Local Singularity Theory for Ricci and Harmonic Ricci Flows

by

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To my mum and dad

The wind speaks not more sweetly to the giant oaks than to the least of all the blades of grass; And he alone is great who turns the voice of the wind into a song made sweeter by his own loving. Work is love made visible.

Khalil Gibran

Declaration

I, Gianmichele Di Matteo, confirm that the research included within this thesis is my own work or that where it has been carried out in collaboration with, or supported by others, that this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below.

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List of publications

Many of the ideas, chapters and sections of this thesis are based on manuscripts which are either published or submitted for publication, these are listed below.

- G. Di Matteo. Analysis of Type I Singularities in the Harmonic Ricci Flow. *ArXiv:1811.09563*.
- G. Di Matteo. Mixed Integral Norms for Ricci flow. *J. Geom. Anal. (online)*, https://doi.org/10.1007/s12220-020-00456-5, 2020.
- R. Buzano, G. Di Matteo. A Local Singularity Analysis for the Ricci Flow and its Applications to Ricci Flows with Bounded Scalar Curvature. *ArXiv:2006.16227v2*.

Abstract

In this thesis, we study the analytical properties of harmonic Ricci flows and Ricci flows in presence of a finite time singularity. After recalling some well-known results from the theories of these flows, we start our analysis considering Type I harmonic Ricci flows. We can characterise the pointwise singular behaviour of the flow by means of several natural curvatures associated to the flow, generalising results by Enders, Müller[1] and Topping. Next, we move our attention to Ricci flows, and prove some extension results for Ricci flows under suitable space-time integral curvature assumptions, extending results by Wang. For this purpose, a new and delicate parabolic regularity theory argument is used. Then, a refined local singularity analysis is developed, introduced jointly with R. Buzano in order to get rid of the Type I condition assumed in the work of Enders, Müller and Topping, obtaining thus a theory for general Ricci flows. This relies on some new concepts of regularity scales and their continuity properties. We give some applications of this new theory to the case of Ricci flows with scalar curvature bounded uniformly along the flow. Finally, we outline a plan for future research on the topic, shortly presenting some research in progress.

[1] Reto Müller changed his surname in Buzano in 2015. In this thesis we will reference works from both before and after this name change, using always the surname under which the cited work has been published.

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A complete list of names of people who supported me during this period of my life would require an other hundred of pages, so I will try to keep my acknowledgments schematic and concise.

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Repeating what I did in my previous theses, I conclude the acknowledgments with a self-reference, meant as a reminder for my future self. Generally speaking, this experience in London has shown me how crucially important is to love oneself and forgive all their own inevitable weaknesses; one needs not to strive against changing, but should instead go with the flow and forge their new identity composed of all their memories, good and bad happenings, using their ideals as waves and their desires as stars. As a wise man once said: “Nearly all people try to prolong their life, whereas they should widen it!”, therefore life needs both good and bad moments. Ending this paragraph with a positive mindset, “Here it comes, a better version of me”.

Finally, I wish to thank my two examiners Professor Rupflin and Professor Schulze for improving the presentation of the thesis, for their precise and detailed corrections, and for extremely valuable and inspiring comments, which will certainly influence my future projects.

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Chapter 1

Introduction and Brief Summary of the Main Results

Geometric flows constitute a general procedure of transforming a given object into an equivalent one with more symmetry or regularity properties. They first appeared in 1964 in [33], where Eells and Sampson introduced the harmonic map flow to solve the homotopy problem in non-positively curved manifolds. Another remarkable example is provided by the solution of the Poincaré and Thurston Geometrization Conjectures by Perelman in [65] and [66] using Ricci flow. In general, geometric flows may develop singularities which may prevent them from being continued smoothly; in these cases, one may hope to understand the geometry at the singular time well enough to be able to either modify the object in consideration without losing any information, and then to start the flow again, or to flow through the singularity in a weak sense. It is therefore natural to study the singularity formation of these kind of flows. Throughout this thesis we will focus on two particular examples, the Ricci flow and the harmonic Ricci flow.

1.1 Extension Theorems for the Ricci Flow

A Ricci flow is a smooth one-parameter family $(g(t))_{t\in I}$ of Riemannian metrics on a n-dimensional smooth manifold M^n, defined on a time interval I and satisfying for every $t \in I$

$$\partial_t g(t) = -2\,\mathrm{Ric}_{g(t)}\,. \tag{1.1.1}$$

Here $\mathrm{Ric}_{g(t)}$ denotes the Ricci tensor of the metric $g(t)$. This flow was introduced by Hamilton in [42], where he showed the remarkable result that a simply connected $3-$manifold admitting a metric of positive Ricci curvature must be a $3-$sphere; this result functioned both as prototype and as inspiration for the subsequent developments of the Ricci flow theory, and ultimately to the resolution of the Poincaré conjecture by Perelman.

A Ricci flow $(M, g(t))$ defined on a time interval of the form $[0, T)$, with $T < \infty$, is said to develop a singularity at time T if it cannot be smoothly extended past T. By Hamilton's original long-time existence criterion [42], if a Ricci flow $(M, g(t))_{t\in[0,T)}$ defined on a closed manifold M develops a singularity at the time $T < \infty$, then the Riemannian curvature tensor Rm must blow-up, i.e.

$$\limsup_{t\nearrow T} \sup_M |\mathrm{Rm}(\cdot, t)|_{g(t)} = \infty \tag{1.1.2}$$

or equivalently

$$(T - t) \sup_M |\mathrm{Rm}(\cdot, t)|_{g(t)} \geq 1/8, \quad \forall t \in [0, T). \tag{1.1.3}$$

Property (1.1.3) follows from (1.1.2) by a maximum principle argument applied to the evolution equation of the Riemannian curvature tensor along the Ricci flow. This result was later extended by Shi to Ricci flows for which $(M, g(t))$ are complete with bounded curvature for all $t \in [0, T)$, see [71]. Let us mention that there exist Ricci flows satisfying (1.1.2), with curvature blow-up at spatial infinity, which can be smoothly extended past time T, see [16, 38] for instance. In particular (1.1.2) is necessary but not sufficient for T to be the maximal time of existence. **In this work however, we always implicitely assume our flows have bounded geometry time-slices, that is they have bounded curvature and their injectivity radius bounded away from zero, up to the maximal time of smooth existence, unless otherwise stated.** These hypotheses are both natural in view of Remark 2.4.10. The reader may be misled to believe that these hypotheses forces the existence of singular points, but we notice that in the very recent [19], the authors constructed Ricci flows satisfying these assumptions and getting singular only at spatial infinity. The result by Hamilton and Shi cited above can be seen as archetypes of *extension theorems*: one can get further smooth existence of the flow as long as a suitable condition (on some curvature) is met. Other remarkable examples of such theorems were proven by Sesum [69] and Ma-Cheng [26], who showed that a Ricci flow can be continued as long as the Ricci curvature remains bounded, respectively in the closed and the complete and bounded curvature time-slices cases, generalising therefore Hamilton's and Shi's results. Extension theorems have been obtained under a wide variety of other pointwise or integral curvature bounds on closed manifolds or complete manifolds with bounded geometry, see for example [17, 26, 45, 50, 51, 83, 84, 90, 91] for a non-exhaustive list.

In Chapter 4 we will present two extension theorems obtained by the author in [32]. Inspired by the results in [84], we will deduce the extendibility of the flow assuming the boundedness of suitable space-time integral norms of some curvature. Our first result, Theorem 4.1.2, says that the flow can be extended past a finite time T if the $(\alpha, \beta)-$integral norm of the Riemann curvature Rm on $M \times [0, T)$ is bounded, under a proper condition on the pair (α, β), see Definition 4.1.1. For the same range of (α, β)'s, we show in our second result, Theorem 4.1.3, that one can weaken this curvature bound to an analogous integral bound on the scalar curvature R, assuming further a global Ricci curvature lower bound along the flow. In [84], Wang considered the case in which the spatial and the temporal integrability exponents match $\alpha = \beta$, and the flow is defined on a closed manifold, whereas we allow them to be different and the flow to be with complete and bounded geometry time-slices, subtleties which lead to several additional difficulties. The first theorem is proven by a standard blow-up argument, using that the space-time integral norm considered constitutes a (sub-)critical quantity for our choice of $(\alpha, \beta)'$s. In order to prove the second theorem, the lack of Rm curvature bounds in the contradiction argument does not allow us to extract a blow-up limit; we therefore develop a Moser iteration under the Ricci flow, and we eventually arrive to a Moser-Harnack type inequality for the scalar curvature. Under our assumptions, we can apply this inequality for a sequence of rescalings in the contradiction argument, comparing the $\mathbb{L}^{\infty}-$norm and the $(\alpha, \beta)-$norm of the rescaled scalar curvature asymptotically, which gives again a contradiction since the latter is again a (sub-)critical quantity. We refer the reader to Chapter 4 for further details.

1.2 Local Singularity Analysis for the Ricci flow

Many of the results currently present in the literature study and classify singularities from a *global* point of view, that is they fit in the framework of the extension theorems discussed above, without considering where the flow becomes singular. Of particular importance is the work of Hamilton, who introduced different notions of finite time singularities, called Type I and Type II singularities, distinguishing them by the rate at which the *maximal curvature* blows-up at the singular time, see [43]. A Ricci flow on $[0, T)$ is said to be of *Type I* if there exists a constant C such that

$$(T - t) \sup_M |\mathrm{Rm}(\cdot, t)|_{g(t)} \leq C, \quad \forall t \in [0, T). \tag{1.2.1}$$

If no such C exists, meaning that

$$\limsup_{t \nearrow T} \, (T - t) \sup_M |\mathrm{Rm}(\cdot, t)|_{g(t)} = \infty, \tag{1.2.2}$$

the Ricci flow is said to be of *Type II*. In view of (1.1.3), these two classes exhaust that of singular Ricci flows. If a Type I Ricci flow develops a singularity, this is forming at the fastest possible rate, so that one expects to be able to carry out the singularity analysis more easily than in the general case. This heuristic idea led Hamilton to make the following conjecture, which we state informally here.

Conjecture 1.2.1 (Hamilton's Conjecture [43])**.** *Singularities in Type I Ricci flows are modelled on non-trivial gradient shrinking soliton solutions.*

We postpone the exact meaning of the verb "modelling", as well as of the words "shrinking soliton" and "trivial" to the concept of tangent flows and the other definitions in Chapter 2. This conjecture was initially approached independently by Enders [34] and Naber [64] who verified the gradient soliton structure of tangent flows to Type I Ricci flows, and Sesum [70] who confirmed the conjecture in case all the tangent flows are closed. The full result was later established by the work [35] of Enders-Müller-Topping, where the authors implemented the first local singularity theory for the Ricci flow, achieving remarkable results which have been of great inspiration, and even more have provided a motivation for the local singularity analysis developed by the author and his supervisor R. Buzano in the joint work [14], which is the focus of Chapter 5. Before proceeding to examine the work [35], let us mention that in [61], a different approach to Conjecture 1.2.1 is presented, consisting of heat kernel bounds and Perelman's entropy preservation along blow-up procedure; this latter work prompted the study of the heat kernel we briefly outline in Chapter 6.

In [35], the authors focused their attention to Type I Ricci flows, for which they proved Conjecture 1.2.1 as well as a really interesting singular points characterisation. First of all, let us recall their general definition of singular points, which we will adopt too, and some related definitions important for their work.

Definition 1.2.2 (Singular Set, Definition 1.5 in [35]). *Suppose $(M, g(t))_{[0,T)}$ is a Ricci flow with complete and bounded geometry time-slices. We say that a point $p \in M$ is a* singular point *if for any neighbourhood U of p, the Riemannian curvature becomes unbounded on U as t approaches T. The* singular set *Σ is the set of all such points and the* regular set *$\mathfrak{Reg}$ consists of the complement of Σ. Given a singular point $p \in \Sigma$, any sequence (p_k, t_k) such that $p_k \to p$, $t_k \nearrow T$ and along which $|\operatorname{Rm}|(p_k, t_k) \to +\infty$ is called an* essential blow-up sequence *(e.b.s. in short) for the point (p, T).*

Definition 1.2.3 (Special Singular Sets, Definitions 1.2, 1.3, 1.4 in [35]). *Suppose $(M, g(t))_{[0,T)}$ is a Ricci flow with complete and bounded geometry time-slices. We define the singular subsets Σ_I^*, Σ_{Ric}^* or Σ_{R}^* as follows*

- *$p \in \Sigma_I^*$ if there exists an e.b.s. for (p, T) with $|\operatorname{Rm}|(p_k, t_k) \geq \frac{c}{T-t_k}$ for some $c > 0$;*
- *$p \in \Sigma_{\mathrm{Ric}}^*$ if $|\operatorname{Ric}|(p, t) \geq \frac{c}{T-t}$ for some $c > 0$ and every $t \in [0, T)$;*
- *$p \in \Sigma_{\mathrm{R}}^*$ if $|\mathrm{R}|(p, t) \geq \frac{c}{T-t}$ for some $c > 0$ and every $t \in [0, T)$.*

Their solution to Hamilton's Conjecture 1.2.1 is presented in the following theorem.

Theorem 1.2.4 (Theorem 1.1 in [35]). *Given a complete Type I Ricci flow $(M, g(t))$, and a point $p \in M$, then any tangent flow at (p, T) carries a gradient shrinking soliton solution structure, which is non-trivial if and only if $p \in \Sigma_I^*$.*

Let us briefly mention the striking generalisation of this result recently proven by Bamler in [8, 9, 10]. This theorem provides an exceptionally strong tool for the singularity analysis of Type I

Ricci flows, which allowed them to deduce the following characterisation result for singular points.

Theorem 1.2.5 (Theorem 1.2 in [35]). *Given a Type I Ricci flow* $(M, g(t))$*, then we have the equivalence* $\Sigma = \Sigma^*_I = \Sigma^*_{\mathrm{Ric}} = \Sigma^*_{\mathrm{R}}$*. In particular, the scalar curvature blows-up at the Type I rate at every singular point.*

Motivated by the theory described above, the accurate study developed by Angenent and Knopf in [2], as well as several recent examples of Type II singularities, e.g. [3, 4, 30, 77, 86], the author and R. Buzano introduced in [14] a local analysis of singularities and curvature blow-up rates, which does not rely on any Type I assumption. We will sketch it here shortly, and postpone a thorough explanation to Chapter 5.

New concepts of singular points are proposed, whose Definitions 5.1.1-5.4.6 carefully take into account all the rates associated to an e.b.s. for one of these points; in particular, one wants to control simultaneously the curvature blow-up rate and how fast the e.b.s. (p_k, t_k) converges to (p, T). Let us mention that the corresponding singularity analysis we derive is actually stronger than a local one, since the domains involved in these definitions typically collapse, but it is still not pointwise as the one in [35], we refer the reader to Chapter 5 for further details. Two of our main results, Theorems 5.1.2, 5.1.8, may be regarded as a generalisation of Theorem 1.2.5 above, and combined they say, roughly speaking, that for any singular point there exists an e.b.s. along which the Ric curvature blows-up at least at the Type I rate (and hence the Rm does the same), further providing information on the convergence of the (p_k, t_k) to (p, T). These results generalise several theorems in the literature and some of those cited above, as we remark in Chapter 5. In order to keep this introduction short, we just say that the proofs of these results are based on some distance distortion estimates under local curvature bounds due to Simon-Topping ([75, 76]), on some new concepts of regularity scales (for the Rm and Ric curvatures) and on a local-to-global blow-up contradiction argument reminiscent of Sesum's original proof of the result we cited above.

Another important result we obtain, Lemma 5.5.5, is an integral concentration for the Ric regularity scale, which we apply to the study of bounded scalar curvature Ricci flows. In view of Sesum's extension theorem, it has been conjectured that a bound on the scalar curvature could potentially also be sufficient to extend the flow. In dimension three, this is a consequence of the Hamilton-Ivey pinching estimate [43, 47] while in higher dimensions it is known to be true for Type I Ricci flows by Theorem 1.2.5 above as well as in the Kähler case by Zhang [95]. In recent years, this conjecture has been the focus of many interesting new developments, see for example [7, 11, 23, 24, 25, 55, 73, 74, 84, 92] and the references therein, but without the Type I or Kähler assumption the conjecture still remains open in dimensions $n \geq 4$. As an application of our singularity theory, we show that, *under a technical assumption*, Ricci flows with bounded scalar curvature cannot develop singularities in dimension n smaller than 8 (Theorem 5.1.10), and the singular set has codimension 8 in a suitable sense if $n \geq 8$ (Theorem 5.1.12). This technical assumption allows us to compare the Ric regularity scale of a singular point to its Ric curvature. Once this is obtained, the proofs are based on contradiction arguments: on one side the Ric regularity scale integral concentration stated above bounds a certain integral away from zero, whereas on the other we have that, for every $\varepsilon > 0$, the $\mathbb{L}^{4-\varepsilon}-$norm of the Rm regularity scale is infinitesimal by a result of Bamler [7], see also [11, 73]. The link between these two integrals comes from our Theorem 5.1.8 stating that any singular point is a Ric singular point, our technical assumption and a general observation by Wang [84] and Bamler-Zhang [11] that for such Ricci flows, one can always control the Ric curvature with the square root of the Rm curvature in a small enough neighbourhood, see Theorem 5.5.4. Further and more precise elements are presented in Chapter 5.

In Chapter 6, we give a brief outline of a joint work in progress by the author and R. Buzano, which is a continuation of the local singularity analysis descrived above. It concerns heat kernel and conjugate heat kernel bounds along the Ricci flow, depending on whether the base point is regular, Type I or Type II singular point. These bounds are then applied to carry out a local version of the proof of Conjecture 1.2.1 given by Mantegazza-Müller in [61].

1.3 Local Singularity Analysis for Type I harmonic Ricci flows

In [63], Müller introduced the harmonic Ricci flow, a coupling between the harmonic map flow and the Ricci flow, developed for it short and long time existence theory and monotonicity formulas for entropy functionals, reduced length and volume, and a non-collapsing theorem reminiscent of Perelman's tools for the Ricci flow from [65].

Suppose we have two Riemannian manifolds (M^n, g), (N^k, γ), with N closed, and a smooth non-negative *coupling function* $\alpha(t)$. We say that a smooth family $(g(t), \phi(t))$ of Riemannian metrics $g(t)$ on M and smooth maps $\phi(t)$ from M to N provides a solution to the harmonic Ricci flow equation if it satisfies

$$\begin{cases} \frac{\partial g(t)}{\partial t} = -2\operatorname{Ric}_{g(t)} + 2\alpha(t)\nabla\phi(t)\otimes\nabla\phi(t), \\ \frac{\partial \phi(t)}{\partial t} = \tau_{g(t)}\phi(t). \end{cases} \tag{1.3.1}$$

Here we have denoted by $\tau_g\phi = \Delta_g\phi - A(\phi)g(\nabla\phi, \nabla\phi)$ the so-called *tension field* of ϕ, where A is the second fundamental form of N, viewed isometrically embedded into $\mathbb{R}^d$ via Nash's embedding theorem. In the special case where $\phi(t)$ is a real valued function (after composing with an isometric map of S^1 into $\mathbb{R}$), this flow reduces to List's flow introduced in [57]. For convenience of the reader, we introduce the $(0,2)-$tensor $\mathcal{S} := \operatorname{Ric} - \alpha\nabla\phi\otimes\nabla\phi$, locally denoted S_{ij}, and its trace $\mathrm{S} := \mathrm{R} - \alpha|\nabla\phi|^2$.

A priori, a solution to (1.3.1) can develop a singularity either in the metric component or the map component only or in both components simultaneously. Surprisingly, it turns out that the metric component dominates the map component and certain types of singularities can be ruled out. In particular, for sufficiently large coupling functions $\alpha(t)$, singularities in the map component can be excluded, see Proposition 5.6 in [63]. (In fact, singularity formation in finite time for a volume-normalised version of (1.3.1) can be excluded completely for large $\alpha(t)$ if the domain manifold M is two dimensional, see [13].) More generally, as long as $\alpha(t)$ is bounded away from zero, the map component cannot become singular without the metric component also developing a singularity. That is, the coupled system has a singularity at a finite time T if and only if the curvature blows-up at T. This was proven by Müller in [63] for closed manifolds, and by Cheng and Zhu in [27] for complete and bounded geometry flows (even under the weaker condition of having bounded Ric curvature). These results should be compared with our results in Section 3.2, where the heuristic dominance of the metric component on the map component is once more confirmed.

The purpose of Chapter 3 is to generalise Theorems 1.2.4 and 1.2.5 to this setting. In order to do so, we first introduce a new notion of Type I harmonic Ricci flows in Definition 3.1.2, depending explicitely only on the metric component. Another definition was yet available and due to Shi [71], but this involved the map component too, so we prefer to give a new one more aligned with the discussion above. Then, we develop some compactness Theorems 3.2.6-3.2.9 and a new pseudolocality Theorem 3.1.5, generalising results in [40, 71], and introduce a reduced length and a reduced volume based at the singular time for the harmonic Ricci flow, adapting the study of Enders [34] to our context, see Theorem 3.1.4. These ingredients are needed in order to prove rigorously Theorem 3.1.3, which says that for a Type I harmonic Ricci flow, any tangent flow can be endowed with a gradient shrinking soliton structure, and this is non-trivial if and only if the point is singular. Once again, this powerful theorem allows us to show a pointwise characterisation for the singular points in the same spirit as Theorem 1.2.5, but this time involving the tensors $\mathcal{S}$ and S introduced above, see Theorem 3.5.3.

Chapter 2

Preliminaries

The purpose of this chapter is to give the reader a general background about Ricci and harmonic Ricci flows, and to recall some preliminary results in their theories which will be used extensively throughout the thesis.

We explicitely remark that the Ricci flow equation (1.1.1) can be recovered from the harmonic Ricci flow one (1.3.1) simply by setting $\alpha = 0$ and $\phi(t) \equiv y_0$ a map to a constant point $y_0 \in N$. For this reason, we will give several definitions only in the harmonic Ricci flow setting, leaving to the reader to deduce their Ricci flow's versions.

2.1 Preliminaries on Differential Geometry

Given a Riemannian manifold (M, g), we always endow it with its Levi-Civita connection ∇, and adopt the following sign convention for its curvature tensors: for any three vectors X, Y, Z belonging to the tangent space T_pM of M at a point $p \in M$, the Riemann curvature tensor Rm is given by the vector defined as

$$\mathrm{Rm}(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \in T_pM. \tag{2.1.1}$$

This is a $(3,1)-$tensor, encoding how the manifold is curved in terms of how the second derivatives of a vector are not commutative. It is sometimes useful to think of it as a $(4,0)-$tensor through the identification $\mathrm{Rm}(X,Y,Z,W) := g(\mathrm{Rm}(X,Y)Z, W)$, where $X, Y, Z, W \in T_pM$. This tensor satisfies some well-known symmetry relations, which we rewrite as they depend on our sign convention: given $X, Y, Z, W \in T_pM$ we have

$$\begin{aligned} &\mathrm{Rm}(X,Y,Z,W) = -\mathrm{Rm}(Y,X,Z,W) = -\mathrm{Rm}(X,Y,W,Z) = \mathrm{Rm}(Z,W,X,Y) \quad [\textit{symmetry}];\\ &\mathrm{Rm}(X,Y,Z,W) + \mathrm{Rm}(Y,Z,X,W) + \mathrm{Rm}(Z,X,Y,W) = 0 \quad [1^{st} \textit{ Bianchi Identity}]; \end{aligned} \tag{2.1.2}$$

$$\nabla_X \operatorname{Rm}(Y,Z)W + \nabla_Y \operatorname{Rm}(Z,X)W + \nabla_Z \operatorname{Rm}(X,Y)W = 0 \quad [2^{nd} \textit{ Bianchi Identity}].$$

Taking the trace of this tensor with respect to the first and third component, we obtain the so called Ricci tensor

$$\operatorname{Ric}(X,Y) := \operatorname{trace}(\operatorname{Rm}(\cdot, X, \cdot, Y)). \tag{2.1.3}$$

The Ricci tensor is a symmetric $(2,0)-$tensor, giving an averaged curvature of the space along the $2-$dimensional plane spanned by X and Y. It can be seen as a local measure of how the volume element differs from the Euclidean one. Tracing the second Bianchi identity we get

$$\operatorname{trace}(\nabla. \operatorname{Rm}(\cdot, Y, Z, W)) = \nabla_Z \operatorname{Ric}(Y,W) - \nabla_W \operatorname{Ric}(Y,Z). \tag{2.1.4}$$

Raising an index with the help of the metric and taking a trace, we obtain the so called scalar curvature $\mathrm{R} := \operatorname{trace}(\operatorname{Ric}(\cdot,\cdot))$. A further trace of (2.1.4) gives

$$\operatorname{trace}(\nabla. \operatorname{Ric}(\cdot, X)) = \frac{1}{2}\nabla_X \mathrm{R}\,. \tag{2.1.5}$$

Assume now (N,h) is another Riemannian manifold, closed and isometrically embedded in $\mathbb{R}^d$ for some large d thanks to Nash's embedding theorem [59]. For any smooth map $\phi : M \longrightarrow N$, we can then identify the map ϕ with a vector $(\phi^\lambda)_{\lambda=1,..,d}$, and consider the pulled-back tangent bundle ϕ^*TN over M, whose fibre at a point $p \in M$ is given by $T_{\phi(p)}N$. The gradient (or any higher covariant derivative) of ϕ^λ will then belong to TM (or higher ranked tensor bundles) for every λ and we can define $|\nabla\phi|^2 := \sum_\lambda \sum_{i,j} g^{ij}\nabla_i\phi^\lambda\nabla_j\phi^\lambda$ in local coordinates (analogously for higher derivatives). A formula of extreme importance in this setting is the Bochner identity:

$$-\Delta_g|\nabla\phi|^2 + 2\langle\nabla\Delta\phi, \nabla\phi\rangle + 2|\nabla^2\phi|^2 + 2\langle \operatorname{Ric}, \nabla\phi\otimes\nabla\phi\rangle = 2\operatorname{Rm}^N(\nabla_i\phi, \nabla_j\phi, \nabla_i\phi, \nabla_j\phi). \tag{2.1.6}$$

Given a constant α, we define a $(0,2)-$tensor $\mathcal{S}$ by $\mathcal{S} := \operatorname{Ric} - \alpha\sum_\lambda \nabla\phi^\lambda \otimes \nabla\phi^\lambda$. We denote its trace as $\mathrm{S} := \mathrm{R} - \alpha\sum_\lambda |\nabla\phi^\lambda|^2$. We set the same definitions in case the metric g and/or the map ϕ depend smoothly on a parameter t, i.e. we compute everything fixing the parameter itself. In that case, we call the triple $(M, g(t), \phi(t))$ *a flow*.

We conclude recalling the orthogonal decomposition for the Riemann tensor

$$\operatorname{Rm} = -\frac{\mathrm{R}}{2(n-1)(n-2)} g \owedge g + \frac{1}{n-2}\operatorname{Ric} \owedge g + \mathrm{W}, \tag{2.1.7}$$

where we denoted by W the Weyl tensor and by $\owedge$ Kulkarni-Nomizu's product of symmetric $(0,2)-$tensors defined as follows: given two symmetric $(0,2)-$tensors h and k, $h \owedge k$ is the

$(0,4)-$tensors defined on four vectors X, Y, Z, W as

$$h \owedge k(X,Y,Z,W) := h(X,Z)k(Y,W) + h(Y,W)k(X,Z) - h(X,W)k(Y,Z) - h(Y,Z)k(X,W).$$

2.2 Evolution equations and Symmetries

In this section, we will focus our attention to families of Riemannian metrics evolving under the Ricci flow equation (1.1.1) or the harmonic Ricci flow equation (1.3.1) and recall some evolution equations for several geometric quantities important for our analysis. Then we will recall the three basic symmetries equation (1.3.1) verifies. More precisely, we will state the equations for the more general harmonic Ricci flow case, and leave to the reader to deduce these in the restricted Ricci flow case. The following evolution equations can be found in [63, 62]. In the formulas below, C depends only on the dimension of M, c_0 depends only on the curvature of N and all the spatial derivatives are computed with respect to the connection of the metric $g(t)$. Under the harmonic Ricci flow (1.3.1) one has

$$\begin{aligned}
\partial_t |\operatorname{Rm}|^2 \leq & \Delta |\operatorname{Rm}|^2 - 2|\nabla \operatorname{Rm}|^2 + C|\operatorname{Rm}|^3 + \alpha C|\nabla\phi|^2|\operatorname{Rm}|^2 + \alpha C|\nabla^2\phi|^2|\operatorname{Rm}| \\
& + \alpha C c_0(N)|\nabla\phi|^4|\operatorname{Rm}|; && (2.2.1)\\
\frac{\partial|\nabla\phi|^2}{\partial t} = & \Delta|\nabla\phi|^2 - 2\alpha|\nabla\phi\otimes\nabla\phi|^2 - 2|\nabla^2\phi|^2 + 2\langle \operatorname{Rm}^N(\nabla_i\phi, \nabla_j\phi)\nabla_j\phi, \nabla_i\phi)\rangle; && (2.2.2)\\
\partial_t|\nabla^2\phi|^2 \leq & \Delta|\nabla^2\phi|^2 - 2|\nabla^3\phi|^2 + C|\operatorname{Rm}||\nabla^2\phi|^2 + \alpha C|\nabla\phi|^2|\nabla^2\phi|^2 + Cc_0|\nabla\phi|^4|\nabla^2\phi| \\
& + Cc_0|\nabla\phi|^2|\nabla^2\phi|^2; && (2.2.3)\\
\partial_t \mathcal{S}_{ij} = & \Delta \mathcal{S}_{ij} + 2\operatorname{Rm}_{ipjq}\mathcal{S}_{pq} - \operatorname{Ric}_{ip}\mathcal{S}_{pj} - \operatorname{Ric}_{jp}\mathcal{S}_{pi} + 2\alpha\tau_g(\phi)\nabla_i\nabla_j\phi - \dot\alpha\nabla_i\phi\nabla_j\phi; && (2.2.4)\\
\partial_t \mathrm{S} = & \Delta \mathrm{S} + 2|\mathcal{S}_{ij}|^2 + 2\alpha|\tau_g\phi|^2 - \dot\alpha|\nabla\phi|^2; && (2.2.5)\\
\partial_t d\mu_{g(t)} = & -\mathrm{S}(t)d\mu_{g(t)}. && (2.2.6)
\end{aligned}$$

In order to recover the evolution equations for the curvatures under the Ricci flow, one can just set $\alpha = 0$ and ϕ any map to a fixed point in N.

Suppose $(M, g(t), \phi(t))$ is a solution of (1.3.1) with coupling function $\alpha(t)$, defined on a time interval I. Then for any value $\lambda > 0$, family of diffeomorphisms $\psi(t)$ of M, and time parameter $t_0 \in \mathbb{R}$, the harmonic Ricci flow equation (1.3.1) satisfies the following symmetries:

- *scaling invariance*: $(M, g_\lambda(t), \phi_\lambda(t))$ defined by $g_\lambda(t) := \lambda g(\frac{t}{\lambda})$ and $\phi_\lambda(t) := \phi(\frac{t}{\lambda})$ solves (1.3.1) with coupling function $\alpha_\lambda(t) := \alpha(\frac{t}{\lambda})$ on the time interval $\lambda \cdot I$;

- *space invariance*: $(M, \psi(t)^*g(t), \psi(t)^*\phi(t))$ solves (1.3.1) with the same coupling function α on the same time interval I;

- *time invariance*: $(M, g(t_0+t), \phi(t_0+t))$ solves (1.3.1) with coupling function $\alpha(t_0+t)$ on the time interval $I - \{t_0\}$.

2.3 Convergence and Compactness of flows

In the study of singular Ricci flows, or even in a more general PDEs context, a major role is played by compactness theorems, since they typically allow to model the singularity as a self-similar solution. Here we introduce the classic concepts of convergence and compactness of manifolds and flows, as well as recall a few results from their theory.

Definition 2.3.1 (Cheeger-Gromov Convergence)**.** *A sequence (M_k, g_k, p_k) of pointed Riemannian manifolds is said to converge in the smooth Cheeger-Gromov sense to a pointed Riemannian manifold $(M_\infty, g_\infty, p_\infty)$ if there exist*

- *a family of relatively compact domains $(\Omega_k)_{k\in\mathbb{N}}$, with $p_k \in \Omega_k$ for every k and exhausting M_∞ in the sense that $\overline{\Omega_k} \subset \Omega_{k+1}$ and $\bigcup_{k\in\mathbb{N}} \Omega_k = M_\infty$;*
- *a family of smooth maps $\Psi_k : \Omega_k \to M_k$ diffeomorphic onto their image, with $\Psi_k(p_\infty) = p_k$,*

such that $\Psi_k^* g_k \to g_\infty$ smoothly locally on M_∞ as $k \to \infty$.

If the sequence considered (M_k, g_k, ϕ_k, p_k) consists of pointed Riemannian manifolds coupled with maps ϕ_k from M_k to a fixed target manifold N, we will require further $\Psi_k^*\phi_k \to \phi_\infty$ smoothly locally, and the limit will be $(M_\infty, g_\infty, \phi_\infty, p_\infty)$.

Finally, if we are considering a sequence of flows $(M_k, g_k(t), \phi_k(t), p_k)$ depending smoothly on a parameter t belonging to a common interval I, we will impose the smooth local convergence of $\Psi_k^* g_k$ and $\Psi_k^*\phi_k$ on compact subsets of $M_\infty \times I$ of the form $\overline{\Omega_k} \times I_k$ exhausting $M_\infty \times I$.

It is clear from the definition of Cheeger-Gromov convergence of flows, that the sequence of time-slices is converging to the corresponding time-slice of the limit, but the convergence defined is stronger. Moreover, this definition admits several natural generalisations, allowing for instance a sequence of target manifolds N_k, converging themselves to a limit target N_∞, or a sequence of intervals I_k of definitions of the flows, converging to a limiting interval I_∞.

We will later need the following compactness theorem of Riemannian manifolds (cfr. Theorem 2.5 in [81]):

Theorem 2.3.2 (Compactness of manifolds)**.** *Suppose that (M_i, g_i, p_i) is a sequence of (smooth) complete, pointed Riemannian manifolds, all of dimension n, satisfying:*

(i) $\forall r > 0$ and $\forall k \in \mathbb{N}$ we have

$$\sup_i \sup_{B_{g_i}(p_i, r)} |\nabla_i^k \operatorname{Rm}(g_i)|_{g_i} < \infty$$

and

(ii) $\inf_i \operatorname{inj}(M_i, g_i, p_i) > 0$.

Then there exists a (smooth) complete, pointed Riemannian manifold $(M_\infty, g_\infty, p_\infty)$ of dimension n such that, after passing to some subsequence, $(M_i, g_i, p_i) \longrightarrow (M_\infty, g_\infty, p_\infty)$ in the pointed Cheeger-Gromov sense.

The following compactness theorem, slightly generalising Hamilton's classical compactness result from [43], is taken from Topping [80]. See also [81] for an expository review and first applications.

Theorem 2.3.3 (Theorem 1.6 in [80])**.** *Let $(M_i, g_i(t), p_i)$ be a sequence of pointed and complete Ricci flows, defined on a common time interval (a, b), with $-\infty \leq a < 0 < b \leq +\infty$. Suppose that*

- $\inf \operatorname{inj}(M_i, g_i(0), p_i) > 0$,
- *there exists a constant M such that for every $r > 0$ there exists $i_r \in \mathbb{N}$, such that for every $i \geq i_r$ and $t \in (a, b)$ we have*

$$\sup_{B_{g_i(0)}(p_i, r)} |\operatorname{Rm}|_{g_i(t)} \leq M. \tag{2.3.1}$$

Then there exists a complete pointed Ricci flow $(M_\infty, g_\infty(t), p_\infty)$ defined on (a, b), which is a pointed smooth Cheeger-Gromov limit of some subsequence of $(M_i, g_i(t), p_i)$.

We will later extend this theorem to the harmonic Ricci flow setting in Chapter 3. We conclude this section with the definition of limit and tangent flows at points in the final time-slice, appearing in the formulation of Theorem 1.2.4.

Definition 2.3.4 (Limit flows and Tangent flows)**.** *Let $(M, g(t), \phi(t), p)$ a pointed harmonic Ricci flow, defined on an interval of the form $[0, T)$, $T < \infty$, with target manifold N. We call the pointed harmonic Ricci flow $(M_\infty, g_\infty(s), \phi_\infty(s), p_\infty)$ a* limit flow *at the point (p, T) if there exist a sequence of parameters $\lambda_k \to \infty$ and space-time points $(p_k, t_k) \to (p, T)$ such that the sequence of rescalings $(M, g_k(s), \phi_k(s), p_k)$ defined by $g_k(s) := \lambda_k g(t_k + \frac{s}{\lambda_k})$ and $\phi_k(s) := \phi(t_k + \frac{s}{\lambda_k})$ is converging to $(M_\infty, g_\infty(s), \phi_\infty(s), p_\infty)$ in the smooth Cheeger-Gromov sense. If we can choose $(p_k, t_k) \equiv (p, T)$ for every k, we say that $(M_\infty, g_\infty(s), \phi_\infty(s), p_\infty)$ is a* tangent flow *at (p, T).*

Notice that limit flows are solutions to (1.3.1) with constant coupling function given by $\lim_{t \nearrow T} \alpha(t)$. Let us mention that a new astonishing *weak* compactness theory has been recently developed by Bamler in [9]. Subsequently, in [10], he has introduced a new weak notion of tangent flow which allows singular limits, which brought major breakthroughs in the Ricci flow theory, such as a thick-thin decomposition generalising Perelman's result from [66] to higher dimensions and an extension of Conjecture 1.2.1 to Type II Ricci flows.

2.4 Preliminaries on the Ricci flow

Mixed Integral Norms

We now want to give a short introduction on mixed integral norms along the Ricci flow, since they will play a central role in Chapter 4. For a measurable function u defined on a smooth flow $(M, g(t))$, we set

$$\left\|\|u\|_{\alpha,\Omega}\right\|_{\beta,I} := \left(\int_I \left(\int_\Omega |u|^\alpha d\mu_{g(t)}\right)^{\beta/\alpha} dt\right)^{1/\beta}, \tag{2.4.1}$$

where $\Omega \subseteq M$ and $I \subseteq [0,T)$. Our first result is a lemma that will be used in Moser's iteration argument, when restricting our attention to Ricci flows, but it is valid in a more general framework.

Lemma 2.4.1. *Let $(M, g(t))$ be a smooth flow defined in $[0,T]$, and fix a subset $\Omega' \subset M$ such that $0 < c \le \mathrm{Vol}_{g(t)}(\Omega') \le C < +\infty$ for every t. Then we have for any measurable function u*

$$\lim_{(a,b)\to(+\infty,+\infty)} \left\|\|u\|_{a,\Omega'}\right\|_{b,[0,T]} = \operatorname{ess\,sup}_{\Omega'\times[0,T]} |u|(x,t). \tag{2.4.2}$$

Before proving Lemma 2.4.1, we show an analogous result on the averages.

