Classical Black Holes from Scattering Amplitudes and the Double Copy

David Peinador Veiga

July 31, 2022

Supervised by
Dr. Ricardo Monteiro

Submitted in partial fulfilment of the requirements of the Degree of Doctor of Philosophy

Queen Mary University of London
School of Physical and Chemical Sciences
Centre for Theoretical Physics

—Ya os he dicho, amigo –replicó el cura–, que esto se hace para entretener nuestros ociosos pensamientos; así como se consiente en las repúblicas bien concertadas que haya juegos de ajedrez, de pelota y de trucos, para entretener a algunos que ni tienen, ni deben, ni pueden trabajar, así se consiente imprimir y que haya tales libros, creyendo, como es verdad, que no ha de haber alguno tan ignorante que tenga por historia verdadera ninguna destos libros.

– Miguel de Cervantes Saavedra, *Don Quijote*

—I have told you, friend –said the curate– that this is done to divert our idle thoughts; and as in well-ordered states games of chess, fives, and billiards are allowed for the diversion of those who do not care, or are not obliged, or are unable to work, so books of this kind are allowed to be printed, on the supposition that, what indeed is the truth, there can be nobody so ignorant as to take any of them for true stories.

– Miguel de Cervantes Saavedra, *Don Quixote*

Declaration

I, David Peinador Veiga, confirm that the research included within this thesis is my own work or that where it has been carried out in collaboration with, or supported by others, that this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below. I attest that I have exercised reasonable care to ensure that the work is original, and does not to the best of my knowledge break any UK law, infringe any third party's copyright or other Intellectual Property Right, or contain any confidential material.

I accept that the College has the right to use plagiarism detection software to check the electronic version of the thesis.

I confirm that this thesis has not been previously submitted for the award of a degree by this or any other university. The copyright of this thesis rests with the author and no quotation from it or information derived from it may be published without the prior written consent of the author.

July 30, 2022

Details of collaboration and publications: This thesis is based on the publications [1–6] and some unpublished notes. The work carried out in [3] was done in collaboration with Ricardo Monteiro, Donal O'Connell and Matteo Sergola. Silvia Nagy joined the same group of authors in [1]. The publications [2, 4] were done in collaboration with Hadi Godazgar, Mahdi Godazgar, Ricardo Monteiro and Christopher Pope. My supervisor, Ricardo Monteiro also co-authored [5], this time with Kwangeon Kim, Kanghoon Lee and Isobel Nicholson. Finally, in [6] I worked with Rashid Alawadhi, David Berman and Bill Spence.

Parts of the introduction and the first sections of chapters 4 and 5 review topics required to understand the material of the thesis. I attest that I strongly believe to have appropriately cited the relevant works therein.

Abstract

For many years, the study of gravitational scattering amplitudes remained a challenging task, due to the inadequacy of traditional quantum field theoretic methods to deal with the increasing complexity of graviton interactions. The relatively low experimental interest also hindered progress in the field. This paradigm changed drastically with the discovery of the double copy and the detection of gravitational waves. The double copy made the calculation of gravity amplitudes much more efficient, by rendering them as the 'square' of gauge theory amplitudes. In its simplest form, it relates amplitudes from Yang-Mills theory to amplitudes in NS-NS gravity, but the map has been extended to numerous other theories. The main topic of this thesis is the classical implications of the double copy. We show how the double copy relations for three-point amplitudes generate 'squaring' relations for classical solutions. This provides a quantum explanation for some classical double copy relations previously known in the literature and identifies the properties that make them local in position space. Two of these known maps, the Weyl and Kerr-Schild double copies, will be studied in detail. We extend the Weyl double copy to non-twisting type N solutions and explore how symmetries in electromagnetism map to gravity. Finally, the Kerr-Schild double copy will allow us to formulate an exact double copy relation between point particle solutions in electromagnetism and NS-NS gravity.

Acknowledgements

I would like to start by thanking my supervisor, Ricardo Monteiro, for his patience and support during these four years. I am sure I would not have enjoyed my time as a PhD student the same way without his encouragement and understanding. Among his many supernatural skills, I was always amazed by his intuition, often much more powerful than my Mathematica notebooks. Similarly, his ability to find collaborations should not go unmentioned. It was particularly helpful during the months of lockdown and online meetings. I feel grateful for everything he taught me.

This thesis would not have been possible without the work of many other collaborators. I would like to thank Rashid Alawadhi, David Berman, Hadi Godazgar, Mahdi Godazgar, Kwangeon Kim, Kanghoon Lee, Silvia Nagy, Isobel Nicholson, Donal O'Connell, Chris Pope, Matteo Sergola and Bill Spence for so many great discussions. They always managed to encourage me as I was learning new topics. Working with them has been both an honour and a pleasure.

I will certainly miss all the members of CTP. Thank you for making my time there so enjoyable. In particular, I would like to thank Joe, Rodolfo, Arnau, Chris, Ray, Zoltán, Linfeng, Luigi, Nadia, Nejc, Ricardo, Gergely, Enrico, Manuel, Marcel, Rajath, Rashid, Shun-Qing, Stefano, Adrian, George, Sam, Graham, Josh, Lewis, Chinmaya, Kymani, Tancredi and Lara for the environment in the office. They not only solved my maths/coding/silly questions, but they also gave me numerous good laughs. I will miss the pizza on Fridays, the lunches in the SCR (shamefully replaced by the seminar room) and the long tea breaks.

There was also a personal cost to my PhD. I could not spend as much time with my family and friends from Spain as I would have liked. I want to thank Sergio and Javi for making me feel as if I had never left every time I visited, Cósima for her support (and showing me around Hamburg) and David for beating me at chess while pretending it was levelled. I owe all my achievements to my family. Special thanks to my parents, my brother and my grandparents; I hope I make you proud. Finally, I want to thank Andrea, my partner, for stubbornly staying by my side these four years. She encouraged me to pursue this challenge in the first place and gave me the strength to achieve it.

This work was supported by a Royal Society studentship.

Contents

Chapter 1

Introduction

The field of scattering amplitudes has become one of the most active subjects in theoretical physics. This success came as spinor-helicity and on-shell techniques unveiled constrained mathematical structures, drastically reducing the computational work needed by traditional diagrammatic approaches. Examples of modern on-shell methods are the BCFW recursion relations [7] and generalised unitarity [8, 9], which provide a systematic way to cut the amplitudes down to their simplest components. Much of this progress was driven by the demand for high precision calculations in gauge theory by particle collider experiments.

Although on-shell techniques were also applicable to gravity theories, they were more computationally challenging and lacked experiential motivation. This paradigm changed with the discovery of the colour-kinematics duality and the double copy by Bern, Carrasco and Johansson (BCJ) [10, 11]. They showed that the kinematic factors of gauge theory amplitudes could be put in a form that satisfies certain algebra-like relations, in analogy with the colour factors. Amplitudes in this form can then be 'double copied' into gravity amplitudes by replacing the colour factors with another set of kinematic factors. The significance of this discovery arises from the simplification that comes with computing gravitational amplitudes from (much simpler) gauge theory amplitudes.

The tree-level relations generated by the BCJ double copy prescription were already known from string theory. Kawai, Lewellen and Tye (KLT) discovered that tree-level bosonic closed-string amplitudes could be written as sums of products of tree-level bosonic open-string amplitudes [12]. More recently, the scattering equation approach to scattering amplitudes developed by Cachazo, He and Yuang (CHY) has provided another formalism to make the double copy manifest [13, 14]. In this prescription, the Parke-Taylor factor of the CHY integrand is replaced with another copy of the numerator. Although this formulation is often more versatile than the BCJ one, it harder to implement at higher loops. The different formulations of the double copy and their vast applications have been recently reviewed in [15–17].

Despite the progress, some of the central concepts of the double copy remain elusive. Perhaps one of the most prominent missing pieces is the nature the kinematic algebra. Its understanding would put colour and kinematics really on the same footing, making the calculation of numerators much simpler. There has been some progress in this direction [18, 19], and it is well understood in special cases, like in self-dual gravity and gauge theory [20, 21]. The self-dual theories also exhibit rather explicit double copy structures at the level of covariant actions [22].

Since its discovery, the concept of the double copy has been expanded to apply beyond gauge and gravity amplitudes. We now know that there is a rich web of theories that are related by double copy relations. Also, there are examples of double copy relations at the level of classical solutions and Lagrangians [23–26]. Therefore, we can define the double copy as the concept of obtaining amplitudes (or amplitude-constructible objects) of one theory by combining the kinematic degrees of freedom of two other theories. For a review on the web of theories related by the double copy, see [15].

The discovery of the double copy came at a time when some of the attention was shifting from particle collider experiments to gravitational wave detectors. The detection of gravitational waves [27] created the need for cost-efficient methods to compute template waveforms of gravitational radiation from binary mergers. The precise perturbative tools of the amplitudes programme are well suited for this task [28–64] and can be used to compute interaction potentials [28–30], or more general effective Lagrangians [31–34], which are relevant to compact binary coalescences. In scattering systems, amplitudes can be used rather directly to evaluate observables of interest [44–46, 65–67], and at least some of these observables may then be analytically continued to the bound regime [52–54, 68].

In these approaches, part of the effort goes into extracting the classical information from the quantum amplitudes. To improve this process, it would be desirable to understand how the double copy manifests itself from a purely classical perspective. One approach, called *classical double copy* is to search for relations between classical solutions of the equations of motion in gauge theory and gravity. The implementation of these relations is often perturbative, including the first approaches [20, 31, 69], constructions based on the local symmetries [70–75], use of the worldline formalism [39, 76–84], and perturbation theory on curved backgrounds [85, 86]. A double copy for classical observables (rather than solutions to the equations of motion) that follows more directly from that of scattering amplitudes has been explored with a view to gravitational phenomenology [28, 33, 34, 36, 43–45, 51, 52, 65, 87–93], a subject of obvious interest for gravitational waves astronomy. For other recent alternative approaches see [94–96]

The perturbative nature of these formulations is expected, given the amplitudes origin of the double copy. However, there are also known examples where the classical double copy applies exactly. The first of them to be discovered was the *Kerr-Schild double copy*, which maps stationary Kerr-Schild metrics to solutions of the Maxwell equations [97].[1] The Schwarzschild solution maps to the Coulomb solution, establishing a relation between the simplest point particle configurations in general relativity and electromagnetism. Some years later, another exact classical double copy, the *Weyl double copy*, was discovered to apply to all vacuum type D spacetimes [99], by rendering the Weyl spinor as the 'square' of a Maxwell spinor. This map was later extended to non-twisting type N solutions [4] and linearly to more algebraically general spacetimes using twistors [100–102].

The existence of exact local classical double copies in position space is striking. Generically, we would expect the amplitudes double copy to generate local perturbative relations in momentum space. But in position space, the momentum space products would turn into convolutions (which motivated the convolution double copy [70, 74, 75, 103–105]). Moreover, although there is evidence that the classical double copy maps are a consequence of the original BCJ double copy, there has been no direct proof of their equivalence [3, 5, 37, 39, 73, 88, 90, 106–108].

The first half of this thesis clarifies some of these questions, following [1, 3]. The key point is the use of 3-point amplitudes to generate the linearised classical field configurations of static point particles in gauge theory and gravity. Then, we show that the double copy relations of the amplitudes directly imply relations for the classical fields. These classical double copy relations are naturally formulated in terms of momentum space spinors. In position space, they involve convolutions and match the convolutional double copy of [70, 74, 75, 103–105]. Moreover, in some cases these convolutions factorise, yielding known Weyl and Kerr-Schild double copy relations, which can be promoted to exact statements. This is possible thanks to a property of the scalar potential which is linked to the point-particle nature of the problem. As a result, we prove that the classical double copy relations for the Kerr-Taub-NUT family follow from the amplitudes double copy of 3-point amplitudes. Our analysis also indicates that a local position space exact double copy might not be always possible.

The domain of applicability of the exact double copy prescriptions is interesting both from the point of view of scattering amplitudes and general relativity. In the second half of the thesis, we will show that non-twisting type N solutions with vanishing twist present exact Weyl double copy relations [4]. A technical limitation of some of the classical notions of the double copy is that they can not accommodate the

[1] Some caution is required when referring to classical double copy prescriptions. Despite being called double copies, they often work best as single-copy procedures. That is, extracting two copies of a gauge theory solution from a gravity solution. This is particularly limiting in the Kerr-Schild prescription, where the relation only works for specific choices of gauge. Also, not all gauge theories solutions can double copy in this manner. The necessary conditions for a Maxwell solution to constitute the single copy of a Kerr-Schild metrics are known in four dimensions [98].

dilaton and axion fields present in the quantum formulation. To overcome this, we generalise the Kerr-Schild double copy in the context of Double Field Theory (DFT) to implement the complete and exact double copy of the Coulomb potential [5]. The result, which was hinted at linearised level by the 3-point amplitudes and other perturbative methods [39, 73, 106], is the spherically symmetric Einstein-dilaton solution found by Janis, Newman and Winicour (JNW) [109].[2]

The remainder of this introduction will be devoted to reviewing some topics that will be needed for the main chapters of this thesis.

1.1 Colour-kinematics and the double copy

The colour-kinematics duality and the double copy were perhaps two of the most remarkable surprises in theoretical physics in recent times. This section will review its origins, which date back to the eighties, and equip the reader with the necessary background for the following chapters. But before we start, it is worth pausing to properly motivate why a duality between gauge theory and gravity is such an outstanding and desirable resource.

Let us start from the simplest gauge theory, pure Yang-Mills (YM):

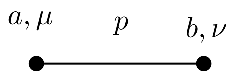
$$S = -\frac{1}{4}\int \mathrm{d}^d x \; \mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}) \; . \tag{1.1}$$

The field strength tensor is defined in terms of the gauge potential A^a_μ as

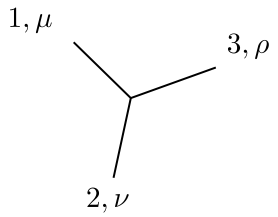
$$F^a_{\mu\nu} = \partial_\mu \, A^a_\nu - \partial_\nu \, A^a_\mu + g \, f^{abc} \, A^b_\mu \, A^c_\nu \; , \tag{1.2}$$

where g is the coupling constant and f^{abc} the structure constants of the gauge algebra. Using standard quantum field theory, one can read off the Feynman rules

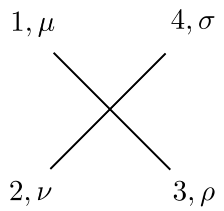
[2] Although this solution is often called JNW, it was first discovered by Fisher [110] and then independently rediscovered by Janis–Newman–Winicour [109] and Wyman [111].



$$\sim \quad -\frac{i\ \eta_{\mu\nu}\ \delta^{ab}}{p^2}\ ,$$



$$\sim \quad g\ f^{a_1 a_2 a_3}\left[\eta_{\mu\nu}(p_1 - p_2)_\rho + \text{cyclic}\right]\ ,$$



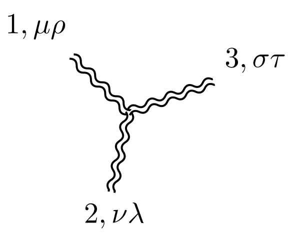
$$\begin{aligned}\sim \quad -i\ g^2\ [&f^{a_1 a_2 b} f^{a_3 a_4 b}(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho})\\ &+ f^{a_1 a_3 b} f^{a_2 a_4 b}(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho})\\ &+ f^{a_1 a_4 b} f^{a_2 a_3 b}(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma})]\ ,\end{aligned}$$

where we have chosen Feynman gauge. We are using a mostly-plus signature for the Minkowski metric $\eta_{\mu\nu}$. These compact rules are all the information needed to write any Feynman diagram integrand in YM theory.

The simplicity of the YM Feynman rules contrasts with the infinite list of vertices in gravity. To illustrate this complexity, consider the Einstein-Hilbert Lagrangian

$$S = \frac{1}{16\pi\, G_N}\int \mathrm{d}^d x\, \sqrt{|g|}\, R\ , \tag{1.3}$$

which describes the dynamics of pure Einstein gravity, G_N being Newton's constant. In the weak-field limit, we can expand the metric as a perturbation around Minkowski, $g_{\mu\nu} = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}$, where κ is the gravitational coupling constant, which is related to Newton's constant via $\kappa^2 = 32\pi\, G_N$. Now, in contrast to YM, (1.3) contains vertices with arbitrarily many external legs. Another difference is that the presence of two indices implies that the structure of the individual vertices is much more elaborate. In de Donder gauge, the 3-point vertex was famously derived by DeWitt [112]



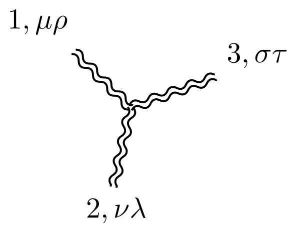
$$\begin{aligned}\sim i\,\kappa\ \mathrm{Sym}\Big[&-\frac{1}{2}P_3\,(p_1\cdot p_2\eta_{\mu\rho}\eta_{\nu\lambda}\eta_{\sigma\tau}) - \frac{1}{2}P_6\,(p_{1\nu}p_{1\lambda}\eta_{\mu\rho}\eta_{\sigma\tau})\\ &+ \frac{1}{2}P_3\,(p_1\cdot p_2\eta_{\mu\nu}\eta_{\rho\lambda}\eta_{\sigma\tau}) + P_6\,(p_1\cdot p_2\eta_{\mu\rho}\eta_{\nu\sigma}\eta_{\lambda\tau})\\ &+ 2P_3\,(p_{1\nu}p_{1\tau}\eta_{\mu\rho}\eta_{\lambda\sigma}) - P_3\,(p_{1\lambda}p_{2\mu}\eta_{\rho\nu}\eta_{\sigma\tau})\\ &+ P_3\,(p_{1\sigma}p_{2\tau}\eta_{\mu\nu}\eta_{\rho\lambda}) + P_6\,(p_{1\sigma}p_{1\tau}\eta_{\mu\nu}\eta_{\rho\lambda})\\ &+ 2P_6\,(p_{1\nu}p_{2\tau}\eta_{\lambda\mu}\eta_{\rho\sigma}) + 2P_3\,(p_{1\nu}p_{2\mu}\eta_{\lambda\sigma}\eta_{\tau\rho})\\ &- 2P_3\,(p_1\cdot p_2\eta_{\rho\nu}\eta_{\lambda\sigma}\eta_{\tau\mu})\Big]\ ,\end{aligned}$$

where the operator Sym symmetrises in each pair of graviton indices (μ,ρ), (ν,λ) and (τ,σ). The operator P_i symmetrises over legs, generating i terms. The complexity of this expression compared to the YM vertex is evident. Even more so when the symmetrisations are done explicitly, resulting in an explosion of terms.

The simple comparison between the Feynman rules of YM and Einstein gravity is enough to show how the diagrammatic expansion, which is tractable at lower orders in YM, becomes practically unapproachable in gravity. This is symptomatic of an apparent leap in complexity from gauge theory to gravity. However, something striking happens when all the redundant (gauge) information is removed from the graviton vertex. If we assume that the legs are on-shell physical states with polarisation matrices satisfying[3]

$$p_{i\,\mu}\,\varepsilon_i^{\mu\nu} = 0\;, \qquad \varepsilon_i^{[\mu\nu]} = 0\;, \qquad \varepsilon^{\mu}_{i\;\mu} = 0\;, \tag{1.4}$$

the vertex simplifies to



$$\sim -i\,\kappa\Big[(p_1-p_2)_\sigma\eta_{\mu\nu} + \text{cyclic}\Big]\Big[(p_1-p_2)_\tau\eta_{\rho\lambda} + \text{cyclic}\Big]\;.$$

The similarities with the YM vertex are striking. Up to constant factors, the on-shell 3-point graviton vertex resembles a gluon vertex where the structure constants have been replaced by another factor of the kinematic variables. The lesson to learn is that taking all off-shell information from gravity might expose structures similar to the ones found in gauge theory. Although the 3-point case is very constrained by symmetries, this simple example motivates hopes for a more general and rigorous relationship. We will now review the steps that were taken in that direction, starting from the KLT relations in string theory and concluding with the BCJ formulation of the double copy.

1.1.1 Once upon a time in string theory

The first evidence of a hidden relation between gravitational and gauge theory amplitudes appeared in the context of string theory. Kawai, Lewellen and Tye (KLT) discovered in 1986 that any closed string tree amplitude can be written as the sum of products of two open string tree amplitudes [12]. Instead of reviewing the general proof, let us illustrate the KLT relations with a simple four-point example.

In a closed string tree amplitude, the external states are represented by operator insertions on a Riemann sphere. The worldsheet $SL(2,\mathbb{C})$ symmetry allows us to fix the location of three of the insertions. The location of the remaining insertion must

[3] We use standard (anti)symmetrisation conventions, e.g. $F_{[\mu\nu]} = \frac{1}{2}\left(F_{\mu\nu} - F_{\nu\mu}\right)$.

be integrated over the Riemann sphere. In bosonic string theory, the simplest case is when all four operator insertions are tachyon states. The result is the celebrated Virasoro-Shapiro amplitude $\mathcal{M}(s,t,u)$ which is, up to overall constant factors [113],

$$\mathcal{M}(s,t,u) \sim \frac{\Gamma(-1-\alpha' s)\,\Gamma(-1-\alpha' t)\,\Gamma(-1-\alpha' u)}{\Gamma(2+\alpha' s)\,\Gamma(2+\alpha' t)\,\Gamma(2+\alpha' u)} \ . \tag{1.5}$$

We have chosen the closed string Regge slope normalisation $\alpha'_{\text{closed}} = 4\,\alpha'$, where α'_{closed} is the closed string inverse tension. The Mandelstam variables have been defined as

$$s = -(p_1+p_2)^2 \ , \qquad t = -(p_1+p_4)^2 \ , \qquad u = -(p_1+p_3)^2 \ . \tag{1.6}$$

The open string counterpart is the Veneziano amplitude, where the open string tachyon insertions are located at the boundary of a disk. As in the closed string case, we can fix the location of three of the insertions based on symmetry grounds. However, now the insertions have a well-defined ordering which must be taken into account when fixing the insertions and integrating over the remaining variable. As a result, the complete amplitude is the sum over inequivalent orderings. For the ordering cyclic$(1,2,3,4)$, the partial amplitude is

$$A(s,t) \sim \frac{\Gamma(-1-\alpha' s)\,\Gamma(-1-\alpha' t)}{\Gamma(2+\alpha' u)} \ , \tag{1.7}$$

with $\alpha'_{\text{open}} = \alpha'$. The normalisation convention has been chosen such that

$$s+t+u = 4\,m^2 = -\frac{4}{\alpha'} \tag{1.8}$$

both for the open and closed string tachyons.

There is a simple relation between (1.5) and (1.7). Using the identity

$$\Gamma(a)\,\Gamma(1-a) = \frac{\sin(\pi\,a)}{\pi} \ , \tag{1.9}$$

it is not hard to show that the Virasoro-Shapiro amplitude can be written as the product of two – differently ordered – Veneziano amplitudes,

$$\mathcal{M}(s,t,u) = \frac{\sin(\pi\,\alpha'\,s)}{\pi\alpha'}\,A(s,t)\,A(s,u) \ . \tag{1.10}$$

In fact, it was shown that similar expressions hold for all higher-point tree amplitudes and all string modes [12].

In particular, the massless modes of the closed string amplitude are a graviton, dilaton ϕ and a two-form B-field (or Kalb-Ramond field, dual to a scalar axion in four dimensions). At low energies, the string length approaches zero ($\alpha' \to 0$) and string

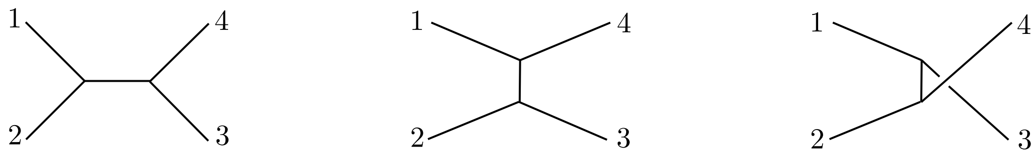


Figure 1.1: Three of the four Feynman diagrams contributing to the YM tree-level 4-point amplitude. The diagrams correspond to the s, t and u channels respectively. The remaining diagram is the contact term generated by the 4-point vertex.

theory approaches quantum field theory. In this limit, the effective dynamics of the massless modes is captured by the Einstein frame action [114, 115]

$$S = \frac{2}{\kappa^2} \int \mathrm{d}^d x \sqrt{|g|} \left(R - \frac{4\kappa^2}{d-2} \nabla_\mu \phi \, \nabla^\mu \phi - \frac{\kappa^2}{12} e^{-\frac{8\kappa\phi}{d-2}} \, H_{\mu\nu\rho} H^{\mu\nu\rho} \right) \, , \tag{1.11}$$

where $H = \mathrm{d}B$. This gravitational theory is often referred to as NS-NS gravity or $\mathcal{N} = 0$ supergravity. Similarly, the $\alpha' \to 0$ limit of the bosonic open string massless modes is pure Yang-Mills (1.1) [116]. Consequently, the generalisation of (1.10) implies a relation between tree-level gravity amplitudes and colour-ordered Yang-Mills amplitudes,

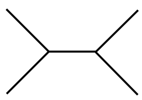
$$\mathcal{M}_4^{\text{tree}}(1,2,3,4) = \left(\frac{\kappa}{2} \right)^2 s \, A^{\text{tree}}(1,2,3,4) \, A^{\text{tree}}(1,2,4,3) \, . \tag{1.12}$$

Although this is a 4-point example of a field theory KLT relation, they can be generalised to arbitrarily many external legs. The main limitation of the KLT relations is that they do not extend to loop level. In order to achieve a loop level gauge-gravity amplitude relation, first we will understand how the tree-level KLT relations arise from the point of view of quantum field theory.

1.1.2 A tree level example

Although the KLT relations were first discovered in the context of string theory, they can also be derived using quantum field theory methods alone, as it was done for the 3-point vertex at the beginning of the section. The derivation of the 4-point relation (1.12) from quantum field theory will provide more insights for the general double copy.

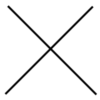
In QFT, the tree-level 4-point amplitude in YM can be computed easily using the Feynman rules given at the beginning of the section. One needs to consider only four diagrams; the three two-vertex diagrams of figure 1.1 plus the contact term of the four-point vertex. The contribution of the s-channel diagram to the amplitude is



$$\sim i \, g^2 \, \frac{f^{a_1 a_2 b} f^{a_3 a_4 b}}{s} \left[(\varepsilon_1 \cdot \varepsilon_2) \, p_1^\mu + 2(\varepsilon_1 \cdot p_2) \varepsilon_2^\mu - (1 \leftrightarrow 2) \right]$$
$$\left[(\varepsilon_3 \cdot \varepsilon_4) \, p_{3\,\mu} + 2(\varepsilon_3 \cdot p_4) \varepsilon_{4\,\mu} - (3 \leftrightarrow 4) \right] \, ,$$

where we have used the on-shell conditions $p_i^2 = 0$ and the gauge choice $\varepsilon_i \cdot p_i = 0$. The t and u channels can be obtained by applying the leg permutations $1 \to 2 \to 3 \to 1$ and $1 \to 3 \to 2 \to 1$ respectively.

The contact term yields



$$\sim -i\, g^2 \Big[c_s \left((\varepsilon_1 \cdot \varepsilon_3)(\varepsilon_2 \cdot \varepsilon_4) - (\varepsilon_1 \cdot \varepsilon_4)(\varepsilon_2 \cdot \varepsilon_3) \right) + c_t \left((\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3 \cdot \varepsilon_4) - (\varepsilon_1 \cdot \varepsilon_3)(\varepsilon_2 \cdot \varepsilon_4) \right) + c_u \left((\varepsilon_1 \cdot \varepsilon_4)(\varepsilon_2 \cdot \varepsilon_3) - (\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3 \cdot \varepsilon_4) \right) \Big]$$

where we have defined the colour factors

$$c_s = f^{a_1 a_2 b} f^{a_3 a_4 b} \,, \qquad c_t = f^{a_1 a_4 b} f^{a_2 a_3 b} \,, \qquad c_u = -f^{a_1 a_3 b} f^{a_2 a_4 b} \,. \tag{1.13}$$

Notice that the three colour factors are related by the Jacobi identity of the gauge algebra

$$c_s + c_t + c_u = f^{a_1 a_2 b} f^{a_3 a_4 b} + f^{a_2 a_3 b} f^{a_1 a_4 b} + f^{a_3 a_1 b} f^{a_2 a_4 b} = 0 \,. \tag{1.14}$$

Adding all the contributions and collecting the terms with the same colour factors, the complete 4-point tree amplitude can be written as

$$i\, \mathcal{A}_4^{\text{tree}} = g^2 \left(\frac{c_s\, n_s}{s} + \frac{c_t\, n_t}{t} + \frac{c_u\, n_u}{u} \right) \,. \tag{1.15}$$

The coefficients n_i are called kinematic factors, in analogy with the colour factors, because they contain the kinematic information of each channel. The s-channel kinematic factor is

$$n_s = [(\varepsilon_1 \cdot \varepsilon_2)\, p_1^\mu + 2(\varepsilon_1 \cdot p_2)\varepsilon_2^\mu - (1 \leftrightarrow 2)][(\varepsilon_3 \cdot \varepsilon_4)\, p_{3\,\mu} + 2(\varepsilon_3 \cdot p_4)\varepsilon_{4\,\mu} - (3 \leftrightarrow 4)] - s \left((\varepsilon_1 \cdot \varepsilon_3)(\varepsilon_2 \cdot \varepsilon_4) - (\varepsilon_1 \cdot \varepsilon_4)(\varepsilon_2 \cdot \varepsilon_3) \right) \,, \tag{1.16}$$

and the remaining kinematic factors can be again obtained by the permutations $1 \to 2 \to 3 \to 1$ for the $t-$channel and $1 \to 3 \to 2 \to 1$ for the u-channel. Remarkably, this set of numerator factors has the striking property of exhibiting a Jacobi-like relation

$$n_s + n_t + n_u = 0 \,. \tag{1.17}$$

Notice that there is no *a priori* reason for this to be the case. This apparent coincidence constitutes a hint of a duality between colour and kinematics. As we shall see, this duality is the ingredient we need to transform a gauge theory amplitude into a gravity amplitude. Intuitively, one needs to relate a gauge field, which has a colour index and a rank-one polarisation vector, to a graviton field with a rank-two polarisation matrix and no colour indices. The existence of a colour-kinematics duality suggests that colour factors can be replaced by kinematic factors, effectively replacing a colour index with another kinematic index. This argument motivates us to consider the tentative amplitude

$$i\,\mathcal{M}_4^{\text{tree}} = -\left(\frac{\kappa}{2}\right)^2 \left(\frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u}\right) \; . \tag{1.18}$$

Now, all the numerators have exactly two powers of every polarisation vector. Hence, we can promote the polarisation vectors to traceless polarisation matrices for graviton states: $\varepsilon_i^\mu \varepsilon_i^\nu \to \varepsilon_i^{\mu\nu}$. However, for this to be a valid on-shell gravitational amplitude, it must be gauge invariant.

In gauge theory, gauge invariance implies that amplitudes must remain be unchanged under $\varepsilon_i \to \varepsilon_i + p_i$. In (1.15), only the kinematic factors transform non-trivially. Without loss of generality, we decide to apply the gauge transformation to the fourth leg, $\varepsilon_4 \to \varepsilon_4 + p_4$,

$$\begin{aligned} n_s &\to n_s + s\left[(\varepsilon_1\cdot\varepsilon_2)(\varepsilon_3\cdot p_1 - \varepsilon_3\cdot p_2) + \text{cyclic}(1,2,3)\right] \\ &= n_s + s\,\alpha(1,2,3)\; . \end{aligned} \tag{1.19}$$

The kinematic function $\alpha(1,2,3)$ is invariant under cyclic permutations of $(1,2,3)$. Hence, applying the gauge transformation to (1.15),

$$\delta\mathcal{A}^{\text{tree}} = g^2\left(c_s\,\alpha(1,2,3) + c_t\,\alpha(2,3,1) + c_u\,\alpha(3,1,2)\right) = (c_s + c_t + c_u)\,\alpha(1,2,3) = 0\; , \tag{1.20}$$

which vanishes due to the Jacobi identity (1.14), indicating that the amplitude is indeed gauge invariant.

We can now check the gauge invariance of (1.18). In linearised general relativity, a linearised diffeomorphism ξ transforms the graviton field as

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu\; , \tag{1.21}$$

which implies that the polarisation tensor transforms according to

$$\varepsilon_{i\,\mu\nu} \to \varepsilon_{i\,\mu\nu} + \varepsilon_{i\,\mu}\,p_{i\,\nu} + \varepsilon_{i\,\nu}\,p_{i\,\mu}\; . \tag{1.22}$$

When this transformation is applied to the fourth external leg of (1.18), the result is

$$i\,\mathcal{M}_4^{\text{tree}} \to i\,\mathcal{M}_4^{\text{tree}} - 2(n_s + n_t + n_u)\,\alpha(1,2,3)\; . \tag{1.23}$$

Consequently, (1.18) is gauge invariant provided that the kinematic Jacobi identity (1.17) is satisfied.

The map that we have obtained from doubling the kinematic factors is equivalent to the KLT relation that we obtained from the string amplitudes. In fact, (1.18) is equivalent to (1.12). To prove it, first we need to compute the colour-ordered YM amplitudes.

The identification of colour-ordered partial amplitudes relies on the decomposition of the colour factors in terms of single traces of gauge algebra generators.[4] For example,

$$c_s = f^{a_1 a_2 b} f^{a_3 a_4 b} \propto \mathrm{Tr}\left([T^{a_1}, T^{a_2}]\,[T^{a_3}, T^{a_4}]\right)\ , \tag{1.24}$$

which can be expanded as

$$\begin{aligned} \mathrm{Tr}\left([T^{a_1}, T^{a_2}]\,[T^{a_3}, T^{a_4}]\right) &= \mathrm{Tr}(T^{a_1}\,T^{a_2}\,T^{a_3}\,T^{a_4}) - \mathrm{Tr}(T^{a_1}\,T^{a_2}\,T^{a_4}\,T^{a_3}) \\ &\quad - \mathrm{Tr}(T^{a_2}\,T^{a_1}\,T^{a_3}\,T^{a_4}) + \mathrm{Tr}(T^{a_2}\,T^{a_1}\,T^{a_4}\,T^{a_3})\ . \end{aligned} \tag{1.25}$$

Similar expansions can be obtained for c_t and c_u. Then, the full amplitude takes the form

$$\mathcal{A}^{\text{tree}} = g^2 \Bigg(\mathrm{Tr}(T^{a_1}\,T^{a_2}\,T^{a_3}\,T^{a_4}) A^{\text{tree}}(1,2,3,4) + \text{perms of (2,3,4)} \Bigg)\ , \tag{1.26}$$

where

$$A^{\text{tree}}(1,2,3,4) = \frac{n_s}{s} - \frac{n_t}{t} \tag{1.27}$$

is the colour-ordered partial amplitude with cyclic ordering (1,2,3,4). To obtain the cyclic ordering $(1,2,4,3)$ we apply the permutation $1 \to 3 \to 2 \to 1$. This permutation transforms $n_s \to n_u$, $n_t \to n_s$, $s \to u$ and $t \to s$, resulting in the partial amplitude

$$A^{\text{tree}}(1,2,4,3) = \frac{n_u}{u} - \frac{n_s}{s}\ . \tag{1.28}$$

These two partial amplitudes are the ingredients needed on the right side of (1.12), which then equals to

$$\begin{aligned} A^{\text{tree}}(1,2,3,4)\,A^{\text{tree}}(1,2,4,3) &= \left(\frac{n_s}{s} - \frac{n_t}{t}\right)\left(\frac{n_u}{u} - \frac{n_s}{s}\right) \\ &= \frac{n_s n_u}{s\,u} - \frac{n_s^2}{s^2} - \frac{n_t n_u}{t\,u} + \frac{n_t n_s}{s\,t} \\ &= -\frac{n_s^2}{s^2} - \frac{n_t^2}{s\,t} - \frac{n_u^2}{s\,u} - n_t\,n_u\left(\frac{1}{s\,u} + \frac{1}{t\,u} + \frac{1}{s\,t}\right) \\ &= -\frac{1}{s}\left(\frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u}\right)\ . \end{aligned} \tag{1.29}$$

The identity (1.17) has been used several times to achieve the final result. Direct comparison between (1.18) and (1.29) yields

$$i\,\mathcal{M}_4^{\text{tree}} = \left(\frac{\kappa}{2}\right)^2 s\,A^{\text{tree}}(1,2,3,4)\,A^{\text{tree}}(1,2,4,3)\ ,$$

[4] At loop level, one needs to consider multi-trace structures also.

recovering the map (1.12) and proving that the squaring of the kinematic factors yields the same relation as the KLT relations.

1.1.3 Colour-kinematics duality and the BCJ double copy

In their full generality, the colour-kinematics duality and the BCJ double copy extend the relations seen in the previous example to higher points, higher loops and more general gauge theories. Although we will provide a review of their main ideas, more comprehensive overviews of the double can be found in [15–17].

To be as general as possible, we will consider a generic m-point L-loop gauge theory amplitude,

$$\mathcal{A}_m^{(L)} = i^{L-1}\, g^{m-2+2L} \sum_{i\in\Gamma} \int \frac{\mathrm{d}^{Ld}\ell}{(2\pi)^{Ld}} \frac{1}{S_i} \frac{c_i\, n_i}{P_i} \ . \tag{1.30}$$

Γ is the set of contributing cubic diagrams, d the spacetime dimension, ℓ is the loop momenta, S_i the symmetry factor associated with the i-th diagram and P_i contains the propagator momenta of each diagram. Like in the tree-level example, the kinematic factors contain momenta, polarisation vectors and spinors whereas the colour factors contain structure constants and algebra generators.

In the previous section, we saw that the entire duality construction relies on the existence of the kinematic Jacobi identity (1.17):

$$c_s + c_t + c_u = 0 \ , \qquad n_s + n_t + n_u = 0 \ .$$

It implied that the kinematic numerators could replace the colour factors while keeping the amplitude gauge invariant. The colour-kinematics duality states that it is always possible to rearrange the kinematic numerators n_i in such a way that they satisfy the same Lie-algebra identities than the corresponding colour factors c_i [10].[5] This not only applies to YM but also to more general gauge theories that may include matter fields. Checking the identity in our simple tree-level 4-point example did not require a lot of work, because the kinematic factors obtained from the Feynman diagrams satisfied the identity automatically. However, this is not always the case, because the numerators are not unique. The non-uniqueness is a consequence of the invariance of the amplitudes under generalised gauge transformations, which are of the form

$$n_i \to n_i + \Delta_i \tag{1.31}$$

[5] There is a subtlety regarding colour factors beyond tree-level. One should not use information about the gauge group to evaluate sums over colour indices. This would create relations between the colour factors that are specific to the particular gauge group. All the algebraic relations must be kept general for any gauge group, otherwise there would be no reason to expect those relations to hold after replacing the colour algebra with the kinematic algebra. More intuitively, summing over colour indices would be equivalent to integrating the loop momenta in the kinematic factors. Since we are not integrating the loop momenta, the sums over colour factors should not be evaluated either.

where the Δ_i are such that

$$\sum_{i\in\Gamma}\int\frac{\mathrm{d}^{Ld}\ell}{(2\pi)^{Ld}}\,\frac{1}{S_i}\,\frac{c_i\,\Delta_i}{P_i}=0\ . \tag{1.32}$$

For example, the amplitude (1.15) remains unchanged under

$$n_s\to n_s+s\,\Delta(p,\varepsilon),\qquad n_t\to n_t+t\,\Delta(p,\varepsilon),\qquad n_u\to n_u+u\,\Delta(p,\varepsilon), \tag{1.33}$$

with the arbitrary function $\Delta(p,\varepsilon)$.

Generically, transformations of this kind are required to put the numerators in duality-ready form. This has been proven to be always possible at tree level [117, 118]. At loop level, it remains a conjecture. A complete proof of the colour kinematics duality would almost certainly require an understanding of the origin of the kinematic algebraic structures. Despite some progress [18, 19], this is currently missing. A more complete understanding is possible by restricting to the self-dual sector, where the kinematic algebra can be identified with the algebra of area-preserving diffeomorphisms [20]. We will explore this particular case in more detail in the next section. More recently, [119] explored a duality between geometry and kinematics, relating theories of massless bosons to the non-linear sigma model.

The most important application of the colour-kinematics duality is the double copy. Whenever we have kinematic factors satisfying the colour-kinematics duality, we can replace the colour factors with another set of the kinematic factors, which we will denote by $\tilde{n}_i$, to obtain a diffeomorphic-invariant amplitude which corresponds to a gravity theory [11]

$$\mathcal{M}_m^{(L)}=i^{L-1}\left(\frac{\kappa}{2}\right)^{m-2+2L}\sum_{i\in\Gamma}\int\frac{\mathrm{d}^{Ld}\ell}{(2\pi)^{Ld}}\,\frac{1}{S_i}\,\frac{\tilde{n}_i\,n_i}{P_i}\ . \tag{1.34}$$

The process of obtaining gravity by duplicating the kinematic degrees of freedom in gauge theory is known as *double copying.* Conversely, we could replace one of the kinematic factors in a given gravity amplitude with a colour factor to obtain a gauge amplitude. This process is called *single copying.* Interestingly, one could go one step further and replace the remaining kinematic factor with another colour factor. The resulting theory is referred to as the *zeroth copy* and, in the case of Yang-Mills, it corresponds to a bi-adjoint scalar field.

It turns out that the double copy map (1.34) is much more robust than it seems. First of all, note that we can perform a generalised gauge transformation (1.31) on only one of the sets of kinematic factors and $\mathcal{M}_m^{(L)}$ will remain invariant. The reason is that equation (1.32) also holds after the replacement $c_i\to\tilde{n}_i$ as long as $\tilde{n}_i$ are in colour-dual form. Therefore, we only need one of the copies to be in colour-dual form.

Moreover, note that the double copy does not require $\tilde{n}_i$ to be equal to n_i, only to satisfy the same algebraic relations. In particular, $\tilde{n}_i$ could be taken to be the kinematic numerators from an amplitude with different external helicities. As a result, in $d = 4$, the gravitational theory not only has states of helicity ± 2, but also two scalars. The explanation can be drawn again from string theory. We saw that at tree level the KLT relations related closed-string amplitudes with open-string amplitudes. The massless sector of the closed string includes not only the graviton, but also the dilaton and the B-field present in (1.11). In the KLT relations, these types of insertions can be obtained by combining gluon amplitudes with different helicities. Here, we are witnessing the same phenomenon in the context of the BCJ double copy,

$$\text{graviton}^{\pm 2} = \text{gluon}^{\pm} \otimes \text{gluon}^{\pm} \ , \qquad \begin{Bmatrix} \text{dilaton} \\ \text{axion} \end{Bmatrix} = \text{gluon}^{\pm} \otimes \text{gluon}^{\mp} \ , \tag{1.35}$$

reminding us that the complete double copy of YM is not Einstein gravity, but NS-NS gravity.

Another possibility is to take the kinematic numerators $\tilde{n}$ from an amplitude from a completely different theory. This is allowed as long as they are paired with the equivalent colour factors. The combination of numerators from two different theories creates an intricate web of relations among different theories. For example, $\mathcal{N} = 8$ supergravity is the double copy of $\mathcal{N} = 4$ super-Yang-Mills. The web of double copy-constructible-theories was extensively reviewed in [15]. Although the concept of the double copy is independent of the number of dimensions, for the most part of this thesis we will work in $d = 4$.

1.1.4 Self-dual double copy

As previously mentioned, our most complete understanding of the double copy occurs in the self-dual sectors of gauge theory and gravity. Imposing self-duality yields consistent truncations of YM and general relativity, which keep a subset of interactions and solutions. From the point of view of quantum field theory, self-dual YM and GR are finite and one-loop exact [120]. For both theories, all tree-level amplitudes vanish except for a three-point amplitude. The physical interpretation of the self-duality condition is the restriction to waves of a single helicity.

In self-dual YM, all solutions of the field equations must satisfy the condition

$$F^a = i \star F^a \ . \tag{1.36}$$

In coordinates

$$\mathrm{d}s^2 = \mathrm{d}u \, \mathrm{d}v - \mathrm{d}w \, \mathrm{d}\bar{w} \tag{1.37}$$

and light-cone gauge ($A^a_u = 0$), the self-dual condition implies that

$$A_w = 0 \ , \qquad A_v = -\frac{1}{4}\partial_w \Phi \ , \qquad A_{\bar{w}} = -\frac{1}{4}\partial_u \Phi \ , \tag{1.38}$$

where Φ is a Lie algebra-valued scalar field. The self-dual condition also implies the equation of motion

$$\Box \Phi + i\, g[\partial_w \Phi, \partial_u \Phi] = 0 \tag{1.39}$$

This equation can be solved perturbatively as $g \to 0$ with a boundary condition that encodes the sources. This process encodes a sum over tree amplitudes, where only one external leg is off-shell. Then, one can easily read off the kinematic factors, which are combinations of "kinematic structure constants"

$$F_{p_1 p_2}{}^{k} = \hat{\delta}(p_1 + p_2 - k)(p_{1w} p_{2u} - p_{1u} p_{2w}) \ , \tag{1.40}$$

which have the same algebraic properties as the gauge structure constants f^{abc}.

This already hints at the colour-kinematics duality. To confirm the duality, one needs to follow the same procedure for self-dual GR,

$$R_{\mu\nu\rho\sigma} = \frac{i}{2}\epsilon_{\mu\nu\delta\lambda} R^{\delta\lambda}{}_{\rho\sigma} \ .$$

This self-dual equation plays the same role as in YM. Using the metric

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}, \tag{1.41}$$

we find that the only non-vanishing components of $h_{\mu\nu}$ must be of the form

$$h_{vv} = -\frac{1}{4}\partial_w^2 \phi, \quad h_{\bar{w}\bar{w}} = -\frac{1}{4}\partial_u^2 \phi, \quad h_{v\bar{w}} = h_{\bar{w}v} = -\frac{1}{4}\partial_w \partial_u \phi \tag{1.42}$$

where ϕ must satisfy the equation

$$\Box \phi + \kappa\, \{\partial_w \phi, \partial_u \phi\} = 0 \ . \tag{1.43}$$

The Poisson brackets are defined as

$$\{f, g\} \equiv (\partial_w f)\,(\partial_u g) - (\partial_u f)\,(\partial_w g) \ . \tag{1.44}$$

Again, we can solve the equation perturbatively. The results are related order-by-order to the ones obtained for the self-dual YM, replacing the colour structure constants with more kinematic structure constants (without duplicating the delta functions). This relation is the manifestation of the double copy in the self-dual sector.

The similarity with (1.39) is striking. In fact, the Poisson version of area-preserving diffeomorphisms in the (u, w) plane generates the kinematic structure constants ${F_{p_1 p_2}}^k$. This identifies the kinematic algebra in the self-dual sector [20]. There are other examples of known kinematic algebras, but perhaps not as well understood as the example studied in this section. A recent example is the discovery of volume-preserving diffeomorphisms as part of the kinematic algebra of three-dimensional Chern-Simons theory [121].

1.2 The KMOC formalism

In the last section, we have reviewed the double copy, and how it hints at hidden similarities between gauge theory and gravity. This thesis aims to explore the classical implications of the double copy. Therefore, we need a means of extracting meaningful classical information from scattering amplitudes, which are inherently quantum objects. The formalism developed by Kosower, Maybee and O'Connell (KMOC) serves this need by giving a prescription for defining appropriate quantum observables in terms of amplitudes and extracting their classical values [44, 45].

At the heart of any classical limit is the $\hbar \to 0$ limit. However, the use of natural units ($\hbar = 1$) hides all the powers of $\hbar$. Of course, they might be restored by dimensional analysis, but there might not be a unique way to do it. For this reason, the goal of any classical prescription is to set rules that fix the factors of $\hbar$ that yield the classical observables of interest. In the KMOC formalism, these rules are

- We will adopt the convention that an n-point scattering amplitude in $d = 4$ has dimensions $[M]^{4-n}$, where $[M]$ denotes dimensions of mass. We will keep relativistic natural units $c = 1$. This corresponds to having single-particle states $|p\rangle = \sqrt{2\,E_p}\,a_p^\dagger|0\rangle$ with dimension $[M]^{-1}$.

- Both in electromagnetism and gravity, the coupling constants e and κ pick up a factor of $\hbar^{-1/2}$ on dimensional grounds. The negative power would suggest that higher points and higher loops are more relevant in the classical limit. This is compensated by the next point.

- When $\hbar = 1$, the momentum of a particle p^μ is equal to its wave-number $\bar{p}^\mu$. However, if $\hbar \neq 1$, we must decide whether the classically relevant vector associated to each line is the momentum or the wave-number $\bar{p}^\mu = p^\mu/\hbar$. For massive external particles, we will choose p^μ to be the relevant variable at classical level, remaining invariant as $\hbar \to 0$. However, for massless particles or momentum transfers between massive lines, the wave-number will be fixed and the momentum will decrease linearly with $\hbar$. This prescription cancels the divergent powers of $\hbar$ that appear from the coupling constants and yields classically well-defined observables.

Besides dimensional analysis, there is another point that requires attention when defining observables with sensible classical limits. The KMOC formalism is often used to study the classical behaviour or interactions of point particles. However, there is no notion of a point particle in quantum field theory. Instead, initial and final states must be regularised as wave-packets, with the appropriate properties to be identified with point particles in the classical limit. For single-particle states, we model the particles as states of a scalar field of mass m,

$$|\psi\rangle = \int \mathrm{d}\Phi(p)\, \varphi(p)\, |p\rangle \ , \tag{1.45}$$

where we integrate over on-shell momentum with positive energy E_p

$$\mathrm{d}\Phi(p) = \hat{\mathrm{d}}^4 p\, \Theta(E_p)\, \hat{\delta}(p^2 + m^2) \ . \tag{1.46}$$

In order to avoid cluttering expressions with factors of 2π, we use the hat notation

$$\hat{\mathrm{d}}x = \frac{\mathrm{d}x}{2\pi} \ , \qquad \hat{\delta}(x) = 2\pi\, \delta(x) \ . \tag{1.47}$$

The wavefunction $\varphi(p)$ has a characteristic spread in momentum space, which we will denote ℓ_w, but it needs to have well-defined classical position and momenta for $|\psi\rangle$ to act as a point particle classically. This puts certain requirements on the ℓ_w. First of all, the Compton wavelength ℓ_c of the particle must be much smaller than ℓ_w, such that quantum effects do not rule the evolution of the wave-packet. That is, the dimensionless parameter

$$\xi := \left(\frac{\ell_c}{\ell_w}\right)^2 \tag{1.48}$$

must approach zero in the classical limit. As it does, $\varphi(p)$ must become sharply peaked around the classical value of the momentum $p = m\, u^\mu$.

At the same time, whenever we have more than one particle, the spread must be much smaller than the modulus of the impact parameter, to ensure that the particles are well-separated and the internal details of the wave-packets do not affect the scattering. In single-particle settings, the distance between the particle and the observer plays the role of the impact parameter. The exhaustive description of the hierarchy of limits in multi-particle states can be found in [44] and expanded to particles with spin in [45].

In summary, the different considerations on $\varphi(p)$ motivate simplifications in the quantum calculations that extract the classical information without the need for explicitly evaluating the wave-function.

1.3 Spinors in electromagnetism and general relativity

Scattering amplitudes are the main building blocks to compute observables in quantum field theory. They store information very efficiently thanks to their properties: their gauge invariance means that they do not contain redundant degrees of freedom. Moreover, they are organised in terms of the helicities of the external particles, which are physical degrees of freedom and in some cases are enough to uniquely determine the amplitudes.[6] On the other hand, classical field theory is written mainly in terms of fields. Although they are more intuitive to work with, they are not as efficient. A gauge potential, for example, contains gauge information that is not physical. To cure this, one may resolve to use gauge-invariant curvatures, like the field strength tensor, but they are still far from the neatness of scattering amplitudes. As with any covariant tensor, curvatures are organised in terms of the components of their indices, which depend on coordinate choices that do not hold physical meaning. We would like to organise the information on the classical curvatures in terms of coordinate-independent and physically relevant degrees of freedom. The spinor formalism serves this very purpose. Instead of storing the physical degrees of freedom scattered in the different components of a tensor, spinors are naturally organised in terms of totally symmetric lower-rank components. For the Weyl curvature and the field strength tensor, these components naturally split the self-dual and anti-self-dual degrees of freedom. These parts can be reduced to scalar components, that can be sorted according to their algebraic properties. As a result, the introduction of spinors in general relativity brought a natural way to classify spacetimes in a coordinate independent manner.

As we shall see in due time, these similarities between spinors and scattering amplitudes are not mere coincidences. In fact, the classical limit of the amplitudes will reveal that the spinors are their natural classical counterparts in position space. For now, we will review the basics of the spinor formalism.

1.3.1 Fundamentals of the spinor formalism

The spinor formalism in general relativity [123] exploits the fact that the universal covering group of $SO(3,1)$ is $SL(2,\mathbb{C})$, whose fundamental and anti-fundamental representations act on the two dimensional complex vector spaces $\mathcal{W}$ and $\bar{\mathcal{W}}$ respectively. The map from $\mathcal{W}$ onto $\bar{\mathcal{W}}$ defines the *complex conjugation.* To avoid confusion with the notation, we will use indices $\alpha,\ \beta,\ \gamma,\ ...$ for elements of $\mathcal{W}$ and $\dot{\alpha},\ \dot{\beta},\ \dot{\gamma}$ for $\bar{\mathcal{W}}$. Tensors defined in these spaces are called spinors. Dotted and undotted indices live in different spaces, so their relative ordering is irrelevant (e.g. $T^{\alpha\beta}{}_{\gamma}{}^{\dot{\delta}} = T^{\alpha\beta\dot{\delta}}{}_{\gamma} \neq T^{\beta\alpha}{}_{\gamma}{}^{\dot{\delta}}$). The spaces of antisymmetric $(0,2)$ tensors over $\mathcal{W}$ or $\bar{\mathcal{W}}$ are one-dimensional and

[6] MHV amplitudes and the Parke-Taylor formula were the starting point of modern on-shell amplitude methods [122].

spanned by

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \qquad \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{1.49}$$

Similarly, for the $(2,0)$ antisymmetric tensors, we choose

$$\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \qquad \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{1.50}$$

Note that ϵ is preserved by the action of $L^{\alpha}{}_{\beta} \in SL(2,\mathbb{C})$,

$$\begin{aligned} & L^{\alpha}{}_{\gamma} L^{\beta}{}_{\delta} \epsilon_{\alpha\beta} \epsilon^{\gamma\delta} = -2\det(L) = -2 = \epsilon_{\gamma\delta}\, \epsilon^{\gamma\delta} \\ \Rightarrow \quad & L^{\alpha}{}_{\gamma} L^{\beta}{}_{\delta} \epsilon_{\alpha\beta} = \epsilon_{\gamma\delta} \; . \end{aligned} \tag{1.51}$$

Consequently, ϵ plays the role of a metric in a spinor space $(\mathcal{W},\ \epsilon)$. However, unlike a normal metric, ϵ is antisymmetric and we need to specify conventions for raising and lowering indices,

$$\xi_\alpha = \epsilon_{\alpha\beta}\, \xi^\beta \; , \qquad \xi^\alpha = \epsilon^{\alpha\beta} \xi_\beta \; . \tag{1.52}$$

The second identity follows from the first one and the fact that $\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = \delta^{\alpha}{}_{\gamma}$. Another consequence of the antisymmetry of ϵ is that $\xi^\alpha\, \xi_\alpha = 0$ for any spinor ξ^α. The same conventions apply to anti-fundamental indices.

Any two spinors, o^α and ι^α such that $o_\alpha\, \iota^\alpha = 1$ define a basis in $(\mathcal{W},\ \epsilon)$. Together with its conjugate basis, they define a real basis for the space of $(1,0;1,0)$ spinor tensors.[7]

This space can be shown to be isomorphic to the tangent space to the Minkowski spacetime by mapping the real spinor basis to a null tetrad in Minkowski. The map is provided by the sigma matrices

$$\sigma^{\mu}{}_{\alpha\dot{\alpha}} = \frac{1}{\sqrt{2}}(\mathbb{1}, \vec{\sigma}) \; , \qquad \tilde{\sigma}^{\mu\dot{\alpha}\alpha} = \frac{1}{\sqrt{2}}(\mathbb{1}, -\vec{\sigma}) \; , \tag{1.53}$$

where $\vec{\sigma}$ are the standard Pauli matrices.[8] These matrices satisfy the Clifford algebra:

$$\sigma^{\mu}{}_{\alpha\dot{\alpha}} \tilde{\sigma}^{\nu\dot{\alpha}\beta} + \sigma^{\nu}{}_{\alpha\dot{\alpha}} \tilde{\sigma}^{\mu\dot{\alpha}\beta} = -\,\eta^{\mu\nu} \delta_{\alpha}{}^{\beta} \; . \tag{1.54}$$

The metric $\eta_{\mu\nu}$ is the flat Minkowski metric, which indicates that the map only works for flat space. However, one can define spinors in a general background by contracting the matrices (1.53) by the vielbein of the metric. Then, the flat metric in the Clifford algebra is replaced by the full curved metric. By doing so, one can obtain the spinor

[7] A tensor is said to be of type $(i,j;k,l)$ if it has i contravariant fundamental indices, j covariant fundamental indices, k contravariant anti-fundamental indices and l covariant anti-fundamental indices.

[8] The tilde is just a matter of notation, $\tilde{\sigma}^\mu$ is the transpose (or complex conjugate) of the σ^μ with the indices raised.

representation of the metric

$$\begin{aligned} g^{\mu\nu} &= -\frac{1}{2}(\sigma^{\mu}{}_{\alpha\dot{\alpha}}\tilde{\sigma}^{\nu\dot{\alpha}\alpha} + \sigma^{\nu}{}_{\alpha\dot{\alpha}}\tilde{\sigma}^{\mu\dot{\alpha}\alpha}) \\ &= -\sigma^{\mu}{}_{\alpha\dot{\alpha}}\sigma^{\nu\alpha\dot{\alpha}} = \sigma^{\mu}_{\alpha\dot{\alpha}}\sigma^{\nu}_{\beta\dot{\beta}}\left(-\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\right) \ . \end{aligned}$$

The map defined by (1.53) also applies to derivatives. In flat space, the standard derivative is defined with respect to a basis

$$\partial_{\alpha\dot{\alpha}}\psi^{\beta} = \sigma^{\mu}{}_{\alpha\dot{\alpha}}\frac{\partial\,\psi^{\beta}}{\partial\,x^{\mu}} \ . \tag{1.55}$$

In curved spacetimes, the definition of the spinor analogous of the covariant derivative is more involved. For a detailed discussion, see [124]. Basically, the spinor covariant derivative $\nabla_{\alpha\dot{\alpha}}$ is defined as a map that applies to any tensor T with any number of dotted and undotted indices

$$\nabla : T \mapsto \nabla_{\alpha\dot{\alpha}}T \ ,$$

with the following properties:

(i) Linearity: $\nabla_{\alpha\dot{\alpha}}\,(a\,T + b\,U) = a\,\nabla_{\alpha\dot{\alpha}}T + b\,\nabla_{\alpha\dot{\alpha}}U$, with U a tensor of the same type as T and a, b constants.

(ii) Leibniz's rule: $\nabla_{\alpha\dot{\alpha}}(T\,V) = T\,\nabla_{\alpha\dot{\alpha}}V + V\,\nabla_{\alpha\dot{\alpha}}T$, with V a tensor of any type.

(iii) Commutation with complex conjugation $\bar{\nabla}_{\alpha\dot{\alpha}} = \nabla_{\alpha\dot{\alpha}}$.

(iv) Preserves the ϵ tensor $\nabla_{\alpha\dot{\alpha}}\,\epsilon_{\beta\gamma} = 0$.

(v) It is torsionless: $(\nabla_{\alpha\dot{\alpha}}\nabla_{\beta\dot{\beta}} - \nabla_{\beta\dot{\beta}}\nabla_{\alpha\dot{\alpha}})\phi = 0$.

This connection is equivalent to the Levi-Civita connection in spinor space.

1.3.2 Spinors in electromagnetism

To expose the virtues of the spinors, let us apply them in the context of electromagnetism. Any antisymmetric $(0,2)$ field strength tensor $F_{\mu\nu}$ can be converted into a (0, 2; 0, 2) spinor tensor $F_{\alpha\dot{\alpha}\beta\dot{\beta}}$ using (1.53). However, the symmetries present in the indices of $F_{\mu\nu}$ translate into symmetries in the spinor indices,

$$F_{\alpha\dot{\alpha}\beta\dot{\beta}} = \frac{1}{2}(F_{\alpha\dot{\alpha}\beta\dot{\beta}} - F_{\beta\dot{\beta}\alpha\dot{\alpha}}) = F_{(\alpha\beta)[\dot{\alpha}\dot{\beta}]} + F_{[\alpha\beta](\dot{\alpha}\dot{\beta})} \ .$$

At this point, we must recall that ϵ spans the space of antisymmetric (0, 2) spinor tensors. That implies that the antisymmetrised indices can be replaced by ϵ times a constant. Thus, all the degrees of freedom are in the symmetrised indices and can be factorised into spinors $\phi_{\alpha\beta}$ and $\tilde{\phi}_{\dot{\alpha}\dot{\beta}}$

$$F_{\alpha\dot{\alpha}\beta\dot{\beta}} = \phi_{\alpha\beta}\, \epsilon_{\dot{\alpha}\dot{\beta}} + \tilde{\phi}_{\dot{\alpha}\dot{\beta}}\, \epsilon_{\alpha\beta} \ . \tag{1.56}$$

Hence, the spinor counterpart of the field strength tensor are the symmetric *Maxwell spinors* $\phi_{\alpha\beta}$ and $\tilde{\phi}_{\dot{\alpha}\dot{\beta}}$. Instead of computing the Maxwell spinors from (1.56), it is often more convenient to contract F with the $SL(2,\mathbb{C})$ generators

$$\sigma^{\mu\nu}_{\alpha\beta} = -\sigma^{[\mu}_{\alpha\dot{\alpha}}\, \tilde{\sigma}^{\nu]\dot{\alpha}}{}_{\beta} \ , \tag{1.57}$$

$$\phi_{\alpha\beta} = \frac{1}{2}\, F_{\mu\nu}\, \sigma^{\mu\nu}_{\alpha\beta} \ . \tag{1.58}$$

More generally, the spinor formulation allows the decomposition of tensors into totally symmetric spinors, which represent the irreducible components of the tensor.

