-algebra. Then there is a unique topology on $\mathcal{A} = \bigsqcup_{x \in X} A_x$ such that $\mathcal{A} \to X$ is a Banach bundle with $x \mapsto a_x$ continuous for each $a \in A$. Furthermore, $\mathcal{A} \to X$ is a C*-bundle. We call $\mathcal{A} \to X$ the *bundle associated to* A.

Conversely, suppose we have a C*-bundle $\mathcal{A} \to X$. Then the space $\Gamma_0(X, \mathcal{A})$ of continuous sections vanishing at infinity taken with pointwise operations, the sup-norm and the pointwise action of $C_0(X)$ forms a $C_0(X)$-algebra, called the *$C_0(X)$-algebra associated to* $\mathcal{A} \to X$.

Remark 1.43. These constructions are inverse to each other. If we start with a C*-bundle $\mathcal{A} \to X$, then the bundle associated to $\Gamma_0(X, \mathcal{A})$ is $\mathcal{A} \to X$, and if we start with a $C_0(X)$-algebra A with associated bundle $\mathcal{A} \to X$, then $A \cong \Gamma_0(X, \mathcal{A})$. We obtain an equivalence between the categories of C*-bundles over X and $C_0(X)$-algebras.

$$A \mapsto \bigsqcup_{x \in X} A_x$$

$$\Gamma_0(X, \mathcal{A}) \leftarrow\!\shortmid \mathcal{A}$$

We could in principle stick to studying one of these pictures of fibred C*-algebras. In practice, certain ideas are clearer in one picture than the other, and it is extremely useful to have both pictures around and to be able to convert freely between them. All of our constructions of fibred objects will have two versions: a bundle version and a section space version. We will often switch between the two pictures, using Roman (e.g. A) and calligraphic (e.g. $\mathcal{A}$) fonts on the same letter to indicate the section space and the bundle respectively.

Proposition 1.40 motivates the idea of a collection of continuous sections in $\Gamma(X, \mathcal{A})$ being sufficiently large to control the topology of a Banach bundle $\mathcal{A} \to X$.

Definition 1.44 (Sufficiently many continuous sections of a Banach bundle)**.** Let $\mathcal{A} \to X$ be a Banach bundle, and let $\Gamma \subseteq \Gamma(X, \mathcal{A})$ be a collection of continuous sections. We say that Γ is *sufficient* for $\mathcal{A}$ if for each $x \in X$, the set $\{\gamma(x) \mid \gamma \in \Gamma\}$ has dense span in A_x. If some property is true for each γ in some sufficient Γ, we may say that the property is true for *sufficiently many* continuous sections.

Proposition 1.45 (Continuity condition for maps out of bundles)**.** *A map $\varphi \colon \mathcal{A} \to \mathcal{B}$ of Banach bundles over a space X is continuous (and therefore a morphism of Banach bundles) if and only if $\varphi \circ \gamma$ is continuous for sufficiently many $\gamma \in \Gamma(X, \mathcal{A})$.*

Proof. The "only if" direction is immediate. For the "if" direction, let $x \in X$, $a \in A_x$ and let $a_i \to a$ be a net converging to a in $\mathcal{A}$, with $a_i \in A_{x_i}$. We aim to show that $\varphi(a_i) \to \varphi(a)$ with the use of Proposition 1.38, so let $\epsilon > 0$. Let $C > 0$ be an upper bound for $\sup_{y \in X}\|\varphi_y\|$. By sufficiency, there is some $\gamma \in \Gamma(X, \mathcal{A})$ such that $\varphi \circ \gamma$ is continuous and $\|a - \gamma(x)\| < \epsilon/C$. We set $u = \varphi(\gamma(x))$ and $u_i = \varphi(\gamma(x_i))$. Continuity of $\varphi \circ \gamma$ ensures that $u_i \to u$. By continuity of γ and addition, we have $a_i - \gamma(x_i) \to a - \gamma(x)$. By upper-semicontinuity of $\|-\|$, we have $\|a_i - \gamma(x_i)\| < \epsilon/C$ for large i. It follows that $\|\varphi(a) - u)\| < \epsilon$ and that $\|\varphi(a_i) - u_i\| < \epsilon$ for large i. We may conclude that $\varphi(a_i) \to \varphi(a)$. □

A very similar argument gives us a continuity condition for bilinear maps $\varphi\colon \mathcal{A} \times_X \mathcal{B} \to \mathcal{C}$ of Banach bundles over X.

Proposition 1.46 (Continuity condition for bilinear maps of bundles)**.** *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be Banach bundles over a space X, and let $\varphi\colon \mathcal{A} \times_X \mathcal{B} \to \mathcal{C}$ be a function respecting the fibres such that the fibre $\varphi_x\colon A_x \times B_x \to C_x$ at $x \in X$ is a bounded bilinear map, which is uniformly bounded in the sense that $\|\varphi\| := \sup_{x\in X}\|\varphi_x\| < \infty$. Suppose $\Gamma_1 \subseteq \Gamma(X, \mathcal{A})$ and $\Gamma_2 \subseteq \Gamma(X, \mathcal{B})$ are sufficient collections of sections for $\mathcal{A}$ and $\mathcal{B}$ such that $x \mapsto \varphi(\gamma_1(x), \gamma_2(x))\colon X \to \mathcal{C}$ is a continuous section for each $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$. Then φ is continuous.*

Proof. Let $(a_i, b_i) \to (a, b)$ be a convergent net in $\mathcal{A} \times_X \mathcal{B}$ with $a_i \in A_{x_i}$ and $a \in A_x$, and let $\epsilon > 0$. Let $C > \max((\|a\|+\|b\|+2)\|\varphi\|, 1/\epsilon)$. By sufficiency, there are $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$ such that $\|a-\gamma_1(x)\| < \epsilon/C$ and $\|b-\gamma_2(x)\| < \epsilon/C$. Setting $u = \varphi(\gamma_1(x), \gamma_2(x))$ and $u_i = \varphi(\gamma_1(x_i), \gamma_2(x_i))$, we have $u_i \to u$. By our choice of C, we may calculate $\|\varphi(a, b) - u\| = \|\varphi(a - \gamma_1(x), b) + \varphi(\gamma_1(x), b - \gamma_2(x))\| < \epsilon$. For large i, we have $\|b_i\| < \|b\| + 1$, $\|\gamma_1(x_i)\| < \|a\| + 1$, $\|a_i - \gamma_1(x_i)\| < \epsilon/C$ and $\|b_i - \gamma_2(x_i)\| < \epsilon/C$. It follows that $\|\varphi(a_i, b_i) - u_i\| < \epsilon$. By Proposition 1.38, we may conclude that $\varphi(a_i, b_i) \to \varphi(a, b)$. □

Remark 1.47. Proposition 1.46 also applies to maps which are conjugate linear in one or both variables through the consideration of conjugate Banach bundles which are identical except for the action of $\mathbb{C}$ on each fibre.

The idea that a collection of sections of a Banach bundle $\mathcal{A} \to X$ is enough to determine the continuity of Banach bundle maps out of $\mathcal{A}$ is seen at the level of the associated section space $\Gamma_0(X, \mathcal{A})$ in terms of density. The following result is a reformulation of [89, Proposition C.24], which says that a subspace $\Gamma \subseteq \Gamma_0(X, \mathcal{A})$ closed under the action of $C_0(X)$ is dense if and only if the evaluation $\{a(x) \mid a \in \Gamma\}$ is dense in A_x for each $x \in X$. This is stated for C*-bundles, but again the proof applies to all Banach bundles.

Proposition 1.48 (Condition for density in the section space of a Banach bundle)**.** *Suppose that Γ is a sufficient collection of sections for a Banach bundle $\mathcal{A} \to X$. Then $C_0(X)\Gamma$ has dense span in the Banach space $\Gamma_0(X, \mathcal{A})$.*

To obtain an isomorphism of Banach bundles, it suffices to have fibre-wise isometric isomorphisms which form a continuous map of Banach bundles in one direction, as the continuity of the inverse is automatic. More generally we can say that a fibre-wise isometric morphism of Banach bundles is a homeomorphism onto its image.

Lemma 1.49 (Condition for isomorphism of Banach bundles, see Remark 3.8 in [61])**.** *Let $p\colon \mathcal{A} \to X$ and $\mathcal{B} \to X$ be Banach bundles and let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a morphism of Banach bundles such that $\varphi_x\colon A_x \to B_x$ is an isometry for each $x \in X$. Then φ is a homeomorphism onto its image. In particular, if φ_x is an isometric isomorphism for each $x \in X$, then φ is an isomorphism of Banach bundles.*

Proof. Let $a \in \mathcal{A}$ and let $(a_i)_i$ be a net in $\mathcal{A}$ such that $\varphi(a_i) \to \varphi(a)$. By Proposition 1.41, there is a continuous section $\xi\colon X \to \mathcal{A}$ such that $\xi(p(a)) = a$. Let $u_i = \xi(p(a_i))$. Then $u_i \to a$, and therefore $\varphi(a_i) - \varphi(u_i) \to 0$ by continuity of φ and addition. By fibre-wise isometry and upper-semicontinuity we get $\|a_i - u_i\| = \|\varphi(a_i) - \varphi(u_i)\| \to 0$, and so $a_i - u_i \to 0$. It follows that $a_i \to a$. □

We now turn to Hilbert modules. For an excellent introduction to Hilbert modules, we recommend Lance's book [44]. A Hilbert module E over a $C_0(X)$-algebra A is automatically fibred over X. We can define an action of $C_0(X)$ on E by adjointable operators as follows. Given $\xi \in C_0(X)$, $e \in E$ and $a \in A$, we define

$$\Phi(\xi)(e \cdot a) := e \cdot (\xi a).$$

The operator $\Phi(\xi)$ is well-defined on the dense subspace $EA \subseteq E$ and extends to a unique adjointable operator $\Phi(\xi)$ on E which acts centrally. We obtain a $*$-homomorphism $\Phi\colon C_0(X) \to Z\mathcal{L}(E)$ called the *structure map* which is non-degenerate in the sense that $\overline{\Phi(C_0(X))E} = E$. Once again, we will usually omit the structure map from the notation and write ξe for $\Phi(\xi)(e)$. We define the *fibre* E_x at $x \in X$ to be the quotient of E by the closed subspace $\overline{C_0(X \setminus \{x\})E}$. This is a Hilbert A_x-module and can be identified with $E \otimes_A A_x$ [8, Remark 4.1]. For $e \in E$ the *fibre* e_x is the image of e in the fibre E_x. We may use Proposition 1.40 once again to construct a Banach bundle $\mathcal{E} \to X$ whose fibre at $x \in X$ is E_x such that $x \mapsto e_x$ is continuous for each $e \in E$. We refer to this as the *Hilbert bundle associated to E*.

Definition 1.50 (Hilbert bundle)**.** Let $\mathcal{A} \to X$ be a C*-bundle. A *Hilbert $\mathcal{A}$-bundle* is a Banach bundle $\mathcal{E} \to X$ with the structure of a Hilbert A_x-module on each fibre E_x such that the inner product and Hilbert module action maps

$$\langle -, - \rangle\colon \mathcal{E} \times_X \mathcal{E} \to \mathcal{A} \qquad\qquad - \cdot -\colon \mathcal{E} \times_X \mathcal{A} \to \mathcal{E}$$

are continuous.

Let A be a $C_0(X)$-algebra and let E and F be Hilbert A-modules. An adjointable operator $T \in \mathcal{L}(E,F)$ is automatically $C_0(X)$-linear in the sense that $\xi \cdot T(e) = T(\xi \cdot e)$ for each $\xi \in C_0(X)$ and $e \in E$. This means that we obtain an operator $T_x \in \mathcal{L}(E_x, F_x)$ which we call the *fibre* of T at x for each $x \in X$ such that $T(e)_x = T_x(e_x)$ for each $e \in E$. Because each $e \in E$ is determined by its fibres, the operator T is also determined by its fibres. Moreover, as E_x inherits the quotient norm from E for each $x \in X$, we obtain $\|T\| = \sup_{x \in X} \|T_x\|$. Given $e \in E$ and $f \in F$, the fibre of the compact operator $\Theta_{f,e}$ at $x \in X$ is Θ_{f_x, e_x}.

By the canonical identification $M(\mathcal{K}(E)) \cong \mathcal{L}(E)$ [44, Theorem 2.4], the structure map $\Phi\colon C_0(X) \to Z\mathcal{L}(E)$ induces a $C_0(X)$-algebra structure on $\mathcal{K}(E)$, as $\Phi\colon C_0(X) \to ZM(\mathcal{K}(E))$ is non-degenerate [7, Proposition 3.2.1]. The following proposition shows that the fibre of $\mathcal{K}(E)$ at $x \in X$ can be identified with $\mathcal{K}(E_x)$, and the $C_0(X)$-algebra fibre of an operator $T \in \mathcal{K}(E)$ at x is the fibre $T_x \in \mathcal{K}(E_x)$. We may even construct a Banach bundle when there are two Hilbert modules:

Proposition 1.51 (Bundle for the compact operators)**.** *Let A be a $C_0(X)$-algebra and let E and F be Hilbert A-modules. Then there is a Banach bundle $\mathcal{K}(\mathcal{E},\mathcal{F}) \to X$ whose fibre at $x \in X$ is $\mathcal{K}(E_x, F_x)$ such that the assignment mapping $T \in \mathcal{K}(E,F)$ to the section $x \mapsto T_x\colon X \to \mathcal{K}(\mathcal{E},\mathcal{F})$ is an isomorphism $\mathcal{K}(E,F) \cong \Gamma_0(X, \mathcal{K}(\mathcal{E},\mathcal{F}))$. Furthermore, the map $f, e \mapsto \Theta_{f,e}\colon \mathcal{F} \times_X \mathcal{E} \to \mathcal{K}(\mathcal{E},\mathcal{F})$ is continuous.*

Proof. We will use Proposition 1.40 to construct a Banach bundle $\mathcal{K}(\mathcal{E},\mathcal{F}) \to X$ with fibre $\mathcal{K}(E_x, F_x)$ at $x \in X$ such that $x \mapsto \Theta_{f(x),e(x)}\colon X \to \mathcal{K}(\mathcal{E},\mathcal{F})$ is continuous for each $e \in E$ and $f \in F$. Let Γ be the complex vector space of sections of the form $x \mapsto \sum_{i=1}^n \Theta_{f_i(x),e_i(x)}$ for $n \in \mathbb{N}$, $e_1, \ldots, e_n \in E$ and $f_1, \ldots, f_n \in F$. For each $x \in X$ the set $\{\Theta_{f(x),e(x)} \mid e \in E,\ f \in F\}$ has dense span in $\mathcal{K}(E_x, F_x)$ by the surjectivity of $E \to E_x$ and $F \to F_x$. Given a section $x \mapsto \sum_{i=1}^n \Theta_{f_i(x),e_i(x)}$ in Γ, we must check that $x \mapsto \|\sum_{i=1}^n \Theta_{f_i(x),e_i(x)}\|$ is upper-semicontinuous.

Let $R \in \mathcal{L}(A^n, E)$ be the operator defined by $R(a) = \sum_{i=1}^n e_i \cdot a_i$ for $a \in A^n$, with adjoint given by $R^*(e) = \sum_{i=1}^n \langle e_i, e\rangle$ for $e \in E$. We define $S \in \mathcal{L}(A^n, F)$ by $S(a) = \sum_{i=1}^n f_i \cdot a_i$ for $a \in A^n$. We have $\sum_{i=1}^n \Theta_{f_i,e_i} = SR^*$, and for each $x \in X$ we may calculate

$$\begin{aligned}
\left\| \sum_{i=1}^n \Theta_{f_i(x),e_i(x)} \right\|^2 &= \left\| S_x R_x^* \right\|^2 \\
&= \left\| R_x S_x^* S_x R_x^* \right\| \\
&= \left\| R_x (S_x^* S_x)^{\frac{1}{2}} \right\|^2 \\
&= \left\| (R_x^* R_x)^{\frac{1}{2}} (S_x^* S_x)^{\frac{1}{2}} \right\|^2 .
\end{aligned}$$

The operator $R^*R \in \mathcal{L}(A^n)$ is given by the matrix $(\langle e_i, e_j\rangle)_{i,j} \in M_n(A)$, and similarly $S^*S \in M_n(A)$ with $(S^*S)_{i,j} = \langle f_i, f_j\rangle$. The entry-wise action of $C_0(X)$ on $M_n(A)$ turns $M_n(A)$ into a $C_0(X)$-algebra. The fibre $M_n(A)_x$ at $x \in X$ is given by $M_n(A_x)$ because $C_0(X \setminus \{x\})M_n(A) = M_n(C_0(X \setminus \{x\})A)$, with the fibre map $M_n(A) \to M_n(A_x)$ given by applying the fibre map $A \to A_x$ entry-wise. Furthermore, this fibre map is consistent with the operator fibre map $\mathcal{L}(A^n) \to \mathcal{L}(A_x^n)$. We can conclude by Proposition 1.33 that $x \mapsto \|\sum_{i=1}^n \Theta_{f_i(x),e_i(x)}\| = \left\|((R^*R)^{\frac{1}{2}}(S^*S)^{\frac{1}{2}})_x\right\|$ is upper-semicontinuous.

By Proposition 1.40 we obtain a Banach bundle $\mathcal{K}(\mathcal{E}, \mathcal{F}) \to X$ such that $x \mapsto \Theta_{f(x),e(x)}$ is continuous for each $e \in E$ and $f \in F$. This extends to an isometric inclusion $\mathcal{K}(E, F) \hookrightarrow \Gamma_0(X, \mathcal{K}(\mathcal{E}, \mathcal{F}))$ sending $T \in \mathcal{K}(E, F)$ to $x \mapsto T_x$. The fibre map $\mathcal{K}(E, F) \to \mathcal{K}(E_x, F_x)$ has dense span for each x as we may lift $\Theta_{e,f}$ for $e \in E_x$ and $f \in F_x$, so by Proposition 1.48 the isometric inclusion $\mathcal{K}(E, F) \hookrightarrow \Gamma_0(X, \mathcal{K}(\mathcal{E}, \mathcal{F}))$ is surjective.

The continuity of the map $f, e \mapsto \Theta_{f,e} \colon \mathcal{F} \times_X \mathcal{E} \to \mathcal{K}(\mathcal{E}, \mathcal{F})$ follows immediately from Proposition 1.46. $\square$

We may hope for a Banach bundle $\mathcal{L}(\mathcal{E}, \mathcal{F}) \to X$ whose fibre at $x \in X$ is $\mathcal{L}(E_x, F_x)$ such that the assignment mapping $T \in \mathcal{L}(E, F)$ to the section $x \mapsto T_x \colon X \to \mathcal{L}(\mathcal{E}, \mathcal{F})$ is an isomorphism $\mathcal{L}(E, F) \cong \Gamma_b(X, \mathcal{L}(\mathcal{E}, \mathcal{F}))$. Unfortunately this is not always possible, as $x \mapsto \|T_x\| \colon X \to \mathbb{R}$ may fail to be upper semicontinuous [89, Remark C.14]. Nevertheless, it is still useful to consider a topology on the total space $\mathcal{L}(\mathcal{E}, \mathcal{F}) = \bigsqcup_{x\in X} \mathcal{L}(E_x)$.

Definition 1.52 (Strict topology for bundles of adjointable operators)**.** Let A be a $C_0(X)$-algebra and let E and F be Hilbert A-modules. The *strict topology* on $\mathcal{L}(\mathcal{E}, \mathcal{F}) = \bigsqcup_{x\in X} \mathcal{L}(E_x, F_x)$ is the weakest topology such that the maps

$$\begin{aligned} \mathcal{L}(\mathcal{E}, \mathcal{F}) &\to \mathcal{F} & \qquad \mathcal{L}(\mathcal{E}, \mathcal{F}) &\to \mathcal{E} \\ T &\mapsto T(e_{p(T)}) & T &\mapsto T^*(f_{p(T)}) \end{aligned}$$

are continuous for each $e \in E$ and $f \in F$, where $p \colon \mathcal{L}(\mathcal{E}, \mathcal{F}) \to X$ picks out the index of the disjoint union. We write $\Gamma_b(X, \mathcal{L}(\mathcal{E}, \mathcal{F}))$ for the space of bounded strictly continuous sections $X \to \mathcal{L}(\mathcal{E}, \mathcal{F})$.

Remark 1.53. Strict continuity of a section $x \mapsto T_x \colon X \to \mathcal{L}(\mathcal{E}, \mathcal{F})$ is equivalent to the continuity of the maps $e \mapsto T_{p(e)}(e) \colon \mathcal{E} \to \mathcal{F}$ and $f \mapsto T^*_{p(f)}(f) \colon \mathcal{F} \to \mathcal{E}$.

Proposition 1.54. *Let A be a $C_0(X)$-algebra and let E and F be Hilbert A-modules. Then for each adjointable operator $T \in \mathcal{L}(E, F)$, the section $x \mapsto T_x \colon X \to \mathcal{L}(\mathcal{E}, \mathcal{F})$ is a bounded strictly continuous section. Moreover, each bounded strictly continuous section determines an adjointable operator in $\mathcal{L}(E, F)$.*

Proof. Given $T \in \mathcal{L}(E,F)$, to see that the section $x \mapsto T_x$ is strictly continuous we need only check that $x \mapsto T_x(e_x)\colon X \to \mathcal{F}$ and $x \mapsto T_x^*(f_x)\colon X \to \mathcal{E}$ are continuous for each $e \in E$ and $f \in F$. These sections correspond to the elements $T(e) \in F$ and $T^*(f) \in E$, so are indeed continuous.

Now suppose that $x \mapsto T_x\colon X \to \mathcal{L}(\mathcal{E},\mathcal{F})$ is a bounded strictly continuous section. For each $e \in E$ and $f \in F$ we define $T(e)\colon X \to \mathcal{F}$ as the section $x \mapsto T_x(e_x)$ and $T^*(f)\colon X \to \mathcal{E}$ as the section $x \mapsto T_x^*(f_x)$. These sections are continuous by strict continuity of $x \mapsto T_x$ and vanish at infinity by boundedness, so $T(e) \in F$ and $T^*(f) \in E$. We have $\langle T(e), f\rangle_x = \langle T_x(e_x), f_x\rangle = \langle e_x, T_x^*(f_x)\rangle = \langle e, T^*(f)\rangle_x$ for each $x \in X$. We may conclude that $e \mapsto T(e)\colon E \to F$ defines an adjointable operator with adjoint T^*. ☐

Proposition 1.55. *Let A be a $C_0(X)$-algebra and let E and F be Hilbert A-modules. Then the application map $(T,e) \mapsto T(e)\colon \mathcal{L}(\mathcal{E},\mathcal{F}) \times_X \mathcal{E} \to \mathcal{F}$ and the adjoint application map $(T,f) \mapsto T^*(f)\colon \mathcal{L}(\mathcal{E},\mathcal{F}) \times_X \mathcal{F} \to \mathcal{E}$ are continuous. Moreover, for any topological space Y (not necessarily LCH), a map $g\colon Y \to \mathcal{L}(\mathcal{E},\mathcal{F})$ is continuous if and only if the maps*

$$\begin{array}{cc} Y \times_X \mathcal{E} \to \mathcal{F} & Y \times_X \mathcal{F} \to \mathcal{E} \\ y, e \mapsto g(y)(e) & y, f \mapsto g(y)^*(f) \end{array} \tag{1.1}$$

are continuous.

Proof. Let $(T_i, e_i) \to (T,e)$ be a convergent net in $\mathcal{L}(\mathcal{E},\mathcal{F}) \times_X \mathcal{E}$ over the convergent net $x_i \to x$ in X. Let $\xi \in E$ be an element with $\xi(x) = e$. Then $T_i(\xi(x_i)) \to T(e)$ and $e_i - \xi(x_i) \to 0$.

For each i let $q_i\colon E \to E_{x_i}$ be the fibre map. Then the norm of $T_i \in \mathcal{L}(E_{x_i}, F_{x_i})$ is $\|T_i \circ q_i\|$, and an application of the uniform boundedness principle shows that $(T_i \circ q_i)_i$ is a bounded family of operators. We may conclude that $T_i(e_i) - T_i(\xi(x_i)) \to 0$ and therefore that $T_i(e_i) \to T(e)$. The application map is therefore continuous, and continuity of the adjoint map $S \mapsto S^*\colon \mathcal{L}(\mathcal{F},\mathcal{E}) \to \mathcal{L}(\mathcal{E},\mathcal{F})$ implies continuity of the adjoint application map.

If $g\colon Y \to \mathcal{L}(\mathcal{E},\mathcal{F})$ is continuous, then continuity of the application maps implies continuity of the maps in (1.1). Conversely, if these maps are continuous, then to verify continuity of g we need to check the continuity of the maps

$$\begin{array}{cc} Y \to \mathcal{F} & Y \to \mathcal{E} \\ y \mapsto g(y)(e_{p(g(y))}) & y \mapsto g(y)^*(f_{p(g(y))}) \end{array}$$

for each $e \in E$ and $f \in F$. These can be written as compositions $Y \to Y \times_X \mathcal{E} \to \mathcal{F}$ and $Y \to Y \times_X \mathcal{F} \to \mathcal{E}$, whose latter maps are continuous by assumption. ☐

Remark 1.56. The inclusion $\mathcal{K}(\mathcal{E},\mathcal{F}) \hookrightarrow \mathcal{L}(\mathcal{E},\mathcal{F})$ is continuous from the Banach bundle topology on $\mathcal{K}(\mathcal{E},\mathcal{F})$ to the strict topology on $\mathcal{L}(\mathcal{E},\mathcal{F})$, which means that continuous sections $X \to \mathcal{K}(\mathcal{E},\mathcal{F})$ are automatically strictly continuous into $\mathcal{L}(\mathcal{E},\mathcal{F})$. However, this is typically not the subspace topology, as there are strictly continuous sections $X \to \mathcal{K}(\mathcal{E},\mathcal{F})$ which are not continuous [89, Example C.13].

When we have C*-algebras and their Hilbert modules fibred over spaces, we often want to change the space we are working over. Pullbacks and pushforwards allow us to do this.

Definition 1.57 (Pullback Banach bundle)**.** Let $f\colon Y \to X$ be a continuous map and let $\mathcal{A} \to X$ be a Banach bundle. The *pullback bundle* $f^*\mathcal{A} \to Y$ is defined to be $f^*\mathcal{A} := Y \times_X \mathcal{A} = \{(y,a) \in Y \times \mathcal{A} \mid a \in A_{f(y)}\}$. This is a Banach bundle over Y. The fibre $(f^*A)_y$ at $y \in Y$ is given by $A_{f(y)}$.

Remark 1.58. A section $a\colon Y \to f^*\mathcal{A}$ can be identified with a function $a\colon Y \to \mathcal{A}$ such that $a(y) \in A_{f(y)}$ for each $y \in Y$, and we will frequently make this identification.

Remark 1.59. The restriction $\mathcal{A}{\restriction_Y} \to Y$ of a Banach bundle $\mathcal{A} \to X$ to a subspace $Y \subseteq X$ may be described as the pullback of $\mathcal{A} \to X$ with respect to the inclusion $Y \hookrightarrow X$.

The pullback of a C*-bundle or a Hilbert bundle over X is a C*-bundle or Hilbert bundle over Y. It is straightforward to check when a map into $f^*\mathcal{A}$ is continuous, because we need only check that the composed maps into Y and $\mathcal{A}$ are continuous. As a result, we get the following continuity condition for Banach bundle maps into a pullback bundle:

Proposition 1.60 (Maps into pullbacks)**.** *Let $f\colon Y \to X$ be a continuous map and consider Banach bundles $\mathcal{A} \to X$ and $\mathcal{B} \to Y$. Let $\varphi\colon \mathcal{B} \to f^*\mathcal{A}$ be a map of Banach bundles over Y. Then φ is continuous if and only if the composition $\mathcal{B} \to f^*\mathcal{A} \to \mathcal{A}$ sending $b \in B_y$ to $\varphi(b) \in A_{f(y)}$ is continuous.*

Proof. The composition $\mathcal{B} \to f^*\mathcal{A} \to Y$ is the structure map $\mathcal{B} \to Y$ which is continuous. The result then follows from the universal properties of the product and subspace topologies on $f^*\mathcal{A} = Y \times_X \mathcal{A}$. □

By Proposition 1.45, to understand when a map out of $f^*\mathcal{A}$ is continuous, we need only consider sufficiently many continuous sections $Y \to f^*\mathcal{A}$. We can obtain a sufficient collection of continuous sections for $f^*\mathcal{A} \to Y$ from a sufficient collection for $\mathcal{A} \to X$. The *pullback* $f^*a \in \Gamma(Y, f^*\mathcal{A})$ of a section $a \in \Gamma(X,\mathcal{A})$ is the section $y \mapsto (f(y), a(f(y)))$. Note that as in Remark 1.58 we will often identify f^*a with the map $a \circ f\colon Y \to \mathcal{A}$.

Proposition 1.61 (Sufficiently many sections for pullbacks)**.** *Let $f\colon Y \to X$ be a continuous map and let $\mathcal{A} \to X$ be a Banach bundle. Suppose $\Gamma \subseteq \Gamma(X, \mathcal{A})$ is a sufficient collection of continuous sections. Then*

$$f^*\Gamma = \{f^*a \mid a \in \Gamma\}$$

is a sufficient collection of continuous sections for the bundle $f^*\mathcal{A} \to Y$.

Proof. For each $y \in Y$ the set $\{f^*a(y) \mid a \in \Gamma\}$ is equal to $\{a(f(y)) \mid a \in \Gamma\}$, which has dense span in $A_{f(y)} = (f^*A)_y$ by sufficiency of Γ. □

Definition 1.62 (Pullback of section spaces)**.** Suppose we have a continuous map $f\colon Y \to X$ and a Banach bundle $\mathcal{A} \to X$. The *pullback* f^*A of the section space $A = \Gamma_0(X, \mathcal{A})$ is the section space $\Gamma_0(Y, f^*\mathcal{A})$. If A is a $C_0(X)$-algebra then f^*A is a $C_0(Y)$-algebra, and if E is a Hilbert A-module, $f^*E = \Gamma_0(Y, f^*\mathcal{E})$ is a Hilbert f^*A-module.

A $C_0(X)$-linear map of $C_0(X)$-algebras $\varphi\colon A \to B$ induces a $C_0(Y)$-linear map $f^*\varphi\colon f^*A \to f^*B$, giving us a functor $f^*\colon \mathsf{C}^*\text{-}\mathsf{alg}^X \to \mathsf{C}^*\text{-}\mathsf{alg}^Y$. An adjointable operator $T\colon E \to F$ of Hilbert A-modules induces an adjointable operator $f^*T\colon f^*E \to f^*F$ of Hilbert f^*A-modules, giving us a functor $f^*\colon \mathsf{Hilb}_A \to \mathsf{Hilb}_{f^*A}$. We call each of these functors the *pullback* by f.

Proposition 1.63 (Adjointable operators on pullback modules)**.** *Let $g\colon Y \to X$ be a continuous map, let A be a $C_0(X)$-algebra and let E and F be a Hilbert A-modules. Then the fibre-wise isomorphisms $g^*\mathcal{K}(\mathcal{E}, \mathcal{F}) \cong \mathcal{K}(g^*\mathcal{E}, g^*\mathcal{F})$ and $g^*\mathcal{L}(\mathcal{E}, \mathcal{F}) := Y \times_X \mathcal{L}(\mathcal{E}, \mathcal{F}) \cong \mathcal{L}(g^*\mathcal{E}, g^*\mathcal{F})$ are homeomorphisms. We obtain identifications $\mathcal{K}(g^*E, g^*F) \cong g^*\mathcal{K}(E, F)$ and $\mathcal{L}(g^*E, g^*F) \cong \Gamma_b(Y, g^*\mathcal{L}(\mathcal{E}, \mathcal{F}))$.*

Proof. To check that $\psi\colon g^*\mathcal{K}(\mathcal{E}, \mathcal{F}) \to \mathcal{K}(g^*\mathcal{E}, g^*\mathcal{F})$ is continuous, it suffices by Propositions 1.45 and 1.61 to check that $y \mapsto \Theta_{f_{g(y)}, e_{g(y)}}\colon Y \to \mathcal{K}(g^*\mathcal{E}, g^*\mathcal{F})$ is continuous for each $e \in E$ and $f \in F$. This map can be written as a composition $Y \to g^*\mathcal{F} \times_Y g^*\mathcal{E} \to \mathcal{K}(g^*\mathcal{E}, g^*\mathcal{F})$, the latter of which is continuous by Proposition 1.51. As a continuous map of Banach bundles which is an isometric isomorphism at each fibre, ψ is a homeomorphism by Proposition 1.49.

To check that $\varphi\colon g^*\mathcal{L}(\mathcal{E}, \mathcal{F}) \to \mathcal{L}(g^*\mathcal{E}, g^*\mathcal{F})$ is continuous, we use Proposition 1.55. It therefore suffices to show that the fibre-wise application maps $g^*\mathcal{L}(\mathcal{E}, \mathcal{F}) \times_Y g^*\mathcal{E} \to g^*\mathcal{F}$ and $g^*\mathcal{L}(\mathcal{E}, \mathcal{F}) \times_Y g^*\mathcal{F} \to g^*\mathcal{E}$ are continuous. We need only check continuity of the composed maps through to $\mathcal{F}$ and $\mathcal{E}$ respectively, which follow from the commutativity of the diagrams below:

$$\begin{array}{ccc} g^*\mathcal{L}(\mathcal{E}, \mathcal{F}) \times_Y g^*\mathcal{E} & \longrightarrow & g^*\mathcal{F} \\ \downarrow & & \downarrow \\ \mathcal{L}(\mathcal{E}, \mathcal{F}) \times_X \mathcal{E} & \longrightarrow & \mathcal{F} \end{array} \qquad \begin{array}{ccc} g^*\mathcal{L}(\mathcal{E}, \mathcal{F}) \times_Y g^*\mathcal{F} & \longrightarrow & g^*\mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{L}(\mathcal{E}, \mathcal{F}) \times_X \mathcal{F} & \longrightarrow & \mathcal{E} \end{array}$$

To check that $\varphi^{-1}\colon \mathcal{L}(g^*\mathcal{E}, g^*\mathcal{F}) \to g^*\mathcal{L}(\mathcal{E},\mathcal{F})$ is continuous, we need only check that the composed map through to $\mathcal{L}(\mathcal{E},\mathcal{F})$ is continuous. Given $e \in E$ and $f \in F$, consider the maps

$$\begin{aligned} \mathcal{L}(g^*\mathcal{E}, g^*\mathcal{F}) &\to \mathcal{F} & \qquad \mathcal{L}(g^*\mathcal{E}, g^*\mathcal{F}) &\to \mathcal{E} \\ T &\mapsto T(e_{p(T)}) & T &\mapsto T^*(f_{p(T)}), \end{aligned}$$

where $p\colon \mathcal{L}(g^*\mathcal{E}, g^*\mathcal{F}) \to X$ picks out the fibre. The first can be written as a composition

$$\mathcal{L}(g^*\mathcal{E}, g^*\mathcal{F}) \xrightarrow{\mathrm{id}\times(g^*e)\circ p} \mathcal{L}(g^*\mathcal{E}, g^*\mathcal{F}) \times_Y g^*\mathcal{E} \xrightarrow{\mathrm{app}} g^*\mathcal{F} \xrightarrow{\pi_{\mathcal{F}}} \mathcal{F}$$

of continuous maps, and similarly for the second. We conclude that φ is a homeomorphism. □

Through a continuous map $f\colon Y \to X$ we may directly view algebras and modules that are fibred over Y as fibred over X instead. This is called the pushforward.

Definition 1.64 (Pushforward of algebras and modules)**.** Let $f\colon Y \to X$ be a continuous map and let A be a $C_0(Y)$-algebra with structure map $\Phi\colon C_0(Y) \to ZM(A)$. Let $\tilde{\Phi}\colon C_b(Y) \to ZM(A)$ be the extension of Φ to the multiplier algebra $C_b(Y)$. The *pushforward* $C_0(X)$*-algebra* f_*A has underlying C*-algebra A, with structure map given by the composition

$$C_0(X) \xrightarrow{f^*} C_b(Y) \xrightarrow{\tilde{\Phi}} ZM(A).$$

This is non-degenerate [8, Proposition 3.5] so f^*A is indeed a $C_0(X)$-algebra. If E is a Hilbert A-module, we define the *pushforward Hilbert module* f_*E to be the Hilbert f_*A-module with the same underlying space. The *pushforward bundles* $f_*\mathcal{A} \to X$ and $f_*\mathcal{E} \to X$ are the associated bundles. The fibres at $x \in X$ of the pushforwards are given by $(f_*A)_x \cong \Gamma_0(Y_x, \mathcal{A})$ and $(f_*E)_x \cong \Gamma_0(Y_x, \mathcal{E})$ (see [8, Proposition 3.6]). The fibre maps $f_*A \to (f_*A)_x$ and $f_*E \to (f_*E)_x$ are given by the restriction maps $\Gamma_0(Y, \mathcal{A}) \to \Gamma_0(Y_x, \mathcal{A})$ and $\Gamma_0(Y, \mathcal{E}) \to \Gamma_0(Y_x, \mathcal{E})$.

A $C_0(Y)$-linear map of $C_0(Y)$-algebras $A \to B$ is $C_0(X)$-linear as a map of $C_0(X)$-algebras $f_*A \to f_*B$, giving us a functor $f_*\colon \mathsf{C^*\text{-}alg}^Y \to \mathsf{C^*\text{-}alg}^X$. Similarly, we get a functor $f_*\colon \mathsf{Hilb}_A \to \mathsf{Hilb}_{f_*A}$ for each $C_0(Y)$-algebra A. We call each of these the pushforward.

Proposition 1.65 (Adjointable operators on pushforward modules)**.** *Let $g\colon Y \to X$ be a continuous map, and consider a $C_0(Y)$-algebra A and Hilbert A-modules E and F.*

Then for each $x \in X$, we may identify the fibre $\mathcal{L}((g_*E)_x, (g_*F)_x)$ of the adjointable operators at x with $\Gamma_b(Y_x, \mathcal{L}(\mathcal{E},\mathcal{F}))$ via the isomorphism

$$\varphi\colon \Gamma_b(Y_x, \mathcal{L}(\mathcal{E},\mathcal{F})) \cong \mathcal{L}(\Gamma_0(Y_x,\mathcal{E}), \Gamma_0(Y_x,\mathcal{F}))$$

$$T \mapsto (e \mapsto (y \mapsto T_y(e_y))).$$

The fibre $\mathcal{K}((g_*E)_x, (g_*F)_x)$ of the compact operator bundle at x can be identified with $\Gamma_0(Y_x, \mathcal{K}(\mathcal{E}, \mathcal{F}))$ via the isomorphism

$$\begin{aligned} \psi\colon \Gamma_0(Y_x, \mathcal{K}(\mathcal{E}, \mathcal{F})) &\cong \mathcal{K}(\Gamma_0(Y_x, \mathcal{E}), \Gamma_0(Y_x, \mathcal{F})) \\ T &\mapsto (e \mapsto (y \mapsto T_y(e_y))). \end{aligned}$$

Under these identifications, the fibre $T_x \in \Gamma_b(Y_x, \mathcal{L}(\mathcal{E}, \mathcal{F}))$ of an adjointable operator $T \in \mathcal{L}(g_*E, g_*F)$ is the restriction of T viewed as an element of $\Gamma_b(Y, \mathcal{L}(\mathcal{E}, \mathcal{F}))$ to Y_x.

Proof. The restriction to Y_x is the pullback by the inclusion $Y_x \hookrightarrow Y$, so φ and ψ are well-defined and isomorphisms by Proposition 1.63. The fibre T_x of T at x is given by $T\restriction_{Y_x} \in \Gamma_b(Y_x, \mathcal{L}(\mathcal{E}, \mathcal{F}))$ because the fibre maps $\Gamma_0(Y, \mathcal{E}) \to \Gamma_0(Y_x, \mathcal{E})$ and $\Gamma_0(Y, \mathcal{F}) \to \Gamma_0(Y_x, \mathcal{F})$ are given by restriction, so the following diagram commutes.

$$\begin{tikzcd} \Gamma_0(Y, \mathcal{E}) \arrow[r, "T"] \arrow[d] & \Gamma_0(Y, \mathcal{F}) \arrow[d] \\ \Gamma_0(Y_x, \mathcal{E}) \arrow[r, "T\restriction_{Y_x}"] & \Gamma_0(Y_x, \mathcal{F}) \end{tikzcd}$$

□

Remark 1.66 (Pushforwards for arbitrary Banach bundles). Given $f\colon Y \to X$, we may define the pushforward $f_*\mathcal{A} \to X$ for any Banach bundle $\mathcal{A} \to Y$. We take the fibre at $x \in X$ to be $(f_*A)_x = \Gamma_0(Y_x, \mathcal{A})$, and consider the set Γ of sections $\eta\colon X \to f_*\mathcal{A}$ such that $y \mapsto \eta(f(y))(y)\colon Y \to \mathcal{A}$ is in $\Gamma_0(Y, \mathcal{A})$. By Proposition 1.40 there is a unique Banach bundle structure on $f_*\mathcal{A}$ such that each element of Γ is a continuous section. Note that Γ contains $x \mapsto \xi\restriction_{Y_x}\colon X \to f_*\mathcal{A}$ for each $\xi \in \Gamma_0(Y, \mathcal{A})$, which form a sufficient collection of sections for $f_*\mathcal{A}$. This pushforward bundle construction of $f_*\mathcal{A}$ for a $C_0(Y)$-algebra A therefore agrees with our earlier definition.

The pushforward and pullback interact in the following way, see Proposition 3.7 in [8].

Proposition 1.67 (Interplay between pushforward and pullback)**.** *Let $f\colon Y \to X$ and $g\colon Z \to X$ be continuous maps of locally compact Hausdorff spaces, and let $\mathcal{A} \to Y$ be a Banach bundle. Let $\pi_Z\colon Z \times_X Y \to Z$ and $\pi_Y\colon Z \times_X Y \to Y$ be the projection maps. Then there is an isomorphism of Banach bundles $\varphi\colon g^*f_*\mathcal{A} \cong (\pi_Z)_*\pi_Y^*\mathcal{A}$ given fibre-wise at $z \in Z$ by*

$$\begin{aligned} \Gamma_0(Y_{g(z)}, \mathcal{A}) &\to \Gamma_0((Z \times_X Y)_z, \pi_Y^*\mathcal{A}) \\ \eta &\mapsto \eta \circ \pi_Y. \end{aligned}$$

Proof. This is clearly an isomorphism at each fibre, so by Proposition 1.49, we need only check that it is continuous. Ranging over $a \in \Gamma_0(Y, \mathcal{A})$ and $\gamma \in C_0(Z)$, the sections $\gamma g^* f_* a \colon z \mapsto \gamma(z) a\restriction_{Y_{g(z)}} \colon Z \to g^* f_* \mathcal{A}$ form a sufficient collection of continuous sections. Composing with φ, the section $z \mapsto (\gamma(z) a\restriction_{Y_{g(z)}}) \circ \pi_Y \colon Z \to (\pi_Z)_* \pi_Y^* \mathcal{A}$ is continuous because its associated map $Z \times_X Y \to \pi_Y^* \mathcal{A}$ is the c_0 section $(z, y) \mapsto \gamma(z) a(y)$. Because $\varphi \circ \gamma g^* f_* a$ is continuous for each $a \in \Gamma_0(Y, \mathcal{A})$ and $\gamma \in C_0(Z)$ we can conclude by Proposition 1.45 that φ is continuous. □

One consequence of this is that we can check the continuity of a function $\xi \colon Z \to f_* \mathcal{A}$ into a pushforward bundle by checking the continuity of the associated "uncurried" bivariate function $\tilde{\xi} \colon (z, y) \mapsto \xi(z)(y) \colon Z \times_X Y \to \mathcal{A}$.

Proposition 1.68 (Continuity of maps into pushforward bundles)**.** *Let $f \colon Y \to X$ be a continuous map of locally compact Hausdorff spaces, let $\mathcal{A} \to Y$ be a Banach bundle with pushforward $p \colon f_* \mathcal{A} \to X$ and let Z be a locally compact Hausdorff space. Then a map $\xi \colon Z \to f_* \mathcal{A}$ which vanishes at infinity is continuous if and only if*

- *the map $g = p \circ \xi \colon Z \to X$ is continuous,*
- *and the map $\tilde{\xi} \colon (z, y) \mapsto \xi(z)(y) \colon Z \times_X Y \to \mathcal{A}$ is c_0.*

Proof. First suppose that ξ is continuous. Then clearly g is continuous. Through the identification $g^* f_* \mathcal{A} \cong (\pi_Z)_* \pi_Y^* \mathcal{A}$, the section $\xi \colon Z \to g^* f_* \mathcal{A}$ is mapped to $z \mapsto \tilde{\xi}\restriction_{(Z \times_X Y)_z} \colon Z \to (\pi_Z)_* \pi_Y^* \mathcal{A}$. The c_0 section corresponding to this is $\tilde{\xi} \colon Z \times_X Y \to \pi_Y^* \mathcal{A}$.

Conversely, if g is continuous and $\tilde{\xi}$ is c_0, we may walk this backwards so that $\tilde{\xi} \colon Z \times_X Y \to \pi_Y^* \mathcal{A}$ corresponds through the pushforward $(\pi_Z)_*$ and the identification $g^* f_* \mathcal{A} \cong (\pi_Z)_* \pi_Y^* \mathcal{A}$ to $\xi \colon Z \to g^* f_* \mathcal{A}$. □

1.3. **Groupoid actions on C*-algebras and crossed products.** The action of a groupoid on an object requires that it be fibred over the unit space. With Banach bundles, we can discuss how étale groupoids act on Banach spaces.

Definition 1.69 (Groupoid action on a Banach bundle)**.** Let G be an étale groupoid with unit space X, and let $p \colon \mathcal{A} \to X$ be a Banach bundle over X. A *(left) Banach bundle action* $G \curvearrowright \mathcal{A}$ is an action of G on $\mathcal{A}$ as a topological space with anchor map $p \colon \mathcal{A} \to X$ such that for each $g \in G$, the map

$$a \mapsto g \cdot a \colon A_{s(g)} \to A_{r(g)}$$

is a linear isometric isomorphism. We call $\mathcal{A}$ a *Banach G-bundle.*

Remark 1.70. The action map $G \times_X \mathcal{A} \to \mathcal{A}$ can be considered as a map $s^* \mathcal{A} \to r^* \mathcal{A}$ of Banach bundles, and continuity can be checked using Proposition 1.61. This means that a candidate action map is continuous if for sufficiently many sections $\xi \in \Gamma_0(X, \mathcal{A})$, the map $g \mapsto g \cdot \xi(s(g)) \colon G \to \mathcal{A}$ is continuous.

Definition 1.71 (Groupoid actions on algebras and modules)**.** Let G be an étale groupoid with unit space X.

A *(left) action* $G \curvearrowright A$ on a C*-algebra A is a $C_0(X)$-algebra structure on A along with a Banach bundle action $G \curvearrowright \mathcal{A}$ on the associated bundle $\mathcal{A} \to X$ such that for each $g \in G$,

$$a \mapsto g \cdot a \colon A_{s(g)} \to A_{r(g)}$$

is a $*$-isomorphism. We call A a *G-C*-algebra* and $\mathcal{A}$ a *G-C*-bundle*. A *compatible G-action* on a Hilbert A-module E is a Banach bundle action $G \curvearrowright \mathcal{E}$ on the associated bundle $\mathcal{E} \to X$ that is compatible with the module action and inner product in the sense that for each $g \in G$, $a \in A_{s(g)}$ and $e_1, e_2 \in E_{s(g)}$, we have:

$$g \cdot (a \cdot e_1) = (g \cdot a) \cdot (g \cdot e_1) \in E_{r(g)}, \qquad g \cdot \langle e_1, e_2 \rangle = \langle g \cdot e_1, g \cdot e_2 \rangle \in A_{r(g)}.$$

We call E a *G-Hilbert A-module* and $\mathcal{E}$ a *G-Hilbert $\mathcal{A}$-bundle*.

Remark 1.72 (Action of an open bisection)*.* Suppose we have a G-C*-algebra A with action $\alpha \colon G \curvearrowright A$. For each open bisection $U \subseteq G$, we obtain a $*$-homomorphism $\alpha_U \colon s(U)A \to r(U)A$ given by

$$\begin{aligned} \alpha_U \colon \Gamma_0(s(U), \mathcal{A}) &\to \Gamma_0(r(U), \mathcal{A}) \\ a &\mapsto (r(u) \mapsto u \cdot a_{s(u)}). \end{aligned}$$

Similarly, a compatible action $\epsilon \colon G \curvearrowright E$ on a Hilbert A-module E induces a map $\epsilon_U \colon \Gamma_0(s(U), \mathcal{E}) \to \Gamma_0(r(U), \mathcal{E})$.

Definition 1.73 (Equivariant $*$-homomorphisms)**.** Let G be an étale groupoid with unit space X and let A and B be G-C*-algebras. A $*$-homomorphism $\varphi \colon A \to B$ is *G-equivariant* if it is $C_0(X)$-linear and $g \cdot \varphi_{s(g)}(a) = \varphi_{r(g)}(g \cdot a)$ for each $g \in G$ and $a \in A_{s(g)}$.

We write $\mathsf{C^*\text{-}alg}^G$ for the category of G-C*-algebras with morphisms given by G-equivariant $*$-homomorphisms.

An action of an étale groupoid G on a Hilbert module over a G-C*-algebra leads naturally to actions on the adjointable operators, and in particular the algebra of compact operators becomes a G-C*-algebra.

Proposition 1.74 (Actions on the spaces of adjointable operators)**.** *Let G be an étale groupoid with unit space X, let A be a G-C*-algebra and let E and F be G-Hilbert A-modules. Then there is an action $G \curvearrowright \mathcal{L}(\mathcal{E}, \mathcal{F})$ as a topological space over X which restricts to an action $G \curvearrowright \mathcal{K}(\mathcal{E}, \mathcal{F})$ as a Banach bundle over X, given at*

$g \in G$ by

$$\begin{aligned} \mathcal{K}(E_{s(g)}, F_{s(g)}) &\to \mathcal{K}(E_{r(g)}, F_{r(g)}) & \qquad \mathcal{L}(E_{s(g)}, F_{s(g)}) &\to \mathcal{L}(E_{r(g)}, F_{r(g)}) \\ T &\mapsto v_g T u_g^{-1} & T &\mapsto v_g T u_g^{-1}, \end{aligned}$$

where $u_g \colon E_{s(g)} \to E_{r(g)}$ is the action map $e \mapsto g \cdot e$ and $v_g \colon F_{s(g)} \to F_{r(g)}$ is the action map $f \mapsto g \cdot f$. We typically write $g \cdot T$ for the operator $v_g T u_g^{-1}$ given $g \in G$ and $T \in \mathcal{L}(E_{s(g)}, F_{s(g)})$.

Proof. We first check that the map $(g, T) \mapsto v_g T u_g^{-1} \colon G \times_X \mathcal{L}(\mathcal{E}, \mathcal{F}) \to \mathcal{L}(\mathcal{E}, \mathcal{F})$ is continuous using Proposition 1.55. The composition

$$\begin{aligned} (G \times_X \mathcal{L}(\mathcal{E}, \mathcal{F})) \times_X \mathcal{E} = \{(g, T, e) \mid g \in G,\ T \in \mathcal{L}(E_{s(g)}, F_{s(g)}),\ e \in E_{r(g)}\} \\ \to \mathcal{F} \\ (g, T, e) \mapsto v_g T u_g^{-1}(e) \end{aligned}$$

with the application map is continuous by continuity of the inverse action map $r^*\mathcal{E} \to s^*\mathcal{E}$, the application map $\mathcal{L}(\mathcal{E}, \mathcal{F}) \times_X \mathcal{E} \to \mathcal{F}$ and the action map $s^*\mathcal{F} \to r^*\mathcal{F}$. The composition with the adjoint application map is similarly continuous, and we may conclude that the candidate action map $G \times_X \mathcal{L}(\mathcal{E}, \mathcal{F}) \to \mathcal{L}(\mathcal{E}, \mathcal{F})$ is continuous.

We use Remark 1.70 to check that the restriction of the operator bundle action map to the map $(g, T) \mapsto v_g T u_g^{-1} \colon G \times_X \mathcal{K}(\mathcal{E}, \mathcal{F}) \to \mathcal{K}(\mathcal{E}, \mathcal{F})$ is continuous by checking against enough continuous sections for $\mathcal{K}(\mathcal{E}, \mathcal{F})$. For each $f \in F$ and $e \in E$, we have $v_g \Theta_{f_{s(g)}, e_{s(g)}} u_g^{-1} = \Theta_{g \cdot f_{s(g)}, g \cdot e_{s(g)}}$ for each $g \in G$. This is a continuous function of g by Proposition 1.51, and so we may conclude that the candidate action map $G \times_X \mathcal{K}(\mathcal{E}, \mathcal{F}) \to \mathcal{K}(\mathcal{E}, \mathcal{F})$ is continuous, and therefore defines a G-Banach bundle structure on $\mathcal{K}(\mathcal{E}, \mathcal{F})$. □

One of the main reasons operator algebraists are interested in groupoid actions on C*-algebras and Hilbert modules is to construct groupoid crossed products. This is a little delicate so we will require the following lemma. For a continuous map $f \colon X \to Y$, a Banach bundle $\mathcal{A} \to Y$ and a continuous section $\xi \in \Gamma(X, f^*\mathcal{A})$, we say that ξ has *proper support with respect to* f if the restriction $f\restriction_{\mathrm{supp}(\xi)} \colon \mathrm{supp}(\xi) \to Y$ is proper.

Lemma 1.75. *Let $f \colon X \to Y$ be a local homeomorphism and let $\mathcal{A} \to Y$ be a Banach bundle. Let $\xi \in \Gamma(X, f^*\mathcal{A})$ be a bounded continuous section with proper support with respect to f. Then the section $f_*\xi \colon Y \to \mathcal{A}$ given by*

$$f_*\xi(y) = \sum_{x \in f^{-1}(y)} \xi(x)$$

is well-defined and continuous. If ξ is compactly supported then so is $f_*\xi$.

Proof. In the special case of the range local homeomorphism $r\colon G \to G^0$ of an étale groupoid G and with $\mathcal{A}$ being scalar this is [8, Lemma 2.3]. This proof carries over almost word for word into our setting. $\square$

We may now describe the convolution $*$-algebra of a G-C*-algebra A that can be completed to form the crossed product $G \ltimes A$.

Proposition 1.76 (Convolution algebras and modules)**.** *Let G be an étale groupoid, let A be a G-C*-algebra and let E be a G-Hilbert A-module. There is a $*$-algebra structure on the space of compactly supported sections $\Gamma_c(G, s^*\mathcal{A})$, given by*

$$\begin{aligned}
- * -\colon \Gamma_c(G, s^*\mathcal{A}) \times \Gamma_c(G, s^*\mathcal{A}) &\to \Gamma_c(G, s^*\mathcal{A}) \\
(\xi, \eta) &\mapsto \xi * \eta \\
&\qquad g \mapsto \sum_{g_1 g_2 = g} (g_2^{-1} \cdot \xi(g_1))\eta(g_2) \\
(-)^*\colon \Gamma_c(G, s^*\mathcal{A}) &\to \Gamma_c(G, s^*\mathcal{A}) \\
\eta &\mapsto \eta^* \\
&\qquad g \mapsto g^{-1} \cdot (\eta(g^{-1}))^*.
\end{aligned}$$

Furthermore, $\Gamma_c(G, s^*\mathcal{E})$ has the structure of a right $\Gamma_c(G, s^*\mathcal{A})$-module and a sesquilinear form valued in $\Gamma_c(G, s^*\mathcal{A})$ given by

$$\begin{aligned}
\Gamma_c(G, s^*\mathcal{E}) \times \Gamma_c(G, s^*\mathcal{A}) &\to \Gamma_c(G, s^*\mathcal{E}) \\
(\xi, \nu) &\mapsto \xi \cdot \nu \\
&\qquad g \mapsto \sum_{g_1 g_2 = g} (g_2^{-1} \cdot \xi(g_1)) \cdot \nu(g_2) \\
\Gamma_c(G, s^*\mathcal{E}) \times \Gamma_c(G, s^*\mathcal{E}) &\to \Gamma_c(G, s^*\mathcal{A}) \\
(\xi, \eta) &\mapsto \langle \xi, \eta \rangle \\
&\qquad g \mapsto \sum_{g_1 g_2 = g} \langle g^{-1} \cdot \xi(g_1^{-1}), \eta(g_2) \rangle.
\end{aligned}$$

The sesquilinear form satisfies $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$ for each $\xi, \eta \in \Gamma_c(G, s^*\mathcal{E})$ and is compatible with the right action of $\Gamma_c(G, s^*\mathcal{A})$ in the sense that $\langle \xi, \eta \cdot \nu \rangle = \langle \xi, \eta \rangle * \nu$ for each $\nu \in \Gamma_c(G, s^*\mathcal{A})$.

Proof. Once it is clear that these operations are well-defined, it is straightforward to check all of the required algebraic identities.

The adjoint map is well-defined by continuity of inversion $G \to G$, the adjoint map $\mathcal{A} \to \mathcal{A}$ and the continuity of the action $G \curvearrowright \mathcal{A}$. To justify the continuity of each other map we will apply Lemma 1.75 to the local homeomorphism $m\colon (g_1, g_2) \mapsto g_1 g_2 \colon G^2 \to G$. For convolution on $\Gamma_c(G, s^*\mathcal{A})$, consider elements ξ and η of $\Gamma_c(G, s^*\mathcal{A})$. The map $(g_1, g_2) \mapsto (g_2^{-1} \cdot \xi(g_1))\eta(g_2) \colon G^2 \to m^* s^* \mathcal{A}$ is continuous by continuity of the action $G \curvearrowright \mathcal{A}$ and multiplication on $\mathcal{A}$, and compactly supported because ξ and η are compactly supported. The element $\xi * \eta \in \Gamma_c(G, s^*\mathcal{A})$ is therefore well-defined by Lemma 1.75. The right module structure and sesquilinear form on $\Gamma_c(G, s^*\mathcal{E})$ can be justified in a very similar manner. □

Remark 1.77. Other authors may use the alternative convention to fibre over the range map r instead of the source map s, and define a $*$-algebra structure on $\Gamma_c(G, r^*\mathcal{A})$ and module structure on $\Gamma_c(G, r^*\mathcal{E})$. These are completely equivalent approaches to the crossed products, but some formulae change slightly depending on which convention is used.

Definition 1.78 (The reduced crossed product)**.** Let G be an étale groupoid and let A be a G-C*-algebra. The Hilbert A-module $L^2(G, A)$ is the completion of $\Gamma_c(G, r^*\mathcal{A})$ under the inner product and right A-module action given by

$$\begin{aligned}\langle -, - \rangle \colon \Gamma_c(G, r^*\mathcal{A}) \times \Gamma_c(G, r^*\mathcal{A}) &\to A \\ (\xi, \eta) &\mapsto \langle \xi, \eta \rangle \\ x &\mapsto \sum_{g \in G^x} \xi(g)^* \eta(g) \\ \Gamma_c(G, r^*\mathcal{A}) \times A &\to \Gamma_c(G, r^*\mathcal{A}) \\ (\xi, a) &\mapsto \xi \cdot a \\ g &\mapsto \xi(g) \cdot a(r(g)).\end{aligned}$$

We can then define a $*$-representation of $\Gamma_c(G, s^*\mathcal{A})$ on $L^2(G, A)$ as follows:

$$\begin{aligned}\Gamma_c(G, s^*\mathcal{A}) \times \Gamma_c(G, r^*\mathcal{A}) &\to \Gamma_c(G, r^*\mathcal{A}) \\ (\xi, \eta) &\mapsto \xi \cdot \eta \\ g &\mapsto \sum_{g_1 g_2 = g} g_1 \cdot (\xi(g_1)\eta(g_2))\end{aligned}$$

For $\xi \in \Gamma_c(G, s^*\mathcal{A})$ supported in an open bisection $U \subseteq G$ and $\eta \in \Gamma_c(G, r^*\mathcal{A}) \subseteq L^2(G, A)$, we have $\|\xi \cdot \eta\| \leq \|\xi\|_\infty \|\eta\|$. The above map therefore extends to a $*$-homomorphism $\lambda \colon \Gamma_c(G, s^*\mathcal{A}) \to \mathcal{L}(L^2(G, A))$, which we may call the *left regular representation.* The *reduced crossed product* $G \ltimes_r A$ is the closure of $\lambda(\Gamma_c(G, s^*\mathcal{A}))$ in $\mathcal{L}(L^2(G, A))$. Furthermore, if E is a G-Hilbert A-module, we may complete $\Gamma_c(G, s^*\mathcal{E})$ under the norm $\|\xi\| := \|\lambda(\langle \xi, \xi \rangle)\|^{\frac{1}{2}}$ for $\xi \in \Gamma_c(G, s^*\mathcal{E})$ to obtain the *reduced crossed product* Hilbert $G \ltimes_r A$-module $G \ltimes_r E$. We note that the positivity of the element $\lambda(\langle \xi, \xi \rangle)$ may be checked straightforwardly by checking that for $\eta \in \Gamma_c(G, r^*\mathcal{A})$, we have $\langle \eta, \lambda(\langle \xi, \xi \rangle)\eta \rangle \geq 0$.

To define the universal crossed products $G \ltimes A$ and $G \ltimes E$, we need the following fact.

Proposition 1.79. *Let G be an étale groupoid and let A be a G-C*-algebra. For any element $\xi \in \Gamma_c(G, s^*\mathcal{A})$ which is supported on an open bisection and any $*$-representation π of $\Gamma_c(G, s^*\mathcal{A})$, we have $\|\pi(\xi)\| \le \|\xi\|_\infty$. Furthermore, for general $\xi \in \Gamma_c(G, s^*\mathcal{A})$, the quantity $\|\pi(\xi)\|$ is bounded independently of π.*

Proof. First suppose that ξ is supported on an open bisection $U \subseteq G$ and supported on a compact set $C \subseteq U$. The element $\xi^* * \xi$ is supported on the compact set $s(C)$. The $*$-representation π restricts to a $*$-representation of the C*-algebra $\Gamma(s(C), \mathcal{A})$, which is automatically contractive, and we obtain $\|\pi(\xi)\| = \|\pi(\xi^* * \xi)\|^{\frac{1}{2}} \le \|\xi^* * \xi\|_\infty^{\frac{1}{2}} = \|\xi\|_\infty$.

A general element $\xi \in \Gamma_c(G, s^*\mathcal{A})$ is supported on a compact set $K \subseteq G$. We may cover K by a finite set of open bisections $U_1, \dots U_n$. Using a partition of unity argument we may write $\xi = \sum_{i=1}^n \xi_i$ such that $\xi_i \in \Gamma_c(U_i, s^*\mathcal{A})$ and $\|\xi_i\|_\infty \le \|\xi\|_\infty$. It follows that $\|\pi(\xi)\| \le n\|\xi\|_\infty$. □

Definition 1.80 (Crossed products)**.** Let G be an étale groupoid, let A be a G-C*-algebra and let E be a G-Hilbert A-module. The *universal crossed product C*-algebra* $G \ltimes A$ is the completion of $\Gamma_c(G, s^*\mathcal{A})$ under the *universal* norm given for $\xi \in \Gamma_c(G, s^*\mathcal{A})$ by

$$\|\xi\| = \sup\{\|\pi(\xi)\| \mid \pi\colon \Gamma_c(G, s^*\mathcal{A}) \to \mathcal{L}(\mathcal{H}) \text{ is a } *\text{-representation}\}.$$

The inclusion $\Gamma_c(G, s^*\mathcal{A}) \subseteq G \ltimes A$ and the $\Gamma_c(G, s^*\mathcal{A})$-valued sesquilinear form from Proposition 1.76 give rise to a $G \ltimes A$-valued sesquilinear form on $\Gamma_c(G, s^*\mathcal{E})$ which is positive definite in the sense that $\langle \xi, \xi \rangle \ge 0$ in $G \ltimes A$ for each $\xi \in \Gamma_c(G, s^*\mathcal{E})$, with equality if and only if $\xi = 0$. The universal norm on $\Gamma_c(G, s^*\mathcal{A})$ therefore allow us to define a norm on $\Gamma_c(G, s^*\mathcal{E})$ by $\|\xi\| := \|\langle \xi, \xi \rangle\|^{\frac{1}{2}}$ for $\xi \in \Gamma_c(G, s^*\mathcal{E})$. The *universal crossed product Hilbert module* $G \ltimes E$ is the completion of $\Gamma_c(G, s^*\mathcal{E})$ with respect to this norm. It is a Hilbert $G \ltimes A$-module.

We will often refer to universal crossed products simply as crossed products. We obtain a functor $G \ltimes -\colon \mathsf{C}^*\text{-}\mathsf{alg}^G \to \mathsf{C}^*\text{-}\mathsf{alg}$ called the *crossed product*.

Proposition 1.81. *Let G be an étale groupoid with unit space X and let A be a G-C*-algebra. Then the inclusion $\Gamma_c(X, \mathcal{A}) \to \Gamma_c(G, s^*\mathcal{A})$ completes to an inclusion $A \subseteq G \ltimes A$. Furthermore, A generates $G \ltimes A$ as an ideal.*

Proof. The space $\Gamma_c(X, \mathcal{A})$ carries the uniform norm inside $G \ltimes A$ by Proposition 1.79 so completes to $A = \Gamma_0(X, \mathcal{A})$ inside $G \ltimes A$. Any approximate unit for A acts as an approximate unit for any element of $\Gamma_c(G, s^*\mathcal{A})$ which is supported on an open bisection. It is therefore an approximate unit for $G \ltimes A$ and so $\overline{A(G \ltimes A)} = G \ltimes A$. □

Proposition 1.82. *Let G be an étale groupoid, let A be a G-C^*-algebra and let E be a G-Hilbert A-module. The $G \ltimes A$-valued sesquilinear form on $\Gamma_c(G, s^*\mathcal{E})$ described in Proposition 1.76 is positive definite, and so we may form the Hilbert $G \ltimes A$-module $G \ltimes E$ by completion. Furthermore, the bilinear map*

$$\begin{aligned} \Phi\colon \Gamma_c(G^0, \mathcal{E}) \times \Gamma_c(G, s^*\mathcal{A}) &\to \Gamma_c(G, s^*\mathcal{E}) \\ e, a &\mapsto \Phi(e, a) \\ &\qquad g \mapsto (g^{-1} \cdot e_{r(g)}) \cdot a(g) \end{aligned}$$

induces isomorphisms $E \otimes_A G \ltimes_r A \cong G \ltimes_r E$ and $E \otimes_A G \ltimes A \cong G \ltimes E$.

Proof. Given $e_1, e_2 \in \Gamma_c(G^0, \mathcal{E})$ and $a_1, a_2 \in \Gamma_c(G, s^*\mathcal{A})$, we have

$$\langle \Phi(e_1, a_1), \Phi(e_2, a_2) \rangle = a_1^* * \langle e_1, e_2 \rangle * a_2 \in \Gamma_c(G, s^*\mathcal{A}).$$

This is equal to the inner product $\langle e_1 \otimes a_1, e_2 \otimes a_2 \rangle$ whether this is taken in $E \otimes_A G \ltimes_r A$ or in $E \otimes_A G \ltimes A$, viewing $\Gamma_c(G, s^*\mathcal{A})$ as a subspace of both $G \ltimes_r A$ and $G \ltimes A$. Positivity of the sesquilinear form on $\Gamma_c(G, s^*\mathcal{E})$ therefore holds on the span of the image of Φ.

Given $\xi, \eta \in \Gamma_c(G, s^*\mathcal{E})$ which are supported on open bisections of G, the element $\langle \xi, \eta \rangle \in \Gamma_c(G, s^*\mathcal{A})$ is supported on an open bisection and $\|\langle \xi, \eta \rangle\|_\infty \leq \|\xi\|_\infty \|\eta\|_\infty$. By Proposition 1.79, this means that $\|\langle \xi, \eta \rangle\|_{G \ltimes A} \leq \|\xi\|_\infty \|\eta\|_\infty$.

We now let $\xi \in \Gamma_c(G, s^*\mathcal{E})$ be a general element. Using a partition of unity argument, we may find $n \in \mathbb{N}$ and open bisections $U_1, \dots, U_n$ of G and $\xi_i \in \Gamma_c(U_i, \mathcal{E})$ such that $\xi = \sum_{i=1}^n \xi_i$. For each i, there are sequences $(\xi_{i,j})_{j \in \mathbb{N}}$ in $\Gamma_c(U_i, \mathcal{E}) \cap \operatorname{im} \Phi$ which converge to ξ_i in the uniform norm. The estimates from the previous paragraph imply that $\langle \xi, \xi \rangle$ can be approximated by positive elements in $G \ltimes A$ and is therefore positive.

Our arguments have shown that Φ extends to inner product preserving maps $E \otimes_A G \ltimes_r A \to G \ltimes_r E$ and $E \otimes_A G \ltimes A \to G \ltimes E$ with dense image, which are therefore isomorphisms. □

The C*-algebras and Hilbert modules that we deal with are often defined as suitable completions of dense subspaces that we can get more of a handle on. However, the definition of the universal crossed product C*-algebra $G \ltimes A$ requires us to consider genuine representations of the $*$-algebra $\Gamma_c(G, s^*\mathcal{A})$ on Hilbert spaces, rather than simply dense subspaces of Hilbert spaces. The following lemma allows us to build such representations even when we only know how to act on certain dense subspaces of a Hilbert space or module. This is known as a pre-representation of $\Gamma_c(G, s^*\mathcal{A})$, see [15, Definition 5.1].

Lemma 1.83. *Suppose that G is an étale groupoid, A is a G-C^*-algebra, D is a C^*-algebra and F is a Hilbert D-module. Suppose there is a vector space F_0 and a* *linear map* $i\colon F_0 \to F$*, along with a bilinear map* $L\colon \Gamma_c(G, s^*\mathcal{A}) \times F_0 \to F$ *satisfying the following properties, under which we say* (F, F_0, L) *is a pre-representation.*

- *the image of* L *has dense span.*
- L *respects* $*$*-algebra structure in that for each* $a_1, a_2 \in \Gamma_c(G, s^*\mathcal{A})$ *and* $f_1, f_2 \in F_0$,

$$\langle L(a_1, f_1), L(a_2, f_2)\rangle = \langle i(f_1), L(a_1^* a_2, f_2)\rangle. \tag{1.2}$$

Then there is a unique non-degenerate $*$*-representation* $\pi\colon \Gamma_c(G, s^*\mathcal{A}) \to \mathcal{L}(F)$ *such that* $\pi(a)i(f) = L(a, f)$ *and* $\pi(a)L(a', f') = L(aa', f')$ *for each* $a, a' \in \Gamma_c(G, s^*\mathcal{A})$ *and* $f, f' \in F_0$.

Proof. Once existence is established uniqueness is clear because the image of L has dense span. We wish to define $\pi\colon \Gamma_c(G, s^*\mathcal{A}) \to \mathcal{L}(F)$ by setting

$$\pi(a)L(a', f') := L(aa', f') \tag{1.3}$$

for each $a \in \Gamma_c(G, s^*\mathcal{A})$. To see that this is well-defined on the span of the image of L, we show that there is $C \geq 0$ such that $\|\sum_{i=1}^n L(aa_i, f_i)\| \leq C\|\sum_{i=1}^n L(a_i, f_i)\|$ for any $a_i \in \Gamma_c(G, s^*\mathcal{A})$ and $f_i \in F_0$. We may assume that a is supported on a bisection, so that $a^*a \in \Gamma_c(G^0, \mathcal{A})$, and we will be able to take $C = \|a\| = \|a^*a\|_\infty^{\frac{1}{2}}$. The multiplier algebra $M(A)$ acts on the left and right of $\Gamma_c(G, s^*\mathcal{A})$, and there is $b \in M(A)$ such that for each $a' \in \Gamma_c(G, s^*\mathcal{A})$ we have $a^*aa' = \|a^*a\|_\infty a' - b^*ba'$, namely $b := \sqrt{\|a^*a\|_\infty - a^*a}$. We then calculate:

$$\begin{aligned}
&\left\langle \sum_i L(aa_i', f_i'), \sum_j L(aa_j', f_j') \right\rangle \\
&= \sum_{i,j} \left\langle L(aa_i', f_i'), L(aa_j', f_j')\right\rangle \\
&= \sum_{i,j} \left\langle i(f_i'), L((a_i')^* a^* aa_j', f_j')\right\rangle \\
&= \sum_{i,j} \left\langle L(a_i', f_i'), L(a^* aa_j', f_j')\right\rangle \\
&= \sum_{i,j} \left\langle L(a_i', f_i'), L(\|a^*a\|_\infty a_j', f_j') - L(b^*ba_j', f_j')\right\rangle \\
&= \sum_{i,j} \|a^*a\|_\infty \left\langle L(a_i', f_i'), L(a_j', f_j')\right\rangle - \sum_{i,j} \left\langle i(f_i'), L((a_i')^*(b^*ba_j'), f_j')\right\rangle \\
&= \|a^*a\|_\infty \left\langle \sum_i L(a_i', f_i'), \sum_j L(a_j', f_j') \right\rangle - \left\langle \sum_i L(ba_i', f_i'), \sum_j L(ba_j', f_j') \right\rangle \\
&\leq \|a^*a\|_\infty \left\langle \sum_i L(a_i', f_i'), \sum_j L(a_j', f_j') \right\rangle
\end{aligned}$$

Therefore for each $a \in \Gamma_c(G, s^*\mathcal{A})$, $\pi(a)$ is a well-defined, bounded linear map from the span of $\operatorname{im} L$ to F, and so extends to a bounded operator on F. Furthermore, its adjoint is given by $\pi(a^*)$: this can be verified for elements in the image of L using (1.2), and extends by linearity and continuity to all of F. The fact that $\pi\colon \Gamma_c(G, s^*\mathcal{A}) \to \mathcal{L}(F)$ is a $*$-homomorphism is direct from the definition (1.3). Non-degeneracy follows from the image of L having dense span. We just need to check that $\pi(a)i(f) = L(a, f)$ for each $a \in \Gamma_c(G, s^*\mathcal{A})$ and $f \in F_0$. To do this we check against $L(a', f')$ using (1.2):

$$\begin{aligned}\langle \pi(a)i(f), L(a', f')\rangle &= \langle i(f), \pi(a^*)L(a', f')\rangle \\ &= \langle i(f), L(a^*a', f')\rangle \\ &= \langle L(a, f), L(a', f')\rangle\end{aligned}$$

As elements of the form $L(a', f')$ have dense span, this is enough to determine that $\pi(a)i(f) = L(a, f)$. □

Remark 1.84. Typically we will be in the special case where F_0 is a dense subspace of F and $\Gamma_c(G, s^*\mathcal{A})$ acts by linear maps on F_0 such that the action

- is non-degenerate: the image $\Gamma_c(G, s^*\mathcal{A}) \cdot F_0$ is dense in F_0,
- respects composition: for $a, a' \in \Gamma_c(G, s^*\mathcal{A})$ and $f \in F_0$, we have $a\cdot(a'\cdot f) = (a * a') \cdot f$,
- respects adjoints: for $a \in \Gamma_c(G, s^*\mathcal{A})$ and $f, f' \in F_0$, we have $\langle a \cdot f, f'\rangle = \langle f, a^* \cdot f'\rangle$.

When we have a proper, principal étale groupoid G with unit space X, its orbit space X/G is Hausdorff and closely related to G. This means that quotients by G are well-behaved.

Definition 1.85 (Quotient bundles and generalised invariant section spaces)**.** Let G be a proper, principal étale groupoid with unit space X. Given a right Banach G-bundle $\mathcal{A} \to X$, the quotient space $\mathcal{A}/G$ has the structure of a Banach bundle over X/G which we call the *quotient bundle*. The associated section space constructions for a G-C*-algebra A and a G-Hilbert A-module E are called the *generalised invariant subalgebra* A^G and the *generalised invariant submodule* E^G.

Proposition 1.86 (The quotient Banach bundle is a Banach bundle)**.** *Let G be a proper, principal étale groupoid with unit space X and let $p\colon \mathcal{A} \to X$ be a right Banach G-bundle. Then the quotient map $\pi\colon \mathcal{A} \to \mathcal{A}/G$ is a local homeomorphism and the quotient space $\mathcal{A}/G$ carries the structure of a Banach bundle over X/G by setting the norm of an element $[a]_G \in \mathcal{A}/G$ to be $\|a\|$.*

Proof. Let $q\colon X \to X/G$ be the quotient map and let $\overline{p}\colon \mathcal{A}/G \to X/G$ be the structure map for the quotient bundle. Because G is proper and principal, q is a local homeomorphism by Proposition 1.11, and p is also open. The structure map

$\overline{p}\colon \mathcal{A}/G \to X/G$ is therefore open because for an open set $U \subseteq \mathcal{A}/G$, we have $\overline{p}(U) = qp(\pi^{-1}(U))$.

The norm $\|[a]_G\| = \|a\|$ for $[a]_G \in \mathcal{A}/G$ is well-defined as G acts by isometries. For each $x \in X$ there is an isometric isomorphism $a \mapsto [a]_G\colon A_x \to (\mathcal{A}/G)_{[x]_G}$ which shows that each fibre of $\mathcal{A}/G \to X/G$ is a Banach space.

Local injectivity of $\pi\colon \mathcal{A} \to \mathcal{A}/G$ is inherited from $q\colon X \to X/G$. To see that π is open, let $U \subseteq \mathcal{A}$ be open. The action map $\alpha\colon \mathcal{A} \times_X G \to \mathcal{A}$ is a local homeomorphism, and so $\pi^{-1}(\pi(U)) = \alpha(U \times_X G)$ is open. Thus $\pi(U)$ is open.

