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Sequential parametrized motion planning and its complexity ☆

Michael Farber *, Amit Kumar Paul

School of Mathematical Sciences, Queen Mary University of London, E1 4NS London, UK

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A B S T R A C T

In this paper we develop a theory of *sequential parametrized* motion planning generalising the approach of *parametrized* motion planning, which was introduced recently in [3]. A sequential parametrized motion planning algorithm produced a motion of the system which is required to visit a prescribed sequence of states, in a certain order, at specified moments of time. The sequential parametrized algorithms are universal as the external conditions are not fixed in advance but rather constitute part of the input of the algorithm. In this article we give a detailed analysis of the sequential parametrized topological complexity of the Fadell - Neuwirth fibration. In the language of robotics, sections of the Fadell - Neuwirth fibration are algorithms for moving multiple robots avoiding collisions with other robots and with obstacles in the Euclidean space. In the last section of the paper we introduce the new notion of TC-*generating function of a fibration*, examine examples and raise some exciting general questions about its analytic properties.

1. Introduction

Autonomously functioning systems in robotics are controlled by motion planning algorithms. Such an algorithm takes as input the initial and the final states of the system and produces a motion of the system from the initial to final state, as output. The theory of algorithms for robot motion planning is a very active field of contemporary robotics and we refer the reader to the monographs [17], [18] for further references.

A topological approach to the robot motion planning problem was developed in [8], [9]; the topological techniques explained relationships between instabilities occurring in robot motion planning algorithms and topological features of robots' configuration spaces.

A new *parametrized* approach to the theory of motion planning algorithms was suggested recently in [3]. The parametrized algorithms are more universal and flexible, they can function in a variety of situations involving external conditions which are viewed as parameters and are part of the input of the algorithm.

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* Corresponding author.
E-mail addresses: m.farber@qmul.ac.uk (M. Farber), amit.paul@qmul.ac.uk, amitkrpaul23@gmail.com (A.K. Paul).

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A typical situation of this kind arises when we are dealing with collision-free motion of many objects (robots) moving in the 3-space avoiding a set of obstacles, and the positions of the obstacles are a priori unknown. This specific problem was analysed in full detail in [3], [4].

In this paper we develop a more general theory of *sequential parametrized* motion planning algorithms. In this approach the algorithm produces a motion of the system which is required to visit a prescribed sequence of states in a certain order. The sequential parametrized algorithms are also universal as the external conditions are not a priori fixed but constitute a part of the input of the algorithm.

In the first part of this article we develop the theory of sequential parametrized motion planning algorithms while the second part consists of a detailed analysis of the sequential parametrized topological complexity of the Fadell - Neuwirth fibration. In the language of robotics, the sections of the Fadell - Neuwirth bundle are exactly the algorithms for moving multiple robots avoiding collisions with each other and with multiple obstacles in the Euclidean space.

Our results depend on the explicit computations of the cohomology algebras of certain configuration spaces. We describe these computations in full detail, they employ the classical Leray - Hirsch theorem from algebraic topology of fibre bundles.

In the last section of the paper we introduce the new notion of a TC-*generating function of a fibration*, discuss a few examples, and raise interesting questions about analytic properties of this function.

In a forthcoming publication (which is now in preparation) we shall describe the explicit sequential parametrized motion planning algorithm for collision free motion of multiple robots in the presence of multiple obstacles in $\mathbb{R}^d$, generalising the ones presented in [12]. These algorithms are optimal as they have minimal possible topological complexity.

2. Preliminaries

In this section we recall the notions of sectional category and topological complexity; we refer to [1,3,4,8,13,19,20] for more information.

Sectional category. Let $p : E \to B$ be a Hurewicz fibration. The *sectional category* of p, denoted $\mathsf{secat}[p : E \to B]$ or $\mathsf{secat}(p)$, is defined as the least non-negative integer k such that there exists an open cover $\{U_0, U_1, \dots, U_k\}$ of B with the property that each open set U_i admits a continuous section $s_i : U_i \to E$ of p. We set $\mathsf{secat}(p) = \infty$ if no finite k with this property exists.

The *generalized sectional category* of a Hurewicz fibration $p : E \to B$, denoted $\mathsf{secat}_g[p : E \to B]$ or $\mathsf{secat}_g(p)$, is defined as the least non-negative integer k such that B admits a partition

$$B = F_0 \sqcup F_1 \sqcup ... \sqcup F_k, \quad F_i \cap F_j = \emptyset \text{ for } i \neq j$$

with each set F_i admitting a continuous section $s_i : F_i \to E$ of p. We set $\mathsf{secat}_g(p) = \infty$ if no such finite k exists.

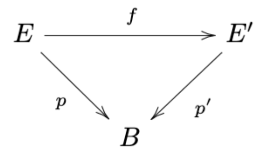
It is obvious that $\mathsf{secat}(p) \geq \mathsf{secat}_g(p)$ in general. However, as was established in [13], in many interesting situations there is an equality:

2.1 Theorem. *Let $p : E \to B$ be a Hurewicz fibration with E and B metrizable absolute neighbourhood retracts (ANRs). Then* $\mathsf{secat}(p) = \mathsf{secat}_g(p)$.

In the sequel the term “fibration” will always mean “Hurewicz fibration”, unless otherwise stated explicitly.

The following Lemma will be used later in the proofs.

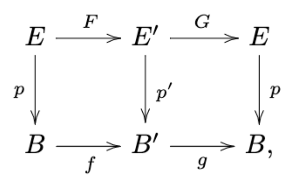
2.2 Lemma. *(A) If for two fibrations $p : E \to B$ and $p' : E' \to B$ over the same base B there exists a continuous map f shown on the following digram*



then $\mathsf{secat}(p) \geq \mathsf{secat}(p')$.

(B) If a fibration $p : E \to B$ can be obtained as a pull-back from another fibration $p' : E' \to B'$ then $\mathsf{secat}(p) \leq \mathsf{secat}(p')$.

(C) Suppose that for two fibrations $p : E \to B$ and $p' : E' \to B'$ there exist continuous maps f, g, F, G shown on the commutative diagram



such that $g \circ f : B \to B$ is homotopic to the identity map $\mathsf{Id}_B : B \to B$. Then $\mathsf{secat}(p) \leq \mathsf{secat}(p')$.

Proof. Statements (A) and (B) are well-known and follow directly from the definition of sectional category. Below we give the proof of (C) which uses (A) and (B).

Consider the fibration $q : \bar{E} \to B$ induced by $f : B \to B'$ from $p' : E' \to B'$. Here $\bar{E} = \{(b, e') \in B \times E'; f(b) = p'(e')\}$ and $q(b, e') = b$. Then

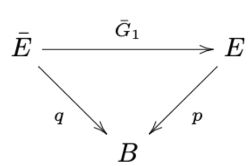
$$\mathsf{secat}(q) \leq \mathsf{secat}(p') \tag{1}$$

by statement (B).

Consider the map $\bar{G} : \bar{E} \to E$ given by $\bar{G}(b, e') = G(e')$ for $(b, e') \in \bar{E}$. Then

$$(p \circ \bar{G})\ (b, e') = p(G(e')) = g(p'(e')) = g(f(b)) = ((g \circ f) \circ q)\ (b, e')$$

and thus $p \circ \bar{G} = (g \circ f) \circ q$ and using the assumption $g \circ f \simeq \mathsf{Id}_B$ we obtain $p \circ \bar{G} \simeq q$. Let $h_t : B \to B$ be a homotopy with $h_0 = g \circ f$ and $h_1 = \mathsf{Id}_B$, $t \in I$. Using the homotopy lifting property, we obtain a homotopy $\bar{G}_t : \bar{E} \to E$, such that $\bar{G}_0 = \bar{G}$ and $p \circ \bar{G}_t = h_t \circ q$. The map $\bar{G}_1 : \bar{E} \to E$ satisfies $p \circ \bar{G}_1 = q$; in other words, $\bar{G}_1$ appears in the commutative diagram



Applying to this diagram statement (A) we obtain the inequality $\mathsf{secat}(p) \leq \mathsf{secat}(q)$ which together with inequality (1) implies $\mathsf{secat}(p) \leq \mathsf{secat}(p')$, as claimed. □

Topological complexity. Let X be a path-connected topological space. Consider the path space X^I (i.e. the space of all continuous maps $I = [0, 1] \to X$ equipped with compact-open topology) and the fibration

$$\pi : X^I \to X \times X, \quad \alpha \mapsto (\alpha(0), \alpha(1)).$$

The *topological complexity* $\mathsf{TC}(X)$ of X is defined as $\mathsf{TC}(X) := \mathsf{secat}(\pi)$, cf. [8]. For information on recent developments related to the notion of $\mathsf{TC}(X)$ we refer the reader to [15], [5].

For any $r \geq 2$, fix r points $0 \leq t_1 < t_2 < \cdots < t_r \leq 1$ (which we shall call the *"time schedule"*) and consider the evaluation map

$$\pi_r : X^I \to X^r, \quad \alpha \mapsto (\alpha(t_1), \alpha(t_2), \ldots, \alpha(t_r)), \quad \alpha \in X^I. \tag{2}$$

Typically, one takes $t_i = (i-1)(r-1)^{-1}$. The *r-th sequential topological complexity* is defined as $\mathsf{TC}_r(X) := \mathsf{secat}(\pi_r)$; this invariant was originally introduced by Rudyak [19]. It is known that $\mathsf{TC}_r(X)$ is a homotopy invariant, it vanishes if and only if the space X is contractible. Moreover, $\mathsf{TC}_{r+1}(X) \geq \mathsf{TC}_r(X)$. Besides, $\mathsf{TC}(X) = \mathsf{TC}_2(X)$.