Lemma 2.4.2. *Let $(M, g(t))$ be a smooth flow defined in $[0,T]$, and fix a subset $\Omega' \subset M$ such that $\mathrm{Vol}_{g(t)}(\Omega') \le C < +\infty$ for every t. Then we have for any measurable u*

$$\lim_{(a,b)\to(+\infty,+\infty)} \phi(a,b) = \operatorname{ess\,sup}_{\Omega'\times[0,T]} |u|(x,t), \tag{2.4.3}$$

where

$$\phi(a,b) := \left(\frac{1}{T}\int_0^T \left(\frac{1}{\mathrm{Vol}_{g(t)}(\Omega')}\int_{\Omega'} |u|^a d\mu_{g(t)}\right)^{\frac{b}{a}} dt\right)^{\frac{1}{b}}. \tag{2.4.4}$$

Proof. Without loss of generality we can assume $u \ge 0$ and $u \in \mathbb{L}^\infty$ (otherwise we prove it for the truncation $u_M := \max\{\min\{u, M\}, -M\}$ and then let M to infinity). A simple application of Hölder inequality gives that ϕ is a non-decreasing function of both a and b, from which we easily deduce

$$\lim_{(a,b)\to(+\infty,+\infty)} \phi(a,b) = \sup_{(a,b)\in[1,+\infty)\times[1,+\infty)} \phi(a,b) = \sup_{a\in[1,+\infty)} \phi(a,a) = \lim_{a\to+\infty} \phi(a,a). \tag{2.4.5}$$

Set $D' := \Omega' \times [0,T]$. We can bound u from above with its essential supremum, so that we obtain

$$\lim_{a\to+\infty} \phi(a,a) \le \operatorname{ess\,sup}_{D'} u(x,t). \tag{2.4.6}$$

For any $\varepsilon > 0$, by definition of essential supremum we get the existence of a $\delta > 0$ such that, if we set $E := \{(x,t) \in D' \mid u(x,t) \geq \operatorname{ess\,sup}_{D'} u - \varepsilon\}$, we have

$$|E| := \int_0^T \left(\int_{E_t} d\mu_{g(t)} \right) dt > \delta, \tag{2.4.7}$$

where $E_t := E \cap (M \times \{t\})$. Therefore we obtain:

$$\begin{aligned} \phi(a,a) &\geq \left(\frac{1}{T} \int_0^T \frac{1}{\operatorname{Vol}_{g(t)}(\Omega')} \int_{E_t} (\operatorname{ess\,sup}_{D'} u - \varepsilon)^a d\mu_{g(t)} dt \right)^{\frac{1}{a}} \\ &\geq (\operatorname{ess\,sup}_{D'} u - \varepsilon) \frac{1}{(CT)^{\frac{1}{a}}} |E|^{\frac{1}{a}} \geq (\operatorname{ess\,sup}_{D'} u - \varepsilon) \frac{1}{(CT)^{\frac{1}{a}}} \delta^{\frac{1}{a}}. \end{aligned} \tag{2.4.8}$$

It suffices now to first let a go to infinity, and then ε to zero to conclude the proof. □

Proof of Lemma 2.4.1. We easily compute

$$\begin{aligned} \big\| \|u\|_{a,\Omega'} \big\|_{b,[0,T]} &= \left(T T^{-1} \int_0^T \operatorname{Vol}_{g(t)}(\Omega') \operatorname{Vol}_{g(t)}(\Omega')^{-1} \int_{\Omega'} |u|^a d\mu \right)^{\frac{b}{a}} dt \Bigg)^{\frac{1}{b}} \\ &\leq T^{\frac{1}{b}} (\sup_{[0,T]} \operatorname{Vol}_{g(t)}(\Omega'))^{\frac{1}{a}} \phi(a,b) \leq T^{\frac{1}{b}} C^{\frac{1}{a}} \phi(a,b). \end{aligned} \tag{2.4.9}$$

Similarly, we get

$$\big\| \|u\|_{a,\Omega'} \big\|_{b,[0,T]} \geq T^{\frac{1}{b}} c^{\frac{1}{a}} \phi(a,b). \tag{2.4.10}$$

The conclusion follows taking the limit for a and b going to infinity. □

In Moser's iteration, we will also need some control on the volume of the domain in consideration. Below we prove a generalization of Property 2.3 in [84].

Lemma 2.4.3. *Let $(M, g(t))$ be a Ricci flow defined on $[0,T]$. For a fixed point $p \in M$ and radius r, we set $\Omega := B_{g(T)}(p,r)$. Suppose there exists a constant $B > 0$ such that $\operatorname{Ric}(x,t) \geq -Bg(t)$ on the set $\big(\Omega \times [0,T]\big) \cup \big(M \times \{T\}\big)$. Then there exists a constant $\tilde{V} = \tilde{V}(n,r,T,B) \geq 1$ such that*

$$\big\| \|1\|_{\alpha,\Omega} \big\|_{\beta,[0,T]} \leq \tilde{V} \tag{2.4.11}$$

for every $\alpha, \beta \geq 1$. Moreover, $\tilde{V}$ is bounded as long as n, r, T and B remain bounded as well.

Proof. From the lower Ricci bound on the region $\Omega \times [0,T]$, and the evolution equation for the volume element (2.2.6), we get

$$\operatorname{Vol}_{g(t)}(\Omega) \leq e^{(n-1)BT} \operatorname{Vol}_{g(T)}(\Omega) \tag{2.4.12}$$

for every $t \in [0,T]$. The lower bound on the whole final time-slice allows us to use Bishop-

Gromov's inequality (see Theorem A.3 in [28] for instance), which gives the existence of a constant $V = V(n, r, B)$ such that

$$\mathrm{Vol}_{g(T)}(\Omega) \le V. \tag{2.4.13}$$

Therefore, we easily obtain the following chain of inequalities

$$\big\|\|1\|_{\alpha,\Omega}\big\|_{\beta,[0,T]} \le \|(e^{(n-1)BT}V)^{\frac{1}{\alpha}}\|_{\beta,[0,T]} \le \tilde{V}, \tag{2.4.14}$$

where we have set $\tilde{V} = \max\{e^{(n-1)BT}T^{\frac{1}{\beta}}V^{\frac{1}{\alpha}}, 1\}$. □

Remark 2.4.4. *Alternatively, one could assume a scalar curvature bound on the cylinder considered and take $\tilde{V} = C(n, B)r^{\frac{n}{\alpha}}$ by the results of Zhang [92], and Chen and Wang [26], after possibly reducing T and r. Our choice is adapted to the structure of the proof of the main Theorem in Chapter 4.*

An easy application of Hölder's inequality both in space and time yields the following inequality.

$$\int_0^T \int_M fg\,d\mu dt \le \big\|\|f\|_{\alpha,M}\big\|_{\beta,[0,T)}\big\|\|g\|_{\alpha',M}\big\|_{\beta',[0,T)}, \tag{2.4.15}$$

where $\alpha' = \alpha/(\alpha-1)$ and $\beta' = \beta/(\beta-1)$ are the Hölder conjugate exponents of α and β respectively. One can use this inequality to produce an interpolation inequality, which we recall.

Proposition 2.4.5. *Let $1 \le p \le q \le r < +\infty$, $1 \le P \le Q \le R < +\infty$. Suppose that for some $\theta \in [0, 1]$ we have*

$$\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{r} \text{ and } \frac{1}{Q} = \frac{\theta}{P} + \frac{1-\theta}{R}. \tag{2.4.16}$$

Then for every measurable function v we have $\big\|\|v\|_{q,M}\big\|_{Q,[0,T)} \le \big\|\|v\|_{p,M}\big\|^{\theta}_{P,[0,T)}\big\|\|v\|_{r,M}\big\|^{1-\theta}_{R,[0,T)}$.

In the proof of Lemma 4.3.3 we will be interested in extrapolating an optimal pair of exponents given a super-optimal one (see Definition 4.1.1) and the pair $(1, 1)$, in the conjugate Hölder exponents plane. The following lemma characterises such a pair.

Lemma 2.4.6. *For any pair (a, b) such that $a > \frac{n}{2}\frac{b}{b-1}$, there exists a unique pair (α_*, β_*) with $\alpha_* = \frac{n}{2}\frac{\beta_*}{\beta_*-1}$, such that the pair (a', b') is interpolated from $(1, 1)$ and (α'_*, β'_*), that is we have*

$$\big\|\|w\|_{a',M}\big\|_{b',[0,T)} \le \big\|\|w\|_{\alpha'_*,M}\big\|^{\theta}_{\beta'_*,[0,T)}\big\|\|w\|_{1,M}\big\|^{1-\theta}_{1,[0,T)}, \tag{2.4.17}$$

for every measurable function w. Moreover, the interpolating parameter θ satisfies $\theta := \frac{1}{b} + \frac{n}{2a}$.

Proof. Multiplying the inequality satisfied by the pair (a, b) by a factor $\frac{b-1}{ab}$, we obtain the following equivalent condition

$$\frac{1}{b}+\frac{n}{2a}<1. \tag{2.4.18}$$

Similarly, we need to find a unique pair (α_*, β_*) such that

$$\frac{1}{\beta_*}+\frac{n}{2\alpha_*}=1, \tag{2.4.19}$$

and verifying for some θ the interpolation identities

$$\frac{1}{a'}=\frac{\theta}{\alpha'_*}+\frac{(1-\theta)}{1},\ \frac{1}{b'}=\frac{\theta}{\beta'_*}+\frac{(1-\theta)}{1}. \tag{2.4.20}$$

Applying the interpolation Theorem 2.4.5 would then allow us to conclude. Using the definition of Hölder's conjugate exponents we get

$$1-\frac{1}{a}=\frac{1}{a'}=\frac{\theta}{\alpha'_*}+(1-\theta)=\theta-\frac{\theta}{\alpha_*}+(1-\theta) \iff \frac{1}{\alpha_*}=\frac{1}{\theta a}. \tag{2.4.21}$$

Analogously, we can infer $\frac{1}{\beta_*}=\frac{1}{\theta b}$, hence substituting in (2.4.19)

$$\frac{1}{\theta b}+\frac{n}{2\theta a}=1 \iff \theta:=\frac{1}{b}+\frac{n}{2a}. \tag{2.4.22}$$

This θ is clearly greater than zero, and smaller than 1 by inequality (2.4.18). Moreover, θ determines uniquely the solution to the system, and we have

$$\alpha_*=\frac{a}{b}+\frac{n}{2} \text{ and } \beta_*=1+\frac{nb}{2a}. \tag{2.4.23}$$

□

We conclude this subsection recalling Theorem 4.1 in Wang's paper [84]; it regards the existence of a uniform Sobolev constant in a parabolic region, which will play a key role in the argument used in showing Theorem 4.1.3.

Definition 2.4.7. *We say that a subset $N \subset M$ admits a uniform Sobolev constant σ at each time-slice if*

$$\left(\int_N |v|^{\frac{2n}{n-2}} d\mu_{g(t)}\right)^{\frac{n-2}{n}} \leq \sigma \int_N |\nabla v|^2_{g(t)} d\mu_{g(t)}, \tag{2.4.24}$$

for every function $v \in W^{1,2}_0(N)$ and $t \in [0,T]$.

Theorem 2.4.8 (Theorem 4.1 in [84])**.** *Suppose $(M, g(t))$ is a complete Ricci flow defined on $[0,1]$. Fix a point $p \in M$ and suppose that*

- $\mathrm{Ric}(x,t) \geq -(n-1)g(t)$ *for every $(x,t) \in M \times [0,1]$;*

- $\mathrm{Ric}(x,t) \leq (n-1)g(t)$ *for every* $(x,t) \in B_{g(1)}(p,1) \times [0,1]$*;*
- *there exists a constant* κ *such that* $\mathrm{Vol}_{g(1)}(B_{g(1)}(p,1)) \geq \kappa$*.*

Then there exist a radius $r = r(n,\kappa) > 0$ *and a uniform Sobolev constant* $\sigma = \sigma(n,\kappa) > 0$ *for* $B_{g(1)}(p, r(n,\kappa))$ *on the time interval* $[0,1]$*.*

A different method to develop Sobolev constant bounds along the Ricci flow was obtained by Zhang in [93], and had great impact on the study of bounded scalar curvature Ricci flows, for instance see [7, 11, 73, 74].

Standard Results for the Singularity Analysis

Let us recall the fundamental local noncollapsing Theorem of Perelman ([48, 65]) as well as an extension of it (see [89]). We need the following definition.

Definition 2.4.9 (κ-Noncollapsing at Curvature Scales)**.** *Let* $\kappa > 0$*. We say that a Ricci flow* $g(t)$ *is* κ*-noncollapsed on the scale* ϱ *if every metric ball* $B_{g(t)}(p,r)$ *of radius* $r < \varrho$ *that satisfies for every* $(x,t) \in B_{g(t)}(p,r) \times (t-r^2, t]$ *the curvature bound* $|\mathrm{Rm}(x,t)|_{g(t)} \leq r^{-2}$*, has volume at least* κr^n*. We say that the flow is* κ*-noncollapsed on the scale* ϱ *relative to the scalar curvature if for every metric ball* $B_{g(t)}(p,r)$ *of radius* $r < \varrho$ *that satisfies the scalar curvature bound* $|\mathrm{R}(\cdot,t)|_{g(t)} \leq n(n-1)r^{-2}$ *on* $B_{g(t)}(p,r)$*, has volume at least* κr^n*.*

For a complete Ricci flow $(M, g(t))$ defined on a finite time interval $[0,T)$, with bounded curvature time-slices and a lower injectivity radius bound at the initial time, Perelman's Noncollapsing Theorem [65] guarantees for every ϱ the existence of a constant $\kappa = \kappa(n, g(0), T, \varrho)$ such that the flow is κ-noncollapsed on the scale ϱ. Furthermore, if the flows has uniformly bounded scalar curvature, $|\mathrm{R}| \leq n(n-1)R_0$ on $M \times [0,T)$, it is κ_1-noncollapsed on the scale $\varrho = \min\{R_0^{-1/2}, \sqrt{T}\}$ relative to the scalar curvature. In fact, by Aubin's classical result we have bounds on the Sobolev constants of the initial metric $g(0)$ (see [5]). These bounds extend to later times thanks to Lemma A.3 in [94], which yields the claimed statement using Lemma A.4 in the same paper. For another approach to these noncollapsing results see [89]. In particular, for any r_0 smaller than ϱ, we have the volume bound

$$\mu_{g(t_0)}\Big(B_{g(t_0)}(p_0, r_0)\Big) \geq \kappa_1 r_0^n. \tag{2.4.25}$$

Here κ_1 depends on the dimension n, the initial metric $g(0)$, the time T and the scalar curvature bound R_0. It can be shown that the dependence of κ_1 on the initial metric $g(0)$ is just through initial lower injectivity radius and Ricci curvature bounds. We will use these noncollapsing theorems several times throughout the thesis, given their primary importance in the context of the singularity analysis of the Ricci flow.

Remark 2.4.10. *Suppose we have a Ricci flow $(M, g(t))$ defined on a finite time interval $[0, T)$, with bounded curvature time-slices for every $t \in [0, T)$ and such that the initial time-slice $(M, g(0))$ is complete and has a positive lower bound on the injectivity radius. A combination of distance distortion estimates (see Lemma 3.2.1 for instance) and Hopf-Rinow's theorem guarantees that the flow remains complete for every time $t \in [0, T)$; therefore we can apply Perelman's Noncollapsing Theorem [65] as above, combined with Cheeger-Gromov-Taylor's injectivity radius bound [20] to deduce that the injectivity radius of $(M, g(t))$ remains bounded away from zero for every $t \in [0, T)$. Finally, Shi's estimates from [71] imply smooth bounds on the metric and the curvatures for every $(M, g(t))$ with $t \in (0, T)$. In other words, the Ricci flow has complete and bounded geometry time-slices as long as the curvature remains bounded, if the initial time-slice has those properties. This argument shows to which extent the bounded geometry assumption introduced in the Introduction chapter is natural.*

2.5 Preliminaries on the harmonic Ricci flow

In this section, we briefly review some definitions and tools from harmonic Ricci flow theory developed by Müller in [63]. Regarding all the results in this section, we would like to stress the dominance of the geometric component of the flow over the map one. Once again, one can recover the Ricci flow case by choosing $\alpha = 0$ and ϕ a constant map, so that the entire section can be viewed as a further preliminary section on Ricci flow.

Using a simple maximum principle argument, together with equation (2.2.5) above, Müller obtained that $\mathrm{S}_{min}(t)$ is monotonically increasing, thus $\mathrm{R} = \alpha|\nabla\phi|^2 + \mathrm{S} \geq \alpha|\nabla\phi|^2 + \mathrm{S}_{min}(0)$. A consequence of this result is the following Lemma.

Lemma 2.5.1 (Corollary 2.8 in [63])**.** *Suppose $(M, g(t), \phi(t))$ is a harmonic Ricci flow on a closed manifold, with $\alpha(t) \geq \underline{\alpha} > 0$. If there exist a sequence $(x_k, t_k) \subset M \times [0, T)$, with $t_k \nearrow T$ and $|\nabla\phi|^2(x_k, t_k) \to +\infty$, then also $\mathrm{R}(x_k, t_k) \to +\infty$, thus the flow becomes singular at $T_{max} \leq T$.*

Based on the inequalities (2.2.1), (2.2.2), (2.2.3), Müller developed an iterative scheme of bounds which yielded a long time existence result.

Theorem 2.5.2 (Theorem 3.12 in [63])**.** *Given a harmonic Ricci flow $(M, g(t), \phi(t))$ on $[0, T)$, $T < \infty$, defined on a closed manifold M and with a non-increasing coupling function $\alpha(t) \in [\underline{\alpha}, \overline{\alpha}]$, $0 < \underline{\alpha} \leq \overline{\alpha} < +\infty$. Then T is maximal, i.e. the flow cannot be extended past T if and only if*

$$\limsup_{t \nearrow T} \big(\sup_{x \in M} |\operatorname{Rm}(x, t)|^2\big) = +\infty. \tag{2.5.1}$$

As already mentioned above, in Chapter 3 we will prove new compactness results for sequences of harmonic Ricci flows. Their proof will rely on the following results.

Definition 2.5.3. *A sequence $(M_i, g_i(t), \phi_i(t), p_i)$ of $n-$dimensional, pointed, harmonic Ricci flows, with fixed target manifold N, defined for $t \in (a', b')$, where $-\infty \leq a' < 0 < b' \leq +\infty$, is said to satisfy uniform derivative curvature bounds if $\forall s > 0$, $\forall k \in \mathbb{N}$, there exist constants $M(k, s)$ and $C(s)$ such that $\forall t \in (a', b')$ and $\forall i$ we have*

$$\sup_{B_{g_i(0)}(p_i,s)} |\nabla_i \phi_i|_{g_i}(t) \leq C(s), \sup_{B_{g_i(0)}(p_i,s)} \left(|\nabla_i^{(k)} \operatorname{Rm}_i|_{g_i}(t) + |\nabla_i^{(k+2)} \phi_i|_{g_i}(t)\right) \leq M(k, s). \tag{2.5.2}$$

Recall the following proposition from [63]:

Proposition 2.5.4 (Proposition *A*.5, [63])**.** *Let $(M, g(t), \phi(t))$ be a complete solution of the harmonic Ricci flow defined on $[0, T)$, with a non-increasing coupling function $\alpha(t) \in [\underline{\alpha}, \overline{\alpha}]$, where $0 < \underline{\alpha} \leq \overline{\alpha} < \infty$ and let $T' < T < +\infty$. Define $B := B_{g(T')}(x, r)$, and assume that $|\operatorname{Rm}| \leq R_0$ on $B \times [0, T']$ for some constant R_0. Then there exist constants $K = K(\underline{\alpha}, \overline{\alpha}, R_0, T, n, N)$ and $C_k = C_k(k, \overline{\alpha}, n, N)$ for $k \in \mathbb{N}$, $C_0 = 1$, such that for all $k \geq 0$ we have*

$$|\nabla\phi|^2 \leq \frac{K}{t}, |\operatorname{Rm}| \leq \frac{K}{t}, |\nabla^k \operatorname{Rm}|^2 + |\nabla^{k+2}\phi|^2 \leq C_k \left(\frac{K}{t}\right)^{k+2} \tag{2.5.3}$$

for every $(x, t) \in B_{g(T')}(x, r/2) \times (0, T')$.

Remark 2.5.5. *We first remark that this theorem gives local estimates depending only on the curvature bound and on the (possibly singular) time T. In the case T is not singular, which is the case we will be interested in, we can take $T' = T$.*

Secondly, given the curvature bound R_0, we know $B_{g(0)}(x, \sigma_1 r) \subset B_{g(T')}(x, r) \subset B_{g(0)}(x, \sigma_2 r)$ for constants σ_1, σ_2 which depend only on R_0, the dimension n and on the finiteness of the time T (see Lemma 3.2.1), thus we can assume that both the bounds in the hypothesis and the conclusion hold on time $(t = 0)-$balls crossed with the time interval $(0, T)$.

In the proof of our compactness Theorem 3.2.6 we will first extract a limit of the central time $t = 0-$time-slices, and then extend the convergence at later times. In order to do so, we will need some uniform bounds on the curvature, given by the following (Lemma 2.10 in [71] or Lemma 3.5 in [85]):

Lemma 2.5.6. *Let (M, h) be a Riemannian manifold, $K \subset\subset M$ and $(g_i(t), \phi_i(t))$ a sequence of solutions to (1.3.1) defined on a neighbourhood of $K \times [a', b']$ with $[a', b']$ containing 0. Suppose that for each $k \in \mathbb{N}$, we have the following conditions:*

(a) $C^{-1}h \leq g_i(0) \leq Ch$ on K for every i,

(b) $|\nabla^k g_i(0)|_h + |\nabla^k \phi_i(0)|_h \leq C_k$ on K, for every i,

(c) $|\nabla_i \phi_i|_i \le C', |\nabla_i^{(k)} \operatorname{Rm}_i|_i + |\nabla_i^{(k+2)} \phi_i|_i \le C'_k$ *on* $K \times [a', b']$,

where the constants are independent of i*, but may depend on* K, a', b' *and* k*. Then for all* $k, s \in \mathbb{N}$ *there exist constants* $\tilde{C}, \tilde{C}_{k,s}$ *independent of* i *such that*

- $\tilde{C}^{-1} h \le g_i(t) \le \tilde{C} h$ *on* $K \times [a', b']$ *for every* i*,*
- *We have on* $K \times [a', b']$ *for every* i

$$\left| \frac{\partial^s}{\partial t^s} \nabla^k g_i \right|_h + \left| \frac{\partial^s}{\partial t^s} \nabla^k \phi_i \right|_h \le \tilde{C}_{k,s}$$

Let us conclude this subsection with the following *Perelman-Li-Yau-Harnack type* inequality due to Băileşteanu and Tran. It relates fundamental solutions of the conjugate heat operator under Ricci flow, defined as $\Box^* u := (-\partial_t - \Delta_{g(t)} + R)u$.

Theorem 2.5.7 (Theorem 1.1 in [6])**.** *Let* $(M, g(t), \phi(t))$ *be a harmonic Ricci flow on* $[0, T]$*, with complete and bounded geometry time-slices, and non-increasing coupling function* $\alpha(t)$*. Suppose* $u := \frac{e^{-f}}{(T-t)^{\frac{n}{2}}}$ *is a fundamental solution of the conjugate heat equation* $\Box^* u = 0$*, i.e. it tends to* δ_p *as* $t \to T$*, where* δ_p *is the Dirac delta at a certain point* $p \in M$*. Then the function* v *defined by* $v := ((T-t)(2\Delta f - |\nabla f|^2 + \mathrm{S}) + f - n)u$*, satisfies* $v \le 0$ *in* $(0, T]$*.*

Self-Similar Solutions

This subsection contains the definitions of self-similar solutions for the harmonic Ricci flow as well as some of their properties shown in [63].

Definition 2.5.8. *Given a coupling function* $\alpha(t)$*, we say that a harmonic Ricci flow* $(M, g(t), \phi(t))$ *defined on* $[0, T)$ *is a soliton solution if there exist a family of diffeomorphisms* $\psi_t \colon M \longrightarrow M$ *with* $t \in [0, T)$*, and a function* $c \colon [0, T) \longrightarrow \mathbb{R}^+$ *such that*

$$\begin{cases} g(t) = c(t) \psi_t^* g(0), \\ \phi(t) = \psi_t^* \phi(0). \end{cases} \tag{2.5.4}$$

If the derivative of c *verifies one of* $dc/dt = \dot{c} < 0$*,* $\dot{c} = 0$ *or* $\dot{c} > 0$*, the solution is called shrinking, steady or expanding solution respectively. If the diffeomorphisms* ψ_t *generate a family of vector fields* $X(t)$ *with* $X(t) = \nabla^{g(t)} f(t)$ *for some function* $f(t)$ *on* M*, we call the solution a gradient soliton solution, and the function* $f(t)$ *is called a potential of the soliton solution.*

Self-similar solutions naturally arise from the symmetry properties of the harmonic Ricci flow equations (1.3.1) seen in Section 2.2. One important example is the so-called *Gaussian soliton*: consider the time independent flat solution given by $(M, g(t), \phi(t), N) \equiv (\mathbb{R}^n, g_{euc}, y_0, N)$, where

g_{euc} is the standard Euclidean metric on $\mathbb{R}^n$, and y_0 is a constant map from $\mathbb{R}^n$ to N with image given by $y_0 \in \mathbb{R}^n$. For any coupling function $\alpha(t)$, we can consider this solution as a gradient shrinking solution, defining $g_{euc} = g(t) = c(t)\psi_t^* g_{euc}$, $\phi(t) = \psi_t^* y_0 \equiv y_0$, where $c(t)$ is any decreasing function with $c(0) = 1$ and ψ_t is the family of diffeomorphisms generated by the complete vector field $\nabla f(t)$ with $f(t) = \frac{-\dot{c}(t)}{4c(t)}|x|^2$. In the particular case in which $c(t) = T - t$ for some T, we will call it *Gaussian soliton in canonical form.* We will see later in which sense the Gaussian soliton models any harmonic Ricci flow near regular space-time points.

Now we recall the following result which describes a system of elliptic equations solved by a gradient soliton, written differently from its presentation in [63].

Lemma 2.5.9 (Lemma 2.2 in [63])**.** *Let $(M, g(t), \phi(t))$ be a gradient soliton solution with potential $f(t)$ defined on $M \times [0, T)$. Then for any $t \in [0, T)$, the solution satisfies*

$$\begin{cases} \mathrm{Ric}_{g(t)} - \alpha(t)\nabla\phi(t) \otimes \nabla\phi(t) + \mathrm{Hess}(f(t)) + \sigma(t)g(t) = 0, \\ \tau_{g(t)}\phi(t) - \langle \nabla\phi(t), \nabla f(t) \rangle = 0, \end{cases} \tag{2.5.5}$$

where $\sigma(t) = \frac{\dot{c}(t)}{2c(t)}$. Conversely, fix smooth functions f on $M \times [0, T)$, α and σ on $[0, T)$. Given a smooth solution $(g(t), \phi(t))$ of (2.5.5) on $M \times [0, T)$ (also smooth in t), suppose that $\nabla f(t)$ is a complete vector field with respect to $g(t)$ for any $t \in [0, T)$. Then we can write the solution $(M, g(t), \phi(t))$ as a gradient soliton solution to the harmonic Ricci flow with potential $f(t)$, coupling function $\alpha(t)$, and with $c(t) = \exp(2\Sigma(t))$, where $\Sigma(t) = \int_0^t \sigma(s)ds$.

Remark 2.5.10. *The converse direction of this lemma was stated differently in [63], namely as follows: given a function f and a solution of (2.5.5) at time $t = 0$, then there exists a family of diffeomorphisms ψ_t, $\psi_0 = id$, such that if we define $(M, g(t), \phi(t))$ as in (2.5.4), with linear scaling function $c(t) = 1 + 2\sigma(0)t$, then $(M, g(t), \phi(t))$ solves the harmonic Ricci flow with (constant) coupling function $\alpha = \alpha(0)$. Clearly, this defines a soliton solution. Our version allows non-constant α and σ.*

Definition 2.5.11. *We say that a gradient soliton solution is in canonical form if the scaling function $c(t)$ is linear, in which case (2.5.5) becomes*

$$\begin{cases} \mathrm{Ric}_{g(t)} - \alpha(t)\nabla\phi(t) \otimes \nabla\phi(t) + \mathrm{Hess}(f(t)) + \frac{a}{2(T-t)}g(t) = 0, \\ \tau_{g(t)}\phi(t) - \langle \nabla\phi(t), \nabla f(t) \rangle = 0, \end{cases} \tag{2.5.6}$$

where (possibly after scaling) $a = +1, 0, -1$ respectively in the expanding, steady and shrinking case.

Differently from what happens in the Ricci flow case, it is easy to construct a soliton solution which is not isometric (as a family) to one in canonical form; for instance, take $M = N = S^2$, $\alpha(t) = 1 - t$, $g(t) = (1 - t^2)g_S$, $\phi(t) = id_{S^2}$ where g_S is the multiple of the standard metric with (constant) scalar curvature 2.

Definition 2.5.12. *Given constants α and σ and a function f on M, then a solution (g, ϕ) of*

$$\begin{cases} \mathrm{Ric}_g - \alpha \nabla\phi \otimes \nabla\phi + \mathrm{Hess}(f) + \sigma g = 0, \\ \tau_g \phi - \langle \nabla\phi, \nabla f \rangle = 0 \end{cases} \tag{2.5.7}$$

is called gradient soliton. The case $\sigma < 0, \sigma = 0, \sigma > 0$ are called respectively shrinking, steady and expanding soliton. The function f is called a potential of the soliton.

Lemma 2.5.9 gives the relation between gradient soliton solutions and (complete) solitons. Recall the following simple consequence of the soliton equations.

Proposition 2.5.13 (Section 2 in [63])**.** *Let (g, ϕ) be a gradient soliton, i.e. a solution of (2.5.7). Then we have*

$$\begin{cases} \mathrm{R} - \alpha|\nabla\phi|^2 + \Delta f + \sigma n = 0 \\ \mathrm{R} - \alpha|\nabla\phi|^2 + |\nabla f|^2 + 2\sigma f = constant. \end{cases} \tag{2.5.8}$$

Definition 2.5.14. *We call a gradient soliton normalised, if the constant in the proposition above is 0. We call a gradient soliton solution normalised if its corresponding solitons are normalised at any time-slice.*

Any gradient soliton can be normalised by adding a constant to its potential.

Reduced Length and Volume at Regular times

We conclude the preliminary chapter with Müller's notions of reduced length and volume and some results about them from [63]. The concepts of reduced length and volume in Ricci flow were introduced by Perelman in [65]; he proved the monotonicity of this volume, a property which played a key role in his proof of the local $\kappa-$non-collapsing theorem we discussed before, essential for his resolution of the Poincaré and Thurston Geometrization Conjectures.

Definition 2.5.15 (Definition 1.2 in [62])**.** *Let $(M, g(t), \phi(t))$ be a harmonic Ricci flow defined on $[0, T)$. For any $0 \le \bar{t} < t_0 < T$, we define the $\mathcal{L}$-length of a curve $\gamma \colon [\bar{t}, t_0] \longrightarrow M$ as*

$$\mathcal{L}(\gamma) \coloneqq \int_{\bar{t}}^{t_0} \sqrt{t_0 - t}\left(|\dot\gamma(t)|^2_{g(t)} + \mathrm{S}_{g(t)}(\gamma(t))\right) dt. \tag{2.5.9}$$

We also define two functions $l_{p,t_0}, L_{p,t_0} \colon M \times (0, t_0) \longrightarrow \mathbb{R}$, for a fixed space-time point $(p, t_0) \in M \times [0, T)$, as

$$l_{p,t_0}(q, \bar{t}) \coloneqq \inf_{\gamma} \frac{1}{2\sqrt{t_0 - \bar{t}}} \mathcal{L}(\gamma) \eqqcolon \frac{1}{2\sqrt{t_0 - \bar{t}}} L_{p,t_0}(q, \bar{t}), \tag{2.5.10}$$

where γ ranges in the set of curves such that $\gamma(\bar{t}) = q$ and $\gamma(t_0) = p$. The function l_{p,t_0} is called reduced length based at (p, t_0).

Definition 2.5.16 (Definition 1.2 in [62]). *Let $(M, g(t), \phi(t))$ be a harmonic Ricci flow defined on $[0, T)$. For any $0 \leq \bar{t} < t_0 < T$, we define its reduced volume based at the point (p, t_0) as the function of $\bar{t}$ given by*

$$V_{(p,t_0)}(\bar{t}) := \int_M (4\pi(t_0 - \bar{t}))^{-\frac{n}{2}} e^{-l_{p,t_0}(q,\bar{t})} d\mu_{g(\bar{t})} =: \int_M v_{p,t_0}(q, \bar{t}) d\mu_{g(\bar{t})}. \tag{2.5.11}$$

From the singularities analysis point of view, this concept has the limitation of being based at a regular time: even in the restricted Ricci flow case, a suitable blow-up limit of a Ricci flow has constant reduced volume, thus from the theory of Perelman it is a gradient shrinking soliton solution, but if the blow-up procedure is done at a regular space-time point, the curvature boundedness implies directly that this limit is a Gaussian soliton, hence a *trivial* model. Enders (in [34]) and Naber (in [64]) therefore independently introduced a concept of reduced length based at the singular time for Ricci flow, which allows to rescale around singular points where the blow-up limits can be non-flat. In Chapter 3 we will transpose their concept to the harmonic Ricci flow setting, and then extend the subsequent analysis of Enders-Müller-Topping to this context.

In order to do so, we will need the following two results, proved by Müller in [63, 62].

Theorem 2.5.17 (Theorem 1.4 in [62]). *Let $(M, g(t), \phi(t))$ be a harmonic Ricci flow on $[0, T)$, $T < \infty$, with complete and bounded geometry time-slices and let $(p, t_0) \in M \times (0, T)$. Then the reduced length based at (p, t_0) verifies the following inequalities in the sense of distributions and in the smooth sense where smooth:*

$$\begin{cases} -\dfrac{\partial l_{p,t_0}(q,\bar{t})}{\partial \bar{t}} - \Delta l_{p,t_0}(q,\bar{t}) + |\nabla l_{p,t_0}(q,\bar{t})|^2 - \mathrm{S}_{g(\bar{t})} + \dfrac{n}{2(t_0 - \bar{t})} \geq 0 \iff \Box^* v_{p,t_0}(q,\bar{t}) \leq 0 \\ -|\nabla l_{p,t_0}(q,\bar{t})|^2 + \mathrm{S}_{g(\bar{t})} + \dfrac{l_{p,t_0}(q,\bar{t}) - n}{(t_0 - \bar{t})} + 2\Delta l_{p,t_0}(q,\bar{t}) \leq 0 \\ -2\dfrac{\partial l_{p,t_0}(q,\bar{t})}{\partial \bar{t}} + |\nabla l_{p,t_0}(q,\bar{t})|^2 - \mathrm{S}_{g(\bar{t})} + \dfrac{l_{p,t_0}(q,\bar{t})}{(t_0 - \bar{t})} = 0. \end{cases} \tag{2.5.12}$$

The last result we cite is not explicitely stated in this form but is derived from [63].

Theorem 2.5.18. *Let $(M, g(t), \phi(t))$ be a harmonic Ricci flow on $[0, T)$, $T < \infty$, with complete and bounded geometry time-slices and let $(p, t_0) \in M \times (0, T)$. Then the reduced volume based at (p, t_0) satisfies*

- *$V_{p,t_0}(\bar{t}) \leq 1$, with equality at one time $\bar{t} < t_0$ if and only if $(M, g(t), \phi(t))$ is a Gaussian soliton solution in canonical form;*

- $V_{p,t_0}(\bar{t})$ *is a non-decreasing function of* $\bar{t}$*; moreover, we have* $V_{p,t_0}(\bar{t}_1) = V_{p,t_0}(\bar{t}_2)$ *for some times* $0 \leq \bar{t}_1 < \bar{t}_2 \leq t_0$ *if and only if* $(M, g(t), \phi(t))$ *is a Gaussian soliton solution in canonical form;*

- $\lim_{\bar{t} \nearrow t_0} V_{p,t_0}(\bar{t}) = 1$.

The proof of this result is based on the fact that the reduced volume element satisfies $\Box^* v_{p,t_0} \leq 0$, with equality if and only if $(M, g(t), \phi(t))$ is a shrinking soliton solution in canonical form with potential given by the reduced length l_{p,t_0}. One can then argue that, since t_0 is a regular time, then the only possibility is that this soliton is a Gaussian soliton solution in canonical form, as we shall see later in Chapter 3.

Chapter 3

Local Singularity Theory for Type I Harmonic Ricci Flows

In [35] Enders, Müller and Topping showed that a Type I Ricci flow admits a non-trivial tangent flow at any singular point, with the structure of a gradient Ricci soliton, leading them to conclude that for such flows several reasonable definitions of singular points agree with each other. In this chapter, we present a proof of the analogous result for the harmonic Ricci flow, generalising in particular results of Guo, Huang and Phong [40] and Shi [72]. In order to obtain our result, we develop refined compactness theorems, a new pseudolocality theorem, and a notion of reduced length and volume based at the singular time for the harmonic Ricci flow. The arguments follow closely those in [31] by the author.

3.1 Introduction

In the present chapter we are dealing with Type I harmonic Ricci flows, which we define as follows.

Definition 3.1.2. *We say that a solution* $(M, g(t), \phi(t))$ *of (1.3.1) on* $[0, T)$*,* $T < \infty$*, is a Type I harmonic Ricci flow if there exists a constant* $C > 0$ *such that*

$$|\operatorname{Rm}|(x,t) \leq \frac{C}{(T-t)}. \tag{3.1.1}$$

In [72], Shi gave a definition of Type I harmonic Ricci flow in terms of the so-called AC curvature, which depends both on the evolving metric and the map. Our definition is different and depends only on the metric component of the flow, in agreement with the remarks in Chapter 1.

The main reason for introducing this concept resides in the fact that if such a flow develops a finite time singularity, this is forming as fast as possible, implying heuristically an easier blow-up analysis; more generally, these singularities are called Type I singularities, compare with the results in Chapter 5.

In Ricci flow theory, Type I Ricci flows are very well studied. The work most relevant for us is [35], where the authors, we recall, proved self-similarity and non-triviality of tangent flows at singular space-time points for these flows, confirming Hamilton's Conjecture 1.2.1 from [43]. Moreover, they also showed an equivalence result for some natural definitions of singular point for Type I Ricci flows. We will see in Chapter 5 how the situation becomes more convoluted without this assumption. Their methods of proof rely upon Perelman's pseudolocality theorem [65] and an extension of Perelman's reduced length, obtained independently by Enders [34] and Naber [64]. This extension has the crucial property of being based at the singular time, so one can use it to analyze the blow-up limit at a singular point, forcing the limit to be a gradient shrinking soliton in canonical form. The former is then used to prevent the trivial limit to occur in case the blow-up is done at a singular point. The other results stated are also consequences of the pseudolocality theorem. The aim of the present chapter is to prove an analogue of these results for the harmonic Ricci flow. Our main result is the following.