The remaining conditions for $\overline{q}\colon \mathcal{A}/G \to X/G$ to be a Banach bundle are straightforwardly verified using that π is a local homeomorphism. □

The space of continuous sections $\Gamma(X/G, \mathcal{A}/G)$ can be identified with the space of G-equivariant continuous sections in $\Gamma(X, \mathcal{A})$, identifying a section $\xi\colon X \to \mathcal{A}$ with the section $[x]_G \mapsto [\xi(x)]_G$. Consider again a proper, principal étale groupoid G with unit space X, a right G-C*-algebra A with associated bundle $\mathcal{A} \to X$ and a (right) G-Hilbert A-module E with associated bundle $\mathcal{E} \to X$. Then the generalised invariant subalgebra A^G is given concretely by the set of bounded continuous sections $a \in \Gamma_b(X, \mathcal{A})$ such that:

- the section a is G-invariant in that $a(r(g)) \cdot g = a(s(g))$ for each $g \in G$,
- the map $[x]_G \mapsto \|a(x)\|\colon X/G \to \mathbb{R}_{\geq 0}$ vanishes at infinity.

The action of $C_0(X/G)$ on A^G is given for $\xi \in C_0(X/G)$ and $a \in A^G$ by $\xi \cdot a(x) = \xi([x]_G)a(x)$. For each $x \in X$, the evaluation at x map $A^G \to A_x$ induces a $*$-isomorphism $A^G_{[x]_G} \cong A_x$ of the fibre at $[x]_G$. Similarly, the generalised invariant submodule E^G is given concretely by the set of bounded continuous sections $e \in \Gamma_b(X, \mathcal{E})$ such that:

- the section e is G-invariant in that $e(r(g)) \cdot g = e(s(g))$ for each $g \in G$,
- the map $[x]_G \mapsto \|e(x)\|\colon X/G \to \mathbb{R}_{\geq 0}$ vanishes at infinity.

For each $x \in X$, the evaluation at x map $E^G \to E_x$ induces an isomorphism $E^G_{[x]_G} \cong E_x$ of the fibre at $[x]_G$.

We obtain a functor $(-)^G\colon \mathsf{C}^*\text{-}\mathsf{alg}^G \to \mathsf{C}^*\text{-}\mathsf{alg}^{X/G}$ which we may refer to as the generalised invariant subalgebra or quotient functor. For each G-C*-algebra A, an equivariant adjointable operator $T \in \mathcal{L}(E, F)$ between G-Hilbert A-modules E and F induces an adjointable operator $T^G \in \mathcal{L}(E^G, F^G)$. In fact, each adjointable operator is of this form:

Proposition 1.87 (Adjointable operators on generalised invariant submodules)**.** *Let G be a proper, principal étale groupoid with unit space X, let A be a right G-C*-algebra, let E and F be (right) G-Hilbert A-modules, and let $T \in \mathcal{L}(E^G, F^G)$ be an adjointable operator.*

Then for each $x \in X$, there is a unique adjointable operator $T_x \in \mathcal{L}(E_x, F_x)$ such that for each $e \in E^G$, $(T(e))(x) = T_x(e(x))$. Through this we can identify the set of adjointable operators $\mathcal{L}(E^G, F^G)$ with the set of G-equivariant operators in $\mathcal{L}(E, F) \cong \Gamma_b(X, \mathcal{L}(\mathcal{E}, \mathcal{F}))$. The compact operators $\mathcal{K}(E^G, F^G)$ are identified with the set of G-equivariant operators $T \in \Gamma_b(X, \mathcal{K}(\mathcal{E}, \mathcal{F}))$ such that $[x]_G \mapsto \|T_x\|: X/G \to \mathbb{R}_{\geq 0}$ vanishes at infinity.

Proof. The fibres of $T \in \mathcal{L}(E^G, F^G)$ and its adjoint over X/G define continuous maps $\mathcal{E}/G \to \mathcal{F}/G$ and $\mathcal{F}/G \to \mathcal{E}/G$. For each $x \in X$ the identification of fibres $(\mathcal{E}/G)_{[x]_G} \cong E_x$ and $(\mathcal{F}/G)_{[x]_G} \cong F_x$ enable us to construct $T_x \in \mathcal{L}(E_x, F_x)$ such that $(T(e))(x) = T_x(e(x))$ for each $e \in E^G$. This determines T_x uniquely, and ensures that the section $x \mapsto T_x: X \to \mathcal{L}(\mathcal{E}, \mathcal{F})$ is bounded and G-equivariant. To check continuity, we may check that the fibre-wise defined map $\mathcal{E} \to \mathcal{F}$ and its fibre-wise adjoint $\mathcal{F} \to \mathcal{E}$ is continuous by Remark 1.53. This may be lifted using the local homeomorphisms $\mathcal{E} \to \mathcal{E}/G$ and $\mathcal{F} \to \mathcal{F}/G$ from the continuous maps $\mathcal{E}/G \to \mathcal{F}/G$ and $\mathcal{F}/G \to \mathcal{E}/G$. Thus we have defined a map from $\mathcal{L}(E^G, F^G)$ to the G-equivariant operators in $\Gamma_b(X, \mathcal{L}(\mathcal{E}, \mathcal{F}))$. Its inverse is defined by setting $(T(e))(x) = T_x(e(x))$ for each $x \mapsto T_x \in \Gamma_b(X, \mathcal{L}(\mathcal{E}, \mathcal{F}))$ and $e \in E^G$.

The identification of the compact operators $\mathcal{K}(E^G, F^G)$ with the G-equivariant operators in $\Gamma_b(X, \mathcal{K}(\mathcal{E}, \mathcal{F}))$ which vanish at infinity follows from the fibre-wise identification $\mathcal{K}(\mathcal{E}/G, \mathcal{F}/G) \cong \mathcal{K}(\mathcal{E}, \mathcal{F})/G$ being an isomorphism. We need only check that it is continuous in one direction by Proposition 1.49. For $e \in E^G$ and $f \in F^G$, the continuous section $[x]_G \mapsto \Theta_{[f(x)]_G, [e(x)]_G}: X/G \to \mathcal{K}(\mathcal{E}/G, \mathcal{F}/G)$ is sent to the section $[x]_G \mapsto [\Theta_{f(x), e(x)}]_G: X/G \to \mathcal{K}(\mathcal{E}, \mathcal{F})/G$. This is continuous by continuity of the maps $e: X \to \mathcal{E}$, $f: X \to \mathcal{F}$ and $\mathcal{F} \times_X \mathcal{E} \to \mathcal{K}(\mathcal{E}, \mathcal{F})$ (see Proposition 1.51). By Proposition 1.45, $\mathcal{K}(\mathcal{E}/G, \mathcal{F}/G) \to \mathcal{K}(\mathcal{E}, \mathcal{F})/G$ is continuous and therefore an isomorphism of Banach bundles. □

1.4. **Equivariant correspondence categories of C*-algebras.** The correspondence categories are categories of C*-algebras with morphisms that are more flexible than $*$-homomorphisms. These morphisms are known as *correspondences*, and can be thought of as the data needed to induce a representation of one C*-algebra from another. They are the building blocks of Kasparov's KK-theory, and can be equipped with étale groupoid actions to form the morphisms of the equivariant correspondence categories.

A *correspondence* of C*-algebras from A to B is a pair (E, φ), where E is a Hilbert B-module and $\varphi: A \to \mathcal{L}(E)$ is a $*$-homomorphism called the *structure map* that is non-degenerate in the sense that $\overline{\varphi(A)E} = E$. Unless there is a specific need to refer to the structure map φ, we omit it from the notation, writing $a \cdot e$ instead of $\varphi(a)(e)$ for $a \in A$ and $e \in E$. We may refer to just the underlying Hilbert module, writing $E: A \to B$ for the correspondence (E, φ). We may also call E a

C-correspondence*. We say that E is a *proper* correspondence if A acts by compact operators in the sense that $\varphi(A) \subseteq \mathcal{K}(E)$.

Definition 1.88 ($C_0(X)$-correspondence)**.** A C*-correspondence $(E, \varphi)\colon A \to B$ between $C_0(X)$-algebras A and B is a $C_0(X)$*-correspondence* if $\varphi\colon A \to \mathcal{L}(E)$ is $C_0(X)$-linear in the sense that $\xi\varphi(a) = \varphi(\xi a)$ for each $\xi \in C_0(X)$ and $a \in A$. This induces a C*-correspondence $(E_x, \varphi_x)\colon A_x \to B_x$ for each $x \in X$ which we call the *fibre* of (E, φ) at x.

Proposition 1.89. *Let $(E, \varphi)\colon A \to B$ be a $C_0(X)$-correspondence. Then the map $\mathcal{A} \to \mathcal{L}(\mathcal{E})$ given by $\varphi_x\colon A_x \to \mathcal{L}(E_x)$ at $x \in X$ is continuous.*

Proof. Let p denote the structure maps of the Banach bundles $\mathcal{A} \to X$ and $\mathcal{E} \to X$. By Proposition 1.55, we need only check that for each $e \in E$, the maps $a \mapsto \varphi_{p(a)}(a)(e_{p(a)})\colon \mathcal{A} \to \mathcal{E}$ and $a \mapsto \varphi_{p(a)}(a)^*(e_{p(a)})\colon \mathcal{A} \to \mathcal{E}$ are continuous. This follows from Proposition 1.45 because for each $a \in A$, the section $x \mapsto \varphi_x(a_x)(e_x)$ is equal to the element $\varphi(a)(e) \in E$, and similarly for the adjoint a^*. □

Definition 1.90 (Equivariant correspondence)**.** Let G be an étale groupoid with unit space X and let A and B be G-C*-algebras. A G*-equivariant correspondence* $(E, \varphi)\colon A \to B$ is a $C_0(X)$-correspondence from A to B with an action of G on E such that E is a G-Hilbert B-module and $\varphi\colon A \to \mathcal{L}(E)$ is G-equivariant in the sense that $g \cdot (a \cdot e) = (g \cdot a) \cdot (g \cdot e)$ for each $g \in G$, $a \in A_{s(g)}$ and $e \in E_{s(g)}$.

In order to discuss correspondence categories of C*-algebras we need to discuss the composition of C*-correspondences, which involves the interior tensor product of Hilbert modules. Given a C*-correspondence $F\colon B \to C$ and a Hilbert B-module E, the *interior tensor product* $E \otimes_B F$ is a Hilbert C-module with the following properties. There is a bilinear map $(e, f) \mapsto e \otimes f\colon E \times F \to E \otimes_B F$ whose image has dense span. The inner product is given on these simple tensors by

$$\langle e_1 \otimes f_1, e_2 \otimes f_2\rangle = \langle f_1, \langle e_1, e_2\rangle \cdot f_2\rangle.$$

An adjointable operator $T \in \mathcal{L}(E)$ can act on $E \otimes_B F$ by sending $e \otimes f$ to $Te \otimes f$, from which we obtain a $*$-homomorphism $T \mapsto T \otimes 1\colon \mathcal{L}(E) \to \mathcal{L}(E \otimes_B F)$. Similarly, for any other Hilbert B-module E', there is a $*$-homomorphism $T \mapsto T \otimes 1\colon \mathcal{L}(E, E') \to \mathcal{L}(E \otimes_B F, E' \otimes_B F)$. We refer to Lance's book [44] for further details on these constructions. We can then define the *composition* $F \circ E$ of correspondences $E\colon A \to B$ and $F\colon B \to C$ as the interior tensor product $E \otimes_B F$ equipped with the non-degenerate action of A given by $a \cdot (e \otimes f) = (a \cdot e) \otimes f$ for $a \in A$ and $(e, f) \in E \times F$. If F is a proper correspondence then $- \otimes 1\colon \mathcal{L}(E) \to \mathcal{L}(E \otimes_B F)$ maps $\mathcal{K}(E)$ into $\mathcal{K}(E \otimes_B F)$, and so the composition of proper correspondences is proper.

Any C*-algebra A may be considered as a Hilbert module over itself, with the inner product $\langle a, b\rangle = a^*b$ and right module action given by right multiplication. With the left action of left multiplication, this turns A into a correspondence $A\colon A \to A$ called the *identity correspondence*. Identity correspondences act as identities for composition up to canonical isomorphisms: given a correspondence $E\colon A \to B$, we have isomorphisms $e \otimes b \mapsto e \cdot b\colon E \otimes_B B \cong E$ and $a \otimes e \mapsto a \cdot e\colon A \otimes_A E \cong E$. Composition is also associative up to canonical isomorphisms: if $E_1\colon A \to B$, $E_2\colon B \to C$ and $E_3\colon C \to D$ are correspondences, we have an isomorphism

$$(e_1 \otimes e_2) \otimes e_3 \mapsto e_1 \otimes (e_2 \otimes e_3)\colon (E_1 \otimes_B E_2) \otimes_C E_3 \cong E_1 \otimes_B (E_2 \otimes_C E_3).$$

The *correspondence category* Corr is the category of C*-algebras with morphisms given by (isomorphism classes of) C*-correspondences, and composition given as above. The *proper correspondence category* Corr_p is the subcategory whose morphisms are (isomorphism classes of) proper correspondences. We may now turn our attention to the composition of $C_0(X)$-correspondences and equivariant correspondences.

Proposition 1.91 (Interior tensor product bundles). *Let A and B be $C_0(X)$-algebras, let E be a Hilbert A-module and let $F\colon A \to B$ be a $C_0(X)$-correspondence and consider the interior tensor product $E \otimes_A F$. Let $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F} \to X$ be the corresponding Hilbert $\mathcal{B}$-bundle.*

Then the fibre $(E \otimes_A F)_x$ at $x \in X$ of the interior tensor product can be identified with $E_x \otimes_{A_x} F_x$ via the map $(e \otimes f)_x \mapsto e_x \otimes f_x$ for $e \in E$ and $f \in F$. If $\Gamma_1 \subseteq \Gamma(X, \mathcal{E})$ and $\Gamma_2 \subseteq \Gamma(X, \mathcal{F})$ are sufficient, then

$$\Gamma_1 \otimes \Gamma_2 := \{e \otimes f \mid e \in \Gamma_1,\ f \in \Gamma_2\}$$

is sufficient for $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}$, where $e \otimes f\colon X \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}$ is defined by $x \mapsto e(x) \otimes f(x)$. This means that for any Banach bundle $\mathcal{B} \to X$, a map of Banach bundles $\varphi\colon \mathcal{E} \otimes_{\mathcal{A}} \mathcal{F} \to \mathcal{B}$ is continuous if and only if its composition $(e, f) \mapsto \varphi(e \otimes f)\colon \mathcal{E} \times_X \mathcal{F} \to \mathcal{B}$ with the tensor product map is continuous.

Proof. Elements of the form $(e \otimes f)_x$ for $e \in E$ and $f \in F$ have dense span in $(E \otimes_A F)_x$. The assignment $(e \otimes f)_x \mapsto e_x \otimes f_x$ preserves the B_x-valued inner product, and is therefore well-defined and extends to a map $(E \otimes_A F)_x \to E_x \otimes_{A_x} F_x$ which preserves the inner product. This is an isomorphism of Hilbert B_x-modules because elements of the form $e_x \otimes f_x$ have dense span in $E_x \otimes_{A_x} F_x$.

The tensor product map $(e, f) \mapsto e \otimes f\colon \mathcal{E} \times_X \mathcal{F} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}$ is continuous by Proposition 1.46. Each element of $\Gamma_1 \otimes \Gamma_2$ is therefore continuous. For each $x \in X$, elements of the form $e(x) \otimes f(x)$ over $e \in \Gamma_1$ and $f \in \Gamma_2$ have dense span in $E_x \otimes_{A_x} F_x$, so $\Gamma_1 \otimes \Gamma_2$ is sufficient. □

Given an étale groupoid G with unit space X with G-C*-algebras A and B, a G-Hilbert A-module E and a G-equivariant correspondence $F\colon A \to B$, we may form a diagonal action of G on the interior tensor product $E \otimes_A F$. This is given for

$g \in G$ on simple tensors $e \otimes f \in E_{s(g)} \otimes_{B_{s(g)}} F_{s(g)}$ by $g \cdot (e \otimes f) = (g \cdot e) \otimes (g \cdot f) \in E_{r(g)} \otimes_{B_{r(g)}} F_{r(g)}$. By Remark 1.70 it is enough to check continuity of the map $g \mapsto g \cdot e_{s(g)} \otimes g \cdot f_{s(g)} \colon G \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}$ for each $e \in E$ and $f \in F$. This follows from the continuity of the tensor product map $\mathcal{E} \times_X \mathcal{F} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}$. This enables us to define the composition of G-equivariant correspondences.

Definition 1.92 (Composition of equivariant correspondences)**.** Let A, B and C be G-C*-algebras, and let $E\colon A \to B$ and $F\colon B \to C$ be G-equivariant correspondences. Then the diagonal action of G on the interior tensor product $E \otimes_B F$ gives $F \circ E\colon A \to C$ the structure of a G-equivariant correspondence. Just as in the non-equivariant setting, composition of equivariant correspondences is associative up to canonical isomorphisms and identity correspondences act as identities up to canonical isomorphisms.

The *G-equivariant correspondence category* Corr^G of an étale groupoid G is the category of G-C*-algebras with morphisms given by (isomorphism classes of) G-equivariant correspondences. The G-equivariant *proper* correspondence category Corr^G_p is the subcategory whose morphisms are (isomorphism classes of) proper G-equivariant correspondences. Any equivariant $*$-homomorphism can be viewed as an equivariant correspondence, giving us a large class of examples.

Definition 1.93 (Equivariant $*$-homomorphisms as equivariant correspondences)**.** A G-equivariant $*$-homomorphism $\varphi\colon A \to B$ induces a G-equivariant correspondence $\mathsf{Corr}^G(\varphi) := \overline{\varphi(A)B}\colon A \to B$. This defines a functor $\mathsf{Corr}^G\colon \mathsf{C^*\text{-}alg}^G \to \mathsf{Corr}^G$.

Most of the constructions we have discussed can be viewed as functors between correspondences categories.

Proposition 1.94 (Pullback functor)**.** *Let $g\colon Y \to X$ be a continuous map. The pullback construction for C*-algebras and Hilbert modules extends to a pullback functor $g^*\colon \mathsf{Corr}^X \to \mathsf{Corr}^Y$ of correspondence categories. Let $(E, \varphi)\colon A \to B$ be a $C_0(X)$-correspondence and let $g^*\varphi\colon g^*A \to \mathcal{L}(g^*E)$ send $a \in g^*A$ to the section $y \mapsto \varphi_{g(y)}(a_y)\colon Y \to \mathcal{L}(g^*\mathcal{E})$. The pullback functor maps $(E, \varphi)\colon A \to B$ to the $C_0(Y)$-correspondence $(g^*E, g^*\varphi)\colon g^*A \to g^*B$.*

Proof. We first note that the section $y \mapsto \varphi_{g(y)}(a_y)\colon Y \to \mathcal{L}(g^*\mathcal{E})$ is continuous by Propositions 1.63 and 1.89. The structure map $g^*\varphi$ is therefore well-defined, and is clearly $C_0(Y)$-linear.

We then need to check that $g^*\colon \mathsf{Corr}^X \to \mathsf{Corr}^Y$ is functorial, so let $F\colon B \to C$ be another $C_0(X)$-correspondence. The fibre-wise identification $g^*\mathcal{E} \otimes_{g^*\mathcal{B}} g^*\mathcal{F} \to g^*(\mathcal{E} \otimes_{\mathcal{B}} \mathcal{F})$ is continuous by Proposition 1.91 because the composition $g^*\mathcal{E} \times_Y g^*\mathcal{F} \to g^*(\mathcal{E} \otimes_{\mathcal{B}} \mathcal{F})$ with the tensor product map is continuous. This provides an isomorphism of Hilbert $g^*\mathcal{C}$-bundles which identifies $g^*F \circ g^*E$ with $g^*(F \circ E)$. □

Proposition 1.95 (Pushforward functor)**.** *Let $f\colon Y \to X$ be a continuous map. The pushforward construction for C*-algebras and Hilbert modules extends to a pushforward functor $f_*\colon \mathsf{Corr}^Y \to \mathsf{Corr}^X$ of correspondence categories. The pushforward functor maps a $C_0(Y)$-correspondence $(E, \varphi)\colon A \to B$ is mapped to the $C_0(X)$-correspondence $(f_*E, f_*\varphi)\colon f_*A \to f_*B$, where $f_*\varphi\colon f_*A \to \mathcal{L}(f_*E)$ is given by $\varphi\colon A \to \mathcal{L}(E)$.*

Proof. The structure map $f_*\varphi\colon f_*A \to \mathcal{L}(f_*E)$ is automatically $C_0(X)$-linear, so $(f_*E, f_*\varphi)$ is a $C_0(X)$-correspondence. Functoriality is automatic as the underlying C*-algebras and Hilbert modules do not change. ☐

Proposition 1.96 (Generalised invariant subalgebra functor)**.** *Let G be a proper, principal étale groupoid with unit space X. The generalised invariant subspace construction for C*-algebras and Hilbert modules extends to a functor $(-)^G\colon \mathsf{Corr}^G \to \mathsf{Corr}^{X/G}$ of correspondence categories. A G-equivariant correspondence $E\colon A \to B$ is mapped to the $C_0(X/G)$-correspondence $E^G\colon A^G \to B^G$. The structure map $A^G \to \mathcal{L}(E^G)$ is given for $a \in A^G \subseteq \Gamma_b(X, \mathcal{A})$ and $e \in E^G \subseteq \Gamma_b(X, \mathcal{E})$ by pointwise multiplication: for $x \in X$, we have $(a \cdot e)(x) = a(x) \cdot e(x)$.*

Proof. The structure map is clearly $C_0(X/G)$-linear, which makes $E^G\colon A^G \to B^G$ a $C_0(X/G)$-correspondence. In order to prove that $(-)^G\colon \mathsf{Corr}^G \to \mathsf{Corr}^{X/G}$ is functorial, consider G-equivariant correspondences $E\colon A \to B$ and $F\colon B \to C$. There is a G-equivariant isomorphism $E^G \otimes_{B^G} F^G \to (E \otimes_B F)^G$ which maps a simple tensor $e \otimes f$ to the section $x \mapsto e_x \otimes f_x\colon X \to \mathcal{E} \otimes_{\mathcal{B}} \mathcal{F}$. This provides an identification $F^G \circ E^G \cong (F \circ E)^G$. ☐

Proposition 1.97 (Crossed product functor)**.** *Let G be an étale groupoid. The crossed product construction for C*-algebras and Hilbert modules extends to a functor $G \ltimes -\colon \mathsf{Corr}^G \to \mathsf{Corr}$ of correspondence categories. A G-equivariant correspondence $E\colon A \to B$ is mapped to the C*-correspondence $G \ltimes E\colon G \ltimes A \to G \ltimes B$. The action $G \ltimes A \curvearrowright G \ltimes E$ is given for $\xi \in \Gamma_c(G, s^*\mathcal{A}) \subseteq G \ltimes A$ and $\eta \in \Gamma_c(G, s^*\mathcal{E}) \subseteq G \ltimes E$ by*

$$\xi \cdot \eta\colon g \mapsto \sum_{g_1 g_2 = g} (g_2^{-1} \cdot \xi(g_1)) \cdot \eta(g_2).$$

Furthermore, if E is proper then $G \ltimes E$ is also proper.

Proof. We first note that the action map $\Gamma_c(G, s^*\mathcal{A}) \times \Gamma_c(G, s^*\mathcal{E}) \to \Gamma_c(G, s^*\mathcal{E})$ has dense range, as for any open bisection $U \subseteq G$, we may consider the restriction $\Gamma_c(G^0, \mathcal{A}) \curvearrowright \Gamma_c(U, s^*\mathcal{E})$ for which dense range follows from the non-degeneracy of $A \curvearrowright E$. By Lemma 1.83, the action $\Gamma_c(G, s^*\mathcal{A}) \curvearrowright \Gamma_c(G, s^*\mathcal{E})$ extends to an action $\Gamma_c(G, s^*\mathcal{A}) \curvearrowright G \ltimes E$, which by the universal property of the universal crossed product extends to an action $G \ltimes A \curvearrowright G \ltimes E$. To see that this respects composition of correspondences $E\colon A \to B$ and $F\colon B \to C$, we consider the map $\eta, \nu \mapsto \eta \otimes$

$\nu\colon \Gamma_c(G, s^*\mathcal{E}) \times \Gamma_c(G, s^*\mathcal{F}) \to \Gamma_c(G, s^*(\mathcal{E} \otimes_{\mathcal{B}} \mathcal{F}))$. This map extends to a unitary isomorphism $G \ltimes E \otimes_{G \ltimes B} G \ltimes F \cong G \ltimes (E \otimes_B F)$.

If E is proper the action $G \ltimes A \curvearrowright G \ltimes E$ factors through the $*$-homomorphism $G \ltimes A \to G \ltimes \mathcal{K}(E)$. We therefore just need to check that $\mathcal{K}(E)$ acts by compact operators on $G \ltimes E$. This follows from considering $\xi, \eta \in \Gamma_c(G^0, \mathcal{E})$, which give us elements $\Theta_{\xi,\eta} \in \mathcal{K}(E)$ with dense span. The operator $\Theta_{\xi,\eta}$ acts as the operator $\Theta_{\xi,\eta} \in \mathcal{K}(G \ltimes E)$, where we consider ξ and η as elements of $\Gamma_c(G, s^*\mathcal{E}) \subseteq G \ltimes E$ supported on G^0. □

Corollary 1.98. *Let A be a G-C^*-algebra and let E be a G-Hilbert A-module. There is a non-degenerate $*$-homomorphism $\beta\colon G \ltimes \mathcal{K}(E) \to \mathcal{K}(G \ltimes E)$ which is given on the dense subspaces $\Gamma_c(G, s^*\mathcal{K}(\mathcal{E})) \subseteq G \ltimes \mathcal{K}(E)$ and $\Gamma_c(G, s^*\mathcal{E}) \subseteq G \ltimes E$ by*

$$\begin{aligned} \Gamma_c(G, s^*\mathcal{K}(\mathcal{E})) \times \Gamma_c(G, s^*\mathcal{E}) &\to \Gamma_c(G, s^*\mathcal{E}) \\ (\xi, \eta) &\mapsto \beta(\xi)(\eta) \\ g &\mapsto \sum_{g_1 g_2 = g} (g_2^{-1} \cdot \xi(g_1)) \cdot \eta(g_2). \end{aligned}$$

Proof. The correspondence $E\colon \mathcal{K}(E) \to A$ is proper and G-equivariant, so applying the crossed product functor gives us the proper correspondence $G \ltimes E\colon G \ltimes \mathcal{K}(E) \to G \ltimes A$. The structure map for this correspondence is β. □

The reduced crossed product also gives us a functor of correspondence categories.

Proposition 1.99 (Reduced crossed product functor)**.** *Let G be an étale groupoid. The reduced crossed product construction for C^*-algebras and Hilbert modules extends to a functor $G \ltimes_r -\colon \mathsf{Corr}^G \to \mathsf{Corr}$ of correspondence categories. A G-equivariant correspondence $E\colon A \to B$ is mapped to $G \ltimes_r E\colon G \ltimes_r A \to G \ltimes_r B$. The action $G \ltimes_r A \curvearrowright G \ltimes_r E$ is given for $\xi \in \Gamma_c(G, s^*\mathcal{A}) \subseteq G \ltimes_r A$ and $\eta \in \Gamma_c(G, s^*\mathcal{E}) \subseteq G \ltimes_r E$ by*

$$\xi \cdot \eta\colon g \mapsto \sum_{g_1 g_2 = g} (g_2^{-1} \cdot \xi(g_1)) \cdot \eta(g_2).$$

Furthermore, if E is proper then $G \ltimes_r E$ is also proper.

Proof. The proof largely follows the same ideas as for the previous proposition, but we need to additionally argue that the action $\Gamma_c(G, s^*\mathcal{A}) \curvearrowright G \ltimes_r E$ extends to an action of $G \ltimes_r A$. This follows from the commutativity of the following diagram.

$$\begin{tikzcd}
\Gamma_c(G, s^*\mathcal{A}) \arrow[r] \arrow[dd] & \mathcal{L}(G \ltimes_r E) \arrow[r, hook] & \mathcal{L}(G \ltimes_r E \otimes_{G \ltimes_r B} L^2(G, B)) \arrow[d, "\cong"] \\
 & & \mathcal{L}(L^2(G, E)) \arrow[d, "\cong"] \\
G \ltimes_r A \arrow[r] \arrow[uur, dashed] & \mathcal{L}(L^2(G, A)) \arrow[r] & \mathcal{L}(L^2(G, A) \otimes_A E)
\end{tikzcd}$$

□

An important feature of C*-correspondences is that proper correspondences induce a map in K-theory. Typically this is justified through Kasparov's KK-theory by constructing a Kasparov cycle from the proper correspondence, which then induces a map in K-theory. However, this is highly non-constructive and relies upon certain countability assumptions which are not strictly necessary to obtain a map in K-theory. We instead follow Exel's approach in [26] to the problem of building an isomorphism in K-theory from a Morita equivalence, which was also motivated by the non-constructiveness and reliance on σ-unitality of previously known techniques [12]. Morita equivalences may be viewed as the invertible correspondences of C*-algebras, and are necessarily proper.

We will use Exel's Fredholm picture of K-theory [26] for a C*-algebra A. This is highly related to Kasparov's picture of $K_0(A)$ as the Kasparov group $\mathrm{KK}(\mathbb{C}, A)$. An element in $K_0(A)$ is given by the *index* $\operatorname{ind} T$ [26, Definition 3.10] of an A-Fredholm operator $T \in \mathcal{L}(M, N)$ [26, Definition 3.1] between Hilbert A-modules M and N. An A-Fredholm operator is an adjointable operator such that there exists $S \in \mathcal{L}(N, M)$ with $1 - ST$ and $1 - TS$ compact. The definition of the index $\operatorname{ind} T \in K_0(A)$ is fairly involved, but every element of $K_0(A)$ is realised as the index of some A-Fredholm operator T such that $TT^* - 1$ and $T^*T - 1$ are compact [26, Proposition 3.14][3]. Furthermore, A-Fredholm operators T_1 and T_2 have the same index if and only if there is some n such that

$$T_1 \oplus T_2^* \oplus 1_n \in \mathcal{L}(M_1 \oplus N_2 \oplus A^n, N_1 \oplus M_2 \oplus A^n)$$

can be written as the sum of an invertible operator and a compact operator [26, Proposition 3.16]. Now let $E \colon A \to B$ be a proper correspondence. For any A-Fredholm operator $T \in \mathcal{L}(M, N)$, the operator $T \otimes 1 \in \mathcal{L}(M \otimes_A E, N \otimes_A E)$ is B-Fredholm. The mapping $K_0(E) \colon K_0(A) \to K_0(B)$ defined by

$$\operatorname{ind} T \mapsto \operatorname{ind} T \otimes 1 \colon K_0(A) \to K_0(B)$$

is a well-defined group homomorphism. This is functorial: if E is the identity correspondence then $K_0(E)$ is the identity, and if E and F are composable proper correspondences then $K_0(F \circ E) = K_0(F) \circ K_0(E)$. To get a map in K_1, we use

[3] In the proof given, S is the adjoint of T.

the identification $K_1(A) = K_0(SA)$, where $SA := C_0(\mathbb{R}, A)$ is the *suspension* of A. Then $SE := C_0(\mathbb{R}, E)\colon SA \to SB$ is a proper correspondence in the obvious way and we define $K_1(E)\colon K_1(A) \to K_1(B)$ to be $K_0(SE)\colon K_0(SA) \to K_0(SB)$. Furthermore, this extends the functoriality of K_* with respect to $*$-homomorphisms. We obtain a commutative diagram of functors.

$$\begin{tikzcd} \mathsf{C^*\text{-}alg} \arrow[r, "\mathsf{Corr}"] \arrow[dr, "K_*"'] & \mathsf{Corr}_p \arrow[d, "K_*"] \\ & \mathsf{Ab}_* \end{tikzcd}$$

2. Induction from groupoid correspondences in correspondence categories

On the path to understanding how a correspondence $\Omega\colon G \to H$ of étale groupoids connects the equivariant Kasparov categories KK^G and KK^H, the first step is to understand the situation at the level of the equivariant correspondence categories Corr^G and Corr^H. We introduce the induction functor $\mathrm{Ind}_\Omega\colon \mathsf{Corr}^H \to \mathsf{Corr}^G$ and explain how an induced algebra $\mathrm{Ind}_\Omega B$ relates to the original algebra B via the evaluation correspondence.

2.1. **Induced algebras and modules.** At the end of Section 1.1 we discussed how a correspondence $\Omega\colon G \to H$ of étale groupoids induces a G-space $\mathrm{Ind}_\Omega Y := \Omega \times_H Y$ from an H-space Y. We now construct induced C*-algebras and Hilbert modules; from an H-C*-algebra B and an H-Hilbert B-module E we construct a G-C*-algebra $\mathrm{Ind}_\Omega B$ and a G-Hilbert $\mathrm{Ind}_\Omega B$-module $\mathrm{Ind}_\Omega E$. This is based on the subgroupoid induction functor in [8], which already contains most of the key ingredients for our construction. We also describe how this construction respects the composition in the category of groupoid correspondences.

Definition 2.1 (Induced algebra and module)**.** Let $\Omega\colon G \to H$ be a correspondence of étale groupoids, let B be an H-C*-algebra with associated bundle $\mathcal{B}$, and let E be an H-Hilbert B-module with associated bundle $\mathcal{E}$. We define the *induced G-C*-algebra* $\mathrm{Ind}_\Omega B$, its bundle $\mathrm{Ind}_\Omega \mathcal{B}$, the *induced G-Hilbert* $\mathrm{Ind}_\Omega B$*-module* $\mathrm{Ind}_\Omega E$ and its bundle $\mathrm{Ind}_\Omega \mathcal{E}$ to be:

$$\mathrm{Ind}_\Omega B := \overline{\rho}_*(\sigma^* B)^{\Omega\rtimes H}, \qquad \mathrm{Ind}_\Omega \mathcal{B} := \overline{\rho}_*((\sigma^*\mathcal{B})/H)$$
$$\mathrm{Ind}_\Omega E := \overline{\rho}_*(\sigma^* E)^{\Omega\rtimes H}, \qquad \mathrm{Ind}_\Omega \mathcal{E} := \overline{\rho}_*((\sigma^*\mathcal{E})/H)$$

We will usually view $\mathrm{Ind}_\Omega B$ and $\mathrm{Ind}_\Omega E$ concretely as the spaces of bounded continuous sections ξ in $\Gamma_b(\Omega, \sigma^*\mathcal{B})$ or $\Gamma_b(\Omega, \sigma^*\mathcal{E})$ such that

- for any $\omega \in \Omega$ and $h \in H^{\sigma(\omega)}$ we have $\xi(\omega \cdot h) = h^{-1} \cdot \xi(\omega)$,
- the function $[\omega]_H \mapsto \|\xi(\omega)\|\colon \Omega/H \to \mathbb{R}$ vanishes at infinity.

These algebras and modules are fibred over G^0, with fibres $(\mathrm{Ind}_\Omega B)_x$ and $(\mathrm{Ind}_\Omega E)_x$ at $x \in G^0$ given concretely by the spaces of bounded continuous sections ξ in $\Gamma_b(\Omega^x, \sigma^*\mathcal{B})$ or $\Gamma_b(\Omega^x, \sigma^*\mathcal{E})$ such that

- for any $\omega \in \Omega^x$ and $h \in H^{\sigma(\omega)}$ we have $\xi(\omega \cdot h) = h^{-1} \cdot \xi(\omega)$,
- the function $[\omega]_H \mapsto \|\xi(\omega)\|\colon \Omega^x/H \to \mathbb{R}$ vanishes at infinity.

The left actions $G \curvearrowright \mathrm{Ind}_\Omega B$ and $G \curvearrowright \mathrm{Ind}_\Omega E$ are induced by the action $G \curvearrowright \Omega$. Concretely, for each $g \in G$ we have the following maps.

$$\Gamma_b(\Omega^{s(g)}, \sigma^*\mathcal{B}) \supseteq (\mathrm{Ind}_\Omega B)_{s(g)} \to (\mathrm{Ind}_\Omega B)_{r(g)} \subseteq \Gamma_b(\Omega^{r(g)}, \sigma^*\mathcal{B})$$
$$\xi \mapsto g \cdot \xi$$

$$\begin{aligned} \omega &\mapsto \xi(g^{-1}\cdot\omega) \\ \Gamma_b(\Omega^{s(g)}, \sigma^*\mathcal{E}) \supseteq (\operatorname{Ind}_\Omega E)_{s(g)} &\to (\operatorname{Ind}_\Omega E)_{r(g)} \subseteq \Gamma_b(\Omega^{r(g)}, \sigma^*\mathcal{E}) \\ \xi &\mapsto g\cdot\xi \\ \omega &\mapsto \xi(g^{-1}\cdot\omega) \end{aligned}$$

By Remark 1.70, to see that the action map $G \times_{G^0} \operatorname{Ind}_\Omega \mathcal{B} \to \operatorname{Ind}_\Omega \mathcal{B}$ is continuous, we need only check that for each $\xi \in \operatorname{Ind}_\Omega B$, the map $g \mapsto g\cdot\xi_{s(g)} \colon G \to \operatorname{Ind}_\Omega \mathcal{B}$ is continuous. This can be checked using the following lemma:

Lemma 2.2 (Continuity of maps into induced bundles)**.** *Let $\Omega\colon G \to H$ be a correspondence of étale groupoids, let B be an H-C*-algebra and consider the induced bundle $p\colon \operatorname{Ind}_\Omega \mathcal{B} \to G^0$. A map $\xi\colon Z \to \operatorname{Ind}_\Omega \mathcal{B}$ from a locally compact Hausdorff space Z is continuous if and only if*

- *the composition $g = p\circ\xi\colon Z \to G^0$ is continuous,*
- *and the map $\tilde{\xi}\colon (z,\omega) \mapsto \gamma(z)\xi(z)(\omega)\colon Z \times_{G^0} \Omega \to \mathcal{B}$ is c_0 with respect to $Z \times_{G^0} \Omega/H$ for any $\gamma \in C_c(Z)$.*

Proof. This follows from the description of continuous maps into pushforward bundles in Proposition 1.68 and the H-equivariance of $\tilde{\xi}$. □

For each $\xi \in \operatorname{Ind}_\Omega B$ the map $(g,\omega) \mapsto g\cdot\xi(g^{-1}\cdot\omega)\colon G \times_{r,G^0,\rho} \Omega \to \mathcal{B}$ is continuous. For each $\gamma \in C_c(G)$ and $\epsilon > 0$, consider the set $K = \{\omega \in \Omega \mid \|\gamma\|_\infty \|\xi(\omega)\| \geq \epsilon\}$ which is the pre-image of a compact set in Ω/H. Then $(\operatorname{supp}\gamma)^{-1}\cdot K \subseteq \Omega$ is also the pre-image of a compact set in Ω/H. We have $\|g\cdot\xi(g^{-1}\cdot\omega)\| < \epsilon$ for $(g,\omega) \in G \times_{G^0} \Omega$ outside the set $\operatorname{supp}\gamma \times_{G^0} (\operatorname{supp}\gamma)^{-1}\cdot K$ which is the pre-image of a compact set in $G \times_{G^0} \Omega/H$. The map $(g,\omega) \mapsto g\cdot\xi(g^{-1}\cdot\omega)\colon G \times_{r,G^0,\rho} \Omega \to \mathcal{B}$ is therefore c_0 with respect to Ω/H. We can conclude that $g \mapsto g\cdot\xi_{s(g)}\colon G \to \operatorname{Ind}_\Omega \mathcal{B}$ is continuous, and so the action map $G \curvearrowright \operatorname{Ind}_\Omega \mathcal{B}$ is continuous. Similarly, the action map on $\operatorname{Ind}_\Omega \mathcal{E}$ is continuous.

Remark 2.3 (Density in induced algebras and modules)*.* Let $\Omega\colon G \to H$ be a correspondence of étale groupoids, and let B be an H-C*-algebra. A subset $\Gamma \subseteq \operatorname{Ind}_\Omega B$ closed under the action of $C_0(\Omega/H)$ has dense span if and only if $\{\xi(\omega) \mid \xi \in \Gamma\}$ has dense span in $B_{\sigma(\omega)}$ for each $\omega \in \Omega$. This follows from Proposition 1.48. The same also holds for Hilbert modules.

We typically think of the elements in induced algebras and modules as continuous H-equivariant sections defined on the groupoid correspondence $\Omega\colon G \to H$. We may also identify adjointable operators between induced Hilbert modules with continuous H-equivariant sections of adjointable operators, which act pointwise on the sections representing the elements of the induced module. This also holds for the fibre of an adjointable operator at $x \in G^0$, which is identified with a section of operators defined on Ω^x:

Proposition 2.4 (Adjointable operators on induced modules)**.** *Let $\Omega\colon G \to H$ be a groupoid correspondence, let B be an H-C*-algebra and consider H-Hilbert B-modules E and F.*

For each adjointable operator $T \in \mathcal{L}(\operatorname{Ind}_\Omega E, \operatorname{Ind}_\Omega F)$ and each $\omega \in \Omega$, there is a unique operator $T_\omega \in \mathcal{L}(E_{\sigma(\omega)}, F_{\sigma(\omega)})$ such that $(T\xi)(\omega) = T_\omega \xi(\omega)$ for each $\xi \in \operatorname{Ind}_\Omega E$. The section $\omega \mapsto T_\omega\colon \Omega \to \sigma^*\mathcal{L}(\mathcal{E}, \mathcal{F})$ is strictly continuous, bounded and H-equivariant. The map sending T to the section $\omega \mapsto T_\omega$ defines an isomorphism from $\mathcal{L}(\operatorname{Ind}_\Omega E, \operatorname{Ind}_\Omega F)$ to the subspace of H-equivariant sections in $\Gamma_b(\Omega, \sigma^*\mathcal{L}(\mathcal{E}, \mathcal{F}))$. The operator T is compact if and only if the section $\omega \mapsto T_\omega$ is an element of $\Gamma_b(\Omega, \sigma^*\mathcal{K}(\mathcal{E}, \mathcal{F}))$ and the map $[\omega]_H \mapsto \|T_\omega\|\colon \Omega/H \to \mathbb{R}_{\geq 0}$ vanishes at infinity.

Similarly, for each $x \in G^0$, we identify the space $\mathcal{L}((\operatorname{Ind}_\Omega E)_x, (\operatorname{Ind}_\Omega F)_x)$ of adjointable operators with the space of H-equivariant sections in $\Gamma_b(\Omega^x, \sigma^*\mathcal{L}(\mathcal{E}, \mathcal{F}))$, which operate pointwise over Ω^x on the elements of $(\operatorname{Ind}_\Omega E)_x$ considered as sections in $\Gamma_b(\Omega^x, \sigma^*\mathcal{E})$. The fibre S_ω of an operator $S \in \mathcal{L}((\operatorname{Ind}_\Omega E)_x, (\operatorname{Ind}_\Omega F)_x)$ at $\omega \in \Omega^x$ is the unique operator satisfying $(S\xi)(\omega) = S_\omega(\xi(\omega))$ for each $\xi \in (\operatorname{Ind}_\Omega E)_x$. The operator S is compact if and only if the section $\omega \mapsto S_\omega$ is an element of $\Gamma_b(\Omega^x, \sigma^*\mathcal{K}(\mathcal{E}, \mathcal{F}))$ and the map $[\omega]_H \mapsto \|S_\omega\|\colon \Omega^x/H \to \mathbb{R}_{\geq 0}$ vanishes at infinity.

Identifying adjointable operators with their associated sections of operators, the fibre $T_x \in \mathcal{L}((\operatorname{Ind}_\Omega E)_x, (\operatorname{Ind}_\Omega F)_x)$ of an adjointable operator $T \in \mathcal{L}(\operatorname{Ind}_\Omega E, \operatorname{Ind}_\Omega F)$ at $x \in G^0$ is the restriction of T to the closed subspace $\Omega^x \subseteq \Omega$.

Proof. The identification of the space of adjointable operators $\mathcal{L}(\operatorname{Ind}_\Omega E, \operatorname{Ind}_\Omega F)$ with the space of H-equivariant sections in $\Gamma_b(\Omega, \sigma^*\mathcal{L}(\mathcal{E}, \mathcal{F}))$ follows directly from the descriptions of the adjointable operators on quotient and pullback modules in Propositions 1.87 and 1.63. The identification of the space of compact operators $\mathcal{K}(\operatorname{Ind}_\Omega E, \operatorname{Ind}_\Omega F)$ with the space of H-equivariant sections in $\Gamma_b(\Omega, \sigma^*\mathcal{K}(\mathcal{E}, \mathcal{F}))$ that vanish at infinity with respect to Ω/H also follows directly from these propositions.

For the description of the space of adjointable operators at each $x \in G^0$, we turn to Proposition 1.65, which says that $\mathcal{L}((\operatorname{Ind}_\Omega E)_x, (\operatorname{Ind}_\Omega F)_x)$ may be identified with $\Gamma_b(\Omega^x/H, \mathcal{L}((\sigma^*\mathcal{E})/H, (\sigma^*\mathcal{F})/H))$. Once again using Propositions 1.87 and 1.63, the bundle $\mathcal{L}((\sigma^*\mathcal{E})/H, (\sigma^*\mathcal{F})/H) \to \Omega/H$ may be identified with $(\sigma^*\mathcal{L}(\mathcal{E}, \mathcal{F}))/H \to \Omega/H$. After lifting to the unique H-equivariant sections, this completes the identification of $\mathcal{L}((\operatorname{Ind}_\Omega E)_x, (\operatorname{Ind}_\Omega F)_x)$ with the space of H-equivariant sections in $\Gamma_b(\Omega^x, \sigma^*\mathcal{L}(\mathcal{E}, \mathcal{F}))$.

For each $\omega \in \Omega^x$, we must check that $T_\omega = (T_x)_\omega \in \mathcal{L}(E_{\sigma(\omega)}, F_{\sigma(\omega)})$. For each $\xi \in \operatorname{Ind}_\Omega E$, we have $T_\omega(\xi(\omega)) = (T\xi)(\omega)$. The fibre at x of an element of $\operatorname{Ind}_\Omega E$ is its restriction to Ω^x, so we may compute $(T_x)_\omega(\xi(\omega)) = (T_x(\xi\restriction_{\Omega^x}))(\omega) = ((T\xi)\restriction_{\Omega^x})(\omega) = (T\xi)(\omega)$. □

It is often convenient to work locally in a groupoid correspondence Ω. For each $\omega \in \Omega$, we may evaluate elements $\xi \in (\operatorname{Ind}_\Omega B)_{\rho(\omega)} \subseteq \Gamma_b(\Omega^{\rho(\omega)}, \sigma^*\mathcal{B})$ at ω, giving us a map $\mathrm{ev}_\omega\colon (\operatorname{Ind}_\Omega B)_{\rho(\omega)} \to B_{\sigma(\omega)}$. This leads to a bundle of evaluation maps $\mathrm{ev} := (\mathrm{ev}_\omega\colon (\operatorname{Ind}_\Omega B)_{\rho(\omega)} \to B_{\sigma(\omega)})_{\omega\in\Omega}$.

Proposition 2.5 (Evaluation bundle map). *Let $\Omega\colon G \to H$ be a groupoid correspondence, let B be an H-C*-algebra and let E be an H-Hilbert B-module. Then the collections $(\mathrm{ev}_\omega)_{\omega\in\Omega}$ of evaluation maps*

$$\mathrm{ev}_\omega\colon (\operatorname{Ind}_\Omega B)_{\rho(\omega)} \to B_{\sigma(\omega)} \qquad\qquad \mathrm{ev}_\omega\colon (\operatorname{Ind}_\Omega E)_{\rho(\omega)} \to E_{\sigma(\omega)}$$

form surjective morphisms $\mathrm{ev}\colon \rho^*\operatorname{Ind}_\Omega \mathcal{B} \to \sigma^*\mathcal{B}$ and $\mathrm{ev}\colon \rho^*\operatorname{Ind}_\Omega \mathcal{E} \to \sigma^*\mathcal{E}$ of Banach bundles over Ω.

Proof. Recall that the set $\{\rho^*\xi \mid \xi \in \operatorname{Ind}_\Omega E\}$ is a sufficient collection of continuous sections for $\rho^*\operatorname{Ind}_\Omega \mathcal{E}$. For each $\xi \in \operatorname{Ind}_\Omega E \subseteq \Gamma_b(\Omega, \sigma^*\mathcal{E})$, the composition $\mathrm{ev}\circ\rho^*\xi\colon \Omega \to \sigma^*\mathcal{E}$ is given by the continuous section ξ, and therefore $\mathrm{ev}\colon \rho^*\operatorname{Ind}_\Omega \mathcal{E} \to \sigma^*\mathcal{E}$ is continuous.

To check surjectivity, let $\omega \in \Omega$ and let $e \in E_{\sigma(\omega)} = (\sigma^*E)_\omega$. There exists a section $\eta \in \Gamma_0(\Omega/H, \sigma^*\mathcal{E}/H)$ such that $\eta([\omega]_H) = [e]_H$, which corresponds to an element $\xi \in \operatorname{Ind}_\Omega E$ which evaluates to e at ω. The argument for B is identical. □

We may also work locally on an open bisection U of a correspondence $\Omega\colon G \to H$, by considering the correspondence $UH\colon q(U) \to H$. Then

$$\operatorname{Ind}_{UH} B = \{\xi \in \Gamma_b(UH, \sigma^*\mathcal{B}) \mid \xi \text{ is } H\text{-invariant and } c_0 \text{ with respect to } q(U)\}$$

can be identified with the subalgebra of sections in $\operatorname{Ind}_\Omega B$ that are supported on UH. It is equal to $q(U)\operatorname{Ind}_\Omega B$, considering $\operatorname{Ind}_\Omega B$ as a $C_0(\Omega/H)$-algebra. Evaluation gives us a $*$-isomorphism $\mathrm{ev}_U\colon \operatorname{Ind}_{UH} B \cong \sigma(U)B$ given by

$$\begin{aligned} \mathrm{ev}_U\colon \operatorname{Ind}_{UH} B &\to \sigma(U)B = \Gamma_0(\sigma(U), \mathcal{B}) \\ \xi &\mapsto \mathrm{ev}_U(\xi) \\ &\qquad \sigma(u) \mapsto \xi(u). \end{aligned} \tag{2.1}$$

This shows that $\operatorname{Ind}_\Omega B$ is generated by subalgebras $\operatorname{Ind}_{UH} B$ which are isomorphic to subalgebras $\sigma(U)B$ of B, indexed by the open bisections $U \subseteq \Omega$.

We now address the functoriality of the induced C*-algebra construction $\operatorname{Ind}_\Omega$ given a correspondence $\Omega\colon G \to H$. Let A and B be H-C*-algebras and suppose we have an H-equivariant $*$-homomorphism $\varphi\colon A \to B$. Applying φ pointwise, we get a $*$-homomorphism $\Gamma_b(\Omega, \sigma^*\mathcal{A}) \to \Gamma_b(\Omega, \sigma^*\mathcal{B})$ that restricts to a G-equivariant $*$-homomorphism $\operatorname{Ind}_\Omega \varphi\colon \operatorname{Ind}_\Omega A \to \operatorname{Ind}_\Omega B$. We obtain a functor $\operatorname{Ind}_\Omega\colon \mathsf{C^*\text{-}alg}^H \to \mathsf{C^*\text{-}alg}^G$. Similarly, given an H-equivariant correspondence $E\colon A \to B$, we can apply operations pointwise to construct a G-equivariant correspondence $\operatorname{Ind}_\Omega E\colon \operatorname{Ind}_\Omega A \to \operatorname{Ind}_\Omega B$.

Proposition 2.6 (Induction functor)**.** *Let $\Omega\colon G \to H$ be a correspondence of étale groupoids. Then the map $\operatorname{Ind}_\Omega\colon \mathsf{Corr}^H \to \mathsf{Corr}^G$ between the equivariant correspondence categories sending an H-equivariant correspondence $E\colon A \to B$ to the G-equivariant correspondence $\operatorname{Ind}_\Omega E\colon \operatorname{Ind}_\Omega A \to \operatorname{Ind}_\Omega B$ is a functor. Furthermore, this restricts to a functor $\operatorname{Ind}_\Omega\colon \mathsf{Corr}_p^H \to \mathsf{Corr}_p^G$ between the proper equivariant correspondence categories.*

Proof. It is clear that an identity correspondence $A\colon A \to A$ is mapped to the identity correspondence $\operatorname{Ind}_\Omega A\colon \operatorname{Ind}_\Omega A \to \operatorname{Ind}_\Omega A$. Now suppose we have composable H-equivariant correspondences $E\colon A \to B$ and $F\colon B \to C$. Consider the following map of Hilbert $\operatorname{Ind}_\Omega C$-modules.

$$\begin{aligned} \operatorname{Ind}_\Omega E \otimes_{\operatorname{Ind}_\Omega B} \operatorname{Ind}_\Omega F &\to \operatorname{Ind}_\Omega(E \otimes_B F) \\ \xi \otimes \eta &\mapsto (\omega \mapsto \xi(\omega) \otimes \eta(\omega)) \end{aligned}$$

This is well-defined because it preserves the inner product. The image of the simple tensors $\xi \otimes \eta$ has dense span by Remark 2.3, so this is an isometric isomorphism of Banach spaces. It intertwines the left actions of $\operatorname{Ind}_\Omega A$ and is G-equivariant, so we get an isomorphism $\operatorname{Ind}_\Omega E \otimes_{\operatorname{Ind}_\Omega B} \operatorname{Ind}_\Omega F \cong \operatorname{Ind}_\Omega(E \otimes_B F)$ of G-equivariant correspondences from $\operatorname{Ind}_\Omega A$ to $\operatorname{Ind}_\Omega B$. The induction construction therefore respects the composition of the correspondence categories and defines a functor. If $E\colon A \to B$ is a proper H-equivariant correspondence, the left action $A \curvearrowright E$ is given by a continuous map $\varphi\colon \mathcal{A} \to \mathcal{K}(\mathcal{E})$ of C*-bundles. The pointwise action $\operatorname{Ind}_\Omega A \curvearrowright \operatorname{Ind}_\Omega E$ sends a section $a \in \operatorname{Ind}_\Omega A$ to the section $(\sigma^*\varphi) \circ a\colon \Omega \to \sigma^*\mathcal{K}(\mathcal{E})$. This section is continuous, H-equivariant and vanishes at infinity with respect to Ω/H, so by Proposition 2.4 defines a compact operator on $\operatorname{Ind}_\Omega E$. Thus $\operatorname{Ind}_\Omega E\colon \operatorname{Ind}_\Omega A \to \operatorname{Ind}_\Omega B$ is also proper. ☐

We now turn to considering how the assignment $\Omega \mapsto \operatorname{Ind}_\Omega$ behaves with respect to the category of groupoid correspondences.

Proposition 2.7 (Compatibility of the induction functor with composition)**.** *Let $G \curvearrowright \Omega \curvearrowleft H$ and $H \curvearrowright \Lambda \curvearrowleft K$ be groupoid correspondences, and let B be a K-C*-algebra. Then there is a G-equivariant $*$-isomorphism of G-C*-algebras*

$$\varphi_B\colon \operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda B \cong \operatorname{Ind}_{\Lambda\circ\Omega} B$$

given by the following maps.

$$\begin{aligned}
\varphi_B\colon \operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda B &\to \operatorname{Ind}_{\Lambda\circ\Omega} B \subseteq \Gamma_b(\Lambda\circ\Omega, \sigma^*\mathcal{B})\\
\xi &\mapsto \varphi_B(\xi)\\
&\qquad [\omega,\lambda]_H \mapsto \xi(\omega)(\lambda)\\
\varphi_B^{-1}\colon \operatorname{Ind}_{\Lambda\circ\Omega} B &\to \operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda B \subseteq \Gamma_b(\Omega, \sigma_\Omega^* \operatorname{Ind}_\Lambda \mathcal{B})\\
\eta &\mapsto \varphi_B^{-1}(\eta)\\
&\qquad \omega \mapsto \varphi_B^{-1}(\eta)(\omega)\\
&\qquad\qquad \lambda \mapsto \eta([\omega,\lambda]_H)
\end{aligned}$$

Suppose C is another K-C^*-algebra and $E\colon B \to C$ is a K-equivariant correspondence. We may define $\varphi_E\colon \operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda E \to \operatorname{Ind}_{\Lambda\circ\Omega} E$ in the same way. Through this we have a natural isomorphism $\varphi\colon \operatorname{Ind}_\Omega\circ\operatorname{Ind}_\Lambda \cong \operatorname{Ind}_{\Lambda\circ\Omega}\colon \mathsf{Corr}^K \rightrightarrows \mathsf{Corr}^G$.

Proof. Let us first make a technical remark that we will need for this proof. There is a closed surjection $\pi\colon (\Lambda\circ\Omega)/K \to \Omega/H$ given by $\pi([[\omega,\lambda]_H]_K) = [\omega]_H$. Recall that a closed map with compact fibres is proper. A closed subset $S \subseteq (\Lambda\circ\Omega)/K$ is therefore compact if and only if its image $\pi(S)$ is compact and each fibre $S_{[\omega]_H} = \pi^{-1}([\omega]_H)\cap S$ is compact. For each $\omega\in\Omega$, the fibre $\pi^{-1}([\omega]_H)$ can be identified with $\Lambda^{\sigma_\Omega(\omega)}/K$ via $[[\omega,\lambda]_H]_K \mapsto [\lambda]_K$, so we may view $S_{[\omega]_H}$ inside $\Lambda^{\sigma_\Omega(\omega)}/K$.

Let $\xi\in\operatorname{Ind}_\Omega\operatorname{Ind}_\Lambda B$. For any $(\omega,\lambda)\in\Omega\times_{H^0}\Lambda$ and $h\in H^{\sigma_\Omega(\omega)}$, we may calculate that $\xi(\omega\cdot h)(h^{-1}\cdot\lambda) = (h^{-1}\cdot\xi(\omega))(h^{-1}\cdot\lambda) = \xi(\omega)(\lambda)$. The map $\varphi_B(\xi)\colon [\omega,\lambda]_H \mapsto \xi(\omega)(\lambda)$ is therefore a well-defined bounded section $(\Omega\times_{H^0}\Lambda)/H \to \sigma^*\mathcal{B}$, and it is clearly K-equivariant. It is continuous by Lemma 2.2 and the continuity of the map $\xi\colon\Omega\to\operatorname{Ind}_\Lambda\mathcal{B}$. To show that it vanishes at infinity with respect to $(\Lambda\circ\Omega)/K$, let $\epsilon>0$ and let $S = \{[[\omega,\lambda]_H]_K \in (\Lambda\circ\Omega)/K \mid \|\xi(\omega)(\lambda)\|\geq\epsilon\}$. As ξ vanishes at infinity with respect to Ω/H, the image $\pi(S)$ is compact. For each $\omega\in\Omega$, the fibre $S_{[\omega]_H}$ is compact because $\xi(\omega)$ vanishes at infinity with respect to $\Lambda^{\sigma_\Omega(\omega)}/K$. Therefore S is compact, and we can finally conclude that $\varphi_B(\xi)\in\operatorname{Ind}_{\Lambda\circ\Omega}B$. It is then clear that φ_B is a G-equivariant $*$-homomorphism.

Now let $\eta\in\operatorname{Ind}_{\Lambda\circ\Omega}B$. Then for each $\omega\in\Omega$, the map $\lambda\mapsto\eta([\omega,\lambda]_H)\colon\Lambda^{\sigma_\Omega(\omega)}\to\sigma_\Lambda^*\mathcal{B}$ is continuous, bounded and K-equivariant. Through the closed continuous injection $[\lambda]_K\mapsto[[\omega,\lambda]_H]_K\colon\Lambda^{\sigma_\Omega(\omega)}/K\to(\Lambda\circ\Omega)/K$ and the vanishing at infinity of η with respect to $(\Lambda\circ\Omega)/K$ we can deduce that $[\lambda]_K\mapsto\|\eta([\omega,\lambda]_H)\|$ vanishes at infinity. We therefore obtain an element $\varphi_B^{-1}(\eta)(\omega)\colon\lambda\mapsto\eta([\omega,\lambda]_H)$ of $(\operatorname{Ind}_\Lambda B)_{\sigma_\Omega(\omega)}$. Applying Lemma 2.2, the map $\varphi_B^{-1}(\eta)\colon\Omega\to\sigma_\Omega^*\operatorname{Ind}_\Lambda\mathcal{B}$ is continuous because of the continuity of $(\omega,\lambda)\mapsto\eta([\omega,\lambda]_H)\colon\Omega\times_{H^0}\Lambda\to\mathcal{B}$. It is clearly also bounded and H-equivariant. Through the continuous surjection $\pi\colon(\Lambda\circ\Omega)/K\to\Omega/H$ and the vanishing at infinity of η, we can see that $[\omega]_H\mapsto\|\varphi_B^{-1}(\eta)(\omega)\|$ vanishes at infinity. Therefore $\varphi_B^{-1}(\eta)\in\operatorname{Ind}_\Omega\operatorname{Ind}_\Lambda B$, and it is then clear that φ_B^{-1} is an inverse to φ_B.

The G-equivariant isomorphism $\varphi_E\colon \operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda E \cong \operatorname{Ind}_{\Lambda\circ\Omega} E$ is defined and justified in exactly the same way. To check naturality of $B \mapsto \mathsf{Corr}^G(\varphi_B)$, we need to check that the following diagram commutes in the G-equivariant correspondence category Corr^G.

$$\begin{tikzcd}
\operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda B \arrow[r, "\operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda E"] \arrow[d, "\mathsf{Corr}^G(\varphi_B)"] & \operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda C \arrow[d, "\mathsf{Corr}^G(\varphi_C)"] \\
\operatorname{Ind}_{\Lambda\circ\Omega} B \arrow[r, "\operatorname{Ind}_{\Lambda\circ\Omega} E"] & \operatorname{Ind}_{\Lambda\circ\Omega} C
\end{tikzcd}$$

The $\urcorner\!\!\downarrow$ composition of correspondences can be described by the space $\operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda E$, with the G-Hilbert $\operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda C$-module structure modified to a G-Hilbert $\operatorname{Ind}_{\Lambda\circ\Omega} C$-module structure via φ_C. The $\hookrightarrow$ composition is given by $\operatorname{Ind}_{\Lambda\circ\Omega} E$ with the left action of $\operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda B$ induced by φ_B. It is straightforward to check that $\varphi_E\colon \operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda E \to \operatorname{Ind}_{\Lambda\circ\Omega} E$ provides the required G-equivariant isomorphism. $\square$

Proposition 2.8 (The induction functor of an identity correspondence)**.** *Let G be an étale groupoid and let A be a G-C^*-algebra. Consider the identity correspondence $G\colon G \to G$. There is a G-equivariant $*$-isomorphism*

$$\psi_A\colon \operatorname{Ind}_G A \to A$$
$$\operatorname{Ind}_G A \subseteq \Gamma_b(G, s^*\mathcal{A}) \to \Gamma_0(G^0, \mathcal{A})$$
$$\xi \mapsto \xi\restriction_{G^0}.$$

Suppose B is another G-C^*-algebra and let $E\colon A \to B$ be a G-equivariant correspondence. We may define $\psi_E\colon \operatorname{Ind}_G E \to E$ in the same way. Through this we have a natural isomorphism $\psi\colon \operatorname{Ind}_G \cong \mathrm{id}_{\mathsf{Corr}^G}\colon \mathsf{Corr}^G \rightrightarrows \mathsf{Corr}^G$.

Proof. The vanishing condition on elements of $\operatorname{Ind}_G A$ ensures that ψ_A is well-defined, and the G-equivariance of the elements ensures that it is a G-equivariant $*$-homomorphism. The inverse is given by $\psi_A^{-1}(\eta)(g) = g^{-1}\cdot\eta(r(g))$ for $\eta \in \Gamma_0(G^0, \mathcal{A})$ and $g \in G$.

Similarly, $\psi_E\colon \operatorname{Ind}_G E \to E$ is well-defined with inverse ψ_E^{-1}. To check naturality of $A \mapsto \mathsf{Corr}^G(\varphi_A)$, we need to check that the following diagram commutes in the G-equivariant correspondence category Corr^G.

$$\begin{tikzcd}
\operatorname{Ind}_G A \arrow[r, "\operatorname{Ind}_G E"] \arrow[d, "\mathsf{Corr}^G(\psi_A)"] & \operatorname{Ind}_G B \arrow[d, "\mathsf{Corr}^G(\psi_B)"] \\
A \arrow[r, "E"] & B
\end{tikzcd}$$

The $\urcorner\!\!\downarrow$ composition is the correspondence with Hilbert module $\operatorname{Ind}_G E$ given the structure of a Hilbert B-module through the G-equivariant isomorphism ψ_B. The $\hookrightarrow$ composition is the correspondence with Hilbert B-module E with a left action of

$\operatorname{Ind}_G A$ through ψ_A. It is straightforward to check that $\psi_E\colon \operatorname{Ind}_G E \to E$ provides the required G-equivariant isomorphism. □

2.2. **Groupoid correspondences with C*-coefficients.** We wish to relate an induced G-C*-algebra $\operatorname{Ind}_\Omega B$ to the original H-C*-algebra B given a groupoid correspondence $\Omega\colon G \to H$. To do this we introduce the notion of groupoid correspondences with C*-coefficients, viewing $\operatorname{Ind}_\Omega B$ and B as C*-coefficients for the étale groupoids G and H respectively. This is based upon the Morita equivalences of groupoid C*-dynamical systems in [61].

Definition 2.9 (Correspondence of étale groupoids with C*-coefficients)**.** Let G and H be étale groupoids with unit spaces $X = G^0$ and $Y = H^0$, let A be a G-C*-algebra and let B be an H-C*-algebra.

A *groupoid correspondence bundle with C*-coefficients* from $(\mathcal{A}, G)$ to $(\mathcal{B}, H)$ is a Banach bundle $p\colon \mathcal{E} \to \Omega$ over a groupoid correspondence $\Omega\colon G \to H$ together with the structure of a correspondence from $A_{\rho(\omega)}$ to $B_{\sigma(\omega)}$ on the fibre E_ω at each $\omega \in \Omega$, and commuting continuous actions of G and H on the left and right respectively of $\mathcal{E}$ such that the following hold.

- (Continuity) The maps $\mathcal{A} \times_X \mathcal{E} \to \mathcal{E}$, $\mathcal{E} \times_\Omega \mathcal{E} \to \mathcal{B}$ and $\mathcal{E} \times_\Omega \mathcal{B} \to \mathcal{E}$ induced by the C*-correspondence structures on the fibres $(E_\omega)_{\omega\in\Omega}$ are continuous.

$$\mathcal{A} \times_X \mathcal{E} \to \mathcal{E} \qquad\qquad \mathcal{E} \times_Y \mathcal{B} \to \mathcal{E}$$
$$\mathcal{E} \times_\Omega \mathcal{E} \to \mathcal{B}$$

  This ensures that $\mathcal{E} \to \Omega$ is the bundle associated to a $C_0(\Omega)$-correspondence from ρ^*A to σ^*B.

- (Equivariance) The bundle map $p\colon \mathcal{E} \to \Omega$ is equivariant with respect to the groupoid actions.

$$g \cdot p(e) = p(g \cdot e) \qquad \begin{tikzcd} & \mathcal{E} \arrow[dl] \arrow[d, "p"] \arrow[dr] & \\ X & \Omega \arrow[l, "\rho"'] \arrow[r, "\sigma"] & Y \end{tikzcd} \qquad p(e) \cdot h = p(e \cdot h)$$

  This says that the actions are really actions of $G \ltimes \Omega$ and $\Omega \rtimes H$.

- (Compatibility) The left G-action on $\mathcal{E}$ is compatible with the left $\mathcal{A}$-action and the right H-action on $\mathcal{E}$ is compatible with the right $\mathcal{B}$-action and the $\mathcal{B}$-valued inner product.

$$g \cdot (a \cdot e) = (g \cdot a) \cdot (g \cdot e) \qquad\qquad (e \cdot b) \cdot h = (e \cdot h) \cdot (h^{-1} \cdot b)$$
$$\langle e \cdot h, f \cdot h\rangle_{\mathcal{B}} = h^{-1} \cdot \langle e, f\rangle_{\mathcal{B}}$$

- (Invariance) The left G-action on $\mathcal{E}$ commutes with the right $\mathcal{B}$-action and the right H-action commutes with the left $\mathcal{A}$-action.

$$g \cdot (e \cdot b) = (g \cdot e) \cdot b \qquad\qquad (a \cdot e) \cdot h = a \cdot (e \cdot h)$$

The associated section space $E \cong \Gamma_0(\Omega, \mathcal{E})$ is called a *groupoid correspondence with C*-coefficients* from (A, G) to (B, H) over Ω. This could alternatively be defined as a $C_0(\Omega)$-correspondence from $\rho^* A$ to $\sigma^* B$ with actions of G and H satisfying analogues of the above conditions. We often refer to (E, Ω) and $(\mathcal{E}, \Omega)$ simply as a correspondence and a correspondence bundle, and we may omit Ω if the underlying groupoid correspondence is understood. To express that $(\mathcal{E}, \Omega)$ is a correspondence bundle from $(\mathcal{A}, G)$ to $(\mathcal{B}, H)$ or that (E, Ω) is a correspondence from (A, G) to (B, H) we may write one of the following:

$$(\mathcal{E}, \Omega) \colon (\mathcal{A}, G) \to (\mathcal{B}, H) \qquad (\mathcal{A}, G) \xrightarrow[\Omega]{\mathcal{E}} (\mathcal{B}, H) \qquad (\mathcal{A}, G) \curvearrowright (\mathcal{E}, \Omega) \curvearrowleft (\mathcal{B}, H)$$

$$(E, \Omega) \colon (A, G) \to (B, H) \qquad (A, G) \xrightarrow[\Omega]{E} (B, H) \qquad (A, G) \curvearrowright (E, \Omega) \curvearrowleft (B, H)$$

While the symbol $\cdot$ is used for many different actions, it should always be clear which action we mean. As ever when introducing an object "with coefficients", we can see that it reduces to our original object when we have *trivial* coefficients.

Example 2.10 (Groupoid correspondence with trivial coefficients)**.** Let $\Omega \colon G \to H$ be a correspondence of étale groupoids. The correspondence with trivial coefficients is a correspondence from $(C_0(G^0), G)$ to $(C_0(H^0), H)$ given by the trivial bundle $\Omega \times \mathbb{C} \to \Omega$.

In the same way that every étale groupoid G has an identity correspondence, every G-C*-algebra A has an identity correspondence with coefficients.

Example 2.11 (Identity correspondence bundle)**.** Let G be an étale groupoid with unit space X and let A be a G-C*-algebra with anchor map $\tau \colon \mathcal{A} \to X$. Consider the Banach bundle $s^* \mathcal{A} \to G$ over the identity correspondence $G \colon G \to G$. The fibre $A_{s(g)}$ at $g \in G$ can be equipped with the structure of a correspondence from $A_{r(g)}$ to $A_{s(g)}$ via the action map $a \mapsto g^{-1} \cdot a \colon A_{r(g)} \to A_{s(g)}$. We define a left action $G \curvearrowright s^* \mathcal{A}$ by $g_1 \cdot (g_2, a_1) := (g_1 g_2, a_1)$ and a right action $s^* \mathcal{A} \curvearrowleft G$ by $(g_1, a_2) \cdot g_2 := (g_1 g_2, g_2^{-1} \cdot a_2)$ when $s(g_1) = r(g_2)$, $\tau(a_1) = s(g_2)$ and $\tau(a_2) = s(g_1)$.

To make sense of why these are the right actions, we may think of the action $G \curvearrowright \mathcal{A}$ as an action of conjugation by unitaries with $g \cdot a = u_g a u_g^*$. The above action maps are then determined by requiring consistency with the formal concatenation $(g, a) \mapsto u_g a$. We can view this as constructing a correspondence with C*-coefficients from the G-equivariant correspondence $A \colon A \to A$. More generally, we can view any equivariant correspondence in the framework of groupoid correspondences with C*-coefficients.

Example 2.12 (Equivariant correspondences as groupoid correspondences with C*-coefficients)**.** Let G be an étale groupoid with unit space X, let A and B be G-C*-algebras and let $E\colon A \to B$ be a G-equivariant correspondence from A to B. Consider the Banach bundle $s^*\mathcal{E} \to G$ over the identity correspondence $G\colon G \to G$. The fibre $E_{s(g)}$ at $g \in G$ is a Hilbert $B_{s(g)}$-module and we can equip s^*G with the following left actions of $\mathcal{A}$ and G and right action of G. For $g_1, g_2 \in G$ with $s(g_1) = r(g_2)$, $a \in A_{r(g_1)}$, $e_1 \in E_{s(g_1)}$ and $e_2 \in E_{s(g_2)}$, we define:

$$\mathcal{A} \curvearrowright G \times_X \mathcal{E}$$
$$a \cdot (g_1, e_1) := (g_1, (g_1^{-1} \cdot a) \cdot e_1)$$
$$G \curvearrowright G \times_X \mathcal{E} \qquad\qquad G \times_X \mathcal{E} \curvearrowleft G$$
$$g_1 \cdot (g_2, e_2) := (g_1 g_2, e_2) \qquad\qquad (g_1, e_1) \cdot g_2 := (g_1 g_2, g_2^{-1} \cdot e_1)$$

We write $i_G(E)\colon (A, G) \to (B, G)$ for the associated correspondence with C*-coefficients.

The following example of a correspondence with C*-coefficients is our motivation for the definition and gives us a way of relating an induced G-C*-algebra $\operatorname{Ind}_\Omega B$ to the original H-C*-algebra B.

Example 2.13 (Evaluation correspondence)**.** Let $\Omega\colon G \to H$ be a correspondence of étale groupoids and let B be an H-C*-algebra. Consider the Banach bundle $\sigma^*\mathcal{B} \to \Omega$. The fibre $B_{\sigma(\omega)}$ at $\omega \in \Omega$ can be equipped with the structure of a $(\operatorname{Ind}_\Omega B)_{\rho(\omega)}$-$B_{\sigma(\omega)}$ correspondence via the evaluation map

$$\mathrm{ev}_\omega\colon \operatorname{Ind}_\Omega B_{\rho(\omega)} \to B_{\sigma(\omega)}.$$

We define actions $G \curvearrowright \sigma^*\mathcal{B} \curvearrowleft H$ for $(\omega, b) \in \sigma^*\mathcal{B}$, $g \in G_{\rho(\omega)}$ and $h \in H^{\sigma(\omega)}$ by

$$g \cdot (\omega, b) = (g \cdot \omega, b) \qquad\qquad (\omega, b) \cdot h = (\omega \cdot h, h^{-1} \cdot b).$$

We obtain a groupoid correspondence $\Theta_{\Omega,B}$ with C*-coefficients called the *evaluation correspondence* from $(\operatorname{Ind}_\Omega B, G)$ to (B, H) over Ω. The continuity of the left action $\operatorname{Ind}_\Omega \mathcal{B} \times_{G^0} \sigma^*\mathcal{B} \to \sigma^*\mathcal{B}$ follows from the continuity of the evaluation bundle map $\rho^* \operatorname{Ind}_\Omega \mathcal{B} \to \sigma^*\mathcal{B}$ in Proposition 2.5.

Just as with correspondences of groupoids and C*-algebras, we want a notion of properness, which will eventually enable us to induce maps in K-theory.

Definition 2.14 (Proper correspondence with C*-coefficients)**.** A groupoid correspondence with C*-coefficients $(E, \Omega)\colon (A, G) \to (B, H)$ is *proper* if the groupoid correspondence $\Omega\colon G \to H$ is proper and the left action map $\mathcal{A} \times_{G^0} \mathcal{E} \to \mathcal{E}$ defines a continuous map of bundles $\rho^*\mathcal{A} \to \mathcal{K}(\mathcal{E})$ over Ω.

We can build a correspondence with C*-coefficients entirely from the "right hand side" of the data, as in Example 1.22.

Example 2.15. Let H be an étale groupoid with a free, proper, étale right H-space Ω with anchor map $\sigma\colon \Omega \to H^0$ and let B be an H-C*-algebra. Let E be a $\Omega \rtimes H$-Hilbert σ^*B-module with associated bundle $\mathcal{E} \to \Omega$. Then the action $\mathcal{K}(\mathcal{E})/H \times_{\Omega/H} \mathcal{E} \to \mathcal{E}$ of the C*-bundle $\mathcal{K}(\mathcal{E})/H \to \Omega/H$ defines a correspondence bundle $(\mathcal{E}, \Omega)\colon (\mathcal{K}(\mathcal{E})/H, \Omega/H) \to (\mathcal{B}, H)$. The underlying groupoid correspondence $\Omega\colon \Omega/H \to H$ is proper. The bundle $\mathcal{K}(\mathcal{E})/H$ acts by compact operators on $\mathcal{E}$, and the map of bundles $q^*(\mathcal{K}(\mathcal{E})/H) \to \mathcal{K}(\mathcal{E})$ is continuous by Proposition 1.49 as it is pointwise an isometric isomorphism and has a continuous inverse. Thus the correspondence $(E, \Omega)\colon (\mathcal{K}(E)^H, \Omega/H) \to (B, H)$ is proper.

To compose groupoid correspondences with C*-coefficients, we must construct a bundle $\mathcal{F} \circ \mathcal{E} \to \Lambda \circ \Omega$ from a pair of correspondences $(E, \Omega)\colon (A, G) \to (B, H)$ and $(F, \Lambda)\colon (B, H) \to (C, K)$. The fibre $(F \circ E)_{[\omega,\lambda]_H}$ at $[\omega, \lambda]_H \in \Lambda \circ \Omega$ should be a C*-correspondence from $A_{\rho(\omega)}$ to $C_{\sigma(\lambda)}$. The obvious choice is the composition $E_\omega \otimes_{B_{\sigma(\omega)}} F_\lambda$, but this depends on the representative $(\omega, \lambda) \in \Omega \times_Y \Lambda$. This is not a huge problem, because given a different representative $(\omega \cdot h, h^{-1} \cdot \lambda)$, there is a canonical isomorphism $E_{\omega \cdot h} \otimes_{B_{s(h)}} F_{h^{-1} \cdot \lambda} \cong E_\omega \otimes_{B_{\sigma(\omega)}} F_\lambda$ through the actions of H. We may instead construct a Banach bundle $\mathcal{F} \circ \mathcal{E} \to \Lambda \circ \Omega$ such that for each representative $(\omega, \lambda) \in \Omega \times_Y \Lambda$, there is a canonical isomorphism $(F \circ E)_{[\omega,\lambda]_H} \cong E_\omega \otimes_{B_{\sigma(\omega)}} F_\lambda$.