In order to understand the physical meaning of the Maxwell spinors, let us define the Hodge-dual electromagnetic tensor[9]

$$\star F_{\mu\nu} = \frac{1}{2}\, \epsilon_{\mu\nu\sigma\rho}\, F^{\sigma\rho} \ , \tag{1.59}$$

where ϵ is the volume-form. Its spinor counterpart is also determined by its symmetries

$$\epsilon_{\mu\nu\sigma\rho} \mapsto \epsilon_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\dot{\gamma}\delta\dot{\delta}} = i(\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}\epsilon_{\dot{\alpha}\dot{\delta}}\epsilon_{\dot{\beta}\dot{\gamma}} - \epsilon_{\alpha\delta}\epsilon_{\beta\gamma}\epsilon_{\dot{\alpha}\dot{\gamma}}\epsilon_{\dot{\beta}\dot{\delta}}) \ . \tag{1.60}$$

The spinor representation of the dual field strength is obtained from (1.56), (1.59) and (1.60),

$$\begin{aligned} \star F_{\alpha\dot{\alpha}\beta\dot{\beta}} &= i(\epsilon_{\alpha\gamma}\epsilon_{\dot{\beta}\delta}\epsilon_{\dot{\alpha}\dot{\delta}}\epsilon_{\dot{\beta}\dot{\gamma}} - \epsilon_{\alpha\delta}\epsilon_{\dot{\beta}\gamma}\epsilon_{\dot{\alpha}\dot{\gamma}}\epsilon_{\dot{\beta}\dot{\delta}})(\phi^{\gamma\delta}\, \epsilon^{\dot{\delta}\dot{\gamma}} + \tilde{\phi}^{\dot{\gamma}\dot{\delta}}\, \epsilon^{\delta\gamma}) \\ &= -i\phi_{\alpha\beta}\, \epsilon_{\dot{\alpha}\dot{\beta}} + i\tilde{\phi}_{\dot{\alpha}\dot{\beta}}\, \epsilon_{\alpha\beta} \ . \end{aligned} \tag{1.61}$$

Hodge duality naturally decomposes any field strength tensor into a self-dual part F^+ and an anti-self-dual part F^- such that $F = F^+ + F^-$ and

$$\star F^{\pm} = \pm i\, F^{\pm} \ . \tag{1.62}$$

The spinor representation of the anti-self-dual and self-dual parts of the field strength are then

$$F^- = \frac{1}{2}\,(F + i\, \star F) \to \phi_{\alpha\beta}\, \epsilon_{\dot{\alpha}\dot{\beta}} \ , \tag{1.63}$$

$$F^+ = \frac{1}{2}\,(F - i\, \star F) \to \tilde{\phi}_{\dot{\alpha}\dot{\beta}}\, \epsilon_{\alpha\beta} \ , \tag{1.64}$$

[9] A comment on notation. In later chapters, we will also refer to the (Hodge) dual field strength tensor as $\tilde{F}$. This will be useful once we introduce the notion of dual Weyl tensor $\tilde{W}_{\mu\nu\rho\sigma}$, which is not the equivalent to Hodge duality because the Weyl tensor is not a form.

implying that $\phi_{\alpha\beta}$ and $\tilde{\phi}_{\dot{\alpha}\dot{\beta}}$ correspond to the anti-self-dual and the self-dual degrees of freedom in F respectively. In particular, for real $F_{\mu\nu}$, the spinor $\tilde{\phi}_{\dot{\alpha}\dot{\beta}}$ is the complex conjugate of $\phi_{\alpha\beta}$, which then uniquely determines $F_{\mu\nu}$.

Petrov classification of the Maxwell spinor

A major advantage of spinors is that they give rise to a natural classification of fields according to their algebraic properties. This classification is based on the fact that any completely symmetric spinor of rank n can be decomposed as the symmetrisation of n *principal spinors*

$$\xi_{\alpha_1 \dots \alpha_n} = \alpha_{(\alpha_1} \cdots \gamma_{\alpha_n)} \ . \tag{1.65}$$

The proof follows from the fact that both sides of the identity are symmetric rank-n expressions with the maximal number of degrees of freedom. ξ has n completely symmetric 2-dimensional indices, so it has $n+1$ degrees of freedom. On the right side, the degrees of freedom could be split up into an overall factor times n normalised spinors with one degree of freedom each. The result is again $n+1$. Each of the principal spinors defines a real null direction, which is referred to as *principal null direction* (PND).

If we apply (1.65) to the Maxwell spinor, we obtain two fundamental spinors

$$\phi_{\alpha\beta} = \alpha_{(\alpha}\, \beta_{\beta)} \ . \tag{1.66}$$

We distinguish three scenarios

- Type I: the two principal spinors are independent, so $\phi_{\alpha\beta} = \alpha_{(\alpha}\, \beta_{\beta)}$. These solutions are algebraically general.
- Type II: the two principal spinors are proportional, so $\phi_{\alpha\beta} = \alpha_\alpha\, \alpha_\beta$. These solutions are algebraically special.
- Type O is the trivial case $\phi_{\alpha\beta} = 0$.

The algebraic classification of the spinors in terms of the multiplicities of their principal spinors receives the name of Petrov classification.

Maxwell equations

We have seen that the spinors reorganise the algebraic degrees of freedom of the field strength tensor in a compact manner. We will now translate the Maxwell equations

$$\nabla^\mu F_{\mu\nu} = J_\nu \ , \tag{1.67}$$

$$\nabla_{[\mu} F_{\nu\rho]} = 0 \Leftrightarrow \nabla^\mu \star F_{\mu\nu} = 0 \tag{1.68}$$

into spinor language. First, we note that Eq. (1.68) implies that

$$\nabla^{\alpha}{}_{\dot\beta}\phi_{\alpha\beta} = \nabla_{\beta}{}^{\dot\alpha}\tilde\phi_{\dot\alpha\dot\beta}$$

and (1.67) that

$$\nabla^{\alpha}{}_{\dot\beta}\phi_{\alpha\beta} + \nabla_{\beta}{}^{\dot\alpha}\tilde\phi_{\dot\alpha\dot\beta} = J_{\beta\dot\beta} \ .$$

Combining both expressions, we conclude that the spinor form of Maxwell's equations can be written as

$$\nabla^{\alpha}{}_{\dot\beta}\phi_{\alpha\beta} = \frac{1}{2}\, J_{\beta\dot\beta} \ . \tag{1.69}$$

In absence of sources the equation takes the simpler form

$$\nabla^{\alpha}{}_{\dot\beta}\phi_{\alpha\beta} = 0 \ . \tag{1.70}$$

1.3.3 Spinors in general relativity

Spinors offer similar advantages in the context of general relativity. From the definition of the covariant derivative, we can define the Riemann tensor $R_{abcd} \to R_{\alpha\dot\alpha\beta\dot\beta\gamma\dot\gamma\delta\dot\delta}$. As we did in (1.56), we would like to use symmetries to decompose the Riemann tensor into lower rank spinors. Following this reasoning [124],

$$\begin{aligned} R_{\alpha\dot\alpha\beta\dot\beta\gamma\dot\gamma\delta\dot\delta} =& X_{\alpha\beta\gamma\delta}\, \epsilon_{\dot\alpha\dot\beta}\, \epsilon_{\dot\gamma\dot\delta} + \phi_{\alpha\beta\dot\gamma\dot\delta}\, \epsilon_{\dot\alpha\dot\beta}\, \epsilon_{\gamma\delta} \\ &+ \tilde X_{\dot\alpha\dot\beta\dot\gamma\dot\delta}\, \epsilon_{\alpha\beta}\, \epsilon_{\gamma\delta} + \tilde\phi_{\dot\alpha\dot\beta\gamma\delta}\, \epsilon_{\alpha\beta}\, \epsilon_{\dot\gamma\dot\delta} \ , \end{aligned} \tag{1.71}$$

where the *curvature spinors*

$$\begin{aligned} X_{\alpha\beta\gamma\delta} &= \frac{1}{4}\, R_{\alpha\dot\alpha\beta}{}^{\dot\alpha}{}_{\gamma\dot\gamma\delta}{}^{\dot\gamma} = \frac{1}{4}\, R_{\mu\nu\rho\sigma}\, \sigma^{\mu\nu}_{\alpha\beta}\, \sigma^{\rho\sigma}_{\gamma\delta} \ , \\ \phi_{\alpha\beta\dot\gamma\dot\delta} &= \frac{1}{4}\, R_{\alpha\dot\alpha\beta}{}^{\dot\alpha}{}_{\gamma\dot\gamma}{}^{\gamma}{}_{\dot\delta} = \frac{1}{4}\, R_{\mu\nu\rho\sigma}\, \sigma^{\mu\nu}_{\alpha\beta}\, \tilde\sigma^{\rho\sigma}_{\dot\gamma\dot\delta}, \end{aligned}$$

are the spinor equivalent of the Riemann tensor. They have the symmetries $X_{(\alpha\beta)(\gamma\delta)}$ and $\phi_{(\alpha\beta)(\dot\gamma\dot\delta)}$. Additionally, the symmetry under exchange of pairs of indices $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$ implies that

$$X_{\alpha\beta\gamma\delta} = X_{\gamma\delta\alpha\beta} \ , \qquad \phi_{\alpha\beta\dot\alpha\dot\beta} = \tilde\phi_{\dot\alpha\dot\beta\alpha\beta} \ . \tag{1.72}$$

Contraction of the relevant indices yields the spinor equivalent for the Ricci tensor

$$R_{\alpha\dot\alpha\beta\dot\beta} = -X_{\alpha\gamma\beta}{}^{\gamma}\, \epsilon_{\dot\alpha\dot\beta} + \phi_{\alpha\beta\dot\beta\dot\alpha} - \tilde X_{\dot\alpha\dot\gamma\dot\beta}{}^{\dot\gamma}\, \epsilon_{\alpha\beta} + \tilde\phi_{\dot\alpha\dot\beta\beta\alpha} \ .$$

Similarly, the curvature scalar is

$$R = -24\,\Lambda \ , \qquad \Lambda = \frac{1}{6}\,X_{\alpha\beta}{}^{\alpha\beta} \ .$$

The first Bianchi identity $R_{\mu[\nu\rho\sigma]} = 0$ implies that $\Lambda = \tilde{\Lambda}$. Finally, the – completely symmetric – Weyl spinor is defined as $\Psi_{\alpha\beta\gamma\delta} = X_{(\alpha\beta\gamma\delta)}$. It is the spinor equivalent of the Weyl tensor W_{abcd}

$$W_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\dot{\gamma}\delta\dot{\delta}} = \Psi_{\alpha\beta\gamma\delta}\,\epsilon_{\dot{\alpha}\dot{\beta}}\,\epsilon_{\dot{\gamma}\dot{\delta}} + \tilde{\Psi}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}\,\epsilon_{\alpha\beta}\,\epsilon_{\dot{\gamma}\dot{\delta}} \ . \tag{1.73}$$

Hence, the Weyl spinor can also be computed as

$$\Psi_{\alpha\beta\gamma\delta} = \frac{1}{4}\,W_{\mu\nu\rho\sigma}\sigma^{\mu\nu}_{\alpha\beta}\sigma^{\rho\sigma}_{\gamma\delta} \ . \tag{1.74}$$

The second Bianchi identity $\nabla_{[e}R_{ab]cd} = 0$ implies that the Weyl spinor must satisfy

$$\begin{aligned} \nabla^{\alpha\dot{\alpha}}\Psi_{\alpha\beta\gamma\delta} &= \nabla^{\dot{\beta}}{}_{(\beta}\phi_{\gamma\delta)\dot{\beta}}{}^{\dot{\alpha}} \ . \\ &= 4\pi\nabla^{\dot{\beta}}{}_{(\beta}T_{\gamma\delta)\dot{\beta}}{}^{\dot{\alpha}} \ . \end{aligned} \tag{1.75}$$

In the last line we have used the Einstein equations to write ϕ in terms of the stress energy tensor. In the vacuum, the equation simplifies to

$$\nabla^{\alpha\dot{\alpha}}\Psi_{\alpha\beta\gamma\delta} = 0 \ . \tag{1.76}$$

As a remark, notice that massless fields of different spin have very similar spinor equations of motion. Indeed, the equation we obtained for the Weyl spinor in (1.76) resembles the Maxwell equation (1.70) and the massless Dirac equation for a spin-1/2 field.

Petrov classification of the Weyl spinor

Just as we did for the Maxwell spinor, we can classify the Weyl spinor according to the multiplicity of its principal spinors

$$\Psi_{\alpha\beta\gamma\delta} = \alpha_{(\alpha}\beta_{\beta}\gamma_{\gamma}\delta_{\delta)} \ . \tag{1.77}$$

The Petrov types, in increasing algebraic speciality are

- Type I: $\Psi_{\alpha\beta\gamma\delta} = \alpha_{(\alpha}\beta_{\beta}\gamma_{\gamma}\delta_{\delta)}$, algebraically general.
- Type II: $\Psi_{\alpha\beta\gamma\delta} = \alpha_{(\alpha}\alpha_{\beta}\beta_{\gamma}\gamma_{\delta)}$.
- Type D: $\Psi_{\alpha\beta\gamma\delta} = \alpha_{(\alpha}\beta_{\beta}\alpha_{\gamma}\beta_{\delta)}$. They have two principal null directions and corre-

Petrov type	NP scalars
Type I	$\phi_1\ \phi_2$
Type II	ϕ_2

Petrov type	NP scalars
Type I	$\Psi_1\ \Psi_2\ \Psi_3\ \Psi_4$
Type II	$\Psi_2\ \Psi_3\ \Psi_4$
Type D	Ψ_2
Type III	$\Psi_3\ \Psi_4$
Type N	Ψ_4

Table 1.1: Non-vanisning NP scalars for the different types of Maxwell and Weyl spinors.

spond to isolated sources, including the Kerr-Taub-NUT family and the C-metric.

- Type III: $\Psi_{\alpha\beta\gamma\delta} = \alpha_{(\alpha}\alpha_\beta\alpha_\gamma\beta_{\delta)}$.
- Type N: $\Psi_{\alpha\beta\gamma\delta} = \alpha_\alpha\alpha_\beta\alpha_\gamma\alpha_\delta$. Only one principal null direction, this is the most special type and represents pure radiation.
- Type O: $\Psi_{\alpha\beta\gamma\delta} = 0$ corresponds to flat space.

1.3.4 Newman-Penrose scalars and peeling theorem

Given a generic spinor basis $\{o, \iota\}$, we can expand the Maxwell and Weyl spinors as

$$\phi_{\alpha\beta} = \phi_0\, \iota_\alpha\iota_\beta - 2\,\phi_1\, \iota_{(\alpha}o_{\beta)} + \phi_2\, o_\alpha o_\beta \ , \tag{1.78}$$

$$\begin{aligned}\Psi_{\alpha\beta\gamma\delta} = \Psi_0\, \iota_\alpha\iota_\beta\iota_\gamma\iota_\delta - 4\,\Psi_1\, \iota_{(\alpha}\iota_\beta\iota_\gamma o_{\delta)} \\ + 6\,\Psi_2\, \iota_{(\alpha}\iota_\beta o_\gamma o_{\delta)} - 4\,\Psi_3\, \iota_{(\alpha}o_\beta o_\gamma o_{\delta)} + \Psi_4\, o_\alpha o_\beta o_\gamma o_\delta \ .\end{aligned} \tag{1.79}$$

The sets of spin coefficients $\{\phi_i\}$ and $\{\Psi_i\}$ are called Newman-Penrose (NP) scalars [125]. Depending on the algebraic type of the spinor, aligning the spinor basis with the principal spinors sets a subset of the NP scalar to zero. The remaining non-vanishing scalars are listed on table 1.1. Trivially, all the coefficients vanish for type O solutions. Remarkably, Type N and Type D spacetimes are characterised by a single NP scalar and its dual.

In asymptotically flat spacetimes, the NP scalars have an important property known as peeling [125, 126]. This is a hierarchy in their fall-off with large distance r between the observer and the localised source, provided they admit an analytic expansion.[10] In electrodynamics, we have

$$\begin{aligned}\phi_0(x) &= \phi_0^{(0)}(\bar{x})\, \frac{1}{r^3} + \mathcal{O}(1/r^4)\,,\\ \phi_1(x) &= \phi_1^{(0)}(\bar{x})\, \frac{1}{r^2} + \mathcal{O}(1/r^3)\,,\\ \phi_2(x) &= \phi_2^{(0)}(\bar{x})\, \frac{1}{r^1} + \mathcal{O}(1/r^2)\,,\end{aligned} \tag{1.80}$$

[10] The nature of the coordinate r will be made more precise later in the text.

where $\bar{x}$ denotes non-radial dependence. Thus, the scalar $\phi_2(x)$ is the dominant component of the field at large distances: it describes the asymptotic radiation field. Meanwhile, $\phi_1(x)$ is Coulombic. In gravity, the situation is very similar:

$$\begin{aligned}
\Psi_0(x) &= \Psi_0^{(0)}(\bar{x})\,\frac{1}{r^5} + \mathcal{O}(1/r^6)\,,\\
\Psi_1(x) &= \Psi_1^{(0)}(\bar{x})\,\frac{1}{r^4} + \mathcal{O}(1/r^5)\,,\\
\Psi_2(x) &= \Psi_2^{(0)}(\bar{x})\,\frac{1}{r^3} + \mathcal{O}(1/r^4)\,,\\
\Psi_3(x) &= \Psi_3^{(0)}(\bar{x})\,\frac{1}{r^2} + \mathcal{O}(1/r^3)\,,\\
\Psi_4(x) &= \Psi_4^{(0)}(\bar{x})\,\frac{1}{r^1} + \mathcal{O}(1/r^2)\,.
\end{aligned} \tag{1.81}$$

As a consequence, asymptotic gravitational radiation is described by $\Psi_4(x)$, while $\Psi_2(x)$ describes a potential-type contribution, as in Schwarzschild.

1.4 Outline

The remaining of this thesis is organised as follows. Chapters 2 and 3 are aimed at deriving classical double copy results directly from amplitudes. In chapter 2 we show how the KMOC formalism can be used to generate the classical electromagnetic fields sourced by static particle charges from 3-point amplitudes. Section 2.3 reviews how magnetic charge and angular momentum can be added to the amplitudes by simple transformations, resulting in fields that correspond to the $\sqrt{\text{Kerr}}$-dyon solution. Momentum conservation implies that the 3-point amplitude vanishes for real momenta in Lorentzian signature, an issue that we avoid by working in split (2,2) signature in chapters 2 and 3.

In chapter 3, we apply the same tools to gravity. To make the double copy structure explicit at the level of the fields, we introduce a Riemann-Cartan generalised connection in section 3.1. This time, we double copy the gauge amplitudes to obtain seeds for gravitational solutions, as section 3.2 demonstrates. The curvature spinors and tensors of explicit solutions are obtained in section 3.3. Section 3.4 concludes the chapter by exploring the double copy relations exhibited by the classical fields, identifying the previously known Weyl, convolutional and Kerr-Schild prescriptions.

The first two chapters motivate a deeper exploration of the Weyl and Kerr-Schild double copies in chapters 4 and 5. After reviewing the original formulation of the Weyl double copy in section 4.1, we study the effect that the electromagnetic duality has on the double copy. Then, we move on to generalise the Weyl double copy to type N solutions in section 4.2. In the final section of the chapter, we explore the Weyl double copy from future null infinity and study the relation between asymptotic symmetries in gravity and large diffeomorphisms in electromagnetism.

The Kerr-Schild double copy is the object of study in chapter 5. Again, we start by reviewing its basics in section 5.1. Then, in section 5.2, we present a double field theory generalisation that enables us to establish a non-perturbative double copy relation between JNW and Coulomb.

We conclude with some final remarks and possible future directions in chapter 6.

Chapter 2

Classical point charges from amplitudes

The renewed interest in gravitational wave physics has motivated the search for new analytic computational methods in perturbative general relativity. Methods based on scattering amplitudes have proven successful, although they require efficient prescriptions to extract classical information from the quantum amplitudes. The KMOC formalism serves this purpose [44,45]. It starts with quantum observables written in terms of amplitudes. Then, by following the criteria in section 1.2, it returns the associated classical observables. The impulse and emmited radiation in two-body processes are examples of observables that can be obtained by this technique [44–46, 66, 68, 127–130], but it extends to other applications [131–134]. A particularly relevant development for gravitational wave physics was the calculation of waveforms from amplitudes [67] to arbitrary order in perturbation theory.

This chapter demonstrates the process of extracting classical field configurations from scattering amplitudes. More precisely, it shows how to apply the KMOC formalism to a wave packet representing a charged static point particle to obtain the classical electromagnetic fields. This also constitutes a prelude to the next chapter, where the same concepts are applied in the context of gravity. In essence, the goal is to bridge two worlds: quantum amplitudes and classical fields, which often speak different languages. Although the spinor formulation will be an effective translator, there is an additional point of friction that needs to be addressed.

Three-point scattering amplitudes are the atoms in our modern approach to computing interactions between particles in quantum field theory. Using modern on-shell methods, it is possible to construct the complete S-matrix for Yang-Mills theory and (up to ultraviolet divergences) for general relativity from their respective three-point amplitudes. This is done by recursively decreasing the number of loops thanks to unitarity cuts [8, 9] and applying BCFW recursions [7] to cut tree amplitudes down to three-point amplitudes. These amplitudes are gauge invariant and beautifully simple objects, completely specified by the helicities of the massless gluons and gravitons [135]. This basic simplicity carries over to the case of massive particles, for any spin [136]. But despite all these virtues, three-point amplitudes have one big defect: they do not exist in Minkowski space. As for any n-point amplitude, the external particles involved in a three-point amplitude must all be on-shell. But there is no solution to the on-shell conditions in Minkowski space for three particles with different momenta. This obstruction has prevented the three-point amplitude from receiving a classical interpretation so far. A contrasting example is the four-point amplitude between massive particles in gravity, which is closely related to the classical potential [137, 138].

Of course, the fact that the three-point amplitude vanishes in Minkowski space is no obstacle for the programme of determining more complicated amplitudes. BCFW taught us a simple trick: we analytically continue the momenta so that the on-shell conditions *do* have a solution. We can take the momenta to be complex-valued, or else continue to a spacetime with metric signature $(+,+,-,-)$.[1] This second option has some conceptual virtues from the point of view of spinors. The Lorentz group in split signature is locally isomorphic to $\mathrm{SL}(2,\mathbb{R})\otimes\mathrm{SL}(2,\mathbb{R})$, and the spinor representations of $\mathrm{SO}(2,2)$ are the (real) two-dimensional fundamental representations of each $\mathrm{SL}(2,\mathbb{R})$ factor. The upshot is that, instead of having two sets of spinors related by complex conjugation, real solutions in split signature have two sets of independent, real spinors. For related discussions of field theory in split signature, see [139–144].

Another virtue of a real spacetime with signature $(+,+,-,-)$ is that real classical equations exist in this spacetime and their solutions can be studied. In this chapter, we find a classical interpretation for the three-point amplitude in a split-signature spacetime: it computes the Newman-Penrose scalars for the classical solution that is generated by the massive particle in the amplitude. For example, the three-point amplitude between a massive scalar and a gauge boson computes the electromagnetic field strength of a static point charge in split signature. In Einstein gravity, the three-point amplitude between a massive scalar and a graviton computes the linearised Weyl spinor of the split-signature analogue of the Schwarzschild solution. Solutions in split signature which are determined by three-point amplitudes are, from the perspective of scattering amplitudes, the simplest non-trivial classical solutions.

2.1 Initial and final states

Let us consider a static particle source in split signature. We will use coordinates (t_1, t_2, x, y), with signature $(+,+,-,-)$.[2] Since we have two time directions, we should

[1] In our split signature conventions, the $(+)$-directions are timelike and the $(-)$-directions are space-like.

[2] We will also denote two dimensional space-like vectors in bold, eg. $\mathbf{x} = (x, y)$.

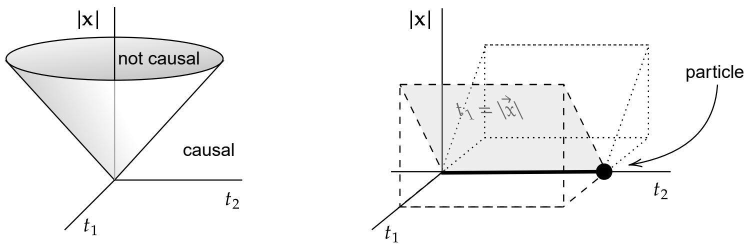


Figure 2.1: The left image shows the 4D light-cone in split signature. The "inner" part of the cone contains all the events that are not causally connected to the vertex, whereas the events in the "exterior" can be reached by causal curves. On the right, the diagram shows the support of the Green's function for our choice of t_1-retarded boundary conditions. The point particle trajectory is represented by the thick line moving along the t_2 axis. The shaded surface is $t_1 - |\vec{x}| = 0$, which contains the radiation. The dashed lines enclose the region where the retarded Green's function is non-zero, i.e. the t_1-future of the particle. The dotted volume is the t_1-past of the particle.

specify the worldline of the particle, which we choose to be the t_2 axis with tangent vector $u^\mu = (0, 1, 0, 0)$. We will model this massive particle as a non-dynamical scalar wave packet, following the KMOC prescription. The expectation value of the momentum of the wave packet should then be $\langle p^\mu \rangle = m\, u^\mu$. We define the quantum state as

$$|\psi\rangle = \int \mathrm{d}\Phi(p)\, \varphi(p)\, |p\rangle, \qquad \mathrm{d}\Phi(p) = \hat{\mathrm{d}}^4 p\, \hat{\delta}(p^2 - m^2)\Theta(E_2)\,, \tag{2.1}$$

where the wave function $\varphi(p)$ is sharply-peaked around the classical momentum $m\, u^\mu$. Notice that the theta function inside $\mathrm{d}\Phi(p)$ enforces positive energy along t_2, the worldline direction of the particle. The existence of the other time direction t_1 implies that there is another energy, E_1.[3]

To connect three-point amplitudes to Newman-Penrose scalars, all that is needed is a direct computation using the methods of quantum field theory. The first order of business, then, is to couple the scalar particle to a quantum electromagnetic field. The existence of two time dimensions implies that we need to specify boundary conditions for the fields. We choose to impose that the "messenger" fields (photons and gravitons) must be in a vacuum state for $t_1 \to -\infty$.

This endows the (t_1, x, y) codimension-1 space with a sense of time ordering in which fields are sourced at $t_1 = 0$ by the instantaneous appearance of the particle. Figure 2.1 diagrammatically represents the causal lightcone in split signature as well as the domain of the gauge field for our boundary conditions. Accordingly, we use the mode expansion for the gauge field operator

$$A^\mu(x) = \sum_{\eta=\pm} \int \mathrm{d}\Phi(k)\, \hbar^{-\frac{1}{2}} \left(a_\eta(k) \varepsilon^\mu_\eta(k) e^{-i\frac{k\cdot x}{\hbar}} + a^\dagger_\eta(k) \varepsilon^\mu_\eta(k) e^{i\frac{k\cdot x}{\hbar}} \right), \tag{2.2}$$

[3]Note that, in the KMOC formalism, the momentum carried by messengers is of order $\hbar$, so in the classical limit our massive particle is indeed static.

where this time the measure is

$$\mathrm{d}\Phi(k) = \hat{\mathrm{d}}^4k\,\hat{\delta}(k^2)\Theta(E^1)\,. \tag{2.3}$$

Note that now the theta function enforces positive energy along the t_1 direction. We will assume from now on that $\mathrm{d}\Phi(k)$ carries a $\Theta(E_1)$ for the gauge field while $\mathrm{d}\Phi(p)$ carries $\Theta(E_2)$ for the massive particle. The associated field strength tensor is

$$F^{\mu\nu}(x) = -2\,i\sum_{\eta=\pm}\int \mathrm{d}\Phi(k)\hbar^{-\frac{3}{2}}\left(a_\eta(k)k^{[\mu}\varepsilon_\eta^{\nu]}e^{-i\frac{k\cdot x}{\hbar}} - a_\eta^\dagger(k)k^{[\mu}\varepsilon_\eta^{\nu]}e^{i\frac{k\cdot x}{\hbar}}\right)\,. \tag{2.4}$$

We want to obtain the field sourced by our particle when it is coupled to the electromagnetic field with a charge Q. For $t_1 < 0$, we impose that there must be no messengers, so the field vanishes,

$$\langle\psi|F^{\mu\nu}|\psi\rangle = 0\,. \tag{2.5}$$

For positive t_1, the state evolves with

$$|\psi_{\text{out}}\rangle = \lim_{t^1\to\infty} U(-t^1, t^1)|\psi\rangle = S|\psi\rangle, \tag{2.6}$$

and the goal is to compute the expectation value of the field

$$\langle F^{\mu\nu}\rangle \equiv \langle\psi|S^\dagger F^{\mu\nu}S|\psi\rangle\,. \tag{2.7}$$

Similarly, we can obtain the spinor counterpart of the field strength tensor

$$\phi_{\alpha\beta}(x) = \sigma^{\mu\nu}{}_{\alpha\beta}F_{\mu\nu}(x)\,. \tag{2.8}$$

In order to have real spinors in split signature, we must choose different conventions for the sigma matrices. Appendix A reviews our split signature spinor conventions as well as other particularities of the signature. In analogy with the spinor-helicity language of scattering amplitudes, we introduce the notation

$$|k\rangle \leftrightarrow \lambda_\alpha\,,\quad \langle k| \leftrightarrow \lambda^\alpha\,,\quad |k] \leftrightarrow \tilde{\lambda}^{\dot\alpha}\,,\quad [k| \leftrightarrow \tilde{\lambda}_{\dot\alpha}\,, \tag{2.9}$$

to pass between momenta k and spinors λ, $\tilde{\lambda}$, where

$$k\cdot\sigma_{\alpha\dot\alpha} = \lambda_\alpha\tilde{\lambda}_{\dot\alpha}\,. \tag{2.10}$$

2.1.1 Coherent final state

In the classical limit, the final state exhibits a dramatic simplification, which enables us to compute the expectation values to all orders of the coupling. To show this, we start by expanding the S matrix

$$S|\psi\rangle = \frac{1}{\mathcal{N}}(1 + iT_3 + iT_4 + \cdots)|\psi\rangle\,, \tag{2.11}$$

where $\mathcal{N}$ is a normalisation constant and the T_n are defined by

$$\begin{aligned} T_{n+2} = \frac{1}{n!}\sum_{\eta_1,\ldots,\eta_n}\int &\mathrm{d}\Phi(p')\mathrm{d}\Phi(p)\prod_{i=1}^{n}\mathrm{d}\Phi(k_i)\,\mathcal{A}^{(n+2)}_{-\eta_1,\ldots,-\eta_n}(p\to p',k_1\cdots k_n) \\ &\times \hat{\delta}^4\left(p - p' - \sum k_i\right)\, a^\dagger_{\eta_1}(k_1)\cdots a^\dagger_{\eta_n}(k_n)\,a^\dagger(p')a(p)\,. \end{aligned} \tag{2.12}$$

That is, the T_{n+2} are projections of the transition matrix T onto final states with n photons, in addition to the massive particle. We denote the creation and annihilation operators for the massive scalar state by $a^\dagger(p')$ and $a(p)$, respectively, as opposed to the photon creation operators $a^\dagger_{\eta_i}(k_i)$. In our conventions, the helicity labels η are for incoming messengers. Note that we include precisely one creation and one annihilation operator for our scalar, which is consistent with treating it as a probe source. We omit all terms in T_{n+2} containing photon annihilation operators since these would annihilate the initial state $|\psi\rangle$. The factor $n!$ in equation (2.12) is a symmetry factor associated with n identical photons in the final state.

We begin by computing the action of T_3 and T_4 on $|\psi\rangle$ explicitly. It will then be a small step to the general case and the exponential structure. First, the case of T_3 is straightforward:

$$\begin{aligned} iT_3|\psi\rangle &= \sum_\eta\int \mathrm{d}\Phi(p')\mathrm{d}\Phi(p)\mathrm{d}\Phi(k)\,\varphi(p)\,i\mathcal{A}_{-\eta}(k)|p',k^\eta\rangle\,\hat{\delta}^4\left(p - p' - k\right) \\ &= \sum_\eta\int \mathrm{d}\Phi(p)\mathrm{d}\Phi(k)\,\varphi(p+k)\,\Theta(E^2+k^2)\hat{\delta}(2p\cdot k)\,i\mathcal{A}_{-\eta}(k)|p,k^\eta\rangle\,, \end{aligned} \tag{2.13}$$

where, in the second line, we integrated over p with the help of a four-fold delta function, and we relabelled p' to p. This expression simplifies when we compute in the domain of validity of the classical approximation. As argued by KMOC [44], the classical approximation reviewed in section 1.2 is valid when the scales in our problem satisfy $x \gg \ell_w \gg \ell_c$, where ℓ_w is the length scale associated with the finite size of the spatial wave packet, which controls the quantum uncertainty in the position of our source particle, while $\ell_c = \hbar/m$ is the (reduced) Compton wavelength of the particle.[4] Working in Fourier space, we require that $k \ll 1/\ell_w \ll m$ (where k is a messenger momentum). It is only when these inequalities are satisfied that our classical expressions are valid. We assume that the integrals appearing in the equations are defined (e.g. with cutoffs) so that these inequalities are satisfied.

Taking advantage of the classical approximation, we can ignore the explicit theta

[4] Recall that the role of the observer position x was played by an impact parameter b.

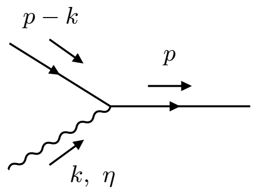


Figure 2.2: The three-point electromagnetic amplitude. Notice that the photon with polarization η is incoming.

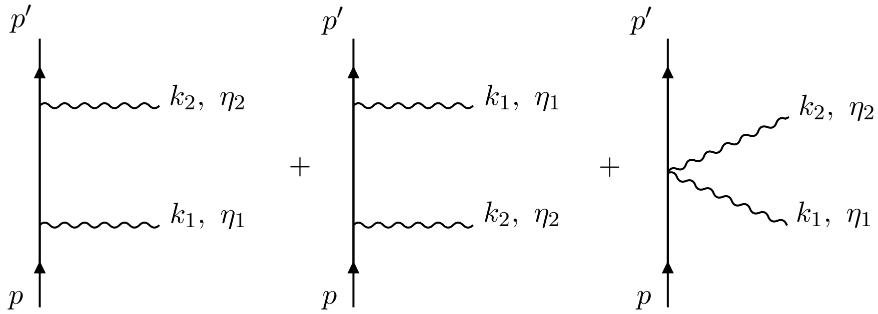


Figure 2.3: The familiar Feynman diagrams for the four point scalar QED amplitude. In this figure, the photons are outgoing.

function in equation (2.13), since k^2 is a small momentum component compared to the large, positive classical energy E^2 of the massive particle, which is of order m. Similarly, we can ignore the shift k in the wave function $\varphi(p+k) \simeq \varphi(p)$, because this shift is small on the scale $1/\ell_w$ of the wave function. Thus, we find

$$iT_3|\psi\rangle = \sum_\eta \int \mathrm{d}\Phi(p)\mathrm{d}\Phi(k)\,\varphi(p)\,\hat{\delta}(2p\cdot k)\,i\mathcal{A}_{-\eta}(k)\,a^\dagger_\eta(k)|p\rangle\,. \tag{2.14}$$

For our static charge in electromagnetism, the three-point amplitude is the scalar QED vertex,

$$\begin{aligned}\mathcal{A}_-(k) &= -2\frac{Q}{\sqrt{\hbar}}\,p\cdot\varepsilon_-(k),\\ \mathcal{A}_+(k) &= -2\frac{Q}{\sqrt{\hbar}}\,p\cdot\varepsilon_+(k).\end{aligned} \tag{2.15}$$

Notice that the amplitude depends on k only through the polarisation vector $\varepsilon_\eta(k)$: it therefore does not depend on whether we treat k as a momentum or as a wave vector.

The factor $1/\sqrt{\hbar}$ in the amplitude cancels out with other factors of $\hbar$ in the expectation value of $F_{\mu\nu}$. This is obviously consistent with the computation of a classical quantity. Since all factors of $\hbar$ will similarly disappear for classical quantities in the remainder of the thesis, we will henceforth set $\hbar = 1$, restoring it only when necessary.

The four-point case requires a little more work on the actual amplitude. Working at the textbook level of Feynman diagrams (using the notation in figure 2.3), we find

$$i\mathcal{A}^{(4)} = -iQ^2 \frac{4p\cdot\varepsilon_{-\eta_1}(k_1)\, p'\cdot\varepsilon_{-\eta_2}(k_2)}{2k_1\cdot p + i\epsilon} + iQ^2 \frac{4p\cdot\varepsilon_{-\eta_2}(k_2)\, p'\cdot\varepsilon_{-\eta_1}(k_1)}{2k_1\cdot p' - i\epsilon} + 2iQ^2 \varepsilon_{-\eta_1}(k_1)\cdot\varepsilon_{-\eta_2}(k_2)\,. \tag{2.16}$$

The superscript (4) emphasises that now this is a four-point amplitude. Now, of these three terms, the last is suppressed relative to the other two in the classical approximation. The suppression factor is of order $p\cdot k/m^2$, which is of order the energy of a single photon in units of the mass of the particle. (Equivalently, the suppression factor is $\hbar\,\bar{k}/m$, where $\bar{k}$ is a typical component of the wave vector of the photon. From this perspective, the contact term is explicitly down by a factor $\hbar$.) Therefore, we neglect the contact diagram. In terms of a more modern unitarity-based construction of the amplitude, this means that we can simply "sew" three-point amplitudes to compute the dominant part of the four-point amplitude relevant for this computation.[5]

We can make this sewing completely manifest in our four-point amplitude by writing

$$k_1\cdot p' = k_1\cdot p + \mathcal{O}(\hbar)\,, \qquad p'\cdot\varepsilon(k) = p\cdot\varepsilon(k) + \mathcal{O}(\hbar)\,, \tag{2.17}$$

and neglecting the $\hbar$ corrections. (In dimensionless terms, these corrections are again suppressed by factors of the photon energy over the particle mass.) It is then a matter of algebra to see that

$$\begin{aligned} i\mathcal{A}^{(4)} &= \hat{\delta}(2p\cdot k_1)\,(-2iQ\,p\cdot\varepsilon_{-\eta_1}(k_1))(-2iQ\,p\cdot\varepsilon_{-\eta_2}(k_2)) \\ &= \hat{\delta}(2p\cdot k_1)\, i\mathcal{A}_{-\eta_1}(k_1)\, i\mathcal{A}_{-\eta_2}(k_2)\,. \end{aligned} \tag{2.18}$$

We picked up a delta function from the sum of two propagators. It is perhaps worth pausing to note that the two photon emissions are completely uncorrelated from one another.

Now we can compute the action of T_4 on our initial state. Using the definition (2.12) of T_4 and the fact that

$$a(p)|\psi\rangle = \varphi(p)|0\rangle\,, \tag{2.19}$$

we find

$$iT_4|\psi\rangle = \frac{1}{2}\sum_{\eta_1,\eta_2}\int \mathrm{d}\Phi(p')\mathrm{d}\Phi(p)\mathrm{d}\Phi(k_1)\mathrm{d}\Phi(k_2)\,\varphi(p)\, i\mathcal{A}^{(4)}_{-\eta_1,-\eta_2}(p\to p', k_1^{\eta_1}k_2^{\eta_2}) \times \hat{\delta}^4(p-p'-k_1-k_2)|p'\, k_1^{\eta_1}k_2^{\eta_2}\rangle\,. \tag{2.20}$$

[5] It may be worth emphasising that a one-loop computation of a classical observable such as the impulse also involves the four-point tree amplitude. But in that case, the contact term is absolutely necessary to recover the correct classical result, and in fact the terms we are concentrating on cancel.

The integration over the momentum p is trivial using the explicit four-fold delta function. The measure $\mathrm{d}\Phi(p)$ contains a theta function, requiring that the E^2 component of $p' + k_1 + k_2$ is positive. Since the $\mathrm{d}\Phi(p')$ measure already requires the relevant energy of p' to be positive, and the photon energies are small compared to the mass, we can ignore this theta function. We also encounter the wave function evaluated at $p' + k_1 + k_2$; since the photon energies are small compared to the width of the wave function, we may approximate $\varphi(p' + k_1 + k_2) \simeq \varphi(p')$. Finally, $\mathrm{d}\Phi(p)$ contains a delta function requiring

$$p^2 = (p' + k_1 + k_2)^2 = m^2 \,. \tag{2.21}$$

Since $p'^2 = m^2$, this becomes a factor

$$\hat{\delta}(2p' \cdot (k_1 + k_2) + (k_1 + k_2)^2)$$

in $T_4|\psi\rangle$. Once again, we may neglect this shift of the delta function, as it is small compared to the width of the broadened δ_{ℓ_w} function resulting from integrating against the wave function [44]. Neglecting this width ℓ_w, we find

$$\begin{aligned} iT_4|\psi\rangle = \frac{1}{2} \sum_{\eta_1,\eta_2} \int \mathrm{d}\Phi(p)\mathrm{d}\Phi(k_1)\mathrm{d}\Phi(k_2)\, \varphi(p)\, i\mathcal{A}^{(4)}_{-\eta_1,-\eta_2}(p + k_1 + k_2 \to p, k_1 k_2) \\ \times \hat{\delta}(2p \cdot (k_1 + k_2))|p\, k_1^{\eta_1} k_2^{\eta_2}\rangle \,, \end{aligned} \tag{2.22}$$

where we relabelled the momentum p' to p. Now we may use our result (2.18) for the four-point amplitude, arriving at

$$\begin{aligned} iT_4|\psi\rangle &= \frac{1}{2} \sum_{\eta_1,\eta_2} \int \mathrm{d}\Phi(p)\mathrm{d}\Phi(k_1)\mathrm{d}\Phi(k_2)\, \varphi(p)\, \hat{\delta}(2p \cdot k_1)\hat{\delta}(2p \cdot k_2) \\ &\qquad \times i\mathcal{A}_{-\eta_1}(k_1)\, i\mathcal{A}_{-\eta_2}(k_2)|p\, k_1^{\eta_1} k_2^{\eta_2}\rangle \\ &= \frac{1}{2} \int \mathrm{d}\Phi(p)\, \varphi(p) \left(\sum_{\eta} \int \mathrm{d}\Phi(k)\hat{\delta}(2p \cdot k) i\mathcal{A}_{-\eta}(k) a^\dagger_\eta(k) \right)^2 |p\rangle \,, \end{aligned} \tag{2.23}$$

Note that we have obtained the same result as in (2.14), but with a factor of 1/2 and the integrand squared.

Now we turn to the general term, evaluating $T_{n+2}|\psi\rangle$. We can make use of the knowledge gained from the four-point example, including the fact that the leading term in the $(n+2)$-point amplitude can be obtained by sewing n three-point amplitudes. We must nevertheless sum over permutations of the external photon momenta as shown in

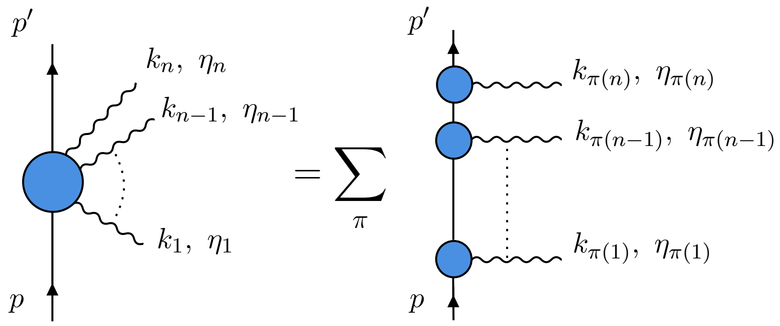


Figure 2.4: The dominant term in the $n+2$ point amplitude can be obtained by sewing n three-point amplitudes. The full amplitude is obtained by summing over permutations π of the n outgoing photon lines.

figure 2.4. The dominant term in the amplitude is

$$\begin{aligned} i\mathcal{A}^{(n+2)} = \left(\prod_{i=1}^{n} i\mathcal{A}_{-\eta_i}(k_i)\right) \sum_{\pi} \frac{i}{2p\cdot k_{\pi(1)}+i\epsilon}\frac{i}{2p\cdot(k_{\pi(1)}+k_{\pi(2)})+i\epsilon}\cdots \\ \times \frac{i}{2p\cdot(k_{\pi(1)}+k_{\pi(2)}+\cdots k_{\pi(n-1)})+i\epsilon}\,. \end{aligned} \tag{2.24}$$

The sum is over permutations π of the n final-state photons.

At four points, the sum over sewings led to a delta function, and the same happens here. We can state the result most simply at the level of $T_{n+2}|\psi\rangle$, which can be written as

$$iT_{n+2}|\psi\rangle = \frac{1}{n!}\sum_{\eta_1,\ldots,\eta_n}\int \mathrm{d}\Phi(p)\prod_{i=1}^{n}\mathrm{d}\Phi(k_i)\,\varphi(p)\,\hat{\delta}\left(2p\cdot\sum_{j=1}^{n}k_j\right) i\mathcal{A}^{(n+2)}\,|p\,k_1^{\eta_1}\cdots k_n^{\eta_n}\rangle\,, \tag{2.25}$$

using the properties of the wave function, and neglecting terms suppressed in the classical region. We may now simplify the sum in equation (2.24) using the result

$$\begin{aligned} \hat{\delta}\left(\sum_{i=1}^{n}\omega_i\right)\sum_{\pi}\frac{i}{\omega_{\pi(1)}+i\epsilon}\frac{i}{\omega_{\pi(1)}+\omega_{\pi(2)}+i\epsilon}\cdots\frac{i}{\omega_{\pi(1)}+\omega_{\pi(2)}+\cdots\omega_{\pi(n-1)}+i\epsilon} \\ = \hat{\delta}(\omega_1)\hat{\delta}(\omega_2)\cdots\hat{\delta}(\omega_n)\,. \end{aligned} \tag{2.26}$$

This result, which is an on-shell analogue of the eikonal identity, is proven (for example) in appendix A of reference [69]. We find that

$$iT_{n+2}|\psi\rangle = \frac{1}{n!}\int \mathrm{d}\Phi(p)\,\varphi(p)\left(\sum_{\eta}\int \mathrm{d}\Phi(k)\,\hat{\delta}(2p\cdot k)\,i\mathcal{A}_{-\eta}(k)\,a_{\eta}^{\dagger}(k)\right)^{n}|p\rangle\,, \tag{2.27}$$

generalising the square found in (2.23). Performing the sum over n, we obtain an exponential structure in the final state

$$S|\psi\rangle = \frac{1}{\mathcal{N}} \int \mathrm{d}\Phi(p)\varphi(p) \exp\left[\sum_\eta \int \mathrm{d}\Phi(k)\, \hat{\delta}(2p\cdot k)\, i\mathcal{A}_{-\eta}(k)\, a^\dagger_\eta(k)\right] |p\rangle\,, \tag{2.28}$$

where $\mathcal{N}$ is the normalisation factor ensuring that $\langle\psi|S^\dagger S|\psi\rangle = 1$. The exponential structure of the state captures the intuition that the outgoing field contains a great many photons. It is also consistent with the intuition that coherent states are the natural description of classical wave phenomena in quantum field theory. The coherence of the state could also be demonstrated by taking advantage of the linear coupling between the gauge field A_μ and a massive probe source worldline, so it comes as no surprise. However, it is satisfying to see that the state is completely controlled by the on-shell three-point amplitude.

Now that we have seen that the final state is given by equation (2.28), let us return to the evaluation of the expectation value of the field strength. The computation is simplified when we recall that (as usual for a coherent state) the annihilation operator acts as a derivative on the state:

$$\begin{aligned} a_\eta(k)S|\psi\rangle &= \hat{\delta}(2p\cdot k)\, i\mathcal{A}_{-\eta}(k)\, S|\psi\rangle \\ &= \frac{\delta}{\delta a^\dagger_\eta(k)} S|\psi\rangle\,. \end{aligned} \tag{2.29}$$

As a result, annihilation operators can be replaced by amplitudes.

The field strength is therefore

$$\begin{aligned} \langle\psi|S^\dagger\, F^{\mu\nu}(x)\, S|\psi\rangle &= -4\ \mathrm{Re}\ i\sum_\eta \int \mathrm{d}\Phi(k)\, \langle\psi|S^\dagger\, a_\eta(k)\, S|\psi\rangle\, k^{[\mu}\varepsilon^{\nu]}_\eta\, e^{-ik\cdot x} \\ &= \frac{2}{m}\,\mathrm{Re}\sum_\eta \int \mathrm{d}\Phi(k)\, \hat{\delta}(u\cdot k)\mathcal{A}_{-\eta}(k)\, k^{[\mu}\varepsilon^{\nu]}_\eta\, e^{-ik\cdot x}\,. \end{aligned} \tag{2.30}$$

Similarly, the Maxwell spinor is obtained making use of (A.8) and (A.9),

$$\langle\psi|S^\dagger\, \phi_{\alpha\beta}(x)\, S|\psi\rangle = -\frac{\sqrt{2}}{m}\,\mathrm{Re}\int \mathrm{d}\Phi(k)\, \hat{\delta}(u\cdot k)\, |k\rangle_\alpha |k\rangle_\beta\, e^{-ik\cdot x}\mathcal{A}_+(k), \tag{2.31}$$

while the conjugate one reads

$$\langle\psi|S^\dagger \tilde{\phi}_{\dot\alpha\dot\beta}(x)\, S|\psi\rangle = \frac{\sqrt{2}}{m}\,\mathrm{Re}\int \mathrm{d}\Phi(k)\, \hat{\delta}(u\cdot k)\, [k|_{\dot\alpha}[k|_{\dot\beta}\, e^{-ik\cdot x}\, \mathcal{A}_-(k)\,. \tag{2.32}$$

Equations (2.31) and (2.32) show a direct relation between the Maxwell spinors and the 3-point amplitudes. The positive helicity amplitude controls the anti-self-dual part of the electromagnetic field, while the negative helicity amplitude controls the self-dual part.

2.2 Coulomb potential

So far we have obtained the classical expectation values of the fields as on-shell momentum integrals of the three-point amplitudes. Such expressions have the attribute of explicitly relating classical fields to amplitudes, but eventually we would like to perform the integrals to obtain position space solutions. Since we are dealing with a simple static electric charge, we can expect to obtain a split signature analogue of the Coulomb potential.

The first step towards this end is to evaluate the amplitudes in (2.30) with (2.15),

$$\begin{aligned}\langle F^{\mu\nu}(x)\rangle &\equiv \langle\psi|S^\dagger\, F^{\mu\nu}(x)\, S|\psi\rangle \\ &= -4\, Q\, \mathrm{Re} \sum_\eta \int \mathrm{d}\Phi(k)\, \hat{\delta}(k\cdot u)\, e^{-ik\cdot x}\, k^{[\mu}\varepsilon_\eta^{\nu]}\, \varepsilon_{-\eta}\cdot u\,.\end{aligned} \tag{2.33}$$

This expression can be simplified by resolving the proper velocity onto a Newman-Penrose-like basis of vectors given by k^μ, $\varepsilon_\pm^\mu$ and a gauge choice n^μ, such that $k\cdot n = 1$ while $n\cdot\varepsilon_\pm = 0$. The metric can be then written as

$$\eta^{\mu\nu} = 2\, k^{(\mu}n^{\nu)} - 2\, \varepsilon_+^{(\mu}\varepsilon_-^{\nu)}\ ,\ . \tag{2.34}$$

Since $k\cdot u = 0$ on the support of the integration, the velocity can be written in the tetrad basis as

$$u^\mu = (u\cdot n)\, k^\mu - (\varepsilon_-\cdot u)\, \varepsilon_+^\mu - (\varepsilon_+\cdot u)\, \varepsilon_-^\mu\,. \tag{2.35}$$

Consequently, the field strength is given by the simple formula

$$\langle F^{\mu\nu}(x)\rangle = 4\, Q\, \mathrm{Re} \int \mathrm{d}\Phi(k)\, \hat{\delta}(k\cdot u)\, e^{-ik\cdot x}\, k^{[\mu}u^{\nu]}\,. \tag{2.36}$$

Before we perform any integrations, let us pause to interpret this formula. Note that we may write

$$\langle F^{\mu\nu}(x)\rangle = 4\, Q\, \partial^{[\mu}u^{\nu]}\, \mathrm{Re}\, i \int \mathrm{d}\Phi(k)\, \hat{\delta}(k\cdot u)\, e^{-ik\cdot x}\,. \tag{2.37}$$

We recognise the definition of the field strength as the (antisymmetrised) derivative of the gauge potential,

$$\langle A^\mu(x)\rangle = 2\, Q\, \mathrm{Re}\, i\, u^\mu \int \mathrm{d}\Phi(k)\, \hat{\delta}(k\cdot u)\, e^{-ik\cdot x}\,. \tag{2.38}$$

Similarly, from the gauge potential we can recognise a scalar potential

$$\langle A^\mu(x)\rangle = Q\, u^\mu \langle S(x)\rangle \,, \tag{2.39}$$

with

$$\langle S(x)\rangle = 2\,\mathrm{Re}\, i \int \mathrm{d}\Phi(k)\hat{\delta}(k\cdot u)e^{-ik\cdot x} \,. \tag{2.40}$$

$$= i\int \hat{\mathrm{d}}^4 k\, \hat{\delta}(k_1^2 - \mathbf{k}^2)\Theta(k_1)\hat{\delta}(k\cdot u)\left(e^{-ik\cdot x} - e^{ik\cdot x}\right) \,. \tag{2.41}$$

To interpret these formulas, it's worth digressing briefly to discuss our situation from a classical perspective.

2.2.1 Classical calculation

Although in this chapter we are attempting to compute classical fields using scattering amplitudes, they can also be computed by solving their classical equations of motion. For instance, consider solving the Maxwell equation with a static point charge

$$\partial_\mu F^{\mu\nu}(x) = \int d\tau\, Q u^\nu\, \delta^4(x - u\tau)\,, \tag{2.42}$$

where $u^\mu = (0,1,0,0)$, with the boundary condition that the electromagnetic field vanishes for $t^1 < 0$. Choosing Lorenz gauge, we can write the solution as a familiar Fourier integral:

$$A^\mu(x) = -\int \hat{\mathrm{d}}^4 k\, \hat{\delta}(k\cdot u)\, e^{-ik\cdot x}\frac{1}{k^2} Q u^\mu \,. \tag{2.43}$$

As usual, we need to define the k integral taking our boundary conditions into account. These boundary conditions are also familiar: they are just traditional retarded boundary conditions. The only novelty lies in the signature of the metric. But even the unfamiliar pattern of signs in split signature disappears for the problem at hand, because of the factor

$$\hat{\delta}(k\cdot u) = \hat{\delta}(k_2)$$

in the measure. Consequently, the second component of the wave vector k^μ is guaranteed to be zero. We end up with an integral of Minkowskian type, but in $1+2$ dimensions. This is a consequence of translation invariance in the t^2 direction.

Treating the k integration as a contour integral, the only poles in the integration of equation (2.43) occur when

$$(k^1)^2 = \mathbf{k}^2 \,, \tag{2.44}$$

where $\mathbf{k} = (k^3, k^4)$ are the spatial components of the wave vector. Taking the sign of the exponent in equation (2.43) into account, retarded boundary conditions are obtained by displacing the poles below the real axis:

$$\frac{1}{k^2} \to \frac{1}{k^2_{\rm ret}} = \frac{1}{(k^1 + i\epsilon)^2 + (k^2)^2 - (k^3)^2 - (k^4)^2}\,, \tag{2.45}$$

while advanced boundary conditions correspond to

$$\frac{1}{k^2} \to \frac{1}{k^2_{\rm adv}} = \frac{1}{(k^1 - i\epsilon)^2 + (k^2)^2 - (k^3)^2 - (k^4)^2}\,. \tag{2.46}$$

Notice that

$$\frac{1}{k^2_{\rm ret}} - \frac{1}{k^2_{\rm adv}} = \frac{1}{k^2 + i(k^1)\epsilon} - \frac{1}{k^2 - i(k^1)\epsilon} \quad = -i\,\mathrm{sign}(k^1)\hat{\delta}(k^2)\,, \tag{2.47}$$

where, in the first equality, we have written (k^1) for the first component of the 4-vector k and have freely rescaled ϵ by positive quantities (as is conventional, we take $\epsilon \to 0$ from above at the end of our calculation).

Returning to the gauge field of equation (2.43), we have

$$\begin{aligned} A^\mu(x) &= -\int \hat{\mathrm{d}}^4k\,\hat{\delta}(k\cdot u)\,e^{-ik\cdot x}\left(-i\,\mathrm{sign}(k^1)\hat{\delta}(k^2) + \frac{1}{k^2_{\rm adv}}\right) Qu^\mu \\ &= i\int \hat{\mathrm{d}}^4k\,\hat{\delta}(k\cdot u)\,e^{-ik\cdot x}\;\mathrm{sign}(k^1)\hat{\delta}(k^2)\,Qu^\mu \\ &= i\int \mathrm{d}\Phi(k)\,\hat{\delta}(k\cdot u)\,Qu^\mu\left(e^{-ik\cdot x} - e^{ik\cdot x}\right)\,. \end{aligned} \tag{2.48}$$

We dropped the advanced term because, with our boundary conditions, the position x has positive t^1. But equation (2.48) is just the result we found from the quantum expectation (2.38). Thus, our quantum mechanical methods are computing the complete gauge field, as expected.

Given that we have made contact with a classical situation, we can use classical intuition to perform the Fourier integrals. The integrals to be performed in equation (2.43) are the same as the integrals in the computation of the retarded Green's function in $1+2$ dimensions. We discuss this Green's function in appendix A.2. We find

$$A^\mu(x) = \frac{Qu^\mu}{2\pi}\Theta(t^1)\frac{\Theta(x^2 - (x\cdot u)^2)}{\sqrt{x^2 - (x\cdot u)^2}}\,. \tag{2.49}$$

In many respects, this result is familiar: it is just the usual 1/'distance' fall-off. There is no other possibility: the dimensional analysis requires this behaviour with distance. Although the obtained the solution in split signature, it can be analytically continued back to Minkowski space. The continuation requires a prescription to complexify the contour that gives rise to our choice of propagator. The details of this process are spelt out in A.3.

2.3 $\sqrt{\text{Kerr}}$ dyon

In section 2.2, we only considered the most basic amplitude in QED for a static point particle. In gauge theory, there exist two non-trivial deformations of the Coulomb amplitude. The first of these introduces a $e^{\eta\,\theta}$ factor in the amplitude,

$$\mathcal{A}_\eta(k) \to \mathcal{A}_\eta(k)\, e^{\eta\,\theta}\ , \tag{2.50}$$

where η is the helicity of the photon [90]. Notice that the rotation parameter has been continued from Lorentzian space $\theta \to -i\theta$, as motivated in appendix A.4. This deformation has the interpretation of an electric-magnetic duality rotation, allowing us to introduce magnetic charges. The second deformation is slightly more complicated. For a photon with momentum k, it introduces a factor $e^{i\,\eta k\cdot a}$ in the amplitude,

$$\mathcal{A}_\eta(k) \to \mathcal{A}_\eta(k)\, e^{i\,\eta\, k\cdot a}\ . \tag{2.51}$$

The vector a^μ is a four-vector parameter related to the classical angular momentum. It will be taken to lie along the Wick rotated coordinate: $a^\mu = (a, 0, 0, 0)$. Consequently, the Lorentzian exponent $-\eta\, k \cdot a$ has been analytically continued to split signature as $i\,\eta\, k\cdot a$. Rather remarkably, this deformation leads to an amplitude describing a particle with large classical spin a^μ interacting with a photon. It may be derived [88] by studying the large spin limit of the "minimally coupled" amplitudes of Arkani-Hamed, Huang and Huang [136], and is known to be a form of the Newman-Janis shift [145]. Both the electromagnetic rotation and spin shift were previously considered in [146] in the context of the double copy, but without the tools to compute curvatures from the amplitudes directly. This can be done now thanks to the exponentiation leading to the coherent state reviewed in section 2.1. The transformations do not obstruct the derivation, and one just needs to replace the amplitudes following (2.50) and (2.51).

The next subsections are devoted to the effects of these deformations on the field strength tensor and its spinors. Later on, these transformations will be carried over to gravity via the double copy. The duality angle will be associated to the NUT charge whereas a will be the Kerr angular momentum parameter. The transformation effects on the electromagnetic field is summarised in table 2.1. The table also shows the effects on the double copy, which are derived in the next chapter.

2.3.1 Deformed amplitude

Both transformations can be applied simultaneously to the same amplitude,

$$\mathcal{A}_\eta \to \mathcal{A}_\eta e^{\,\eta\,(ik\cdot a+\theta)}\ . \tag{2.52}$$

Transformation	Gauge theory	Pure gravity
None	Coulomb	Schwarzschild
EM rotation	dyon	Taub-NUT
Newman-Janis shift	$\sqrt{\text{Kerr}}$	Kerr
EM rotation + NJ shift	spinning dyon	Kerr-Taub-NUT

Table 2.1: Effect of the transformations (2.50) *and* (2.51).