Parametrized topological complexity. For a Hurewicz fibration $p : E \to B$ denote by $E^I_B \subset E^I$ the space of all paths $\alpha : I \to E$ such that $p \circ \alpha : I \to B$ is a constant path. Let $E^2_B \subset E \times E$ denote the space of pairs $(e_1, e_2) \in E^2$ satisfying $p(e_1) = p(e_2)$. Consider the fibration

$$\Pi : E^I_B \to E^2_B = E \times_B E, \quad \alpha \mapsto (\alpha(0), \alpha(1)).$$

The fibre of Π is the loop space ΩX where X is the fibre of the original fibration $p : E \to B$. The following notion was introduced in a recent paper [3]:

2.3 Definition. The *parametrized topological complexity* $\mathsf{TC}[p : E \to B]$ of the fibration $p : E \to B$ is defined as

$$\mathsf{TC}[p : E \to B] = \mathsf{secat}[\Pi : E^I_B \to E^2_B].$$

Parametrized motion planning algorithms are universal and flexible, they are capable to function under a variety of external conditions which are parametrized by the points of the base B. We refer to [3] for more detail and examples.

If $B' \subset B$ and $E' = p^{-1}(B')$ then obviously $\mathsf{TC}[p : E \to B] \geq \mathsf{TC}[p' : E' \to B']$ where $p' = p|_{E'}$. In particular, restricting to a single fibre we obtain

$$\mathsf{TC}[p : E \to B] \geq \mathsf{TC}(X).$$

3. The concept of sequential parametrized topological complexity

In this section we define a new notion of sequential parametrized topological complexity and establish its basic properties.

Let $p : E \to B$ be a Hurewicz fibration with fibre X. Fix an integer $r \geq 2$ and denote

$$E^r_B = \{(e_1, \cdots, e_r) \in E^r;\, p(e_1) = \cdots = p(e_r)\}.$$

Let $E^I_B \subset E^I$ be as above the space of all paths $\alpha : I \to E$ such that $p \circ \alpha : I \to B$ is constant. Fix r points

$$0 \leq t_1 < t_2 < \cdots < t_r \leq 1$$

in I (for example, one may take $t_i = (i-1)(r-1)^{-1}$ for $i = 1, 2, \ldots, r$), which will be called the *time schedule*. Consider the evaluation map

$$\Pi_r : E_B^I \to E_B^r, \quad \Pi_r(\alpha) = (\alpha(t_1), \alpha(t_2), \dots, \alpha(t_r)). \tag{3}$$

Π_r is a Hurewicz fibration, see [4, Appendix], the fibre of Π_r is $(\Omega X)^{r-1}$. A section $s : E_B^r \to E^I$ of the fibration Π_r can be interpreted as a parametrized motion planning algorithm, i.e. as a function which assigns to every sequence of points $(e_1, e_2, \dots, e_r) \in E_B^r$ a continuous path $\alpha : I \to E$ (motion of the system) satisfying $\alpha(t_i) = e_i$ for every $i = 1, 2, \dots, r$ and such that the path $p \circ \alpha : I \to B$ is constant. The latter condition means that the system moves under constant external conditions (such as positions of the obstacles).

Typically Π_r does not admit continuous sections; then the motion planning algorithms are necessarily discontinuous.

The following definition gives a measure of complexity of sequential parametrized motion planning algorithms. This concept is the main character of this paper.

3.1 Definition. The *r-th sequential parametrized topological complexity* of the fibration $p : E \to B$, denoted $\mathsf{TC}_r[p : E \to B]$, is defined as the sectional category of the fibration Π_r, i.e.

$$\mathsf{TC}_r[p : E \to B] := \mathsf{secat}(\Pi_r). \tag{4}$$

In more detail, $\mathsf{TC}_r[p : E \to B]$ is the minimal integer k such that there is a open cover $\{U_0, U_1, \dots, U_k\}$ of E_B^r with the property that each open set U_i admits a continuous section $s_i : U_i \to E_B^I$ of Π_r.

Let $B' \subset B$ be a subset and let $E' = p^{-1}(B')$ be its preimage, then obviously

$$\mathsf{TC}_r[p : E \to B] \geq \mathsf{TC}_r[p' : E' \to B']$$

where $p' = p|_{E'}$. In particular, taking B' to be a single point, we obtain

$$\mathsf{TC}_r[p : E \to B] \geq \mathsf{TC}_r(X),$$

where X is the fibre of p.

3.2 Example. Let $p : E \to B$ be a trivial fibration with fibre X, i.e. $E = B \times X$. In this case we have $E_B^r = B \times X^r$, $E_B^I = B \times X^I$ and the map $\Pi_r : E_B^I \to E_B^r$ becomes

$$\Pi_r : B \times X^I \to B \times X^r, \quad \Pi_r = \mathsf{Id}_B \times \pi_r,$$

where $\mathsf{Id}_B : B \to B$ is the identity map and π_r is the fibration (2). Thus we obtain in this example

$$\mathsf{TC}_r[p : E \to B] = \mathsf{TC}_r(X),$$

i.e. for the trivial fibration the sequential parametrized topological complexity equals the sequential topological complexity of the fibre.

3.3 Proposition. *Let $p : E \to B$ be a principal bundle with a connected topological group G as fibre. Then*

$$\mathsf{TC}_r[p : E \to B] = \mathsf{cat}(G^{r-1}) = \mathsf{TC}_r(G).$$

Proof. Let $0 \leq t_1 < t_2 < \dots < t_r \leq 1$ be the fixed time schedule. Denote by $P_0G \subset G^I$ the space of paths α satisfying $\alpha(t_1) = e$ where $e \in G$ denotes the unit element. Consider the evaluation map $\pi_r' : P_0G \to G^{r-1}$ where $\pi_r'(\alpha) = (\alpha(t_2), \alpha(t_3), \dots, \alpha(t_r))$. We obtain the commutative diagram

$$\begin{array}{ccc} P_0G \times E & \xrightarrow{F} & E_B^I \\ {\scriptstyle \pi'_r \times \mathsf{Id}} \downarrow & & \downarrow {\scriptstyle \Pi_r} \\ G^{r-1} \times E & \xrightarrow[F']{} & E_B^r \end{array}$$

where $F : P_0G \times E \to E_B^I$ and $F' : G^{r-1} \times E \to E_B^r$ are homeomorphisms given by

$$F(\alpha, x)(t) = \alpha(t)x, \quad F'(g_2, g_3, \dots, g_r, x) = (x, g_2x, g_3x, \dots, g_rx),$$

where $\alpha \in P_0G$, $x \in E$, $t \in I$ and $g_i \in G$. Thus we have

$$\mathsf{TC}_r[p : E \to B] = \mathsf{secat}(\Pi_r) = \mathsf{secat}(\pi'_r \times \mathsf{Id}) = \mathsf{secat}(\pi'_r).$$

Clearly, $\mathsf{secat}(\pi'_r) = \mathsf{cat}(G^{r-1})$ since P_0G is contractible. And finally $\mathsf{cat}(G^{r-1}) = \mathsf{TC}_r(G)$, see [1, Theorem 3.5]. □

3.4 Example. As a specific example consider the Hopf fibration $p : S^3 \to S^2$ with fibre S^1. Applying the result of the previous Proposition we obtain

$$\mathsf{TC}_r[p : S^3 \to S^2] = \mathsf{TC}_r(S^1) = r - 1$$

for any $r \geq 2$.

Alternative descriptions of sequential parametrized topological complexity. Let K be a path-connected finite CW-complex and let $k_1, k_2, \cdots, k_r \in K$ be a collection of r pairwise distinct points of K, where $r \geq 2$. For a Hurewicz fibration $p : E \to B$, consider the space E_B^K of all continuous maps $\alpha : K \to E$ such that the composition $p \circ \alpha : K \to B$ is a constant map. We equip E_B^K with the compact-open topology induced from the function space E^K. Consider the evaluation map

$$\Pi_r^K : E_B^K \to E_B^r, \quad \Pi_r^K(\alpha) = (\alpha(k_1), \alpha(k_2), \cdots, \alpha(k_r)) \quad \text{for} \quad \alpha \in E_B^K.$$

It is known that Π_r^K is a Hurewicz fibration, see Appendix to [4].