Theorem 3.1.3. *Let $(M, g(t), \phi(t), p)$ be a complete pointed Type I harmonic Ricci flow on $[0, T)$, $T < \infty$, with non-increasing coupling function $\alpha(t) \in [\underline{\alpha}, \overline{\alpha}]$, where $0 < \underline{\alpha} \leq \overline{\alpha} < \infty$. Then for any given sequence $\lambda_j \nearrow +\infty$, and any point $p \in M$, the sequence $(M, g_j(t), \phi_j(t), p)$ of harmonic Ricci flows, where $g_j(t) := \lambda_j g(T + \frac{t}{\lambda_j})$, $\phi_j(t) := \phi(T + \frac{t}{\lambda_j})$ on $[-\lambda_j T, 0)$, with associated coupling functions $\alpha_j(t) = \alpha(T + \frac{t}{\lambda_j})$, admits a subsequence converging in the pointed Cheeger-Gromov sense to a normalised gradient shrinking harmonic Ricci soliton solution in canonical form $(M_\infty, g_\infty(t), \phi_\infty(t), p_\infty)$, $t \in (-\infty, 0)$, with constant coupling function $\lim_{t \nearrow T} \alpha(t)$. Moreover, any subsequential limit is of this form. Finally, if the point $p \in \Sigma_I^H$ is a Type I singular point (see Definition 3.5.1), then the limit is non-trivial, i.e. not locally isometric to the Gaussian soliton.*

This theorem generalises Theorem 4.6 in [71], where Shi considered only the case of tangent flows defined on closed manifolds, and Theorem 2 in [40], in which the authors considered the case $N = \mathbb{R}$, so List's flow.

In order to prove Theorem 3.1.3, we choose to adapt the strategy from the Ricci flow case outlined above, that is to introduce a reduced length based at a singular time for the harmonic Ricci flow and to show a pseudolocality theorem. These two results might be of independent interest. We firstly extend the reduced length definition to admit a singular space-time base point (p, T).

Theorem 3.1.4. *Let $(M, g(t), \phi(t))$ be a complete Type I harmonic Ricci flow on $[0, T)$, $T < \infty$. Fix $t_i \nearrow T$ and a point $p \in M$. Then there exists a locally Lipschitz function*

$$l_{p,T}\colon M \times (0, T) \to \mathbb{R}, \tag{3.1.2}$$

which is a subsequential limit of the reduced lengths l_{p,t_i} in $C^{0,1}_{loc}$. Moreover, defining

$$v_{p,T}(q, \bar{t}) := (4\pi(T - \bar{t}))^{-\frac{n}{2}} e^{-l_{p,T}(q,\bar{t})}, \tag{3.1.3}$$

these functions verify in the sense of distributions the following inequalities:

$$\begin{cases} -\dfrac{\partial l_{p,T}(q,\bar{t})}{\partial \bar{t}} - \Delta l_{p,T}(q,\bar{t}) + |\nabla l_{p,T}(q,\bar{t})|^2 - \mathrm{S}_{g(\bar{t})} + \dfrac{n}{2(T-\bar{t})} \geq 0 \iff \Box^* v_{p,T}(q,\bar{t}) \leq 0 \\ -|\nabla l_{p,T}(q,\bar{t})|^2 + \mathrm{S}_{g(\bar{t})} + \dfrac{l_{p,T}(q,\bar{t}) - n}{(T-\bar{t})} + 2\Delta l_{p,T}(q,\bar{t}) \leq 0 \\ -2\dfrac{\partial l_{p,T}(q,\bar{t})}{\partial \bar{t}} + |\nabla l_{p,T}(q,\bar{t})|^2 - \mathrm{S}_{g(\bar{t})} + \dfrac{l_{p,T}(q,\bar{t})}{(T-\bar{t})} = 0. \end{cases} \tag{3.1.4}$$

After obtaining the proper compactness theorems for sequences of harmonic Ricci flows, we show the following pseudolocality theorem:

Theorem 3.1.5. *For every $\beta > 0$, N, $\underline{\alpha}$ and $\overline{\alpha}$, where $0 < \underline{\alpha} \leq \overline{\alpha} < \infty$, there exist $\delta, \varepsilon > 0$ with the following property. Suppose that we have a pointed complete harmonic Ricci flow $(M, g(t), \phi(t), p)$ defined for $t \in [0, (\varepsilon r_0)^2]$, with non-increasing coupling function $\alpha(t) \in [\underline{\alpha}, \overline{\alpha}]$. Suppose the following:*

- $\mathrm{S}(0) \geq -r_0^2$ *on* $B_{g(0)}(p, r_0)$*;*
- $\mathrm{Area}_{g(0)}(\partial\Omega)^n \geq (1 - \delta) c_n \,\mathrm{Vol}_{g(0)}(\Omega)^{n-1}$*, for any* $\Omega \subset B_{g(0)}(p, r_0)$*, where* c_n *is the Euclidean isoperimetric constant.*

Then $|\mathrm{Rm}|(x, t) < \beta t^{-1} + (\varepsilon r_0)^{-2}$ *whenever* $0 < t \leq (\varepsilon r_0)^2$ *and* $d_{g(t)}(x, p) \leq \varepsilon r_0$*.*

Notice that we had to substitute the lower bound on the initial scalar curvature, originally present in the Ricci flow analogue of this result (see Theorem 10.1 in [65]), with the *stronger* lower bound on the initial S curvature. Moreover, we want to stress that this theorem generalizes Theorem 1 in [40], where the authors prove a pseudolocality theorem for List's flow, under the further assumption of having bounded curvature time-slices; this is due to their application of the compactness Theorem 7.5 in [57] which only holds under this additional hypothesis. However, we can prove the theorem in this generality thanks to our stronger compactness Theorem 3.2.9, see Section 3.4 for details on the proof of the pseudolocality theorem above.

This chapter is organised as follows. In Section 3.2, we develop distance distortion estimates, a compactness theory for the harmonic Ricci flow and a rigidity result for solitons. We also motivate the concept of Type I harmonic Ricci flow given in Definition 3.1.2. The reduced length and volume based at singular time are studied in Section 3.3, where Theorem 3.1.4 is proved. We show Theorem 3.1.5 in Section 3.4. Finally, in Section 3.5 we bring all these results together to obtain our main result, Theorem 3.1.3.

3.2 Some Basic new Results in harmonic Ricci flow Theory

Distance Distortion Estimates

In this subsection we carry out distance distortion estimates for the harmonic Ricci flow.

Lemma 3.2.1. *Let $(M, g(t), \phi(t))$ be a harmonic Ricci flow with target manifold N. If for some $t_0 < t_1$ we have $\mathcal{S}(\cdot, t) \leq K g(t)$ on $M \times [t_0, t_1]$, then for every two points $x_0, x_1 \in M$ we have*

$$\frac{d_{g(t_1)}(x_0, x_1)}{d_{g(t_0)}(x_0, x_1)} \geq e^{-K(t_1 - t_0)}.$$

Proof. For a fixed curve $\gamma\colon [0, a] \longrightarrow M$, its length evolves by

$$\frac{d}{dt}\,\mathrm{Length}(\gamma) = \frac{d}{dt}\int_0^a \sqrt{\Big\langle \frac{d\gamma}{ds}(s), \frac{d\gamma}{ds}(s)\Big\rangle_t}\,ds = -\int_0^a \mathcal{S}\Big(\frac{d\gamma}{ds}(s), \frac{d\gamma}{ds}(s)\Big)\frac{ds}{\big|\frac{d\gamma}{ds}\big|} \geq -K\,\mathrm{Length}(\gamma).$$

Integrating this gives

$$\frac{\mathrm{Length}(\gamma)\big|_{t_1}}{\mathrm{Length}(\gamma)\big|_{t_0}} \geq e^{-K(t_1 - t_0)}$$

so we deduce the conclusion by approximating the $g(t_1)-$distance between x_0 and x_1 with γ. □

Remark 3.2.2. *Similarly, if $\mathcal{S}(\cdot, t) \geq -K g(t)$ on $M \times [t_0, t_1]$, we get*

$$\frac{d_{g(t_1)}(x_0, x_1)}{d_{g(t_0)}(x_0, x_1)} \leq e^{K(t_1 - t_0)}.$$

Lemma 3.2.3. *Let $(M, g(t), \phi(t))$ be a complete harmonic Ricci flow with target manifold N. Suppose that $x_0, x_1 \in M$ are two points with $d_{g(t_0)}(x_0, x_1) \geq 2r_0$ and $\mathrm{Ric}(x, t_0) \leq K g(t_0)$ for all points $x \in B_{t_0}(x_0, r_0) \cup B_{t_0}(x_1, r_0)$. Then $\frac{d}{dt} d_{g(t)}(x_0, x_1) \geq -2(\frac{2}{3}K r_0 + (n-1) r_0^{-1})$ at $t = t_0$.*

Proof. Let γ be a normalised minimal geodesic from x_0 to x_1 in $(M, g(t_0))$, and set $X(s) := \frac{d\gamma}{ds}(s)$. Then for every vector field V along γ such that $V(x_0) = V(x_1) = 0$, from the second variation of length we have

$$\int_0^{d_{g(t_0)}(x_0,x_1)} \Big(|\nabla_X V|^2 - \langle \mathrm{R}(V,X)V,X\rangle\Big)ds \geq 0. \tag{3.2.1}$$

Extend X to a parallel orthonormal frame along γ with $\{e_i(s)\}_{i=1}^{n-1}$. Put $V_i(s) = f(s)e_i(s)$ where

$$f(s) = \begin{cases} \frac{s}{r_0} & \text{if } 0 \leq s \leq r_0, \\ 1 & \text{if } r_0 \leq s \leq d_{g(t_0)}(x_0,x_1) - r_0, \\ \frac{d_{g(t_0)}(x_0,x_1)-s}{r_0} & \text{if } d_{g(t_0)}(x_0,x_1) - r_0 \leq s \leq d_{g(t_0)}(x_0,x_1). \end{cases} \tag{3.2.2}$$

Thus we have $|\nabla_X V_i| = |f'(s)|$ and

$$\int_0^{d_{g(t_0)}(x_0,x_1)} |\nabla_X V_i|^2 ds = 2\int_0^{r_0} \frac{1}{r_0} = \frac{2}{r_0}.$$

Next, we compute

$$\begin{aligned}\int_0^{d_{g(t_0)}(x_0,x_1)} \langle \mathrm{R}(V_i,X)V_i,X\rangle ds &= \int_0^{r_0} \frac{s^2}{r_0^2}\langle \mathrm{R}(e_i,X)e_i,X\rangle ds + \int_{r_0}^{d_{g(t_0)}(x_0,x_1)-r_0} \langle \mathrm{R}(e_i,X)e_i,X\rangle ds \\ &+ \int_{d_{g(t_0)}(x_0,x_1)-r_0}^{d_{g(t_0)}(x_0,x_1)} \frac{(d_{g(t_0)}(x_0,x_1)-s)^2}{r_0^2}\langle \mathrm{R}(e_i,X)e_i,X\rangle ds\end{aligned}$$

so summing over i we get

$$\begin{aligned}0 \leq \sum_{i=1}^{n-1}\int_0^{d_{g(t_0)}(x_0,x_1)} \big(|\nabla_X V_i|^2 - \langle \mathrm{R}(V_i,X)V_i,X\rangle\big)ds &= \frac{2(n-1)}{r_0} - \int_0^{d_{g(t_0)}(x_0,x_1)} \mathrm{Ric}(X,X)ds \\ - \int_0^{r_0}\left(\frac{s^2}{r_0^2}-1\right)\mathrm{Ric}(X,X)ds - \int_{d_{g(t_0)}(x_0,x_1)-r_0}^{d_{g(t_0)}(x_0,x_1)} &\left(\frac{(d_{g(t_0)}(x_0,x_1)-s)^2}{r_0^2}-1\right)\mathrm{Ric}(X,X)ds.\end{aligned}$$

Taking a geodesic variation of γ with fixed endpoints for t near t_0 (using the completeness assumption) we have

$$\begin{aligned}\frac{d}{dt}d_{g(t)}(x_0,x_1)\big|_{t=t_0} &= -\int_0^{d_{g(t_0)}(x_0,x_1)} \mathcal{S}(X,X)ds \geq -\int_0^{d_{g(t_0)}(x_0,x_1)} \mathrm{Ric}(X,X)ds \\ &\geq -\frac{2(n-1)}{r_0} - 2K\frac{2}{3}r_0.\end{aligned}$$

□

Corollary 3.2.4. *Given a complete harmonic Ricci flow $(M, g(t), \phi(t))$, if $\mathrm{Ric}(\cdot,t) \leq Kg(t)$ on M for every $t \in [0,T)$ then there exists a constant $c = c(n) > 0$ such that for every $x_0, x_1 \in M$, $\frac{d}{dt}d_{g(t)}(x_0,x_1) \geq -cK^{1/2}$.*

Proof. Put $r_0 = K^{-\frac{1}{2}}$. If $d_{g(t)}(x_0,x_1) \leq 2r_0$ then use Lemma 3.2.1, otherwise use Lemma 3.2.3. □

With a very similar proof as the one of Lemma 3.2.3 one can prove (see Lemma 4 in [40]):

Lemma 3.2.5. *Let $(M, g(t), \phi(t))$ be a complete harmonic Ricci flow. If $\mathrm{Ric}(\cdot, t_0) \leq Kg(t_0)$ on $B_{g(t_0)}(x_0, r_0)$ then for every $x \notin B_{g(t_0)}(x_0, r_0)$ we have*

$$\frac{d}{dt} d_{g(t)}(x_0, x)\Big|_{t=t_0} - \Delta_{t_0} d_{g(t_0)}(x_0, x) \geq -\left(\frac{2}{3}Kr_0 + (n-1)r_0^{-1}\right).$$

Compactness Theorems

We now provide several refinements of the compactness results for harmonic Ricci flows obtained by Shi in [71] and Williams in [85]. One of these improvements is needed in the proof of the pseudo-locality theorem where only local bounds on the curvature are available, and therefore Shi's and Williams' results do not directly apply.

We follow the argument developed by Topping in [80] for the Ricci flow case (extending Hamilton's compactness result in [43]), see also [81] for an expository review. The first theorem we want to prove is the following.

Theorem 3.2.6 (Compactness of harmonic Ricci flows: Extension 1)**.** *Consider a sequence of pointed, $n-$dimensional, complete harmonic Ricci flows $(M_i, g_i(t), \phi_i(t), p_i)$, with fixed target manifold N, all defined for $t \in (a, b)$, where $-\infty \leq a < 0 < b \leq +\infty$, with (possibly different) non-increasing coupling functions $\alpha_i(t) \in [\underline{\alpha}, \overline{\alpha}]$, where $0 < \underline{\alpha} \leq \overline{\alpha} < \infty$. Assume further that*

(i) $\forall r > 0$ there exists a constant $M = M(r)$ such that $\forall t \in (a, b)$ and $\forall i$

$$\sup_{B_{g_i(0)}(p_i, r)} |\mathrm{Rm}(g_i(t))|_{g_i(t)} \leq M,$$

(ii) $\inf_i \mathrm{inj}(M_i, g_i(0), p_i) > 0$.

Then there exist an $n-$dimensional manifold M_∞, a limit solution $(g_\infty(t), \phi_\infty(t))$ of (1.3.1) on $M_\infty \times (a, b)$ with target manifold N and non-increasing coupling function $\alpha_\infty(t)$, and finally a point $p_\infty \in M_\infty$, such that $(M_\infty, g_\infty(0))$ is complete, and $(M_i, g_i(t), \phi_i(t), p_i) \longrightarrow (M_\infty, g_\infty(t), \phi_\infty(t), p_\infty)$ in the pointed Cheeger-Gromov sense (after passing to a subsequence) and $\alpha_i(t) \longrightarrow \alpha_\infty(t)$ pointwise in (a, b).

Remark 3.2.7. *We notice that in our definition of harmonic Ricci flow (1.3.1), the coupling function must be smooth, whereas the limit $\alpha_\infty(t)$ obtained above is a-priori not regular at all. We still chose to state the theorem this way, to emphasize the geometric hypotheses, over the technical uniform local smooth bounds on the coupling functions α_i one would need. However, in our applications of the compactness theorem, the convergence will always turn out to be smooth, so* *that the limit is a proper harmonic Ricci flow with a smooth coupling function.*

Remark 3.2.8. *As Hamilton remarked in [43] for the corresponding Ricci flow result, it is enough to prove Theorem 3.2.6 in the case $a, b \in \mathbb{R}$. Once proved on finite intervals, a standard diagonal argument yields the other cases. Furthermore, from the assumption (i) in the main Theorem 3.2.6, we get that the sequence verifies the uniform derivative curvature bounds (2.5.2) in a slightly smaller finite interval $[a+\varepsilon, b-\varepsilon]$ for any arbitrarily small $\varepsilon > 0$ from Proposition 2.5.4. So if we can prove the theorem with the extra assumption of having uniform derivative curvature bounds, we can take a sequence ε_j tending to 0, and for any such j we have a limit harmonic Ricci flow that agrees with the ones obtained for smaller j (on the interval where both limits exist), and so, by a diagonal argument, we obtain a limit flow on the interval (a, b) as required.*

The plan of the proof is the same as in [43]: we first extract a limit manifold of the $t = 0$ time-slices, and then use uniform bounds to obtain convergence at the other times.

Proof of Theorem 3.2.6. Without loss of generality, we can assume (possibly after extracting a subsequence, still denoted by i) that the coupling functions $\alpha_i(t)$ converge pointwise in (a, b) to a certain coupling function $\alpha_\infty(t)$, which is still non-increasing and verifies the same bounds as the α_i's.

We can assume that our sequence satisfies *uniform derivative curvature bounds* by Remark 3.2.8. By Theorem 2.3.2 we can extract a subsequence such that $(M_i, g_i(0), p_i) \longrightarrow (M_\infty, g_\infty(0), p_\infty)$ in the pointed Cheeger-Gromov sense, for a certain pointed manifold (M_∞, p_∞) and metric $g_\infty(0)$. Let us remark that $(M_\infty, g_\infty(0))$ is complete by the theorem. Call Ψ_i the diffeomorphisms given by the Cheeger-Gromov convergence. Since we have $\sup_{B_{g_i(0)}(p_i,s)} |\nabla_i^k \phi_i|_{g_i}(0) \leq M(k, s)$, extracting a further subsequence we can assume $\Psi_i^* \phi_i(0) \longrightarrow \phi_\infty(0)$ in $C^\infty_{loc}(M_\infty; N)$ for a certain map $\phi_\infty(0)$. To extend the convergence at the other times, it is sufficient to apply Lemma 2.5.6 to $\Psi_i^* g_i(t)$ and $\Psi_i^* \phi_i(t)$, so by Arzelá-Ascoli's theorem we get a limit flow $g_\infty(t)$ for $t \in (a, b)$ and a limit $\phi_\infty(t)$ of the maps $\Psi_i^* \phi_i(t)$ in $C^\infty_{loc}(M_\infty \times (a, b); N)$, which agree with the metric $g_\infty(0)$ and the map $\phi_\infty(0)$ at $t = 0$ respectively, and form a harmonic Ricci flow. □

Once we have a limit harmonic Ricci flow, we can inquire its completeness. As remarked during the proof of the main Theorem 3.2.6, we already have completeness at the $(t = 0)-$time-slice. A good idea to get completeness at other time-slices is to exploit the length distortion estimates obtained before in this section together with the Hopf-Rinow Theorem, as in [80].

Theorem 3.2.9 (Compactness of harmonic Ricci flows: Extension 2)**.** *Under the same assumptions as in Theorem 3.2.6, suppose instead of having (i) we have the stronger assumption that*

(i') There exists $M < \infty$ with the following property: $\forall r > 0$ there exists $K = K(r) \in \mathbb{N}$ such

that $\forall t \in (a, b)$ *and* $i \geq K$ *we have*

$$\sup_{B_{g_i(0)}(p_i, r)} |\operatorname{Rm}(g_i(t))|_{g_i(t)} \leq M. \tag{3.2.3}$$

Then the harmonic Ricci flow constructed in Theorem 3.2.6 has complete time-slices.

Proof. Since $(M_\infty, g_\infty(0))$ is complete, for every $r > 0$ we have $B_{g_\infty(0)}(p_\infty, \frac{r}{2}) \subset\subset M_\infty$. By the assumption (i') and the convergence, we have $\sup_{B_{g_\infty(0)}(p_\infty, \frac{r}{2})} |\operatorname{Rm}(g_\infty(t))|_{g_\infty(t)} \leq M$. Since M is independent of r, the limit harmonic Ricci flow has bounded curvature at every time-slice. Finally, for every $s > 0$ we have $B_{g_\infty(t)}(p_\infty, s) \subseteq B_{g_\infty(0)}(p_\infty, C(M, t)s) \subset\subset M_\infty$, using the length distortion estimate developed in the previous subsection. □

Remark 3.2.10. *We remark that this version of the compactness theorem will be used in the proof of the pseudolocality theorem.*

In the theorem above we used the strong bound on the curvature to apply the length distortion estimate in Lemma 3.2.1. Since the hypothesis of the latter are weaker, we can improve the result, showing that a uniform unilateral bound on the tensors $\mathcal{S}_{g_i(t)}$ implies completeness in the past or the future.

Theorem 3.2.11 (Compactness of harmonic Ricci flows: Extension 3)**.** *Under the same assumptions as in Theorem 3.2.6, if there exists a constant* $C > 0$ *such that*

$$\inf_{M_i} \mathcal{S}_{g_i(t)} \geq -C g_i(t) \quad \forall t \in (a, 0] \quad \forall i$$

then $(M_\infty, g_\infty(t))$ *is complete* $\forall t \in (a, 0]$*. Analogously, the bound*

$$\sup_{M_i} \mathcal{S}_{g_i(t)} \leq C g_i(t) \quad \forall t \in [0, b) \quad \forall i$$

implies completeness of time-slices of the limit for $t \in [0, b)$*.*

Type I and Type A Conditions

In this subsection we define what is a Type I singularity for the harmonic Ricci flow, and give motivation for our Definition 3.1.2. It is worth mentioning that in [71], Shi gave a definition of Type I singularity expressed in terms of the so-called *AC Curvature*, a quantity he defined as $Q(x, t) \coloneqq (|\operatorname{Rm}| + |\nabla^2\phi| + |\nabla\phi|^2)(x, t)$. In particular, he proved that if a harmonic Ricci flow develops a singularity at a time $T < +\infty$, then there exists a constant $c > 0$ such that

$$\sup_{x\in M} Q(x,t) \geq \frac{c}{T-t}. \tag{3.2.4}$$

However, motivated by Theorem 2.5.2, we would like a definition focusing only on the metric component. Therefore our Definition 3.1.2 is different from this and coherent with the heuristic dominance of the metric component over the map one. Throughout this subsection we assume for simplicity that the domain manifold M is closed, though we only need to use the strong maximum principle.

Lemma 3.2.12. *Let $(M, g(t), \phi(t))$ be a harmonic Ricci flow defined on $[0, T)$ with a non-increasing coupling function $\alpha(t) \in [\underline{\alpha}, \overline{\alpha}]$, $0 < \underline{\alpha} \leq \overline{\alpha} < +\infty$. Suppose $T < +\infty$ is maximally chosen. Then there exists a constant $c > 0$ and a sequence $(t_k) \nearrow T$ such that*

$$\sup_{x\in M} |\operatorname{Rm}|(x,t_k) \geq \frac{c}{T-t_k}. \tag{3.2.5}$$

Proof. Let us set

$$y(t) := \max_{x\in M} |\nabla\phi|^2(x,t),\ \ w(t) := \max_{x\in M} |\nabla^2\phi|^2(x,t) \ \mathit{and}\ z(t) := \max_{x\in M} |\operatorname{Rm}|^2(x,t). \tag{3.2.6}$$

From the evolution inequalities (2.2.1), (2.2.2) and (2.2.3) they satisfy, we deduce

$$\partial_t y \leq C(N)y^2; \tag{3.2.7}$$
$$\partial_t w \leq C\sqrt{z}w + C(\alpha + c_0)yw + Cc_0 y^2\sqrt{w} \tag{3.2.8}$$
$$\partial_t z \leq Cz^{\frac{3}{2}} + \alpha Cyz + \alpha Cw\sqrt{z} + \alpha Cc_0 y^2\sqrt{z}. \tag{3.2.9}$$

The proof is split in two cases. Firstly, if we have that y is unbounded, an application of the maximum principle to the equation (3.2.7) yields $y \geq \frac{c}{T-t}$, thus using (2.2.5) as in the proof of Lemma 2.5.1 we get the desired bound. In the case y is bounded, we can absorb it in the constants appearing in (3.2.8) and (3.2.9). Writing the differential inequalities for $\sqrt{w}$ and $\sqrt{z}$ and summing them up, we obtain $2 \cdot max\{z(t), w(t)\} \geq (z+w)(t) \geq \frac{c_1}{(T-t)^2}$, again via a maximum principle argument. Proceeding now by contradiction, suppose that for every $\varepsilon > 0$ there exists a $T(\varepsilon)$ such that $z(t) \leq \frac{\varepsilon}{T-t}$ for $t \in [T(\varepsilon), T)$. Then rewrite (3.2.8) with this new bound to get:

$$\frac{\partial w}{\partial t} \leq c\sqrt{z}w + cw \leq c\sqrt{z}w \leq \frac{c\varepsilon}{T-t}w, \quad \mathit{in}\ \ [T(\varepsilon), T). \tag{3.2.10}$$

One can apply Gronwall's Lemma to get $w(t) \leq \frac{c'}{(T-t)^{c\varepsilon}}$ which is contradictory for $\varepsilon < c^{-1}\min\{c_1, 2\}$. □

Motivated by this Lemma, we make the following definition.

Definition 3.2.13. *We say that a harmonic Ricci flow $(M, g(t), \phi(t))$ develops a Type I singularity at the finite time T if there exist a constant $c > 0$ and a sequence $(t_k) \nearrow T$ such that*

$$\sup_{x\in M} |\operatorname{Rm}|(x, t_k) \sim \frac{c}{T - t_k}. \tag{3.2.11}$$

We are naturally tempted to define a Type I harmonic Ricci flow by "reversing" the inequality (3.2.5); since our main scope will be to perform blow-up arguments, we need to control all the components of the flow. Fortunately we have:

Theorem 3.2.14. *Let $(M, g(t), \phi(t))$ be a harmonic Ricci flow defined on $[0, T)$ with a non-increasing coupling function $\alpha(t) \in [\underline{\alpha}, \overline{\alpha}]$, $0 < \underline{\alpha} \le \overline{\alpha} < +\infty$. Suppose that there exist constants $r \ge 1$ and $C > 0$ such that*

$$|\operatorname{Rm}|(x, t) \le \frac{C}{(T - t)^r}. \tag{3.2.12}$$

Then the same inequality, with a different constant $\tilde{C} = \tilde{C}(\underline{\alpha}, \overline{\alpha}, n, N, C, r)$ holds for $|\nabla\phi|^2$ and $|\nabla^2\phi|$. Moreover

$$|\nabla \operatorname{Rm}|(x, t) \le \frac{C}{(T - t)^{\frac{3}{2}r}}. \tag{3.2.13}$$

Proof. The proof is straight-forward and we therefore only give a brief sketch. Using the bound on the Riemann tensor, we get the same kind of bound for the scalar curvature, thus for S since $\mathrm{S}(x, t) \le \mathrm{R}(x, t)$. Since $\mathrm{S}_{min}(t)$ is increasing, we get the same bound on $|\nabla\phi|^2$. Using the bounds on y and z defined as in (3.2.6) to rewrite (3.2.8), we get the same bound on $|\nabla^2\phi|$ by applying the maximum principle to the function $f(x, t) := (T - t)^{2r}|\nabla^2\phi|^2(A + (T - t)^r|\nabla\phi|^2)$ for a large enough constant A. This proves the first statement. A similar argument leads to the second statement. □

Definition 3.2.15. *We say that a solution $(M, g(t), \phi(t))$ of (1.3.1) on $[0, T)$ is a Type A harmonic Ricci flow if there exist constants $r \in [1, \frac{3}{2})$ and $C > 0$ such that*

$$|\operatorname{Rm}|(x, t) \le \frac{C}{(T - t)^r}. \tag{3.2.14}$$

A priori it might be that the supremum of the curvature oscillates between different rates, e.g. the Type I rate and a smaller one. We will exclude this phenomenon in the Type I flow case, see Corollary 3.5.4, where we will see that every singular harmonic Ricci flow satisfying Definition 3.1.2 necessarily develops a Type I singularity in the stronger sense that

$$\sup_{x\in M} |\operatorname{Rm}|(x, t) \ge \frac{c}{T - t} \tag{3.2.15}$$

for every $t \in [0, T)$.

A soliton rigidity theorem

Motivated by Zhang's paper [96], we prove the following theorem. It is worth mentioning that in [40] the authors proved the first part of this result for List's flow, and our argument closely follows theirs. In addition, we also include a rigidity remark in the same spirit of [67]. Together, these results will be used to prove the equivalence between the singular sets we mentioned in the introduction of the chapter, see Theorem 3.5.3.

Theorem 3.2.16. *Suppose $(M, g(t), \phi(t), f(t))$ is a complete gradient shrinking soliton solution defined for $t \in (-\infty, T)$. Then we have $\mathrm{S}(t) \geq 0$ and $\nabla f(t)$ is a complete vector field for any $t \in (-\infty, T)$. Moreover, if there exist $p \in M$ and $t \in (-\infty, T)$ such that $\mathrm{S}(p, t) = 0$, then $(M, g(t), \phi(t), f(t))$ is isometric to the Gaussian soliton solution.*

Before proceeding with the proof we state a lemma, which is easy to prove using Proposition 2.5.13 and the Bochner identity (Section 4 in [63]) and whose proof is therefore left to the reader.

Lemma 3.2.17. *For a normalised gradient shrinking soliton (g, ϕ, f) we have*

$$\Delta \mathrm{S} - \langle \nabla \mathrm{S}, \nabla f \rangle = -2\sigma \mathrm{S} - 2|S_{ij}|^2 - 2\alpha(\tau_g \phi)^2. \tag{3.2.16}$$

Proof of Theorem 3.2.16. For the first part of the theorem we will work in a fixed time-slice, so without loss of generality we can suppose to have (g, ϕ, f) satysfying (2.5.7) with $\sigma = -1/2$, and we can drop the time dependence. Rewrite Lemma 3.2.5 in the following way: (for any fixed time-slice), if $\mathrm{Ric} \leq Kg$ in the ball $B(p, r_0)$, then

$$\Delta d(p, \cdot) \leq \left((n-1)r_0^{-1} + \frac{2}{3}Kr_0\right) - \int_0^{d(p,\cdot)} \mathrm{Ric}(\dot\gamma(s), \dot\gamma(s))ds,$$

where γ is a normalised geodesic starting at p. Applying the first of the soliton equations to the pair $(\dot\gamma, \dot\gamma)$ we get $\mathcal{S}(\dot\gamma, \dot\gamma) + \nabla^2 f(\dot\gamma, \dot\gamma) = 1/2$. Moreover, we have

$$\begin{aligned} \frac{d}{ds} f(\gamma(s)) &= \langle \nabla f, \dot\gamma \rangle, \\ \frac{d^2}{ds^2} f(\gamma(s)) &= \langle \nabla_{\dot\gamma} \nabla f, \dot\gamma \rangle = \nabla^2 f(\dot\gamma, \dot\gamma), \end{aligned}$$

therefore for every $x \in M$

$$\begin{aligned} \int_0^{d(p,x)} \mathrm{Ric}(\dot\gamma(s), \dot\gamma(s))ds &= \frac{1}{2}d(p,x) - \int_0^{d(p,x)} \frac{d^2}{ds^2} f(\gamma(s))ds + \alpha \int_0^{d(p,x)} d\phi \otimes d\phi(\dot\gamma, \dot\gamma)ds \\ &\geq \frac{1}{2}d(p,x) - \langle \nabla f, \dot\gamma \rangle\Big|_{s=0}^{s=d(p,x)} \geq \frac{1}{2}d(p,x) - \langle \nabla f(x), \nabla_x d(p,x) \rangle - |\nabla f(p)|. \end{aligned}$$

In other words, for any fixed point $p \in M$ the function $d(x) := d(p,x)$ satisfies

$$\Delta d - \langle \nabla f, \nabla d\rangle \leq \left((n-1)r_0^{-1} + \frac{2}{3}Kr_0\right) - \frac{1}{2}d + |\nabla f(p)|. \tag{3.2.17}$$

For every point p there exists a small enough radius $r_0 = r_0(p) > 0$ such that $\mathrm{Ric}(g) \leq (n-1)r_0^{-2}$ on the ball $B(p, r_0)$. Fix a cut-off function $\psi(x)$ which is equal to 1 if $|x| \leq 1$ and zero for $|x| > 2$, is non-increasing on the positive axis and such that $|\psi'|^2/\psi \leq 4$, $|\psi''|, |\psi'| \leq 2$. Define then the function on M

$$\eta(x) := \psi\left(\frac{d(p,x)}{Ar_0}\right),$$

where A is a large constant. Define $u := \mathrm{S}\,\eta$. It easily follows that

$$\Delta u = \eta \Delta\, \mathrm{S} + \frac{\mathrm{S}\,\psi'}{Ar_0}\Delta d + \frac{\mathrm{S}\,\psi''}{(Ar_0)^2} + 2\langle \nabla\, \mathrm{S}, \nabla \eta\rangle. \tag{3.2.18}$$

Now if the minimum of u, which exists because it is a function of compact support, is non negative, then $\mathrm{S}(p) \geq 0$ and we have the first assertion since the point p can be chosen arbitrarily. Therefore it is enough to consider the case in which this minimum is strictly negative. Thus the point of minimum p_{min} must be in the support of η, so it must be in $B(p, 2Ar_0)$. At p_{min} we have $\nabla u = 0$, $\Delta u \geq 0$, so $\nabla\, \mathrm{S} = \frac{-\mathrm{S}\,\nabla \eta}{\eta}$. Clearly, we also have $u\psi' \geq 0$. Using Lemma 3.2.17 above, we get

$$\begin{aligned}
\Delta u &= \eta(\langle \nabla\, \mathrm{S}, \nabla f\rangle + \mathrm{S} - 2|S_{ij}|^2 - 2\alpha(\tau_g\phi)^2) + \frac{u\psi'}{Ar_0\eta}\Delta d + \frac{u\psi''}{(Ar_0)^2\eta} + 2\langle \nabla\, \mathrm{S}, \nabla\eta\rangle \\
&\leq \eta(\langle \nabla\, \mathrm{S}, \nabla f\rangle + \mathrm{S} - 2|S_{ij}|^2) + \frac{u\psi'}{Ar_0\eta}\Delta d + \frac{u\psi''}{(Ar_0)^2\eta} - 2S\frac{|\nabla\eta|^2}{\eta} \\
&= \eta(\langle \nabla\, \mathrm{S}, \nabla f\rangle + \mathrm{S} - 2|S_{ij}|^2) + \frac{u\psi'}{Ar_0\eta}\Delta d + \frac{u\psi''}{(Ar_0)^2\eta} - 2u\frac{|\psi'|^2}{\eta^2(Ar_0)^2} \\
&\leq \eta(\langle \nabla\, \mathrm{S}, \nabla f\rangle + \mathrm{S} - 2|S_{ij}|^2) + \frac{u\psi'}{Ar_0\eta}(\langle \nabla f, \nabla d\rangle + 2nr_0^{-1} + |\nabla f(p)|) + \frac{u\psi''}{(Ar_0)^2\eta} - 2u\frac{|\psi'|^2}{\eta^2(Ar_0)^2}.
\end{aligned}$$

Remark that at the point p_{min}, the first term on the right hand side is $-\eta\, \mathrm{S}\, \frac{\langle \nabla\eta, \nabla f\rangle}{\eta} = -\frac{u\psi'}{Ar_0\eta}\langle \nabla f, \nabla d\rangle$, so we have a cancellation. Rewrite the inequality as

$$u - 2\eta|S_{ij}|^2 + \frac{2nu\psi'}{Ar_0^2\eta} + \frac{u\psi'}{Ar_0\eta}|\nabla f(p)| + \frac{u\psi''}{(Ar_0)^2\eta} - 2u\frac{|\psi'|^2}{\eta^2(Ar_0)^2} \geq 0. \tag{3.2.19}$$

Using Cauchy-Schwarz $-2\eta|S_{ij}|^2 \leq -2\eta\, \mathrm{S}^2/n = -2u^2/(n\eta)$, thus multiplying (3.2.19) by η we get

$$\eta u - \frac{2u^2}{n} + \frac{2nu\psi'}{Ar_0^2} + \frac{u\psi'}{Ar_0}|\nabla f(p)| + \frac{u\psi''}{(Ar_0)^2} - 2u\frac{|\psi'|^2}{\eta(Ar_0)^2} \geq 0,$$

and exploiting the bound on the cut-off function

$$-\frac{2u^2}{n} + \frac{4nu}{Ar_0^2} + \frac{2|u|}{Ar_0}|\nabla f(p)| + 6\frac{|u|}{(Ar_0)^2} \geq 0,$$

so we must have

$$|u| \leq \frac{C(|\nabla f(p)|, n)}{Ar_0^2},$$

for any constant A, so $\mathrm{S}(p) = u(p) \geq u(p_{min}) \geq -\frac{C(|\nabla f(p)|,n)}{Ar_0^2}$ and since this estimate is uniform in A, passing to the limit $A \to \infty$ we get $\mathrm{S}(p) \geq 0$. Since p can be chosen arbitrarily this is valid on the whole M, concluding the proof of the first assertion.

Recall that Proposition 2.5.13 ensures $\mathrm{S} + |\nabla f|^2 = f$, so the previous step yields $|\nabla f|^2 \leq f$. Equivalently, we have that $|\nabla(\sqrt{f})| \leq \frac{1}{2}$, therefore $\sqrt{f}$ grows at most linearly being a Lipschitz function, and so does $|\nabla f| \leq \sqrt{f}$; clearly, this implies that ∇f is a complete vector field.

Finally, suppose there exists a $p \in M$ such that $\mathrm{S}(p) = 0$. From the first part of the theorem, such a point is a minimum for the u defined above for every A. By the strong maximum principle, which we can apply since the completeness of the metric implies u has compact support, we have $u \equiv 0$ for every A, so $\mathrm{S} \equiv 0$ on $B(p, Ar_0)$ for every A, that is $\mathrm{S} \equiv 0$ on M. Restoring the time dependence, from the equation (3.2.16) we have $\mathcal{S} = 0$ and $\tau_g \phi = 0$, so $g(t) = g_0$ and $\phi(t) = \phi_0$. Rewriting the soliton solutions equation we get

$$\mathrm{Hess}(f(t)) = -\sigma(t) g_0.$$

Being a shrinking soliton solution, we know that $\sigma(t) < 0$ for any t, thus $\Sigma(t) = \int_0^t \sigma(s)ds$ is monotone decreasing as is $c(t) := \exp(2\Sigma(t))$. Defining Ψ_t as the diffeomorphism induced by $\nabla f(t)/c(t)$ (recall it is a complete vector field), we have $\hat{g} = (\Psi_t^*)^{-1} g_0 = id$, thus $\hat{g}$ is flat. From the Killing-Hopf Theorem, its universal cover is the Euclidean space $(\mathbb{R}^n, g_{can})$, with covering map π. Pulling back the above equation on $\mathbb{R}^n$ via π, we obtain that the function $\pi^* f$ is strictly convex. Therefore π must be trivial, otherwise $\pi^* f$ would be periodic, and thus a global isometry. Since $\mathrm{S} \equiv \mathrm{R} \equiv 0$ and $\alpha \neq 0$, we deduce that $\phi_0 = y_0$ is a constant map. □

3.3 Reduced Length and Volume based at Singular Time

In this section we develop concepts of reduced length and volume based at singular time for the harmonic Ricci flow. Throughout the rest of the chapter the harmonic Ricci flows we are considering are *complete* unless otherwise stated.