Performing this replacement in (2.31) yields the transformed Maxwell spinor

$$\langle \phi_{\alpha\beta}(x) \rangle = -\,\mathrm{Re} \int \mathrm{d}\Phi(k)\, \hat{\delta}(2p \cdot k)\, 2\sqrt{2}\, |k\rangle_\alpha |k\rangle_\beta\, \mathcal{A}_+(k)\, e^{ik\cdot a + \theta}\, e^{-ik\cdot x}\ . \tag{2.53}$$

It is immediately clear that

$$\langle \phi_{\alpha\beta}(x) \rangle = e^{\theta} \langle \phi^{\text{Coul.}}_{\alpha\beta}(x-a) \rangle\,, \tag{2.54}$$

where $\langle \phi^{\text{Coul.}}_{\alpha\beta}(x) \rangle$ is the Maxwell spinor of the Coulomb solution. A similar expression can be obtained for the conjugate spinor,

$$\langle \tilde{\phi}_{\dot{\alpha}\dot{\beta}}(x) \rangle = e^{-\theta} \langle \tilde{\phi}^{\text{Coul.}}_{\dot{\alpha}\dot{\beta}}(x+a) \rangle \tag{2.55}$$

The interpretation of these transformations in terms of the Newman-Janis shift and of electric-magnetic duality is now more manifest. This will be even more transparent if we apply the transformation to the field strength tensor, which is our next goal.

Scalar potential

As a preliminary step, we will study the combined effect of the transformations on the scalar potential

$$\begin{aligned} S_{a,\theta}(x) &:= 2\,\mathrm{Re}\, i \int \mathrm{d}\Phi(k) \hat{\delta}(k\cdot u)\, e^{-i\,k\cdot(x-a)} e^{\theta} \\ &= \frac{e^{\theta}}{2\pi} \frac{\Theta((t_1-a)^2 - r^2)}{\sqrt{(t_1-a)^2 - r^2}} = e^{\theta} S_{0,0}(x-a) \end{aligned} \tag{2.56}$$

where r is defined as the $2d$ radius $\sqrt{x^2+y^2}$. The first line implies that the effect of the spin vector a^μ is merely a shift in the spacetime coordinates x^μ along t_1. The last line was obtained by introducing an $i\epsilon$ prescription for convergence, in the same sense as in (A.15).

Field strength tensor

The deformed classical field strength tensor can be obtained as the expectation value

$$\begin{aligned}\langle F^{\mu\nu}(x)\rangle &\equiv \langle\psi|S^\dagger\, F^{\mu\nu}(x)\, S|\psi\rangle \\ &= -4\,Q\,\mathrm{Re}\sum_\eta \int \mathrm{d}\Phi(k)\,\hat{\delta}(k\cdot u)\,e^{-ik\cdot(x+\eta\,a)}\,k^{[\mu}\varepsilon_\eta^{\nu]}\,e^{-\theta\,\eta}\,\varepsilon_{-\eta}\cdot u\,.\end{aligned} \tag{2.57}$$

In the second line, we have substituted the amplitude (2.52) into (2.30). The integrand can be expanded as

$$\begin{aligned}\langle F^{\mu\nu}(x)\rangle = -4\,Q\,\mathrm{Re}\int \mathrm{d}\Phi(k)\,\hat{\delta}(k\cdot u)\,e^{-ik\cdot x} \\ \times\left(k^{[\mu}\varepsilon_+^{\nu]}\,e^{-\theta-i\,k\cdot a}\,\varepsilon_-\cdot u + k^{[\mu}\varepsilon_-^{\nu]}\,e^{\theta+i\,k\cdot a}\,\varepsilon_+\cdot u\right)\,.\end{aligned} \tag{2.58}$$

Using the null tetrad (2.34) and (2.35), the above expression can be rearranged as

$$\begin{aligned}\langle F^{\mu\nu}(x)\rangle = 4\,Q\,\mathrm{Re}\int \mathrm{d}\Phi(k)\,\hat{\delta}(k\cdot u)\,e^{-ik\cdot x} \\ \times\left(\cos(k\cdot a-i\theta)\,k^{[\mu}u^{\nu]} + i\,\sin(k\cdot a-i\theta)\left(k^{[\mu}\varepsilon_+^{\nu]}\,\varepsilon_-\cdot u - k^{[\mu}\varepsilon_-^{\nu]}\,\varepsilon_+\cdot u\right)\right)\,.\end{aligned} \tag{2.59}$$

This can be further simplified to

$$\begin{aligned}\langle F^{\mu\nu}(x)\rangle = 4\,Q\,\mathrm{Re}\int \mathrm{d}\Phi(k)\,\hat{\delta}(k\cdot u)\,e^{-ik\cdot x} \\ \times\left(\cos(k\cdot a-i\theta)\,k^{[\mu}u^{\nu]} - \frac{i\,\sin(k\cdot a-i\theta)}{2}\epsilon^{\mu\nu\rho\sigma}k_{[\rho}u_{\sigma]}\right),\end{aligned} \tag{2.60}$$

making use of the identity

$$\epsilon^{\mu\nu\rho\sigma}k_\rho\,u_\sigma = -2\left(\varepsilon_-\cdot u\,k^{[\mu}\varepsilon_+^{\nu]} - \varepsilon_+\cdot u\,k^{[\mu}\varepsilon_-^{\nu]}\right)\,. \tag{2.61}$$

Expanding the sine and cosine and recalling the definition of $S_{a,\theta}(x)$, we obtain

$$\langle F^{\mu\nu}(x)\rangle = 2\,Q\,\partial^{[\mu}u^{\nu]}\left[\frac{S_{a,\theta}+S_{-a,-\theta}}{2}\right] - Q\,\epsilon^{\mu\nu\rho\sigma}\partial_{[\rho}u_{\sigma]}\left[\frac{S_{a,\theta}-S_{-a,-\theta}}{2}\right]. \tag{2.62}$$

In this expression, the derivatives act on the terms in brackets, since u^μ has constant components. We have obtained the fields for generic a and θ but in the next two subsections we will focus on each transformation individually.

2.3.2 Newman-Janis shift

A pure Newman-Janis shift has $\theta = 0$,

$$S_{a,0}(x) = \frac{1}{2\pi} \frac{\Theta\left((t_1 - a)^2 - r^2\right)}{\sqrt{(t_1 - a)^2 - r^2}} \ . \tag{2.63}$$

It is worth commenting on some features of this field. First of all, it encodes the complex zeroth copy of Kerr. This can be checked by rotating to (1,3) signature as $t_1 \to i\,z$

$$(t_1 - a)^2 - r^2 \to (i\,z - a)^2 - r^2 = -(z + ia)^2 - r^2 = -\left(\tilde{R} + i\,a\,\cos\vartheta\right)^2 \ . \tag{2.64}$$

The "Kerr radius" $\tilde{R}$ and polar angle ϑ are implicitly defined by

$$\frac{r^2}{\tilde{R}^2 + a^2} + \frac{z^2}{\tilde{R}^2} = 1 \ , \qquad \cos\vartheta = \frac{z}{\tilde{R}} \ . \tag{2.65}$$

Thus,

$$\frac{1}{\sqrt{(t_1 - a)^2 - r^2}} \quad \to \quad \frac{i}{\tilde{R} + i\,a\,\cos\vartheta} \tag{2.66}$$

which is proportional to the scalar (3.32) in [99]. The fact that we recover the complex scalar supports the idea of associating the double copy for amplitudes in (2,2) signature to the Weyl double copy.

Secondly, notice that the spin a appears only in the combination $(t_1 - a)$ in equation (2.63). This is the Newman-Janis shift at work: in (2,2) signature, the shift is a *real* translation, in the t_1 direction, at the level of the field strength. (The shift acts in a more subtle way on the potential. At the level of the effective action, the shift can be interpreted as replacing the usual worldline action with a worldsheet structure [108].)

To obtain the shifted field strength tensor, we just need to set $\theta = 0$ in (2.62) and find

$$\langle F^{\mu\nu}(x)\rangle = 2\,Q\,\partial^{[\mu}u^{\nu]}\left[\frac{S_{a,0} + S_{-a,0}}{2}\right] - Q\,\epsilon^{\mu\nu\rho\sigma}\partial_{[\rho}u_{\sigma]}\left[\frac{S_{a,0} - S_{-a,0}}{2}\right] . \tag{2.67}$$

Note that in Minkowski signature the first bracket corresponds to the real part of S while the second corresponds to the imaginary part.

The field (2.67) is the split signature equivalent of the $\sqrt{\text{Kerr}}$ solution, the single copy of the Kerr black hole [88,97]. Instead of checking this claim by direct comparison, which would be tedious due to coordinate transformations, we can derive (2.60) with $\theta = 0$ in a purely classical way. To do so, we solve the Maxwell equations in the presence of a $\sqrt{\text{Kerr}}$ source,

$$\begin{aligned}\partial_\mu F^{\mu\nu}(x) &= Q\int \mathrm{d}\tau \exp(a*\partial)^\nu{}_\rho u^\rho \delta^{(4)}(x-u\tau) \equiv j^\nu_{\sqrt{\text{Kerr}}}(x),\\ (a*b)_{\mu\nu} &:= \epsilon_{\mu\nu\rho\sigma}a^\rho b^\sigma.\end{aligned} \tag{2.68}$$

Note that the source for $\sqrt{\text{Kerr}}$ is formally the one for Coulomb (see for instance [147] or [108]) but acted upon by the differential operator $\exp(a*\partial)^\nu{}_\rho$. We observe that the exponential $\exp(a*\partial)$ remains invariant under analytic continuation to split signature. This is because the derivative picks up a factor of $-i$, which is cancelled out by the i picked up by the volume form.

We solve (2.68) by Fourier transform with the boundary conditions outlined in section 2.1. We get

$$A^\mu(x) = 2Q\,\mathrm{Re}\, i\int \mathrm{d}\Phi(k)\,\hat{\delta}(k\cdot u)e^{-ik\cdot x}\exp(-ia*k)^\mu{}_\rho u^\rho. \tag{2.69}$$

The action of the exponential matrix can be further simplified. In fact, on the support of the on-shell measure, it can be shown that

$$\exp(-ia*k)^\mu{}_\rho u^\rho = u^\mu \cos a\cdot k - i\epsilon^\mu(a,k,u)\frac{\sin a\cdot k}{a\cdot k}, \tag{2.70}$$

where we defined $\epsilon^\mu(a,b,c) := \epsilon^{\mu\alpha\beta\gamma}a_\alpha b_\beta c_\gamma$. We obtain finally

$$A^\mu(x) = 2\,Q\,\mathrm{Re}\, i\int \mathrm{d}\Phi(k)\,\hat{\delta}(k\cdot u)e^{-ik\cdot x}\left(u^\mu\cos a\cdot k - i\epsilon^\mu(a,k,u)\frac{\sin a\cdot k}{a\cdot k}\right). \tag{2.71}$$

The Maxwell tensor is then easily computed,

$$F^{\mu\nu}(x) = 4\,Q\,\mathrm{Re}\int \mathrm{d}\Phi(k)\,\hat{\delta}(k\cdot u)e^{-ik\cdot x}\left(k^{[\mu}u^{\nu]}\cos a\cdot k - \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}k_{[\rho}u_{\sigma]}\sin a\cdot k\right), \tag{2.72}$$

which is equal to the $F_{\mu\nu}$ we had in the purely spinning case with $\theta = 0$.

Furthermore, starting from (2.72) we can also confirm the expressions (2.54) and (2.55) in the $\theta = 0$ case. Projecting on a spinor basis, these are found to be

$$\begin{aligned}\phi^{\sqrt{\text{Kerr}}}_{\alpha\beta}(x) &= \sigma^{\mu\nu}{}_{\alpha\beta}F_{\mu\nu}(x)\\ &= 2\sqrt{2}Q\ \mathrm{Re}\int \mathrm{d}\Phi(k)\hat{\delta}(k\cdot u)e^{-ik\cdot(x-a)}\varepsilon_+\cdot u\,|k\rangle_\alpha|k\rangle_\beta\\ &= \phi^{\text{Coul.}}_{\alpha\beta}(x-a),\end{aligned} \tag{2.73}$$

we report the negative-helicity spinor too

$$\begin{aligned}\tilde{\phi}^{\sqrt{\text{Kerr}}}_{\dot{\alpha}\dot{\beta}}(x) &= \tilde{\sigma}^{\mu\nu}{}_{\dot{\alpha}\dot{\beta}} F_{\mu\nu}(x) \\ &= -2\sqrt{2}Q\ \text{Re}\int \mathrm{d}\Phi(k)\hat{\delta}(k\cdot u) e^{-ik\cdot(x+a)} \varepsilon_- \cdot u\, [k|_{\dot{\alpha}}[k|_{\dot{\beta}} \\ &= \tilde{\phi}^{\text{Coul.}}_{\dot{\alpha}\dot{\beta}}(x+a),\end{aligned} \tag{2.74}$$

matching the expressions first obtained in [108].

Notice again that the action of the Newman-Janis translation on the Maxwell spinors is beautifully simple: $\phi^{\sqrt{\text{Kerr}}}_{\alpha\beta}$ is a translation of $\phi^{\text{Coul.}}_{\alpha\beta}$ in one direction, while $\tilde{\phi}^{\sqrt{\text{Kerr}}}_{\dot{\alpha}\dot{\beta}}$ is a translation of $\tilde{\phi}^{\text{Coul.}}_{\dot{\alpha}\dot{\beta}}$ in the opposite direction. This is in contrast to the more complicated structure at the level of the field strength (2.67). We see that the notion of chirality is intimately related to the structure of the Newman-Janis shift.

2.3.3 Duality rotation

To investigate the effect of the EM rotation on the fields, we take a to zero,

$$\langle F^{\mu\nu}(x)\rangle = 2\,Q\,\partial^{[\mu}u^{\nu]}\left[\frac{S_{0,\theta}+S_{0,-\theta}}{2}\right] - Q\,\epsilon^{\mu\nu\rho\sigma}\partial_{[\rho}u_{\sigma]}\left[\frac{S_{0,\theta}-S_{0,-\theta}}{2}\right]. \tag{2.75}$$

Substituting the value of the scalar integrals,

$$\langle F^{\mu\nu}(x)\rangle = 2\,Q\,\cosh\theta\,\partial^{[\mu}u^{\nu]}\left[\frac{\Theta(\rho^2)}{2\pi\,\rho}\right] - Q\,\sinh\theta\,\epsilon^{\mu\nu\rho\sigma}\partial_{[\rho}u_{\sigma]}\left[\frac{\Theta(\rho^2)}{2\pi\rho}\right], \tag{2.76}$$

where $\rho^2 \equiv x^2 - (x\cdot u)^2$. In the region $\rho^2 > 0$, we can Wick-rotate back to Lorentzian signature. The duality angle transforms as $\theta \to i\theta$ and so $\cosh\theta \to \cos\theta$ and $\sinh\theta \to -i\sin\theta$. This factor of i is absorbed by the continuation of the volume form to Lorentzian signature, yielding

$$\langle F^{\mu\nu}\rangle = \cos\theta\, F^{\mu\nu}_{\text{Coul.}} + \sin\theta\, \star F^{\mu\nu}_{\text{Coul.}}\,. \tag{2.77}$$

As expected from a electric-magnetic duality rotation, we find that (2.50) mixes the field strength tensor with its Hodge dual. The result is a dyonic field where some of the electric charge has been traded for magnetic charge.

Chapter 3

Black holes from the double copy

In the previous chapter, we saw how three-point amplitudes in QED give rise to the classical electromagnetic fields. This serves us in two different ways. First, the arguments used in electromagnetism will be translated into gravity in the present chapter, exposing the relation between three-point amplitudes in gravity and the linearised gravitational field. Secondly, the gauge amplitudes and fields derived in the previous chapter provide the necessary building blocks to obtain gravitational equivalents via the double copy. At the amplitudes level, one can obtain the gravitational 3-point amplitudes as simple products of 3-point electromagnetic amplitudes. The resulting amplitudes represent gravitational fields in the classical limit. Consequently, the amplitudes double copy will generate classical double copy relations at the level of the fields. Within these classical relations, we can identify previously known classical double copy structures that had not been directly related to the amplitudes double copy. For example, we provide an on-shell momentum space version of the convolutional prescription of [70, 74, 75, 103–105]. Another natural question is why some solutions admit a double copy interpretation that is local in position space (i.e. not written as a convolution), as in the Kerr-Schild double copy [97] and the Weyl double copy [99]. We provide a connection between the properties that allow some solutions to have local double copy structures and their point particle nature.

One difference with the previous chapter is that we will retrieve the gravitational amplitude from the double copy. Our previous work showed how the electromagnetic amplitude (2.52), which includes both spin and duality angle parameters, generates classical fields sourced by isolated static point-like particles. Analogously, the double copy of (2.52) yields the three-point gravitational amplitudes that seed linearised gravity solutions sourced by static point-like objects. For three-point amplitudes, the double copy takes a particularly simple form in which the gravity amplitude is the ordinary product of two gauge amplitudes,

$$\text{Gravity} \sim \mathcal{A}_{\eta_L} e^{\eta_L\,(ik\cdot a_L+\theta_L)} \times \mathcal{A}_{\eta_R} e^{\eta_R\,(ik\cdot a_R+\theta_R)} \tag{3.1}$$

Recall that, generically, the left and right copies can be different. In particular, the left and right instances of (2.52) can have different helicities and different duality angles and angular momentum, generating an interesting interplay of helicities and parameters. As motivated in the introduction, the combination $(\eta_L, \eta_R) = (\pm 1, \pm 1)$ corresponds to a graviton emission amplitude and generates a linearised graviton field. If both helicities contribute equally, the resulting field will be real. The mixed helicities, $(\eta_L, \eta_R) = (\pm 1, \mp 1)$ source the dilaton and B-field. Since we will restrict to $d = 4$, the B-field can be thought of as a pseudoscalar axion field σ that can be combined with the dilaton into a complex scalar field.

There is one complication regarding the scalar fields. At linearised level, they decouple from the graviton meaning that they do not appear in the linearised Riemann curvature tensor. In different words, the usual Riemann tensor misses half of the degrees of freedom in (3.1). We propose a solution which relies on encoding the dilaton and axion as torsion degrees of freedom in a metric-affine connection. The resulting connection gives rise to a Riemann tensor with fewer symmetries but which contains all the NS-NS fields at linearised level. Intuitively, the prescription assigns a geometric meaning to the scalar fields, putting them on the same footing as the graviton.

In a similar fashion to the gauge amplitude, the left and right duality and spin parameters give rise to different transformations on the gravitational side. Let us define $\bar{a} = a_L + a_R$ and $\Delta a = a_L - a_R$, and likewise $\bar{\theta} = \theta_L + \theta_R$ and $\Delta\theta = \theta_L - \theta_R$. Of these four parameters appearing in the linearised gravity solution, only the parameters $\bar{a}$ and $\bar{\theta}$ appear in the graviton components, whereas only the parameters Δa and $\Delta\theta$ appear in the complex scalar and its conjugate. Due to the fact that $\mathcal{A}_+ \mathcal{A}_-$ is a constant (i.e. k-independent), as we will review later, the complex scalar is generated by $e^{i\, k \cdot \Delta a + \Delta\theta}$, and its conjugate by $e^{-i\, k \cdot \Delta a - \Delta\theta}$. Focusing on the duality parameters θ, the effect of $\bar{\theta}$ is to perform a gravitational Ehlers-type 'electric-magnetic' duality transformation of the metric [6, 90, 148], whereas the effect of $\Delta\theta$ is to perform an axion-dilaton duality transformation, well known from supergravity. Likewise, $\bar{a}$ performs a Newman-Janis shift on the graviton, whereas Δa performs a similar transformation to the complex field.

We believe the notion of the double copy present in this chapter is the most faithful representation of the double copy at a classical level (perhaps with the exception of the self-dual double copy). Its quantum origin ensures agreement with the amplitude formulations of the double copy. It admits dilaton and axion fields, which can be sourced by picking different left and right single copies. Also, it can explain some of the known examples of classical double copy procedures. Unfortunately, this notion of classical double copy is limited by the class of point-particle solutions chosen, and the fact that our exploration does not go beyond the linear level.

3.1 Generalised curvature and NS-NS fields

As we have seen, the full double copy of Yang-Mills theory is not only pure Einstein gravity, but NS-NS gravity. Besides the graviton, this theory includes a scalar field ϕ, the dilaton, and a two-form field $B_{\mu\nu}$ known as the B-field or the Kalb-Ramond field. A complete classical double copy map should include all three fields on its gravitational side. Examples of such maps have been found using double field theory, both for certain exact solutions [5,149–152] and for perturbative solutions [153,154]. In all these studies, the maps are written in terms of fields, in contrast to the Weyl double copy, where the map relates curvatures, which are gauge invariant at the linearised level. In this section, we will address this challenge by defining a generalised curvature that packages all the NS-NS fields in geometric degrees of freedom, yielding a natural object from a double copy perspective.

The standard notion of geometry in general relativity, a (pseudo-)Riemannian manifold (M, g) endowed with the Levi-Civita connection ∇, can be generalised by relaxing the requirements on the connection. If we allow the connection to have torsion, while insisting on metric compatibility, the result is classified as a Riemann-Cartan geometry.

Consider a d-dimensional manifold M equipped with a metric $g_{\mu\nu}$ and an affine connection $\mathfrak{D}$. In a coordinate basis, the covariant derivative acts on a vector V as

$$\mathfrak{D}_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu{}_{\nu\rho} V^\rho \,. \tag{3.2}$$

In general, the affine symbols $\Gamma^\mu{}_{\nu\rho}$ do not have to be symmetric. Their anti-symmetric part is the *torsion* tensor, $T^\mu{}_{\nu\rho} \equiv \frac{1}{2}(\Gamma^\mu{}_{\nu\rho} - \Gamma^\mu{}_{\rho\nu}) = \Gamma^\mu{}_{[\nu\rho]}$. We will take $(M, g, \mathfrak{D})$ to be a Riemann-Cartan manifold by requiring the connection to be metric-compatible,

$$\mathfrak{D}_\lambda \, g_{\mu\nu} = 0 \,.$$

This condition constrains the affine symbols to take the form

$$\Gamma^\mu{}_{\nu\rho} = \left\{ {\mu \atop \nu\rho} \right\} + K^\mu{}_{\nu\rho} \,, \tag{3.3}$$

where the first term denotes the standard Christoffel symbols of the Levi-Civita connection and the second, a tensor called *contorsion*, must satisfy $K_{\mu\nu\rho} = -K_{\rho\nu\mu}$. It can be written uniquely in terms of the torsion as

$$K^\mu{}_{\nu\rho} = \frac{1}{2} \, g^{\mu\lambda} \left(g_{\nu\tau} \, T^\tau{}_{\lambda\rho} + g_{\rho\tau} \, T^\tau{}_{\lambda\nu} + g_{\lambda\tau} \, T^\tau{}_{\nu\rho} \right) \,. \tag{3.4}$$

The generalised connection defines a generalised Riemann tensor, which in our conventions we write as

$$\mathfrak{R}_{\mu\nu\rho}{}^{\lambda} = \mathfrak{D}_\nu \Gamma^\lambda{}_{\mu\rho} - \mathfrak{D}_\mu \Gamma^\lambda{}_{\nu\rho} + \Gamma^\lambda{}_{\nu\tau}\Gamma^\tau{}_{\mu\rho} - \Gamma^\lambda{}_{\mu\tau}\Gamma^\tau{}_{\nu\rho} \ . \tag{3.5}$$

It is important to note that this tensor does not have the symmetries of the usual Riemann tensor. It satisfies $\mathfrak{R}_{\mu\nu\rho\sigma} = \mathfrak{R}_{[\mu\nu]\rho\sigma} = \mathfrak{R}_{\mu\nu[\rho\sigma]}$, but $\mathfrak{R}_{\mu\nu\rho\sigma} \neq \mathfrak{R}_{\rho\sigma\mu\nu}$ due to the lack of symmetry in the last two indices of the contorsion. Using (3.3), it can be shown that

$$\mathfrak{R}_{\mu\nu\rho}{}^{\lambda} = R_{\mu\nu\rho}{}^{\lambda} + \nabla_\nu K^\lambda{}_{\mu\rho} - \nabla_\mu K^\lambda{}_{\nu\rho} + K^\lambda{}_{\nu\tau}K^\tau{}_{\mu\rho} - K^\lambda{}_{\mu\tau}K^\tau{}_{\nu\rho} \ , \tag{3.6}$$

where ∇ denotes the Levi-Civita connection and $R_{\mu\nu\sigma}{}^{\lambda}$ its Riemann tensor. In general, $\mathfrak{R}$ will denote curvatures with torsion, whereas R is reserved for the standard Riemannian curvatures of the metric.

Riemann-Cartan manifolds have extra geometrical degrees of freedom in the contorsion. These degrees of freedom can be used to accommodate the NS-NS fields, giving them a geometric status in analogy with the metric. The dilaton is assigned to the trace of the contorsion while the B-field is related to its fully antisymmetric component

$$K^\mu{}_{\nu\rho} = \frac{\kappa}{2\sqrt{3}} e^{-\frac{4\kappa\phi}{d-2}}\, H^\mu{}_{\nu\rho} - \frac{2\,\kappa}{(d-2)\sqrt{d-1}} \left(\delta^\mu{}_\nu\, \partial_\rho \phi \ - \ g_{\nu\rho}\, g^{\mu\sigma}\, \partial_\sigma\phi \right) \ , \tag{3.7}$$

where $H = \mathrm{d}B$ is the curvature of the B-field and κ is the gravitational coupling constant. The contorsion (3.7) was chosen such that the Ricci scalar is

$$\mathfrak{R} = R - \frac{4\,\kappa^2}{d-2} \nabla_\mu\phi\, \nabla^\mu\phi - \frac{\kappa^2}{12} e^{-\frac{8\kappa\phi}{d-2}}\, H_{\mu\nu\rho}H^{\mu\nu\rho} + \frac{4\,\kappa\,\sqrt{d-1}}{d-2} \nabla^\mu\nabla_\mu\phi \ , \tag{3.8}$$

the motivation being that $\sqrt{|g|}\,\mathfrak{R}$ is equivalent to the usual NS-NS Lagrangian density in the Einstein frame, up to a boundary term:

$$S = \frac{2}{\kappa^2}\int \mathrm{d}^d x \sqrt{|g|} \left(R - \frac{4\kappa^2}{d-2}\nabla_\mu\phi\,\nabla^\mu\phi - \frac{\kappa^2}{12} e^{-\frac{8\kappa\phi}{d-2}}\, H_{\mu\nu\rho}H^{\mu\nu\rho} \right) \ , \tag{3.9}$$

$$= \frac{2}{\kappa^2}\int \mathrm{d}^d x \sqrt{|g|}\ \mathfrak{R} \ . \tag{3.10}$$

Similar constructions have been proposed since the discovery of the NS-NS action [114]. Although the originally proposed connections were not metric compatible, they also assigned connection degrees of freedom to the dilaton and B-fields. This motivated a series of works trying to recast higher-order terms of the bosonic string Lagrangian exclusively in terms of generalised curvature invariants [155–158]. A similar connection is also used in the context of double field theory [159]. A metric-compatible connection was later introduced in [160], which, together with a non-parallel volume element, reproduces the NS-NS Lagrangian in the string frame. Other generalised connections, also metric-compatible, have been used to endow Einstein-dilaton gravity with a geometric interpretation [161,162]. A drawback of these geometric formulations of NS-NS gravity is that, in order to obtain the correct equations of motion, one needs to impose artificial constraints on the torsion [163]. For example, the totally antisymmetric component, which we set proportional to $H_{\mu\nu\rho}$, is not completely free, since H must be exact. Hence, the geometric interpretation of the massless modes is not entirely clear [164].

We are primarily interested in the curvature at linear order in the fields. Starting from $g_{\mu\nu} = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}$, and expanding to linearised order, we obtain

$$\mathfrak{R}_{\mu\nu}{}^{\rho\sigma} = -2\,\kappa\,\partial_{[\mu}\partial^{[\rho}h_{\nu]}{}^{\sigma]} + \frac{4\,\kappa}{(d-2)\sqrt{d-1}}\,\delta_{[\mu}{}^{[\rho}\partial_{\nu]}\partial^{\sigma]}\phi + \frac{2\,\kappa}{\sqrt{3}}\partial_{[\mu}\partial^{[\rho}B_{\nu]}{}^{\sigma]} \ . \tag{3.11}$$

In $d = 4$, the field redefinitions

$$\phi \to \frac{\sqrt{3}}{2}\phi\ , \qquad\qquad B \to \sqrt{3}B\ , \tag{3.12}$$

simplify the factors to reduce the linearised Riemann tensor to

$$\mathfrak{R}_{\mu\nu}{}^{\rho\sigma} = -2\,\kappa\,\left(\partial_{[\mu}\partial^{[\rho}h_{\nu]}{}^{\sigma]} - \delta_{[\mu}{}^{[\rho}\partial_{\nu]}\partial^{\sigma]}\phi - \partial_{[\mu}\partial^{[\rho}B_{\nu]}{}^{\sigma]}\right)\ . \tag{3.13}$$

This expression highlights the fact that the generalised Riemann packages all the NS-NS fields. At this order, the packaging can be taken one step further by using the 'fat graviton' defined in [73] [1]

$$\mathfrak{H}_{\mu\nu} = \mathfrak{h}_{\mu\nu} - B_{\mu\nu} - P^q_{\mu\nu}\,(2\phi + \mathfrak{h})\ , \tag{3.14}$$

where $\mathfrak{h}_{\mu\nu}$ is the trace-reversed graviton and $P^q_{\mu\nu}$ is a projector

$$\mathfrak{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\,h\,\eta_{\mu\nu}\ , \qquad P^q_{\mu\nu} = \frac{1}{2}\left(\eta_{\mu\nu} - \frac{q_\mu\partial_\nu + q_\nu\partial_\mu}{q\cdot\partial}\right)\ . \tag{3.15}$$

The constant auxiliary null vector q^μ is related to gauge choices. In fact, the terms involving q^μ drop out of the gauge-invariant curvature, which can be written as the compact expression

$$\mathfrak{R}_{\mu\nu}{}^{\rho\sigma} = -2\,\kappa\,\partial_{[\mu}\partial^{[\rho}\mathfrak{H}_{\nu]}{}^{\sigma]}\ . \tag{3.16}$$

In this sense, our generalised curvature is the 'fat Riemann' associated with the 'fat graviton'.

There is yet another way to rewrite (3.13). In four dimensions, the two-form $B_{\mu\nu}$ can be traded for a pseudoscalar axion σ, defined by

$$H_{\mu\nu\rho} = -e^{2\sqrt{3}\,\phi}\,\epsilon_{\mu\nu\rho\sigma}\partial^\sigma\sigma\ . \tag{3.17}$$

[1] Some factors differ from [73] due to different normalisation conventions.

At linearised order, the exponential in the expression above equals 1, and the fat Riemann is

$$\mathfrak{R}_{\mu\nu}{}^{\rho\sigma} = -2\,\kappa\,\left(\partial_{[\mu}\partial^{[\rho}h_{\nu]}{}^{\sigma]} - \,\delta_{[\mu}{}^{[\rho}\partial_{\nu]}\partial^{\sigma]}\phi + \epsilon^{\rho\sigma\lambda}{}_{[\mu}\partial_{\nu]}\partial_\lambda\sigma\right)\ . \tag{3.18}$$

Later in the chapter, we will see how the different products of gauge theory amplitudes are related to the different components of the generalised curvature. We will work in $d = 4$, where it is convenient to use the spinor-helicity formalism for the amplitudes. The relation between the amplitudes and the generalised curvature is, therefore, much clearer if we also express the latter spinorially. As described in appendix B, the generalised Riemann tensor can be decomposed into spinors as

$$\begin{aligned}\mathfrak{R}_{\alpha\dot\alpha\beta\dot\beta\gamma\dot\gamma\delta\dot\delta} &= \mathbf{X}_{\alpha\beta\gamma\delta}\,\epsilon_{\dot\alpha\dot\beta}\,\epsilon_{\dot\gamma\dot\delta} + \tilde{\mathbf{X}}_{\dot\alpha\dot\beta\dot\gamma\dot\delta}\,\epsilon_{\alpha\beta}\,\epsilon_{\gamma\delta}\\ &\quad + \mathbf{\Phi}_{\alpha\beta\dot\gamma\dot\delta}\,\epsilon_{\dot\alpha\dot\beta}\,\epsilon_{\gamma\delta} + \tilde{\mathbf{\Phi}}_{\dot\alpha\dot\beta\gamma\delta}\,\epsilon_{\alpha\beta}\,\epsilon_{\dot\gamma\dot\delta}\ ,\end{aligned} \tag{3.19}$$

where we use the bold typeface in order to distinguish the spinors from those of R. The spinors $\mathbf{X}_{\alpha\beta\gamma\delta}$, $\mathbf{\Phi}_{\alpha\beta\dot\gamma\dot\delta}$ and their duals are symmetric in their first and second pairs of indices. Recall that, generically, $\mathfrak{R}_{\mu\nu\rho\sigma} \neq \mathfrak{R}_{\rho\sigma\mu\nu}$. This asymmetry implies that $\mathbf{X}_{\alpha\beta\gamma\delta} \neq \mathbf{X}_{\gamma\delta\alpha\beta}$ and $\mathbf{\Phi}_{\alpha\beta\dot\gamma\dot\delta} \neq \tilde{\mathbf{\Phi}}_{\dot\gamma\dot\delta\alpha\beta}$. Since $\mathbf{X}_{\alpha\beta\gamma\delta}$ is not completely symmetric, it can be reduced further as

$$\mathbf{X}_{\alpha\beta\gamma\delta} = \mathbf{\Psi}_{\alpha\beta\gamma\delta} - 2\left(\mathbf{\Sigma}_{\alpha(\gamma}\,\epsilon_{\delta)\beta} + \mathbf{\Sigma}_{\beta(\gamma}\,\epsilon_{\delta)\alpha}\right) + \mathbf{\Lambda}(\epsilon_{\alpha\gamma}\,\epsilon_{\beta\delta} + \epsilon_{\alpha\delta}\,\epsilon_{\beta\gamma})\ , \tag{3.20}$$

where $\mathbf{\Psi}_{\alpha\beta\gamma\delta}$ and $\mathbf{\Sigma}_{\alpha\beta}$ are completely symmetric. A similar decomposition holds for $\tilde{\mathbf{X}}_{\dot\alpha\dot\beta\dot\gamma\dot\delta}$. Restricting to linearised level, we can compare the right-hand side of (3.19) to the right-hand side of (3.13): the first line of the former corresponds to the graviton contribution, whereas the second line corresponds to contributions from combinations of the dilaton and the axion (which is the single degree of freedom of the B-field in $d = 4$). We will make this more explicit in a later section.

3.2 Double copy map

Armed with the appropriate background on the geometry of the generalised connection, we are ready to relate it to scattering amplitudes. Following the spirit of (3.1), the gravitational amplitudes will be obtained from the double copy map for three-point amplitudes

$$\mathcal{M}_{\eta_L\eta_R} = -\frac{\kappa}{4\,Q^2}\,c_{\eta_L\eta_R}\,\mathcal{A}^{(L)}_{\eta_L}\,\mathcal{A}^{(R)}_{\eta_R}\ , \tag{3.21}$$

where there are four choices for (η_L, η_R):

$$(+,+)\,,\quad(-,-)\,,\quad(+,-)\,,\quad(-,+)\,. \tag{3.22}$$

These correspond, respectively, to the gravity field being: positive-helicity graviton, negative-helicity graviton, complex scalar (dilaton and axion), and conjugate complex scalar. In general, we allow for four independent couplings $c_{\eta_L\eta_R}$ of our massive particle to these gravity fields. Any choice of these couplings will lead to a linearised gravity solution. In practice, we will be most interested in the case where the particle couples equally to the two chiralities, in which case we take $c_{++} = c_{--}$ and $c_{+-} = c_{-+}$.

The generalise curvature tensor will play the role that the field strength played in gauge theory. The similarities between the two objects can be already seen from the operator mode expansion

$$\mathfrak{R}^{\mu\nu\rho\sigma} = 4\,\kappa\ \mathrm{Re}\int \mathrm{d}\Phi(k)\Big[\sum_{\eta_L\eta_R} a_{\eta_L\eta_R}\,\varepsilon^{[\mu}_{\eta_L}(k)k^{\nu]}\varepsilon^{[\rho}_{\eta_R}(k)k^{\sigma]}\Big]e^{-ik\cdot x}\ . \tag{3.23}$$

The operator version of the linearised spinor coefficients is computed by contracting the curvature with the sigma matrices [165]

$$\mathbf{X}_{\alpha\beta\gamma\delta} = \sigma^{\mu\nu}{}_{\alpha\beta}\sigma^{\rho\sigma}{}_{\gamma\delta}\,\mathfrak{R}_{\mu\nu\rho\sigma}\ , \qquad \tilde{\mathbf{X}}_{\dot\alpha\dot\beta\dot\gamma\dot\delta} = \tilde\sigma^{\mu\nu}{}_{\dot\alpha\dot\beta}\tilde\sigma^{\rho\sigma}{}_{\dot\gamma\dot\delta}\,\mathfrak{R}_{\mu\nu\rho\sigma}\ , \tag{3.24}$$

$$\mathbf{\Phi}_{\alpha\beta\dot\gamma\dot\delta} = \sigma^{\mu\nu}{}_{\alpha\beta}\tilde\sigma^{\rho\sigma}{}_{\dot\gamma\dot\delta}\,\mathfrak{R}_{\mu\nu\rho\sigma}\ , \qquad \tilde{\mathbf{\Phi}}_{\dot\alpha\dot\beta\gamma\delta} = \tilde\sigma^{\mu\nu}{}_{\dot\alpha\dot\beta}\sigma^{\rho\sigma}{}_{\gamma\delta}\,\mathfrak{R}_{\mu\nu\rho\sigma}\ . \tag{3.25}$$

These contractions are easily computed applying (A.8) and (A.9). The resulting spinors are

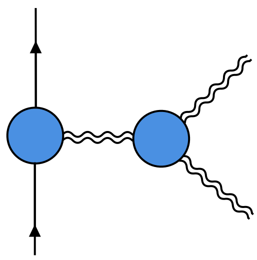
$$\mathbf{X}_{\alpha\beta\gamma\delta} = 2\,\kappa\ \mathrm{Re}\int \mathrm{d}\Phi(k)\ a_{--}|k\rangle_\alpha|k\rangle_\beta|k\rangle_\gamma|k\rangle_\delta\, e^{-ik\cdot x}\ , \tag{3.26}$$

$$\tilde{\mathbf{X}}_{\dot\alpha\dot\beta\dot\gamma\dot\delta} = 2\,\kappa\ \mathrm{Re}\int \mathrm{d}\Phi(k)\ a_{++}[k|_{\dot\alpha}[k|_{\dot\beta}[k|_{\dot\gamma}[k|_{\dot\delta}\, e^{-ik\cdot x}\ , \tag{3.27}$$

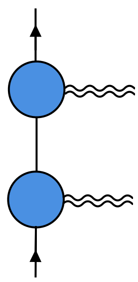
$$\mathbf{\Phi}_{\alpha\beta\dot\gamma\dot\delta} = -2\,\kappa\ \mathrm{Re}\int \mathrm{d}\Phi(k)\ a_{-+}\,|k\rangle_\alpha|k\rangle_\beta[k|_{\dot\gamma}[k|_{\dot\delta}\, e^{-ik\cdot x}\ , \tag{3.28}$$

$$\tilde{\mathbf{\Phi}}_{\dot\alpha\dot\beta\gamma\delta} = -2\,\kappa\ \mathrm{Re}\int \mathrm{d}\Phi(k)\ a_{+-}\,[k|_{\dot\alpha}[k|_{\dot\beta}|k\rangle_\gamma|k\rangle_\delta\, e^{-ik\cdot x}\ . \tag{3.29}$$

To link these objects to the amplitudes (3.21), we would like to use the equivalent of (2.29) for the NS-NS fields. Graviton self-interactions could spoil the exponentiation present in electromagnetism. Clearly, there are additional diagrams in gravity, for example at four points we could encounter the diagram



which involves a graviton three-point interaction. However, self-interactions of gravitons are suppressed compared to the dominant diagram



where the gravitons connect directly to the massive line. The reason is simply that the graviton self-interaction involves powers of the momenta of the gravitons, while the coupling to the massive line involves the particle mass. Since the particle mass is large compared to the graviton momenta, we may neglect graviton self-interactions. We may also neglect contact vertices (as in electromagnetism) for the same reason.

This does not mean that all self-interactions of the gravitational field are eliminated. The metric quantum operator has a perturbative expansion which includes these self-interactions. The expectation value of this all-order operator on our coherent state reproduces the classical metric. Notice that the coherent state is gauge invariant, while the quantum operator may not be (in quantum gravity, only asymptotic observables may be associated with gauge-invariant operators). This procedure would allow us to perturbatively construct the Schwarzschild metric, along the lines of [166–168] but in a manifestly on-shell formalism; see also [31] for an alternative approach based on an intermediate matching with an effective theory of sources coupled to gravitons. We restrict ourselves to the first order and leave this programme for future work.

In conclusion, it can be shown that the same arguments described in section 2.1.1 apply also to linearised gravity [1, 3]. From this, it follows that the final state is also coherent,

$$S|\psi\rangle = \frac{1}{\mathcal{N}}\int \mathrm{d}\Phi(p)\varphi(p)\exp\left[\int \mathrm{d}\Phi(k)\, i\,\hat{\delta}(2p\cdot k) \times \left(\sum_{\eta_L\eta_R}\mathcal{M}_{-\eta_L,-\eta_R}(k)\, a^\dagger_{\eta_L\eta_R}(k)\right)\right]|p\rangle\,, \tag{3.30}$$

which is analogous to (2.28). The exponentiation of the gravitational amplitude implies that

$$\begin{aligned} a_{\eta_L\eta_R}(k)S|\psi\rangle &= \hat{\delta}(2p\cdot k)\, i\mathcal{M}_{-\eta_L,-\eta_R}(k)\, S|\psi\rangle \\ &= \frac{\delta}{\delta a^\dagger_{\eta_L\eta_R}(k)}S|\psi\rangle\,. \end{aligned} \tag{3.31}$$

Equation (3.31) implies that we can easily exchange annihilation operators for amplitudes inside expectation values, so that we find

$$\langle \mathfrak{R}^{\mu\nu\rho\sigma}\rangle \;=\; 4\,\kappa\ \mathrm{Re}\ i\int \mathrm{d}\Phi(k)\hat{\delta}(2k\,\cdot\,p)\left[\sum_\eta \mathcal{M}_{-\eta_L,-\eta_R}\,\varepsilon^{[\mu}_{\eta_L}(k)k^{\nu]}\varepsilon^{[\rho}_{\eta_R}(k)k^{\sigma]}\right]e^{-ik\cdot x}\,. \tag{3.32}$$

The same can be done in the spinor coefficients. The application of the map (3.21) results in

$$\langle \mathbf{X}_{\alpha\beta\gamma\delta}\rangle = -\frac{\kappa^2 c_{++}}{2Q^2}\ \mathrm{Re}\ i\int \mathrm{d}\Phi(k)\hat{\delta}(2p\cdot k)\,\mathcal{A}^{(L)}_+\mathcal{A}^{(R)}_+\ |k\rangle_A|k\rangle_B|k\rangle_C|k\rangle_D\, e^{-ik\cdot x}\ , \tag{3.33}$$

$$\langle \tilde{\mathbf{X}}_{\dot\alpha\dot\beta\dot\gamma\dot\delta}\rangle = -\frac{\kappa^2 c_{--}}{2Q^2}\ \mathrm{Re}\ i\int \mathrm{d}\Phi(k)\hat{\delta}(2p\cdot k)\,\mathcal{A}^{(L)}_-\mathcal{A}^{(R)}_-\ [k|_{\dot\alpha}[k|_{\dot\beta}[k|_{\dot\gamma}[k|_{\dot\delta}\, e^{-ik\cdot x}\ , \tag{3.34}$$

$$\langle \mathbf{\Phi}_{\alpha\beta\dot\gamma\dot\delta}\rangle = +\frac{\kappa^2 c_{+-}}{2\,Q^2}\ \mathrm{Re}\, i\int \mathrm{d}\Phi(k)\hat{\delta}(2p\cdot k)\,\mathcal{A}^{(L)}_+\mathcal{A}^{(R)}_-\ |k\rangle_A|k\rangle_B[k|_{\dot\gamma}[k|_{\dot\delta}\, e^{-ik\cdot x}\ , \tag{3.35}$$

$$\langle \tilde{\mathbf{\Phi}}_{\dot\alpha\dot\beta\gamma\delta}\rangle = +\frac{\kappa^2 c_{-+}}{2\,Q^2}\ \mathrm{Re}\, i\int \mathrm{d}\Phi(k)\hat{\delta}(2p\cdot k)\,\mathcal{A}^{(L)}_-\mathcal{A}^{(R)}_+\ [k|_{\dot\alpha}[k|_{\dot\beta}|k\rangle_C|k\rangle_D\, e^{-ik\cdot x}\ . \tag{3.36}$$

The above expressions make it clear that every quadratic term in the amplitudes sources a different component of the spinor curvature. The double copy structure is remarkably explicit when comparing with the gauge field spinors (2.31) and (2.32). Moreover, we can obtain the equivalent of (2.54) and (2.55) by considering the gauge amplitudes

$$\begin{aligned} \mathcal{A}^{(L)}_\eta &= -2Q(p\cdot\varepsilon_\eta)e^{\eta(\theta_L+ik\cdot a_L)}, \\ \mathcal{A}^{(R)}_\eta &= -2Q(p\cdot\varepsilon_\eta)e^{\eta(\theta_R+ik\cdot a_R)}, \end{aligned} \tag{3.37}$$

which under the double copy map implies that

$$\begin{aligned}
\langle \mathbf{X}_{\alpha\beta\gamma\delta}(x)\rangle &= e^{\bar{\theta}}\langle \mathbf{X}^{\mathrm{JNW}}_{\alpha\beta\gamma\delta}(x-\bar{a})\rangle \ , \\
\langle \tilde{\mathbf{X}}_{\dot\alpha\dot\beta\dot\gamma\dot\delta}(x)\rangle &= e^{-\bar{\theta}}\langle \tilde{\mathbf{X}}^{\mathrm{JNW}}_{\dot\alpha\dot\beta\dot\gamma\dot\delta}(x+\bar{a})\rangle \ , \\
\langle \mathbf{\Phi}_{\alpha\beta\dot\gamma\dot\delta}(x)\rangle &= e^{\Delta\theta}\langle \mathbf{\Phi}^{\mathrm{JNW}}_{\alpha\beta\dot\gamma\dot\delta}(x-\Delta a)\rangle \ , \\
\langle \tilde{\mathbf{\Phi}}_{\dot\alpha\dot\beta\gamma\delta}(x)\rangle &= e^{-\Delta\theta}\langle \tilde{\mathbf{\Phi}}^{\mathrm{JNW}}_{\dot\alpha\dot\beta\gamma\delta}(x+\Delta a)\rangle \ ,
\end{aligned} \tag{3.38}$$

where we have defined

$$\bar{\theta} := \theta_L + \theta_R \ , \qquad \Delta\theta := \theta_L - \theta_R \ , \tag{3.39}$$

$$\bar{a} := a_L + a_R, \qquad \Delta a := a_L - a_R \, . \tag{3.40}$$

The superscript JNW refers to the solution where both single copies are Coulomb.[2] Notice that, at linearised level, the first two spinors in (3.38) match those of the Schwarzschild solution. The various parameters are elegantly distributed over the different spinors. The parameter $\bar{a}$ corresponds to the spin of Kerr, and appears as expected via the Newman-Janis shift, while $\bar{\theta}$ corresponds to the split signature version of the rotation between the mass and the NUT parameter; together, these two parameters correspond to the Kerr-Taub-NUT solution. The interpretation of the spin parameter as a translation, complex in Lorentzian signature, was discussed in more detail in [108]. Note that the limit $\bar{\theta} \to \pm\infty$ kills one of the chiralities of the Weyl spinor, and the Weyl tensor becomes (anti-)self-dual.[3] The same happens for the parameters $\theta_{L,R}$ of the single copies. If one of the single copies, say the left one, has $\theta_L \to \pm\infty$, the double copy will also be (anti-)self-dual unless θ_R scales in the opposite way.

The parameters Δa and $\Delta\theta$ correspond, respectively, to a novel type of Newman-Janis shift for the axion and dilaton, and to the standard axion-dilaton supergravity duality transformation.

The spinor language is better fitted for displaying the double copy, but it is instructive to think about the dilaton and axion. We can map (3.32) to the field degrees of freedom using (3.13), together with the mode expansions of the fields

$$h^{\mu\nu} = 2\ \mathrm{Re} \sum_\eta \int \mathrm{d}\Phi(k) a_{\eta\eta}(k) \varepsilon^\mu_\eta(k) \varepsilon^\nu_\eta(k)\, e^{-ik\cdot x} \ , \tag{3.41}$$

$$\phi = 2\ \mathrm{Re} \int \mathrm{d}\Phi(k)\, a_\phi(k)\, e^{-ik\cdot x} \ , \tag{3.42}$$

[2]The meaning of the superscript will become clear in section 3.3.
[3]Other parameters must be suitably scaled to keep the solution finite.

$$B^{\mu\nu} = 2\,\mathrm{Re}\int \mathrm{d}\Phi(k)\, a_B(k)\left(\varepsilon_+^\mu(k)\varepsilon_-^\nu(k) - \varepsilon_-^\mu(k)\varepsilon_+^\nu(k)\right)\, e^{-ik\cdot x}\ . \tag{3.43}$$

Substituting in (3.13) implies that equation (3.32) can be re-expressed as

$$\begin{aligned}\mathfrak{R}^{\mu\nu\rho\sigma} = 4\,\kappa\,\mathrm{Re}\int \mathrm{d}\Phi(k)\Bigg[&\sum_\eta a_\eta \varepsilon_\eta^{[\mu}(k)k^{\nu]}\varepsilon_\eta^{[\rho}(k)k^{\sigma]}\\ &+ a_\phi\, k^{[\mu}\eta^{\nu][\rho}k^{\sigma]} + a_B\, k^{[\mu}(\varepsilon_+^{\nu]}\varepsilon_-^{[\rho} - \varepsilon_-^{\nu]}\varepsilon_+^{[\rho})k^{\sigma]}\Bigg]e^{-ik\cdot x}\ .\end{aligned} \tag{3.44}$$

The first term in the second line of (3.44) needs simplification. This is achieved by expanding the flat metric in terms of the null tetrad

$$k^{[\mu}\eta^{\nu][\rho}k^{\sigma]} = -k^{[\mu}\varepsilon_+^{\nu]}\varepsilon_-^{[\rho}k^{\sigma]} - k^{[\mu}\varepsilon_-^{\nu]}\varepsilon_+^{[\rho}k^{\sigma]}\,. \tag{3.45}$$

Comparison to (3.23) implies the following relations between annihilation operators

$$\begin{aligned} a_{++} &= a_+\ , \qquad & a_{-+} &= a_\phi + a_B\ ,\\ a_{--} &= a_-\ , \qquad & a_{-+} &= a_\phi - a_B\ , \end{aligned} \tag{3.46}$$

and consequently the gauge and gravity amplitudes are related by

$$\begin{aligned}\mathcal{M}_{\eta\eta} &= -\frac{\kappa}{4\,Q^2}\, c_{\eta\eta}\,\mathcal{A}_\eta^{(L)}\,\mathcal{A}_\eta^{(R)}\ ,\\ \mathcal{M}_\phi &= -\frac{\kappa}{4\,Q^2}\,\frac{1}{2}\left(c_{+-}\mathcal{A}_+^{(L)}\mathcal{A}_-^{(R)} + c_{-+}\mathcal{A}_-^{(L)}\mathcal{A}_+^{(R)}\right)\ ,\\ \mathcal{M}_B &= -\frac{\kappa}{4\,Q^2}\,\frac{1}{2}\left(c_{+-}\mathcal{A}_+^{(L)}\mathcal{A}_-^{(R)} - c_{-+}\mathcal{A}_-^{(L)}\mathcal{A}_+^{(R)}\right)\ .\end{aligned} \tag{3.47}$$

In the next sections, this prescription will be put into practice to compute the classical fields obtained by double copying the amplitudes discussed in section 2.3. From now on, we will restrict to the case $c_{++} = c_{--}$ and $c_{+-} = c_{-+}$ since these solutions naturally continue to real solutions in Lorentzian signature.

3.3 Explicit solutions

3.3.1 Duality rotation

It is time to provide concrete example solutions. Consider left and right amplitudes that differ in their EM duality angle,

$$\begin{aligned}\mathcal{A}_\eta^{(L)} &= -2Q(p\cdot\varepsilon_\eta)e^{\theta_L\eta},\\ \mathcal{A}_\eta^{(R)} &= -2Q(p\cdot\varepsilon_\eta)e^{\theta_R\eta},\end{aligned} \tag{3.48}$$

the effect of this difference will be a rotation between dilaton and axion. The double copied amplitudes are obtained by applying the map (3.47),

$$\begin{aligned} \mathcal{M}_\eta &= -c_{++}\kappa\, m^2 (u\cdot\varepsilon_\eta)^2\, e^{\bar\theta\eta}\ , \\ \mathcal{M}_\phi &= \frac{\kappa\, c_{+-}}{2}\, p^2 \cosh\Delta\theta = \frac{\tilde c\, m}{2}\ \cosh\Delta\theta\ , \\ \mathcal{M}_B &= \frac{\kappa\, c_{+-}}{2}\, p^2 \sinh\Delta\theta = \frac{\tilde c\, m}{2}\ \sinh\Delta\theta\ , \end{aligned} \tag{3.49}$$

where we use $\tilde c$ as a shorthand for $\kappa\, c_{+-}\, m$. To test the effect of the rotation on the metric, let us compute the transformed Weyl tensor,

$$\begin{aligned} \langle W^{\mu\nu\rho\sigma}(x)\rangle = -4\,\kappa^2\,\mathrm{Re}\, i\, c_{++}\, m^2 \int \mathrm{d}\Phi(k)\, \hat\delta(2k\cdot p)\, e^{-ik\cdot x} \Big[&(\varepsilon_+\cdot u)^2 k^{[\mu}\varepsilon_-^{\nu]} k^{[\rho}\varepsilon_-^{\sigma]}\, e^{\bar\theta} \\ &+ (\varepsilon_-\cdot u)^2 k^{[\mu}\varepsilon_+^{\nu]} k^{[\rho}\varepsilon_+^{\sigma]}\, e^{-\bar\theta} \Big]\ . \end{aligned} \tag{3.50}$$

A little algebra shows that this can be rewritten as

$$\begin{aligned} \langle W^{\mu\nu\rho\sigma}(x)\rangle &= -4\,\kappa^2\,\mathrm{Re}\, i\, c_{++}\, m^2 \int \mathrm{d}\Phi(k)\, \hat\delta(2k\cdot p)\, e^{-ik\cdot x} \\ \times &\left[\cosh\bar\theta \left(k^{[\mu}u^{\nu]}k^{[\rho}u^{\sigma]} + \frac{1}{2}k^{[\mu}\eta^{\nu][\rho}k^{\sigma]} \right) - \frac{1}{2}\sinh\bar\theta\, \epsilon^{\mu\nu\tau\lambda} \left(k_{[\tau}u_{\lambda]}k^{[\rho}u^{\sigma]} + \frac{1}{2}k_{[\tau}\delta^{[\rho}_{\lambda]}k^{\sigma]} \right) \right]\ . \end{aligned} \tag{3.51}$$

The first term, with the hyperbolic cosine, corresponds to the Schwarzschild solution, which we will denote $W^{\mu\nu\rho\sigma}_{\text{Schw.}}$. Making use of this notation leads to the compact result

$$\langle W^{\mu\nu\rho\sigma}\rangle = \cosh\bar\theta\ W^{\mu\nu\rho\sigma}_{\text{Schw.}} - \sinh\bar\theta\,\frac{1}{2}\epsilon^{\mu\nu\tau\lambda}\ W^{\text{Schw.}\ \rho\sigma}_{\ \tau\lambda}. \tag{3.52}$$

The second term represents the dual of $W_{\text{Schw.}}$, in analogy with the result of section 2.3.3. We conclude that the angle $\bar\theta$ indeed rotates the mass and the NUT charge of the solution [6, 90].

The Weyl tensor we have computed represents the graviton degrees of freedom in $\mathfrak{R}^{\mu\nu\rho\sigma}$. The next step is to obtain the classical expectation value of the dilaton and axion degrees of freedom. Instead of computing the corresponding components of $\mathfrak{R}^{\mu\nu\rho\sigma}$, we will obtain the field profiles $\langle\phi\rangle$ and $\langle\sigma\rangle$ directly.

Let us start with the classical expectation value of the dilaton field, $\langle\phi\rangle = \langle\psi|S^\dagger\phi S|\psi\rangle$. The result is obtained by application of the field operator (3.42), together with the coherent state exponentiation (3.30) and the amplitude given in (3.49):

$$\langle\phi(x)\rangle = \tilde c\, m\ \cosh\Delta\theta\ \ \mathrm{Re}\, i \int \mathrm{d}\Phi(k)\, \hat\delta(2p\cdot k) e^{-ik\cdot x}\ . \tag{3.53}$$

Performing the integration as in (2.56), we obtain

$$\langle\phi(x)\rangle = \tilde{c}\cosh\Delta\theta\,\frac{\Theta(\rho^2)}{8\pi}\frac{1}{\rho}\;. \tag{3.54}$$

The axion field requires a bit more work. Recalling (3.43) and taking a derivative, we quickly find

$$\langle H_{\mu\nu\rho}(x)\rangle = \langle 3\,\partial_{[\mu}B_{\nu\rho]}(x)\rangle = 3\,\tilde{c}\sinh\Delta\theta\,\mathrm{Re}\int \mathrm{d}\Phi(k)\hat{\delta}(u\cdot k)k_{[\mu}\varepsilon^+_{\nu}\varepsilon^-_{\rho]}\,e^{-ik\cdot x}\;. \tag{3.55}$$

At this stage, it is very helpful to note that

$$\epsilon^{\mu\nu\rho\sigma}k_{[\nu}\varepsilon^+_{\rho}\varepsilon^-_{\sigma]} = 4!\,k^{[\mu}n^{\nu}\varepsilon^{\rho}_{+}\varepsilon^{\sigma]}_{-}k_{[\nu}\varepsilon^+_{\rho}\varepsilon^-_{\sigma]} = -\,k^{\mu}\;. \tag{3.56}$$

Hence,

$$\begin{aligned}\Rightarrow \langle\epsilon^{\mu\nu\rho\sigma}H_{\nu\rho\sigma}(x)\rangle &= -3\,\tilde{c}\sinh\Delta\theta\;\mathrm{Re}\int \mathrm{d}\Phi(k)\,\hat{\delta}(k\cdot u)\,k^{\mu}\,e^{-ik\cdot x}\\ &= -3\,\tilde{c}\sinh\Delta\theta\,\partial^{\mu}\left(\frac{\Theta(\rho^2)}{4\pi}\frac{1}{\rho}\right)\;.\end{aligned} \tag{3.57}$$

This expression provides direct information on the axion σ. To see how, note from equation (3.17), expanded to leading order, that the relation between H and σ is simply

$$H_{\mu\nu\rho} = -\epsilon_{\mu\nu\rho\sigma}\,\partial^{\sigma}\sigma \Rightarrow \epsilon^{\mu\nu\rho\sigma}H_{\nu\rho\sigma} = -3!\,\partial^{\mu}\sigma\;. \tag{3.58}$$

Comparing with the previous expression, we find

$$\langle\sigma(x)\rangle = \tilde{c}\sinh\Delta\theta\,\frac{\Theta(\rho^2)}{8\pi}\frac{1}{\rho}\;. \tag{3.59}$$

3.3.2 Newman-Janis shift

Now we turn our attention to the spin parameter. Just as we did in the previous section, we can use the prescription (3.21) to source an axion and a dilaton. However, we now consider products of gauge theory amplitudes with different spins

$$\begin{aligned}\mathcal{A}^{(L)}_{\eta} &= -2Q(p\cdot\varepsilon_{\eta})e^{i\,\eta\,a_L\cdot k},\\ \mathcal{A}^{(R)}_{\eta} &= -2Q(p\cdot\varepsilon_{\eta})e^{i\,\eta\,a_R\cdot k}.\end{aligned} \tag{3.60}$$

These yield the following gravity amplitudes

$$\begin{aligned}\mathcal{M}_{\eta} &= -\kappa\,c_{++}\,m^2(u\cdot\varepsilon_{\eta})^2\,e^{i\,\eta\,\bar{a}\cdot k}\;,\\ \mathcal{M}_{\phi} &= \frac{\tilde{c}\,m}{2}\;\cos(\Delta a\cdot k)\;,\\ \mathcal{M}_{B} &= \frac{\tilde{c}\,m}{2}\;\sin(\Delta a\cdot k)\;.\end{aligned} \tag{3.61}$$

Once more, the graviton components of the fat curvature tensor found in (3.32) reduce to the Weyl tensor

$$W^{\mu\nu\rho\sigma}(x) = 4\,\kappa\, c_{++}\ \mathrm{Re}\ i \int \mathrm{d}\Phi(k)\hat{\delta}(2k\cdot p)e^{-ik\cdot x}\sum_{\eta}\mathcal{M}_{\eta}\varepsilon_{-\eta}^{[\mu}k^{\nu]}\varepsilon_{-\eta}^{[\rho}k^{\sigma]}e^{i\eta k\cdot\bar{a}}. \tag{3.62}$$

It is not difficult to see that this matches the classical computation with a spinning source. The linearised equations of motion are

$$\partial^2 h^{\mu\nu}(x) = -\kappa\,\mathsf{P}^{\mu\nu}{}_{\alpha\beta}T^{\alpha\beta}(x), \qquad \mathsf{P}^{\mu\nu}{}_{\alpha\beta} = \delta^{(\mu}{}_{(\alpha}\delta^{\nu)}{}_{\beta)} - \frac{1}{2}\eta^{\mu\nu}\eta_{\alpha\beta}, \tag{3.63}$$

with the following stress-energy tensor for Kerr [108, 147]

$$T^{\mu\nu}(x) = 4\,m\int \mathrm{d}\tau\, u^{(\mu}\exp(\bar{a}*\partial)^{\nu)}{}_{\rho}u^{\rho}\delta^{(4)}(x-u\tau)\,. \tag{3.64}$$

Solving (3.63) with the usual boundary conditions, we find the linearised metric

$$\begin{aligned} h^{\mu\nu}(x) &= -2\,\kappa\, m^2 \mathrm{Re}\, i\int \mathrm{d}\Phi(k)\hat{\delta}(p\cdot k)e^{-ik\cdot x}\,\mathsf{P}^{\mu\nu}{}_{\alpha\beta}\,u^{(\alpha}\exp(-i\bar{a}*k)^{\beta)}{}_{\rho}u^{\rho} \\ &= -\kappa m^2 \mathrm{Re}\, i\int \mathrm{d}\Phi(k)\hat{\delta}(p\cdot k)e^{-ik\cdot x}\left[\left(u^{\mu}u^{\nu} - \frac{1}{2}\eta^{\mu\nu}\right)\cos(\bar{a}\cdot k)\right. \\ &\qquad\qquad \left. - i\, u^{(\mu}\epsilon^{\nu)}(\bar{a},k,u)\,\frac{\sin(\bar{a}\cdot k)}{\bar{a}\cdot k}\right], \end{aligned} \tag{3.65}$$

from which the curvature can be computed. After some tedious but straightforward algebra one finds

$$W^{\mu\nu\rho\sigma}(x) = -4\,\kappa^2 m^2 \mathrm{Re}\, i\sum_{\eta}\int \mathrm{d}\Phi(k)\hat{\delta}(2p\cdot k)e^{-ik\cdot x}(\varepsilon_{\eta}\cdot u)^2 k^{[\mu}\varepsilon_{-\eta}^{\nu]}k^{[\rho}\varepsilon_{-\eta}^{\sigma]}e^{i\eta k\cdot\bar{a}}. \tag{3.66}$$

The result matches the one we obtained from amplitudes upon setting $c_{++} = 1$.