3.5 Lemma. *For any path-connected finite CW-complex K and a set of pairwise distinct points $k_1, \dots, k_r \in K$ one has*

$$\mathsf{secat}(\Pi_r^K) = \mathsf{TC}_r[p : E \to B].$$

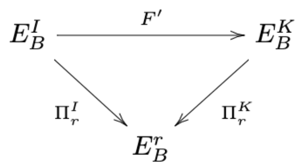
Proof. Let $0 \leq t_1 < t_2 < \cdots < t_r \leq 1$ be a given time schedule used in the definition of the map $\Pi_r = \Pi_r^I$ given by (3). Since K is path-connected we may find a continuous map $\gamma : I \to K$ with $\gamma(t_i) = k_i$ for all $i = 1, 2, \dots, r$. We obtain a continuous map $F_\gamma : E_B^K \to E_B^I$ acting by the formula $F_\gamma(\alpha) = \alpha \circ \gamma$. It is easy to see that the following diagram commutes

$$\begin{array}{ccc} E_B^K & \xrightarrow{F} & E_B^I \\ & {\scriptstyle \Pi_r^K} \searrow \quad \swarrow {\scriptstyle \Pi_r^I} & \\ & E_B^r & \end{array}$$

Using statement (A) of Lemma 2.2 we obtain

$$\mathsf{TC}_r[p : E \to B] = \mathsf{secat}(\Pi_r^I) \leq \mathsf{secat}(\Pi_r^K).$$

To obtain the inverse inequality note that any locally finite CW-complex is metrisable. Applying Tietze extension theorem we can find continuous functions $\psi_1, \dots, \psi_r : K \to [0, 1]$ such that $\psi_i(k_j) = \delta_{ij}$, i.e. $\psi_i(k_j)$ equals 1 for $j = i$ and it equals 0 for $j \neq i$. The function $f = \min\{1, \sum_{i=1}^r t_i \cdot \psi_i\} : K \to [0, 1]$ has the property that $f(k_i) = t_i$ for every $i = 1, 2, \dots, r$. We obtain a continuous map $F' : E_B^I \to E_B^K$, where $F'(\beta) = \beta \circ f$, $\beta \in E_B^I$, which appears in the commutative diagram



By Lemma 2.2 this implies the opposite inequality $\mathsf{secat}(\Pi_r^K) \leq \mathsf{secat}(\Pi_r^I)$ and completes the proof. □

The following proposition is an analogue of [3, Proposition 4.7].

3.6 Proposition. *Let E and B be metrisable separable ANRs and let $p : E \to B$ be a locally trivial fibration. Then the sequential parametrized topological complexity $\mathsf{TC}_r[p : E \to B]$ equals the smallest integer n such that E_B^r admits a partition*

$$E_B^r = F_0 \sqcup F_1 \sqcup ... \sqcup F_n, \quad F_i \cap F_j = \emptyset \ \text{ for } i \neq j,$$

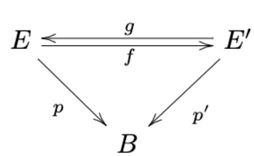
with the property that on each set F_i there exists a continuous section $s_i : F_i \to E_B^I$ of Π_r. In other words,

$$\mathsf{TC}_r[p : E \to B] = \mathsf{secat}_g[\Pi_r : E_B^I \to E_B^r].$$

Proof. From the results of [2, Chapter IV] it follows that the fibre X of $p : E \to B$ is ANR and hence X^r is also ANR. Now, E_B^r is the total space of the locally trivial fibration $E_B^r \to B$ with fibre X^r. Thus, applying [2, Chapter IV, Theorem 10.5], we obtain that the space E_B^r is ANR. Using [3, Proposition 4.7] we see that E_B^I is ANR. Finally, using Theorem 2.1, we conclude that $\mathsf{TC}_r[p : E \to B] = \mathsf{secat}_g[\Pi_r : E_B^I \to E_B^r]$. □

4. Fibrewise homotopy invariance

4.1 Proposition. *Let $p : E \to B$ and $p' : E' \to B$ be two fibrations and let $f : E \to E'$ and $g : E' \to E$ be two continuous maps such the following diagram commutes*



i.e. $p = p' \circ f$ and $p' = p \circ g$. If the map $g \circ f : E \to E$ is fibrewise homotopic to the identity map $\mathsf{Id}_E : E \to E$ then

$$\mathsf{TC}_r[p : E \to B] \leq \mathsf{TC}_r[p' : E' \to B].$$

Proof. Denote by $f^r : E^r_B \to E'^r_B$ the map given by $f^r(e_1, \dots, e_r) = (f(e_1), \dots, f(e_r))$ and by $f^I : E^I_B \to E'^I_B$ the map given by $f^I(\gamma)(t) = f(\gamma(t))$ for $\gamma \in E^I_B$ and $t \in I$. One defines similarly the maps $g^r : E'^r_B \to E^r_B$ and $g^I : E'^I_B \to E^I_B$. This gives the commutative diagram

$$\begin{array}{ccccc} E^I_B & \xrightarrow{f^I} & E'^I_B & \xrightarrow{g^I} & E^I_B \\ \downarrow{\scriptstyle \Pi_r} & & \downarrow{\scriptstyle \Pi'_r} & & \downarrow{\scriptstyle \Pi_r} \\ E^r_B & \xrightarrow[f^r]{} & E'^r_B & \xrightarrow[g^r]{} & E^r_B, \end{array}$$

in which $g^r \circ f^r \simeq \mathsf{Id}_{E^r_B}$. Applying statement (C) of Lemma 2.2 we obtain

$$\begin{aligned} \mathsf{TC}_r[p : E \to B] &= \mathsf{secat}[\Pi_r : E^I_B \to E^r_B] \\ &\leq \mathsf{secat}[\Pi'_r : E'^I_B \to E'^r_B] \\ &= \mathsf{TC}_r[p' : E' \to B]. \quad \square \end{aligned}$$

Proposition 4.1 obviously implies the following property of $\mathsf{TC}_r[p : E \to B]$:

4.2 Corollary. *If fibrations $p : E \to B$ and $p' : E' \to B$ are fibrewise homotopy equivalent then*

$$\mathsf{TC}_r[p : E \to B] = \mathsf{TC}_r[p' : E' \to B].$$

5. Further properties of $\mathsf{TC}_r[p : E \to B]$

Next we consider products of fibrations:

5.1 Proposition. *Let $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$ be two fibrations where the spaces E_1, E_2, B_1, B_2 are metrisable. Then for any $r \geq 2$ we have*

$$\mathsf{TC}_r[p_1 \times p_2 : E_1 \times E_2 \to B_1 \times B_2] \leq \mathsf{TC}_r[p_1 : E_1 \to B_1] + \mathsf{TC}_r[p_2 : E_2 \to B_2].$$

Proof. The proof is essentially identical to the proof of [3, Proposition 6.1] where it is done for the case $r = 2$. $\square$

5.2 Proposition. *Let $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ be two fibrations where the spaces E_1, E_2, B are metrisable. Consider the fibration $p : E \to B$ where $E = E_1 \times_B E_2 = \{(e_1, e_2) \in E_1 \times E_2 \mid p_1(e_1) = p_2(e_2)\}$ and $p(e_1, e_2) = p(e_1) = p(e_2)$. Then*

$$\mathsf{TC}_r[p : E \to B] \leq \mathsf{TC}_r[p_1 : E_1 \to B] + \mathsf{TC}_r[p_2 : E_2 \to B].$$

Proof. Viewing B as the diagonal of $B \times B$ gives

$$\mathsf{TC}_r[p : E \to B] \leq \mathsf{TC}_r[p_1 \times p_2 : E_1 \times E_2 \to B \times B].$$

Combining this inequality with the result of Proposition 5.1 completes the proof. $\square$

5.3 Lemma. *For any fibration $p : E \to B$ one has*

$$\mathsf{TC}_{r+1}[p : E \to B] \geq \mathsf{TC}_r[p : E \to B].$$

Proof. We shall apply Lemma 3.5 and consider the interval $K = [0, 2]$ and the time schedule $0 \le t_1 < t_2 < \cdots < t_r \le 1$ and the additional point $t_{r+1} = 2$. We have the following diagram

$$\begin{array}{ccccc} E_B^I & \xrightarrow{F} & E_B^K & \xrightarrow{G} & E_B^I \\ \downarrow{\scriptstyle \Pi_r} & & \downarrow{\scriptstyle \Pi_{r+1}^K} & & \downarrow{\scriptstyle \Pi_r} \\ E_B^r & \xrightarrow[f]{} & E_B^{r+1} & \xrightarrow[g]{} & E_B^r, \end{array}$$

where f acts by the formula $f(e_1, e_2, \dots, e_r) = (e_1, e_2, \dots, e_r, e_r)$ and F sends a path $\gamma : I \to E$, $\gamma \in E_B^I$, to the path $\bar{\gamma} : K = [0, 2] \to E$ where $\bar{\gamma}|_{[0,1]} = \gamma$ and $\bar{\gamma}(t) = \gamma(1)$ for any $t \in [1, 2]$. The vertical maps are evaluations at the points $t_1, \dots, t_r$ and at the points $t_1, \dots, t_r, t_{r+1}$, for Π_r and Π_{r+1}^K correspondingly. The map G is the restriction, it maps $\alpha : K \to E$ to $\alpha_I : I \to E$. Similarly, the map $g : E_B^{r+1} \to E_B^r$ is given by $(e_1, \dots, e_r, e_{r+1}) \mapsto (e_1, \dots, e_r)$. The diagram commutes and besides the composition $g \circ f : E_B^r \to E_B^r$ is the identity map. Applying statement (C) of Lemma 2.2 we obtain

$$\begin{aligned} \mathsf{TC}_r[p : E \to B] &= \mathsf{secat}[\Pi_r : E_B^I \to E_B^r] \\ &\le \mathsf{secat}[\Pi_{r+1}^K : E_B^K \to E_B^{r+1}] \\ &= \mathsf{TC}_{r+1}[p : E \to B]. \quad \square \end{aligned}$$

6. Upper and lower bounds for $\mathsf{TC}_r[p : E \to B]$

In this section we give upper and lower bound for sequential parametrized topological complexity.