Reduced Length Based at Singular Time

We start with a proof of Theorem 3.1.4.

Proof of Theorem 3.1.4. Fix a sequence of time $t_i \nearrow T$ and a point $p \in M$. To simplify notation let $l_i \coloneqq l_{p,t_i}$ and $L_i \coloneqq L_{p,t_i}$. Fix an arbitrary compact subset of the form $K = K_1 \times [a,b] \subset M \times (0,T)$. The plan of the proof is to apply Arzelà-Ascoli's theorem to the sequence, once we have proved the necessary bounds. By definition of l_i it suffices to uniformly bound L_i on K. Let $\eta\colon [0,1] \to M$ be a $g(0)-$geodesic with $\eta(0) = q \in K_1$, $\eta(1) = p$. Fix $k \in (b,T)$ and consider

$$\gamma(t) \coloneqq \begin{cases} \eta\left(\frac{t-\bar{t}}{k-\bar{t}}\right) & t \in [\bar{t},k], \\ p & t \in (k,t_i]. \end{cases} \tag{3.3.1}$$

Because $|\eta'(s)|^2_{g(0)}$ is constant (depending on $d_0(p,q)$, and hence only on K), there exists a constant $D = D(C,K)$ such that $|\eta'(s)|^2_{g(t)} \le D$ because of the uniform equivalence of the metrics along the harmonic Ricci flow on $[0,k]$. An application of Theorem 3.2.14, using the Type A assumption, implies the existence of a constant $C = C(\underline{\alpha}, \overline{\alpha}, n, N, C, r)$ such that $|\mathrm{S}|(x,t) \le \frac{C}{(T-t)^r}$, so:

$$\begin{aligned} |L_i(q,\bar{t})| &\le \left| \int_{\bar{t}}^{t_i} \sqrt{t_i - t}\left(|\dot{\gamma}(t)|^2_{g(t)} + \mathrm{S}_{g(t)}(\gamma(t))\right) dt \right| \\ &\le \int_{\bar{t}}^{k} \frac{\sqrt{t_i-t}}{(k-\bar{t})^2} \left|\eta'\left(\frac{t-\bar{t}}{k-\bar{t}}\right)\right|^2_{g(t)} dt + C\int_{\bar{t}}^{t_i} \frac{\sqrt{t_i-t}}{(T-t)^r} dt \\ &\le \frac{D\sqrt{T}}{k-b} + \frac{2C}{3-2r} T^{\frac{3}{2}-r} =: E(\underline{\alpha}, \overline{\alpha}, n, N, C, r, K) \end{aligned} \tag{3.3.2}$$

thus the uniform bound is proved.

Now we would like a uniform bound on the derivatives of the L_i. Let γ_i be an $\mathcal{L}-$minimising $\mathcal{L}-$geodesic from $(q,\bar{t})$ to (p,t_i), where $(q,\bar{t}) \in K$.

We claim there exists a constant $G = G(\underline{\alpha}, \overline{\alpha}, n, N, C, r, K) > 0$ independent of i such that for all $t \in [\bar{t},k]$, $|\sqrt{t_i - t}\dot{\gamma}_i(t)|^2_{g(t)} \le G$. Denote by $V_i(t) \coloneqq \sqrt{t_i - t}\dot{\gamma}_i(t)$. Using the $\mathcal{L}-$geodesic equation (cfr. formula (4.3) in [62])

$$\nabla_{V_i(t)} V_i(t) - 2\sqrt{t_i - t}\mathcal{S}_{g(t)}(V_i(t), \cdot)^{\#} - \frac{1}{2}(t_i - t)\nabla \mathrm{S}_{g(t)} = 0, \tag{3.3.3}$$

we obtain

$$\begin{aligned} \frac{d}{dt}|V_i(t)|^2_{g(t)} &= -2\mathcal{S}(V_i(t), V_i(t)) + 2\langle \nabla_{\dot{\gamma}_i(t)} V_i(t), V_i(t)\rangle_{g(t)} \\ &= -2\mathcal{S}(V_i(t), V_i(t)) + \frac{2}{\sqrt{t_i - t}}\langle \nabla_{V_i(t)} V_i(t), V_i(t)\rangle_{g(t)} \end{aligned}$$

$$
\begin{aligned}
&= -2\mathcal{S}(V_i(t), V_i(t)) + \frac{2}{\sqrt{t_i - t}}\langle 2\sqrt{t_i - t}\mathcal{S}_{g(t)}(V_i(t), \cdot)^{\#} + \frac{1}{2}(t_i - t)\nabla \mathrm{S}_{g(t)}, V_i(t)\rangle_{g(t)} \\
&= 2\mathcal{S}(V_i(t), V_i(t)) + \sqrt{t_i - t}\langle \nabla \mathrm{S}_{g(t)}, V_i(t)\rangle_{g(t)} \qquad (3.3.4) \\
&\leq \frac{C_1}{(T-t)^r}|V_i(t)|^2_{g(t)} + \frac{C_2}{(T-t)^{\frac{3}{2}r - \frac{1}{2}}}|V_i(t)|_{g(t)}.
\end{aligned}
$$

Here the constants depend upon the Type A constant C as in Theorem 3.2.14 but are independent of i. Having a uniform bound on $V_i(t)$ for t in a compact set of time is necessary to exploit this inequality. We have

$$
\int_{\bar{t}}^{t_i} \frac{1}{\sqrt{t_i - t}}|V_i(t)|^2_{g(t)}dt = \mathcal{L}(\gamma_i) - \int_{\bar{t}}^{t_i} \sqrt{t_i - t}\,\mathrm{S}_{g(t)}(\gamma_i(t))dt \leq \mathcal{L}(\gamma_i) + \frac{2C_1}{3-2r}T^{\frac{3}{2}-r} \qquad (3.3.5)
$$

by the Type A assumption, thus the integral mean value theorem gives the existence of $\hat{t_i} \in [\bar{t}, k]$ such that for i large

$$
\frac{1}{\sqrt{t_i - \hat{t_i}}}|V_i(\hat{t_i})|^2_{g(\hat{t_i})} = \frac{1}{k - \bar{t}}\int_{\bar{t}}^{k} \frac{1}{\sqrt{t_i - t}}|V_i(t)|^2_{g(t)}dt \leq \frac{1}{k - \bar{t}}\left(\mathcal{L}(\gamma_i) + \frac{2C_1}{3-2r}T^{\frac{3}{2}-r}\right) \qquad (3.3.6)
$$

and since $\sqrt{t_i - \hat{t_i}} \leq \sqrt{T}$, we get $|V_i(\hat{t_i})|^2_{g(\hat{t_i})} \leq F$ for some constant $F = F(\underline{\alpha}, \overline{\alpha}, n, N, C, r, K)$. Without loss of generality we can assume that $|V_i(t)|^2_{g(t)} \geq 1$, so the inequality (3.3.4) becomes for $t \in [a, k]$

$$
\frac{d}{dt}|V_i(t)|^2_{g(t)} \leq \left(\frac{C_1}{(T-k)^r} + \frac{C_2}{(T-k)^{\frac{3}{2}r - \frac{1}{2}}}\right)|V_i(t)|^2_{g(t)} = C_3|V_i(t)|^2_{g(t)}. \qquad (3.3.7)
$$

The constant C_3 depends on $\underline{\alpha}, \overline{\alpha}, n, N, C, r, K$. Integration implies that for all $t \in [a, b] \subset [a, k]$

$$
|V_i(t)|^2_{g(t)} \leq F e^{C_3(t - \hat{t_i})} \leq F e^{C_3 T} =: G = G(\underline{\alpha}, \overline{\alpha}, n, N, C, r, K). \qquad (3.3.8)
$$

In order to get uniform gradient bounds for L_i, we use the first variation formula of the functional $\mathcal{L}$ to obtain $\nabla L_i(q, \bar{t}) = -2\sqrt{t_i - \bar{t}}\dot{\gamma}_i(\bar{t})$, thus $|\nabla L_i(q, \bar{t})| \leq \sqrt{2G}$. Regarding the time derivative bounds we proceed as follows

$$
\begin{aligned}
\frac{\partial}{\partial \bar{t}}L_i(q, \bar{t}) &= \frac{d}{d\bar{t}}L_i(q, \bar{t}) - \langle \nabla L_i(q, \bar{t}), \dot{\gamma}_i(\bar{t})\rangle_{g(\bar{t})} \\
&= -\sqrt{t_i - \bar{t}}(|\dot{\gamma}_i(\bar{t})|^2_{g(\bar{t})} + \mathrm{S}_{g(\bar{t})}(\gamma_i(\bar{t}))) + 2\sqrt{t_i - \bar{t}}|\dot{\gamma}_i(\bar{t})|^2_{g(\bar{t})} \\
&= \frac{1}{\sqrt{t_i - \bar{t}}}|\sqrt{t_i - \bar{t}}\dot{\gamma}_i(\bar{t})|^2_{g(\bar{t})} - \sqrt{t_i - \bar{t}}\,\mathrm{S}_{g(\bar{t})}(\gamma_i(\bar{t}))
\end{aligned}
$$

therefore using the Type A bound and what we obtained before, we have for any $(q, \bar{t}) \in K$

$$
\frac{\partial}{\partial \bar{t}}L_i(q, \bar{t}) \leq \frac{G}{\sqrt{k-b}} + \frac{C}{(T-b)^{r-\frac{1}{2}}} =: H = H(\underline{\alpha}, \overline{\alpha}, n, N, C, r, K). \qquad (3.3.9)
$$

Thus for any compact set K, the $C^{0,1}-$norm of l_i are uniformly bounded (in terms of $\underline{\alpha}, \overline{\alpha}, n, N, C, r$ and K) and we can extract a limit by Arzelà-Ascoli's Theorem. It remains to show that the differential inequalities for l_i pass to the limit. This follows analogously to the Ricci flow case in [34] and we therefore leave this part of the proof to the reader. □

Remark 3.3.1. *We stress that the uniform bound of the C^1_{loc}-norm of the l_i, which brings another bound on the $C^{0,1}_{loc}$-norm of the $l_{p,T}$, depends on $\underline{\alpha}, \overline{\alpha}, n, N$ and on the Type A constants C, r (and on the compact set in consideration).*

Reduced Volume Based at Singular Time

Definition 3.3.2. *Let $(M, g(t), \phi(t))$ be a Type A harmonic Ricci flow on $(0, T)$. Fix a point $p \in M$, a sequence $t_i \nearrow T$ and any $l_{p,T}$ and $v_{p,T}$ given by the Theorem 3.1.4. We define a reduced volume based at the singular time (p, T) to be the function*

$$V_{p,T}(\bar{t}) := \int_M v_{p,T}(q, \bar{t})\ dvol_{g(\bar{t})}(q) = \int_M (4\pi(T - \bar{t}))^{-\frac{n}{2}} e^{-l_{p,T}(q,\bar{t})}\ dvol_{g(\bar{t})}(q). \tag{3.3.10}$$

From Theorem 2.5.18 and Fatou's Lemma, we know that $V_{p,T}(\bar{t}) \leq 1$. The proof of the next result reads the same as in [34], hence we skip it here.

Corollary 3.3.3. *We have $\frac{d}{dt}V_{p,T}(\bar{t}) \geq 0$ and $\lim_{\bar{t}\nearrow T} V_{p,T}(\bar{t}) \leq 1$.*

We still need an analogous result in the case the reduced volume is constant in an interval. In order to derive it, we notice that if $V(t_1) = V(t_2)$ then $\square^* v_{p,T} = 0$ in (t_1, t_2), so the parabolic regularity theory implies $l_{p,T}$ is smooth. For the convenience of the reader we recall that the equation $\square^* v_{p,T} = 0$ is equivalent to

$$-\frac{\partial l_{p,T}(q,\bar{t})}{\partial \bar{t}} - \Delta l_{p,T}(q,\bar{t}) + |\nabla l_{p,T}(q,\bar{t})|^2 - \mathrm{S}_{g(\bar{t})} + \frac{n}{2(T - \bar{t})} = 0. \tag{3.3.11}$$

Combining with the last equality in Theorem 3.1.4, we deduce

$$w_{p,T} := \big((T - \bar{t})(2\Delta l_{p,T} - |\nabla l_{p,T})|^2 + \mathrm{S}) + l_{p,T} - n\big) v_{p,T} \equiv 0. \tag{3.3.12}$$

Theorem 3.3.4. *Suppose $v(q,t) = (4\pi(T-t))^{-\frac{n}{2}} e^{-l(q,t)}$ solves the conjugate heat equation $\square^* v = 0$ under the harmonic Ricci flow. Then defining $w := \big((T - t)(2\Delta l - |\nabla l|^2 + \mathrm{S}) + l - n\big)v$ as above, we have*

$$\square^* w = -2(T - t)\left[\left|S_{ij} + \nabla_i\nabla_j l - \frac{g_{ij}}{2(T - t)}\right|^2 + \alpha|\tau_g\phi - \nabla_i\phi\nabla_i l|^2 - \dot{\alpha}|\nabla\phi|^2\right]v. \tag{3.3.13}$$

Proof. Recall that

$$\frac{d\Delta}{dt} = 2S_{ij}\nabla_i\nabla_j - 2\alpha\tau_g\phi\nabla_i\phi\nabla_i. \tag{3.3.14}$$

By definition of v, we know that $v^{-1}\nabla v = \nabla l$. Hence we get

$$\begin{aligned}
v^{-1}\Box^* w &= v^{-1}(-\partial_t - \Delta + \mathrm{S})\Big(\big((T-t)(2\Delta l - |\nabla l|^2 + \mathrm{S}) + l - n\big)v\Big)\\
&= -(\partial_t + \Delta)\big((T-t)(2\Delta l - |\nabla l|^2 + \mathrm{S}) + l\big) - 2\langle \nabla\big((T-t)(2\Delta l - |\nabla l|^2 + \mathrm{S}) + l\big), v^{-1}\nabla v\rangle\\
&= 2\Delta l - |\nabla l|^2 + \mathrm{S} - (T-t)(\partial_t + \Delta)(2\Delta l - |\nabla l|^2 + \mathrm{S}) - (\partial_t + \Delta) l\\
&\quad + 2(T-t)\langle \nabla(2\Delta l - |\nabla l|^2 + \mathrm{S}), \nabla l\rangle + 2|\nabla l|^2.
\end{aligned}$$

Let us analyze more carefully the term $(\partial_t + \Delta)(2\Delta l - |\nabla l|^2 + \mathrm{S})$ (for convenience in the time derivative calculation, note that $|\nabla l|^2 = |dl|^2$):

$$\begin{aligned}
&(\partial_t + \Delta)(2\Delta l - |\nabla l|^2 + \mathrm{S}) = \ 2\partial_t(\Delta) l + 2\Delta(\partial_t + \Delta) l - (\partial_t + \Delta)|dl|^2 + (\partial_t + \Delta)\,\mathrm{S}\\
&= 4S_{ij}\nabla_i\nabla_j l - 4\alpha\tau_g\phi\nabla_i\phi\nabla_i l + 2\Delta(|\nabla l|^2 - \mathrm{S}) - 2\mathcal{S}(dl, dl) - 2\langle \nabla(\partial_t l), \nabla l\rangle\\
&\quad - \Delta(|\nabla l|^2) + \Delta\,\mathrm{S} + \Delta\,\mathrm{S} + 2|S_{ij}|^2 + 2\alpha|\tau_g\phi|^2 - \dot\alpha|\nabla\phi|^2\\
&= 4S_{ij}\nabla_i\nabla_j l - 4\alpha\tau_g\phi\nabla_i\phi\nabla_i l + \Delta(|\nabla l|^2) - 2\mathcal{S}(dl, dl)\\
&\quad - 2\langle \nabla(-\Delta l + |\nabla l|^2 - \mathrm{S}), \nabla l\rangle + 2|S_{ij}|^2 + 2\alpha|\tau_g\phi|^2 - \dot\alpha|\nabla\phi|^2,
\end{aligned}$$

where we used $\Box^* v = 0$ in the last equality (written in term of l). Substituting this we obtain

$$\begin{aligned}
v^{-1}\Box^* w &= 2\Delta l - |\nabla l|^2 + \mathrm{S} - (T-t)\Big[4S_{ij}\nabla_i\nabla_j l - 4\alpha\tau_g\phi\nabla_i\phi\nabla_i l + \Delta(|\nabla l|^2) - 2\mathcal{S}(dl, dl)\\
&\quad - 2\langle \nabla(-\Delta l + |\nabla l|^2 - \mathrm{S}), \nabla l\rangle + 2|S_{ij}|^2 + 2\alpha|\tau_g\phi|^2 - \dot\alpha|\nabla\phi|^2\Big] - \frac{n}{2(T-t)} + \mathrm{S} - |\nabla l|^2\\
&\quad + 2(T-t)\langle \nabla(2\Delta l - |\nabla l|^2 + \mathrm{S}), \nabla l\rangle + 2|\nabla l|^2\\
&= -\frac{n}{2(T-t)} + \Big[2\Delta l - |\nabla l|^2 + \mathrm{S} + \mathrm{S} - |\nabla l|^2\Big] - (T-t)\Big[4S_{ij}\nabla_i\nabla_j l - 4\alpha\tau_g\phi\nabla_i\phi\nabla_i l\\
&\quad + \underline{\Delta(|\nabla l|^2)} - \underline{2\mathcal{S}(dl, dl)} - \underline{2\langle \nabla(\Delta l), \nabla l\rangle} + 2|S_{ij}|^2 + 2\alpha|\tau_g\phi|^2 - \dot\alpha|\nabla\phi|^2\Big]\\
&= -\frac{n}{2(T-t)} + 2(\Delta l + \mathrm{S}) - (T-t)\Big[4S_{ij}\nabla_i\nabla_j l - 4\alpha\tau_g\phi\nabla_i\phi\nabla_i l + 2|S_{ij}|^2 + 2\alpha|\tau_g\phi|^2\\
&\quad - \dot\alpha|\nabla\phi|^2 + I\Big],
\end{aligned}$$

where I is the sum of the underlined terms, i.e.

$$\begin{aligned}
I &= \Delta(|\nabla l|^2) - 2\mathcal{S}(dl, dl) - 2\langle \nabla(\Delta l), \nabla l\rangle = \nabla_i\nabla_i(\nabla_j l\nabla_j l) - 2S_{ij}\nabla_i l\nabla_j l - 2\nabla_i\nabla_j\nabla_j l\nabla_i l\\
&= 2\nabla_i\nabla_i\nabla_j l\nabla_j l + 2\nabla_i\nabla_j l\nabla_i\nabla_j l - 2R_{ij}\nabla_i l\nabla_j l + 2\alpha\nabla_i\phi\nabla_i l\nabla_j\phi\nabla_j l - 2\nabla_i\nabla_i\nabla_j l\nabla_j l\\
&\quad - 2R_{ijjk}\nabla_k l\nabla_i l = 2|\operatorname{Hess}(l)|^2 + 2\alpha\nabla_i\phi\nabla_i l\nabla_j\phi\nabla_j l.
\end{aligned}$$

Now we can plug this expression in the equation above, obtaining

$$\begin{aligned}
v^{-1}\square^* w = &-\frac{n}{2(T-t)} + 2(\Delta l + \mathrm{S}) - (T-t)\Big[4S_{ij}\nabla_i\nabla_j l - 4\alpha\tau_g\phi\nabla_i\phi\nabla_i l + 2|S_{ij}|^2 + 2\alpha|\tau_g\phi|^2 \\
&- \dot\alpha|\nabla\phi|^2 + 2|\operatorname{Hess}(l)|^2 + 2\alpha\nabla_i\phi\nabla_i l\nabla_j\phi\nabla_j l\Big] \\
= &-2(T-t)\bigg[\bigg(|S_{ij}|^2 + |\nabla_i l\nabla_j l|^2 + \frac{n}{4(T-t)^2} + 2S_{ij}\nabla_i\nabla_j l - \frac{\Delta l}{T-t} - \frac{\mathrm{S}}{T-t}\bigg) \\
&+ \alpha\big(\nabla_i\phi\nabla_i l\nabla_j\phi\nabla_j l - 2\tau_g\phi\nabla_i\phi\nabla_i l + |\tau_g\phi|^2\big) - \dot\alpha|\nabla\phi|^2\bigg] \\
= &-2(T-t)\bigg[\bigg|S_{ij} + \nabla_i\nabla_j l - \frac{g_{ij}}{2(T-t)}\bigg|^2 + \alpha|\tau_g\phi - \nabla_i\phi\nabla_i l|^2 - \dot\alpha|\nabla\phi|^2\bigg]. \qquad \square
\end{aligned}$$

In the next theorem, we follow the argument in [28] for the similar Ricci flow case.

Theorem 3.3.5. *Suppose we are given a complete gradient shrinking soliton solution in canonical form, i.e. a solution of (2.5.6) with $a = 1$. Then for any point $p \in M$, its reduced length based at the singular space-time point (p, T) equals the soliton potential plus the normalisation constant. In particular, it is independent of the point p.*

Proof. We know that $g(t) = (T-t)\psi_t^* g(0)$, $\phi(t) = \psi_t^*\phi(0)$ and $f(t) = \psi_t^* f(0)$, where ψ_t is the diffeomorphism generated by $\nabla f(0)/(T-t)$. Therefore, defining $\tau = T - t$, we know that f solves $\frac{\partial f}{\partial\tau} = -|\nabla f|^2$. Without loss of generality, we can suppose the soliton solution to be normalised (note that this requires adding a constant independent of time), so $\mathrm{S} + |\nabla f|^2 - \frac{1}{\tau}f = 0$. Given any curve $\gamma\colon [0,\bar\tau] \longrightarrow M$, such that $\gamma(0) = p$, $\gamma(\bar\tau) = q$, we have

$$\begin{aligned}
\frac{d}{d\tau}(\sqrt{\tau}f(\gamma(\tau),\tau)) &= \sqrt{\tau}\bigg(\frac{f}{2\tau} + \frac{\partial f}{\partial\tau} + \nabla f\cdot\dot\gamma\bigg) = \sqrt{\tau}\Big(\frac{f}{2\tau} + \frac{\partial f}{\partial\tau} + \frac{1}{2}|\nabla f|^2 + \frac{1}{2}|\dot\gamma|^2 - \frac{1}{2}|\dot\gamma - \nabla f|^2\Big) \\
&= \frac{1}{2}\sqrt{\tau}\Big(\mathrm{S} + |\dot\gamma|^2 - |\dot\gamma - \nabla f|^2\Big).
\end{aligned}$$

Hence integrating from 0 to $\bar\tau$

$$f(\gamma(\bar\tau),\bar\tau)) = \frac{1}{2\sqrt{\bar\tau}}\mathcal{L}(\gamma) - \frac{1}{2\sqrt{\bar\tau}}\int_0^{\bar\tau}\sqrt{\tau}|\dot\gamma(\tau) - \nabla f(\gamma(\tau),\tau)|^2_{g(\tau)}d\tau. \tag{3.3.15}$$

Now choosing γ such that $\dot\gamma = \nabla f(\tau)$, we get $f(q,\bar\tau) \geq l_{p,0}(q,\bar\tau)$. For the opposite inequality, it is sufficient to apply (3.3.15) to a minimal $\mathcal{L}$-geodesic since the integrand is non-negative. $\square$

Combining the two theorems above we get the following result.

Corollary 3.3.6. *Any complete normalised gradient shrinking soliton solution in canonical form has constant reduced volume based at singular time.*

We now want to enquire for a converse of this statement. Suppose the coupling function is non increasing, and suppose that the reduced volume based at the singular time T is constant. From Theorem 3.3.4 we then deduce

$$\begin{cases} \operatorname{Ric} - \alpha \nabla\phi \otimes \nabla\phi + \operatorname{Hess}(l_{p,T}) - \frac{g}{2(T-t)} = 0, \\ \tau_g\phi - \langle \nabla\phi, \nabla l_{p,T} \rangle = 0, \\ \dot{\alpha}|\nabla\phi|^2 = 0. \end{cases} \tag{3.3.16}$$

The first two equations are the equations characterising a normalised gradient shrinking soliton solution in canonical form with potential function $l_{p,T}$.

Suppose now that α is not constant, then there exists $\bar{t} \in (0, T)$ such that $\dot{\alpha}(\bar{t}) < 0$. Therefore the third equation yields $\nabla\phi = 0$, i.e. $\phi(\bar{t}) \equiv y_0$ where $y_0 \in N$ (here we are assuming connectedness of M). Writing the harmonic Ricci flow at $\bar{t}$ we get

$$\begin{cases} \partial_t g\big|_{t=\bar{t}} = -2 \operatorname{Ric}_{g(\bar{t})} = 2 \operatorname{Hess}(l_{p,T}) - \frac{g}{(T-t)}, \\ \partial_t \phi\big|_{t=\bar{t}} = \tau_g(y_0) = 0. \end{cases} \tag{3.3.17}$$

Thus from the time $\bar{t}$ on, the solution coincides with a gradient shrinking Ricci soliton solution and a constant map. Since the flow is smooth on the whole interval $[0, \bar{t}]$, we also get that ϕ_0 is isotopic to a constant map. This strongly restricts the class of initial data (M, g_0, N, ϕ_0) that gives rise to a harmonic gradient shrinking Ricci soliton solution *in canonical form*. For example, if we have $M = N = S^n$, with $\phi_0 = id$, and non-constant coupling function, then for *any metric* on the domain and target, the solution of the harmonic Ricci flow will not be a harmonic gradient shrinking Ricci soliton solution in canonical form! Compare this with the discussion in Chapter 2.

Reassuringly, if we blow-up a Type I harmonic Ricci flow, we get a harmonic gradient shrinking Ricci soliton solution in canonical form, but the limit of the coupling functions $\alpha(T - \frac{t}{\lambda})$ will be a constant because λ tends to infinity, and we cannot apply the argument above.

Remark 3.3.7. *We point out that if we blow-up a complete harmonic Ricci flow at a regular time, then we get the Gaussian soliton in canonical form and a constant map from $\mathbb{R}^n \to N$. Indeed suppose we have constant reduced volume based at a regular time $t_0 < T$, then we can assume without loss of generality that α is constant: otherwise we get as above that $\phi \equiv y_0$ is a constant map, and $(M, g(t), l_{p,t_0})$ is a gradient shrinking Ricci soliton solution, which verifies the Ricci soliton equations with respect to a regular time, thus from Ricci flow theory it must be the Gaussian soliton in canonical form.*

In the case α is constant, we get the more general definition of a gradient shrinking harmonic Ricci soliton solution in canonical form, i.e. $(M, g(t), \phi(t), l_{p,t_0})$ solves the system:

$$\begin{cases} \mathrm{Ric} - \alpha \nabla\phi \otimes \nabla\phi + \mathrm{Hess}(l_{p,t_0}) - \frac{g}{2(t_0-t)} = 0, \\ \tau_g \phi - \langle \nabla\phi, \nabla l_{p,t_0} \rangle = 0. \end{cases} \tag{3.3.18}$$

We claim that this can only be the case if $(M, g(t), y_0, l_{p,t_0})$ is the Gaussian soliton in canonical form. Recall that by the long time existence result in [63], we have bounded curvature up to t_0, so we can use the maximum principle. By Theorem 3.2.16, we have $\mathrm{R} \geq \mathrm{S} \geq 0$. Moreover, by the same theorem we know that there exists a family ψ_t of diffeomorphisms induced by $\nabla l_{p,t_0}(t)$, so $\mathrm{S}(t,x) = (t_0 - t)^{-1}\,\mathrm{S}(0, \psi_t(x))$. Taking the limit $t \to t_0$, if there exists an x_0 such that $\mathrm{S}(x_0, 0) > 0$, then $\mathrm{S}(t_0, \psi_{t_0}(x_0)) = \infty$, which is contradictory to the assumption that t_0 is a regular time. Thus $\mathrm{S} \equiv 0$ and we conclude using the rigidity part in Theorem 3.2.16.

3.4 Pseudolocality Theorem for harmonic Ricci Flow

In this section we want to prove our pseudolocality Theorem 3.1.5 for the harmonic Ricci flow in the same spirit as [40] where this is done for List's flow (i.e. harmonic Ricci flow with target $N = \mathbb{R}$). The proof is similar to the analoguous Ricci flow case (see [65],[48]), and we want to stress that we had to build all the machinery in the previous sections to get rid of the curvature boundedness assumed in [40]. For this reason we will emphasize the parts of the proof which needed a refinement and refer the reader back to [40] for the remaining ones.

Proof of Theorem 3.1.5. First of all, we can assume without loss of generality that $\beta < 1/100n$ and $r_0 = 1$. Arguing by contradiction, suppose that there exist $\varepsilon_k, \delta_k \longrightarrow 0$ such that for each k, there exist a complete pointed harmonic Ricci flow (M_k, g_k, ϕ_k, p_k), with fixed target manifold N, such that $\mathrm{S}_{g_k}(0) \geq -1$ on a ball $B_{g_k(0)}(p_k, 1)$ and for any subset $\Omega \subset B_{g_k(0)}(p_k, 1)$ we have $\mathrm{Area}_{g_k(0)}(\partial\Omega)^n \geq (1 - \delta_k) c_n Vol_{g_k(0)}(\Omega)^{n-1}$, but there are points (x_k, t_k) such that $0 < t_k \leq \varepsilon_k^2$, $d_{g_k(t_k)}(x_k, p_k) \leq \varepsilon_k$, and

$$|\mathrm{Rm}|(x_k, t_k) \geq \beta t_k^{-1} + \varepsilon_k^{-2}.$$

Moreover, we can choose ε_k small enough such that

$$|\mathrm{Rm}|(x, t) \leq \beta t_k^{-1} + 2\varepsilon_k^{-2}, \tag{3.4.1}$$

whenever $0 < t \leq \varepsilon_k^2$ and $d_{g_k(t)}(x, p_k) \leq \varepsilon_k$. We can use precisely the same point selection argument as in [48] to get the following.

Lemma 3.4.1. *For any sequence $A_k \longrightarrow \infty$, there exist space-time points $(\overline{x}_k, \overline{t}_k) \in M_k \times (0, \varepsilon_k]$ with $d_{g_k(\overline{t}_k)}(\overline{x}_k, p_k) \leq (1 + 2A_k)\varepsilon_k$, such that*

$$|\operatorname{Rm}|_{g_k(t)}(x,t) \leq 4Q_k := 4|\operatorname{Rm}|(\overline{x}_k, \overline{t}_k),$$

for any (x,t) in the backward parabolic region

$$\Omega_k := \left\{(x,t)\middle| d_{g(\overline{t}_k)}(x, \overline{x}_k) \leq \frac{1}{10}A_k Q_k^{-1/2}, \overline{t}_k - \frac{1}{2}\beta Q_k^{-1} \leq t \leq \overline{t}_k\right\}.$$

For each k, let $u_k := (4\pi(T-t))^{-n/2}e^{-f_k}$ be the fundamental solution of the conjugate heat equation $\Box_k^* u_k = 0$, which tends to $\delta_{\overline{x}_k}$ as $t \to T$. Define as above $v_k := ((T-t)(2\Delta f_k - |\nabla f_k|^2 + \mathrm{S}_k) + f_k - n)u_k$. Then we know by the Theorem 2.5.7 that $v_k \leq 0$. The proof of the next lemma requires our Compactness Theorem 3.2.9 since no curvature boundedness of the time-slices is assumed, unlike the case treated in [40].

Lemma 3.4.2. *There exist a constant $b > 0$ independent of k and times $\tilde{t}_k \in (t_k - \frac{1}{2}\beta Q_k^{-1}, \overline{t}_k)$ such that*

$$\int_{B_k} v_k dV_{g_k(\tilde{t_k})} \leq -b < 0,$$

where $B_k = B_{g_k(\tilde{t_k})}\left(\overline{x}_k, \sqrt{\overline{t}_k - \tilde{t}_k}\right)$.

Proof. This proof is by contradiction. Suppose that for any $\tilde{t}_k \in (t_k - \frac{1}{2}\beta Q_k^{-1}, \overline{t}_k)$ there exists a subsequence (not relabelled) such that

$$\liminf_{k\to\infty} \int_{B_k} v_k dV_{g_k(\tilde{t_k})} \geq 0. \tag{3.4.2}$$

Consider the rescaling $\hat{g}_k(t) = Q_k g_k(Q_k^{-1}t + \overline{t}_k)$, $\hat{\phi}_k(t) = \phi_k(Q_k^{-1}t + \overline{t}_k)$ for $t \in [-Q_k\overline{t}_k, 0]$. Note that under this rescaling $\alpha_k(t)$ becomes $\hat{\alpha}_k(t) = \alpha_k(Q_k^{-1}t + \overline{t}_k)$, while the set Ω_k becomes the parabolic region $B_{\hat{g}_k(0)}\left(\overline{x}_k, \frac{1}{10}A_k\right) \times [-\frac{1}{2}\beta, 0]$, where we have curvature bounded by 4. We split the proof in two cases:

Case 1: Suppose first that the injectivity radii of the $\hat{g}_k$ are uniformly bounded away from zero. We can use Theorem 3.2.9 to extract a subsequence of $(M_k, \hat{g}_k(t), \hat{\phi}_k(t), \overline{x}_k)$ converging to a harmonic Ricci flow $(M_\infty, g_\infty(t), \phi_\infty(t), x_\infty)$ on $[-\frac{1}{2}\beta, 0]$. We remark that the use of Theorem 3.2.9 is forced by the lack of a global uniform curvature bound for the time-slices, compare to [40].This limit flow is complete, has $|\operatorname{Rm}| \leq 4$, and $|\operatorname{Rm}|(x_\infty, 0) = 1$. Notice that the limit coupling function α_∞ is constant, hence we can improve the pointwise convergence given by the Theorem 3.2.9 to a $C^\infty([-\frac{1}{2}\beta, 0])$-convergence because of the particular form of $\hat{\alpha}_k(t)$.

The fundamental solutions $\hat{u}_k$ based at $(\overline{x}_k, 0)$ of the rescaled flows will converge smoothly to u_∞, the fundamental solution to the conjugate heat equation on the limit flow, based at $(x_\infty, 0)$. So the respective $\hat{v}_k$ converge to the respective $v_\infty \leq 0$ since it is a pointwise limit of non-positive functions. On the other hand, from the inequality (3.4.2), it follows that for any fixed $t_0 \in [-\frac{1}{2}\beta, 0]$ we have

$$\int_{B_{g_\infty(t_0)}\left(x_\infty, \sqrt{-t_0}\right)} v_\infty(\cdot, t_0) dV_{g_\infty(t_0)} \geq 0.$$

Therefore it must be $v_\infty(\cdot, t_0) = 0$ on $B_{g_\infty(t_0)}\big(x_\infty, \sqrt{-t_0}\big)$. A strong maximum principle argument yields $v_\infty \equiv 0$ on $M_\infty \times (t_0, 0]$. In particular, we obtain that $\Box^* w_\infty = 0$, where w_∞ is the function defined by $w_\infty := \big((T-t)(2\Delta f_\infty - |\nabla f_\infty)|^2 + \mathrm{S}) + f_\infty - n\big)v_\infty$, so Theorem 3.3.4 allows us conclude that $(M_\infty, g_\infty(t), \phi_\infty(t), f_\infty(t))$ is a gradient harmonic Ricci shrinking soliton solution. Since it has bounded curvature and is complete, the results in Chapter 2 and Section 3.2 ensure it is the Gaussian soliton in canonical form; this is contradictory to $|\operatorname{Rm}|(x_\infty, 0) = 1$.

Case 2: In the case the injectivity radii of the rescaled metrics $\hat{g}_k$ at the point $\overline{x}_k$ tend to zero, denote them by $r_k = inj(\overline{x}_k, \hat{g}_k(0)) \to 0$. Rescale further by $\tilde{g}_k(t) = r_k^{-2}\hat{g}_k(r_k^2 t)$, $\tilde{\phi}_k(t) = \hat{\phi}_k(r_k^2 t)$, $\tilde{\alpha}_k(t) = \hat{\alpha}_k(r_k^2 t)$, defined for $t \in [-\frac{1}{2}\beta r_k^{-2}, 0]$. Clearly, the injectivity radius of $\tilde{g}_k(0)$ at $\overline{x}_k$ is 1 for every k. The region Ω_k becomes $B_{\tilde{g}_k(0)}\big(\overline{x}_k, \frac{1}{10}A_k r_k^{-1}\big) \times \big[-\frac{1}{2}\beta r_k^{-2}, 0\big]$. Remark that on these regions, which are larger and larger (both in space and time) with k increasing, we have curvature bounded by $4r_k^2 < \infty$ uniformly in k, so we can use Theorem 3.2.9 to get a subsequence of $(M_k, \tilde{g}_k(t), \tilde{\phi}_k(t), \overline{x}_k)$ converging in the pointed Cheeger-Gromov sense to a complete pointed harmonic Ricci flow $(M_\infty, g_\infty(t), \phi_\infty(t), x_\infty)$ for $(-\infty, 0]$, with constant coupling function (again with smooth convergence). We want to stress once again the difference with [40]. Moreover, the curvature bound gives that g_∞ is a flat metric and hence ϕ_∞ is a constant map since the limit flow is ancient and complete.

Passing to the limit (3.4.2), we get as above that $(M_\infty, g_\infty(t), \phi_\infty(t), f_\infty(t))$ is a gradient harmonic Ricci shrinking soliton solution. Again, the completeness and the boundedness of the curvature imply by the discussion in Section 3.2 that the limit flow has to be the Gaussian soliton in canonical form. In particular, the injective radius $inj_{g_\infty}(x_\infty, 0) = +\infty$, which is contradictory. □

From here on the proof of Theorem 3.1.5 reads the same as in [40]: One can first show for every k a similar uniform integral bound as in Lemma 3.4.2 for the time $\bar{t}_k$-slice and certain compactly supported functions, using the completeness assumption and the Harnack inequality. Then asymptotically confronting this bound with the log-Sobolev inequality, which is valid for domains close to the euclidean space as by assumption, one obtains a contradiction. We refer the reader to [40] for details. □

Remark 3.4.3. *Remark that the hypothesis* (c) *appearing in Theorem 1 of [40] is absorbed here in the assumption that* N *is closed.*

We present a slightly modified version of the Pseudolocality Theorem, which is a corollary of Theorem 3.1.5.