For the dilaton and the axion, the calculations are formally analogous to the ones outlined in 3.3.1, except that now we have momentum-dependent trigonometric functions which characterise the spin mixing. For this reason, we omit the explicit computations and report the final results. We find for the dilaton

$$\begin{aligned} \langle\phi(x)\rangle &= \frac{\tilde{c}}{2}\,\mathrm{Re}\, i\int \mathrm{d}\Phi(k)\,\hat{\delta}(u\cdot k)e^{-ik\cdot x}\cos(\Delta a\cdot k) \\ &= \frac{\tilde{c}}{8}\left(S_{\Delta a,0}(x) + S_{-\Delta a,0}(x)\right), \end{aligned} \tag{3.67}$$

referencing the definition of the scalar potential (2.56). The axion is instead given by

$$\langle\epsilon^{\mu\nu\rho\sigma}H_{\nu\rho\sigma}(x)\rangle = -3\,\tilde{c}\,\partial^{\mu}\,\mathrm{Re}\, i\int \mathrm{d}\Phi(k)\,\hat{\delta}(k\cdot u)\,e^{-ik\cdot x}\sin(\Delta a\cdot k), \tag{3.68}$$

telling us that the scalar σ is, at leading order,

$$\begin{aligned}\langle\sigma(x)\rangle &= \frac{\tilde{c}}{2}\,\mathrm{Re}\, i\int \mathrm{d}\Phi(k)\,\hat{\delta}(u\cdot k)e^{-ik\cdot x}\sin(\Delta a\cdot k)\\ &= \frac{\tilde{c}}{8}\left(S_{\Delta a,0}(x) - S_{-\Delta a,0}(x)\right).\end{aligned} \tag{3.69}$$

3.3.3 Comparison with known solutions

Some of the linearised solutions that we have found from double copying amplitudes are known exactly in the literature.

When all four deformation parameters are zero, both single copies represent Coulomb particles. The resulting gravitational solution has vanishing axion and NUT charge and corresponds to a linearised Schwarzschild metric and a '1/distance' dilaton profile. The solution discovered by Janis, Newman and Winicour (JNW) [109] has the same characteristics at linear level. The JNW solution is a static, spherically symmetric deformation of the Schwarzschild black hole with a dilaton field.[4] These characteristics make JNW a natural candidate for the double copy of the point charge, a relation that has been confirmed using a variety of perturbative methods [5,39,73,106]. In chapter 5 we will show how to establish an exact double copy relation between JNW and Coulomb using tools from double field theory [5]. Additionally, the simplicity of JNW will prove useful to check for the existence of a double copy relation in position space in section 3.4.

The linearised solution of subsection 3.3.1, which has generic $\bar{\theta}$ and $\Delta\theta$ corresponds to an axion-dilaton Taub-NUT black hole, which is known exactly; see (17), (19) in [169]. It is instructive to check that our results agree with the linearisation of the known solution. There, dilaton and axion are given as[5]

$$e^{-\phi} = (1+\epsilon^2)\frac{\Lambda^\delta}{\epsilon^2\Lambda^{2\delta}+1}\,, \qquad \sigma = \frac{\epsilon(\Lambda^{2\delta}-1)}{\epsilon^2\Lambda^{2\delta}+1}\,, \tag{3.70}$$

where

$$\Lambda = 1 - \frac{R_0}{R}\,,$$

$\delta\, R_0$ is the charge of the dilaton and ϵ is a duality rotation parameter between dilaton and axion. Notice that R is the Lorentzian equivalent of ρ. At linearised level, the fields decouple and the metric is equivalent to Taub-NUT. Expanding at linear order the other fields and defining $\epsilon = -\tan\frac{\Delta\theta}{2}$, we find

$$\phi = \cos\Delta\theta\,\frac{\delta\, R_0}{R}\,, \qquad \sigma = \sin\Delta\theta\,\frac{\delta\, R_0}{R}\,. \tag{3.71}$$

[4] Note that for non-vanishing dilaton charge the JNW spacetime contains a naked singularity. This naked singularity is not surprising because the uniqueness theorems prevent a scalar-hair deformation of the Schwarzschild solution.

[5] Ignoring factors of $\sqrt{3}$ that can be absorbed into δ at linear order.

Our solution (3.54), (3.59) agrees with this up to an overall constant ($\tilde{c} = 16\pi\, \delta\, R_0$).[6] Then, $\Delta\theta$ is just the parameter inside $\mathrm{SL}(2,\mathbb{R})$ that generates linear rotations between dilaton and axion.

In the special case where $\theta_L = \theta_R$ both single copies are identical, and we have no mixing: $\Delta\theta = 0$. From (3.49), we see that this implies that the axion will vanish, leaving a linearised solution that would be the equivalent to Taub-NUT plus the dilaton. In [169], this corresponds to (17) and (18).

On the contrary, if $\theta_R = -\theta_L$, $\bar{\theta}$ vanishes and the resulting metric has vanishing NUT charge. The result is a linearised Schwarzschild metric plus axion plus dilaton, corresponding to the linearisation of (10) and (13) in [169].

The solutions considered in subsection 3.3.2 involving spin are not so well understood in the literature. There have been attempts to apply a Newman-Janis shift to the JNW solution, with the prospects of obtaining a spinning generalisation. However, these claimed generalisations fail to satisfy the Einstein-dilaton equations of motion [170]. Although linear, our solution might help to find a satisfactory generalisation of the JNW metric with spin. For pure Einstein gravity, $\Delta\theta = \Delta a = 0$, the metric corresponds to the linearised equivalent of the Kerr-Taub-NUT metric in split signature. Recently, this metric was derived from the usual Lorentzian line element by a Wick rotation [143, 144].

3.4 The classical double copy in position space

3.4.1 Weyl double copy

We have found expressions that exhibit clear double copy relations in on-shell momentum space, but we are interested in finding out whether these straightforward double copy relations can be carried over to position space. In particular, our goal is to see how the position-space Weyl double copy relations [99] emerge from amplitudes map (3.21). In most of this section, we will set $a_{L,R} = \theta_{L,R} = 0$, since this simple scenario is enough to illustrate most of the points. To simplify the discussion, each term in the expectation value of (3.44) will be analysed individually. We will omit the axion term since it vanishes when $a_{L,R} = \theta_{L,R} = 0$.

First, we shall consider the terms associated to the graviton amplitude, which we will denote by $\mathfrak{R}^{(h)}{}_{\mu\nu\rho\sigma}$. This is precisely the Schwarzschild Riemann (and Weyl) tensor

[6] After rotating back to Lorentzian signature, the hyperbolic trigonometric functions turn into standard trigonometric functions. Additionally, one has to continue $\sigma \to i\sigma$ due to its pseudo-scalar nature, which cancels the factor of i from the sine. The factor of 16 takes into account a factor of 2 generated by the analytic continuation of the propagator (A.25).

considered in [3]. It can be obtained from (3.51), setting $\bar{\theta} = 0$,

$$\langle \mathfrak{R}^{(h)\,\mu\nu\rho\sigma} \rangle = -4 \;\mathrm{Re}\; i\, m^2 \kappa^2 \int \mathrm{d}\Phi(k) \hat{\delta}(2k\cdot p) e^{-ik\cdot x} \left(k^{[\mu} u^{\nu]} k^{[\rho} u^{\sigma]} + \frac{1}{2} k^{[\mu} \eta^{\nu][\rho} k^{\sigma]} \right) \; . \tag{3.72}$$

The second term in the brackets makes the expression traceless. Alternatively, we can write

$$\langle \mathfrak{R}^{(h)\,\mu\nu\rho\sigma} \rangle = -4\, \mathcal{P}^{\mu\nu\rho\sigma}_{\tau\lambda\eta\omega} \;\mathrm{Re}\; i\, m^2 \kappa^2 \int \mathrm{d}\Phi(k) \hat{\delta}(2k\cdot p) e^{-ik\cdot x} k^{[\tau} u^{\lambda]} k^{[\eta} u^{\omega]} \; , \tag{3.73}$$

where $\mathcal{P}^{\mu\nu\rho\sigma}_{\tau\lambda\eta\omega}$ projects out the trace, as in the definition of the Weyl tensor

$$\begin{aligned} W^{\mu\nu\nu\rho} &= \mathcal{P}^{\mu\nu\rho\sigma}_{\tau\lambda\eta\omega} \, R^{\tau\lambda\eta\omega} \; , \\ \mathcal{P}^{\mu\nu\rho\sigma}_{\tau\lambda\eta\omega} &= \frac{3}{2} \Big(\delta^{\mu}_{\tau} \delta^{\nu}_{\lambda} \delta^{\rho}_{\eta} \delta^{\sigma}_{\omega} - \delta^{\mu}_{\tau} \delta^{[\rho}_{\lambda} \delta^{\sigma]}_{\eta} \delta^{\nu}_{\omega} \Big) + 2\, g_{\tau\eta} \delta^{[\mu}_{\lambda} g^{\nu][\rho} \delta^{\sigma]}_{\omega} + \frac{1}{3}\, g_{\tau\eta}\, g_{\lambda\omega}\, g^{\mu[\rho}\, g^{\sigma]\nu} \; . \end{aligned} \tag{3.74}$$

Next, we can take the factors of k outside the integral as derivatives

$$\begin{aligned} \langle \mathfrak{R}^{(h)\,\mu\nu\rho\sigma} \rangle &= 4\, m^2 \kappa^2 \, \mathcal{P}^{\mu\nu\rho\sigma}_{\tau\lambda\eta\omega} \, \partial^{[\tau} u^{\lambda]} \partial^{[\eta} u^{\omega]} \;\mathrm{Re}\; i \int \mathrm{d}\Phi(k) \hat{\delta}(2k\cdot p) e^{-ik\cdot x} \\ &= 2\, m\, \kappa^2 \, \mathcal{P}^{\mu\nu\rho\sigma}_{\tau\lambda\eta\omega} \, \partial^{[\tau} u^{\lambda]} \partial^{[\eta} u^{\omega]} \;\mathrm{Re}\; i \int \mathrm{d}\Phi(k) \hat{\delta}(k\cdot u) e^{-ik\cdot x} \; . \end{aligned} \tag{3.75}$$

For positive t^1, it can be checked that

$$\mathrm{Re}\; i \int \mathrm{d}\Phi(k) \hat{\delta}(k\cdot u) e^{-ik\cdot x} = -\frac{1}{2} \int \hat{\mathrm{d}}^3 k \, \frac{e^{-ik\cdot x}}{k^2} \; . \tag{3.76}$$

The integral on the right is performed over the three-dimensional subspace of momenta orthogonal to the worldline $k\cdot u = 0$, so $k_\mu = (k_1, 0, k_3, k_4)$. This prescription will be used for the rest of this section. Additionally, the divergence is resolved by the $i\epsilon$ prescription $(k)^2 = (k^1 + i\epsilon)^2 - |\vec{k}|^2$, which selects the retarded contour. Substituting in the Riemann tensor and taking the derivatives inside the integral, we get

$$\langle \mathfrak{R}^{(h)\,\mu\nu\rho\sigma} \rangle = \; m\, \kappa^2 \, \mathcal{P}^{\mu\nu\rho\sigma}_{\tau\lambda\eta\omega} \int \hat{\mathrm{d}}^3 k \, \frac{e^{-ik\cdot x}}{k^2} k^{[\tau} u^{\lambda]} k^{[\eta} u^{\omega]} \; . \tag{3.77}$$

This already looks like a double copy of the field strength tensor

$$F^{\mu\nu}(x) = -2\, Q \int \hat{\mathrm{d}}^3 k \, \frac{e^{-ik\cdot x}}{k^2} k^{[\mu} u^{\nu]} \; . \tag{3.78}$$

To obtain a concrete double copy expression, we introduce a delta function

$$\begin{aligned}
\langle \mathfrak{R}^{(h)\,\mu\nu\rho\sigma} \rangle &= m\kappa^2\, \mathcal{P}^{\mu\nu\rho\sigma}_{\tau\lambda\eta\omega} \int \hat{\mathrm{d}}^3k\, \hat{\mathrm{d}}^3q\, \hat{\delta}^3(k-q) \frac{e^{-ik\cdot x}}{k^2} k^{[\tau}u^{\lambda]} q^{[\eta}u^{\omega]} \\
&= m\kappa^2\, \mathcal{P}^{\mu\nu\rho\sigma}_{\tau\lambda\eta\omega} \int \hat{\mathrm{d}}^3k\, \hat{\mathrm{d}}^3q\, \mathrm{d}^3y\; e^{-iy\cdot(q-k)}\, \frac{e^{-ik\cdot x}}{k^2} k^{[\tau}u^{\lambda]} q^{[\eta}u^{\omega]} \\
&= -m\kappa^2\, \mathcal{P}^{\mu\nu\rho\sigma}_{\tau\lambda\eta\omega} \int \mathrm{d}^3y \int \hat{\mathrm{d}}^3k\; \frac{e^{-ik\cdot(x-y)}}{k^2} k^{[\tau}u^{\lambda]} \int \hat{\mathrm{d}}^3q\;\; e^{-iy\cdot q} q^{[\eta}u^{\omega]}\;.
\end{aligned} \tag{3.79}$$

The last line is already a convolution of $F^{\tau\lambda}$ with the integral in q. To complete the calculation we will need the scalar field and its formal inverse

$$S(x) = -\int \hat{\mathrm{d}}^3k\, \frac{e^{-ik\cdot x}}{k^2}\;, \qquad S^{-1}(x) = -\int \hat{\mathrm{d}}^3k\, e^{-ik\cdot x}\, k^2\;, \tag{3.80}$$

satisfying[7]

$$\left(S \circ S^{-1}\right)(x) = \delta(t_1)\delta^2(\mathbf{x})\,. \tag{3.81}$$

We emphasise that the expression for S^{-1} is only formal, due to the divergence of the integral. S^{-1} is really defined distributionally, acting via the convolution. It will always appear in convolutions where this divergence is cancelled, yielding a finite result. The calculation can be carried out following a strategy similar to the previous one. Inserting a delta function and a factor which equals 1 on its support, we have

$$\begin{aligned}
\langle \mathfrak{R}^{(h)\,\mu\nu\rho\sigma} \rangle = &- m\kappa^2\, \mathcal{P}^{\mu\nu\rho\sigma}_{\tau\lambda\eta\omega} \int \mathrm{d}^3y \int \hat{\mathrm{d}}^3k\; \frac{e^{-ik\cdot(x-y)}}{k^2} k^{[\tau}u^{\lambda]} \\
&\qquad \int \hat{\mathrm{d}}^3q\, \hat{\mathrm{d}}^3l\, \hat{\delta}(l-q)\, \frac{l^2}{q^2}\, e^{-iy\cdot q} q^{[\eta}u^{\omega]} \\
= &- m\kappa^2\, \mathcal{P}^{\mu\nu\rho\sigma}_{\tau\lambda\eta\omega} \int \mathrm{d}^3y\; \mathrm{d}^3z \int \hat{\mathrm{d}}^3k\; \frac{e^{-ik\cdot(x-y)}}{k^2} k^{[\tau}u^{\lambda]} \\
&\qquad \int \hat{\mathrm{d}}^3q\, \frac{1}{q^2}\, e^{-iq\cdot(y-z)} q^{[\eta}u^{\omega]} \int \hat{\mathrm{d}}^3l\, l^2\, e^{-il\cdot z}\;.
\end{aligned} \tag{3.82}$$

Finally, we recognise another convolution,

$$\langle \mathfrak{R}^{(h)\,\mu\nu\rho\sigma}(x) \rangle = -\frac{m\,\kappa^2}{4\,Q^2} \mathcal{P}^{\mu\nu\rho\sigma}_{\tau\lambda\eta\omega} \left(F^{\tau\lambda} \circ S^{-1} \circ F^{\eta\omega}\right)(x)\;. \tag{3.83}$$

We conclude that this contribution of the Riemann tensor is the convolution of two copies of the field strength tensor with an inverse power of the scalar field. We remark that the convolutions are performed over a 3-dimensional subspace of spacetime, reflecting the fact that all our solutions are independent of t_2.

The convolution we have obtained is the most natural operation from the point of

[7]The symbol $\circ$ denotes convolution: $(f \circ g)(x) = \int \mathrm{d}y f(y) g(x-y)$.

view of the double copy [70, 74, 75, 103–105]. However, we know that for some cases (like Schwarzschild) the relation must factorise in position space, turning convolutions into ordinary products. In this factorisation, the projector plays an important role. On its own, $F \circ S^{-1} \circ F$ does not factorise, but the offending terms are pure traces that are projected out by $\mathcal{P}$, leaving a neat factorised expression. In order to gain a better understanding of this, let us go back to (3.77) and take the derivatives out of the integral:

$$\begin{aligned} \langle \mathfrak{R}^{(h)\,\mu\nu\rho\sigma} \rangle &= -\, m\kappa^2\, \mathcal{P}^{\mu\nu\rho\sigma}_{\tau\lambda\eta\omega} \partial^{[\tau} u^{\lambda]} \partial^{[\eta} u^{\omega]} \int \hat{\mathrm{d}}^3 k \frac{e^{-ik\cdot x}}{k^2} \\ &= m\kappa^2\, \mathcal{P}^{\mu\nu\rho\sigma}_{\tau\lambda\eta\omega} \partial^{[\tau} u^{\lambda]} \partial^{[\eta} u^{\omega]}\, S(x)\ . \end{aligned} \tag{3.84}$$

The crucial point now is that the double derivative acting on $S(x)$ factorises into single derivatives under the contraction of the projector $\mathcal{P}$, in the sense to be described below. This happens in the strict interior of the split signature light-cone, where the curvature is non-vanishing and type D. It is simpler if we first perform an analytic continuation to (1,3) signature. Then, the interpretation of Kerr-Schild vectors in the context of radiating point particles (see [37, 171]) provides the necessary tools to show the factorisation. First, we note that the analytic continuation of the scalar $S(x)$ is

$$S(x) = \frac{1}{4\pi R}\ , \tag{3.85}$$

where R can be interpreted as the retarded null distance between a point x^μ and a static worldline $y^\mu(\tau)$ tangent to u^μ. Then, the Kerr-Schild vector is defined as

$$K^\mu = \frac{[x^\mu - y^\mu(\tau)]_{\mathrm{ret}}}{R}\ . \tag{3.86}$$

It is not hard to prove that

$$\partial_\mu R = u_\mu - K_\mu\ , \qquad \partial_\mu K_\nu = \frac{1}{R}\left(\eta_{\mu\nu} + K_\mu K_\nu - K_\mu u_\nu - K_\nu u_\mu\right)\ . \tag{3.87}$$

These two identities imply

$$\begin{aligned} \partial_\mu S &= -4\pi\, S^2\, (u_\mu - K_\mu) \\ \partial_\mu \partial_\nu S &= 3(4\pi)^2\, S^3\, (u_\mu - K_\mu)(u_\nu - K_\nu) + (4\pi)^2 S^3\, (\eta_{\mu\nu} - u_\mu u_\nu)\ . \end{aligned} \tag{3.88}$$

The last line can be rewritten as

$$\partial_\mu \partial_\nu S = 3\, \frac{\partial_\mu S\, \partial_\nu S}{S} + (4\pi)^2\, S^3\, (\eta_{\mu\nu} - u_\mu u_\nu)\ . \tag{3.89}$$

Upon substitution in (3.84), the contribution from the last term on the right-hand side of the expression above vanishes. Hence, we arrive at

$$\mathcal{P}^{\mu\nu\rho\sigma}_{\tau\lambda\eta\omega}\partial^{[\tau}u^{\lambda]}\partial^{[\eta}u^{\omega]}\,S(x) = \frac{3}{S(x)}\mathcal{P}^{\mu\nu\rho\sigma}_{\tau\lambda\eta\omega}\left(\partial^{[\tau}u^{\lambda]}S(x)\right)\left(\partial^{[\eta}u^{\omega]}S(x)\right)\ . \tag{3.90}$$

This expression completes the argument because the factors in parenthesis equal the field strength tensor up to some constants,

$$\langle\mathfrak{R}^{(h)\,\mu\nu\rho\sigma}\rangle = \frac{3\kappa^2\,m}{4}\,\mathcal{P}^{\mu\nu\rho\sigma}_{\tau\lambda\eta\omega}\,\frac{F^{\tau\lambda}F^{\eta\omega}}{S}\ . \tag{3.91}$$

This relation is very reminiscent of the double copy, as the gravitational object (the Weyl curvature) is constructed as the product of two gauge theory objects (field strength tensors). The scalar S plays the role of the propagators in the amplitude double copy and is often referred to as the zeroth copy.[8] The existence of such a simple relation in position space implies that, at least for Schwarzschild, the double copy structure of the amplitudes permeates to the classical solution. In fact, this double copy relation was already discovered in the spinor structure of vacuum type D spacetimes [99] and receives the name of *Weyl double copy.* Nonetheless, it had not been derived or linked directly to the amplitudes double copy. It is worth remarking that, unlike for other type D solutions, the Schwarzschild metric satisfies – a generalisation of – (3.91) in any dimension, with the appropriate overall constant factor.

Above, we have presented the position space factorisation in the simplest example, where the deformation parameters $\bar{\theta}$ and $\bar{a}$ are set to zero. Since the Kerr-Taub-NUT solution is also type D, it satisfies the Weyl double copy and a similar factorisation must happen for generic $\bar{\theta}$ and $\bar{a}$. The proof is rather straightforward. The key observation is that (3.90) holds also for $S_{\bar{a},\bar{\theta}}$, since the effects of $\bar{a}$ and $\bar{\theta}$ are a translation and a constant scaling respectively. Then, after some algebra, one can prove that

$$\langle\mathfrak{R}^{(h)\,\mu\nu\rho\sigma}\rangle = \frac{3\kappa^2\,m}{4}\mathcal{P}^{\mu\nu\rho\sigma}_{\tau\lambda\eta\omega}\left(\frac{F_-^{\tau\lambda}F_-^{\eta\omega}}{S_{\bar{a},\bar{\theta}}}+\frac{F_+^{\tau\lambda}F_+^{\eta\omega}}{S_{-\bar{a},-\bar{\theta}}}\right)\ , \tag{3.92}$$

where F_+ and F_- are the self-dual and anti-self-dual parts of the field strength as defined in (A.29). The result (3.92) is equivalent to the original Weyl double copy but written in terms of curvature tensors instead of spinors. The tensorial formulation of the Weyl double copy was also studied in [6, 172].

Having seen how the graviton degrees of freedom in the curvature factorise in position space, one might ask whether these similar position-space relations exist for the dilaton

[8] As mentioned in the introduction, the zeroth copy is a bi-adjoint scalar field, but since we are considering the linearised theory it corresponds to a bi-Abelian bi-adjoint field.

contribution, which reads

$$\langle \mathfrak{R}^{(\phi)\,\mu\nu\rho\sigma}(x)\rangle = 2\,\tilde{c}\,m\,\kappa\ \mathrm{Re}\ i \int \mathrm{d}\Phi(k)\hat{\delta}(2k\cdot p)e^{-ik\cdot x}k^{[\mu}\eta^{\nu][\rho}k^{\sigma]}\ . \tag{3.93}$$

On the support of the delta functions, we can write this as

$$\langle \mathfrak{R}^{(\phi)\,\mu\nu\rho\sigma}(x)\rangle = -2\,\tilde{c}\,m\,\kappa\ \mathrm{Re}\ i \int \mathrm{d}\Phi(k)\hat{\delta}(2k\cdot p)e^{-ik\cdot x}f^{\lambda[\mu}\eta^{\nu][\rho}f^{\sigma]}{}_{\lambda}\ , \tag{3.94}$$

where $f^{\mu\nu} = 2\,k^{[\mu}u^{\nu]}$. We can follow the same steps as before to obtain a position space convolutional double copy

$$\langle \mathfrak{R}^{(\phi)\,\mu\nu}{}_{\rho\sigma}(x)\rangle = -\frac{\tilde{c}\,\kappa}{2\,Q^2}\left(F^{\lambda[\mu}\circ S^{-1}\circ F_{\lambda[\rho}\right)(x)\ \ \delta^{\nu]}{}_{\sigma]}\ . \tag{3.95}$$

In contrast to what happened for the graviton contribution, these convolutions cannot be turned into products. To understand why, we could proceed as for the graviton and rewrite (3.94) as a second order differential operator acting on the Lorentz continuation of $S(x)$. Then, we would like to use (3.89) to accomplish the factorisation. The result would be

$$\langle \mathfrak{R}^{(\phi)\,\mu\nu}{}_{\rho\sigma}(x)\rangle = \frac{\tilde{c}\,\kappa}{2}\left(3\frac{F^{\lambda[\mu}\,\delta^{\nu]}_{[\rho}\,F_{\sigma]\lambda}}{S} + (4\pi)^2 S^3\left(\delta^{[\mu}_{[\rho}\delta^{\nu]}_{\sigma]} - \delta^{[\mu}_{[\rho}u^{\nu]}\,u_{\sigma]}\right)\right)\ . \tag{3.96}$$

While the first term exhibits a local position-space double copy form, the others do not. Thus, the double copy of the dilatonic contribution is natural only in terms of convolutions and it is non-local in position space. Interestingly, the JNW solution admits an exact double copy interpretation in position space, based on a Kerr-Schild-like construction in double field theory that we will review in chapter 5 [5]. However, unlike the Kerr-Schild double copy for Schwarzschild, the dilatonic deformation makes the relation non-local in position space.

3.4.2 Kerr-Schild double copy

Since we have performed a linearised gravity calculation, one might expect the classical double copy relations obtained in the previous subsection to hold only at first order. This is probably the case in the most general scenarios, when the axion and dilaton are non-zero. However, if the scalar fields are set to zero, the Weyl double copy is promoted to an exact statement for the Kerr-Taub-NUT family. The reason is that the Kerr-Taub-NUT is part of a special class of metrics that linearises the equations of motion. Hence, the linear solution is exact and all our linearised statements can be regarded as exact. At the same time, this class of metrics exhibit a prominent classical double copy structure, known as the *Kerr-Schild* double copy. Whereas the

Weyl double copy applies to curvature tensors, the Kerr-Schild double copy applies to the metric and the gauge potential. Once again, we will unveil the double copy structure in the simplest example first, Schwarzschild.

The first step is to compare the linearised Weyl tensor (3.72) with (3.13) to deduce the linearised metric in de Donder gauge

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + \kappa\, h_{\mu\nu} \ , \\ h_{\mu\nu} &= -\kappa\, m \ \mathrm{Re}\, i \int \mathrm{d}\Phi(k)\, \hat{\delta}(k\cdot u)\, e^{-ik\cdot x} \left(u_\mu u_\nu - \frac{1}{2}\,\eta_{\mu\nu} \right) \\ &= -\frac{\kappa\, m}{2}\, S \left(u_\mu u_\nu - \frac{1}{2}\,\eta_{\mu\nu} \right) \ . \end{aligned} \tag{3.97}$$

Recall that S contains a Heaviside theta function $\Theta(t_1 - r)$ which determines the physically meaningful region. We will restrict ourselves to the strict interior of this region to avoid spurious divergences at the boundary. Under that consideration, S is replaced by

$$\hat{S} = \frac{1}{2\pi\,\sqrt{t_1^2 - x^2 - y^2}} \ . \tag{3.98}$$

Then, we apply the diffeomorphism generated by

$$\xi = -\frac{\kappa^2\, m}{16\pi} \left(\mathrm{d}\,\sqrt{t_1^2 - x^2 - y^2} - i \log \frac{t_1^2 - x^2 - y^2}{r_0^2}\, \mathrm{d}t_2 \right) \ , \tag{3.99}$$

where r_0 is a constant needed for dimensional purposes. This complex diffeomorphism is acceptable as a means of putting the metric in our desired form, but one can always choose coordinates where the metric is real. The result is the linear metric

$$t_1 > \sqrt{x^2 + y^2} \ : \qquad g_{\mu\nu} = \eta_{\mu\nu} - \frac{\kappa^2 m}{4}\, \hat{S}\, L\, L \ , \tag{3.100}$$

where

$$L = \mathrm{d}t_2 + i\, \frac{t_1 \mathrm{d}t_1 - x\mathrm{d}x - y\mathrm{d}y}{\sqrt{t_1^2 - x^2 - y^2}} \ . \tag{3.101}$$

The vector L has the properties of being null and geodesic with respect to $\eta_{\mu\nu}$ (and hence also $g_{\mu\nu}$). Spacetimes of the form (3.100) with L null and geodesic are known as *Kerr-Schild* spacetimes. Their most important characteristic is that they linearise the Einstein equation. Hence, (3.100) solves the Einstein equations exactly because higher-order corrections vanish identically.

Alternatively, the exact metric can be expressed in terms of real coordinates as

$$t_1 > \sqrt{x^2+y^2} \; :$$
$$\mathrm{d}s^2 = \left(1 - \frac{\kappa^2 m}{4}\,\hat{S}\right)\mathrm{d}t_2'^{\,2} + \mathrm{d}t_1^2 - \mathrm{d}x^2 - \mathrm{d}y^2 + \frac{\frac{\kappa^2 m}{4}\,\hat{S}}{1-\frac{\kappa^2 m}{4}\,\hat{S}}\,\frac{(t_1\mathrm{d}t_1 - x\mathrm{d}x - y\mathrm{d}y)^2}{t_1^2 - x^2 - y^2}\,, \tag{3.102}$$

using

$$\mathrm{d}t_2' = \mathrm{d}t_2 - i\,\frac{\frac{\kappa^2 m}{4}\,\hat{S}}{1-\frac{\kappa^2 m}{4}\,\hat{S}}\,\frac{t_1\mathrm{d}t_1 - x\mathrm{d}x - y\mathrm{d}y}{\sqrt{t_1^2 - x^2 - y^2}}\,. \tag{3.103}$$

Now it is clear how to extend the solution beyond $t_1 > \sqrt{x^2+y^2}$,

$$\mathrm{d}s^2 = \left(1 - \frac{\kappa^2 m}{4}\,S\right)\mathrm{d}t_2'^{\,2} + \mathrm{d}t_1^2 - \mathrm{d}x^2 - \mathrm{d}y^2 + \frac{\frac{\kappa^2 m}{4}\,S}{1-\frac{\kappa^2 m}{4}\,S}\,\frac{(t_1\mathrm{d}t_1 - x\mathrm{d}x - y\mathrm{d}y)^2}{t_1^2 - x^2 - y^2}\,. \tag{3.104}$$

This gives us the final answer of the exact gravity solution, which represents the split signature equivalent of the Schwarzschild black hole.[9]

The gauge potential can be put in a gauge resembling (3.100). Recall that in the previous chapter it was found that

$$A^\mu(x) = Q\,u^\mu\,S(x)\;. \tag{3.107}$$

Then, a complex gauge transformation inside the three-dimensional lightcone takes us to 'Kerr-Schid' gauge

$$t_1 > \sqrt{x^2+y^2} \; : \qquad A^{(KS)} = Q\,\hat{S}\,\mathrm{d}t_2 - \frac{Q}{2\pi i}\,\mathrm{d}\log\frac{\sqrt{t_1^2-x^2-y^2}}{r_0} \tag{3.108}$$
$$= Q\,\hat{S}\,L\;, \tag{3.109}$$

where L is the same vector that appears in the Kerr-Schild metric (3.101).

There is a clear parallelism between (3.100) and (3.109). By doubling the vector L and changing the coupling constants of the gauge potential one obtains the second term

[9] We could also write the line element in coordinates analogous to the Schwarzschild spherical coordinates, but would have to split into spacetime regions. Inside the light-cone, with $\chi = \sqrt{t_1^2-x^2-y^2}$ and $\hat{S} = (2\pi\chi)^{-1}$, we pick $t_1 = \chi\cosh\psi$ inside the future light-cone and $t_1 = -\chi\cosh\psi$ inside the past light-cone, obtaining

$$t_1^2 > x^2+y^2 \; : \quad \mathrm{d}s^2 = \left(1 - \frac{\kappa^2 m}{4}\,\Theta(t_1)\hat{S}\right)\mathrm{d}t_2'^{\,2} + \frac{d\chi^2}{1-\frac{\kappa^2 m}{4}\,\Theta(t_1)\hat{S}} - \chi^2(d\psi^2 + \sinh^2\psi\,d\phi^2)\,. \tag{3.105}$$

Outside the light-cone, with $\tilde{\chi} = \sqrt{x^2+y^2-t_1^2}$, we can write

$$t_1^2 < x^2+y^2 \; : \quad \mathrm{d}s^2 = \mathrm{d}t_2'^{\,2} - d\tilde{\chi}^2 + \tilde{\chi}^2(-d\tilde{\psi}^2 + \sin^2\tilde{\psi}\,d\phi^2)\,. \tag{3.106}$$

of the Kerr-Schild metric. In this sense, we obtain the double copy of the Coulomb potential by doubling the kinematic information (represented by L). This is an example of another instance of the (exact) classical double copy, the Kerr-Schild double copy, which applies not only to Schwarzschild/Coulomb but to more general (multi-)Kerr-Schild solutions [97,173]. Like the Weyl double copy, these classical relations have been known for some years, but they had not been connected directly to the amplitudes double copy. Here, we have derived the Coulomb/Schwarzschild example from the double copy of the corresponding three-point amplitudes. An equivalent argument would hold for the Kerr-Taub-NUT family, which is of complex double-Kerr-Schild type. That means that there are two Kerr-Schild terms in the metric, with different null vectors, but the equations still linearise.

3.5 Summary and outlook

Chapters 2 and 3 explored the classical limit of the double copy certain three-point amplitudes in split signature. The KMOC prescription was used to compute the classical electromagnetic field of a $\sqrt{\text{Kerr}}$-dyon particle from the simple photon emission amplitude with massive scalar legs. Then, the double copied amplitude (3.21) was used in tandem with a metric-affine connection to produce linearised axion-dilaton Kerr-Taub-NUT solutions. Although we performed the calculation in split signature to avoid the cancellation of the three-point amplitudes in real kinematics, the final solutions can be traced back to Lorentzian signature, matching previously known solutions. More importantly, the double copy relation at the level of the amplitudes implies double copy relations at the level of the classical fields. In particular, we identified two types of classical double copies that are local in position space, the Weyl double copy and the Kerr-Schild double copy. Importantly, in the absence of dilaton and axion, both prescriptions can be promoted to exact relations owing to the existence of Kerr-Schild coordinates. Both exact classical double copy prescriptions were available in the literature, but we have shown a precise connection to the amplitudes double copy for the first time, pinning down the properties that explain the unexpected locality in position space.

The remaining of this thesis is devoted to exploring the classical double copy. Our previous findings motivate us to review the Weyl and Kerr-Schild double copies and expand their domains of applicability. In this chapter, we derived the Weyl double copy for Kerr-Taub-NUT spacetime, but it also holds for other type-D spacetimes. The C-metric is another example, which we use to investigate the nature of the Weyl double copy map in the vicinity of future null infinity. Moreover, we show how the relation can be extended to non-twisting type N solutions, which represent radiation spacetimes. In chapter 5, we review the Kerr-Schild double copy. As already mentioned, the original formulation holds exclusively in the absence of matter fields. To extend the Kerr-Schild double copy to the entire NS-NS spectrum, the formulation of Double Field Theory (DFT) is helpful. In this setting, we obtain an exact Kerr-Schild double copy relation for JNW.

Chapter 4

The Weyl double copy

In the previous chapter, we saw how the Weyl spinors of certain spacetimes have a clear double copy structure in momentum space. This structure extends to position space, where we wrote the Kerr-Taub-NUT Weyl tensor as a sum of quadratic terms in field strength tensors. The quadratic relation between the Weyl tensors of certain classes of spacetimes and field strength tensors is known as the Weyl double copy, and it was originally shown to apply exactly to all type D vacuum solutions and pp-waves [99]. The original formulation was written in spinor language, which clears out all the redundant index contractions and symmetrisations to yield much simpler expressions. It also provides tools to prove that the gauge fields that stem from the Weyl tensors automatically satisfy the Maxwell equations. In this chapter, we study the Weyl double copy from a purely classical point of view. From this point on, the signature of the metrics will be Lorentzian $(-,+,+,+)$, as there will be no more 3-point amplitudes involved.

The origins of the Weyl double copy date back to the golden age of general relativity, when Walker and Penrose discovered that all vacuum type D solutions admitted a Killing rank-two spinor [174]. Together with Hughston and Sommers, the same authors showed that the Killing spinor could be used to generate a non-backreacting electromagnetic field on the curved background [175]. These facts were revisited and given a double copy interpretation much more recently [99], where the electromagnetic field was moved to a flat background. At the same time, the prescription was extended to the Eguchi-Hanson instanton (see also [176]) and pp-waves, which also exhibit a Killing spinor. The generalisation to all non-twisting type N solutions was provided in [4]. Extensions to non-vacuum solutions were given in [94, 177]. In [178] the Weyl double copy was applied to fluid mechanics; specifically to flow configurations that are associated with algebraically special spacetimes under the fluid-gravity duality.

The search for a deeper understanding of the Weyl double copy led to its reformulation in twistorial language [100–102, 144, 179]. This construction, which relies on the properties of the Penrose transform, is able to extend the Weyl double copy beyond type D and type N solutions, yet only at linearised level.

Although the spinor and twistor languages expose the structure of the double copy more clearly, they also have limitations. One of them is their dimensional dependency, making the generalisation beyond four dimensions not straightforward. Some steps were given in this direction in [180], where an algebraic classification of higher-dimensional spacetimes based on spinors was proposed. There have also been recent developments in three dimensions, where the vanishing of the Weyl tensor brings extra challenges. The solution found by the authors of [181,182] was to use the Cotton tensor as the analogue of the Weyl tensor in topologically massive gravity. Ref. [181] followed an approach very similar to the one described in chapter 3, deriving the double copy relation from the squaring of 3-point amplitudes. The authors of [182] followed a purely classical approach instead, providing a derivation along the lines of this chapter. The Cotton double copy is closer in spirit to the original double copy than some earlier attempts at the double copy in three dimensions [183], despite applying to topologically massive gravity. Additionally, it might provide useful insight for formulating a Weyl double copy in higher dimensions. However, it would be helpful to understand to what extent the Cotton double copy is just a consequence of the Weyl double copy under dimensional reduction.

Our next goal will be to review and explore the limits of the Weyl double copy from the point of view of classical general relativity. After reviewing the basic aspects of the map, we provide a more complete picture of what are the effects of the electric-magnetic duality under the double copy. On the gravity side, it corresponds to an Ehlers transformation, a solution-generating transformation that rotates and rescales the mass and charge parameters [6]. We will also follow [4] to extend the Weyl double copy to type N solutions. The subjects of radiation and symmetries intersect in the study of asymptotic boundaries. In the last part of this chapter, we will explore the Weyl double copy in the vicinity of future null infinity. The asymptotic expansion provides an alternative perturbation scheme for the double copy which extends its domain of applicability to less special solutions [2]. At the same time, it enables the study of asymptotic symmetries and their relation to large gauge transformations. To answer this question, we will focus on the C-metric, a type D solution that can be regarded as a superrotation for large retarded times. The single copy of the C-metric is the Liénard-Wiechert potential generated by an uniformly accelerated charge, and contains a large gauge transformation in a similar sense, giving a precise relation between both transformations.

4.1 Type D

4.1.1 Spinor calculus

Consider a Type D metric. According to (1.79) and table 1.1, its Weyl spinor can be written as

$$\Psi_{\alpha\beta\gamma\delta} = \psi\, o_{(\alpha}\iota_\beta o_\gamma \iota_{\delta)} \ , \tag{4.1}$$

where $\psi = 6\Psi_2$ and o, ι are chosen to be a normalised ($o_\alpha \iota^\alpha = 1$) dyad basis.

First, consider that our spacetime is a vacuum solution and impose the Bianchi identity (1.76)

$$\nabla^{\alpha\dot{\alpha}}(\psi\, o_{(\alpha}\iota_\beta o_\gamma \iota_{\delta)}) = 0 \ . \tag{4.2}$$

This has to hold for every component in the (o, ι) basis. In particular, it must hold for the projection

$$\begin{aligned} 2\iota^\beta o^\gamma o^\delta \nabla^{\alpha\dot{\alpha}}\Psi_{\alpha\beta\gamma\delta} =& \iota^\beta o^\gamma o^\delta \nabla^{\alpha\dot{\alpha}}(\psi o_\alpha o_{(\beta}\iota_\gamma \iota_{\delta)} + \psi \iota_\alpha o_{(\beta} o_\gamma \iota_{\delta)}) \\ =& \frac{1}{3}\nabla^{\alpha\dot{\alpha}}(\psi o_\alpha) + \iota^\beta o^\gamma o^\delta \psi [o_\alpha \nabla^{\alpha\dot{\alpha}}(o_{(\beta}\iota_\gamma\iota_{\delta)}) + \iota_\alpha \nabla^{\alpha\dot{\alpha}}(o_{(\beta} o_\gamma \iota_{\delta)})] \\ =& \frac{1}{3}\nabla^{\alpha\dot{\alpha}}(\psi o_\alpha) + \frac{1}{3}\psi[\iota^\beta o_\alpha \nabla^{\alpha\dot{\alpha}} o_\beta - 2\, o^\beta o_\alpha \nabla^{\alpha\dot{\alpha}}\iota_\beta - 2\, o^\beta \iota_\alpha \nabla^{\alpha\dot{\alpha}} o_\beta] \ , \end{aligned}$$

expanding and using the identities

$$\begin{aligned} \epsilon_{\alpha\beta} &= \iota_\alpha o_\beta - o_\alpha \iota_\beta \ , \\ \epsilon^{\alpha\beta} &= o^\alpha \iota^\beta - \iota^\alpha o^\beta \ , \\ \delta^\alpha_\beta &= \iota^\alpha o_\beta - o^\alpha \iota_\beta \ , \end{aligned} \tag{4.3}$$

we find that

$$\frac{1}{3} o_\alpha \nabla^{\alpha\dot{\alpha}}\psi - \psi \iota_\alpha o^\beta \nabla^{\alpha\dot{\alpha}} o_\beta = 0 \ . \tag{4.4}$$

The component obtained by contraction with $\iota^\beta \iota^\gamma o^\delta$ gives

$$\frac{1}{3}\iota_\alpha \nabla^{\alpha\dot{\alpha}}\psi + \psi o_\alpha \iota^\beta \nabla^{\alpha\dot{\alpha}}\iota_\beta = 0 \ . \tag{4.5}$$

The components obtained projecting by $o^\beta o^\gamma o^\delta$ and $\iota^\beta \iota^\gamma \iota^\delta$ imply

$$o_\alpha o_\beta \nabla^{\alpha\dot{\alpha}} o^\beta = 0 \ , \qquad \iota_\alpha \iota_\beta \nabla^{\alpha\dot{\alpha}}\iota^\beta = 0 \ , \tag{4.6}$$

which mean that the principal directions are tangent to shear-free null geodesic congruences, as expected from the Goldberg-Sachs theorem [184].

Consider now a Maxwell spinor of the form

$$\phi_{\alpha\beta} = \psi^n o_{(\alpha}\iota_{\beta)} \ . \tag{4.7}$$

Let us compute the o^β component of the vacuum Maxwell equations (1.70),

$$\begin{aligned} o^\beta \nabla^{\alpha\dot{\alpha}}(\psi^n o_{(\alpha}\iota_{\beta)}) &= -\frac{n}{2}\psi^{n-1} o_\alpha \nabla^{\alpha\dot{\alpha}}\psi + \frac{1}{2}\psi^n o^\beta \nabla^{\alpha\dot{\alpha}}(o_\alpha \iota_\beta + o_\beta \iota_\alpha) \\ &= -\frac{n}{2}\psi^{n-1}\, o_\alpha \nabla^{\alpha\dot{\alpha}}\psi + \psi^n\, \iota_\alpha\, o^\beta \nabla^{\alpha\dot{\alpha}} o_\beta = 0 \ . \end{aligned} \tag{4.8}$$

Comparing (4.4) and (4.8) we conclude that they are equivalent if $n = 2/3$. Similarly, the ι^β component of the Maxwell equation is equivalent to (4.5) for $n = 2/3$.

The conclusion is that

$$\nabla^{\alpha\dot{\alpha}}(\psi\, o_{(\alpha}\iota_\beta o_\gamma \iota_{\delta)}) = 0 \quad \Rightarrow \quad \nabla^{\alpha\dot{\alpha}}(\psi^{2/3}\, o_{(\alpha}\iota_{\beta)}) = 0 \ . \tag{4.9}$$

Based on this fact, it is natural to decompose the Weyl spinor as

$$\Psi_{\alpha\beta\gamma\delta} = \frac{1}{S}\phi_{(\alpha\beta}\phi_{\gamma\delta)} \ , \tag{4.10}$$

where $S = \psi^{1/3}$. This complex scalar is associated to the zeroth copy, a solution of the linearised bi-adjoint scalar theory. At linear level, the bi-adjoint scalar theory becomes Abelian, and the equation of motion reduces to the wave equation. It can be checked that S is a solution:

$$\Box\psi^{1/3} = \frac{1}{3}\, \nabla^{\alpha\dot{\alpha}}(\psi^{-2/3}\epsilon_{\alpha\beta}\nabla^\beta{}_{\dot{\alpha}}\psi) \tag{4.11}$$

$$= \nabla^{\alpha\dot{\alpha}}[\psi^{1/3}\iota_\alpha\iota_\beta o^\gamma \nabla^\beta{}_{\dot{\alpha}} o_\gamma + (o \leftrightarrow \iota)] \ , \tag{4.12}$$

where we have used (4.3), (4.4) and (4.5). The same equations reduce the expanded expression to

$$\begin{aligned} \Box\psi^{1/3} = \psi^{1/3}\Big[& 2\, (o^\beta \iota_\alpha \nabla^{\alpha\dot{\alpha}} o_\beta)(\iota^\delta o_\gamma \nabla^\gamma{}_{\dot{\alpha}} \iota_\delta) \\ & + \Big(\nabla^{\alpha\dot{\alpha}}(\iota_\alpha\iota_\beta o^\gamma)\nabla^\beta{}_{\dot{\alpha}} o_\gamma + \iota_\alpha\iota_\beta o^\gamma \nabla^{\alpha\dot{\alpha}}\nabla^\beta{}_{\dot{\alpha}} o_\gamma + (o \leftrightarrow \iota)\Big)\Big] \end{aligned} \tag{4.13}$$

It can be shown, by application of (4.3) and (4.6) that

$$\nabla^{\alpha\dot{\alpha}}(\iota_\alpha\iota_\beta o^\gamma)\nabla^\beta{}_{\dot{\alpha}} o_\gamma + (o \leftrightarrow \iota) = -2\, (o^\beta \iota_\alpha \nabla^{\alpha\dot{\alpha}} o_\beta)(\iota^\delta o_\gamma \nabla^\gamma{}_{\dot{\alpha}} \iota_\delta) \ . \tag{4.14}$$

Thus, defining

$$\nabla_{(\alpha|\dot{\alpha}}\nabla_{\beta)}{}^{\dot{\alpha}} := \Box_{\alpha\beta} \tag{4.15}$$

we can write

$$\Box\psi^{1/3} = -\iota_\alpha\iota_\beta o^\gamma \Box_{\alpha\beta} o_\gamma - o_\alpha o_\beta \iota^\gamma \Box_{\alpha\beta}\iota_\gamma \ . \tag{4.16}$$

For a generic spinor ξ^α, we have – (4.9.8) in [124] –

$$\Box_{\alpha\beta}\xi_\gamma = X_{\alpha\beta\gamma\delta}\,\xi^\delta \ , \tag{4.17}$$

where $X_{\alpha\beta\gamma\delta}$ is the curvature spinor of the underlying space. Hence, in the flat space limit $X_{\alpha\beta\gamma\delta} = 0$ and we find

$$\Box\, S = 0 \ . \tag{4.18}$$

Let us explain what is meant by the 'flat space limit'. The double copy maps objects from a theory with gravity (spacetime curvature) to a gauge theory on a flat background. Hence, for this spinor map to be considered a double copy, the Maxwell and wave equations should be understood to hold on a flat background. This requires the existence of a limit where the metric becomes flat, keeping the spinor basis invariant. The simplest example is the massless limit of Schwarzschild. To prove that the Maxwell spinor equations hold, we only required that the basis (o, ι) was tangent to share-free null congruences, which also holds in the flat space limit. As a result, $\phi_{\alpha\beta}$ solves the Maxwell equations both on the curved and flat backgrounds. The 'flat' field strength tensor is obtained by contracting $\phi_{\alpha\beta}$ with the Pauli matrices and the flat-limit vielbein. However, to prove that S solves the wave equation we needed to set the background curvature to zero, implying that generically it is only a solution on the flat background.

What lies behind these relations is the existence of the well-known hidden symmetry for type D vacuum solutions: the existence of a Killing 2-spinor [174, 175, 185], defined as a solution of

$$\nabla_{(\alpha}{}^{\dot\alpha}\chi_{\beta\gamma)} = 0 \ . \tag{4.19}$$

The presence of the Killing spinor is a consequence of the integrability properties of the solutions under consideration, whose Weyl spinor can be written as

$$\Psi_{\alpha\beta\gamma\delta} = [\chi]^{-5}\chi_{(\alpha\beta}\chi_{\gamma\delta)} \ , \qquad \text{where} \qquad [\chi] = \left(\chi_{\alpha\beta}\chi^{\alpha\beta}\right)^{1/2} \ . \tag{4.20}$$

One can also formulate the Maxwell spinor and scalar in terms of this object

$$\phi_{\alpha\beta} = [\chi]^{-3}\chi_{\alpha\beta} \ , \qquad S = [\chi]^{-1}. \tag{4.21}$$

See [6, 94, 98, 172, 176, 178, 186–192] for related works.

The double copy structure of (4.10) is rather transparent. The Weyl spinor of any vacuum type D spacetime can be decomposed into two copies of a Maxwell spinor and a massless scalar field, which play the role of the single and zeroth copies. This structure was already present in the momentum-space integrals of chapter 3, from which the position-space relation originates.

4.1.2 The Plebanski-Demianski metric

Although in [99] this prescription was also successfully applied to pp-waves and the self-dual Eguchi-Hanson instanton, the main domain of application of (4.10) is real type D solutions of the vacuum Einstein equation without cosmological constant. Surprisingly, all the members of this family can be derived as limits of a single metric [193], known as the Plebanski-Demianski solution.[1] Using a complex change of coordinates, the metric can be written in double-Kerr-Schild form [99]:

$$\mathrm{d}s^2 = \frac{1}{(1-pq)^2}\Bigg[2i(\mathrm{d}u+q^2\mathrm{d}v)\mathrm{d}p - 2(\mathrm{d}u-p^2\mathrm{d}v)\mathrm{d}q + \frac{P(p)}{p^2+q^2}(\mathrm{d}u+q^2\mathrm{d}v)^2 - \frac{Q(q)}{p^2+q^2}(\mathrm{d}u-p^2\mathrm{d}v)^2\Bigg], \quad (4.22)$$

with

$$\begin{aligned} P(p) &= \gamma(1-p^4) + 2np - \epsilon p^2 + 2mp^3 \,, \\ Q(q) &= \gamma(1-q^4) - 2mq + \epsilon q^2 - 2nq^3 \,, \end{aligned} \quad (4.23)$$

where the parameters m, n, γ, ϵ are related to the mass, NUT charge, angular momentum and acceleration (see [194] for a discussion of the various limits and definitions which enable the identifications in different cases). In particular, different scaling limits turn (4.22) into the Kerr-Taub-NUT family or the C-metric, whose single copy is the Liénard-Wiechert potential. The latter metric was not obtained from the amplitudes double copy in chapter (3), but will prove useful to study the properties of the Weyl double copy close to the asymptotic boundary. A double-Kerr-Schild metric is a generalisation of the Kerr-Schild metric mentioned in section 3.4.2. The main difference is that it has two "Kerr-Schild terms" instead of just one. The double-Kerr-Schild form of the Plebanski-Demianski metric also linearises the field equations.

The metric (4.22) justifies the flat space limit mentioned above. The Kerr-Schild vectors

$$K = \mathrm{d}u + q^2\mathrm{d}v \,, \qquad L = \mathrm{d}u - p^2\mathrm{d}v \quad (4.24)$$

are the principal null directions. The associated spinor basis does not depend on the parameters m, n, γ, ϵ, in particular they are independent of the *dynamical parameters* m and n. These parameters, related to mass and NUT charge, generate the geometric curvature. Therefore, the flat limit $m \to 0$, $n \to 0$ provides a valid background for the

[1] The general Plebanski-Demianski family extends to solutions of the Einstein-Maxwell equations with a cosmological constant.

gauge and scalar equations of motion.

4.1.3 Tensorial Weyl double copy

The spinor expression (4.10) is remarkably compact compared to the tensorial relation (3.92). Nonetheless, a tensorial expression has some advantages. First of all, it avoids the need for explicitly computing the tetrad, which can be somewhat tedious for non-diagonal metrics. Secondly, a tensorial expression is much easier to generalise to higher dimensions. This will motivate us to study the tensorial map (3.92) in more detail.

Instead of directly deriving the tensorial expression from the spinor Weyl double copy, it will prove easier to bootstrap it. From (4.10), we know that we are looking for an expression for the Weyl tensor as a sum of terms involving two copies of the field strength. The absence of traces and the index symmetries (including $W_{\mu[\nu\rho\sigma]} = 0$) of the Weyl tensor constrain the possible quadratic combinations of the field strength tensor that can appear. Keeping these in mind, in four dimensions there are two independent expressions that one can write down:

$$C_{\mu\nu\rho\sigma}[F] = F_{\mu\nu}\, F_{\rho\sigma} - \, F_{\mu[\rho}\, F_{\sigma]\nu} + \tfrac{1}{2} g_{\mu[\rho}\, g_{\sigma]\nu}\, F^2 + 3\, F_{\lambda[\mu} g_{\nu][\rho} F^{\lambda}{}_{\sigma]} \ , \tag{4.25}$$

$$D_{\mu\nu\rho\sigma}[F] = \tfrac{3}{4}(F_{\mu\nu}\, \tilde{F}_{\rho\sigma} + \tilde{F}_{\mu\nu}\, F_{\rho\sigma}) - \tfrac{1}{4} g_{\mu[\rho}\, g_{\sigma]\nu}\, \tilde{F}^{\alpha\beta}\, F_{\alpha\beta} - \tfrac{1}{8}\epsilon_{\mu\nu\rho\sigma}\, F^2 \,, \tag{4.26}$$

where $F^2 = F^{\lambda\delta} F_{\lambda\delta}$ and $\tilde{F} = \star F$. In d dimensions there is no equivalent of $D_{\mu\nu\rho\sigma}[F]$ and one just has

$$\begin{aligned} C^{(d)}_{\mu\nu\rho\sigma}[F] &= F_{\mu\nu}\, F_{\rho\sigma} - \, F_{\mu[\rho}\, F_{\sigma]\nu} \\ &\quad + \frac{3}{(d-1)(d-2)} g_{\mu[\rho}\, g_{\sigma]\nu}\, F^2 + \frac{6}{d-2}\, F_{\lambda[\mu} g_{\nu][\rho} F^{\lambda}{}_{\sigma]} \ . \end{aligned} \tag{4.27}$$

This higher-dimensional structure is enough to capture the higher dimensional generalisation of Schwarzschild, although it cannot be applied to less symmetric spacetimes like a Myers-Perry black hole.

In $d = 4$, the Weyl double copy must involve a linear sum of the two expressions $C_{\mu\nu\rho\sigma}$ and $D_{\mu\nu\rho\sigma}$, with suitable coefficients which will generically be functions of the coordinates and constants. We note some useful properties of these expressions:

$$C_{\mu\nu\rho\sigma}[\tilde{F}] = -C_{\mu\nu\rho\sigma}[F] \ , \tag{4.28}$$

$$D_{\mu\nu\rho\sigma}[\tilde{F}] = -D_{\mu\nu\rho\sigma}[F] \ , \tag{4.29}$$

$$\tilde{C}_{\mu\nu\rho\sigma}[F] = D_{\mu\nu\rho\sigma}[F] \ , \tag{4.30}$$

$$\tilde{D}_{\mu\nu\rho\sigma}[F] = -C_{\mu\nu\rho\sigma}[F] \tag{4.31}$$

and

$$C_{\mu\nu\rho\sigma}[a\, F + b\, \tilde{F}] = (a^2 - b^2)\, C_{\mu\nu\rho\sigma}[F] + 2\, a\, b\, D_{\mu\nu\rho\sigma}[F] \ , \tag{4.32}$$

where the dual of a rank-four tensor with Weyl symmetries was defined as

$$\tilde{W}_{\mu\nu\rho\sigma} = \tfrac{1}{2}\,\epsilon_{\mu\nu\alpha\beta}\,W^{\alpha\beta}{}_{\rho\sigma}\ . \tag{4.33}$$

As a consequence, there is a unique linear combination of $C_{\mu\nu\rho\sigma}[F]$ and $D_{\mu\nu\rho\sigma}[F]$ that is (anti-)self-dual. Those combinations must determine the corresponding chiral components of the Weyl tensor up to a proportionality factor:

$$W^{\pm}_{\mu\nu\rho\sigma} = \frac{c}{S_{\pm}}\left(C_{\mu\nu\rho\sigma}[F] \mp i\,D_{\mu\nu\rho\sigma}[F]\right) = \frac{2\,c}{S_{\pm}}\,C_{\mu\nu\rho\sigma}[F^{\pm}]\ . \tag{4.34}$$

The – possibly complex – functional coefficients $S_{\pm}$ are related to the complex scalar field S in (4.10). Matching by chiralities, we find that $S_{-} = S$ and $S_{+} = \bar{S}$. The overall proportionality constant c, which was not explicitly written in (4.10), replaces gauge couplings and charges by their gravitational counterparts. It is also simple to check that (4.34) is equivalent to (3.92), since

$$C^{\mu\nu\rho\sigma}[F] = \frac{3}{2}\,\mathcal{P}^{\mu\nu\rho\sigma}_{\tau\lambda\eta\omega}\,F^{\tau\lambda}\,F^{\eta\omega}\ . \tag{4.35}$$

Although in general c/S is complex, it is real in some cases, like Schwarzschild and the C-metric, where one has

$$W_{\mu\nu\rho\sigma} = \frac{c}{S}\left\{2\,F_{\mu\nu}\,F_{\rho\sigma} - 2\,F_{\mu[\rho}\,F_{\sigma]\nu} + g_{\mu[\rho}\,g_{\sigma]\nu}\,F^2 + 6\,F_{\lambda[\mu}g_{\nu][\rho}F^{\lambda}{}_{\sigma]}\right\}. \tag{4.36}$$

4.1.4 The Ehlers group and EM duality

The tensorial form of the Weyl double copy is particularly suited to study how symmetries in gravity are mapped to symmetries in electromagnetism. An example of a symmetry mapping was given in section 3.3.1, where we saw the effect of turning on an electric-magnetic duality rotation in the single copy. For the gauge field, this rotates the electric and magnetic charges of a dyon. When double copied, the effect is to rotate the mass and NUT parameters of the Taub-NUT solution. One can ask what this transformation corresponds to in the general relativity literature. Our proposal is that it corresponds to an Ehlers transformation [195].

The Ehlers transformation can be best described in the context of the projection formalism developed by Geroch [196]. This formalism defines a map from a stationary spacetime to a three-dimensional manifold using the trajectories of the timelike Killing vector ξ_{μ}. This decomposition provides advantageous simplifications at the level of the field equations. It also highlights symmetries of the solution on the lower dimensional space, that can be used to generate new stationary spacetimes. In particular, the Ehlers transformation is a one-parameter transformation that maps static vacuum solutions to stationary vacuum solutions. In general, the transformed spacetime suffers from a

NUT-like singularity.[2]

The original Ehlers transformation is limited to spacetimes admitting a Killing vector field that is timelike everywhere. This limits its applicability to the outside region of rotating black holes, for example. There exists a generalisation of the Ehlers prescription that bypasses the projection formalism and allows for more general Killing vector fields: the *spacetime Ehlers group* [197], which we now summarise briefly.

Given a Killing vector field $\xi = \xi^\mu \partial_\mu$ and one-form $W = W_\mu dx^\mu$ on a Lorentzian manifold with metric $g_{\mu\nu}$, satisfying the vacuum Einstein equations, the spacetime Ehlers group is defined in [197] by the transformation

$$g_{\mu\nu} \to \Omega^2 g_{\mu\nu} - 2\xi_{(\mu}W_{\nu)} - \frac{\lambda}{\Omega^2} W_\mu W_\nu, \tag{4.37}$$

where $\Omega^2 \equiv \xi^\mu W_\mu + 1 \geq 1$, and the inequality holds over the whole geometry. Define the two-form $F_{\mu\nu} = 2\,\partial_{[\mu}\xi_{\nu]}$ (note that we have a factor of 2 here, and a factor of 1/2 in the definitions of the (anti-)self-dual parts of F, in comparison with [197]). Define also the twist potential $\omega_\mu = \epsilon_{\mu\nu\sigma\rho}\xi^\nu \nabla^\sigma \xi^\rho$ and the Killing vector norm $\lambda = -\xi^\mu \xi_\mu$. Then the Ernst one-form

$$\sigma_\mu := 2\xi^\nu F^+_{\nu\mu} = \nabla_\mu \lambda - i\omega_\mu \tag{4.38}$$

is closed, following from the vanishing of the Ricci tensor, and so locally $\sigma_\mu = \nabla_\mu \sigma$ for some complex function σ called Ernst scalar potential. To define the spacetime Ehlers group we need this to hold also globally.