6.1 Proposition. *Let $p : E \to B$ be a locally trivial fibration with fiber X, where E, B, X are CW-complexes. Assume that the fiber X is k-connected, where $k \ge 0$. Then*

$$\mathsf{TC}_r[p : E \to B] < \frac{\mathsf{hdim}(E_B^r) + 1}{k + 1} \le \frac{r \cdot \mathsf{dim} X + \mathsf{dim} B + 1}{k + 1}. \tag{5}$$

Proof. Since X is k-connected, the loop space ΩX is $(k - 1)$-connected and hence the space $(\Omega X)^{r-1}$ is also $(k - 1)$-connected. Thus, the fibre of the fibration $\Pi_r : E_B^I \to E_B^r$ is $(k - 1)$-connected and applying Theorem 5 from [20] we obtain:

$$\mathsf{TC}_r[p : E \to B] \; = \; \mathsf{secat}(\Pi_r) \; < \; \frac{\mathsf{hdim}(E_B^r) + 1}{k + 1}, \tag{6}$$

where $\mathsf{hdim}(E_B^r)$ denotes homotopical dimension of E_B^r, i.e. the minimal dimension of a CW-complex homotopy equivalent to E_B^r,

$$\mathsf{hdim}(E_B^r) := \min\{\mathsf{dim}\, Z |\, Z \text{ is a CW-complex homotopy equivalent to } E_B^r\}.$$

Clearly, $\mathsf{hdim}(E_B^r) \le \mathsf{dim}(E_B^r)$. The space E_B^r is the total space of a locally trivial fibration with base B and fibre X^r. Hence, $\mathsf{dim}(E_B^r) \le \mathsf{dim}(X^r) + \mathsf{dim} B = r \cdot \mathsf{dim} X + \mathsf{dim} B$. Combining this with (6), we obtain (5). $\square$

Below we shall use the following classical result of A.S. Schwarz [20]:

6.2 Lemma. *For any fibration $p : E \to B$ and coefficient ring R, if there exist cohomology classes $u_1, \cdots, u_k \in \ker[p^* : H^*(B;R) \to H^*(E;R)]$ such that their cup-product is nonzero, $u_1 \cup \cdots \cup u_k \neq 0 \in H^*(B;R)$, then $\mathsf{secat}(p) \geq k$.*

The following Proposition gives a simple and powerful lower bound for sequential parametrized topological complexity.

6.3 Proposition. *For a fibration $p : E \to B$, consider the diagonal map $\Delta : E \to E^r_B$ where $\Delta(e) = (e, e, \cdots, e)$, and the induced by Δ homomorphism in cohomology $\Delta^* : H^*(E^r_B;R) \to H^*(E;R)$ with coefficients in a ring R. If there exist cohomology classes*

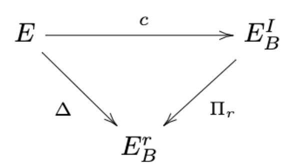
$$u_1, \cdots, u_k \in \ker[\Delta^* : H^*(E^r_B;R) \to H^*(E;R)]$$

such that

$$u_1 \cup \cdots \cup u_k \neq 0 \in H^*(E^r_B;R)$$

then $\mathsf{TC}_r[p : E \to B] \geq k$.

Proof. Define the map $c : E \to E^I_B$ where $c(e)(t) = e$ is the constant path. Note that the map $c : E \to E^I_B$ is a homotopy equivalence. Besides, the following diagram commutes



and thus, $\ker[\Pi_r^* : H^*(E^r_B;R) \to H^*(E^I_B;R)] = \ker[\Delta^* : H^*(E^r_B;R) \to H^*(E;R)]$. The result now follows from Lemma 6.2 and from the definition $\mathsf{TC}_r[p : E \to B] = \mathsf{secat}(\Pi_r)$. $\square$

7. Cohomology algebras of certain configuration spaces

In this section we present auxiliary results about cohomology algebras of relevant configuration spaces which will be used later in this paper for computing the sequential parametrized topological complexity of the Fadell - Neuwirth fibration.

All cohomology groups will be understood as having the integers as coefficients although the symbol $\mathbb{Z}$ will be skipped from the notations.

We start with the following well-known fact, see [6, Chapter V, Theorem 4.2]:

7.1 Lemma. *The integral cohomology ring $H^*(F(\mathbb{R}^d, m+n))$ contains $(d-1)$-dimensional cohomology classes ω_{ij}, where $1 \leq i < j \leq m+n$, which multiplicatively generate $H^*(F(\mathbb{R}^d, m+n))$ and satisfy the following defining relations*

$$(\omega_{ij})^2 = 0 \quad \text{and} \quad \omega_{ip}\omega_{jp} = \omega_{ij}(\omega_{jp} - \omega_{ip}) \quad \text{for all } i < j < p.$$

The cohomology class ω_{ij} arises as follows. For $1 \leq i < j \leq m+n$, mapping a configuration $(u_1, \ldots, u_{m+n}) \in F(\mathbb{R}^d, m+n)$ to the unit vector

$$\frac{u_i - u_j}{||u_i - u_j||} \in S^{d-1},$$

defines a continuous map $\phi_{ij} : F(\mathbb{R}^d, m+n) \to S^{d-1}$, and the class

$$\omega_{ij} \in H^{d-1}(F(\mathbb{R}^d, m+n))$$

is defined by $\omega_{ij} = \phi_{ij}^*(v)$ where $v \in H^{d-1}(S^{d-1})$ is the fundamental class.

Below we shall denote by E the configuration space $E = F(\mathbb{R}^d, n+m)$. A point of E will be understood as a configuration

$$(o_1, o_2, \dots, o_m, z_1, z_2, \dots, z_n)$$

where the first m points $o_1, o_2, \dots, o_m$ represent "obstacles"while the last n points $z_1, z_2, \dots, z_n$ represent "robots". The map

$$p : F(\mathbb{R}^d, m+n) \to F(\mathbb{R}^d, m), \tag{7}$$

where

$$p(o_1, o_2, \dots, o_m, z_1, z_2, \dots, z_n) = (o_1, o_2, \dots, o_m),$$

is known as the Fadell - Neuwirth fibration. This map was introduced in [7] where the authors showed that p is a locally trivial fibration. The fibre of p over a configuration $\mathcal{O}_m = \{o_1, \dots, o_m\} \in F(\mathbb{R}^d, m)$ is the space $X = F(\mathbb{R}^d - \mathcal{O}_m, n)$, the configuration space of n pairwise distinct points lying in the complement of the set $\mathcal{O}_m = \{o_1, \dots, o_m\}$ of m fixed obstacles.

We plan to use Lemma 6.3 to obtain lower bounds for the topological complexity, and for this reason our first task will be to calculate the integral cohomology ring of the space E^r_B. Here E denotes the space $E = F(\mathbb{R}^d, m+n)$ and B denotes the space $B = F(\mathbb{R}^d, m)$ and $p : E \to B$ is the Fadell - Neuwirth fibration (7); hence E^r_B is the space of r-tuples $(e_1, e_2, \dots, e_r) \in E^r$ satisfying $p(e_1) = p(e_2) = \dots = p(e_r)$. Explicitly, a point of the space E^r_B can be viewed as a configuration

$$(o_1, o_2, \cdots o_m, z^1_1, z^1_2, \cdots z^1_n, z^2_1, z^2_2, \cdots, z^2_n, \cdots, z^r_1, z^r_2, \cdots, z^r_n) \tag{8}$$

of $m + rn$ points o_i, $z^l_j \in \mathbb{R}^d$ (for $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$ and $l = 1, 2, \dots, r$), such that

(1) $o_i \neq o_{i'}$ for $i \neq i'$,
(2) $o_i \neq z^l_j$ for $1 \le i \le m$, $1 \le j \le n$ and $1 \le l \le r$,
(3) $z^l_j \neq z^l_{j'}$ for $j \neq j'$.

The following statement is a generalisation of Proposition 9.2 from [3].

7.2 Proposition. *The integral cohomology ring $H^*(E^r_B)$ contains cohomology classes ω^l_{ij} of degree $d-1$, where $1 \le i < j \le m+n$ and $1 \le l \le r$, satisfying the relations*

(a) $\omega^l_{ij} = \omega^{l'}_{ij}$ *for* $1 \le i < j \le m$ *and* $1 \le l \le l' \le r$,
(b) $(\omega^l_{ij})^2 = 0$ *for* $i < j$ *and* $1 \le l \le r$,
(c) $\omega^l_{ip}\omega^l_{jp} = \omega^l_{ij}(\omega^l_{jp} - \omega^l_{ip})$ *for* $i < j < p$ *and* $1 \le l \le r$.