Definition 2.16 (Composition of groupoid correspondences with C*-coefficients)**.** Let $(E, \Omega)\colon (A, G) \to (B, H)$ and $(F, \Lambda)\colon (B, H) \to (C, K)$ be correspondences of étale groupoids with C*-coefficients. Let the unit spaces of G, H and K be X, Y and Z respectively, and let π_Ω, π_Y and π_Λ be the projections from $\Omega \times_Y \Lambda$ to Ω, Y and Λ respectively. The *composition bundle* $\mathcal{F} \circ \mathcal{E} \to \Lambda \circ \Omega$ is given by the following Banach bundle.

$$\mathcal{F} \circ \mathcal{E} := \left(\pi_\Omega^* \mathcal{E} \otimes_{\pi_Y^* \mathcal{B}} \pi_\Lambda^* \mathcal{F}\right)/H \to (\Omega \times_Y \Lambda)/H = \Lambda \circ \Omega$$

Indeed, for each $(\omega, \lambda) \in \Omega \times_Y \Lambda$, the fibre at $[\omega, \lambda]_H$ is isomorphic to $E_\omega \otimes_{B_{\sigma(\omega)}} F_\lambda$, through which we equip it with the structure of a C*-correspondence from $A_{\rho(\omega)}$ to $C_{\sigma(\lambda)}$. For an element $e \otimes f \in E_\omega \otimes_{B_{\sigma(\omega)}} F_\lambda$, we write $[e \otimes f]_H$ for the associated element of $(F \circ E)_{[\omega,\lambda]_H}$. We endow $\mathcal{F} \circ \mathcal{E}$ with commuting actions of G and K as follows. For $g \in G_{\rho(\omega)}$, $k \in K^{\sigma(\lambda)}$ and $e \otimes f \in E_\omega \otimes_{B_{\sigma(\omega)}} F_\lambda$, we define

$$\begin{aligned} G \times_X \mathcal{F} \circ \mathcal{E} &\to \mathcal{F} \circ \mathcal{E} & \qquad \mathcal{F} \circ \mathcal{E} \times_Z K &\to \mathcal{F} \circ \mathcal{E} \\ g \cdot [e \otimes f]_H &:= [g \cdot e \otimes f]_H & [e \otimes f] \cdot k &:= [e \otimes f \cdot k]. \end{aligned}$$

It is straightforward to see that the actions of $\mathcal{A}$, G, $\mathcal{C}$ and K on $\mathcal{F} \circ \mathcal{E}$ and the inner product $\mathcal{F} \circ \mathcal{E} \times_{\Lambda \circ \Omega} \mathcal{F} \circ \mathcal{E} \to \mathcal{C}$ are well-defined and satisfy equivariance, compatibility and invariance. By construction, a map $\varphi\colon \mathcal{F} \circ \mathcal{E} \to \mathcal{D}$ of Banach bundles over $\Lambda \circ \Omega$ is continuous if and only if the H-invariant map $\mathcal{E} \times_Y \mathcal{F} \to \mathcal{D}$ given by $(e, f) \mapsto \varphi([e \otimes f]_H)$ is continuous. This can be used to verify each required continuity condition. We therefore obtain a correspondence $(F\circ E, \Lambda\circ\Omega)\colon (A,G)\to (C,K)$ which is the *composition* of (E,Ω) and (F,Λ).

Proposition 2.17 (Associativity of composition of groupoid correspondences with C*-coefficients)**.** *Composition of groupoid correspondences with C*-coefficients is associative up to canonical isomorphisms.*

Proof. Let $(E_1,\Omega_1)\colon (A_1,G_1)\to(A_2,G_2)$, $(E_2,\Omega_2)\colon(A_2,G_2)\to(A_3,G_3)$ and $(E_3,\Omega_3)\colon(A_3,G_3)\to(A_4,G_4)$ be correspondences, and let X_i be the unit space of G_i. We may identify the composition correspondences with the following maps, writing $[-]$ instead of $[-]_{G_i}$ to avoid clutter.

$$\begin{aligned} &\Omega_3\circ(\Omega_2\circ\Omega_1)\cong(\Omega_3\circ\Omega_2)\circ\Omega_1 && \alpha\colon \mathcal{E}_3\circ(\mathcal{E}_2\circ\mathcal{E}_1)\cong(\mathcal{E}_3\circ\mathcal{E}_2)\circ\mathcal{E}_1\\ &[[\omega_1,\omega_2],\omega_3]\mapsto[\omega_1,[\omega_2,\omega_3]] && [[e_1\otimes e_2]\otimes e_3]\mapsto[e_1\otimes[e_2\otimes e_3]]\\ &(\omega_1,\omega_2,\omega_3)\in\Omega_1\times_{X_2}\Omega_2\times_{X_3}\Omega_3, && e_1\in(E_1)_{\omega_1},\ e_2\in(E_2)_{\omega_2},\ e_3\in(E_3)_{\omega_3}.\end{aligned}$$

Both modules $(E_3\circ(E_2\circ E_1))_{[[\omega_1,\omega_2],\omega_3]}$ and $((E_3\circ E_2)\circ E_1)_{[\omega_1,[\omega_2,\omega_3]]}$ are isomorphic to $(E_1)_{\omega_1}\otimes_{(A_2)_{\rho(\omega_2)}}(E_2)_{\omega_2}\otimes_{(A_3)_{\rho(\omega_3)}}(E_3)_{\omega_3}$. Under these isomorphisms both elements $[[e_1\otimes e_2]\otimes e_3]$ and $[e_1\otimes[e_2\otimes e_3]]$ map to $e_1\otimes e_2\otimes e_3$, so α is fibre-wise an isomorphism. To check that it is continuous, consider the following commutative diagram.

$$\begin{tikzcd} (\mathcal{E}_1\times_{X_2}\mathcal{E}_2)\times_{X_3}\mathcal{E}_3 \arrow[r] \arrow[d, "\cong"] \arrow[drr, "\gamma"] & (\mathcal{E}_2\circ\mathcal{E}_1)\times_{X_3}\mathcal{E}_3 \arrow[r] \arrow[dr, "\beta"] & \mathcal{E}_3\circ(\mathcal{E}_2\circ\mathcal{E}_1) \arrow[d, "\alpha"] \\ \mathcal{E}_1\times_{X_2}(\mathcal{E}_2\times_{X_3}\mathcal{E}_3) \arrow[r] & \mathcal{E}_1\times_{X_2}(\mathcal{E}_3\circ\mathcal{E}_2) \arrow[r] & (\mathcal{E}_3\circ\mathcal{E}_2)\circ\mathcal{E}_1 \end{tikzcd}$$

The continuity of α reduces to the continuity of β which reduces to the continuity of γ. We know that the horizontal maps are continuous, so we can conclude that α is continuous. □

To describe how the identity correspondences with C*-coefficients behave under composition, we will work in the more general setting of correspondences with C*-coefficients coming from equivariant correspondences.

Proposition 2.18 (Composition with equivariant correspondences)**.** *Let E be a G-equivariant correspondence from A to B, and let $(F,\Omega)\colon(B,G)\to(C,H)$ be a correspondence of groupoids with C*-coefficients. Then the composition $F\circ i_G(E)\colon(A,G)\to(C,H)$ is given by the correspondence bundle*

$$\mathcal{F}\circ i_G(\mathcal{E})\cong\rho^*\mathcal{E}\otimes_{\rho^*\mathcal{B}}\mathcal{F},\qquad (F\circ i_G(E))_\omega\cong E_{\rho(\omega)}\otimes_{B_{\rho(\omega)}}F_\omega.$$

The actions are given by

$$\begin{aligned} &G\curvearrowright\rho^*\mathcal{E}\otimes_{\rho^*\mathcal{B}}\mathcal{F} && \rho^*\mathcal{E}\otimes_{\rho^*\mathcal{B}}\mathcal{F}\curvearrowleft H\\ &g\cdot((\omega,e)\otimes f)=(g\cdot\omega,g\cdot e)\otimes(g\cdot f), && ((\omega,e)\otimes f)\cdot h=(\omega\cdot h,e)\otimes(f\cdot h),\end{aligned}$$

for $\omega \in \Omega$, $e \in E_{\rho(\omega)}$, $f \in F_\omega$, $g \in G_{\rho(\omega)}$ and $h \in H^{\sigma(\omega)}$. Similarly if $E' \colon C \to D$ is an H-equivariant correspondence, the composition $i_H(E') \circ F \colon (B, G) \to (D, H)$ is given by

$$i_H(\mathcal{E}') \circ \mathcal{F} \cong \mathcal{F} \otimes_{\sigma^*\mathcal{C}} \sigma^*\mathcal{E}', \qquad (i_H(E') \circ F)_\omega \cong F_\omega \otimes_{C_{\sigma(\omega)}} E'_{\sigma(\omega)}.$$

The fibre $E_{\rho(\omega)} \otimes_{B_{\rho(\omega)}} F_\omega$ at $\omega \in \Omega$ is already a correspondence from $A_{\rho(\omega)}$ to $C_{\sigma(\omega)}$. The groupoid actions are given by

$$\begin{gathered} G \curvearrowright \mathcal{F} \otimes_{\sigma^*\mathcal{C}} \sigma^*\mathcal{E}' \qquad\qquad \mathcal{F} \otimes_{\sigma^*\mathcal{C}} \sigma^*\mathcal{E}' \curvearrowleft H \\ g \cdot (f \otimes (\omega, e')) = (g \cdot f) \otimes (g \cdot \omega, e'), \quad (f \otimes (\omega, e')) \cdot h = (f \cdot h) \otimes (\omega \cdot h, h^{-1} \cdot e'), \end{gathered}$$

for $\omega \in \Omega$, $f \in F_\omega$, $e' \in E'_{\sigma(\omega)}$, $g \in G_{\rho(\omega)}$ and $h \in H^{\sigma(\omega)}$.

Proof. Under the identification $[g, \omega]_G \mapsto g \cdot \omega \colon \Omega \circ G \cong \Omega$, we have an isomorphism of Banach bundles over Ω which is described below at $g \cdot \omega \in \Omega$ for $(g, e) \in (s^*E)_g$ and $f \in F_\omega$.

$$\mathcal{F} \circ i_G(\mathcal{E}) \cong \rho^*\mathcal{E} \otimes_{\rho^*\mathcal{B}} \mathcal{F} \qquad \begin{aligned} (F \circ i_G(E))_{g\cdot\omega} &\cong E_{r(g)} \otimes_{B_{r(g)}} F_{g\cdot\omega} \\ [(g, e) \otimes f]_G &\mapsto (g \cdot e) \otimes (g \cdot f) \end{aligned}$$

This is an isomorphism at each fibre because $[(g, e) \otimes f]_G = [(r(g), g \cdot e) \otimes g \cdot f]_G$, and it is well-defined and continuous because the map

$$\begin{aligned} s^*\mathcal{E} \times_{G^0} \mathcal{F} &\to \rho^*\mathcal{E} \otimes_{\rho^*\mathcal{B}} \mathcal{F} \\ ((g, e), f) &\mapsto (g \cdot e) \otimes (g \cdot f) \end{aligned}$$

is continuous and invariant for the diagonal G-action on $s^*\mathcal{E} \times_{G^0} \mathcal{F}$. It is straightforward to see that the already described actions of $\mathcal{A}$, G, $\mathcal{C}$ and H coincide under this identification. Similarly, we have an isomorphism described below at $\omega \cdot h \in \Omega$ for $f \in F_\omega$ and $(h, e') \in (s^*E')_h$.

$$i_H(\mathcal{E}') \circ \mathcal{F} \cong \mathcal{F} \otimes_{\sigma^*\mathcal{C}} \sigma^*\mathcal{E}' \qquad \begin{aligned} (i_H(E') \circ F)_{\omega\cdot h} &\cong F_{\omega\cdot h} \otimes_{C_{s(h)}} E'_{s(h)} \\ [f \otimes (h, e')]_H &\mapsto (f \cdot h) \otimes e' \end{aligned}$$

□

As a special case of this, we can conclude that the identity correspondence at (A, G) acts as an identity for composition of groupoid correspondences with C*-coefficients. We can now introduce the correspondence category $\mathsf{GpdCorr}_{C^*}$ of groupoids with C*-coefficients.

Definition 2.19 (Correspondence category of groupoids with C*-coefficients)**.** Let $\mathsf{GpdCorr}_{C^*}$ be the category whose objects are pairs (A, G) consisting of an étale groupoid G and a G-C*-algebra A, and whose morphisms are the isomorphism classes of groupoid correspondences with C*-coefficients.

The isomorphisms in this category recover the Morita equivalences of groupoid C*-dynamical systems in [61].

Proposition 2.20. *A correspondence $(E,\Omega)\colon (A,G) \to (B,H)$ is invertible if and only if Ω is a Morita equivalence and for each $\omega \in \Omega$, the correspondence $E_\omega\colon A_{\rho(\omega)} \to B_{\sigma(\omega)}$ is a Morita equivalence. We call (E,Ω) a Morita equivalence.*

Proof. If (E,Ω) is invertible then Ω is invertible and each E_ω is invertible, thus they are Morita equivalences. Conversely, if Ω is invertible and each E_ω is invertible, we may write Ω^{-1} and E^*_ω for the inverses. The space Ω^{-1} consists of formal inverses ω^{-1} of elements of Ω, topologised so that $\omega \mapsto \omega^{-1}$ is a homeomorphism. The actions $H \curvearrowright \Omega^{-1} \curvearrowleft G$ are given by $h \cdot \omega^{-1} = (\omega \cdot h^{-1})^{-1}$ and $\omega^{-1} \cdot g = (g^{-1} \cdot \omega)^{-1}$. Similarly, the correspondence E_ω is isometrically isomorphic to E^*_ω, with a mapping $e \mapsto e^*$. The actions are given by $b \cdot e^* = (e \cdot b^*)^*$ and $e^* \cdot a = (a^* \cdot e)^*$, while the inner product goes through the identification $A_{\rho(\omega)} \cong \mathcal{K}(E_\omega)$, identifying $\langle e^*, f^*\rangle$ with $\Theta_{e,f}$. As a result, we may form a correspondence bundle $(E^*, \Omega^{-1})\colon (B,H) \to (A,G)$, topologised using the topology on (E,Ω). There are bundle maps $(E \circ E^*, \Omega \circ \Omega^{-1}) \to (i_G(A), G)$ and $(E^* \circ E, \Omega^{-1} \circ \Omega) \to (i_H(B), H)$ from the compositions to the identity correspondences which are isomorphisms on each fibre and respect the isomorphisms $\Omega \circ \Omega^{-1} \cong G$ and $\Omega^{-1} \circ \Omega \cong H$. The continuity of the bundle map $\mathcal{E} \circ \mathcal{E}^* \to s^*\mathcal{B}$ follows from the continuity of the inner product on $\mathcal{E}$, and similarly for the bundle map $\mathcal{E}^* \circ \mathcal{E} \to s^*\mathcal{A}$. These bundle maps are therefore isomorphisms. □

Proposition 2.21 (Functor sending equivariant correspondences to correspondences of groupoids with C*-coefficients)**.** *The construction of the correspondence with C*-coefficients $(i_G(E), G)$ from a G-equivariant correspondence $E\colon A \to B$ from Example 2.12 is functorial. We obtain a functor $i_G\colon \mathsf{Corr}^G \to \mathsf{GpdCorr}_{C^*}$. Furthermore, if E is proper then $i_G(E)$ is proper.*

Proof. We have already seen that $i_G\colon \mathsf{Corr}^G \to \mathsf{GpdCorr}_{C^*}$ maps identities to identities. To check that it respects composition, let $E\colon A \to B$ and $F\colon B \to C$ be G-equivariant correspondences. By Proposition 2.18, $i_G(\mathcal{F}) \circ i_G(\mathcal{E}) \cong i_G(\mathcal{E}) \otimes_{s^*\mathcal{B}} s^*\mathcal{F}$ which is in turn isomorphic to $s^*(\mathcal{E} \otimes_{\mathcal{B}} \mathcal{F}) = i_G(\mathcal{E} \otimes_{\mathcal{B}} \mathcal{F})$. It is straightforward to check that this isomorphism $i_G(\mathcal{F}) \circ i_G(\mathcal{E}) \cong i_G(\mathcal{E} \otimes_{\mathcal{B}} \mathcal{F})$ is compatible with the left actions of $\mathcal{A}$ and G and the right actions of $\mathcal{C}$ and G.

If $E\colon A \to B$ is proper, then for each $g \in G$, $E_{s(g)}\colon A_{r(g)} \to B_{s(g)}$ is proper, as it is the composition of the proper correspondence $E_{s(g)}\colon A_{s(g)} \to B_{s(g)}$ with the $*$-isomorphism $A_{r(g)} \to A_{s(g)}$ induced by g. The bundle map $r^*\mathcal{A} \to i_G(\mathcal{E})$ is therefore given by the composition $r^*\mathcal{A} \to s^*\mathcal{A} \to s^*\mathcal{E}$ of continuous bundle maps, and so $(i_G(E), G)$ is proper. The identity correspondence $G\colon G \to G$ is proper, so $i_G(E)$ is a proper correspondence of groupoids with C*-coefficients. □

2.3. **The crossed product construction for correspondences.** The crossed product of a groupoid correspondence with C*-coefficients builds on Holkar's original construction of a C*-correspondence from a groupoid correspondence. This construction is slightly cleaner in the setting of étale groupoids [3]. Muhly and Williams considered C*-coefficients [61] in the setting of Morita equivalences, which are exactly the invertible correspondences.

Let $(E,\Omega)\colon (A,G) \to (B,H)$ be a groupoid correspondence with C*-coefficients. The aim is to construct a C*-correspondence $\Omega \ltimes E\colon G \ltimes A \to H \ltimes B$. This is built from the space $\Gamma_c(\Omega,\mathcal{E})$ of compactly supported continuous sections of $\mathcal{E}$. Consider the following structure on $\Gamma_c(\Omega,\mathcal{E})$.

- A $\Gamma_c(H,s^*\mathcal{B})$-valued inner product on $\Gamma_c(\Omega,\mathcal{E})$. For ξ and $\eta \in \Gamma_c(\Omega,\mathcal{E})$, we define $\langle \xi,\eta\rangle \in \Gamma_c(H,s^*\mathcal{B})$ for $h \in H$ by

$$\langle \xi,\eta\rangle\colon h \mapsto \sum_{\omega\in\Omega_{r(h)}} \langle \xi(\omega)\cdot h, \eta(\omega\cdot h)\rangle \in B_{s(h)}. \tag{2.2}$$

- A right action $\Gamma_c(\Omega,\mathcal{E}) \curvearrowleft \Gamma_c(H,s^*\mathcal{B})$. For $\xi \in \Gamma_c(\Omega,\mathcal{E})$ and $b \in \Gamma_c(H,s^*\mathcal{B})$ we define $\xi\cdot b \in \Gamma_c(\Omega,\mathcal{E})$ for $\omega\in\Omega$ by

$$\xi\cdot b\colon \omega \mapsto \sum_{h\in H^{\sigma(\omega)}} \Big(\xi(\omega\cdot h)\cdot h^{-1}\Big)\cdot b(h^{-1}) \in E_\omega. \tag{2.3}$$

- A left action $\Gamma_c(G,s^*\mathcal{A}) \curvearrowright \Gamma_c(\Omega,\mathcal{E})$. For $\xi\in\Gamma_c(\Omega,\mathcal{E})$ and $a \in \Gamma_c(G,s^*\mathcal{A})$ we define $a\cdot\xi \in \Gamma_c(\Omega,\mathcal{E})$ for $\omega\in\Omega$ by

$$a\cdot\xi\colon \omega \mapsto \sum_{g\in G_{\rho(\omega)}} g^{-1}\cdot \Big(a(g^{-1})\cdot \xi(g\cdot\omega)\Big) \in E_\omega. \tag{2.4}$$

The elements $\langle\xi,\eta\rangle$, $\xi\cdot b$ and $a\cdot\xi$ are indeed compactly supported continuous sections by applications of Lemma 1.75 to the local homeomorphisms $\pi_H\colon \Omega\rtimes H \to H$, $r\colon \Omega\rtimes H\to\Omega$ and $s\colon G\ltimes\Omega\to\Omega$ respectively.

Theorem 2.22 (The crossed product construction for a groupoid correspondence with C*-coefficients)**.** *In the above setting, there is a C*-correspondence*

$$\Omega\ltimes E\colon G\ltimes A \to H\ltimes B$$

containing $\Gamma_c(\Omega,\mathcal{E})$ as a dense subspace whose operations restrict to the above formulae.

We emphasise that for a groupoid correspondence $\Omega\colon G\to H$ without coefficients, plugging the correspondence with trivial coefficients from Example 2.10 into the above construction recovers the C*-correspondence

$$C^*(\Omega)\colon C^*(G)\to C^*(H)$$

as introduced in [38] and explored in the étale setting in [3]. The section spaces $\Gamma_c(G, s^*\mathcal{A})$, $\Gamma_c(\Omega, \mathcal{E})$ and $\Gamma_c(H, s^*\mathcal{B})$ become the spaces $C_c(G)$, $C_c(\Omega)$ and $C_c(H)$.

We will break the proof of Theorem 2.22 down into the construction of the Hilbert module and the construction of the structure map. Note that the formulae (2.2) and (2.3) depend only on the data of the right action $(\mathcal{E}, \Omega) \curvearrowleft (\mathcal{B}, H)$.

Proposition 2.23 (The crossed product Hilbert module)**.** *Let H be an étale groupoid, let B be an H-C*-algebra, let Ω be a free, proper étale right H-space and let $\mathcal{E} \to \Omega$ be a $\Omega \rtimes H$-Hilbert $\sigma^*\mathcal{B}$-bundle. Then there is a Hilbert $H \ltimes B$-module $\Omega \ltimes E$ containing a dense copy of $\Gamma_c(\Omega, \mathcal{E})$ whose inner product and right module structure extend (2.2) and (2.3).*

Proof. The inner product $\Gamma_c(\Omega, \mathcal{E}) \times \Gamma_c(\Omega, \mathcal{E}) \to \Gamma_c(H, s^*\mathcal{B})$ is clearly linear in the second argument and the right action $\Gamma_c(\Omega, \mathcal{E}) \times \Gamma_c(H, s^*\mathcal{B}) \to \Gamma_c(\Omega, \mathcal{E})$ is clearly bilinear. It is straightforward to verify that the right action respects composition, that the inner product is conjugate symmetric and that the right action and inner product are compatible:

$$\begin{aligned} \xi \cdot (b * c) &= (\xi \cdot b) \cdot c \quad \text{for all } \xi \in \Gamma_c(\Omega, \mathcal{E}) \text{ and } b, c \in \Gamma_c(H, s^*\mathcal{B}) \\ \langle \xi, \eta \rangle^* &= \langle \eta, \xi \rangle \qquad\qquad\qquad\qquad\quad \text{for all } \xi, \eta \in \Gamma_c(\Omega, \mathcal{E}) \\ \langle \xi, \eta \cdot b \rangle &= \langle \xi, \eta \rangle * b \quad \text{for all } \xi, \eta \in \Gamma_c(\Omega, \mathcal{E}) \text{ and } b \in \Gamma_c(H, s^*\mathcal{B}) \end{aligned} \tag{2.5}$$

For each $\xi \in \Gamma_c(\Omega, \mathcal{E})$ and $\omega \in \Omega$, we have an inequality $\langle \xi, \xi \rangle(\sigma(\omega)) \geq \langle \xi(\omega), \xi(\omega) \rangle$, so $\langle \xi, \xi \rangle = 0$ implies that $\xi = 0$. The main missing ingredient is the positivity of the inner product. To prove this, we follow the proof of [3, Lemma 7.9], adding techniques from [61] to handle the C*-coefficients.

Let $\xi \in \Gamma_c(\Omega, \mathcal{E})$. We aim to show that $\langle \xi, \xi \rangle$ is a positive element of $H \ltimes B$. There are finitely many open bisections $U_i \subseteq \Omega$ and $\xi_i \in \Gamma_c(U_i, \mathcal{E})$ with $\|\xi_i\|_\infty \leq \|\xi\|_\infty$ such that $\xi = \sum_{i=1}^n \xi_i$. As in the proof of [3, Lemma 7.9], we take $\varphi_i \in C_c(q(U_i)) \subseteq C_c(\Omega/H)$ such that $\sum_{i=1}^n |\varphi_i(z)|^2 = 1$ for $z \in \bigcup_{k=1}^n \operatorname{supp}(q(\xi_k))$. Consider the Hilbert $(\sigma^* B)^{\Omega \rtimes H}$-module $E^{\Omega \rtimes H}$ of H-equivariant sections in $\Gamma_b(\Omega, \mathcal{E})$ that vanish at infinity with respect to Ω/H, whose operations are taken pointwise over Ω. For each i, the inner product induces the supremum norm on $\Gamma_c(U_i, \mathcal{E})$ and there is an isometric linear map $\Phi_i \colon \Gamma_c(U_i, \mathcal{E}) \to E^{\Omega \rtimes H}$ given by H-equivariant extension. The image of Φ_i is the set of equivariant sections supported in $U_i \cdot H$ which have compact support when restricted to U_i, and Φ_i^{-1} is just the restriction map. Applying [61, Lemma 6.3], for any $\epsilon > 0$ there are finitely many elements $\nu_j \in E^{\Omega \rtimes H}$ such that

$$\sum_{j=1}^m \nu_j \cdot \langle \nu_j, \Phi_k(\xi_k) \rangle \sim_\epsilon \Phi_k(\xi_k)$$

for each $1 \leq k \leq n$. We use $x \sim_\epsilon y$ to mean that $d(x, y) < \epsilon$. We then define

$$\eta_{i,j} := \Phi_i^{-1}(\varphi_i \cdot \nu_j) \in \Gamma_c(U_i, \mathcal{E}),$$

which maps $u \in U_i$ to $\varphi_i([u]_H)\nu_j(u)$. We claim that $\langle \xi, \xi \rangle$ is approximated by the positive operator $\sum_{i=1}^n \sum_{j=1}^m \langle \eta_{i,j}, \xi \rangle^* \langle \eta_{i,j}, \xi \rangle$. Unfolding the definition of the inner product and action shows that $\eta_{i,j} \cdot \langle \eta_{i,j}, \xi_k \rangle \in \Gamma_c(\Omega, \mathcal{E})$ is non-zero only on $\omega \in U_k$ for which there is a (necessarily unique) $h \in H^{\sigma(\omega)}$ such that $\omega \cdot h \in U_i$, and in this case

$$\begin{aligned} \eta_{i,j} \cdot \left\langle \eta_{i,j}, \xi_k \right\rangle (\omega) &= (\eta_{i,j}(\omega \cdot h) \cdot h^{-1}) \cdot \left\langle \eta_{i,j}(\omega \cdot h) \cdot h^{-1}, \xi_k(\omega) \right\rangle \\ &= (\varphi_i([\omega]_H)\nu_j(\omega)) \cdot \left\langle \varphi_i([\omega]_H)\nu_j(\omega), \xi_k(\omega) \right\rangle \\ &= (\varphi_i \cdot \nu_j) \cdot \left\langle \varphi_i \cdot \nu_j, \Phi_k(\xi_k) \right\rangle (\omega). \end{aligned}$$

It follows that $\Phi_k\left(\eta_{i,j} \cdot \left\langle \eta_{i,j}, \xi_k \right\rangle\right) = (\varphi_i \cdot \nu_j) \cdot \left\langle \varphi_i \cdot \nu_j, \Phi_k(\xi_k) \right\rangle$. Summing over i and j,

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^m \Phi_k\left(\eta_{i,j} \cdot \langle \eta_{i,j}, \xi_k \rangle\right) &= \sum_{i=1}^n \sum_{j=1}^m (\varphi_i \cdot \nu_j) \cdot \left\langle \varphi_i \cdot \nu_j, \Phi_k(\xi_k) \right\rangle \\ &= \sum_{j=1}^m \nu_j \cdot \langle \nu_j, \Phi_k(\xi_k) \rangle \\ &\sim_\epsilon \Phi_k(\xi_k), \end{aligned}$$

and as Φ_k is isometric we get $\sum_{i=1}^n \sum_{j=1}^m \eta_{i,j} \cdot \langle \eta_{i,j}, \xi_k \rangle \sim_\epsilon \xi_k$. For $e_i \in \Gamma_c(U_i, \mathcal{E})$ and $e_j \in \Gamma_c(U_j, \mathcal{E})$, the inner product $\langle e_i, e_j \rangle \in \Gamma_c(H, s^*\mathcal{B})$ is supported on a bisection and $\|\langle e_i, e_j \rangle\| \le \|e_i\| \|e_j\|$. We may calculate

$$\begin{aligned} \langle \xi, \xi \rangle &= \sum_{k=1}^n \sum_{l=1}^n \langle \xi_k, \xi_l \rangle \\ &\sim_{n^2 \|\xi\|_\infty \epsilon} \sum_{k=1}^n \sum_{l=1}^n \left\langle \xi_k, \sum_{i=1}^n \sum_{j=1}^m \eta_{i,j} \cdot \langle \eta_{i,j}, \xi_l \rangle \right\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^m \langle \eta_{i,j}, \xi \rangle^* \langle \eta_{i,j}, \xi \rangle. \end{aligned}$$

Sending ϵ to 0, we conclude that $\langle \xi, \xi \rangle \ge 0$.

We may now complete $\Gamma_c(\Omega, \mathcal{E})$ under the norm $\|\xi\| = \|\langle \xi, \xi \rangle\|^{\frac{1}{2}}$ to obtain $\Omega \ltimes E$, which comes with an inner product valued in $H \ltimes B$. For each $b \in \Gamma_c(H, s^*\mathcal{B})$ and $\xi \in \Gamma_c(\Omega, \mathcal{E})$, the inequality $\|\xi \cdot b\| \le \|\xi\| \|b\|$ follows from (2.5) and positivity of $\langle \xi, \xi \rangle$. We can therefore extend the action of each $b \in \Gamma_c(H, s^*\mathcal{B})$ to a bounded operator on $\Omega \ltimes E$. By the same inequality, this extends to all of $H \ltimes B$, so we obtain a continuous bilinear map $\Omega \ltimes E \times H \ltimes B \to \Omega \ltimes E$. Each condition for being a Hilbert module extends by continuity from the dense subspaces, so the right action and inner product define a $H \ltimes B$-Hilbert module structure on $\Omega \ltimes E$. $\square$

Proposition 2.24. *Let $(E, \Omega)\colon (A, G) \to (B, H)$ be a groupoid correspondence with C^*-coefficients. Then the left action of $\Gamma_c(G, s^*\mathcal{A})$ on $\Gamma_c(\Omega, \mathcal{E})$ described by formula (2.4) extends to a non-degenerate action $G \ltimes A \curvearrowright E \ltimes \Omega$ by adjointable operators.*

Proof. It is straightforward to verify that $\Gamma_c(G, s^*\mathcal{A})$ acts by adjointable operators on $\Gamma_c(\Omega, \mathcal{E})$ via (2.4) in the following sense.

$$\begin{aligned} \langle a \cdot \xi, \eta \rangle &= \langle \xi, a^* \cdot \eta \rangle \quad \text{for all } \xi, \eta \in \Gamma_c(\Omega, \mathcal{E}) \text{ and } a \in \Gamma_c(G, s^*\mathcal{A}) \\ a \cdot (b \cdot \xi) &= (a * b) \cdot \xi \quad \text{for all } a, b \in \Gamma_c(G, s^*\mathcal{A}) \text{ and } \xi \in \Gamma_c(\Omega, \mathcal{E}) \end{aligned} \tag{2.6}$$

Extending this to a non-degenerate $*$-homomorphism $G \ltimes A \to \mathcal{L}(\Omega \ltimes E)$ is a direct application of Lemma 1.83, with $D = H \ltimes B$, $F = \Omega \ltimes E$, $F_0 = \Gamma_c(\Omega, \mathcal{E})$ and $L\colon \Gamma_c(G, s^*\mathcal{A}) \times F_0 \to F$ given by the left action $\Gamma_c(G, s^*\mathcal{A}) \curvearrowright \Gamma_c(\Omega, \mathcal{E})$. The condition that L respects the $*$-algebra structure follows from (2.6), and the proof that the image of L has dense span follows from fibre-wise non-degeneracy of $\mathcal{A} \curvearrowright \mathcal{E}$: Let $U \subseteq \Omega$ be an open bisection, let $\xi \in \Gamma_c(U, \mathcal{E})$ and let $\epsilon > 0$. For each $u \in U$ there is some $a_u \in \Gamma_c(G^0, \mathcal{A})$ such that $(a_u)_{\rho(u)} \cdot \xi(u) \sim_\epsilon \xi(u)$. This relation holds on an open neighbourhood of u, and by compactness of $\operatorname{supp} \xi$ we obtain an open cover $U_1, \ldots, U_n$ of $\operatorname{supp} \xi$ with elements $a_i \in \Gamma_c(G^0, \mathcal{A})$ satisfying $(a_i)_{\rho(u)} \cdot \xi(u) \sim_\epsilon \xi(u)$ for $u \in U_i$. Let $\varphi_i \in C_c(U_i)$ be such that $\sum_{i=1}^n \varphi_i = 1$ on $\operatorname{supp} \xi$. Then ξ is ϵ-close to $\sum_{i=1}^n a_i \cdot (\varphi_i \xi_i) = \sum_{i=1}^n L(a_i, \varphi_i \xi_i)$. Such ξ span $\Gamma_c(\Omega, \mathcal{E})$ which is dense in $\Omega \ltimes E$, so the image of L has dense span. □

Proposition 2.25. *If $(E, \Omega)\colon (A, G) \to (B, H)$ is a proper correspondence, then $\Omega \ltimes E$ is a proper C*-correspondence from $G \ltimes A$ to $H \ltimes B$.*

Proof. As A is embedded non-degenerately in $G \ltimes A$, it is enough to see that A acts by compact operators. We first reduce to the special case where $(A, G) = (\mathcal{K}(E^{\Omega \rtimes H}), \Omega/H)$ as in Example 2.15. For $a \in A$ and $\eta \in \mathcal{K}(E^{\Omega \rtimes H}) \subseteq \Gamma_b(\Omega, \mathcal{K}(\mathcal{E}))$ and $\xi \in \Gamma_c(\Omega, \mathcal{E})$, the left actions are given by

$$\begin{aligned} a \cdot \xi &\colon \omega \mapsto a_{\rho(\omega)} \cdot \xi(\omega) \\ \eta \cdot \xi &\colon \omega \mapsto \eta(\omega) \cdot \xi(\omega). \end{aligned}$$

Through composition with the continuous map of bundles $\rho^*\mathcal{A} \to \mathcal{K}(\mathcal{E})$, there is a $*$-homomorphism $\varphi\colon A \to \Gamma_b(\Omega, \mathcal{K}(\mathcal{E}))$ such that for each $\omega \in \Omega$, $a \in A$ and $e \in E_\omega$, we have $\varphi(a)(\omega) \cdot e = a_{\rho(\omega)} \cdot e$. By the properness of $\Omega\colon G \to H$, the operator $\varphi(a)$ vanishes at infinity with respect to Ω/H. By Proposition 1.87, this means that $\varphi(a) \in \mathcal{K}(E^{\Omega \rtimes H})$. By design, for each $\xi \in \Gamma_c(\Omega, \mathcal{E})$ we have $a \cdot \xi = \varphi(a) \cdot \xi$, so it suffices to show that $K(E^{\Omega \rtimes H})$ acts by compact operators.

Let $U \subseteq \Omega$ be an open bisection and consider the inclusion $\Phi\colon \Gamma_c(U, \mathcal{E}) \to E^{\Omega \rtimes H}$ given by H-equivariant extension. For each $\xi_1, \xi_2 \in \Gamma_c(U, \mathcal{E})$, the compact operator $\Theta_{\Phi(\xi_1), \Phi(\xi_2)}$ acts on $\Omega \ltimes E$ as the compact operator Θ_{ξ_1, ξ_2}. Such operators $\Theta_{\Phi(\xi_1), \Phi(\xi_2)}$ generate $q(U)\mathcal{K}(E^{\Omega \rtimes H})$, which over all open bisections $U \subseteq \Omega$ generate $\mathcal{K}(E^{\Omega \rtimes H})$. We can conclude that $\mathcal{K}(E^{\Omega \rtimes H})$, and hence $G \ltimes A$, acts by compact operators.

□

Proposition 2.26 (Compatibility of composition and crossed product)**.** *Given composable groupoid correspondences with C*-coefficients*

$$(A, G) \xrightarrow[\Omega]{E} (B, H) \xrightarrow[\Lambda]{F} (C, K)$$

the crossed product is compatible with composition in that the following diagram commutes up to canonical isomorphism.

$$\begin{tikzcd} G \ltimes A \arrow[rr, "(\Lambda\circ\Omega)\ltimes(F\circ E)"] \arrow[dr, "\Omega\ltimes E"'] & & K \ltimes C \\ & H \ltimes B \arrow[ur, "\Lambda\ltimes F"'] & \end{tikzcd}$$

This means that the crossed product gives us a functor $\ltimes\colon \mathsf{GpdCorr}_{C^*} \to \mathsf{Corr}$.

Proof. First we prove that the Hilbert $K \ltimes C$-modules are isomorphic, by defining the following map.

$$\begin{aligned} \Omega \ltimes E \otimes_{H\ltimes B} \Lambda \ltimes F &\to (\Lambda \circ \Omega) \ltimes (F \circ E) \\ u\colon \Gamma_c(\Omega, \mathcal{E}) \times \Gamma_c(\Lambda, \mathcal{F}) &\to \Gamma_c(\Lambda \circ \Omega, \mathcal{F} \circ \mathcal{E}) \\ (e, f) &\mapsto u(e, f) \\ [\omega, \lambda]_H &\mapsto \sum_{h\in H^{\sigma(\omega)}} \Big[e(\omega \cdot h) \cdot h^{-1} \otimes h \cdot f(h^{-1} \cdot \lambda)\Big]_H \end{aligned}$$

The elements $u(e, f)$ are compactly supported and continuous because they are for e and f supported on open bisections $U \subseteq \Omega$ and $V \subseteq \Lambda$. The image of u has dense span, which we check by again considering U and V. The sets $V \circ U$ cover $\Lambda \circ \Omega$, so it's enough to check that $u(\Gamma_c(U, \mathcal{E}) \times \Gamma_c(V, \mathcal{F}))$ has dense span in $\Gamma_c(V \circ U, \mathcal{F} \circ \mathcal{E})$. We note that on $V \circ U$, the norm is the sup-norm, so we may apply Proposition 1.48 to check for dense span in $\Gamma_0(V \circ U, \mathcal{F} \circ \mathcal{E})$. By design we can observe pointwise dense span, and closure under the action of $C_0(V \circ U)$ is straightforward to show because the element of $C_0(V \circ U)$ can be absorbed into the element of $\Gamma_c(V, \mathcal{F})$ via the isomorphism $V \circ U \cong V$. Finally, u preserves the inner products, so it extends to a unitary isomorphism $\tilde{u}\colon \Omega \ltimes E \otimes_{H\ltimes B} \Lambda \ltimes F \to (\Lambda \circ \Omega) \ltimes (F \circ E)$. The bilinear map u interwines the left actions of $\Gamma_c(G, s^*\mathcal{A})$ on $\Gamma_c(\Omega, \mathcal{E})$ and $\Gamma_c(\Lambda \circ \Omega, \mathcal{F} \circ \mathcal{E})$. Therefore $\tilde{u}$ intertwines the left actions of $G \ltimes A$, and so we get an isomorphism of C*-correspondences. □

2.4. **The evaluation natural transformation.** Let $\Omega\colon G \to H$ be a groupoid correspondence. For each H-C*-algebra B, the evaluation correspondence $\Theta_{\Omega,B}$ from Example 2.13 is a correspondence of groupoids with C*-coefficients from $(\operatorname{Ind}_\Omega B, G)$ to (B, H) over Ω. We care about the evaluation correspondence because we can take its crossed product to get back to the more familiar land of C*-correspondences. For each H-algebra B we obtain the crossed product

$$\Omega \ltimes \Theta_{\Omega,B}\colon G \ltimes \operatorname{Ind}_\Omega B \to H \ltimes B. \tag{2.7}$$

This is proper because it is isomorphic to the crossed product of the proper correspondence $\Theta_{\Omega,B}\colon (\operatorname{Ind}_\Omega B, G \ltimes \Omega/H) \to (B, H)$. We can then take the K-theory to get a map $K_*(G \ltimes \operatorname{Ind}_\Omega B) \to K_*(H \ltimes B)$. In order to do computations with this map in K-theory, it will help to see the correspondence $\Omega \ltimes \Theta_{\Omega,B}$ at the local level with respect to Ω.

Lemma 2.27 (Local picture of the evaluation correspondence)**.** *Let $\Omega\colon G \to H$ be a correspondence of groupoids and let B be an H-C*-algebra. For every open bisection $U \subseteq \Omega$, the following diagram commutes in the (proper) correspondence category. The horizontal maps are induced by the respective inclusions of C*-algebras.*

$$\begin{tikzcd} \operatorname{Ind}_{U\cdot H} B \arrow[r] \arrow[d, "\mathit{Corr}(\mathrm{ev}_U)"] & G \ltimes \operatorname{Ind}_\Omega B \arrow[d, "\Omega\ltimes\Theta_{\Omega,B}"] \\ \sigma(U)B \arrow[r] & H \ltimes B \end{tikzcd}$$

Proof. The Hilbert $H \ltimes B$-module for the $\urcorner\!\downarrow$ composition is $\overline{\Gamma_c(UH, \sigma^*\mathcal{B})} \subseteq \Omega \ltimes \Theta_{\Omega,B}$. The $\downarrow\!\!\rightarrow$ composition is $\overline{(\sigma(U)B)H \ltimes B} = \overline{\Gamma_c(H^{\sigma(U)}, s^*\mathcal{B})}$. The homeomorphism $UH \cong H^{\sigma(U)}$ induces a unitary equivalence of these correspondences. □

Describing the crossed product is also easier when we start with an H-space Y. In this setting we can view the crossed product of the evaluation correspondence as the C*-correspondence induced by a groupoid correspondence.

Proposition 2.28 (The crossed product of the evaluation correspondence for a commutative C*-algebra)**.** *Let $\Omega\colon G \to H$ be an étale correspondence and let Y be an H-space. Then the crossed product*

$$\Omega \ltimes \Theta_{\Omega,C_0(Y)}\colon G \ltimes \operatorname{Ind}_\Omega C_0(Y) \to H \ltimes C_0(Y)$$

of the evaluation correspondence is induced by the groupoid correspondence

$$G \ltimes \Omega \times_H Y \curvearrowright \Omega \times_{H^0} Y \curvearrowleft H \ltimes Y.$$

The anchor maps are defined by $\rho(\omega, y) = [\omega, y]_H$ and $\sigma(\omega, y) = y$, with actions given by $(g, [\omega, y]_H) \cdot (\omega, y) = (g \cdot \omega, y)$ and $(\omega, h \cdot y) \cdot (h, y) = (\omega \cdot h, y)$.

Proof. We first note that we can describe both C*-algebras as the appropriate groupoid C*-algebras. We have $H \ltimes C_0(Y) = C^*(H \ltimes Y)$ and $\operatorname{Ind}_\Omega C_0(Y) = C_0(\Omega \times_H Y)$ so that $G \ltimes \operatorname{Ind}_\Omega C_0(Y) = C^*(G \ltimes \Omega \times_H Y)$.

Let $\mathcal{B} \to H^0$ be the C*-bundle associated to $C_0(Y)$, whose fibre at $z \in H^0$ is $C_0(Y_z)$. We wish to relate $C_c(\Omega \times_{H^0} Y)$ with $\Gamma_c(\Omega, \sigma^*\mathcal{B})$. Consider the inclusion

$$\begin{aligned} C_c(\Omega \times_{H^0} Y) &\hookrightarrow \Gamma_c(\Omega, \sigma^*\mathcal{B}) \\ \xi &\mapsto (\omega \mapsto (y \mapsto \xi(\omega, y))). \end{aligned}$$

The similar inclusion $C_c(H \ltimes Y) \hookrightarrow \Gamma_c(H, s^*\mathcal{B})$ completes to the identification $C^*(H \ltimes Y) = H \ltimes C_0(Y)$. Under this identification the inner products and module actions agree on $C_c(\Omega \times_{H^0} Y)$. To see that the image of $C_c(\Omega \times_{H^0} Y)$ is dense in $\Gamma_c(\Omega, \sigma^*\mathcal{B})$, pick an open bisection $U \subseteq \Omega$, and note that $C_c(U \times_{H^0} Y) \hookrightarrow \Gamma_c(U, \sigma^*\mathcal{B})$ is isomorphic to the dense inclusion $C_c(Y_{\sigma(U)}) \hookrightarrow C_0(Y_{\sigma(U)})$. We obtain an isomorphism of Hilbert modules $C^*(\Omega \times_{H^0} Y) \cong \Omega \ltimes \Theta_{\Omega, C_0(Y)}$. The identification $C^*(G \ltimes \Omega \times_H Y) = G \ltimes \operatorname{Ind}_\Omega C_0(Y)$ is obtained from the dense inclusion

$$\begin{aligned} C_c(G \ltimes \Omega \times_H Y) &\hookrightarrow \Gamma_c(G, s^* \operatorname{Ind}_\Omega \mathcal{B}) \\ a &\mapsto (g \mapsto (\omega \mapsto (y \mapsto a(g, [\omega, y]_H)))). \end{aligned}$$

For each $a \in C_c(G \ltimes \Omega \times_H Y)$ and $\xi \in C_c(\Omega \times_{H^0} Y)$, we may perform the left action $a \cdot \xi$ with the correspondence structure on $C^*(\Omega \times_{H^0} Y)$ or the structure on $\Omega \ltimes \Theta_{\Omega, C_0(Y)}$. It is straightforward to check that these agree, giving the same element of $\Gamma_c(\Omega, \sigma^*\mathcal{B})$. □

We are now equipped with the language to talk about the naturality of the evaluation correspondence $\Theta_{\Omega,B}$ in B. The algebra B sits in the H-equivariant correspondence category Corr^H, and $\Theta_{\Omega,B}$ is a correspondence in $\mathsf{GpdCorr}_{C^*}$ from $i_G(\operatorname{Ind}_\Omega B)$ to $i_H(B)$. Altogether, the assignment $\Theta_\Omega \colon B \mapsto \Theta_{\Omega,B}$ is a natural transformation from $i_G \circ \operatorname{Ind}_\Omega \colon \mathsf{Corr}^H \to \mathsf{GpdCorr}_{C^*}$ to $i_H \colon \mathsf{Corr}^H \to \mathsf{GpdCorr}_{C^*}$.

$$\Theta_\Omega \colon i_G \circ \operatorname{Ind}_\Omega \Rightarrow i_H \colon \mathsf{Corr}^H \rightrightarrows \mathsf{GpdCorr}_{C^*}.$$

We call Θ_Ω the *evaluation natural transformation*, and it is indeed natural:

Proposition 2.29 (Evaluation natural transformation)**.** *Let $\Omega \colon G \to H$ be a groupoid correspondence. The assignment $\Theta_\Omega \colon B \mapsto \Theta_{\Omega,B}$ is a natural transformation $i_G \circ \operatorname{Ind}_\Omega \Rightarrow i_H \colon \mathsf{Corr}^H \rightrightarrows \mathsf{GpdCorr}_{C^*}$.*

Proof. Suppose we have an H-equivariant correspondence $(E, \varphi) \colon B \to C$, and consider the following diagram.

$$\begin{tikzcd} (\operatorname{Ind}_\Omega B, G) \arrow[r, "i_G(\operatorname{Ind}_\Omega E)", "G"'] \arrow[d, "\Omega"', "\Theta_{\Omega,B}"] & (\operatorname{Ind}_\Omega C, G) \arrow[d, "\Omega"', "\Theta_{\Omega,C}"] \\ (B, H) \arrow[r, "i_H(E)", "H"'] & (C, H) \end{tikzcd}$$

Both compositions in this diagram involve an equivariant correspondence, so we can use Proposition 2.18 to describe the underlying $C_0(\Omega)$-correspondences from $\rho^* \operatorname{Ind}_\Omega B$ to $\sigma^* C$. For the $\urcorner$ composition, we have $\rho^* \operatorname{Ind}_\Omega E \otimes_{\rho^* \operatorname{Ind}_\Omega C} \sigma^* C$, and for the $\hookrightarrow$ composition we have $\sigma^* B \otimes_{\sigma^* B} \sigma^* E$. These are both isomorphic to the correspondence $(\sigma^* E, (\sigma^* \varphi) \circ \mathrm{ev})$. The isomorphism $\sigma^* B \otimes_{\sigma^* B} \sigma^* E \cong \sigma^* E$ is given by $b \otimes e \mapsto b \cdot e$, which intertwines the actions of G and H because the G-action only switches the fibre and H acts diagonally on the tensor product. The isomorphism $\rho^* \operatorname{Ind}_\Omega E \otimes_{\rho^* \operatorname{Ind}_\Omega C} \sigma^* C \cong \sigma^* E$ is given by $\eta \otimes c \mapsto \mathrm{ev}(\eta) \cdot c$, which intertwines the actions of G because given $\omega \in \Omega$, $\eta \in (\operatorname{Ind}_\Omega E)_{\rho(\omega)}$, $c \in C_{\sigma(\omega)}$ and $g \in G_{\rho(\omega)}$, we have $g \cdot ((\omega, \eta) \otimes (\omega, c)) = (g \cdot \omega, (g \cdot \eta) \otimes c)$, which maps to $(g \cdot \omega, \eta(\omega) \cdot c) = g \cdot (\omega, \eta(\omega) \cdot c)$. It is H-equivariant because given $h \in H^{\sigma(\omega)}$, we have $((\omega, \eta) \otimes (\omega, c)) \cdot h = (\omega \cdot h, \eta) \otimes (\omega \cdot h, h^{-1} \cdot c)$ and $(h^{-1} \cdot (\eta(\omega) \cdot c) = (h^{-1} \cdot \eta(\omega)) \cdot (h^{-1} \cdot c) = \eta(\omega \cdot h) \cdot (h^{-1} \cdot c)$. The induced isomorphism

$$\rho^* \operatorname{Ind}_\Omega E \otimes_{\rho^* \operatorname{Ind}_\Omega C} \sigma^* C \cong \sigma^* B \otimes_{\sigma^* B} \sigma^* E$$

of $C_0(\Omega)$-correspondences therefore intertwines the actions of G and H, giving us an isomorphism of correspondences $(\operatorname{Ind}_\Omega B, G) \to (C, H)$ over Ω. $\square$

Proposition 2.30 (Compatibility of the evaluation natural transformation with composition)**.** *Let $G \curvearrowright \Omega \curvearrowleft H$ and $H \curvearrowright \Lambda \curvearrowleft K$ be groupoid correspondences, and let C be a K-C*-algebra. Let $\varphi_C \colon \operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda C \to \operatorname{Ind}_{\Lambda \circ \Omega} C$ be the G-equivariant $*$-isomorphism from Proposition 2.7. Then the following diagram in $\mathsf{GpdCorr}_{C^*}$ commutes.*

$$\begin{tikzcd}
(\operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda C, G) \arrow[r, "\Theta_{\Omega, \operatorname{Ind}_\Lambda C}", "\Omega"'] \arrow[d, "i_G(\mathsf{Corr}^G(\varphi_C))"', "G"] & (\operatorname{Ind}_\Lambda C, H) \arrow[d, "\Theta_{\Lambda, C}"', "\Lambda"] \\
(\operatorname{Ind}_{\Lambda \circ \Omega} C, G) \arrow[r, "\Theta_{\Lambda \circ \Omega, C}", "\Lambda \circ \Omega"'] & (C, K)
\end{tikzcd} \tag{2.8}$$

Proof. Let X, Y and Z be the unit spaces of G, H and K respectively. Let the following commuting diagram describe the range and source maps of the three groupoid correspondences Ω, Λ and $\Lambda \circ \Omega$, as well as projection maps from the space $\Omega \times_Y \Lambda$.

$$\begin{tikzcd}
\Omega \arrow[r, "\sigma_\Omega"] \arrow[d, "\rho_\Omega"'] & Y & \Lambda \arrow[l, "\rho_\Lambda"'] \arrow[d, "\sigma_\Lambda"] \\
X & \Omega \times_Y \Lambda \arrow[l, "\pi_X"'] \arrow[r, "\pi_Z"] \arrow[ul, "\pi_\Omega"'] \arrow[u, "\pi_Y"'] \arrow[ur, "\pi_\Lambda"] \arrow[d, "q"] & Z \\
& \Lambda \circ \Omega \arrow[ul, "\rho"] \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

By Proposition 2.18, the bundle for the ↳ composition in (2.8) is given by $\sigma^* \mathcal{C} \to \Lambda \circ \Omega$, with the left action of $\operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda \mathcal{C}$ induced by φ_C and the evaluation correspondence $\Theta_{\Lambda \circ \Omega, C}$. Therefore the action $\operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda \mathcal{C} \curvearrowright \sigma^* \mathcal{C}$ is given by $(\rho([\omega, \lambda]_H), \xi) \cdot ([\omega, \lambda]_H, c) = ([\omega, \lambda]_H, \xi(\omega)(\lambda) c)$. The actions of G and K are clear.

The ↳ composition in (2.8) is given by the bundle $\Theta_{\Lambda, \mathcal{C}} \circ \Theta_{\Omega, \operatorname{Ind}_\Lambda \mathcal{C}}$. For $(\omega, \lambda) \in \Omega \times_Y \Lambda$, the fibre at $[\omega, \lambda]_H$ is isomorphic to $(\operatorname{Ind}_\Lambda C)_{\rho_\Lambda(\lambda)} \otimes_{(\operatorname{Ind}_\Lambda C)_{\rho_\Lambda(\lambda)}} C_{\sigma_\Lambda(\lambda)}$, which is isomorphic to $C_{\sigma([\omega, \lambda]_H)}$ via $\xi \otimes c \mapsto \xi(\lambda) c$. The continuity of this fibrewise isomorphic map of Banach bundles $\Theta_{\Lambda, \mathcal{C}} \circ \Theta_{\Omega, \operatorname{Ind}_\Lambda \mathcal{C}} \to \sigma^* \mathcal{C}$ follows from the continuity of the following H-invariant bilinear map.

$$\begin{aligned}
\sigma_\Omega^* \operatorname{Ind}_\Lambda \mathcal{C} \times_Y \sigma_\Lambda^* \mathcal{C} &\to \sigma^* \mathcal{C} \\
((\omega, \xi), (\lambda, c)) &\mapsto ([\omega, \lambda]_H, \xi(\lambda) c)
\end{aligned}$$

It is straightforward to verify that the actions of each of $\operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda \mathcal{C}$, G and K are all compatible with this isomorphism, so we obtain an isomorphism of correspondences in $\mathsf{GpdCorr}_{C^*}$. $\square$

Proposition 2.31 (The natural transformation of an identity correspondence)**.** *Let G be an étale groupoid and consider the identity correspondence $G \curvearrowright G \curvearrowleft G$. Then the correspondence $\Theta_{G,A}\colon (\operatorname{Ind}_G A, G) \to (A, G)$ is isomorphic to $i_G(\mathsf{Corr}^G(\psi_A))$, where $\psi_A\colon \operatorname{Ind}_G A \cong A$ is the G-equivariant $*$-isomorphism from Proposition 2.8.*

Proof. The underlying Banach bundle for both of these correspondences is $s^*\mathcal{A}$. All the actions agree, so these are the same groupoid correspondences with C*-coefficients. $\square$

In the case without coefficients, i.e. when we have a correspondence $\Omega\colon G \to H$ and $B = C_0(H^0)$, the proper correspondence $\Omega \ltimes \Theta_{\Omega,B}$ is from $G \ltimes \operatorname{Ind}_\Omega B = C^*(G \ltimes \Omega/H)$ to $C^*(H)$. We can view this as a factor of the correspondence $C^*(\Omega)\colon C^*(G) \to C^*(H)$, as there is a G-equivariant correspondence $\Delta\colon C_0(G^0) \to C_0(\Omega/H)$ induced by $\overline{\rho}\colon \Omega/H \to G^0$ such that $C^*(\Omega) \cong G \ltimes \Delta \otimes_{C^*(G \ltimes \Omega/H)} \Omega \ltimes \Theta_{\Omega,B}$. It turns out that we can do this more generally for a correspondence with C*-coefficients. We phrase this as a universal property.

Proposition 2.32 (Universal property of induction)**.** *Suppose we have a groupoid correspondence $\Omega\colon G \to H$ and an H-C*-algebra B. Then the evaluation correspondence $(\Theta_{\Omega,B}, \Omega)\colon (\operatorname{Ind}_\Omega B, G) \to (B, H)$ has the following universal property.*

For any correspondence of groupoids with C-coefficients $E\colon (A, G) \to (B, H)$ over Ω, there is a G-equivariant correspondence $\Delta(E)\colon A \to \operatorname{Ind}_\Omega B$, unique up to G-equivariant unitary equivalence, such that $\Theta_{\Omega,B} \circ i_G(\Delta(E)) \cong E$.*

$$
\begin{tikzcd}
(A,G) \arrow[rr, "E"', "\Omega"] \arrow[dr, dashed, "i_G(\Delta(E))", "G"'] & & (B,H) \\
& (\operatorname{Ind}_\Omega B, G) \arrow[ur, "\Theta_{\Omega,B}", "\Omega"'] &
\end{tikzcd}
$$

Furthermore, if $(E, \Omega)\colon (A, G) \to (B, H)$ is a proper correspondence, then $\Delta(E)$ is a proper G-equivariant correspondence.

Proof. Recall that E carries the structure of a $\Omega \rtimes H$-equivariant correspondence from ρ^*A to σ^*B. Furthermore, the actions $G \curvearrowright \Omega \curvearrowleft \Omega \rtimes H$ define the structure of a groupoid correspondence from G to $\Omega \rtimes H$ which we will denote $\tilde{\Omega}\colon G \to \Omega \rtimes H$ to distinguish it from the correspondence $\Omega\colon G \to H$. The G-C*-algebras $\operatorname{Ind}_{\tilde{\Omega}} \sigma^* B$ and $\operatorname{Ind}_\Omega B$ are canonically isomorphic as they are both given concretely by the subalgebra of H-equivariant sections in $\Gamma_b(\Omega, \sigma^*\mathcal{B})$ that vanish at infinity with respect to Ω/H. Through this canonical isomorphism we may view $\operatorname{Ind}_{\tilde{\Omega}} E$ as a

G-Hilbert $\operatorname{Ind}_\Omega B$-module. The left action $\mathcal{A} \times_{G^0} \mathcal{E} \to \mathcal{E}$ of A on E induces a left action of A on $\operatorname{Ind}_{\tilde{\Omega}} E$ by pointwise operations:

$$\begin{aligned} A \times \operatorname{Ind}_{\tilde{\Omega}} E &\to \operatorname{Ind}_{\tilde{\Omega}} E \\ \Gamma_0(G^0, \mathcal{A}) \times \Gamma_b(\Omega, \mathcal{E}) &\to \Gamma_b(\Omega, \mathcal{E}) \\ (a, e) &\mapsto a \cdot e \\ &\qquad\quad \omega \mapsto a_{\rho(\omega)} \cdot e_\omega \end{aligned}$$

This is well-defined and G-equivariant, and we obtain a G-equivariant correspondence $\Delta(E)\colon A \to \operatorname{Ind}_\Omega B$. If (E, Ω) is proper, each element of A acts as an element of $\operatorname{Ind}_{\tilde{\Omega}}(\mathcal{K}(E)) \cong \mathcal{K}(\Delta(E))$, so $\Delta(E)$ is proper.

To check that $\Theta_{\Omega,B} \circ i_G(\Delta(E)) \cong E$, we first note that as we are composing with a G-equivariant correspondence, Proposition 2.18 tells us that

$$\Theta_{\Omega,B} \circ i_G(\Delta(E)) \cong \rho^* \Delta(E) \otimes_{\rho^* \operatorname{Ind}_\Omega B} \Theta_{\Omega,B}.$$

We can now construct the following map of $\sigma^* B$-modules.

$$\begin{aligned} V\colon \rho^* \Delta(E) \otimes_{\rho^* \operatorname{Ind}_\Omega B} \Theta_{\Omega,B} &\to E \\ \rho^* \operatorname{Ind}_{\tilde{\Omega}} E \times \sigma^* B &\to \Gamma_0(\Omega, \mathcal{E}) \\ (\eta, b) &\mapsto V(\eta \otimes b) \\ &\qquad\quad \omega \mapsto \operatorname{ev}_\omega(\eta_\omega) \cdot b_\omega \end{aligned}$$

By Lemma 2.5, $\omega \mapsto \operatorname{ev}_\omega(\eta_\omega)$ is continuous and for each $\omega \in \Omega$, $\{\operatorname{ev}_\omega(\eta_\omega) \cdot b_\omega \colon \eta \in \rho^* \Delta(E),\ b \in \sigma^* B\}$ is dense in E_ω. The map V preserves the inner product on linear combinations of simple tensors and so is in particular well-defined. Combining these facts, V is an isomorphism of Hilbert $\sigma^* B$-modules. Verifying that the bundle map associated to V intertwines the actions of $\mathcal{A}$, G and H on $\mathcal{E}$ and $\Theta_{\Omega,\mathcal{B}} \circ i_G(\Delta(\mathcal{E}))$ is straightforward as it may be checked on simple tensors in each fibre of $\Theta_{\Omega,\mathcal{B}} \circ i_G(\Delta(\mathcal{E})) \to \Omega$. The map V is therefore an isomorphism of correspondences from (A, G) to (B, H) over Ω.

For uniqueness, suppose we have a G-equivariant correspondence $F\colon A \to \operatorname{Ind}_\Omega B$ with composition $\Theta_{\Omega,B} \circ i_G(F) \cong E$. We will demonstrate that $F \cong \Delta(\Theta_{\Omega,B} \circ i_G(F))$, and therefore $F \cong \Delta(E)$. Consider the following densely defined map on F.

$$\begin{aligned} U\colon F \cdot \operatorname{Ind}_\Omega B &\to \Delta(\Theta_{\Omega,B} \circ i_G(F)) \\ F \times \operatorname{Ind}_\Omega B &\to \Gamma_b(\Omega, \rho^* \mathcal{F} \otimes_{\rho^* \operatorname{Ind}_\Omega \mathcal{B}} \sigma^* \mathcal{B}) \\ (f, \eta) &\mapsto \rho^* f \otimes \eta \end{aligned}$$

Recall from Remark 1.91 that $\rho^* f \otimes \eta$ is the continuous section $\omega \mapsto f_{\rho(\omega)} \otimes \eta(\omega)$. The map U preserves the inner product, and is therefore well-defined on $F \cdot \operatorname{Ind}_\Omega B$ and extends to all of F. To check the density of the image of U in the Hilbert $\operatorname{Ind}_\Omega B$-module $\Delta(\Theta_{\Omega,B} \circ i_G(F))$, we view $\operatorname{Ind}_\Omega B$ as a $C_0(\Omega/H)$-algebra. The set

$\{\rho^* f \otimes \eta \mid f \in F,\ \eta \in \operatorname{Ind}_\Omega B\}$ is closed under the action of $C_0(\Omega/H)$ and for each $\omega \in \Omega$ its evaluation $\{f_{\rho(\omega)} \otimes \eta(\omega) \mid f \in F,\ \eta \in \operatorname{Ind}_\Omega B\}$ has dense span in $F_{\rho(\omega)} \otimes_{(\operatorname{Ind}_\Omega B)_{\rho(\omega)}} B_{\sigma(\omega)}$. By Proposition 1.48, the image of U is dense. Therefore $U\colon F \to \Delta(\Theta_{\Omega,B} \circ i_G(F))$ is a unitary isomorphism of Hilbert $\operatorname{Ind}_\Omega B$-modules. It is furthermore G-equivariant so that $F \cong \Delta(\Theta_{\Omega,B} \circ i_G(F))$, and we conclude that $F \cong \Delta(E)$. ☐

Proposition 2.33 (Compatibility of the universal property of induction with composition)**.** *Let $(E,\Omega)\colon (A,G) \to (B,H)$ and $(F,\Lambda)\colon (B,H) \to (C,K)$ be correspondences of groupoid with C^*-coefficients. Consider the G-equivariant $*$-isomorphism $\varphi_C\colon \operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda C \cong \operatorname{Ind}_{\Lambda\circ\Omega} C$ from Proposition 2.7. Then there is an isomorphism of G-equivariant correspondences from A to $\operatorname{Ind}_{\Lambda\circ\Omega} C$:*

$$\Delta(E) \otimes_{\operatorname{Ind}_\Omega B} \operatorname{Ind}_\Omega \Delta(F) \otimes_{\operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda C} \mathsf{Corr}^G(\varphi_C) \cong \Delta(F \circ E).$$

Proof. We could check this directly, but we can instead get it "for free" from our previous results. Consider the following diagram in $\mathsf{GpdCorr}_{C^*}$. Our aim is to show that the square of G-equivariant correspondences that maps via i_G to 6 commutes.

$$\begin{tikzcd}
(A,G) \arrow[rr, "E", "\Omega"'] \arrow[rrrr, bend left, "F\circ E", "\Lambda\circ\Omega"'] \arrow[dr, "i_G(\Delta(E))", "G"'] \arrow[ddrr, bend right, "i_G(\Delta(F\circ E))", "G"'] \arrow[rr, phantom, shift left=6ex, "1"] & & (B,H) \arrow[rr, "F", "\Lambda"'] \arrow[dr, "i_G(\Delta(F))"', "H"] & & (C,K) \\
& (\operatorname{Ind}_\Omega B, G) \arrow[ur, "\Theta_{\Omega,B}", "\Omega"'] \arrow[dr, "i_G(\operatorname{Ind}_\Omega \Delta(F))", "G"'] \arrow[u, phantom, "2"] & 4 & (\operatorname{Ind}_\Lambda C, H) \arrow[ur, "\Theta_{\Lambda,C}", "\Lambda"'] \arrow[u, phantom, "3"] & \\
6 & & (\operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda C, G) \arrow[ur, "\Theta_{\Omega,\operatorname{Ind}_\Lambda C}", "\Omega"'] \arrow[d, "i_G(\mathsf{Corr}^G(\varphi_C))", "G"'] & & 5 \\
& & (\operatorname{Ind}_{\Lambda\circ\Omega} C, G) \arrow[uurr, bend right, "\Theta_{\Lambda\circ\Omega,C}", "\Lambda\circ\Omega"'] & &
\end{tikzcd}$$

The triangle 1 commutes by definition. The triangles 2 and 3 commute by the universal properties of $\operatorname{Ind}_\Omega$ and $\operatorname{Ind}_\Lambda$ respectively (Proposition 2.32). The square 4 commutes by naturality of Θ_Ω (Proposition 2.29). The square 5 commutes by the compatibility of the evaluation natural transformations with composition (Proposition 2.30). Putting this all together, we see that $\Delta(E) \otimes_{\operatorname{Ind}_\Omega B} \operatorname{Ind}_\Omega \Delta(F) \otimes_{\operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda C} \mathsf{Corr}^G(\varphi_C)$ satisfies the universal property for $\operatorname{Ind}_{\Lambda\circ\Omega}$, and so by uniqueness (Proposition 2.32), we may conclude that

$$\Delta(E) \otimes_{\operatorname{Ind}_\Omega B} \operatorname{Ind}_\Omega \Delta(F) \otimes_{\operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda C} \mathsf{Corr}^G(\varphi_C) \cong \Delta(F \circ E).$$

$\square$

3. Induction in groupoid equivariant KK-theory

We start this section with a brief summary of groupoid equivariant KK-theory. This was developed by Le Gall in [46], with further accounts in [8, 70]. Many of the difficulties of Kasparov's theory are present even without groupoids, and we point to the standard reference [6] for a treatment of non-equivariant KK-theory.

3.1. **Groupoid equivariant KK-theory.** Kasparov's equivariant bivariant K-theory, also known as KK-theory, assigns to each pair (A, B) of G-C*-algebras an abelian group $\mathrm{KK}^G(A, B)$ of homotopy classes of Kasparov cycles. KK-theory has been a key tool in studying the K-theory of C*-algebras, which may be recovered from Kasparov's theory with $\mathrm{KK}(\mathbb{C}, A) \cong K_0(A)$ and $\mathrm{KK}(\mathbb{C}, SA) \cong K_1(A)$. Kasparov introduced the theory for G a locally compact group [40], but for us G is an étale groupoid.

The key feature of equivariant KK-theory is the Kasparov product, which allows us to compose a class in $\mathrm{KK}^G(A, B)$ with a class in $\mathrm{KK}^G(B, C)$ to get a class in $\mathrm{KK}^G(A, C)$. This allows us to view elements of $\mathrm{KK}^G(A, B)$ as "generalised G-equivariant morphisms" from A to B in a category KK^G which we call the G-equivariant Kasparov category. The construction of the Kasparov product is highly technical and comes at the price of assuming that certain algebras are separable, so we require that G-C*-algebras in KK^G be separable. We therefore also restrict attention to second countable étale groupoids G so that $C_0(G^0)$ and $C^*(G)$ are separable C*-algebras.

Standing assumption. When doing KK-theory we work only with second countable étale groupoids and separable C*-algebras, aside from algebras of adjointable operators and multiplier algebras. We consider only countably generated Hilbert modules, which therefore have separable algebras of compact operators.

Many accounts of KK-theory involve graded C*-algebras. However, as remarked by Meyer and Nest in [58], the Kasparov category of graded C*-algebras does not form a triangulated category, so we work instead with ungraded C*-algebras. However, we will still need to work with graded Hilbert modules and correspondences.

Definition 3.1 (Graded Hilbert module)**.** Let B be a C*-algebra. A *graded Hilbert B-module* is a Hilbert B-module E with distinguished Hilbert submodules E^+ and E^- such that $E = E^+ \oplus E^-$. We say that elements of E^+ have *degree* 0 and elements of E^- have *degree* 1, and we refer to these collectively as *homogeneous* elements. An adjointable operator $T \in \mathcal{L}(E, F)$ between graded Hilbert B-modules E and F is said to be of degree 0 if $TE^+ \subseteq F^+$ and $TE^- \subseteq F^-$, and it is said to be of degree 1 if $TE^+ \subseteq F^-$ and $TE^- \subseteq F^+$. Each adjointable operator T can be written uniquely as the sum $T^+ + T^-$ of a degree 0 operator T^+ and a degree 1 operator T^-.

Definition 3.2 (Graded correspondence)**.** Let A and B be C*-algebras. A *graded C*-correspondence* from A to B is a C*-correspondence $E\colon A \to B$ with a grading on E such that the left action of A respects the grading on E in that $A \cdot E^+ \subseteq E^+$ and $A \cdot E^- \subseteq E^-$.

Suppose we have a graded Hilbert B-module E and an étale groupoid G with unit space X. If B is a $C_0(X)$-algebra, then the fibres over X of E automatically respect the grading. If B is a G-C*-algebra, then for E to be a graded G-Hilbert B-module we ask additionally that the action of G respects the grading. This means that for each $g \in G$ and $e \in E^+_{s(g)}$, we have $g \cdot e \in E^+_{r(g)}$, and similarly for E^-.

The basic objects of study in KK-theory are Kasparov cycles, which are used to build the morphisms in the equivariant Kasparov category KK^G.

Definition 3.3 (Equivariant Kasparov cycle)**.** Let G be an étale groupoid and let A and B be G-C*-algebras. A *G-equivariant Kasparov A-B cycle* is a pair (E, T) where $E\colon A \to B$ is a graded G-equivariant correspondence with structure map φ, and $T \in \mathcal{L}(E)$ is an adjointable operator of degree 1 which is:

- *almost self-adjoint* meaning that $\varphi(a)(T - T^*) \in \mathcal{K}(E)$ for each $a \in A$,
- *almost unitary* meaning that $\varphi(a)(T^*T - 1) \in \mathcal{K}(E)$ for each $a \in A$,
- *almost commuting with* φ meaning that $[T, \varphi(a)] \in \mathcal{K}(E)$ for each $a \in A$,
- *almost invariant* meaning that $\varphi_{s(g)}(a)(g^{-1} \cdot T_{r(g)} - T_{s(g)}) \in \mathcal{K}(E_{s(g)})$ for each $g \in G$ and $a \in A_{s(g)}$, and this defines a continuous map $s^*\mathcal{A} \to s^*\mathcal{K}(\mathcal{E})$.

We call such operators *Fredholm operators.* When G, A and B are understood, we may simply call (E, T) a Kasparov cycle. We say that Kasparov cycles (E_1, T_1) and (E_2, T_2) are *unitarily equivalent*, written $(E_1, T_1) \sim_u (E_2, T_2)$, if there is a G-equivariant unitary operator $U \in \mathcal{L}(E_1, E_2)$ of degree 0 intertwining the actions of A and the Fredholm operators T_i. We write $\mathbb{E}^G(A, B)$ for the set of unitary equivalence classes of G-equivariant Kasparov A-B cycles. We frequently abuse notation and write $(E, T) \in \mathbb{E}^G(A, G)$, implicitly identifying unitarily equivalent Kasparov cycles.

We may take $E = 0$ and $T = 0$ to get the trivial Kasparov cycle $(0, 0)$. Given a graded G-equivariant correspondence $E\colon A \to B$, then $(E, 0)$ is a Kasparov cycle if and only if E is a proper correspondence. As a special case, every G-equivariant $*$-homomorphism $\varphi\colon A \to B$ gives rise to a Kasparov cycle $\mathbb{E}^G(\varphi) \in \mathbb{E}^G(A, B)$. In particular, for each G-C*-algebra A, there is an identity cycle $(A, 0)$. The direct sum $(E_1 \oplus E_2, T_1 \oplus T_2)$ of Kasparov cycles is again a Kasparov cycle.

Definition 3.4 (Functoriality of Kasparov cycles for $*$-homomorphisms)**.** Let A, B and C be G-C*-algebras, let (E, T) be a G-equivariant Kasparov A-B cycle and let $\varphi\colon B \to C$ be a non-degenerate G-equivariant $*$-homomorphism. Then $\varphi_*(E,T) := (E \otimes_B C, T \otimes 1)$ is a G-equivariant Kasparov A-C cycle.

We use this to define a notion of homotopy for Kasparov cycles. Given a G-C*-algebra B, we may consider the G-C*-algebra $C([0,1],B)$ which comes with an evaluation map $\mathrm{ev}_t\colon C([0,1],B) \to B$ for each $t \in [0,1]$ which is a non-degenerate G-equivariant $*$-homomorphism.

Definition 3.5 (Homotopy of Kasparov cycles)**.** We say that two G-equivariant Kasparov A-B cycles (E_1,T_1) and (E_2,T_2) are *homotopic*, written $(E_1,T_1) \sim_h (E_2,T_2)$, if there is a G-equivariant Kasparov A-$C([0,1],B)$ cycle (E,T) such that $\mathrm{ev}_{0*}(E,T) \sim_u (E_1,T_1)$ and $\mathrm{ev}_{0*}(E,T) \sim_u (E_2,T_2)$.

Homotopy is an equivalence relation on Kasparov cycles, and we write $[E,T]$ for the class of a Kasparov cycle (E,T). We define the Kasparov group $\mathrm{KK}^G(A,B)$ to be the set of homotopy classes of G-equivariant Kasparov A-B cycles. It is clear that homotopy of Kasparov cycles is compatible with direct sums of Kasparov cycles, so the direct sum $[E_1,T_1] \oplus [E_2,T_2] = [E_1 \oplus E_2, T_1 \oplus T_2]$ is a well-defined commutative binary operation on $\mathrm{KK}^G(A,B)$. In fact, this is an abelian group. The inverse of $[E,T]$ is given by $[E^{\mathrm{op}}, -T]$, where E^{op} is the Hilbert module E with the opposite grading.

We will not often construct homotopies of Kasparov cycles directly, as there are more straightforward notions which often suffice. A homotopy may be thought of as a continuous path of Kasparov cycles, whereas in an *operator homotopy*, see [6, Definition 17.2.2], the correspondence is instead fixed and only the Fredholm operator may vary. An important source of homotopies is from so-called compact perturbations. As in the non-equivariant setting [6, Corollary 17.2.6], the straight line segment between compact perturbations defines an operator homotopy and so a homotopy.

Proposition 3.6 (Compact perturbations are homotopic)**.** *Let $(E,\varphi)\colon A \to B$ be a graded G-equivariant correspondence and suppose that $T_0, T_1 \in \mathcal{L}(E)$ are Fredholm operators. Suppose that T_0 and T_1 are compact perturbations in the sense that for each $a \in A$, we have $\varphi(a)(T_0 - T_1) \in \mathcal{K}(E)$ and $(T_0 - T_1)\varphi(a) \in \mathcal{K}(E)$. Then $(E,T_0) \sim_h (E,T_1)$.*

The key technical feature of KK-theory is the Kasparov product, which serves as the composition in the equivariant Kasparov category KK^G. The construction of this product is involved and requires technical assumptions on the C*-algebras. However, it is possible to treat this construction as a black box, and instead recognise when a Kasparov cycle is the Kasparov product of two other Kasparov cycles. We know how to compose correspondences, so the difficulty lies in the Kasparov product of the Fredholm operators.

Definition 3.7 (Kasparov product)**.** Let $(E_1, T_1) \in \mathbb{E}^G(A, B)$ and $(E_2, T_2) \in \mathbb{E}^G(B, C)$ and $(E, T) \in \mathbb{E}^G(A, C)$ be Kasparov cycles, and let $\varphi\colon A \to \mathcal{L}(E)$ be the structure map of $E\colon A \to C$.