Theorem 3.4.4 (HRF Pseudolocality Theorem: Version 2)**.** *There exist* $\varepsilon, \delta > 0$ *depending as above on* $\underline{\alpha}, \overline{\alpha}$ *and* N *with the following property. Let* $(M, g(t), \phi(t), p)$ *be a smooth complete pointed harmonic Ricci flow solution defined for* $t \in [0, (\varepsilon r_0)^2]$*. Assume further the following conditions:*

- $|\operatorname{Rm}|(0) \leq r_0^{-2}$ *on* $B_{g(0)}(p, r_0)$*;*
- $\operatorname{Vol}_{g(0)}(B_{g(0)}(p, r_0)) \geq (1-\delta)\omega_n r_0^n$*, where* ω_n *is the volume of the Euclidean unit ball.*

Then $|\operatorname{Rm}|(x,t) < (\varepsilon r_0)^{-2}$ *whenever* $0 \leq t \leq (\varepsilon r_0)^2$ *and* $d_{g(0)}(x,p) \leq \varepsilon r_0$*.*

Proof. First of all, we notice that Theorem 3.1.5 guarantees the existence of ε' and δ such that for every flow verifying the hypothesis in that theorem we have $|\operatorname{Rm}|(x,t) \leq \beta t^{-1} + (\varepsilon r_0)^{-2}$ for $0 \leq t \leq (\varepsilon r_0)^2$ and $d_{g(t)}(x,p) \leq \varepsilon r_0$. In order to apply that Theorem, one needs a bound on the isoperimetric ratio inside a ball of definite size; this can be done arguing as in [58], that is reducing the problem to a smaller ball in the injectivity domain for p, which can be chosen of definite size thanks to Cheeger-Gromov-Taylor argument [20] and the assumptions, then appealing to the Euclidean isoperimetric inequality. Using the continuity and the initial data bound, for ε'' smaller than 2 (say), we have $|\operatorname{Rm}|(x,t) < (\varepsilon'' r_0)^{-2}$ for $0 \leq t \leq t_0(\varepsilon'')$ and $d_{g(t)}(x,p) \leq r_0$. Choosing a possibly smaller ε (depending only on the previous ε' and ε''), we get $|\operatorname{Rm}|(x,t) < (\varepsilon r_0)^{-2}$ for $0 \leq t \leq (\varepsilon r_0)^2$ and $d_{g(t)}(x,p) \leq \varepsilon r_0$. Finally, the length distortion Lemma 3.2.1 gives us the conclusion after possibly further decreasing ε in dependence of the previous ε. □

3.5 Proof of the Main Theorem

In this section we apply the theory developed previously in this chapter to the study of Type I singularities in the harmonic Ricci flow. We follow the structure in [35]. Let us give a new definition of essential blow-up sequences, that we will consider *only* throughout this section.

Definition 3.5.1. *Given* $(M, g(t), \phi(t))$ *a harmonic Ricci flow on* $[0,T)$*, a sequence of space-time points* $(p_i, t_i) \in M \times [0,T)$ *with* $t_i \nearrow T$ *is called an essential blow-up sequence if there exists a constant* $c > 0$ *such that*

$$|\operatorname{Rm}_{g(t_i)}|_{g(t_i)}(p_i) \geq \frac{c}{T - t_i}.$$

If $(M, g(t), \phi(t))$ *is a Type I harmonic Ricci flow, a point* $p \in M$ *is called a Type I singular point if there exists an essential blow-up sequence* (p_i, t_i) *with* $p_i \to p$*. We denote by* Σ_I^H *the set of Type*

I singular points.

Note that this definition slightly differs from the one given in Definition 1.2.2, and is set in a more coherent way to the analysis in [35]. In any case, our Theorem 3.5.3 below shows that for Type I harmonic Ricci flows the two definitions agree with each other, so that we do not commit any error in working with this new notion.

Proof of Theorem 3.1.3. Fix a sequence $\lambda_j \longrightarrow +\infty$. Notice that it suffices to show that we can extract a subsequence converging to a limit as claimed in order to conclude that every possible limit flow verifies the same structural properties. Therefore we would like to use one of the compactness theorems obtained in Section 3.2. First of all, we note that the Type I assumption is preserved by the rescaling considered:

$$|\operatorname{Rm}_{g_j(t)}|_{g_j(t)}(x) = \frac{1}{\lambda_j}|\operatorname{Rm}_{g\left(T+\frac{t}{\lambda_j}\right)}|_{g\left(T+\frac{t}{\lambda_j}\right)}(x) \leq \frac{C}{\lambda_j\left(T-\left(T+\frac{t}{\lambda_j}\right)\right)} = \frac{C}{-t}. \tag{3.5.1}$$

This gives uniform curvature bounds only on compact subsets of $(-\infty, 0)$. In order to have convergence on the full time domain, we may use the compactness Theorem 3.2.9 on the time interval $[-n, 1/n]$ for every $n \in \mathbb{N}^+$ and then use a diagonal argument to obtain a limit flow on $(-\infty, 0)$. We need to check the uniform injectivity radii bound at the time, say, $t = -1$. We proceed as follows.

Let $l_{p,T}$ be any fixed reduced length based at the singular space-time point (p, T) for the harmonic Ricci flow $(M, g(t), \phi(t))$, whose existence is guaranteed by Theorem 3.1.4. For every space-time point $(q, \bar{t}) \in M \times (-\infty, 0)$, it makes sense to consider for large enough j

$$l^j_{p,0}(q, \bar{t}) := l_{p,T}\Big(q, T + \frac{\bar{t}}{\lambda_j}\Big),$$

which is, by the scaling properties of the reduced length, a reduced length based at the singular time $t = 0$ for the rescaled harmonic Ricci flow. We stress that these functions are locally uniformly Lipschitz continuous, since Remark 3.3.1 guarantees that their Lipschitz norm depend only upon the compact subset chosen, the Type I bound (which is uniform in j by (3.5.1)) and other quantities that depend uniformly on j . The corresponding reduced volumes verify $V^j_{p,0}(\bar{t}) = V_{p,T}\big(T + \frac{\bar{t}}{\lambda_j}\big)$, and are uniformly bounded on compact subsets of $(-\infty, 0)$. Thus we can assume, possibly taking a subsequence, that $V^j_{p,0}(\bar{t})$ is pointwise convergent in $(-\infty, 0)$. Because of the monotonicity of the reduced volume based at the singular time, this limit is the constant $\lim_{t\nearrow T} V_{p,T}(t) \in (0, 1]$, which is continuous, so the convergence is uniform on compact subsets of $(-\infty, 0)$. In particular, $V^j_{p,0}(-1)$ is uniformly bounded away from zero, hence we obtain the uniform injectivity radii bound needed because of the $\kappa-$non-collapsing Theorem 6.13 in [63].

Therefore we can use Theorem 3.2.9 to get a complete limit flow $(M_\infty, g_\infty(t), \phi_\infty(t), p_\infty)$ on $(-\infty, 0)$ (after a diagonal argument). By the discussion above we can assume that the sequence $l^j_{p,0}$ is converging in $C^{0,1}_{loc}(M_\infty \times (-\infty, 0))$ to a certain function $l^\infty_{p_\infty,0}$ (here we are pulling-back through the diffeomorphisms given by the Cheeger-Gromov convergence for considering every function as a function on M_∞). Since its corresponding formal reduced volume is constant, we can conclude as in Section 3.3 that $(M_\infty, g_\infty(t), \phi_\infty(t), l^\infty_{p_\infty,0})$ is a normalised gradient shrinking harmonic Ricci soliton solution in canonical form. This proves the first statement.

Suppose now that $p \in \Sigma^H_I$. Arguing by contradiction, we assume that the limit $g_\infty(t)$ is flat for all $t < 0$ and that the map ϕ_∞ is constant. In particular, they are independent of time, and we denote them by $\hat{g}$ and $\hat{\phi}$. Fix any r_0 smaller than the injectivity radius of $\hat{g}$ at p_∞ (which could be shown to be infinite using the gradient shrinking soliton structure), so that $B_{\hat{g}}(p_\infty, r_0)$ is a Euclidean ball. By Cheeger-Gromov convergence, for any ε smaller than one, we have that $B_{g_j(-(\varepsilon r_0)^2)}(p, r_0)$ is as close as we want to a Euclidean ball for every j large enough, as well as ϕ_j is as close as we want to the constant map $\hat{\phi}$. Now for any β, we can pick ε and δ given by Theorem 3.1.5; after possibly reducing ε we can assume that $B_{g_j(-(\varepsilon r_0)^2)}(p, r_0)$ verifies the assumptions in Theorem 3.1.5, hence we have

$$|\operatorname{Rm}|_{g_j}(t,x) \leq \beta(t + (\varepsilon r_0)^2)^{-1} + (\varepsilon r_0)^{-2} \quad for \quad -(\varepsilon r_0)^2 \leq t < 0,\ x \in B_{g_j(-(\varepsilon r_0)^2)}(p, \varepsilon r_0). \quad (3.5.2)$$

Using that p is a Type I singular point, we get the existence of a sequence (p_i, t_i), with $p_i \to p$, $t_i \nearrow T$ and of a constant $c > 0$ such that

$$|\operatorname{Rm}|_{g_j(\lambda_j(t_i - T))}(p_i) \geq \frac{c}{\lambda_j(T - t_i)}.$$

Therefore, for i large enough we can use both the inequalities to get

$$\frac{c}{\lambda_j(T - t_i)} \leq \frac{\beta}{(\lambda_j(T - t_i) + (\varepsilon r_0)^2)} + (\varepsilon r_0)^{-2},$$

which yields a contradiction for i large enough, since $T - t_i$ is tending to zero. □

Definition 3.5.2. *Define $\Sigma^H_S \subset \Sigma^H_I$ to be the set of points p in M for which there exists a constant $c > 0$ such that (for t close to T)*

$$|\operatorname{S}_{g(t)}|(p) \geq \frac{c}{T - t}.$$

Moreover, we say that $p \in M$ is a singular point if there does not exist any neighbourhood $U_p \ni p$ on which $|\operatorname{Rm}_{g(t)}|_{g(t)}$ remains bounded as t approaches T. The set of such points is denoted by Σ^H.

Notice that the definition of Σ^H matches exactly the Ricci flow's analogue given in Definition 1.2.2. It is clear from the definitions above that $\Sigma^H_S \subseteq \Sigma^H_I \subseteq \Sigma^H$. More interestingly, we have the following result.

Theorem 3.5.3. *Let $(M, g(t), \phi(t))$ be a complete Type I harmonic Ricci flow on $[0, T)$ with finite singular time T, with non-increasing coupling function $\alpha(t) \in [\underline{\alpha}, \overline{\alpha}]$, where $0 < \underline{\alpha} \leq \overline{\alpha} < \infty$. Then $\Sigma^H = \Sigma^H_S$, so every definition of singular set given above agrees with the others.*

Proof. Firstly, we prove that $\Sigma^H_I \subseteq \Sigma^H_S$. Suppose that $p \in M \setminus \Sigma^H_S$. Then, there exist a sequence of number $c_j \searrow 0$ and $t_j \in [T - c_j, T)$ such that $\mathrm{S}(p, t_j) < \frac{c_j}{T - t_j}$. Let $\lambda_j := (T - t_j)^{-1}$, and rescale the harmonic Ricci flow as in Theorem 3.1.3. The same theorem gives us that $(M, g_j(t), \phi_j(t), p)$ subconverges to a normalised complete gradient shrinking harmonic Ricci soliton in canonical form $(M_\infty, g_\infty(t), \phi_\infty(t), p_\infty)$ on $(-\infty, 0)$, with

$$0 \leq \mathrm{S}_{g_\infty(-1)}(p_\infty) = \lim \lambda_j^{-1} \mathrm{S}_{g(t_j)}(p) \leq \lim c_j = 0.$$

Here we have used the Type I assumption as well as the first part of Theorem 3.2.16. The second conclusion in the latter theorem ensures that $(M_\infty, g_\infty(t), \phi_\infty(t))$ is the Gaussian soliton. By Theorem 3.1.3, we conclude that p must not be a Type I singular point. We have reached the contradiction, so we conclude that $\Sigma^H_I = \Sigma^H_S$.

In order to prove $\Sigma^H \subseteq \Sigma^H_I$ we claim the following: Given any $p \in M \setminus \Sigma^H_I$, there exists a neighbourhood U_p of p on which the curvature remains bounded. Indeed, given such a point p, for any $\lambda_j \to \infty$, the rescaled harmonic Ricci flow (as in Theorem 3.1.3) subconverges to the Gaussian soliton in canonical form. As we did in the proof of Theorem 3.1.3 above, for large enough $j \geq j_0$ we can assume that the hypothesis in the Theorem 3.1.5 are verified with $r_0 = 1$. This yields the existence of constants ε and δ such that, calling $K = \lambda_{j_0}$, we get

$$|\operatorname{Rm}_{g_{j_0}(t)}|_{g_{j_0}(t)} \leq \beta(t + \varepsilon^2)^{-1} + \varepsilon^{-2} \quad for \quad -\varepsilon^2 \leq t < 0,\ x \in B_{g_{j_0}(\varepsilon^2)}(p, \varepsilon),$$

which is equivalent to

$$|\operatorname{Rm}_{g(t)}|_{g(t)} \leq K(\beta(t + \varepsilon^2)^{-1} + \varepsilon^{-2}) \quad \forall t \in \left[T - \frac{\varepsilon^2}{K}, T\right)$$

on the neighbourhood $U_p = B_{g(T - \frac{\varepsilon^2}{K})}(p, \varepsilon/\sqrt{K})$ of p. The curvature bound for times $t < T - \frac{\varepsilon^2}{K}$ can be easily derived from the Type I condition. □

As a corollary we get the following non-oscillation result:

Corollary 3.5.4. *Let $(M, g(t), \phi(t))$ be a complete Type I harmonic Ricci flow on $[0, T)$ with finite singular time T, with non-increasing coupling function $\alpha(t) \in [\underline{\alpha}, \overline{\alpha}]$, where $0 < \underline{\alpha} \leq \overline{\alpha} < \infty$. Then for every $p \in \Sigma_I^H$ there exists a constant $c > 0$ such that*

$$|\operatorname{Rm}_g|(p,t) \geq \frac{c}{T-t}. \tag{3.5.3}$$

Theorem 3.5.5. *Let $(M, g(t), \phi(t))$ be a complete Type I harmonic Ricci flow on $[0, T)$ with finite singular time T, with non-increasing coupling function $\alpha(t) \in [\underline{\alpha}, \overline{\alpha}]$, where $0 < \underline{\alpha} \leq \overline{\alpha} < \infty$. Then* $\operatorname{Vol}_{g(0)}(\Sigma^H) < \infty$ *implies that* $\operatorname{Vol}_{g(t)}(\Sigma^H) \longrightarrow 0$ *as $t \nearrow T$.*

Proof. By the Type I assumption we can apply the maximum principle to the evolution equation of S to get the existence of a constant $\tilde{C}$ such that $\inf_M \mathrm{S}(t) \geq -\tilde{C}$ for all $t \in [0, T)$. Define the following sets

$$\Sigma_{S,k}^H \coloneqq \left\{ p \in M \,\middle|\, \mathrm{S}(p,t) \geq \frac{1/k}{T-t}, \forall t \in \left(T - \frac{1}{k}, T\right) \right\} \subseteq \Sigma_S^H = \Sigma^H,$$

for every $k \in \mathbb{N}^+$, and $\Sigma_{S,0}^H \coloneqq \emptyset$. We claim that for every point $x \in \Sigma_{S,k}^H$ and for all $t \in [0, T)$ we have

$$\int_0^t \mathrm{S}(x,s)ds \geq -\tilde{C}T + \log\left(\frac{1/k}{T-t}\right)^{1/k}.$$

Clearly, this holds for $t \leq T - \frac{1}{k}$, since the argument of the logarithm is less than one and by the initial consideration $\int_0^t \mathrm{S}(x,s)ds \geq -\tilde{C}t \geq -\tilde{C}T$. Instead for $t \in \left(T - \frac{1}{k}, T\right)$, the definition of $\Sigma_{S,k}^H$ yields

$$\begin{aligned}\int_0^t \mathrm{S}(x,s)ds &= \int_0^{T-1/k} \mathrm{S}(x,s)ds + \int_{T-1/k}^t \mathrm{S}(x,s)ds \geq -\tilde{C}\left(T - \frac{1}{k}\right) + \int_{T-1/k}^t \frac{1/k}{T-s}ds \\ &\geq -\tilde{C}T + \log\left(\frac{1/k}{T-t}\right)^{1/k}.\end{aligned}$$

Using the evolution equation for the volume form, as well as the inequality $k \leq 2^k$ for all $k \in \mathbb{N}$, we get

$$\operatorname{Vol}_{g(t)}(\Sigma_{S,k}^H \setminus \Sigma_{S,k-1}^H) = \int_{\Sigma_{S,k}^H \setminus \Sigma_{S,k-1}^H} e^{-\int_0^t \mathrm{S}(x,s)ds} dvol_{g(0)}(x) \leq 2e^{\tilde{C}T}(T-t)^{1/k} \operatorname{Vol}_{g(0)}(\Sigma_{S,k}^H \setminus \Sigma_{S,k-1}^H).$$

Notice that

$$\sum_k \operatorname{Vol}_{g(0)}(\Sigma_{S,k}^H \setminus \Sigma_{S,k-1}^H) = \operatorname{Vol}_{g(0)}(\Sigma_S^H) < \infty,$$

so we can use the (discrete) dominated convergence theorem to conclude that

$$\begin{aligned}\limsup_{t\to T}\operatorname{Vol}_{g(t)}(\Sigma_S^H) &= \limsup_{t\to T}\sum_k \operatorname{Vol}_{g(t)}(\Sigma_{S,k}^H \setminus \Sigma_{S,k-1}^H)\\ &\le 2e^{\tilde{C}T}\limsup_{t\to T}\sum_k (T-t)^{1/k}\operatorname{Vol}_{g(0)}(\Sigma_{S,k}^H \setminus \Sigma_{S,k-1}^H) = 0.\end{aligned}$$

□

Chapter 4

Global Extension Results for Ricci Flows

In this chapter, we prove that a Ricci flow cannot develop a finite time singularity assuming the boundedness of a suitable space-time integral norm of the curvature tensor. Moreover, the extensibility of the flow is proved under a Ricci lower bound and the boundedness of a space-time integral norm of the scalar curvature. The present chapter consists of an exposition of the paper [32] by the author.

4.1 Introduction

Extension theorems are abundant in the Ricci flow literature as already seen in Chapter 1; these guarantee the extendibility of a Ricci flow, as long as a suitable bound on the curvature is assumed. Let us just recall the results by Hamilton [42] and Sesum [69] as examples, where a global pointwise bound on the curvature or respectively Ricci curvature tensors are assumed in order to extend the flow. A different approach was adopted by Wang in [84] and consists of considering integral bounds rather than point-wise ones. *Wang's first extension theorem* (Theorem 1.1 in [84]) states that if $(M, g(t))$ is a Ricci flow on a closed manifold M satisfying the bound

$$\|\mathrm{Rm}\|_{\alpha, M\times[0,T)} < \infty \text{ for some } \alpha \geq \frac{n+2}{2}, \tag{4.1.1}$$

then the flow can be extended past time T. We will generalise this result in Theorem 4.1.2 below.

Wang's theorem is proved via a blow-up argument exploiting the scaling invariance of the integral norm above for $\alpha = \frac{n+2}{2}$. His result was extended by Ma and Cheng to complete manifolds in [60]; similar results were then obtained, for Ricci and Mean Curvature flow, see [45, 52, 53, 55, 54, 90, 91, 87]. In particular, we remark that Theorem 1.6 in [53] considers mixed integral norms for the Mean Curvature flow case.

In the same paper, Wang was able to pass from a Riemann curvature integral bound to a scalar curvature one assuming a Ricci lower bound. More precisely, *Wang's second extension theorem* (Theorem 1.2 in [84]) states that if $(M, g(t))$ is a Ricci flow on a closed manifold M satisfying

$$\|\mathrm{R}\|_{\alpha, M\times[0,T)} < \infty \text{ for some } \alpha \geq \frac{n+2}{2}, \tag{4.1.2}$$

and Ric is uniformly bounded from below along the flow up to the time T, then the flow can be extended past time T. Our Theorem 4.1.3 below is an extension of this result.

Similar results for (compact) Mean Curvature Flow have then been independently obtained by Le and Sesum in [54] and Xu, Ye and Zhao in [87].

Here we generalise these results using mixed integral norms.

Definition 4.1.1. *Given a pair (α, β) of integrability exponents in $(1, \infty)$ and a dimension $n \in \mathbb{N}$, we say that the pair is optimal (respectively super-optimal, sub-optimal) if*

$$\alpha = \frac{n}{2}\frac{\beta}{\beta - 1} \quad (\textit{resp.} \ \geq, \ \leq). \tag{4.1.3}$$

The main reason for introducing the concept of optimal pair is that the mixed integral norm of the Riemann tensor with respect to it is invariant under the parabolic scaling of the Ricci flow. Our first theorem is a generalisation of Wang's first extension theorem to mixed integral norms and complete non compact flows.

Theorem 4.1.2. *Let $(M, g(t))$ be a Ricci flow on a manifold M of dimension n, defined on $[0, T)$, $T < +\infty$, and such that $(M, g(t))$ is complete and has bounded geometry for every t in $[0, T)$. Assume the integral bound $\big\| \|\mathrm{Rm}\|_{\alpha, M} \big\|_{\beta, [0,T)} < +\infty$ for some super-optimal pair (α, β). Then the flow can be extended past the time T.*

In this theorem we assumed a control on the geometry of the Ricci flow in order to set up a blow-up procedure near the singular time T. In the case the underlying manifold M is closed this control comes for free, so we obtain an extension result under the sole integral bound.

It is worth noticing that we can include the "endpoint" $(\infty, 1)$ but not the one $(n/2, \infty)$, even on closed manifolds, see Remark 4.2.1. Moreover, this case is of particular importance after the integrability results obtained in [7, 8, 73] for flows with bounded scalar curvature: in fact, for such flows we have a $(\frac{n}{2}, \infty)-$bound on the curvature tensor, so any slight improvement of integrability (perhaps relaxing the time integrability) would rule out the formation of such singularities. We

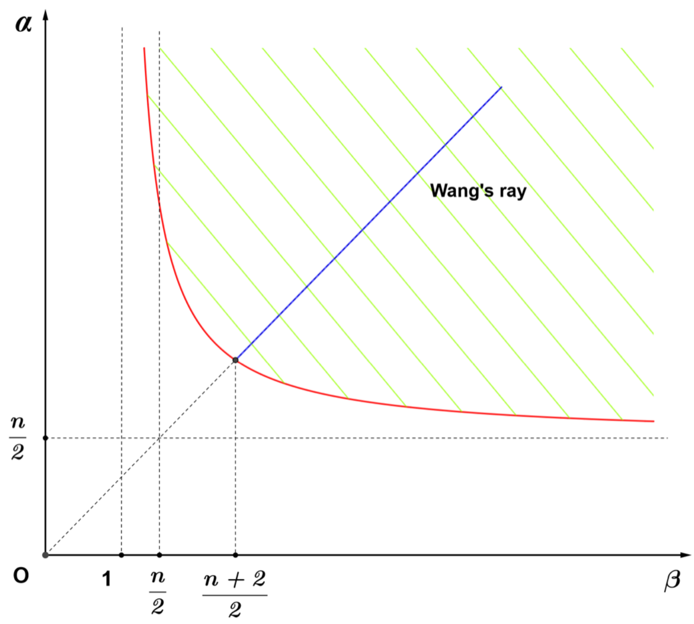

Figure 4.1: Integrability Exponents Graphic

will develop further this idea in Chapter 5, as an application of our local theory to this problem. We generalise Wang's second extension theorem as follows.

Theorem 4.1.3. *Let $(M, g(t))$ be a Ricci flow on a manifold M of dimension n, defined on $[0, T)$, $T < +\infty$, and such that $(M, g(t))$ is complete and has bounded geometry for every t in $[0, T)$. Assume the following conditions are satisfied*

- *there exists a positive constant B such that $\mathrm{Ric}(x, t) \geq -Bg(t)$ on $M \times [0, T)$;*
- $\left\| \|\mathrm{R}\|_{\alpha,M} \right\|_{\beta,[0,T)} < +\infty$ *for some super-optimal pair (α, β).*

Then the flow can be extended past the time T.

As for Wang's result, the method of the proof relies on developing a Moser iteration along the Ricci flow, in order to get a Moser-Harnack inequality for the scalar curvature. Then one can apply this inequality to a rescaled sequence of flows and deduce the result by a contradiction argument. The main additional difficulty that we have to overcome compared to Wang's case, is that we have to deal with a temporal integrability exponent lower than $n/2$, compare with Figure 4.1. In order to deal with it, we need to develop a further iteration procedure for showing the better integrability result in Proposition 4.3.5.

The chapter is organised as follows. In Section 4.2, we give a proof of Theorem 4.1.2 and some similar results. In Section 4.3 we first set up the necessary Moser iteration argument and then deduce a Moser-Harnack type inequality for the scalar curvature suitable for our purposes. Finally in Section 4.4 we show Theorem 4.1.3.

4.2 Extension under Integral Curvature bound

Our proof of Theorem 4.1.2 follows now directly from a blow-up argument, exploiting the scaling behaviour of the norm considered in the statement.

Proof. If the pair (α,β) is super-optimal but not optimal, a straightforward application of Hölder's inequality in time gives the existence of an optimal pair $(\alpha^*,\beta^*)=(\alpha,\beta^*)$, where $\beta^*<\beta$, for which we have

$$\big\|\|\mathrm{Rm}\|_{\alpha,M}\big\|_{\beta^*,[0,T)} \leq T^{\frac{1}{(\beta^*)'}}\big\|\|\mathrm{Rm}\|_{\alpha,M}\big\|_{\beta,[0,T)} < \infty, \tag{4.2.1}$$

so it is sufficient to prove the claim in the optimal case. Arguing by contradiction, if the flow is not extensible, Shi's Theorem implies that $|\mathrm{Rm}|$ is unbounded on $M\times[0,T)$. From the boundedness assumption for times smaller than T, we can pick a sequence of space-time points (x_i,t_i) such that $t_i\nearrow T$, and for some constant C greater than 1 we obtain

$$|\mathrm{Rm}|(x_i,t_i) \geq C^{-1}\sup_{M\times[0,t_i]}|\mathrm{Rm}|(x,t). \tag{4.2.2}$$

Set $Q_i := |\mathrm{Rm}|(x_i,t_i)\to+\infty$ and $P_i := B_{g(t_i)}(x_i,Q_i^{-\frac{1}{2}})\times[t_i-Q_i^{-1},t_i]$. Clearly, $|\mathrm{Rm}|\leq CQ_i$ on the region $M\times[t_i-Q_i^{-1},t_i]$. Consider a sequence of Ricci flows on $M\times[-Q_it_i,0]$ defined as $g_i(t):=Q_ig(Q_i^{-1}t+t_i)$. We can now apply Perelman's $\kappa-$noncollapsing theorem [65] for the parabolic region P_i, with any scale ρ for i large enough, which guarantees that the injectivity radii of the rescaled metrics g_i at (x_i,t_i) are uniformly bounded away from zero. Then by the compactness result in Theorem 2.3.3 we can extract a subsequence converging in the pointed smooth Cheeger-Gromov sense to a complete Ricci flow $(M_\infty,g_\infty(t),x_\infty)$ defined on $(-\infty,0]$, whose curvature is uniformly bounded by C and such that $|\mathrm{Rm}_{g_\infty}|(x_\infty,0)=1$. On the other hand, if the pair (α,β) is optimal, we compute

$$\begin{aligned}
&\int_{-1}^{0}\left(\int_{B_{g_\infty(0)}(x_\infty,1)}|\mathrm{Rm}_{g_\infty(t)}|^{\alpha}d\mu_{g_\infty(t)}\right)^{\frac{\beta}{\alpha}}dt\\
&=\lim_{i\to\infty}\int_{-1}^{0}\left(\int_{B_{g_i(0)}(\bar{x}_i,1)}|\mathrm{Rm}_{g_i(t)}|^{\frac{n}{2}\frac{\beta}{\beta-1}}d\mu_{g_i(t)}\right)^{\frac{2}{n}(\beta-1)}dt\\
&=\lim_{i\to\infty}\int_{t_i-Q_i^{-1}}^{t_i}\left(\int_{B_{g(t_i)}(x_i,Q_i^{-\frac{1}{2}})}|\mathrm{Rm}_{g(t)}|^{\frac{n}{2}\frac{\beta}{\beta-1}}Q_i^{\frac{n}{2}-\frac{n}{2}\frac{\beta}{\beta-1}}d\mu_{g(t)}\right)^{\frac{2}{n}(\beta-1)}Q_i dt\\
&\leq\lim_{i\to\infty}\int_{t_i-Q_i^{-1}}^{t_i}\left(\int_{M}|\mathrm{Rm}_{g(t)}|^{\frac{n}{2}\frac{\beta}{\beta-1}}d\mu_{g(t)}\right)^{\frac{2}{n}(\beta-1)}dt=0,
\end{aligned} \tag{4.2.3}$$

where the last step is justified by Lebesgue's dominated convergence theorem and the assumption $\big\| \|\mathrm{Rm}\|_{\alpha,M} \big\|_{\beta,[0,T)} < \infty$. Since the limit flow $g_\infty(t)$ is smooth, this chain of inequalities implies that $\mathrm{Rm}_{g_\infty(t)} \equiv 0$ on the parabolic region $B_{g_\infty(0)}(x_\infty, 1) \times [-1, 0]$, in particular, $\mathrm{Rm}_\infty(x_\infty, 0) = 0$, a contradiction. □

Remark 4.2.1. *It is interesting to analyse the "endpoints" case.*

Firstly, consider $\alpha = \infty$ and $\beta = 1$. Even in the closed case, Hamilton's theorem in [42] guarantees that the sectional curvature blows up at the finite time singularity T, and a maximum principle argument yields $|\,\mathrm{Rm}\,| \geq \frac{1}{8(T-t)} \notin L^1$. Moreover, the boundedness of the L^1-norm of the maximum of the Ricci curvature is sufficient to extend the flow, as shown in [84] and subsequently in [45].

In the case $\alpha = \frac{n}{2}$ and $\beta = \infty$, the Ricci flow of the standard sphere shows that one cannot expect to extend the flow even if $\big\| \|\mathrm{Rm}\|_{\frac{n}{2},S^n} \big\|_{\infty,[0,T)} < +\infty$. Interestingly, an extension theorem is proven in [90] under a smallness assumption on the $(n/2, \infty)$-mixed norm.

Corollary 4.2.2. *Let $(M, g(t))$ be a Ricci flow on a closed manifold M of dimension n, defined on $[0, T)$, with $T < +\infty$. Assume the integral bound $\big\| \|\mathrm{Rm}\|_{\alpha,M} \big\|_{\beta,[0,T)} < +\infty$ for some super-optimal pair (α, β). Then the flow can be extended past the time T.*

The following result generalises Theorem 1.1 in [60] to mixed norms along complete (possibly non-compact) Ricci flows. The proof strictly follows the one in [60].

Theorem 4.2.3. *Let $(M, g(t))$ be a Ricci flow on a manifold M of dimension n, defined on $[0, T)$, $T < +\infty$, and such that $(M, g(t))$ is complete and has bounded geometry for every t in $[0, T)$. Assume the integral bounds $\big\| \|\mathrm{R}\|_{\alpha,M} \big\|_{\beta,[0,T)} < +\infty$ and $\big\| \|\mathrm{W}\|_{\alpha,M} \big\|_{\beta,[0,T)} < +\infty$ for some super-optimal pair (α, β). Then the flow can be extended past the time T.*

Proof. Arguing by contradiction, if the flow is not extensible, Shi's Theorem implies that $|\,\mathrm{Rm}\,|$ is unbounded on $M \times [0, T)$. From the boundedness assumption for times smaller than T, we can pick a sequence of space-time points (x_i, t_i) such that $t_i \nearrow T$, and for some constant C greater than 1 we have

$$|\,\mathrm{Rm}\,|(x_i, t_i) \geq C^{-1} \sup_{M \times [0,t_i]} |\,\mathrm{Rm}\,|(x, t). \tag{4.2.4}$$

Set $Q_i := |\,\mathrm{Rm}\,|(x_i, t_i) \to +\infty$ and $P_i := B_{g(t_i)}(x_i, Q_i^{-\frac{1}{2}}) \times [t_i - Q_i^{-1}, t_i]$. Clearly, $|\,\mathrm{Rm}\,| \leq CQ_i$ on the region $M \times [t_i - Q_i^{-1}, t_i]$. Consider a sequence of Ricci flows on $M \times [-Q_i t_i, 0]$ defined as $g_i(t) := Q_i g(Q_i^{-1} t + t_i)$. We can argue as in the proof of Theorem 4.1.2 to extract a subsequence converging in the pointed smooth Cheeger-Gromov sense to a complete Ricci flow $(M_\infty, g_\infty(t), x_\infty)$ defined on $(-\infty, 0]$, whose curvature is uniformly bounded by C and such that $|\,\mathrm{Rm}_{g_\infty}|(x_\infty, 0) = 1$.

Again, if the pair (α, β) is optimal, the scaling properties of R and W and the finiteness of their mixed integral norms give us

$$\int_{-1}^{0}\left(\int_{B_{g_\infty(0)}(x_\infty,1)} |\,\mathrm{R}_{g_\infty(t)}\,|^\alpha d\mu_{g_\infty(t)}\right)^{\frac{\beta}{\alpha}} dt = 0, \tag{4.2.5}$$

and

$$\int_{-1}^{0}\left(\int_{B_{g_\infty(0)}(x_\infty,1)} |\,\mathrm{W}_{g_\infty(t)}\,|^\alpha d\mu_{g_\infty(t)}\right)^{\frac{\beta}{\alpha}} dt = 0. \tag{4.2.6}$$

Once more we deduce from the smoothness of the limit flow $g_\infty(t)$, together with these equations, that $\mathrm{R}_{g_\infty(t)} \equiv 0$, thus also $\mathrm{Ric}_{g_\infty(t)} \equiv 0$ from the evolution equation of the scalar curvature, and $\mathrm{W}_{g_\infty(t)} \equiv 0$ on the parabolic region $B_{g_\infty(0)}(x_\infty, 1) \times [-1, 0]$, from which we deduce $\mathrm{Rm}_{g_\infty(t)} \equiv 0$ through (2.1.7); in particular, $\mathrm{Rm}_{g_\infty}(x_\infty, 0) = 0$, a contradiction. We argue exactly as in the proof of Theorem 4.1.2 in the case (α, β) is super-optimal but not optimal. □

4.3 Parabolic Moser Iteration

In this section we develop the theory needed to show Theorem 4.1.3. The idea of the proof is similar to the one of Theorem 4.1.2 in the previous section, but this time we will rescale with scalar curvature rather than Riemannian curvature. Consequently, we do not have the full curvature bounds needed to extract a smooth limit flow. Hence we will need to prove the necessary estimates for elements of the sequence of rescaled flows rather than for the limit. For this reason, we develop a Moser iteration along the Ricci flow in order to obtain a Moser-Harnack type inequality. This is the main technical part of this chapter. The main method of the proof resembles the one in Wang's paper [84]; however, several modifications are necessary. We first settle the super-optimal case (Theorem 4.3.3) and then we prove the optimal case (Theorem 4.3.7) with the help of a better integrability result (Theorem 4.3.5). The main difficulty to overcome - which is not present in Wang's case - is given by the possibly low temporal integrability case, when $\beta < \frac{n}{2}$. We will deal with this constructing an iterative scheme of reverse Hölder inequalities.

Moser Iteration in the Super-optimal Case

Throughout this section, we consider a fixed complete Ricci flow $(M, g(t))$ on a $n-$dimensional manifold M, with $n \geq 3$, defined on $[0, T]$.

Definition 4.3.1. *For any given point $p \in M$ and radius $r > 0$, we define the sets*

$$\begin{aligned} \Omega &:= B_{g(T)}(p, r), \quad & \Omega' &:= B_{g(T)}\left(p, \tfrac{r}{2}\right), \\ D &:= \Omega \times [0, T], \quad & D' &:= \Omega' \times \left[\tfrac{T}{2}, T\right]. \end{aligned} \tag{4.3.1}$$

From now on we suppose to have a uniform Sobolev constant for the domain Ω, and also the bound $0 < \mathrm{Vol}_{g(t)}(\Omega') < +\infty$ for $t \in [\frac{T}{2}, T]$. We have the following analogue of Property 4.1 in Wang's paper [84].

Lemma 4.3.2. *Under the above assumptions, consider a function $v \in C^1(D)$ with $v(\cdot, t) \in C^1_0(\Omega)$ for every $t \in [0, T]$. Then for any (α, β) optimal pair we have*

$$\big\| \|v^2\|_{\alpha',\Omega} \big\|_{\beta',[0,T]} \leq \sigma^{\frac{1}{\beta'}} \big\| \|v\|_{2,\Omega} \big\|^{s}_{\infty,[0,T]} \|\nabla v\|^{\frac{2}{\beta'}}_{2,D}, \tag{4.3.2}$$

where $s := \frac{2}{\beta} \in (0, 2)$.

Proof. For the convenience of the reader we remark that, arguing exactly as in Lemma 2.4.6, we can get

$$1 = \frac{1}{\beta} + \frac{n}{2\alpha} = 1 - \frac{1}{\beta'} + \frac{n}{2}(1 - \frac{1}{\alpha'}) \iff \frac{n}{2} = \frac{1}{\beta'} + \frac{n}{2\alpha'}. \tag{4.3.3}$$

Let us write

$$\frac{1}{2\alpha'} = \frac{1-\theta}{2} + \frac{\theta}{2^*} = \frac{1}{2} - \frac{\theta}{n}. \tag{4.3.4}$$

where we have adopted the usual notation $2^* := \frac{2n}{n-2}$ for Sobolev's conjugate exponent of 2. Dividing equation (4.3.3) by a factor n, we get

$$\frac{1}{2} - \frac{1}{n\beta'} = \frac{1}{2\alpha'} = \frac{1}{2} - \frac{\theta}{n} \iff \theta = \frac{1}{\beta'}. \tag{4.3.5}$$

The standard interpolation inequality applied to the time-slice $(M, g(t))$ and Sobolev's inequality, give

$$\|v\|_{2\alpha',\Omega} \leq \|v\|^{1-\theta}_{2,\Omega} \|v\|^{\theta}_{2^*,\Omega} \leq \|v\|^{1-\frac{1}{\beta'}}_{2,\Omega} \sigma^{\frac{1}{\beta'}} \|\nabla v\|^{\frac{1}{\beta'}}_{2,\Omega} = \sigma^{\frac{1}{\beta'}} \|v\|^{\frac{1}{\beta}}_{2,\Omega} \|\nabla v\|^{\frac{1}{\beta'}}_{2,\Omega}. \tag{4.3.6}$$

Squaring this inequality and integrating in time, we finally arrive at

$$\big\| \|v^2\|_{\alpha',\Omega} \big\|_{\beta',[0,T]} \leq \sigma^{\frac{1}{\beta'}} \max_{t\in[0,T]} \|v(\cdot,t)\|^{\frac{2}{\beta}}_{2,\Omega} \|\nabla v\|^{\frac{2}{\beta'}}_{2,\Omega}, \tag{4.3.7}$$

which concludes the proof, once set $s := \frac{2}{\beta}$. □

We start showing a Moser-Harnack inequality for super-optimal pairs.