Proof. For $1 \le l \le r$, consider the projection map $q_l : E^r_B \to E$ which acts as follows: the configuration (8) is mapped into

$$(u_1, u_2, \ldots, u_{m+n}) \in E = F(\mathbb{R}^d, m+n)$$

where

$$u_i = \begin{cases} o_i & \text{for} \quad i \le m, \\ z^l_{i-m} & \text{for} \quad i > m. \end{cases}$$

Using Lemma 7.1 and the cohomology classes $\omega_{ij} \in H^{d-1}(E)$, we define

$$(q_l)^*(\omega_{ij}) = \omega^l_{ij} \in H^*(E^r_B).$$

Relations (a), (b), (c) are obviously satisfied. This completes the proof. □

For $1 \le i < j \le m$ we shall denote the class $\omega^l_{ij} \in H^{d-1}(E^r_B)$ simply by ω_{ij}; this is justified because of the relation (a) above.

We shall introduce notations for the classes which arise as the cup-products of the classes ω^l_{ij}. For $p \ge 1$ consider two sequences of integers $I = (i_1, i_2, \cdots, i_p)$ and $J = (j_1, j_2, \cdots, j_p)$ where $i_s, j_s \in \{1, 2, \ldots, m+n\}$ for $s = 1, 2, \ldots, p$. We shall say that the sequence J is *increasing* if either $p = 1$ or $j_1 < j_2 < \cdots < j_p$. Besides, we shall write $I < J$ if $i_s < j_s$ for all $s = 1, 2, \ldots, p$.

A pair of sequences $I < J$ of length p as above determines the cohomology class

$$\omega^l_{IJ} = \omega^l_{i_1 j_1}\omega^l_{i_2 j_2} \cdots \omega^l_{i_p j_p} \in H^{(d-1)p}(E^r_B)$$

for any $l = 1, 2, \ldots, r$. Note that the order of the factors is important in the case when the dimension d is even. Because of the property (a) of Proposition 7.2 this class is independent of l assuming that $j_p \le m$; for this reason, if $j_p \le m$, we shall denote ω^l_{IJ} simply by ω_{IJ}.

For formal reasons we shall allow $p = 0$. In this case the symbol ω^l_{IJ} will denote the unit $1 \in H^0(E^r_B)$.

The next result is a generalisation of [3, Proposition 9.3] where the case $r = 2$ was studied.

7.3 Proposition. *An additive basis of $H^*(E^r_B)$ is formed by the following set of cohomology classes*

$$\omega_{IJ}\omega^1_{I_1 J_1}\omega^2_{I_2 J_2} \cdots \omega^r_{I_r J_r} \in H^*(E^r_B) \quad \text{with} \quad I < J \quad \text{and} \quad I_i < J_i, \tag{9}$$

where:

(i) *the sequences $J, J_1, J_2, \cdots J_r$ are increasing,*
(ii) *the sequence J takes values in $\{2, 3, \cdots, m\}$,*
(iii) *the sequences $J_1, J_2, \ldots, J_r$ take values in $\{m+1, \cdots m+n\}$.*

Proof. Recall our notations: $E = F(\mathbb{R}^d, m+n)$, $B = F(\mathbb{R}^d, m)$ and $p : E \to B$ is the Fadell - Neuwirth fibration (7). Consider the fibration

$$p_r : E^r_B \to B \quad \text{where} \quad p_r(e_1, \ldots, e_r) = p(e_1) = \cdots = p(e_r).$$

Its fibre over a configuration $\mathcal{O}_m = (o_1, \ldots, o_m) \in B$ is X^r, the Cartesian product of r copies of the space X, where $X = F(\mathbb{R}^d - \mathcal{O}_m, n)$.

We shall apply Leray-Hirsch theorem to the fibration $p_r : E^r_B \to B$. The classes ω_{ij} with $i < j \le m$ originate from the base of this fibration. Moreover, from Lemma 7.1 it follows that a free additive basis of $H^*(B)$ forms the set of the classes ω_{IJ} where $I < J$ run over all sequences of elements of the set $\{1, 2, ..., m\}$ such that the sequence $J = (j_1, j_2, ..., j_p)$ is increasing.

Next consider the classes of the form

$$\omega^1_{I_1J_1}\omega^2_{I_2J_2}\cdots\omega^r_{I_rJ_r} \in H^*(E^r_B),$$

with increasing sequences $J_1, J_2, \dots, J_r$ satisfying (iii) above. Using the known results about the cohomology algebras of configuration spaces (see [6], Chapter V, Theorems 4.2 and 4.3) as well as the Kuñneth theorem, we see that the restrictions of the family of these classes onto the fiber X^r form a free basis in the cohomology of the fiber $H^*(X^r)$.

Hence, Leray-Hirsch theorem [16] is applicable and we obtain that a free basis of the cohomology $H^*(E^r_B)$ is given by the set of classes described in the statement of Proposition 7.3. This completes the proof. □

Proposition 7.3 implies:

7.4 Corollary. *Consider two basis elements $\alpha, \beta \in H^*(E^r_B)$*

$$\alpha = \omega_{IJ}\omega^1_{I_1J_1}\omega^2_{I_2J_2}\cdots\omega^r_{I_rJ_r} \quad \text{and} \quad \beta = \omega_{I'J'}\omega^1_{I'_1J'_1}\omega^2_{I'_2J'_2}\cdots\omega^r_{I'_rJ'_r},$$

satisfying the properties (i), (ii), (iii) *of Proposition 7.3. The product*

$$\alpha \cdot \beta \in H^*(E^r_B)$$

is another basis element up to sign (and hence is nonzero) if the sequences J and J' are disjoint and for every $k = 1, 2, \dots, r$ the sequences J_k and J'_k are disjoint.

There is a one-to-one correspondence between increasing sequences and subsets; this explains the meaning of the term "disjoint" applied to two increasing sequences.

Next we consider the situation when the product of basis elements is not a basis element but rather a linear combination of basis elements.

Let $J = (j_1, j_2, \dots, j_p)$ be an increasing sequence of positive integers, where $p \geq 2$, and let j be an integer satisfying $j_p < j$. Our goal is to represent the product

$$\omega^l_{j_1j}\omega^l_{j_2j}\dots\omega^l_{j_pj} \in H^{p(d-1)}(E^r_B), \quad l = 1, \dots, r,$$

as a linear combination of the basis elements of Proposition 7.3.

We say that a sequence $I = (i_1, i_2, \dots, i_p)$ with $p \geq 2$ is a *J-modification* if $i_1 = j_1$ and for $s = 2, 3, \dots, p$ each number i_s equals either i_{s-1} or j_s. An increasing sequence of length p has 2^{p-1} modifications. For example, for $p = 3$ the sequence $J = (j_1, j_2, j_3)$ has the following 4 modifications

$$(j_1, j_2, j_3),\ (j_1, j_1, j_3),\ (j_1, j_2, j_2),\ (j_1, j_1, j_1). \tag{10}$$

For a J-modification I we shall denote by $r(I)$ the number of repetitions in I. For instance, the numbers of repetitions of the modifications (10) are 0, 1, 1, 2 correspondingly.

The following statement is equivalent to Proposition 3.5 from [4]. Lemma 9.5 from [3] gives the answer in a recurrent form.

7.5 Lemma. *For a sequence $j_1 < j_2 < \cdots < j_p < j$ of positive integers, where $p \geq 2$, denote $J = (j_1, j_2, \dots, j_p)$ and $J' = (j_2, j_3, \dots, j_p, j)$. In the cohomology algebra $H^*(E^r_B)$ associated to the Fadell - Neuwirth fibration, one has the following relation*

$$\omega^l_{j_1j}\omega^l_{j_2j}\dots\omega^l_{j_pj} = \sum_I (-1)^{r(I)}\omega^l_{IJ'}, \tag{11}$$

where I runs over 2^{p-1} J-modifications and $l = 1, 2, \dots, r$.

Proof. First note that for any J-modification I one has $I < J'$ and hence the terms in the RHS of (11) make sense. We shall use induction in p. For $p = 2$ the statement of Lemma 7.5 is

$$\omega^l_{j_1 j}\omega^l_{j_2 j} = \omega^l_{j_1 j_2}\omega^l_{j_2 j} - \omega^l_{j_1 j_2}\omega^l_{j_1 j},$$

which is the familiar 3-term relation, see Proposition 7.2, statement (c). The first term on the right corresponds to the sequence $I = (j_1, j_2)$ and the second term corresponds to $I = (j_1, j_1)$; the latter has one repetition and appears with the minus sign.