We say that a Fredholm operator $T \in \mathcal{L}(E_1 \otimes_B E_2)$ is *a Kasparov product* of T_1 and T_2 if

- the Fredholm operator T is a T_2*-connection for* E_1. This means that for each homogeneous $e_1 \in E_1$,
$$\theta_{e_1} T_2 - (-1)^{\deg(e_1)} T \theta_{e_1} \in \mathcal{K}(E_2, E)$$
$$\theta_{e_1} T_2^* - (-1)^{\deg(e_1)} T^* \theta_{e_1} \in \mathcal{K}(E_2, E)$$
where $\theta_{e_1} \in \mathcal{L}(E_2, E)$ is defined by $\theta_{e_1}(e_2) = e_1 \otimes e_2$,
- for each $a \in A$, we have $\varphi(a)[T_1 \otimes 1, T]\varphi(a)^* \geq 0 \bmod \mathcal{K}(E)$.

The set of Kasparov products of T_1 and T_2 is written $T_1 \#_B T_2$. We say that $(E, T) \in \mathbb{E}^G(A, C)$ is *a Kasparov product* of (E_1, T_1) and (E_2, T_2) if there is a unitary equivalence $E_1 \otimes_B E_2 \cong E$ of graded G-equivariant correspondences and up to this equivalence, T is a Kasparov product of T_1 and T_2. We write $(E_1, T_1) \#_B (E_2, T_2)$ for the set of (unitary equivalence classes of) Kasparov products of (E_1, T_1) and (E_2, T_2).

Remark 3.8. The commutator $[T_1 \otimes 1, T]$ is the graded commutator, and since both of these operators have degree 1, it is equal to $(T_1 \otimes 1)T + T(T_1 \otimes 1)$. All of our commutators are graded, but if one of the entries is degree 0 then the graded commutator agrees with the standard commutator, so this has not previously been relevant.

Example 3.9. The functoriality of Kasparov cycles is an example of the Kasparov product. Let A, B and C be G-C*-algebras, let $(E, T) \in \mathbb{E}^G(A, B)$ and let $\varphi\colon B \to C$ a non-degenerate G-equivariant $*$-homomorphism. Then $\varphi_*(E, T) \in (E, T) \#_B (C, 0)$.

Kasparov products exist and are unique up to homotopy, so it makes sense to talk of "the" Kasparov product at the level of Kasparov groups.

Theorem 3.10 (Existence and uniqueness of the Kasparov Product (see Chapitre 5 of [45]))**.** *Let* $(E_1, T_1) \in \mathbb{E}^G(A, B)$ *and* $(E_2, T_2) \in \mathbb{E}^G(B, C)$ *be Kasparov cycles. Then there is a Kasparov product* $(E, T) \in (E_1, T_1) \#_B (E_2, T_2)$ *which is unique up to homotopy. Furthermore, this descends to a bilinear operation which we call the Kasparov product*
$$- \otimes_B -\colon \mathrm{KK}^G(A, B) \times \mathrm{KK}^G(B, C) \to \mathrm{KK}^G(A, C)$$

that sends a pair $[E_1, T_1] \in \mathrm{KK}^G(A, B)$ *and* $[E_2, T_2] \in \mathrm{KK}^G(B, C)$ *of Kasparov classes to the class* $[E, T] \in \mathrm{KK}^G(A, C)$ *of their Kasparov product* (E, T)*. The Kasparov product is associative and has identities* $\mathrm{id}_A = [A, 0]$ *for each* G*-C*-algebra* A*.*

Definition 3.11 (The equivariant Kasparov categories)**.** Let G be a second countable étale groupoid. The G-equivariant Kasparov category KK^G is the category whose objects are separable G-C*-algebras and whose morphism sets are the Kasparov groups $\mathrm{KK}^G(A, B)$, with composition given by the Kasparov product.

One of the main reasons that we study the equivariant Kasparov categories KK^G is to better understand the crossed product $G \ltimes A$ of a G-C*-algebra A. Working in the equivariant category KK^G allows us to effectively deal with G-equivariant data, which we can then apply to the crossed product through the descent functor $\mathrm{KK}^G \to \mathrm{KK}$. Consider the crossed product functor $G \ltimes -\colon \mathsf{Corr}^G \to \mathsf{Corr}$ and reduced crossed product functor $G \ltimes_r -\colon \mathsf{Corr}^G \to \mathsf{Corr}$ between correspondence categories from Propositions 1.97 and 1.99. Given a Kasparov cycle $(E, T) \in \mathbb{E}^G(A, B)$, we aim to construct Kasparov cycles $(G \ltimes E, G \ltimes T) \in \mathbb{E}(G \ltimes A, G \ltimes B)$ and $(G \ltimes_r E, G \ltimes_r T) \in \mathbb{E}(G \ltimes_r A, G \ltimes_r B)$. Consider the following map on $\Gamma_c(G, s^*\mathcal{E})$.

$$\begin{aligned} \Gamma_c(G, s^*\mathcal{E}) &\to \Gamma_c(G, s^*\mathcal{E}) \\ \nu &\mapsto (g \mapsto (g^{-1} \cdot T_{r(g)}) \cdot \nu(g)) \end{aligned}$$

This extends to an adjointable operator $G \ltimes T \in \mathcal{L}(G \ltimes E)$ and an adjointable operator $G \ltimes_r T \in \mathcal{L}(G \ltimes_r E)$. These may be identified with $T \otimes 1$ under the isomorphisms $G \ltimes E \cong E \otimes_B G \ltimes B$ and $G \ltimes_r E \cong E \otimes_B G \ltimes_r B$ in Proposition 1.82.

Theorem 3.12 (The descent functors (see Chapitre 7 of [45]))**.** *Let* G *be an étale groupoid. For each* $A, B \in \mathrm{KK}^G$*, the assignment*

$$[E, T] \mapsto [G \ltimes E, G \ltimes T]\colon \mathrm{KK}^G(A, B) \to \mathrm{KK}(G \ltimes A, G \ltimes B)$$

is a well-defined homomorphism of groups. Furthermore, this defines a functor $G \ltimes -\colon \mathrm{KK}^G \to \mathrm{KK}$ *called the descent functor. Similarly, the assignment*

$$[E, T] \mapsto [G \ltimes_r E, G \ltimes_r T]\colon \mathrm{KK}^G(A, B) \to \mathrm{KK}(G \ltimes_r A, G \ltimes_r B)$$

is a well-defined homomorphism of groups, and defines a functor $G \ltimes_r -\colon \mathrm{KK}^G \to \mathrm{KK}$ *called the reduced descent functor.*

After passing from KK^G to KK, we may pass further to K-theory. The following result is due to Kasparov, see also [6, Corollary 18.5.4].

Proposition 3.13 (Relation of KK-theory to K-theory)**.** *There is a natural isomorphism* $K_0(A) \cong \mathrm{KK}(\mathbb{C}, A)$ *for each* $A \in \mathrm{KK}$*.*

We obtain a homomorphism $\mathrm{KK}(A, B) \to \mathrm{Hom}(K_*(A), K_*(B))$ through the Kasparov product, and therefore a K-theory functor $K_*\colon \mathrm{KK} \to \mathsf{Ab}_*$ which is naturally isomorphic to $(\mathrm{KK}(\mathbb{C}, -), \mathrm{KK}(\mathbb{C}, S-))$.

Another powerful result in K-theory is Bott periodicity, which says that $K_n \cong K_{n+2}$. At the level of Kasparov categories, we may say that the suspension functor $A \mapsto SA = C_0(\mathbb{R}, A)$ is an auto-equivalence.

Theorem 3.14 (Bott periodicity)**.** *There are natural* KK^G*-equivalences* $S^2A \cong A$ *for* $A \in \mathrm{KK}^G$*.*

For a proof of this in the non-equivariant setting, see [6, Corollary 19.2.2].

3.2. **The KK-theoretic induction functor.** We return to the setting of a correspondence $\Omega\colon G \to H$ of étale groupoids. We wish to extend the induction functor $\mathrm{Ind}_\Omega\colon \mathsf{Corr}^H \to \mathsf{Corr}^G$ to the KK-theoretic induction functor $\mathrm{Ind}_\Omega\colon \mathrm{KK}^H \to \mathrm{KK}^G$. Again, most of the details for this construction appear already in [8]. We place this in the framework of groupoid correspondences and describe the compatibility with composition of correspondences. Given a Kasparov cycle $(E, T) \in \mathbb{E}^H(B, C)$, we need to construct an induced Fredholm operator on $\mathrm{Ind}_\Omega E$. We use a cutoff function for Ω to construct an H-equivariant operator on σ^*E related to $\sigma^*T \in \mathcal{L}(\sigma^*E)$.

Definition 3.15 (Cutoff function)**.** Let G be an étale groupoid. A *cutoff function* for G is a continuous function $c\colon G^0 \to \mathbb{R}$ such that:

- for each $u \in G^0$, $c(u) \geq 0$.
- for each $u \in G^0$, we have $\sum_{g \in G_u} c(r(g)) = 1$.
- the map $c \circ r$ has proper support with respect to $s\colon G \to G^0$.

Recall that the last condition means that the map $s\colon \mathrm{supp}(c \circ r) \to G^0$ is proper. A *cutoff function for a groupoid correspondence* $\Omega\colon G \to H$ is a cutoff function for $\Omega \rtimes H$.

We first collect a couple of relevant facts.

Remark 3.16. For any étale groupoid G and continuous function $c\colon G^0 \to \mathbb{R}$ we have $\mathrm{supp}(c \circ r) = r^{-1}(\mathrm{supp}(c))$. The $\subseteq$ inclusion holds in general, and the $\supseteq$ inclusion holds because r is open.

Lemma 3.17. *Let G be an étale groupoid and let $c\colon G^0 \to \mathbb{R}$ be continuous. If $c \circ r$ has proper support with respect to $s\colon G \to G^0$ then $c\colon G^0 \to \mathbb{R}$ has proper support with respect to $q\colon G^0 \to G^0/G$. If G is proper, the converse also holds.*

Proof. First suppose that $s\colon \operatorname{supp}(c\circ r)\to G^0$ is proper, and let $K\subseteq G^0/G$ be compact. Pick $K'\subseteq G^0$ compact with $q(K')=K$. Then $q^{-1}(K)=K'G=r(s^{-1}(K'))$. Therefore

$$\begin{aligned} q^{-1}(K)\cap \operatorname{supp}(c) &= r(s^{-1}(K'))\cap \operatorname{supp}(c)\\ &= r(s^{-1}(K')\cap r^{-1}(\operatorname{supp}(c)))\\ &= r(s^{-1}(K')\cap \operatorname{supp}(c\circ r)),\end{aligned}$$

which is compact by the properness of s on $\operatorname{supp}(c\circ r)$. Now suppose that G is proper and that $q\colon \operatorname{supp}(c)\to G^0/G$ is proper, and let $K\subseteq G^0$ be compact. Then we have the equalities

$$s^{-1}(K)\cap \operatorname{supp}(c\circ r)=G_K^{\operatorname{supp}(c)}=G_K^{\operatorname{supp}(c)\cap KG}.$$

The set $\operatorname{supp}(c)\cap KG$ is compact by the properness of q on $\operatorname{supp}(c)$, as $KG=q^{-1}(q(K))$. Therefore by the properness of G, $s^{-1}(K)\cap\operatorname{supp}(c\circ r)$ is compact. ☐

Proposition 3.18 (Tu, Propositions 6.10 and 6.11 in [85])**.** *If an étale groupoid G admits a cutoff function, it is proper. Conversely, if G is proper with a σ-compact orbit space, then it admits a cutoff function.*

Proof. First, suppose G admits a cutoff function $c\colon G^0\to\mathbb{R}$. Then the map $s\colon \operatorname{supp}(c\circ r)\to G^0$ is proper, and due to the inversion homeomorphism $G\to G$, the map $r\colon \operatorname{supp}(c\circ s)\to G^0$ is also proper. Therefore, given $K_1,K_2\subseteq G^0$ compact, then both $r^{-1}(K_1)\cap\operatorname{supp}(c\circ s)$ and $s^{-1}(K_2)\cap\operatorname{supp}(c\circ r)$ are compact.

The summation condition implies that c does not vanish on any orbit in G^0, and therefore each $g\in G$ can be expressed as g_1g_2 with $g_1\in\operatorname{supp}(c\circ s)$ and $g_2\in\operatorname{supp}(c\circ r)$. Therefore we get the equation

$$G_{K_2}^{K_1}=(r^{-1}(K_1)\cap\operatorname{supp}(c\circ s))\cdot(s^{-1}(K_2)\cap\operatorname{supp}(c\circ r)),$$

which is compact, so G is proper.

Conversely, suppose G is proper with G^0/G σ-compact. Then G^0/G is locally compact, Hausdorff and σ-compact and is therefore paracompact, so we may take a locally finite open cover $\{U_i\}$ of G^0/G such that each U_i comes with a relatively compact open set $V_i\subseteq G^0$ with $q\colon V_i\to U_i$ a homeomorphism, and $0\le\psi_i\in C_c(G^0)$ with $V_i=\{x\in G^0\mid \psi_i(x)\neq 0\}$. We can then define a continuous function $f\colon G^0\to\mathbb{R}$ by

$$f(x)=\sum_i\psi_i(x).$$

It is crucial that $\{U_i\}$ is locally finite for this to be a pointwise finite sum and continuous. Given a compact neighbourhood K of a point $x\in G^0$, there are finitely many U_i covering $q(K)$ and only finitely many U_j can intersect each of those, so only finitely many V_j can intersect K. Therefore on K, f is a finite sum of continuous functions. By construction, $f\colon G^0\to\mathbb{R}$ has proper support with respect to q, so by Proposition 1.75, we obtain the continuous function $f_*\colon G^0/G \to \mathbb{R}$. We may finally define $c\colon G^0 \to \mathbb{R}$ by

$$c(x) := \frac{f(x)}{f_*(q(x))}$$

By construction c is positive and sums to 1 on each orbit. It has proper support with respect to q and G is proper, so by Lemma 3.17, $c \circ r$ is proper with respect to $s\colon G \to G^0$. □

In particular, if we have a groupoid correspondence $\Omega\colon G \to H$ such that Ω/H is σ-compact, then Ω admits a cutoff function. For the purposes of KK-theory, all the correspondences we consider are second countable, so this is automatic. This allows us to define the induced Fredholm operator, see the proof of [8, Proposition 4.7].

Definition 3.19 (Induced Fredholm operator)**.** Let $\Omega\colon G \to H$ be a groupoid correspondence and let $(E, T) \in \mathbb{E}^H(B, C)$ be a Kasparov cycle. Let $c\colon \Omega \to \mathbb{R}$ be a cutoff function. Recall that we may identify $\mathcal{L}(\operatorname{Ind}_\Omega E)$ with the H-equivariant sections in $\Gamma_b(\Omega, \sigma^*\mathcal{L}(\mathcal{E}))$. Under this identification, we define the *induced Fredholm operator* $\operatorname{Ind}_{\Omega,c} T \in \mathcal{L}(\operatorname{Ind}_\Omega E)$ by

$$\operatorname{Ind}_{\Omega,c} T\colon \omega \mapsto \sum_{h \in H^{\sigma(\omega)}} c(\omega \cdot h)(h \cdot T_{s(h)}) \in \mathcal{L}(E_{\sigma(\omega)}).$$

This is H-equivariant by construction, and bounded by the summation condition of c. To justify continuity, we may by Remark 1.53 check that $\operatorname{Ind}_{\Omega,c} T$ and its adjoint define continuous maps $\sigma^*\mathcal{E} \to \sigma^*\mathcal{E}$. We may take $\xi \in \Gamma_c(\Omega, \sigma^*\mathcal{E})$ to be compactly supported and consider the continuous section

$$\begin{aligned} \Omega \rtimes H &\to r^*\sigma^*\mathcal{E} \\ (\omega, h) &\mapsto c(\omega \cdot h)(h \cdot T_{s(h)})(\xi(\omega)). \end{aligned}$$

This is compactly supported by the properness condition of c. We may therefore apply Lemma 1.75 to deduce that $\omega \mapsto \operatorname{Ind}_{\Omega,c} T(\omega)(\xi(\omega))\colon \Omega \to \sigma^*\mathcal{E}$ is continuous. By Proposition 1.45 the induced map $\sigma^*\mathcal{E} \to \sigma^*\mathcal{E}$ is continuous, and similarly for the adjoint. The induced Fredholm operator $\operatorname{Ind}_{\Omega,c} T\colon \Omega \to \sigma^*\mathcal{L}(\mathcal{E})$ is therefore strictly continuous.

Lemma 3.20. *In the above setting, for each $\omega \in \Omega$, the operator*

$$\operatorname{Ind}_{\Omega,c} T(\omega) = \sum_{h \in H^{\sigma(\omega)}} c(\omega \cdot h)(h \cdot T_{s(h)}) \in \mathcal{L}(E_{\sigma(\omega)})$$

is a compact perturbation of $T_{\sigma(\omega)}$. Furthermore, the compact perturbation is continuous in ω in the sense that the maps

$$\begin{aligned} \sigma^*\mathcal{B} &\to \sigma^*\mathcal{K}(\mathcal{E}) \\ b &\mapsto \varphi_{\sigma(\omega)}(b)(\operatorname{Ind}_{\Omega,c} T(\omega) - T_{\sigma(\omega)}) \end{aligned} \qquad \begin{aligned} \sigma^*\mathcal{B} &\to \sigma^*\mathcal{K}(\mathcal{E}) \\ b &\mapsto (\operatorname{Ind}_{\Omega,c} T(\omega) - T_{\sigma(\omega)})\varphi_{\sigma(\omega)}(b) \end{aligned}$$

are continuous, where $\varphi\colon B \to \mathcal{L}(E)$ *is the structure map for the correspondence* $E\colon B \to C$.

Proof. That $\operatorname{Ind}_{\Omega,c} T(\omega)$ is a compact perturbation of $T_{\sigma(\omega)}$ for each $\omega \in \Omega$ follows straightforwardly from the Fredholm properties of T and the summation condition of the cutoff function c. To check then that the above maps are continuous, we may by Proposition 1.45 check that any $\xi \in \Gamma_c(\Omega, \sigma^*\mathcal{B})$ is mapped to a continuous section $\Omega \to \sigma^*\mathcal{K}(\mathcal{E})$. The section

$$\begin{aligned}&\Omega \rtimes H \to r^*\sigma^*\mathcal{K}(\mathcal{E})\\&(\omega, h) \mapsto c(\omega \cdot h)\varphi_{\sigma(\omega)}(\xi(\omega))(h \cdot T_{s(h)} - T_{\sigma(\omega)}) \in \mathcal{K}(E_{\sigma(\omega)})\end{aligned}$$

is continuous by almost invariance of T and compactly supported by the properness condition of c. Therefore by Lemma 1.75, the section

$$\begin{aligned}&\Omega \to \sigma^*\mathcal{K}(\mathcal{E})\\&\omega \mapsto \sum_{h \in H^{\sigma(\omega)}} c(\omega \cdot h)\varphi_{\sigma(\omega)}(\xi(\omega))(h \cdot T_{s(h)} - T_{\sigma(\omega)}) \in \mathcal{K}(E_{\sigma(\omega)})\end{aligned}$$

is well-defined, continuous and compactly supported. We may follow the same argument for the section with the action of ξ from the right. We may conclude that both maps of Banach bundles are continuous. □

We check that the induced Fredholm operator really does define a Fredholm operator:

Proposition 3.21 (Induced Fredholm operator)**.** *Let* $\Omega\colon G \to H$ *be a groupoid correspondence, let* $c\colon \Omega \to \mathbb{R}$ *be a cutoff function for* Ω *and let* $(E, T) \in \mathbb{E}^H(B, C)$ *be a Kasparov cycle. Then* $(\operatorname{Ind}_\Omega E, \operatorname{Ind}_{\Omega,c} T)$ *is a* G*-equivariant Kasparov* $\operatorname{Ind}_\Omega B$-$\operatorname{Ind}_\Omega C$ *cycle.*

Proof. By Proposition 2.4, for an operator on $\operatorname{Ind}_\Omega E$ (respectively $(\operatorname{Ind}_\Omega E)_x$) to be compact, it must have compact fibres at each $\omega \in \Omega$ (respectively Ω^x) which vary continuously and vanish at infinity with respect to Ω/H (respectively Ω^x/H). Let $\varphi\colon B \to \mathcal{L}(E)$ be the structure map for $E\colon B \to C$, and let $\hat{\varphi}\colon \operatorname{Ind}_\Omega B \to \mathcal{L}(\operatorname{Ind}_\Omega E)$ be the structure map for $\operatorname{Ind}_\Omega E\colon \operatorname{Ind}_\Omega B \to \operatorname{Ind}_\Omega C$. We first check that for each $\xi \in \operatorname{Ind}_\Omega B$ the following operators in $\mathcal{L}(\operatorname{Ind}_\Omega E)$ are compact:

$$\begin{gathered}\hat{\varphi}(\xi)((\operatorname{Ind}_{\Omega,c} T)^* - \operatorname{Ind}_{\Omega,c} T),\\\hat{\varphi}(\xi)((\operatorname{Ind}_{\Omega,c} T)^* \operatorname{Ind}_{\Omega,c} T - 1),\\{[\operatorname{Ind}_{\Omega,c} T, \hat{\varphi}(\xi)]}.\end{gathered}$$

These all vanish at infinity with respect to Ω/H because $\hat{\varphi}(\xi)$ does. The compactness of each fibre over Ω and the continuity of the associated sections $\Omega \to \sigma^*\mathcal{K}(\mathcal{E})$ follow from combining the Fredholm properties of T, the properness condition of c and Lemma 1.75 in much the same way as in the proof of Lemma 3.20.

To check almost invariance of $\operatorname{Ind}_{\Omega,c} T$, we have to show that for each $g \in G$ and each $\xi \in \operatorname{Ind}_\Omega B$ compactly supported with respect to Ω/H, the operator

$$\hat{\varphi}_{s(g)}(\xi_{s(g)})\left(g^{-1} \cdot (\operatorname{Ind}_{\Omega,c} T)_{r(g)} - (\operatorname{Ind}_{\Omega,c} T)_{s(g)}\right) \in \mathcal{L}((\operatorname{Ind}_\Omega E)_{s(g)}) \tag{3.1}$$

is compact and that it varies continuously in g. This operator vanishes at infinity with respect to $\Omega^{s(g)}/H$ because $\hat{\varphi}_{s(g)}(\xi_{s(g)})$ does. We then check that the evaluation at ω in $\mathcal{L}(E_{\sigma(\omega)})$ is compact for each $\omega \in \Omega^{s(g)}$. Using that $(g^{-1} \cdot (\operatorname{Ind}_{\Omega,c} T)_{r(g)})(\omega) = (\operatorname{Ind}_{\Omega,c} T)_{r(g)}(g \cdot \omega)$, this evaluation reduces to

$$(\hat{\varphi}(\xi)(\omega))(\operatorname{Ind}_{\Omega,c} T(g \cdot \omega) - \operatorname{Ind}_{\Omega,c} T(\omega)) \in \mathcal{L}(E_{\sigma(\omega)}). \tag{3.2}$$

This is compact because by Lemma 3.20, both $\operatorname{Ind}_{\Omega,c} T(g \cdot \omega)$ and $\operatorname{Ind}_{\Omega,c} T(\omega)$ are compact perturbations of $T_{\sigma(\omega)}$. To help us justify continuity, we consider the local homeomorphisms $\alpha\colon (g,\omega,h) \mapsto (g,\omega)\colon G \ltimes \Omega \rtimes H \to G \ltimes \Omega$ and $\beta\colon (g,\omega) \mapsto \sigma(\omega)\colon G \ltimes \Omega \to H^0$. Continuity of (3.2) in both g and ω as a function of $G \ltimes \Omega$ follows from applying Lemma 1.75 to the local homeomorphism α and the section

$$\begin{aligned} &G \ltimes \Omega \rtimes H \to \alpha^*\beta^*\mathcal{K}(\mathcal{E}) \\ &\quad (g,\omega,h) \mapsto (c(g \cdot \omega \cdot h) - c(\omega \cdot h))(\hat{\varphi}(\xi)(\omega))(h \cdot T_{s(h)} - T_{\sigma(\omega)}), \end{aligned}$$

which is continuous and well-defined by almost invariance of T. It is proper with respect to α because we chose ξ to be compactly supported with respect to Ω/H and $c\colon \Omega \to \mathbb{R}_{\geq 0}$ has proper support with respect to $q\colon \Omega \to \Omega/H$ by Lemma 3.17. With g fixed, continuity in ω shows that the operator (3.1) is compact. By Lemma 2.2, the fact that these compact operators vary continuously in g follows from the joint continuity of (3.2) with respect to g and ω. The pair $(\operatorname{Ind}_\Omega E, \operatorname{Ind}_{\Omega,c} T)$ is therefore a Kasparov cycle. □

We can now define the KK-theoretic induction functor.

Definition 3.22 (The KK-theoretic induction functor)**.** Let $\Omega\colon G \to H$ be a second countable correspondence of second countable étale groupoids. The *KK-theoretic induction functor* $\operatorname{Ind}_\Omega\colon \mathrm{KK}^H \to \mathrm{KK}^G$ is given by the following.

- The H-C*-algebra $B \in \mathrm{KK}^H$ is mapped to $\operatorname{Ind}_\Omega B \in \mathrm{KK}^G$.
- The class $[E,T] \in \mathrm{KK}^H(B,C)$ is mapped to the class $[\operatorname{Ind}_\Omega E, \operatorname{Ind}_{\Omega,c} T] \in \mathrm{KK}^G(\operatorname{Ind}_\Omega B, \operatorname{Ind}_\Omega C)$, where $c\colon \Omega \to \mathbb{R}$ is any cutoff function for Ω.

Recall that by Proposition 3.18, there is at least one cutoff function c. Thankfully, it does not matter which one we pick.

Proposition 3.23 (Well-definition of the KK-theoretic induction functor)**.** *The above definition is well-defined for each cutoff function and independent of the cutoff function.*

Proof. Fix a cutoff function c and suppose that (E_0, T_0) and (E_1, T_1) are homotopic Kasparov cycles via a Kasparov cycle $(F, S) \in \mathbb{E}^H(B, C([0,1], C))$. We aim to find a homotopy from $(\operatorname{Ind}_\Omega E_0, \operatorname{Ind}_{\Omega,c} T_0)$ to $(\operatorname{Ind}_\Omega E_1, \operatorname{Ind}_{\Omega,c} T_1)$. After identifying $\operatorname{Ind}_\Omega C([0,1], C)$ with $C([0,1], \operatorname{Ind}_\Omega C)$, we can take the homotopy to be the Kasparov cycle

$$(\operatorname{Ind}_\Omega F, \operatorname{Ind}_{\Omega,c} S) \in \mathbb{E}^G(\operatorname{Ind}_\Omega B, \operatorname{Ind}_\Omega C([0,1], C)).$$

Therefore $[E, T] \mapsto [\operatorname{Ind}_\Omega E, \operatorname{Ind}_{\Omega,c} T] \colon \mathrm{KK}^H(B, C) \to \mathrm{KK}^G(\operatorname{Ind}_\Omega B, \operatorname{Ind}_\Omega C)$ is well-defined for each cutoff function c.

Now suppose that c_0 and c_1 are two different cutoff functions. For any $\xi \in \operatorname{Ind}_\Omega B$, the operator $\xi(\omega) \cdot (\operatorname{Ind}_{\Omega,c_0} T(\omega) - \operatorname{Ind}_{\Omega,c_1} T(\omega))$ is compact for each $\omega \in \Omega$ and varies continuously in ω by Lemma 3.20. By Proposition 2.4 this section defines a compact operator and so $\operatorname{Ind}_{\Omega,c_0} T$ and $\operatorname{Ind}_{\Omega,c_1} T$ are compact perturbations. Therefore by Proposition 3.6 $(\operatorname{Ind}_\Omega E, \operatorname{Ind}_{\Omega,c_0} T)$ and $(\operatorname{Ind}_\Omega E, \operatorname{Ind}_{\Omega,c_1} T)$ are homotopic, so $[\operatorname{Ind}_\Omega E, \operatorname{Ind}_{\Omega,c} T] \in \mathrm{KK}^G(\operatorname{Ind}_\Omega B, \operatorname{Ind}_\Omega C)$ is independent of the cutoff function c. □

Theorem 3.24 (The KK-theoretic induction functor)**.** *The map* $\operatorname{Ind}_\Omega \colon \mathrm{KK}^H \to \mathrm{KK}^G$ *defines a homomorphism of Kasparov groups and respects the Kasparov product and identity classes. In other words,* $\operatorname{Ind}_\Omega$ *is an additive functor.*

Proof. The assignment $(E, T) \mapsto (\operatorname{Ind}_\Omega E, \operatorname{Ind}_{\Omega,c} T)$ preserves direct sums of Kasparov cycles, so defines a homomorphism of the Kasparov groups. The identity class at $B \in \mathrm{KK}^H$ is represented by the Kasparov cycle $(B, 0) \in \mathbb{E}^H(B, B)$, which is mapped to the identity class $[\operatorname{Ind}_\Omega B, 0] \in \mathrm{KK}^G(\operatorname{Ind}_\Omega B, \operatorname{Ind}_\Omega B)$. To show that $\operatorname{Ind}_\Omega$ respects the Kasparov product, we will show that if $(E_1, T_1) \in \mathbb{E}^H(A, B)$, $(E_2, T_2) \in \mathbb{E}^H(B, C)$ and $(E, T) \in (E_1, T_1) \#_B (E_2, T_2)$, then

$$(\operatorname{Ind}_\Omega E, \operatorname{Ind}_{\Omega,c} T) \in (\operatorname{Ind}_\Omega E_1, \operatorname{Ind}_{\Omega,c} T_1) \#_{\operatorname{Ind}_\Omega B} (\operatorname{Ind}_\Omega E_2, \operatorname{Ind}_{\Omega,c} T_2).$$

Let $\varphi \colon A \to \mathcal{L}(E)$ and $\varphi_1 \colon A \to \mathcal{L}(E_1)$ be the structure maps for E and E_1. We need to show that:

- for each homogeneous $\xi_1 \in \operatorname{Ind}_\Omega E_1$, the operator $\theta_{\xi_1} \in \mathcal{L}(\operatorname{Ind}_\Omega E_2, \operatorname{Ind}_\Omega E)$ given by $\xi_2 \mapsto \xi_1 \otimes \xi_2$ under the identification $\operatorname{Ind}_\Omega E_1 \otimes_{\operatorname{Ind}_\Omega B} \operatorname{Ind}_\Omega E_2 \cong \operatorname{Ind}_\Omega E$ satisfies

$$\begin{aligned} \theta_{\xi_1}(\operatorname{Ind}_{\Omega,c} T_2) - (-1)^{\deg(\xi_1)}(\operatorname{Ind}_{\Omega,c} T)\theta_{\xi_1} &\in \mathcal{K}(\operatorname{Ind}_\Omega E_2, \operatorname{Ind}_\Omega E), \\ \theta_{\xi_1}(\operatorname{Ind}_{\Omega,c} T_2)^* - (-1)^{\deg(\xi_1)}(\operatorname{Ind}_{\Omega,c} T)^*\theta_{\xi_1} &\in \mathcal{K}(\operatorname{Ind}_\Omega E_2, \operatorname{Ind}_\Omega E). \end{aligned} \tag{3.3}$$

- for each $\eta \in \operatorname{Ind}_\Omega A$,

$$\operatorname{Ind}_\Omega(\varphi)(\eta) \left[\operatorname{Ind}_{\Omega,c} T_1 \otimes 1, \operatorname{Ind}_{\Omega,c} T\right] \operatorname{Ind}_\Omega(\varphi)(\eta^*) \geq 0 \bmod \mathcal{K}(\operatorname{Ind}_\Omega E). \tag{3.4}$$

By Proposition 2.4, in order to show that the operators in (3.3) are compact, we may show that they vanish at infinity with respect to Ω/H and that their fibres at $\omega \in \Omega$ are compact and vary continuously in ω. They vanish at infinity because θ_{ξ_1} does. Using the symbol $\sim$ to denote two operators differing by a compact operator, we make the following calculation in $\mathcal{L}((E_2)_{\sigma(\omega)}, E_{\sigma(\omega)})$.

$$\begin{aligned}
(\theta_{\xi_1}(\operatorname{Ind}_{\Omega,c} T_2))(\omega) &= \theta_{\xi_1(\omega)} \sum_{h\in H^{\sigma(\omega)}} c(\omega\cdot h)(h\cdot (T_2)_{s(h)}) \\
&= \sum_{h\in H^{\sigma(\omega)}} c(\omega\cdot h)\left[h\cdot\left((h^{-1}\cdot\theta_{\xi_1(\omega)})(T_2)_{s(h)}\right)\right] \\
&= \sum_{h\in H^{\sigma(\omega)}} c(\omega\cdot h)\left[h\cdot\left(\theta_{h^{-1}\cdot\xi_1(\omega)}(T_2)_{s(h)}\right)\right] \\
&\sim (-1)^{\deg(\xi_1)} \sum_{h\in H^{\sigma(\omega)}} c(\omega\cdot h)\left[h\cdot\left(T_{s(h)}\theta_{h^{-1}\cdot\xi_1(\omega)}\right)\right] \\
&= (-1)^{\deg(\xi_1)} \sum_{h\in H^{\sigma(\omega)}} c(\omega\cdot h)\left(h\cdot T_{s(h)}\right)\theta_{\xi_1(\omega)} \\
&= (-1)^{\deg(\xi_1)}((\operatorname{Ind}_{\Omega,c} T)\theta_{\xi_1})(\omega).
\end{aligned}$$

Here, we are using the fact that T is a T_2-connection for E_1. To justify continuity of the section $\omega \mapsto (\theta_{\xi_1}(\operatorname{Ind}_{\Omega,c} T_2))(\omega) - (-1)^{\deg(\xi_1)}((\operatorname{Ind}_{\Omega,c} T)\theta_{\xi_1})(\omega) \colon \Omega \to \sigma^*\mathcal{K}(\mathcal{E})$, we may assume that ξ_1 is compactly supported with respect to Ω/H and then apply Lemma 1.75 to the local homeomorphism $r\colon \Omega \rtimes H \to \Omega$ and the continuous section

$$\begin{aligned}
&\Omega \rtimes H \to r^*\sigma^*\mathcal{K}(\mathcal{E}) \\
&(\omega, h) \mapsto c(\omega\cdot h)\left[h\cdot\left(\theta_{h^{-1}\cdot\xi_1(\omega)}(T_2)_{s(h)} - T_{s(h)}\theta_{h^{-1}\cdot\xi_1(\omega)}\right)\right].
\end{aligned}$$

This is compactly supported because c has proper support with respect to $q\colon \Omega \to \Omega/H$ by Lemma 3.17. The same calculation holds for the adjoints, so we have shown (3.3). To prove (3.4), consider the operator

$$R := \operatorname{Ind}_\Omega(\varphi)(\eta)\left[\operatorname{Ind}_{\Omega,c} T_1 \otimes 1, \operatorname{Ind}_{\Omega,c} T\right]\operatorname{Ind}_\Omega(\varphi)(\eta^*).$$

To check that $R \gtrsim 0$, we claim that it is enough to find bounded continuous sections $k_U \colon U \to \sigma^*\mathcal{K}(\mathcal{E})$ and $p_U \colon U \to \sigma^*\mathcal{L}(\mathcal{E})$ for each U in some open cover $\mathcal{U}$ of Ω such that for each $U \in \mathcal{U}$ and $\omega \in U$, we have $R(\omega) = p_U(\omega) + k_U(\omega)$ and $p_U(\omega) \geq 0$. This is because by Proposition 2.4, the restrictions $\mathcal{L}(\operatorname{Ind}_\Omega E) \to \Gamma_b(U, \sigma^*\mathcal{L}(\mathcal{E}))$ induce an embedding of C*-algebras

$$\overline{C_0(\Omega/H)\mathcal{L}(\operatorname{Ind}_\Omega E)}/\mathcal{K}(\operatorname{Ind}_\Omega E) \hookrightarrow \prod_{U\in\mathcal{U}} \Gamma_b(U, \sigma^*\mathcal{L}(\mathcal{E}))/\Gamma_b(U, \sigma^*\mathcal{K}(\mathcal{E})).$$

Positivity of R up to the compact operators may be checked in the larger C*-algebra, which is exactly our claimed sufficient condition. Let $\mathcal{U}$ be the open cover of open sets $U \subseteq \Omega$ on which σ is injective and thus a homeomorphism onto its image. Let $a = \operatorname{Ind}_\Omega(\varphi)(\eta)$ so that $a_\omega = \varphi_{\sigma(\omega)}(\eta(\omega))$ for $\omega \in \Omega$ and let $U \in \mathcal{U}$. By the positivity condition for T and T_1, the section $R_1 := a[\sigma^* T_1 \otimes 1, \sigma^* T]a^* : \Omega \to \sigma^* \mathcal{L}(\mathcal{E})$

$$\begin{aligned} R_1 : \Omega &\to \sigma^* \mathcal{L}(\mathcal{E}) \\ \omega &\mapsto a_\omega [(T_1)_{\sigma(\omega)} \otimes 1, T_{\sigma(\omega)}] a_\omega^* \end{aligned}$$

may be written on U as the sum $p_U + k$ of a bounded continuous section $k : U \to \sigma^* \mathcal{K}(\mathcal{E})$ and a positive bounded continuous section $p_U : U \to \sigma^* \mathcal{L}(\mathcal{E})$. Our aim now is to show that the section $R - R_1 : \Omega \to \sigma^* \mathcal{L}(\mathcal{E})$ lands in the compacts bundle and is continuous into it. We will write $\sim$ to denote that the difference of two sections in $\Gamma_b(\Omega, \sigma^* \mathcal{L}(\mathcal{E}))$ lies in $\Gamma_b(\Omega, \sigma^* \mathcal{K}(\mathcal{E}))$. Let

$$R_2 := a[\operatorname{Ind}_{\Omega,c} T_1 \otimes 1, \sigma^* T]a^* : \Omega \to \sigma^* \mathcal{L}(\mathcal{E}).$$

By Lemma 3.20, the sections $\operatorname{Ind}_{\Omega,c} T$ and $\sigma^* T$ are compact perturbations, and therefore $R \sim R_2$. By the Fredholm properties of T, we have $[a^*, \sigma^* T] \sim 0$ and $[a, \sigma^* T] \sim 0$. Setting $b := \operatorname{Ind}_\Omega(\varphi_1)(\eta)$, we may then calculate:

$$\begin{aligned} R - R_1 &\sim R_2 - R_1 \\ &= a \left[(\operatorname{Ind}_{\Omega,c} T_1 - \sigma^* T_1) \otimes 1, \sigma^* T \right] a^* \\ &\sim \left[a((\operatorname{Ind}_{\Omega,c} T_1 - \sigma^* T_1) \otimes 1)a^*, \sigma^* T \right] \\ &= \left[b(\operatorname{Ind}_{\Omega,c} T_1 - \sigma^* T_1) b \otimes 1, \sigma^* T \right]. \end{aligned}$$

The section $b(\operatorname{Ind}_{\Omega,c} T_1 - \sigma^* T_1)b : \Omega \to \sigma^* \mathcal{L}(\mathcal{E}_1)$ is pointwise compact and continuous into the compact operators bundle $\sigma^* \mathcal{K}(\mathcal{E}_1)$ again by Lemma 3.20. For any $\nu \in \Gamma_b(\Omega, \sigma^* \mathcal{K}(\mathcal{E}_1))$ of degree 1, the graded commutator $[\eta \otimes 1, \sigma^* T] = (\nu \otimes 1)\sigma^* T + \sigma^* T(\nu \otimes 1)$ satisfies $[\nu \otimes 1, \sigma^* T] \sim 0$. This holds because for any homogeneous $e, f \in \sigma^* E_1$ of opposite degree, it follows from T being a T_2 connection for E_1 that

$$\begin{aligned} (\Theta_{e,f} \otimes 1)\sigma^* T &= \theta_e \theta_f^* (\sigma^* T) \\ &\sim (-1)^{\deg(f)} \theta_e (\sigma^* T_2) \theta_f^* \\ &\sim (-1)^{\deg(e) + \deg(f)} (\sigma^* T) \theta_e \theta_f^* \\ &= -\sigma^* T (\Theta_{e,f} \otimes 1). \end{aligned}$$

As a result, $R - R_1 \sim 0$, and so R may be written as the required sum $p_U + k_U$ on U, with $p_U : U \to \sigma^* \mathcal{L}(\mathcal{E})$ positive and $k_U : U \to \sigma^* \mathcal{K}(\mathcal{E})$ continuously compact. We may conclude that $R \gtrsim 0$ and therefore that $\operatorname{Ind}_{\Omega,c} T$ is a Kasparov product of $\operatorname{Ind}_{\Omega,c} T_1$ and $\operatorname{Ind}_{\Omega,c} T_2$. □

Proposition 3.25 (Cutoff function for the composition of correspondences)**.** *Suppose we have groupoid correspondences $\Omega : G \to H$ and $\Lambda : H \to K$ with cutoff functions $c_\Omega : \Omega \to \mathbb{R}$ and $c_\Lambda : \Lambda \to \mathbb{R}$ for $\Omega \rtimes H$ and $\Lambda \rtimes K$. Define $c : \Lambda \circ \Omega \to \mathbb{R}$ by*

$$c([\omega, \lambda]_H) := \sum_{h \in H^{\sigma(\omega)}} c_\Omega(\omega \cdot h) c_\Lambda(h^{-1} \cdot \lambda)$$

Then c is a cutoff function for $\Lambda \circ \Omega$, which we may refer to as the product cutoff function.

Proof. We must check that c is well-defined, continuous and satisfies the summation properness conditions. The summation condition is clear and well-definition follows.

To check continuity, we first check that $c_\Omega \times c_\Lambda \colon \Omega \times_Y \Lambda \to \mathbb{R}$ has proper support with respect to the local homeomorphism $q\colon \Omega \times_Y \Lambda \to \Lambda \circ \Omega$. Let $V \subseteq \Lambda \circ \Omega$ be compact, and let $W \subseteq \Omega \times_Y \Lambda$ be a compact set with $q(W) = V$. Then we must check that $q^{-1}(V) \cap \operatorname{supp}(c_\Omega \times c_\Lambda)$ is compact. Noting that $\operatorname{supp}(c_\Omega \times c_\Lambda) = \operatorname{supp} c_\Omega \times_Y \operatorname{supp} c_\Lambda$, we have $q^{-1}(V) \cap \operatorname{supp}(c_\Omega \times c_\Lambda) = W \cdot H \cap (\operatorname{supp}(c_\Omega) \times_Y \operatorname{supp}(c_\Lambda))$. The compactness of this space follows from the following equality

$$W \cdot H \cap (\operatorname{supp}(c_\Omega) \times_Y \operatorname{supp}(c_\Lambda)) = \Big(W \cdot (\Omega \rtimes H)^{\pi_\Omega(W)}_{\operatorname{supp}(c_\Omega)}\Big) \cap \Omega \times_Y \operatorname{supp}(c_\Lambda),$$

noting that $W \cdot (\Omega \rtimes H)^{\pi_\Lambda(W)}_{\operatorname{supp}(c_\Omega)}$ is compact because c_Ω is a cutoff function and $\Omega \times_Y \operatorname{supp}(c_\Lambda)$ is closed. Therefore $c_\Omega \times c_\Lambda$ has proper support with respect to $q\colon \Omega \times_Y \Lambda \to \Lambda \circ \Omega$. The function $c\colon \Lambda \circ \Omega \to \mathbb{R}$ is obtained by summing the values of $c_\Omega \times c_\Lambda$ over the fibres of the local homeomorphism $q\colon \Omega \times_Y \Lambda \to \Lambda \circ \Omega$, and is therefore continuous by Lemma 1.75.

Now we need to check that $c\colon \Lambda \circ \Omega \to \mathbb{R}$ has proper support with respect to the local homeomorphism $q_K\colon \Lambda \circ \Omega \to (\Lambda \circ \Omega)/K$. Again we consider $V \subseteq \Lambda \circ \Omega$ compact. Our aim is to show that $VK \cap \operatorname{supp}(c)$ is compact. As $q^{-1}(VK) = WHK$ and $q(\operatorname{supp}(c_\Omega \times c_\Lambda)) = \operatorname{supp}(c)$, we have the following equality.

$$VK \cap \operatorname{supp}(c) = q\left(WHK \cap \operatorname{supp}(c_\Omega \times c_\Lambda)\right)$$

Let $W' = WH \cap (\operatorname{supp}(c_\Omega) \times_Y \Lambda)$, which is compact because c_Ω is a cutoff function. We then have the following equality

$$W'' := WHK \cap \operatorname{supp}(c_\Omega \times_Y c_\Lambda) = W' \cdot (\Lambda \rtimes K)^{\pi_\Lambda(W')}_{\operatorname{supp}(c_\Lambda)},$$

which is compact because c_Λ is a cutoff function. Finally, $VK \cap \operatorname{supp}(c) = q(W'')$ is compact, so we are done. $\square$

This allows us to show that the KK-theoretic induction functors are compatible with composition of correspondences in that there is a natural isomorphism $\operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda \cong \operatorname{Ind}_{\Lambda \circ \Omega}$ for composable correspondences Ω and Λ:

Proposition 3.26 (Compatibility of the induction functor with composition of correspondences)**.** *Let $\Omega\colon G \to H$ and $\Lambda\colon H \to K$ be correspondences, and consider for each K-C*-algebra C the G-equivariant $*$-isomorphism $\varphi_C\colon \operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda C \cong \operatorname{Ind}_{\Lambda\circ\Omega} C$ from Proposition 2.7. This induces a KK^G-equivalence which is natural in C with respect to KK^K.*

Proof. The map $\varphi_C\colon \operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda C \to \operatorname{Ind}_{\Lambda\circ\Omega} C$ is defined for $\xi \in \operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda C$ by $\varphi_C(\xi)\colon [\omega,\lambda]_H \mapsto \xi(\omega)(\lambda)$. Consider an element $[E,T] \in \mathrm{KK}^K(C,D)$. We aim to show that the following diagram in KK^G commutes.

$$\begin{tikzcd}
\operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda C \arrow[r, "\operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda[E,T]"] \arrow[d, "\mathrm{KK}^G(\varphi_C)"] & \operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda D \arrow[d, "\mathrm{KK}^G(\varphi_D)"] \\
\operatorname{Ind}_{\Lambda\circ\Omega} C \arrow[r, "\operatorname{Ind}_{\Lambda\circ\Omega}[E,T]"] & \operatorname{Ind}_{\Lambda\circ\Omega} D
\end{tikzcd} \tag{3.5}$$

Let c_Ω and c_Λ be cutoff functions for Ω and Λ and let $c\colon \Lambda\circ\Omega \to \mathbb{R}$ be the product cutoff function, defined as

$$c([\omega,\lambda]_H) := \sum_{h\in H^{\sigma(\omega)}} c_\Omega(\omega\cdot h)c_\Lambda(h^{-1}\cdot\lambda).$$

Then the induced elements of KK are given by

$$\begin{aligned}
\operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda[E,T] &= [\operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda E, \operatorname{Ind}_{\Omega,c_\Omega} \operatorname{Ind}_{\Lambda,c_\Lambda} T],\\
\operatorname{Ind}_{\Lambda\circ\Omega}[E,T] &= [\operatorname{Ind}_{\Lambda\circ\Omega} E, \operatorname{Ind}_{\Lambda\circ\Omega,c} T].
\end{aligned}$$

Adjusting the left action on $\operatorname{Ind}_{\Lambda\circ\Omega} E$ via φ_C and the Hilbert module structure on $\operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda E$ via φ_D, the respective Kasparov products in (3.5) are represented by $[\operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda E, \operatorname{Ind}_{\Omega,c_\Omega} \operatorname{Ind}_{\Lambda,c_\Lambda} T]$ and $[\operatorname{Ind}_{\Lambda\circ\Omega} E, \operatorname{Ind}_{\Lambda\circ\Omega,c} T]$. By Proposition 2.7, there is a G-equivariant unitary isomorphism $\varphi_E\colon \operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda E \cong \operatorname{Ind}_{\Lambda\circ\Omega} E$ of the underlying correspondences. We will show that under this isomorphism we can identify the Fredholm operators, in that the following diagram commutes:

$$\begin{tikzcd}
\operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda E \arrow[r, "\operatorname{Ind}_{\Omega,c_\Omega} \operatorname{Ind}_{\Lambda,c_\Lambda} T"] \arrow[d, "\varphi_E"] & \operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda E \arrow[d, "\varphi_E"] \\
\operatorname{Ind}_{\Lambda\circ\Omega} E \arrow[r, "\operatorname{Ind}_{\Lambda\circ\Omega,c} T"] & \operatorname{Ind}_{\Lambda\circ\Omega} E
\end{tikzcd}$$

To check this, let $\xi \in \operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda E$. Then $\varphi_E(\xi) \in \operatorname{Ind}_{\Lambda\circ\Omega} E \subseteq \Gamma_b(\Lambda\circ\Omega, \sigma^*\mathcal{E})$ is given by

$$\varphi_E(\xi)\colon [\omega,\lambda]_H \mapsto \xi(\omega)(\lambda),$$

and therefore $\operatorname{Ind}_{\Lambda\circ\Omega,c} T\varphi_E(\xi) \in \operatorname{Ind}_{\Lambda\circ\Omega} E \subseteq \Gamma_b(\Lambda\circ\Omega, \sigma^*\mathcal{E})$ is given by

$$\operatorname{Ind}_{\Lambda\circ\Omega,c} T\varphi_E(\xi)\colon [\omega,\lambda]_H \mapsto \sum_{k\in K^{\sigma(\lambda)}} c([\omega,\lambda]_H\cdot k)(k\cdot T_{s(k)})(\xi(\omega)(\lambda)).$$

On the other hand, $\varphi_E((\operatorname{Ind}_{\Omega,c_\Omega} \operatorname{Ind}_{\Lambda,c_\Lambda} T)(\xi)) \in \operatorname{Ind}_{\Lambda\circ\Omega} E \subseteq \Gamma_b(\Lambda\circ\Omega, \sigma^*\mathcal{E})$ is given by

$$\begin{aligned}
[\omega,\lambda]_H &\mapsto \operatorname{Ind}_{\Omega,c_\Omega} \operatorname{Ind}_{\Lambda,c_\Lambda} T(\xi)(\omega)(\lambda)\\
&= \sum_{h\in H^{\sigma(\omega)}} c_\Omega(\omega\cdot h)(h\cdot(\operatorname{Ind}_{\Lambda,c_\Lambda} T)_{s(h)})(\xi(\omega))(\lambda)\\
&= \sum_{h\in H^{\sigma(\omega)}} c_\Omega(\omega\cdot h)\left(h\cdot((\operatorname{Ind}_{\Lambda,c_\Lambda} T)_{s(h)}(h^{-1}\cdot\xi(\omega)))\right)(\lambda)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{h \in H^{\sigma(\omega)}} c_\Omega(\omega \cdot h) \Big((\operatorname{Ind}_{\Lambda, c_\Lambda} T)_{s(h)} (h^{-1} \cdot \xi(\omega))(h^{-1} \cdot \lambda) \Big) \\
&= \sum_{\substack{h \in H^{\sigma(\omega)}, \\ k \in K^{\sigma(\lambda)}.}} c_\Omega(\omega \cdot h) \Big(c_\Lambda(h^{-1} \cdot \lambda \cdot k)(k \cdot T_{s(k)}) \Big((h^{-1} \cdot \xi(\omega))(h^{-1} \cdot \lambda) \Big) \Big) \\
&= \sum_{k \in K^{\sigma(\lambda)}} c([\omega, \lambda]_H \cdot k)(k \cdot T_{s(k)})(\xi(\omega)(\lambda)).
\end{aligned}
$$

This demonstrates that $\varphi_E\colon \operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda E \cong \operatorname{Ind}_{\Lambda \circ \Omega} E$ identifies the Fredholm operators $\operatorname{Ind}_{\Omega, c_\Omega} \operatorname{Ind}_{\Lambda, c_\Lambda} T$ and $\operatorname{Ind}_{\Lambda \circ \Omega, c} T$. It follows that (3.5) commutes. □

The KK-theoretic induction functor of an identity correspondence $G\colon G \to G$ is naturally isomorphic to the identity functor on KK^G:

Proposition 3.27 (The identification of the identity correspondence induction functor with the identity)**.** *Let G be an étale groupoid. Consider the functor $\operatorname{Ind}_G\colon \mathrm{KK}^G \to \mathrm{KK}^G$ associated to the identity correspondence $G\colon G \to G$ and for each $A \in \mathrm{KK}^G$ the G-equivariant $*$-isomorphism $\psi_A\colon \operatorname{Ind}_G A \cong A$ from Proposition 2.8. This induces a KK^G-equivalence which is natural in A with respect to KK^G.*

Proof. The map ψ_A is defined by restricting elements of $\operatorname{Ind}_G A \subseteq \Gamma_b(G, s^*\mathcal{A})$ to the unit space G^0. Let $(E, T) \in \mathbb{E}^G(A, B)$ be a G-equivariant Kasparov A-B cycle. We aim to show that the following diagram in KK^G commutes.

$$
\begin{tikzcd}
\operatorname{Ind}_G A \arrow[r, "\operatorname{Ind}_G[E,T]"] \arrow[d, "\mathrm{KK}^G(\psi_A)"] & \operatorname{Ind}_G B \arrow[d, "\mathrm{KK}^G(\psi_B)"] \\
A \arrow[r, "{[E,T]}"] & B
\end{tikzcd}
\tag{3.6}
$$

The characteristic function of the unit space $c_G = \chi_{G^0}\colon G \to \mathbb{R}$ is a cutoff function for the identity correspondence, so the induced Kasparov element is given by $\operatorname{Ind}_G([E,T]) = [\operatorname{Ind}_G E, \operatorname{Ind}_{G, c_G} T] \in \mathrm{KK}^G(\operatorname{Ind}_G A, \operatorname{Ind}_G B)$. The induced Fredholm operator is given by $\operatorname{Ind}_{G, c_G} T\colon g \mapsto g^{-1} \cdot T_{r(g)}$.

Adjusting the left action on E via ψ_A and the Hilbert module structure on $\operatorname{Ind}_G E$ via ψ_B, the respective Kasparov products in (3.6) are represented by $[E, T]$ and $[\operatorname{Ind}_G E, \operatorname{Ind}_{G, c_G} T]$. By Proposition 2.8, there is a G-equivariant unitary isomorphism $\psi_E\colon \operatorname{Ind}_G E \cong E$ of the underlying correspondences also given by restriction to the unit space. The restriction of $\operatorname{Ind}_{G, c_G} T$ to the unit space G^0 is just T, so these Kasparov products agree. □

3.3. **The induction natural transformation.** Let $\Omega\colon G \to H$ be a correspondence of étale groupoids. The induction functor $\operatorname{Ind}_\Omega\colon \mathrm{KK}^H \to \mathrm{KK}^G$ gives us a relation between the equivariant Kasparov categories. Our aim is to complement this by relating the functors $K_*(G \ltimes -)\colon \mathrm{KK}^G \to \mathsf{Ab}_*$ and $K_*(H \ltimes -)\colon \mathrm{KK}^H \to \mathsf{Ab}_*$.

This takes the form of the *induction natural transformation*

$$\alpha_\Omega\colon K_*(G \ltimes \operatorname{Ind}_\Omega -) \Rightarrow K_*(H \ltimes -)\colon \mathrm{KK}^H \rightrightarrows \mathsf{Ab}_*.$$

For each separable H-C*-algebra $B \in \mathrm{KK}^H$ we consider again the crossed product $\Omega \ltimes \Theta_{\Omega,B}\colon G \ltimes \operatorname{Ind}_\Omega B \to H \ltimes B$ of the evaluation correspondence. This is proper (see the discussion after (2.7)) and so induces a map

$$\alpha_\Omega(B) := K_*(\Omega \ltimes \Theta_{\Omega,B})\colon K_*(G \ltimes \operatorname{Ind}_\Omega B) \to K_*(H \ltimes B) \tag{3.7}$$

in K-theory. The assignment $B \mapsto \alpha_\Omega(B)$ defines the induction natural transformation α_Ω. Proposition 2.29 tells us that $\Theta_{\Omega,B}$ is natural in B with respect to H-equivariant correspondences. Our aim now is to show that α_Ω is indeed natural with respect to morphisms in KK^H, justifying its name.

Proposition 3.28 (The induction natural transformation)**.** *Let $\Omega\colon G \to H$ be a correspondence of étale groupoids. Then the induction natural transformation $\alpha_\Omega\colon B \mapsto \alpha_\Omega(B)$ is natural with respect to KK^H.*

$$\alpha_\Omega\colon K_*(G \ltimes \operatorname{Ind}_\Omega -) \Rightarrow K_*(H \ltimes -)\colon \mathrm{KK}^H \rightrightarrows \mathit{Ab}_*$$

Proof. Let $(E,T) \in \mathbb{E}^H(B,C)$ be a Kasparov cycle, and let $c\colon \Omega \to [0,1]$ be a cutoff function for $\Omega\colon G \to H$. We will show that the diagram of Kasparov cycles

$$\begin{tikzcd} G \ltimes \operatorname{Ind}_\Omega B \arrow[r, "(G\ltimes \operatorname{Ind}_\Omega E, G\ltimes \operatorname{Ind}_{\Omega,c} T)"] \arrow[d, "(\Omega\ltimes\Theta_{\Omega,B},0)"] & G \ltimes \operatorname{Ind}_\Omega C \arrow[d, "(\Omega\ltimes\Theta_{\Omega,C},0)"] \\ H \ltimes B \arrow[r, "(H\ltimes E, H\ltimes T)"] & H \ltimes C \end{tikzcd}$$

commutes at the level of KK, from which it follows that α_Ω is a natural transformation.

By the naturality of the transformation Θ_Ω of the functors between the correspondence categories (Proposition 2.29) and the functoriality of the crossed product functor on correspondences with C*-coefficients (Proposition 2.26), we may identify the $G \ltimes \operatorname{Ind}_\Omega B$-$H \ltimes C$ correspondences

$$G \ltimes \operatorname{Ind}_\Omega E \otimes_{G\ltimes \operatorname{Ind}_\Omega C} \Omega \ltimes \Theta_{\Omega,C} \cong \Omega \ltimes \Theta_{\Omega,B} \otimes_{H\ltimes B} H \ltimes E \cong \Omega \ltimes \sigma^* E.$$

The isomorphism $\Phi\colon G \ltimes \operatorname{Ind}_\Omega E \otimes_{G\ltimes \operatorname{Ind}_\Omega C} \Omega \ltimes \Theta_{\Omega,C} \cong \Omega \ltimes \sigma^* E$ is induced by the map

$$\begin{aligned} \Gamma_c(G, s^* \operatorname{Ind}_\Omega \mathcal{E}) \times \Gamma_c(\Omega, \sigma^*\mathcal{C}) &\to \Gamma_c(\Omega, \sigma^*\mathcal{E}) \\ (\xi,\eta) &\mapsto \Phi(\xi\otimes\eta) \\ &\qquad \omega \mapsto \sum_{g\in G_{\rho(\omega)}} ((g^{-1}\cdot \xi(g^{-1}))(\omega))\cdot(\eta(g\cdot\omega)), \end{aligned}$$

and the isomorphism $\Psi\colon \Omega \ltimes \Theta_{\Omega,B} \otimes_{H\ltimes B} H \ltimes E \cong \Omega \ltimes \sigma^* E$ is induced by the map

$$\Gamma_c(\Omega, \sigma^*\mathcal{B}) \times \Gamma_c(H, s^*\mathcal{E}) \to \Gamma_c(\Omega, \sigma^*\mathcal{E})$$

$$\begin{aligned}(\mu,\nu) &\mapsto \Psi(\mu\otimes\nu)\\ &\qquad \omega \mapsto \sum_{h\in H^{\sigma(\omega)}} (h\cdot\mu(\omega\cdot h))\cdot\nu(h^{-1}).\end{aligned}$$

We can immediately see that $(\Omega\ltimes\sigma^*E, \Phi(G\ltimes \operatorname{Ind}_{\Omega,c} T\otimes 1)\Phi^{-1})$ is a Kasparov product for $(G\ltimes\operatorname{Ind}_\Omega E, G\ltimes\operatorname{Ind}_{\Omega,c} T)$ and $(\Omega\ltimes\Theta_{\Omega,C},0)$. Therefore it remains to show that

$$(\Omega\ltimes\sigma^*E, \Phi(G\ltimes\operatorname{Ind}_{\Omega,c}T\otimes 1)\Phi^{-1}) \in (\Omega\ltimes\Theta_{\Omega,B},0)\ \#_{H\ltimes B}\ (H\ltimes E, H\ltimes T).$$

This boils down to checking that $\Phi(G\ltimes\operatorname{Ind}_{\Omega,c}T\otimes 1)\Phi^{-1}$ is an $H\ltimes T$-connection. For each $\mu\in\Omega\ltimes\Theta_{\Omega,B}$ let $T_\mu\in\mathcal{L}(H\ltimes E,\Omega\ltimes\sigma^*E)$ be given by $\nu\mapsto\Psi(\mu\otimes\nu)$. Ultimately, we need to check that the operators

$$\begin{aligned}S_\mu &:= T_\mu(H\ltimes T) - \Phi(G\ltimes\operatorname{Ind}_{\Omega,c}T\otimes 1)\Phi^{-1}T_\mu &&\in\mathcal{L}(H\ltimes E,\Omega\ltimes\sigma^*E)\\ S'_\mu &:= T_\mu(H\ltimes T)^* - \Phi(G\ltimes\operatorname{Ind}_{\Omega,c}T\otimes 1)^*\Phi^{-1}T_\mu &&\in\mathcal{L}(H\ltimes E,\Omega\ltimes\sigma^*E)\end{aligned}$$

are compact. To help us compute S_μ, we claim that for each $R\in\mathcal{L}(\operatorname{Ind}_\Omega E)$, $\lambda\in\Gamma_c(\Omega,\sigma^*\mathcal{E})$ and $\omega\in\Omega$, we have

$$\Phi(G\ltimes R\otimes 1)\Phi^{-1}(\lambda)(\omega) = R(\omega)\lambda(\omega).$$

We may first define a $*$-homomorphism $\psi\colon\mathcal{L}(\operatorname{Ind}_\Omega E)\to\mathcal{L}(\Omega\ltimes\sigma^*E)$ by defining $\psi(R)(\lambda)(\omega)=R(\omega)\lambda(\omega)$ for each $R\in\mathcal{L}(\operatorname{Ind}_\Omega E)$, $\lambda\in\Gamma_c(\Omega,\sigma^*\mathcal{E})$ and $\omega\in\Omega$. To justify that $\|\psi(R)(\lambda)\|\le\|R\|\|\lambda\|$, we may apply Lemma 1.83, or calculate:

$$\begin{aligned}\langle\psi(R)\lambda,\psi(R)\lambda\rangle &= \langle\lambda,\psi(R^*R)\lambda\rangle\\ &= \|R\|^2\langle\lambda,\lambda\rangle - \langle\lambda,\psi(\|R\|^2-R^*R)\lambda\rangle\\ &= \|R\|^2\langle\lambda,\lambda\rangle - \left\langle\psi\left((\|R\|^2-R^*R)^{\frac12}\right)\lambda,\psi\left((\|R\|^2-R^*R)^{\frac12}\right)\lambda\right\rangle\\ &\le \|R\|^2\langle\lambda,\lambda\rangle.\end{aligned}$$

It is straightforward to verify that $\Phi\circ(G\ltimes R\otimes 1)$ agrees with $\psi(R)\circ\Phi$ on simple tensors $\xi\otimes\eta\in G\ltimes\operatorname{Ind}_\Omega E\otimes_{G\ltimes\operatorname{Ind}_\Omega C}\Omega\ltimes\Theta_{\Omega,C}$ with $\xi\in\Gamma_c(G,s^*\operatorname{Ind}_\Omega\mathcal{E})$ and $\eta\in\Gamma_c(\Omega,\sigma^*\mathcal{C})$, and therefore $\Phi(G\ltimes R\otimes 1)\Phi^{-1}=\psi(R)$ as claimed. We will show that

$$S_\mu = T_\mu(H\ltimes T) - \psi(\operatorname{Ind}_{\Omega,c}T)T_\mu$$

is compact by showing that $S_\mu^*S_\mu\in\mathcal{L}(H\ltimes E)$ is compact. To simplify notation we set $b_\omega := \varphi_{\sigma(\omega)}(\mu(\omega))$ for $\omega\in\Omega$, where $\varphi\colon B\to\mathcal{L}(E)$ is the structure map for $E\colon B\to C$. Assume for now that $\mu\in\Gamma_c(\Omega,\sigma^*\mathcal{B})\subseteq\Omega\ltimes\Theta_{\Omega,B}$, and consider the following.

- the element $\zeta\in\Gamma_c(\Omega,\sigma^*\mathcal{K}(\mathcal{E}))$ of the Hilbert $H\ltimes\mathcal{K}(E)$-module $\Omega\ltimes\Theta_{\Omega,\mathcal{K}(E)}$ given by

$$\zeta(\omega) := b_\omega T_{\sigma(\omega)} - \operatorname{Ind}_{\Omega,c}T(\omega)b_\omega\in\mathcal{K}(E_{\sigma(\omega)}),$$

- the $*$-homomorphism $\beta\colon H\ltimes\mathcal{K}(E)\to\mathcal{K}(H\ltimes E)$ from Corollary 1.98.

Note that ζ is continuous by Lemma 3.20 and the Fredholm property of T which implies that $\omega \mapsto [b_\omega, T_{\sigma(\omega)}]\colon \Omega \to \sigma^*\mathcal{K}(\mathcal{E})$ is continuous. We claim that

$$\beta(\langle\zeta,\zeta\rangle) = S_\mu^* S_\mu.$$

While we may directly compute S_μ and $\beta(\langle\zeta,\zeta\rangle)$ on an element $\nu \in \Gamma_c(H, s^*\mathcal{E}) \subseteq H \ltimes E$, we do not have such a closed form expression for S_μ^*. Instead we will show that for each $\nu_1, \nu_2 \in \Gamma_c(H, s^*\mathcal{E})$ we have

$$\langle S_\mu(\nu_1), S_\mu(\nu_2)\rangle = \langle \beta(\langle\zeta,\zeta\rangle)(\nu_1), \nu_2\rangle \in \Gamma_c(H, s^*B),$$

both sides of which we may directly compute. In order to compute $\langle S_\mu(\nu_1), S_\mu(\nu_2)\rangle$, we first compute for $\nu \in \Gamma_c(H, s^*\mathcal{E})$ and $\omega \in \Omega$:

$$\begin{aligned}
(T_\mu(H \ltimes T)(\nu))(\omega) &= \sum_{h \in H^{\sigma(\omega)}} (h \cdot b_{\omega\cdot h})\left(((H \ltimes T)(\nu))(h^{-1})\right)\\
&= \sum_{h \in H^{\sigma(\omega)}} (h \cdot b_{\omega\cdot h})\,(h \cdot T_{s(h)})\nu(h^{-1})\\
&= \sum_{h \in H^{\sigma(\omega)}} \left(h \cdot \left(b_{\omega\cdot h} T_{s(h)}\right)\right)\nu(h^{-1}),\\
(\psi(\operatorname{Ind}_{\Omega,c} T)T_\mu(\nu))(\omega) &= (\operatorname{Ind}_{\Omega,c} T)(\omega)(T_\mu(\nu))(\omega)\\
&= \sum_{h \in H^{\sigma(\omega)}} \operatorname{Ind}_{\Omega,c} T(\omega)(h \cdot b_{\omega\cdot h})\nu(h^{-1}).
\end{aligned}$$

For each $h \in H$, the element $\langle S_\mu(\nu_1), S_\mu(\nu_2)\rangle(h)$ breaks down into four terms which we will compute separately.

$$\begin{aligned}
&\left\langle T_\mu(H \ltimes T)(\nu_1), T_\mu(H \ltimes T)(\nu_2)\right\rangle(h)\\
&= \sum_{\omega \in \Omega_{r(h)}} \left\langle h^{-1} \cdot \left((T_\mu(H \ltimes T)(\nu_1))(\omega)\right), \left(T_\mu(H \ltimes T)(\nu_2)\right)(\omega \cdot h)\right\rangle\\
&= \sum_{\substack{\omega \in \Omega_{r(h)},\\ h_1 \in H^{r(h)},\\ h_2 \in H^{s(h)}}} \left\langle h^{-1} \cdot \left(\left(h_1 \cdot \left(b_{\omega\cdot h_1} T_{s(h_1)}\right)\right)\nu_1(h_1^{-1})\right), \left(h_2 \cdot \left(b_{\omega\cdot hh_2} T_{s(h_2)}\right)\right)\nu_2(h_2^{-1})\right\rangle,\\
&\left\langle T_\mu(H \ltimes T)(\nu_1), \psi(\operatorname{Ind}_{\Omega,c} T)T_\mu(\nu_2)\right\rangle(h)\\
&= \sum_{\omega \in \Omega_{r(h)}} \left\langle h^{-1} \cdot \left((T_\mu(H \ltimes T)(\nu_1))(\omega)\right), \left(\psi(\operatorname{Ind}_{\Omega,c} T)T_\mu(\nu_2)\right)(\omega \cdot h)\right\rangle\\
&= \sum_{\substack{\omega \in \Omega_{r(h)},\\ h_1 \in H^{r(h)},\\ h_2 \in H^{s(h)}}} \Big\langle h^{-1} \cdot \left(\left(h_1 \cdot \left(b_{\omega\cdot h_1} T_{s(h_1)}\right)\right)\nu_1(h_1^{-1})\right),\\
&\qquad\qquad \operatorname{Ind}_{\Omega,c} T(\omega \cdot h)(h_2 \cdot b_{\omega\cdot hh_2})\nu_2(h_2^{-1})\Big\rangle,\\
&\left\langle \psi(\operatorname{Ind}_{\Omega,c} T)T_\mu(\nu_1), T_\mu(H \ltimes T)(\nu_2)\right\rangle(h)\\
&= \sum_{\omega \in \Omega_{r(h)}} \left\langle h^{-1} \cdot \left((\psi(\operatorname{Ind}_{\Omega,c} T)T_\mu(\nu_1))(\omega)\right), \left(T_\mu(H \ltimes T)(\nu_2)\right)(\omega \cdot h)\right\rangle
\end{aligned}$$

$$= \sum_{\substack{\omega\in\Omega_{r(h)},\\ h_1\in H^{r(h)},\\ h_2\in H^{s(h)}}} \Big\langle h^{-1}\cdot\Big(\mathrm{Ind}_{\Omega,c}\,T(\omega)(h_1\cdot b_{\omega\cdot h_1})\nu_1(h_1^{-1})\Big), \qquad \big(h_2\cdot\big(b_{\omega\cdot hh_2}T_{s(h_2)}\big)\big)\,\nu_2(h_2^{-1})\Big\rangle,$$

$$\big\langle\psi(\mathrm{Ind}_{\Omega,c}\,T)T_\mu(\nu_1),\psi(\mathrm{Ind}_{\Omega,c}\,T)T_\mu(\nu_2)\big\rangle\,(h)$$
$$= \sum_{\omega\in\Omega_{r(h)}} \Big\langle h^{-1}\cdot\big((\psi(\mathrm{Ind}_{\Omega,c}\,T)T_\mu(\nu_1))(\omega)\big)\,,\big(\psi(\mathrm{Ind}_{\Omega,c}\,T)T_\mu(\nu_2)\big)\,(\omega\cdot h)\Big\rangle$$
$$= \sum_{\substack{\omega\in\Omega_{r(h)},\\ h_1\in H^{r(h)},\\ h_2\in H^{s(h)}}} \Big\langle h^{-1}\cdot\Big(\mathrm{Ind}_{\Omega,c}\,T(\omega)(h_1\cdot b_{\omega\cdot h_1})\nu_1(h_1^{-1})\Big), \qquad \mathrm{Ind}_{\Omega,c}\,T(\omega\cdot h)(h_2\cdot b_{\omega\cdot hh_2})\nu_2(h_2^{-1})\Big\rangle.$$

For $h\in H$, we may calculate:

$$\begin{aligned}
&\beta(\langle\zeta,\zeta\rangle)(\nu)(h)\\
&= \sum_{h_2\in H^{s(h)}} (h_2\cdot(\langle\zeta,\zeta\rangle(hh_2)))\nu(h_2^{-1})\\
&= \sum_{\substack{h_2\in H^{s(h)},\\ \omega\in\Omega_{r(h)}}} \Big(h_2\cdot\Big(((hh_2)^{-1}\cdot\zeta(\omega)^*)(\zeta(\omega\cdot hh_2)\Big)\Big)\,\nu(h_2^{-1})\\
&= \sum_{\substack{h_2\in H^{s(h)},\\ \omega\in\Omega_{r(h)}}} \Big(h^{-1}\cdot\zeta(\omega)^*\Big)\,(h_2\cdot\zeta(\omega\cdot hh_2))\,\nu(h_2^{-1}),
\end{aligned}$$

and therefore

$$\begin{aligned}
&\langle\nu_1,\beta(\langle\zeta,\zeta\rangle)(\nu_2)\rangle\,(h)\\
&= \sum_{h_1\in H^{r(h)}} \Big\langle h^{-1}\cdot\nu_1(h_1^{-1}),\beta(\langle\zeta,\zeta\rangle)(\nu_2)(h_1^{-1}h)\Big\rangle\\
&= \sum_{\substack{h_1\in H^{r(h)},\\ h_2\in H^{s(h)},\\ \omega\in\Omega_{s(h_1)}}} \Big\langle h^{-1}\cdot\nu_1(h_1^{-1}),\Big(h^{-1}h_1\cdot\zeta(\omega)^*\Big)\Big(h_2\cdot\zeta(\omega\cdot h_1^{-1}hh_2)\Big)\,\nu_2(h_2^{-1})\Big\rangle\\
&= \sum_{\substack{h_1\in H^{r(h)},\\ h_2\in H^{s(h)},\\ \omega\in\Omega_{r(h)}}} \Big\langle h^{-1}\cdot\nu_1(h_1^{-1}),\Big(h^{-1}h_1\cdot\zeta(\omega\cdot h_1)^*\Big)\,(h_2\cdot\zeta(\omega\cdot hh_2))\,\nu_2(h_2^{-1})\Big\rangle\\
&= \sum_{\substack{h_1\in H^{r(h)},\\ h_2\in H^{s(h)},\\ \omega\in\Omega_{r(h)}}} \Big\langle h^{-1}\cdot\Big((h_1\cdot\zeta(\omega\cdot h_1))\,\nu_1(h_1^{-1})\Big)\,,(h_2\cdot\zeta(\omega\cdot hh_2))\,\nu_2(h_2^{-1})\Big\rangle\,.
\end{aligned}$$

Expanding out $\zeta(\omega)=b_\omega T_{\sigma(\omega)}-\mathrm{Ind}_{\Omega,c}\,T(\omega)b_\omega$ and using the H-invariance of $\mathrm{Ind}_{\Omega,c}\,T$, this expression is identical to the expression for $\langle S_\mu(\nu_1),S_\mu(\mu_2)\rangle(h)$ we have already computed in four steps. We conclude that $S_\mu^*S_\mu=\beta(\langle\zeta,\zeta\rangle)$ and S_μ is therefore compact. As S_μ varies continuously in μ, it follows that S_μ is compact for all $\mu\in\Omega\ltimes\Theta_{\Omega,B}$. Similarly, S'_μ is compact, and so we have verified that

$G \ltimes \mathrm{Ind}_{\Omega,c} T \otimes 1$ is an $H \ltimes T$-connection. Our original diagram therefore commutes at the level of KK. $\square$

4. Ample groupoid homology and proper correspondences

An *ample* groupoid is an étale groupoid with a totally disconnected unit space. In this document, we assume it is locally compact and Hausdorff, although Hausdorffness is not essential for much of what follows. In this chapter we aim to construct as explicit a map as possible in groupoid homology from a proper correspondence of ample groupoids. This provides new perspective on the preservation of homology under Morita equivalences of ample groupoids.

4.1. **Ample groupoid modules.** We take a module-theoretic approach to the homology of an ample groupoid, as described in [10] for groupoids with σ-compact unit spaces. We remove this condition by emphasizing flat modules rather than projective modules. This involves incorporating some standard arguments which appear in homological algebra over unital rings, which we justify by introducing the multiplier ring of a locally unital ring. The bar resolution is then used to equate our definition of groupoid homology with Matui's concrete definition [54] and the definition appearing in [71].

Definition 4.1 (Groupoid ring)**.** Let G be an ample groupoid. The *groupoid ring* $\mathbb{Z}[G]$ is the convolution ring $C_c(G,\mathbb{Z})$ of compactly supported integer valued continuous functions on G. The convolution $\xi * \eta$ of elements ξ and η in $\mathbb{Z}[G]$ is given at $g \in G$ by

$$\xi * \eta(g) = \sum_{g_1 g_2 = g} \xi(g_1)\eta(g_2).$$

In this document, a ring need not be commutative nor even unital. In place of unitality, $\mathbb{Z}[G]$ is *locally unital*. This means that for any finite collection $\xi_1, \cdots, \xi_n$ of elements in $\mathbb{Z}[G]$, there is an idempotent $e \in \mathbb{Z}[G]$ such that $e\xi_i = \xi_i e = \xi_i$ for each i. In this case the idempotent may be taken to be the indicator function χ_U on a compact open set $U \subseteq G^0$. For a locally unital ring R, we require our (left) R-modules M to be *non-degenerate* or *unitary* in the sense that $RM = M$. The categories of left and right R-modules are written R-Mod and Mod-R respectively. These are both abelian categories, and there is an isomorphism $\mathsf{Mod}\text{-}R \cong R^{\mathrm{op}}\text{-}\mathsf{Mod}$ so any result for left modules will also hold for right modules. When $R = \mathbb{Z}[G]$, we refer to R-modules as G-modules and write G-Mod and Mod-G for the left and right module categories.