There is a second global integrability condition. Consider the two-form

$$-\,4\gamma\,\mathrm{Re}[(\gamma\bar{\sigma} + i\delta)\,F^+_{\mu\nu}\ , \tag{4.39}$$

where the bar denotes complex conjugation, and γ and δ are non-simultaneously vanishing real constants. This form is closed by construction, but it must be globally exact for any value of γ and δ. That is, we require the existence of a form W satisfying

$$2\nabla_{[\mu}W_{\nu]} = -4\gamma\,\mathrm{Re}[(\gamma\bar{\sigma} + i\delta)\,F^+_{\mu\nu}]\,, \tag{4.40a}$$

$$\Omega^2 := \xi^\mu W_\mu + 1 = (i\gamma\sigma + \delta)(-i\gamma\bar{\sigma} + \delta)\,, \tag{4.40b}$$

where γ and δ, as a pair, fix the gauge of W. After repeated action, the transformation

[2] There is not a global agreement in the literature regarding what is referred to as Ehlers transformation. Some authors prefer to reserve the term for the transformation that applies exclusively to vacuum solutions, while others also include transformations for electrovac solutions. We refer the reader to section 34.1 of [171] for a more complete discussion.

defines an $SL(2,\mathbb{R})$ group action on the Ernst scalar by the Möbius map

$$\sigma \to \frac{\alpha\sigma + i\beta}{i\gamma\sigma + \delta}, \qquad \text{where} \quad \beta\gamma + \alpha\delta = 1. \tag{4.41}$$

The self-dual part of the Killing tensor transforms as

$$F^+_{\mu\nu} \to \frac{1}{(i\gamma\sigma + \delta)^2}\Big(\Omega^2 F^+_{\mu\nu} - W_{[\mu}\sigma_{\nu]}\Big)\,, \tag{4.42}$$

where W, σ are the one-forms defined above. The self-dual part of the Weyl tensor transforms as

$$W^+_{\mu\nu\rho\sigma} \to \frac{1}{(i\gamma\sigma + \delta)^2} P^{\alpha\beta}_{\mu\nu} P^{\gamma\delta}_{\rho\sigma}\left(W^+_{\alpha\beta\gamma\delta} - \frac{6i\gamma}{i\gamma\sigma + \delta}\Big(F^+_{\alpha\beta}F^+_{\gamma\delta} - \frac{1}{3}I_{\alpha\beta\gamma\delta}(F^+)^2\Big)\right)\,, \tag{4.43}$$

where in our conventions $I_{\mu\nu\rho\sigma} = \frac{1}{4}(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma} + \epsilon_{\mu\nu\rho\sigma})$ is the canonical metric in the space of self-dual two-forms and $P^{\alpha\beta}_{\mu\nu} = \Omega^2\delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\mu\xi^\beta W_\nu - \xi^\alpha W_\mu\delta^\beta_\nu$. This formalism can be used to check that the Kerr-Taub-NUT family is invariant under the Ehlers group [197].

Taub-NUT black holes

Let us now consider applying these arguments in the context of the Weyl double copy described earlier. Let us start from the Taub-NUT metric, in the real form

$$ds^2 = -f(r)(dt - 2N\cos\theta d\phi)^2 + f(r)^{-1}dr^2 + (r^2 + N^2)d\Omega^2_{(2)}, \tag{4.44}$$

where

$$f(r) = \frac{(r - r_+)(r - r_-)}{r^2 + N^2}, \qquad r_\pm = M \pm r_0, \qquad r_0^2 = M^2 + N^2. \tag{4.45}$$

Next, consider the Killing vector $\xi = \partial_t$. Its associated two-form $F_{\mu\nu} = 2\,\partial_{[\mu}\xi_{\nu]}$ is

$$\begin{aligned} F = {}& \frac{2M\left(r^2 - N^2\right) + 4N^2 r}{\left(N^2 + r^2\right)^2}\, dt \wedge dr + \frac{4N\cos(\theta)\left(M\left(r^2 - N^2\right) + 2N^2 r\right)}{\left(N^2 + r^2\right)^2}\, dr \wedge d\phi \\ & + \frac{2\,N\sin(\theta)\left(r(2M - r) + N^2\right)}{N^2 + r^2}\, d\theta \wedge d\phi\ . \end{aligned} \tag{4.46}$$

This solves the Maxwell equations on the Taub-NUT background. In the flat limit, it gives the single copy of Taub-NUT, found in [99, 173] and solves the flat-background Maxwell equations. The Ernst one-form is obtained from its definition

$$\sigma_\mu := 2\xi^\nu F^+_{\nu\mu} = \frac{2(M - i\,N)}{(r - i\,N)^2}\,\delta^r_\mu\ . \tag{4.47}$$

In [197], it was proven that the Ernst one-form is exact, $\sigma_\mu = \partial_\mu \sigma$, and the integration constant can be chosen such that $\mathrm{Re}(\sigma) = -\xi^\mu \xi_\mu$, giving

$$\sigma = 1 - \frac{2(N + iM)}{N + i\,r} \ . \tag{4.48}$$

Additionally, the fact that (4.44) has a Weyl double copy structure implies that

$$\begin{aligned} W^+_{\alpha\beta\gamma\delta} &= -\frac{6}{c-\sigma}\left(F^+_{\alpha\beta}F^+_{\gamma\delta} - \frac{(F^+)^2}{3} I_{\alpha\beta\gamma\delta}\right) \ , \\ (F^+)^2 &= A(c-\sigma)^4 \ , \end{aligned} \tag{4.49}$$

with $c = 1$ and $A = -(4(M - iN))^{-1}$. Next, W is found by solving (4.40). After this, we can transform the original metric into (4.37)

$$g'_{\mu\nu} = \Omega^2\, g_{\mu\nu} - 2\xi_{(\mu}W_{\nu)} + \frac{\xi^\sigma \xi_\sigma}{\Omega^2} W_\mu W_\nu \ . \tag{4.50}$$

In order to interpret this new metric, it is convenient to define polar coordinates in the parameter space

$$\rho = \sqrt{\delta^2 + \gamma^2} \ , \qquad \tan\zeta = \frac{\delta}{\gamma} \ . \tag{4.51}$$

Performing a charge redefinition and a change of coordinates

$$\begin{aligned} \begin{pmatrix} M' \\ N' \end{pmatrix} &= \begin{pmatrix} \cos 2\zeta & -\sin 2\zeta \\ \sin 2\zeta & \cos 2\zeta \end{pmatrix} \begin{pmatrix} \rho\, M \\ \rho\, N \end{pmatrix} \ , \\ t' = \frac{t}{\rho} \ , &\qquad r' = \rho\, r + M'(1 - \cos 2\zeta) - N' \sin 2\zeta \ , \end{aligned} \tag{4.52}$$

the metric simplifies to

$$\begin{aligned} ds'^2 &= -f(r')(dt' - 2N' \cos\theta\, d\phi)^2 + \frac{dr'^2}{f(r')} + (r'^2 + N'^2)d\Omega_2^2 \ , \\ f(r') &= \frac{r'^2 - 2M'\, r' - N'^2}{r'^2 + N'^2} \ . \end{aligned} \tag{4.53}$$

Hence, it is still a member of the Taub-NUT family. The self-dual part of $F_{\mu\nu}$ transforms as (4.42). The integrated Ernst one-form transforms as

$$\sigma' = \frac{1}{\delta^2 + \gamma^2} \frac{\delta\sigma + i\,\gamma}{i\gamma\sigma + \delta} \ . \tag{4.54}$$

After the transformation, (4.49) also holds with

$$c' = \frac{1}{\gamma^2 + \delta^2} \qquad A' = -\frac{(\delta + i\,\gamma)^4}{4(M - i\,N)} \ , \tag{4.55}$$

in agreement with (54) in [197].

Let us now study the implications for the single copy. The single copy of (4.44) can be written in flat spherical coordinates $(\tilde{t}, \tilde{r}, \theta, \phi)$

$$F_{\rm sc} = -\frac{M}{\tilde{r}^2}\, d\tilde{t} \wedge d\tilde{r} - N\, \sin\theta\, d\theta \wedge d\phi\ . \tag{4.56}$$

Hence, the single copy of the transformed space-time, on the same background reads

$$F'_{\rm sc} = -\frac{M'}{\tilde{r}^2}\, d\tilde{t} \wedge d\tilde{r} - N'\, \sin\theta\, d\theta \wedge d\phi\ . \tag{4.57}$$

Using (4.52), it can be checked that the transformation in terms of the Ehlers group parameters is

$$F'_{\rm sc} = \rho\cos(2\zeta)\, F_{\rm sc} + \rho\sin(2\zeta) \star F_{sc}\ . \tag{4.58}$$

This corresponds to an electric-magnetic duality rotation and a rescaling by ρ. Both transformations are contained in the electric-magnetic duality, where the rescaling can be interpreted as the transformation of the gauge coupling [198].

The zeroth copy is affected similarly, transforming using $M \to M'$, $N \to N'$, meaning that the Weyl double copy relation

$$W^+_{\mu\nu\rho\sigma} = \frac{2}{\sigma^+} C_{\mu\nu\rho\sigma}[F^+]\,, \tag{4.59}$$

where $\sigma^+ = \sigma - c = -2(N + iM)/(N + ir)$, is preserved: (4.59) transforms directly to the double copy in the transformed spacetime

$$W'^{+}_{\mu\nu\rho\sigma} = \frac{2}{\sigma^{+\prime}} C_{\mu\nu\rho\sigma}[F'^{+}]\,, \tag{4.60}$$

where F^+ is the self-dual part of the single-copy Killing tensor, now defined using the shifted metric (and the same Killing vector, although note of course that the co-vector differs in the new spacetime), and the transformed Ernst scalar is

$$\sigma'^{+} = -\frac{2(M - iN)}{(\gamma - i\delta)(2\gamma M + N(\delta - i\gamma) + r(i\delta - \gamma))}. \tag{4.61}$$

We see that in terms of the action on the fields, a restricted set of the $SL(2, \mathbb{R})$ transformations act in this case, and the orbit is within the Taub-NUT class of metrics. The two degrees of freedom are realised by the rotation parameter ζ and scaling ρ. This result reproduces the behaviour observed in section 3.3.1 for $\rho = 1$ and generalises it for more general electromagnetic transformations. In [6], the authors discuss the action of the Ehlers group on more general type D examples, as well as a self-dual solution. In all cases, the double copy structure is preserved under the transformation. Reference [90] also studies the effects of the electric-magnetic duality on the double copy. Asymptotically, their findings suggest that it induces a supertranslation on the spacetime, which results in the same rotation between mass and NUT charge present in the Ehlers transformation. More recently, [199] proposed a new notion of gravitational duality that might extend the scope of the Ehlers transformation beyond the space of metrics that admit a Killing vector.

4.2 Type N

Type N spacetimes are another class of algebraically special metrics with prominent physical significance, as they describe the radiation region of isolated gravitational systems. As a necessary step in applying the classical double copy tools to gravitational-wave physics, we provide a systematic understanding of the status of the double copy for type-N radiative solutions, beyond the most special example of pp-waves. More precisely, we extend the curved Weyl double copy relation to most type N vacuum solutions, for which we show that $\Psi_{\alpha\beta\gamma\delta} = S^{-1}\phi_{(\alpha\beta}\phi_{\gamma\delta)}$; $\phi_{\alpha\beta}$ being a degenerate Maxwell spinor and S a scalar that satisfies the wave equation. Interestingly, this construction does not lead to a unique Maxwell field, since there is functional freedom associated with the scalar S.

For non-twisting radiative spacetimes, the Maxwell field and the scalar field solve the Maxwell equation and the wave equation, respectively, on Minkowski spacetime. This extends the Weyl double copy for this large class of spacetimes.

For twisting spacetimes, the Maxwell field and the scalar depend generically on the metric functions. Hence, they are solutions only on the curved spacetime. However, the standard double copy interpretation applies at the linearised level. This may be indicative of the fact that twisting solutions have an intrinsic non-Abelian nature, as one needs to go beyond linear (i.e. Abelianised) gauge solutions to capture the full metric. This is a speculative statement, as we lack a non-Abelian extension of the Weyl double copy to prove it.

Note that, while the double copy for scattering amplitudes involves two copies of non-Abelian gauge theory, the first step in that procedure is to consider the double copy of the asymptotic states, which for linearised gauge theory are solutions to the Maxwell equation. The fact that certain exact gravity solutions can be interpreted as a double copy of a Maxwell field means that they should be interpreted as coherent states, an exact extension of the linearised asymptotic states in scattering amplitudes. We saw how this works for a subfamily of type D spacetimes, but this reasoning indicates that a similar description should be possible for type N examples.

4.2.1 Spinor calculus

A lot of the intuition for the type D Weyl double copy came from the fact that the Weyl spinor has two principal spinors, both of them with multiplicity two. Each of the multiplicities corresponds to a copy of the Maxwell spinor. Type N can be seen as a degeneration of type D where the principal spinors align, yielding a single principal spinor with multiplicity four and one principal null direction $k^\mu \sim o_\alpha \bar{o}_{\dot\alpha}$. Choosing a spinor basis adapted to the principal null direction, the Newman-Penrose Weyl scalars all vanish except Ψ_4, and the Weyl spinor takes the simple form

$$\Psi_{\alpha\beta\gamma\delta} = \Psi_4\, o_\alpha o_\beta o_\gamma o_\delta. \tag{4.62}$$

In spinor language, the curved background Weyl double copy relation is

$$\Psi_{\alpha\beta\gamma\delta} = \frac{1}{S}\, \phi_{(\alpha\beta}\phi_{\gamma\delta)}, \tag{4.63}$$

for some scalar S and Maxwell spinor $\phi_{\alpha\beta}$. Note that $\phi_{\alpha\beta}$ satisfies the Maxwell equation (1.70) in the fixed curved background metric, but it is viewed as a test field that does not back-react on the geometry. From (4.62), it follows that the NP Maxwell scalars all vanish except ϕ_2, and we have $\phi_{\alpha\beta} = \phi_2\, o_{(\alpha} o_{\beta)}$. Thus the type N double copy relation is

$$\Psi_4 = \frac{1}{S}\, (\phi_2)^2. \tag{4.64}$$

The Maxwell 2-spinor is degenerate, which means that the electromagnetic field is null, i.e. the electric and magnetic fields are perpendicular and of equal magnitude. An example of a null electromagnetic field is that of a plane electromagnetic wave in flat spacetime. Now we must consider whether such a relation (4.63) exists. Expanding out the Bianchi identity (1.76) by substituting (4.62) gives two equations:

$$o_\alpha \nabla^{\alpha\dot\alpha} \log \Psi_4 + 4\, o_\alpha \iota^\beta \nabla^{\alpha\dot\alpha} o_\beta - \iota_\alpha o^\beta \nabla^{\alpha\dot\alpha} o_\beta = 0 \tag{4.65}$$

and $o_\alpha o^\beta \nabla^{\alpha\dot\alpha} o_\beta = 0$. The second equation is again the statement that the null congruence generated by the PND is geodesic and shear-free, i.e. $\kappa = \sigma = 0$, which follows from the Goldberg-Sachs theorem.[3] Expanding out the Maxwell equation in a similar fashion gives

$$o_\alpha \nabla^{\alpha\dot\alpha} \log \phi_2 + 2 o_\alpha \iota^\beta \nabla^{\alpha\dot\alpha} o_\beta - \iota_\alpha o^\beta \nabla^{\alpha\dot\alpha} o_\beta = 0, \tag{4.66}$$

as well as the same equation above that is equivalent to $\kappa = \sigma = 0$. Now, substituting

[3] For later convenience, let us introduce the NP spin coefficients κ, σ, τ and ρ [125]

$$\kappa = o^\alpha o^\beta \tilde{o}^{\dot\alpha} \nabla_{\alpha\dot\alpha} o_\beta\ , \qquad \sigma = o^\alpha o^\beta \tilde{\iota}^{\dot\alpha} \nabla_{\alpha\dot\alpha} o_\beta\ ,$$
$$\rho = o^\alpha \iota^\beta \tilde{o}^{\dot\alpha} \nabla_{\alpha\dot\alpha} o_\beta\ , \qquad \tau = o^\alpha \iota^\beta \tilde{\iota}^{\dot\alpha} \nabla_{\alpha\dot\alpha} o_\beta\ .$$

$\Psi_4 = (\phi_2)^2/S$ into (4.65) and simplifying this using (4.66) gives

$$o_\alpha \nabla^{\alpha\dot{\alpha}} \log S - \iota_\alpha o^\beta \nabla^{\alpha\dot{\alpha}} o_\beta = 0. \tag{4.67}$$

There is a clear structure in equations (4.65)–(4.67), where the coefficient of the middle term is the rank of the respective spinor. Equation (4.67) translates to

$$k \cdot \nabla \log S - \rho = 0, \qquad m \cdot \nabla \log S - \tau = 0, \tag{4.68}$$

where $(k, n, m, \bar{m})$ form an NP null frame. A simple calculation shows that the integrability condition on equations (4.68) are satisfied, which means that they are simple integral equations that can always be solved. Thus, we are guaranteed the existence of a scalar S satisfying these equations, which then gives a Maxwell field $\phi_2 = \sqrt{\Psi_4 \, S}$. In tensor language, this Maxwell spinor translates to a field strength (the single copy) of the form[4]

$$F = \phi_2 \, k^\flat \wedge m^\flat + \bar{\phi}_2 \, k^\flat \wedge \bar{m}^\flat. \tag{4.69}$$

This establishes the curved Weyl double copy relation for type N vacuum solutions.

Furthermore, it is simple to show using (4.67) that S solves the wave equation. First, we write

$$\Box S = \nabla^{\alpha\dot{\alpha}} \nabla_{\alpha\dot{\alpha}} S = \epsilon_{\alpha\beta} \nabla^{\alpha\dot{\alpha}} \nabla^\beta{}_{\dot{\alpha}} S = 2 \, \iota_\alpha o_\beta \nabla^{\alpha\dot{\alpha}} \nabla^\beta{}_{\dot{\alpha}} S. \tag{4.70}$$

Then, we can integrate by parts to obtain

$$\Box S = 2 \, \iota_\alpha \nabla^{\alpha\dot{\alpha}} (S \, \iota_\beta o^\gamma \nabla^\beta{}_{\dot{\alpha}} o_\gamma) - 2 \, \iota_\alpha \nabla^{\alpha\dot{\alpha}} o_\beta \, \nabla^\beta{}_{\dot{\alpha}} S \ , \tag{4.71}$$

where in the first term we have applied (4.67). Expanding the derivative and using (4.3) in the second term

$$\begin{aligned} \Box S = 2 \, S \Big[& - \iota_\alpha \iota_\beta o^\gamma \Box_{\alpha\beta} o_\gamma + (\iota_\alpha \nabla^{\alpha\dot{\alpha}} o^\gamma)(\iota_\beta \nabla^\beta{}_{\dot{\alpha}} o_\gamma) \\ & + (\iota_\alpha \nabla^{\alpha\dot{\alpha}} \iota_\gamma)(o^\beta \nabla^\gamma{}_{\dot{\alpha}} o_\beta) - (\iota_\alpha \iota^\beta \nabla_{\alpha\dot{\alpha}} o_\beta)(o_\delta \nabla^\delta{}_{\dot{\alpha}} S) \Big] \ . \end{aligned} \tag{4.72}$$

The last three terms cancel out upon application of (4.67) in the last term and (4.3) in the second and third terms to expand the contractions with index γ. Finally, making use of of (4.17), we find

$$\Box S = -2 \, S \, X_{\alpha\beta\gamma\delta} \iota^\alpha \iota^\beta o^\gamma o^\delta \ . \tag{4.73}$$

Following the type D Weyl double copy, we might want to solve the wave equation on a flat background. In that case, $X_{\alpha\beta\gamma\delta} = 0$. However, we could also choose the type N

[4] The musical notation emphasises that $k^\flat$ is the covector (or one-form) with components $k^\flat_\mu = g_{\mu\nu} k^\nu$.

Type D

$$\nabla^{\alpha\dot{\alpha}}\Psi_{\alpha\beta\gamma\delta} = 0$$

$$\Downarrow$$

$$\nabla^{\alpha\dot{\alpha}}\phi_{\alpha\beta} = 0 \quad \text{(flat and curved)}$$
$$\Box S = 0 \quad \text{(flat)}$$

Type N

$$\nabla^{\alpha\dot{\alpha}}\Psi_{\alpha\beta\gamma\delta} = 0$$
$$o_\alpha \nabla^{\alpha\dot{\alpha}} S = \iota_\alpha o^\beta \nabla^{\alpha\dot{\alpha}} o_\beta$$

$$\Downarrow$$

$$\nabla^{\alpha\dot{\alpha}}\phi_{\alpha\beta} = 0 \quad \text{(flat* and curved)}$$
$$\Box S = 0 \quad \text{(flat* and curved)}$$

Figure 4.1: Comparison between the Weyl double copy for type D and type N solutions. For type D, a Weyl spinor satisfying the Bianchi identity determines a Maxwell spinor and a harmonic scalar field, whereas for type N the extra condition (4.67) *makes the relation non-unique. *There is no flat background interpretation for twisting type N solutions.*

background $X_{\alpha\beta\gamma\delta} = \Psi_4 o_\alpha o_\beta o_\gamma o_\delta$.[5] In either case, we obtain that

$$\Box S = 0 \, . \tag{4.74}$$

These results mirror those that exist for type D solutions, see figure 4.1 for a comparison. Although the double copy formula (4.10) holds for both type D and type N solutions, there are some differences. A type D Weyl spinor uniquely determines a Maxwell spinor and a scalar field. On the other hand, a type N Weyl spinor only determines a Maxwell spinor once we specify a scalar field satisfying (4.67). This introduces a non-uniqueness which will be discussed at the end of the section.

Another difference is that the type N scalar solves both the flat and curved wave equations, whereas its type D counterpart is only valid on flat space. However, it is not guaranteed that there exists an appropriate flat background for all type N solutions. In the next section, we will study all subclasses of type N solutions to check the double copy more explicitly and discuss the flat space interpretation.

4.2.2 Type N vacuum solutions

Type N vacuum solutions are classified in terms of the optical properties of the congruence generated by the PND. We have that $\kappa = \sigma = 0$; the properties that remain are parametrised by $\rho = -(\Theta + i\,\omega)$ with Θ denoting the expansion of the congruence and ω denoting its twist. The different cases lead to three distinct classes of solutions:

- Kundt solutions: $\Theta = 0$, which implies that $\omega = 0$ [6].
- Robinson-Trautman solutions: $\Theta \neq 0,\ \omega = 0$.
- Twisting solutions: $\Theta \neq 0,\ \omega \neq 0$.

[5] We are assuming that there is no cosmological constant.

[6] Substituting $\Theta = \sigma = R_{\mu\nu} = 0$ into the Raychaudhuri equation $k\cdot\nabla\Theta - \omega^2 + \Theta^2 + \sigma\bar{\sigma} + \frac{1}{2}R_{\mu\nu}k^\mu k^\nu = 0$ gives $\omega = 0$.

Choosing a null frame for which k is the PND, so that $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$, we consider each case separately.

Kundt solutions

There are two kinds of type N Kundt solutions, both corresponding to plane-fronted wave solutions [7]. Plane-fronted waves with parallel propagation (pp-waves) are given by the metric

$$\mathrm{d}s^2 = -2\mathrm{d}u\,(\mathrm{d}v + H\mathrm{d}u) + 2\mathrm{d}z\mathrm{d}\bar{z}, \tag{4.75}$$

with $H(u, z, \bar{z}) = f(u, z) + \bar{f}(u, \bar{z})$ for general functions f. Choosing

$$k = \partial_v, \quad n = \partial_u - H\,\partial_v, \quad m = \partial_z, \tag{4.76}$$

one has $\rho = \tau = 0$ and so (4.68) implies $S = S(u, \bar{z})$, while the Weyl scalar $\Psi_4 = \partial_{\bar{z}}^2 \bar{f}$, so (4.64) implies that

$$\phi_2 = \sqrt{\partial_{\bar{z}}^2 \bar{f}\, S(u, \bar{z})}. \tag{4.77}$$

The other class of plane-fronted waves is given by

$$\mathrm{d}s^2 = -2\mathrm{d}u\Big(\mathrm{d}v + W\mathrm{d}z + \bar{W}\mathrm{d}\bar{z} + H\mathrm{d}u\Big) + 2\mathrm{d}z\mathrm{d}\bar{z}, \tag{4.78}$$

with $W(v, z, \bar{z}) = -2v\,(z + \bar{z})^{-1}$ and

$$H(u, v, z, \bar{z}) = \big[f(u, z) + \bar{f}(u, \bar{z})\big]\,(z + \bar{z}) - \frac{v^2}{(z + \bar{z})^2};$$

again $f(u, z)$ is arbitrary. Choosing

$$k = \partial_v,\ n = \partial_u - (H + W\bar{W})\,\partial_v + \bar{W}\partial_z + W\partial_{\bar{z}},\ m = \partial_z,$$

one has $\rho = 0$, $\tau = 2\beta = -(z + \bar{z})^{-1}$, so (4.68) gives $S = \zeta(u, \bar{z})/(z + \bar{z})$. The Weyl scalar $\Psi_4 = (z + \bar{z})\,\partial_{\bar{z}}^2 \bar{f}$, so (4.64) implies that

$$\phi_2 = \sqrt{\partial_{\bar{z}}^2 \bar{f}\,\zeta(u, \bar{z})}. \tag{4.79}$$

Given that the only non-zero components of $F_{\mu\nu}$ are for $\mu\nu = [uz]$ and $[u\bar{z}]$, the simple form of the relevant components of $g^{\mu\nu}$ and the fact that $g = 1$ give

$$\begin{aligned}\nabla_\nu F^{\mu\nu} &= \frac{1}{\sqrt{|g|}}\partial_\nu\left(\sqrt{|g|}\,g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma}\right)\\ &= \partial_\nu\left(\eta^{\mu\rho}\eta^{\nu\sigma}F_{\rho\sigma}\right) = 0.\end{aligned} \tag{4.80}$$

[7] See Theorem 31.2 of [171].

On the other hand, S does not depend on $f(u,z)$ or $\bar{f}(u,\bar{z})$, meaning that it must solve the wave equation on any member of the family. In particular, it solves the wave equation on Minkowski spacetime. This implies that the Maxwell and the scalar fields also satisfy their equations on Minkowski spacetime, establishing the Weyl double copy for type N Kundt solutions.

Robinson-Trautman solutions

Type N Robinson-Trautman solutions take the form [8]

$$\mathrm{d}s^2 = -H\,\mathrm{d}u^2 - 2\mathrm{d}u\,\mathrm{d}r + \frac{2r^2}{P^2}\,\mathrm{d}z\,\mathrm{d}\bar{z}\,, \tag{4.81}$$

with $H(u,r,z,\bar{z}) = \mathcal{K} - 2r\,\partial_u \log P$ (where $\mathcal{K} = 0, \pm 1$) and $2P^2\partial_z\partial_{\bar{z}} \log P(u,z,\bar{z}) = \mathcal{K}$. Choosing

$$k = \partial_r, \quad n = \partial_u - \tfrac{1}{2}H\partial_r, \quad m = -\frac{P}{r}\,\partial_z, \tag{4.82}$$

one has $\rho = -r^{-1}$, $\tau = 0$, so (4.68) gives $S = -\zeta(u,\bar{z})/r$. Now $\Psi_4 = -\frac{P^2}{r}\partial_u\left(\frac{\partial_{\bar{z}}^2 P}{P}\right)$, so (4.64) determines that

$$\phi_2 = \frac{P}{r}\sqrt{\zeta(u,\bar{z})\,\partial_u\left(\partial_{\bar{z}}^2 P/P\right)}. \tag{4.83}$$

As an example, consider Robinson-Trautman solutions with $\mathcal{K} = 0$ in (4.81). Writing $P = e^W$ we have $\partial_z\partial_{\bar{z}}\,W = 0$ and hence $W = w(u,z) + \bar{w}(u,\bar{z})$, implying that $\Psi_4 = -P^2/r\,\partial_u[\partial_{\bar{z}}^2\bar{w}(u,\bar{z}) + (\partial_{\bar{z}}\bar{w}(u,\bar{z})^2]$. We can obtain type N solutions of the Maxwell equation in the Robinson-Trautman background by taking

$$A = \gamma(u,z,\bar{z})\,du, \tag{4.84}$$

where $\partial_z\partial_{\bar{z}}\,\gamma = 0$ and hence $\gamma = h(u,z) + \bar{h}(u,\bar{z})$. Thus from (4.69) we have $\phi_2 = -P/r\,\partial_{\bar{z}}\,\bar{h}(u,\bar{z})$. Plugging into (4.64) we have

$$\partial_u[\partial_{\bar{z}}^2\bar{w}(u,\bar{z}) + \left(\partial_{\bar{z}}\bar{w}(u,\bar{z})\right)^2] = -\frac{1}{rS}\left(\partial_{\bar{z}}\bar{h}(u,\bar{z})\right)^2, \tag{4.85}$$

and so indeed we have that $S = -\zeta/r$, where ζ is a function only of u and $\bar{z}$, as required in the general result stated above.

As with Kundt solutions, the only non-zero components of $F_{\mu\nu}$ are for $\mu\nu = [uz]$ and $[u\bar{z}]$. As before, using the fact that $\sqrt{|g|} = r^2/P^2$ and the relevant components of $g^{\mu\nu}$, it can be shown that (4.80) holds. Once again, S is independent of P and solves the wave equation on any member of the family (4.81), including Minkowski. Hence, both $F_{\mu\nu}$ and S satisfy their equations also on the flat background, establishing the Weyl double copy for Robinson-Trautman solutions.

[8] See Theorem 28.1 and Section 28.1.2 of [171].

Twisting solutions

Type N solutions with non-vanishing twist are more complicated, with only one explicit solution known [200]. The general metric is given by [9]

$$\mathrm{d}s^2 = -2(\mathrm{d}u + L\,\mathrm{d}z + \bar{L}\,\mathrm{d}\bar{z})\Big[\mathrm{d}r + W\mathrm{d}z + \bar{W}\mathrm{d}\bar{z} + H\left(\mathrm{d}u + L\,\mathrm{d}z + \bar{L}\,\mathrm{d}\bar{z}\right)\Big] + \frac{2}{P^2|\rho|^2}\mathrm{d}z\mathrm{d}\bar{z}, \tag{4.86}$$

$$\begin{aligned}
&\rho^{-1} = -(r + i\Sigma), \quad 2i\,\Sigma(u,z,\bar{z}) = P^2(\bar{\partial}L - \partial\bar{L}),\\
&W(u,r,z,\bar{z}) = \rho^{-1}\,\partial_u L + i\partial\Sigma, \quad \partial = \partial_z - L\partial_u,\\
&H(u,r,z,\bar{z}) = \frac{1}{2}K - r\partial_u \log P,
\end{aligned}$$

with $K = 2P^2\,\mathrm{Re}\left[\partial(\bar{\partial}\log P - \partial_u\bar{L})\right]$. There exists a residual gauge freedom to choose $P = 1$, but we shall not yet impose this choice. The solution is determined by the complex scalar L, for which the field equations and type N condition impose

$$\Sigma K + P^2\,\mathrm{Re}\left[\partial\bar{\partial}\Sigma - 2\partial_u\bar{L}\partial\Sigma - \Sigma\partial_u\partial\bar{L}\right] = 0, \quad \partial I = 0,$$

and $\partial_u I \neq 0$, with $I = \bar{\partial}(\bar{\partial}\log P - \partial_u\bar{L}) + (\bar{\partial}\log P - \partial_u\bar{L})^2$. Choosing

$$k = \partial_r, \quad n = \partial_u - H\,\partial_r, \quad m = -P\bar{\rho}\,(\partial - W\partial_r), \tag{4.87}$$

ρ is as defined above, while $\tau = 0$. Equation (4.68) then implies that $S = \rho\,\chi(u,z,\bar{z})$, with χ satisfying

$$\partial\chi - \partial_u L\,\chi = 0. \tag{4.88}$$

Defining new coordinates $(v,w) = (I,z)$, the above equation can be solved using the method of characteristics ($I =$ constant correspond to the characteristics)

$$\chi(v,w) = \zeta(I)\,e^{\int\left[\left(\frac{\partial I(u,z)}{\partial u}\right)(v,w')\times\frac{\partial L(v,w')}{\partial v}\right]dw'}, \tag{4.89}$$

with $\zeta(I)$ arbitrary. The Weyl scalar $\Psi_4 = \rho\,P^2\,\partial_u I$, and so (4.64) implies

$$\phi_2 = \rho\,P\,\sqrt{\partial_u I\,\chi(u,z,\bar{z})}. \tag{4.90}$$

Only one twisting type N solution, found by Hauser [200], is known explicitly. It lacks a clear physical interpretation as it is not asymptotically flat and can not be

[9] See Chapter 29 of Ref. [171].

interpreted as the radiation of an isolated source. The metric functions are given by

$$P = (z+\bar{z})^{3/2}\, f(t), \quad t \equiv \frac{u}{(z+\bar{z})^2}, \quad L = 2i(z+\bar{z}),$$

where f satisfies $16(1+t^2)f''(t) + 3f(t) = 0$, which is a hypergeometric equation, and I turns out to be given by

$$I = \frac{3}{2[(z+\bar{z})^2 - iu]}. \tag{4.91}$$

The solution to (4.68) is

$$S = \rho\,\zeta(I)\ , \tag{4.92}$$

where $\zeta(I)$ is arbitrary. As expected, this is consistent with the general result (4.89). The Weyl scalar is $\Psi_4 = (2i/3)\,\rho P^2\, I^2$, implying that

$$\phi_2 = \rho\, P\, I \sqrt{\frac{2\,i\,\zeta(I)}{3}}\ . \tag{4.93}$$

As a further remark about the twisting type N solutions, we note that if the gauge freedom to set $P = 1$ is employed, the metric is specified purely in terms of the function $L(u, z, \bar{z})$, and the type N and Ricci flat conditions may be succinctly condensed down to just

$$\partial I = 0, \quad \mathrm{Im}(\bar{\partial}\bar{\partial}\partial\partial L) = 0, \quad \text{where} \quad I = -\partial_u \bar{\partial}\bar{L}. \tag{4.94}$$

The Weyl curvature is given by $\Psi_4 = \rho\, \partial_u I$.

In contrast to non-twisting solutions, the second equality in (4.80) does not hold for twisting solutions. Therefore, while there is a curved Weyl double copy relation, in this case it does not translate to a relation where the Maxwell field and the scalar can be thought of as Minkowski fields, unless we consider all the fields (gravity, Maxwell and scalar) at the linearised level. This may be indicative of the fact that twisting solutions have an intrinsic non-Abelian nature.

4.2.3 Non-uniqueness

In all the cases above, neither the Maxwell field nor the scalar field are uniquely determined. They are fixed only up to an arbitrary function of some of the coordinates, which we are free to choose. This contrasts with the Weyl double copy for vacuum type D solutions, for which, in a spinor basis adapted to the principal null directions, we have $S^3 \propto (\phi_2)^{3/2} \propto \Psi_4$, where the proportionality is up to complex parameters [99]; hence the Maxwell and scalar fields are functionally fixed. This feature is related to the fact that vacuum type D spacetimes are fully determined up to a few parameters, whereas vacuum type N spacetimes (of any class, as seen above) have functional freedom. By analogy, there is additional freedom in the Maxwell and scalar fields in the curved background.

In considering a special choice, we may ask whether it is possible to choose ϕ_2 and S to be given by specific powers of Ψ_4, as in the type D case, i.e. there exists some constant a such that $\phi_2 \propto (\Psi_4)^a$ and $S \propto (\Psi_4)^{2a-1}$. The functional dependence of the results above implies that this possibility holds only for Kundt solutions. For pp-waves, the power is actually undetermined, i.e. the relation above holds for any a. A simple choice is $a = 1/2$, where S is constant, and in fact this choice implies that Maxwell plane waves double copy to gravitational plane waves (ϕ_2 and Ψ_4 are functions of u only). For the other plane-fronted Kundt solutions, such a relation is possible for $a = 0$, in which case $S \propto (\Psi_4)^{-1}$. Analogously simple choices for the other type N classes are: $S \propto 1/r$ for Robinson-Trautman solutions and $S \propto \rho$ for twisting solutions.

Interestingly, pp-waves are the only type N solutions admitting a Killing 2-spinor [201], another feature that they share with type D solutions.

4.3 Asymptotic formulation of the Weyl double copy

In recent years there has been a surge in the interest in asymptotic symmetries in gauge theory and gravity. Particularly, since the developments reviewed in [202], which relate asymptotic symmetries to soft theorems and the memory effect. Given a particular family of metrics, with specific fall-off conditions near the conformal boundary, an asymptotic symmetry is a diffeomorphism that acts tangentially to the family. Although the usual diffeomorphisms do not have physical significance, asymptotic symmetries change the asymptotic data, modifying the boundary data of the characteristic value formulation of general relativity. Asymptotic symmetries also provide a way to define charges in general relativity. For asymptotically flat spacetimes, the standard boundary conditions are specified using Bondi coordinates. The group of diffeomorphisms that preserve the Bondi metric is bigger than the Poincaré group. It is denoted as the BMS group and contains supertranslations and superrotations. Supertransaltions are redefinitions of the retarded (or advanced) time coordinate on the celestial 2-sphere. On the other hand, superrotations correspond to conformal Killing generators of the 2-sphere that are not globally well-defined. Consequently, they relate different locally asymptotically flat backgrounds. Physically speaking, we will see that their effect can be interpreted as the appearance/snapping of cosmic strings.

Some relations between asymptotic symmetries and the double copy have been explored in the literature. Supertranslations have been related via the double copy to electric-magnetic duality [90,189]. Large diffeomorphisms in self-dual gravity and large gauge transformations in self-dual gauge theory have also been related [203], based on the fact that the self-dual theories provide a simple setting for the double copy [20]. The notions of celestial operators and amplitudes that arose recently are also revealing their own versions of the double copy [204–206]. Closer to the approach to be taken here, the double copy has been seen to arise in the characteristic value formulation of general relativity, particularly in the example of the Taub-NUT spacetime [189]. In [189], it was shown that the Dirac monopole solution can be viewed as a seed solution for the Taub-NUT spacetime in a characteristic value formulation and, moreover, that the proper and large gauge transformations of the monopole solution map onto proper and asymptotic diffeomorphisms of the Taub-NUT solution. More recently, [179] presented two prescriptions for realising the double copy asymptotically. The first one applies to radiative fields (Yang-Mills and NS-NS gravity fields), which are fully determined by characteristic data at null infinity. The second one is based on the dynamics of asymptotically flat solutions, deriving a map between the asymptotic Maxwell equations to the asymptotic Einstein equations. This map is motivated by the squaring of the 3-point amplitude seen in chapter 3, which makes it more closely related to the amplitudes double copy.

In this section, we will explore aspects of the classical double copy as seen asymptotically, near null infinity, to gain new insights into the connections between these various subjects. Instead of deriving a new double copy map, we will be guided by the asymptotic form of the Weyl double copy. We will see how the Weyl double copy relates in a simple manner various quantities that are familiar from the literature on asymptotic symmetries and the characteristic value formulation. While the Weyl double copy has only been successfully applied to certain algebraically special spacetimes, the fact that a much wider class of spacetimes is *asymptotically special* allows to extend its application, albeit in a restricted asymptotic framework.[10] An example of this was presented in [2], where the authors considered rotating STU supergravity black holes [207], which are algebraically general but asymptotically of type D; the Kerr-Newman solution with equal dyonic charges is a particular case. While a double copy interpretation of STU supergravity is not known, the asymptotic properties of these solutions were sufficient to show the presence of the double copy asymptotically.

We will study in detail the C-metric, which is exactly type D, so the Weyl double copy applies directly [99]. The metric can be interpreted as describing a pair of black holes uniformly accelerated in opposite directions. The double copy relates the C-metric to the Liénard-Wiechert field for uniformly accelerated point charges. The Kerr-Schild double copy is insufficient to deal on its own with this example (even in the multi-Kerr-Schild framework) due to its time dependence, but the Weyl double copy provides a complete prescription. Here, we will revisit the C-metric example of the double copy as seen asymptotically, based on Bondi coordinates, whose construction for the C-metric is not straightforward. Our study is also motivated by the interpretation of the C-metric as a non-linear solution associated to a superrotation [208]. This suggests that

[10] By asymptotically special we mean solutions which are algebraically special in the vicinity of null infinity, up to a given order in the Bondi expansion.

its single copy can be interpreted analogously as a large gauge transformation, a result that we are able to confirm.

4.3.1 Weyl double copy in Bondi coordinates

In the previous sections of this chapter, we argued that for type D and N vacuum solutions the Weyl double copy takes the form,[11]

$$\Psi_{\alpha\beta\gamma\delta} = \frac{3c}{S}\,\phi_{(\alpha\beta}\,\phi_{\gamma\delta)}, \tag{4.95}$$

where we have introduced the proportionality constant c of section 4.1.3, designed to absorb parameters. In the type D case

$$\frac{S}{3} = (-2\phi^{\alpha\beta}\phi_{\alpha\beta})^{1/4}\ , \tag{4.96}$$

which only satisfies the wave equation in a (at least locally) flat spacetime, rather than the curved background. For type N spacetimes, S solves the wave equation on the curved background, and also the wave equation on a Minkowski background in the case of non-twisting solutions.

We may introduce a Newman-Penrose null frame [125] $(k, n, m, \bar{m})$, where k and n are null vectors such that $k \cdot n = -1$ and m is a complex null vector, orthogonal to k and n, that parametrises the remaining two spacelike directions, so that $m \cdot \bar{m} = 1$. Therefore,

$$g^{\mu\nu} = -2\,k^{(\mu}n^{\nu)} + 2\,m^{(\mu}\bar{m}^{\nu)}, \tag{4.97}$$

or, in vielbein language

$$g_{\mu\nu} = E^a_\mu\,E^b_\nu\,\eta_{ab}, \qquad \eta_{ab} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{4.98}$$

with

$$E^0 = -n^\flat, \quad E^1 = -k^\flat, \quad E^m = \bar{m}^\flat, \quad E^{\bar{m}} = m^\flat. \tag{4.99}$$

Equivalently,

$$E_0 = k, \quad E_1 = n, \quad E_m = m, \quad E_{\bar{m}} = \bar{m}\,, \tag{4.100}$$

where $E^a = E^a_\mu\,\mathrm{d}x^\mu$ and $n^\flat = n_\mu\,\mathrm{d}x^\mu$, etc., and $E_a = E^\mu_a\,\partial_\mu$ and $n = n^\mu\,\partial_\mu$, etc. For convenience, we can use the above notation to project any tensor into this null frame.

[11] Some type III examples, in the linearised approximation, were discussed in [100, 101].

For example, for any 1-form X,

$$X_0 \equiv k^\mu X_\mu = -X^1, \quad X_1 \equiv n^\mu X_\mu = -X^0, \\ X_m \equiv m^\mu X_\mu = X^{\bar{m}}, \quad X_{\bar{m}} \equiv \bar{m}^\mu X_\mu = X^m. \tag{4.101}$$

A corresponding spinor basis (o, ι) may be constructed, so that

$$k \sim o^\alpha \bar{o}^{\dot{\alpha}}, \qquad n \sim \iota^\alpha \bar{\iota}^{\dot{\alpha}}, \qquad m \sim o^\alpha \bar{\iota}^{\dot{\alpha}}. \tag{4.102}$$

Note that this time we will not require the spinor basis to be aligned with principal null directions. In such a null frame, Maxwell and Weyl scalars may be defined by projecting the Maxwell field strength and the Weyl tensor into the null frame. In particular, we have

$$\phi_0 = F_{0m} = \phi_{\alpha\beta}\, o^\alpha\, o^\beta\,, \quad \phi_1 = \tfrac{1}{2}(F_{01} - F_{m\bar{m}}) = \phi_{\alpha\beta}\, o^\alpha\, \iota^\beta\,, \\ \phi_2 = F_{\bar{m}1} = \phi_{\alpha\beta}\, \iota^\alpha\, \iota^\beta \tag{4.103}$$

and the Weyl scalars as

$$\Psi_0 = C_{0m0m} = \Psi_{\alpha\beta\gamma\delta}\, o^\alpha\, o^\beta\, o^\gamma\, o^\delta\,, \quad \Psi_1 = C_{010m} = \Psi_{\alpha\beta\gamma\delta}\, o^\alpha\, o^\beta\, o^\gamma\, \iota^\delta\,, \\ \Psi_2 = C_{0m\bar{m}1} = \Psi_{\alpha\beta\gamma\delta}\, o^\alpha\, o^\beta\, \iota^\gamma\, \iota^\delta\,, \\ \Psi_3 = C_{101\bar{m}} = \Psi_{\alpha\beta\gamma\delta}\, o^\alpha\, \iota^\beta\, \iota^\gamma\, \iota^\delta\,, \quad \Psi_4 = C_{1\bar{m}1\bar{m}} = \Psi_{\alpha\beta\gamma\delta}\, \iota^\alpha\, \iota^\beta\, \iota^\gamma\, \iota^\delta\,. \tag{4.104}$$

Therefore, translating the Weyl double copy equation (4.95) into the null frame constructed above gives

$$\Psi_0 = 3c\,\frac{(\phi_0)^2}{S}\,, \qquad \Psi_1 = 3c\,\frac{\phi_0\,\phi_1}{S}\,, \qquad \Psi_2 = c\,\frac{\phi_0\,\phi_2 + 2(\phi_1)^2}{S}\,, \\ \Psi_3 = 3c\,\frac{\phi_1\,\phi_2}{S}\,, \qquad \Psi_4 = 3c\,\frac{(\phi_2)^2}{S}\,. \tag{4.105}$$

Having re-expressed the Weyl double copy equation in a generic null frame, we choose coordinates that will provide a direct relation to a characteristic value formulation of the Einstein equation. We begin by assuming that the spacetime is locally asymptotically flat. [12] Locally asymptotically flat spacetimes provide a mathematical model of an isolated gravitational system that may be emitting radiation that is measured by an observer at infinity. We choose Bondi coordinates $(u, r, x^I = \{\theta, \phi\})$, where

[12] By locally asymptotically flat we mean spacetimes that can be put into a Bondi form as described below, but with metric components that are not necessarily regular on the 2-sphere. Examples where there are singularities on the sphere include the Taub-NUT solution and the C-metric.

u is a timelike coordinate, r is a radial null coordinate and x^I correspond to angular coordinates. In such a coordinate system, the metric takes the Bondi form [13]

$$ds^2 = -Fe^{2\beta}\mathrm{d}u^2 - 2e^{2\beta}\mathrm{d}u\mathrm{d}r + r^2 h_{IJ}(\mathrm{d}x^I - C^I\mathrm{d}u)(\mathrm{d}x^J - C^J\mathrm{d}u), \tag{4.106}$$

where we assume the following large-r fall-off conditions for the metric components:

$$F(u,r,x^I) = 1 + \sum_{i=0}^{\infty}\frac{F_i(u,x^I)}{r^{i+1}}, \qquad \beta(u,r,x^I) = \sum_{i=0}^{\infty}\frac{\beta_i(u,x^I)}{r^{i+2}}, \tag{4.107}$$

$$C^I(u,r,x^I) = \sum_{i=0}^{\infty}\frac{C_i^I(u,x^I)}{r^{i+2}}, \qquad h_{IJ}(u,r,x^I) = \omega_{IJ} + \frac{C_{IJ}}{r} + \frac{C^2\,\omega_{IJ}}{4\,r^2} + \sum_{i=1}^{\infty}\frac{D_{IJ}^{(i)}(u,x^I)}{r^{i+2}}$$

with ω_{IJ} the metric on the round 2-sphere. Note that $C^2 = C_{IJ}C^{IJ}$, where we always lower/raise indices on tensors defined on the 2-sphere using ω_{IJ} and its inverse. Furthermore, we fix a residual coordinate freedom in the definition of the radial coordinate r by requiring that

$$\det(h_{IJ}) = \det(\omega_{IJ}) = \sin^2\theta. \tag{4.108}$$

Following Ref. [209], we may choose a parametrisation of h_{IJ} that is adapted to this gauge choice

$$2\,h_{IJ}\mathrm{d}x^I\mathrm{d}x^J = (e^{2f}+e^{2g})\mathrm{d}\theta^2 + 4\sin\theta\sinh(f-g)\mathrm{d}\theta\mathrm{d}\phi + \sin^2\theta(e^{-2f}+e^{-2g})\mathrm{d}\phi^2, \tag{4.109}$$

where

$$f(u,r,x^I) = \frac{f_0(u,x^I)}{r} + \sum_{i=2}^{\infty}\frac{f_i(u,x^I)}{r^{i+1}}, \qquad g(u,r,x^I) = \frac{g_0(u,x^I)}{r} + \sum_{i=2}^{\infty}\frac{g_i(u,x^I)}{r^{i+1}}. \tag{4.110}$$

The tensor C_{IJ} is parametrised by f_0 and g_0, while the higher f_i, g_i (with $i \geq 2$) parametrise the $D_{IJ}^{(i-2)}(u,x^I)$ tensors.

Assuming appropriate fall-off conditions for the energy-momentum tensor, there are equations relating the various metric tensor components; see Ref. [211]. However, here we will keep the discussion general by not assuming any fall-off conditions on the energy-momentum tensor.

Above, we have assumed an analytic expansion in the metric components. This is a consistent assumption from an initial value problem perspective, in the sense that assuming an analytic fall-off for initial data will guarantee that the evolved solution will remain analytic [209]. However, it does preclude some physically interesting cases

[13]In fact, this form of the metric is due to Sachs [209]. The form of the metric that appears in [210] is restricted to axisymmetric solutions. Moreover, the choice of coordinates that was made in [210] is not well-adapted to solutions with angular momentum. Therefore, except in section 4.3.2, we shall use the Sachs form even when dealing with axisymmetric solutions.

[212–214]. Another more general class of consistent fall-offs that one may consider are polyhomogenous spacetimes [215–218]. Nevertheless, the analytic expansion we assume here will be sufficient for our purposes.

We choose the following null frame associated with the metric (4.106): [211]

$$k = \frac{\partial}{\partial r}, \quad n = e^{-2\beta}\left[\frac{\partial}{\partial u} - \frac{1}{2}F\frac{\partial}{\partial r} + C^I\frac{\partial}{\partial x^I}\right], \quad m = \frac{\hat{m}^I}{r}\frac{\partial}{\partial x^I}, \tag{4.111}$$

where

$$2\,\hat{m}^{(I}\bar{\hat{m}}^{J)} = h^{IJ} \tag{4.112}$$

with h^{IJ} the matrix inverse of h_{IJ}. In particular, here, we choose

$$\hat{m} = \frac{(e^{-f} + i\,e^{-g})}{2}\,\partial_\theta - \frac{i(e^f + i\,e^g)}{2\sin\theta}\,\partial_\phi. \tag{4.113}$$

In this formulation, the Einstein equation divides into three sets of equations (see, for example, [125, 189]): hypersurface equations, which hold in each u = constant hypersurface, evolution equations, which are first order equations in time derivatives, and finally conservation equations that are satisfied on r = constant hypersurfaces. One major advantage of the characteristic formulation of the Einstein equation is that there are no constraint equations, unlike the situation in the initial value formulation. $C_{IJ}(u, x^I)$ constitutes free data, while $F_0(u_0, X^I)$, $C_1^I(u_0, x^I)$ and $D_{IJ}^{(i)}(u_0, x^I)$ are unconstrained initial data with associated evolution equations. All other metric functions can then be solved from these functions and their form at time step $u_0 + \Delta u$, derived via their evolution equations; see [211, 219–222].

The frame (4.111) provides a precise formulation of the peeling property mentioned in section 1.3.4. In such a frame, the Weyl scalars can be written in a $1/r$ expansion, where they take the form

$$\Psi_i = \mathcal{O}\left(\frac{1}{r^{5-i}}\right). \tag{4.114}$$

More precisely, given our assumptions of analyticity, one has the expansions

$$\Psi_i = \sum_{j\geq 0} \Psi_i^{(j)}\,\frac{1}{r^{5+j-i}}\,. \tag{4.115}$$

The Ψ_i have the form

$$\Psi_0 = \left[-3(1+i)(f_2+ig_2) - \frac{3}{2}\sigma^0[(\sigma^0)^2 + |\sigma^0|^2 - (\bar{\sigma}^0)^2] + \frac{1}{2}(\bar{\sigma}^0)^3\right]\frac{1}{r^5}$$
$$- [6(1+i)(f_3+ig_3)]\frac{1}{r^6} + \mathcal{O}(\frac{1}{r^7})$$
$$\Psi_1 = \left[\frac{3(1+i)}{4}(C_1^\theta - i\sin\theta C_1^\phi) + \frac{3}{4}\eth|\sigma^0|^2 + 3\sigma^0\eth\bar{\sigma}^0\right]\frac{1}{r^4} + \mathcal{O}(\frac{1}{r^5}),$$
$$\Psi_2 = \frac{1}{2}\left[F_0 - 2\sigma^0\partial_u\bar{\sigma}^0 + \bar{\eth}^2\sigma^0 - \eth^2\bar{\sigma}^2\right]\frac{1}{r^3} + \left[F_1 + \frac{(1+i)}{2}\bar{\eth}(C_1^\theta - i\sin\theta\, C_1^\phi)\right.$$
$$\left. - \frac{(1-i)}{4}\eth(C_1^\theta + i\sin\theta\, C_1^\phi) - \frac{3}{4}\eth(\bar{\sigma}^0\bar{\eth}\sigma^0) + \frac{9}{4}\sigma^0\bar{\eth}\eth\bar{\sigma}^0 + \frac{1}{4}\bar{\eth}\bar{\sigma}^0\eth\sigma^0\right]\frac{1}{r^4} + \mathcal{O}(\frac{1}{r^5}),$$
$$\Psi_3 = \eth\partial_u\bar{\sigma}^0\frac{1}{r^2} + \mathcal{O}(\frac{1}{r^3}),$$
$$\Psi_4 = -\partial_u^2\bar{\sigma}^0\frac{1}{r} + \bar{\eth}\eth\partial_u\bar{\sigma}^0\frac{1}{r^2} + \mathcal{O}(\frac{1}{r^3}), \tag{4.116}$$

where

$$\sigma^0 = \frac{(1+i)}{2}(f_0 + ig_0)\,. \tag{4.117}$$

Acting on a scalar of spin n, we have

$$\eth\eta = -\frac{(1+i)}{2}\sin^n\theta\left(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right)\left(\frac{\eta}{\sin^n\theta}\right). \tag{4.118}$$

In a similar fashion, we may consider a $1/r$ expansion of the gauge potential components, which for physically reasonable matter take the form

$$A_u(u,r,x^I) = \sum_{i=0}^{\infty}\frac{A_u^{(i)}(u,x^I)}{r^{i+1}}, \quad A_r(u,r,x^I) = \sum_{i=0}^{\infty}\frac{A_r^{(i)}(u,x^I)}{r^{i+2}},$$
$$A_I(u,r,x^I) = \sum_{i=0}^{\infty}\frac{A_I^{(i)}(u,x^I)}{r^i}, \tag{4.119}$$

where, again, we assume an analytic form for the dependence of the gauge fields on $1/r$. The analogue of the Bondi gauge in this case is to use gauge freedom to set A_r to zero:

$$A \to A - d\Lambda, \qquad \Lambda = \int_r^\infty A_r(u,r',x^I)dr' + \lambda(x^I). \tag{4.120}$$

In this gauge, we have

$$A_u(u,r,x^I) = \sum_{i=0}^{\infty} \frac{A_u^{(i)}(u,x^I)}{r^{i+1}}, \quad A_r(u,r,x^I) = 0, \quad A_I(u,r,x^I) = \sum_{i=0}^{\infty} \frac{A_I^{(i)}(u,x^I)}{r^i}, \tag{4.121}$$

with a residual gauge freedom parametrised by $\lambda(x^I)$, which corresponds to a so-called large gauge transformation. This large gauge transformation is the single copy analogue of the gravitational BMS generator; a statement that we shall make more precise in section 4.3.3.

The corresponding Maxwell field strengths are of the form

$$F_{ur} = -\partial_r A_u = \frac{A_u^{(0)}}{r^2} + \mathcal{O}(1/r^3), \qquad F_{uI} = \partial_u A_I - \partial_I A_u = \partial_u A_I^{(0)} + \mathcal{O}(1/r),$$

$$F_{rI} = \partial_r A_I = -\frac{A_I^{(1)}}{r^2} + \mathcal{O}(1/r^3), \qquad F_{IJ} = 2\partial_{[I}A_{J]} = 2\partial_{[I}A_{J]}^{(0)} + \mathcal{O}(1/r). \tag{4.122}$$

The Bianchi identity $dF = 0$ is trivially satisfied, while the Maxwell equation

$$\mathrm{d} \star F = 0 \tag{4.123}$$

is equivalent to

$$\partial_\mu(\sqrt{-g}\, g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma}) = 0. \tag{4.124}$$

In Bondi coordinates

$$\sqrt{-g} = r^2 e^{2\beta}\sqrt{\omega}, \tag{4.125}$$

while the inverse metric takes the form

$$g^{\mu\nu} = \begin{pmatrix} 0 & -e^{-2\beta} & 0 \\ -e^{-2\beta} & e^{-2\beta}F & -e^{-2\beta}C^J \\ 0 & -e^{-2\beta}C^I & r^{-2}h^{IJ} \end{pmatrix}. \tag{4.126}$$

In fact, for type D and for non-twisting type N solutions, the Maxwell field appearing in the Weyl double copy satisfies the Maxwell equation also on Minkowski spacetime [4, 99].

Using equations (4.103), (4.111), (4.113) and (4.122), we can derive the appropriate Maxwell scalars ϕ_0, ϕ_1 and ϕ_2 in a $1/r$ expansion. For type D solutions, the scalar S is then given by

$$\frac{S}{3} = \sqrt{2}\left(\phi_1^2 - \phi_0\phi_2\right)^{1/4} = \mathcal{O}(1/r). \tag{4.127}$$

Comparing the $1/r$ expansions of the Weyl scalars (4.116) with the Maxwell scalars and the scalar given by (4.127) via the double copy relations (4.105) gives an asymptotic formulation of the Weyl double copy.

It is important to stress that in formulating the Weyl double copy in the characteristic value formulation, the single copy must be expressed in Bondi coordinates on a flat Minkowski background. However, the Maxwell scalars must be defined with respect to the curved null frame.

4.3.2 C-metric and the Liénard-Wiechert solution

In this section, we demonstrate the conclusions of the previous section using the C-metric as an example. The C-metric is a type D solution that can be interpreted as a pair of causally disconnected black holes accelerating in opposite directions. The Schwarzschild-like metric

$$\begin{gathered} \mathrm{d}s^2 = \frac{1}{(1-\alpha\hat{r}\cos\hat{\theta})^2}\left[-f(\hat{r})\mathrm{d}\hat{t}^2 + \frac{\mathrm{d}\hat{r}^2}{f(\hat{r})} + \hat{r}^2\left(\frac{\mathrm{d}\hat{\theta}}{g(\hat{\theta})} + g(\hat{\theta})\,\sin^2\hat{\theta}\,\mathrm{d}\hat{\phi}^2\right)\right] , \\ f(\hat{r}) = \left(1-\frac{2\,m}{\hat{r}}\right)(1-\alpha^2\hat{r}^2) , \qquad g(\hat{\theta}) = 1 - 2\,\alpha\,m\,\cos\hat{\theta} \end{gathered} \tag{4.128}$$

describes a patch corresponding to one of the black holes. The parameters m and α are related to the total mass and the acceleration of the black holes. In the $\alpha = 0$ limit, we recover the standard Schwarzschild metric. There is an acceleration horizon at $\hat{r} = 1/\alpha$ and an event horizon at $\hat{r} = 2\,m$. Note that whenever $\alpha\,m \neq 0$ there must be a conical deficit along the vertical axis. As we will see, the periodicity of the axial coordinate can be chosen such that the conical deficit stretches from each black hole to infinity, but leaves the segment between the black holes singularity-free. In that scenario, the conical singularities could be viewed as cosmic strings pulling the black holes towards null infinity. The metric (4.128) is useful to interpret the C-metric, but it is not the most appropriate for the purpose of this chapter. Its major drawback is that its domain patch does not include the entirety of the null asymptotic boundary. To overcome this, we will resort to another common, yet less intuitive, form of the C-metric (C.1). For a detailed review of the different patches, maximal extension and interpretation of the C-metric see [223].