Lemma 4.3.3. *Given $(M, g(t))$ as above, suppose there exists a constant $B \geq 0$ such that on D we have $\mathrm{Ric}(x, t) \geq -Bg(t)$ and let (a, b) be a strictly super-optimal pair. Assume that for two measurable functions f and h there exists a non-negative function $u \in C^\infty(D)$ satisfying*

$$\frac{\partial u}{\partial t} \leq \Delta u + fu + h \tag{4.3.8}$$

in the sense of distributions, where $\big\| \|f\|_{a,\Omega} \big\|_{b,[0,T]} + \big\| \|\mathrm{R}_-\|_{a,\Omega} \big\|_{b,[0,T]} + 1 \le C_0$*, where we have set* $\mathrm{R}_- := \max\{0, -\mathrm{R}\}$*. Then there exists a constant* $C = C(n, a, b, \sigma, C_0, r, T, B)$ *such that*

$$\|u\|_{\infty,D'} \le C\big(\big\| \|u\|_{\alpha',\Omega} \big\|_{\beta',[0,T]} + \big\| \|h\|_{a,\Omega} \big\|_{b,[0,T]} \cdot \big\| \|1\|_{\alpha',\Omega} \big\|_{\beta',[0,T]}\big), \tag{4.3.9}$$

where $\alpha = \alpha_*(a, b, n)$ *and* $\beta = \beta_*(a, b, n)$ *are the optimal integrability exponents given by Lemma 2.4.6.*

Proof. Consider a cut-off function $\eta \in C^\infty(D)$ such that $\eta(\cdot, t) \in C_0^\infty(\Omega)$ for every $t \in [0, T]$, $\eta(x, 0) \equiv 0$ and $\eta(x, \cdot)$ is a non-decreasing function for every $x \in \Omega$. Set $\kappa := \big\| \|h\|_{a,\Omega} \big\|_{b,[0,T]}$ and $v := u + \kappa$. Rewriting (4.3.8) in terms of v we simply have

$$\frac{\partial v}{\partial t} - \Delta v \le f(v - \kappa) + h. \tag{4.3.10}$$

For a fixed $\lambda > 1$, it makes sense to consider $\eta^2(u + \kappa)^{\lambda-1}$ as a test function, so we get for any $s \in (0, T]$

$$\begin{aligned} \int_0^s \int_\Omega (-\Delta v)\eta^2 v^{\lambda-1} d\mu dt + \int_0^s \int_\Omega \frac{\partial v}{\partial t}\eta^2 v^{\lambda-1} d\mu dt &\le \int_0^s \int_\Omega (fu + h)\eta^2 (u + \kappa)^{\lambda-1} d\mu dt \\ &\le \int_0^s \int_\Omega \Big(|f| + \frac{|h|}{\kappa}\Big)\eta^2 v^\lambda d\mu dt. \end{aligned} \tag{4.3.11}$$

Using the equation for the volume element under the Ricci flow and integrating by parts, we deduce

$$\begin{aligned} &\int_0^s \int_\Omega (2\eta\langle\nabla\eta, \nabla v\rangle v^{\lambda-1} + (\lambda - 1)\eta^2 v^{\lambda-2}|\nabla v|^2) d\mu dt \\ &+ \frac{1}{\lambda}\Big(\int_\Omega \eta^2 v^\lambda d\mu\Big|_s - \int_0^s \int_\Omega 2\eta\frac{\partial\eta}{\partial t} v^\lambda d\mu dt + \int_0^s \int_\Omega \eta^2 v^\lambda \,\mathrm{R}\, d\mu dt\Big) \le \int_0^s \int_\Omega \Big(|f| + \frac{|h|}{\kappa}\Big)\eta^2 v^\lambda d\mu dt. \end{aligned} \tag{4.3.12}$$

Schwartz's inequality yields the following estimate

$$\int_0^s \int_\Omega (2\eta\langle\nabla\eta, \nabla v\rangle v^{\lambda-1} d\mu dt \ge -\varepsilon^2 \int_0^s \int_\Omega \eta^2 v^{\lambda-2}|\nabla v|^2 d\mu dt - \frac{1}{\varepsilon^2}\int_0^s \int_\Omega v^\lambda |\nabla\eta|^2 d\mu dt. \tag{4.3.13}$$

Substituting in the previous one we obtain, after reordering, that

$$\begin{aligned} (\lambda - 1 - \varepsilon^2)\int_0^s \int_\Omega \eta^2 v^{\lambda-2}|\nabla v|^2 d\mu dt + \frac{1}{\lambda}\int_\Omega \eta^2 v^\lambda d\mu\Big|_s &\le \int_0^s \int_\Omega \Big(|f| + \frac{|h|}{\kappa}\Big)\eta^2 v^\lambda d\mu dt \\ + \frac{1}{\varepsilon^2}\int_0^s \int_\Omega v^\lambda |\nabla\eta|^2 d\mu dt + \frac{1}{\lambda}\Big(\int_0^s \int_\Omega 2\eta\frac{\partial\eta}{\partial t} v^\lambda d\mu dt &- \int_0^s \int_\Omega \eta^2 v^\lambda \,\mathrm{R}\, d\mu dt\Big). \end{aligned} \tag{4.3.14}$$

Choose $\varepsilon^2 = \frac{\lambda-1}{2}$. Since $|\nabla v^{\frac{\lambda}{2}}|^2 = \frac{\lambda^2}{4} v^{\lambda-2}|\nabla v|^2$, we compute

$$\begin{aligned} 2\Big(1-\frac{1}{\lambda}\Big)\int_0^s\int_\Omega \eta^2|\nabla v^{\frac{\lambda}{2}}|^2 d\mu dt + \int_\Omega \eta^2 v^\lambda d\mu\Big|_s &\le \lambda \int_0^s\int_\Omega \Big(|f| + \frac{|h|}{\kappa} + \mathrm{R}_-\Big)\eta^2 v^\lambda d\mu dt \\ &+ \frac{2\lambda}{\lambda-1}\int_0^s\int_\Omega v^\lambda |\nabla\eta|^2 d\mu dt + \int_0^s\int_\Omega 2\eta\frac{\partial\eta}{\partial t} v^\lambda d\mu dt. \end{aligned} \tag{4.3.15}$$

Now we use $|\nabla(\eta v^{\frac{\lambda}{2}})|^2 \le 2\eta^2|\nabla v^{\frac{\lambda}{2}}|^2 + 2v^\lambda|\nabla\eta|^2$ to infer that

$$\begin{aligned} \Big(1-\frac{1}{\lambda}\Big)\int_0^s\int_\Omega |\nabla(\eta v^{\frac{\lambda}{2}})|^2 d\mu dt + \int_\Omega \eta^2 v^\lambda d\mu\Big|_s &\le \lambda \int_0^s\int_\Omega \Big(|f| + \frac{|h|}{\kappa} + \mathrm{R}_-\Big)\eta^2 v^\lambda d\mu dt \\ &+ 2\Big(\frac{\lambda}{\lambda-1} + \frac{\lambda-1}{\lambda}\Big)\int_0^s\int_\Omega v^\lambda |\nabla\eta|^2 d\mu dt + \int_0^s\int_\Omega 2\eta\frac{\partial\eta}{\partial t} v^\lambda d\mu dt. \end{aligned} \tag{4.3.16}$$

Therefore we have

$$\begin{aligned} &\int_0^s\int_\Omega |\nabla(\eta v^{\frac{\lambda}{2}})|^2 d\mu dt + \int_\Omega \eta^2 v^\lambda d\mu\Big|_s \\ &\le \Lambda(\lambda)\Big(\int_0^s\int_\Omega \Big(|f| + \frac{|h|}{\kappa} + \mathrm{R}_-\Big)\eta^2 v^\lambda d\mu dt + \int_0^s\int_\Omega v^\lambda|\nabla\eta|^2 d\mu dt + \int_0^s\int_\Omega 2\eta\frac{\partial\eta}{\partial t} v^\lambda d\mu dt\Big) \\ &\le \Lambda(\lambda)\Big(\Big\|\Big\||f| + \frac{|h|}{\kappa} + \mathrm{R}_-\Big\|_{a,\Omega}\Big\|_{b,[0,T]}\Big\|\|\eta^2 v^\lambda\|_{a',\Omega}\Big\|_{b',[0,T]} + \int_0^s\int_\Omega v^\lambda|\nabla\eta|^2 d\mu dt + \int_0^s\int_\Omega 2\eta\frac{\partial\eta}{\partial t} v^\lambda d\mu dt\Big) \\ &\le \Lambda(\lambda)\Big(C_0\Big\|\|\eta^2 v^\lambda\|_{a',\Omega}\Big\|_{b',[0,T]} + \int_0^s\int_\Omega v^\lambda|\nabla\eta|^2 d\mu dt + \int_0^s\int_\Omega 2\eta\frac{\partial\eta}{\partial t} v^\lambda d\mu dt\Big). \end{aligned} \tag{4.3.17}$$

The constant $\Lambda(\lambda)$ can be chosen as follows:

$$\Lambda(\lambda) = \begin{cases} 4\frac{\lambda^2}{(\lambda-1)^2} & \text{if } 1<\lambda<2, \\ 6\lambda & \text{if } \lambda \ge 2. \end{cases} \tag{4.3.18}$$

In particular, we get

$$\begin{aligned} \int_0^T\int_\Omega |\nabla(\eta v^{\frac{\lambda}{2}})|^2 d\mu dt &\le \Lambda(\lambda)\Big(C_0\Big\|\|\eta^2 v^\lambda\|_{a',\Omega}\Big\|_{b',[0,T]} + \Big\|\Big(|\nabla\eta|^2 + 2\eta\frac{\partial\eta}{\partial t}\Big)v^\lambda\Big\|_{1,D}\Big), \\ \max_{0\le s\le T}\int_\Omega \eta^2 v^\lambda d\mu\Big|_s &\le \Lambda(\lambda)\Big(C_0\Big\|\|\eta^2 v^\lambda\|_{a',\Omega}\Big\|_{b',[0,T]} + \Big\|\Big(|\nabla\eta|^2 + 2\eta\frac{\partial\eta}{\partial t}\Big)v^\lambda\Big\|_{1,D}\Big). \end{aligned} \tag{4.3.19}$$

Applying the inequality (4.3.2) to the function $w = \eta v^{\frac{\lambda}{2}}$, we arrive to

$$\Big\|\|\eta^2 v^\lambda\|_{\alpha',\Omega}\Big\|_{\beta',[0,T]} \le \sigma^{\frac{1}{\beta'}}\Lambda(\lambda)\Big(C_0\Big\|\|\eta^2 v^\lambda\|_{a',\Omega}\Big\|_{b',[0,T]} + \Big\|\Big(|\nabla\eta|^2 + 2\eta\frac{\partial\eta}{\partial t}\Big)v^\lambda\Big\|_{1,D}\Big), \tag{4.3.20}$$

where we have chosen the optimal integrability pair $(\alpha,\beta) = (\alpha^*,\beta^*)$ given by Lemma 2.4.6.

Coherently to the notation in that Lemma, we have used that

$$s + \frac{2}{\beta'} = \frac{2}{\beta} + \frac{2}{\beta'} = 2. \tag{4.3.21}$$

Combining Lemma 2.4.6 and Young's inequality, we obtain

$$\begin{aligned}
\big\| \|\eta^2 v^\lambda\|_{a',\Omega} \big\|_{b',[0,T]} &\leq \big\| \|\eta^2 v^\lambda\|_{\alpha',\Omega} \big\|^{\theta}_{\beta',[0,T]} \big\| \|\eta^2 v^\lambda\|_{1,\Omega} \big\|^{1-\theta}_{1,[0,T]} \\
&\leq \big\| \|\eta^2 v^\lambda\|_{\alpha',\Omega} \big\|_{\beta',[0,T]} \varepsilon^{\frac{1}{\theta}} \theta + \big\| \|\eta^2 v^\lambda\|_{1,\Omega} \big\|_{1,[0,T]} \varepsilon^{-\frac{1}{1-\theta}} (1-\theta) \\
&\leq \big\| \|\eta^2 v^\lambda\|_{\alpha',\Omega} \big\|_{\beta',[0,T]} \varepsilon^{\frac{1}{\theta}} + \big\| \|\eta^2 v^\lambda\|_{1,\Omega} \big\|_{1,[0,T]} \varepsilon^{-\frac{1}{1-\theta}} \\
&= \big\| \|\eta^2 v^\lambda\|_{\alpha',\Omega} \big\|_{\beta',[0,T]} \varepsilon' + \big\| \|\eta^2 v^\lambda\|_{1,\Omega} \big\|_{1,[0,T]} \varepsilon'^{-\frac{\theta}{1-\theta} =: -\nu},
\end{aligned} \tag{4.3.22}$$

where $\varepsilon' = \varepsilon^{\frac{1}{\theta}}$. This allows us to absorb the first term in the right hand side of the inequality (4.3.20) to the left hand side:

$$\big(1 - \Lambda(\lambda)\sigma^{\frac{1}{\beta'}} C_0 \varepsilon'\big) \big\| \|\eta^2 v^\lambda\|_{\alpha',\Omega} \big\|_{\beta',[0,T]} \leq \sigma^{\frac{1}{\beta'}} \Lambda(\lambda) \big(C_0 \varepsilon'^{-\nu} \|\eta^2 v^\lambda\|_{1,D} + \big\| \big(|\nabla \eta|^2 + 2\eta \partial_t \eta\big) v^\lambda \big\|_{1,D} \big).$$

Setting $\varepsilon' = (2\Lambda(\lambda)\sigma^{\frac{1}{\beta'}} C_0)^{-1}$, we have

$$\big\| \|\eta^2 v^\lambda\|_{\alpha',\Omega} \big\|_{\beta',[0,T]} \leq 2\sigma^{\frac{1}{\beta'}} \Lambda(\lambda) \Big(C_0 \big(2\Lambda(\lambda)\sigma^{\frac{1}{\beta'}} C_0\big)^{\nu} \|\eta^2 v^\lambda\|_{1,D} + \Big\| \Big(|\nabla \eta|^2 + 2\eta \frac{\partial \eta}{\partial t} \Big) v^\lambda \Big\|_{1,D} \Big).$$

Since we can always choose $\Lambda(\lambda) \geq 1$, we get

$$\big\| \|\eta^2 v^\lambda\|_{\alpha',\Omega} \big\|_{\beta',[0,T]} \leq C_1(n, a, b, \sigma, C_0) \Lambda(\lambda)^{1+\nu} \int_D \Big(|\nabla \eta|^2 + 2\eta \frac{\partial \eta}{\partial t} + \eta^2 \Big) v^\lambda d\mu dt. \tag{4.3.23}$$

The inequality (4.3.23) is the basis for the Moser iteration. As in Wang's paper [84], we will construct a nested sequence of cylindrical sets and test functions on them, and we will eventually arrive at an $L^\infty-$bound. Define for every natural number k

$$\begin{aligned}
t_k &:= \frac{T}{2} - \frac{T}{4^{k+1}}, \quad r_k := \Big(\frac{1}{2} + \frac{1}{2^{k+1}} \Big) r, \\
\Omega_k &:= B_{g(T)}(p, r_k), \quad D_k := \Omega_k \times [t_k, T].
\end{aligned} \tag{4.3.24}$$

Notice that $t_k \nearrow \frac{T}{2}$, $r_k \searrow \frac{1}{2} r$ and $\Omega_k, D_k \searrow \Omega', D'$ respectively. Fix two functions $\gamma, \rho \in C^\infty(\mathbb{R})$ such that

$$0 \leq \gamma' \leq 2, \quad \gamma(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 1 & \text{if } t \geq 1 \end{cases}, \quad -2 \leq \rho' \leq 0, \quad \rho(s) = \begin{cases} 1 & \text{if } s \leq 0 \\ 0 & \text{if } s \geq 1 \end{cases}. \tag{4.3.25}$$

Set $\gamma_k(t) := \gamma(\frac{t - t_{k-1}}{t_k - t_{k-1}})$ and $\rho_k(s) := \rho(\frac{s - r_k}{r_{k-1} - r_k})$, and consider the sequence of cut-off functions given by

$$\eta_k(x,t) \coloneqq \gamma_k(t)\rho_k(d_{g(T)}(x,p)). \tag{4.3.26}$$

Clearly, η_k is a smooth function such that $0 \leq \eta_k \leq 1$, $\eta_k \equiv 1$ on D_k and $\eta_k \equiv 0$ outside D_{k-1}. The lower Ricci bound together with the Ricci flow equation gives us

$$|\frac{\partial \eta_k}{\partial t}| \leq \frac{2}{3T}4^{k+1}, \quad \text{and} \quad |\nabla \eta_k|_{g(t)} \leq e^{2BT}2^{k+2}r^{-1} \quad \forall t \in [0,T]. \tag{4.3.27}$$

If $\lambda \geq 2$, recall $\Lambda(\lambda) = 6\lambda$, we compute from (4.3.23)

$$\begin{aligned} \big\| \|v^\lambda\|_{\alpha',\Omega_k} \big\|_{\beta',[t_k,T]} &= \big\| \|\eta_k^2 v^\lambda\|_{\alpha',\Omega_k} \big\|_{\beta',[t_k,T]} \leq \big\| \|\eta_k^2 v^\lambda\|_{\alpha',\Omega_{k-1}} \big\|_{\beta',[t_{k-1},T]} \\ &\leq C_2(n,a,b,\sigma,C_0)\lambda^{1+\nu} \int_{D_{k-1}} \Big(|\nabla \eta_k|^2 + 2\eta_k \frac{\partial \eta_k}{\partial t} + \eta_k^2 \Big) v^\lambda d\mu dt \\ &\leq 4^{k+2} C_3(r,T,B) C_2(n,a,b,\sigma,C_0)\lambda^{1+\nu} \int_{D_{k-1}} v^\lambda d\mu dt \\ &\leq C_4(n,a,b,\sigma,C_0,r,T,B) 4^{k-1}\lambda^{1+\nu} \|v^\lambda\|_{1,D_{k-1}}. \end{aligned} \tag{4.3.28}$$

Let us remark that $C_3(r,T,B)$ can be chosen to be $\max\{e^{4BT}/r^2, 1/(3T), 1\}$. It is convenient to rewrite this as

$$\big\| \|v\|_{\alpha'\cdot\lambda,\Omega_k} \big\|_{\beta'\cdot\lambda,[t_k,T]} \leq C_4^{\frac{1}{\lambda}} 4^{\frac{k-1}{\lambda}} \lambda^{\frac{1+\nu}{\lambda}} \|v\|_{\lambda,D_{k-1}}. \tag{4.3.29}$$

Let us set $\zeta = \min\{\alpha',\beta'\}$, $\alpha_k = \zeta^{k-1}\alpha'$, and $\beta_k = \zeta^{k-1}\beta'$. Applying the inequality above with $\lambda = \zeta^{k-1}$ we obtain

$$\big\| \|v\|_{\alpha_k,\Omega_k} \big\|_{\beta_k,[t_k,T]} \leq C_4^{\frac{1}{\lambda}} 4^{\frac{k-1}{\lambda}} \lambda^{\frac{1+\nu}{\lambda}} \|v\|_{\zeta^{k-1},D_{k-1}} \leq C_4^{\frac{1}{\lambda}} 4^{\frac{k-1}{\lambda}} \lambda^{\frac{1+\nu}{\lambda}} \big\| \|v\|_{\alpha_{k-1},\Omega_{k-1}} \big\|_{\beta_{k-1},[t_{k-1},T]}. \tag{4.3.30}$$

Using $\lambda = \zeta^{k-1}, \zeta^{k-2}, ..., \zeta^{k_0}$, where k_0 is the smallest integer such that $\zeta^{k_0} \geq 2$, a simple iteration implies

$$\begin{aligned} \big\| \|v\|_{\alpha_k,\Omega_k} \big\|_{\beta_k,[t_k,T]} \leq & C_4^{\frac{1}{\zeta^{k-1}} + \frac{1}{\zeta^{k-2}} + ... + \frac{1}{\zeta^{k_0}}} 4^{\frac{k-1}{\zeta^{k-1}} + \frac{k-2}{\zeta^{k-2}} + ... + \frac{k_0}{\zeta^{k_0}}} \zeta^{(1+\nu)\left(\frac{k-1}{\zeta^{k-1}} + \frac{k-2}{\zeta^{k-2}} + ... + \frac{k_0}{\zeta^{k_0}}\right)} \cdot \\ & \cdot \big\| \|v\|_{\alpha_{k_0},\Omega_{k_0}} \big\|_{\beta_{k_0},[t_{k_0},T]} \leq C_5(n,a,b,\sigma,C_0,r,T,B) \big\| \|v\|_{\alpha_{k_0},\Omega_{k_0}} \big\|_{\beta_{k_0},[t_{k_0},T]}, \end{aligned} \tag{4.3.31}$$

where in the last step we used that any of the exponents in consideration can be bounded by a summable series, whose value is independent of k. To cover the left cases $\lambda < 2$, if needed we can iterate directly (4.3.23) (a finite amount of times independent of k) with the different definition of $\Lambda(\lambda)$, and we obtain

$$\big\| \|v\|_{\alpha_{k_0},\Omega_{k_0}} \big\|_{\beta_{k_0},[t_{k_0},T]} \leq C_6(n,a,b,\sigma,C_0,r,T,B) \big\| \|v\|_{\alpha',\Omega_0} \big\|_{\beta',[\frac{T}{4},T]}. \tag{4.3.32}$$

Resuming, we have

$$\big\|\|v\|_{\alpha'_k,\Omega_k}\big\|_{\beta'_k,[t_k,T]} \leq C_7(n,a,b,\sigma,C_0,r,T,B)\big\|\|v\|_{\alpha',\Omega_0}\big\|_{\beta',[\frac{T}{4},T]}, \tag{4.3.33}$$

therefore from the inclusions of the domains

$$\big\|\|v\|_{\alpha'_k,\Omega'}\big\|_{\beta'_k,[\frac{T}{2},T]} \leq \big\|\|v\|_{\alpha'_k,\Omega_k}\big\|_{\beta'_k,[t_k,T]} \leq C_7\big\|\|v\|_{\alpha',\Omega_0}\big\|_{\beta',[\frac{T}{4},T]} \leq C_7\big\|\|v\|_{\alpha',\Omega}\big\|_{\beta',[0,T]}. \tag{4.3.34}$$

With k going to infinity, both α'_k and β'_k tend to infinity, so we obtain

$$\|v\|_{\infty,D'} \leq C\big\|\|v\|_{\alpha',\Omega}\big\|_{\beta',[0,T]}, \tag{4.3.35}$$

using Lemma 2.4.1. Notice that the conditions in this Lemma are satisfied by the domain Ω', because the flow is smooth on the whole time interval. Since $u \geq 0$, using the definition of v we finally get

$$\begin{aligned}\|u\|_{\infty,D'} \leq \|v\|_{\infty,D'} \leq C\big\|\|v\|_{\alpha',\Omega}\big\|_{\beta',[0,T]} &\leq C(\big\|\|u\|_{\alpha',\Omega}\big\|_{\beta',[0,T]} + \kappa\big\|\|1\|_{\alpha',\Omega}\big\|_{\beta',[0,T]}) \\ &\leq C(\big\|\|u\|_{\alpha',\Omega}\big\|_{\beta',[0,T]} + \big\|\|h\|_{a,\Omega}\big\|_{b,[0,T]}\big\|\|1\|_{\alpha',\Omega}\big\|_{\beta',[0,T]}).\end{aligned} \tag{4.3.36}$$

□

Remark 4.3.4. *We remark that, following the same iteration scheme, we could arrive at*

$$\|v\|_{\infty,D'} \leq C_7\big\|\|v\|_{\alpha_l,\Omega}\big\|_{\beta_l,[0,T]} \tag{4.3.37}$$

for every $l \in \mathbb{N}$. In particular, choosing l large enough, we can get $\alpha_l \geq \alpha$ and $\beta_l \geq \beta$, hence

$$\|v\|_{\infty,D'} \leq C_7\big\|\|v\|_{\alpha,\Omega}\big\|_{\beta,[0,T]}. \tag{4.3.38}$$

Improved Integrability and Moser-Harnack Inequality in the Optimal Case

In showing a Moser-Harnack type inequality for the scalar curvature in the optimal case, we cannot directly appeal to the absorption scheme used in Lemma 4.3.3, but we need to impose a certain smallness of the data, see Theorem 4.3.7; this feature is common among differential equations with super-linear forcing terms, where one can exploit a point-wise smallness of the solution to reduce the study to the linear case. Here the smallness assumption is given in an integral form rather than point-wise, thus we develop once again a Moser's iteration, this time to link the problem to the strictly super-optimal case, compare with Lemma 4.4 in [84]. Furthermore, the proof of the following proposition directly implies an improved integrability result, see Remark 4.3.6.

Proposition 4.3.5. *Given $(M, g(t))$ as above, suppose there exists a constant $B \geq 0$ such that on D we have $\mathrm{Ric}(x,t) \geq -Bg(t)$ and let (α, β) be an optimal pair. If a non-negative function $u \in C^\infty(D)$ satisfies*

$$\frac{\partial u}{\partial t} \leq \Delta u + fu + h \tag{4.3.39}$$

in the sense of distributions, where $\big\|\|f\|_{\alpha,\Omega}\big\|_{\beta,[0,1]} < +\infty$, then there exist a strictly super-optimal pair (a, b) and constants $C = C(n, \alpha, \beta, \sigma, r, T, B)$ and $\delta = \delta(n, \sigma, \alpha, \beta)$ such that if the smallness assumption $\big\|\|f\|_{\alpha,\Omega}\big\|_{\beta,[0,T]} + \big\|\|\mathrm{R}_-\|_{\alpha,\Omega}\big\|_{\beta,[0,T]} \leq \delta$ is satisfied, then we have

$$\big\|\|u\|_{a,\Omega'}\big\|_{b,[\frac{T}{2},T]} \leq C(\big\|\|u\|_{\alpha,\Omega}\big\|_{\beta,[0,T]} + \big\|\|h\|_{\alpha,\Omega}\big\|_{\beta,[0,T]} \cdot \big\|\|1\|_{\alpha,\Omega}\big\|_{\beta,[0,T]}). \tag{4.3.40}$$

Proof. Let $v = u + \kappa$, where $\kappa = l \cdot \big\|\|h\|_{\alpha,\Omega}\big\|_{\beta,[0,T]}$ for some positive constant l. Then v solves

$$\partial_t v \leq \Delta v + fu + h. \tag{4.3.41}$$

We choose a test function of the form $\eta^2 v^{\lambda-1}$, and proceed as in the proof of Lemma 4.3.3 to get

$$\begin{aligned}\big\|\|\eta^2 v^\lambda\|_{\alpha',\Omega}\big\|_{\beta',[0,T]} \leq& \sigma^{\frac{1}{\beta'}}\Lambda(\lambda)\Big(\Big(\big\|\|f\|_{\alpha,\Omega}\big\|_{\beta,[0,T]} + \big\|\|\mathrm{R}_-\|_{\alpha,\Omega}\big\|_{\beta,[0,T]} + \frac{1}{l}\Big)\big\|\|\eta^2 v^\lambda\|_{a',\Omega}\big\|_{b',[0,T]} \\ &+ \Big\|\Big(|\nabla\eta|^2 + 2\eta\frac{\partial\eta}{\partial t}\Big)v^\lambda\Big\|_{1,D}\Big),\end{aligned} \tag{4.3.42}$$

Suppose now that $\big\|\|f\|_{\alpha,\Omega}\big\|_{\beta,[0,T]} + \big\|\|\mathrm{R}_-\|_{\alpha,\Omega}\big\|_{\beta,[0,T]} \leq \delta_\lambda := (4\sigma^{\frac{1}{\beta'}}\Lambda(\lambda))^{-1}$; in order to absorb the first term of the right hand side, we choose $l_\lambda := 4\sigma^{\frac{1}{\beta'}}\Lambda(\lambda) + 1$, so we get

$$\big\|\|\eta^2 v^\lambda\|_{\alpha',\Omega}\big\|_{\beta',[0,T]} \leq 2\sigma^{\frac{1}{\beta'}}\Lambda(\lambda)\Big\|\Big(|\nabla\eta|^2 + 2\eta\frac{\partial\eta}{\partial t}\Big)v^\lambda\Big\|_{1,D}. \tag{4.3.43}$$

We have arrived at a situation analogous to (4.3.23). Choosing properly the function η we get a reverse Hölder inequality, so a better integrability. The inequality above ensures that, given the optimal pair (α, β), we can bound the $(\alpha'\lambda, \beta'\lambda)-$norm in terms of the $(\alpha, \beta)-$norm if we have $\lambda \leq \min\{\alpha, \beta\}$. On the other hand, the condition on the pair $(\alpha'\lambda, \beta'\lambda)$ to be strictly super-optimal is

$$\alpha'\lambda > \frac{n}{2}\frac{\beta'\lambda}{\beta'\lambda - 1} \iff \lambda > \frac{n}{2}. \tag{4.3.44}$$

These conditions imply that $\min\{\alpha, \beta\} =: \zeta > \frac{n}{2}$, which is restrictive. In order to cover the general case, we consider the same nested family of cylinders and cut-off functions as in Lemma 4.3.3. Set $\alpha_k = (\alpha')^k\zeta$ and $\beta_k = (\beta')^k\zeta$ for every k. Iterating the inequality (4.3.43) with different values of $\lambda = \alpha_{k-1} > \alpha_{k-2} > ... > \alpha_1 > \beta$ if $\beta < \alpha$, or $\lambda = \beta_{k-1} > \beta_{k-2} > ... > \beta_1 > \alpha$ if $\beta > \alpha$, we obtain

$$\big\|\|v\|_{\alpha_k,\Omega_k}\big\|_{\beta_k,[t_k,T]} \leq C\big\|\|v\|_{\alpha_{k-1},\Omega_{k-1}}\big\|_{\beta_{k-1},[t_{k-1},T]} \leq ... \leq C\big\|\|v\|_{\alpha'\zeta,\Omega_1}\big\|_{\beta'\zeta,[t_1,T]} \leq C\big\|\|v\|_{\alpha,\Omega}\big\|_{\beta,[0,T]}.$$

Here the value of C varies from inequality to inequality, but it never approaches infinity. Since $D' \subset D_k$ for every k, in order to conclude it suffices to show that for the fixed pair (α, β), there exists a finite k such that (α_k, β_k) is super-optimal. By definition, we need to check

$$(\alpha')^k \zeta > \frac{n}{2} \frac{(\beta')^k \zeta}{(\beta')^k \zeta - 1}. \tag{4.3.45}$$

Since the limit for $k \to +\infty$ of the left hand side is infinite, we deduce the existence of a finite k satisfying this inequality, and hence the claim. □

Remark 4.3.6 (Improved Integrability)**.** *Notice that the above iteration process does not reach the endpoint case $(\alpha, \beta) = (\infty, 1)$ unless $n = 1$. Moreover, by taking a further larger k we may assume a and b to be as large as we want.*

We conclude the subsection using the integrability result just obtained to deduce a Moser-Harnack inequality for the scalar curvature.

Theorem 4.3.7. *Under the above assumptions there exist a constant $C = C(n, \alpha, \beta, \sigma, r, T, B)$ and a small constant $\delta = \delta(n, \sigma, \alpha, \beta)$, such that if $\big\| \|\mathrm{R}\|_{\alpha,\Omega} \big\|_{\beta,[0,T]} + B \leq \delta$, then we have*

$$\|\mathrm{R}_+\|_{\infty, D'} \leq C(\big\| \|\mathrm{R}\|_{\alpha,\Omega} \big\|_{\beta,[0,T]} + B) \leq C\delta. \tag{4.3.46}$$

Proof. Set $\hat{\mathrm{R}} := \mathrm{R} + nB$. By the lower bound on the Ricci tensor, with same argument as in [84], we get the following inequality in D:

$$\frac{\partial \hat{\mathrm{R}}}{\partial t} \leq \Delta \hat{\mathrm{R}} + 2(\hat{\mathrm{R}} - 2B)\hat{\mathrm{R}} + 2nB^2. \tag{4.3.47}$$

Remark that we are in case where $\big\| \|1\|_{\alpha,\Omega} \big\|_{\beta,[0,T]} \leq \tilde{V}$ by Lemma 2.4.3. Referring to Proposition 4.3.5, we set $u = \hat{\mathrm{R}}$, $f = 2(\hat{\mathrm{R}} - 2B)$, $h = 2nB^2$, and we call C' and δ' the constants given by the Proposition. Let $\delta := \frac{\delta'}{3n\tilde{V}}$, and compute

$$\begin{aligned}
\big\| \|f\|_{\alpha,\Omega} \big\|_{\beta,[0,T]} + \big\| \|\mathrm{R}_-\|_{\alpha,\Omega} \big\|_{\beta,[0,T]} &= \big\| \|2(\hat{\mathrm{R}} - 2B)\|_{\alpha,\Omega} \big\|_{\beta,[0,T]} + \big\| \|\mathrm{R}_-\|_{\alpha,\Omega} \big\|_{\beta,[0,T]} \\
&= \big\| \|2(\mathrm{R} + (n-2)B)\|_{\alpha,\Omega} \big\|_{\beta,[0,T]} + \big\| \|\mathrm{R}_-\|_{\alpha,\Omega} \big\|_{\beta,[0,T]} \\
&\leq 3 \big\| \|\mathrm{R}\|_{\alpha,\Omega} \big\|_{\beta,[0,T]} + 2(n-2)B \big\| \|1\|_{\alpha,\Omega} \big\|_{\beta,[0,T]} \\
&\leq 3n\tilde{V}(\big\| \|\mathrm{R}\|_{\alpha,\Omega} \big\|_{\beta,[0,T]} + B) \leq \delta'.
\end{aligned} \tag{4.3.48}$$

We can apply Proposition 4.3.5 to get the existence of a super-optimal pair (a, b) such that

$$\big\| \|\hat{\mathrm{R}}\|_{a,\Omega'} \big\|_{b,[\frac{T}{2},T]} \leq C(\big\| \|\hat{\mathrm{R}}\|_{\alpha,\Omega} \big\|_{\beta,[0,T]} + \big\| \|2nB^2\|_{\alpha,\Omega} \big\|_{\beta,[0,T]} \big\| \|1\|_{\alpha,\Omega} \big\|_{\beta,[0,T]}) \leq C. \tag{4.3.49}$$

We can bound

$$\big\|\|2(\hat{\mathrm{R}}-2B)\|_{a,\Omega'}\big\|_{b,[\frac{T}{2},T]}+\big\|\|\mathrm{R}_-\|_{a,\Omega'}\big\|_{b,[\frac{T}{2},T]}+1\leq 3\big\|\|\hat{\mathrm{R}}\|_{a,\Omega'}\big\|_{b,[\frac{T}{2},T]}+(n+4)B\big\|\|1\|_{a,\Omega'}\big\|_{b,[\frac{T}{2},T]}+1$$
$$\leq C_0(n,\alpha,\beta,\sigma,r,T,B). \tag{4.3.50}$$

Notice that we can assume $\alpha'_*\leq a$ and $\beta'_*\leq b$ by Remark 4.3.6, where α_* and β_* are the exponents given by Lemma 2.4.6, for which (4.3.39) holds. Now we apply the super-optimal Moser iteration Lemma 4.3.3 to get the existence of a constant $C=C(n,\alpha,\beta,\sigma,r,T,B)$ such that (we use Hölder's inequality, (4.3.49), $\alpha'_*\leq a, \beta'_*\leq b$ and the definition of $\hat{\mathrm{R}}$)

$$\begin{aligned}
\|\hat{\mathrm{R}}\|_{\infty,D'} &\leq C\big(\big\|\|\hat{\mathrm{R}}\|_{\alpha'_*,\Omega}\big\|_{\beta'_*,[0,1]}+\big\|\|h\|_{a,\Omega}\big\|_{b,[0,1]}\cdot\big\|\|1\|_{\alpha'_*,\Omega}\big\|_{\beta'_*,[0,1]}\big)\\
&= C\big(\big\|\|\hat{\mathrm{R}}\|_{\alpha'_*,\Omega}\big\|_{\beta'_*,[0,1]}+2nB^2\big\|\|1\|_{a,\Omega}\big\|_{b,[0,1]}\cdot\big\|\|1\|_{\alpha'_*,\Omega}\big\|_{\beta'_*,[0,1]}\big)\\
&\leq C\big(C\big\|\|\hat{\mathrm{R}}\|_{a,\Omega}\big\|_{b,[0,1]}+2nB^2\big\|\|1\|_{a,\Omega}\big\|_{b,[0,1]}\cdot\big\|\|1\|_{\alpha'_*,\Omega}\big\|_{\beta'_*,[0,1]}\big)\\
&\leq C\big(\big\|\|\mathrm{R}\|_{\alpha,\Omega}\big\|_{\beta,[0,1]}+B\big).
\end{aligned} \tag{4.3.51}$$

It suffices to notice that $\|\mathrm{R}_+\|_{\infty,D'}\leq\|\hat{\mathrm{R}}\|_{\infty,D'}$ to conclude the proof. □

4.4 Extension under Integral Scalar Curvature bound

In this Section we use the results of the previous sections to give a proof of Theorem 4.1.3. Inspired by the proof of Theorem 4.1.2 we argue by contradiction, assuming that the flow is not extensible, and deduce a contradiction from an asymptotic analysis for a sequence of rescalings. The hypotheses assumed in the statement of Theorem 4.1.3 naturally lead to rescale the flow with the scalar curvature. Were we rescaling at the maximal Riemann curvature scale, we would have the right bounds needed to extract a blow-up limit, hence we could get a contradiction similar to the one obtained in the proof of Theorem 4.1.2. Unfortunately, the choice of the scaling factors introduces the technical problem of the lack of compactness for the sequence. In order to deal with this issue, we apply the Moser-Harnack inequality given by Theorem 4.3.7, which holds uniformly for the sequence of rescalings considered in view of the assumed lower Ricci bound; in fact, the uniformity of the Sobolev constant is guaranteed by Theorem 2.4.8, and we consider appropriate parabolic regions for the inequality, see below. Finally, we deduce from the finiteness of the mixed integral norm considered that the sequence of rescalings uniformly approaches scalar flatness in a region containing a normalised scalar curvature point, a contradiction.

Proof. Without loss of generality, we can assume that (α,β) is an optimal pair: indeed similar to what we did in the proof of Theorem 4.1.2, it is sufficient to apply Hölder's inequality in time to reduce the problem to the optimal case.