Example 4.2 (G-space module)**.** Let X be a G-space with anchor map $\tau\colon X \to G^0$. Then the abelian group $\mathbb{Z}[X] := C_c(X,\mathbb{Z})$ is a G-module with $\mathbb{Z}[G]$-action given by

$$\begin{aligned} \mathbb{Z}[G] \times \mathbb{Z}[X] &\to \mathbb{Z}[X] \\ \xi, m &\mapsto \xi \cdot m \\ &\qquad x \mapsto \sum_{g \in G_{\tau(x)}} \xi(g^{-1}) m(g \cdot x). \end{aligned}$$

Similarly, for a right G-space Z with anchor map $\pi\colon Z \to G^0$, the abelian group $\mathbb{Z}[Z]$ is a right G-module via the map

$$\begin{aligned} \mathbb{Z}[Z] \times \mathbb{Z}[G] &\to \mathbb{Z}[Z] \\ m, \xi &\mapsto m \cdot \xi \\ & z \mapsto \sum_{g \in G^{\pi(z)}} m(z \cdot g)\xi(g^{-1}). \end{aligned}$$

Given a G-equivariant local homeomorphism $f\colon X \to Y$ of (left) G-spaces, we obtain a G-module homomorphism $f_*\colon \mathbb{Z}[X] \to \mathbb{Z}[Y]$ which for $\xi \in \mathbb{Z}[X]$ is given by

$$f_*(\xi)(y) = \sum_{x \in f^{-1}(y)} \xi(x).$$

This defines a functor $\mathbb{Z}[-]\colon \mathsf{LCH}^G_{\mathrm{loc}} \to G\text{-}\mathsf{Mod}$.

The following lemma will help us to work with the abelian groups $\mathbb{Z}[X]$ for totally disconnected locally compact Hausdorff spaces X, and will be key for constructing G-modules.

Lemma 4.3. *Let X be a totally disconnected locally compact Hausdorff space and let A be an abelian group. Suppose that $\mathcal{U}$ is a basis of compact opens for X that is closed under compact open subsets. Let $\varphi\colon \mathcal{U} \to A$ be a function such that $\varphi(U_1) + \varphi(U_2) = \varphi(U)$ whenever $U_1 \sqcup U_2 = U$ with U_1, U_2 and U in $\mathcal{U}$.*

Then $\varphi\colon \mathcal{U} \to A$ extends uniquely to a group homomorphism $\hat{\varphi}\colon \mathbb{Z}[X] \to A$ such that $\hat{\varphi}(\chi_U) = \varphi(U)$ for each $U \in \mathcal{U}$.

Proof. The group $\mathbb{Z}[X]$ is generated by the indicator functions χ_U for $U \in \mathcal{U}$, so if φ has an extension it is unique. It suffices to check that the obvious extension $\sum_i a_i\chi_{U_i} \mapsto \sum_i a_i\varphi(U_i)$ is well-defined. Suppose that $\sum_i a_i\chi_{U_i} = \sum_j b_j\chi_{V_j}$. Let $W_1, \dots, W_N$ be the "smallest pieces" that can be made out of the U_i and V_j, which means that they are intersections over all i and all j of either U_i or U_i^c (V_j or V_j^c), taking at least one to not be a complement. As $\mathcal{U}$ is closed under compact open subsets, each W_k is in $\mathcal{U}$, and they are disjoint. There are $c_k \in \mathbb{Z}$ such that $\sum_i a_i\chi_{U_i} = \sum_k c_k\chi_{W_k}$, and from the condition on φ it follows that $\sum_i a_i\varphi(U_i) = \sum_k c_k\varphi(W_k) = \sum_j b_j\varphi(V_j)$. □

For each G-C*-algebra A, its K-theory groups $K_*(A)$ can be canonically equipped with the structure of a G-module. To do this we first note that the K-theory of a $C_0(X)$-algebra A over a totally disconnected space X can be understood in terms of restrictions to clopen subsets $U \subseteq X$. For each clopen set $U \subseteq X$, we have $UA := C_0(U)A = \Gamma_0(U, \mathcal{A})$. The inclusion and restriction maps $UA \to A \to UA$ induce inclusion and restriction maps $K_*(UA) \to K_*(A) \to K_*(UA)$, so in particular we can consider $K_*(UA)$ as a subgroup of $K_*(A)$. Moreover, by additivity and continuity of K-theory, the groups $K_*(UA)$ over U in a clopen cover of X will generate $K_*(A)$.

Example 4.4 (K-theory groups as groupoid modules)**.** Let G be an ample groupoid and let A be a G-C*-algebra. Then for $i = 0, 1$ the K-theory group $K_i(A)$ is a G-module. For each compact open bisection $U \subseteq G$, consider the $*$-isomorphism $\alpha_U : s(U)A \to r(U)A$ induced by the action map $\alpha : G \curvearrowright A$ (Remark 1.72). By abuse of notation we may also write $\alpha_U : A \to A$ for the composition with the restriction $A \to s(U)A$ and the inclusion $r(U)A \to A$. This induces a homomorphism $K_i(\alpha_U) : K_i(A) \to K_i(A)$. By Lemma 4.3, the assignment $\chi_U \mapsto K_*(\alpha_U)$ extends to a ring homomorphism $\mathbb{Z}[G] \to \mathrm{Hom}_{\mathsf{Ab}}(K_i(A), K_i(A))$. The unitarity of $\mathbb{Z}[G] \curvearrowright K_i(A)$ follows from the fact that $K_i(A)$ is generated by $K_i(VA)$ for compact open $V \subseteq G^0$.

If X is a totally disconnected G-space, the K-theory group $K_0(C_0(X))$ is given by $\mathbb{Z}[X]$ and we recover Example 4.2.

Proposition 4.5. *Let G be an ample groupoid. Then the induced map in K-theory $K_*(E) : K_*(A) \to K_*(B)$ from a G-equivariant proper correspondence $E : A \to B$ is G-equivariant. If G is second countable, A and B are separable and E is countably generated, the induced map in K-theory $K_*([E, T]) : K_*(A) \to K_*(B)$ from a morphism $[E, T] \in \mathrm{KK}^G(A, B)$ is G-equivariant.*

Proof. Let $(E, \psi) : A \to B$ be a G-equivariant correspondence, and let $\alpha : G \curvearrowright A$, $\beta : G \curvearrowright B$ and $\gamma : G \curvearrowright E$ be the respective actions of G. Let $U \subseteq G$ be a compact open bisection, inducing $*$-homomorphisms $\alpha_U : A \to A$ and $\beta_U : B \to B$. The G-equivariance of $E : A \to B$ implies that the following diagram commutes in the correspondence category Corr.

$$\begin{tikzcd} A \arrow[r, "E"] \arrow[d, "\mathsf{Corr}(\alpha_U)"] & B \arrow[d, "\mathsf{Corr}(\beta_U)"] \\ A \arrow[r, "E"] & B \end{tikzcd}$$

The composition is given by $(r(U)E, \psi \circ \alpha_U)$. When (E, ψ) is proper, this implies that $K_*(E) \circ K_*(\alpha_U) = K_*(\beta_U) \circ K_*(E)$, and so $K_*(E)$ is G-equivariant. Now suppose that G is second countable, A and B are separable, E is countably generated and graded and that there is a Fredholm operator $T \in \mathcal{L}(E)$ so that $[E, T] \in \mathrm{KK}^G(A, B)$. Consider the following diagram in KK.

$$\begin{tikzcd} A \arrow[r, "{[E,T]}"] \arrow[d, "\mathrm{KK}(\alpha_U)"] & B \arrow[d, "\mathrm{KK}(\beta_U)"] \\ A \arrow[r, "{[E,T]}"] & B \end{tikzcd}$$

The $\hookrightarrow$ Kasparov product is represented by $(r(U)E, T\restriction_{r(U)E})$, while the $\rightarrow\!\downarrow$ product is represented by $(r(U)E, \gamma_U(T\restriction_{s(U)E})\gamma_U^{-1})$. These Fredholm operators are compact perturbations by the almost invariance of T, so these define the same KK element. By taking K-theory, we conclude that $K_*([E,T])$ is G-equivariant. □

In analogy to the G-C*-bundle associated to each G-C*-algebra, there is a *G-sheaf* associated to any G-module. We will not formally use this picture of G-modules, but it provides good intuition. A G-sheaf $p\colon \mathcal{E} \to G^0$ is similar to a G-bundle, but each fibre is an abelian group. The section space $\Gamma_c(G^0, \mathcal{E})$ is naturally a G-module. For each G-module M there is a G-sheaf of abelian groups $(\mathcal{E}, p)$ such that $M \cong \Gamma_c(G^0, \mathcal{E})$. This defines an equivalence of categories between G-Mod and the category of G-sheaves of abelian groups. See [82] for more details. A takeaway from this perspective is that we can understand the structure of G-modules locally. Given a G-module M and an open set $U \subseteq G^0$, we write M_U for the subgroup $\mathbb{Z}[U]M$ of M. For a G-C*-algebra A and a compact open set $U \subseteq G^0$, we have $K_i(A)_U = K_i(UA)$.

Lemma 4.6. *Let G be an ample groupoid, let $f\colon M \to N$ be a map of G-modules and let $\mathcal{U}$ be an open cover of G^0. Then f is injective/surjective if and only if the restriction $f_U\colon M_U \to N_U$ is injective/surjective for each U in $\mathcal{U}$.*

Proof. It is clear that if f is injective/surjective then for each $U \in \mathcal{U}$ the restriction f_U is injective/surjective. For the other direction, consider the open cover $\mathcal{V}$ of G^0 of compact open sets contained in some element of the open cover $\mathcal{U}$. Each $\xi \in \mathbb{Z}[G^0]$ can be written as a sum of elements supported on disjoint elements of $\mathcal{V}$. It follows that N is generated by N_U for $U \in \mathcal{U}$ and surjectivity of f follows from surjectivity of f_U for each $U \in \mathcal{U}$. If $m = \sum_{i=1}^n m_i$ is in the kernel of f with $m_i \in M_{V_i}$ for pairwise disjoint $V_i \in \mathcal{V}$, then $f(m_i) \in N_{V_i}$ must vanish for each i. If f_U is injective for each $U \in \mathcal{U}$, then each m_i is zero and so f must be injective. □

The *tensor product* $M \otimes_R N$ of a right R-module M and a left R-module N over a locally unital ring R is an abelian group equipped with a bilinear map $M \times N \to M \otimes_R N$ sending (m,n) to $m \otimes n$. This map is *balanced* in the sense that $(m \cdot r) \otimes n = m \otimes (r \cdot n)$ for each $m \in M$, $n \in N$ and $r \in R$. The tensor product satisfies the following universal property. For any abelian group A and balanced bilinear map $f\colon M \times N \to A$, there is a unique homomorphism $g\colon M \otimes_R N \to A$ such that $g(m \otimes n) = f(m,n)$ for each $m \in M$ and $n \in N$.

$$\begin{tikzcd} M \times N \arrow[r, "f"] \arrow[d] & A \\ M \otimes_R N \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

Let S be another locally unital ring. An *R-S-bimodule* X is an abelian group equipped with a left R-module structure and a right S-module structure that commute in the sense that $r \cdot (x \cdot s) = (r \cdot x) \cdot s$ for each $r \in R$, $x \in X$ and $s \in S$. Let Q be another locally unital ring. The tensor product $M \otimes_R N$ of a Q-R-bimodule

M and an R-S-bimodule N inherits the structure of a Q-S-bimodule. The tensor product is associative up to canonical isomorphisms and for each locally unital ring R the R-R-bimodule R acts as an identity (up to canonical isomorphisms) for the tensor product $\otimes_R$. We obtain a category Bimod of locally unital rings whose morphisms are (isomorphism classes of) bimodules with composition given by the tensor product. Note that any left R-module can be viewed as an R-$\mathbb{Z}$-bimodule, so left and right modules also fit into the bimodule picture.

For any R-module N, we obtain a functor $-\otimes_R N\colon \mathsf{Mod}\text{-}R \to \mathsf{Ab}$. This functor is *right-exact* in the sense that given an exact sequence $0 \to A \to B \to C \to 0$ of right R-modules, the chain complex $0 \to A \otimes_R N \to B \otimes_R N \to C \otimes_R N \to 0$ is exact at $B \otimes_R N$ and $C \otimes_R N$. If N is an R-S-bimodule, this is a functor $\mathsf{Mod}\text{-}R \to \mathsf{Mod}\text{-}S$.

We now wish to do homological algebra. We have to be slightly careful working with modules over locally unital rings, and some concepts such as free modules don't have obvious analogues in this setting. However, with the right approach much goes through in exactly the same way as for unital rings.

Definition 4.7 (Projective module)**.** Let R be a locally unital ring. An R-module P is *projective* if whenever there are homomorphisms of R-modules $f\colon P \to B$ and $\pi\colon A \to B$ with π surjective, f lifts through π to an R-module homomorphism $\tilde{f}\colon P \to A$.

$$\begin{tikzcd} & A \arrow[d, "\pi"] \\ P \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

Example 4.8. For any idempotent $e \in R$, the left R-module Re is projective. To pick a lift of $f\colon Re \to B$ through a surjective R-module map $\pi\colon A \to B$, take any $a \in A$ such that $\pi(a) = f(e)$, and define $\tilde{f}\colon Re \to A$ by $\tilde{f}(r) = r \cdot a$.

This is one of the places where we have to be careful about the fact that our ring R only has local units. As a module over itself, R may fail to be projective. As discussed in [10], this can happen with $\mathbb{Z}[X]$ for totally disconnected spaces X which are not σ-compact. However, we may still say that R is a *flat* module over itself.

Definition 4.9 (Flat module)**.** Let R be a locally unital ring. An R-module F is *flat* if the tensor product functor

$$-\otimes_R F\colon \mathsf{Mod}\text{-}R \to \mathsf{Ab}$$

is *exact*. This means that it preserves the exactness of exact sequences. Equivalently, for any injective right R-module map $i\colon A \to B$, the map $i \otimes \mathrm{id}\colon A \otimes_R F \to B \otimes_R F$ is injective.

Example 4.10. Clearly R is flat as a module over itself as $A \otimes_R R \cong A$. Furthermore, for any idempotent $e \in R$, the left R-module Re is flat. For each right R-module N, there is an isomorphism $n \otimes r \mapsto n \cdot r \colon N \otimes_R Re \cong Ne$. Given an injective right R-module map $i \colon A \to B$, the map $i \otimes \mathrm{id} \colon A \otimes_R Re \to B \otimes_R Re$ corresponds to the restriction $i \colon Ae \to Be$, which is injective.

A direct limit of flat modules is flat, and using this we can construct many flat G-modules.

Proposition 4.11. *Let X be a free, proper, étale G-space. Then $\mathbb{Z}[X]$ is a flat $\mathbb{Z}[G]$-module.*

Proof. Let $\mathcal{U}$ be the basis of compact open sets $U \subseteq X$ on which the quotient map $q \colon U \to G\backslash X$ and the anchor map $\tau \colon U \to G^0$ are injective. For each $U \in \mathcal{U}$, the map $g \cdot u \mapsto g \colon G \cdot U \to G_{\tau(U)}$ is an isomorphism of G-spaces. The G-module $\mathbb{Z}[X]$ is the sum over $U \in \mathcal{U}$ of its submodules $\mathbb{Z}[G \cdot U]$, each of which is flat as $\mathbb{Z}[G \cdot U] \cong \mathbb{Z}[G_{\tau(U)}] \cong \mathbb{Z}[G]\chi_{\tau(U)}$. For any finite collection $U_1, \ldots, U_n \in \mathcal{U}$, we may write the union $\bigcup_{i=1}^n G \cdot U_i$ as a (topological) disjoint union $\bigsqcup_{i=1}^n G \cdot U_i'$ for sets $U_i' \in \mathcal{U}$ by setting $U_i' := U_i \cap q^{-1}(q(U_i) \setminus \cup_{j<i} q(U_j))$. The sum $\sum_{i=1}^n \mathbb{Z}[G \cdot U_i]$ inside $\mathbb{Z}[X]$ is therefore a direct sum $\bigoplus_{i=1}^n \mathbb{Z}[G \cdot U_i']$ of flat modules, hence flat. The G-module $\mathbb{Z}[X]$ is the direct limit of the finite sums $\sum_{i=1}^n \mathbb{Z}[G \cdot U_i]$, so is flat. □

In order to do homological algebra in R-Mod, we need to introduce projective resolutions.

Definition 4.12 (Projective resolution)**.** A *resolution* $P_\bullet \to M$ of an R-module M is an exact sequence of R-modules

$$\cdots \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0.$$

It is a *projective resolution* if each P_n is projective.

The notions of projective modules and resolutions are categorical - they make sense for any abelian category $\mathfrak{C}$. One of the fundamental ideas in homological algebra is that projective resolutions are unique up to chain homotopy equivalence. This means that given two projective resolutions $P_\bullet \to M$ and $Q_\bullet \to M$, there are chain maps $f \colon P_\bullet \to Q_\bullet$ and $g \colon Q_\bullet \to P_\bullet$ and chain homotopies $fg \simeq \mathrm{id}$ and $gf \simeq \mathrm{id}$. This follows from the fundamental lemma of homological algebra (see [88, Comparison Theorem 2.2.6]).

Lemma 4.13 (Fundamental lemma of homological algebra)**.** *Let $\mathfrak{C}$ be an abelian category and let $f \colon A \to B$ be a map in $\mathfrak{C}$. Suppose $P_\bullet \to A$ is a projective resolution and suppose $Q_\bullet \to B$ is a resolution of B in $\mathfrak{C}$. Then there is a chain map $P_\bullet \to Q_\bullet$ over $f \colon A \to B$, and it is unique up to chain homotopy.*

To build projective resolutions of R-modules, it is useful to know that there are *enough projectives* in the sense that for every R-module M, there is a projective R-module P with a surjective homomorphism $\pi\colon P \to M$. It follows that every R-module has a projective resolution.

Proposition 4.14 (Module categories have enough projectives)**.** *Let R be a locally unital ring. Then the category of R-modules has enough projectives.*

Proof. Let M be an R-module and let $M_0 \subseteq M$ generate M as an R-module. For each $m \in M_0$, pick an idempotent $e_m \in R$ such that $e_m \cdot m = m$. Then there is a surjective homomorphism given by

$$\bigoplus_{m\in M_0} Re_m \to M$$
$$re_m \mapsto r \cdot m$$

This module is projective because it is a direct sum of projective modules. □

Our proof shows further that this statement remains true if we restrict attention to countably generated R-modules. To define groupoid homology, we introduce the following analogue of taking the quotient of a G-module by an ample groupoid G:

Definition 4.15 (Coinvariants functor)**.** Let G be an ample groupoid and let M be a G-module. The *coinvariants* M_G of M is the abelian group

$$M_G := \mathbb{Z}[G^0] \otimes_G M.$$

This gives us a functor $\mathrm{Coinv}_G\colon G\text{-}\mathsf{Mod} \to \mathsf{Ab}$. We often think of M_G as a quotient of M via the surjective homomorphism $\pi_G\colon m \mapsto [m]\colon M \to M_G$ that sends m to $e \otimes m$ for any idempotent $e \in \mathbb{Z}[G^0]$ such that $e \cdot m = m$. The kernel of π_G is generated by elements of the form $\xi \cdot m - (s^*\xi) \cdot m$ for $\xi \in \mathbb{Z}[G]$ and $m \in M$.

Example 4.16. The coinvariants of the G-module $\mathbb{Z}[G^{n+1}]$ is isomorphic to $\mathbb{Z}[G^n]$. The quotient map $\mathbb{Z}[G^{n+1}] \to \mathbb{Z}[G^n]$ is induced by the local homeomorphism $(g_0, \ldots, g_n) \mapsto (g_1, \ldots, g_n)\colon G^{n+1} \to G^n$.

The coinvariants quotient map $\pi_G\colon M \to M_G$ can be thought of as the universal G-invariant map out of M. Given a G-module M and an abelian group N, we may call a homomorphism $f\colon M \to N$ *G-invariant* if $f(\xi \cdot m) = f(s^*\xi \cdot m)$ for each $\xi \in \mathbb{Z}[G]$ and $m \in M$. If X is a G-space and $\varphi\colon X \to Y$ is a G-invariant local homeomorphism, then $\varphi_*\colon \mathbb{Z}[X] \to \mathbb{Z}[Y]$ is G-invariant. We now introduce the module-theoretic definition of ample groupoid homology.

Definition 4.17 (Groupoid homology groups)**.** Let G be an ample groupoid and let M be a G-module. Consider any projective resolution $P_\bullet \to M$.

$$\cdots \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$$

We apply the coinvariants functor to obtain a chain complex of abelian groups.

$$\cdots \xrightarrow{(d_{n+1})_G} (P_n)_G \xrightarrow{(d_n)_G} \cdots \xrightarrow{(d_2)_G} (P_1)_G \xrightarrow{(d_1)_G} (P_0)_G \to 0$$

The *groupoid homology* $H_\bullet(G;M)$ *with coefficients in* M is the homology of this chain complex, so that $H_0(G;M) \cong (P_0)_G/\operatorname{im}((d_1)_G) \cong M_G$ and $H_n(G;M) \cong \ker((d_n)_G)/\operatorname{im}((d_{n+1})_G)$ for $n > 0$. This is independent of the choice of projective resolution because projective resolutions are unique up to chain homotopy equivalence. The groupoid homology *without coefficients* is $H_n(G) := H_n(G;\mathbb{Z}[G^0])$.

The functors $H_n(G;-)\colon G\text{-}\mathsf{Mod} \to \mathsf{Ab}$ are an example of *left derived functors*.

Definition 4.18 (Left derived functors)**.** Let $\mathfrak{C}$ be an abelian category with enough projectives (e.g. R-Mod or Mod-R), and let $F\colon \mathfrak{C} \to \mathsf{Ab}$ be a right exact functor. For an object A, let $P_\bullet \to A$ be a projective resolution. For $n \geq 0$, the *nth left derived functor* $\mathbb{L}_n F(A)$ of F at A is the homology group $H_n(F(P_\bullet))$ given by the chain complex

$$\cdots \to F(P_n) \to \cdots \to F(P_1) \to F(P_0) \to 0.$$

By the fundamental lemma of homological algebra, this is independent the choice of projective resolutions up to canonical isomorphisms and a morphism $A \to B$ induces a canonical map $\mathbb{L}_n F(A) \to \mathbb{L}_n F(B)$.

By right exactness of F we have $\mathbb{L}_0 F(A) \cong F(A)$, and if F is exact then $\mathbb{L}_n F(A) = 0$ for each $n > 0$. The left derived functors are in some sense a measure of the failure of exactness of a right exact functor.

The groupoid homology groups $H_n(G;M)$ are therefore the left derived functors $\mathbb{L}_n \operatorname{Coinv}_G(M)$ of the coinvariants functor $\operatorname{Coinv}_G\colon G\text{-}\mathsf{Mod} \to \mathsf{Ab}$. For each right R-module M, the functor $T_M = M \otimes_R -\colon R\text{-}\mathsf{Mod} \to \mathsf{Ab}$ is right exact, and its left derived functors are the *Tor groups* $\operatorname{Tor}_n^R(M,N) = \mathbb{L}_n T_M(N)$. Our definition of groupoid homology can be summarised as $H_n(G;M) := \operatorname{Tor}_n^G(\mathbb{Z}[G^0], M)$ and $H_n(G) := \operatorname{Tor}_n^G(\mathbb{Z}[G^0], \mathbb{Z}[G^0])$.

There are many facts of homological algebra that are well-known for unital rings that we would like to draw upon for locally unital rings. For example, in the setting of unital rings it suffices to use flat resolutions to calculate Tor, rather than projective resolutions. In the setting of locally unital rings this result is especially useful, because many modules we might expect to be projective (such as R itself) may not be projective, but are still flat. It should be possible to prove this result, along with many others, by adapting the proofs from the unital setting. However, there is a way to deduce results in the locally unital setting from their (known) specialisations to the unital setting. For a locally unital ring R, we introduce the *multiplier ring* $M(R)$ which is a unital ring in which R embeds, inspired by the multiplier algebra of a C*-algebra.

Definition 4.19 (Multiplier ring)**.** Let R be a locally unital ring. The *multiplier ring* $M(R) := \mathrm{Hom}_{\mathsf{Mod}\text{-}R}(R, R)$ is the endomorphism ring of the right R-module R. This contains a copy of the locally unital ring R which acts by left multiplication on R.

We can now view each R-module A as an $M(R)$-module by defining $m \cdot (r \cdot a) = m(r) \cdot a$ for $m \in M(R)$ and $r \cdot a \in RA = A$. Conversely, for each $M(R)$-module B, RB is an R-module. Through this we can view R-modules as precisely those $M(R)$-modules B for which $RB = B$.

Proposition 4.20. *Let R be a locally unital ring and let P be an R-module. Then P is projective/flat as an R-module if and only if it is projective/flat as an $M(R)$-module.*

Proof. We first remark that for a right R-module A and a left R-module B, there is a canonical isomorphism $A \otimes_R B \cong A \otimes_{M(R)} B$ of the tensor products over each ring. Therefore it is clear that if A is flat as an $M(R)$-module then it is flat as an R-module. Now suppose that P is flat as an R-module and let $i\colon A \to B$ be an injective right $M(R)$-module map. Then $i \otimes \mathrm{id}\colon A \otimes_{M(R)} P \to B \otimes_{M(R)} P$ is isomorphic to $i \otimes \mathrm{id}\colon AR \otimes_R P \to BR \otimes_R P$, which is injective by flatness of P.

It is clear that if P is projective as an $M(R)$-module then it is projective as an R-module. Conversely, suppose P is projective as an R-module and consider $M(R)$-module maps $f\colon P \to B$ and $\pi\colon A \to B$ with π surjective. Then f lands in the R-module RB and we may lift this through the (surjective) restriction $\pi\colon RA \to RB$ of π to RA. $\square$

From this we can deduce that projective R-modules are flat. We may also deduce the following fact about Tor which is well-known in the unital case [88, Lemma 3.2.8]. This says both that Tor is *balanced* in that we may consider resolutions of either entry and that it suffices to consider flat resolutions.

Proposition 4.21. *For a locally unital ring R, a right R-module A and a left R-module B, the group $\mathrm{Tor}_n^R(A, B)$ can be computed as follows. Let $P_\bullet \to A$ and $Q_\bullet \to B$ be flat resolutions of right and left R-modules respectively. Then $\mathrm{Tor}_n^R(A, B) \cong H_n(P_\bullet \otimes_R B) \cong H_n(A \otimes_R Q_\bullet)$. Furthermore, the isomorphism is induced by the unique (up to chain homotopy) chain map from any projective resolution of A to $P_\bullet$, and similarly for $Q_\bullet \to B$.*

Proof. We know that R-projective resolutions are $M(R)$-projective resolutions and always exist by Proposition 4.14, and that the tensor products over R and $M(R)$ coincide. It follows that $\mathrm{Tor}_n^R(A, B) \cong \mathrm{Tor}_n^{M(R)}(A, B)$. This proposition follows from the unital version because flat resolutions of R-modules are flat resolutions of $M(R)$-modules. $\square$

Example 4.22 (Bar resolution)**.** Let G be an ample groupoid. There is an explicit flat resolution $(\mathbb{Z}[G^{\bullet+1}], \partial_\bullet)$ of the (left) G-module $\mathbb{Z}[G^0]$

$$\cdots \xrightarrow{\partial_{n+1}} \mathbb{Z}[G^{n+1}] \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} \mathbb{Z}[G^2] \xrightarrow{\partial_1} \mathbb{Z}[G^1] \xrightarrow{\partial_0} \mathbb{Z}[G^0] \to 0 \tag{4.1}$$

called the *bar resolution*. For $n \geq 0$ we consider the space G^{n+1} of composable $n+1$-tuples as a left G-space. The G-module $\mathbb{Z}[G^{n+1}]$ is flat because G^{n+1} is a free, proper, étale G-space. We set $\partial_0 := s_*\colon \mathbb{Z}[G^1] \to \mathbb{Z}[G^0]$. For $n > 0$ and $0 \leq i \leq n$ we define face maps $\partial_i^n\colon G^{n+1} \to G^n$ by

$$\partial_i^n\colon (g_0, \ldots, g_n) \mapsto \begin{cases} (g_0, \ldots, g_i g_{i+1}, \ldots, g_n) & \text{if } i < n, \\ (g_0, \ldots, g_{n-1}) & \text{if } i = n. \end{cases}$$

The face maps are G-equivariant local homeomorphisms and therefore induce G-module maps $(\partial_i^n)_*\colon \mathbb{Z}[G^{n+1}] \to \mathbb{Z}[G^n]$. The boundary maps $\partial_n\colon \mathbb{Z}[G^{n+1}] \to \mathbb{Z}[G^n]$ are given for $n > 0$ by

$$\partial_n := \sum_{i=0}^{n} (-1)^i (\partial_i^n)_*.$$

The exactness of the bar resolution (4.1) is witnessed by a chain homotopy induced by local homeomorphisms $h_n\colon G^n \to G^{n+1}$. These are defined for $n > 0$ by

$$h_n\colon (g_0, \ldots, g_{n-1}) \mapsto (r(g_0), g_0, \ldots, g_{n-1})$$

and h_0 is the inclusion $G^0 \subseteq G^1$. By Proposition 4.21, we can use the bar resolution to compute the groupoid homology $H_*(G)$. Taking the coinvariants of the bar resolution, we obtain the chain complex

$$\cdots \xrightarrow{(\partial_{n+1})_G} \mathbb{Z}[G^n] \xrightarrow{(\partial_n)_G} \cdots \xrightarrow{(\partial_2)_G} \mathbb{Z}[G^1] \xrightarrow{(\partial_1)_G} \mathbb{Z}[G^0] \to 0. \tag{4.2}$$

The boundary maps are given by $(\partial_n)_G = \sum_{i=0}^n (-1)^i (\epsilon_i^n)_*$, where for $n > 0$, the face map $\epsilon_i^n\colon G^n \to G^{n-1}$ is defined by

$$\epsilon_i^n\colon (g_1, \ldots, g_n) \mapsto \begin{cases} (g_2, \ldots, g_n) & \text{if } i = 0 \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) & \text{if } 0 < i < n \\ (g_1, \ldots, g_{n-1}) & \text{if } i = n \end{cases}$$

The homology of the chain complex $(\mathbb{Z}[G^\bullet], (\partial_\bullet)_G)$ computes the groupoid homology $H_*(G) = \operatorname{Tor}_*^G(\mathbb{Z}[G^0], \mathbb{Z}[G^0])$. To compute the groupoid homology $H_*(G; M) = \operatorname{Tor}_*^G(\mathbb{Z}[G^0], M)$ with coefficients in a G-module M, we may use a symmetric bar resolution of right G-modules

$$\cdots \xrightarrow{\partial'_{n+1}} \mathbb{Z}[G^{n+1}] \xrightarrow{\partial'_n} \cdots \xrightarrow{\partial'_2} \mathbb{Z}[G^2] \xrightarrow{\partial'_1} \mathbb{Z}[G^1] \xrightarrow{\partial'_0} \mathbb{Z}[G^0] \to 0$$

and take the homology of the complex $(\mathbb{Z}[G^{\bullet+1}] \otimes_G M, \partial'_\bullet \otimes \mathrm{id})$. This chain complex is isomorphic to the complex defining $H_*(G; M)$ in [71].

There are resolutions even more general than flat resolutions that can be used to compute groupoid homology in particular. A well-known characterisation of flatness for a left R-module F is that $\operatorname{Tor}_n^R(M, F) = 0$ for each right R-module M and $n > 0$. For the purposes of groupoid homology, the only right G-module we are interested in is $\mathbb{Z}[G^0]$. We call a left G-module A *left* Coinv_G*-acyclic* if $\operatorname{Tor}_n^G(\mathbb{Z}[G^0], A) = \mathbb{L}_n \operatorname{Coinv}_G(A) = 0$ for each $n > 0$. It turns out that left Coinv_G-acyclic resolutions are good enough to compute groupoid homology:

Theorem 4.23 (Acyclic resolutions for derived functors)**.** *Let $\mathfrak{C}$ be an abelian category with enough projectives and let $F\colon \mathfrak{C} \to \mathsf{Ab}$ be a right exact functor. Then for any left F-acyclic resolution $Q_\bullet \to M$ of an object M and any projective resolution $P_\bullet \to M$, the unique (up to homotopy) chain map $P_\bullet \to Q_\bullet$ induces an isomorphism $\mathbb{L}_n F(M) \cong H_n(F(Q_\bullet))$.*

Proof. For unital rings, see [88, 2.4.3], including the linked exercise. The dual version of this theorem is covered in the full generality of abelian categories in several lecture notes, see [77, Theorem 3.60] and [76, Theorem 4.6.7]. □

4.2. **The induced map in homology from a groupoid correspondence.** Given a correspondence $\Omega\colon G \to H$ of ample groupoids, we construct a module-theoretic induction functor $\operatorname{Ind}_\Omega\colon H\text{-}\mathsf{Mod} \to G\text{-}\mathsf{Mod}$. As Ω is a G-H-bispace, the abelian group $\mathbb{Z}[\Omega]$ is a G-H-bimodule. The tensor product by $\mathbb{Z}[\Omega]$ yields the induction functor $\operatorname{Ind}_\Omega$.

$$\operatorname{Ind}_\Omega := \mathbb{Z}[\Omega] \otimes_H -\colon H\text{-}\mathsf{Mod} \to G\text{-}\mathsf{Mod}$$

The right H-space Ω is free, proper and étale, so by Proposition 4.11, $\mathbb{Z}[\Omega]$ is a flat right H-module. It follows that $\operatorname{Ind}_\Omega$ is exact.

Remark 4.24. If we were to consider the H-sheaf $\mathcal{M}$ associated to an H-module M, it is possible to identify $\mathbb{Z}[\Omega] \otimes_H M$ with the abelian group of H-equivariant sections in $\Gamma_b(\Omega, \sigma^*\mathcal{M})$ that are compactly supported with respect to Ω/H. This makes the analogy with the C*-algebraic induction functor much more explicit.

The G-H-bimodule $\mathbb{Z}[\Omega]$ is compatible with composition of groupoid correspondences in the following sense.

Proposition 4.25. *Let G, H and K be ample groupoids and let $\Omega\colon G \to H$ and $\Lambda\colon H \to K$ be correspondences. Then there is an isomorphism $\kappa_{\Omega,\Lambda}\colon \mathbb{Z}[\Omega] \otimes_H \mathbb{Z}[\Lambda] \cong \mathbb{Z}[\Lambda \circ \Omega]$ of G-K-bimodules given as follows.*

$$\begin{aligned}\kappa_{\Omega,\Lambda}\colon \mathbb{Z}[\Omega] \otimes_H \mathbb{Z}[\Lambda] &\cong \mathbb{Z}[\Lambda \circ \Omega]\\ \xi \otimes \eta &\mapsto \kappa_{\Omega,\Lambda}(\xi \otimes \eta)\\ [\omega, \lambda]_H &\mapsto \sum_{h \in H^{\sigma(\omega)}} \xi(\omega \cdot h)\eta(h^{-1} \cdot \lambda)\end{aligned}$$

Proof. Checking that the above bilinear map is balanced is straightforward. We therefore obtain a well-defined map $\kappa_{\Omega,\Lambda}\colon \mathbb{Z}[\Omega]\otimes_H \mathbb{Z}[\Lambda]\to\mathbb{Z}[\Lambda\circ\Omega]$, which we can further check is a homomorphism of G-K bimodules. Now, let $\mathcal{S}$ be the set of pairs (U,V) of compact open subsets $U\subseteq\Omega$ and $V\subseteq\Lambda$ such that U is a bisection and $\rho(V)\subseteq\sigma(U)$. The sets $U\times_Y V$ with $(U,V)\in\mathcal{S}$ form a basis of compact opens in $\Omega\times_Y\Lambda$, so their images $q(U\times_Y V)$ under the local homeomorphism $q\colon \Omega\times_Y\Lambda\to\Lambda\circ\Omega$ form a basis of compact opens in $\Lambda\circ\Omega$. By construction, $\kappa_{\Omega,\Lambda}(\chi_U\otimes\chi_V)=\chi_{q(U\times_Y V)}$, and surjectivity of $\kappa_{\Omega,\Lambda}$ follows.

Now, let $\mathcal{R}=\{q(U\times_Y V)\mid (U,V)\in\mathcal{S}\}$. This is a basis of compact opens in $\Lambda\circ\Omega$ that is closed under compact open subsets. Using Lemma 4.3, we define an inverse $\psi\colon \mathbb{Z}[\Lambda\circ\Omega]\to\mathbb{Z}[\Omega]\otimes_H\mathbb{Z}[\Lambda]$ to $\kappa_{\Omega,\Lambda}$ by setting

$$\psi(\chi_{q(U\times_Y V)}):=\chi_U\otimes\chi_V.$$

We need to check that this is well-defined and respects disjoint unions within $\mathcal{R}$. First, suppose that $(U_1,V_1),(U_2,V_2)\in\mathcal{S}$ with $q(U_1\times_Y V_1)=q(U_2\times_Y V_2)$. Define

$$W:=\{h\in H\mid \text{there are } u_1\in U_1 \text{ and } u_2\in U_2 \text{ such that } u_1\cdot h=u_2\}.$$

The action $\Omega\curvearrowleft H$ is free and proper and σ restricts to homeomorphisms on U_1 and U_2, from which it follows that W is a compact open bisection in H. Furthermore, because $q(U_1\times_Y V_1)=q(U_2\times_Y V_2)$ we obtain that $\rho(V_1)\subseteq r(W)$, $\rho(V_2)\subseteq s(W)$ and $W\cdot V_2=V_1$. We may then calculate

$$\chi_{U_1}\otimes\chi_{V_1}=\chi_{U_1}\otimes\chi_W\cdot\chi_{V_2}=\chi_{U_1}\cdot\chi_W\otimes\chi_{V_2}=\chi_{U_2}\cdot\chi_{r(W)}\otimes\chi_{V_2}=\chi_{U_2}\otimes\chi_{V_2}.$$

Now suppose that $q(U_1\times_Y V_1)\sqcup q(U_2\times_Y V_2)=q(U\times_Y V)$. By the above argument we may assume that $U_1\times_Y V_1\sqcup U_2\times_Y V_2=U\times_Y V$, so $V_1\sqcup V_2=V$ and we can write $\chi_U\otimes\chi_V=\chi_U\otimes\chi_{V_1}+\chi_U\otimes\chi_{V_2}=\chi_{U_1}\otimes\chi_{V_1}+\chi_{U_2}\otimes\chi_{V_2}$. By Lemma 4.3, ψ extends uniquely to a homomorphism $\psi\colon\mathbb{Z}[\Lambda\circ\Omega]\to\mathbb{Z}[\Omega]\otimes_{\mathbb{Z}[H]}\mathbb{Z}[\Lambda]$ such that $\kappa_{\Omega,\Lambda}\circ\psi=1$ by construction. The elements $\chi_U\otimes\chi_V$ for $(U,V)\in\mathcal{S}$ generate $\mathbb{Z}[\Omega]\otimes_H\mathbb{Z}[\Lambda]$, so therefore ψ is an inverse to $\kappa_{\Omega,\Lambda}$. □

The assignment $\mathsf{GpdCorr}\to\mathsf{Bimod}$ sending the groupoid G to the ring $\mathbb{Z}[G]$ and the correspondence $\Omega\colon G\to H$ to the G-H-bimodule $\mathbb{Z}[\Omega]$ is therefore functorial. We can conclude that the induction functor is compatible with composition of correspondences in that $\mathrm{Ind}_\Omega\circ\mathrm{Ind}_\Lambda\cong\mathrm{Ind}_{\Lambda\circ\Omega}$. Next we want to relate the coinvariants functors Coinv_G and Coinv_H via the induction functor Ind_Ω. We do this by relating the trivial right modules $\mathbb{Z}[G^0]$ and $\mathbb{Z}[H^0]$ via $\mathbb{Z}[\Omega]$.

Proposition 4.26. *Let $\Omega\colon G\to H$ be a correspondence of ample groupoids. There is a map $\delta_\Omega\colon\mathbb{Z}[\Omega]_G\to\mathbb{Z}[H^0]$ of right H-modules such that the following diagram* *commutes.*

$$\begin{tikzcd} \mathbb{Z}[\Omega] \arrow[r, "\sigma_*"] \arrow[d, "\pi_G"'] & \mathbb{Z}[H^0] \\ \mathbb{Z}[\Omega]_G \arrow[ur, "\delta_\Omega"] & \end{tikzcd}$$

The assignment $\Omega \mapsto \delta_\Omega$ respects composition of groupoid correspondences in that for any additional correspondence $\Lambda\colon H \to K$, the following diagram commutes.

$$\begin{tikzcd} \mathbb{Z}[G^0] \otimes_{\mathbb{Z}[G]} \mathbb{Z}[\Omega] \otimes_{\mathbb{Z}[H]} \mathbb{Z}[\Lambda] \arrow[r, "\delta_\Omega \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \kappa_{\Omega,\Lambda}"] & \mathbb{Z}[H^0] \otimes_{\mathbb{Z}[H]} \mathbb{Z}[\Lambda] \arrow[d, "\delta_\Lambda"] \\ \mathbb{Z}[G^0] \otimes_{\mathbb{Z}[G]} \mathbb{Z}[\Lambda \circ \Omega] \arrow[r, "\delta_{\Lambda\circ\Omega}"] & \mathbb{Z}[K^0] \end{tikzcd}$$

It also respects identities in that for the identity correspondence $G\colon G \to G$, the map $\delta_G\colon \mathbb{Z}[G^0] \otimes_G \mathbb{Z}[G] \to \mathbb{Z}[G^0]$ is the canonical isomorphism induced by the right action of $\mathbb{Z}[G]$ on $\mathbb{Z}[G^0]$.

Proof. We define $\delta_\Omega\colon \mathbb{Z}[G^0] \otimes_G \mathbb{Z}[\Omega] \to \mathbb{Z}[H^0]$ on simple tensors by

$$\begin{aligned} \delta_\Omega\colon \mathbb{Z}[G^0] \otimes_G \mathbb{Z}[\Omega] &\to \mathbb{Z}[H^0] \\ \eta \otimes \xi &\mapsto \sigma_*(\eta \cdot \xi). \end{aligned}$$

The balancedness of the bilinear map $\mathbb{Z}[G^0] \times \mathbb{Z}[\Omega] \to \mathbb{Z}[H^0]$ follows from the G-invariance of $\sigma\colon \Omega \to H^0$. It follows that $\delta_\Omega\colon \mathbb{Z}[G^0] \otimes_G \mathbb{Z}[\Omega] \to \mathbb{Z}[H^0]$ is well-defined. It is a map of right H-modules because σ is H-equivariant.

Compatibility with composition can be checked on simple tensors. For $\eta \in \mathbb{Z}[G^0]$, $\xi \in \mathbb{Z}[\Omega]$ and $\nu \in \mathbb{Z}[\Lambda]$, the simple tensor $\eta \otimes \xi \otimes \nu$ is sent to the following element of $\mathbb{Z}[K^0]$ under both the $\hookrightarrow$ and $\urcorner$ routes round the diagram.

$$z \mapsto \sum_{\lambda \in \Lambda_z} \sum_{\omega \in \Omega_{\rho(\lambda)}} \eta(\rho(\omega))\xi(\omega)\nu(\lambda)$$

To avoid ambiguity, let $\star$ refer to the right action $\mathbb{Z}[G^0] \curvearrowleft \mathbb{Z}[G]$ to distinguish it from the left action $\mathbb{Z}[G^0] \curvearrowright \mathbb{Z}[G]$. The canonical isomorphism $\mathbb{Z}[G^0] \otimes_G \mathbb{Z}[G] \to \mathbb{Z}[G^0]$ sends $\eta \otimes \xi$ to $\eta \star \xi$, and this is indeed equal to $\delta_\Omega(\eta \otimes \xi) = s_*(\eta \cdot \xi)$. □

Given a proper correspondence $\Omega\colon G \to H$ of ample groupoids, the exact functor $\mathrm{Ind}_\Omega\colon H\text{-}\mathsf{Mod} \to G\text{-}\mathsf{Mod}$ and the H-equivariant map $\delta_\Omega\colon \mathbb{Z}[\Omega]_G \to \mathbb{Z}[H^0]$ are the ingredients we need to construct a map in homology $H_*(\Omega)\colon H_*(G) \to H_*(H)$. The compatibility of Ind_Ω and δ_Ω with composition of correspondences and identity correspondences lead to compatibility for $H_*(\Omega)$. Now suppose that on top of that we have C*-coefficients in the form of a proper correspondence $(E, \Omega)\colon (A, G) \to (B, H)$. In order to build a map $H_{*,i}(E, \Omega)\colon H_*(G; K_i(A)) \to H_*(H; K_i(B))$ we will need the further ingredient of a G-equivariant map $K_i(A) \to \mathrm{Ind}_\Omega K_i(B)$. We obtain this through an isomorphism $\mathrm{Ind}_\Omega K_*(B) \cong K_*(\mathrm{Ind}_\Omega B)$. Recall that for each open bisection $U \subseteq \Omega$ there is a $*$-isomorphism $\mathrm{ev}_U\colon \mathrm{Ind}_{UH} B \cong \sigma(U)B$ (see

(2.1)). When U is clopen, we can view $K_*(\operatorname{Ind}_{UH} B)$ and $K_*(\sigma(U)B)$ as subgroups of $K_*(\operatorname{Ind}_\Omega B)$ and $K_*(B)$ respectively.

Proposition 4.27 (K-theory intertwines the induction functors)**.** *For every H-C^*-algebra B, there is an isomorphism of G-modules*

$$\zeta_{\Omega,B}\colon K_*(\operatorname{Ind}_\Omega B) \cong \operatorname{Ind}_\Omega K_*(B)$$

such that $\zeta_{\Omega,B}(y) = \chi_U \otimes K_*(\mathrm{ev}_U)(y)$ for each compact open bisection $U \subseteq \Omega$ and $y \in K_*(\operatorname{Ind}_{UH} B)$.

Proof. It will be slightly more straightforward to define the inverse of $\zeta_{\Omega,B}$. For each compact open bisection $U \subseteq \Omega$, let $\epsilon_U\colon B \to \operatorname{Ind}_\Omega B$ be the composition of the following $*$-homomorphisms.

$$\epsilon_U\colon B \to \sigma(U)B \xrightarrow{\mathrm{ev}_U^{-1}} \operatorname{Ind}_{UH} B \to \operatorname{Ind}_\Omega B$$

Consider the assignment $(\chi_U, x) \mapsto K_*(\epsilon_U)(x)$ for a compact open bisection $U \subseteq \Omega$ and an element $x \in K_*(B)$. By Lemma 4.3, this extends to a bilinear map $\psi\colon \mathbb{Z}[\Omega] \times K_*(B) \to K_*(\operatorname{Ind}_\Omega B)$. Consider the actions $\beta\colon H \curvearrowright B$ and $\alpha\colon G \curvearrowright \operatorname{Ind}_\Omega B$. Given compact open bisections $V \subseteq H$ and $W \subseteq G$ we have induced $*$-homomorphisms $\beta_V\colon B \to B$ and $\alpha_W\colon \operatorname{Ind}_\Omega B \to \operatorname{Ind}_\Omega B$ (see Example 4.4). Consider the following diagram given a compact open bisection $U \subseteq \Omega$.

$$\begin{tikzcd}
B \arrow[r, "\epsilon_{U\cdot V}"] \arrow[d, "\beta_V"'] & \operatorname{Ind}_\Omega B \arrow[d, "\alpha_W"] \\
B \arrow[ur, "\epsilon_U"] \arrow[r, "\epsilon_{W\cdot U}"'] & \operatorname{Ind}_\Omega B
\end{tikzcd}$$

Both triangles in the above diagram commute. This is because $\beta_V \circ \mathrm{ev}_{U\cdot V} = \mathrm{ev}_U$ on their common domain and similarly $\mathrm{ev}_{U\cdot W} \circ \alpha_W = \mathrm{ev}_U$. It follows that $K_*(\epsilon_{U\cdot V})(x) = K_*(\epsilon_U)K_*(\beta_V)(x)$ for each $x \in K_*(B)$, and therefore ψ is balanced and induces a homomorphism $\tilde{\psi}\colon \operatorname{Ind}_\Omega K_*(B) \to K_*(\operatorname{Ind}_\Omega B)$ such that $\tilde{\psi}(\chi_U \otimes x) = K_*(\epsilon_U)(x)$ for each compact open bisection $U \subseteq \Omega$ and $x \in K_*(B)$. We also get that $K_*(\epsilon_{W\cdot U})(x) = K_*(\alpha_W)K_*(\epsilon_U)(x)$ from which we can conclude that $\tilde{\psi}$ is G-equivariant.

As the construction of $\tilde{\psi}$ is independent of G, we may replace G by Ω/H and apply Lemma 4.6, so that we need only check injectivity and surjectivity by restricting to $q(U) \subseteq \Omega/H$ for compact open bisections $U \subseteq \Omega$. The elements of $q(U)\mathbb{Z}[\Omega] \otimes_H K_*(B)$ can be written as $\chi_U \otimes x$ for $x \in K_*(\sigma(U)B)$. Applying $\tilde{\psi}$, we get $K_*(\mathrm{ev}_U^{-1})(x)$. The map $\mathrm{ev}_U\colon \sigma(U)B \to \operatorname{Ind}_{UH} B$ is an isomorphism, so if $K_*(\mathrm{ev}_U^{-1})(x) = 0$ then $x = 0$, and we also hit all of $K_*(\operatorname{Ind}_{UH} B) = q(U)K_*(\operatorname{Ind}_\Omega B)$. It follows that $\tilde{\psi}$ is an isomorphism. We set $\zeta_{\Omega,B}\colon K_*(\operatorname{Ind}_\Omega B) \to \operatorname{Ind}_\Omega K_*(B)$ to be the inverse of $\tilde{\psi}$. This satisfies $\zeta_{\Omega,B}(y) = \chi_U \otimes K_*(\mathrm{ev}_U)(y)$ for each compact open bisection $U \subseteq \Omega$ and $y \in K_*(\operatorname{Ind}_{UH} B)$ by construction. $\square$

Proposition 4.28 (Naturality of $\zeta_{\Omega,B}$)**.** *Let $\Omega\colon G \to H$ be a correspondence of ample groupoids. Then the isomorphism $\zeta_{\Omega,B}\colon K_*(\operatorname{Ind}_\Omega B) \cong \operatorname{Ind}_\Omega K_*(B)$ is natural in B with respect to proper H-equivariant correspondences, and when H, G and Ω are second countable it is natural with respect to morphisms in* KK^H.

Proof. Let $E\colon B \to C$ be an H-equivariant correspondence. For each compact open bisection $U \subseteq \Omega$ we obtain a commutative diagram of C*-correspondences, with horizontal maps defined by $*$-homomorphisms.

$$\begin{tikzcd}
\operatorname{Ind}_\Omega B \arrow[d, "\operatorname{Ind}_\Omega E"] & \operatorname{Ind}_{UH} B \arrow[l] \arrow[r, "\mathrm{ev}_U"] \arrow[d, "\operatorname{Ind}_{UH} E"] & \sigma(U)B \arrow[r] \arrow[d, "\sigma(U)E"] & B \arrow[d, "E"] \\
\operatorname{Ind}_\Omega C & \operatorname{Ind}_{UH} C \arrow[l] \arrow[r, "\mathrm{ev}_U"] & \sigma(U)C \arrow[r] & C
\end{tikzcd}$$

When E is a proper correspondence, this is a diagram of proper correspondences which induces a diagram of K-theory groups. It follows that for $y \in K_*(\operatorname{Ind}_{UH} B)$, we have $K_*(\mathrm{ev}_U)K_*(\operatorname{Ind}_\Omega E)(y) = K_*(E)K_*(\mathrm{ev}_U)(y)$. Now consider the following diagram, whose commutativity describes the naturality of $\zeta_{\Omega,-}$ with respect to proper H-equivariant correspondences.

$$\begin{tikzcd}
K_*(\operatorname{Ind}_\Omega B) \arrow[r, "K_*(\operatorname{Ind}_\Omega E)"] \arrow[d, "\zeta_{\Omega,B}"] & K_*(\operatorname{Ind}_\Omega C) \arrow[d, "\zeta_{\Omega,C}"] \\
\operatorname{Ind}_\Omega K_*(B) \arrow[r, "\operatorname{Ind}_\Omega K_*(E)"] & \operatorname{Ind}_\Omega K_*(C)
\end{tikzcd}$$

Applying ⤵ to y yields $\chi_U \otimes K_*(\mathrm{ev}_U)K_*(\operatorname{Ind}_\Omega E)(y)$ and applying ↳ yields $\chi_U \otimes K_*(E)K_*(\mathrm{ev}_U)(y)$. These are equal and such y generate $K_*(\operatorname{Ind}_\Omega B)$, so the diagram commutes. Now suppose that G, H and Ω are second countable, B and C are separable, E is countably generated and we have a Fredholm operator $T \in \mathcal{L}(E)$ so that $[E, T] \in \mathrm{KK}^H(B, C)$. We apply the same strategy to show that $\zeta_{\Omega,-}$ is natural with respect to $[E, T]$, and so we consider the following diagram in KK.

$$\begin{tikzcd}
\operatorname{Ind}_\Omega B \arrow[d, "\operatorname{Ind}_\Omega {[E,T]}"] & \operatorname{Ind}_{UH} B \arrow[l] \arrow[r, "\mathrm{KK}(\mathrm{ev}_U)"] \arrow[d, "\operatorname{Ind}_{UH}{[E,T]}"] & \sigma(U)B \arrow[r] \arrow[d, "\sigma(U){[E,T]}"] & B \arrow[d, "{[E,T]}"] \\
\operatorname{Ind}_\Omega C & \operatorname{Ind}_{UH} C \arrow[l] \arrow[r, "\mathrm{KK}(\mathrm{ev}_U)"] & \sigma(U)C \arrow[r] & C
\end{tikzcd}$$

The left hand square commutes because any cutoff function $c\colon \Omega \to \mathbb{R}$ restricts to a cutoff function on UH. The right hand square commutes because the restriction $\sigma(U)[E, T] = [\sigma(U)E, T\restriction_{\sigma(U)E}]$ of the element $[E, T] \in \mathrm{KK}^H(B, C)$ induces the same map $K_*(\sigma(U)B) \to K_*(\sigma(U)C)$ in K-theory as the restriction of $K_*([E, T])\colon K_*(B) \to K_*(C)$. The middle square commutes because up to the isomorphism $\sigma(U)E \cong \operatorname{Ind}_{UH} E$, the Fredholm operators $T\restriction_{\sigma(U)E}$ and $\operatorname{Ind}_{UH,c} T$ are compact perturbations. This can be checked by viewing our algebras and modules over U and checking compactness fibre-wise. Similarly, the commutativity of this diagram shows that for $y \in K_*(\operatorname{Ind}_{UH} B)$ we have $K_*(\mathrm{ev}_U)K_*(\operatorname{Ind}_\Omega[E, T])(y) = K_*([E, T])K_*(\mathrm{ev}_U)(y)$ and we can conclude that $\zeta_{\Omega,-}$ is natural with respect to morphisms in KK^H.

$$\begin{tikzcd}
K_*(\operatorname{Ind}_\Omega B) \arrow[r, "K_*(\operatorname{Ind}_\Omega[E,T])"] \arrow[d, "\zeta_{\Omega,B}"] & K_*(\operatorname{Ind}_\Omega C) \arrow[d, "\zeta_{\Omega,C}"] \\
\operatorname{Ind}_\Omega K_*(B) \arrow[r, "\operatorname{Ind}_\Omega K_*([E,T])"] & \operatorname{Ind}_\Omega K_*(C)
\end{tikzcd}$$

□

Now that we have shown that the K-theory functors $K_*\colon \mathrm{KK}^G \to G\text{-}\mathsf{Mod}$ intertwine the induction functors $\operatorname{Ind}_\Omega$ naturally, there are some features of these induction functors that we want to check are compatible under K_*.

Proposition 4.29 (Compatibility of ζ with composition)**.** *Let $\Omega\colon G \to H$ and $\Lambda\colon H \to K$ be correspondences of ample groupoids and let C be a K-C^*-algebra. Consider the G-equivariant $*$-isomorphism $\varphi_C\colon \operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda C \to \operatorname{Ind}_{\Lambda\circ\Omega} C$ introduced in Proposition 2.7 and the isomorphism $\kappa_{\Omega,\Lambda}\colon \mathbb{Z}[\Omega] \otimes_H \mathbb{Z}[\Lambda] \cong \mathbb{Z}[\Lambda \circ \Omega]$ of G-K-bimodules introduced in Proposition 4.25. Then the following diagram commutes.*

$$\begin{tikzcd}
K_*(\operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda C) \arrow[r, "\zeta_{\Omega,\operatorname{Ind}_\Lambda C}"] \arrow[d, "K_*(\varphi_C)"] & \operatorname{Ind}_\Omega K_*(\operatorname{Ind}_\Lambda C) \arrow[r, "\operatorname{Ind}_\Omega(\zeta_{\Lambda,C})"] & \operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda K_*(C) \arrow[d, "\kappa_{\Omega,\Lambda}\otimes\mathrm{id}"] \\
K_*(\operatorname{Ind}_{\Lambda\circ\Omega} C) \arrow[rr, "\zeta_{\Lambda\circ\Omega,C}"] & & \operatorname{Ind}_{\Lambda\circ\Omega} K_*(C)
\end{tikzcd} \tag{4.3}$$

Proof. The crux of this is the following diagram for each pair of compact open bisections $U \subseteq \Omega$ and $V \subseteq \Lambda$ with $\rho(V) \subseteq \sigma(U)$.

$$\begin{tikzcd}
q(V\circ U)\operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda C \arrow[r, "\mathrm{ev}_U"] \arrow[d, "\varphi_C"] & q(V)\operatorname{Ind}_\Lambda C \arrow[d, "\mathrm{ev}_V"] \\
q(V\circ U)\operatorname{Ind}_{\Lambda\circ\Omega} C \arrow[r, "\mathrm{ev}_{V\circ U}"'] & \sigma(V)C
\end{tikzcd} \tag{4.4}$$

This is commutative because by definition, for $\xi \in q(V\circ U)\operatorname{Ind}_\Omega \operatorname{Ind}_\Lambda C$ and $[u,v]_H \in V \circ U$, we have $\varphi_C(\xi)([u,v]_H) = \xi(u)(v)$. Now consider an element $x \in K_*(\operatorname{Ind}_{(V\circ U)K} C) \subseteq K_*(\operatorname{Ind}_{\Lambda\circ\Omega} C)$. This is mapped along the bottom of the diagram (4.3) to $\chi_{V\circ U} \otimes K_*(\mathrm{ev}_{V\circ U})(x)$. Following along the other route round the diagram, we can see that it is mapped to $\kappa_{\Omega,\Lambda}(\chi_U \otimes \chi_V) \otimes K_*(\mathrm{ev}_V \circ \mathrm{ev}_U \circ \varphi_C^{-1})(x)$. By construction of $\kappa_{\Omega,\Lambda}$ and by the commutativity of (4.4), this is equal to $\chi_{V\circ U} \otimes K_*(\mathrm{ev}_{V\circ U})(x)$. Such x generate $K_*(\operatorname{Ind}_{\Lambda\circ\Omega} C)$ and $K_*(\varphi_C)$ is an isomorphism, we can conclude that (4.3) commutes. □

Proposition 4.30 (Compatibility of ζ with identity)**.** *Let G be an ample groupoid and let A be a G-C^*-algebra. Consider the canonical isomorphism $\varphi\colon \mathbb{Z}[G] \otimes_G K_*(A) \cong K_*(A)$ and the G-equivariant $*$-isomorphism $\psi_A\colon \operatorname{Ind}_G A \to A$ from*

Proposition 2.8. Then the following diagram commutes.

$$\begin{tikzcd} K_*(\mathrm{Ind}_G A) \arrow[r, "\zeta_{G,A}"] \arrow[dr, "K_*(\psi_A)"'] & \mathrm{Ind}_G K_*(A) \arrow[d, "\varphi"] \\ & K_*(A) \end{tikzcd}$$

Proof. For each $x \in K_*(A)$, there is a compact open set $U \subseteq G^0$ such that $x \in K_*(UA)$. The $*$-isomorphism $\psi_A\colon \mathrm{Ind}_G A \to A$ restricts to the $*$-isomorphism $\mathrm{ev}_U\colon U(\mathrm{Ind}_G A) \to UA$, and therefore $\zeta_{G,A}(K_*(\psi_A)^{-1}(x)) = \chi_U \otimes x$. This is equal to $\varphi^{-1}(x)$, so the diagram commutes. $\square$

We are now able to introduce the final ingredient that enables us to induce a map in homology with coefficients from a proper groupoid correspondence with coefficients.

Definition 4.31. Let $(E,\Omega)\colon (A,G) \to (B,H)$ be a proper correspondence. We define the map $\varphi_E\colon K_*(A) \to \mathrm{Ind}_\Omega K_*(B)$ of G-modules to be the composition

$$\varphi_E\colon K_*(A) \xrightarrow{K_*(\Delta(E))} K_*(\mathrm{Ind}_\Omega B) \xrightarrow{\zeta_{\Omega,B}} \mathrm{Ind}_\Omega K_*(B).$$

This respects composition of correspondences in that the following diagram commutes.

$$\begin{tikzcd} K_*(A) \arrow[rr, "\varphi_{F\circ E}"] \arrow[d, "\varphi_E"] & & \mathrm{Ind}_{\Lambda\circ\Omega}(K_*(C)) \\ \mathrm{Ind}_\Omega(K_*(B)) \arrow[rr, "\mathrm{Ind}_\Omega(\varphi_F)"] & & \mathrm{Ind}_\Omega \mathrm{Ind}_\Lambda(K_*(C)) \arrow[u, "\cong"', "\kappa_{\Omega,\Lambda}\otimes\mathrm{id}"] \end{tikzcd}$$

This follows directly from Propositions 4.29 and 2.33. In the case of trivial coefficients with $A = C_0(G^0)$, $B = C_0(H^0)$ and $E = C_0(\Omega)$, the map $\varphi_E\colon K_*(A) \to \mathrm{Ind}_\Omega K_*(B)$ is given by $\overline{\rho}^*\colon \mathbb{Z}[G^0] \to \mathbb{Z}[\Omega/H]$.

Given a proper correspondence $(E,\Omega)\colon (A,G) \to (B,H)$ of ample groupoids with C*-coefficients, we have the following three ingredients.

- an exact functor $\mathrm{Ind}_\Omega\colon H\text{-}\mathsf{Mod} \to G\text{-}\mathsf{Mod}$,
- an H-equivariant map $\delta_\Omega\colon \mathbb{Z}[G^0] \otimes_G \mathbb{Z}[\Omega] \to \mathbb{Z}[H^0]$ of right H-modules,
- and a G-equivariant map $\varphi_E\colon K_j(A) \to \mathbb{Z}[\Omega] \otimes_H K_j(B)$ of left G-modules.

We use these to build a map $H_{i,j}(E,\Omega)\colon H_i(G;K_j(A)) \to H_i(H;K_j(B))$ in homology, which is a map $\mathrm{Tor}_i^G(\mathbb{Z}[G^0],K_j(A)) \to \mathrm{Tor}_i^H(\mathbb{Z}[H^0],K_j(B))$. By Theorem 4.23, the groupoid homology of an ample groupoid G can be computed using left Coinv_G-acyclic resolutions, of which flat (and projective) resolutions are a special case. In this generality we construct a map in homology $H_{i,j}(E,\Omega)\colon H_i(G;K_j(A)) \to H_i(H;K_j(B))$.

Theorem 4.32 (Induced map in homology from a proper correspondence)**.** *Let $(E,\Omega)\colon (A,G)\to(B,H)$ be a proper correspondence. Then there are maps*

$$H_{i,j}(E,\Omega)\colon H_i(G;K_j(A))\to H_i(H;K_j(B))$$

such that for any

- *left* Coinv_G*-acyclic resolution* $P_\bullet\to K_j(A)$*,*
- *left* Coinv_H*-acyclic resolution* $Q_\bullet\to K_j(B)$*,*
- *and chain map* $f\colon P_\bullet\to \mathrm{Ind}_\Omega Q_\bullet$ *lifting* $\varphi_E\colon K_j(A)\to \mathrm{Ind}_\Omega K_j(B)$*,*

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & P_n & \longrightarrow & \cdots \longrightarrow & P_0 & \longrightarrow & K_j(A) & \longrightarrow 0\\
 & & \big\downarrow{\scriptstyle f_n} & & & \big\downarrow{\scriptstyle f_0} & & \big\downarrow{\scriptstyle \varphi_E} & \\
\cdots & \longrightarrow & \mathrm{Ind}_\Omega Q_n & \longrightarrow & \cdots \longrightarrow & \mathrm{Ind}_\Omega Q_0 & \longrightarrow & \mathrm{Ind}_\Omega K_j(B) & \longrightarrow 0
\end{array}$$

the chain map $(\delta_\Omega\otimes\mathrm{id})\circ f_G\colon (P_\bullet)_G\to(Q_\bullet)_H$ shown below induces $H_{i,j}(E,\Omega)$ in homology.

$$\begin{array}{ccccc}
\cdots \longrightarrow & (P_n)_G & \longrightarrow \cdots \longrightarrow & (P_0)_G\\
 & \big\downarrow{\scriptstyle (f_n)_G} & & \big\downarrow{\scriptstyle (f_0)_G}\\
\cdots \longrightarrow & (\mathrm{Ind}_\Omega Q_n)_G & \longrightarrow \cdots \longrightarrow & (\mathrm{Ind}_\Omega Q_0)_G\\
 & \big\downarrow{\scriptstyle \delta_\Omega\otimes\mathrm{id}} & & \big\downarrow{\scriptstyle \delta_\Omega\otimes\mathrm{id}}\\
\cdots \longrightarrow & (Q_n)_H & \longrightarrow \cdots \longrightarrow & (Q_0)_H
\end{array}$$

Proof. To construct the map $H_{i,j}(E,\Omega)$, we may consider arbitrary projective resolutions $P'_\bullet\to K_j(A)$ and $Q'_\bullet\to K_j(B)$, which exist because the categories have enough projectives. We obtain a resolution $\mathrm{Ind}_\Omega Q'_\bullet\to\mathrm{Ind}_\Omega K_j(B)$. A chain map $f'\colon P'_\bullet\to\mathrm{Ind}_\Omega Q'_\bullet$ lifting $\varphi_E\colon K_j(A)\to\mathrm{Ind}_\Omega K_j(B)$ exists by the fundamental lemma of homological algebra (Lemma 4.13). We may then define $H_{i,j}(E,\Omega)$ as $H_i((\delta_\Omega\otimes\mathrm{id})\circ f')\colon H_i((P'_\bullet)_G)\to H_i((Q'_\bullet)_H)$ after identifying these with $H_i(G;K_j(A))$ and $H_i(H;K_j(B))$.

Now given $P_\bullet\to K_j(A)$, $Q_\bullet\to K_j(B)$ and $f'\colon P_\bullet\to\mathrm{Ind}_\Omega Q_\bullet$ as in the statement of the theorem, there are chain maps $\pi\colon P'_\bullet\to P_\bullet$ and $\tau\colon Q'_\bullet\to Q_\bullet$. By the fundamental lemma of homological algebra, $f\circ\pi$ is chain homotopic to $\tau\circ f'$, and therefore $(\delta\otimes\mathrm{id})\circ f'$ and $(\delta_\Omega\otimes\mathrm{id})\circ f$ induce the same map in homology. □

This map respects the composition of correspondences: if (F,Λ) is another proper correspondence of ample groupoids with C*-coefficients which is composable with (E,Ω), then $H_{i,j}(F,\Lambda)\circ H_{i,j}(E,\Omega)=H_{i,j}(F\circ E,\Lambda\circ\Omega)$ and if (E,Ω) is an identity correspondence then $H_{i,j}(E,\Omega)$ is the identity. This follows from the compatibility of Ind_Ω, δ_Ω and φ_E with composition and identities. We recover the Morita invariance of ample groupoid homology.

Example 4.33 (Induced map in homology from an étale homomorphism)**.** Let $\varphi\colon G \to H$ be an étale homomorphism of ample groupoids. Recall from Example 4.1 that the chain complex $(\mathbb{Z}[G^\bullet], (\partial_\bullet)_G)$ (4.2) computes the homology of G. For each $n \geq 0$ the local homeomorphism $\varphi^n\colon G^n \to H^n$ induces a homomorphism $(\varphi^n)_*\colon \mathbb{Z}[G^n] \to \mathbb{Z}[H^n]$. These form a chain map which induces a map in homology $H_*(\varphi)\colon H_*(G) \to H_*(H)$.

The associated correspondence $\Omega_\varphi\colon G \to H$ is the space $G^0 \times_{H^0} H$. Consider the bar resolutions $\mathbb{Z}[G^{\bullet+1}] \to \mathbb{Z}[G^0]$ and $\mathbb{Z}[H^{\bullet+1}] \to \mathbb{Z}[H^0]$. For each $n \geq 0$ the induced module $\operatorname{Ind}_{\Omega_\varphi} \mathbb{Z}[H^{n+1}]$ has underlying abelian group $\mathbb{Z}[G^0 \times_{H^0} H^{n+1}]$, with the G-module structure coming from the action $G \curvearrowright G^0 \times_{H^0} H^{n+1}$. Consider the following local homeomorphisms.

$$
\begin{aligned}
\psi_n\colon G^{n+1} &\to G^0 \times_{H^0} H^{n+1} & \mu_n\colon G^0 \times_{H^0} H^{n+1} &\to H^n\\
(g_0, \ldots, g_n) &\mapsto (r(g), \varphi(g_0), \ldots, \varphi(g_n)) & (x, h_0, \ldots, h_n) &\mapsto (h_1, \ldots, h_n)\\
\epsilon_{G,n}\colon G^{n+1} &\to G^n & \epsilon_{H,n}\colon H^{n+1} &\to H^n\\
(g_0, \ldots, g_n) &\mapsto (g_1, \ldots, g_n) & (h_0, \ldots, h_n) &\mapsto (h_1, \ldots, h_n)
\end{aligned}
$$

The map ψ_n is G-equivariant, while μ_n and $\epsilon_{G,n}$ are G-invariant and $\epsilon_{H,n}$ is H-invariant. We obtain a chain map

$$f := (\psi_\bullet)_*\colon \mathbb{Z}[G^{\bullet+1}] \to \operatorname{Ind}_{\Omega_\varphi} \mathbb{Z}[H^{\bullet+1}]$$

over the identity $\mathbb{Z}[G^0] \to \mathbb{Z}[G^0] = \operatorname{Ind}_{\Omega_\varphi} \mathbb{Z}[H^0]$. By Theorem 4.32, $H_*(\Omega_\varphi)$ is induced by the chain map $(\delta_{\Omega_\varphi} \otimes \mathrm{id}) \circ f_G\colon \mathbb{Z}[G^{\bullet+1}]_G \to \mathbb{Z}[H^{\bullet+1}]_H$. The coinvariants $\mathbb{Z}[G^{n+1}]_G$ and $\mathbb{Z}[H^{n+1}]_H$ are given by $\mathbb{Z}[G^n]$ and $\mathbb{Z}[H^n]$ with quotient maps induced by $\epsilon_{G,n}$ and $\epsilon_{H,n}$. Under these identifications, $\delta_{\Omega_\varphi} \otimes \mathrm{id}\colon (\operatorname{Ind}_{\Omega_\varphi} \mathbb{Z}[H^{n+1}])_G \to \mathbb{Z}[H^{n+1}]_H$ is induced by the G-invariant map μ_n.

$$
\begin{tikzcd}
\mathbb{Z}[G^{n+1}] \arrow[r, "(\psi_n)_*"] \arrow[d, "\pi_G"', "(\epsilon_{G,n})_*"] & \mathbb{Z}[G^0 \times_{H^0} H^{n+1}] \arrow[d, "\pi_G"'] \arrow[ddr, bend left, "(\mu_n)_*"] & \\
\mathbb{Z}[G^n] \arrow[r, "f_G"] \arrow[drr, "(\varphi^n)_*"'] & (\mathbb{Z}[G^0 \times_{H^0} H^{n+1}])_G \arrow[dr, "\delta_{\Omega_\varphi} \otimes \mathrm{id}"] & \\
 & & \mathbb{Z}[H^n]
\end{tikzcd}
$$

The equality $\mu_n \circ \psi_n = \varphi^n \circ \epsilon_{G,n}\colon G^{n+1} \to H^n$ of local homeomorphisms implies that $(\delta_{\Omega_\varphi} \otimes \mathrm{id}) \circ ((\psi_n)_*)_G = (\varphi^n)_*\colon \mathbb{Z}[G^n] \to \mathbb{Z}[H^n]$. It follows that $H_*(\Omega_\varphi) = H_*(\varphi)$, so Theorem 4.32 recovers the standard functoriality of groupoid homology with respect to étale homomorphisms.

Example 4.34 (Induced map in homology from an action correspondence)**.** Let G be an ample groupoid, let X be a totally disconnected G-space with a proper anchor map $\tau\colon X \to G^0$ and let $H = G \ltimes X$ be the action groupoid. For each n the map $\tau_n\colon H^n \to G^n$ picking out the elements of G is proper and therefore induces a map $(\tau_n)^*\colon \mathbb{Z}[G^n] \to \mathbb{Z}[H^n]$. This gives us a chain map $(\tau_\bullet)^*\colon \mathbb{Z}[G^\bullet] \to \mathbb{Z}[H^\bullet]$ which induces a map in homology $H_*(\tau)\colon H_*(G) \to H_*(H)$.

The associated correspondence $\Omega\colon G \to H$ is the space $H = G \times_{G^0} X$. We again consider the bar resolutions $\mathbb{Z}[G^{\bullet+1}] \to \mathbb{Z}[G^0]$ and $\mathbb{Z}[H^{\bullet+1}] \to \mathbb{Z}[X]$. For each $n \geq 0$ the induced module $\operatorname{Ind}_\Omega \mathbb{Z}[H^{n+1}]$ has underlying abelian group $\mathbb{Z}[H^{n+1}]$, with the G-module structure coming from the action $G \curvearrowright H^{n+1}$. The proper G-equivariant maps $\tau_n\colon G^{n+1} \to H^{n+1}$ induce a chain map

$$f := (\tau_{\bullet+1})^*\colon \mathbb{Z}[G^{\bullet+1}] \to \operatorname{Ind}_\Omega \mathbb{Z}[H^{\bullet+1}]$$

over $\tau^*\colon \mathbb{Z}[G^0] \to \mathbb{Z}[X] = \operatorname{Ind}_\Omega \mathbb{Z}[X]$. By Theorem 4.32, $H_*(\Omega)$ is induced by the chain map $(\delta_\Omega \otimes \mathrm{id}) \circ f_G\colon \mathbb{Z}[G^{\bullet+1}]_G \to \mathbb{Z}[H^{\bullet+1}]_H$. The coinvariants $(\operatorname{Ind}_\Omega \mathbb{Z}[H^{n+1}])_G$ is given by $\mathbb{Z}[H^n]$, and $\delta_\Omega \otimes \mathrm{id}\colon \mathbb{Z}[H^n] \to \mathbb{Z}[H^n]$ is simply the identity. The chain map $f_G\colon (\mathbb{Z}[G^{\bullet+1}])_G \to (\operatorname{Ind}_\Omega \mathbb{Z}[H^{\bullet+1}])_G$ is given at n by $(\tau_n)^*\colon \mathbb{Z}[G^n] \to \mathbb{Z}[H^n]$. Therefore $(\delta_\Omega \otimes \mathrm{id}) \circ f_G$ is simply $(\tau_n)^*\colon \mathbb{Z}[G^n] \to \mathbb{Z}[H^n]$, and we obtain that $H_*(\Omega) = H_*(\tau)$.

4.3. **Interaction between the coinvariants and C*-algebras.** In module theory, the coinvariants Coinv_G of an ample groupoid is an analogue of taking the quotient by an action. In C*-theory, the crossed product $G\ltimes$ is also an analogue of the quotient. Given a G-C*-algebra A, we may relate the coinvariants of the K-theory $K_*(A)_G$ with the K-theory of the crossed product $K_*(G \ltimes A)$. This will be useful later when we want to recover groupoid homology from a purely C*-algebraic context, see Proposition 7.27.

Proposition 4.35 (Comparison between the coinvariants and the crossed product)**.** *Let G be an ample groupoid and let A be a G-C*-algebra. Then the inclusion $A \subseteq G \ltimes A$ induces a map $\gamma_A\colon K_*(A)_G \to K_*(G \ltimes A)$. Furthermore, this is natural with respect to proper G-equivariant correspondences and (when G is second countable and A is separable) morphisms in* KK^G.

Proof. Let $\iota_A\colon A \to G \ltimes A$ be the inclusion and let $\alpha\colon G \curvearrowright A$ be the action of G on A. We first check that $K_*(\iota_A)\colon K_*(A) \to K_*(G \ltimes A)$ vanishes on the kernel of $\pi_G\colon K_*(A) \to K_*(A)_G$. To see this, let $U \subseteq G$ be an open bisection, and consider $x \in K_*(A)$. We need to check that $K_*(\iota_A)(\chi_U \cdot x) = K_*(\iota_A)(\chi_{s(U)} \cdot x)$. Consider the $*$-homomorphisms $\alpha_U\colon A \to A$ and $\alpha_{s(U)}\colon A \to A$ induced by the action. The two maps $\iota_A \circ \alpha_{s(U)}, \iota_A \circ \alpha_U\colon A \rightrightarrows G \ltimes A$ induce unitarily equivalent (proper) correspondences. These are $s(U)G \ltimes A = \overline{\Gamma_c(G^{s(U)}, s^*\mathcal{A})}$ and $r(U)G \ltimes A = \overline{\Gamma_c(G^{r(U)}, s^*\mathcal{A})}$ respectively. Then $s(U)G \ltimes A$ is unitarily equivalent to $r(U)G \ltimes A$ via precomposition with the canonical homeomorphism $G^{r(U)} \cong G^{s(U)}$. It follows that $K_*(i \circ \alpha_U) = K_*(\iota_A \circ \alpha_{s(U)})$, and so $K_*(\iota_A)(\chi_U \cdot x) = K_*(\iota_A)(\chi_{s(U)} \cdot x)$. Therefore $\gamma_A\colon K_*(A)_G \to K_*(G \ltimes A)$ given by $[x] \mapsto K_*(\iota_A)(x)$ is well-defined.