The single copy of the C-metric is the analogous Liénard-Wiechert solution [99], describing an accelerated electric charge. Following the prescription given above, we need to transform the coordinates for the C-metric to Bondi coordinates and also transform the coordinates for the Liénard-Wiechert solution to those corresponding to Minkowski in accelerated coordinates (i.e. the C-metric with the mass parameter m set to zero). In appendix C, we derive the Bondi form of the C-metric. In the original C-metric coordinates in which the metric takes the form (C.1), the Liénard-Wiechert gauge potential reads

$$A = Q\,y\,dt, \tag{4.129}$$

which can be easily checked to solve the Maxwell equations. In particular, this remains true if we turn off the mass parameter, so that the metric just describes Minkowski spacetime in accelerating coordinates. Thus A has the interpretation of being the Liénard-Wiechert potential for an accelerating charge. After rewriting it in terms of our new Bondi coordinates (see appendix C), we will simply have that A is given by

$$A = Q\left(\frac{1}{\Omega\alpha} - x - T\right)\left(\mathrm{d}w - \frac{\mathrm{d}y}{F(y)}\right), \tag{4.130}$$

where we then implement the various substitutions and expansions detailed in appendix C. After doing this we find that

$$\begin{aligned} A_u &= -\frac{Q\,x\,\cos\theta}{r\,\sin^2\theta} + \mathcal{O}\Big(\frac{1}{r^2}\Big), \qquad & A_r &= \frac{Q\,x\,Gj(x)\,\cos\theta}{\alpha\,r^2\,\sin\theta} + \mathcal{O}\Big(\frac{1}{r^3}\Big), \\ A_\theta &= -Q\,x\,\csc\theta + \mathcal{O}\Big(\frac{1}{r}\Big), & A_\phi &= 0\,. \end{aligned} \tag{4.131}$$

After a compensating gauge transformation to restore the $A_r = 0$ gauge choice, we have

$$\begin{aligned} A_u &= \frac{Q\,\cos\theta}{r\,\sin^2\theta}\Big(1 - x + G^{3/2}Gj\Big) + \mathcal{O}\Big(\frac{1}{r^2}\Big), \quad & A_r &= 0\,, \\ A_\theta &= -Q\,x\,\csc\theta + \mathcal{O}\Big(\frac{1}{r}\Big), & A_\phi &= 0\,. \end{aligned} \tag{4.132}$$

These expressions are valid in the general C-metric, but we actually want them just in the flat spacetime limit, which can be obtained by setting m to zero in the expressions in appendix C.1. Thus x and $Gj(x)$ are then just given by the expansions in (C.19) and (C.20) with m set to 0, and so we have

$$\begin{aligned} A_u &= \frac{Q\,\cos\theta}{r\,\sin^2\theta}\left(1 - \frac{u^3\alpha^3}{(u^2\alpha^2 + \sin^2\theta)^{3/2}}\right) + \mathcal{O}\Big(\frac{1}{r^2}\Big), \quad & A_r &= 0\,, \\ A_\theta &= -\frac{Q\,u\alpha}{\sqrt{u^2\alpha^2 + \sin^2\theta}\,\sin\theta} + \mathcal{O}\Big(\frac{1}{r}\Big), & A_\phi &= 0\,. \end{aligned} \tag{4.133}$$

Defining the null tetrad (k, n, m) and the scalar components of the Weyl tensor as before, we find that for the C-metric written in Bondi coordinates as described in appendix C, we have

$$\begin{aligned}
\Psi_0 &= \frac{i(1-\cos^2\theta\, G\, Gj^2)^2\sqrt{G}\, G'''}{16\alpha^3\,\sin^3\theta\, r^5} + \mathcal{O}(r^{-6})\,,\\
\Psi_1 &= \frac{(1+i)\cos\theta\,(1-\cos^2\theta\, G\, Gj^2)\, G^{3/2}\, Gj\, G'''}{16\alpha^2\,\sin^3\theta\, r^4} + \mathcal{O}(r^{-5})\,,\\
\Psi_2 &= -\frac{(1-3\cos^2\theta\, G\, Gj^2)\, G^{3/2}\, G'''}{24\alpha\,\sin^3\theta\, r^3} + \mathcal{O}(r^{-4})\,,\\
\Psi_3 &= -\frac{(1-i)\cos\theta\, G^{5/2}\, Gj\, G'''}{8\sin^3\theta\, r^2} + \mathcal{O}(r^{-3})\,, \qquad \Psi_4 = -\frac{i\alpha\, G^{5/2}\, G'''}{4\sin^3\theta\, r} + \mathcal{O}(r^{-2})\,.
\end{aligned} \tag{4.134}$$

We now make the small-m expansion described in the appendix to give the leading-order terms in the expansions of the Weyl tensor in (4.134):

$$\begin{aligned}
\Psi_0^{(0)} &= -\frac{3i\, m\,\sin^2\theta\,(u^2\alpha^2+1)^2}{4\alpha^2\,(u^2\alpha^2+\sin^2\theta)^{5/2}} + \mathcal{O}(m^2)\,,\\
\Psi_1^{(0)} &= -\frac{3(1+i)\, m\, u\,\sin\theta\,\cos\theta\,(u^2\alpha^2+1)}{4(u^2\alpha^2+\sin^2\theta)^{5/2}} + \mathcal{O}(m^2)\,,\\
\Psi_2^{(0)} &= \frac{m\,[(3u^2\alpha^2+1)\sin^2\theta - 2u^2\alpha^2]}{2(u^2\alpha^2+\sin^2\theta)^{5/2}} + \mathcal{O}(m^2)\,,\\
\Psi_3^{(0)} &= \frac{3(1-i)\, m\, u\,\alpha^2\,\sin\theta\,\cos\theta}{2(u^2\alpha^2+\sin^2\theta)^{5/2}} + \mathcal{O}(m^2)\,, \quad \Psi_4^{(0)} = \frac{3im\,\alpha^2\,\sin^2\theta}{(u^2\alpha^2+\sin^2\theta)^{5/2}} + \mathcal{O}(m^2)\,.
\end{aligned} \tag{4.135}$$

(See eqn (4.115) for the definition of the $\Psi_i^{(j)}$.)

Now we calculate also the field strength for the Liénard-Wiechert potential, and thence the Newman-Penrose scalars

$$\phi_0 = F_{\mu\nu}\, k^\mu\, m^\nu\,, \qquad \phi_1 = \tfrac{1}{2}F_{\mu\nu}\,(k^\mu\, n^\nu + \bar{m}^\mu\, m^\nu)\,, \qquad \phi_2 = F_{\mu\nu}\,\bar{m}^\mu\, n^\nu\,. \tag{4.136}$$

We find that at leading order in the $1/r$ expansion, these are given by

$$\begin{aligned}
\phi_0 &= -\frac{(1+i)\, Q\,(1-\cos^2\theta\, G\, Gj^2)\sqrt{G}}{4\alpha\sin^2\theta\, r^3} + \mathcal{O}(r^{-4})\,,\\
\phi_1 &= -\frac{Q\,\cos\theta\, G^{3/2}\, Gj}{2\sin^2\theta\, r^2} + \mathcal{O}(r^{-3})\,, \qquad \phi_2 = \frac{(1-i)\;\alpha\, Q\, G^{3/2}}{2\sin^2\theta\, r} + \mathcal{O}(r^{-2})\,.
\end{aligned} \tag{4.137}$$

As with the Weyl scalars, the expressions can be made more explicit in a small-m expansion in which

$$\begin{aligned}
G(x) = 1 - x^2 + \mathcal{O}(m), \qquad Gj(x) = x\,(1-x^2)^{-1/2} + \mathcal{O}(m),\\
x = u\alpha\,(u^2\alpha^2+\sin^2\theta)^{-1/2} + \mathcal{O}(m).
\end{aligned} \tag{4.138}$$

Thus we have

$$\phi_0 = -\frac{(1+i)Q\sin\theta\,(u^2\alpha^2+1)}{4\alpha r^3\,(u^2\alpha^2+\sin^2\theta)^{3/2}} + \mathcal{O}(r^{-4})\,, \tag{4.139}$$

$$\phi_1 = -\frac{\alpha Q u\cos\theta}{2r^2\,(u^2\alpha^2+\sin^2\theta)^{3/2}} + \mathcal{O}(r^{-3})\,, \qquad \phi_2 = \frac{(1-i)\alpha Q\sin\theta}{2r\,(u^2\alpha^2+\sin^2\theta)^{3/2}} + \mathcal{O}(r^{-2})\,.$$

We can see from the results for the Weyl scalars in (4.135) and the Maxwell scalars in (4.139) that a relation of the form seen in (4.105) holds. If we define

$$R_0 = \frac{\phi_0^2}{\Psi_0}\,, \quad R_1 = \frac{\phi_0\,\phi_1}{\Psi_1}\,, \quad R_2 = \frac{\phi_0\,\phi_2+2\phi_1^2}{3\Psi_2}\,, \quad R_3 = \frac{\phi_1\,\phi_2}{\Psi_3}\,, \quad R_4 = \frac{\phi_2^2}{\Psi_4}\,, \tag{4.140}$$

then to leading order in $1/r$ these are all the same:

$$R_a = \frac{S}{3c} = -\frac{Q^2}{6mr(u^2\alpha^2+\sin^2\theta)^{1/2}} + \mathcal{O}(r^{-2})\,, \qquad \text{for all } a\,. \tag{4.141}$$

From the Weyl double copy, we know that the scalar potential is [99]

$$S = \widetilde{Q}\,(\hat{x}+y) \tag{4.142}$$

for some constant $\widetilde{Q}$. In Bondi coordinates this has the large-r expansion

$$S = \frac{\widetilde{Q}\,\sqrt{G}}{\alpha\sin\theta\, r} - \frac{\widetilde{Q}\,(2\sqrt{G}\,Gj+\cos^2\theta\, GG'\,Gj^2)}{4\alpha^2\sin^2\theta\, r^2} + \mathcal{O}(r^{-3}) \tag{4.143}$$

and in the Minkowski background, it reduces to

$$S = \frac{\widetilde{Q}}{\alpha r\,(u^2\alpha^2+\sin^2\theta)^{1/2}} + \mathcal{O}(r^{-2})\,. \tag{4.144}$$

Comparing this expression with (4.141), we find that the two expressions agree once we choose

$$c = -\frac{2m\widetilde{Q}}{\alpha Q^2}. \tag{4.145}$$

Let us make a clarifying remark. We used the asymptotic Weyl scalars with coefficients given in (4.135), which correspond to the linearised order in m. On the other hand, the Weyl double copy interpretation of the C-metric is exact [99]. The linearisation in m is actually equivalent to an alternative, but exact, procedure. In [99], double-Kerr-Schild coordinates were used for the exact double copy, and in these coordinates the Weyl spinor is proportional to m. The advantage of multi-Kerr-Schild coordinates for the double copy is that they allow us to map the gravitational curved spacetime to a flat spacetime where the gauge field and the scalar live. Asymptotically, however, we are interested in using the Bondi coordinates. So the alternative procedure would be to start with double-Kerr-Schild coordinates for gravity, and then transform these into 'flat spacetime Bondi coordinates', which we are using for the gauge field and the scalar. In this way, the Weyl coefficients will indeed be linear in m. We chose to proceed as in (4.135) for brevity.

Axisymmetric Weyl double copy

We have discussed above how to express the Weyl double copy (4.105) asymptotically starting from the Bondi form of the metric (4.106). The relation becomes lengthy if one attempts to write it down in terms of the $1/r$ expansions for the metric and the gauge field described in that section. For illustrative purposes, particularly regarding the discussion of the next section, we will now consider the restriction to the axisymmetric case, which simplifies the map considerably. The original axisymmetric Bondi metric reads [210]

$$\begin{aligned} \mathrm{d}s^2 = -\left(\frac{V}{r}e^{2\beta} - U^2 r^2 e^{2\gamma}\right)\mathrm{d}u^2 - 2e^{2\beta}\mathrm{d}u\mathrm{d}r \\ -2Ur^2 e^{2\gamma}\mathrm{d}u\mathrm{d}\theta + r^2(e^{2\gamma}\mathrm{d}\theta^2 + e^{-2\gamma}\sin^2\theta\,\mathrm{d}\phi^2)\ , \end{aligned} \tag{4.146}$$

with fall-off conditions

$$\gamma(u,r,x^I) = \frac{c(u,x^I)}{r} + \mathcal{O}(r^{-3})\ , \tag{4.147}$$

$$\beta(u,r,x^I) = -\frac{c(u,x^I)^2}{4\,r^2} + \mathcal{O}(r^{-3})\ , \tag{4.148}$$

$$U(u,r,x^I) = -(c_{,\theta} + 2c\cot\theta)\frac{1}{r^2} + (2\,N(u,x^I) + 3c\,c_{,\theta} + 4c^2\cot\theta)\frac{1}{r^3} + \mathcal{O}(r^{-4})\ , \tag{4.149}$$

$$V(u,r,x^I) = r - 2M_B(u,x^I) + \mathcal{O}(r^{-1})\ . \tag{4.150}$$

This Bondi form is a subclass of the more general form (4.106), to which it is related by

$$F = \frac{V}{r} \quad \Rightarrow \quad F_0 \ = -2M_B\ , \tag{4.151}$$

$$C^I = (U,0)\ , \tag{4.152}$$

$$h_{IJ} = \begin{pmatrix} e^{2\gamma} & 0 \\ 0 & e^{-2\gamma}\sin^2\theta \end{pmatrix} \quad \Rightarrow \quad C_{\theta\theta} \ = 2\,c\ . \tag{4.153}$$

For the gauge field, the axial symmetry allows us to set $A_\phi = 0$ in (4.121). Finally, we introduce an additional simplification, by taking the scalar S to be real in (4.34), which leads to (4.36); this applies for instance to the C-metric.[14] We can plug these expressions directly into (4.36) and compare the components to obtain the neat relations

$$\frac{(A^{(0)}_{\theta,u})^2}{2\,S} = -c_{,uu} \ , \tag{4.154a}$$

$$\frac{A^{(0)}_{\theta,u}\,A^{(1)}_{u}}{2\,S} = -\frac{\partial_\theta(\sin^2\theta\, c_{,u})}{\sin^2\theta} \ , \tag{4.154b}$$

$$\frac{(A^{(1)}_{u})^2 + A^{(1)}_{\theta}\,A^{(0)}_{\theta,u}}{6\,S} = -M - c\,c_{,u} \ , \tag{4.154c}$$

$$\frac{A^{(1)}_{u}\,A^{(1)}_{\theta}}{6\,S} = N \ . \tag{4.154d}$$

In principle, these expressions could be used to obtain (up to constants of integration) a metric tensor from any axisymmetric gauge potential in Bondi gauge. However, the system (4.154) is over-complete, which gives rise to the integrability condition

$$\partial_u\left(\frac{A^{(1)}_{u}A^{(0)}_{\theta,u}}{S}\right) = \frac{1}{\sin^2\theta}\partial_\theta\left(\sin^2\theta\,\frac{(A^{(0)}_{\theta,u})^2}{S}\right) \ . \tag{4.155}$$

Alternatively, this equation can be obtained by imposing the vacuum Bianchi identities, $\nabla^\mu W_{\mu\nu\rho\sigma} = 0$, on (4.36), assuming that the gauge field satisfies the Maxwell equations.

The expressions above can easily be checked to hold for the example of the C-metric.

4.3.3 Asymptotic symmetries and the Weyl double copy

The classical double copy is fundamentally about relating solutions in gravity and gauge theory. An important aspect of both gravitational and gauge theories is their symmetry structure. In gravity, this is given by diffeomorphisms, while in gauge theory it is gauge transformations. Proper diffeomorphisms and gauge transformations, while not physical in the sense that they parametrise redundancies in the description of the same physics, are important aspects of the theories. In addition, there exist also improper transformations, which generally change boundary conditions and hence are physical. Any claim towards a new understanding of such theories ought to give some insight into how the respective symmetry structures arise and how they relate to one another. No fully general relation has been found yet, but much progress has been made on both global and linearised local symmetries, e.g. [70, 74, 224–226], and more

[14] We also assume here the reality of the couplings-absorbing constant c in (4.36). It should not be confused with the metric function c in the present section, so here we will choose that constant to take the numerical value 1/3.

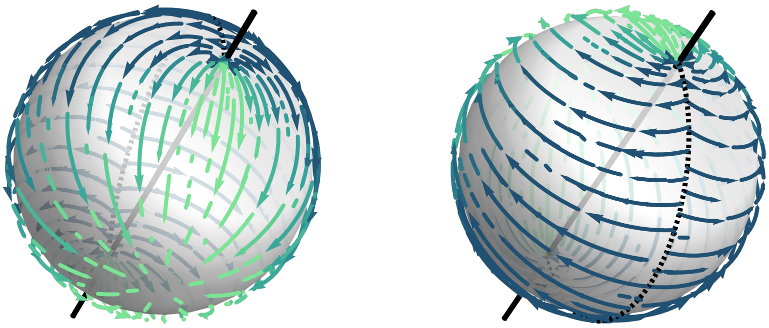


Figure 4.2: Representation of a superrotation generator acting on S^2. Both images show the same transformation, seen from the front and the back. Darker arrows indicate bigger modulus. The black rod corresponds to the vertical axis ($\theta = 0, \pi$). The diffeomorphism vanishes at the contraction point seen in the first image. Note the singular points along on the poles and dashed line.

recently on asymptotic symmetries [6, 90, 148, 179, 189, 203, 206].

The difficulty arises from the nature of the Weyl double copy due to its gauge independence. The leading-order asymptotic Weyl double copy is completely insensitive to symmetry transformations. This can be easily checked for the simplified case of axial symmetry and real scalar considered in (4.154).

Despite these challenges, some progress has been made on improper or asymptotic transformations. In [189], it was shown that (proper and improper) supertranslations on the Taub-NUT background correspond to (proper and improper/large) gauge transformations of the Dirac monopole field. This relation, which has been explored also in [6, 90], relied heavily on the time-independence of the background. Recently, a more general relation has been given in the context of the self-dual sectors of the respective theories in a light-cone formulation [203]. Within the self-dual sector, all fields within both the gravitational and Yang-Mills theory can be described in terms of scalars, and this formulation helps in relating the symmetries; in fact, it is known to help make the double copy fully manifest at the level of the equations of motion or the Lagrangian [20]. It is not clear how similar ideas can be used more generally.

In this section, we study this problem by focusing on the particular example of the C-metric and its single copy, the associated Liénard-Wiechert solution. In [208], it was argued that there is a superrotation embedded in the C-metric. From a double copy perspective, this result indicates that there ought to be an analogous large gauge transformation embedded in the Liénard-Wiechert potential. Indeed, we find that the superrotation on the gravitational side maps to a large residual gauge transformation at leading order.

The C-metric as a superrotation

First, we review the results of [208], where it is argued that superrotations are to be viewed as a "memory effect" related to the appearance/disappearance of cosmic strings piercing null infinity. The intuition behind this lies in the singular nature of the superrotations, which are elements of the BMS group which are generated by conformal Killing vectors of the celestial sphere that are not well defined globally. Therefore, superrotations relate spacetimes that are only locally asymptotically flat. Figure 4.2 depicts the flow generated by a superrotation generator that preserves axial symmetry. Note how the transformation 'unwraps' the sphere, creating an infinitesimal conical deficit, and conical singularities at the poles. The appearance of the conical deficit and singularities can be interpreted as the formation of a cosmic string. The C-metric, as an exact solution, provides a non-linear realisation for such a process; see figure 4.3.

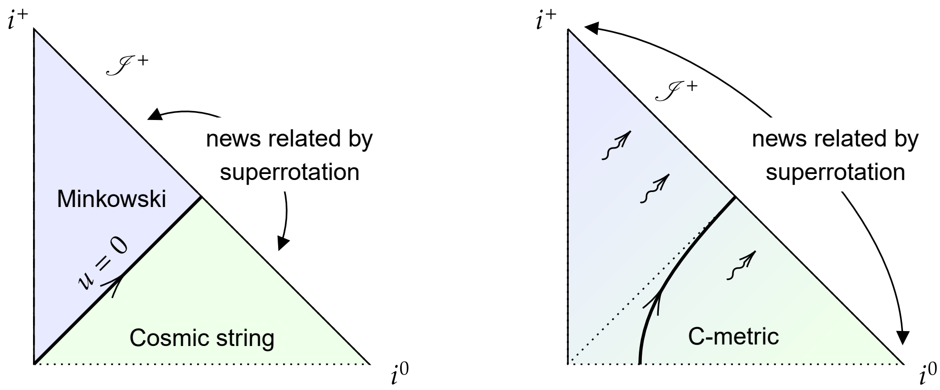


Figure 4.3: A portion of the Penrose diagram for the snapping cosmic string considered in [208] is shown on the left. The green area represents the infinite cosmic string metric, which is glued to a Minkowski patch (in blue) along $u = 0$. The trajectory of the endpoints is represented by the thick line. On the right is the corresponding picture for the (linearised) C-metric. In this case, we have radiation approaching $\mathscr{I}^+$. Note that the true Penrose diagram for the C-metric is more involved [223].

Consider the Bondi news of the C-metric [15]

$$N_{\theta\theta} \equiv \partial_u C_{\theta\theta} = -\frac{1}{4\sin^2\theta}\left(4 + 2\kappa^2 G(x)G''(x) - \kappa^2 G'(x)^2\right), \tag{4.156}$$

[15] In [208], the solution chosen was the charged C-metric, since it admits a thermodynamic interpretation: a relation between α and q can be imposed such that the event and acceleration horizons have equal surface gravities, hence equal temperatures. This is not needed for our purposes. In fact, we will set $q = 0$ since the Weyl double copy is only known to apply in vacuum. On the other hand, the single copy gauge field to be considered later, which lives in flat spacetime, coincides precisely with the gauge field of the charged C-metric after taking $m \to 0$.

where κ parametrises the deficit angle. Note that in appendix C, we have normalised κ to one. The expression above can be derived by making the κ dependence explicit in the metric (C.1), i.e. multiplying the $\mathrm{d}\phi^2$ term by κ^2, and using equations (C.10) and (C.11), as well as the expression for $C_{\theta\theta}$ in (C.18).

We are interested in the limits $u \to \pm\infty$. Recall that x is implicitly defined in (C.10) in terms of the Bondi u and θ coordinates. Using the parametrisation for $G(x)$ porposed by Hong and Teo [227] in equation (C.4), these limits correspond simply to $x = \pm 1$. The asymptotic structure of (C.10) gives

$$x \to 1 - \frac{\sin^2\theta}{2\alpha^2\kappa^2\,(1+2\alpha m)^3}\,\frac{1}{u^2} + \mathcal{O}(u^{-4}) \quad \text{as} \quad u \to \infty \ , \tag{4.157}$$

$$x \to -1 + \frac{\sin^2\theta}{2\alpha^2\kappa^2\,(1-2\alpha m)^3}\,\frac{1}{u^2} + \mathcal{O}(u^{-4}) \quad \text{as} \quad u \to -\infty \ . \tag{4.158}$$

Fixing κ so that the segment between the two black holes is regular gives

$$\kappa = \frac{2}{|G'(1)|} = \frac{1}{1+2\,\alpha\, m} \ . \tag{4.159}$$

This then implies that

$$\lim_{u\to-\infty} N_{\theta\theta} = -\frac{8\,\alpha\, m}{(1+2\,\alpha\, m)^2}\,\frac{1}{\sin^2\theta} \ , \tag{4.160a}$$

$$\lim_{u\to+\infty} N_{\theta\theta} = 0 \ . \tag{4.160b}$$

The $u \to \pm\infty$ limits of the Bondi news, (4.160b) and (4.160a), are related by a superrotation. To show this, we start from a Minkowski background, for which $N_{\theta\theta} = 0$. Superrotations are generated by 2-dimensional vector fields Y^I that are conformal Killing vectors of the celestial sphere and independent of u and r.[16] We are interested in the subgroup that preserves ∂_ϕ as a Killing vector of the metric. This restricts the superrotations to a three-parameter subgroup

$$Y^\theta = \left(\beta + \mu \ln\tan\frac{\theta}{2}\right)\sin\theta \ , \qquad Y^\phi = \mu\,\phi + \vartheta \ . \tag{4.161}$$

Setting $\vartheta = 0$ and $\beta = 0$, the effect on the Bondi news of the flat Minkowski metric is [17]

$$N_{\theta\theta} \to -\mu\,\frac{1}{\sin^2\theta} \ . \tag{4.162}$$

[16]In other words, they must generate transformations of the type

$$z \to f(z) \ , \qquad z = e^{i\phi}\cot\frac{\theta}{2} \ ,$$

with $f(z)$ a holomorphic function.

[17]ϑ generates a standard ϕ-rotation and β a boost along the z-axis.

A comparison with (4.160a) reveals that the two limits of the Bondi news of the C-metric (4.160) are related by the superrotation

$$Y^{\theta} = \frac{8\,\alpha\,m}{(1+2\,\alpha\,m)^2}\,\sin\theta\,\ln\tan\frac{\theta}{2}\,, \qquad Y^{\phi} = \frac{8\,\alpha\,m}{(1+2\,\alpha\,m)^2}\,\phi\,. \tag{4.163}$$

The Liénard-Wiechert potential as a large gauge transformation

Having made the case for the C-metric as a superrotation, it is reasonable to expect that its single copy, the Liénard-Wiechert potential, has a similar interpretation in terms of a large gauge transformation.

To investigate this, we need to put the Liénard-Wiechert potential on a background in which Minkowski spacetime is written in Bondi coordinates and in a gauge in which $A_r = 0$. We have already done this in section 4.3.2, with the appropriate expression given by (4.133). In these coordinates, a large gauge transformation corresponds to

$$A_{\theta}^{(0)} \to A_{\theta}^{(0)} - \partial_{\theta}\lambda(\theta)\,. \tag{4.164}$$

These gauge transformations are the electromagnetic analogues of the BMS asymptotic symmetries [228, 229].

As we did in the previous section, we will compare the two limits of the gauge potential

$$A_{\pm} := \lim_{u\to\pm\infty} A\,. \tag{4.165}$$

Taking this limit in the expression for the gauge potential (4.133) gives

$$A_{+} = \mathcal{O}(r^{-3})\,\mathrm{d}u + \left(-\frac{Q}{\sin\theta} + \mathcal{O}(r^{-2})\right)\,\mathrm{d}\theta\,, \tag{4.166a}$$

$$A_{-} = \mathcal{O}(r^{-3})\,\mathrm{d}u + \left(\frac{Q}{\sin\theta} + \mathcal{O}(r^{-2})\right)\,\mathrm{d}\theta\,, \tag{4.166b}$$

Taking the difference between these gauge potentials we find that

$$A_{+} - A_{-}\Big|_{r^0} = -\frac{2\,Q}{\sin\theta}\mathrm{d}\theta = \mathrm{d}\lambda(\theta)\,, \tag{4.167}$$

where

$$\lambda(\theta) = -2\,Q\,\ln\tan\frac{\theta}{2}\,. \tag{4.168}$$

This is indeed a large gauge transformation; compare with (4.164). Note the similarities between (4.163) and (4.168): it is not just that the solutions can be thought of as large diffeomorphisms or gauge transformations, but the corresponding parameters are also related. It would be interesting to find a fully general relation, beyond the example studied here.

Chapter 5

Kerr-Schild double copy

At the end of chapter 3 we touched on how the Kerr-Schild (KS) nature of the Schwarzschild metric provided a base for the prescription of another classical double copy relation. Far from being a particularity of Schwarzschild, this relation extends to a broader class of stationary spacetimes in what is called the Kerr-Schild double copy, and constituted the first exact classical formulation of the double copy [97].

Since its discovery, the KS double copy has been extended to some multi-KS spacetimes [173] and curved backgrounds [230–232]. The KS ansatz is dimension-agnostic, but the lack of propagating gravitational degrees of freedom in three dimensions has interesting consequences for the double copy [183,233,234]. The KS double copy has been applied to a wide range of solutions, including static spherically-symmetric solutions [235], radiating particles [37], Kerr-Schild-Kundt spacetimes [236], self-dual solutions [187, 237], Born-Infeld point charges [238], non-singular black holes [239, 240], shock-waves and monopoles [190]. Besides mapping solutions, the KS double copy can also be used to understand how properties of the gravitational solutions map to gauge theory. The exact nature of the map allowed the authors of [191] to study Wilson line operators and non-trivial topologies, shedding some light on the global aspects of the double copy. The behaviour of homotopy and the Ricci flow under the double copy were also studied in [241,242]. Another example is the extension of the KS double copy for probe-particle geodesics in a Kerr background formulated in [243].

One of the limitations of the original KS double copy is that it is limited to Einstein gravity, while we know that the complete double copy of Yang-Mills also includes dilaton and B-field. An extension of the KS class of solutions that is well suited to deal with these fields was proposed in [149], based on the formalism of double field theory (DFT) [159,244–249]. This DFT KS prescription applies to solutions with non-trivial configurations for the dilaton and the B-field. Similarly, there are extensions to other gravitational theories, which include heterotic gravity [150, 250] and exceptional field theory [151, 251]. These developments clearly demonstrate that the 'double' in double copy and double field theory are indeed related; see [252,253] for earlier insights and [154] for a more recent connection at Lagrangian level. Moreover, the double field theory approach explicitly relates the left-/right-moving factorisation in string theory, which is at the origin of the double copy, to the KS ansatz.

In chapter 3 we saw that the most general double copy of a Coulomb charge was the JNW spacetime. Since this solution has a non-vanishing dilaton field, the original KS double copy can not reach its entire parameter space. Hence, in order to reproduce the result using an exact classical double copy, we must employ a DFT generalisation. Unfortunately, the JNW fields do not satisfy the DFT KS ansatz provided in [149]. In the present chapter, we relax the requirements of the DFT KS ansatz to capture the JNW metric. By inspecting the equations of motion, we will identify the Maxwell equations for its single copy, the Coulomb potential [5]. This argument provides a purely classical alternative to the double copy of the three-point amplitudes studied in chapter 3. Moreover, in contrast with the calculation resulting from amplitudes, the DFT KS double copy is exact.[1] Our calculation will complement other works that identified JNW as the double copy of Coulomb using perturbative methods [5, 39, 73], also confirmed in the context of the convolutional double copy [106].

5.1 Kerr-Schild spacetimes

As a first step, we will review the KS double copy prescription outlined in [97]. This will generalise the example seen in section 3.4.2 and provide an explanation from the point of view of the equations of motion.

Writing the metric in KS form is a crucial step in making the double/single copy manifest. A solution is Kerr-Schild if it is possible to find a set of coordinates such that the spacetime metric $g_{\mu\nu}$ is put in the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \varphi\, k_\mu k_\nu \,, \tag{5.1}$$

where φ is a scalar field and k_μ is a co-vector satisfying

$$\eta^{\mu\nu} k_\mu k_\nu = 0 = g^{\mu\nu} k_\mu k_\nu \,, \tag{5.2}$$

i.e. it is null with respect to both the full and background metric. Note that (5.1) is a full, non-perturbative metric, where the second term does not need to be small. The inverse metric then takes the form

$$g^{\mu\nu} = \eta^{\mu\nu} - \varphi k^\mu k^\nu \,. \tag{5.3}$$

[1] Recall that the (standard) multi-KS form of the Kerr-Taub-NUT implied that the classical double copy map from the 3-point amplitudes was also exact. However, the JNW solution does not satisfy the KS ansatz and the relation to the Coulomb solution that originates only from the 3-point amplitudes must be regarded as a first-order approximation.

The co-vector must also be geodesic with respect to the background metric $k^\nu \partial_\nu k_\mu = 0$, which also implies that it is geodesic with respect to the Levi-Civita connection of g: $k^\nu \nabla_\nu k_\mu = 0$.

In terms of the scalar field φ and co-vector k_μ, the Ricci tensor and Ricci scalar are

$$\begin{aligned} R^\mu_\nu &= \frac{1}{2}(\partial^\mu \partial_\alpha(\varphi k^\alpha k_\nu) + \partial_\nu \partial^\alpha(\varphi k_\alpha k^\mu) - \partial^2(\varphi k^\mu k_\nu))\,, \\ R &= \partial_\mu \partial_\nu(\varphi k^\mu k^\nu)\,, \end{aligned} \tag{5.4}$$

where $\partial^\mu = \eta^{\mu\nu}\partial_\nu$. In the stationary spacetime case ($\partial_0\varphi = \partial_0 k^\mu = 0$) one may take the time component of the KS vector as $k^0 = 1$. Then, the time-time component is completely determined by φ. As a consequence, the components of the Ricci tensor simplify to

$$R^0_0 = \frac{1}{2}\partial^i \partial_i\, \varphi\,, \tag{5.5}$$

$$R^i_0 = -\frac{1}{2}\partial_j[\partial^i(\varphi k^j) - \partial^j(\varphi k^i)]\,, \tag{5.6}$$

$$R^i_j = \frac{1}{2}\partial_l[\partial^i(\varphi k^l k_j) + \partial_j(\varphi k^l k^i) - \partial^l(\varphi k^i k_j)]\,, \tag{5.7}$$

$$R = \partial_i \partial_j(\varphi k^i k^j)\,, \tag{5.8}$$

where Latin indices indicate the spatial components.

Now define a gauge field $A_\mu = \varphi\, k_\mu$, with the Maxwell field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu$. If the metric satisfies the vacuum Einstein equations $R_{\mu\nu} = 0$, (5.5) and (5.6) imply that the gauge field automatically satisfies the Abelian Maxwell equations

$$\partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu(\varphi k^\nu) - \partial^\nu(\varphi k^\mu)) = 0\,. \tag{5.9}$$

One could also obtain a non-Abelian gauge field A^a_μ with the gauge group index a, from the double copy prescription. The recipe is to take the quantity $\varphi k_\mu k_\nu$ of a given gravity solution, strip off one of the KS vectors, and dress it with a gauge group vector to get the corresponding gauge field $A^a_\mu = c^a \varphi k_\mu$. Thus, the basic statement of the double/single copy we will be applying is: If $g_{\mu\nu} = \eta_{\mu\nu} + \varphi\, k_\mu k_\nu$ is a stationary solution of Einstein's equations, then $A^a_\mu = c^a \varphi k_\mu$ is a solution of the Yang-Mills equations (linearised by the factorisation of the colour index in the constant vector c^a).

It is worth noting the role of the scalar φ. In analogy to the complex scalar S in the Weyl double copy, it satisfies the wave equation on the flat background (5.5). In fact, φ corresponds to the real part of S [99]. The scalar field φ can be interpreted as the zeroth copy: if we were to repeat the double copy procedure to A^a_μ, and replace the remaining kinematic vector with another colour vector, we would obtain

$$\Phi^{aa'} = c^a\, \tilde{c}^{a'}\, \varphi\ , \tag{5.10}$$

where the primed indices could correspond to a different colour group. The field $\Phi^{aa'}$ is a solution of the linearised (i.e. Abelian) equations of motion of the biadjoint scalar theory [76, 254–256].

For illustrative purposes, we can revisit the simplest example, Schwarzschild, but this time in Lorentzian signature. In ingoing Eddington-Finkelstein coordinates, the metric reads

$$\mathrm{d}s^2 = -\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2\mathrm{d}\Omega_2^2 + \frac{2\,m}{r}\,\mathrm{d}v^2 \ . \tag{5.11}$$

Comparing to (5.1), the first three terms represent the flat metric and from the last one we deduce

$$k^\flat = \mathrm{d}v \ , \qquad \varphi = \frac{2\,m}{r} \ . \tag{5.12}$$

Under the KS map, we then find the Coulomb gauge potential

$$A^a = \frac{Q\,c^a}{r}\,\mathrm{d}v \ , \tag{5.13}$$

where the Schwarzschild radius has been replaced by a charge parameter. A gauge transformation can be introduced to recast A^a in a more familiar way by replacing $\mathrm{d}v$ with $\mathrm{d}t$.

It is also worth mentioning that the self-dual sectors also exhibit useful properties in this context [97, 187]. In self-dual solutions, k^μ can be promoted to be a differential operator in position space $k^\mu \to \hat{k}^\mu$, and the metric can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + \hat{k}_\mu \hat{k}_\nu(\phi) \ . \tag{5.14}$$

This description is more closely related to the amplitudes double copy, given the momentum space nature of the operator $\hat{k}^\mu$.

5.2 JNW as the double copy of Coulomb

We have seen how the KS double copy relates the Schwarzschild solution to the Coulomb potential. However, previous chapters have pointed at the JNW solution as the complete double copy of Coulomb. The JNW solution is a static, spherically symmetric, asymptotically flat deformation of Schwarzschild with a minimally coupled dilaton field [109]. Due to the presence of the dilaton, the standard KS prescription described in the previous section does not apply. In the present section, we will discuss how to define an exact double-copy map based on double field theory (DFT). To this end, we introduce an ansatz for the generalised metric in DFT, by relaxing the null condition in the KS formalism, and derive a pair of Maxwell solutions as the two factors in the double copy. We apply this general formalism to the JNW case and show that both Maxwell solutions are the Coulomb potential, which is therefore the single copy of

JNW.

5.2.1 The JNW solution

As a preliminary step, we review the JNW metric and proceed to put in in a form that will be useful later on. In Einstein frame, the JNW metric is

$$\mathrm{d}s^2 = -\left(1-\frac{r_0}{r}\right)^{\frac{a}{r_0}}\mathrm{d}t^2 + \left(1-\frac{r_0}{r}\right)^{-\frac{a}{r_0}}\mathrm{d}r^2 + \left(1-\frac{r_0}{r}\right)^{1-\frac{a}{r_0}} r^2\mathrm{d}\Omega_2^2, \tag{5.15}$$

with the dilaton field

$$\phi = \frac{1}{2}\frac{b}{r_0}\log\left(1-\frac{r_0}{r}\right), \tag{5.16}$$

where

$$r_0 = \sqrt{a^2+b^2}\ . \tag{5.17}$$

The two parameters a and b parametrise the mass and the dilaton charge respectively. The special case for which $b = 0$ and $a > 0$ is the Schwarzschild solution. If $a > 0$ and the dilaton field is non-vanishing (i.e., $|b| > 0$), the solution is still asymptotically flat, but there is a naked singularity at zero radius, which corresponds to $r = r_0$ since the 2-sphere factor vanishes in the line element. This naked singularity is not surprising because the uniqueness theorems prevent a scalar-hair deformation of the Schwarzschild solution.

We have introduced the JNW metric that solves the Einstein-dilaton equations of motion in the Einstein frame. However, for the remaining sections it will be more convenient to work in the string frame. This is achieved by performing the following field redefinition

$$\begin{aligned} g^E_{\mu\nu} \ \rightarrow \ g^S_{\mu\nu} &= e^{2(\phi-\phi_0)} g^E_{\mu\nu}\ , \\ \phi_0 &= \lim_{r\to\infty}\phi\ . \end{aligned} \tag{5.18}$$

In this section, since we are not working in perturbation theory, we suppress the coupling constant κ, and use instead a common string-frame normalisation convention for the fields. The action in the string frame then reads

$$S = 2\int d^4x\sqrt{-g^S}e^{-2\phi}\left(R - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + 4\,\partial_\mu\phi\,\partial^\mu\phi\right)\ , \tag{5.19}$$

which corresponds to the low energy effective action of string theory. In the string frame, the JNW metric is given by

$$\begin{aligned} \mathrm{d}s^2 &= e^{2\phi}\left[-\left(1-\frac{r_0}{r}\right)^{\frac{a}{r_0}}\mathrm{d}t^2 + \left(1-\frac{r_0}{r}\right)^{\frac{-a}{r_0}}\left(\mathrm{d}r^2 + r(r-r_0)\,\mathrm{d}\Omega_2^2\right)\right]\ , \\ e^{2\phi} &= \left(1-\frac{r_0}{r}\right)^{\frac{b}{r_0}}\ , \qquad r_0 = \sqrt{a^2+b^2}\ . \end{aligned} \tag{5.20}$$

While JNW does not admit KS coordinates, we can express it in a similar manner, inspired by the generalised KS form of double field theory [149]. We start by defining the area-radius coordinate

$$R^2 = e^{2\phi}\left(1-\frac{r_0}{r}\right)^{\frac{-a}{r_0}} r(r-r_0)\ , \tag{5.21}$$

such that the metric reads

$$\begin{aligned} \mathrm{d}s^2 &= -f_t(r)\,\mathrm{d}t^2 + f_R(r)\,\mathrm{d}R^2 + R^2\,\mathrm{d}\Omega_2^2\ , \\ f_t(r) = \left(1-\frac{r_0}{r}\right)^{\frac{a+b}{r_0}}\ , &\qquad f_R(r) = \frac{4r(r-r_0)}{(2\,r-a+b-r_0)^2}\ . \end{aligned} \tag{5.22}$$

Changing to ingoing Eddington-Finkelstein coordinates,

$$\mathrm{d}v = \mathrm{d}t + \sqrt{\frac{f_R(r)}{f_t(r)}}\,\mathrm{d}R\ ,$$

$$\begin{aligned} \mathrm{d}s^2 &= -\mathrm{d}v^2 + 2\sqrt{f_t(r)\,f_R(r)}\,\mathrm{d}v\,\mathrm{d}R + R^2\,\mathrm{d}\Omega_2^2\ , \\ &= -\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}R + R^2\,\mathrm{d}\Omega_2^2 + (1-f_t(r))\mathrm{d}v\left(\mathrm{d}v + \frac{2\sqrt{f_t(r)\,f_R(r)}}{1-f_t(r)}\mathrm{d}R\right)\ , \end{aligned} \tag{5.23}$$

where the first three terms are the flat background metric. Let us define two auxiliary variables and another change of coordinates:

$$\begin{aligned} v &= T + R\ , \\ V &:= 1 - f_t(r) = 1 - \left(1-\frac{r_0}{r}\right)^{\frac{a+b}{r_0}}\ , \end{aligned} \tag{5.24}$$

$$\begin{aligned} \omega &:= 1 - \frac{2}{V}\left(1-\sqrt{f_t(r)f_R(r)}\right) \\ &= 1 - \frac{2}{V}\left[1-\left(1-\frac{r_0}{r}\right)^{\frac{r_0+a+b}{r_0}}\left(1-\frac{r_0+a-b}{2r}\right)^{-1}\right]\ . \end{aligned} \tag{5.25}$$

The line element is transformed into

$$\begin{aligned} \mathrm{d}s^2 &= -\mathrm{d}T^2 + \mathrm{d}R^2 + R^2\,\mathrm{d}\Omega_2^2 + V\,l^\flat\,\bar{l}^\flat\ , \\ l^\flat &= \mathrm{d}T + \mathrm{d}R\ , \qquad \bar{l}^\flat = \mathrm{d}T + \omega\,\mathrm{d}R\ , \end{aligned} \tag{5.26}$$

which is reminiscent of KS. Note, however, that only in the Schwarzschild case (i.e., $b = 0$, $\omega = 1$) do l and $\bar{l}$ coincide, and we recover the standard KS form of Schwarschild. Moreover, the metric does not even admit the DFT generalisation of the KS metric [149], because $\bar{l}$ is not null unless $a = 0$ or $b = 0$. A relaxation of the DFT KS form is therefore required. Some of the properties of the vectors still hold:

$$l^\mu l_\mu = 0 \ , \qquad \bar{l}^\mu \partial_\mu l_\nu = 0 \ , \qquad \bar{l}^\mu \partial_\nu l_\mu = 0 \ , \tag{5.27}$$

$$l^\mu \bar{l}_\mu \neq 0 \ , \qquad l^\mu \partial_\mu l_\nu = 0 \ , \qquad l^\mu \partial_\nu l_\mu = 0 \ , \tag{5.28}$$

where we have used the flat metric to contract indices. Finally, we can express it in Cartesian coordinates,

$$\begin{gathered} \mathrm{d}s^2 = -\mathrm{d}T^2 + \mathrm{d}X_i \, \mathrm{d}X_i + V \, l^\flat \, \bar{l}^\flat \ , \\ l^\flat = \mathrm{d}T + \frac{X_i}{R} \, \mathrm{d}X_i \ , \qquad \bar{l}^\flat = \mathrm{d}T + \omega \, \frac{X_i}{R} \, \mathrm{d}X_i \ . \end{gathered} \tag{5.29}$$

Setting

$$\varphi = -V \left(1 - \frac{r_0}{r}\right)^{-\frac{r_0+a+b}{2r_0}} \left(1 - \frac{r_0 + a - b}{2\, r}\right) \ , \tag{5.30}$$

the metric can be written in the form

$$\mathrm{d}s^2 = -\mathrm{d}T^2 + \mathrm{d}X_i \, \mathrm{d}X_i - \frac{\varphi}{1 + \frac{\varphi}{2}(l \cdot \bar{l})} \, l^\flat \, \bar{l}^\flat \ , \tag{5.31}$$

which obeys the KS-like ansatz (5.46) to be used in section 5.2.2. In this coordinate system the following relations also hold

$$\det g = -\left(\frac{V(1-\omega)}{2} - 1\right)^2 \ , \tag{5.32}$$

$$\varphi = -\frac{V}{\sqrt{-\det g}} \ , \tag{5.33}$$

$$\omega = 1 - 2\,(V^{-1} + \varphi^{-1}) \ . \tag{5.34}$$

5.2.2 Double field theory and the relaxed Kerr-Schild ansatz

Double field theory is a closed string effective field theory in D spacetime dimensions with manifest T-duality, where the latter is expressed by $O(D, D)$ covariance in a 'doubled spacetime' where points are labelled as $(x^\mu, \tilde{x}_\mu)$ [159, 244–249]. It provides a unified geometric framework for the entire massless NS-NS sector, encoded in an $O(D, D)$ covariant manner in the DFT fields, which are the generalised metric $\mathcal{H}_{MN}$ and the DFT dilaton d.

Throughout this chapter, $M, N, \ldots = 1, \ldots, 2D$ are $O(D, D)$ vector indices.

The generalised metric is a symmetric rank-2 $O(D, D)$ tensor satisfying the $O(D, D)$ constraint,

$$\mathcal{H}_{MP} \mathcal{J}^{PQ} \mathcal{H}_{QN} = \mathcal{J}_{MN} \, , \tag{5.35}$$

where $\mathcal{J}_{MN}$ is the $O(D,D)$ metric

$$\mathcal{J}_{MN} = \begin{pmatrix} 0 & \delta^{\mu}{}_{\nu} \\ \delta_{\mu}{}^{\nu} & 0 \end{pmatrix} , \qquad \mathcal{J}^{MN} = \begin{pmatrix} 0 & \delta_{\mu}{}^{\nu} \\ \delta^{\mu}{}_{\nu} & 0 \end{pmatrix} , \tag{5.36}$$

which defines the inner product and raises and lowers the $O(D,D)$ vector indices. One can solve the $O(D,D)$ constraint such that the generalised metric $\mathcal{H}$ and the DFT dilaton d encode the usual string-frame massless NS-NS fields as follows:

$$\mathcal{H}_{MN} = \begin{pmatrix} g^{\mu\nu} & -g^{\mu\rho}B_{\rho\nu} \\ B_{\mu\rho}g^{\rho\nu} & g_{\mu\nu} - B_{\mu\rho}g^{\rho\sigma}B_{\sigma\nu} \end{pmatrix} , \qquad e^{-2d} = \sqrt{-g}e^{-2\phi} . \tag{5.37}$$

In general, $O(D,D)$ vectors unify a D-dimensional vector and form field pair into a single object. For example, an arbitrary $O(D,D)$ vector V_M is parametrised in terms of a D-dimensional vector v^μ and a form field k_μ as

$$V_M = \begin{pmatrix} v^{\mu} \\ k_{\mu} \end{pmatrix} , \qquad \text{and} \qquad V^M = \mathcal{J}^{MN}V_N = \begin{pmatrix} k_{\mu} \\ v^{\mu} \end{pmatrix} . \tag{5.38}$$

An important feature of DFT related to the double copy is the doubled local Lorentz group, $O(1,D-1)_L \times O(1,D-1)_R$, which is the maximally compact subgroup of $O(D,D)$ including the Lorentz group. The doubled local Lorentz group originates in the left-right mode decomposition of the closed string, and shares the same origin as the KLT relations [12] in string scattering amplitudes, which underlie the double copy. This structure is transparent if we introduce a chiral and anti-chiral basis in the doubled vector space. One may recast the $O(D,D)$ constraint as $\mathcal{H}_M{}^P\mathcal{H}_P{}^N = \delta_M{}^N$, and it defines a pair of projection operators,

$$P_M{}^N = \frac{1}{2}\big(\delta_M{}^N + \mathcal{H}_M{}^N\big) , \qquad \bar{P}_M{}^N = \frac{1}{2}\big(\delta_M{}^N - \mathcal{H}_M{}^N\big) . \tag{5.39}$$

These project the doubled vector space into chiral and anti-chiral sectors which correspond to the left- and right-moving sectors, respectively.

Motivated by the KS-like form of the JNW metric (5.29), we introduce an ansatz for $\mathcal{H}$ and d in terms of two $O(D,D)$ vectors, K_M and $\bar{K}_M$, where K is null but $\bar{K}$ does not have to be null in general,

$$K_M K^M = 0 , \qquad \bar{K}_M \bar{K}^M \neq 0 . \tag{5.40}$$

Let us consider a flat background, $g_{0\mu\nu} = \eta_{\mu\nu}$, $B_{\mu\nu} = 0$ and $\phi = \text{constant}$, and denote the corresponding background DFT fields as $\mathcal{H}_0$ and d_0, where

$$\mathcal{H}_{0MN} = \begin{pmatrix} \eta^{\mu\nu} & 0 \\ 0 & \eta_{\mu\nu} \end{pmatrix}, \qquad d_0 = \text{constant}\,. \tag{5.41}$$

We associate to $\mathcal{H}_0$ a pair of background projection operators P_0 and $\bar{P}_0$ via (5.39). As we have described above, the chiralities are closely related to the underlying structure of the double copy, hence we require definite chiralities on K_M and $\bar{K}_M$ for the manifest left and right mode decomposition,

$$P_{0M}{}^{N} K_N = K_M\,, \qquad \bar{P}_{0M}{}^{N} \bar{K}_N = \bar{K}_M\,. \tag{5.42}$$

This implies that K and $\bar{K}$ are orthogonal, $K_M \bar{K}^M = 0$. One may solve the above chirality conditions explicitly using (5.41), which yields

$$K_M = \frac{1}{\sqrt{2}} \begin{pmatrix} l^\mu \\ \eta_{\mu\nu} l^\nu \end{pmatrix}, \qquad \bar{K}_M = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{l}^\mu \\ -\eta_{\mu\nu} \bar{l}^\nu \end{pmatrix}. \tag{5.43}$$

Now we are ready to write down a KS-like ansatz for the generalised metric:

$$\begin{aligned} \mathcal{H}_{MN} &= \mathcal{H}_{0MN} + \kappa\varphi\big(K_M \bar{K}_N + K_N \bar{K}_M\big) - \frac{\kappa^2}{2}\varphi^2 \bar{K}^2 K_M K_N\,, \\ d &= d_0 + \kappa f\,, \end{aligned} \tag{5.44}$$

where κ is an expansion parameter. We refer to this form as the 'relaxed KS ansatz' because the null condition for the DFT KS ansatz of [149] is partially relaxed; the latter is recovered when $\bar{K}$ is a null vector. Though the null condition is relaxed, the new ansatz satisfies the $O(D,D)$ constraint (5.35) automatically without further truncation. Substituting the parametrisation of K and $\bar{K}$ in (5.43) into (5.40), we obtain conditions on l and $\bar{l}$:

$$l_\mu l^\mu = 0\,, \qquad \bar{l}_\mu \bar{l}^\mu \neq 0\,, \qquad l_\mu \bar{l}^\mu \neq 0\,, \tag{5.45}$$

which are consistent with the JNW geometry as expressed in (5.29). Interestingly, the feature of the partially relaxed null condition is analogous to previous studies such as the 'extended' KS ansatz [257] and the heterotic KS ansatz [250]. From the parametrisation of $\mathcal{H}$, we can easily read off the corresponding ansatz for the metric and Kalb-Ramond field:

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} - \frac{\kappa\,\varphi}{1 + \frac{\kappa\,\varphi}{2}(l\cdot\bar{l})} l_{(\mu}\bar{l}_{\nu)}\,, \\ g^{\mu\nu} &= \eta^{\mu\nu} + \kappa\,\varphi l^{(\mu}\bar{l}^{\nu)} + \frac{\kappa^2\varphi^2\bar{l}^2}{4} l^\mu l^\nu\,, \\ B_{\mu\nu} &= \frac{\kappa\,\varphi}{1 + \frac{\kappa\,\varphi}{2}(l\cdot\bar{l})} l_{[\mu}\bar{l}_{\nu]}\,. \end{aligned} \tag{5.46}$$

It can easily be seen that the JNW solution fits this ansatz. The JNW metric was written precisely in this form in (5.31); we kept $\kappa \neq 1$ here for clarity. As for the Kalb-Ramond field, given the JNW expressions for φ, l and $\bar{l}$, it is of the form $B = B(r)\,\mathrm{d}R \wedge \mathrm{d}T$. Since r is a function of R only, $B_{\mu\nu}$ is pure gauge and it can be set to zero.

5.2.3 DFT equations of motion and the single copy

The field equations of DFT are given by the generalised curvatures, analogously to general relativity.[2] The generalised curvature scalar $\mathcal{R}$ and tensor $\mathcal{R}_{MN}$ defined in (D.10) are the equations of motion of the DFT dilaton and the generalised metric, respectively,

$$\mathcal{R} = 0\,, \qquad \mathcal{R}_{\mu\nu} = 0\,, \tag{5.47}$$

where $\mathcal{R}_{\mu\nu}$ is a pullback of $\mathcal{R}_{MN}$ into the D-dimensional spacetime. Note that $\mathcal{R}_{\mu\nu}$ is not symmetric nor antisymmetric: the symmetric and antisymmetric parts are the equations of motion for the metric and the Kalb-Ramond field, respectively. These reproduce the supergravity equations of motion for the massless NS-NS fields in the string frame,

$$\begin{aligned} R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\phi - \frac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} &= 0\ , \\ R + 4\,\Box\phi - 4\,\nabla^\mu\phi\nabla_\mu\phi - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\sigma} &= 0\ , \\ \nabla^\rho H_{\rho\mu\nu} - 2H_{\rho\mu\nu}\nabla^\rho\phi &= 0\ , \end{aligned} \tag{5.48}$$

which follow from the action (5.19).

Let us now discuss the field equations subject to the relaxed KS ansatz (5.44). Recall that, in the KS ansatz, an additional constraint is required in order to linearise the equations of motion, which in the case of general relativity is the geodesic condition on the null vector field. Such a constraint is obtained by contracting the null vectors with the free indices of the (generalised) curvature tensor. In the case of our relaxed KS ansatz, however, it is very cumbersome to work with this constraint. Therefore, we will assume a stronger constraint, which is satisfied in a class of solutions that includes JNW. We impose

$$\bar{K}^M\partial_M K_N = 0\,, \tag{5.49}$$

which reduces to the second equation of (5.27). Note that, while the analogous condition also appeared in the KS ansatz of DFT [149], in our ansatz we allow for $K^M\partial_M\bar{K}_N \neq 0$. Given the null condition on l and the constraint (5.49), the gen-

[2] See appendix D for a concise review of the equations of motion in DFT.

eralised curvature tensor reduces to

$$\begin{aligned}\mathcal{R}_{\mu\nu} &= \frac{1}{4}e^{2\kappa f}\partial^\rho\Big[e^{-2\kappa f}\Big(\partial_\rho\big(\kappa\varphi l_\mu \bar{l}_\nu\big) - \partial_\mu\big(\kappa\,\varphi l_\rho \bar{l}_\nu\big) - \partial_\nu\,\big(\kappa\,\varphi l_\mu \bar{l}_\rho\big) - \kappa^2\varphi^2 l_\mu l^\sigma \bar{l}_{[\nu}\partial_{|\sigma|}\bar{l}_{\rho]}\Big)\Big] \\ &\quad + \kappa\partial_\mu\partial_\nu f + \frac{\kappa^2}{2}\Big(\,\varphi l^\rho \bar{l}_\nu \partial_\mu\partial_\rho f + \varphi l_\mu \bar{l}^\rho \partial_\nu \partial_\rho f\,\Big) - \frac{\kappa^2}{8}\varphi^2 \bar{l}^2 \partial_\mu l_\rho \partial_\nu l^\rho \\ &\quad + \frac{\kappa^2}{4}\partial^\rho\big(\varphi^2\bar{l}^2 l_{[\mu}\partial_{|\nu|}l_{\rho]}\big) - \frac{\kappa^2}{8}\varphi l_\mu \bar{l}^\sigma \partial_\nu\partial_\rho(\varphi l^\rho \bar{l}_\sigma) + \frac{\kappa^3}{4}\varphi l^\rho \bar{l}^\sigma \partial_\nu\big(\varphi l_\mu \bar{l}_\sigma \partial_\rho f\big) = 0\,. \end{aligned} \tag{5.50}$$

We now discuss how to extract the single copy from $\mathcal{R}_{\mu\nu}$. By carrying out the same procedure described in the conventional KS formalism, it can be shown that one obtains the Maxwell equations from the gravity equations of motion. Suppose that the relaxed KS geometry admits at least one Killing vector ξ. We also assume that the Killing vector is constant in our choice of coordinates, and satisfies $\xi^\nu\partial_\nu\mathcal{F}_{\mu_1\cdots\mu_n} = 0$, where $\mathcal{F}_{\mu_1\cdots\mu_n}$ is an arbitrary tensor field. We will be interested in the timelike Killing vector $\xi = \partial_T$ for JNW. The single copy can be realised by contracting the Killing vector ξ with one of the free indices of the field equations of the generalised metric, $\mathcal{R}_{\mu\nu}$. We further require that l, $\bar{l}$ and ξ are normalised as $\xi\cdot l = \xi\cdot\bar{l} = 1$, which is directly the case for JNW in (5.29). Such normalisation is always possible since the KS form is preserved under the rescaling of l and $\bar{l}$. Recall that $\mathcal{R}_{\mu\nu}$ is not symmetric nor antisymmetric, thus there are two distinct equations:

$$\begin{aligned}\xi^\nu\mathcal{R}_{\mu\nu} &= \frac{1}{4}e^{2f}\partial^\rho\Big[2\partial_{[\rho}(\tilde{\varphi}l_{\mu]}) + 4\tilde{\varphi}l_{[\mu}\partial_{\rho]}f - \frac{1}{2}e^{2f}\tilde{\varphi}^2 l^\sigma l_\mu \partial_\sigma \bar{l}_\rho\Big)\Big] + \frac{1}{2}e^{2f}\tilde{\varphi}l^\rho\partial_\rho\partial_\mu f\,, \\ \xi^\mu\mathcal{R}_{\mu\nu} &= \frac{1}{4}e^{2f}\Big[\,\partial^\rho\big(2\partial_{[\rho}(\tilde{\varphi}\bar{l}_{\nu]}) + 4\tilde{\varphi}\bar{l}_{[\nu}\partial_{\rho]}f - e^{2f}\tilde{\varphi}^2 l^\sigma \bar{l}_{[\nu}\partial_{|\sigma|}\bar{l}_{\rho]}\big) + 2\tilde{\varphi}\bar{l}^\sigma\partial_\sigma\partial_\nu f \\ &\quad + \frac{1}{2}\tilde{\varphi}\bar{l}^\sigma\big(\partial_\rho(e^{2f}\tilde{\varphi}\bar{l}_\sigma)\partial_\nu l^\rho - \partial_\rho(\partial_\nu(e^{2f}\tilde{\varphi}\bar{l}_\sigma)l^\rho) + 2l^\rho\partial_\nu(e^{2f}\tilde{\varphi}\bar{l}_\sigma\partial_\rho f)\big)\Big]\,, \end{aligned} \tag{5.51}$$

where we defined $\tilde{\varphi} = e^{-2f}\varphi$ and we set $\kappa = 1$ for simplicity.

It is not immediately obvious how to extract the single copy from (5.51) due to the higher-order terms in κ, as opposed to the simpler case of the DFT KS ansatz. However, the terms linear in κ in overlap with the analogous computation in the DFT KS case. Thus one may guess that the higher-order terms would be extra contributions over the KS single copy relation, where the two gauge fields are proportional to l_μ and $\bar{l}_\mu$. Let us collect the higher-order terms, and express them with the help of a pair of auxiliary vector fields C_μ and $\bar{C}_\mu$, obeying

$$\begin{aligned}\partial^\rho\partial_{[\rho}C_{\mu]} &= -\partial^\rho\Big(\frac{1}{4}e^{2f}\tilde{\varphi}^2 l_\mu l^\sigma\partial_\sigma\bar{l}_\rho - 2\tilde{\varphi}l_{[\mu}\partial_{\rho]}f\Big) + \tilde{\varphi}l^\rho\partial_\rho\partial_\mu f\,, \\ \partial^\rho\partial_{[\rho}\bar{C}_{\mu]} &= -\partial^\rho\Big(\frac{1}{2}e^{2f}\tilde{\varphi}^2 l^\sigma\bar{l}_{[\nu}\partial_{|\sigma|}\bar{l}_{\rho]} - 2\tilde{\varphi}\bar{l}_{[\nu}\partial_{\rho]}f\Big) + \tilde{\varphi}\bar{l}^\sigma\partial_\sigma\partial_\nu f \\ &\quad + \frac{1}{4}\tilde{\varphi}\bar{l}^\sigma\Big(\partial_\rho(e^{2f}\tilde{\varphi}\bar{l}_\sigma)\partial_\nu l^\rho - \partial_\rho\big(\partial_\nu(e^{2f}\tilde{\varphi}\bar{l}_\sigma)l^\rho\big) + 2l^\rho\partial_\nu(e^{2f}\tilde{\varphi}\bar{l}_\sigma\partial_\rho f)\Big)\,. \end{aligned} \tag{5.52}$$

Notice that these definitions are possible because the currents on the right-hand side are conserved, by virtue of the equations of motion, i.e., $\partial^\mu\partial^\rho\partial_{[\rho}C_{\mu]} = \partial^\mu\partial^\rho\partial_{[\rho}\bar{C}_{\mu]} = 0$. Even though the equations look rather complicated to solve, the Killing direction components can be easily integrated as

$$\begin{aligned} \partial^\rho\Big(\partial_\rho C_\xi + \frac{1}{2}e^{2f}\tilde{\varphi}^2 l^\sigma\partial_\sigma\bar{l}_\rho - 2\tilde{\varphi}\partial_\rho f\Big) &= 0\,, \\ \partial^\rho\Big(\partial_\rho \bar{C}_\xi + \frac{1}{2}e^{2f}\tilde{\varphi}^2 l^\sigma\partial_\sigma\bar{l}_\rho - 2\tilde{\varphi}\partial_\rho f\Big) &= 0\,, \end{aligned} \tag{5.53}$$

where $C_\xi = \xi^\mu C_\mu$ and $\bar{C}_\xi = \xi^\mu\bar{C}_\mu$, and we have used the normalisation, $l_\xi = \bar{l}_\xi = 1$. This indicates that C_ξ and $\bar{C}_\xi$ should be identified; indeed, that will be required by the uniqueness of the 'zeroth' copy to be discussed shortly. As for the other components of C and $\bar{C}$, we have to treat them case by case. We will discuss the JNW example in the next subsection.