Suppose now that Lemma 7.5 is true for all sequences J of length p. Consider an increasing sequence $J = (j_1, j_2, \dots, j_{p+1})$ of length $p+1$ and an integer j satisfying $j > j_{p+1}$. Denote by $K = (j_1, j_2, \dots, j_p)$ the shortened sequence and let $I = (i_1, i_2, \dots, i_p)$ be a modification of K. As in (11), denote $K' = (j_2, j_3, \dots, j_p, j)$. Consider the product

$$\begin{aligned}
\omega^l_{IK'}\omega^l_{j_{p+1}j} &= \omega^l_{i_1 j_2}\omega^l_{i_2 j_3}\dots\omega^l_{i_{p-1}j_p}\omega^l_{i_p j}\cdot\omega^l_{j_{p+1}j}\\
&= \left[\omega^l_{i_1 j_2}\omega^l_{i_2 j_3}\dots\omega^l_{i_{p-1}j_p}\right]\cdot\omega^l_{i_p j_{p+1}}\cdot\left[\omega^l_{j_{p+1}j} - \omega^l_{i_p j}\right]\\
&= \omega^l_{I_1 J'} - \omega^l_{I_2 J'}
\end{aligned}$$

where $I_1 = (i_1, \dots, i_p, j_{p+1})$ and $I_2 = (i_1, \dots, i_p, i_p)$ are the only two modifications of J extending I. The equality of the second line is obtained by applying the relation (c) of Proposition 7.2. Note that $r(I_1) = r(I)$ and $r(I_2) = r(I_1) + 1$ which is consistent with the minus sign. Thus, we see that Lemma follows by induction. □

Since each term in the RHS of (11) is a $\pm$ multiple of a basis element we obtain:

7.6 Corollary. *Any basis element (9) which appears with nonzero coefficient in the decomposition of the monomial*

$$\omega^l_{j_1 j}\omega^l_{j_2 j}\dots\omega^l_{j_p j}, \quad \text{where} \quad j_1 < j_2 < \dots < j_p < j, \tag{12}$$

contains a factor of the form $\omega^l_{j_s j}$, where $s \in \{1, 2, \dots, p\}$. Moreover,

$$\omega^l_{j_1 j_2}\omega^l_{j_1 j_3}\dots\omega^l_{j_1 j_p}\omega^l_{j_1 j}$$

is the only basis element in the decomposition of (12) which contains the factor $\omega^l_{j_1 j}$.

Consider the diagonal map

$$\Delta : E \to E^r_B, \quad \Delta(e) = (e, e, \dots, e), \quad e \in E.$$

7.7 Lemma. *The kernel of the homomorphism $\Delta^* : H^*(E^r_B) \to H^*(E)$ contains the cohomology classes of the form*

$$\omega^l_{ij} - \omega^{l'}_{ij}.$$

Proof. This follows directly from the definition of the classes ω^l_{ij}; compare the proof of Proposition 9.4 from [3]. □

8. Sequential parametrized topological complexity of the Fadell-Neuwirth bundle; the odd-dimensional case

Our goal is to compute the sequential parametrized topological complexity of the Fadell - Neuwirth bundle. As we shall see, the answers in the cases of odd and even dimension d are slightly different. When d is odd the cohomology algebra has only classes of even degree and is therefore commutative; in the case when d is even the cohomology algebra is skew-commutative which imposes major distinction in treating these two cases.

The main result of this section is:

8.1 Theorem. *For any odd $d \geq 3$, and for any $n \geq 1$, $m \geq 2$ and $r \geq 2$, the sequential parametrized topological complexity of the Fadell - Neuwirth bundle (7) equals $rn + m - 1$.*

This result was obtained in [3] for $r = 2$. Note that the special case of $d = 3$ is most important for robotics applications.

As in the previous section, we shall denote the Fadell - Neuwirth bundle (7) by $p : E \to B$ for short; this convention will be in force in this and in the following sections.

We start with the following statement which is valid without imposing restriction on the parity of the dimension $d \geq 3$. Note that for $d = 2$ we shall have a stronger upper bound in §9.

8.2 Proposition. *For any $d \geq 3$ and $m \geq 2$ one has*

$$\mathsf{TC}_r[p : E \to B] \leq rn + m - 1.$$

Proof. The space E^r_B is $(d-2)$-connected and in particular it is simply connected (since $d \geq 3$). By Proposition 7.3 the top dimension with nonzero cohomology is $(rn+m-1)(d-1)$. Hence the homotopical dimension of the configuration space $\mathsf{hdim}(E^r_B)$ equals $(rn+m-1)(d-1)$. Here we use the well-known fact that the homotopical dimension of a simply connected space with torsion free integral cohomology equals its cohomological dimension. The fibre $X = F(\mathbb{R}^d - \mathcal{O}_m, n)$ of the Fadell - Neuwirth bundle $p : E \to B$ is $(d-2)$-connected. Applying Proposition 6.1 we obtain

$$\mathsf{TC}_r[p : E \to B] < rn + m - 1 + \frac{1}{d-1},$$

which is equivalent to our statement. □

To complete the proof of Theorem 8.1 we only need to establish the lower bound:

8.3 Proposition. *For any odd $d \geq 3$ and $m \geq 2$ one has*

$$\mathsf{TC}_r[p : E \to B] \geq rn + m - 1.$$

Note that the assumption of this Proposition that the dimension d is odd is essential as Proposition 8.3 is false for d even, see below.

Proof. We shall use Lemma 6.3 and Propositions 7.2 and 7.3 and Lemma 7.7.

Consider the cohomology classes

$$x_1 = \prod_{i=2}^{m} (\omega^1_{i(m+1)} - \omega^2_{i(m+1)}) \in H^{(m-1)(d-1)}(E^r_B),$$

$$x_2 = \prod_{j=m+1}^{m+n} (\omega_{1j}^2 - \omega_{1j}^1)^2 \; = \; -2 \prod_{j=m+1}^{m+n} \omega_{1j}^1 \omega_{1j}^2 \; \in \; H^{2n(d-1)}(E_B^r),$$

$$x_3 = \prod_{l=3}^{r} \prod_{j=m+1}^{m+n} (\omega_{1j}^l - \omega_{1j}^1) \; \in H^{n(r-2)(d-1)}(E_B^r).$$

Each of these classes is a product of elements of the kernel of the homomorphism $\Delta^* : H^*(E_B^r) \to H^*(E)$, by Lemma 7.7. Proposition 8.3 would follow once we show that the product

$$x_1 x_2 x_3 \neq 0 \; \in \; H^*(E_B^r)$$

is nonzero. By Proposition 7.3, the product $x_1x_2x_3$ is a linear combination of the basis cohomology classes and it is nonzero if at least one coefficient in this decomposition does not vanish.

According to [3], cf. page 248, the product x_1x_2 contains the basis element

$$\omega_{I_0 J_0} \omega_{IJ}^1 \omega_{I'J}^2 \; \in \; H^{(2n+m-1)(d-1)}(E_B^r) \tag{13}$$

with a nonzero coefficient; here

$$I_0 = (1, 2, 2, \ldots, 2), \quad J_0 = (2, 3, \ldots, m),$$

and

$$I = (1, 1, \ldots, 1), \quad I' = (2, 1, 1, \ldots, 1), \quad J = (m+1, m+2, \ldots, m+n),$$

with $|I_0| = |J_0| = m - 1$ and $|I| = |I'| = |J| = n$.

The product representing x_3 can be expanded into a sum. This sum contains the class $\prod_{l=3}^{r} \omega_{IJ}^l$ and each of the other terms contains a factor of type ω_{1j}^1. Since obviously $x_1x_2\omega_{1j}^1 = 0$, we obtain that the product $x_1x_2x_3$ contains the basis element

$$\omega_{I_0 J_0} \cdot \omega_{IJ}^1 \cdot \omega_{I'J}^2 \cdot \prod_{l=3}^{r} \omega_{IJ}^l$$

with a nonzero coefficient. Hence $x_1x_2x_3 \neq 0$ is nonzero. This completes the proof of Proposition 8.3. □

8.4 Remark. The lower bound estimate of Proposition 8.3 fails to work in the case when the dimension d of the ambient Euclidean space is even. Indeed, then the classes ω_{ij}^l have odd degree (which equals $d - 1$) and the square of any class of odd degree vanishes (since the cohomology algebra $H^*(E_B^r)$ with integral coefficients is torsion free). Thus, in the case of even dimension d the product x_2 vanishes. In the following section we shall suggest a different estimate for d even.

9. Sequential parametrized topological complexity of the Fadell-Neuwirth bundle; the even-dimensional case

In this section we give a lower bound for $\mathsf{TC}_r[p : E \to B]$ for the Fadell - Neuwirth bundle (7) in the case when the dimension d of the Euclidean space $\mathbb{R}^d$ is even. We also prove a matching upper bound for the planar case $d = 2$. Such an upper bound can be obtained for any even d by a totally different method; this material will be presented in another publication.

First we establish the following lower bound which is valid for any d regardless of its parity.

9.1 Proposition. *For any $d \geq 2$, $r \geq 2$ and $m \geq 2$, the sequential parametrized topological complexity of the Fadell - Neuwirth bundle satisfies*

$$\mathsf{TC}_r[p : E \to B] \geq rn + m - 2. \tag{14}$$

Proof. As an illustration, consider first the special case $m = 2$ and $n = 1$, i.e. the situation when we have one robot and two obstacles. Then the product of r classes

$$(\omega_{23}^1 - \omega_{23}^2) \cdot \prod_{l=2}^{r} (\omega_{13}^l - \omega_{13}^1) \tag{15}$$

lying in the kernel of Δ^* contains the basis element

$$\omega_{23}^1 \cdot \prod_{l=2}^{r} \omega_{13}^l \tag{16}$$

with a nonzero coefficient. Indeed, (15) equals $(\omega_{23}^1 - \omega_{23}^2) \cdot \left[\prod_{l=2}^{r} \omega_{13}^l - \omega_{13}^1 \cdot \alpha\right]$ where α is a polynomial in the classes ω_{13}^l with $l \in \{2, \dots, r\}$. Opening the brackets gives

$$\omega_{23}^1 \cdot \prod_{l=2}^{r} \omega_{13}^l - \omega_{23}^2 \cdot \prod_{l=2}^{r} \omega_{13}^l - \omega_{23}^1 \omega_{13}^1 \alpha + \omega_{23}^2 \omega_{13}^1 \alpha.$$

Here the second and the third terms are the sums of basis elements each containing the factor ω_{12} and hence distinct from (16). The basis elements of the fourth term all contain the factor ω_{13}^1 and therefore are also distinct from (16). Thus, (15) is nonzero and Lemma 6.3 gives the desired lower bound in the case $m = 2$, $n = 1$.