Naturality of γ_A reduces to naturality of $K_*(\iota_A)\colon K_*(A) \to K_*(G \ltimes A)$. Now let $E\colon A \to B$ be a G-equivariant correspondence. There is a unitary equivalence $G \ltimes E \cong E \otimes_B G \ltimes B$ from which we obtain a commutative diagram in the correspondence category.

$$\begin{tikzcd} A \arrow[r, "\mathsf{Corr}(\iota_A)"] \arrow[d, "E"] & G \ltimes A \arrow[d, "G\ltimes E"] \\ B \arrow[r, "\mathsf{Corr}(\iota_B)"] & G \ltimes B \end{tikzcd}$$

This implies naturality for proper correspondences. To see that this is natural with respect to KK^G, we assume G is second countable and A is separable, that E is countably generated and graded, and that we have a Fredholm operator $T \in \mathcal{L}(E)$ such that $(E, T) \in \mathbb{E}^G(A, B)$. Under the identification of $G \ltimes E$ with $E \otimes_B G \ltimes B$, the operators $G \ltimes T$ and $T \otimes 1$ are the same. The Kasparov A-$G \ltimes B$ cycles $(G \ltimes E, G \ltimes T)$ and $(E \otimes_B G \ltimes B, T \otimes 1)$ are therefore unitarily equivalent. We deduce naturality of $\mathrm{KK}(\iota_A)$ with respect to KK^G and hence naturality of γ_A. $\square$

The comparison map $\gamma_A\colon K_*(A)_G \to K_*(G \ltimes A)$ is compatible with the maps $K_*(\Omega \ltimes \Theta_{\Omega,B})\colon K_*(G \ltimes \mathrm{Ind}_\Omega B) \to K_*(H \ltimes B)$ and $\delta_\Omega \otimes \mathrm{id}\colon (\mathrm{Ind}_\Omega K_*(B))_G \to K_*(B)_H$ in the following way.

Proposition 4.36. *Let $\Omega\colon G \to H$ be a correspondence of ample groupoids and let B be an H-C^*-algebra. Then the following diagram commutes.*

$$\begin{tikzcd} K_*(\mathrm{Ind}_\Omega B)_G \arrow[rr, "\gamma_{\mathrm{Ind}_\Omega B}"] \arrow[d, "\cong"', "(\zeta_{\Omega,B})_G"] & & K_*(G \ltimes \mathrm{Ind}_\Omega B) \arrow[d, "K_*(\Omega\ltimes\Theta_{\Omega,B})"] \\ (\mathrm{Ind}_\Omega K_*(B))_G \arrow[r, "\delta_\Omega\otimes\mathrm{id}"] & K_*(B)_H \arrow[r, "\gamma_B"] & K_*(H \ltimes B) \end{tikzcd}$$

Proof. We may start with an element $[\chi_U \otimes x] \in (\mathrm{Ind}_\Omega K_*(B))_G$, where $U \subseteq \Omega$ is a compact open bisection and $x \in K_*(\sigma(U)B)$. Consider the inclusions $\iota\colon \sigma(U)B \to H \ltimes B$ and $\epsilon\colon \mathrm{Ind}_{UH} B \to G \ltimes \mathrm{Ind}_\Omega B$. We may calculate that $(\delta_\Omega \otimes \mathrm{id})([\chi_U \otimes x]) = [x]$ and that $\gamma_B([x]) = K_*(\iota)(x)$. Going the other way, our element is mapped to $K_*(\Omega \ltimes \Theta_{\Omega,B}) \circ K_*(\epsilon) \circ K_*(\mathrm{ev}_U^{-1})(x)$. By Lemma 2.27, $\mathsf{Corr}(\epsilon) \otimes_{G\ltimes \mathrm{Ind}_\Omega B} \Omega \ltimes \Theta_{\Omega,B} \cong \mathsf{Corr}(\mathrm{ev}_U) \otimes_{\sigma(U)B} \mathsf{Corr}(\iota)$, so we can conclude that $K_*(\iota)(x) = K_*(\Omega \ltimes \Theta_{\Omega,B}) \circ K_*(\epsilon) \circ K_*(\mathrm{ev}_U^{-1})(x)$. $\square$

The groupoid correspondence $G\colon G \to G^0$ gives us a way to induce a G-C*-algebra $\mathrm{Ind}_{G^0}^G B = \mathrm{Ind}_G B$ from a $C_0(G^0)$-algebra B. For these induced algebras, the comparison map is an isomorphism.

Proposition 4.37. *Let B be a $C_0(G^0)$-algebra and let $A = \mathrm{Ind}_{G^0}^G B$ be the induced G-C^*-algebra. Then the induced map $\gamma_A\colon K_*(A)_G \to K_*(G \ltimes A)$ is an isomorphism.*

Proof. Applying Proposition 4.36 we obtain the following commutative diagram.

$$\begin{tikzcd}
K_*(\operatorname{Ind}_G B)_G \arrow[rr, "\gamma_A"] \arrow[d, "\cong"', "(\zeta_{G,B})_G"] & & K_*(G \ltimes \operatorname{Ind}_G B) \arrow[d, "K_*(G \ltimes \Theta_{G,B})"] \\
(\operatorname{Ind}_G K_*(B))_G \arrow[r, "\delta_G \otimes \mathrm{id}"] & K_*(B)_{G^0} \arrow[r, "\gamma_B"] & K_*(B)
\end{tikzcd}$$

It suffices to check that each of the maps other than γ_A is an isomorphism. The map $\delta_G \colon \mathbb{Z}[G^0] \otimes_G \mathbb{Z}[G] \to \mathbb{Z}[G^0]$ is the canonical isomorphism induced by the right action of $\mathbb{Z}[G]$ on $\mathbb{Z}[G^0]$, and therefore $\delta_G \otimes \mathrm{id}$ is an isomorphism. The comparison map $\gamma_B \colon K_*(B)_{G^0} \to K_*(B)$ is the inverse of the coinvariants quotient map π_{G^0}. Finally, we claim that $G \ltimes \Theta_{G,B}$ is a Morita equivalence, and therefore $K_*(G \ltimes \Theta_{G,B})$ is an isomorphism. It can be checked directly that $G \ltimes \Theta_{G,B} \cong L^2(G,B)$ and that the structure map is an isomorphism $G \ltimes \operatorname{Ind}_G B \cong \mathcal{K}(L^2(G,B))$, witnessing the Morita equivalence.

Alternatively, we can view $G \ltimes \Theta_{G,B}$ as the crossed product of the evaluation correspondence $(\operatorname{Ind}_G B, G \ltimes G) \to (B, G^0)$ for the Morita equivalence $G \colon G \ltimes G \to G^0$. By the compatibility of the evaluation natural transformation and the induction functor with composition and identities (Propositions 2.30, 2.31, 2.7 and 2.8), this is a Morita equivalence. □

The homology of modules induced by subgroupoids is also very tractable. This follows from a version of Shapiro's lemma, which is usually stated for subgroups.

Lemma 4.38 (Shapiro's Lemma)**.** *Let G be an ample groupoid and let $H \subseteq G$ be a closed subgroupoid containing G^0. Let $\operatorname{Ind}_H^G = \mathbb{Z}[G] \otimes_H - \colon H\text{-}\mathsf{Mod} \to G\text{-}\mathsf{Mod}$ be the subgroupoid induction functor, with the bimodule $\mathbb{Z}[G]$ constructed from the actions $G \curvearrowright G \curvearrowleft H$ by left and right multiplication. Then for any H-module M, we have an isomorphism*

$$H_*(G; \operatorname{Ind}_H^G M) \cong H_*(H; M)$$

in homology. In particular, if $H = G^0$ then $H_n(G; \operatorname{Ind}_{G^0}^G M) \cong 0$ for $n > 0$, because $\operatorname{Coinv}_{G^0} \colon G^0\text{-}\mathsf{Mod} \to \mathsf{Ab}$ is just the forgetful functor, which is exact.

Proof. The right action $G \curvearrowleft H$ is proper because H is closed, and so $G \curvearrowright G \curvearrowleft H$ is a groupoid correspondence, and in particular $\operatorname{Ind}_H^G \colon H\text{-}\mathsf{Mod} \to G\text{-}\mathsf{Mod}$ is the exact induction functor of this groupoid correspondence. Let $F_\bullet \to \mathbb{Z}[G^0]$ be a flat resolution of right G-modules. We obtain a resolution of right H-modules $F_\bullet \otimes_G \mathbb{Z}[G] \to \mathbb{Z}[G^0]$. Moreover, these are flat, as $F_i \otimes_G \mathbb{Z}[G] \otimes_H -$ is the composition of two exact functors. By Proposition 4.21, the homology of the chain complex

$$F_\bullet \otimes_G (\mathbb{Z}[G] \otimes_H M) \cong (F_\bullet \otimes_G \mathbb{Z}[G]) \otimes_H M$$

computes both $H_*(G; \operatorname{Ind}_H^G M)$ and $H_*(H; M)$. □

5. Spectral sequences

Spectral sequences are a key part of the topologist's toolkit, but may be unfamiliar to an operator algebraist. We cover the basics of spectral sequences in this short chapter, so that we may later deal with the ABC spectral sequence. We refer to Weibel [88] for a more detailed account.

5.1. Spectral sequences.

Spectral sequences organise together lots of homological objects in a powerful way. For us, they will be systems of abelian groups which are sorted into *sheets* or *pages*, each of which consists of a two dimensional array of abelian groups called a *bigraded* abelian group.

Definition 5.1 (Graded abelian group)**.** A *graded* abelian group G is a collection of abelian groups G_n indexed by integers n. A *bigraded* abelian group H is a collection of abelian groups $H_{p,q}$ indexed by integers p, q.

Many of the abelian groups we are interested in fit into the context of graded abelian groups naturally, for example K-theory groups K_* and homology groups H_*.

Definition 5.2 (Morphisms of graded abelian groups)**.** A *morphism* $f\colon G \to G'$ *of graded abelian groups* with *degree* d is a morphism $f_n\colon G_n \to G'_{n+d}$ for each integer n. Similarly, a *morphism* $g\colon H \to H'$ *of bigraded groups* with *bidegree* (m, n) is a morphism $g_{p,q}\colon H_{p,q} \to H'_{p+m,q+n}$ for each p, q. If we do not mention a degree or a bidegree, we mean that the (bi)degree is 0 or $(0, 0)$.

We can now introduce the notion of a spectral sequence.

Definition 5.3 (Spectral sequence)**.** Let r_0 be a non-negative integer. A *spectral sequence starting at the* r_0*th sheet* is a pair $(E, d) = (E^r, d^r)_{r\geq r_0}$ consisting of:

- bigraded abelian groups E^r called *sheets* or *pages*. This means we have an abelian group $E^r_{p,q}$ for each $p, q \in \mathbb{Z}$.
- maps $d^r : E^r \to E^r$ with bidegree $(-r, r-1)$ satisfying $d^r \circ d^r = 0$ called *differentials*. This means that we have maps $d^r_{p,q}\colon E^r_{p,q} \to E^r_{p-r,q+r-1}$ such that $d^r_{p-r,q+r-1} \circ d^r_{p,q} = 0$.

along with specified isomorphisms $E^{r+1} \cong H(E^r, d^r)$ which consist of isomorphisms $E^{r+1}_{p,q} \cong \ker d^r_{p,q}/\operatorname{im} d^r_{p+r,q-r+1}$ for each $p, q \in \mathbb{Z}$.

Definition 5.4 (Morphism of spectral sequences)**.** A *morphism* $f\colon (E, d) \to (E', d')$ *of spectral sequences* that start at the r_0th sheet is a collection of maps $f^r : E^r \to E'^r$ respecting the bigraded structure such that

- f^r is compatible with the differentials, i.e. $f^r \circ d^r = d'^r \circ f^r$.

- $f^{r+1}\colon E^{r+1} \to E'^{r+1}$ coincides with the map $f^r_*\colon H(E^r, d^r) \to H(E'^r, d'^r)$ that f^r induces in homology under the isomorphisms $E^{r+1} \cong H(E^r, d^r)$ and $E'^{r+1} \cong H(E'^r, d'^r)$.

Note that the second bullet point implies that a morphism of spectral sequences is determined by its morphism of the r_0th sheets $f^{r_0}\colon E^{r_0} \to E'^{r_0}$. This has the following important consequence (see [88, Mapping Lemma 5.2.4]):

Proposition 5.5 (Mapping lemma)**.** *Suppose $f\colon (E, d) \to (E', d')$ is a morphism of spectral sequences such that $f^{r_1}\colon E^{r_1} \to E'^{r_1}$ is an isomorphism for some $r_1 \geq r_0$. Then $f^r\colon E^r \to E'^r$ is an isomorphism for every $r \geq r_1$.*

We can form the category $\mathsf{SpecSeq}^{\geq r_0}$ of spectral sequences starting at the r_0th sheet. Technically, we should specify that these are homological spectral sequences in the category of abelian groups. Other authors might consider cohomological spectral sequences or alternative abelian categories. Usually, r_0 will be either 1 or 2.

5.2. **Cycles, boundaries and limit sheets.** Given a spectral sequence (E, d) starting at r_0, we are interested in the behavour of (E^r, d^r) as r grows larger. It can be useful to think of these later pages in terms of groups of *cycles* and *boundaries* which live in E^{r_0}. We can construct (bigraded abelian) groups B^r and Z^r of boundaries and cycles as subgroups of E^{r_0}, such that

$$\operatorname{im} d^{r_0} = B^{r_0} \subseteq B^{r_0+1} \subseteq \cdots \subseteq B^r \subseteq \cdots \subseteq Z^r \subseteq \cdots \subseteq Z^{r_0+1} \subseteq Z^{r_0} = \ker d^{r_0},$$
$$E^{r+1} \cong Z^r/B^r.$$

Assuming we have already constructed Z^r and B^r with an isomorphism $E^{r+1} \cong Z^r/B^r$, we define B^{r+1} and Z^{r+1} to be the preimages of $\operatorname{im} d^{r+1}$ and $\ker d^{r+1}$ under the map $Z^r \to E^{r+1}$:

$$\begin{tikzcd}
Z^{r+1} \arrow[dr, hook] & & & & \ker d^{r+1} \arrow[dl, hook'] \\
& Z^r \arrow[r] & Z^r/B^r \arrow[r, "\cong"] & E^{r+1} & \\
B^{r+1} \arrow[ur, hook] & & & & \operatorname{im} d^{r+1} \arrow[ul, hook']
\end{tikzcd}$$

We can then check that

$$Z^{r+1}/B^r \cong \ker d^{r+1},$$
$$B^{r+1}/B^r \cong \operatorname{im} d^{r+1},$$
$$Z^{r+1}/B^{r+1} \cong \ker d^{r+1}/\operatorname{im} d^{r+1} \cong E^{r+2}.$$

We can then define

$$Z^\infty := \cap_{r\geq r_0} Z^r \qquad \text{the group of } \textit{infinite cycles},$$
$$B^\infty := \cup_{r\geq r_0} B^r \qquad \text{the group of } \textit{infinite boundaries},$$

$$E^\infty := Z^\infty / B^\infty \qquad\qquad \text{the } \textit{limit sheet}.$$

If we truncate (E, d) to start at a later sheet $r_1 > r_0$, we would obtain different groups of cycles and boundaries Z'^r and B'^r as subgroups of E^{r_1}. However, we can check that

$$Z'^r \cong Z^r / B^{r_1 - 1}, \qquad B'^r \cong B^r / B^{r_1-1}, \qquad \text{and so} \qquad Z'^\infty / B'^\infty \cong Z^\infty / B^\infty.$$

This means that the limit sheet is independent of the starting sheet, up to canonical isomorphism. The limit sheet is also functorial with respect to spectral sequence morphisms (see [88, Exercise 5.2.3]):

Proposition 5.6 (Mapping lemma at infinity)**.** *Suppose $f\colon (E, d) \to (E', d')$ is a morphism of spectral sequences. Then we obtain a morphism $f^\infty\colon E^\infty \to E'^\infty$ of the limit sheets. Furthermore, if $f^{r_1}\colon E^{r_1} \to E'^{r_1}$ is an isomorphism for some $r_1 \geq r_0$, then $f^\infty\colon E^\infty \to E'^\infty$ is an isomorphism.*

5.3. **Convergence of spectral sequences.** When discussing the convergence of a spectral sequence (E, d) to a graded abelian group G, we need a filtration $(\mathcal{F}_k G)_k$ of the graded group, as the limit sheet E^∞ of the spectral sequence is a bigraded object. The filtration lets us break the graded group down into a bigraded object $\mathcal{F}_{p+1} G_{p+q} / \mathcal{F}_p G_{p+q}$ which we can compare with the limit sheet $E^\infty_{p,q}$. Extra properties of the filtration can then help us to reconstruct G from this bigraded object.

Definition 5.7 (Filtration)**.** An *ascending filtration* $(\mathcal{F}_k G)_k$ on a (graded) abelian group G is a nested series of (graded) subgroups of G indexed by $\mathbb{Z}$.

$$0 \subseteq \cdots \subseteq \mathcal{F}_{-1} G \subseteq \mathcal{F}_0 G \subseteq \mathcal{F}_1 G \subseteq \cdots \subseteq G$$

We say G is a *filtered* (graded) abelian group.

- The filtration is *exhaustive*, or *exhausts* G, if $\cup_k \mathcal{F}_k G = G$.
- The filtration is *Hausdorff* if $\cap_k \mathcal{F}_k G = \emptyset$.
- The filtration is *complete* if every Cauchy sequence $(x_n)_{n \in \mathbb{N}}$ in G converges.

A sequence $(x_n)_{n\in\mathbb{N}}$ in G is *Cauchy* if for any k, $x_n - x_m \in \mathcal{F}_k G$ for sufficiently large n, m. It converges to $x \in G$ if for any k, we have $x_n - x \in \mathcal{F}_k G$ for sufficiently large n. Note that if $\mathcal{F}_k G = 0$ for any k, then the filtration is automatically complete and Hausdorff.

When considering the convergence of a spectral sequence (E, d) to a filtered graded abelian group G, we may write

$$E^{r_0}_{p,q} \Rightarrow G$$

to indicate that the target group being considered is G, without implying that any convergence is actually achieved. Convergence is usually stated alongside this notation.

Definition 5.8 (Convergence of a spectral sequence)**.** Given a spectral sequence (E, d) and a target filtered graded group G, we say that the spectral sequence:

- *converges weakly* to G if the filtration exhausts G and we have isomorphisms
$$E^{\infty}_{p,q} \cong \mathcal{F}_{p+1}G_{p+q}/\mathcal{F}_pG_{p+q}$$
for each $p, q \in \mathbb{Z}$.
- *converges* to G if furthermore the filtration is Hausdorff.
- *converges strongly* to G if again further the filtration is complete.

The reason we ask for these conditions on the filtration is so that we can rebuild the group from subquotients coming from the filtration. This is demonstrated by the following proposition.

Proposition 5.9. *Let G be a (graded) group with an ascending filtration $(\mathcal{F}_kG)_k$ that is exhaustive, complete and Hausdorff. Then*
$$G \cong \lim_{s\to-\infty} \frac{G}{\mathcal{F}_sG} = \lim_{s\to-\infty} \bigcup_{t\geq s} \frac{\mathcal{F}_tG}{\mathcal{F}_sG}$$

Proof. The isomorphism on the left is due to the filtration being complete and Hausdorff, and the equality on the right is from exhaustiveness. □

Reconstructing $\mathcal{F}_tG/\mathcal{F}_sG$ for $t > s$ from subquotients of the form $\mathcal{F}_{k+1}G/\mathcal{F}_kG$ comes down to solving several extension problems. For example, we have an exact sequence
$$0 \longrightarrow \mathcal{F}_{s+1}G/\mathcal{F}_sG \longrightarrow \mathcal{F}_tG/\mathcal{F}_sG \longrightarrow \mathcal{F}_tG/\mathcal{F}_{s+1}G \longrightarrow 0.$$
Given only the isomorphism classes of $\mathcal{F}_{s+1}G/\mathcal{F}_sG$ and $\mathcal{F}_tG/\mathcal{F}_{s+1}G$, there may be multiple groups which fit as the middle term of a short exact sequence with them. However, given a morphism of short exact sequences
$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_{s+1}G/\mathcal{F}_sG \arrow[r] \arrow[d, "\alpha"] & \mathcal{F}_tG/\mathcal{F}_sG \arrow[r] \arrow[d, "\beta"] & \mathcal{F}_tG/\mathcal{F}_{s+1}G \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & \mathcal{F}_{s+1}G'/\mathcal{F}_sG' \arrow[r] & \mathcal{F}_tG'/\mathcal{F}_sG' \arrow[r] & \mathcal{F}_tG'/\mathcal{F}_{s+1}G' \arrow[r] & 0,
\end{tikzcd}$$
then if α and γ are isomorphisms then so is β. Using this idea, we obtain the following.

Proposition 5.10. *Suppose $f\colon G \to G'$ is a morphism of filtered (graded) groups, such that*

- *The filtrations on G and G' are exhaustive.*
- *The filtrations on G and G' are Hausdorff.*
- *The filtration on G is complete.*
- *The maps $\mathcal{F}_{k+1}G/\mathcal{F}_kG \to \mathcal{F}_{k+1}G'/\mathcal{F}_kG'$ induced by f are isomorphisms for each k.*

Then $f\colon G \to G'$ is an isomorphism.

To relate this to convergence of spectral sequences, we need to relate morphisms of spectral sequences with morphisms of the target filtered graded abelian groups.

Definition 5.11 (Compatibility of spectral sequence morphisms with target group homomorphisms)**.** Suppose that $f\colon (E,d) \to (E',d')$ is a morphism of spectral sequences, $g\colon G \to G'$ is a morphism of filtered graded abelian groups, (E,d) converges weakly to G and (E',d') converges weakly to G'. Then $g\colon G \to G'$ is *compatible* with $f\colon (E,d) \to (E',d')$ if the following diagram commutes for each p, q:

$$\begin{tikzcd} E^{\infty}_{p,q} \arrow[r, "\cong"] \arrow[d, "f^{\infty}_{p,q}"] & \frac{\mathcal{F}_{p+1}G_{p+q}}{\mathcal{F}_pG_{p+q}} \arrow[d, "g_{p,q}"] \\ {E'}^{\infty}_{p,q} \arrow[r, "\cong"] & \frac{\mathcal{F}_{p+1}G'_{p+q}}{\mathcal{F}_pG'_{p+q}} \end{tikzcd}$$

Here the horizontal maps are the isomorphisms from the weak convergence of the spectral sequences to the groups, and $g_{p,q}$ is the map of the subquotients induced by g.

Combining Proposition 5.6 and Proposition 5.10, we obtain the following useful consequence of convergence with respect to morphisms of spectral sequences (see [88, Comparison Theorem 5.2.12]):

Theorem 5.12. *Suppose that $f\colon (E,d) \to (E',d')$ is a morphism of spectral sequences and $g\colon G \to G'$ is a morphism of filtered graded abelian groups, such that:*

- *(E,d) converges strongly to G.*
- *(E',d') converges to G'.*
- *$g\colon G \to G'$ is compatible with $f\colon (E,d) \to (E',d')$.*
- *$f^{r_1}\colon E^{r_1} \to {E'}^{r_1}$ is an isomorphism for some $r_1 \geq r_0$.*

Then $g\colon G \to G'$ is an isomorphism.

5.4. **The spectral sequence of an exact couple.** An important source of spectral sequences is the construction of a spectral sequence from an exact couple. The groups of boundaries and cycles are reasonably explicit in this picture.

Definition 5.13 (Exact couple)**.** An *exact couple* (C, D, i, j, k) is a diagram of the following form, where C and D are bigraded abelian groups.

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & C \arrow[ul, "k"] & \end{tikzcd} \qquad \begin{aligned} \deg i &= (1, -1) \\ \deg j &= (0, 0) \\ \deg k &= (-1, 0) \end{aligned} \qquad \begin{aligned} \ker i &= \operatorname{im} k \\ \ker j &= \operatorname{im} i \\ \ker k &= \operatorname{im} j \end{aligned}$$

A *morphism of exact couples* $(C, D, i, j, k) \to (C', D', i', j', k')$ is a pair of bidegree $(0, 0)$ morphisms $C \to C'$ and $D \to D'$ intertwining the maps i, j, k with i', j', k' respectively.

From an exact couple (C, D, i, j, k) we will construct a spectral sequence $(E, d) = (E^r, d^r)_{r \geq 1}$ with $(E^1, d^1) = (C, jk)$. For each $r \geq 0$, we consider the r-fold composition $i^{(r)} \colon D \to D$, and define

$$\begin{aligned} Z^r &:= k^{-1}(\operatorname{im} i^{(r)}) \subseteq C && \text{the subgroup of } r\text{-cycles}, \\ B^r &:= j(\ker i^{(r)}) \subseteq C && \text{the subgroup of } r\text{-boundaries}, \\ E^{r+1} &:= Z^r / B^r && \text{the } r+1\text{th sheet of the spectral sequence.} \end{aligned}$$

Note that we have

$$0 = B^0 \subseteq B^1 \subseteq B^2 \subseteq \cdots \subseteq \operatorname{im} j = \ker k \subseteq \cdots \subseteq Z^2 \subseteq Z^1 \subseteq Z^0 = C.$$

To construct the $r+1$th differential $d^{r+1} \colon E^{r+1} \to E^{r+1}$, we can chase the following diagram.

$$\begin{tikzcd} Z^r = k^{-1}(\operatorname{im} i^{(r)}) \arrow[r, "k"] \arrow[d, two heads, "q"] & \operatorname{im} i^{(r)} \arrow[d, dashed, "j \circ i^{(-r)}"] & D \arrow[l, two heads, "i^{(r)}"'] \arrow[d, "j"] \\ \frac{Z^r}{B^r} = \frac{k^{-1}(\operatorname{im} i^{(r)})}{j(\ker i^{(r)})} \arrow[r, dashed, "d^{r+1}"] & \frac{Z^r}{B^r} = \frac{k^{-1}(\operatorname{im} i^{(r)})}{j(\ker i^{(r)})} & k^{-1}(\operatorname{im} i^{(r)}) \arrow[l, two heads, "q"] \end{tikzcd}$$

The map $j \circ i^{(-r)}$ is induced by the vanishing of $q \circ j$ on $\ker i^{(r)}$. The differential d^{r+1} is induced by the vanishing of $j \circ i^{(-r)} \circ k$ on $j(\ker i^{(r)})$. The differential satisfies

$$d^{r+1} \circ d^{r+1} = 0, \qquad \ker d^{r+1} = Z^{r+1}/B^r, \qquad \operatorname{im} d^{r+1} = B^{r+1}/B^r,$$

and so $E^{r+2} \cong \ker d^{r+1} / \operatorname{im} d^{r+1}$. We can do this for each $r \geq 0$, so we obtain a spectral sequence (E, d). This process is functorial: a morphism of exact couples induces a morphism of their spectral sequences.

6. Relative homological algebra in triangulated categories

This is a summary of the relative homological algebra in triangulated categories that we will need. Most of the material in this section is from [57–59], with some developments in the context of groupoid equivariant KK-theory from [11, 71].

6.1. Triangulated categories.

To do homology in equivariant Kasparov categories, we want to view them as triangulated categories. We collect some basic definitions:

Definition 6.1 (Additive category/functor)**.** An *additive category* is a category whose morphism sets are abelian groups such that composition is bilinear and finite coproducts exist (which we then call direct sums). Such a category automatically has a zero object, the empty coproduct, which we call 0. A functor $F\colon \mathfrak{C} \to \mathfrak{D}$ between additive categories is *additive* if it is a homomorphism of abelian groups on each morphism set.

Definition 6.2 (Stable additive category)**.** A *stable additive category* is an additive category $\mathfrak{C}$ with an additive auto-equivalence $\Sigma\colon \mathfrak{C} \to \mathfrak{C}$, called the *suspension*.

We may write Σ^{-1} to refer to an inverse to the equivalence Σ, and although $\Sigma^{-1}\Sigma$ and $\Sigma\Sigma^{-1}$ are only naturally isomorphic to the identity, we often pretend that they are identical for convenience.

Example 6.3. The category $\mathsf{Ab}^{\mathbb{Z}/2\mathbb{Z}}$ of pairs of abelian groups is a stable additive category, with a suspension that swaps the groups. The category Ab_* of $\mathbb{Z}$-graded abelian groups is also stable, with a suspension that shifts the groups.

Definition 6.4 (Triangulated category)**.** A triangulated category is a stable additive category $\mathfrak{T}$ with suspension $\Sigma\colon \mathfrak{T} \to \mathfrak{T}$ equipped with a class of *exact triangles* subject to a collection of axioms. Each exact triangle is a diagram $A \to B \to C \to \Sigma A$ in $\mathfrak{T}$, which we call a triangle.

$$\begin{tikzcd} A \arrow[rr] & & B \arrow[dl] \\ & C \arrow[ul, "\circ"] & \end{tikzcd}$$

The circled arrow $C \to\!\!\!\circ\, A$ signifies a *degree* 1 map, i.e. a map $C \to \Sigma A$. There are a number of ways to formulate the axioms of a triangulated category, here is one:

TR0: The class of exact triangles is closed under isomorphism of triangles.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f", "\cong"'] & B \arrow[r] \arrow[d, "g", "\cong"'] & C \arrow[r] \arrow[d, "h", "\cong"'] & \Sigma A \arrow[d, "\Sigma f", "\cong"'] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & \Sigma A' \end{tikzcd}$$

TR1: Every morphism $f\colon A \to B$ is the first map in an exact triangle $A \to B \to C \to \Sigma A$. The object C is called a cone for the morphism f, and this exact triangle is determined up to isomorphism by the other axioms. The cone of an identity morphism is the 0 object.

$$
\begin{tikzcd}
A \arrow[rr, "1_A"] & & A \arrow[dl] \\
& 0 \arrow[ul, "\circ" marking] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, "f"] & & B \arrow[dl] \\
& C \arrow[ul, "\circ" marking] &
\end{tikzcd}
$$

TR2: A triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ is exact if and only if its rotation $B \xrightarrow{g} C \xrightarrow{h} \Sigma A \xrightarrow{-\Sigma f} \Sigma B$ is exact.

$$
\begin{tikzcd}
A \arrow[rr, "f"] & & B \arrow[dl, "g"] \\
& C \arrow[ul, "\circ" marking, "h"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[rr, "g"] & & C \arrow[dl, "h"] \\
& \Sigma A \arrow[ul, "\circ" marking, "-\Sigma f"] &
\end{tikzcd}
$$

TR3: Given a commutative diagram

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] & \Sigma A \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & \Sigma A'
\end{tikzcd}
$$

whose rows are exact triangles, there exists a morphism $h\colon C \to C'$ such that the following diagram commutes.

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, dashed, "h"] & \Sigma A \arrow[d, "\Sigma f"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & \Sigma A'
\end{tikzcd}
$$

TR4: Given exact triangles

$$
\begin{tikzcd}
A \arrow[rr, "u"] & & B \arrow[dl, "i"] \\
& Z \arrow[ul, "\circ" marking, "l"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[rr, "v"] & & C \arrow[dl, "k"] \\
& X \arrow[ul, "\circ" marking, "n"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, "vu"] & & C \arrow[dl, "j"] \\
& Y \arrow[ul, "\circ" marking, "m"] &
\end{tikzcd}
$$

there exists an exact triangle $Z \xrightarrow{f} Y \xrightarrow{g} X \xrightarrow{h} \Sigma Z$ such that the following diagram commutes.

$$
\begin{tikzcd}
A \arrow[rr, bend left, "vu"] \arrow[dr, "u"] & & C \arrow[rr, bend left, "k"] \arrow[dr, "j"] & & X \arrow[rr, dashed, bend left, "h"] \arrow[dr, "n"] & & \Sigma Z \\
& B \arrow[ur, "v"] \arrow[dr, "i"] & & Y \arrow[ur, dashed, "g"] \arrow[dr, "m"] & & \Sigma B \arrow[ur, "\Sigma i"] & \\
& & Z \arrow[ur, dashed, "f"] \arrow[rr, bend right, "l"] & & \Sigma A \arrow[ur, "\Sigma u"] & &
\end{tikzcd}
$$

This is called the octahedral axiom, because in a certain form the diagram can be viewed as an octahedron. If we think of cones as like quotients, this is analogous to the third isomorphism theorem: $(C/A)/(B/A) \cong C/B$.

Although a triangulated category involves the data of an additive category $\mathfrak{T}$, a suspension Σ and a class of exact triangles, we will commonly refer to the triangulated category as $(\mathfrak{T}, \Sigma)$ or simply as $\mathfrak{T}$.

Here are some basics about triangulated categories that we may use.

- For any consecutive morphisms $A \xrightarrow{u} B \xrightarrow{v} C$ in an exact triangle, $vu = 0$. This uses the following diagram with TR1 and TR3, with TR2 and TR0 to rotate the triangle if necessary.

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "1_A"] & A \arrow[r] \arrow[d, "u"] & 0 \arrow[r] \arrow[d, dashed] & \Sigma A \arrow[d, "1_{\Sigma A}"] \\ A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r] & \Sigma A \end{tikzcd}$$

- The two-out-of-three property: given a morphism of exact triangles with two out of three vertical maps isomorphisms, f and g in this diagram,

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f", "\cong"'] & B \arrow[r] \arrow[d, "g", "\cong"'] & C \arrow[r] \arrow[d, "h"] & \Sigma A \arrow[d, "\Sigma f", "\cong"'] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & \Sigma A' \end{tikzcd}$$

  the third map h is also an isomorphism.

- The uniqueness of cones up to isomorphism, which follows from the two-out-of-three property.

- Given an exact triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ and a morphism $i\colon D \to B$ such that $gi = 0$, there is a lift of i through f to a morphism $k\colon D \to A$. This follows from applying a rotated version of TR3 to the diagram on the left. Dually, given a morphism $l\colon B \to D$ such that $lf = 0$, there is a morphism $m\colon C \to D$ such that $l = mg$.

$$\begin{tikzcd} D \arrow[r, "1_D"] \arrow[d, dashed, "k"] & D \arrow[r] \arrow[d, "i"] & 0 \arrow[r] \arrow[d] & \Sigma D \arrow[d, dashed, "\Sigma k"] \\ A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r, "h"] & \Sigma A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[r, "g"] \arrow[d, "l"] & C \arrow[r, "h"] \arrow[d, dashed, "m"] & \Sigma A \arrow[d] \\ 0 \arrow[r] & D \arrow[r, "1_D"] & D \arrow[r] & 0 \end{tikzcd}$$

  The first describes a way in which A acts like $\ker g$, and the second describes a way in which C acts like $B/\ker g$.

Example 6.5 (Equivariant Kasparov category)**.** Let G be a second countable étale groupoid and consider its equivariant Kasparov category KK^G. The abelian group structures on the morphism sets and bilinearity of the Kasparov product turn KK^G into an additive category. We take the functor $S\colon \mathrm{KK}^G \to \mathrm{KK}^G$ mapping a C*-algebra A to its suspension $SA = C_0(\mathbb{R}, A)$ as our suspension functor, and it is its own inverse (up to natural isomorphism, by Bott periodicity). We take the class of exact triangles to be the triangles isomorphic to mapping cone triangles:

$$SB \to \mathrm{cone}(f) \to A \xrightarrow{f} B.$$

Here $f\colon A \to B$ is a G-equivariant $*$-homomorphism and the *mapping cone* of f is

$$\mathrm{cone}(f) := \{(a,b) \in A \oplus C_0((0,1],B) \mid f(a) = b(1)\},$$

which contains SB and quotients onto A. This defines a triangulated category structure on KK^G.

In the setting of group action groupoids, this is proved in [58, Appendix A]. The general case largely follows the same procedure and is discussed in [11, Section 1.1].

Definition 6.6 (Triangulated functor)**.** A morphism of triangulated categories from $(\mathfrak{T},\Sigma)$ to $(\mathfrak{T}',\Sigma')$ is called a *triangulated functor* or an *exact functor*, and is an additive functor $\Phi\colon \mathfrak{T} \to \mathfrak{T}'$ with a natural isomorphism $\Phi \circ \Sigma \Rightarrow \Sigma' \circ \Phi$ such that every exact triangle is mapped to an exact triangle.

Example 6.7 (The restriction functor)**.** Let G be an étale groupoid and let $H \subseteq G$ be a subgroupoid. Then each G-C*-algebra naturally restricts to an H-C*-algebra, giving rise to a triangulated functor $\mathrm{Res}_G^H\colon \mathrm{KK}^G \to \mathrm{KK}^H$ called the *restriction* functor.

Example 6.8 (The descent functors)**.** Let G be an étale groupoid. Then the descent functor $G \ltimes -\colon \mathrm{KK}^G \to \mathrm{KK}$ is triangulated, as is the reduced descent functor $G \ltimes_r -\colon \mathrm{KK}^G \to \mathrm{KK}$.

Proposition 6.9 (The induction functor is triangulated)**.** *Let $\Omega : G \to H$ be a groupoid correspondence. Then the induction functor $\mathrm{Ind}_\Omega\colon \mathrm{KK}^H \to \mathrm{KK}^G$ is triangulated.*

Proof. It is additive because it is a group homomorphism on each $\mathrm{KK}^H(B,C)$ by Theorem 3.24. It is straightforward to see that Ind_Ω is compatible with suspensions. For each H-equivariant $*$-homomorphism $f\colon A \to B$, we have a G-equivariant $*$-isomorphism

$$\mathrm{Ind}_\Omega(\mathrm{cone}(f)) \cong \mathrm{cone}(\mathrm{Ind}_\Omega f)$$
$$\xi \mapsto (\xi_1,\xi_2),$$

where $\xi_1 \in \mathrm{Ind}_\Omega A$ maps $\omega \in \Omega$ to $\pi_{A_{\sigma(\omega)}}(\xi(\omega))$ and $\xi_2 \in C_0((0,1],\mathrm{Ind}_\Omega B)$ maps $t \in (0,1]$ to the section $\omega \mapsto \pi_{B_{\sigma(\omega)}}(\xi(\omega))(t)\colon \Omega \to \sigma^*\mathcal{B}$. The induction functor therefore sends mapping cone triangles to mapping cone triangles up to equivariant $*$-isomorphism. $\square$

We note that restriction functors of open subgroupoids are special cases of (correspondence) induction functors. Given an open subgroupoid $H \subseteq G$, the space G^{H^0} forms a correspondence from H to G, with actions by left and right multiplication respectively.

6.2. **Homological ideals.** The price for doing homological algebra in a triangulated category is that it has to be relative to something called a homological ideal. These are defined using homological functors on the category.

Definition 6.10 (Homological functor)**.** Let $\mathfrak{T}$ be a triangulated category and let $\mathfrak{C}$ be an abelian category. A *homological functor* is an additive functor $F\colon \mathfrak{T} \to \mathfrak{C}$ such that for every exact triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$, the sequence $F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$ is exact at $F(B)$.

$$\begin{tikzcd} A \arrow[rr, "f"] & & B \arrow[dl, "g"] \\ & C \arrow[ul, "h", "\circ" marking] & \end{tikzcd} \qquad F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

We typically will only use the abelian category Ab of abelian groups and the abelian category Ab^I of collections of abelian groups indexed by a set I.

Remark 6.11. Using TR2 to rotate the exact triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$, we can see that if F is homological then $F(B) \xrightarrow{F(g)} F(C) \xrightarrow{F(h)} F(\Sigma A)$ is exact at $F(C)$, and by rotating repeatedly we get a long exact sequence

$$\cdots \to F(\Sigma^{-1}C) \xrightarrow{F(\Sigma^{-1}h)} F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \xrightarrow{F(h)} F(\Sigma A) \to \cdots$$

Example 6.12. For any object $A \in \mathfrak{T}$, the functor $\mathrm{Hom}(A, -)\colon \mathfrak{T} \to \mathsf{Ab}$ is a homological functor. In particular, when $\mathfrak{T} = \mathrm{KK}$ is the Kasparov category and $A = \mathbb{C}$ is the complex numbers, $\mathrm{Hom}(A, -)$ is naturally isomorphic to the K-theory functor $K_0\colon \mathrm{KK} \to \mathsf{Ab}$, which is therefore a homological functor.

Remark 6.13. The composition of a homological functor with a triangulated functor is again homological.

Example 6.14. Let $\mathfrak{T} = \mathrm{KK}^G$. Then the functor $F_G := K_0(G \ltimes -)\colon \mathrm{KK}^G \to \mathsf{Ab}$ is homological.

Definition 6.15 (Stable homological functor)**.** Let $\mathfrak{T}$ be a triangulated category with suspension $\Sigma_{\mathfrak{T}}$ and $\mathfrak{C}$ a stable abelian category with suspension $\Sigma_{\mathfrak{C}}$. A *stable homological functor* is an additive functor $F\colon \mathfrak{T} \to \mathfrak{C}$ with a natural isomorphism $F \circ \Sigma_{\mathfrak{T}} \Rightarrow \Sigma_{\mathfrak{C}} \circ F$.

Definition 6.16 (Stabilisation)**.** Given any homological functor $F\colon \mathfrak{T} \to \mathfrak{C}$, there is a stable homological functor $F_*\colon \mathfrak{T} \to \mathfrak{C}_*$ given by $F_*(A) = (F_n(A))_{n\in\mathbb{Z}}$ where $F_n :=$

$F \circ \Sigma^{-n}$. The target of this is the stable abelian category $\mathfrak{C}_*$ of $\mathbb{Z}$-indexed sequences in $\mathfrak{C}$, with suspension given by the (right) shift map $(A_n)_{n\in\mathbb{Z}} \mapsto (A_{n-1})_{n\in\mathbb{Z}}$. We call F_* the *stabilisation* of F.

Also note that the precomposition of a (stable) homological functor with a triangulated functor is again a (stable) homological functor.

Example 6.17. The stabilisation of $K_0\colon \mathrm{KK} \to \mathsf{Ab}$ is the functor $K_*\colon \mathrm{KK} \to \mathsf{Ab}_*$ which is given by $A \mapsto (K_n(A))_{n\in\mathbb{Z}}$. Similarly, the stabilisation of $F_G\colon \mathrm{KK}^G \to \mathsf{Ab}$ is given by $A \mapsto (K_n(G \ltimes A))_{n\in\mathbb{Z}}$.

If we have a stable homological functor $F\colon \mathfrak{T} \to \mathfrak{C}$, we can naïvely define homological properties in $\mathfrak{T}$ relative to F by mapping through F to $\mathfrak{C}$ and asking for the relevant homological property in the stable abelian category $\mathfrak{C}$. Often, this naïve definition will depend only on the kernel $\ker F$ of morphisms which vanish under F, although this notably fails for projectivity. The approach we take ensures that relative homological properties depend only on $\ker F$, which we call a homological ideal.

Definition 6.18 (Homological ideal)**.** A *homological ideal* $\mathfrak{I} \lhd \mathfrak{T}$ in a triangulated category $\mathfrak{T}$ is the kernel $\mathfrak{I} = \ker F$ of a stable homological functor $F\colon \mathfrak{T} \to \mathfrak{C}$.

Remark 6.19 (Kernels of triangulated functors are homological ideals). It is noted in [58, Remark 2.21] that the kernel of a triangulated functor between triangulated categories is exact, due to Freyd's theorem which constructs a universal homological functor associated to a triangulated category.

Example 6.20 (Homological ideal from a subgroupoid)**.** Let G be an étale groupoid and let $H \leq G$ be a subgroupoid. Then the kernel $\mathfrak{I}_H := \ker \mathrm{Res}_G^H$ of the restriction functor $\mathrm{Res}_G^H\colon \mathrm{KK}^G \to \mathrm{KK}^H$ is a homological ideal.

Remark 6.21 (Intersections of homological ideals are homological). The intersection of a family of homological ideals is homological, because we may consider the collection of stable homological functors as a single functor to the product of the stable abelian categories. This means that given a family $\mathcal{F}$ of subgroupoids of an étale groupoid G, the ideal $\mathfrak{I}_{\mathcal{F}} := \bigcap_{F\in\mathcal{F}} \mathfrak{I}_F$ is homological.

Definition 6.22 (Homological notions relative to a homological ideal)**.** Let $\mathfrak{I}$ be a homological ideal in a triangulated category $\mathfrak{T}$, the kernel of a stable homological functor $F\colon \mathfrak{T} \to \mathfrak{C}$. Let $f\colon A \to B$ be a morphism and embed it in an exact triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$.

- We say the morphism f is

- $\mathfrak{I}$-*phantom* if $f \in \mathfrak{I}$, or equivalently $F(f) = 0$,
- $\mathfrak{I}$-*monic* if $h \in \mathfrak{I}$, or equivalently $F(f)$ is monic,
- $\mathfrak{I}$-*epic* if $g \in \mathfrak{I}$, or equivalently $F(f)$ is epic,
- an $\mathfrak{I}$-*equivalence* if it is both $\mathfrak{I}$-monic and $\mathfrak{I}$-epic, or equivalently $F(f)$ is an isomorphism.

There are equivalent versions of these definitions based on rotating the position of f in the exact triangle.

- We say the object A is
  - $\mathfrak{I}$-*contractible* if $1_A \in \mathfrak{I}$, or equivalently $F(A) = 0$,
  - $\mathfrak{I}$-*projective* if $\mathfrak{I}(A, D) = 0$ for every $D \in \mathfrak{T}$.
- We say the exact triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ is $\mathfrak{I}$-*exact* if $h \in \mathfrak{I}$. Equivalently, f is $\mathfrak{I}$-monic, g is $\mathfrak{I}$-epic, or

$$0 \longrightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \longrightarrow 0$$

  is a short exact sequence.
- We say a chain complex $C_\bullet = (C_n, d_n)_{n\in\mathbb{Z}}$ (i.e. a diagram $= \cdots \to C_n \xrightarrow{d_n} C_{n-1} \to \cdots$ with $d_n d_{n+1} = 0$ for each n) is
  - $\mathfrak{I}$-*exact in degree* n or $\mathfrak{I}$-*exact at* C_n if when we embed d_n and d_{n+1} in exact triangles (which are unique up to isomorphism of triangles)

$$C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{f_n} X_n \xrightarrow{g_n} \Sigma C_n \qquad C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{f_{n+1}} X_{n+1} \xrightarrow{g_{n+1}} \Sigma C_n,$$

    the map $X_n \xrightarrow{g_n} \Sigma C_n \xrightarrow{\Sigma f_{n+1}} \Sigma X_{n+1}$ belongs to $\mathfrak{I}(X_n, \Sigma X_{n+1})$. Equivalently,

$$F(C_{n+1}) \xrightarrow{F(d_{n+1})} F(C_n) \xrightarrow{F(d_n)} F(C_{n-1})$$

    is exact at $F(C_n)$.
  - an $\mathfrak{I}$-*exact sequence* if it is $\mathfrak{I}$-exact in degree n for each n. Equivalently, $F(C_\bullet) = (F(C_n), F(d_n))_{n\in\mathbb{Z}}$ is an exact sequence.

Monomorphisms (monic morphisms) and epimorphisms (epic morphisms) are categorical versions of injections and surjections. They are left-cancellative and right-cancellative morphisms respectively, and coincide with injections/surjections in many categories whose morphisms are functions.

Remark 6.23. Given an $\mathfrak{I}$-projective object P, a morphism $f\colon P \to A$ and an $\mathfrak{I}$-epimorphism $\pi\colon B \to A$, there is a lift of f through π to a morphism $g\colon P \to B$.

$$\begin{tikzcd} & B \arrow[d, "\pi"] \\ P \arrow[ur, dashed, "g"] \arrow[r, "f"'] & A \end{tikzcd}$$

This lifting property is closer to the standard definition of being projective, and is in fact equivalent to being $\mathfrak{I}$-projective. Note also that P is $\mathfrak{I}$-projective if and only if ΣP is.

Remark 6.24. Another equivalent definition of $\mathfrak{I}$-projectivity is that $\mathrm{Hom}(P, -)$ maps $\mathfrak{I}$-exact sequences to exact sequences of abelian groups - we call such functors $\mathfrak{I}$-exact functors. This means that given an $\mathfrak{I}$-exact sequence $\cdots \to A \xrightarrow{f} B \xrightarrow{g} C \to \cdots$ and a morphism $p\colon P \to B$ such that $pg = 0$, the morphism p lifts through f to a morphism $q\colon P \to A$.

$$\begin{tikzcd} & & P \arrow[dl, dashed, "q"'] \arrow[d, "p"] \arrow[dr, "0"] & & \\ \cdots \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & \cdots \end{tikzcd}$$

Example 6.25 (Projective objects from an adjoint)**.** A standard source of $\mathfrak{I}$-projective objects is from adjointness. A left adjoint of a triangulated functor $F\colon \mathfrak{T} \to \mathfrak{T}'$ is a triangulated functor $E\colon \mathfrak{T}' \to \mathfrak{T}$ with natural isomorphisms

$$\mathfrak{T}(EA, B) \cong \mathfrak{T}'(A, FB), \qquad A \in \mathfrak{T}', B \in \mathfrak{T}.$$

In this setting, each object EA is $\ker F$-projective.

We are particularly interested in ideals constructed from restriction functors for subgroupoids, and will make heavy use of the induction-restriction adjunction [11, Theorem 2.3]:

Theorem 6.26 (Induction-restriction adjunction)**.** *Let $H \subseteq G$ be a proper open subgroupoid of a second countable étale groupoid G. Then there is a left adjoint*

$$\mathrm{Ind}_H^G\colon \mathrm{KK}^H \to \mathrm{KK}^G$$

to the restriction functor $\mathrm{Res}_G^H\colon \mathrm{KK}^G \to \mathrm{KK}^H$.

6.3. **Projective resolutions relative to a homological ideal.** In order to construct derived functors and the ABC spectral sequence we need projective resolutions.

Definition 6.27 (Projective resolutions)**.** Given a homological ideal $\mathfrak{I}$ in a triangulated category, an $\mathfrak{I}$-*resolution* $P_\bullet \to A$ of an object A is a chain complex

$\cdots \xrightarrow{\delta_3} P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0$ with a map $\pi_0 \colon P_0 \to A$ such that the combined sequence is $\mathfrak{I}$-exact:

$$\cdots \xrightarrow{\ \delta_3\ } P_2 \xrightarrow{\ \delta_2\ } P_1 \xrightarrow{\ \delta_1\ } P_0 \xrightarrow{\ \pi_0\ } A \longrightarrow 0.$$

If each P_n is $\mathfrak{I}$-projective, we call this an *$\mathfrak{I}$-projective resolution.* $\mathfrak{I}$-exactness at A is equivalent to π_0 being $\mathfrak{I}$-epic, and we call a map $\pi \colon P \to A$ a *1-step $\mathfrak{I}$-projective resolution* of A if P is projective and π is $\mathfrak{I}$-epic. We say that $\mathfrak{I}$ has *enough projectives* or *$\mathfrak{T}$ has enough $\mathfrak{I}$-projectives* if every object has a 1-step $\mathfrak{I}$-projective resolution.

It is extremely useful for a homological ideal to have enough projectives. This will allow us to define derived functors, but it also ensures that products of homological ideals are again homological.

Proposition 6.28 (Proposition 2.2 in [57], Proposition 3.3 in [18])**.** *Let $\mathfrak{I}$ and $\mathfrak{J}$ be homological ideals with enough projectives in a triangulated category $\mathfrak{T}$. Then the product $\mathfrak{I} \circ \mathfrak{J} = \{fg \mid f \in \mathfrak{I},\ g \in \mathfrak{J}\}$ is a homological ideal with enough projectives.*

We refer to the n-fold product $\mathfrak{I} \circ \mathfrak{I} \circ \cdots \circ \mathfrak{I}$ as $\mathfrak{I}^n$.

The existence of $\mathfrak{I}$-projective resolutions for each object could alternatively be taken as the definition of having enough $\mathfrak{I}$-projectives, but the construction of $\mathfrak{I}$-projective resolutions from 1-step $\mathfrak{I}$-projective resolutions is illuminating, so we present it as a proposition rather than as a definition.

Proposition 6.29 (See Proposition 3.26 in [59])**.** *If $\mathfrak{I}$ has enough projectives, then every object has a $\mathfrak{I}$-projective resolution.*

Proof. We can build a projective resolution of A by inductively building the following diagram.

$$\begin{tikzcd}
A \arrow[r, equal] & N_0 \arrow[rr, "\iota_0", "\circ"{marking}] & & N_1 \arrow[dl, "\epsilon_0"] \arrow[rr, "\iota_1", "\circ"{marking}] & & N_2 \arrow[dl, "\epsilon_1"] \arrow[rr, "\circ"{marking}] & & \cdots \arrow[dl] & \\
& & P_0 \arrow[ul, "\pi_0"] & & P_1 \arrow[ll, "\delta_1"] \arrow[ul, "\pi_1"] & & P_2 \arrow[ll, "\delta_2"] \arrow[ul, "\pi_2"] & & \cdots \arrow[ll] \arrow[ul]
\end{tikzcd}$$

We start by finding a 1-step $\mathfrak{I}$-projective resolution $\pi_0 \colon P_0 \to N_0$ of $N_0 = A$. We can then embed π_0 in an exact triangle $N_1 \xrightarrow{\epsilon_0} P_0 \xrightarrow{\pi_0} N_0 \xrightarrow{\iota_0} \Sigma N_1$, which is $\mathfrak{I}$-exact because π_0 is $\mathfrak{I}$-epic. We can then find a 1-step $\mathfrak{I}$-projective resolution $\pi_1 \colon P_1 \to N_1$ of N_1 and continue this process inductively, defining $\delta_{n+1} = \epsilon_n \pi_{n+1}$, so that the top triangles are exact and the bottom triangles commute. Finally, the sequence

$$\cdots \xrightarrow{\ \delta_3\ } P_2 \xrightarrow{\ \delta_2\ } P_1 \xrightarrow{\ \delta_1\ } P_0 \xrightarrow{\ \pi_0\ } A \longrightarrow 0$$

is $\mathfrak{I}$-exact, because if we apply a stable homological functor F with kernel $\mathfrak{I}$, the chain complex $F(P_\bullet)$ decomposes into the short exact sequences

$$0 \longrightarrow F(N_{n+1}) \xrightarrow{F(\epsilon_n)} F(P_n) \xrightarrow{F(\pi_n)} F(N_n) \longrightarrow 0.$$

$\square$

Note that we have constructed more than just an $\mathfrak{I}$-projective resolution here. The extra structure is related to what we call an $\mathfrak{I}$-phantom tower over A.

Definition 6.30 (Phantom tower)**.** A *pre-$\mathfrak{I}$-phantom tower* over A is a diagram

$$\begin{tikzcd}
A \arrow[r, equal] & N_0 \arrow[rr, "\iota_0^1"] & & N_1 \arrow[rr, "\iota_1^2"] \arrow[dl, "\circ" marking, "\epsilon_0"] & & N_2 \arrow[rr] \arrow[dl, "\circ" marking, "\epsilon_1"] & & \cdots \arrow[dl, "\circ" marking] \\
& & P_0 \arrow[ul, "\pi_0"] & & P_1 \arrow[ul, "\pi_1"] \arrow[ll, "\circ" marking, "\delta_1"] & & P_2 \arrow[ul, "\pi_2"] \arrow[ll, "\circ" marking, "\delta_2"] & & \cdots \arrow[ul] \arrow[ll, "\circ" marking]
\end{tikzcd}$$

such that:

- The top triangles are exact, i.e. $P_n \xrightarrow{\pi_n} N_n \xrightarrow{\iota_n^{n+1}} N_{n+1} \xrightarrow{\epsilon_n} \Sigma P_n$ is an exact triangle for each n.
- The bottom triangles commute, i.e. $\delta_n = \epsilon_{n-1} \circ \pi_n$ for each n.
- The top maps are $\mathfrak{I}$-phantom, i.e. $\iota_n^{n+1} \in \mathfrak{I}$ for each n.
- The sequence $\cdots \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\pi_0} N_0 \to 0$ is $\mathfrak{I}$-exact.

It is worth noting that the last two conditions are equivalent in the presence of the other conditions. We say that the tower is *$\mathfrak{I}$-phantom* if P_n is $\mathfrak{I}$-projective for each n. We may refer to the (pre)-$\mathfrak{I}$-phantom tower as $\mathcal{P} = (P_\bullet, N_\bullet, \delta, \pi, \iota, \epsilon)$ or just $(P_\bullet, N_\bullet)$.

Remark 6.31. This differs from the construction in Proposition 6.29 only in the degrees of the maps involved. In particular, if $\mathfrak{T}$ has enough $\mathfrak{I}$-projectives, we can construct a phantom tower over any object by a similar construction. We may also identify the sequence $\cdots \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\pi_0} N_0 \to 0$ with the equivalent sequence $\cdots \xrightarrow{\Sigma^{-2}\delta_2} \Sigma^{-1}P_1 \xrightarrow{\Sigma^{-1}\delta_1} P_0 \xrightarrow{\pi_0} N_0 \to 0$, which is a genuine $\mathfrak{I}$-projective resolution. This is what we mean when we say the former is $\mathfrak{I}$-exact. The reason for this degree change is that it will be convenient later on for the top arrows in an $\mathfrak{I}$-phantom tower to be degree 0 morphisms.

Definition 6.32 (Tower map)**.** A *tower map* $\mathcal{P} \to \mathcal{Q}$ of pre-$\mathfrak{I}$-phantom towers $\mathcal{P} = (P_\bullet, M_\bullet)$ and $\mathcal{Q} = (Q_\bullet, N_\bullet)$ is a collection of morphisms shown in red such that the following diagram commutes. We call this a tower map *over*, or *a lift of* the morphism $A \to B$.

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[dd, red] & M_0 \arrow[rr] \arrow[dd, red] & & M_1 \arrow[rr] \arrow[dl, "\circ" marking] \arrow[dd, red] & & M_2 \arrow[rr] \arrow[dl, "\circ" marking] \arrow[dd, red] & & \cdots \arrow[dl, "\circ" marking] \\
& & P_0 \arrow[ul] \arrow[dd, red] & & P_1 \arrow[ll, "\circ" marking] \arrow[ul] \arrow[dd, red] & & P_2 \arrow[ll, "\circ" marking] \arrow[ul] \arrow[dd, red] & & \cdots \arrow[ll, "\circ" marking] \arrow[ul] \\
B \arrow[r, equal] & N_0 \arrow[rr] & & N_1 \arrow[rr] \arrow[dl, "\circ" marking] & & N_2 \arrow[rr] \arrow[dl, "\circ" marking] & & \cdots \arrow[dl, "\circ" marking] \\
& & Q_0 \arrow[ul] & & Q_1 \arrow[ll, "\circ" marking] \arrow[ul] & & Q_2 \arrow[ll, "\circ" marking] \arrow[ul] & & \cdots \arrow[ll, "\circ" marking] \arrow[ul]
\end{tikzcd}
$$

The fundamental lemma of homological algebra also has a version for homological algebra in a triangulated category relative to a homological ideal:

Lemma 6.33 (The fundamental lemma of homological algebra, see Proposition 3.26 in [59])**.** *Let $\mathfrak{T}$ be a triangulated category and $\mathfrak{I}$ be a homological ideal. If A and B are objects of $\mathfrak{T}$, $P_\bullet \to A$ is an $\mathfrak{I}$-projective resolution of A and $Q_\bullet \to B$ is an $\mathfrak{I}$-resolution of B, then any morphism $f\colon A \to B$ lifts to a chain map $\tilde{f}\colon P_\bullet \to Q_\bullet$ that is unique up to chain homotopy. In particular, any two $\mathfrak{I}$-projective resolutions of an object are chain homotopy equivalent.*

Proof. We wish to fill in the dashed arrows to make the following diagram commutative.

$$
\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "d_2^P"] \arrow[d, dashed, "\tilde{f}_2"] & P_1 \arrow[r, "d_1^P"] \arrow[d, dashed, "\tilde{f}_1"] & P_0 \arrow[r, "d_0^P"] \arrow[d, dashed, "\tilde{f}_0"] & A \arrow[r] \arrow[d, "f"] & 0 \arrow[d] \\
\cdots \arrow[r] & Q_2 \arrow[r, "d_2^Q"] & Q_1 \arrow[r, "d_1^Q"] & Q_0 \arrow[r, "d_0^Q"] & B \arrow[r] & 0
\end{tikzcd}
$$

First we may lift $f \circ d_0^P$ through d_0^Q to $\tilde{f}_0$ because P_0 is $\mathfrak{I}$-projective and d_0^Q is $\mathfrak{I}$-epic. Next we may lift $\tilde{f}_0 \circ d_1^P$ through d_1^Q to obtain $\tilde{f}_1$ because $\cdots \to Q_1 \xrightarrow{d_1^Q} Q_0 \xrightarrow{d_0^Q} B \to 0$ is $\mathfrak{I}$-exact and $d_0^Q \circ \tilde{f}_0 \circ d_1^P = 0$, by Remark 6.24. We can continue this process by induction. Next, given two lifts of f to chain maps $g, h\colon P_\bullet \to Q_\bullet$, we wish to construct a chain homotopy $\psi\colon g \simeq h$.

$$
\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "d_2^P"] \arrow[d, shift right, "g_2"'] \arrow[d, shift left, "h_2"] \arrow[dl, dashed, "\psi_2"'] & P_1 \arrow[r, "d_1^P"] \arrow[d, shift right, "g_1"'] \arrow[d, shift left, "h_1"] \arrow[dl, dashed, "\psi_1"'] & P_0 \arrow[r, "d_0^P"] \arrow[d, shift right, "g_0"'] \arrow[d, shift left, "h_0"] \arrow[dl, dashed, "\psi_0"'] & A \arrow[r] \arrow[d, "f"] & 0 \arrow[d] \\
\cdots \arrow[r] & Q_2 \arrow[r, "d_2^Q"'] & Q_1 \arrow[r, "d_1^Q"'] & Q_0 \arrow[r, "d_0^Q"'] & B \arrow[r] & 0
\end{tikzcd}
$$

First we may lift $g_0 - h_0$ through d_1^Q to obtain ψ_0 because $d_0^Q \circ (g_0 - h_0) = 0$. We need ψ_1 to satisfy

$$g_1 - h_1 = d_2^Q \circ \psi_1 + \psi_0 \circ d_1^P$$

so we lift $g_1 - h_1 - \psi_0 \circ d_1^P$ through d_2^Q, which is possible because $d_1^Q \circ (g_1 - h_1 - \psi_0 \circ d_1^P) = 0$. Again, we can continue this process by induction, constructing the required chain homotopy $\psi\colon g \simeq h$.

Any two $\mathfrak{I}$-projective resolutions $P_\bullet, Q_\bullet \to A$ of an object are homotopy equivalent, as we can lift the identity $1_A\colon A \to A$ to chain maps $g\colon P_\bullet \to Q_\bullet$, $h\colon Q_\bullet \to P_\bullet$, and we get $hg \simeq 1_{P_\bullet}$ and $gh \simeq 1_{Q_\bullet}$ by uniqueness of lifts up to chain homotopy. □

This allows us to make sense of derived functors in the setting of a homological ideal in a triangulated category. Let $\mathfrak{I}$ be a homological ideal in a triangulated category $\mathfrak{T}$ with enough projective objects. Let $F\colon \mathfrak{T} \to \mathfrak{C}$ be an additive functor with values in an abelian category $\mathfrak{C}$ and let A be an object of $\mathfrak{T}$. Let $P_\bullet \to A$ and $Q_\bullet \to A$ be $\mathfrak{I}$-projective resolutions. Then there is a lift $\tilde{\mathrm{id}}\colon P_\bullet \to Q_\bullet$ of $\mathrm{id}\colon A \to A$ which is unique up to chain homotopy, and induces a canonical isomorphism $H_n(F(\tilde{\mathrm{id}}))\colon H_n(F(P_\bullet)) \cong H_n(F(Q_\bullet))$.

Definition 6.34 (Left derived functors)**.** In this setting, we define the *left derived functor* $\mathbb{L}_nF(A)$ to be $H_n(F(P_\bullet))$ for any $\mathfrak{I}$-projective resolution $P_\bullet \to A$, noting that it is only defined up to canonical isomorphism. Given a morphism $f\colon A \to A'$ in $\mathfrak{T}$ and $\mathfrak{I}$-projective resolutions $P_\bullet \to A$ and $P'_\bullet \to A'$, there is a lift $\tilde{f}\colon P_\bullet \to P'_\bullet$ of f. This induces a map $H_n(F(\tilde{f}))\colon H_n(F(P_\bullet)) \to H_n(F(P'_\bullet))$ in homology. To check this is well-defined, suppose we have $\mathfrak{I}$-projective resolutions $Q_\bullet \to A$ and $Q'_\bullet \to A'$, a lift $\hat{f}\colon Q_\bullet \to Q'_\bullet$ and lifts $\tilde{\mathrm{id}}\colon P_\bullet \to Q_\bullet$ and $\tilde{\mathrm{id}}'\colon P'_\bullet \to Q'_\bullet$. By Lemma 6.33, $\tilde{\mathrm{id}}' \circ \tilde{f}$ is chain homotopic to $\hat{f} \circ \tilde{\mathrm{id}}$, and therefore the following diagram commutes.

$$\begin{tikzcd}
H_n(F(P_\bullet)) \arrow[r, "H_n(F(\tilde{f}))"] \arrow[d, "\cong"', "H_n(F(\tilde{\mathrm{id}}))"] & H_n(F(P'_\bullet)) \arrow[d, "\cong"', "H_n(F(\tilde{\mathrm{id}}'))"] \\
H_n(F(Q_\bullet)) \arrow[r, "H_n(F(\hat{f}))"] & H_n(F(Q'_\bullet))
\end{tikzcd}$$

This shows that $H_n(F(\tilde{f}))$ gives us a morphism from $\mathbb{L}_nF(A)$ to $\mathbb{L}_nF(B)$ that is independent of the choice of $\mathfrak{I}$-projective resolutions and lift of f, which we may call $\mathbb{L}_nF(f)$. This is functorial, and we call $\mathbb{L}_nF\colon \mathfrak{T} \to \mathfrak{C}$ the *nth left $\mathfrak{I}$-derived functor* of F. If we want to stress the homological ideal $\mathfrak{I}$ we may write $\mathbb{L}_n^{\mathfrak{I}}F$ instead of $\mathbb{L}_nF$. If $f\colon A \to A'$ is an $\mathfrak{I}$-equivalence, we may use the same $\mathfrak{I}$-projective resolution for A and A', and therefore $\mathbb{L}_n(f)$ is an isomorphism.

For the ABC spectral sequence, which is in a sense built on top of the left derived functors, we will need to work with phantom towers.

Lemma 6.35 (see Lemma 3.3 in [57])**.** *Any $\mathfrak{I}$-projective resolution $P_\bullet \to A$ can be embedded in an $\mathfrak{I}$-phantom tower, which is unique up to non-canonical isomorphism.*

Furthermore, if $\mathcal{P}$ is a pre-$\mathfrak{I}$-phantom tower over A, $\mathcal{Q}$ is a pre-$\mathfrak{I}$-phantom tower over B, and $P_\bullet$ and $Q_\bullet$ are the $\mathfrak{I}$-resolutions of A and B embedded in $\mathcal{P}$ and $\mathcal{Q}$

respectively, then any chain map from $P_\bullet$ to $Q_\bullet$ over a map $f\colon A \to B$ extends to a tower map $\mathcal{P} \to \mathcal{Q}$. In particular, if $\mathcal{P}$ is $\mathfrak{I}$-phantom, any map $f\colon A \to B$ lifts (non-canonically) to a tower map $\mathcal{P} \to \mathcal{Q}$.

Proof. We wish to find N_n and fill in the dashed arrows to construct the following $\mathfrak{I}$-phantom tower.

$$\begin{tikzcd}
A \arrow[r, equal] & N_0 \arrow[rr, dashed, "\iota_0^1"] \arrow[dr, dashed, phantom] & & N_1 \arrow[rr, dashed, "\iota_1^2"] \arrow[dl, dashed, "\epsilon_0"] & & N_2 \arrow[rr, dashed] \arrow[dl, dashed, "\epsilon_1"] & & \cdots \arrow[dl, dashed] \\
& & P_0 \arrow[ul, "\pi_0"] & & P_1 \arrow[ll, "\delta_1"] \arrow[ul, dashed, "\pi_1"] & & P_2 \arrow[ll, "\delta_2"] \arrow[ul, dashed, "\pi_2"] & & \cdots \arrow[ll] \arrow[ul, dashed]
\end{tikzcd}$$

First, we let $N_0 = A$ and embed $\pi_0\colon P_0 \to N_0$ in an exact triangle $P_0 \xrightarrow{\pi_0} N_0 \xrightarrow{\iota_0^1} N_1 \xrightarrow{\epsilon_0} \Sigma P_0$. Using exactness of this triangle and that $\Sigma\pi_0 \circ \delta_1 = 0$, we can lift δ_1 through ϵ_0 to obtain π_1.

$$\begin{tikzcd}
& P_1 \arrow[dl, dashed, "\pi_1"'] \arrow[d, "\delta_1"] \arrow[dr, "0"] & \\
N_1 \arrow[r, "\epsilon_0"'] & \Sigma P_0 \arrow[r, "\Sigma\pi_0"'] & \Sigma N_0
\end{tikzcd}$$

Similarly, we can use exactness of the triangle and the vanishing of $\Sigma\epsilon_0 \circ \Sigma\pi_1 \circ \delta_2 = \Sigma\delta_1 \circ \delta_2$ to lift $\Sigma\pi_1 \circ \delta_2$ through ι_0^1.

$$\begin{tikzcd}
& P_2 \arrow[dl, dashed, bend right] \arrow[d, "\Sigma\pi_1\circ\delta_2"] \arrow[dr, bend left, "0"] & \\
N_0 \arrow[r, "\iota_0^1"'] & N_1 \arrow[r, "\epsilon_0"'] & \Sigma P_0
\end{tikzcd}$$

The morphism ι_0^1 is $\mathfrak{I}$-phantom because π_0 is $\mathfrak{I}$-epic, so $\Sigma\pi_1 \circ \delta_2$ is $\mathfrak{I}$-phantom and therefore 0 by $\mathfrak{I}$-projectivity of P_2. To show that π_1 is $\mathfrak{I}$-epic, let $F\colon \mathfrak{T} \to \mathfrak{C}$ be a homological functor with $\ker F = \mathfrak{I}$.

$$\begin{tikzcd}
F(\Sigma N_0) & & F(N_1) \arrow[dl, "F(\epsilon_0)"] & \\
& F(\Sigma P_0) \arrow[ul, "F(\Sigma\pi_0)"] & & F(P_1) \arrow[ll, "F(\delta_1)"] \arrow[ul, "F(\pi_1)"']
\end{tikzcd}$$

By exactness at $F(\Sigma P_0)$, the image of $F(\delta_1)$ is the kernel of $F(\Sigma\pi_0)$, which is $F(N_1)$. This means that $F(\pi_1)$ is epic, so π_1 is $\mathfrak{I}$-epic. Finally, the sequence $P_2 \xrightarrow{\delta_2} \Sigma P_1 \xrightarrow{\Sigma\pi_1} \Sigma N_1$ is $\mathfrak{I}$-exact at ΣP_1 because ϵ_0 is $\mathfrak{I}$-monic, so $\ker F(\pi_1) = \ker F(\delta_1) = \operatorname{im} F(\delta_2)$.

We may now continue the construction of the $\mathfrak{I}$-phantom tower inductively, noting that the top triangles are exact and the bottom triangles commutative by construction, and ι_n^{n+1} is $\mathfrak{I}$-phantom because π_n is $\mathfrak{I}$-epic.

Now let $\mathcal{P} = (P_\bullet, M_\bullet)$ be a pre-$\mathfrak{I}$-phantom tower over A, and let $\mathcal{Q} = (Q_\bullet, N_\bullet)$ be a pre-$\mathfrak{I}$-phantom tower over B. Let $\tilde{f}\colon P_\bullet \to Q_\bullet$ be a chain map over $f\colon A \to B$.

We wish to construct a tower map from $\mathcal{P}$ to $\mathcal{Q}$ by filling in the dashed arrows g_n.

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[dd, "f"] & M_0 \arrow[rr] \arrow[dd, "f"] & & M_1 \arrow[rr] \arrow[dl, "\circ" description] \arrow[dd, dashed, "g_1"] & & M_2 \arrow[rr] \arrow[dl, "\circ" description] \arrow[dd, dashed, "g_2"] & & \cdots \arrow[dl, "\circ" description] \\
& & P_0 \arrow[ul] \arrow[dd, "\tilde{f}_0"] & & P_1 \arrow[ll, "\circ" description] \arrow[ul] \arrow[dd, "\tilde{f}_1"] & & P_2 \arrow[ll, "\circ" description] \arrow[ul] \arrow[dd, "\tilde{f}_2"] & & \cdots \arrow[ll, "\circ" description] \arrow[ul] \\
B \arrow[r, equal] & N_0 \arrow[rr] & & N_1 \arrow[rr] \arrow[dl, "\circ" description] & & N_2 \arrow[rr] \arrow[dl, "\circ" description] & & \cdots \arrow[dl, "\circ" description] \\
& & Q_0 \arrow[ul] & & Q_1 \arrow[ll, "\circ" description] \arrow[ul] & & Q_2 \arrow[ll, "\circ" description] \arrow[ul] & & \cdots \arrow[ll, "\circ" description] \arrow[ul]
\end{tikzcd}$$

This is possible by repeated use of TR3, first finding g_1, and then g_2, and so on.

If $\mathcal{P}$ and $\mathcal{Q}$ are pre-$\mathfrak{I}$-phantom towers over the same object A with embeddings of the same $\mathfrak{I}$-resolution $P_\bullet$, we may let $\tilde{f}\colon P_\bullet \to P_\bullet$ be the identity chain map. By the two-out-of-three property, each g_n is an isomorphism. This means that $\mathcal{P}$ and $\mathcal{Q}$ are isomorphic, so we obtain uniqueness of pre-$\mathfrak{I}$-phantom towers that a $\mathfrak{I}$-resolution embeds in.

Finally, if $\mathcal{P} = (P_\bullet, M_\bullet)$ is an $\mathfrak{I}$-phantom tower over A and $\mathcal{Q} = (Q_\bullet, N_\bullet)$ is a pre-$\mathfrak{I}$-phantom tower over B and $f\colon A \to B$ is a morphism, we may first lift f to a chain map $\tilde{f}\colon P_\bullet \to Q_\bullet$ by Lemma 6.33 and then lift $\tilde{f}$ to a map of towers $\mathcal{P} \to \mathcal{Q}$. $\square$

Definition 6.36 (Phantom filtration)**.** Let $\mathfrak{I}$ be a homological ideal in a triangulated category $\mathfrak{T}$, let $F\colon \mathfrak{T} \to \mathfrak{C}$ be a functor with $\mathfrak{C} = \mathsf{Ab}$ or $\mathfrak{C} = \mathsf{Ab}_*$ and let A be an object of $\mathfrak{T}$. For each $k \geq 0$, we define a subgroup $F\colon \mathfrak{I}^k(A)$ of $F(A)$ by

$$F\colon \mathfrak{I}^k(A) := \left\{ x \in F(A) \;\middle|\; F(f)(x) = 0 \text{ for all } f \in \mathfrak{I}^k(A,B),\ B \in \mathfrak{T} \right\}$$

This gives us an ascending filtration of $F(A)$ called the $\mathfrak{I}$-*phantom filtration* of F at A.

$$0 = F\colon \mathfrak{I}^0(A) \subseteq F\colon \mathfrak{I}^1(A) \subseteq F\colon \mathfrak{I}^2(A) \subseteq \cdots \subseteq F(A)$$

This is related to $\mathfrak{I}$-phantom towers over A in the following way.

Lemma 6.37 (see 3.4-3.7 in [57])**.** *Let $\mathfrak{T}$ be a triangulated category with a homological ideal $\mathfrak{I}$ and an object A, let $F\colon \mathfrak{T} \to \mathfrak{C}$ be a functor with $\mathfrak{C} = \mathsf{Ab}$ or $\mathfrak{C} = \mathsf{Ab}_*$, and let the following diagram be an $\mathfrak{I}$-phantom tower over A.*

$$\begin{tikzcd}
A \arrow[r, equal] & N_0 \arrow[rr, "\iota_0^1"] & & N_1 \arrow[rr, "\iota_1^2"] \arrow[dl, "\circ\ \epsilon_0" description] & & N_2 \arrow[rr] \arrow[dl, "\circ\ \epsilon_1" description] & & \cdots \arrow[dl, "\circ" description] \\
& & P_0 \arrow[ul, "\pi_0"] & & P_1 \arrow[ll, "\circ" description, "\delta_1"'] \arrow[ul, "\pi_1"] & & P_2 \arrow[ll, "\circ" description, "\delta_2"'] \arrow[ul, "\pi_2"] & & \cdots \arrow[ll, "\circ" description] \arrow[ul]
\end{tikzcd}$$

Then $\ker F(\iota_m^n) = F\colon \mathfrak{I}^{n-m}(N_m)$ *for every* $n \geq m \geq 0$*, where* $\iota_m^n := \iota_{n-1}^n \circ \iota_{n-2}^{n-1} \circ \cdots \circ \iota_m^{m+1}$.

Proof. Let $x \in F\colon \mathfrak{I}^{n-m}(N_m)$. Then because $\iota_m^n \in \mathfrak{I}^{n-m}(N_m, N_n)$, we know $F(\iota_m^n)(x) = 0$. Therefore $\ker F(\iota_m^n) \supseteq F\colon \mathfrak{I}^{n-m}(N_m)$.