Suppose by contradiction that the flow cannot be extended; thus Theorem 1.4 in [60] implies that $|\,\mathrm{Ric}\,|$ is unbounded on $M \times [0,T)$. The assumed lower Ricci bound implies the unboundedness of the scalar curvature, $\sup_{M\times[0,T)} \mathrm{R} = +\infty$. Since the curvature tensor is bounded up to the singular time T, we can pick a sequence of space-time points (x_i,t_i) such that $t_i \nearrow T$, and for some constant C greater than 1

$$\mathrm{R}(x_i,t_i) \geq C^{-1} \sup_{M\times[0,t_i]} \mathrm{R}(x,t). \tag{4.4.1}$$

Set $Q_i := \mathrm{R}(x_i,t_i) \to +\infty$ and $P_i := B_{g(t_i)}(x_i, Q_i^{-\frac{1}{2}}) \times [t_i - Q_i^{-1}, t_i]$. Clearly, $\mathrm{R} \leq CQ_i$ on the parabolic region P_i. Consider a sequence of Ricci flows defined as $g_i(t) := Q_i g(Q_i^{-1}(t-1)+t_i)$ on $M \times [0,1]$. By construction we get

$$\begin{aligned} \mathrm{R}_i(x,t) \leq C \quad &\forall (x,t) \in B_{g_i(1)}(x_i,1)\times[0,1] \quad \text{and} \quad \mathrm{R}_i(x_i,1) = 1; \\ \mathrm{Ric}_i(x,t) \geq -\frac{B}{Q_i} g_i(t) \qquad &\forall (x,t) \in M \times [0,1]. \end{aligned} \tag{4.4.2}$$

From this we deduce that the tensor $\mathrm{Ric}_i + \frac{B}{Q_i}$ is non-negative, thus

$$\mathrm{Ric}_i + \frac{B}{Q_i} \leq tr(\mathrm{Ric}_i + \frac{B}{Q_i}) g_i = \big(\mathrm{R}_i + \frac{nB}{Q_i}\big) g_i. \tag{4.4.3}$$

In particular, choosing $C \leq n - \frac{3}{2}$, we get for i large enough

$$\begin{aligned} \mathrm{Ric}_i(x,t) \leq (n-1) g_i(t) \quad &\forall (x,t) \in B_{g_i(1)}(x_i,1)\times[0,1]; \\ \mathrm{Ric}_i(x,t) \geq -\frac{B}{Q_i} g_i(t) \qquad &\forall (x,t) \in M \times [0,1]. \end{aligned} \tag{4.4.4}$$

Notice that this choice is allowed since $n \geq 3$. Moreover, for i large enough, $-nB \leq \mathrm{R}(x,t) \leq CQ_i$ on P_i implies $|\,\mathrm{R}_i\,| \leq C$ on $B_{g_i(1)}(x_i,1)$, so the κ-non local collapsing theorem [65] applies (with scale 2) yielding for every scale ρ the existence of a constant $\kappa = \kappa(g(0),n,T)$ such that we have the uniform lower bound

$$\mathrm{Vol}_{g_i(1)}(B_{g_i(1)}(x_i,1)) = \frac{\mathrm{Vol}_{g(t_i)}(B_{g(t_i)}(x_i, Q_i^{-\frac{1}{2}}))}{Q_i^{-\frac{n}{2}}} \geq \kappa. \tag{4.4.5}$$

Theorem 2.4.8 now guarantees the existence of a radius $r = r(\kappa,n)$ and a uniform Sobolev constant $\sigma = \sigma(n,r)$ for every time-slice $t \in [0,1]$ on the ball $B_{g_i(1)}(x_i,r)$. We can therefore use Theorem 4.3.7, by setting

$$\begin{aligned} \Omega_i := B_{g_i(1)}(x_i,r), \quad &\Omega_i' := B_{g_i(1)}(x_i,\frac{r}{2}), \\ D_i := \Omega_i \times [0,1], \quad &D_i' := [\frac{1}{2},1]. \end{aligned} \tag{4.4.6}$$

In the following we exploit the finiteness of $\big\|\|\mathrm{R}\|_{\alpha,M}\big\|_{\beta,[0,T)}$, as well as its scaling invariance in the case (α, β) is an optimal pair, to compute

$$\begin{aligned}\lim_{i\to+\infty} \big\| \|\mathrm{R}\|_{\alpha,B_{g_i(1)}(x_i,r)} \big\|^{\beta}_{\beta,[0,1)} &= \lim_{i\to+\infty} \int_0^1 \left(\int_{\Omega_i} |\,\mathrm{R}_{g_i(t)}\,|^{\alpha} d\mu_{g_i(t)} \right)^{\frac{\beta}{\alpha}} dt \\ &= \lim_{i\to+\infty} \int_{t_i - Q_i^{-1}}^{t_i} \left(\int_{B_{g(t_i)}(x_i, rQ_i^{-\frac{1}{2}})} |\,\mathrm{R}_{g(t)}\,|^{\alpha} d\mu_{g(t)} \right)^{\frac{\beta}{\alpha}} dt = 0.\end{aligned} \tag{4.4.7}$$

Since $\big\| \|\mathrm{R}\|_{\alpha,B_{g_i(1)}(x_i,r)} \big\|_{\beta,[0,1)} + \frac{B}{Q_i} \leq \delta$ for i large, where $\delta = \delta(n, \sigma, \alpha, \beta)$ is the one given by Theorem 4.3.7, we can apply the theorem to obtain

$$\|(\mathrm{R}_i)_+\|_{\infty,D_i'} \leq C(n, \alpha, \beta, \sigma)\big(\big\| \|\mathrm{R}_i\|_{\alpha,\Omega_i} \big\|_{\beta,[0,1]} + \frac{B}{Q_i}\big). \tag{4.4.8}$$

Let us remark, that here we dropped the dependence of the constant C given by Theorem 4.3.7 on the lower bound $\frac{B}{Q_i}$ because this last one can be uniformly bounded by any constant asymptotically. However, the points x_i were selected in such a way that $\|(\mathrm{R}_i)_+\|_{\infty,D_i'} \geq \mathrm{R}_i(x_i, 1) = 1$, so the inequality just obtained gives a contradiction for i large enough. □

Chapter 5

Local Singularity Theory for Ricci Flows

In this chapter, we develop a refined singularity analysis for the Ricci flow by investigating curvature blow-up rates locally. We first introduce general definitions of Type I and Type II singular points and show that these are indeed the only possible types of singular points. In particular, near any singular point the Riemannian curvature tensor has to blow-up at least at a Type I rate, partially generalising Theorem 1.2.5 that relied on a global Type I assumption. We also prove analogous results for the Ricci tensor, as well as a localised version of Sesum's result, namely that the Ricci curvature must blow-up near every singular point of a Ricci flow, again at least at a Type I rate. Finally, we show some applications of the theory to Ricci flows with bounded scalar curvature. The theory presented here is based on the paper [13], a joint work by the author and his supervisor R. Buzano.

5.1 Introduction

Most of the results presented in this Chapter do not assume the flows considered to have bounded geometry time-slices, with the aim in mind of proving our theorems for the widest class of Ricci flows as possible. Their proofs will therefore not rely upon blow-up arguments.

A Refined Local Singularity Analysis

As mentioned in Chapter 1, singular Ricci flows can be divided among two classes, Type I or Type II Ricci flows, depending on the rate at which the maximal curvature is blowing-up at the singular time. Property (1.1.3) excludes any possible lower rate of divergence. In our first definition we localise these concepts.

Definition 5.1.1 (Type I and Type II Singular Points). *Let $(M, g(t))$ be a Ricci flow maximally defined on the time interval $[0, T)$, $T < \infty$ and assume that $(M, g(t))$ has bounded curvatures for all $t \in [0, T)$. For any fixed $t \in [0, T)$, we consider the parabolically rescaled Ricci flow $\widetilde{g}_t(s)$ defined for $s \in [-\frac{t}{T-t}, 1)$ as $\widetilde{g}_t(s) := (T-t)^{-1} g(t + (T-t)s)$.*

i) We say that a point $p \in M$ is a Type I singular point *if there exist constants $c_I, C_I, r_I > 0$ such that we have*

$$c_I < \limsup_{t \nearrow T} \sup_{B_{\widetilde{g}_t(0)}(p, r_I) \times (-r_I^2, r_I^2)} |\mathrm{Rm}_{\widetilde{g}_t}|_{\widetilde{g}_t} \leq C_I. \tag{5.1.1}$$

We denote the set of such points by Σ_I and call it the Type I singular set.

ii) We say that a point p is a Type II singular point *if for any $r > 0$ we have*

$$\limsup_{t \nearrow T} \sup_{B_{\widetilde{g}_t(0)}(p, r) \times (-r^2, r^2)} |\mathrm{Rm}_{\widetilde{g}_t}|_{\widetilde{g}_t} = \infty. \tag{5.1.2}$$

We denote the set of such points by Σ_{II} and call it the Type II singular set.

A somewhat related local definition of Type I and Type II singular points for the mean curvature flow has appeared in a very recent preprint [68], but it differs from ours in the sense that it uses backwards parabolic cylinders based at the singular time, while we use forwards and backwards parabolic cylinders based at regular times. This subtle difference turns out to be crucial for our results below.

We know by Definition 1.2.2 that for any singular point $p \in \Sigma$ there exists an essential blow-up sequence for (p, T). Our Definition 5.1.1 does not only consider the rate of curvature blow-up, but also takes into account the rate of convergence of essential blow-up sequences to the point (p, T). In Section 5.2, we give a heuristic explanation for the precise choices of Σ_I and Σ_{II}.

Note that if the Ricci flow in consideration is globally of Type I in the sense of (1.2.1), then so are all the rescaled flows $(M, \widetilde{g}_t)$ – with the same constant C – and therefore we immediately obtain the upper bound in (5.1.1) for any radius $r_I < 1$ with $C_I = \frac{C}{1-r_I^2}$. As promised, we extend Theorem 1.2.5 to great generality in the following theorem.

Theorem 5.1.2 (Decomposition of the Singular Set). *Let $(M, g(t))$ be a Ricci flow on a manifold M of dimension n, maximally defined on $[0, T)$, $T < +\infty$, and such that $(M, g(t))$ has bounded curvature for every t in $[0, T)$. Then $\Sigma = \Sigma_I \cup \Sigma_{II}$.*

We point out that this theorem not only shows that for every singular point $p \in \Sigma$ there is an essential blow-up sequence (p_i, t_i) such that the curvature blows-up with

$$\lim_{i \to \infty} (T - t_i) |\mathrm{Rm}(p_i, t_i)|_{g(t_i)} > 0, \tag{5.1.3}$$

but this blow-up sequence can be chosen to satisfy also

$$\limsup_{i\to\infty} \frac{d^2_{g(t_i)}(p_i,p)}{T-t_i} < \infty. \tag{5.1.4}$$

In fact, if p is a Type II singular point, then there exists an essential blow-up sequence (p_i, t_i) with

$$\lim_{i\to\infty} (T-t_i)|\mathrm{Rm}(p_i,t_i)|_{g(t_i)} = \infty \qquad \text{and} \qquad \lim_{i\to\infty} \frac{d^2_{g(t_i)}(p_i,p)}{T-t_i} = 0. \tag{5.1.5}$$

While the proof of Theorem 1.2.5 in [35] relies on the crucial Theorem 1.2.4 and on Perelman's pseudolocality theorem from [65], our Theorem 5.1.2 is proved using much more elementary estimates. The main technical tool is the following notion of Riemann (or regularity) scale.

Definition 5.1.3 (Parabolic Cylinders and Riemann Scales)**.** *Let $(M, g(t))$ be a Ricci flow defined on $[0,T)$ and let $(p,t) \in M \times [0,T)$ be a space-time point.*

i) For $r > 0$, we define the parabolic cylinder *$\mathcal{P}(p,t,r)$ with centre (p,t) and radius r by*

$$\mathcal{P}(p,t,r) := B_{g(t)}(p,r) \times (\max\{t-r^2, 0\}, \min\{t+r^2, T\}). \tag{5.1.6}$$

ii) We define the Riemann scale *$r_{\mathrm{Rm}}(p,t)$ at (p,t) by*

$$r_{\mathrm{Rm}}(p,t) := \sup\{r > 0 \mid |\mathrm{Rm}| < r^{-2} \text{ on } \mathcal{P}(p,t,r)\}. \tag{5.1.7}$$

If $(M, g(t))$ is flat for every $t \in [0,T)$, we set $r_{\mathrm{Rm}}(p,t) = +\infty$. Moreover, by slight abuse of notation, we may sometimes write $\mathcal{P}(p,t,r_{\mathrm{Rm}})$ for $\mathcal{P}(p,t,r_{\mathrm{Rm}}(p,t))$.

iii) We define the time-slice Riemann scale *$\widetilde{r}_{\mathrm{Rm}}(p,t)$ at (p,t) by*

$$\widetilde{r}_{\mathrm{Rm}}(p,t) := \sup\{r > 0 \mid |\mathrm{Rm}| < r^{-2} \text{ on } B_{g(t)}(p,r)\}. \tag{5.1.8}$$

When the flow is flat at time t, we set $\widetilde{r}_{\mathrm{Rm}}(p,t) = +\infty$. Clearly $\widetilde{r}_{\mathrm{Rm}}(p,t) \geq r_{\mathrm{Rm}}(p,t)$.

Notions of Riemann scales similar to the ones above have first been defined for static manifolds (see e.g. [1]) and both (5.1.8) as well as a definition involving backwards parabolic cylinders have appeared in various results about the Ricci flow (see e.g. [7, 11, 46]), but our definition (5.1.7) using *forwards and backwards* parabolic cylinders seems to have some advantages. In particular, we can prove that this Riemann scale is Lipschitz continuous in space and Hölder continuous in time, see Theorem 5.2.2, a result which has several interesting corollaries, for example a local Harnack-type inequality (Corollary 5.2.3), estimating the infimum and supremum of r_{Rm} on a smaller parabolic cylinder $\mathcal{P}(p,t,a_1\, r_{\mathrm{Rm}}(p,t))$, with $a_1 \in (0,1)$, by its value at the center of the cylinder. A similar definition of regularity scale using forwards and backwards cylinders has previously been given for mean curvature flow in [21].

Using the Riemann scale, we can give an alternative characterisation of the different types of singular points which should be compared to their global counterparts in (1.1.2)–(1.1.3)–(1.2.2).

Theorem 5.1.4 (Alternative Characterisation of Singular Sets). *Let $(M, g(t))$ be a Ricci flow on a manifold M of dimension n, maximally defined on $[0, T)$, $T < +\infty$, and with bounded curvature time-slices. Then*

i) $p \in \Sigma$ if and only if $\limsup_{t \nearrow T} r_{\mathrm{Rm}}^{-2}(p, t) = \infty$.

ii) $p \in \Sigma_I$ if and only if for some $0 < \widetilde{c}_I, \widetilde{C}_I$ we have $\widetilde{c}_I < \limsup_{t \nearrow T} (T-t) r_{\mathrm{Rm}}^{-2}(p, t) \leq \widetilde{C}_I$.

iii) $p \in \Sigma_{II}$ if and only if $\limsup_{t \nearrow T} (T-t) r_{\mathrm{Rm}}^{-2}(p, t) = \infty$.

We also note that the Riemann scale cannot oscillate between the Type I rate and a lower rate in the sense that if $p \in \Sigma_I$ then we can also obtain the stronger bound $1 \leq \liminf_{t \nearrow T} (T-t) r_{\mathrm{Rm}}^{-2}(p, t)$. This is basically a feature of the definition in (5.1.7) and will be proven in Corollary 5.2.4. We expect that a similar result should also be true for Type II singular points and thus conjecture that in Theorem 5.1.4, we can replace each instance of $\limsup_{t \nearrow T}$ with $\liminf_{t \nearrow T}$. We provide some evidence for this at the end of Section 5.2.

An Integral Concentration Result and a Density Function

Next, we study an integral characterisations of the different types of singular points. In view of the results in [32], discussed in Chapter 4, one would expect that a space-time integral norm of the curvature, with exponents (α, β) forming an optimal pair, should concentrate in neighbourhoods of a singular point. Such a result would in theory be equivalent to a Harnack inequality for $|\mathrm{Rm}|$ on parabolic cylinders near the singular point and would in particular guarantee that $|\mathrm{Rm}|$ blows up *at* (rather than *near*) every singular point, a result that seems too hard to achieve with the tools developed here. Moreover, for regular points one must clearly weaken any such claim: in fact, considering flat points in the Ricci flow starting at an immersed two-torus, we see that they are surrounded by non-flat points, so no Harnack inequality in any parabolic cylinder centred at them can hold without a correction term. Hence, we instead consider space-time integral norms of r_{Rm}^{-2} on parabolic cylinders $\mathcal{P}(p, t, a_1 r_{\mathrm{Rm}})$ on which our Harnack-type inequality for the Riemann scale holds, obtaining the following ε-regularity result: if $p \in \Sigma$, then for t sufficiently close to T, we have

$$0 < C_2 \leq \|r_{\mathrm{Rm}}^{-2}\|_{\alpha, \beta, \mathcal{P}(p, t, a_1 r_{\mathrm{Rm}})} \leq C_3 < \infty, \tag{5.1.9}$$

where $a_1 \in (0, 1)$ is the constant from Corollary 5.2.3. See Theorem 5.3.1 for the precise statement and the dependence of the constants C_2 and C_3. In order to distinguish between the different types of singular points, we then compare their Riemann scale with the Type I rate, so we consider the following.

Definition 5.1.5 (Singular Density). *Given a Ricci flow $(M, g(t))$ maximally defined on the time interval $[0, T)$, $T < \infty$, and an optimal pair (α, β), we define* the singular density function at time T *of the flow to be the function $\Theta\colon M \to [0, +\infty]$ given by*

$$\Theta(p) := \liminf_{t \nearrow T} \left\| \frac{1}{T-s} \right\|_{\alpha,\beta,\mathcal{P}(p,t,a_1\, r_{\mathrm{Rm}})} \tag{5.1.10}$$

where $a_1 \in (0,1)$ is again the constant from the Harnack-type result in Corollary 5.2.3.

It is then easy to prove the following alternative characterisation of the singular sets.

Theorem 5.1.6 (Integral Classification of the Singular Sets). *Let $(M, g(t))$ be a Ricci flow maximally defined on the time interval $[0, T)$, $T < \infty$, and (α, β) an optimal pair. Suppose that for every time $t \in [0, T)$, $(M, g(t))$ is complete with bounded geometry. Then we have that $\Sigma_I = \{p \in M \mid \Theta(p) \in (0, \infty)\}$ and $\Sigma_{II} = \{p \in M \mid \Theta(p) = 0\}$.*

We want to remark that one of the main reasons for introducing an integral concept of a density function is that we expect the following conjecture to be true.

Conjecture 5.1.7. *The density Θ is lower semi continuous with respect to the topology of $(M, g(t))$, and therefore, in particular, its zero-level set Σ_{II} is closed.*

The Ricci Singular Sets and a Localised Version of Sesum's Result

In Definition 5.4.1, we give definitions of Ricci scale $r_{\mathrm{Ric}}(p,t)$ and time-slice Ricci scale $\widetilde{r}_{\mathrm{Ric}}(p,t)$ similar to Definition 5.1.3 above. In a similar way, in Section 5.4, we then also introduce versions of Definition 5.1.1 involving the Ricci curvature tensor instead of the full Riemannian curvature tensor, i.e. we define the *Ricci singular set* Σ^{Ric} as well as the *Type I* and *Type II Ricci singular sets* Σ_I^{Ric} and $\Sigma_{II}^{\mathrm{Ric}}$ in Definition 5.4.6. As it turns out, the theorems above all have a direct Ricci curvature counterpart: in particular, we prove the Lipschitz-Hölder continuity of the Ricci scale in Theorem 5.4.4, the alternative characterisation of singular sets in Theorem 5.4.7, and finally the decomposition of the Ricci singular set $\Sigma^{\mathrm{Ric}} = \Sigma_I^{\mathrm{Ric}} \cup \Sigma_{II}^{\mathrm{Ric}}$ in Corollary 5.4.10. We also obtain an ε-regularity similar to (5.1.9), see Theorem 5.4.12.

All of these results follow rather similarly to their Riemann scale versions. The main new result of Section 5.4 instead is a localised version of the result that the Ricci curvature blows-up at a singularity of the Ricci flow, proven by Sesum in [69] and generalised to bounded geometry flows by Ma-Cheng in [60], as we have seen in Chapter 1. Let us mention that in [84], Wang strengthened this result, showing that, similar to (1.1.3), if T is the singular time of a closed Ricci flow $(M, g(t))$, one has

$$(T-t)\sup_M |\mathrm{Ric}(\cdot,t)|_{g(t)} \geq \eta_1, \quad \forall t\in[0,T). \tag{5.1.11}$$

Here $\eta_1 = \eta_1(n,\kappa)$ is a constant depending on the dimension of M and the non-collapsing constant κ of the flow. We generalise these results to the local setting, showing that any singular point is also a Ricci singular point – the other direction is obviously true – hence obtaining the following theorem.

Theorem 5.1.8 (Singular Points are Ricci Singular Points)**.** *Let $(M,g(t))$ be a Ricci flow on a manifold M of dimension n, maximally defined on $[0,T)$, $T<+\infty$ and with complete and bounded geometry time-slices. Then $\Sigma = \Sigma^{\mathrm{Ric}}$.*

We remark that this theorem is not a direct consequence of the local curvature bounds obtained in [26], [84], and [51]. While their results bound *the oscillations* of Rm (locally), we instead require a bound on the absolute value. Our proof relies both on the original ideas of Sesum [69] as well as on the characterisation of Ricci singular points in Theorem 5.4.7. A direct corollary of this result is the following.

Corollary 5.1.9. *Let $(M,g(t))$ be a Ricci flow on a manifold M of dimension n, maximally defined on $[0,T)$, $T<+\infty$ and such that $(M,g(t))$ is complete and has bounded geometry for every t in $[0,T)$. Then we have the inclusions $\Sigma_I \subseteq \Sigma_I^{\mathrm{Ric}}$ and $\Sigma_{II}\supseteq \Sigma_{II}^{\mathrm{Ric}}$ as well as the identity*

$$\Sigma_{II}\setminus\Sigma_{II}^{\mathrm{Ric}} = \Sigma_I^{\mathrm{Ric}}\setminus\Sigma_I.$$

Applications to Bounded Scalar Curvature Ricci Flows

In the final section we discuss how and to which extent the local theory described above can be applied to the study of bounded scalar curvature Ricci flows. In order for our theory to apply, we need to exclude badly behaved singular points, at which the Ricci curvature blows-up at a lower rate than their Ricci curvature scale. We will show that well behaved singularities cannot occur in dimensions lower than eight and that in higher dimensions the well-behaved singular set has codimension at least eight.

In the context of Ricci flow, the lack of an ambient space with respect to which we can measure this dimension forces us to consider dimensional bounds in terms of volume estimates on the singular set. In fact, the dimension of the singular set is related to the rate of convergence of its volume to zero as t approaches the singular time. This approach has been adopted in the study of Type I Ricci flows by Gianniotis (see [36, 37]) and we briefly recall the heuristic behind it. An actual estimate on the (intrinsic) Minkowski content cannot be available in general. Indeed, if we consider the Ricci flow of a round sphere $\mathbb{S}^n$, we see that the singular set coincides with the entire manifold, so for every time t the singular set is n-dimensional. On the other hand, the flow collapses the sphere to a single point as t approaches the final time, and one can easily see that the volume $\mu_{g(t)}(\mathbb{S}^n) \sim (\sqrt{T-t})^n$ as $t \to T$, so that the volume of the singular set goes to zero at the fastest possible rate, which means that it may be interpreted as (Minkowski) 0-dimensional. We phrase our codimension eight result in the sense of such a decay estimate.

It is worth mentioning that addressing the issue from an extrinsic point of view is in principle also possible: we could study bounds on the dimension either in the space-time structure developed by Kleiner and Lott in [49], or in the final time-slice of the flow, which can be endowed with a pseudo-metric structure by the works of Bamler-Zhang [12] (one might want to pass to the quotient metric space). We could also follow the work of Bamler [7] to pass to a singular limit and estimate the singular set there. Our decay estimates should essentially be equivalent to this last approach. Finally, let us mention that a new notion of $*-$Minkowsky dimension has been introduced by Bamler in [8, 9, 10] and applied to the study of Ricci flows (more generally to the so-called metric flows); it would be interesting to see how our results transpose in his terminology.

As mentioned above, for these results we need to exclude badly behaved singular points. This technical assumption allows us to compare the Ricci scale to the square root of the Riemann scale at the points in consideration. It would be interesting to remove this assumption or in fact to rule out such badly behaved points not only in the bounded scalar curvature case but possibly even for general Ricci flows. To phrase our results precisely, let us consider a Ricci flow $(M, g(t))$ maximally defined on $[0, T)$, and let us assume that $|\mathrm{Ric}|$ is not identically 0. For every $\delta \in (0, 1)$, we can consider the set of δ-well behaved points

$$G_\delta = G_{\delta,t_1} := \left\{q \in M \mid \delta a_0 r_{\mathrm{Ric}}^{-2}(q,t) < |\mathrm{Ric}|(q,t), \text{ for all } t \in [t_1, T)\right\}. \tag{5.1.12}$$

(The constant $a_0 = \sqrt{n}(n-1)$ is added for convenience, since it implies $r_{\mathrm{Ric}} \geq r_{\mathrm{Rm}}$, compare with Remark 5.4.2, and the value of t_1 will be chosen suitably below.) We point out that points in these sets are well behaved in the above mentioned sense *in a uniform way* near the singular time T and obviously the size of these sets increases as δ goes to 0. Set $\Sigma_\delta := \Sigma \cap G_\delta$. Implicit in our method of proof is the fact that $\Sigma_\delta \subseteq \Sigma_{II}$ for any $\delta > 0$. It is not clear whether we also have

$$\Sigma_{II} = \overline{\bigcup_{\delta \in (0,1)} \Sigma_\delta}. \tag{5.1.13}$$

Similarly, we define well-behaved blow-up sequences as follows: A sequence (p_i, t_i) with $t_i \nearrow T$ and $r_{\mathrm{Ric}}(p_i, t_i) \to 0$ is said to be δ-well behaved, if for sufficiently large i, the $\sqrt{\delta}\widetilde{r}_{\mathrm{Ric}}(p_i, t_i)$-ball around (p_i, t_i) contains only δ-well behaved points, that is for all $q \in B_{g(t_i)}(p_i, t_i, \sqrt{\delta}\widetilde{r}_{\mathrm{Ric}}(p_i, t_i))$ we have $\delta a_0 r_{\mathrm{Ric}}^{-2}(q, t_i) < |\mathrm{Ric}|(q, t_i)$.

Our first result is a non-existence result of well-behaved singularities in dimensions $n < 8$.

Theorem 5.1.10 (No Well-Behaved Singularities in Dimensions $n < 8$)**.** *Let $(M, g(t))$ be a Ricci flow on a closed manifold M of dimension $n < 8$, maximally defined on $[0, T)$, $T < +\infty$. Assume that the scalar curvature is uniformly bounded, $|\mathrm{R}| \leq n(n-1)R_0 < \infty$ on $M \times [0, T)$. Then for any $t_1 \in (0, T)$ and $\delta \in (0, 1)$ there cannot be any δ-well-behaved blow-up sequences. Moreover, if $M = G_\delta$ for some $\delta > 0$, then the flow can be smoothly extended past time T.*

As a corollary, we obtain an extension result under the slightly stronger assumption of an injectivity radius bound.

Corollary 5.1.11 (No Singularities in Dimensions $n < 8$ Under an Injectivity Radius Bound)**.** *Let $(M, g(t))$ be a Ricci flow on a closed manifold M of dimension $n < 8$, maximally defined on $[0, T)$, $T < +\infty$. Assume that the scalar curvature is uniformly bounded, $|\mathrm{R}| \leq n(n-1)R_0 < \infty$ on $M \times [0, T)$ and the injectivity radius is bounded from below by*

$$\operatorname{inj}(M, g(t)) \geq \alpha \Big(\sup_{M \times [0,t]} |\mathrm{Ric}| \Big)^{-1/2} \tag{5.1.14}$$

for some $\alpha > 0$. Then the flow can be smoothly extended past time T.

In our last result, we give a codimension estimate for the well-behaved singular set.

Theorem 5.1.12 (The Well-Behaved Singular Set has Codimension 8)**.** *For any natural number $n \in \mathbb{N}$, real numbers $T < +\infty$ $R_0 > 0$, $i_0 > 0$, $k_0 > 0$, $\delta \in (0, 1)$ and $d \in (0, 8)$ there exist a constant $E = E(n, R_0, i_0, k_0, T, \delta, d)$ and a time $t_1 = t_1(n, R_0, i_0, k_0, T, \delta) \in (0, T)$ such that the following statement holds. Let $(M, g(t))$ be a Ricci flow on a closed manifold M of dimension n, maximally defined on $[0, T)$. Assume that the scalar curvature is uniformly bounded, $|\mathrm{R}| \leq n(n-1)R_0 < \infty$ on $M \times [0, T)$, and that the initial metric satisfies $\operatorname{inj}(M, g(0)) > i_0$ and $\mathrm{Ric}_{g(0)} \geq -(n-1)k_0 g(0)$. Set $\Sigma_\delta = \Sigma \cap G_{\delta, t_1}$. Then for any $p \in M$, $d \in (0, 8)$, $\delta \in (0, 1)$ and $t \in [t_1, T)$ we have*

$$\mu_{g(t)}\big(\Sigma_\delta \cap B_{g(t)}(p, \tfrac{1}{2})\big) \leq E(\sqrt{T-t})^d. \tag{5.1.15}$$

The constant E in Theorem 5.1.12 degenerates as d approaches 8 or δ approaches 0. This is due to our application of Proposition 6.4 of [7], which plays a crucial role in our argument. Preventing this degeneration for $d = 8$ would correspond in this context to the finiteness of the $(n-8)$-dimensional measure of Σ_δ.

The importance of considering well-behaved points in G_δ consists on the fact that they verify a property (see Theorem 5.5.4) which is one of the key ingredients of our proof of Theorem 5.1.12, namely we can compare the *square* of the parabolic Ricci scale with the parabolic Riemann scale ($r_{\mathrm{Ric}}^2 \gtrsim r_{\mathrm{Rm}}$) at these points. We note that such an estimate also implies the corresponding estimate $\widetilde{r}_{\mathrm{Ric}}^2 \gtrsim \widetilde{r}_{\mathrm{Rm}}$ for the time-slice scales.

Let us briefly describe the proofs of these results. The proof of Theorem 5.1.10 is via an argument by contradiction. If the flow contains a δ-well-behaved blow-up sequence, then by the powerful integral bound of Theorem 1.7 in [7] by Bamler (see also [73] for a similar result in dimension four), and the estimate $\widetilde{r}^2_{\mathrm{Ric}} \gtrsim \widetilde{r}_{\mathrm{Rm}}$, we see that $\widetilde{r}^{-1}_{\mathrm{Ric}}$ has infinitesimal $L^{8-4\varepsilon}$-norm along the sequence (p_i, t_i) for $\varepsilon > 0$ small enough. On the other hand this norm is bounded away from zero by our Ricci Scale Concentration Lemma 5.5.5, yielding the desired contradiction for the first statement. To obtain the second statement, assume that $M = G_\delta$ for some $\delta > 0$. If the flow develops a singularity, we can pick a sequence of space-time points (p_i, t_i) along which the Ricci curvature blows-up by Sesum's result [69], in particular $r_{\mathrm{Ric}}(p_i, t_i) \to 0$. Since $M = G_\delta$, this blow-up sequence must be δ-well behaved. But such a sequence cannot exist by the first part of the theorem.

The main idea to prove the codimension eight estimate is to use our localised version of Sesum's result from Theorem 5.1.8 and to apply suitably Proposition 6.4 of [7], which gives a bound on the volumes of lower level sets of the time-slice Riemann scale (or equivalently of the parabolic Riemann scale). Theorem 5.1.8 ensures that the singular set is contained in small lower level sets of the Ricci scale. Since $r^2_{\mathrm{Ric}} \gtrsim r_{\mathrm{Rm}}$, these level sets are comparable to drastically smaller lower level sets of the Riemann scale yielding a significant improvement in the volume bound (for well behaved points). It is worth remarking that a straightforward application of Proposition 6.4 of [7] would give a codimension 4 result on the entire singular set Σ in the sense of (5.1.15).

We conclude this section outlining the structure of the Chapter. In Section 5.2 we carry out the pointwise analysis of the Riemann scale described above. The results involving mixed integral norms will then be proven in Section 5.3. The results involving the Ricci scale can be found in Section 5.4. Finally, in Section 5.5, we prove our result about bounded scalar curvature flows, Theorems 5.1.10 and 5.1.12.

5.2 Pointwise Analysis of the Singular Sets

Let us start this section with some heuristic ideas behind Definition 5.1.1 and a comparison to existing results in the literature. In [35], the authors gave the alternative definition of the Type I singular set Σ^*_I in Definition 1.2.3. In particular, they do not impose *explicitily* any restriction on the rate of convergence of (p_i, t_i) to (p, T). Their analysis shows that, under the global Type I assumption (1.2.1), this set coincides with the entire singular set Σ as we have seen in Theorem 1.2.5. Moreover, the analysis of the precise asymptotic behaviour of the neckpinch singularity developed by Angenent and Knopf [2] shows that, given a singular point p, the curvature tensor actually *cannot* blow-up along a sequence of regular space-time points (p_i, t_i) with $(T - t_i) = \mathfrak{o}(d^2_{g(t_i)}(p_i, p))$. Such a sequence is in fact sent to infinity by rescaling parabolically with $(T - t_i)$. This suggests that the rate of convergence of any essential blow-up sequence should be such that (5.1.4) holds.

The situation becomes more convoluted when the flow is not Type I, and one needs to carefully check the balance between the different rates involved. A good example to have in mind is one of the degenerate neck-pinch solutions constructed by Angenent, Isenberg, and Knopf in [3]. They prove that any blow-up sequence (p, t_i) based at a fixed point p in the smallest neck, and with rescaling factor $(T - t_i)$, converges to a shrinking cylinder while any blow-up sequence (q, t_i) based at the tip q, with scaling factor given by the curvature at the tip at time t_i, converges to a Bryant soliton. From the first convergence result it is clear that $(T - t_i) = \mathfrak{o}(d^2_{g(t_i)}(p, q))$, which geometrically corresponds to the tip being sent to infinity for this blow-up sequence. Heuristically, we may think that a sequence of points converging to the tip will have curvature going to infinity at a Type I or Type II rate depending on how fast it converges to the tip. These argument suggests that suitable definitions of the Type I and Type II singular sets for general flows should involve the rate of convergence of the essential blow-up sequences, and the condition (5.1.4) should give the right scale.

Before starting the proofs of the theorems from the introduction, we need the following adaptation of the "shrinking and expanding balls lemmas" in [75, 76] to our Riemann scale.

Lemma 5.2.1. *Suppose $(M, g(t))$ is a complete Ricci flow defined on $[0, T)$. Then there exists a constant $C_1 = C_1(n)$ such that for any point $(p, t) \in M \times [0, T)$ we have the inclusion*

$$B_{g(t)}(p, r_{\mathrm{Rm}}(p, t)) \supseteq B_{g(s)}(p, r_{\mathrm{Rm}}(p, t) - C_1^2 r_{\mathrm{Rm}}^{-1}(p, t)|s - t|), \tag{5.2.1}$$

for all $s \in (t - r^2_{\mathrm{Rm}}(p, t), t + r^2_{\mathrm{Rm}}(p, t))$.

Proof. We set $C_1 = 2\sqrt[4]{2/3}\sqrt{n-1}$. Since $\mathrm{Ric}_{g(t)} \leq (n-1) r^{-2}_{\mathrm{Rm}}(p, t) g(t)$ on $\mathcal{P}(p, t, r_{\mathrm{Rm}})$, and the function $r^{-2}_{\mathrm{Rm}}(p, t)$ is constant, hence in particular continuous and integrable on the interval $[t - r^2_{\mathrm{Rm}}(p, t), t + r^2_{\mathrm{Rm}}(p, t)]$, we can appeal to Lemma 3.2 in [75] (note that their constant $\beta/2$ is equal to our C_1^2) for any $s \in [t, t + r^2_{\mathrm{Rm}}(p, t))$ to obtain

$$B_{g(t)}(p, r_{\mathrm{Rm}}(p, t)) \supseteq B_{g(s)}(p, r_{\mathrm{Rm}}(p, t) - C_1^2 r^{-1}_{\mathrm{Rm}}(p, t)(s - t)).$$

In order to prove the inclusion for times $s \in (t - r^2_{\mathrm{Rm}}(p, t), t]$, we notice that due to the lower bound $\mathrm{Ric}_{g(t)} \geq -(n-1) r^{-2}_{\mathrm{Rm}}(p, t_0) g(t)$ on $\mathcal{P}(p, t, r_{\mathrm{Rm}})$, we can use Lemma 2.1 from [76] to obtain

$$B_{g(t)}(p, r_{\mathrm{Rm}}(p, t)) \supseteq B_{g(s)}(p, r_{\mathrm{Rm}}(p, t) e^{-(n-1) r^{-2}_{\mathrm{Rm}}(p,t)(t-s)}).$$

Using the elementary inequalities $e^{-x} \geq 1 - x$ and $C_1^2 \geq (n-1)$ we get

$$\begin{aligned} B_{g(t)}(p, r_{\mathrm{Rm}}(p, t)) &\supseteq B_{g(s)}(p, r_{\mathrm{Rm}}(p, t) - (n-1) r^{-1}_{\mathrm{Rm}}(p, t)(t - s)) \\ &\supseteq B_{g(s)}(p, r_{\mathrm{Rm}}(p, t) - C_1^2 r^{-1}_{\mathrm{Rm}}(p, t)(t - s)). \end{aligned}$$

□

We can now deduce the following estimate for the Riemann scale.

Theorem 5.2.2 (Lipschitz-Hölder Continuity of Riemann Scale)**.** *Suppose $(M, g(t))$ is a complete Ricci flow defined on $[0, T)$. Then for any pair of space-time points (p, t) and (q, s) we have*

$$|r_{\mathrm{Rm}}(p,t) - r_{\mathrm{Rm}}(q,s)| \leq \min\{d_{g(t)}(p,q), d_{g(s)}(p,q)\} + C_1|t-s|^{\frac{1}{2}}, \tag{5.2.2}$$

where C_1 is a constant depending only on n.

Proof. We first prove that the Riemann scale is Lipschitz continuous in space and then prove the Hölder continuity in time.

Step 1: We show that for fixed $t \in [0, T)$ the Riemann scale $r_{\mathrm{Rm}}(\cdot, t)$ is 1-Lipschitz continuous with respect to the metric $g(t)$. The proof of this part is a close adaptation of the analogous result in [11], but since its proof is as short as enlightening, we quickly recall it here.

Suppose towards a contradiction that for some t the function $r_{\mathrm{Rm}}(\cdot, t)$ is not 1-Lipschitz with respect to $g(t)$, that is we can find p and q such that

$$r_{\mathrm{Rm}}(p,t) - r_{\mathrm{Rm}}(q,t) > d_{g(t)}(p,q).$$

Here we assumed without loss of generality that $r_{\mathrm{Rm}}(p,t) \geq r_{\mathrm{Rm}}(q,t)$. We then define the radius $r := r_{\mathrm{Rm}}(p,t) - d_{g(t)}(p,q) > r_{\mathrm{Rm}}(q,t)$, so that $B_{g(t)}(q, r) \subseteq B_{g(t)}(p, r_{\mathrm{Rm}}(p,t))$ by the triangle inequality as well as $[t - r^2, t + r^2] \subseteq [t - r^2_{\mathrm{Rm}}(p,t), t + r^2_{\mathrm{Rm}}(p,t)]$, hence

$$\mathcal{P}(q, t, r) \subseteq \mathcal{P}(p, t, r_{\mathrm{Rm}}(p,t)).$$

In particular, by definition of the Riemann scale at (p,t), we have $|\mathrm{Rm}| \leq r^{-2}_{\mathrm{Rm}}(p,t) \leq r^{-2}$ on $\mathcal{P}(q,t,r)$, and therefore $r_{\mathrm{Rm}}(q,t) \geq r$ by definition of the Riemann scale at (q,t). This yields the desired contradiction with the definition of r.

Step 2: We show that $r_{\mathrm{Rm}}(p, \cdot)$ is $\frac{1}{2}$-Hölder continuous with constant $C_1 = C_1(n)$ being given by Lemma 5.2.1.