Making the use of the auxiliary fields, (5.51) reduces to the following compact form,

$$\begin{aligned} 4e^{-2f}\xi^\nu\mathcal{R}_{\mu\nu} &= \partial^\rho\Big[\,\partial_\rho(\tilde{\varphi}l_\mu + C_\mu) - \partial_\mu(\tilde{\varphi}l_\rho + C_\rho)\Big] = 0\,, \\ 4e^{-2f}\xi^\mu\mathcal{R}_{\mu\nu} &= \partial^\rho\Big[\,\partial_\rho(\tilde{\varphi}\bar{l}_\nu + \bar{C}_\nu) - \partial_\nu(\tilde{\varphi}\bar{l}_\rho + \bar{C}_\rho)\Big] = 0\,. \end{aligned} \tag{5.54}$$

This can be interpreted as a pair of Maxwell equations

$$\partial^\mu F_{\mu\nu} = 0\,, \qquad \partial^\mu\bar{F}_{\mu\nu} = 0\,, \tag{5.55}$$

by identifying the gauge fields as the single copy

$$A_\mu = \tilde{\varphi}\,l_\mu + C_\mu\,, \qquad \bar{A}_\mu = \tilde{\varphi}\,\bar{l}_\mu + \bar{C}_\mu\,. \tag{5.56}$$

Here, $F_{\mu\nu}$ and $\bar{F}_{\mu\nu}$ are the field strengths of the A_μ and $\bar{A}_\mu$ respectively. This ensures that solutions of (5.48) with the form of (5.46), subject to the constraint (5.49) and the stationary condition, can be represented by a pair of Maxwell gauge fields.

Finally, we can consider also the 'zeroth copy'. It is obtained in our formalism by contracting the Killing vector into both free indices of $\mathcal{R}_{\mu\nu}$, leading to a scalar equation of motion. One may use the result of (5.54) to get a pair of d'Alembertian equations,

$$\Box(\tilde{\varphi} + C_\xi) = 0\,, \qquad \Box(\tilde{\varphi} + \bar{C}_\xi) = 0\,. \tag{5.57}$$

As we mentioned, C_ξ and $\bar{C}_\xi$ should be identified. The zeroth copy can therefore be recognised as

$$\Phi = \tilde{\varphi} + C_\xi = \tilde{\varphi} + \bar{C}_\xi\,. \tag{5.58}$$

5.2.4 JNW and Coloumb

So far we have considered a general construction of the single copy for the relaxed KS ansatz (5.46). We now apply the previous formalism to the JNW case and show that the corresponding single copy is the Coulomb potential (i.e., both A_μ and $\bar{A}_\mu$ are Coulomb). As noted before, we need to determine the auxiliary vector fields C_μ and $\bar{C}_\mu$ to spell out the single copy. Since the JNW geometry is static, with timelike Killing vector $\xi = \partial_T$, C_T and $\bar{C}_T$ can be solved straightforwardly from (5.53). If we substitute all the necessary data, we get

$$\partial_r C_T(r) = \partial_r \bar{C}_T(r) = -e^{-2f}\Big(\frac{\varphi^2}{V^2}\partial_r V + \partial_r \varphi + 2e^{-2f}\partial_r f\Big) \tag{5.59}$$

in the asymptotically decaying case. The field strengths associated to (5.56) satisfy

$$F_{iT} = \bar{F}_{iT} = (a+b)r^{-2}\Big(1-\frac{r_0}{r}\Big)^{\frac{-r_0+a-b}{r_0}} l_i = \frac{a+b}{R^2}\, l_i = \frac{(a+b)}{R^3}\, X_i\,. \tag{5.60}$$

These are nothing but the electric field for the Coulomb potential, and it turns out that all other components of the field strengths vanish. In particular, we can easily show that the spatial components of the static, spherically symmetric gauge fields A_μ and $\bar{A}_\mu$ are pure gauge. This is better seen in spherical coordinates, where the only non-vanishing spatial component of l, $\bar{l}$, C or $\bar{C}$ is the radial one, and it only depends on the radial coordinate, which is also the case for $\tilde{\varphi}$. Therefore, the relevant spatial vector fields are all curl-free,

$$\partial_{[i}(\tilde{\varphi} l_{j]}) = \partial_{[i}(\tilde{\varphi}\bar{l}_{j]}) = \partial_{[i}C_{j]} = \partial_{[i}\bar{C}_{j]} = 0\,. \tag{5.61}$$

Hence, A_i and $\bar{A}_i$ are pure gauge, and only A_T and $\bar{A}_T$ contribute to the field strength.

This shows that the single copy for the JNW solution is given by a point electric charge as expected. The corresponding electric charge parameter is associated to the linear sum of the mass and dilaton coupling in the string frame, $a+b$. As argued earlier, the two parameters in gravity reduce to one via the single copy.

One interesting point is that the single copy exists and is the same whether the gravity solution is a naked singularity ($b \neq 0$) or the Schwarzschild solution ($b = 0$). This highlights the fact that the single copy does not reflect the causal structure of the gravity solution. Some reflection indicates, however, that this is to be expected. The single copy does not apply to the full metric, but only to the deviation from the Minkowski metric. It is from the interplay between the Minkowski metric and the deviation that the causal structure arises.

Finally, using (5.58), it is straightforward to consider the zeroth copy. As expected, the associated linearised bi-adjoint field for the JNW solution is a Coulombic potential,

$$\Phi = \tilde{\varphi} + C_T = \frac{a+b}{R}\,, \tag{5.62}$$

which is the static, spherically symmetric solution that decays asymptotically.

Therefore, both the single and zeroth copies for the JNW solution coincide with those of the Schwarzschild solution, up to irrelevant constant factors, as anticipated at linear level in chapter 3. The standard KS procedure can only explore the region of the parameter space where the dilaton vanishes. One remarkable feature of the DFT approach is that it exhibits the double copy origin of Kerr-Schild-type maps of solutions between gravity and gauge theory, by associating the pair of Kerr-Schild-type vectors to left- and right-movers in closed string theory. Moreover, it shows that, when the dilaton is turned on, the exact double copy is best expressed in the string frame, rather than the Einstein frame. These features are reminders of the string theory origin of the KLT relations and the double copy.

Chapter 6

Concluding remarks

The nature of the double copy is truly fascinating. It is the manifestation of the secrets gravity stubbornly holds back from us. All the evidence in its favour, contrasted with our inability to prove it, is an indication of the gaps that we might be missing something about gravity. That on its own is a very good reason to carry on exploring the double copy, but not the only one. In its relatively short life, the double copy has found applications in many other fields, from state-of-the-art gravitational wave calculations (e.g. [258]) to fluid dynamics [178, 259]. The journey of this thesis has also shown how the implications of the double copy span from the most fundamental interactions of quantum field theory to black holes.

Classical from quantum

Let us summarise our results in more detail. We used the building block of the on-shell approach to scattering amplitudes, the three-point amplitude, to study classical solutions in electromagnetism and gravity. The three-point amplitudes studied correspond to the emission of a messenger (photon or graviton) by a charged/massive particle, and the classical solutions are precisely the solutions sourced by the massive particle. In order for the three-point amplitude to be non-trivial, we worked with a split-signature spacetime. The alternative would have been to consider complexified momenta in Lorentzian signature, as often done in the scattering amplitudes literature, but we found the split-signature choice more straightforward, given that relevant quantities like spinors are real. Moreover, split signature is interesting in its own right, particularly regarding boundary conditions and the meaning of causality. We discussed how our results are related via analytic continuation to Lorentzian signature.

Building on the KMOC formalism [44], we used the three-point amplitude to determine the coherent state generated by the massive particle, which is associated to the split-signature versions of $\sqrt{\text{Kerr}}$-dyon and axion-dilaton Kerr-Taub-NUT solutions, for electromagnetism and NS-NS gravity respectively.

We described how to extract from that a classical field, namely via the expectation value of a quantum operator on the coherent state. As operators, we considered the 'curvatures': the field strength in electromagnetism and the generalised spacetime curvature in NS-NS gravity. These are gauge-invariant quantities (for gravity, in the linearised approximation). We found that the vacuum expectation value of these curvatures is an on-shell Fourier transform of the corresponding three-point amplitudes. This is easier to verify when we express the curvatures in terms of spinors, namely the Maxwell and Weyl spinors. As anticipated by previous work on the classical double copy [90, 97, 99, 146, 173], we saw that magnetic charge in gauge theory indeed double copies to NUT charge in gravity. Furthermore, our methods confirmed the three-point amplitudes associated by more indirect arguments [90, 146] to Taub-NUT and its spinning generalisation, Kerr-Taub-NUT. Indeed a split-signature form of Kerr-Taub-NUT and its scattering amplitude also appeared in [143, 144].

The expressions we obtained for the Maxwell and Weyl spinors exhibit a Weyl-type classical double copy in on-shell momentum space. Although many previous results support these ideas, we have provided here the ultimate connection to the double copy of scattering amplitudes. The Weyl double copy in on-shell momentum space *is* the amplitudes double copy. The simplicity of the map, together with its neat connection to the amplitudes double copy allowed us to clarify the role of the axion and dilaton on the classical double copy.

The most basic example of a classical double copy relates a Coulomb charge to a Schwarzschild black hole [97]. However, the literature also contains a *different* double copy of Coulomb: namely the JNW solution [39, 73]. The origin of this non-uniqueness was discussed in [5], where a Kerr-Schild-type exact double copy interpretation of JNW was also presented, and in [106], where an off-shell convolutional approach based on the BRST formulation (including Fadeev-Popov ghosts) [74] was used. We were able to understand the origin of the non-uniqueness using the framework introduced in [1]. It arises directly from choices inherent in the standard double copy of scattering amplitudes. At the level of amplitudes, it is always possible to *define* the gravitational theory by declaring that its three-point amplitudes are either the double copy of two same-helicity gluons (resulting in Einstein gravity) or the double copy of two same-helicity gluons *and* two opposite-helicity gluons (resulting in NS-NS gravity). Making the former choice, the double copy of Coulomb is indeed Schwarzschild. The latter choice, by contrast, leads to the JNW solution. So we are free to choose the couplings of the massive particle.

It is fascinating that the Kerr solution and its single copy, $\sqrt{\text{Kerr}}$, correspond to particularly simple three-point amplitudes [88]. Clearly, this fact is related to the Newman-Janis shift, which is an all-orders property of Kerr [145]. Turning on a magnetic charge in addition to the spin leads to a spinning dyonic solution which is, to date, the most general known three-point amplitude in pure gauge theory. The double copy of this amplitude in pure gravity is the Kerr-Taub-NUT solution [146]. However, it is also possible to perform the double copy of these amplitudes in NS-NS gravity where, as we have seen, the resulting class of solutions is of the type Kerr-Taub-NUT-dilaton-axion. This generalises the previous discussion of the double copy from Coulomb to the JNW solution to the more general three-point amplitudes.

We showed that our double copy prescription is formally equivalent to the convolutional double copy [70, 75, 106], but with the advantage, from our perspective, of being supported on on-shell momentum space, with a direct connection to scattering amplitudes. It also leads to the previously known Weyl double copy in position space, clarifying another apparent mystery in the classical double copy. For scattering amplitudes, the double copy is clearly a creature of momentum space: it is local in that context. Yet the classical double copy is frequently presented in position space – the question is then how does locality in position space somehow become locality in momentum space? In section 3.4, we showed that this locality arises non-trivially only in specific cases which include the Kerr-Taub-NUT solution. Finally, the Kerr-Schild nature of some of the solutions allowed us to promote the linearised expressions to exact relations. We illustrated this point with the split signature equivalent of Schwarzschild.

Despite this progress, there is still much to be understood. The classical double copy obviously applies to solutions which are not (yet) connected to scattering amplitudes. An important example is the (A)dS-Schwarzschild metric, which is related by the Kerr-Schild double copy to an electromagnetic solution with a point charge immersed in a background of constant charge density [173]. Given that the classical double copy connects to scattering amplitudes as well as configurations with a cosmological constant, perhaps it can lead to some insight into the amplitudes double copy in the presence of a cosmological constant.

Another topic that deserves more research is the characterisation of spacetimes that admit a local, position-space double copy. Although we explored the reasons behind locality in the Kerr-Taub-NUT family, we know that more solutions exhibit local maps. That being said, it is not realistic to expect a local map that works for any solution. We argued, for example, that the JNW metric does not exhibit a local double copy map. A more realistic expectation would be a momentum space double copy that works for all solutions that can be described by amplitudes. This map does not need to be one-to-one. From our discussion, it is simple to see that different combinations of single copy amplitudes can lead to the same gravity amplitude. Still, finding a general rule to achieve this would mean having a direct link between classical and quantum maps, and unifying all the different prescriptions present in the literature. It could also help understand which of the many classical maps do not agree with the amplitudes double copy. There are known examples of maps between classical solutions in electromagnetism and gravity that disagree with the amplitudes double copy [179, 260].

There are several obvious and exciting avenues for future research. The most obvious one is the consideration of self-interactions in gravity, i.e. going beyond linear order, using the on-shell formalism of the coherent state. This will be particularly illuminating in the cases of non-vanishing axion and dilaton, where the Kerr-Schild linearisation of the equations of motion does not apply. It will be interesting to explore this result better in position space, without resorting to Kerr-Schild coordinates beyond linearised order, to understand what the Kerr-Schild condition means from the point of view of amplitudes.

From our work, it would seem like the existence of an exact classical double copy is tightly linked to Kerr-Schild metrics. However, as we saw in chapter 4, type N solutions also exhibit exact classical double copy structures. This might indicate that the Kerr-Schild nature of the solutions obtained from amplitudes is a consequence of the point-particle source and not a necessary feature for the exact classical double copy. Perhaps the most interesting avenue would be to extend our results beyond static point particles. There are related KMOC formulations in the literature linking coherent states to radiation fields [67]. Obtaining amplitude representations of the type N Weyl double copy would provide a novel understanding of the double copy beyond linearised level.

Weyl double copy

The Weyl double copy was first introduced [99] as a procedure to decompose the Weyl spinors of vacuum Type D spacetimes into Maxwell spinors. This map is completely on-shell, which means that the resulting Maxwell spinors automatically satisfy their equations of motion. We confirmed that the Weyl map obtained from amplitudes coincides with the standard Weyl double copy. This was done by expressing the Weyl double copy directly in terms of tensors. The tensorial map also provided an intuitive way to study how the amplitude transformations of the first chapter translate into the exact language of the classical double copy. We showed how the Ehlers group transforms the mass parameter and NUT charges. This induced an electromagnetic duality transformation in the single copy, indicating the $\bar{\theta}$ parameter introduced in the amplitude double copy map generates a linearised Ehlers rotation.

Our argument was limited to a single example, which was not able to explore all the freedom of the Ehlers group. Therefore, it would be interesting to apply the transformation to other solutions to obtain more general results. Very recently, another transformation between mass and NUT parameter was proposed [199]. This transformation does not require the existence of a Killing vector, and it is isomorphic to $U(1)$, making it a candidate for the double copy of electromagnetic duality rotations in less symmetric spacetimes.

Another of our goals was to generalise the Weyl double copy beyond type D solutions. The authors of [99] already hinted at this by considering pp-waves. We showed that other vacuum type N solutions also exhibit double copy relations. For any non-twisting solution, both the Maxwell field and the scalar satisfy their equations of motion on flat backgrounds, establishing a traditional double copy relation. We performed explicit checks for Kundt and Robinson-Trautman solutions. The main difference with respect to type D metrics is that there is a non-uniqueness in the splitting between the scalar field and the Maxwell spinor. We argued that this was a consequence of the functional degrees of freedom in type N solutions, as opposed to the parametric degrees of freedom of type D solutions. Type N solutions with twist are more complicated. The lack of an appropriate flat space limit implies that the gauge field does not satisfy flat Maxwell equations, hindering the standard double copy interpretation. However, the map is still valid if one keeps the fields on the curved background. At linearised level, these fields live on Minkowski space, but the map is no longer exact.

We have also taken steps toward providing an asymptotic understanding of the classical double copy. We have shown how the Weyl double copy can be formulated asymptotically in the neighbourhood of null infinity, where it applies to a wider class of spacetimes, including algebraically general ones [2]. Our study of the asymptotic symmetries could help to make connections with recent advances in celestial (i.e. flat-space) holography. We showed how the Weyl double copy of the C-metric provides a link between superrotations and large gauge transformations. This extends the known relations between them beyond the self-dual sector [203].

The next natural step would be to understand this formulation at sub-leading orders, i.e. moving from asymptotic infinity into the bulk. This may assist us in generalising the Weyl double copy beyond algebraically special solutions in an appropriate expansion.

It would also be interesting to understand how our formulation fits in with the story of conformally primary metrics on the celestial sphere and their double copy interpretation [206], where the C-metric still provides a puzzling example.

Kerr-Schild double copy

The last chapter was devoted to the Kerr-Schild double copy. We showed that the most general double copy of the Coulomb solution is the JNW solution, which includes a mass parameter and a dilaton parameter. This is consistent with previous perturbative works [5, 39, 73, 106]. However, we provide exact evidence, extending the double field theory Kerr-Schild ansatz of [149]. One remarkable feature of our approach is that it exhibits the double copy origin of Kerr-Schild-type maps of solutions between gravity and gauge theory, by associating the pair of Kerr-Schild-type vectors to left and right movers in closed string theory.

Our method was only applied to JNW, a special case of the more general linearised solution obtained from amplitudes in chapter 3. An extension of this analysis would allow us to study the double copy interpretation of the most general known static, spherically symmetric and asymptotically flat solution to NS-NS gravity [169]. It is more general than the JNW solution, in that it admits a B-field whose field strength is spherically symmetric. The 'single copy' is not, however, the Coulomb solution, since two distinct gauge-theory solutions are required to introduce the antisymmetric B-field via the double copy.

On a similar note, it might be possible to use this formalism to obtain a rotating JNW solution, which is currently unknown. The linearised version of the solution was obtained in chapter 3 already, and it could be used in conjunction with the generalised DFT ansatz to generate the new solution.

Another interesting point is the connection between the double field theory and the double copy. In particular, double field theory could provide an interpretation for the generalised connection of section 3.1 in terms of DFT generalised curvatures. The absence of an analogue of the Weyl tensor in double field theory makes this point particularly intriguing.

Final words

During the last four years, we have witnessed how the classical double copy became a dominant field attracting interest from the scattering amplitudes and general relativity communities. As a consequence, it has been developed at an incredible pace, giving birth to a growing number of relations and prescriptions. This work has tried to blur the edges between some of them. Although the proliferation of so many different schemes might be at times dizzying, it might be regarded as evidence for a more general structure that is yet to be discovered. Hopefully, a complete description of the kinematic algebra will be achieved, getting a handle on the inner workings of the duality. This would also help to understand which theories admit a double copy description and which do not. On the classical side, it would be desirable to understand how to relate more classical solutions to amplitudes, in a way that allows us to apply the double copy, also beyond linear level. Even if a one-to-one local map is not a realistic expectation, a more general position space map would already help understand the duality better and learn more about gravity.

Appendix A

Split signature

A.1 Spinor conventions in split signature

In coordinates (t^1, t^2, x^1, x^2), we work with a metric of signature $(+1, +1, -1, -1)$. Since this signature may be unfamiliar, we gather here a list of spinor-helicity conventions appropriate for working in this signature.

The Clifford algebra is

$$\sigma^\mu \tilde{\sigma}^\nu + \sigma^\nu \tilde{\sigma}^\mu = \eta^{\mu\nu} \mathbb{1} \,. \tag{A.1}$$

In our signature, it is possible to choose a real basis of σ^μ matrices. Our choice is

$$\sigma^\mu = \frac{1}{\sqrt{2}}(1, i\sigma_y, \sigma_z, \sigma_x) \tag{A.2}$$

where $\sigma_{x,y,z}$ are the usual Pauli matrices. The $\tilde{\sigma}^\mu$ are obtained by raising spinor indices, as usual:

$$\tilde{\sigma}^{\mu\dot{\alpha}\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\beta\dot{\beta}} \,. \tag{A.3}$$

The conventions for ϵ and raising/lowering indices are kept the same as in Lorentzian signature, (1.49), (1.50) and (1.52).

To pass between momenta k and spinors λ, $\tilde{\lambda}$, we define

$$k \cdot \sigma_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} \,. \tag{A.4}$$

In analogy with the spinor-helicity formalism, we use the symbols $|k\rangle$, $\langle k|$, $[k|$, and $|k]$ to indicate the spinors with the indices in various positions as follows:

$$|k\rangle \leftrightarrow \lambda_\alpha \,, \quad \langle k| \leftrightarrow \lambda^\alpha \,, \quad |k] \leftrightarrow \tilde{\lambda}^{\dot{\alpha}} \,, \quad [k| \leftrightarrow \tilde{\lambda}_{\dot{\alpha}} \,. \tag{A.5}$$

As usual, we choose a basis of polarisation vectors of definite helicity $\eta = \pm$. Unlike the Minkowski case, these vectors can be chosen to be real, and we make such a choice.

Given a momentum k and gauge choice q satisfying $k \cdot q \neq 0$, $k^2 = 0 = q^2$, we define

$$\varepsilon_-^\mu = -\frac{\langle k|\sigma^\mu|q]}{[kq]}\,, \qquad \varepsilon_+^\mu = \frac{[k|\tilde{\sigma}^\mu|q\rangle}{\langle kq\rangle}\,. \tag{A.6}$$

These polarisation vectors have the properties:

$$\begin{aligned}(\varepsilon_h^\mu(k))^* &= \varepsilon_h^\mu(k)\,,\\ \varepsilon_\pm^2(k) &= 0\,,\\ \varepsilon_+(k)\cdot\varepsilon_-(k) &= -1\,,\end{aligned} \tag{A.7}$$

assuming that both k and q are real.

A plane wave with negative polarisation has a self-dual field strength in our conventions:

$$\begin{aligned}\sigma_{\mu\nu}\, k^{[\mu}\varepsilon_-^{\nu]} &= -\frac{1}{\sqrt{2}}\,|k\rangle\langle k|\,,\\ \tilde{\sigma}_{\mu\nu}\, k^{[\mu}\varepsilon_-^{\nu]} &= 0\,.\end{aligned} \tag{A.8}$$

Meanwhile, a positive helicity plane wave has anti-self dual field strength given by

$$\begin{aligned}\sigma_{\mu\nu}\, k^{[\mu}\varepsilon_+^{\nu]} &= 0\,,\\ \tilde{\sigma}_{\mu\nu}\, k^{[\mu}\varepsilon_+^{\nu]} &= \frac{1}{\sqrt{2}}\,|k][k|\,.\end{aligned} \tag{A.9}$$

A.2 The retarded Green's function in $1+2$ dimensions

Because of the translation symmetry in the t^2 direction, much of our discussion really takes place in a three-dimensional space with signature $(+,-,-)$. In this appendix, we compute the retarded Green's function (for the wave operator) in this space. We use the familiar notation $x = (t, \vec{x})$ for points in this spacetime, and write wave vectors as $k = (E, \vec{k})$.

The Green's function is defined to satisfy

$$\partial^2 G_{\text{ret}}(x) = \delta^{(3)}(x)\,, \tag{A.10}$$

with the boundary condition that

$$G_{\text{ret}}(x) = 0\,, \quad t < 0\,. \tag{A.11}$$

It is easy to express the Green's function in Fourier space as

$$G_{\text{ret}}(x) = -\int \hat{\mathrm{d}}^3 k\, e^{-ik\cdot x}\,\frac{1}{k_{\text{ret}}^2}\,. \tag{A.12}$$

The instruction 'ret' indicates that we must define the integral to enforce the retarded boundary condition (A.11). As usual, we interpret the integral over the first component E of k^μ as a contour integral, and (as in the main text) we impose the boundary condition by displacing the poles below the real E axis. It is easy to compute the value of the E integral using the residue theorem, with the result that

$$\begin{aligned} G_{\text{ret}}(x) &= \frac{-i}{8\pi^2}\Theta(t)\int \mathrm{d}^2k\, e^{i\vec{k}\cdot\vec{x}}\frac{e^{i|\vec{k}|t}-e^{-i|\vec{k}|t}}{|\vec{k}|} \\ &= \frac{-i}{8\pi^2}\Theta(t)\int_0^\infty \mathrm{d}k\int_0^{2\pi}\mathrm{d}\theta\, e^{ikr\cos\theta}\left(e^{ikt}-e^{-ikt}\right)\,, \end{aligned} \tag{A.13}$$

where, in the second equality, we defined $r = |\vec{x}|$ and introduced polar coordinates for the $\vec{k}$ integration.

Our integral is still not completely well-defined. Notice that if we perform the k integral in equation (A.13) first, we encounter oscillatory factors which do not converge. The solution is again familiar: we introduce $ik\epsilon$ convergence factors in the exponents, adjusting the signs to make the integrals well-defined. The result is

$$G_{\text{ret}}(x) = \frac{-i}{8\pi^2}\Theta(t)\int_0^\infty \mathrm{d}k\int_0^{2\pi}\mathrm{d}\theta\, e^{ikr\cos\theta}\left(e^{ik(t+i\epsilon)}-e^{-ik(t-i\epsilon)}\right)\,. \tag{A.14}$$

Recognising the definition of the Bessel function, it is easy to perform the θ integration next, yielding

$$G_{\text{ret}}(x) = \frac{-i}{4\pi}\Theta(t)\int_0^\infty \mathrm{d}k\, J_0(kr)\left(e^{ik(t+i\epsilon)}-e^{-ik(t-i\epsilon)}\right)\,. \tag{A.15}$$

We can perform the final integral using the result

$$\int_0^\infty \mathrm{d}u\, J_0(u)e^{iuv} = \frac{1}{\sqrt{1-v^2}}\,, \tag{A.16}$$

so that

$$G_{\text{ret}}(x) = \frac{i}{4\pi}\Theta(t)\left(\frac{1}{\sqrt{r^2-t^2+i\epsilon}}-\frac{1}{\sqrt{r^2-t^2-i\epsilon}}\right)\,. \tag{A.17}$$

At this point, the $i\epsilon$ factors come into their own. Evidently, the Green's function vanishes when we can ignore the ϵ's: this occurs when $r^2 - t^2$ is positive. But when $r^2 - t^2 < 0$, then the ϵ's control which side of the branch cut in the square root function we must choose. We have

$$
\begin{aligned}
G_{\text{ret}}(x) &= \frac{i}{4\pi}\Theta(t)\Theta(t^2-r^2)\left(\frac{1}{\sqrt{-|t^2-r^2|+i\epsilon}} - \frac{1}{\sqrt{-|t^2-r^2|-i\epsilon}}\right) \\
&= \frac{i}{4\pi}\Theta(t)\Theta(t^2-r^2)\left(\frac{1}{i\sqrt{|t^2-r^2|}} - \frac{1}{(-i)\sqrt{|t^2-r^2|}}\right) \\
&= \frac{1}{2\pi}\Theta(t)\Theta(t^2-r^2)\frac{1}{\sqrt{t^2-r^2}}\,.
\end{aligned} \tag{A.18}
$$

As discussed in more detail in section 2.2, this Green's function is a Lorentzian version of the familiar Euclidean Green's function $\sim 1/r$. The theta functions are a result of our boundary conditions.

A.3 Analytic continuation of propagators

All the classical fields we have obtained are written as integrals of three-point amplitudes over on-shell momentum space. Therefore, these integrals have no support in Lorentzian signature for real kinematics. We have avoided this problem by using split signature. Alternatively, we could have proceeded in Lorentzian signature provided that we integrate over complex momenta. To illustrate these two alternatives, consider the scalar potential introduced in (2.40). It can be shown that the scalar potential is related to the retarded and advanced Green's functions,

$$
S(x) = G_{\text{ret}}(x) - G_{\text{adv}}(x)\ , \tag{A.19}
$$

where

$$
\begin{aligned}
G_{\text{ret}}(x) &= -\int \hat{\mathrm{d}}^4k \frac{e^{-i\,k\cdot x}\hat{\delta}(k\cdot u)}{(k_1+i\epsilon)^2-\mathbf{k}^2} = \frac{\Theta(t_1)\Theta(t_1^2-r^2)}{2\pi\sqrt{t_1^2-r^2}}\ , \\
G_{\text{adv}}(x) &= -\int \hat{\mathrm{d}}^4k \frac{e^{-i\,k\cdot x}\hat{\delta}(k\cdot u)}{(k_1-i\epsilon)^2-\mathbf{k}^2} = \frac{\Theta(-t_1)\Theta(t_1^2-r^2)}{2\pi\sqrt{t_1^2-r^2}}\ .
\end{aligned} \tag{A.20}
$$

The existence of different Green's functions is linked to the freedom to choose boundary conditions. Our choice is that the field should vanish for $t_1 < 0$, which selects the retarded propagator. Hence, under these boundary conditions, we can write

$$
S(x) = 0 \quad \text{for } t_1 < 0\,, \quad S(x) = G_{\text{ret}}(x) \quad \text{for } t_1 > 0\,. \tag{A.21}
$$

In Lorentzian signature, the time coordinate t_1 is replaced by another space coordinate, z, dual to k_1. Now, all three coordinates orthogonal to t_2 have the same signature and no $i\epsilon$ prescription is needed. Consequently, the only Green's function is

$$
G(x) = -\int \hat{\mathrm{d}}^4k\, \hat{\delta}(k\cdot u)\, \frac{e^{-ik\cdot x}}{k^2} = \frac{1}{4\pi\sqrt{r^2+z^2}}\ . \tag{A.22}
$$

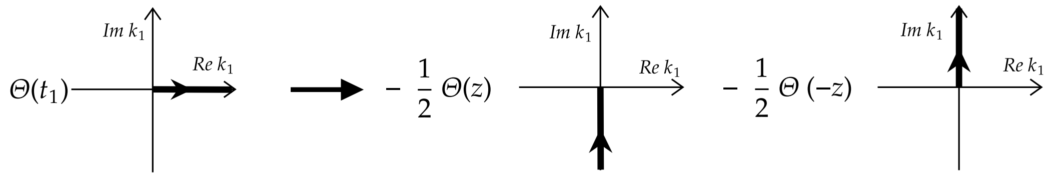


Figure A.1: Analytical continuation of the split signature contour to Minkowski signature.

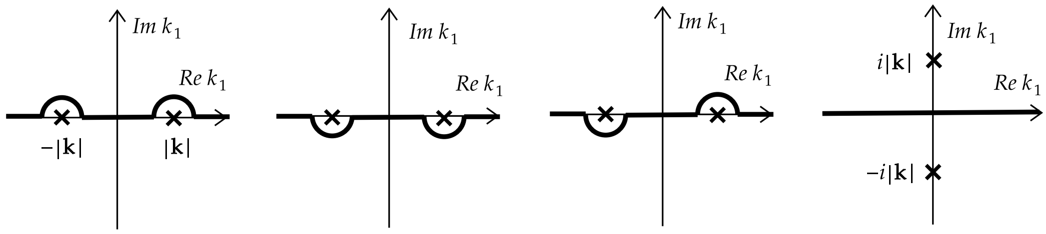


Figure A.2: Contour of the different Green's functions. From left to right: retarded, advanced, Feynman and the last one corresponds to Lorentzian signature.

Analogously to the split signature case, we can recast this Green's function into an integral of the form (2.40). However, the Lorentzian delta function $\hat{\delta}(k_1^2 + \mathbf{k}^2)$ has no roots on the real line of k_1. As a result, the integration contour on k_1 must be deformed in the complex plane. The appropriate contour is

$$\begin{aligned} S(x) = -\Theta(z)\ \mathrm{Re}\ i \int_{-i\infty}^{0} \hat{\mathrm{d}}k_1 \int \hat{\mathrm{d}}^2\mathbf{k}\, \hat{\delta}(k_1^2 + \mathbf{k}^2) e^{-ik\cdot x} \\ - \Theta(-z)\ \mathrm{Re}\ i \int_{0}^{i\infty} \hat{\mathrm{d}}k_1 \int \hat{\mathrm{d}}^2\mathbf{k}\, \hat{\delta}(k_1^2 + \mathbf{k}^2) e^{-ik\cdot x}\ . \end{aligned} \tag{A.23}$$

The prescription to analytically continue the retarded term in (2.40) to Lorentzian signature is summarised in figure A.1. The splitting or doubling of the contour might seem surprising. Ultimately, it is a reminder that the correct split signature propagator to analytically continue to Lorentzian signature is the Feynman propagator,

$$G_{\mathrm{F}} = -\int \hat{\mathrm{d}}^4 k \frac{e^{-i\,k\cdot x}\,\hat{\delta}(k\cdot u)}{k_1^2 - \mathbf{k}^2 + i\epsilon}\ , \tag{A.24}$$

which is time-symmetric. This is shown graphically in figure A.2, where only the Feynman propagator contour can be deformed into the Lorentzian contour without crossing the poles. In position space, the correct analytic continuation of the scalar potential is [1]

$$\frac{\Theta(t_1)\Theta(t_1^2 - r^2)}{2\pi\sqrt{t_1^2 - r^2}} \quad \to \quad \frac{1}{4\pi\sqrt{r^2 + z^2}}\ . \tag{A.25}$$

[1]This statement is explained more extensively in section 5 of [3].

A.4 Electromagnetic duality

Another difference in Lorentzian and split signature appears in the electromagnetic duality. In (1,3) signature, we defined the self-dual and anti-self-dual electromagnetic tensors as

$$\begin{aligned} F^{+}_{\mu\nu} &= \frac{1}{2}\left(F_{\mu\nu} - i \star F_{\mu\nu}\right) \ , \\ F^{-}_{\mu\nu} &= \frac{1}{2}\left(F_{\mu\nu} + i \star F_{\mu\nu}\right) \ , \end{aligned} \tag{A.26}$$

such that $\star F^{\pm}_{\mu\nu} = \pm i F^{\pm}_{\mu\nu}$. The electromagnetic stress-energy tensor can be expressed as

$$T_{\mu}{}^{\nu} = F^{+}_{\mu\rho} F^{-\rho\nu} + F^{-}_{\mu\rho} F^{+\rho\nu} \ . \tag{A.27}$$

Under electromagnetic duality with parameter θ,

$$\begin{aligned} F_{\mu\nu} &\to \cos\theta\, F_{\mu\nu} + \sin\theta\ \star F_{\mu\nu} \ , \\ \star F_{\mu\nu} &\to \cos\theta\ \star F_{\mu\nu} - \sin\theta\, F_{\mu\nu} \ . \end{aligned} \tag{A.28}$$

The self-dual and anti-self-dual tensors pick up a phase, $F^{\pm}_{\mu\nu} \to e^{\pm i\theta} F^{\pm}_{\mu\nu}$, implying that the stress-energy tensor is preserved.

In split signature, however, the self-dual and anti-self-dual field strength tensors are

$$\begin{aligned} F^{+}_{\mu\nu} &= \frac{1}{2}\left(F_{\mu\nu} + \star F_{\mu\nu}\right) \ , \\ F^{-}_{\mu\nu} &= \frac{1}{2}\left(F_{\mu\nu} - \star F_{\mu\nu}\right) \ , \end{aligned} \tag{A.29}$$

such that $\star F^{\pm}_{\mu\nu} = \pm F^{\pm}_{\mu\nu}$. The stress-energy tensor is still (A.27). On this occasion, to keep it invariant we need to have

$$\begin{aligned} F_{\mu\nu} &\to \cosh\theta\, F_{\mu\nu} + \sinh\theta\ \star F_{\mu\nu} \ , \\ \star F_{\mu\nu} &\to \cosh\theta\ \star F_{\mu\nu} + \sinh\theta\, F_{\mu\nu} \ , \end{aligned} \tag{A.30}$$

such that $F^{\pm}_{\mu\nu} \to e^{\pm\theta} F^{\pm}_{\mu\nu}$. This difference in duality transformations (A.28) and (A.30) can be interpreted as $\theta \to -i\theta$ under analytic continuation.

Appendix B

2-Spinors in Riemann-Cartan geometries

In this appendix, we will study the Riemann-Cartan objects defined in section 3.1 under the light of the 2-spinor formalism. The generalisation of spinors to spacetimes with torsion was also addressed in [124, 165].

The procedure to define the spinor structure does not differ from the Riemannian case. Tensors are mapped to spinors using the Pauli matrices (also called Infeld-van der Waerden symbols) ${\sigma_\mu}^{\alpha\dot{\alpha}}$ and ${\tilde{\sigma}_\mu}^{\dot{\alpha}\alpha}$. The metric on spinor space is the anti-symmetric two by two matrix $\epsilon_{\alpha\beta}$ (and $\epsilon_{\dot{\alpha}\dot{\beta}}$). The conventions for raising and lowering spinors are

$$\xi^\alpha = \epsilon^{\alpha\beta}\xi_\beta \ , \qquad \xi_\alpha = \xi^\beta \epsilon_{\beta\alpha} \ , \tag{B.1}$$

$$\epsilon^{\alpha\gamma}\epsilon_{\gamma\beta} = {\epsilon^\alpha}_\beta = {\delta^\alpha}_\beta \ , \qquad \epsilon^{\gamma\alpha}\epsilon_{\gamma\beta} = {\epsilon_\beta}^\alpha = -{\delta^\alpha}_\beta \ . \tag{B.2}$$

Similar expressions hold for $\epsilon_{\dot{\alpha}\dot{\beta}}$. For conciseness, the σ-matrices will be used implicitly every time indices are translated from the spacetime tangent bundle to the spinor bundles. In this way, we write

$$K_{\mu\nu\rho} \to K_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\dot{\gamma}} \ . \tag{B.3}$$

In going to spinor space, we lengthen the list of indices, but we gain extra simplification power. This is because tensor symmetries imply that the spinor counterparts must decompose into lower-rank symmetric spinors and epsilon matrices. Moreover, since every pair of indices is the sum of their symmetrisation plus their antisymmetrisation, any spinor can be expressed as a sum of totally symmetric spinors combined with epsilon matrices. The most famous example is the reduction of the Riemann spinor to its irreducible components

$$\begin{aligned} R_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\dot{\gamma}\delta\dot{\delta}} = \Psi_{\alpha\beta\gamma\delta}\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\dot{\gamma}\dot{\delta}} + \tilde{\Psi}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} \\ + \Phi_{\alpha\beta\dot{\gamma}\dot{\delta}}\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\gamma\delta} + \tilde{\Phi}_{\dot{\alpha}\dot{\beta}\gamma\delta}\epsilon_{\alpha\beta}\epsilon_{\dot{\gamma}\dot{\delta}} \\ + 2\Lambda\left(\epsilon_{\alpha\gamma}\,\epsilon_{\beta\delta}\,\epsilon_{\dot{\alpha}\dot{\gamma}}\,\epsilon_{\dot{\beta}\dot{\delta}} - \epsilon_{\alpha\delta}\,\epsilon_{\beta\gamma}\,\epsilon_{\dot{\alpha}\dot{\delta}}\,\epsilon_{\dot{\beta}\dot{\gamma}}\right) . \end{aligned} \tag{B.4}$$

The symmetry under the exchange of pairs of indices and the first Bianchi identity impose $\tilde{\Phi}_{\dot{\alpha}\dot{\beta}\gamma\delta} = \Phi_{\gamma\delta\dot{\alpha}\dot{\beta}}$ and $\tilde{\Lambda} = \Lambda$ respectively.

However, this result does not hold for Riemann-Cartan manifolds. One of our goals is to see explicitly how the above expression changes in the presence of contorsion. As a preliminary step, we have to study the contorsion itself from the point of view of spinors.

B.1 Contorsion spinors

The natural first step for decomposing the contorsion spinor is to exploit the antisymmetry of $K_{\mu\nu\rho}$ in the first and third indices,

$$K_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\dot{\gamma}} = \Theta_{\alpha\beta\gamma\dot{\beta}}\epsilon_{\dot{\alpha}\dot{\gamma}} + \tilde{\Theta}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\beta}\epsilon_{\alpha\gamma} \ , \tag{B.5}$$

where $\Theta_{\alpha\beta\gamma\dot{\beta}} = \Theta_{(\alpha|\beta|\gamma)\dot{\beta}}$. The resulting spinor is still not totally symmetric, implying that it can be separated into two irreducible parts

$$\Theta_{\alpha\beta\gamma\dot{\beta}} = \Xi_{\gamma\dot{\beta}}\,\epsilon_{\alpha\beta} + \Xi_{\alpha\dot{\beta}}\,\epsilon_{\gamma\beta} + \Omega_{\alpha\beta\gamma\dot{\beta}} \ , \tag{B.6}$$

where $\Omega_{\alpha\beta\gamma\dot{\beta}} = \Omega_{(\alpha\beta\gamma)\dot{\beta}}$. The spinors $\Xi_{\alpha\dot{\beta}}$ and $\Omega_{\alpha\beta\gamma\dot{\beta}}$ constitute the irreducible spinor decomposition of the contorsion.

Now that we have pinned down the spinor degrees of freedom of the contorsion, we can map them to the tensor degrees of freedom. These tensor degrees of freedom are arranged into three components: a completely antisymmetric tensor $\breve{K}_{\mu\nu\rho}$, a trace $\bar{K}_\mu$ and a traceless tensor $\hat{K}_{\mu\nu\rho}$

$$\breve{K}_{\mu\nu\rho} = K_{[\mu\nu\rho]} \ , \tag{B.7a}$$

$$\bar{K}_\mu = K^{\nu}{}_{\nu\mu} \ , \tag{B.7b}$$

$$\hat{K}_{\mu\nu\rho} = \frac{2}{3}(K_{(\mu\nu)\rho} + K_{\mu(\nu\rho)}) - \frac{1}{3}\,g_{\mu\nu}\,K^{\sigma}{}_{\sigma\rho} + \frac{1}{3}\,g_{\nu\rho}\,K^{\sigma}{}_{\sigma\mu} \ . \tag{B.7c}$$

For completeness, the inverse relation is

$$K_{\mu\nu\rho} = \breve{K}_{\mu\nu\rho} + \hat{K}_{\mu\nu\rho} + \frac{1}{3}\,g_{\mu\nu}\,\bar{K}_\rho - \frac{1}{3}\,g_{\nu\rho}\,\bar{K}_\mu \ . \tag{B.8}$$

	Tensor components	Spinor components	d.o.f
Antisymmetric	$\breve{K}_{\mu\nu\rho}$	$\Xi_{\alpha\dot{\alpha}} - \tilde{\Xi}_{\dot{\alpha}\alpha}$	4
Trace	$\bar{K}_{\mu}$	$\Xi_{\alpha\dot{\alpha}} + \tilde{\Xi}_{\dot{\alpha}\alpha}$	4
Traceless	$\hat{K}_{\mu\nu\rho}$	$\Omega_{\alpha\beta\gamma\dot{\alpha}}$, $\tilde{\Omega}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\alpha}$	16

Table B.1: d.o.f in the different components of the contorsion.

Upon applying the sigma matrices to the right hand side of (B.7), we obtain[1]

$$\begin{aligned}\breve{K}_{\mu\nu\rho} \;\to\; & (\epsilon_{\alpha\gamma}\epsilon_{\beta\zeta}\epsilon_{\dot{\alpha}\dot{\zeta}}\;\epsilon_{\dot{\beta}\cdot} - \epsilon_{\alpha\zeta}\epsilon_{\beta\gamma}\epsilon_{\dot{\alpha}\dot{\gamma}}\epsilon_{\dot{\beta}\dot{\zeta}})(\Xi^{\zeta\dot{\zeta}} - \tilde{\Xi}^{\dot{\zeta}\zeta}) \\ & = -i\,\epsilon_{\alpha\beta\gamma\zeta\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\zeta}}\,(\Xi^{\zeta\dot{\zeta}} - \tilde{\Xi}^{\dot{\zeta}\zeta})\end{aligned} \quad , \tag{B.9a}$$

$$\bar{K}_{\mu} \;\to\; 3(\Xi_{\alpha\dot{\alpha}} + \tilde{\Xi}_{\dot{\alpha}\alpha}) \;, \tag{B.9b}$$

$$\hat{K}_{\mu\nu\rho} \;\to\; \epsilon_{\dot{\alpha}\dot{\gamma}}\,\Omega_{\alpha\beta\gamma\dot{\beta}} + \;\epsilon_{\alpha\gamma}\,\tilde{\Omega}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\beta}\;. \tag{B.9c}$$

The factor of i in (B.9a) appears in Lorentzian signature only. The rest of this section is signature agnostic. Table B.1 summarises the share of degrees of freedom among the different tensor and spinor components. Under the map (3.7), $\Omega_{\alpha\beta\gamma\dot{\gamma}} = \tilde{\Omega}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\beta} = 0$, $\Xi_{\alpha\dot{\alpha}} + \tilde{\Xi}_{\dot{\alpha}\alpha}$ is related to $\partial_\mu\phi$ and $\Xi_{\alpha\dot{\alpha}} - \tilde{\Xi}_{\dot{\alpha}\alpha}$ to $\partial_\mu\sigma$.

B.2 Riemann spinors

A similar process can be followed to decompose the spinor equivalent of $\mathfrak{R}_{\mu\nu\rho\sigma}$. First, we will exhaust the – now smaller – symmetry group of the tensor to identify its irreducible spinor components. Then, employing (3.6), we will relate the newly found spinors to the contorsion spinors and the usual curvature spinors of $R_{\mu\nu\rho\sigma}$.

We begin the spinoral reduction of the generalised Riemann tensor by implementing its only symmetries: the antisymmetry of both pairs of indices

$$\begin{aligned}\mathfrak{R}_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\dot{\gamma}\delta\dot{\delta}} = \;& \mathbf{X}_{\alpha\beta\gamma\delta}\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\dot{\gamma}\dot{\delta}} + \tilde{\mathbf{X}}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} \\ & + \mathbf{\Phi}_{\alpha\beta\dot{\gamma}\dot{\delta}}\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\gamma\delta} + \tilde{\mathbf{\Phi}}_{\dot{\alpha}\dot{\beta}\gamma\delta}\epsilon_{\alpha\beta}\epsilon_{\dot{\gamma}\dot{\delta}}\;.\end{aligned} \tag{B.10}$$

The curvature spinors of $\mathfrak{R}$ are printed in bold typeface in order to distinguish them from those of R. Recall that $\mathfrak{R}_{\mu\nu\rho\sigma} \neq \mathfrak{R}_{\rho\sigma\mu\nu}$. The lack of this symmetry implies that $\mathbf{X}_{\alpha\beta\gamma\delta} \neq \mathbf{X}_{\gamma\delta\alpha\beta}$ and $\mathbf{\Phi}_{\alpha\beta\dot{\gamma}\dot{\delta}} \neq \tilde{\mathbf{\Phi}}_{\dot{\gamma}\dot{\delta}\alpha\beta}$ in general. The spinor $\mathbf{X}_{\alpha\beta\gamma\delta}$ is not completely symmetric and must be further reduced

$$\mathbf{X}_{\alpha\beta\gamma\delta} = \mathbf{\Psi}_{\alpha\beta\gamma\delta} - 2\left(\mathbf{\Sigma}_{\alpha(\gamma}\epsilon_{\delta)\beta} + \mathbf{\Sigma}_{\beta(\gamma}\epsilon_{\delta)\alpha}\right) + \mathbf{\Lambda}(\epsilon_{\alpha\gamma}\epsilon_{\beta\delta} + \epsilon_{\alpha\delta}\epsilon_{\beta\gamma})\;, \tag{B.11}$$

[1]The identity $\epsilon_{\alpha[\beta}\epsilon_{\gamma\delta]} = 0$ is needed to simplify the result of the calculation.

Spinor component	$\mathfrak{R}_{\mu\nu\rho\sigma}$	$R_{\mu\nu\rho\sigma}$
Ψ, $\tilde{\Psi}$	2×5	2×5
Σ, $\tilde{\Sigma}$	2×4	0
Λ, $\tilde{\Lambda}$	2×1	1
Φ, $\tilde{\Phi}$	2×9	9
Total	38	20

Table B.2: d.o.f counting for the curvature spinnors.

where $\mathbf{\Psi}_{\alpha\beta\gamma\delta}$ and $\mathbf{\Sigma}_{\alpha\beta}$ are completely symmetric. Putting tildes and dots yields the analogous expression for $\tilde{\mathbf{X}}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}$. It might be worth remarking that $\tilde{\mathbf{\Lambda}} \neq \mathbf{\Lambda}$, because $\mathfrak{R}_{\mu[\nu\rho\sigma]} \neq 0$. All the remaining spinors are completely symmetric and hence irreducible. Table B.2 shows how the degrees of freedom encoded in the irreducible spinors add up to 36, the number of independent (real) components of $\mathfrak{R}_{\mu\nu\rho\sigma}$ [165]. These degrees of freedom also include the Ricci tensor and the Ricci scalar, which can be obtained from the same spinor components

$$\mathfrak{R}_{\mu\rho\nu}{}^{\rho} \to -\mathbf{\Phi}_{\alpha\beta\dot{\alpha}\dot{\beta}} - \tilde{\mathbf{\Phi}}_{\dot{\alpha}\dot{\beta}\alpha\beta} + 4(\mathbf{\Sigma}_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} + \tilde{\mathbf{\Sigma}}_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\delta}) + 3(\mathbf{\Lambda} + \tilde{\mathbf{\Lambda}})\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} \tag{B.12}$$

$$\mathfrak{R}_{\mu\nu}{}^{\mu\nu} = 12(\mathbf{\Lambda} + \tilde{\mathbf{\Lambda}}) \ . \tag{B.13}$$

The inverse spinor identities

$$\mathbf{\Phi}_{\alpha\beta\dot{\gamma}\dot{\delta}} = \frac{1}{4}\mathfrak{R}_{\alpha\dot{\alpha}\beta}{}^{\dot{\alpha}}{}_{\gamma\dot{\gamma}}{}^{\gamma}{}_{\dot{\delta}} \ , \tag{B.14}$$

$$\mathbf{X}_{\alpha\beta\gamma\delta} = \frac{1}{4}\mathfrak{R}_{\alpha\dot{\alpha}\beta}{}^{\dot{\alpha}}{}_{\gamma\dot{\gamma}\delta}{}^{\dot{\gamma}} \ , \tag{B.15}$$

$$\mathbf{\Psi}_{\alpha\beta\gamma\delta} = \mathbf{X}_{(\alpha\beta\gamma\delta)} \ , \tag{B.16}$$

$$\mathbf{\Sigma}_{\alpha\beta} = \frac{1}{4}\mathbf{X}_{(\alpha|\gamma|\beta)}{}^{\gamma} \ , \tag{B.17}$$

$$\mathbf{\Lambda} = \frac{1}{6}\mathbf{X}_{\alpha\beta}{}^{\alpha\beta} \ , \tag{B.18}$$

are better suited expressions for computing the spinors of a given solution.

So far, we have identified the irreducible parts that make up the contorsion and the Riemann tensor. However, the Riemann tensor and the contorsion are not independent. Their relation is made explicit in equation (3.6). Hence, the bold curvature spinors must be functions of the contorsion spinors and the usual curvature spinors. The procedure to establish these relations is straightforward. First, the spinor equivalent of the right-hand side of (3.6) must be obtained. Then, applying (B.14)–(B.18) yields the desired expressions:

$$\begin{aligned}\mathbf{\Phi}_{\alpha\beta\dot\gamma\dot\delta} &= \Phi_{\alpha\beta\dot\gamma\dot\delta} + \frac{1}{4}\,\nabla_{(\alpha|\dot\alpha}\tilde\Theta_{\dot\gamma}{}^{\dot\alpha}{}_{\dot\delta|\beta)} - \frac{1}{4}\tilde\Theta_{\dot\gamma}{}^{\dot\alpha\dot\epsilon}{}_{(\alpha|}\,\tilde\Theta_{\dot\delta\dot\alpha\dot\epsilon|\beta)} \\ &= \Phi_{\alpha\beta\dot\gamma\dot\delta} + \nabla_{(\alpha|\dot\alpha}\tilde\Omega_{\dot\gamma}{}^{\dot\alpha}{}_{\dot\delta|\beta)} + \nabla_{\alpha(\dot\gamma}\tilde{\bar\Xi}_{\dot\delta)\beta} + \nabla_{\beta(\dot\gamma}\tilde{\bar\Xi}_{\dot\delta)\alpha} \\ &\qquad - 4\,\tilde{\bar\Xi}_{\dot\gamma(\alpha|}\tilde{\bar\Xi}_{\dot\delta|\beta)} - 2\,\tilde{\bar\Xi}^{\dot\alpha}{}_{(\alpha|}\tilde\Omega_{\dot\gamma\dot\delta\dot\alpha|\beta)} - \tilde\Omega_{\dot\gamma}{}^{\dot\alpha\dot\epsilon}{}_{(\alpha|}\tilde\Omega_{\dot\delta\dot\alpha\dot\epsilon|\beta)}\ ,\end{aligned} \tag{B.19}$$

$$\begin{aligned}\mathbf{\Psi}_{\alpha\beta\gamma\delta} &= \Psi_{\alpha\beta\gamma\delta} + \nabla_{\dot\alpha(\alpha}\Theta_{\beta\gamma\delta)}{}^{\dot\alpha} - \Theta_{(\alpha\beta}{}^{\epsilon\dot\alpha}\Theta_{\gamma\delta)\epsilon\dot\alpha} \\ &= \Psi_{\alpha\beta\gamma\delta} + \nabla_{\dot\alpha(\alpha}\Omega_{\beta\gamma\delta)}{}^{\dot\alpha} + 2\,\Xi_{(\alpha}{}^{\dot\alpha}\,\Omega_{\beta\gamma\delta)\dot\alpha} - \Omega_{(\alpha\beta}{}^{\epsilon\dot\alpha}\,\Omega_{\gamma\delta)\epsilon\dot\alpha}\end{aligned} \tag{B.20}$$

$$\mathbf{\Sigma}_{\alpha\beta} = -\frac{1}{2}\nabla_{\dot\alpha(\alpha}\Xi_{\beta)}{}^{\dot\alpha} + \frac{1}{8}\nabla_{\gamma\dot\alpha}\,\Omega_{\alpha\beta}{}^{\gamma\dot\alpha} + \frac{3}{4}\,\Xi^{\ \gamma\dot\alpha}\,\Omega_{\alpha\beta\gamma\dot\alpha}\ , \tag{B.21}$$

$$\begin{aligned}\mathbf{\Lambda} &= \Lambda + \frac{1}{6}\,\nabla_{\beta\dot\alpha}\Theta^{\alpha}{}_{\alpha}{}^{\beta\dot\alpha} + \frac{1}{12}\,\Theta_{\alpha\beta\gamma\dot\alpha}\,\Theta^{\alpha\beta\gamma\dot\alpha} + \frac{1}{12}\,\Theta^{\alpha}{}_{\alpha}{}^{\beta\dot\alpha}\,\Theta_{\beta}{}^{\epsilon}{}_{\epsilon\dot\alpha} \\ &= \Lambda - \frac{1}{2}\nabla^{\alpha\dot\alpha}\Xi_{\alpha\dot\alpha} - \Xi_{\alpha\dot\alpha}\,\Xi^{\alpha\dot\alpha} + \frac{1}{12}\,\Omega_{\alpha\beta\gamma\dot\alpha}\,\Omega^{\alpha\beta\gamma\dot\alpha}\ .\end{aligned} \tag{B.22}$$

Appendix C

C-metric in Bondi coordinates

In this appendix, we shall derive the Bondi form of the C-metric. There exists a variety of papers addressing the radiative properties of the C-metric in the literature. An early attempt to study the C-metric using the Bondi method was carried out by Bičák [261], who gave an expression for the Bondi news (see also [262–264]). Other studies of the asymptotic properties of the C-metric include [223, 265–268]. Here, we provide a systematic procedure for deriving Bondi coordinates for the C-metric to any desired order in a $1/r$ expansion.

The most common form of the C-metric is

$$\mathrm{d}s^2 = \frac{1}{\alpha^2(x+y)^2}\left[-F(y)\mathrm{d}t^2 + \frac{\mathrm{d}y^2}{F(y)} + \frac{\mathrm{d}x^2}{G(x)} + G(x)\mathrm{d}\phi^2\right] , \tag{C.1}$$

where

$$G(x) = 1 - x^2 - 2m\,\alpha\, x^3 , \qquad F(y) = -G(-y) . \tag{C.2}$$

Here, $0 < 2\,\alpha\, m < 1$, $y \in (-x, \infty)$ and $x \in (x_2, x_3)$, with x_2 and x_3 the largest two roots of $G(x)$. The periodicity of the coordinate ϕ determines the location of the conical singularity. The coordinate domain $\phi \in (0, 2\pi\,\kappa)$ is equivalent to setting $\phi \in (0, 2\pi)$ and replacing $\mathrm{d}\phi \to \kappa\,\mathrm{d}\phi$ in the metric. For simplicity, we will set $\kappa = 1$ in this appendix, but we will reintroduce it in the main text.

An alternative form of the C-metric, given by Hong and Teo [227], has the same metric (C.1) but with a different form for the functions $F(y)$ and $G(x)$ with a simplified root structure:

$$G(x) = (1-x^2)(1 + 2\,m\,\alpha\, x) , \qquad F(y) = -G(-y) , \tag{C.3}$$

where $y \in (-x, \infty)$ and $x \in (-1, 1)$, $G(x) > 0$. This patch covers both the static and asymptotic regions, which is given by the limit $x \to -y$ [223].

In addition, the charged C-metric solution is given by same metric (C.1), but now with

$$G(x) = (1 - x^2)(1 + r_+ \alpha x)(1 + r_- \alpha x) , \qquad F(y) = -G(-y) , \tag{C.4}$$

where $r_\pm = m \pm \sqrt{m^2 - q^2}$ and $0 < r_-\alpha < r_+\alpha < 1$. The coordinate ranges are the same as those of the uncharged Hong-Teo coordinate system.

Given the fact that the form of the metric is the same in both coordinate systems, one can derive a Bondi form from both coordinate systems in one go; this is what we now proceed to do. We begin by relabelling x as $\hat{x}$ and defining coordinates Ω and w by

$$\Omega = \frac{1}{\alpha(\hat{x} + y)}, \quad w = t + \int \mathrm{d}y/F(y) . \tag{C.5}$$

The metric now takes the form

$$\mathrm{d}s^2 = \Omega^2 \left[-F(y)\mathrm{d}w^2 - \frac{2}{\alpha\Omega^2}\mathrm{d}w\mathrm{d}\Omega - 2dw\mathrm{d}\hat{x} + \frac{\mathrm{d}\hat{x}^2}{G(\hat{x})} + G(\hat{x})\mathrm{d}\phi^2 \right] , \tag{C.6}$$

with the understanding that y is given in terms of $\hat{x}$ and Ω by

$$y = -\hat{x} + \frac{1}{\alpha\,\Omega} . \tag{C.7}$$

Now, replacing $\hat{x}$ by α, with

$$\frac{\mathrm{d}\hat{x}}{G(\hat{x})} = \mathrm{d}w - \frac{\mathrm{d}\alpha}{\sin\alpha} \tag{C.8}$$

gives

$$\mathrm{d}s^2 = -\Omega^2 \left[F(y) + G(\hat{x}) \right] \mathrm{d}w^2 - \frac{2\mathrm{d}w\,\mathrm{d}\Omega}{\alpha} + \frac{\Omega^2\, G(\hat{x})}{\sin^2\alpha} \left(\mathrm{d}\alpha^2 + \sin^2\alpha\, \mathrm{d}\phi^2 \right) . \tag{C.9}$$

To summarise, we have transformed the C-metric into coordinates given by $(w, \Omega, \alpha, \phi)$ with an auxiliary coordinate $\hat{x}$ given implicitly in terms of the other coordinates via equation (C.8).

We can now proceed perturbatively, order by order in an inverse distance expansion, to put the metric into the Bondi form with coordinates (u, r, θ, ϕ). However, before we do this, it will prove useful to define a new auxiliary coordinate x implicitly in terms of the new coordinates u and θ by

$$u = \frac{Gj(x)\, \sin\theta}{\alpha} , \tag{C.10}$$

where

$$Gj(x) \equiv \int^x \frac{\mathrm{d}x'}{G(x')^{3/2}} . \tag{C.11}$$

The coordinate transformation will define the old coordinates in terms of the new coordinates, i.e. we have

$$w = w(u,r,\theta), \quad \Omega = \Omega(u,r,\theta), \quad \alpha = \alpha(u,r,\theta) \tag{C.12}$$

as well as a relation between the old auxiliary coordinate $\hat{x}$ and the new coordinates

$$\hat{x} = x + T(r,x,\theta)\,. \tag{C.13}$$

Writing these relations in terms of $1/r$ expansions

$$\begin{aligned}
\Omega &= \bar{g}_1(x,\theta)\,r + g_0(x,\theta) + \frac{g_1(x,\theta)}{r} + \frac{g_2(x,\theta)}{r^2} + \cdots\,,\\
\alpha &= \theta + \frac{\bar{h}_1(x,\theta)}{r} + \frac{h_0(x,\theta)}{r^2} + \frac{h_1(x,\theta)}{r^3} + \cdots\\
w &= \bar{f}_1(x,\theta) + \frac{f_0(x,\theta)}{r} + \frac{f_1(x,\theta)}{r^2} + \frac{f_2(x,\theta)}{r^3} + \cdots\\
T &= \frac{k_0(x,\theta)}{r} + \frac{k_2(x,\theta)}{r^2} + \frac{k_3(x,\theta)}{r^3} + \cdots\,,
\end{aligned} \tag{C.14}$$

we proceed in a systematic manner, requiring that the metric expressed in terms of the new coordinates (u,r,θ,ϕ) have the Bondi form (4.106) and satisfying the fall-offs (4.107) as well as the gauge condition (4.108). Note that in addition, we have the constraint that the old auxiliary coordinate $\hat{x}$ must satisfy equation (C.8).

We have chosen to define x so that it is equal to $\hat{x}$ up to a perturbative correction $T = \mathcal{O}(1/r)$, as can be seen from eqns (C.13) and (C.14). This means that $G(\hat{x})$ and $G(y)$ can simply be written in terms of x using a Taylor expansion. In particular,

$$G(\hat{x}) = G(x) + TG'(x) + \frac{1}{2}T^2G''(x) + \frac{1}{6}T^3G^{(3)}(x) + \frac{1}{24}T^4G^{(4)}(x), \tag{C.15}$$

where primes denote derivatives with respect to x, and we have used the fact that G is a quartic polynomial in its argument.