Conversely, let $x \in \ker F(\iota_m^n)$. Let $f \in \mathfrak{I}^{n-m}(N_m, B)$. Then there are morphisms $f_s \in \mathfrak{I}(B_s, B_{s+1})$ for $m \le s < n$ with $B_m = N_m$ and $B_n = B$ that compose to form f. We will inductively build morphisms g_s from N_s to B_s such that the following diagram commutes, starting with the identity $N_m \to B_m$.

$$\begin{tikzcd} N_m \arrow[r, "\iota_m^{m+1}"] \arrow[d, "g_m"] & N_{m+1} \arrow[r, "\iota_{m+1}^{m+1}"] \arrow[d, "g_{m+1}"] & \cdots \arrow[r, "\iota_{n-1}^{n}"] & N_n \arrow[d, "g_n"] \\ B_m \arrow[r, "f_m"] & B_{m+1} \arrow[r, "f_{m+1}"] & \cdots \arrow[r, "f_{n-1}"] & B_n \end{tikzcd}$$

Supposing we have found $g_s\colon N_s \to B_s$, consider the following diagram.

$$\begin{tikzcd} P_s \arrow[r, "\pi_s"] & N_s \arrow[r, "\iota_s^{s+1}"] \arrow[d, "g_s"] & N_{s+1} \arrow[d, dashed, "g_{s+1}"] \\ & B_s \arrow[r, "f_s"] & B_{s+1} \end{tikzcd}$$

Since $f_s \in \mathfrak{I}$, we know the morphism $f_s g_s \pi_s$ is in $\mathfrak{I}(P_s, B_{s+1})$ and is therefore equal to 0 by the $\mathfrak{I}$-projectivity of P_s. By exactness of the triangle (P_s, N_s, N_{s+1}), $f_s g_s$ factorises as $g_{s+1}\iota_s^{s+1}$ for some morphism $g_{s+1}\colon N_{s+1} \to B_{s+1}$. We can then inductively build the whole commutative diagram. In particular, $f = g_n \iota_m^n$. As $x \in \ker F(\iota_m^n)$, we have $F(f)(x) = 0$. Therefore $x \in F\colon \mathfrak{I}^{n-m}(N_m)$, and so $\ker F(\iota_m^n) \subseteq F\colon \mathfrak{I}^{n-m}(N_m)$. □

6.4. **Localisation.** One of the key features of the theory of triangulated categories we will make use of is the concept of *localisation* with respect to a *complementary pair* of subcategories.

Definition 6.38 (Complementary pair of subcategories)**.** Let $\mathfrak{P}$ and $\mathfrak{N}$ be thick (i.e. closed under direct summands) full triangulated subcategories of a triangulated category $\mathfrak{T}$. We call the pair $(\mathfrak{P}, \mathfrak{N})$ *complementary* if

- $\mathfrak{T}(P, N) = 0$ for each $P \in \mathfrak{P}$, $N \in \mathfrak{N}$,
- for each $A \in \mathfrak{T}$ there is an exact triangle $P \to A \to N \to \Sigma P$ with $P \in \mathfrak{P}$, $N \in \mathfrak{N}$.

Proposition 6.39 (Basic results on complementary subcategories, see Proposition 2.9 in [58])**.** *Let $(\mathfrak{P}, \mathfrak{N})$ be a complementary pair of subcategories of a triangulated category $\mathfrak{T}$.*

- *We have $N_0 \in \mathfrak{N}$ if and only if $\mathfrak{T}(P, N_0) = 0$ for each $P \in \mathfrak{P}$, and $P_0 \in \mathfrak{P}$ if and only if $\mathfrak{T}(P_0, N) = 0$ for each $N \in \mathfrak{N}$. Thus $\mathfrak{P}$ and $\mathfrak{N}$ determine each other.*

- *For each A, the exact triangle $\Delta_A = P_A \to A \to N_A \to \Sigma P_A$ with $P_A \in \mathfrak{P}$ and $N_A \in \mathfrak{N}$ is uniquely determined up to isomorphism and furthermore the triangle depends functorially on A. In particular, we get functors $A \mapsto P_A : \mathfrak{T} \to \mathfrak{P}$, $A \mapsto N_A : \mathfrak{T} \to \mathfrak{N}$.*
- *The functors $A \mapsto P_A : \mathfrak{T} \to \mathfrak{P}$, $A \mapsto N_A : \mathfrak{T} \to \mathfrak{N}$ are triangulated.*
- *$P_P \cong P$ and $N_N \cong N$ for $P \in \mathfrak{P}$, $N \in \mathfrak{N}$.*
- *Any triangulated functor $\Phi : \mathfrak{T} \to \mathfrak{T}'$ vanishing on $\mathfrak{N}$ factors uniquely up to natural isomorphism through the functor $A \mapsto P_A : \mathfrak{T} \to \mathfrak{P}$.*

This final property of the functor $A \mapsto P_A$ is the defining feature of the *localisation* of $\mathfrak{T}$ at $\mathfrak{N}$.

Definition 6.40 (Localisation)**.** Let $(\mathfrak{P}, \mathfrak{N})$ be a complementary pair of subcategories of a triangulated category $\mathfrak{T}$. The functor $A \mapsto P_A$ is called the *localisation functor* $L : \mathfrak{T} \to \mathfrak{T}$. Furthermore, by the naturality of the exact triangle Δ_A, it comes with a natural transformation $\mu : L \Rightarrow \mathrm{id}_{\mathfrak{T}}$ which we call the *localisation natural transformation.* For any functor $F : \mathfrak{T} \to \mathfrak{C}$, the functor $\mathbb{L}F := F \circ L : \mathfrak{T} \to \mathfrak{C}$ is called the *localisation* of F at $\mathfrak{N}$, and the natural transformation $\mathbb{L}F \Rightarrow F$ is the *assembly map.*

Definition 6.41 (Localising subcategory)**.** Let $\mathfrak{T}$ be a triangulated category with countable direct sums. A subcategory $\mathfrak{P}$ is *localising* (more precisely, $\aleph_0$*-localising*) if it is a triangulated subcategory and closed under countable direct sums.

Definition 6.42. Let $\mathfrak{I}$ be a homological ideal in a triangulated category $\mathfrak{T}$ with countable direct sums. We write $\mathfrak{P}_{\mathfrak{I}}$ for the class of $\mathfrak{I}$-projective objects, and $\langle \mathfrak{P}_{\mathfrak{I}} \rangle$ for the localising subcategory generated by $\mathfrak{P}_{\mathfrak{I}}$. We write $\mathfrak{N}_{\mathfrak{I}}$ for the full subcategory of $\mathfrak{I}$-contractible objects.

For each $\mathfrak{I}$-projective object P and each $\mathfrak{I}$-contractible object N, we have $\mathfrak{T}(P, N) = 0$ because $1_N \in \mathfrak{I}$ and $\mathfrak{I}(P, N) = 0$. This extends to each P in the localising subcategory $\langle \mathfrak{P}_{\mathfrak{I}} \rangle$ generated by the $\mathfrak{I}$-projectives. We want $(\langle \mathfrak{P}_{\mathfrak{I}} \rangle, \mathfrak{N}_{\mathfrak{I}})$ to be a complementary pair, and for this we have the following theorem.

Theorem 6.43 (Theorem 3.21 in [57])**.** *Let $\mathfrak{T}$ be a triangulated category with countable direct sums, and let $\mathfrak{I}$ be a homological ideal of $\mathfrak{T}$ compatible with countable direct sums. Suppose $\mathfrak{T}$ has enough $\mathfrak{I}$-projectives. Then $(\langle \mathfrak{P}_{\mathfrak{I}} \rangle, \mathfrak{N}_{\mathfrak{I}})$ is a complementary pair of subcategories of $\mathfrak{T}$.*

In this setting we write $L_{\mathfrak{I}} : \mathfrak{T} \to \langle \mathfrak{P}_{\mathfrak{I}} \rangle$ for the localisation functor, $N_{\mathfrak{I}} : \mathfrak{T} \to \mathfrak{N}_{\mathfrak{I}}$ for the functor $A \mapsto N_A$ and $\mu_{\mathfrak{I}} : L_{\mathfrak{I}} \Rightarrow 1_{\mathfrak{T}} : \mathfrak{T} \to \mathfrak{T}$ for the localisation natural transformation. We write $\mathbb{L}_{\mathfrak{I}} F$ for the localisation $F \circ L_{\mathfrak{I}}$ of a functor F.

In concrete situations we can aim to ensure that $\mathfrak{I}$ has enough projectives with the help of an adjunction. Everything becomes nicer with the right adjunction: we can construct $\mathfrak{I}$-projective resolutions concretely and better understand the complementary pair $(\langle\mathfrak{P}_{\mathfrak{I}}\rangle, \mathfrak{N}_{\mathfrak{I}})$.

Theorem 6.44 (See Section 3.6 in [59] and Section 2.1 in [71])**.** *Let $\mathfrak{T}$ and $\mathfrak{T}'$ be triangulated categories with countable direct sums and let $F\colon \mathfrak{T} \to \mathfrak{T}'$ be a triangulated functor with a left adjoint $E\colon \mathfrak{T}' \to \mathfrak{T}$ both compatible with countable direct sums. Then $\mathfrak{I} = \ker F$ has enough projectives, and the $\mathfrak{I}$-projective objects are precisely the direct summands of EA for $A \in \mathfrak{T}'$, so that $\langle E\mathfrak{T}'\rangle = \langle\mathfrak{P}_{\mathfrak{I}}\rangle$ is (left) complementary to $\mathfrak{N}_{\mathfrak{I}}$. Furthermore, each object $B \in \mathfrak{T}$ has an $\mathfrak{I}$-projective resolution*

$$\cdots \xrightarrow{\delta_{n+1}} (EF)^{n+1}B \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_2} (EF)^2B \xrightarrow{\delta_1} EFB \xrightarrow{\pi_0} B,$$

where $\pi_0 := \epsilon_B\colon EFB \to B$ is given by the counit ϵ of the adjunction at B and

$$\delta_n := \sum_{i=0}^{n}(-EF)^i(\epsilon_{(EF)^{n-i}B})\colon (EF)^{n+1}B \to (EF)^nB.$$

In many examples of complementary pairs $(\mathfrak{P}, \mathfrak{N})$, we think of the objects in $\mathfrak{N}$ as contractible, they vanish in some sense with respect to a family that we care about.

Definition 6.45. Let $\mathfrak{T}$ be a triangulated category and let $\mathfrak{N}$ be a localising subcategory of $\mathfrak{T}$. Let N be an object and $f\colon A \to B$ a morphism in $\mathfrak{T}$. We say that N is $\mathfrak{N}$-*contractible* if $N \in \mathfrak{N}$ and we say that f is an $\mathfrak{N}$-*equivalence* if its cone is in $\mathfrak{N}$.

When we have a homological ideal $\mathfrak{I}$ in $\mathfrak{T}$, the notions of $\mathfrak{N}_{\mathfrak{I}}$-equivalence and $\mathfrak{N}_{\mathfrak{I}}$-contractibility coincide with $\mathfrak{I}$-equivalence and $\mathfrak{I}$-contractibility. We collect some more useful facts about complementary pairs.

Remark 6.46*.* Given a complementary pair of subcategories $(\mathfrak{P}, \mathfrak{N})$ of a triangulated category $\mathfrak{T}$, the localisation functor $L\colon \mathfrak{T} \to \mathfrak{T}$ vanishes on $\mathfrak{N}$, and so maps $\mathfrak{N}$-equivalences to isomorphisms. The localisation natural transformation μ applied to any object A gives us an $\mathfrak{N}$-equivalence $\mu_A\colon LA \to A$.

Proposition 6.47. *Let $(\mathfrak{P}, \mathfrak{N})$ be a complementary pair of subcategories of a triangulated category $\mathfrak{T}$, and let $L\colon \mathfrak{T} \to \mathfrak{T}$ and $\mu\colon L \Rightarrow \mathrm{id}_{\mathfrak{T}}$ be the localisation functor and natural transformation, and let A be an object of $\mathfrak{T}$.*

Then we have an equality of morphisms $\mu_{LA} = L(\mu_A)\colon LLA \to LA$.

Proof. Apply the naturality of μ to $L(\mu_A)$ and μ_{LA} and the naturality of $L\mu$ to μ_A to obtain the following commutative diagrams.

$$\begin{array}{ccc} LLLA & \xrightarrow[\mu_{LLA}]{} & LLA \\ \downarrow{\scriptstyle LL(\mu_A)} & & \downarrow{\scriptstyle L(\mu_A)} \\ LLA & \xrightarrow[\mu_{LA}]{} & LA \end{array} \qquad \begin{array}{ccc} LLLA & \xrightarrow[\mu_{LLA}]{} & LLA \\ \downarrow{\scriptstyle L(\mu_{LA})} & & \downarrow{\scriptstyle \mu_{LA}} \\ LLA & \xrightarrow[\mu_{LA}]{} & LA \end{array} \qquad \begin{array}{ccc} LLLA & \xrightarrow[L(\mu_{LA})]{} & LLA \\ \downarrow{\scriptstyle LL(\mu_A)} & & \downarrow{\scriptstyle L(\mu_A)} \\ LLA & \xrightarrow[L(\mu_A)]{} & LA \end{array}$$

As each morphism is invertible, we can deduce from right to left that $LL(\mu_A) = L(\mu_{LA}) = \mu_{LLA}$ and therefore $\mu_{LA} = L(\mu_A)$. $\square$

Proposition 6.48 (see Proposition 3.28 in [57])**.** *Let $\mathfrak{T}$ be a triangulated category with countable direct sums, let $\mathfrak{I}$ be a homological ideal of $\mathfrak{T}$ with enough projectives and compatible with countable direct sums. Then, for each $k \geq 0$ and each morphism $f \in \mathfrak{T}(A, B)$, $L_{\mathfrak{I}}(f) \in \mathfrak{I}^k$ if and only if $f \in \mathfrak{I}^k$.*

Proof. Let $F\colon \mathfrak{T} \to \mathfrak{C}$ be a stable homological functor such that $\ker F = \mathfrak{I}^k$. From the morphism of exact triangles $\Delta_A \to \Delta_B$ induced by $f\colon A \to B$, we obtain the following chain map of long exact sequences in $\mathfrak{C}$.

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & F\Sigma^{-1}N_{\mathfrak{I}}(A) & \longrightarrow & FL_{\mathfrak{I}}(A) & \xrightarrow{F((\mu_{\mathfrak{I}})_A)} & F(A) & \longrightarrow & FN_{\mathfrak{I}}(A) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow{\scriptstyle F(L_{\mathfrak{I}}f)} & & \downarrow{\scriptstyle F(f)} & & \downarrow & & \\ \cdots & \longrightarrow & F\Sigma^{-1}N_{\mathfrak{I}}(B) & \longrightarrow & FL_{\mathfrak{I}}(B) & \xrightarrow{F((\mu_{\mathfrak{I}})_B)} & F(B) & \longrightarrow & FN_{\mathfrak{I}}(B) & \longrightarrow & \cdots \end{array}$$

Every $\mathfrak{I}$-contractible object $N \in \mathfrak{N}_{\mathfrak{I}}$ has $1_N \in \mathfrak{I}$ and so $1_N \in \mathfrak{I}^k$. Therefore F vanishes on $\mathfrak{N}_{\mathfrak{I}}$, so the above diagram shows that $F((\mu_{\mathfrak{I}})_A)$ and $F((\mu_{\mathfrak{I}})_B)$ are isomorphisms, so $F(L_{\mathfrak{I}}f) = 0$ if and only if $F(f) = 0$. $\square$

6.5. **The categorical approach to the Baum-Connes conjecture.** The Baum-Connes conjecture for a (Hausdorff) second countable étale groupoid G with coefficients in a separable G-C*-algebra A states that a particular homomorphism called the *Baum-Connes assembly map*

$$\mu_{G,A}\colon K_*^{\mathrm{top}}(G; A) \to K_*(G \ltimes_r A)$$

is an isomorphism of abelian groups. This has been verified whenever G satisfies the Haagerup property [84, Théorèma 9.3], which in particular covers all amenable groupoids. There are known counterexamples [34] to the conjecture even with trivial coefficients $A = C_0(G^0)$, for which the failure of the isomorphism is derived from a failure of exactness.

The topological K-theory $K_*^{\mathrm{top}}(G; A)$ is typically defined as a direct limit of the Kasparov groups $\mathrm{KK}_*^G(C_0(X), A)$ over all second countable G-compact locally compact subspaces $X \subseteq \mathcal{E}(G)$ of a universal proper G-space $\mathcal{E}(G)$. The assembly map $\mu_{G,A}$ is then defined by constructing compatible maps $\mathrm{KK}_*^G(C_0(X), A) \to K_*(G \ltimes_r A)$ for each X. Instead of working directly with these definitions, we make use of the "categorical" or "localisation" reformulation of the Baum-Connes conjecture. This is developed in [58] for groups and then in [11] for étale groupoids. The idea of this approach is to describe the topological K-theory $K_*^{\mathrm{top}}(G;A)$ as a localisation of the operator K-theory $K_*(G \ltimes_r A)$ with respect to some complementary pair in the sense of Definition 6.40. The Baum-Connes assembly map is then given by the assembly map associated to this localisation.

Bönicke and Proietti describe the Baum-Connes assembly map $\mu_{G,A}$ in [11] as the assembly map associated to a complementary pair $(\langle \mathcal{P}r \rangle, \mathfrak{N}_p)$ of subcategories of KK^G. Here, $\langle \mathcal{P}r \rangle$ is the localising subcategory generated by the proper G-C*-algebras and $\mathfrak{N}_p$ is the full subcategory of G-C*-algebras B such that $\mathrm{Res}_G^H B = 0$ for each proper open subgroupoid $H \subseteq G$. More precisely, [11, Theorem B] says that $(\langle \mathcal{P}r \rangle, \mathfrak{N}_p)$ is a complementary pair, while [11, Theorem C] identifies the associated assembly map $\mathbb{L}K_*(G \ltimes_r A) \to K_*(G \ltimes_r A)$ with the Baum-Connes assembly map $\mu_{G,A}\colon K_*^{\mathrm{top}}(G;A) \to K_*(G \ltimes_r A)$.

Theorem 6.49 (Identification of the Baum-Connes assembly map (Theorems B and C in [11]))**.** *Let G be a second countable étale groupoid. Then $(\langle \mathcal{P}r \rangle, \mathfrak{N}_p)$ is a complementary pair of subcategories of KK^G and the assembly map*

$$\mathbb{L}K_*(G \ltimes_r -) \Rightarrow K_*(G \ltimes_r -)$$

for the associated localisation of the reduced K-theory functor $K_*(G \ltimes_r -)$ can be identified with the Baum-Connes assembly map

$$\mu_{G,-}\colon K_*^{top}(G;-) \Rightarrow K_*(G \ltimes_r -).$$

The pair $(\langle \mathcal{P}r \rangle, \mathfrak{N}_p)$ is shown to be complementary by proving that it is equal to the complementary pair $(\langle \mathfrak{P}_{\mathfrak{J}} \rangle, \mathfrak{N}_{\mathfrak{J}})$ constructed from a homological ideal $\mathfrak{J}$ with enough projectives as in Theorem 6.43. Furthermore, $\mathfrak{J}$ is constructed as the kernel of a triangulated functor F with a left adjoint E. Theorem 6.44 ensures that on top of having enough $\mathfrak{J}$-projectives there are explicit $\mathfrak{J}$-projective resolutions, and that $\mathfrak{J}$-projective objects are direct summands of objects in the image of E. The objects in the image of E therefore generate $\langle \mathcal{P}r \rangle$ as a localising subcategory.

As $\mathfrak{N}_p$ is the subcategory of G-C*-algebras B such that $\mathrm{Res}_G^H B = 0$ for each proper open subgroupoid $H \subseteq G$, the natural triangulated functor F to consider is the product of the restriction functors $\mathrm{Res}_G^H\colon \mathrm{KK}^G \to \mathrm{KK}^H$. The problem is that there are almost always uncountably many proper open subgroupoids, and for the adjoint to exist we would need uncountable direct sums, which do not exist in KK^G. Bönicke and Proietti's solution to this problem is to consider a countable subfamily $\mathcal{F}$ of proper open subgroupoids, to obtain the following adjunction:

$$\begin{aligned} \mathrm{Ind}_{\mathcal{F}}\colon \prod_{H \in \mathcal{F}} \mathrm{KK}^H &\to \mathrm{KK}^G & \dashv \qquad \mathrm{Res}_{\mathcal{F}}\colon \mathrm{KK}^G &\to \prod_{H \in \mathcal{F}} \mathrm{KK}^H \\ (B_H)_{H \in \mathcal{F}} &\mapsto \bigoplus_{H \in \mathcal{F}} \mathrm{Ind}_H^G B_H & A &\mapsto (\mathrm{Res}_G^H(A))_{H \in \mathcal{F}}. \end{aligned}$$

It follows from Theorem 6.44 that the localising subcategory $\langle \mathcal{FI} \rangle$ generated by the G-C*-algebras $\mathrm{Ind}_{\mathcal{F}}((B_H)_{H\in\mathcal{F}})$ induced from $\mathcal{F}$ is left complementary to the thick subcategory $\mathfrak{N}_{\mathcal{F}}$ of G-C*-algebras $N \in \mathrm{KK}^G$ such that $\mathrm{Res}_G^H N = 0$ for each $H \in \mathcal{F}$. For any choice of $\mathcal{F}$ we have an inclusion $\mathfrak{N}_p \subseteq \mathfrak{N}_{\mathcal{F}}$, and we can achieve equality with a condition on $\mathcal{F}$ which roughly says that it sees enough of G.

Condition (P)**.** We say that a family $\mathcal{F}$ of proper open subgroupoids of an étale groupoid G satisfies condition (P) if for any proper G-space Z with anchor map $\rho\colon Z \to G^0$ and any element $z \in Z$, there is a subgroupoid $H \in \mathcal{F}$ and an open neighbourhood U of z such that $(G \ltimes Z)_U^U \subseteq H \ltimes \rho^{-1}(H^0)$.

The following theorem is implicit from [11, Theorem 3.10, Lemma 3.15], as condition (P) is the only property of the countable family of "compact actions" that is used in the proof. A *compact action* (see [11, Section 3]) around a point $x \in G^0$ is an open subgroupoid of G that is the union of a finite group of pairwise disjoint open bisections with a common range and domain containing x such that each bisection fixes x.

Theorem 6.50. *Let G be a second countable étale groupoid and let $\mathcal{F}$ be a family of proper open subgroupoids of G satisfying condition (P). Then we have equalities*

$$\langle \mathcal{FI} \rangle = \langle \mathcal{P}r \rangle, \qquad\qquad \mathfrak{N}_{\mathcal{F}} = \mathfrak{N}_p,$$

which express that the G-C-algebras induced from $\mathcal{F}$ generate the localising subcategory $\langle \mathcal{P}r \rangle$ generated by proper G-C*-algebras and that given $N \in \mathrm{KK}^G$ with $\mathrm{Res}_G^H N = 0$ for all $H \in \mathcal{F}$, we have $\mathrm{Res}_G^H N = 0$ for all proper open subgroupoids H. Furthermore, there is a countable family $\mathcal{F}$ satisfying condition (P) consisting of compact actions.*

The existence of a countable family of proper open subgroupoids of G satisfying condition (P) therefore ensures that $(\langle \mathcal{P}r \rangle, \mathfrak{N}_p)$ is a complementary pair. In many situations we can construct a concrete countable family $\mathcal{F}$ satisfying (P). The description of compact actions is somewhat technical, but it is very natural from an inverse semigroup perspective. The open bisections of G form an inverse semigroup S which acts on $X = G^0$ such that $G \cong S \ltimes X$. A compact action is then the transformation groupoid $T \ltimes X$ of a finite inverse subsemigroup $T \leq S$ consisting of pairwise disjoint open bisections with a common domain and range fixing some element $x \in X$.

Suppose that S is a countable inverse semigroup and $G = S \ltimes X$ is the transformation groupoid of an inverse semigroup action $S \curvearrowright X$. Given an inverse subsemigroup $T \leq S$, there is an étale homomorphism $\iota_T\colon T \ltimes X \to S \ltimes X$ given by $[t, x] \mapsto [t, x]$ for $t \in T$ and $x \in \operatorname{dom} t$. This may fail to be injective because the equivalence relation defining $S \ltimes X$ stems from the idempotent semilattice E of S, which may be larger than that of T. If we take the inverse semigroup $\langle T, E\rangle = \{te \mid t \in T,\ e \in E\} \cup E$ generated in S by T and E, then we may identify $\langle T, E\rangle \ltimes X$ as an open subgroupoid of $S \ltimes X$, which also contains the image of ι_T. We will show that the family

$$\mathcal{F} = \{\langle F, E\rangle \ltimes X \mid F \leq S \text{ finite inverse subsemigroup}\}$$

satisfies condition (P). In fact, it is enough to consider only the finite subgroups of stabiliser groups $S_e = \{s \in S \mid s^*s = ss^* = e\}$ for idempotents $e \in E$. For the transformation groupoid $\Gamma \ltimes X$ of an action $\Gamma \curvearrowright X$ of a countable group Γ, this is the family $\{F \ltimes X \mid F \leq \Gamma \text{ finite}\}$ of transformations groupoids of finite subgroups. The following two results are essentially a modification of [11, Proposition 3.2], with the aim of a more explicit description of a family $\mathcal{F}$ to which we may apply Theorem 6.50.

Lemma 6.51. *Let $S \curvearrowright X$ be an inverse semigroup action. Let $x \in X$ and suppose that $\Gamma \leq (S \ltimes X)_x^x$ is a finite subgroup of the isotropy group at x. Then there is an idempotent $e \in E(S)$ with $x \in \operatorname{dom}_X e$ and a finite subgroup $T \leq S_e$ such that $\Gamma = \{[t, x] \mid t \in T\}$.*

Proof. For each $g \in \Gamma$, let $s_g \in S$ represent g in the sense that $g = [s_g, x]$, with an idempotent representing the identity element of Γ. For each pair $g, h \in \Gamma$ let $e_{g,h} \in E(S)$ be an idempotent with $x \in \operatorname{dom} e_{g,h}$, such that $s_g s_h e_{g,h} = s_{gh} e_{g,h}$, $e_{g,h} \leq \operatorname{dom} s_h$, $e_{g,h} \leq \operatorname{dom} s_{gh}$ and $s_h \cdot e_{g,h} \leq \operatorname{dom} s_g$. Let $e_0 = \prod_{g,h\in\Gamma} e_{g,h}$ and let $e = \prod_{g\in\Gamma} s_g \cdot e_0$. For each $g \in \Gamma$, the element $t_g = s_g e$ is an element of S_e because $s_g \cdot e = e$. Setting $T = \{t_g \mid g \in \Gamma\}$ completes the proof. □

Proposition 6.52. *Let $S \curvearrowright X$ be an inverse semigroup action, suppose that $G = S \ltimes X$ is Hausdorff and let $G \curvearrowright Z$ be a proper action with anchor map $\rho\colon Z \to X$. Then for each $z \in Z$ there is an idempotent $e \in E = E(S)$, a finite subgroup $T \leq S_e$ fixing z and an open neighbourhood U of z such that $(G \ltimes Z)_U^U \subseteq (\langle T, E\rangle \ltimes X) \ltimes Z$. In particular, the family*

$$\{\langle T, E\rangle \ltimes X \mid e \in E,\ T \leq S_e \textit{ finite subgroup}\}$$

of proper open subgroupoids of $S \ltimes X$ satisfies condition (P).

Proof. The action $G \curvearrowright Z$ induces an action $S \curvearrowright Z$ and we may identify $G \ltimes Z$ with $S \ltimes Z$. By properness, the isotropy group $(S \ltimes Z)_z^z$ at z is finite, and so by Lemma 6.51 there is an idempotent $e \in E$ and a finite subgroup $T \leq S_e$ such that $(S \ltimes Z)_z^z = \{[t, z] \mid t \in T\}$. It also follows from properness that because $\langle T, E\rangle \ltimes Z$ is an open subspace of $S \ltimes Z$ containing the fibre $(S \ltimes Z)_z^z$, it also contains the restriction of $S \ltimes Z$ to some open neighbourhood U of z. Under the identification $(\langle T, E\rangle \ltimes X) \ltimes Z = \langle T, E\rangle \ltimes Z$ we are done. □

An important special case is when the étale groupoid G has torsion-free isotropy groups. In this setting, the family $\mathcal{F} = \{G^0\}$ satisfies condition (P), and much analysis is considerably simplified.

7. The ABC spectral sequence as a functor

Meyer introduced the ABC spectral sequence in [57], with the following set-up. Let $\mathfrak{T}$ be a triangulated category with countable direct sums, and let $\mathfrak{I}$ be a homological ideal which has enough projectives and is compatible with countable direct sums. Let $F\colon \mathfrak{T} \to \mathsf{Ab}_*$ be a stable homological functor that commutes with countable direct sums and let A be an object in $\mathfrak{T}$. Recall that $\mathbb{L}_{\mathfrak{I}}F$ is the localisation at $\mathfrak{N}_{\mathfrak{I}}$ of F. Meyer describes the second sheet and convergence of the ABC spectral sequence in Proposition 4.8 and Theorem 5.1, summarized in the following theorem.

Theorem 7.1 (The ABC spectral sequence)**.** *There is a spectral sequence $(E, d) = (E^r, d^r)_{r\geq 2}$ which we call the ABC spectral sequence for $(\mathfrak{T}, \mathfrak{I}, F, A)$ that strongly converges towards $\mathbb{L}_{\mathfrak{I}}F(A)$ with the filtration $(\mathbb{L}_{\mathfrak{I}}F\colon \mathfrak{I}^k(A))_{k\geq 0}$. The second sheet is described by the $\mathfrak{I}$-derived functors $\mathbb{L}_pF$ of F with canonical isomorphisms $E^2_{p,q} \cong \mathbb{L}_pF_q(A)$.*

$$E^2_{p,q} \cong \mathbb{L}_pF_q(A) \Rightarrow \mathbb{L}_{\mathfrak{I}}F(A)$$

7.1. The ABC category.

Our aim is to describe how to make this theorem functorial in the variables $(\mathfrak{T}, \mathfrak{I}, F, A)$. Let us call such a quadruple an ABC tuple. We construct a category of ABC tuples called the ABC category.

Definition 7.2 (ABC tuple)**.** An *ABC tuple* is a tuple $(\mathfrak{T}, \mathfrak{I}, F, A)$ consisting of

- a triangulated category $\mathfrak{T}$,
- a homological ideal $\mathfrak{I} \lhd \mathfrak{T}$,
- a stable homological functor $F\colon \mathfrak{T} \to \mathsf{Ab}_*$,
- and an object $A \in \mathfrak{T}$,

such that $\mathfrak{T}$ has enough $\mathfrak{I}$-projectives and countable direct sums which are compatible with $\mathfrak{I}$ and F in the sense that $\mathfrak{I}$ is closed under countable direct sums of morphisms and F commutes with countable direct sums.

Example 7.3. Let G be an étale groupoid and let A be a G-C*-algebra. Then for any homological ideal $\mathfrak{I} \lhd \mathrm{KK}^G$ with enough projectives which is compatible with countable direct sums, we have an ABC tuple

$$(\mathrm{KK}^G, \mathfrak{I}, K_*(G \ltimes -), A).$$

One choice for the ideal $\mathfrak{I}$ is the kernel of the restriction functor $\mathrm{Res}_G^{G^0}\colon \mathrm{KK}^G \to \mathrm{KK}^{G^0}$.

$$\mathfrak{I} = \ker \mathrm{Res}_G^{G^0}$$

Another choice is the intersection of such kernels over the family $\mathcal{F}_p$ of proper open subgroupoids of G.

$$\mathfrak{I} = \bigcap_{H\in\mathcal{F}_p} \ker \mathrm{Res}_G^H$$

In order to work with morphisms of ABC tuples, we will often have to combine natural transformations with functors in various ways, so we set the following notation. Suppose we have functors $F, G\colon \mathfrak{C} \to \mathfrak{D}$, a natural transformation $\alpha\colon F \Rightarrow G$ and further functors $H\colon \mathfrak{B} \to \mathfrak{C}$ and $K\colon \mathfrak{D} \to \mathfrak{E}$. We write αH for the natural transformation $FH \Rightarrow GH$ that sends an object $B \in \mathfrak{B}$ to the morphism $\alpha_{H(B)}\colon FH(B) \to GH(B)$.

$$
\begin{tikzcd}
\mathfrak{B} \arrow[r, "H"] \arrow[rr, bend left=60, "FH"] \arrow[rr, bend right=60, "GH"'] & \mathfrak{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha H \Downarrow\alpha"] & \mathfrak{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha\Downarrow K\alpha"] \arrow[rr, bend left=60, "KF"] \arrow[rr, bend right=60, "KG"'] & \mathfrak{D} \arrow[r, "K"] & \mathfrak{E}
\end{tikzcd}
$$

Similarly, we write $K\alpha\colon KF \Rightarrow KG$ for the natural transformation that sends an object $C \in \mathfrak{C}$ to the morphism $K(\alpha_C)\colon KF(C) \to KG(C)$.

Definition 7.4 (Morphism of ABC tuples)**.** Given ABC tuples $\mathfrak{M} = (\mathfrak{T}, \mathfrak{J}, F, A)$ and $\mathfrak{M}' = (\mathfrak{T}', \mathfrak{J}', F', A')$, an *ABC cycle* from $\mathfrak{M}$ to $\mathfrak{M}'$ is given by a triple (Φ, α, f):

- a triangulated functor $\Phi\colon \mathfrak{T}' \to \mathfrak{T}$ mapping $\mathfrak{J}'$ into $\mathfrak{J}$,
- a natural transformation $\alpha\colon F \circ \Phi \Rightarrow F'$,
- and a morphism $f\colon A \to \Phi(A')$ in $\mathfrak{T}$.

We say that two cycles (Φ_1, α_1, f_1) and (Φ_2, α_2, f_2) from $\mathfrak{M}$ to $\mathfrak{M}'$ are *equivalent* if there is a natural isomorphism $\eta\colon \Phi_1 \cong \Phi_2$ which identifies f_1 with f_2 and α_1 with α_2 in the sense that $\eta_{A'} f_1 = f_2$ and $\alpha_1 = \alpha_2 \circ F\eta$. An *ABC morphism* is an equivalence class $[\Phi, \alpha, f]$ of ABC cycles under this notion of equivalence, and we may write $[\Phi, \alpha, f]\colon \mathfrak{M} \to \mathfrak{M}'$. We will usually work directly with ABC cycles, and it will usually be clear that our constructions depend only on the equivalence class.

A proper correspondence $(E, \Omega)\colon (A, G) \to (B, H)$ of étale groupoids with C*-coefficients allows us to build an ABC morphism with the appropriate countability assumptions and homological ideals. We say that (E, Ω) is *separable* if E, A and B are separable and Ω, G and H are second countable.

Example 7.5 (An ABC morphism from a proper groupoid correspondence)**.** Let $(E, \Omega)\colon (A, G) \to (B, H)$ be a separable proper correspondence of étale groupoids with C*-coefficients. Consider the induction functor $\mathrm{Ind}_\Omega\colon \mathrm{KK}^H \to \mathrm{KK}^G$ (Definition 3.22), natural transformation $\alpha_\Omega\colon K_*(G \ltimes \mathrm{Ind}_\Omega -) \Rightarrow K_*(H \ltimes -)$ (Proposition 3.28) and G-equivariant proper correspondence $\Delta(E)\colon A \to \mathrm{Ind}_\Omega B$ (Proposition 2.32). Suppose we have homological ideals $\mathfrak{I} \lhd \mathrm{KK}^G$ and $\mathfrak{J} \lhd \mathrm{KK}^H$ with enough projectives and compatible with countable direct sums such that $\mathrm{Ind}_\Omega(\mathfrak{J}) \subseteq \mathfrak{I}$. Then setting $f_E := [\Delta(E), 0] \in \mathrm{KK}^G(A, \mathrm{Ind}_\Omega B)$,

$$
[\mathrm{Ind}_\Omega, \alpha_\Omega, f_E]\colon (\mathrm{KK}^G, \mathfrak{I}, K_*(G \ltimes -), A) \to (\mathrm{KK}^H, \mathfrak{J}, K_*(H \ltimes -), B) \tag{7.1}
$$

is an ABC morphism. In the setting without coefficients, we have $A = C_0(G^0)$, $B = C_0(H^0)$, and $f_\Omega\colon A \to \operatorname{Ind}_\Omega B$ is induced by the G-equivariant proper map $\overline{\rho}\colon \Omega/H \to G^0$.

Ultimately, we are interested in ABC tuples $(\mathfrak{T}, \mathfrak{I}, F, A)$ because we are interested in the group $F(A)$, which is the application of $(\mathfrak{T}, \mathfrak{I}, F, A)$. We can view $(\mathfrak{T}, \mathfrak{I}, F, A)$ as "sitting above" $F(A)$. An ABC cycle $(\Phi, \alpha, f)\colon (\mathfrak{T}, \mathfrak{I}, F, A) \to (\mathfrak{T}', \mathfrak{I}', F', A')$ sits above a morphism $\operatorname{App}(\Phi, \alpha, f) := \alpha_{A'} \circ F(f)\colon FA \to F'A'$, which we call its *application*.

$$\begin{tikzcd} FA \arrow[rr, dashed, "\operatorname{App}(\Phi,\alpha,f)"] \arrow[dr, "F(f)"'] & & F'A' \\ & F\Phi A' \arrow[ur, "\alpha_{A'}"'] & \end{tikzcd}$$

This only depends on the equivalence class of (Φ, α, f), so we can define the application of $[\Phi, \alpha, f]$ to be $\operatorname{App}(\Phi, \alpha, f) = \alpha_{A'} \circ F(f)\colon FA \to F'A'$.

Definition 7.6 (Composition in the ABC category)**.** If $(\Phi, \alpha, f)\colon (\mathfrak{T}, \mathfrak{I}, F, A) \to (\mathfrak{T}', \mathfrak{I}', F', A')$ and $(\Psi, \beta, g)\colon (\mathfrak{T}', \mathfrak{I}', F', A') \to (\mathfrak{T}'', \mathfrak{I}'', F'', A'')$ are ABC cycles, then their *composition* $(\Psi, \beta, g) \circ (\Phi, \alpha, f)$ consists of:

- the triangulated functor $\Phi \circ \Psi\colon \mathfrak{T}'' \to \mathfrak{T}$,
$$\mathfrak{T}'' \xrightarrow{\Psi} \mathfrak{T}' \xrightarrow{\Phi} \mathfrak{T}$$
- the natural transformation $\beta \circ (\alpha\Psi)$,
$$F \circ \Phi \circ \Psi \xRightarrow{\alpha\Psi} F' \circ \Psi \xRightarrow{\beta} F''$$
- and the morphism $\Phi(g) \circ f$.
$$A \xrightarrow{f} \Phi(A') \xrightarrow{\Phi(g)} \Phi(\Psi(A''))$$

This respects equivalence of ABC cycles, and we can define composition of ABC morphisms as $[\Psi, \beta, g] \circ [\Phi, \alpha, f] = [\Phi \circ \Psi, \beta \circ (\alpha\Psi), \Phi(g) \circ f]$. This composition is associative at the level of ABC cycles. Given another cycle $(\Xi, \gamma, h)\colon (\mathfrak{T}'', \mathfrak{I}'', F'', A'') \to (\mathfrak{T}''', \mathfrak{I}''', F''', A''')$, both ways to associate $(\Xi, \gamma, h) \circ (\Psi, \beta, g) \circ (\Phi, \alpha, f)$ are given by:

- the triangulated functor $\Phi \circ \Psi \circ \Xi$,
- the natural transformation $\gamma \circ (\beta\Xi) \circ (\alpha\Psi\Xi)$,
- and the morphism $\Phi\Psi h \circ \Phi g \circ f$.

We have identity morphisms given at $(\mathfrak{T}, \mathfrak{I}, F, A)$ by $[\mathrm{id}_{\mathfrak{T}}, \mathrm{id}_F, \mathrm{id}_A]$. We obtain a category ABC called the *ABC category* whose objects are ABC tuples with ABC morphisms as morphisms.

Remark 7.7. We can see why this composition makes sense by considering the applications of ABC cycles.

$$\begin{tikzcd}
FA \arrow[rr, "\mathrm{App}(\Phi,\alpha,f)"] \arrow[rrrr, bend left, "\mathrm{App}(\Psi\circ\Phi,\beta\circ\alpha\Psi,\Phi(g)\circ f)"] \arrow[dr, "F(f)"'] \arrow[ddrr, bend right, "F(\Phi(g)\circ f)"'] & & F'A' \arrow[rr, "\mathrm{App}(\Psi,\beta,g)"] \arrow[dr, "F'(g)"'] & & F''A'' \\
& F\Phi A' \arrow[ur, "\alpha_{A'}"'] \arrow[dr, "F\Phi(g)"] & & F'\Psi A'' \arrow[ur, "\beta_{A''}"'] & \\
& & F\Phi\Psi A'' \arrow[ur, "\alpha_{\Psi A''}"] \arrow[uurr, bend right, "(\beta\circ\alpha\Psi)_{A''}"'] & &
\end{tikzcd}$$

This diagram commutes, and we obtain a functor App: $\mathsf{ABC} \to \mathsf{Ab}_*$.

Proposition 7.8. *The assignment* $[E,\Omega] \mapsto [\mathrm{Ind}_\Omega, \alpha_\Omega, f_E]$ *from Example 7.5 of an ABC morphism to a separable proper correspondence* $(E,\Omega)\colon (A,G) \to (B,H)$ *with appropriate ideals* $\mathfrak{I} \lhd \mathrm{KK}^G$ *and* $\mathfrak{J} \lhd \mathrm{KK}^H$ *is functorial. Here by appropriate we mean that* $\mathfrak{I}$ *and* $\mathfrak{J}$ *have enough projectives, are closed under countable direct sums and* $\mathrm{Ind}_\Omega(\mathfrak{J}) \subseteq \mathfrak{I}$.

Proof. We need to check that this assignment respects composition, and that it respects identities. Let $(E,\Omega)\colon (A,G) \to (B,H)$ and $(F,\Lambda)\colon (B,H) \to (C,K)$ be separable proper correspondences. There is a natural isomorphism $\varphi\colon \mathrm{Ind}_\Omega \circ \mathrm{Ind}_\Lambda \cong \mathrm{Ind}_{\Lambda\circ\Omega}\colon \mathrm{KK}^K \rightrightarrows \mathrm{KK}^G$ by Proposition 3.26. We must also check that the following diagram of natural transformations of functors $\mathrm{KK}^K \to \mathsf{Ab}_*$ commutes.

$$\begin{tikzcd}
K_*(G \ltimes \mathrm{Ind}_\Omega \mathrm{Ind}_\Lambda -) \arrow[r, Rightarrow, "\alpha_\Omega \mathrm{Ind}_\Lambda"] \arrow[d, Rightarrow, "K_*(G\ltimes\varphi)"] & K_*(H \ltimes \mathrm{Ind}_\Lambda -) \arrow[d, Rightarrow, "\alpha_\Lambda"] \\
K_*(G \ltimes \mathrm{Ind}_{\Lambda\circ\Omega} -) \arrow[r, Rightarrow, "\alpha_{\Lambda\circ\Omega}"] & K_*(K \ltimes -)
\end{tikzcd}$$

This is exactly the result of applying the crossed product functor $\mathsf{GpdCorr}_{C^*} \to \mathsf{Corr}$ followed by the K-theory functor $\mathsf{Corr}_p \to \mathsf{Ab}_*$ to diagram (2.8) which Proposition 2.30 says commutes. The final requirement to respect composition is that the following diagram commutes in KK^G. Recall that $f_E = [\Delta(E), 0] \in \mathrm{KK}^G(A, \mathrm{Ind}_\Omega B)$.

$$\begin{tikzcd}
A \arrow[r, "f_{F\circ E}"] \arrow[d, "f_E"] & \mathrm{Ind}_{\Lambda\circ\Omega} C \arrow[d, "\varphi_C^{-1}"] \\
\mathrm{Ind}_\Omega B \arrow[r, "\mathrm{Ind}_\Omega(f_F)"] & \mathrm{Ind}_\Omega \mathrm{Ind}_\Lambda C
\end{tikzcd}$$

This follows from Proposition 2.33, which says that the compositions come from isomorphic G-correspondences.

Now consider the identity morphism $(i_G(A), G)\colon (A,G) \to (A,G)$. Proposition 3.27 says that we have a natural isomorphism $\beta_G\colon \mathrm{Ind}_G \cong \mathrm{id}_{\mathrm{KK}^G}$. We need to check that up to this identification, α_G is the identity natural transformation and $f_{i_G(A)}$ is the identity morphism. This means that we have to check that:

- $K_*(G \ltimes \beta_G(A)) = \alpha_G(A)\colon K_*(G \ltimes \operatorname{Ind}_G A) \to K_*(G \ltimes A)$.
- $f_{i_G(A)} = \beta_G(A)^{-1} \in \mathrm{KK}^G(A, \operatorname{Ind}_G A)$.

The first bullet point comes from Proposition 2.31, which says that $\Theta_{G,A}$ comes from the same G-equivariant $*$-isomorphism $\psi_A\colon \operatorname{Ind}_G A \to A$ that induces $\beta_G(A)$. Given that $\alpha_G(A) := K_*(G \ltimes \Theta_{G,A})$, we recover that $K_*(G \ltimes \beta_G(A)) = \alpha_G(A)$.

For the second, consider the following commutative diagram in $\mathsf{GpdCorr}_{C^*}$.

$$\begin{tikzcd}
(A,G) \arrow[rr, "i_G(A,\mathrm{id}_A)"] \arrow[dr, "i_G(\operatorname{Ind}_G A,\psi_A^{-1})"'] & & (A,G) \\
& (\operatorname{Ind}_G A, G) \arrow[ur, "\Theta_{G,A}=i_G(A,\psi_A)"'] &
\end{tikzcd}$$

By the universal property of induction applied to the evaluation correspondence $\Theta_{G,A}$ (Proposition 2.32), the G-equivariant correspondence $(\operatorname{Ind}_G A, \psi_A^{-1})$ is G-equivariantly isomorphic to $\Delta(i_G(A, \mathrm{id}_A))$, and so they induce the same element of KK^G. Therefore $f_{i_G(A)} = \beta_G(A)^{-1}$.

□

In order to get a version of Theorem 7.1 which is functorial with respect to ABC morphisms, we will need to build a morphism of spectral sequences from an ABC morphism, but we will also need functorial versions of the applications of the left derived functors $\mathbb{L}_n F(A)$ and localisation $\mathbb{L}F(A)$. We now focus our attention on building all of these and proving that they are compatible with each other.

Theorem 7.9 (Derived functor maps of an ABC morphism)**.** *An ABC morphism* $\mathfrak{m} = [\Phi, \alpha, f]\colon (\mathfrak{T}, \mathfrak{I}, F, A) \to (\mathfrak{T}', \mathfrak{I}', F', A')$ *functorially induces a map*

$$\mathbb{L}_n(\mathfrak{m})\colon \mathbb{L}_n F(A) \to \mathbb{L}_n F'(A')$$

for each n with the following property. Let $P_\bullet \to A$ and $P'_\bullet \to A'$ be projective resolutions with respect to $\mathfrak{I}$ and $\mathfrak{I}'$ respectively and let $\tilde{f}\colon P_\bullet \to \Phi(P'_\bullet)$ be a chain map over $f\colon A \to \Phi(A')$. Let $\alpha_{P'}\colon F\Phi(P'_\bullet) \to F'(P'_\bullet)$ be the chain map induced by α. Then the chain map $\alpha_{P'} \circ F(\tilde{f})\colon F(P_\bullet) \to F'(P'_\bullet)$ over $\mathrm{App}(\mathfrak{m})\colon F(A) \to F'(A')$ induces $\mathbb{L}_n(\mathfrak{m})\colon \mathbb{L}_n F(A) \to \mathbb{L}_n F'(A')$ by taking homology.

$$\begin{tikzcd}
\cdots \arrow[r] & F(P_1) \arrow[r] \arrow[d, "F(\tilde{f}_1)"] & F(P_0) \arrow[r] \arrow[d, "F(\tilde{f}_0)"] & F(A) \arrow[r] \arrow[d, "F(f)"] & 0 \\
\cdots \arrow[r] & F\Phi(P'_1) \arrow[r] \arrow[d, "\alpha_{P'_1}"] & F\Phi(P'_0) \arrow[r] \arrow[d, "\alpha_{P'_0}"] & F\Phi(A') \arrow[r] \arrow[d, "\alpha_{A'}"] & 0 \\
\cdots \arrow[r] & F'(P'_1) \arrow[r] & F'(P'_0) \arrow[r] & F'(A') \arrow[r] & 0
\end{tikzcd}$$

Proof. To construct $\mathbb{L}_n(\mathfrak{m})$, we consider arbitrary projective resolutions $P_\bullet \to A$ and $P'_\bullet \to A'$ with respect to $\mathfrak{I}$ and $\mathfrak{I}'$ respectively. The triangulated functor Φ

maps $\mathfrak{J}'$ into $\mathfrak{J}$ and so sends $\mathfrak{J}'$-exact sequences to $\mathfrak{J}$-exact sequences. By Lemma 6.33, there is a lift $\tilde{f}\colon P_\bullet \to \Phi(P'_\bullet)$ of $f\colon A \to \Phi(A')$ which is unique up to chain homotopy. We may then define $\mathbb{L}_n(\mathfrak{m})$ as the map in homology induced by the chain map $\alpha_{P'} \circ F(\tilde{f})$, which is a lift of $\mathrm{App}(\mathfrak{m}) = \alpha_{A'} \circ F(f)$.

To check that this is well-defined, suppose we have some other projective resolutions $Q_\bullet \to A$ and $Q'_\bullet \to A'$ and a lift $\hat{f}\colon Q_\bullet \to \Phi(Q'_\bullet)$ of f, and lifts $\tilde{\mathrm{id}}\colon P_\bullet \to Q_\bullet$ and $\tilde{\mathrm{id}}'\colon P'_\bullet \to Q'_\bullet$ of $\mathrm{id}\colon A \to A$ and $\mathrm{id}'\colon A' \to A'$. Then by Lemma 6.33, $\tilde{\mathrm{id}}' \circ \tilde{f}$ is chain homotopic to $\hat{f} \circ \tilde{\mathrm{id}}$, and therefore the following diagram commutes.

$$\begin{tikzcd} H_n(F(P_\bullet)) \arrow[r, "H_n\left(\alpha_{P'} \circ F(\tilde{f})\right)"] \arrow[d, "\cong"', "H_n(F(\tilde{\mathrm{id}}))"] & H_n(F'(P'_\bullet)) \arrow[d, "\cong"', "H_n(F'(\tilde{\mathrm{id}}'))"] \\ H_n(F(Q_\bullet)) \arrow[r, "H_n\left(\alpha_{Q'} \circ F(\hat{f})\right)"] & H_n(F'(Q'_\bullet)) \end{tikzcd}$$

When $\mathfrak{m} = \mathrm{id}_{\mathfrak{T},\mathfrak{J},F,A}$ is an identity morphism, we can pick any projective resolution $P_\bullet \to A$ and pick $\tilde{\mathrm{id}}\colon P_\bullet \to P_\bullet$ to be the identity chain map. This induces the identity in homology, so $\mathbb{L}_n(\mathrm{id}_{\mathfrak{T},\mathfrak{J},F,A}) = \mathrm{id}_{\mathbb{L}_n F(A)}$.

Suppose $\mathfrak{n} = [\Psi, \beta, g]\colon (\mathfrak{T}', \mathfrak{J}', F', A') \to (\mathfrak{T}'', \mathfrak{J}'', F'', A'')$ is another ABC morphism. Let $P''_\bullet \to A''$ be an $\mathfrak{J}''$-projective resolution, and let $\tilde{g}\colon P'_\bullet \to \Psi(P''_\bullet)$ be a lift of $g\colon A' \to \Psi(A'')$. Then $\Phi(\tilde{g}) \circ \tilde{f}\colon P_\bullet \to \Phi\Psi(P''_\bullet)$ is a lift of $\Phi(g) \circ f\colon A \to \Phi\Psi(A'')$. It follows that $\mathbb{L}_n(\mathfrak{n} \circ \mathfrak{m}) = \mathbb{L}_n(\mathfrak{n}) \circ \mathbb{L}_n(\mathfrak{m})$. ☐

Remark 7.10. If we keep the triangulated category $\mathfrak{T}$, homological ideal $\mathfrak{J}$ and stable homological functor F constant, an ABC morphism specialises to a morphism $f\colon A \to B$ in $\mathfrak{T}$. We can deduce from the above theorem that $\mathbb{L}_p(\mathrm{id}_{\mathfrak{T}}, \mathrm{id}_F, f) = \mathbb{L}_p F(f)\colon \mathbb{L}_p F(A) \to \mathbb{L}_p F(B)$.

We now turn to some technical details to construct functoriality of the localisation $\mathbb{L}F(A)$. Consider triangulated categories $\mathfrak{T}$ and $\mathfrak{T}'$ with homological ideals $\mathfrak{J}$ and $\mathfrak{J}'$ respectively. Let L and L' be the localisation functors $L = L_{\mathfrak{J}}\colon \mathfrak{T} \to \mathfrak{T}$ and $L' = L_{\mathfrak{J}'}\colon \mathfrak{T}' \to \mathfrak{T}'$. Let $\Phi\colon \mathfrak{T}' \to \mathfrak{T}$ be a triangulated functor such that $\Phi(\mathfrak{J}') \subseteq \mathfrak{J}$. Let μ and μ' be the localisation natural transformations $\mu = \mu_{\mathfrak{J}}\colon L \Rightarrow \mathrm{id}_{\mathfrak{T}}$ and $\mu' = \mu_{\mathfrak{J}'}\colon L' \Rightarrow \mathrm{id}_{\mathfrak{T}'}$. Consider the following diagram.

$$LA \xrightarrow{L(f)} L\Phi A' \xleftarrow[\cong]{L\Phi(\mu'_{A'})} L\Phi L'A' \xrightarrow{\mu_{\Phi L'A'}} \Phi L'A'$$

The morphism $L\Phi(\mu'_{A'})\colon L\Phi L'A' \to L\Phi A'$ is an isomorphism because L maps $\mathfrak{J}$-equivalences to isomorphisms. We define the *localisation* of f to be $\mathfrak{L}_\Phi(f)\colon LA \to \Phi(L'A')$ as the composition $\mu_{\Phi L'A'} \circ L\Phi(\mu'_{A'})^{-1} \circ L(f)$. This enables us to apply the localisation functor $A \mapsto LA$ at the level of the ABC category.

Proposition 7.11. *Let $\mathfrak{m} = [\Phi, \alpha, f]\colon (\mathfrak{T}, \mathfrak{J}, F, A) \to (\mathfrak{T}', \mathfrak{J}', F', A')$ be an ABC morphism. Then the mapping sending $[\Phi, \alpha, f]$ to*

$$[\Phi, \alpha, \mathfrak{L}_\Phi(f)]\colon (\mathfrak{T}, \mathfrak{J}, F, LA) \to (\mathfrak{T}', \mathfrak{J}', F', L'A')$$

is functorial. We refer to this as the ABC localisation functor $\mathfrak{L}\colon \mathsf{ABC} \to \mathsf{ABC}$, as on an object $(\mathfrak{T}, \mathfrak{I}, F, A)$ it simply applies the localisation functor $L\colon \mathfrak{T} \to \mathfrak{T}$ to A. Furthermore, when $\Phi = \mathrm{id}_{\mathfrak{T}}$ we recover the localisation functor, as $\mathfrak{L}_{\mathrm{id}_{\mathfrak{T}}}(f) = L(f)$.

Proof. When $\Phi = \mathrm{id}_{\mathfrak{T}}$, $\mathfrak{L}_\Phi$ is given by the composition $\mu_{LA} \circ L(\mu_A)^{-1} \circ L(f)$, which is equal to $L(f)$ because $\mu_{LA} = L(\mu_A)$ by Proposition 6.47. Setting $\alpha = \mathrm{id}_F$ and $f = \mathrm{id}_A$, we see that $\mathfrak{L}$ respects identity morphisms.

Let $\mathfrak{n} = [\Psi, \beta, g]\colon (\mathfrak{T}', \mathfrak{I}', F', A') \to (\mathfrak{T}'', \mathfrak{I}'', F'', A'')$ be another ABC morphism, with localisation functor $L''\colon \mathfrak{T}'' \to \mathfrak{T}''$ and localisation natural transformation $\mu''\colon L'' \Rightarrow \mathrm{id}$. Consider the following diagram.

$$
\begin{tikzcd}
LA \arrow[r, "L(f)"'] \arrow[dr, "L(\Phi(g)f)"'] & L\Phi A' \arrow[d, "L\Phi(g)"] & L\Phi L'A' \arrow[l, "\cong", "L\Phi(\mu'_{A'})"'] \arrow[r, "\mu_{\Phi L'A'}"] \arrow[d, "L\Phi L'(g)"] & \Phi L'A' \arrow[d, "\Phi L'(g)"] \\
& L\Phi\Psi A'' & L\Phi L'\Psi A'' \arrow[l, "\cong", "L\Phi(\mu'_{\Psi A''})"'] \arrow[r, "\mu_{\Phi L'\Psi A''}"'] & \Phi L'\Psi A'' \\
& & L\Phi L'\Psi L''A'' \arrow[u, "\cong", "L\Phi L'\Psi(\mu''_{A''})"'] \arrow[r, "\mu_{\Phi L'\Psi L''A''}"'] \arrow[dl, "L\Phi(\mu'_{\Psi L''A''})", "\cong"'] & \Phi L'\Psi L''A'' \arrow[u, "\cong", "\Phi L'\Psi(\mu''_{A''})"'] \arrow[d, "\Phi(\mu'_{\Psi L''A''})"] \\
& L\Phi\Psi L''A'' \arrow[uu, "L\Phi\Psi(\mu''_{A''})", "\cong"'] \arrow[rr, "\mu_{\Phi\Psi L''A''}"] & & \Phi\Psi L''A''
\end{tikzcd}
$$

Let us move left to right through the diagram. The triangle commutes by functoriality of L. The two squares on the left commute by naturality of μ'. The three squares on the right commute by naturality of μ. We conclude that $\mathfrak{L}_{\Phi\Psi}(\Phi(g) \circ f) = \Phi(\mathfrak{L}_\Psi(g)) \circ \mathfrak{L}_\Phi(f)$, and therefore $\mathfrak{L}(\mathfrak{n} \circ \mathfrak{m}) = \mathfrak{L}(\mathfrak{n}) \circ \mathfrak{L}(\mathfrak{m})$. $\square$

In the same way that the functor $L\colon \mathfrak{T} \to \mathfrak{T}$ comes with natural maps $\mu_A\colon LA \to A$, the functor $\mathfrak{L}\colon \mathsf{ABC} \to \mathsf{ABC}$ comes with natural maps $\mathfrak{L}(\mathfrak{T}, \mathfrak{I}, F, A) \to (\mathfrak{T}, \mathfrak{I}, F, A)$.

Proposition 7.12. *Let $[\Phi, \alpha, f]\colon (\mathfrak{T}, \mathfrak{I}, F, A) \to (\mathfrak{T}', \mathfrak{I}', F', A')$ be an ABC morphism, and let $\mu_A\colon LA \to A$ and $\mu'_{A'}\colon L'A' \to A'$ be the localisation maps. Then the following diagram commutes.*

$$
\begin{tikzcd}
(\mathfrak{T}, \mathfrak{I}, F, LA) \arrow[r, "{[\Phi,\alpha,\mathfrak{L}_\Phi(f)]}"] \arrow[d, "{[\mathrm{id}_{\mathfrak{T}},\mathrm{id}_F,\mu_A]}"] & (\mathfrak{T}', \mathfrak{I}', F', L'A') \arrow[d, "{[\mathrm{id}_{\mathfrak{T}'},\mathrm{id}_{F'},\mu'_{A'}]}"] \\
(\mathfrak{T}, \mathfrak{I}, F, A) \arrow[r, "{[\Phi,\alpha,f]}"] & (\mathfrak{T}', \mathfrak{I}', F', A')
\end{tikzcd}
$$

Proof. By the naturality of $\mu\colon L \Rightarrow \mathrm{id}_{\mathfrak{T}}$, we obtain the commuting diagram below.

$$
\begin{tikzcd}
LA \arrow[r, "L(f)"] \arrow[d, "\mu_A"] & L\Phi A' \arrow[r, "(L\Phi(\mu'_{A'}))^{-1}"] \arrow[d, "\mu_{\Phi A'}"] & L\Phi L'A' \arrow[d, "\mu_{\Phi L'A'}"] \\
A \arrow[r, "f"'] & \Phi A' & \Phi L'A' \arrow[l, "\Phi(\mu'_{A'})"]
\end{tikzcd}
$$

We can deduce that $\Phi(\mu'_{A'}) \circ \mathfrak{L}_\Phi(f) = f \circ \mu_A\colon LA \to \Phi(A')$, and therefore that $[\mathrm{id}_{\mathfrak{T}'}, \mathrm{id}_{F'}, \mu'_{A'}] \circ [\Phi, \alpha, \mathfrak{L}_\Phi f] = [\Phi, \alpha, f] \circ [\mathrm{id}_{\mathfrak{T}}, \mathrm{id}_F, \mu_A]$. $\square$

Definition 7.13 (Localisation application functor)**.** The *localisation application functor* $\mathbb{L}\colon \mathsf{ABC} \to \mathsf{Ab}$ is the composition $\mathrm{App} \circ \mathfrak{L}$. We also recover the localisation $\mathbb{L}F$ of F by holding the category $\mathfrak{T}$, ideal $\mathfrak{I}$ and the functor F constant, because $\mathbb{L}(\mathrm{id}_{\mathfrak{T}}, \mathrm{id}_F, f) = \mathbb{L}F(f)$.

We are now ready to take on the task of proving the following functorial version of Theorem 7.1.

Theorem 7.14. *An ABC morphism* $\mathfrak{m}\colon \mathfrak{M} \to \mathfrak{M}'$ *induces functorially a morphism of ABC spectral sequences* $\mathrm{ABC}(\mathfrak{m})\colon\ \mathrm{ABC}(\mathfrak{M}) \to \mathrm{ABC}(\mathfrak{M}')$, *such that:*

(i) the map on the second sheet is given by $\mathbb{L}_p(\mathfrak{m})_q\colon \mathbb{L}_p(\mathfrak{M})_q \to \mathbb{L}_p(\mathfrak{M}')_q$,

(ii) the map on the limit sheet agrees with $\mathbb{L}(\mathfrak{m})\colon \mathbb{L}(\mathfrak{M}) \to \mathbb{L}(\mathfrak{M}')$.

7.2. **The construction of the ABC spectral sequence.** Before we embark, we summarise Mayer's construction of the ABC spectral sequence from [57]. We start with an ABC tuple $(\mathfrak{T}, \mathfrak{I}, F, A)$. First, we construct an $\mathfrak{I}$-phantom tower over A using Proposition 6.29 and Lemma 6.35.

$$\begin{tikzcd}
A = N_0 \arrow[rr, "\iota_0^1"] & & N_1 \arrow[rr, "\iota_1^2"] \arrow[dl, "\circ" marking, "\epsilon_0"] & & N_2 \arrow[rr] \arrow[dl, "\circ" marking, "\epsilon_1"] & & \cdots \arrow[dl, "\circ" marking] \\
& P_0 \arrow[ul, "\pi_0"] & & P_1 \arrow[ll, "\circ" marking, "\delta_1"] \arrow[ul, "\pi_1"] & & P_2 \arrow[ll, "\circ" marking, "\delta_2"] \arrow[ul, "\pi_2"] & & \cdots \arrow[ll, "\circ" marking] \arrow[ul]
\end{tikzcd}$$

An exact couple is then constructed from the phantom tower. We start by viewing the top collection of exact triangles as a single triangle of $\mathbb{Z}$-graded objects, by defining $N_n = A$, $\iota_n^{n+1} = \mathrm{id}_A$ and $P_n = 0$ for $n < 0$.

$$\begin{tikzcd}
N_\bullet \arrow[rr, "\iota"] & & N_\bullet \arrow[dl, "\circ" marking, "\epsilon"] \\
& P_\bullet \arrow[ul, "\pi"] &
\end{tikzcd}$$

We now consider the stable homological functor $F\colon \mathfrak{T} \to \mathsf{Ab}_*$. This consists of a homological functor $F_p\colon \mathfrak{T} \to \mathsf{Ab}$ for each $p \in \mathbb{Z}$. We use this to construct an exact triangle of bigraded Abelian groups

$$\begin{tikzcd}
D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\
& C \arrow[ul, "k"] &
\end{tikzcd}
\qquad
\begin{aligned}
D_{p,q} &= F_{p+q+1}(N_{p+1}) \\
C_{p,q} &= F_{p+q}(P_p)
\end{aligned}
\tag{7.2}$$

with maps i, j and k defined by

$$\begin{aligned}
i_{p,q} &:= (\iota_{p+1}^{p+2})_*\colon D_{p,q} \to D_{p+1,q-1} & \deg i &= (1, -1) \\
j_{p,q} &:= (\epsilon_p)_*\colon D_{p,q} \to C_{p,q} & \deg j &= (0, 0) \\
k_{p,q} &:= (\pi_p)_*\colon C_{p,q} \to D_{p-1,q} & \deg k &= (-1, 0)
\end{aligned}$$

This is exact because F is a stable homological functor and so for each p, the following sequence is exact.

$$\cdots \longrightarrow F_q(P_p) \xrightarrow{(\pi_p)_*} F_q(N_p) \xrightarrow{(\iota_p^{p+1})_*} F_q(N_{p+1}) \xrightarrow{(\epsilon_p)_*} F_{q-1}(P_p) \longrightarrow \cdots$$

We then apply the construction of a spectral sequence from an exact couple (see Chapter 5.4) to obtain a spectral sequence (E, d) starting from the first sheet. In particular,

- The first sheet E^1 is given by $E^1_{p,q} := F_{p+q}(P_p)$.
- The first differential $d^1_{p,q}\colon E^1_{p,q} \to E^1_{p-1,q}$ is given by $F_{p+q}(\delta_p)\colon F_{p+q}(P_p) \to F_{p+q}(\Sigma P_{p-1})$, noting that $F_{p+q}(\Sigma P_{p-1}) = F_{p+q-1}(P_{p-1})$.
- The second sheet $E^2 := \frac{k^{-1}(\operatorname{im} i)}{j(\ker i)}$ is given by $\frac{\ker F_*(\delta)}{\operatorname{im} F_*(\delta)} = H_*(F_*(P_\bullet), F_*(\delta))$, and therefore $E^2_{p,q} \cong \mathbb{L}_p F_q(A)$.
- The limit sheet $E^\infty := \frac{\cap_{r\geq 1} k^{-1}(\operatorname{im} i^r)}{\cup_{r\geq 1} j(\ker i^r)}$ is described in [57] with isomorphisms $E^\infty_{p,q} \cong \frac{\mathbb{L}F_{p+q}\colon \mathfrak{I}^{p+1}(A)}{\mathbb{L}F_{p+q}\colon \mathfrak{I}^{p}(A)}$.

The *ABC spectral sequence* $\mathrm{ABC}(\mathfrak{T}, \mathfrak{I}, F, A)$ is this spectral sequence $(E^r, d^r)_{r\geq 2}$ starting from the second sheet. As we will shortly see, this is independent of the choice of $\mathfrak{I}$-phantom tower over A up to canonical isomorphism.

7.3. **The morphism of spectral sequences induced by an ABC morphism.** The construction (7.2) of an exact couple from an $\mathfrak{I}$-phantom tower also works for pre-$\mathfrak{I}$-phantom towers and is functorial with respect to tower maps and morphisms of exact couples.

Proposition 7.15. *Let $\mathcal{P}$ be a pre-$\mathfrak{I}$-phantom tower. Then the construction in (7.2) builds an exact couple $EC_{\mathfrak{T},\mathfrak{I},F}(\mathcal{P})$ from $\mathcal{P}$. A tower map $f\colon \mathcal{P} \to \mathcal{Q}$ of pre-$\mathfrak{I}$-phantom towers functorially induces a morphism $EC_{\mathfrak{T},\mathfrak{I},F}(f)\colon EC_{\mathfrak{T},\mathfrak{I},F}(\mathcal{P}) \to EC_{\mathfrak{T},\mathfrak{I},F}(\mathcal{Q})$ of exact couples.*

Proof. We never invoke the $\mathfrak{I}$-projectivity of any P_n, so the construction works equally well for pre-$\mathfrak{I}$-phantom towers. The maps induced by F from a tower map are exactly what is needed to create a morphism of exact couples. It is straightforward to check that this respects composition and identity morphisms. □

This construction of the ABC spectral sequence involves a number of choices, so we recall why it is well-defined. Let $(\mathfrak{T}, \mathfrak{I}, F, A)$ be an ABC tuple and let $\mathcal{P}$ and $\mathcal{Q}$ be two $\mathfrak{I}$-phantom towers over A. By Lemma 6.35 there is a tower map $\widetilde{\mathrm{id}}\colon \mathcal{P} \to \mathcal{Q}$ over the identity map $\mathrm{id}\colon A \to A$. This induces a morphism of exact couples $EC_{\mathfrak{T},\mathfrak{I},F}(\mathcal{P}) \to EC_{\mathfrak{T},\mathfrak{I},F}(\mathcal{Q})$ and therefore a morphism of spectral sequences. Let $\widehat{\mathrm{id}}\colon P_\bullet \to Q_\bullet$ be the restriction of $\widetilde{\mathrm{id}}$ to the projective resolutions within $\mathcal{P}$ and $\mathcal{Q}$, which is unique up to chain homotopy by Lemma 6.33. On the second sheet, the morphism of spectral sequences is given by taking the homology of the chain map $\hat{\mathrm{id}}\colon P_\bullet \to Q_\bullet$, which is a canonical isomorphism independent of $\hat{\mathrm{id}}$. The morphism of spectral sequences is therefore a canonical isomorphism independent of the tower map $\tilde{\mathrm{id}}$ from the second sheet onwards by the mapping lemma. To describe the spectral sequence more absolutely, we may define the second sheet by $E^2_{p,q} = \mathbb{L}_p F_q(A)$ and define the later sheets as subquotients of this.

Lemma 7.16. *Let $(\Phi, \alpha, f)\colon (\mathfrak{T}, \mathfrak{I}, F, A) \to (\mathfrak{T}', \mathfrak{I}', F', A')$ be an ABC cycle and let $\mathcal{P}'$ be an $\mathfrak{I}'$-phantom tower over A'. Then $\Phi(\mathcal{P}')$ is a pre-$\mathfrak{I}$-phantom tower over $\Phi(A')$.*

Proof. The triangulated functor Φ preserves exact triangles, sends $\mathfrak{I}'$-exact sequences to $\mathfrak{I}$-exact sequences and sends $\mathfrak{I}'$-phantom maps to $\mathfrak{I}$-phantom maps. □

Theorem 7.17. *An ABC morphism $\mathfrak{m} = [\Phi, \alpha, f]\colon (\mathfrak{T}, \mathfrak{I}, F, A) \to (\mathfrak{T}', \mathfrak{I}', F', A')$ functorially induces a map*

$$\mathrm{ABC}(\mathfrak{m})\colon\ \mathrm{ABC}(\mathfrak{T}, \mathfrak{I}, F, A) \to \mathrm{ABC}(\mathfrak{T}', \mathfrak{I}', F', A')$$

with the following property. Let $\mathcal{P}$ be an $\mathfrak{I}$-phantom tower over A, let $\mathcal{P}'$ be an $\mathfrak{I}'$-phantom tower over A', and let $\tilde{f}\colon \mathcal{P} \to \Phi(\mathcal{P}')$ be a tower map over $f\colon A \to \Phi(A')$. Let $\alpha_{\mathcal{P}'}\colon EC_{\mathfrak{T},\mathfrak{I},F}(\Phi(\mathcal{P}')) \to EC_{\mathfrak{T}',\mathfrak{I}',F'}(\mathcal{P}')$ be the morphism of exact couples induced by α. Then the morphism of spectral sequences induced by the morphism $\alpha_{\mathcal{P}'} \circ EC_{\mathfrak{T},\mathfrak{I},F}(\tilde{f})\colon EC_{\mathfrak{T},\mathfrak{I},F}(\mathcal{P}) \to EC_{\mathfrak{T}',\mathfrak{I}',F'}(\mathcal{P}')$ of exact couples is given by $\mathrm{ABC}(\mathfrak{m})$ at the second sheet onwards. Furthermore, when restricted to the second sheets, the morphism $\mathrm{ABC}(\mathfrak{m})$ is given by $\mathbb{L}_p(\mathfrak{m})_q\colon \mathbb{L}_p F_q(A) \to \mathbb{L}_p F'_q(A')$.

$$\begin{tikzcd}
N_0 \arrow[rr] \arrow[dd, red] & & N_1 \arrow[rr] \arrow[dl, "\circ" marking] \arrow[dd, red] & & \cdots \arrow[dl, "\circ" marking] & \\
& P_0 \arrow[ul] \arrow[dd, red] & & P_1 \arrow[ll, "\circ" marking] \arrow[ul] \arrow[dd, red] & & \cdots \arrow[ll, "\circ" marking] \arrow[ul] \\
\Phi(N'_0) \arrow[rr] & & \Phi(N'_1) \arrow[rr] \arrow[dl, "\circ" marking] & & \cdots \arrow[dl, "\circ" marking] & \\
& \Phi(P'_0) \arrow[ul] & & \Phi(P'_1) \arrow[ll, "\circ" marking] \arrow[ul] & & \cdots \arrow[ll, "\circ" marking] \arrow[ul]
\end{tikzcd}
\qquad
\begin{tikzcd}
F_*(N_\bullet) \arrow[rr] \arrow[dd, red] & & F_*(N_\bullet) \arrow[dl] \arrow[dd, red] \\
& F_*(P_\bullet) \arrow[ul] \arrow[dd, red] & \\
F_*\Phi(N'_\bullet) \arrow[rr] \arrow[dd, red] & & F_*\Phi(N'_\bullet) \arrow[dl] \arrow[dd, red] \\
& F_*\Phi(P'_\bullet) \arrow[ul] \arrow[dd, red] & \\
F'_*(N'_\bullet) \arrow[rr] & & F'_*(N'_\bullet) \arrow[dl] \\
& F'_*(P'_\bullet) \arrow[ul] &
\end{tikzcd}$$

In red on the left is the tower map $\tilde{f}\colon \mathcal{P} \to \Phi(\mathcal{P}')$, and on the right are the morphisms $EC_{\mathfrak{T},\mathfrak{I},F}(\tilde{f})$ and $\alpha_{\mathcal{P}'}$ of exact couples.

Proof. To construct $\mathrm{ABC}(\mathfrak{m})$, we consider arbitrary phantom towers $\mathcal{P}$ and $\mathcal{P}'$ over A and A' respectively. By Lemma 6.35, there is a tower map $\tilde{f}\colon \mathcal{P} \to \Phi(\mathcal{P}')$ over $f\colon A \to \Phi(A')$. We obtain a morphism of exact couples $\alpha_{\mathcal{P}'} \circ EC_{\mathfrak{T},\mathfrak{I},F}(\tilde{f})$. We may then define $\mathrm{ABC}(\mathfrak{m})$ to be the morphism of spectral sequences induced by this morphism, starting at the second sheet.

Let $\hat{f}\colon P_\bullet \to \Phi(P'_\bullet)$ be the restriction of $\tilde{f}$ to the $\mathfrak{J}$-exact sequence $P_\bullet$ inside $\mathcal{P}$. By the description of the exact couple to spectral sequence functor, the map $\mathrm{ABC}^2_{p,q}(\mathfrak{m})\colon \mathbb{L}_pF_q(A) \to \mathbb{L}_pF_q(A')$ is induced in homology by the chain map $\alpha_{P'} \circ F(\hat{f})$. Theorem 7.9 then tells us that $\mathrm{ABC}^2_{p,q}(\mathfrak{m}) = \mathbb{L}_p(\mathfrak{m})_q$. Morphisms of spectral sequences are determined by their restrictions to the second sheets (Proposition 5.5), and therefore $\mathrm{ABC}(\mathfrak{m})$ is functorial and independent of the choice of $\mathfrak{J}$-phantom tower $\mathcal{P}$ and tower map $\tilde{f}\colon \mathcal{P} \to \Phi(\mathcal{P}')$.

□

When we say that a spectral sequence converges to a graded abelian group, we need a filtration on the graded abelian group to make sense of this. Recall that for a functor $F\colon \mathfrak{T} \to \mathfrak{C}$ (with $\mathfrak{C} = \mathsf{Ab}$ or Ab_*), a homological ideal $\mathfrak{J} \lhd \mathfrak{T}$ and an object A, we have the ascending filtration

$$0 = F\colon \mathfrak{J}^0(A) \subseteq F\colon \mathfrak{J}^1(A) \subseteq F\colon \mathfrak{J}^2(A) \subseteq \cdots \subseteq F(A)$$

of $F(A)$ defined by

$$F\colon \mathfrak{J}^k(A) := \{x \in F(A) \mid F(f)(x) = 0 \text{ for all } f \in \mathfrak{J}^k(A,B), B \in \mathfrak{T}\}.$$

As we want the ABC spectral sequence for $(\mathfrak{T}, \mathfrak{J}, F, A)$ to converge to $\mathbb{L}F(A)$, we will take the filtration $(\mathbb{L}F\colon \mathfrak{J}^k(A))_{k\geq 0}$. The group $\mathbb{L}F(A)$ is defined as $F(L_\mathfrak{J}A)$, so we could alternatively use the filtration $(F\colon \mathfrak{J}^k(L_\mathfrak{J}A))_{k\geq 0}$, but there turns out to be no difference.

Lemma 7.18. *Let $(\mathfrak{T}, \mathfrak{J}, F, A)$ be an ABC tuple and let $L\colon \mathfrak{T} \to \mathfrak{T}$ be its localisation functor. For every $k \geq 0$, we have the equality $\mathbb{L}F\colon \mathfrak{J}^k(A) = F\colon \mathfrak{J}^k(LA)$ of subgroups of $\mathbb{L}F(A)$.*

Proof. First, suppose $x \in F\colon \mathfrak{J}^k(LA)$. Now, let $f \in \mathfrak{J}^k(A,B)$ for some object B. By Proposition 6.48, $Lf \in \mathfrak{J}^k(LA, LB)$ and therefore $F(Lf)(x) = 0$. By definition, $\mathbb{L}F = F \circ L$, so we must have $x \in \mathbb{L}F\colon \mathfrak{J}^k(A)$. Thus $\mathbb{L}F\colon \mathfrak{J}^k(A) \supseteq F\colon \mathfrak{J}^k(LA)$.