Returning to the general case, consider the following three cohomology classes $x_1, x_2, x_3 \in H^*(E_B^r)$, where

$$x_1 = \prod_{i=2}^{m} (\omega_{i(m+1)}^1 - \omega_{i(m+1)}^2) \in H^{(m-1)(d-1)}(E_B^r),$$

$$x_2 = \prod_{j=m+2}^{m+n} (\omega_{(j-1)j}^1 - \omega_{(j-1)j}^2) \in H^{(n-1)(d-1)}(E_B^r),$$

$$x_3 = \prod_{l=2}^{r} \prod_{j=m+1}^{m+n} (\omega_{1j}^l - \omega_{1j}^1) \in H^{n(r-1)(d-1)}(E_B^r).$$

Note that in the case when $n = 1$ the class x_2 is not defined; however, the arguments below show that in the case $n = 1$ the class $x_1 x_3$ (which is the product of $r + m - 2$ classes lying in the kernel of Δ^*) is nonzero.

Each of the classes x_1, x_2, x_3 is the product of elements of the kernel of Δ^*, see Lemma 7.7, and the total number of the factors is $rn + m - 2$. Hence, by Lemma 6.3, our statement (14) will follow once we know that the product $x_1 x_2 x_3 \neq 0 \in H^*(E_B^r)$ is nonzero.

Consider the following sequences

$$\begin{aligned}
I_0 &= (2, 2, \dots, 2), && \text{where} \quad |I_0| = m - 2, \\
J_0 &= (3, 4, \dots, m), && \text{where} \quad |J_0| = m - 2, \\
I &= (1, 1, \dots, 1), && \text{where} \quad |I| = n, \\
K &= (2, m + 1, m + 2, \dots, m + n - 1), && \text{where} \quad |K| = n, \\
J &= (m + 1, m + 2, \dots, m + n), && \text{where} \quad |J| = n.
\end{aligned}$$

We claim that the basis element

$$\omega_{I_0J_0}\omega^1_{KJ}\omega^2_{IJ}\omega^3_{IJ}\dots\omega^r_{IJ} \tag{17}$$

appears in the decomposition of the product $x_1x_2x_3$ with a nonzero coefficient.

In the special case $n = 1$ the class (17) has the form

$$\omega_{23}\omega_{24}\dots\omega_{2m}\cdot\omega^1_{2(m+1)}\cdot\prod_{l=2}^{r}\omega^l_{1(m+1)}. \tag{18}$$

Consider the basis elements which appear in the decomposition of the class x_1. For $m = 2$ the class x_1 equals $\omega^1_{23} - \omega^2_{23}$ and for $m > 2$ we can write

$$x_1 = \sum_{R\subset[m]} \pm\left(\prod_{i\in R}\omega^1_{i(m+1)}\cdot\prod_{i\in R^c}\omega^2_{i(m+1)}\right), \tag{19}$$

where R runs over all subsets (including $R = \emptyset$) of the set $[m] = \{2, 3, \dots, m\}$ and R^c denotes the complement $[m] - R$. The terms of (19) are basis elements for $m = 2$; for $m > 2$ they can be decomposed into basis elements using Lemma 7.5. For example, taking $R = [m]$ and applying Lemma 7.5 we find that one of the 2^{m-2} basis elements which appear in the decomposition of the product $\prod_{i=2}^m \omega^1_{i(m+1)}$ is the class

$$\omega_{23}\omega_{24}\dots\omega_{2m}\omega^1_{2(m+1)} = \omega_{I_0J_0}\omega^1_{2(m+1)}. \tag{20}$$

This class clearly is a factor of (17). For $m > 2$ each other basis elements in the decomposition of $\prod_{i=2}^m \omega^1_{i(m+1)}$ has a factor of type $\omega^1_{i(m+1)}$ with $2 < i \le m$, see Corollary 7.6.

Note also the basic elements of the form

$$\omega_{23}\omega_{24}\dots\omega_{2(m-1)}\omega_{2m}\omega^2_{k(m+1)} = \omega_{I_0J_0}\omega^2_{k(m+1)}, \quad \text{where} \quad 2 \le k \le m, \tag{21}$$

which arise in the basic element decomposition of the summand of (19) with $R = \emptyset$.

The basis element decomposition of x_2 is given by

$$\sum_S \pm\left(\prod_{j\in S}\omega^1_{(j-1)j}\cdot\prod_{j\in S^c}\omega^2_{(j-1)j}\right), \tag{22}$$

where S runs over all subsets $S \subset \{m + 2, m + 3, \dots, m + n\}$, including $S = \emptyset$. The symbol S^c denotes the complement $\{m + 2, m + 3, \dots, m + n\} - S$. Taking $S = \{m + 2, m + 3, \dots, m + n\}$ in (22) gives the class ω^1_{KJ}, without the factor $\omega^1_{2(m+1)}$, which is a factor of (17). Note that the missing factor $\omega^1_{2(m+1)}$ appears in (20).

The basis element decomposition of the class x_3 is given by

$$\sum_{T_2,\dots,T_r} \pm\,\omega^1_{I_1T_1}\omega^2_{I_2T_2}\omega^3_{I_3T_3}\dots\omega^r_{I_rT_r}, \tag{23}$$

where $T_2, T_3, \dots, T_r$ run over subsets of the set $\{m+1, m+2, \dots, m+n\}$ such that every two of these sets cover $\{m+1, m+2, \dots, m+n\}$ and $T_1 = \cup_{j=2}^r T_j^c$ where T_j^c stands for the complement $\{m+1, m+2, \dots, m+n\} - T_j$. We identify the subsets of $\{m + 1, m + 2, \dots, m + n\}$ with increasing sequences in the obvious way. The sequences $I_1, I_2, \dots, I_r$ in (23) all have the form $(1, 1, \dots, 1)$. Taking in (23) $T_2 = T_3 = \dots = T_r = J$ gives the class $\omega^2_{IJ}\omega^3_{IJ}\dots\omega^r_{IJ}$ which is a factor of (17).

We have seen that the class (17) appears as a product of specific basis elements in the decomposition of x_1, x_2 and x_3. We show below that the class (17) appears *only* with the set of choices indicated above and hence it cannot be cancelled.

Firstly, we note that only x_3 involves terms ω^l_{ij} with $l \geq 3$ and $j \geq m+1$. Therefore the only choice $T_3 = T_4 = \cdots = T_r = J$ in (23) may possibly lead to (17).

Secondly, the basis elements in the decompositions of x_2 and x_3 have no factors ω^l_{ij} with $j \leq m$. Hence the factor $\omega_{I_0 J_0}$ of (17) may only arise from the basis elements of the decomposition of x_1. It is clear that this may happen either when $R = [m]$ with (20) corresponding to the modification $(2, 2, \dots, 2)$ of the sequence $(2, 3, \dots, m)$, or with $R = \emptyset$, see above. Any basis element of x_1 distinct from (20) has either a factor of type $\omega^1_{i(m+1)}$ with $3 \leq i \leq m$ or a factor of type $\omega^2_{k(m+1)}$ with $2 \leq k \leq m$. Such factors do not appear in (17). If the set T_1 in (23) contains $m+1$ then we could have the factor

$$\omega^1_{i(m+1)}\omega^1_{1(m+1)} = \pm\omega_{1i}(\omega^1_{i(m+1)} - \omega^1_{1(m+1)})$$

with the factor ω_{1i} missing in (17). Similarly, the set T_2 might contain $m+1$ leading to the product

$$\omega^2_{k(m+1)}\omega^2_{1(m+1)} = \pm\omega_{1k}(\omega^2_{k(m+1)} - \omega^2_{1(m+1)})$$

with the factor ω_{1k} being absent in (17). Thus, we see that (20) is the only basis element of the decomposition of x_1 which can contribute into (17).

Comparing (22) and (17) and using Corollary 7.4 we see that the only basis element of the sum (22) with $S = \{m+2, m+3, \dots, m+n\}$ can contribute into (17). This basis element, together with the factor $\omega^1_{2(m+1)}$, gives ω^1_{KJ}.

Finally, examining (23), we see that the only way obtaining (17) is by taking $T_2 = J$ and hence $T_1 = \emptyset$, since, as we established earlier, one must have $T_3 = \cdots = T_r = J$ and $T_1 = \cup_{j=2}^r T_j^c$.