We now proceed order by order in inverse powers of r, by plugging the expansions (C.14) into the metric (C.9) and the constraint (C.8). First, we solve equation (C.8) at order r^0, obtaining first-order differential equations for $\bar{f}_1(x,\theta)$ whose solution is

$$\bar{f}_1(x,\theta) = Gi(x) + \log\tan\tfrac{1}{2}\theta\,, \qquad Gi(x) \equiv \int^x \frac{dx'}{G(x')}\,. \tag{C.16}$$

From this point on, all the expansion coefficients can be solved purely algebraically, according to the following scheme:

(a) Solve equation (4.108) at order r^0 for $\bar{g}_1(x,\theta)$.

(b) Solve $g_{r\theta} = 0$ at order r^0 for $\bar{h}_1(x,\theta)$.

(c) Solve $g_{rr} = 0$ at order r^{-2} for $f_0(x,\theta)$.

(d) Solve the dr component of equation (C.8) at order r^{-2} for $k_0(x,\theta)$. (Note that the differential equations for $k_0(x,\theta)$ that arise in the dx and dθ components of equation (C.8) at the preceding order r^{-1} are now automatically satisfied.)

One then proceeds by iterating steps (a)–(d) at the next order, solving algebraically for $g_0(x,\theta)$, $h_0(x,\theta)$, $f_1(x,\theta)$, and $k_1(x,\theta)$, and so on ad infinitum. The results for the first few expansion coefficients are:

$$\bar{f}_1 = Gi + \log\tan\tfrac{1}{2}\theta\,, \qquad \bar{g}_1 = \frac{\sin\theta}{\sqrt{G}}\,, \qquad \bar{h}_1 = \frac{\sqrt{G}\,Gj\;\cos\theta - 1}{\alpha\sqrt{G}}\,,$$
$$f_0 = -\frac{[\sqrt{G}\,Gj\;\cos\theta - 1]^2}{2\alpha\sqrt{G}\,\sin\theta}\,, \qquad k_0 = \frac{\sqrt{G}\,[1 - GGj^2\;\cos^2\theta]}{2A\,\sin\theta}\,,$$
$$g_0 = \frac{[2 + \sqrt{G}\,G'\,Gj\;\cos^2\theta]\,Gj}{4\alpha\sqrt{G(x)}}\,,$$
$$h_0 = \frac{[1 - \sqrt{G}\,Gj\;\cos\theta][2\cos\theta + \sqrt{G}\,G'\,Gj\;\cos\theta + 2\sqrt{G}\,Gj\;\sin^2\theta]}{4\alpha^2\,G\,\sin\theta}\,,$$
$$f_1 = \frac{[1 - \sqrt{G}\,Gj\;\cos\theta]^2\,[4\sqrt{G}\,Gj - G' + 2\sqrt{G}\,G'\,Gj\;\cos\theta + G\,G'\,Gj^2\;\cos^2\theta]}{16\alpha^2\,G\,\sin^2\theta}\,,$$
$$k_1 = \frac{[G\,Gj^2\;\cos^2\theta - 1][4\sqrt{G}\,Gj - G' + 3G\,G'\,Gj^2\;\cos^2\theta]}{16\alpha^2\,\sin^2\theta}\,,$$
$$g_1 = \frac{[2\sqrt{G}\,Gj + G']^2 - 2[G\,Gj^2\;\cos^2\theta - 1]^2\,G\,G''}{32\alpha^2\,G^{3/2}\,\sin\theta}\,, \tag{C.17}$$

where the arguments of G, G', G'', Gj and Gi are all x, defined implicitly in terms of u and θ by equation (C.10). We have obtained explicit results also for $(h_1, f_2, k_2, g_2, h_2, f_3, k_3, g_3, h_3)$.

The first few terms in the expansions (4.107) are given by

$$
\begin{aligned}
C_0^\theta &= \frac{[2G' + \sqrt{G}\,G'^2\,Gj - 2G^{3/2}\,G''\,Gj]\,\cos\theta}{8\alpha\,\sqrt{G}\,\sin^2\theta}\,, \qquad C_0^\phi = 0\,,\\
C_1^\theta &= \frac{G'^2\,\cos\theta}{16\alpha^2\,G\,\sin^3\theta} + \frac{Gj\,[9G'^2 - 6G'\,(2+3GG'') + 8G^2\,G''']\,\cos\theta}{96\alpha^2\,\sqrt{G}\,\sin^3\theta}\\
&\quad + \frac{Gj^2\,(3G'^2 - 6GG'' - 8)\,\cos\theta}{16\alpha^2\,\sin^3\theta} + \frac{Gj^3\,G^{5/2}\,G'''\,\cos^3\theta}{12\alpha^2\,\sin^3\theta}\,.\\
C_1^\phi &= 0\,,\\
C_{\theta\theta} &= -\frac{2\sqrt{G}\,Gj + G'}{2\alpha\,\sqrt{G}\,\sin\theta}\,, \qquad C_{\theta\phi} = 0\,, \qquad C_{\phi\phi} = \frac{2\sqrt{G}\,Gj + G'}{2\alpha\,\sqrt{G}}\sin\theta\,,\\
D^{(1)}_{\theta\theta} &= -\frac{Gj^4\,G^{5/2}\,G'''\,\cos^4\theta}{48\alpha^3\,\sin^3\theta} - \frac{Gj^3}{8\alpha^3\,\sin^3\theta} + \frac{Gj^2\,(2G^2\,G'''\,\cos^2\theta - 9G')}{48\alpha^3\,\sqrt{G}\,\sin^3\theta}\\
&\quad - \frac{3Gj\,G'^2}{32\alpha^3\,G\,\sin^3\theta} - \frac{3G'^3 + 4G^2\,G'''}{192\alpha^3\,G^{3/2}\,\sin^3\theta}\,,\\
D^{(1)}_{\phi\phi} &= -D^{(1)}_{\theta\theta}\,\sin^2\theta\,, \qquad D^{(1)}_{\theta\phi} = 0\,,\\
F_0 &= \frac{12G^3\,G'''\,Gj^2\,\cos^2\theta + 6\sqrt{G}\,[4 - G'^2 + 2G\,G'']\,Gj + 6G'\,(2 + G\,G'') - 3G'^3 - 4G^2\,G'''}{48\alpha\,\sqrt{G}\,\sin^3\theta}\,.
\end{aligned}
\tag{C.18}
$$

C.1 Small mass expansion

It is hard to gain much insight from these expressions as they stand, since x is defined implicitly in terms of u and θ by equation (C.10). One thing we can do is to consider an expansion in powers of the mass parameter m (or, to be precise, in powers of the small dimensionless quantity $m\alpha$). Expanding the integrand $G^{-3/2}$ in equation (C.11) in powers of m and then integrating term by term, we then have

$$
Gj(x) = \frac{x}{\sqrt{1-x^2}} + \frac{m\alpha\,(3x^2-2)}{(1-x^2)^{3/2}} + \mathcal{O}(m^2)\,. \tag{C.19}
$$

Conversely, we can then express x perturbatively in m, in terms of u and θ, finding

$$
x = \frac{u\alpha}{\sqrt{u^2\alpha^2 + \sin^2\theta}} + \frac{m\alpha(u^2\alpha^2 - 2\sin^2\theta)}{u^2\alpha^2 + \sin^2\theta} + \mathcal{O}(m^2)\,. \tag{C.20}
$$

It is now straightforward to expand the expressions in (C.18) for C_0^I, C^{IJ} and F_0 in powers of m. In particular, we find

$$
C_0^\theta = -\frac{m\,\sin\theta\,\cos\theta}{(u^2A^2 + \sin^2\theta)^{3/2}} + \mathcal{O}(m^2)\,. \tag{C.21}
$$

The expansion coefficient F_0, which is related to the Bondi mass aspect, is given by

$$F_0 = \frac{m\,[(3u^2\alpha^2 + 1)\,\sin^2\theta - 2u^2\alpha^2]}{(u^2\alpha^2 + \sin^2\theta)^{5/2}} + \mathcal{O}(m^2)\,. \tag{C.22}$$

We also have

$$C_{\theta\theta} = \frac{2m(2u^2\alpha^2 + \sin^2\theta)}{\sqrt{u^2\alpha^2 + \sin^2\theta}\,\sin^2\theta} + \mathcal{O}(m^2)\,, \qquad C_{\phi\phi} = -C_{\theta\theta}\,\sin^2\theta\,. \tag{C.23}$$

Appendix D

Equations of motion in double field theory

We review here the derivation of the double field theory (DFT) equations of motion. The covariant derivative and its curvature tensors are defined with respect to the so-called generalised Lie derivative or generalised diffeomorphism. It plays the role of gauge symmetry in DFT, and acts on the DFT field content as

$$\begin{aligned}(\hat{\mathcal{L}}_X\mathcal{H})_{MN} &= X^P\partial_P\mathcal{H}_{MN} + (\partial_M X^P - \partial^P X_M)\mathcal{H}_{PN} + (\partial_N X^P - \partial^P X_N)\mathcal{H}_{MP}\,,\\ \hat{\mathcal{L}}_X d &= X^M\partial_M d - \frac{1}{2}\partial_M X^M\,,\end{aligned} \tag{D.1}$$

where, with respect to the generalised Lie derivative, the generalised metric $\mathcal{H}$ is a rank-2 tensor and the DFT dilaton d is a scalar density. The gauge parameter X^M combines the diffeomorphism parameter ξ^μ and the one-form gauge parameter Λ_μ for the Kalb-Ramond field in an $O(D,D)$ covariant manner

$$X^M = \{\xi^\mu\,,\Lambda_\mu\}\,. \tag{D.2}$$

For closure of the algebra of generalised diffeomorphisms (i.e., the Jacobi identity for $\hat{\mathcal{L}}$), we have to impose the section condition

$$\partial_M\partial^M\mathcal{F}_1 = 0\,, \qquad \partial_M\mathcal{F}_1\partial^M\mathcal{F}_2 = 0\,, \tag{D.3}$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ are arbitrary functions on doubled space. The section condition is equivalent to ignoring the winding coordinate $\tilde{x}$ dependence,

$$\partial_M = \begin{pmatrix} \tilde{\partial}^\mu \\ \partial_\mu \end{pmatrix} = \begin{pmatrix} 0 \\ \partial_\mu \end{pmatrix}\,. \tag{D.4}$$

As for the covariant differential operator of the generalised Lie derivative (D.1), we define the covariant derivative acting on an $O(D,D)$ tensor as

$$\mathcal{D}_M T_{N_1 N_2 \cdots N_n} = \partial_M T_{N_1 N_2 \cdots N_n} + \sum_{i=1}^{n} \Gamma_{M N_i}{}^{P} T_{N_1 \cdots P \cdots N_n}\,, \tag{D.5}$$

where Γ_{MNP} is the DFT connection [269, 270]. One may try to obtain the DFT connection using the compatibility and torsion-free conditions, analogously to Riemannian geometry. However, it turns out that these conditions are sufficient for determining all the components. Fortunately, one can project out the undetermined part using the projection operators (5.39), and the determined part is

$$\begin{aligned}\Gamma_{PMN} =& 2(P\partial_P P\bar{P})_{[MN]} + 2(\bar{P}_{[M}{}^{Q}\bar{P}_{N]}{}^{R} - P_{[M}{}^{Q}P_{N]}{}^{R})\partial_Q P_{RP} \\ &- \frac{4}{D-1}\big(\bar{P}_{P[M}\bar{P}_{N]}{}^{Q} + P_{P[M}P_{N]}{}^{Q}\big)\big(\partial_Q d + (P\partial^R P\bar{P})_{[RQ]}\big)\,.\end{aligned} \tag{D.6}$$

Let us turn to the curvature tensors $\mathcal{R}$ and $\mathcal{R}_{MN}$ in terms of the DFT connection (D.6). First, we introduce the 4-index object S_{MNPQ} defined as

$$S_{MNPQ} = \frac{1}{2}\big(R_{MNPQ} + R_{PQMN} - \Gamma^{R}{}_{MN}\Gamma_{RPQ}\big)\,, \tag{D.7}$$

where R_{MNPQ} is defined from the standard commutator of the covariant derivatives

$$R_{MNPQ} = \partial_M \Gamma_{NPQ} - \partial_N \Gamma_{MPQ} + \Gamma_{MP}{}^{R}\Gamma_{NRQ} - \Gamma_{NP}{}^{R}\Gamma_{MRQ}\,. \tag{D.8}$$

One can show that S_{MNPQ} satisfies similar symmetry properties as the Riemann tensor, namely $S_{MNPQ} = S_{[MN][PQ]} = S_{[PQ][MN]}$ as well as the Bianchi identity

$$S_{M[NPQ]} = 0\,. \tag{D.9}$$

However, it is not a tensor with respect to the generalised Lie derivative and cannot be a physically meaningful object. Instead, we can obtain meaningful tensors by contracting S_{MNPQ} with the projection operators. The generalised curvature tensor and scalar are defined as

$$\mathcal{R}_{MN} = 2P_{(M}{}^{P}\bar{P}_{N)}{}^{Q}P^{RS}S_{RPSQ}\,, \qquad \mathcal{R} = 2P^{MN}P^{PQ}S_{MPNQ}\,, \tag{D.10}$$

and one can show that these are covariant under $O(D,D)$, as well as under generalised diffeomorphisms. Substituting the parametrisations (5.37), the equations of motion reduce to the conventional supergravity equations of motion (5.48). The generalized curvatures satisfy an similar identity as the Einstein tensor, $\nabla_\mu G^{\mu\nu} = 0$,

$$\mathcal{D}_M\left(4P^{MP}\bar{P}^{NQ}\mathcal{R}_{PQ} - \bar{P}^{MN}\mathcal{R}\right) = 0\,, \qquad \mathcal{D}_M\left(4\bar{P}^{MP}P^{NQ}\mathcal{R}_{PQ} + P^{MN}\mathcal{R}\right) = 0\,. \tag{D.11}$$

Bibliography

[1] R. Monteiro, S. Nagy, D. O'Connell, D. Peinador Veiga and M. Sergola, *NS-NS spacetimes from amplitudes*, *JHEP* **06** (2022) 021 [2112.08336].

[2] H. Godazgar, M. Godazgar, R. Monteiro, D. Peinador Veiga and C.N. Pope, *Asymptotic Weyl double copy*, *JHEP* **11** (2021) 126 [2109.07866].

[3] R. Monteiro, D. O'Connell, D. Peinador Veiga and M. Sergola, *Classical solutions and their double copy in split signature*, *JHEP* **05** (2021) 268 [2012.11190].

[4] H. Godazgar, M. Godazgar, R. Monteiro, D. Peinador Veiga and C.N. Pope, *Weyl Double Copy for Gravitational Waves*, *Phys. Rev. Lett.* **126** (2021) 101103 [2010.02925].

[5] K. Kim, K. Lee, R. Monteiro, I. Nicholson and D. Peinador Veiga, *The Classical Double Copy of a Point Charge*, *JHEP* **02** (2020) 046 [1912.02177].

[6] R. Alawadhi, D.S. Berman, B. Spence and D. Peinador Veiga, *S-duality and the double copy*, *JHEP* **03** (2020) 059 [1911.06797].

[7] R. Britto, F. Cachazo, B. Feng and E. Witten, *Direct proof of tree-level recursion relation in Yang-Mills theory*, *Phys. Rev. Lett.* **94** (2005) 181602 [hep-th/0501052].

[8] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *One loop n point gauge theory amplitudes, unitarity and collinear limits*, *Nucl. Phys. B* **425** (1994) 217 [hep-ph/9403226].

[9] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *Fusing gauge theory tree amplitudes into loop amplitudes*, *Nucl. Phys. B* **435** (1995) 59 [hep-ph/9409265].

[10] Z. Bern, J.J.M. Carrasco and H. Johansson, *New Relations for Gauge-Theory Amplitudes*, *Phys. Rev.* **D78** (2008) 085011 [0805.3993].

[11] Z. Bern, J.J.M. Carrasco and H. Johansson, *Perturbative Quantum Gravity as a Double Copy of Gauge Theory*, *Phys. Rev. Lett.* **105** (2010) 061602 [1004.0476].

[12] H. Kawai, D. Lewellen and S. Tye, *A Relation Between Tree Amplitudes of Closed and Open Strings*, *Nucl. Phys. B* **269** (1986) 1.

[13] F. Cachazo, S. He and E.Y. Yuan, *Scattering equations and Kawai-Lewellen-Tye orthogonality*, *Phys. Rev. D* **90** (2014) 065001 [1306.6575].

[14] F. Cachazo, S. He and E.Y. Yuan, *Scattering of Massless Particles in Arbitrary Dimensions*, *Phys. Rev. Lett.* **113** (2014) 171601 [1307.2199].

[15] Z. Bern, J.J. Carrasco, M. Chiodaroli, H. Johansson and R. Roiban, *The Duality Between Color and Kinematics and its Applications*, 1909.01358.

[16] Z. Bern, J.J. Carrasco, M. Chiodaroli, H. Johansson and R. Roiban, *The SAGEX Review on Scattering Amplitudes, Chapter 2: An Invitation to Color-Kinematics Duality and the Double Copy*, 2203.13013.

[17] T. Adamo, J.J.M. Carrasco, M. Carrillo-González, M. Chiodaroli, H. Elvang, H. Johansson et al., *Snowmass White Paper: the Double Copy and its Applications*, in *2022 Snowmass Summer Study*, 4, 2022 [2204.06547].

[18] C.-H. Fu and K. Krasnov, *Colour-Kinematics duality and the Drinfeld double of the Lie algebra of diffeomorphisms*, *JHEP* **01** (2017) 075 [1603.02033].

[19] A. Brandhuber, G. Chen, H. Johansson, G. Travaglini and C. Wen, *Kinematic Hopf Algebra for Bern-Carrasco-Johansson Numerators in Heavy-Mass Effective Field Theory and Yang-Mills Theory*, *Phys. Rev. Lett.* **128** (2022) 121601 [2111.15649].

[20] R. Monteiro and D. O'Connell, *The Kinematic Algebra From the Self-Dual Sector*, *JHEP* **07** (2011) 007 [1105.2565].

[21] R.H. Boels, R.S. Isermann, R. Monteiro and D. O'Connell, *Colour-Kinematics Duality for One-Loop Rational Amplitudes*, *JHEP* **04** (2013) 107 [1301.4165].

[22] K. Krasnov and E. Skvortsov, *Flat self-dual gravity*, *JHEP* **08** (2021) 082 [2106.01397].

[23] Z. Bern, T. Dennen, Y.-t. Huang and M. Kiermaier, *Gravity as the Square of Gauge Theory*, *Phys. Rev.* **D82** (2010) 065003 [1004.0693].

[24] M. Tolotti and S. Weinzierl, *Construction of an effective Yang-Mills Lagrangian with manifest BCJ duality*, *JHEP* **07** (2013) 111 [1306.2975].

[25] D. Vaman and Y.-P. Yao, *Color kinematic symmetric (BCJ) numerators in a light-like gauge*, *JHEP* **12** (2014) 036 [1408.2818].

[26] P. Mastrolia, A. Primo, U. Schubert and W.J. Torres Bobadilla, *Off-shell currents and color–kinematics duality*, *Phys. Lett. B* **753** (2016) 242 [1507.07532].

[27] LIGO Scientific, Virgo collaboration, *Observation of Gravitational Waves from a Binary Black Hole Merger*, *Phys. Rev. Lett.* **116** (2016) 061102 [1602.03837].

[28] M.-Z. Chung, Y.-T. Huang, J.-W. Kim and S. Lee, *The simplest massive S-matrix: from minimal coupling to Black Holes*, *JHEP* **04** (2019) 156 [1812.08752].

[29] M.-Z. Chung, Y.-T. Huang and J.-W. Kim, *Classical potential for general spinning bodies*, *JHEP* **09** (2020) 074 [1908.08463].

[30] Z. Bern, A. Luna, R. Roiban, C.-H. Shen and M. Zeng, *Spinning black hole binary dynamics, scattering amplitudes, and effective field theory*, *Phys. Rev. D* **104** (2021) 065014 [2005.03071].

[31] D. Neill and I.Z. Rothstein, *Classical Space-Times from the S Matrix*, *Nucl. Phys.* **B877** (2013) 177 [1304.7263].

[32] C. Cheung, I.Z. Rothstein and M.P. Solon, *From Scattering Amplitudes to Classical Potentials in the Post-Minkowskian Expansion*, *Phys. Rev. Lett.* **121** (2018) 251101 [1808.02489].

[33] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M.P. Solon and M. Zeng, *Black Hole Binary Dynamics from the Double Copy and Effective Theory*, *JHEP* **10** (2019) 206 [1908.01493].

[34] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M.P. Solon and M. Zeng, *Scattering Amplitudes and the Conservative Hamiltonian for Binary Systems at Third Post-Minkowskian Order*, *Phys. Rev. Lett.* **122** (2019) 201603 [1901.04424].

[35] N.E.J. Bjerrum-Bohr, J.F. Donoghue and P. Vanhove, *On-shell Techniques and Universal Results in Quantum Gravity*, *JHEP* **02** (2014) 111 [1309.0804].

[36] N.E.J. Bjerrum-Bohr, J.F. Donoghue, B.R. Holstein, L. Planté and P. Vanhove, *Bending of Light in Quantum Gravity*, *Phys. Rev. Lett.* **114** (2015) 061301 [1410.7590].

[37] A. Luna, R. Monteiro, I. Nicholson, D. O'Connell and C.D. White, *The double copy: Bremsstrahlung and accelerating black holes*, *JHEP* **06** (2016) 023 [1603.05737].

[38] T. Damour, *Gravitational scattering, post-Minkowskian approximation and Effective One-Body theory*, *Phys. Rev.* **D94** (2016) 104015 [1609.00354].

[39] W.D. Goldberger and A.K. Ridgway, *Radiation and the classical double copy for color charges*, *Phys. Rev. D* **95** (2017) 125010 [1611.03493].

[40] F. Cachazo and A. Guevara, *Leading Singularities and Classical Gravitational Scattering*, *JHEP* **02** (2020) 181 [1705.10262].

[41] A. Guevara, *Holomorphic Classical Limit for Spin Effects in Gravitational and Electromagnetic Scattering*, *JHEP* **04** (2019) 033 [1706.02314].

[42] T. Damour, *High-energy gravitational scattering and the general relativistic two-body problem*, *Phys. Rev.* **D97** (2018) 044038 [1710.10599].

[43] A. Luna, I. Nicholson, D. O'Connell and C.D. White, *Inelastic Black Hole Scattering from Charged Scalar Amplitudes*, *JHEP* **03** (2018) 044 [1711.03901].

[44] D.A. Kosower, B. Maybee and D. O'Connell, *Amplitudes, Observables, and Classical Scattering*, *JHEP* **02** (2019) 137 [1811.10950].

[45] B. Maybee, D. O'Connell and J. Vines, *Observables and amplitudes for spinning particles and black holes*, *JHEP* **12** (2019) 156 [1906.09260].

[46] R. Aoude and A. Ochirov, *Classical observables from coherent-spin amplitudes*, *JHEP* **10** (2021) 008 [2108.01649].

[47] A. Laddha and A. Sen, *Gravity Waves from Soft Theorem in General Dimensions*, *JHEP* **09** (2018) 105 [1801.07719].

[48] A. Laddha and A. Sen, *Observational Signature of the Logarithmic Terms in the Soft Graviton Theorem*, *Phys. Rev. D* **100** (2019) 024009 [1806.01872].

[49] N.E.J. Bjerrum-Bohr, P.H. Damgaard, G. Festuccia, L. Planté and P. Vanhove, *General Relativity from Scattering Amplitudes*, *Phys. Rev. Lett.* **121** (2018) 171601 [1806.04920].

[50] A. Cristofoli, N. Bjerrum-Bohr, P.H. Damgaard and P. Vanhove, *Post-Minkowskian Hamiltonians in general relativity*, *Phys. Rev. D* **100** (2019) 084040 [1906.01579].

[51] A. Guevara, A. Ochirov and J. Vines, *Black-hole scattering with general spin directions from minimal-coupling amplitudes*, *Phys. Rev. D* **100** (2019) 104024 [1906.10071].

[52] G. Kälin and R.A. Porto, *From Boundary Data to Bound States*, *JHEP* **01** (2020) 072 [1910.03008].

[53] G. Kälin and R.A. Porto, *From boundary data to bound states. Part II. Scattering angle to dynamical invariants (with twist)*, *JHEP* **02** (2020) 120 [1911.09130].

[54] D. Bini, T. Damour and A. Geralico, *Sixth post-Newtonian nonlocal-in-time dynamics of binary systems*, *Phys. Rev. D* **102** (2020) 084047 [2007.11239].

[55] R. Aoude, K. Haddad and A. Helset, *On-shell heavy particle effective theories*, *JHEP* **05** (2020) 051 [2001.09164].

[56] C. Cheung and M.P. Solon, *Classical gravitational scattering at $\mathcal{O}(G^3)$ from Feynman diagrams*, *JHEP* **06** (2020) 144 [2003.08351].

[57] C. Cheung and M.P. Solon, *Tidal Effects in the Post-Minkowskian Expansion*, *Phys. Rev. Lett.* **125** (2020) 191601 [2006.06665].

[58] G. Kälin, Z. Liu and R.A. Porto, *Conservative Dynamics of Binary Systems to Third Post-Minkowskian Order from the Effective Field Theory Approach*, *Phys. Rev. Lett.* **125** (2020) 261103 [2007.04977].

[59] K. Haddad and A. Helset, *Tidal effects in quantum field theory*, *JHEP* **12** (2020) 024 [2008.04920].

[60] G. Kälin, Z. Liu and R.A. Porto, *Conservative Tidal Effects in Compact Binary Systems to Next-to-Leading Post-Minkowskian Order*, *Phys. Rev. D* **102** (2020) 124025 [2008.06047].

[61] P. Di Vecchia, C. Heissenberg, R. Russo and G. Veneziano, *Universality of ultra-relativistic gravitational scattering*, *Phys. Lett. B* **811** (2020) 135924 [2008.12743].

[62] Z. Bern, J. Parra-Martinez, R. Roiban, E. Sawyer and C.-H. Shen, *Leading Nonlinear Tidal Effects and Scattering Amplitudes*, 2010.08559.

[63] M. Accettulli Huber, A. Brandhuber, S. De Angelis and G. Travaglini, *From amplitudes to gravitational radiation with cubic interactions and tidal effects*, *Phys. Rev. D* **103** (2021) 045015 [2012.06548].

[64] A. Buonanno, M. Khalil, D. O'Connell, R. Roiban, M.P. Solon and M. Zeng, *Snowmass White Paper: Gravitational Waves and Scattering Amplitudes*, in *2022 Snowmass Summer Study*, 4, 2022 [2204.05194].

[65] Y.F. Bautista and A. Guevara, *From Scattering Amplitudes to Classical Physics: Universality, Double Copy and Soft Theorems*, 1903.12419.

[66] E. Herrmann, J. Parra-Martinez, M.S. Ruf and M. Zeng, *Radiative classical gravitational observables at $\mathcal{O}(G^3)$ from scattering amplitudes*, *JHEP* **10** (2021) 148 [2104.03957].

[67] A. Cristofoli, R. Gonzo, D.A. Kosower and D. O'Connell, *Waveforms from Amplitudes*, 2107.10193.

[68] E. Herrmann, J. Parra-Martinez, M.S. Ruf and M. Zeng, *Gravitational Bremsstrahlung from Reverse Unitarity*, *Phys. Rev. Lett.* **126** (2021) 201602 [2101.07255].

[69] R. Saotome and R. Akhoury, *Relationship Between Gravity and Gauge Scattering in the High Energy Limit*, *JHEP* **01** (2013) 123 [1210.8111].

[70] A. Anastasiou, L. Borsten, M. Duff, L. Hughes and S. Nagy, *Yang-Mills origin of gravitational symmetries*, *Phys. Rev. Lett.* **113** (2014) 231606 [1408.4434].

[71] G. Cardoso, S. Nagy and S. Nampuri, *A double copy for $\mathcal{N} = 2$ supergravity: a linearised tale told on-shell*, *JHEP* **10** (2016) 127 [1609.05022].

[72] G. Cardoso, S. Nagy and S. Nampuri, *Multi-centered $\mathcal{N} = 2$ BPS black holes: a double copy description*, *JHEP* **04** (2017) 037 [1611.04409].

[73] A. Luna, R. Monteiro, I. Nicholson, A. Ochirov, D. O'Connell, N. Westerberg et al., *Perturbative spacetimes from Yang-Mills theory*, *JHEP* **04** (2017) 069 [1611.07508].

[74] A. Anastasiou, L. Borsten, M.J. Duff, S. Nagy and M. Zoccali, *Gravity as Gauge Theory Squared: A Ghost Story*, *Phys. Rev. Lett.* **121** (2018) 211601 [1807.02486].

[75] L. Borsten, I. Jubb, V. Makwana and S. Nagy, *Gauge $\times$ gauge on spheres*, *JHEP* **06** (2020) 096 [1911.12324].

[76] W.D. Goldberger, S.G. Prabhu and J.O. Thompson, *Classical gluon and graviton radiation from the bi-adjoint scalar double copy*, *Phys. Rev.* **D96** (2017) 065009 [1705.09263].

[77] W.D. Goldberger and A.K. Ridgway, *Bound states and the classical double copy*, *Phys. Rev.* **D97** (2018) 085019 [1711.09493].

[78] D. Chester, *Radiative double copy for Einstein-Yang-Mills theory*, *Phys. Rev.* **D97** (2018) 084025 [1712.08684].

[79] W.D. Goldberger, J. Li and S.G. Prabhu, *Spinning particles, axion radiation, and the classical double copy*, *Phys. Rev.* **D97** (2018) 105018 [1712.09250].

[80] J. Li and S.G. Prabhu, *Gravitational radiation from the classical spinning double copy*, *Phys. Rev.* **D97** (2018) 105019 [1803.02405].

[81] M. Carrillo González, R. Penco and M. Trodden, *Radiation of scalar modes and the classical double copy*, *JHEP* **11** (2018) 065 [1809.04611].

[82] C.-H. Shen, *Gravitational Radiation from Color-Kinematics Duality*, *JHEP* **11** (2018) 162 [1806.07388].

[83] J. Plefka, J. Steinhoff and W. Wormsbecher, *Effective action of dilaton gravity as the classical double copy of Yang-Mills theory*, *Phys. Rev.* **D99** (2019) 024021 [1807.09859].

[84] J. Plefka, C. Shi, J. Steinhoff and T. Wang, *Breakdown of the classical double copy for the effective action of dilaton-gravity at NNLO*, *Phys. Rev. D* **100** (2019) 086006 [1906.05875].

[85] T. Adamo, E. Casali, L. Mason and S. Nekovar, *Scattering on plane waves and the double copy*, *Class. Quant. Grav.* **35** (2018) 015004 [1706.08925].

[86] T. Adamo, E. Casali, L. Mason and S. Nekovar, *Plane wave backgrounds and colour-kinematics duality*, *JHEP* **02** (2019) 198 [1810.05115].

[87] N.E.J. Bjerrum-Bohr, J.F. Donoghue, B.R. Holstein, L. Planté and P. Vanhove, *Light-like Scattering in Quantum Gravity*, *JHEP* **11** (2016) 117 [1609.07477].

[88] N. Arkani-Hamed, Y.-t. Huang and D. O'Connell, *Kerr black holes as elementary particles*, *JHEP* **01** (2020) 046 [1906.10100].

[89] H. Johansson and A. Ochirov, *Double copy for massive quantum particles with spin*, *JHEP* **09** (2019) 040 [1906.12292].

[90] Y.-T. Huang, U. Kol and D. O'Connell, *Double copy of electric-magnetic duality*, *Phys. Rev. D* **102** (2020) 046005 [1911.06318].

[91] Y.F. Bautista and A. Guevara, *On the Double Copy for Spinning Matter*, 1908.11349.

[92] N. Moynihan, *Kerr-Newman from Minimal Coupling*, *JHEP* **01** (2020) 014 [1909.05217].

[93] J. Plefka, C. Shi and T. Wang, *Double copy of massive scalar QCD*, *Phys. Rev. D* **101** (2020) 066004 [1911.06785].

[94] G. Elor, K. Farnsworth, M.L. Graesser and G. Herczeg, *The Newman-Penrose Map and the Classical Double Copy*, 2006.08630.

[95] C. Cheung and J. Mangan, *Covariant color-kinematics duality*, *JHEP* **11** (2021) 069 [2108.02276].

[96] C. Cheung, J. Mangan, J. Parra-Martinez and N. Shah, *Non-perturbative Double Copy in Flatland*, 2204.07130.

[97] R. Monteiro, D. O'Connell and C.D. White, *Black holes and the double copy*, *JHEP* **12** (2014) 056 [1410.0239].

[98] I. Bah, R. Dempsey and P. Weck, *Kerr-Schild Double Copy and Complex Worldlines*, *JHEP* **02** (2020) 180 [1910.04197].

[99] A. Luna, R. Monteiro, I. Nicholson and D. O'Connell, *Type D Spacetimes and the Weyl Double Copy*, *Class. Quant. Grav.* **36** (2019) 065003 [1810.08183].

[100] C.D. White, *Twistorial Foundation for the Classical Double Copy*, *Phys. Rev. Lett.* **126** (2021) 061602 [2012.02479].

[101] E. Chacón, S. Nagy and C.D. White, *The Weyl double copy from twistor space*, *JHEP* **05** (2021) 2239 [2103.16441].

[102] E. Chacón, S. Nagy and C.D. White, *Alternative formulations of the twistor double copy*, 2112.06764.

[103] G. Lopes Cardoso, G. Inverso, S. Nagy and S. Nampuri, *Comments on the double copy construction for gravitational theories*, *PoS* **CORFU2017** (2018) 177 [1803.07670].

[104] L. Borsten and S. Nagy, *The pure BRST Einstein-Hilbert Lagrangian from the double-copy to cubic order*, *JHEP* **07** (2020) 093 [2004.14945].

[105] L. Borsten, I. Jubb, V. Makwana and S. Nagy, *Gauge $\times$ gauge = gravity on homogeneous spaces using tensor convolutions*, *JHEP* **06** (2021) 117 [2104.01135].

[106] A. Luna, S. Nagy and C.D. White, *The convolutional double copy: a case study with a point*, 2004.11254.

[107] A. Cristofoli, *Gravitational shock waves and scattering amplitudes*, 2006.08283.

[108] A. Guevara, B. Maybee, A. Ochirov, D. O'Connell and J. Vines, *A worldsheet for Kerr*, *JHEP* **03** (2021) 201 [2012.11570].

[109] A.I. Janis, E.T. Newman and J. Winicour, *Reality of the Schwarzschild Singularity*, *Phys. Rev. Lett.* **20** (1968) 878.

[110] I.Z. Fisher, *Scalar mesostatic field with regard for gravitational effects*, *Zh. Eksp. Teor. Fiz.* **18** (1948) 636 [gr-qc/9911008].

[111] M. Wyman, *Static Spherically Symmetric Scalar Fields in General Relativity*, *Phys. Rev. D* **24** (1981) 839.

[112] B.S. DeWitt, *Quantum Theory of Gravity. 3. Applications of the Covariant Theory*, *Phys. Rev.* **162** (1967) 1239.

[113] J. Polchinski, *String Theory*, Cambridge University Press (oct, 1998), 10.1017/CBO9780511816079.

[114] J. Scherk and J.H. Schwarz, *Dual Models and the Geometry of Space-Time*, *Phys. Lett. B* **52** (1974) 347.

[115] C.G. Callan, Jr., E.J. Martinec, M.J. Perry and D. Friedan, *Strings in Background Fields*, *Nucl. Phys. B* **262** (1985) 593.

[116] A. Abouelsaood, C.G. Callan, Jr., C.R. Nappi and S.A. Yost, *Open Strings in Background Gauge Fields*, *Nucl. Phys. B* **280** (1987) 599.

[117] N.E.J. Bjerrum-Bohr, P.H. Damgaard, T. Sondergaard and P. Vanhove, *The Momentum Kernel of Gauge and Gravity Theories*, *JHEP* **01** (2011) 001 [1010.3933].

[118] C.R. Mafra, O. Schlotterer and S. Stieberger, *Explicit BCJ Numerators from Pure Spinors*, *JHEP* **07** (2011) 092 [1104.5224].

[119] C. Cheung, A. Helset and J. Parra-Martinez, *Geometry-Kinematics Duality*, 2202.06972.

[120] K. Krasnov, *Self-Dual Gravity*, *Class. Quant. Grav.* **34** (2017) 095001 [1610.01457].

[121] M. Ben-Shahar and H. Johansson, *Off-shell color-kinematics duality for Chern-Simons*, *JHEP* **08** (2022) 035 [2112.11452].

[122] S.J. Parke and T.R. Taylor, *An Amplitude for n Gluon Scattering*, *Phys. Rev. Lett.* **56** (1986) 2459.

[123] R. Penrose, *A Spinor approach to general relativity*, *Annals Phys.* **10** (1960) 171.

[124] R. Penrose and W. Rindler, *Spinors and Space-Time*, Cambridge Monographs on Mathematical Physics, Cambridge Univ. Press, Cambridge, UK (4, 2011), 10.1017/CBO9780511564048.

[125] E. Newman and R. Penrose, *An Approach to gravitational radiation by a method of spin coefficients*, *J. Math. Phys.* **3** (1962) 566.

[126] R. Sachs, *Gravitational waves in general relativity. 6. The outgoing radiation condition*, *Proc. Roy. Soc. Lond. A* **264** (1961) 309.

[127] A. Manu, D. Ghosh, A. Laddha and P.V. Athira, *Soft radiation from scattering amplitudes revisited*, *JHEP* **05** (2021) 056 [2007.02077].

[128] L. de la Cruz, B. Maybee, D. O'Connell and A. Ross, *Classical Yang-Mills observables from amplitudes*, *JHEP* **12** (2020) 076 [2009.03842].

[129] L. de la Cruz, A. Luna and T. Scheopner, *Yang-Mills observables: from KMOC to eikonal through EFT*, *JHEP* **01** (2022) 045 [2108.02178].

[130] Z. Bern, J.P. Gatica, E. Herrmann, A. Luna and M. Zeng, *Scalar QED as a toy model for higher-order effects in classical gravitational scattering*, 2112.12243.

[131] A. Cristofoli, R. Gonzo, N. Moynihan, D. O'Connell, A. Ross, M. Sergola et al., *The Uncertainty Principle and Classical Amplitudes*, 2112.07556.

[132] L. de la Cruz, *Scattering amplitudes approach to hard thermal loops*, *Phys. Rev. D* **104** (2021) 014013 [2012.07714].

[133] Y.F. Bautista and A. Laddha, *Soft Constraints on KMOC Formalism*, 2111.11642.

[134] R. Britto, R. Gonzo and G.R. Jehu, *Graviton particle statistics and coherent states from classical scattering amplitudes*, *JHEP* **03** (2022) 214 [2112.07036].

[135] P. Benincasa and F. Cachazo, *Consistency Conditions on the S-Matrix of Massless Particles*, 0705.4305.

[136] N. Arkani-Hamed, T.-C. Huang and Y.-t. Huang, *Scattering Amplitudes For All Masses and Spins*, 1709.04891.

[137] J.F. Donoghue, *Leading quantum correction to the Newtonian potential*, *Phys. Rev. Lett.* **72** (1994) 2996 [gr-qc/9310024].

[138] J.F. Donoghue, *General relativity as an effective field theory: The leading quantum corrections*, *Phys. Rev.* **D50** (1994) 3874 [gr-qc/9405057].

[139] S. Srednyak and G. Sterman, *Perturbation theory in (2,2) signature*, *Phys. Rev. D* **87** (2013) 105017 [1302.4290].

[140] L. Mason, *Global anti-self-dual Yang-Mills fields in split signature and their scattering*, math-ph/0505039.

[141] J.W. Barrett, G. Gibbons, M. Perry, C. Pope and P. Ruback, *Kleinian geometry and the N=2 superstring*, *Int. J. Mod. Phys. A* **9** (1994) 1457 [hep-th/9302073].

[142] A. Atanasov, A. Ball, W. Melton, A.-M. Raclariu and A. Strominger, *(2, 2) Scattering and the celestial torus*, *JHEP* **07** (2021) 083 [2101.09591].

[143] E. Crawley, A. Guevara, N. Miller and A. Strominger, *Black Holes in Klein Space*, 2112.03954.

[144] A. Guevara, *Reconstructing Classical Spacetimes from the S-Matrix in Twistor Space*, 2112.05111.

[145] E.T. Newman and A.I. Janis, *Note on the Kerr spinning particle metric*, *J. Math. Phys.* **6** (1965) 915.

[146] W.T. Emond, Y.-T. Huang, U. Kol, N. Moynihan and D. O'Connell, *Amplitudes from Coulomb to Kerr-Taub-NUT*, 2010.07861.

[147] J. Vines, *Scattering of two spinning black holes in post-Minkowskian gravity, to all orders in spin, and effective-one-body mappings*, *Class. Quant. Grav.* **35** (2018) 084002 [1709.06016].

[148] A. Banerjee, E. Colgáin, J. Rosabal and H. Yavartanoo, *Ehlers as EM duality in the double copy*, *Phys. Rev. D* **102** (2020) 126017 [1912.02597].

[149] K. Lee, *Kerr-Schild Double Field Theory and Classical Double Copy*, *JHEP* **10** (2018) 027 [1807.08443].

[150] E. Lescano and J.A. Rodríguez, *Higher-derivative heterotic Double Field Theory and classical double copy*, *JHEP* **07** (2021) 072 [2101.03376].

[151] D.S. Berman, K. Kim and K. Lee, *The classical double copy for M-theory from a Kerr-Schild ansatz for exceptional field theory*, *JHEP* **04** (2021) 071 [2010.08255].

[152] S. Angus, K. Cho and K. Lee, *The classical double copy for half-maximal supergravities and T-duality*, *JHEP* **10** (2021) 211 [2105.12857].

[153] K. Cho, K. Kim and K. Lee, *The Off-Shell Recursion for Gravity and the Classical Double Copy for currents*, 2109.06392.

[154] F. Diaz-Jaramillo, O. Hohm and J. Plefka, *Double Field Theory as the Double Copy of Yang-Mills*, 2109.01153.

[155] R.I. Nepomechie, *On the Low-energy Limit of Strings*, *Phys. Rev. D* **32** (1985) 3201.

[156] D.J. Gross and J.H. Sloan, *The Quartic Effective Action for the Heterotic String*, *Nucl. Phys. B* **291** (1987) 41.

[157] M.C. Bento and N.E. Mavromatos, *Ambiguities in the Low-energy Effective Actions of String Theories With the Inclusion of Antisymmetric Tensor and Dilaton Fields*, *Phys. Lett. B* **190** (1987) 105.

[158] Z. Bern, T. Shimada and D. Hochberg, *Incompatibility of Torsion With the Gauss-Bonnet Combination in the Bosonic String*, *Phys. Lett. B* **191** (1987) 267.

[159] C. Hull and B. Zwiebach, *Double Field Theory*, *JHEP* **09** (2009) 099 [0904.4664].

[160] A. Saa, *A geometrical action for dilaton gravity*, *Class. Quant. Grav.* **12** (1995) L85 [hep-th/9307095].

[161] K.A. Dunn, *A scalar-tensor theory of gravitation*, *J. Math. Phys.* **15** (1974) 2229.

[162] J.B. Fonseca-Neto, C. Romero and S.P.G. Martinez, *Scalar torsion and a new symmetry of general relativity*, *Gen. Rel. Grav.* **45** (2013) 1579 [1211.1557].

[163] T. Dereli and R.W. Tucker, *An Einstein-Hilbert action for axidilaton gravity in four-dimensions*, *Class. Quant. Grav.* **12** (1995) L31 [gr-qc/9502018].

[164] F.W. Hehl and Y.N. Obukhov, *Elie Cartan's torsion in geometry and in field theory, an essay*, *Annales Fond. Broglie* **32** (2007) 157 [0711.1535].

[165] R. Penrose, *Spinors and torsion in general relativity*, *Found. Phys.* **13** (1983) 325.

[166] M.J. Duff, *Quantum Tree Graphs and the Schwarzschild Solution*, *Phys. Rev.* **D7** (1973) 2317.

[167] G.U. Jakobsen, *Schwarzschild-Tangherlini Metric from Scattering Amplitudes*, *Phys. Rev. D* **102** (2020) 104065 [2006.01734].

[168] S. Mougiakakos and P. Vanhove, *Schwarzschild-Tangherlini metric from scattering amplitudes in various dimensions*, *Phys. Rev. D* **103** (2021) 026001 [2010.08882].

[169] C.P. Burgess, R.C. Myers and F. Quevedo, *On spherically symmetric string solutions in four-dimensions*, *Nucl. Phys. B* **442** (1995) 75 [hep-th/9410142].

[170] I. Bogush and D. Gal'tsov, *Generation of rotating solutions in Einstein-scalar gravity*, *Phys. Rev. D* **102** (2020) 124006 [2001.02936].

[171] H. Stephani, D. Kramer, M.A.H. MacCallum, C. Hoenselaers and E. Herlt, *Exact solutions of Einstein's field equations*, Cambridge Monographs on Mathematical Physics, Cambridge Univ. Press, Cambridge (2003), 10.1017/CBO9780511535185.

[172] R. Alawadhi, D.S. Berman and B. Spence, *Weyl doubling*, *JHEP* **09** (2020) 127 [2007.03264].

[173] A. Luna, R. Monteiro, D. O'Connell and C.D. White, *The classical double copy for Taub–NUT spacetime*, *Phys. Lett.* **B750** (2015) 272 [1507.01869].

[174] M. Walker and R. Penrose, *On quadratic first integrals of the geodesic equations for type [22] spacetimes*, *Commun. Math. Phys.* **18** (1970) 265.

[175] L.P. Hughston, R. Penrose, P. Sommers and M. Walker, *On a quadratic first integral for the charged particle orbits in the charged kerr solution*, *Commun. Math. Phys.* **27** (1972) 303.

[176] S. Sabharwal and J.W. Dalhuisen, *Anti-Self-Dual Spacetimes, Gravitational Instantons and Knotted Zeros of the Weyl Tensor*, *JHEP* **07** (2019) 004 [1904.06030].

[177] D.A. Easson, T. Manton and A. Svesko, *Sources in the Weyl double copy*, 2110.02293.

[178] C. Keeler, T. Manton and N. Monga, *From Navier-Stokes to Maxwell via Einstein*, *JHEP* **08** (2020) 147 [2005.04242].

[179] T. Adamo and U. Kol, *Classical double copy at null infinity*, 2109.07832.

[180] R. Monteiro, I. Nicholson and D. O'Connell, *Spinor-helicity and the algebraic classification of higher-dimensional spacetimes*, *Class. Quant. Grav.* **36** (2019) 065006 [1809.03906].

[181] W.T. Emond and N. Moynihan, *Scattering Amplitudes and The Cotton Double Copy*, 2202.10499.

[182] M.C. González, A. Momeni and J. Rumbutis, *Cotton Double Copy for Gravitational Waves*, 2202.10476.

[183] M. Carrillo González, B. Melcher, K. Ratliff, S. Watson and C.D. White, *The classical double copy in three spacetime dimensions*, 1904.11001.

[184] J.N. Goldberg and R.K. Sachs, *A theorem on Petrov types*, *Acta Phys. Pol.* **22 (supplement)** (1962) .

[185] B. Carter, *Global structure of the kerr family of gravitational fields*, *Phys. Rev.* **174** (1968) 1559.

[186] A. Ilderton, *Screw-symmetric gravitational waves: a double copy of the vortex*, *Phys. Lett. B* **782** (2018) 22 [1804.07290].

[187] D.S. Berman, E. Chacón, A. Luna and C.D. White, *The self-dual classical double copy, and the Eguchi-Hanson instanton*, *JHEP* **01** (2019) 107 [1809.04063].

[188] K. Andrzejewski and S. Prencel, *From polarized gravitational waves to analytically solvable electromagnetic beams*, *Phys. Rev. D* **100** (2019) 045006 [1901.05255].

[189] H. Godazgar, M. Godazgar and C. Pope, *Taub-NUT from the Dirac monopole*, *Phys. Lett. B* **798** (2019) 134938 [1908.05962].

[190] N. Bahjat-Abbas, R. Stark-Muchão and C.D. White, *Monopoles, shockwaves and the classical double copy*, *JHEP* **04** (2020) 102 [2001.09918].

[191] L. Alfonsi, C.D. White and S. Wikeley, *Topology and Wilson lines: global aspects of the double copy*, *JHEP* **07** (2020) 091 [2004.07181].

[192] T. Adamo and A. Ilderton, *Classical and quantum double copy of back-reaction*, *JHEP* **09** (2020) 200 [2005.05807].

[193] J.F. Plebanski and M. Demianski, *Rotating, charged, and uniformly accelerating mass in general relativity*, *Annals Phys.* **98** (1976) 98.

[194] J.B. Griffiths and J. Podolsky, *A New look at the Plebanski-Demianski family of solutions*, *Int. J. Mod. Phys. D* **15** (2006) 335 [gr-qc/0511091].

[195] J. Ehlers, *Transformations of static exterior solutions of Einstein's gravitational field equations into different solutions by means of conformal mapping*, *Colloq. Int. CNRS* **91** (1962) 275.

[196] R.P. Geroch, *A Method for generating solutions of Einstein's equations*, *J. Math. Phys.* **12** (1971) 918.

[197] M. Mars, *Space-time Ehlers group: Transformation law for the Weyl tensor*, *Class. Quant. Grav.* **18** (2001) 719 [gr-qc/0101020].

[198] T. Ortin, *Gravity and Strings*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, 2nd ed. ed. (7, 2015), 10.1017/CBO9781139019750.

[199] U. Kol, *Duality in Einstein's Gravity*, 2205.05752.

[200] I. Hauser, *Type-N gravitational field with twist*, *Phys. Rev. Lett.* **33** (1974) 1112.

[201] W. Dietz and R. Rüdiger, *Space-times admitting Killing-Yano tensors. I*, *Proc. Roy. Soc. Lond. A* **375** (1981) 361.

[202] A. Strominger, *Lectures on the Infrared Structure of Gravity and Gauge Theory*, 1703.05448.

[203] M. Campiglia and S. Nagy, *A double copy for asymptotic symmetries in the self-dual sector*, *JHEP* **03** (2021) 262 [2102.01680].

[204] E. Casali and A. Puhm, *A Double Copy for Celestial Amplitudes*, *Phys. Rev. Lett.* **126** (2020) 101602 [2007.15027].

[205] E. Casali and A. Sharma, *Celestial double copy from the worldsheet*, *JHEP* **05** (2021) 157 [2011.10052].

[206] S. Pasterski and A. Puhm, *Shifting Spin on the Celestial Sphere*, 2012.15694.

[207] Z.-W. Chong, M. Cvetič, H. Lü and C. Pope, *Charged rotating black holes in four-dimensional gauged and ungauged supergravities*, *Nuclear Physics B* **717** (2005) 246–271.

[208] A. Strominger and A. Zhiboedov, *Superrotations and Black Hole Pair Creation*, *Class. Quant. Grav.* **34** (2017) 064002 [1610.00639].

[209] R.K. Sachs, *Gravitational waves in general relativity: 8. Waves in asymptotically flat space-times*, *Proc. Roy. Soc. Lond.* **A270** (1962) 103.

[210] H. Bondi, M.G.J. van der Burg and A.W.K. Metzner, *Gravitational waves in general relativity: 7. Waves from axisymmetric isolated systems*, *Proc. Roy. Soc. Lond.* **A269** (1962) 21.

[211] H. Godazgar, M. Godazgar and C.N. Pope, *Subleading BMS charges and fake news near null infinity*, *JHEP* **01** (2019) 143 [1809.09076].

[212] D. Bini and T. Damour, *Gravitational spin-orbit coupling in binary systems at the second post-Minkowskian approximation*, *Phys. Rev. D* **98** (2018) 044036 [1805.10809].

[213] D. Christodoulou, *The global initial value problem in general relativity*, in *The 9th Marcel Grossmann meeting*, V.G. Gurzadyan, R.T. Jantzen and R. Ruffini, eds., pp. 44–54, Dec., 2002, DOI.

[214] L.M.A. Kehrberger, *The Case Against Smooth Null Infinity I: Heuristics and Counter-Examples*, *Annales Henri Poincare* **23** (2022) 829 [2105.08079].

[215] P.T. Chrusciel, M.A.H. MacCallum and D.B. Singleton, *Gravitational waves in general relativity xiv. bondi expansions and the 'polyhomogeneity' of i*, *Phil. Trans. R. Soc. A* **350** (1995) 113.

[216] P.T. Chrusciel, J. Jezierski and M.A. MacCallum, *Uniqueness of the Trautman-Bondi mass*, *Phys. Rev. D* **58** (1998) 084001 [gr-qc/9803010].

[217] J.A. Valiente-Kroon, *Conserved quantities for polyhomogeneous space-times*, *Class. Quant. Grav.* **15** (1998) 2479 [gr-qc/9805094].

[218] J.A. Valiente-Kroon, *Polyhomogeneity and zero rest mass fields with applications to Newman-Penrose constants*, *Class. Quant. Grav.* **17** (2000) 605 [gr-qc/9907097].

[219] R.K. Sachs, *On the Characteristic Initial Value Problem in Gravitational Theory*, *J. Math. Phys.* **3** (1962) 908.

[220] H. Friedrich, *The asymptotic characteristic initial value problem for Einstein's vacuum field equations as an initial value problem for a first-order quasilinear symmetric hyperbolic system*, *Proc. Roy. Soc. Lond. A* **378** (1981) 401.

[221] H. Friedrich, *On the Regular and Asymptotic Characteristic Initial Value Problem for Einstein's Vacuum Field Equations*, *Proc. Roy. Soc. Lond. A* **375** (1981) 169.

[222] H. Friedrich, *ON PURELY RADIATIVE SPACE-TIMES*, *Commun. Math. Phys.* **103** (1986) 35.

[223] J.B. Griffiths, P. Krtous and J. Podolsky, *Interpreting the C-metric*, *Class. Quant. Grav.* **23** (2006) 6745 [gr-qc/0609056].

[224] L. Borsten, M.J. Duff, L.J. Hughes and S. Nagy, *Magic Square from Yang-Mills Squared*, *Phys. Rev. Lett.* **112** (2014) 131601 [1301.4176].

[225] A. Anastasiou, L. Borsten, M.J. Hughes and S. Nagy, *Global symmetries of Yang-Mills squared in various dimensions*, *JHEP* **01** (2016) 148 [1502.05359].

[226] A. Anastasiou, L. Borsten, M.J. Duff, M.J. Hughes, A. Marrani, S. Nagy et al., *Twin supergravities from Yang-Mills theory squared*, *Phys. Rev. D* **96** (2017) 026013 [1610.07192].

[227] K. Hong and E. Teo, *A new form of the c-metric*, *Classical and Quantum Gravity* **20** (2003) 3269–3277.

[228] T. He, P. Mitra, A.P. Porfyriadis and A. Strominger, *New symmetries of massless QED*, *Journal of High Energy Physics* **2014** (2014) .

[229] A. Strominger, *Asymptotic Symmetries of Yang-Mills Theory*, *JHEP* **07** (2014) 151 [1308.0589].

[230] N. Bahjat-Abbas, A. Luna and C.D. White, *The Kerr-Schild double copy in curved spacetime*, *JHEP* **12** (2017) 004 [1710.01953].

[231] M. Carrillo-González, R. Penco and M. Trodden, *The classical double copy in maximally symmetric spacetimes*, *JHEP* **04** (2018) 028 [1711.01296].

[232] G. Alkac, M.K. Gumus and M. Tek, *The Kerr-Schild Double Copy in Lifshitz Spacetime*, *JHEP* **05** (2021) 214 [2103.06986].

[233] M.K. Gumus and G. Alkac, *More on the classical double copy in three spacetime dimensions*, *Phys. Rev. D* **102** (2020) 024074 [2006.00552].

[234] G. Alkac, M.K. Gumus and M.A. Olpak, *Kerr-Schild double copy of the Coulomb solution in three dimensions*, *Phys. Rev. D* **104** (2021) 044034 [2105.11550].

[235] A.K. Ridgway and M.B. Wise, *Static Spherically Symmetric Kerr-Schild Metrics and Implications for the Classical Double Copy*, *Phys. Rev.* **D94** (2016) 044023 [1512.02243].

[236] M. Gurses and B. Tekin, *Classical Double Copy: Kerr-Schild-Kundt metrics from Yang-Mills Theory*, *Phys. Rev. D* **98** (2018) 126017 [1810.03411].

[237] K. Armstrong-Williams, C.D. White and S. Wikeley, *Non-perturbative aspects of the self-dual double copy*, 2205.02136.

[238] O. Pasarin and A.A. Tseytlin, *Generalised Schwarzschild metric from double copy of point-like charge solution in Born-Infeld theory*, *Phys. Lett. B* **807** (2020) 135594 [2005.12396].

[239] D.A. Easson, C. Keeler and T. Manton, *Classical double copy of nonsingular black holes*, *Phys. Rev. D* **102** (2020) 086015 [2007.16186].

[240] K. Mkrtchyan and M. Svazas, *Solutions in Nonlinear Electrodynamics and their double copy regular black holes*, 2205.14187.

[241] R. Alawadhi, D.S. Berman, C.D. White and S. Wikeley, *The single copy of the gravitational holonomy*, *JHEP* **10** (2021) 229 [2107.01114].

[242] R. Alawadhi, *Single copy of the Ricci flow*, 2202.09874.

[243] R. Gonzo and C. Shi, *Geodesics from classical double copy*, *Phys. Rev. D* **104** (2021) 105012 [2109.01072].

[244] W. Siegel, *Two vierbein formalism for string inspired axionic gravity*, *Phys. Rev. D* **47** (1993) 5453 [hep-th/9302036].

[245] W. Siegel, *Superspace duality in low-energy superstrings*, *Phys. Rev. D* **48** (1993) 2826 [hep-th/9305073].

[246] C. Hull and B. Zwiebach, *The Gauge algebra of double field theory and Courant brackets*, *JHEP* **09** (2009) 090 [0908.1792].

[247] O. Hohm, C. Hull and B. Zwiebach, *Background independent action for double field theory*, *JHEP* **07** (2010) 016 [1003.5027].

[248] O. Hohm, C. Hull and B. Zwiebach, *Generalized metric formulation of double field theory*, *JHEP* **08** (2010) 008 [1006.4823].

[249] G. Aldazabal, D. Marques and C. Nunez, *Double Field Theory: A Pedagogical Review*, *Class. Quant. Grav.* **30** (2013) 163001 [1305.1907].

[250] W. Cho and K. Lee, *Heterotic Kerr-Schild Double Field Theory and Classical Double Copy*, *JHEP* **07** (2019) 030 [1904.11650].

[251] D.S. Berman, K. Kim and K. Lee, *Double copying Exceptional Field theories*, 2201.10854.

[252] O. Hohm, *On factorizations in perturbative quantum gravity*, *JHEP* **04** (2011) 103 [1103.0032].

[253] C. Cheung and G.N. Remmen, *Twofold Symmetries of the Pure Gravity Action*, *JHEP* **01** (2017) 104 [1612.03927].

[254] C.D. White, *Exact solutions for the biadjoint scalar field*, *Phys. Lett. B* **763** (2016) 365 [1606.04724].

[255] P.-J. De Smet and C.D. White, *Extended solutions for the biadjoint scalar field*, *Phys. Lett. B* **775** (2017) 163 [1708.01103].

[256] N. Bahjat-Abbas, R. Stark-Muchão and C.D. White, *Biadjoint wires*, *Phys. Lett. B* **788** (2019) 274 [1810.08118].

[257] B. Ett and D. Kastor, *An Extended Kerr-Schild Ansatz*, *Class. Quant. Grav.* **27** (2010) 185024 [1002.4378].

[258] Z. Bern, J. Parra-Martinez, R. Roiban, M.S. Ruf, C.-H. Shen, M.P. Solon et al., *Scattering Amplitudes and Conservative Binary Dynamics at $\mathcal{O}(G^4)$*, *Phys. Rev. Lett.* **126** (2021) 171601 [2101.07254].

[259] C. Cheung and J. Mangan, *Scattering Amplitudes and the Navier-Stokes Equation*, 2010.15970.

[260] A. Luna, N. Moynihan and C.D. White, *Why is the Weyl double copy local in position space?*, 2208.08548.

[261] J. Bičák, *Gravitational radiation from uniformly accelerated particles in general relativity*, *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences* **302** (1968) 201.

[262] V. Pravda and A. Pravdova, *Boost-rotation symmetric spacetimes - review*, *Czechoslovak Journal of Physics* **50** (2000) 333 [0003067].

[263] P. Sládek and J.D. Finley, *Asymptotic properties of the C-metric*, *Classical and Quantum Gravity* **27** (2010) [1003.1471].

[264] A. Tomimatsu, *Power law tail of gravitational waves from a uniformly accelerating black hole*, *Phys. Rev. D* **57** (1998) 2613.

[265] H. Farhoosh and R.L. Zimmerman, *KILLING HORIZONS AROUND A UNIFORMLY ACCELERATING AND ROTATING PARTICLE*, *Phys. Rev. D* **22** (1980) 797.

[266] A. Ashtekar and T. Dray, *On the existence of solutions to Einstein's equation with non-zero Bondi news*, *Communications in Mathematical Physics* **79** (1981) 581.

[267] D. Bini, C. Cherubini and B. Mashhoon, *Inertial effects of an accelerating black hole*, *AIP Conference Proceedings* **751** (2004) 37 [0410098].

[268] J.W. Maluf, V.C. Andrade and J.R. Steiner, *Gravitational radiation of accelerated sources*, *International Journal of Modern Physics D* **16** (2007) 857 [0610102].

[269] I. Jeon, K. Lee and J.-H. Park, *Stringy differential geometry, beyond Riemann*, *Phys. Rev. D* **84** (2011) 044022 [1105.6294].

[270] I. Jeon, K. Lee and J.-H. Park, *Differential geometry with a projection: Application to double field theory*, *JHEP* **04** (2011) 014 [1011.1324].