Conversely, suppose $x \in \mathbb{L}F\colon \mathfrak{J}^k(A)$ and let $g \in \mathfrak{J}^k(LA, C)$. By TR1 and TR3, we may extend the commutative diagram on the left to a morphism of exact triangles as on the right.

$$\begin{array}{ccccccc} \Sigma^{-1}N_\mathfrak{J}A & \longrightarrow & LA & \longrightarrow & A & \longrightarrow & N_\mathfrak{J}A \\ \| & & \downarrow{\scriptstyle g} & & & & \\ \Sigma^{-1}N_\mathfrak{J}A & \longrightarrow & C & & & & \end{array} \qquad \begin{array}{ccccccc} \Sigma^{-1}N_\mathfrak{J}A & \longrightarrow & LA & \longrightarrow & A & \longrightarrow & N_\mathfrak{J}A \\ \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \| \\ \Sigma^{-1}N_\mathfrak{J}A & \longrightarrow & C & \xrightarrow{h} & D & \longrightarrow & N_\mathfrak{J}A \end{array}$$

Note that h is an $\mathfrak{I}$-equivalence as its cone $N_{\mathfrak{I}}A$ is $\mathfrak{I}$-contractible. Now consider the following diagram.

$$\begin{tikzcd}
LLA \arrow[r, "\cong", "L\mu_A"'] \arrow[d, "Lg"] & LA \arrow[d, "Lf"'] \arrow[ddr, bend left, "g"] & \\
LC \arrow[r, "\cong", "Lh"'] \arrow[drr, bend right, "\mu_C"] & LD \arrow[dr, dashed] & \\
 & & C
\end{tikzcd}$$

The square with LD commutes as it is an application of L to a commutative square. The square with C commutes due to the naturality of μ combined with the fact that $L\mu_A = \mu_{LA}$. The horizontal maps are isomorphisms because μ_A and h are $\mathfrak{I}$-equivalences. By Proposition 6.48, we have $Lg \in \mathfrak{I}^k$, and therefore $Lf \in \mathfrak{I}^k$ and therefore $f \in \mathfrak{I}^k(A, D)$. By definition of $\mathbb{L}F\colon \mathfrak{I}^k(A)$, x vanishes under $F(Lf)$, and as g factors through Lf, we get that $F(g)(x) = 0$, as required to show that $x \in F\colon \mathfrak{I}^k(LA)$. Therefore $\mathbb{L}F\colon \mathfrak{I}^k(A) \subseteq F\colon \mathfrak{I}^k(LA)$. $\square$

Recall that convergence of a spectral sequence E to a graded group G with filtration $(\mathcal{F}_k G)_{k\geq 0}$ has two components:

- For each p and q, isomorphisms
$$E^\infty_{p,q} \cong \frac{\mathcal{F}_{p+1}G_{p+q}}{\mathcal{F}_p G_{p+q}}.$$
- Properties of the filtration. Exhaustiveness is always required, and for strong convergence we need the filtration to be complete Hausdorff.

The phantom filtration $(\mathbb{L}F\colon \mathfrak{I}^k(A))_{k\geq 0}$ of $\mathbb{L}F(A)$ is complete and Hausdorff because it starts with $\mathbb{L}F\colon \mathfrak{I}^0(A) = \{0\}$. The fact that it is exhaustive comes down to the following theorem, which is part of Theorem 5.1 in [57].

Theorem 7.19. *Let $(\mathfrak{T}, \mathfrak{I}, F, A)$ be an ABC tuple such that $A \in \langle \mathfrak{P}_{\mathfrak{I}} \rangle$, the localising subcategory generated by the $\mathfrak{I}$-projective objects. Then $\cup_{k\in\mathbb{N}} F\colon \mathfrak{I}^k(A) = F(A)$.*

Proof. See the proof of Theorem 5.1 in [57]. $\square$

In [57], Meyer shows that the ABC spectral sequence converges to the graded filtered group $\mathbb{L}F(A)$ by constructing isomorphisms
$$\psi_{p,q}\colon \operatorname{ABC}(\mathfrak{T}, \mathfrak{I}, F, A)^\infty_{p,q} \cong \frac{\mathbb{L}F_{p+q}\colon \mathfrak{I}^{p+1}(A)}{\mathbb{L}F_{p+q}\colon \mathfrak{I}^p(A)}.$$
Our aim is to show that this isomorphism is functorial with respect to ABC morphisms. First we show that ABC morphisms respect these filtrations.

Lemma 7.20. *Let $[\Phi, \alpha, f]\colon (\mathfrak{T}, \mathfrak{I}, F, A) \to (\mathfrak{T}', \mathfrak{I}', F', A')$ be an ABC morphism. Then $\operatorname{App}(\Phi, \alpha, f)\left(F\colon \mathfrak{I}^k(A)\right) \subseteq F'\colon \mathfrak{I}'^k(A')$.*

Proof. Let $x \in F\colon \mathfrak{J}^k(A)$, and suppose $g \in \mathfrak{J}'^k(A', B')$ for some $B' \in \mathfrak{T}'$. Then the left hand diagram in ABC is mapped to the right hand diagram by the application functor $\mathrm{App}\colon \mathsf{ABC} \to \mathsf{Ab}_*$.

$$\begin{array}{ccc} (\mathfrak{T}, \mathfrak{J}, F, A) & \xrightarrow{(\Phi,\alpha,f)} & (\mathfrak{T}', \mathfrak{J}', F', A') \\ \Big\downarrow{\scriptstyle (\mathrm{id}_{\mathfrak{T}},\mathrm{id}_F,\Phi(g)\circ f)} & & \Big\downarrow{\scriptstyle (\mathrm{id}_{\mathfrak{T}'},\mathrm{id}_{F'},g)} \\ (\mathfrak{T}, \mathfrak{J}, F, \Phi(B')) & \xrightarrow{(\Phi,\alpha,\mathrm{id}_{\Phi(B')})} & (\mathfrak{T}', \mathfrak{J}', F', B') \end{array} \qquad \begin{array}{ccc} F(A) & \xrightarrow{\mathrm{App}(\Phi,\alpha,f)} & F'(A') \\ \Big\downarrow{\scriptstyle F(\Phi(g)\circ f)} & & \Big\downarrow{\scriptstyle F'(g)} \\ F\Phi(B') & \xrightarrow{\alpha_{B'}} & F'(B') \end{array}$$

The morphism $\Phi(g) \circ f$ is in $\mathfrak{J}^k$, so $F(\Phi(g) \circ f)(x) = 0$. The diagrams commute, so $F'(g) \circ \mathrm{App}(\Phi, \alpha, f)(x) = 0$. We can conclude that $\mathrm{App}(\Phi, \alpha, f)(x) \in F'\colon \mathfrak{J}'^k(A')$. □

As the application $\mathrm{App}(\Phi, \alpha, f)\colon FA \to F'A'$ respects the filtrations it induces maps of subquotients

$$\mathrm{App}(\Phi, \alpha, f)_{p,q}\colon \frac{\mathbb{L}F_{p+q}\colon \mathfrak{J}^{p+1}(A)}{\mathbb{L}F_{p+q}\colon \mathfrak{J}^{p}(A)} \to \frac{\mathbb{L}F'_{p+q}\colon (\mathfrak{J}')^{p+1}(A')}{\mathbb{L}F'_{p+q}\colon (\mathfrak{J}')^{p}(A')}.$$

The double index distinguishes an induced map of subquotients from the map $\mathrm{App}(\Phi, \alpha, f)_n\colon F_nA \to F'_nA'$ induced by compatibility with the grading. If we take an ABC morphism $\mathfrak{m}$ and apply Lemma 7.20 to the ABC morphism $\mathfrak{L}(\mathfrak{m})$, we deduce that $\mathbb{L}(\mathfrak{m})$ respects the phantom filtrations in the sense that $\mathbb{L}(\mathfrak{m})\left(\mathbb{L}F\colon \mathfrak{J}^k(A)\right) \subseteq \mathbb{L}F'\colon \mathfrak{J}'^k(A')$. We now want to get a handle on the limit sheet of the spectral sequence.

Lemma 7.21. *Let (C, D, i, j, k) be an exact couple and let (E, d) be the associated spectral sequence. Recall that we have the following descriptions.*

$$\begin{tikzcd} D \arrow[rr, "{(1,-1)}", "i"'] & & D \arrow[dl, "j", "{(0,0)}"'] \\ & C \arrow[ul, "k", "{(1,0)}"'] & \end{tikzcd} \qquad E^{\infty}_{p,q} := \frac{\bigcap_{r\geq 1} Z^r_{p,q}}{\bigcup_{r\geq 1} B^r_{p,q}} = \frac{\bigcap_{r\geq 1}(k^{-1}(i^r D))_{p,q}}{\bigcup_{r\geq 1}(j(\ker i^r))_{p,q}}.$$

Then there is a homomorphism

$$E^{\infty}_{p,q} \to \frac{(\ker i^{p+1})_{-1,p+q}}{(\ker i^{p})_{-1,p+q}}$$

that sends each $[x] \in E^{\infty}_{p,q}$ to the unique $[y] \in \frac{(\ker i^{p+1})_{-1,p+q}}{(\ker i^{p})_{-1,p+q}}$ such that $i^p(y) = k(x)$.

Proof. We first note that $[x] \mapsto k(x)$ and $[y] \mapsto i^p(y)$ are well-defined. Uniqueness of $[y]$ follows from the fact that it lives in a quotient by $\ker i^p$. For existence, x is an element of $(k^{-1}(i^pD))_{p,q}$, so there is some $y \in D_{-1,p+q}$ such that $k(x) = i^p(y)$. Because $i \circ k = 0$, it follows that $y \in \ker i^{p+1}$. □

Suppose we have an ABC tuple $\mathfrak{M} = (\mathfrak{T}, \mathfrak{J}, F, A)$ and an $\mathfrak{J}$-phantom tower $\mathcal{P}$ over A, with associated exact couple $(C, D, i, j, k) = EC_{\mathfrak{T},\mathfrak{J},F}(\mathcal{P})$. Lemma 6.37 tells us that

$$\frac{(\ker i^{p+1})_{-1,p+q}}{(\ker i^{p})_{-1,p+q}} = \frac{F_{p+q}\colon \mathfrak{J}^{p+1}(A)}{F_{p+q}\colon \mathfrak{J}^{p}(A)}.$$

We can then apply Lemma 7.21 to obtain a map $\varphi^{\mathfrak{M}}_{p,q}\colon \operatorname{ABC}(\mathfrak{M})^{\infty}_{p,q} \to \frac{F_{p+q}\colon \mathfrak{J}^{p+1}(A)}{F_{p+q}\colon \mathfrak{J}^{p}(A)}$ which a priori might depend on $\mathcal{P}$. One consequence of the following proposition is that it does not.

Proposition 7.22. *Let $\mathfrak{M} = (\mathfrak{T}, \mathfrak{J}, F, A)$ and $\mathfrak{M}' = (\mathfrak{T}', \mathfrak{J}', F', A')$ be ABC tuples and let $\mathfrak{m}\colon \mathfrak{M} \to \mathfrak{M}'$ be an ABC morphism. Then using any choices $\mathcal{P}$ and $\mathcal{P}'$ of phantom towers over A and A' to define $\varphi^{\mathfrak{M}}_{p,q}$ and $\varphi^{\mathfrak{M}'}_{p,q}$ respectively, the following diagram commutes.*

$$\begin{tikzcd} \operatorname{ABC}(\mathfrak{M})^{\infty}_{p,q} \arrow[r, "\varphi^{\mathfrak{M}}_{p,q}"] \arrow[d, "\operatorname{ABC}(\mathfrak{m})^{\infty}_{p,q}"] & \frac{F_{p+q}\colon \mathfrak{J}^{p+1}(A)}{F_{p+q}\colon \mathfrak{J}^{p}(A)} \arrow[d, "\operatorname{App}(\mathfrak{m})_{p,q}"] \\ \operatorname{ABC}(\mathfrak{M}')^{\infty}_{p,q} \arrow[r, "\varphi^{\mathfrak{M}'}_{p,q}"] & \frac{F'_{p+q}\colon \mathfrak{J}'^{p+1}(A')}{F'_{p+q}\colon \mathfrak{J}'^{p}(A')} \end{tikzcd} \tag{7.3}$$

Proof. Suppose $\mathfrak{m} = [\Phi, \alpha, f]$ and let $\tilde{f}\colon \mathcal{P} \to \Phi(\mathcal{P}')$ be a tower map over $f\colon A \to \Phi(A')$. Let (C, D, i, j, k) and (C', D', i', j', k') be the exact couples built from $\mathcal{P}$ and $\mathcal{P}'$ respectively. Let $S\colon C \to C'$ and $T\colon D \to D'$ be the constituent maps of the morphism $\alpha_{\mathcal{P}'} \circ EC_{\mathfrak{T},\mathfrak{J},F}(\tilde{f})$ of exact couples. Then for each n, we have $D_{-1,n} = F_n(A)$ and $D'_{-1,n} = F'_n(A')$. Furthermore, $T_{-1,n}\colon D_{-1,n} \to D'_{-1,n}$ is given by the map $\operatorname{App}(\mathfrak{m})_n\colon F_n(A) \to F'_n(A')$.

$$\begin{tikzcd} D \arrow[rr, "i"] \arrow[dd, "T"] & & D \arrow[dl, "j"] \arrow[dd, "T"] \\ & C \arrow[ul, "k"] \arrow[dd, "S"] & \\ D' \arrow[rr, "i'"] & & D' \arrow[dl, "j'"] \\ & C' \arrow[ul, "k'"] & \end{tikzcd} \qquad \begin{tikzcd} D_{-1,n} \arrow[r, phantom, "="] \arrow[d, "T_{-1,n}"] & F_n(A) \arrow[d, "\operatorname{App}(\mathfrak{m})_n"] \\ D'_{-1,n} \arrow[r, phantom, "="] & F'_n(A') \end{tikzcd}$$

Now, consider $[x] \in E^{\infty}_{p,q} = \frac{\bigcap_{r\geq 1}(k^{-1}(i^r D))_{p,q}}{\bigcup_{r\geq 1}(j(\ker i^r))_{p,q}}$. Let $y \in F_{p+q}\colon \mathfrak{J}^{p+1}(A)$ be such that $\varphi^{\mathfrak{M}}_{p,q}([x]) = [y]$. Then because $y \in D_{-1,p+q} = F_{p+q}(A)$, we may use that $\operatorname{App}(\mathfrak{m})_{p+q} = D_{-1,p+q}$ to calculate that

$$\operatorname{App}(\mathfrak{m})_{p,q}(\varphi^{\mathfrak{M}}_{p,q}([x])) = \operatorname{App}(\mathfrak{m})_{p,q}([y]) = \big[\operatorname{App}(\mathfrak{m})_{p+q}(y)\big] = [T(y)].$$

On the other hand, $\operatorname{ABC}(\mathfrak{m})^{\infty}_{p,q}([x]) = [S(x)]$. We may calculate that $i'^{p}(T(y)) = T(i^p(y)) = T(k(x)) = k'(S(x))$, and therefore by Lemma 7.21, $\varphi^{\mathfrak{M}'}_{p,q}([S(x)]) = [T(y)]$. We conclude that (7.3) commutes. ∎

Lemma 7.23. *Let $\mathfrak{M} = (\mathfrak{T}, \mathfrak{I}, F, A)$ be an ABC tuple such that A is in $\langle \mathfrak{P}_{\mathfrak{I}} \rangle$. Then $\varphi^{\mathfrak{M}}_{p,q}$ is an isomorphism.*

Proof. Let $\mathcal{P} = (P_\bullet, N_\bullet, \epsilon_\bullet, \iota_\bullet, \pi_\bullet)$ be an $\mathfrak{I}$-phantom tower over A, with associated exact couple $(C, D, i, j, k) = EC_{\mathfrak{T},\mathfrak{I},F}(\mathcal{P})$.

First we show that $\varphi^{\mathfrak{M}}_{p,q}$ is surjective. Recall that $D_{p-1,q} = F_{p+q}(N_p)$, and that therefore for $r \geq p$, $i^r = i^q = (\iota^p_0)_*\colon D_{p-r-1,q+r} = F_{p+q}(A) \to D_{p-1,q} = F_{p+q}(N_p)$. Therefore $(i^r D)_{p-1,q} = (i^p D)_{p-1,q}$. Now, for any $y \in (\ker i^{p+1})_{-1,p+q}$, we can pick $z \in (\ker i)_{p-1,q} = (\operatorname{im} k)_{p-1,q}$ such that $i^p(z) = y$. We can then pick an element $x \in (k^{-1}(i^p D))_{p,q}$ with $k(x) = z = i^p(y)$. Then for each $r \geq p$, we have $x \in k^{-1}((i^r D))_{p,q} = k^{-1}((i^p D))_{p,q}$, and so $[x] \in \mathrm{ABC}(\mathfrak{M})^\infty_{p,q}$. Then $\varphi^{\mathfrak{M}}_{p,q}([x]) = [y]$, so we have proved surjectivity.

For injectivity, we first note that because of the exact triangles in the phantom tower, we can inductively prove that $N_n \in \langle \mathfrak{P}_{\mathfrak{I}} \rangle$ because $N_0 = A \in \langle \mathfrak{P}_{\mathfrak{I}} \rangle$ and each $P_n \in \langle \mathfrak{P}_{\mathfrak{I}} \rangle$. Then by Theorem 7.19, $D_{p,q} = \cup_{r\geq 1} F_{p+q+1}\colon \mathfrak{I}^r(N_{p+1})$, which by Lemma 6.37 is $\cup_{r\geq 1}(\ker i^r)_{p,q}$. Now, suppose $x, x' \in (k^{-1}(i^p D))_{p,q}$ are such that $\varphi^{\mathfrak{M}}_{p,q}([x]) = \varphi^{\mathfrak{M}}_{p,q}([x'])$. Then $k(x) = k(x')$, so $x - x' \in (j(D))_{p,q} = \cup_{r\geq 1}(j(\ker i^r))_{p,q}$. We can conclude that $[x] = [x']$. □

Lemma 7.24. *Let $(\mathfrak{T}, \mathfrak{I}, F, A)$ be an ABC tuple, let $L\colon \mathfrak{T} \to \mathfrak{T}$ be the localisation functor and let $\mu\colon L \Rightarrow \mathrm{id}_{\mathfrak{T}}$ be the localisation natural transformation. Then the ABC morphism $[\mathrm{id}_{\mathfrak{T}}, \mathrm{id}_F, \mu_A]\colon (\mathfrak{T}, \mathfrak{I}, F, A) \to (\mathfrak{T}, \mathfrak{I}, F, LA)$ induces an isomorphism of ABC spectral sequences.*

Proof. By Remark 7.10, when we keep the category $\mathfrak{T}$, ideal $\mathfrak{I}$ and functor F constant, we recover the derived functors, so that

$$\mathbb{L}_p(\mathrm{id}_{\mathfrak{T}}, \mathrm{id}_F, \mu_A)_q = \mathbb{L}_p F_q(\mu_A)\colon \mathbb{L}_p F_q(LA) \to \mathbb{L}_p F_q(A).$$

This is an isomorphism because μ_A is an $\mathfrak{I}$-equivalence. The induced map on the second page is therefore an isomorphism. By the mapping lemma, $\mathrm{ABC}(\mathrm{id}_{\mathfrak{T}}, \mathrm{id}_F, \mu_A)$ is an isomorphism of spectral sequences. □

Combining Lemmas 7.18, 7.23 and 7.24, the isomorphism $\psi^{\mathfrak{M}}_{p,q}\colon \mathrm{ABC}(\mathfrak{M})^\infty_{p,q} \cong \frac{\mathbb{L}F_{p+q}\colon \mathfrak{I}^{p+1}(A)}{\mathbb{L}F_{p+q}\colon \mathfrak{I}^{p}(A)}$ is defined as the composition

$$\mathrm{ABC}(\mathfrak{M})^\infty_{p,q} \cong \mathrm{ABC}(\mathfrak{L}(\mathfrak{M}))^\infty_{p,q} \cong \frac{F_{p+q}\colon \mathfrak{I}^{p+1}(LA)}{F_{p+q}\colon \mathfrak{I}^{p}(LA)} = \frac{\mathbb{L}F_{p+q}\colon \mathfrak{I}^{p+1}(A)}{\mathbb{L}F_{p+q}\colon \mathfrak{I}^{p}(A)}.$$

This recovers the result from [57] that the ABC spectral sequence $\mathrm{ABC}(\mathfrak{M})$ strongly converges to the filtered graded group $\mathbb{L}(\mathfrak{M}) = \mathbb{L}F(A)$. We have also shown that $\mathbb{L}(\mathfrak{m})\colon \mathbb{L}(\mathfrak{M}) \to \mathbb{L}(\mathfrak{M}')$ is a morphism of filtered graded groups. We can now combine all of the work in this section to conclude its main result.

Theorem (Theorem 7.14)**.** *An ABC morphism* $\mathfrak{m}\colon \mathfrak{M}\to\mathfrak{M}'$ *induces functorially a morphism of ABC spectral sequences* $\mathrm{ABC}(\mathfrak{m})\colon \mathrm{ABC}(\mathfrak{M})\to\mathrm{ABC}(\mathfrak{M}')$*, such that:*

(i) the map on the second sheet is given by $\mathbb{L}_p(\mathfrak{m})_q\colon \mathbb{L}_p(\mathfrak{M})_q\to\mathbb{L}_p(\mathfrak{M}')_q$*,*

(ii) the map on the limit sheet agrees with $\mathbb{L}(\mathfrak{m})\colon\mathbb{L}(\mathfrak{M})\to\mathbb{L}(\mathfrak{M}')$*.*

Proof. The construction of the morphism $\mathrm{ABC}(\mathfrak{m})\colon \mathrm{ABC}(\mathfrak{M})\to\mathrm{ABC}(\mathfrak{M}')$ was given in Theorem 7.17, where it was shown to be functorial and agree with $\mathbb{L}_p(\mathfrak{m})_q$ on the second sheets.

The map that $\mathrm{ABC}(\mathfrak{m})$ induces on the limit sheet agrees with $\mathbb{L}(\mathfrak{m})$ because the following diagram commutes, where $\mathfrak{M}=(\mathfrak{T},\mathfrak{I},F,A)$ and $\mathfrak{M}'=(\mathfrak{T}',\mathfrak{I}',F',A')$.

$$\begin{tikzcd}
\mathrm{ABC}(\mathfrak{M})^\infty_{p,q} \arrow[d, "\mathrm{ABC}(\mathfrak{m})^\infty_{p,q}"] & \mathrm{ABC}(\mathfrak{L}(\mathfrak{M}))^\infty_{p,q} \arrow[l, "\cong"'] \arrow[r, "\cong", "\varphi^{\mathfrak{M}}_{p,q}"'] \arrow[d, "\mathrm{ABC}(\mathfrak{L}(\mathfrak{m}))^\infty_{p,q}"] & \frac{F_{p+q}\colon \mathfrak{I}^{p+1}(LA)}{F_{p+q}\colon \mathfrak{I}^{p}(LA)} \arrow[r, phantom, "="] \arrow[d, "\mathrm{App}(\mathfrak{L}(\mathfrak{m}))_{p,q}"] & \frac{\mathbb{L}F_{p+q}\colon \mathfrak{I}^{p+1}(A)}{\mathbb{L}F_{p+q}\colon \mathfrak{I}^{p}(A)} \arrow[d, "\mathbb{L}(\mathfrak{m})_{p,q}"] \\
\mathrm{ABC}(\mathfrak{M}')^\infty_{p,q} & \mathrm{ABC}(\mathfrak{L}(\mathfrak{M}'))^\infty_{p,q} \arrow[l, "\cong"'] \arrow[r, "\cong", "\varphi^{\mathfrak{M}'}_{p,q}"'] & \frac{F'_{p+q}\colon \mathfrak{I}'^{p+1}(L'A')}{F'_{p+q}\colon \mathfrak{I}'^{p}(L'A')} \arrow[r, phantom, "="] & \frac{\mathbb{L}'F'_{p+q}\colon \mathfrak{I}'^{p+1}(A')}{\mathbb{L}'F'_{p+q}\colon \mathfrak{I}'^{p}(A')}
\end{tikzcd}$$

The left square commutes by Proposition 7.12, and the middle square commutes by Proposition 7.22. Finally, the right square commutes as $\mathbb{L}(\mathfrak{m})=\mathrm{App}(\mathfrak{L}(\mathfrak{m}))$ by definition. ☐

7.4. **Application to isomorphisms in K-theory.** In general, the ABC spectral sequence converges to the localisation $\mathbb{L}F(A)$ rather than the application $F(A)$ that we are interested in. However, the localisation natural transformation $\mu_{\mathfrak{I}}$ gives us a canonical *assembly map* $\mu_{\mathfrak{M}}$ defined by

$$\mu_{\mathfrak{M}} := F(\mu_{\mathfrak{I}}(A))\colon \mathbb{L}F(A)\to F(A).$$

In favourable settings, this assembly map can be an isomorphism. The Baum-Connes assembly map for an étale groupoid G with coefficients in a C*-algebra A is isomorphic to the assembly map for the ABC tuple $(\mathrm{KK}^G,\mathfrak{I}_{\mathcal{F}},K_*(G\ltimes_r -),A)$, where $\mathcal{F}$ is a countable family of proper open subgroupoids of G satisfying condition (P), as discussed in Section 6.5. The assembly maps are compatible with ABC morphisms in the following way.

Proposition 7.25. *For each ABC morphism* $\mathfrak{m}=[\Phi,\alpha,f]\colon\mathfrak{M}=(\mathfrak{T},\mathfrak{I},F,A)\to\mathfrak{M}'=(\mathfrak{T}',\mathfrak{I}',F',A')$*, the following diagram commutes.*

$$\begin{tikzcd}
\mathbb{L}F(A) \arrow[r, "\mu_{\mathfrak{M}}"] \arrow[d, "\mathbb{L}(\mathfrak{m})"] & F(A) \arrow[d, "\mathrm{App}(\mathfrak{m})"] \\
\mathbb{L}F'(A') \arrow[r, "\mu_{\mathfrak{M}'}"] & F'(A')
\end{tikzcd}$$

Proof. This follows by combining the application functor $\mathrm{App}\colon \mathsf{ABC}\to\mathsf{Ab}_*$ and Proposition 7.12. ☐

To get an ABC morphism from a general proper groupoid correspondence $\Omega\colon G \to H$, we use the universal crossed product because it has greater functoriality properties. However, for the Baum-Connes assembly map we use the reduced crossed product. This is thankfully not a problem, because the reduced and universal crossed products agree on the localising subcategory generated by proper G-C*-algebras, so the localisations $\mathbb{L}K_*(G \ltimes_r A)$ and $\mathbb{L}K_*(G \ltimes A)$ coincide. This also means that any ABC spectral sequences we construct will not see the difference between the full and reduced crossed products. We use this to our advantage, exploiting useful properties of the full crossed product in order to obtain results about the reduced crossed product. We obtain our main corollary of Theorem 7.14:

Corollary 7.26. *Let G and H be étale groupoids satisfying the Baum-Connes conjecture with coefficients in C*-algebras A and B, and let $(E,\Omega)\colon (A,G) \to (B,H)$ be a separable proper correspondence. Let $\mathcal{F}_G$ and $\mathcal{F}_H$ be families of proper open subgroupoids of G and H respectively satisfying condition (P) such that $\mathrm{Ind}_\Omega(\mathfrak{I}_{\mathcal{F}_H}) \subseteq \mathfrak{I}_{\mathcal{F}_G}$, so that we get an ABC morphism*

$$[\mathrm{Ind}_\Omega, \alpha_\Omega, f_E]\colon (\mathrm{KK}^G, \mathfrak{I}_{\mathcal{F}_G}, K_*(G \ltimes -), A) \to (\mathrm{KK}^H, \mathfrak{I}_{\mathcal{F}_H}, K_*(H \ltimes -), B).$$

Suppose that for each $n \geq 0$ the derived functor map

$$\mathbb{L}_n(\mathrm{Ind}_\Omega, \alpha_\Omega, f_E)\colon \mathbb{L}_n K_*(G \ltimes A) \to \mathbb{L}_n K_*(H \ltimes B)$$

is an isomorphism. Then $K_*(G \ltimes_r A) \cong K_*(H \ltimes_r B)$.

This becomes much easier to apply when the groupoids have torsion-free isotropy groups, because we may take the families $\mathcal{F}_G$ and $\mathcal{F}_H$ to be the singletons $\{G^0\}$ and $\{H^0\}$, in which case we set $\mathfrak{I}_0^G$ and $\mathfrak{I}_0^H$ to be the associated homological ideals. When G and H are further ample, this enters the setting of Proietti and Yamashita's spectral sequence from [71]. This has the added advantage that the derived functors $\mathbb{L}_p K_q(G \ltimes A)$ are given by the groupoid homology $H_p(G; K_q(A))$. In order for a proper groupoid correspondence $\Omega\colon G \to H$ to give us an ABC morphism in this setting, we need $\mathrm{Ind}_\Omega(\mathfrak{I}_0^H) \subseteq \mathfrak{I}_0^G$. This turns out to be automatic because of second countability and totally disconnectedness. This means that there is an étale section $s\colon \Omega/H \to \Omega$ of the quotient map, from which we can build a proper correspondence $\Omega/H\colon G^0 \to H^0$ of the unit spaces, with right anchor map $\sigma \circ s\colon \Omega/H \to H^0$. This gives us a commutative diagram

$$\begin{tikzcd}
\mathrm{KK}^H \arrow[r, "\mathrm{Ind}_\Omega"] \arrow[d, "\mathrm{Res}_H^{H^0}"] & \mathrm{KK}^G \arrow[d, "\mathrm{Res}_G^{G^0}"] \\
\mathrm{KK}^{H^0} \arrow[r, "\mathrm{Ind}_{\Omega/H}"] & \mathrm{KK}^{G^0},
\end{tikzcd}$$

which implies that $\mathrm{Ind}_\Omega(\mathfrak{I}_0^H) \subseteq \mathfrak{I}_0^G$. Thus every proper groupoid correspondence $(E,\Omega)\colon (A,G) \to (B,H)$ gives us an ABC morphism

$$[\mathrm{Ind}_\Omega, \alpha_\Omega, f_E]\colon (\mathrm{KK}^G, \mathfrak{I}_0^G, K_*(G \ltimes -), A) \to (\mathrm{KK}^H, \mathfrak{I}_0^H, K_*(H \ltimes -), B).$$

We want to understand the derived functor maps, and we already have a description of the derived functors as groupoid homology in [71]. It turns out that the derived functor maps agree with the induced map in homology from Theorem 4.32.

Proposition 7.27. *Let $(E,\Omega)\colon (A,G)\to(B,H)$ be a separable proper correspondence of ample groupoids with C^*-coefficients. Let $\mathfrak{m}=[\mathrm{Ind}_\Omega,\alpha_\Omega,f_E]$ be the associated ABC morphism. Then the induced map in homology $H_{p,q}(E,\Omega)$ agrees with the derived functor application $\mathbb{L}_p(\mathfrak{m})_q$.*

$$\begin{tikzcd}
H_p(G;K_q(A)) \arrow[r, "H_{p,q}(E,\Omega)"] \arrow[d, "\cong"] & H_p(H;K_q(B)) \arrow[d, "\cong"] \\
\mathbb{L}_pK_q(G\ltimes A) \arrow[r, "\mathbb{L}_p(\mathfrak{m})_q"] & \mathbb{L}_pK_q(H\ltimes B)
\end{tikzcd}$$

Proof. Let $P_n=(\mathrm{Ind}_{G^0}^G\mathrm{Res}_G^{G^0})^{n+1}A$ and $Q_n=(\mathrm{Ind}_{H^0}^H\mathrm{Res}_H^{H^0})^{n+1}B$ and consider the $\mathfrak{I}_0^G$-projective resolutions $P_\bullet\to A$ and $Q_\bullet\to B$ as in [71]. The definition of $\mathbb{L}_n(\mathfrak{m})$ comes from a chain map $f\colon P_\bullet\to\mathrm{Ind}_\Omega Q_\bullet$ over $\Delta(E)\colon A\to\mathrm{Ind}_\Omega B$ as follows.

$$\begin{tikzcd}
\cdots \arrow[r] & P_n \arrow[r] \arrow[d, "f_n"] & \cdots \arrow[r] & P_0 \arrow[d, "f_0"] \\
\cdots \arrow[r] & \mathrm{Ind}_\Omega Q_n \arrow[r] & \cdots \arrow[r] & \mathrm{Ind}_\Omega Q_0
\end{tikzcd}$$

From this we construct the following diagram, applying $K_*(G\ltimes -)$ and α_Ω.

$$\begin{tikzcd}
\cdots \arrow[r] & K_*(G\ltimes P_n) \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d, "K_*(G\ltimes -)"] & K_*(G\ltimes P_0) \arrow[d] \\
\cdots \arrow[r] & K_*(G\ltimes \mathrm{Ind}_\Omega Q_n) \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d, "\alpha_\Omega"] & K_*(G\ltimes \mathrm{Ind}_\Omega Q_0) \arrow[d] \\
\cdots \arrow[r] & K_*(H\ltimes Q_n) \arrow[r] & \cdots \arrow[r] & K_*(H\ltimes Q_0).
\end{tikzcd}$$

By Shapiro's Lemma (Lemma 4.38), $K_*(P_n)$ is a Coinv_G-acyclic G-module. The chain complexes $K_*(P_\bullet)$ and $K_*(\mathrm{Ind}_\Omega Q_\bullet)$ are exact because $\ker K_*\supseteq\mathfrak{I}_0^G$. Therefore

$$\begin{tikzcd}
\cdots \arrow[r] & K_*(P_n) \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d, "K_*"] & K_*(P_1) \arrow[d] \\
\cdots \arrow[r] & K_*(\mathrm{Ind}_\Omega Q_n) \arrow[r] & \cdots \arrow[r] & K_*(\mathrm{Ind}_\Omega Q_1)
\end{tikzcd}$$

is a chain map of the form discussed in Theorem 4.32. To finish we apply Proposition 4.36 and Proposition 4.37 to see that the following diagram commutes.

$$\begin{tikzcd}
K_*(P_n)_G \arrow[r, "(\delta_\Omega\otimes\mathrm{id})\circ K_*(f_n)_G"] \arrow[d, "\cong"] & K_*(Q_n)_H \arrow[d, "\cong"] \\
K_*(G\ltimes P_n) \arrow[r, "\alpha_{\Omega,\mathrm{Ind}_\Omega Q_n}\circ K_*(G\ltimes f_n)"] & K_*(H\ltimes Q_n)
\end{tikzcd}$$

These chain maps induce $H_{p,q}(E,\Omega)$ and $\mathbb{L}_p(\mathfrak{m})_q$ respectively, so we are done. $\square$

We may finally combine Corollary 7.26 and Proposition 7.27 to turn isomorphisms in homology into isomorphisms in K-theory:

Corollary 7.28. *Let $(E, \Omega)\colon (A, G) \to (B, H)$ be a separable proper correspondence of étale groupoids with C^*-coefficients such that G and H are ample groupoids with torsion-free isotropy groups satisfying the Baum-Connes conjecture with coefficients in A and B respectively. Suppose that the induced maps in homology*

$$H_{*,i}(E, \Omega)\colon H_*(G; K_i(A)) \to H_*(H; K_i(B))$$

are isomorphisms. Then there is an isomorphism $K_*(G \ltimes_r A) \cong K_*(H \ltimes_r B)$.

Without coefficients, this says the following.

Corollary 7.29. *Let G and H be second countable ample groupoids with torsion-free isotropy groups satisfying the Baum-Connes conjecture and let $\Omega\colon G \to H$ be a second countable proper correspondence. Suppose that the induced map in homology $H_*(\Omega)\colon H_*(G) \to H_*(H)$ is an isomorphism. Then there is an isomorphism $K_*(C_r^*(G)) \to K_*(C_r^*(H))$.*

8. Orbit-stabiliser K-theory formula

This chapter is dedicated to the proof of the following theorem.

Theorem 8.1 (Orbit-stabiliser K-theory formula)**.** *Let S be a countable inverse semigroup and let I be a countable locally finite weak semilattice, with an action $S \curvearrowright I$ by order-preserving bijections of down-sets. Let Y be the space of filters on I, which inherits an action of S. Suppose that the transformation groupoid $S \ltimes Y$ is Hausdorff, has torsion-free isotropy groups and satisfies the Baum-Connes conjecture. Then we have an isomorphism*

$$K_*(C_r^*(S \ltimes Y)) \cong \bigoplus_{[i] \in S \backslash I^\times} K_*(C_r^*(\mathrm{Stab}_i(S))). \tag{8.1}$$

Let us first discuss the terms that appear in this statement. Weak semilattices are defined by Steinberg in [81], and relate to finite alignment for left cancellative small categories [80].

Definition 8.2 (Locally finite weak semilattice)**.** A *weak semilattice* P is a poset (with a 0 element) such that for each $p, q \in P$, there is a finite set $F \subseteq P$ such that for $r \in P$ we have $r \leq p$ and $r \leq q$ if and only if $r \leq f$ for some $f \in F$. In other words, the intersection of principal down-sets is a finite union of principal down-sets. The *join* $p \downarrow q$ of p and q is the set of maximal elements below both p and q: this must then be finite. We say that P is *locally finite* if the closure of any finite subset $F \subseteq P$ under $\downarrow$ remains finite.

Morally we view locally finite weak semilattices as a modest generalisation of semilattices, with the join binary operation replaced by the finite-set-valued join $\downarrow$. Proofs become a little more technical but not usually more difficult. As result, not much is lost by reading this and replacing every instance of "locally finite weak semilattice" with "semilattice". Our motivating example comes from a higher rank graph Λ, where we write $p \leq q$ if a path $p \in \Lambda$ extends a path $q \in \Lambda$. Finite alignment of Λ says exactly that this is a weak semilattice. Here any join is a finite set of pairwise orthogonal paths, which implies local finiteness. We now turn our attention to the space of filters.

Definition 8.3 (Space of filters)**.** Let I be a weak semilattice. A *filter* on I is a non-empty downward directed up-set not containing 0. We equip the set $\hat{I}$ of filters with the topology inherited from the product topology on $\{0,1\}^I$.

For each non-zero element $i \in I^\times$ the set U_i of filters containing i is a compact open subset of $\hat{I}$. This gives an open cover of $\hat{I}$.

Now suppose we have an action $S \curvearrowright I$ of an inverse semigroup S on a locally finite weak semilattice I by order-preserving bijections of down-sets. This induces an action $S \curvearrowright \hat{I}$ of S by partial homeomorphisms in the following way. The domain $\mathrm{dom}_{\hat{I}}\, s$ of $s \in S$ is $\bigcup_{i \in \mathrm{dom}_I s} U_i$, and the filter $s \cdot x$ is given by the upward closure of the downward directed set $\{s \cdot j \mid j \in \mathrm{dom}\, s \cap x\}$. We may assume without loss of generality that each $i \in I$ is in the domain of some $s \in S$, and that therefore each $x \in \hat{I}$ is in the domain of some $s \in S$.

The stabiliser group $\mathrm{Stab}_i(S)$ of an element $i \in I^\times$ is defined as the isotropy group of $S \ltimes I^\times$ at i. This can often be viewed as a subgroup of S. Thus the right hand side of (8.1) is the K-theory of the C*-algebra of the discrete groupoid $S \ltimes I^\times$. To prove Theorem 8.1 we construct a proper groupoid correspondence $\Omega\colon S \ltimes I^\times \to S \ltimes Y$ and prove that it induces an isomorphism in the K-theory of the C*-algebras. We have a proper topological correspondence given by

$$I^\times \leftarrow \bigsqcup_{i \in I^\times} U_i \to Y,$$

where the left map picks out the indexing element of $I^\times$, and the right map includes each U_i into Y. The left map is proper because each U_i is compact, and the right map is a local homeomorphism because each U_i is open. Furthermore, the inverse semigroup S acts on $\bigsqcup_{i \in I^\times} U_i$ by setting $s \cdot (i, y) := (s \cdot i, s \cdot y)$ whenever $i \in \mathrm{dom}_I\, s$ and $y \in \mathrm{dom}_Y\, s$. Both maps in the proper topological correspondence are S-equivariant, and we obtain a proper correspondence of groupoids

$$S \ltimes I^\times \qquad \curvearrowright \qquad S \ltimes \bigsqcup_{i \in I^\times} U_i \qquad \curvearrowleft \qquad S \ltimes Y.$$

For the rest of this chapter, we set $X = I^\times$, $G = S \ltimes X$, $H = S \ltimes Y$ and $\Omega = S \ltimes \bigsqcup_{i \in X} U_i$. Our aim is to use Corollary 7.29 to show that when $S \ltimes Y$ has torsion-free isotropy groups, is Hausdorff and satisfies Baum-Connes then $\Omega\colon G \to H$ induces an isomorphism in K-theory $C_r^*(G) \cong C_r^*(H)$, thus proving Theorem 8.1.

It suffices to show that Ω induces an isomorphism in homology. We may decompose $\Omega\colon G \to H$ into the action correspondence $\Omega\colon G \to S \ltimes \bigsqcup_{i \in X} U_i$ and the étale homomorphism $S \ltimes \bigsqcup_{i \in X} U_i \to H$. By combining Examples 4.33 and 4.34, we can see that $H_*(\Omega)$ is induced by the chain map

$$\mathbb{Z}[G^n] \to \mathbb{Z}[H^n]$$

$$\chi_{\{\gamma\}} \mapsto \chi_{V_\gamma},$$

where for each $\gamma = [s_n, i_n, \dots, s_1, i_1] \in G^n$, the compact open subset $V_\gamma \subseteq H^n$ is given by

$$V_\gamma := \{[s_n, y_n, \dots, s_1, y_1] \mid i_k \in y_k \text{ for each } k\}.$$

The element γ is determined by $s_1, \dots, s_n$ and the initial element i_1 of X, as $s_k \cdot i_k = i_{k+1}$ for each k. Similarly, each element of V_γ is determined by its initial entry y_1, which defines a homeomorphism $V_\gamma \cong U_{i_1}$. For $n = 0$, we have $G^0 = X$ and $V_i = U_i \subseteq H^0 = Y$.

Proposition 8.4. *For each $n \geq 0$ the map $\chi_{\{\gamma\}} \mapsto \chi_{V_\gamma} \colon \mathbb{Z}[G^n] \to \mathbb{Z}[H^n]$ is an isomorphism.*

Proof. We show that $\mathcal{B} := \{\chi_{V_\gamma} \mid \gamma \in G^n\}$ forms a $\mathbb{Z}$-basis for $\mathbb{Z}H^n$, and make use of the following natural partial order on G^n. We define $[s_n, i_n, \dots, s_1, i_1] \leq [t_n, j_n, \dots, t_1, j_1]$ if $i_1 \leq j_1$ and $[s_n, i_n, \dots, s_1, i_1] = [t_n, i_n, \dots, t_1, i_1]$. This is the same as the partial order induced by inclusion for the compact open sets V_γ.

To each $i \in I^\times$ we associate a principal filter $i^\uparrow = \{j \in I \mid i \leq j\}$. Then for each $\gamma = [s_n, i_n, \dots, s_1, i_1] \in G^n$ we consider the element $\hat{\gamma} = [s_n, i_n^\uparrow, \dots, s_1, i_1^\uparrow] \in H^n$. The map $\gamma \mapsto \hat{\gamma} \colon G^n \to H^n$ induces a linear map $\mathbb{Z}H^n \to C_b(G^n, \mathbb{Z})$. The image of $\mathcal{B}$ under this map is $\{\chi_{\gamma^\downarrow} \mid \gamma \in G^n\}$. This is linearly independent because for a finite subset $J \subseteq G^n$, if $\sum_{\gamma \in J} a_\gamma \chi_{\gamma^\downarrow} = 0$ then $a_\gamma = 0$ for any maximal γ in J (see also [81, Corollary 6.2]).

To prove that $\mathcal{B}$ spans $\mathbb{Z}H^n$, we note that $\{V_\gamma \mid \gamma \in G^n\}$ is an open cover of H^n, so it suffices to span $\mathbb{Z}V_\gamma$ for each $\gamma = [s_n, i_n, \dots, s_1, i_1]$. For each $j_1 \leq i_1$, we may define $j_{k+1} = s_k \cdot j_k$ to obtain an element $\gamma' = [s_n, j_n, \dots, s_1, j_1] \leq \gamma$. The set $V_{\gamma'}$ is mapped to U_{j_1} under the homeomorphism $V_\gamma \cong U_{i_1}$, so we may further reduce the problem to showing that $\{\chi_{U_j} \mid j \leq i\}$ spans $\mathbb{Z}U_i$ for each $i \in I$. Because I is a weak semilattice, the intersection $U_j \cap U_k$ can be expressed as the finite union $\bigcup_{l \in j \downarrow k} U_l$. By inclusion-exclusion, we can express the product $\chi_{U_j}\chi_{U_k}$ as a linear combination of products of χ_{U_l} for strictly smaller elements l in the finite $\downarrow$-closure of $\{j, k\}$. We may iterate this process to express the product $\chi_{U_j}\chi_{U_k}$ as a linear combination of such χ_{U_l}, and conclude that the span of $\{\chi_{U_j} \mid j \leq i\}$ is closed under products. Then we may express an arbitrary compact open $V \subseteq U_i$ as a finite union of basis elements, which are of the form $U_{J_1, J_2} = \bigcap_{j \in J_1} U_j \cap \bigcap_{j \in J_2} U_j^c$ for finite subsets J_1 and J_2 of $i^\downarrow$. Each $\chi_{U_{J_1, J_2}}$ is in the span of $\{\chi_{U_j} \mid j \leq i\}$ because it is closed under products. By inclusion-exclusion the indicator function χ_V is also in this span. Finally, these indicators generate $\mathbb{Z}U_i$, so $\{\chi_{U_j} \mid j \leq i\}$ spans $\mathbb{Z}U_i$. □

Proof of Theorem 8.1. We have built a proper correspondence $\Omega \colon G \to H$ which induces an isomorphism $H_*(\Omega) \colon H_*(G) \to H_*(H)$ by Proposition 8.4. By Corollary 7.29, we obtain an isomorphism $K_*(C_r^*(G)) \cong K_*(C_r^*(H))$ in K-theory. The groupoid $G = S \ltimes I^\times$ is discrete, so we may compute

$$K_*(C_r^*(S \ltimes I^\times)) \cong \bigoplus_{[i] \in S \backslash I^\times} K_*(C_r^*(\mathrm{Stab}_i(S))).$$

As $H = S \ltimes \hat{I}$, the result follows. □

8.1. **Removing the torsion-free condition.** In previous versions of the orbit-stabiliser K-theory formula [20–22, 48, 64], the condition that the groupoid in question has torsion-free isotropy groups is not needed. In our more general setting, it is possible to remove the torsion-free condition, and we plan to address this fully in future work. We sketch the approach here for the rest of this chapter. For general ample groupoids, the groupoid homology is no longer the correct thing to compare to the K-theory groups, and should be replaced by an analogue of Bredon homology (see [60] for the group setting) which takes into account all of the proper open subgroupoids. Without developing a Bredon homology for ample groupoids, we may still use Corollary 7.26 to obtain an orbit-stabiliser K-theory formula in the setting with torsion. Because the correspondence $\Omega\colon G \to H$ comes from an S-equivariant topological correspondence, we may consider the families $\mathcal{F}_G$ and $\mathcal{F}_H$ of proper open subgroupoids of G and H indexed by the family $\mathcal{F}_S$ of finite subgroups of stabiliser groups of S, see Proposition 6.52. These satisfy $\mathrm{Ind}_\Omega(\mathfrak{I}_{\mathcal{F}_H}) \subseteq \mathfrak{I}_{\mathcal{F}_G}$, so the goal becomes to show that for each $n \geq 0$ the derived functor map

$$\mathbb{L}_n(\mathrm{Ind}_\Omega, \alpha_\Omega, f_\Omega)\colon \mathbb{L}_n K_*(G \ltimes C_0(X)) \to \mathbb{L}_n K_*(H \ltimes C_0(Y))$$

is an isomorphism. Morally this should be thought of as asking for an isomorphism in Bredon homology. Projective resolutions $P_\bullet \to C_0(X)$ and $Q_\bullet \to C_0(Y)$ can be constructed using Theorem 6.44. The algebra P_n is a direct sum over all choices $F_0, \dots, F_n \in \mathcal{F}_S$ of finite subgroups of stabiliser groups of S. Each summand is given by an action of the product inverse semigroup $F = \langle F_n, E\rangle \times \cdots \times \langle F_0, E\rangle$ on the space G^{n+1}, with $\langle F_0, E\rangle$ acting on the rightmost factor by right multiplication and $\langle F_i, E\rangle$ acting on the $(i-1)$th and ith factor by left and right multiplication for $i \geq 1$. The key fact of F we will use is that any action of F has finite orbits. There is a similar description for Q_n, and we have

$$P_n = \bigoplus_{F_0,\dots,F_n \in \mathcal{F}_S} C^*(F \ltimes G^{n+1}),$$
$$Q_n = \bigoplus_{F_0,\dots,F_n \in \mathcal{F}_S} C^*(F \ltimes H^{n+1}).$$

Taking crossed products, we may exploit Morita equivalences which arise from $G \ltimes G^{n+1} \sim_M G^n$ to see that

$$K_*(G \ltimes P_n) \cong \bigoplus_{F_0,\dots,F_n \in \mathcal{F}_S} K_*(C^*(F \ltimes G^n)),$$
$$K_*(H \ltimes Q_n) \cong \bigoplus_{F_0,\dots,F_n \in \mathcal{F}_S} K_*(C^*(F \ltimes H^n)).$$

Now $\langle F_n, E\rangle$ just acts by left multiplication on the leftmost factor. For each n and $F_0, \dots, F_n \in \mathcal{F}_S$, there is an F-equivariant proper topological correspondence

$$\Omega^n\colon G^n \to H^n,$$

which together induce a chain map $K_*(G \ltimes P_\bullet) \to K_*(H \ltimes Q_\bullet)$. This induces the left derived maps $\mathbb{L}_n(\mathrm{Ind}_\Omega, \alpha_\Omega, f_\Omega)$ in homology. It therefore suffices to show that $\Omega^n\colon G^n \to H^n$ induces an isomorphism in K-theory $K_*(C^*(F \ltimes G^n)) \cong K_*(C^*(F \ltimes H^n))$.

Proposition 8.5. *The map $K_*(C^*(F \ltimes \Omega^n))\colon K_*(C^*(F \ltimes G^n)) \to K_*(C^*(F \ltimes H^n))$ is injective.*

Proof. We consider $C^*(F \ltimes G^n)$ as the inductive limit of $F \ltimes C_0(J)$ for finite F-invariant subsets $J \subseteq G^n$. If we suppose that $K_*(C^*(F \ltimes \Omega^n))(x) = 0$, then there is some J such that $x \in K_*(F \ltimes C_0(J))$. The map $J \to H^n$ sending $j \in J$ to the element $\hat{j}$ of H^n as defined in Proposition 8.4 is proper and F-equivariant so induces an F-equivariant proper correspondence $\eta\colon H^n \to J$.

$$\begin{tikzcd} G^n \arrow[r, "\Omega^n"] & H^n \arrow[d, "\eta"] \\ J \arrow[u, hook] \arrow[r, "\mu"] & J \end{tikzcd}$$

The resulting F-equivariant correspondence $\mu\colon J \to J$ is given by

$$J \leftarrow \bigsqcup_{j \in J} \{i \in J \mid i \leq j\} \to J.$$

The left map picks out the indexing element of J, and the right map includes each component of the disjoint union into J. This is a unipotent correspondence, in that it is the disjoint union of the identity correspondence and the F-equivariant nilpotent correspondence $\nu\colon J \to J$ given by

$$J \leftarrow \bigsqcup_{j \in J} \{i \in J \mid i < j\} \to J,$$

which satisfies $\nu^{|J|} = 0$. It follows that $K_*(C^*(F \ltimes \mu))$ is an isomorphism, with inverse

$$K_*(C^*(F \ltimes \mu))^{-1} = \sum_{k=0}^{|J|} (-K_*(C^*(F \ltimes \nu)))^k.$$

As μ factors through Ω^n, it follows from $K_*(C^*(F \ltimes \Omega^n))(x) = 0$ that the invertible map $K_*(C^*(F \ltimes \mu))$ also sends x to 0, and hence $x = 0$. □

Proposition 8.6. *The map $K_*(C^*(F \ltimes \Omega^n))\colon K_*(C^*(F \ltimes G^n)) \to K_*(C^*(F \ltimes H^n))$ is surjective, and hence an isomorphism.*

Proof. For each F-invariant finite subset $J \subseteq G^n$, we define $D_J = C^*(p_j \mid j \in J) \subseteq C_0(H^n)$, where $p_j = \chi_{V_j}$ is a projection in $C_0(H^n)$. This is a finite dimensional F-invariant subalgebra. Furthermore, the inductive limit of all such D_J is $C_0(H^n)$. By continuity of K-theory, $\varinjlim_J K_*(F \ltimes D_J) = K_*(C^*(F \ltimes H^n))$, so it suffices to hit the image of $K_*(F \ltimes D_J)$ for each J, which vanishes for $* = 1$. The finite dimensional commutative C*-algebra D_J is the span $\mathbb{C}P_J$ of its minimal projections $p \in P_J$ which are pairwise orthogonal and form an F-invariant set. Consider a minimal projection $p \in P_J$ in the domain of some $f \in F$, and let

$$F_p = \{f \in F \mid p \in \operatorname{dom} f,\ f \cdot p = p\}$$

be the stabiliser inverse subsemigroup of F at p. We obtain a Morita equivalence

$$F \ltimes \mathbb{C}P_J \sim_M \bigoplus_{\operatorname{orb} p \in F \backslash P_J} F_p \ltimes \mathbb{C}p$$

from which we can deduce that $K_0(F \ltimes D_J)$ is generated by the groups $K_0(F_p \ltimes \mathbb{C}p)$ for projections $p \in P_J$ in each F-orbit. By minimality, each such p is below p_{j_0} for some $j_0 \in J$. The poset $j_0^{\downarrow} \cup \{0\}$ is order isomorphic to a down-set in I, so it is a locally finite weak semilattice. By the argument in Proposition 8.4 p can be expressed uniquely as a linear combination $\sum_{i \in J_0} n_i p_i$ for a finite set $J_0 \subseteq j_0^{\downarrow}$. By uniqueness, the set J_0 is F_p-invariant, and its closure J_1 under $\downarrow$ (and 0) is a finite F_p-invariant set such that D_{J_1} is the span of $\{p_j : j \in J_1\}$. Consider the following diagram of inclusions of C*-algebras.

$$\begin{tikzcd} F \ltimes D_J \arrow[r, hook] & C^*(F \ltimes H^n) \\ F_p \ltimes \mathbb{C}p \arrow[u, hook] \arrow[r, hook] & F_p \ltimes D_{J_1} \arrow[u, hook] \end{tikzcd}$$

We have reduced the problem to showing that we hit the image of $K_0(F_p \ltimes D_{J_1})$ with $K_0(C^*(F \ltimes \Omega^n))$. The algebra D_{J_1} can be identified F_p-equivariantly with $C_0(\hat{J}_1)$, such that $p_j \in D_{J_1}$ is identified with the indicator function q_j on the set of filters in $\hat{J}_1$ containing j. We form a F_p-equivariant proper correspondence $\eta\colon J_1^{\times} \to \hat{J}_1$ in the same way as we did for I. We obtain a commutative diagram of F_p-equivariant proper correspondences, which induces a commutative diagram of proper C*-correspondences:

$$\begin{tikzcd} G^n \arrow[r, "\Omega^n"] & H^n \\ J_1^{\times} \arrow[u, hook] \arrow[r, "\eta"] & \hat{J}_1 \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} C^*(F \ltimes G^n) \arrow[r, "C^*(F \ltimes \Omega^n)"] & C^*(F \ltimes H^n) \\ F_p \ltimes C_0(J_1^{\times}) \arrow[u, hook] \arrow[r, "C^*(F_p \ltimes \eta)"] & F_p \ltimes D_{J_1} \arrow[u, hook] \end{tikzcd}$$

Every filter on J_1 is principal because J_1 is finite, so $j \mapsto j^{\uparrow} \colon J_1^{\times} \to \hat{J}_1$ defines an F_p-equivariant bijection $\varphi\colon J_1^{\times} \to \hat{J}_1$. Then $\varphi^{-1} \circ \eta\colon J_1^{\times} \to J_1^{\times}$ is the unipotent F_p-equivariant topological correspondence given by

$$J_1^{\times} \leftarrow \bigsqcup_{j \in J_1^{\times}} \{i \in J_1^{\times} : i \leq j\} \to J_1^{\times}.$$

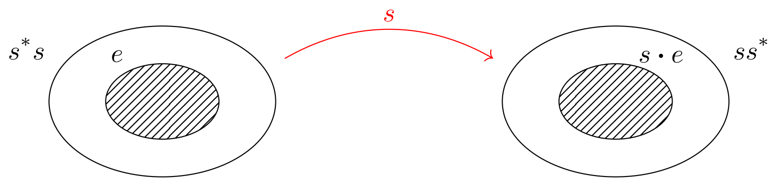
It follows that $K_*(C^*(F_p \ltimes \eta))\colon K_*(F_p \ltimes C_0(J_1^{\times})) \to K_*(F_p \ltimes D_{J_1})$ is invertible, and therefore the image of the map $K_0(C^*(F \ltimes \Omega^n))$ contains the image of the group $K_0(F_p \ltimes D_{J_1})$, and we are done. □

This proves that the derived functor maps $\mathbb{L}_n(\mathrm{Ind}_\Omega, \alpha_\Omega, f_\Omega)$ are isomorphisms.

Proof of Theorem 8.1 without the torsion-free assumption. As discussed, the correspondence $\Omega\colon G \to H$ induces an ABC morphism $[\mathrm{Ind}_\Omega, \alpha_\Omega, f_\Omega]$ given our choice of families $\mathcal{F}_G$ and $\mathcal{F}_H$. The derived functor maps $\mathbb{L}_n(\mathrm{Ind}_\Omega, \alpha_\Omega, f_\Omega)$ are isomorphisms, so by Corollary 7.26 we get an isomorphism $K_*(C_r^*(G)) \cong K_*(C_r^*(H))$. $\square$

9. Applications

How does the orbit-stabiliser K-theory formula (Theorem 8.1) work in practice? It calls for an inverse semigroup S and a locally finite weak semilattice I with an action of S by order isomorphisms of down-sets. This is a bit of a mouthful, but not much is lost by considering the special case where $I = E$ is the idempotent semilattice of S, with the canonical action $S \curvearrowright E$. For each $s \in S$, the domain $\operatorname{dom}_E s$ consists of the idempotents $e \in E$ below the domain s^*s of s in the sense that $e \leq s^*s$. The action is then given by $s \cdot e := ses^*$.



The resulting groupoid $G(S) = S \ltimes \hat{E}$ is the *universal groupoid* of S and when it is Hausdorff it is a groupoid model for the reduced C*-algebra $C^*_r(S)$ of S [22, Theorem 5.5.18]. The formula then says that we can compute the K-theory of $C^*_r(S)$ as the direct sum

$$K_*(C^*_r(S)) \cong \bigoplus_{\operatorname{orb}(e) \in S \backslash E^\times} K_*(C^*_r(\operatorname{Stab}_e(S))), \tag{9.1}$$

where $\operatorname{Stab}_e(S) = \{s \in S \mid s^*s = e = ss^*\}$ is the stabiliser subgroup of S at $e \in E$. The direct sum is taken over the orbits of the action of S restricted to the non-zero idempotents $E^\times$. This reduces the K-theory computation to a problem for group C*-algebras, which are trivial in many cases of interest.

Inverse semigroups can provide the dynamics for any étale groupoid and thus have wide appeal to C*-algebraists, but when are we interested in the reduced C*-algebra of an inverse semigroup itself? The left inverse hull $I_\ell(P)$ of a left cancellative monoid P is an inverse semigroup which can be used to help us understand the left regular C*-algebra $C^*_\lambda(P)$. The reduced C*-algebra $C^*_r(I_\ell(P))$ quotients onto $C^*_\lambda(P)$, and this is an isomorphism in many cases under the independence condition [63]. More generally, we can carry out this construction for a left cancellative small category Λ to help us understand the left regular algebra $C^*_\lambda(\Lambda)$. Spielberg constructs a groupoid G_Λ (G_2 in [80]) which models this algebra when it is Hausdorff or Λ is finitely aligned. Li presents a transformation groupoid ($I_l \ltimes \Omega$ in [49]) by the left inverse hull inverse semigroup $I_\ell(\Lambda)$ of zigzag maps which is isomorphic to G_Λ and models $C^*_\lambda(\Lambda)$ in the Hausdorff or finitely aligned setting. This extends work of Farthing Muhly and Yeend who construct an inverse semigroup S_Λ from a finitely aligned higher rank graph Λ and use it to build a groupoid model $S_\Lambda \ltimes X_\Lambda$ for the Toeplitz algebra $\mathcal{T}C^*(\Lambda)$ [30].

Higher rank graphs and other left cancellative small categories which can be finitely aligned are the inspiration for the study of a locally finite weak semilattice I and the resulting groupoid $S \ltimes \hat{I}$ induced by an action $S \curvearrowright I$. In a weak semilattice, the join operation $a \wedge b$ of a semilattice is replaced by a finite-set-valued join $a \downarrow b$, which should be thought of as representing the union of its elements as the intersection of a and b. Local finiteness means that the weak semilattices generated by a finite subset under $\downarrow$ remain finite. The principal left ideals $\lambda\Lambda$ in a left cancellative small category Λ form a poset by containment, which means that $\mu\Lambda \leq \lambda\Lambda$ if there is $\nu \in \Lambda$ such that $\mu = \lambda\nu$. We take Λ to be finitely aligned, which means exactly that the poset $\mathcal{P}_\Lambda$ formed by adjoining 0 to the principal ideals is a weak semilattice. In many cases local finiteness is automatic, such as for singly aligned categories or for higher rank graphs. Filters on $\mathcal{P}_\Lambda$ correspond to directed hereditary subsets of Λ, which form the unit space X_Λ of G_Λ [80, Theorem 6.8]. The left inverse hull $I_\ell(\Lambda)$ of zigzag maps is the inverse semigroup of partial bijections on Λ generated by the left multiplication maps $\mu \mapsto \lambda\mu \colon s(\lambda)\Lambda \to \lambda\Lambda$. The action $I_\ell(\Lambda) \curvearrowright \Lambda$ by partial bijections induces an action $I_\ell(\Lambda) \curvearrowright \mathcal{P}_\Lambda$ by order isomorphisms of down-sets. We obtain an action $I_\ell(\Lambda) \curvearrowright \hat{\mathcal{P}}_\Lambda$ which coincides with the action $I_l \curvearrowright \Omega$ in [49]. Our transformation groupoid $I_\ell(\Lambda) \ltimes \hat{\mathcal{P}}_\Lambda$ is therefore isomorphic to G_Λ and models the left regular algebra $C^*_\lambda(\Lambda)$. When Λ is a higher rank graph, this is the Toeplitz algebra $\mathcal{T}C^*(\Lambda)$ [79, Remark 8.4].

We have three major assumptions: that the transformation groupoid $S \ltimes \hat{I}$ is Hausdorff, has torsion-free isotropy groups and satisfies the Baum-Connes conjecture.

Conditions for transformation groupoids to be Hausdorff are well-studied. Steinberg shows that Hausdorffness of the universal groupoid $G(S)$ is equivalent to S being a weak semilattice with its natural partial order [81], and in this case every transformation groupoid is Hausdorff. Exel and Pardo present a condition on an inverse semigroup action $S \curvearrowright X$ to have a Hausdorff transformation groupoid [27]. It suffices for S to be 0-E-unitary: any element $s \in S$ which extends a non-zero idempotent e in the sense that $se = e$ must itself be idempotent. In the setting of a finitely aligned left cancellative small category Λ, the groupoid $I_\ell(\Lambda) \ltimes \hat{\mathcal{P}}_\Lambda$ is Hausdorff if and only if for all $\lambda, \mu \in \Lambda$ with equal range and source, there is a finite subset $A \subseteq \Lambda$ such that $\{\nu \in \Lambda \mid \lambda\nu = \mu\nu\} = \bigcup_{\alpha \in A} \alpha\Lambda$ [49, Corollary 4.2]. This is automatic if Λ is right cancellative.

Many of the groupoids we are interested in have torsion-free isotropy groups. Deaconu-Renault groupoids of rank k (see [23, 73, 78]) are built from k commuting local homeomorphisms, and all have torsion-free isotropy groups. The groupoid $S_\Lambda \ltimes X_\Lambda$ associated to a rank k graph can be viewed as a Deaconu-Renault groupoid with the k commuting local homeomorphisms given by shifts in the k directions of the graph. In general, any torsion in the isotropy of $S \ltimes \hat{I}$ must come from torsion in S by Lemma 6.51, so it is enough for each stabiliser group S_e to be torsion-free.

The Baum Connes conjecture with coefficients was shown by Tu in [84] to hold for any second countable Hausdorff groupoid with the Haagerup property, which is a weak version of amenability. Amenability is well-studied for groupoid models of interesting C*-algebras, partially because it is equivalent to nuclearity of the C*-algebra [13, Theorem 5.6.18]. Although typically it is more straightforward to directly verify amenability of the groupoid, the nuclearity of $C_r^*(G)$ can be used to deduce that G satisfies Baum-Connes. Deaconu-Renault groupoids of rank k are always amenable [78, Lemma 3.5]. Conditions for the amenability of the universal groupoid of an inverse semigroup are discussed in [2].

A key outcome of this work is that the inverse semigroup S need not admit an idempotent pure partial homomorphism $S^\times \to \Gamma$ to a group Γ. This is the case that has been covered previously in [48] based on [20–22] and roughly this condition says that the dynamics of S can be described using partial actions of Γ. In the setting of the inverse hull $I_\ell(P)$ of a left cancellative monoid P, this asks for a group embedding $P \subseteq \Gamma$. For left cancellative small categories Λ such as higher rank graphs, this means there is a faithful functor $\Lambda \to \Gamma$. For usual graphs, the category of paths maps faithfully to the free group on the edges, but for higher rank graphs it is too much to ask for a faithful functor to a group, and there are even counterexamples with a single vertex [68].

With these conditions in mind, the orbit-stabiliser K-theory formula for an inverse semigroup is the following.

Corollary 9.1. *Let S be a countable inverse semigroup such that:*

- *the natural order on S has the structure of a weak semilattice (e.g. if S is 0-E-unitary),*
- *the stabiliser subgroups $\mathrm{Stab}_e(S)$ of idempotents $e \in S$ are torsion-free,*
- *and the universal groupoid $G(S)$ satisfies Baum-Connes (e.g. if $C_r^*(S)$ is nuclear).*

Then the K-theory of the reduced C-algebras of S is described by*

$$K_*(C_r^*(S)) \cong \bigoplus_{\mathrm{orb}(e) \in S \setminus E^\times} K_*(C_r^*(\mathrm{Stab}_e(S))).$$

Let us return to the setting of a finitely aligned left cancellative small category Λ. The non-zero elements of $\mathcal{P}_\Lambda$ are the principal ideals $\{\lambda\Lambda \mid \lambda \in \Lambda\}$. The action of the left inverse hull $I_\ell(\Lambda)$ is by left multiplication and left cancellation. We get orbits $\mathrm{orb}(\lambda\Lambda) = \{\mu\Lambda \mid \mu \in \Lambda s(\lambda)\}$. Two principal ideals $\lambda\Lambda$ and $\mu\Lambda$ are in the same orbit if and only if there is an invertible element $\nu \in \Lambda$ with $s(\nu) = s(\lambda)$ and $r(\nu) = s(\mu)$. Thus the orbits are indexed by the isomorphism classes of objects in Λ. The stabiliser group of $\lambda\Lambda$ is isomorphic to the group of invertible elements with range and source $s(\lambda)$. In fact, the discrete groupoid $I_\ell(\Lambda) \ltimes \mathcal{P}_\Lambda^\times$ can be identified with the groupoid Λ^* of invertible elements in Λ.

In order to apply Theorem 8.1 to a finitely aligned left cancellative small category Λ, we ask further that it is *manageably* finitely aligned, in the sense that the weak semilattice $\mathcal{P}_\Lambda \cup \{0\}$ of principal ideals is locally finite. Recall that this means that each finite set of principal ideals is contained in a finite set which is closed under $\downarrow$. In this setting, the orbit-stabiliser formula says:

Corollary 9.2. *Let Λ be a countable manageably finitely aligned left cancellative small category such that:*

- *for all $\lambda, \mu \in \Lambda$ with equal range and source, there is a finite subset $A \subseteq \Lambda$ such that $\{\nu \in \Lambda \mid \lambda\nu = \mu\nu\} = \bigcup_{\alpha \in A} \alpha\Lambda$ (e.g. if Λ is right cancellative),*
- *the transformation groupoid $I_\ell(\Lambda) \ltimes \hat{\mathcal{P}}_\Lambda$ has torsion-free isotropy groups, (e.g. if the left inverse hull $I_\ell(\Lambda)$ of Λ has torsion-free stabiliser subgroups),*
- *and $I_\ell(\Lambda) \ltimes \hat{\mathcal{P}}_\Lambda$ satisfies the Baum-Connes conjecture (e.g. if $C^*_\lambda(\Lambda)$ is nuclear).*

Then the K-theory of the left regular algebra of Λ is described by

$$K_*(C^*_\lambda(\Lambda)) \cong K_*(C^*_r(\Lambda^*)).$$

Example 9.3 (Finitely aligned higher rank graph)**.** Let $d\colon \Lambda \to \mathbb{N}^k$ be a (countable) k-graph. Explicitly, Λ is a countable category equipped with a *degree* functor $d\colon \Lambda \to \mathbb{N}^k$ satisfying the *unique factorisation property*: given $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$, there are unique $\mu, \nu \in \Lambda$ with $\lambda = \mu\nu$ such that $d(\mu) = m$ and $d(\nu) = n$. This implies that Λ is a cancellative small category. Its left regular C*-algebra is the Toeplitz algebra $\mathcal{T}C^*(\Lambda)$.

When $k = 1$, Λ is the category of finite paths on a directed graph, and $d\colon \Lambda \to \mathbb{N}$ measures the length of a path, so, in general, we think of an element of Λ as a path and an element of $\Lambda^0 = d^{-1}(0)$ as a vertex. The poset $\mathcal{P}_\Lambda$ of principal ideals may be identified with Λ itself (reverse) ordered by extension of paths: $\lambda \leq \mu$ if λ extends μ in the sense that $\lambda \in \mu\Lambda$. When $k > 1$, two paths λ and μ may have a common extension even if neither λ nor μ is an extension of the other. However, the unique factorisation property implies that any two distinct minimal common extensions α and β of λ and μ are orthogonal in the sense that α and β have no common extensions. Further suppose that Λ is finitely aligned: for each pair $\lambda, \mu \in \Lambda$ of paths, the set of minimal common extensions of λ and μ is finite. This means that $\Lambda \cup \{0\}$ taken with the (reverse) extension order is a weak semilattice, and it is automatically locally finite because minimal common extensions are orthogonal. Thus Λ is a countable manageably finitely aligned cancellative small category.

The groupoid $I_\ell(\Lambda) \ltimes \hat{\Lambda}$ is isomorphic to the Deaconu-Renault groupoid associated to k commuting shifts on the space $\hat{\Lambda}$ of filters on $\Lambda \cup \{0\}$. It is therefore torsion-free and amenable, so Λ fits into our framework. The only invertible elements Λ^* are the vertices Λ^0 of the graph. We recover Fletcher's computation [32] of the K-theory of the Toeplitz algebra $\mathcal{T}C^*(\Lambda)$:

$$K_0(\mathcal{T}C^*(\Lambda)) \cong \bigoplus_{v\in\Lambda^0} \mathbb{Z}, \qquad\qquad K_1(\mathcal{T}C^*(\Lambda)) = 0.$$

While manageability of a finitely aligned LCSC (i.e. local finiteness of the associated weak semilattice) might seem like a frustrating condition, we do require it.

Example 9.4 (An unmanageably finitely aligned monoid failing the K-theory formula)**.** Consider the multiplicative monoid $R^\times$ of non-zero elements in the ring $R = \mathbb{Z}[\sqrt{-3}]$, as studied in [51, Section 6.6]. This is cancellative and commutative, so the associated groupoid is Hausdorff and satisfies Baum-Connes. Although there is torsion, we have explained in 8.1 that it is possible to remove the torsion-free condition. The group of invertible elements is $R^* = \{\pm 1\}$, so the orbit-stabiliser formula would predict that the K-theory of $C^*_\lambda(R^\times)$ is given by $K_*(C^*_r(R^*))$, which is $(\mathbb{Z}^2, 0)$. However, in [51] the K-theory is computed to be $(\mathbb{Z}^6, \mathbb{Z}^4)$. Once we have verified that this monoid is finitely aligned, the only condition missing from $R^\times$ is manageability, so we will see that it is a necessary assumption to make. In [51] the constructible ideals, which include all the intersections $aR^\times \cap bR^\times$ of principal ideals, are computed to be $\{aR^\times \mid a \in R^\times\} \cup \{2c\overline{R} \mid c \in \overline{R}\} \cup \{\emptyset\}$. Here $\overline{R} = \mathbb{Z}[\alpha]$ is the integral closure of R, with $\alpha = \frac{1}{2}(1+\sqrt{-3})$. They also compute that $\overline{R} = R \cup \alpha R \cup \alpha^2 R$. For each $c \in \overline{R}$, the constructible ideal $2c\overline{R}$ is therefore a finite union of principal ideals, because $2c$, $2c\alpha$ and $2c\alpha^2$ are all elements of R. The monoid $R^\times$ is therefore finitely aligned.

10. Outlook

This work is a snapshot of a larger story in which many interesting questions are as of yet unanswered. This means that there is plenty of scope for future work on this topic.

A problem we discussed in 8.1 is removing the torsion-free condition from the orbit-stabiliser K-theory formula (Theorem 8.1). We have sketched the approach to this, and plan to flesh this out in future work. The derived functors which appear are morally Bredon homology groups of ample groupoids, and this motivates a systematic study which develops a Bredon homology theory for ample groupoids. This should reflect the situation for groups, where Bredon homology is done relative to the family of all finite subgroups of a group. In the ample groupoid setting, we want to consider the family of proper open subgroupoids, but this raises complications. First of all, this family will be uncountable, and it becomes desirable to extract a sufficient countable subfamily - this sufficiency should be captured by condition (P). There is much work to be done to describe how to do Bredon homology relative to this family, and how to show that this approach is independent of the choice of family. At the end of this work, the Bredon homology of an ample groupoid G should be closely linked with the K-theory $K_*(C_r^*(G))$ through the ABC spectral sequence and Baum-Connes. In addition to the application towards the orbit-stabiliser K-theory formula, this will allow us to study variations of the HK conjecture and further homological dimension ideas for general ample groupoids in the spirit of [10].

The reader familiar with the proofs of the predecessors [20–22, 48] of the orbit-stabiliser K-theory formula might ask where the Going Down principle is hidden in our proof. This principle is intimitely tied to the Baum-Connes conjecture. For a group Γ it says that if we have a Kasparov cycle $x \in \mathrm{KK}^\Gamma(A, B)$ and we want to check that it induces a K-theoretic isomorphism of crossed products, it suffices to check this for crossed products by finite subgroups. This has been extended to the setting of étale groupoids in [8, 9, 11], but there is no clear way to apply this directly to the K-theory formula in our setting. The problem is that there is no candidate Kasparov cycle which is equivariant with respect to a single groupoid - we instead have to consider two distinct groupoids and a correspondence between them. Our approach is more closely tied morally to a potential Going Down principle for inverse semigroups. The crux of our argument is an inverse semigroup equivariant correspondence, and the relevant subobjects are the finite inverse subsemigroups, which can even be specialised to those which are themselves finite groups. It is an interesting problem to see if a genuine Going Down principle for inverse semigroups can be formulated and proven along these lines.

The orbit-stabiliser K-theory formula only allows us to compute the K-theory for a specific kind of ample groupoid. The groupoids $S \ltimes Y$ covered can be characterised by an independence condition on the dynamics $S \curvearrowright Y$. Many of the groupoid C*-algebras we are interested in do not come from these "independent" dynamical systems, which almost always have non-simple C*-algebras. However, we can often find a resolution of a C*-algebra A of interest by C*-algebras $\mathcal{T}_n$ coming from such independent dynamical systems, as is done in [50, 51].

$$\cdots \to \mathcal{T}_n \to \cdots \to \mathcal{T}_0 \to A \to 0$$

We can compute the K-theory of each $\mathcal{T}_n$ using the K-theory formula. If this resolution is finite, we can break it down into multiple six-term exact sequences and in principle learn a lot about the K-theory of A. This approach looks particularly promising in the setting of Garside categories [49], with A given by the boundary quotient C*-algebra. This covers Cuntz Krieger algebras of higher rank graphs, C*-algebras of Artin-Tits groups and C*-algebras coming from self-similar groups.

There has been growing interest recently in non-Hausdorff étale groupoids [28, 41, 43, 62], which arise very naturally from inverse semigroup actions. We may conjecture that the orbit-stabiliser K-theory formula still holds in the non-Hausdorff setting if we replace the reduced C*-algebra by the essential C*-algebra. However, there is a lot of work to be done before we can attempt a similar approach to this. Basic properties of non-Hausdorff groupoid equivariant KK-theory have been developed in [52], but a triangulated category approach to Baum-Connes is yet to be investigated. Developing a categorical approach to Baum-Connes for non-Hausdorff étale groupoids is a worthwhile goal in its own right. A starting point for this investigation is the Green-Julg Theorem for non-Hausdorff proper étale groupoids (see [66]). Any successful approach to the Baum-Connes conjecture will have to find a suitable replacement for the subgroupoid induction functors $\mathrm{Ind}_H^G\colon \mathrm{KK}^H \to \mathrm{KK}^G$ which currently rely upon the G-C*-algebra $C_0(G)$. There is hope, because analogues of this G-C*-algebra have been developed in the non-Hausdorff setting in [16]. To get examples of non-Hausdorff étale groupoids satisfying Baum-Connes, we would want to extend Tu's approach [84] to this setting. The good news is that correspondences of groupoids work well in the non-Hausdorff setting [3], and ample groupoid homology is essentially unchanged. In particular, the G-module $\mathbb{Z}G$ associated to the G-C*-algebra $C_0(G)$ is a perfectly good G-module even in the non-Hausdorff setting, making the prospect of a nice C*-algebraic replacement for $C_0(G)$ more promising.

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