Thus, the basis element (17) appears in the decomposition of the product $x_1x_2x_3$ with a nonzero coefficient and hence $x_1x_2x_3 \neq 0$. This completes the proof. □

Next we state the main result of this section:

9.2 Theorem. *For any $m \geq 2$, $n \geq 1$ and $r \geq 2$, the r-th sequential parametrized topological complexity of the Fadell-Neuwirth bundle in the plane is given by*

$$\mathsf{TC}_r[p : F(\mathbb{R}^2, n+m) \to F(\mathbb{R}^2, m)] = rn + m - 2.$$

Proof. Proposition 9.1 gives the lower bound. In the proof below we establish the upper bound. We shall adopt the method developed in [4]. As in [4], we identify $\mathbb{R}^2$ with the set of complex numbers $\mathbb{C}$ and for any $s \geq 3$ consider the homeomorphism

$$h_s : F(\mathbb{C}, s) \to F(\mathbb{C} \setminus \{0, 1\}, s-2) \times F(\mathbb{C}, 2)$$

given by

$$h_s(u_1, u_2, ..., u_s) = \left(\left(\frac{u_3 - u_1}{u_2 - u_1}, \frac{u_4 - u_1}{u_2 - u_1}, ..., \frac{u_s - u_1}{u_2 - u_1}\right), (u_1, u_2)\right),$$

where $u_i \in \mathbb{C}$, $u_i \neq u_j$ for $i \neq j$. Thus, using the algebraic structure of complex numbers we may split the configuration space into a product. We have the following commutative diagram

$$\begin{array}{ccc} F(\mathbb{C}, n+m) & \xrightarrow{h_{n+m}} & F(\mathbb{C} \smallsetminus \{0,1\}, n+m-2) \times F(\mathbb{C}, 2) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q\times \mathsf{Id}} \\ F(\mathbb{C}, m) & \xrightarrow[h_m]{} & F(\mathbb{C} \smallsetminus \{0,1\}, m-2) \times F(\mathbb{C}, 2) \end{array}$$

where p is the Fadell - Neuwirth fibration, q is analogue of the Fadell - Neuwirth bundle for the plane with points $0, 1$ removed and with $m-2$ obstacles, and Id is the identity map. In the case when $m = 2$ we shall consider the space $F(\mathbb{C} \smallsetminus \{0,1\}, m-2)$ as consisting of a single point; then the diagram above will make sense for $m = 2$ (two obstacles only) as well.

Noting that $\mathsf{TC}_r[\mathsf{Id} : F(\mathbb{C}, 2) \to F(\mathbb{C}, 2)] = 0$ and applying Proposition 5.1 we obtain

$$\mathsf{TC}_r[p : F(\mathbb{C}, n+m) \to F(\mathbb{C}, m)] \leq \mathsf{TC}_r[q : E' \to B'],$$

where $E' = F(\mathbb{C}\smallsetminus\{0,1\}, n+m-2)$ and $B' = F(\mathbb{C}\smallsetminus\{0,1\}, m-2)$. The fibre of the fibration $q : E' \to B'$ is the configuration space $F(\mathbb{C} \smallsetminus \mathcal{O}_m, n)$, which is connected and has homotopical dimension n. The homotopical dimension of the base $F(\mathbb{C} \smallsetminus \{0,1\}, m-2)$ is $m-2$. Proposition 6.1 gives $\mathsf{TC}_r[q : E' \to B'] \leq rn + m - 2$. Hence,

$$\mathsf{TC}_r[p : F(\mathbb{C}, n+m) \to F(\mathbb{C}, m)] \leq rn + m - 2.$$

This completes the proof. □

9.3 Remark. Theorems 8.1 and 9.2 leave unanswered the question about the sequential parametrized topological complexity for the Fadell - Neuwirth bundle for $d \geq 4$ even. The upper bound of Proposition 8.2 and the lower bound of Proposition 9.1 specify the answer with indeterminacy one. In a forthcoming publication we shall extend the upper bound $rn + m - 2$ for any $d \geq 2$ even. We shall employ the method which was briefly described in [12], §7 for the case $r = 2$.

10. TC-generating function and rationality

10.1. Definition 3.1 associates with each fibration $p : E \to B$ an infinite sequence of integer numerical invariants,

$$\mathsf{TC}_2[p : E \to B], \quad \mathsf{TC}_3[p : E \to B], \quad \dots, \quad \mathsf{TC}_r[p : E \to B], \quad \dots \tag{24}$$

In order to understand the global behaviour of the sequence (24), it can be organised into a generating function

$$\mathcal{F}(t) = \sum_{r\geq 1} \mathsf{TC}_{r+1}[p : E \to B] \cdot t^r, \tag{25}$$

which we shall call *the* TC*-generating function of the fibration* $p : E \to B$. Various analytic properties of the generating function $\mathcal{F}(t)$ reflect asymptotic behaviour of the sequence (24) and topological structure of the fibration $p : E \to B$. Rationality of the generating function (25) would mean existence of a linear recurrence relation between the integers (24) representing sequential parametrized topological complexities for various values of r.

10.2 Lemma. *The* TC*-generating function (25) depends only on the fiberwise homotopy type of the fibration* $p : E \to B$.

Proof. This is equivalent to Corollary 4.2. □

10.3. In paper [11] the authors introduced the TC-generating function

$$\mathcal{F}_X(t) = \sum_{r\geq 1} \mathsf{TC}_{r+1}(X) \cdot t^r \tag{26}$$

associating with a finite path-connected CW-complex X a formal power series (26). The paper [11] contains many examples when this power series can be explicitly computed and in all these examples $\mathcal{F}_X(t)$ is representable by a rational function of the form

$$\mathcal{F}_X(t) = \frac{A}{(1-t)^2} + \frac{B}{1-t} + p(t), \quad \text{where} \quad p(t) \quad \text{is a polynomial.} \tag{27}$$

This property of $\mathcal{F}_X(t)$ is equivalent to the recurrence relation

$$\mathsf{TC}_{r+1}(X) = \mathsf{TC}_r(X) + A$$

valid for all sufficiently large r; we refer to [10] for more detail. In many examples the principal residue A in (27) equals the Lusternik - Schnirelmann category,

$$A = \mathsf{cat}(X). \tag{28}$$

These examples lead to the Rationality Question of [11]: *for which finite CW-complexes the formal power series (26) represents a rational function of the form (27) with the top residue equal the Lusternik - Schnirelmann category (28)?*

10.4. In the subsequent paper [10] the authors analysed a class of CW-complexes violating the Ganea conjecture and found examples X such that the TC-generating function (26) is a rational function of the form (27) although the top residue A is distinct from $\mathsf{cat}(X)$.

10.5. Next we mention a few examples when the generating function (25) can be computed.

Firstly, suppose that $p : E \to B$ is the trivial fibration with path-connected fibre X. Then the generating function (25) equals $\mathcal{F}_X(t)$.

Secondly, consider the Hopf bundle $p : S^3 \to S^2$. Then, according to Proposition 3.3, we have

$$\mathsf{TC}_{r+1}[p : S^3 \to S^2] = \mathsf{TC}_{r+1}(S^1) = \mathsf{cat}((S^1)^r) = r \quad \text{for any} \quad r \geq 1.$$

Therefore, the TC-generating function of the Hopf bundle equals

$$\sum_{r\geq 1} r \cdot t^r = \frac{t}{(1-t)^2} = \frac{1}{(1-t)^2} - \frac{1}{1-t}.$$

In this case the principal residue equals $A = 1 = \mathsf{cat}(S^1)$.

Exactly the same answer for the TC-generating function can be obtained in the case of a more general Hopf bundle $p : S^{2n+1} \to \mathbb{CP}^n$.

10.6. Consider now the TC-generating function of the Fadell - Neuwirth bundle $p : F(\mathbb{R}^d, m+n) \to F(\mathbb{R}^d, m)$ which was analysed in this paper. We start with the case when the dimension d is odd. Then we have the TC-generating function

$$\mathcal{F}(t) \;=\; \sum_{r=1}[(r+1)n+m-1]\cdot t^r = \frac{n}{(1-t)^2}+\frac{m-1}{1-t}-n-m+1.$$

It is a rational function of the form (27) with the principal residue

$$A = n = \mathsf{cat}(F(\mathbb{R}^d - \mathcal{O}_m, n))$$

equal category of the fibre. Using Theorem 1.3 from [14] we may write the TC-generating function of the fiber $X = F(\mathbb{R}^d - \mathcal{O}_m, n)$ (for any $d \geq 2$) as follows

$$\mathcal{F}_X(t) = n\cdot\sum_{r\geq 1}(r+1)t^r = \frac{n}{(1-t)^2} - n. \tag{29}$$

The TC-generating function of the Fadell - Neuwirth bundle is slightly different in the case when $d = 2$:

$$\mathcal{F}(t) \;=\; \sum_{r=1}[(r+1)n+m-2]\cdot t^r = \frac{n}{(1-t)^2}+\frac{m-2}{1-t}-n-m+2.$$

In this case the power series represents a rational function of form (27) and the principal residue equals the Lusternik - Schnirelmann category of the fibre.

We see that for the Fadell - Neuwirth bundle the TC-generating functions of the bundle and of the fiber have the same principal residue and their difference has a simple pole at $t = 1$. This suggests the following general question:

How the TC*-generating functions of a fibration* $p : E \to B$ *and of its fibre* X *are related?* More specifically we may ask: *For which fibrations* $p : E \to B$ *the differences*

$$\mathsf{TC}_{r+1}[p : E \to B] - \mathsf{TC}_r[p : E \to B] \quad \textit{and} \quad \mathsf{TC}_r[p : E \to B] - \mathsf{TC}_r(X)$$

are eventually constant? This stabilisation happens in the case of the Fadell - Neuwirth fibration.

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