Arguing by contradiction, let us assume that for some fixed $p \in M$ there exist two times t and s in $[0, T)$ such that (assuming without loss of generality that $r_{\mathrm{Rm}}(p,t) \geq r_{\mathrm{Rm}}(p,s)$)

$$r_{\mathrm{Rm}}(p,t) - r_{\mathrm{Rm}}(p,s) > C_1\sqrt{|t-s|}, \tag{5.2.3}$$

with $C_1 = 2\sqrt[4]{2/3}\sqrt{n-1}$ the constant from Lemma 5.2.1. Set $r := r_{\mathrm{Rm}}(p,t) - C_1\sqrt{|t-s|}$, so that $r > r_{\mathrm{Rm}}(p,s)$. We claim that

$$\mathcal{P}(p, s, r) \subseteq \mathcal{P}(p, t, r_{\mathrm{Rm}}(p,t)). \tag{5.2.4}$$

As above, if this claim is true, then we can deduce that $|\mathrm{Rm}| < r_{\mathrm{Rm}}^{-2}(p,t) < r^{-2}$ on $\mathcal{P}(p,s,r)$ from the definition of $r_{\mathrm{Rm}}(p,t)$, therefore by definition of $r_{\mathrm{Rm}}(p,s)$ we obtain $r_{\mathrm{Rm}}(p,s) \geq r$, in contradiction with the definition of r, concluding the proof.

It remains to verify the claim (5.2.4). We first check the time intervals. Note that we can consider the time intervals in $\mathbb{R}$ rather than $[0,T)$, therefore omitting the truncation at 0 and T as this inclusion clearly implies the one between the truncated intervals. By (5.2.3), and using $C_1 \geq 1$, we can estimate

$$2C_1 r_{\mathrm{Rm}}(p,t)\sqrt{|t-s|} > 2C_1^2|t-s| \geq C_1^2|t-s| - t + s,$$

as well as

$$2C_1 r_{\mathrm{Rm}}(p,t)\sqrt{|t-s|} > 2C_1^2|t-s| \geq C_1^2|t-s| + t - s.$$

Therefore, we obtain

$$\begin{aligned} s + r^2 &= s + r_{\mathrm{Rm}}^2(p,t) + C_1^2|t-s| - 2C_1 r_{\mathrm{Rm}}(p,t)\sqrt{|t-s|} \leq t + r_{\mathrm{Rm}}^2(p,t), \\ s - r^2 &= s - r_{\mathrm{Rm}}^2(p,t) - C_1^2|t-s| + 2C_1 r_{\mathrm{Rm}}(p,t)\sqrt{|t-s|} \geq t - r_{\mathrm{Rm}}^2(p,t), \end{aligned}$$

and hence $(s-r^2, s+r^2) \subseteq (t - r_{\mathrm{Rm}}^2(p,t), t + r_{\mathrm{Rm}}^2(p,t))$. To prove $B_{g(s)}(p,r) \subseteq B_{g(t)}(p, r_{\mathrm{Rm}}(p,t))$, we recall that Lemma 5.2.1 implies

$$B_{g(t)}(p, r_{\mathrm{Rm}}(p,t)) \supseteq B_{g(s)}(p, r_{\mathrm{Rm}}(p,t) - C_1^2 r_{\mathrm{Rm}}^{-1}(p,t)|t-s|)$$

and hence we are done if we can show that $B_{g(s)}(p,r) \subseteq B_{g(s)}(p, r_{\mathrm{Rm}}(p,t) - C_1^2 r_{\mathrm{Rm}}^{-1}(p,t)|t-s|)$. To verify this, note that, again using (5.2.3), we can estimate

$$C_1^2 r_{\mathrm{Rm}}^{-1}(p,t)|t-s| < C_1\sqrt{|t-s|}$$

and therefore

$$r_{\mathrm{Rm}}(p,t) - C_1^2 r_{\mathrm{Rm}}^{-1}(p,t)|t-s| > r_{\mathrm{Rm}}(p,t) - C_1\sqrt{|t-s|} = r.$$

This finishes the proof of Step 2.

Step 3: Finishing the proof of Theorem 5.2.2 is now straight-forward: From the above two steps, an easy application of the triangle inequality both in space and time yields (5.2.2) with the constant $C_1 = 2\sqrt[4]{2/3}\sqrt{n-1}$. □

A first corollary of this theorem gives upper and lower bounds for r_{Rm}^{-2} on a parabolic cylinder in terms of its value at the center of this cylinder.

Corollary 5.2.3 (Local Harnack-Type Inequality)**.** *Suppose $(M, g(t))$ is a complete Ricci flow maximally defined on $[0,T)$, and let $(p,t) \in M \times (0,T)$. Then there exists a dimensional constant*

$a_1 = a_1(n) \in (0,1)$ *such that*

$$\frac{1}{4} r_{\mathrm{Rm}}^{-2}(p,t) \leq r_{\mathrm{Rm}}^{-2}(\cdot,\cdot) \leq 4 r_{\mathrm{Rm}}^{-2}(p,t) \quad \text{on } \mathcal{P}(p,t,a_1\, r_{\mathrm{Rm}}(p,t)) \tag{5.2.5}$$

Proof. Let $(q,s) \in \mathcal{P}(p,t,a_1\, r_{\mathrm{Rm}})$, where $a_1 < 1$ will be determined later. We deduce from Theorem 5.2.2 that

$$|r_{\mathrm{Rm}}(p,t) - r_{\mathrm{Rm}}(q,s)| \leq d_{g(t)}(p,q) + C_1|t-s|^{\frac{1}{2}} \leq a_1\, r_{\mathrm{Rm}}(p,t) + C_1 a_1\, r_{\mathrm{Rm}}(p,t).$$

Rearranging this inequality as

$$(1 + a_1 + a_1 C_1) r_{\mathrm{Rm}}(p,t) \geq r_{\mathrm{Rm}}(q,s)$$

we see that the first inequality in (5.4.9) is implied by

$$1 + a_1 + a_1 C_1 \leq 2 \iff a_1 \leq \frac{1}{1+C_1}.$$

Similarly, rearranging differently, we see that

$$(1 - a_1 - a_1 C_1) r_{\mathrm{Rm}}(p,t) \leq r_{\mathrm{Rm}}(q,s),$$

therefore for the second inequality in (5.4.9) it is sufficient to impose

$$1 - a_1 - a_1 C_1 \geq \frac{1}{2} \iff a_1 \leq \frac{1}{2(1+C_1)}.$$

The claim hence follows by setting $a_1 := \frac{1}{2(1+C_1)}$ with C_1 as above. □

We are now ready to prove Theorem 5.1.4. Maybe slightly paradoxically, the most subtle argument is actually needed in the proof of Part i), because the definition of singular and regular points is not set in a parabolic way. Proving the other two statements instead is easier as our definitions of Type I and Type II singular points are already in terms of parabolic neighbourhoods.

Proof of Theorem 5.1.4. Let $(M, g(t))$ be a Ricci flow defined on $[0,T)$, $T < \infty$, such that $(M, g(t))$ has bounded curvature for every $t \in [0,T)$ (possibly incomplete) and let $p \in M$.

i) We prove the following equivalent statement to the claim in the theorem: p is singular if and only if $\liminf_{t \nearrow T} r_{\mathrm{Rm}}(p,t) = 0$.

   To this end, suppose first that p is regular, and let U be a neighbourhood of p and $C > 0$ a constant such that $|\mathrm{Rm}| \leq C$ on $U \times [0,T)$. Let $r_0 > 0$ be such that $B_{g(0)}(p, r_0) \subset\subset U$. The standard multiplicative distance distortion estimate gives the containment

$$B_{g(0)}(p,r_0) \supseteq B_{g(t)}(p, r_0 e^{-(n-1)Ct}) \supseteq B_{g(t)}(p,r_1), \tag{5.2.6}$$

where $r_1 := r_0 e^{-(n-1)CT} > 0$. From this we deduce $r_{\mathrm{Rm}}(p,t) \geq \min\{C^{-1/2}, r_1\} > 0$.

For the converse statement, suppose that there exists some constant $\delta > 0$ such that we have the bound $\liminf_{t\nearrow T} r_{\mathrm{Rm}}(p,t) > \delta$. By the curvature boundedness assumption, we can choose this δ so that we have $r_{\mathrm{Rm}}(p,t) > \delta$ for every $t \in [0,T)$; by definition this means that for every $t \in [\delta^2, T)$ we have $|\mathrm{Rm}| \leq \delta^{-2}$ on $\mathcal{P}(p,t,r_{\mathrm{Rm}}) \supseteq \mathcal{P}(p,t,\delta)$. Fixing any t in this range, we claim that there exists a constant $r_2 = r_2(\delta, n, T)$ such that

$$B_{g(t)}(p,\delta) \supseteq B_{g(\delta^2)}(p, r_2). \tag{5.2.7}$$

In particular, if this claim holds, we have found a (fixed) neighbourhood of p on which the curvature remains bounded (by δ^{-2}), and we can conclude the proof. Hence it remains to prove the claim. In order to do so, for the t we fixed above, we set $k_0 = k_0(t)$ to be the smallest integer such that $t - (k_0+1)\delta^2 < 0$. Notice that $k_0 \leq T/\delta^2$. We are going to implement an iterative scheme of inclusions using the distance distortion estimates in any of the cylinders considered. From the bound on the curvature we obtain

$$\begin{aligned} B_{g(t)}(p,\delta) &\supseteq B_{g(t-\delta^2)}\big(p, \delta e^{-(n-1)\delta^{-2}\cdot\delta^2}\big) = B_{g(t-\delta^2)}\big(p, \delta e^{-(n-1)}\big) \\ &\supseteq B_{g(t-2\delta^2)}\big(p, \delta (e^{-(n-1)})^2\big) = B_{g(t-2\delta^2)}\big(p, \delta e^{-2(n-1)}\big) \\ &\supseteq \ldots. \supseteq B_{g(t-(k_0-1)\delta^2)}\big(p, \delta e^{-(k_0-1)(n-1)}\big) \\ &\supseteq B_{g(\delta^2)}\big(p, \delta e^{-(n-1)[(k_0-1)+\delta^{-2}(t-(k_0-1)\delta^2-\delta^2)]}\big) \\ &\supseteq B_{g(\delta^2)}\big(p, \delta e^{-k_0(n-1)}\big) \supseteq B_{g(\delta^2)}\big(p, \delta e^{-(n-1)(T/\delta^2)}\big), \end{aligned}$$

and (5.2.7) follows by defining $r_2 := \delta e^{-(n-1)(T/\delta^2)}$.

ii) Let us first assume the lower bound in (5.1.1) and let $t_i \nearrow T$ be a sequence realising the lim sup. Then, setting $r := \max\{r_I, 1/\sqrt{c_I}\}$, we have for sufficiently large i

$$\frac{1}{r^2(T-t_i)} \leq \frac{c_I}{T-t_i} < \sup_{\mathcal{P}(p,t_i,r_I\sqrt{T-t_i})} |\mathrm{Rm}| \leq \sup_{\mathcal{P}(p,t_i,r\sqrt{T-t_i})} |\mathrm{Rm}|.$$

This means that we must have $r_{\mathrm{Rm}}(p,t_i) < r\sqrt{T-t_i}$ for all sufficiently large i, because if it was $r_{\mathrm{Rm}}(p,t_j) \geq r\sqrt{T-t_j} =: r'$ for some j large enough for the above inequality to hold, we would obtain

$$\frac{1}{r^2(T-t_j)} < \sup_{\mathcal{P}(p,t_j,r')} |\mathrm{Rm}| \leq \frac{1}{(r')^2} = \frac{1}{r^2(T-t_j)},$$

a contradiction. We therefore conclude that

$$r_{\mathrm{Rm}}(p,t_i)/\sqrt{T-t_i} < \max\{r_I, 1/\sqrt{c_I}\},$$

or equivalently

$$(T-t_i)r_{\mathrm{Rm}}^{-2}(p,t_i) > \min\{r_I^{-2}, c_I\} =: \widetilde{c}_I > 0, \tag{5.2.8}$$

which gives the lower bound we claimed. Supposing instead the upper bound of (5.1.1), for $\varepsilon > 0$, let us set $r := \min\{r_I, 1/\sqrt{C_I+\varepsilon}\}$, so that we can compute for any t sufficiently close to T that

$$\frac{1}{r^2(T-t)} \geq \frac{C_I+\varepsilon}{T-t} \geq \sup_{\mathcal{P}(p,t,r_I\sqrt{T-t})} |\mathrm{Rm}| \geq \sup_{\mathcal{P}(p,t,r\sqrt{T-t})} |\mathrm{Rm}|,$$

so by definition we have $r_{\mathrm{Rm}}(p,t) \geq r\sqrt{T-t}$. In particular, letting $\varepsilon \searrow 0$, we conclude that

$$\liminf_{t\nearrow T} r_{\mathrm{Rm}}(p,t)/\sqrt{T-t} \geq \min\{r_I, 1/\sqrt{C_I}\},$$

or equivalently

$$\limsup_{t\nearrow T}(T-t)r_{\mathrm{Rm}}^{-2}(p,t) \leq \max\{r_I^{-2}, C_I\} =: \widetilde{C}_I. \tag{5.2.9}$$

To prove the converse statement, assume now that $\widetilde{c}_I < \limsup_{t\nearrow T}(T-t)r_{\mathrm{Rm}}^{-2}(p,t)$ for some constant $0 < \widetilde{c}_I$. Set $r_I := (\widetilde{c}_I)^{-\frac{1}{2}}$ and $c_I := \widetilde{c}_I$. We claim that for this choice, the lower bound in (5.1.1) must hold. If not, then we have

$$\limsup_{t\nearrow T} \sup_{B_{\widetilde{g}_t(0)}(p,r_I)\times(-r_I^2,r_I^2)} |\mathrm{Rm}_{\widetilde{g}_t}|_{\widetilde{g}_t} \leq c_I,$$

that is, we have an upper bound as in (5.1.1), but with C_I now replaced by c_I. By what we have just proved above, we thus get the analogue of (5.2.9), namely

$$\limsup_{t\nearrow T}(T-t)r_{\mathrm{Rm}}^{-2}(p,t) \leq \max\{r_I^{-2}, c_I\} = \widetilde{c}_I,$$

which is the desired contradiction. Finally, for the last remaining statement, assume that $\limsup_{t\nearrow T}(T-t)r_{\mathrm{Rm}}^{-2}(p,t) \leq \widetilde{C}_I$. Then given any ε, we have for every t close enough to T that

$$r_{\mathrm{Rm}}(p,t) > \sqrt{\frac{T-t}{\widetilde{C}_I+\varepsilon}} =: r.$$

The definition of the Riemann scale thus implies

$$\sup_{\mathcal{P}(p,t,r)} |\mathrm{Rm}| \leq \sup_{\mathcal{P}(p,t,r_{\mathrm{Rm}})} |\mathrm{Rm}| = r_{\mathrm{Rm}}^{-2}(p,t) \leq r^{-2} = \frac{\widetilde{C}_I+\varepsilon}{T-t}, \tag{5.2.10}$$

which for $\varepsilon \searrow 0$ and after rescaling parabolically gives the upper bound in (5.1.1) with constants $r_I := (\widetilde{C}_I)^{-\frac{1}{2}}$ and $C_I := \widetilde{C}_I$.

iii) The argument used in the first paragraph of the proof of Part *ii)* also works with Type II points, that is verifying (5.1.2), and one shows that for every $r > 0$ and every $c_I > 0$ the analogue of (5.2.8) holds, namely there exists a sequence of $t_i \nearrow T$ such that

$$(T - t_i) r_{\mathrm{Rm}}^{-2}(p, t_i) > \min\{r^{-2}, c_I\}.$$

We can therefore let $r \searrow 0$ and $c_I \nearrow \infty$ to conclude that $\limsup_{t \nearrow T} (T-t) r_{\mathrm{Rm}}^{-2}(p,t) = \infty$.

Conversely, suppose we have $\limsup_{t \nearrow T} (T-t) r_{\mathrm{Rm}}^{-2}(p,t) = \infty$. If (5.1.2) does not hold for all $r > 0$, then there exists some $r_I > 0$ and some constant C_I such that the upper bound in (5.1.1) holds. But we have just proven that this implies $\limsup_{t \nearrow T} (T-t) r_{\mathrm{Rm}}^{-2}(p,t) \le \widetilde{C}_I$, a contradiction. This finishes the proof of the theorem. □

With a slight modification of Part *i)* of Theorem 5.1.4 we obtain a non-oscillation result which in particular, says that, to a certain extent, the curvature of a Ricci flow cannot oscillate between a Type I rate and a lower rate arbitrarily close to the singular time.

Corollary 5.2.4 (Type I non-Oscillation)**.** *Suppose $(M, g(t))$ is a Ricci flow maximally defined on a finite time interval $[0,T)$, with either bounded curvature or complete time-slices. Then $p \in \Sigma$ if and only if*

$$r_{\mathrm{Rm}}^{-2}(p,t) > \frac{1}{T-t}, \quad \forall t \in [0,T). \tag{5.2.11}$$

Proof. If (5.2.11) holds, then by Part *i)* of Theorem 5.1.4 the point p must be singular. Conversely, assume towards a contradiction that $p \in \Sigma$, but that

$$\delta_0 := r_{\mathrm{Rm}}(p, t_0) \ge \sqrt{T - t_0}, \quad \text{for some } t_0 \in [0,T).$$

By definition of the Riemann scale, this means in particular that

$$|\mathrm{Rm}| \le \delta_0^{-2}, \quad \text{on } \mathcal{P}(p, t_0, \delta_0) \supseteq B_{g(t_0)}(p, \delta_0) \times [t_0, T). \tag{5.2.12}$$

Set $U := B_{g(t_0)}(p, \delta_0)$. If the flow has bounded curvature time-slices, we can choose $\delta \le \delta_0$ such that $|\mathrm{Rm}| \le \delta^{-2}$ on $U \times [0, t_0]$. Otherwise, by the completeness assumption, the continuous function $|\mathrm{Rm}|$ is less than or equal to δ^{-2} for some $\delta \le \delta_0$ on the compact set $U \times [0, t_0]$. Hence in any case, combining this with (5.2.12), $|\mathrm{Rm}| \le \delta^{-2}$ on $U \times [0,T)$ and therefore by definition $p \in \mathfrak{Reg}$, yielding the desired contradiction. □

The decomposition of the singular set given by Theorem 5.1.2 follows very easily now.

Proof of Theorem 5.1.2. From the equivalent definitions of Type I and Type II singular points given by Theorem 5.1.4, it is clear that if $p \in \Sigma_I \cup \Sigma_{II}$ then $p \in \Sigma$.

Conversely, if $p \in \Sigma$, then by Corollary 5.2.4 we have

$$r_{\mathrm{Rm}}^{-2}(p,t) > \frac{1}{T-t}, \quad \forall t \in [0,T),$$

and therefore in particular

$$\limsup_{t \nearrow T} (T-t) r_{\mathrm{Rm}}^{-2}(p,t) \geq 1 > 0.$$

By Theorem 5.1.4, we therefore have $p \in \Sigma_I \cup \Sigma_{II}$. □

As an immediate consequence of Theorem 5.1.2, we find the following corollary which is equivalent to Theorem 3.2 in [35].

Corollary 5.2.5. *Let $(M, g(t))$ be a Type I Ricci flow on a manifold M of dimension n, maximally defined on $[0,T)$, $T < +\infty$. Then $\Sigma = \Sigma_I$.*

Next, we give some evidence for the following conjecture.

Conjecture 5.2.6. *In Theorem 5.1.4, we can replace each instance of $\limsup_{t\nearrow T}$ with $\liminf_{t\nearrow T}$.*

A full proof of this conjecture would require ruling out significant oscillations of the curvature between Type I and Type II rates. Intuitively, such an oscillatory behaviour would be in extreme contrast with the parabolic nature of the Ricci flow; nevertheless, it is not clear to the author how to prevent it using only the differential inequalities on the curvature tensor given by Shi's estimates and even the Hölder continuity proved for the Riemannian scale is not strong enough to rule this out completely. The following result is the best we can currently prove in this direction. It studies the convergence rates of an essential blow-up sequence (p_i, t_i) along which the curvature blows-up at a Type I rate but which converges to a Type II singular point.

Proposition 5.2.7. *Let $(M, g(t))$ be a complete Ricci flow on a manifold M of dimension n, maximally defined on a finite time interval $[0,T)$ and with bounded curvature time-slices. Suppose $p \in \Sigma_{II}$, and let (p_i, t_i) be an essential blow-up sequence, with $p_i \to p$ in the topology of $(M, g(0))$, $t_i \nearrow T$, and such that r_{Rm}^{-2} blows-up at a Type I rate along (p_i, t_i), i.e.*

$$r_{\mathrm{Rm}}^{-2}(p_i, t_i) \leq \frac{1}{m^2(T-t_i)}$$

for some $m \in (0,1)$. Then there exists $\delta = \delta(n,m) > 0$ such that

$$\limsup_{i\to\infty} \frac{T-t_{i-1}}{T-t_i} \geq 1+\delta \qquad \textit{or} \qquad \limsup_{i\to+\infty} \frac{d^2_{g(t_i)}(p,p_i)}{T-t_i} \geq \delta.$$

Proof. Suppose instead that

$$\limsup_{i\to+\infty}\frac{T-t_{i-1}}{T-t_i}=1 \qquad \text{and} \qquad \limsup_{i\to+\infty}\frac{d^2_{g(t_i)}(p,p_i)}{T-t_i}=0. \tag{5.2.13}$$

Since by hypothesis $r^2_{\mathrm{Rm}}(p_i,t_i)\geq m^2(T-t_i)$, we infer

$$\bigcup_{i\in\mathbb{N}}\big(t_i-a_1\,r^2_{\mathrm{Rm}}(p_i,t_i),t_i+a_1\,r^2_{\mathrm{Rm}}(p_i,t_i)\big)\supseteq\bigcup_{i\in\mathbb{N}}\big(t_i-m^2a_1^2(T-t_i),t_i+m^2a_1^2(T-t_i)\big).$$

We first check that the set on the right hand side contains (t_{i_1},T) for some i_1. Indeed, the intervals in consideration overlap definitively as

$$t_{i-1}+m^2a_1^2(T-t_{i-1})\geq t_i-m^2a_1^2(T-t_i)\iff T-t_{i-1}\leq\frac{1+m^2a_1^2}{1-m^2a_1^2}(T-t_i),$$

which is satisfied by (5.2.13) for any i large enough. On the other hand, we also have

$$B_{g(t_i)}(p_i,a_1\,r_{\mathrm{Rm}}(p_i,t_i))\supseteq B_{g(t_i)}(p_i,a_1m\sqrt{T-t_i})\ni p$$

for i large enough by (5.2.13). Therefore, for any sequence $\xi_j\nearrow T$ and for every j large enough, there exists $i=i(j)$ such that

$$(p,\xi_j)\in\mathcal{P}(p_i,t_i,a_1\,r_{\mathrm{Rm}}(p_i,t_i)).$$

We can therefore appeal to the local Harnack inequality of Corollary 5.2.3 to compare the values $r^2_{\mathrm{Rm}}(p,\xi_j)$ to $r^2_{\mathrm{Rm}}(p_i,t_i)$, where i depends on j.

Since $p\in\Sigma_{II}$, by Theorem 5.1.4 we have $r^2_{\mathrm{Rm}}(p,\xi_j)=\mathfrak{o}(T-\xi_j)$ along a sequence of times $\xi_j\nearrow T$. From the discussion above we can apply Corollary 5.2.3 to obtain

$$\mathfrak{o}(T-\xi_j)=4r^2_{\mathrm{Rm}}(p,\xi_j)\geq r^2_{\mathrm{Rm}}(p_i,t_{i(j)})\geq m^2(T-t_{i(j)})\geq\frac{m^2}{1+m^2a_1^2}(T-\xi_j), \tag{5.2.14}$$

which gives a contradiction for j large enough. □

Remark 5.2.8. *The proposition above ensures in particular that for a point $p\in\Sigma_{II}$, the existence of a sequence $t_i\nearrow T$ along which $r^{-2}_{\mathrm{Rm}}(p,t_i)$ blows-up at a Type I rate forces the convergence to be exponential. We think that this property is in contrast to the Lipschitz continuity of $r_{\mathrm{Rm}}(p,\cdot)$. It could be instructive to notice that the same is a priori not true if we swap the roles of Type I and Type II. Indeed, consider the function*

$$f(t)=(T-t)^2+(T-t)|\sin(\ln(T-t))|. \tag{5.2.15}$$

This is a Lipschitz function, such that $f(t) = (T - t_k)^2$ on $t_k = T - \exp(-k\pi)$, and $f(t) \sim T - t$ elsewhere as $t \sim T$.

From a more optimistic point of view, if Conjecture 5.2.6 holds, it is natural to ask whether we must have

$$\liminf_{t \nearrow T} r_{\mathrm{Rm}}^{-2}(p,t)(T-t) = \limsup_{t \nearrow T} r_{\mathrm{Rm}}^{-2}(p,t)(T-t),$$

at any singular point $p \in \Sigma$. This would be coherent with the examples in the literature.

For completeness, let us also study the convergence of an essential blow-up sequence along which the curvature blows-up at a Type II rate but which converges to a Type I singular point p. From Definition 5.1.1, it is clear that this convergence cannot be too fast. The following proposition makes this more precise, giving an explicit relation between the blow-up rate of $r_{\mathrm{Rm}}^{-2}(p,\cdot)$ and the convergence rate.

Proposition 5.2.9. *Let $(M, g(t))$ be a complete Ricci flow on a manifold M of dimension n, maximally defined on a finite time interval $[0, T)$ and with bounded curvature time-slices. Let $p \in \Sigma_I$, meaning in particular that there exists $m \in (0,1)$ such that $r_{\mathrm{Rm}}^{-2}(p,t) \leq \frac{1}{m^2(T-t)}$ for all $t \in [0,T)$. Suppose further that (p_i, t_i) is an essential blow-up sequence, with $p_i \to p$ in the topology of $(M, g(0))$, $t_i \nearrow T$, and such that r_{Rm}^{-2} blows-up at a Type II rate along (p_i, t_i), i.e.*

$$r_{\mathrm{Rm}}^{2}(p_i,t_i) = \mathfrak{o}(T - t_i).$$

Then we obtain

$$\limsup_{i\to\infty} \frac{d^2_{g(t_i)}(p,p_i)}{T-t_i} \geq m^2 a_1^2,$$

where $a_1 = a_1(n)$ is the constant from Corollary 5.2.3.

Proof. Suppose by contradiction that

$$\limsup_{i\to\infty} \frac{d^2_{g(t_i)}(p,p_i)}{T-t_i} < m^2 a_1^2,$$

so that we obtain

$$B_{g(t_i)}(p, a_1\, r_{\mathrm{Rm}}(p,t_i)) \supseteq B_{g(t_i)}(p, a_1 m\sqrt{T-t_i}) \ni p_i$$

for i large enough. This means in particular that

$$(p_i,t_i) \in \mathcal{P}(p, t_i, a_1\, r_{\mathrm{Rm}}(p,t_i)),$$

for i large enough and we can therefore use the local Harnack-type inequality of Corollary 5.2.3 to compare the values $r^2_{\mathrm{Rm}}(p_i, t_i)$ to $r^2_{\mathrm{Rm}}(p, t_i)$ to obtain

$$\mathfrak{o}(T - t_i) = 4r^2_{\mathrm{Rm}}(p_i, t_i) \geq r^2_{\mathrm{Rm}}(p, t_i) \geq m^2(T - t_i)$$

which gives a contradiction for i large enough. □

We conclude this section by establishing a link between the two different Riemann scales defined in Definition 5.1.3. As already said, these two scales are equivalent for bounded scalar curvature Ricci flows in view of the pseudolocality result in Proposition 3.2 of [7]. For a general Ricci flow, one should not expect such a strong result, but a weaker infinitesimal analogue holds true.

Proposition 5.2.10 (Characterisation of Singular Set using Fixed Time Slice Scale)**.** *Let $(M, g(t))$ be a Ricci flow defined on $[0, T)$, $T < \infty$, such that $(M, g(t))$ is complete and has bounded curvature for every $t \in [0, T)$. Then $p \in \Sigma$ if and only if $\liminf_{t \nearrow T} \widetilde{r}_{\mathrm{Rm}}(p, t) = 0$.*

Proof. The implication $p \in \mathfrak{Reg} \Rightarrow \liminf_{t\nearrow T} \widetilde{r}_{\mathrm{Rm}}(p, t) > 0$ follows the exact same lines as in the proof of Part $i)$ of Theorem 5.1.4.

Conversely, assume that for a point $p \in M$ there exists some constant $\delta > 0$ such that $\liminf_{t\nearrow T} \widetilde{r}_{\mathrm{Rm}}(p, t) > \delta$. Since the flow has bounded curvature time-slices, for a possibly smaller δ we have $\widetilde{r}_{\mathrm{Rm}}(p, t) > \delta$ for every $t \in [0, T)$; by definition of time-slice Riemann scale, this means that for every $t \in [0, T)$ we have $|\mathrm{Rm}| \leq \delta^{-2}$ on $B_{g(t)}(p, \widetilde{r}_{\mathrm{Rm}}(p, t)) \supseteq B_{g(t)}(p, \delta)$. Thus we obtain

$$|\mathrm{Rm}| \leq \delta^{-2} \quad \text{on} \quad \bigcup_{t\in[0,T)} B_{g(t)}(p, \delta) \times \{t\}.$$

We claim that there exists a constant $a_4 = a_4(n)$ such that for any time $t_0 \in [0, T)$ we have $r_{\mathrm{Rm}}(p, t_0) > a_4\delta$. Once we have proven this, we infer that $\liminf_{t\nearrow T} r_{\mathrm{Rm}}(p, t) \geq a_4\delta$, and we can conclude the proof thanks to Theorem 5.1.4, Part $i)$. Using the Expanding balls Lemma 3.1 in [75] with $R = \delta$ and $r = \frac{\delta}{2}$, the lower bound $\mathrm{Ric}_{g(t)} \geq -(n - 1)\delta^{-2}g(t)$ on the balls $B_{g(t)}(p, \delta)$ ensures

$$B_{g(t_0)}(p, \tfrac{\delta}{2}) \subseteq B_{g(t)}(p, \delta), \quad \forall t \in \Big[t_0, \min\Big\{t_0 + \tfrac{\delta^2}{n-1}\log(2), T\Big\}\Big).$$

On the other hand, the upper bound $\mathrm{Ric}_{g(t)} \leq (n - 1)\delta^{-2}g(t)$ on the balls $B_{g(t)}(p, \delta)$ guarantees that we are in the hypothesis of the Shrinking balls Lemma 3.2 in [75] with our t being their initial time 0, $r = \delta$ and $f = \delta^{-1}$, so that we obtain

$$B_{g(t_0)}(p, \tfrac{\delta}{2}) \subseteq B_{g(t_0)}(p, \delta - C_1^2(t_0 - t)\delta^{-1}) \subseteq B_{g(t)}(p, \delta), \quad \forall t \in \Big(\max\Big\{0, t_0 - \tfrac{\delta^2}{2C_1^2}\Big\}, t_0\Big].$$

Recall that their constant $\beta/2$ is equal to our C_1^2. Therefore, we have obtained the inclusion

$$\mathcal{P}(p,t_0,a_4\delta)\subseteq\bigcup_t B_{g(t)}(p,\delta)\times\{t\},$$

where the union is taken over $t\in(\max\{t_0-a_4^2\delta^2,0\},\min\{t_0+a_4^2\delta^2,T\})$ and where $a_4=a_4(n):=\min\big\{\sqrt{\frac{\log(2)}{n-1}},\frac{1}{\sqrt{2}C_1},\frac{1}{2}\big\}$. By definition of the Riemann scale, we see that $r_{\mathrm{Rm}}(p,t_0)>a_4\delta$, as we wanted to prove. □

5.3 Integral Characterisation of the Singular Sets

The Harnack-type inequality proved in Corollary 5.2.3 implies an integral concentration of the curvature.

Theorem 5.3.1 (Integral Curvature Concentration)**.** *Let $\kappa>0$ and let (α,β) be an optimal pair of integrability exponents as in Definition 4.1.1. Then there exist constants $C_2=C_2(n,\kappa,\alpha)>0$ and $C_3=C_3(n)$ such that the following holds. Let $(M,g(t))$ be a complete Ricci flow defined on $[0,T)$, $T<\infty$, which is κ-non-local-collapsed on a scale ϱ in the sense of Definition 2.4.9. Then for a space-time point $(p,t)\in\Sigma\times(T-\varrho^2,T)$, we have the integral bounds*

$$C_2\leq\|r_{\mathrm{Rm}}^{-2}\|_{\alpha,\beta,\mathcal{P}(p,t,a_1\,r_{\mathrm{Rm}})}\leq C_3,\tag{5.3.1}$$

where $a_1\in(0,1)$ is the constant from Corollary 5.2.3.

This can be seen as an ε-regularity theorem since the lower bound in (5.3.1) shows that if $\|r_{\mathrm{Rm}}^{-2}\|_{\alpha,\beta,\mathcal{P}(p,t,a_1\,r_{\mathrm{Rm}})}\leq\varepsilon<C_2$ as $t\to T$, then p must be a regular point.

Proof. Since $p\in\Sigma$, Corollary 5.2.4 implies that $r_{\mathrm{Rm}}^2(p,t)\leq(T-t)$. On the one hand this implies that $r_{\mathrm{Rm}}(p,t)<\varrho$ for large enough t so that we can use the κ-noncollapsing property for balls of radius $r\leq r_{\mathrm{Rm}}(p,t)$. On the other hand, it also shows that

$$\begin{aligned}t+a_1^2r_{\mathrm{Rm}}^2(p,t)&\leq t+a_1^2(T-t)<t+\frac{1}{2}(T-t)<T,\\ t-a_1^2r_{\mathrm{Rm}}^2(p,t)&\geq t-a_1^2(T-t)>t-\frac{1}{2}(T-t)>0.\end{aligned}\tag{5.3.2}$$

In particular, we have

$$\mathcal{P}(p,t,a_1\,r_{\mathrm{Rm}}(p,t))=B_{g(t)}(p,a_1\,r_{\mathrm{Rm}}(p,t))\times(t-a_1^2r_{\mathrm{Rm}}^2(p,t),t+a_1^2r_{\mathrm{Rm}}^2(p,t))\tag{5.3.3}$$

and we do not need to worry about the truncation in (5.1.6).

We now first prove the upper bound. By definition of the Riemann scale, we have a Riemann upper bound on $\mathcal{P}(p,t,r_{\mathrm{Rm}})$, so after rescaling and using the Bishop-Gromov inequality ([28]) we obtain

$$\mu_{g(t)}(B_{g(t)}(p,a_1\,r_{\mathrm{Rm}}(p,t))) \le a_1^n r_{\mathrm{Rm}}^n(p,t)\mu_{g_{hyp}}(B_{hyp}) = a_1^n C_H(n) r_{\mathrm{Rm}}^n(p,t),$$

where B_{hyp} denotes a unitary ball in the hyperbolic space, and $C_H(n) = \int_0^1 \sinh^{n-1}(s)ds$ is its volume. Considering the evolution equation of the volume element under Ricci flow, and using the curvature bound in the region $\mathcal{P}(p,t,r_{\mathrm{Rm}})$, for every time $s \in (t - a_1^2 r_{\mathrm{Rm}}^2(p,t), t + a_1^2 r_{\mathrm{Rm}}^2(p,t))$ we deduce that

$$\begin{aligned}\mu_{g(s)}(B_{g(t)}(p,a_1\,r_{\mathrm{Rm}}(p,t))) &\le e^{n(n-1)r_{\mathrm{Rm}}^{-2}(p,t)(t-s)}\mu_{g(t)}(B_{g(t)}(p,a_1\,r_{\mathrm{Rm}}(p,t)))\\ &\le e^{n(n-1)a_1^2} a_1^n C_H(n) r_{\mathrm{Rm}}^n(p,t)\\ &= C(n) a_1^n r_{\mathrm{Rm}}^n(p,t).\end{aligned} \tag{5.3.4}$$

Therefore, using the upper bound given by Corollary 5.2.3, we compute

$$\begin{aligned}\|r_{\mathrm{Rm}}^{-2}\|_{\alpha,\beta,\mathcal{P}(p,t,a_1\,r_{\mathrm{Rm}})} &= \left(\int_{t-a_1^2 r_{\mathrm{Rm}}^2}^{t+a_1^2 r_{\mathrm{Rm}}^2}\left(\int_{B_{g(t)}(p,a_1\,r_{\mathrm{Rm}}(p,t))} |r_{\mathrm{Rm}}^{-2}|^\alpha \, d\mu_s\right)^{\beta/\alpha} ds\right)^{1/\beta}\\ &\le \left(\int_{t-a_1^2 r_{\mathrm{Rm}}^2}^{t+a_1^2 r_{\mathrm{Rm}}^2}\left(4^\alpha C(n) a_1^n r_{\mathrm{Rm}}(p,t)^{n-2\alpha}\right)^{\beta/\alpha} ds\right)^{1/\beta}\\ &= 4\cdot 2^{1/\beta} C(n)^{1/\alpha} a_1^{n/\alpha+2/\beta} r_{\mathrm{Rm}}(p,t)^{n/\alpha+2/\beta-2}\\ &= 4\cdot 2^{1/\beta} C(n)^{1/\alpha} a_1^2 \le 8C(n) a_1^2 =: C_3(n).\end{aligned} \tag{5.3.5}$$

Here we used in particular that $\frac{n}{\alpha} + \frac{2}{\beta} - 2 = 0$ for an optimal pair.

The proof of the opposite inequality follows a similar argument. Since $|\mathrm{Rm}|$ is bounded by $r_{\mathrm{Rm}}^{-2}(p,t)$ on $\mathcal{P}(p,t,r_{\mathrm{Rm}})$ and $a_1 \in (0,1)$, we can use the κ-noncollapsedness of the flow to obtain

$$\mu_{g(t)}(B_{g(t)}(x,a_1\,r_{\mathrm{Rm}}(p,t))) \ge \kappa\, a_1^n\, r_{\mathrm{Rm}}^n(p,t).$$

Again, the evolution of the volume element under Ricci flow and the curvature bound in the cylinder considered yield the inequality

$$\begin{aligned}\mu_{g(s)}(B_{g(t)}(x,a_1\,r_{\mathrm{Rm}}(p,t))) &\ge e^{-n(n-1)r_{\mathrm{Rm}}^{-2}(p,t)(t-s)}\mu_{g(t)}(B_{g(t)}(x,a_1\,r_{\mathrm{Rm}}(p,t)))\\ &\ge \kappa\, e^{-n(n-1)a_1^2} a_1^n\, r_{\mathrm{Rm}}^n(p,t)\\ &= c(n,\kappa) a_1^n r_{\mathrm{Rm}}^n(p,t),\end{aligned} \tag{5.3.6}$$

for every $s \in (t - a_1^2 r_{\mathrm{Rm}}^2(p,t), t + a_1^2 r_{\mathrm{Rm}}^2(p,t))$. Corollary 5.2.3 guarantees the lower bound on the integrand $r_{\mathrm{Rm}}^{-2} \ge \frac{1}{4} r_{\mathrm{Rm}}^{-2}(p,t)$ on the region $\mathcal{P}(p,t,a_1\,r_{\mathrm{Rm}})$. We can therefore follow (5.3.5), reversing all inequalities except the very last one, to obtain

$$\|r_{\mathrm{Rm}}^{-2}\|_{\alpha,\beta,\mathcal{P}(p,t,a_1\, r_{\mathrm{Rm}})} \geq \frac{1}{4}\cdot 2^{1/\beta} c(n,\kappa)^{1/\alpha} a_1^2 \geq \frac{1}{4} c(n,\kappa)^{1/\alpha} a_1^2 =: C_2(n,\kappa,\alpha). \qquad \square$$

The proof of Theorem 5.1.6 follows a very similar argument to what we have just seen. Notice, that since the integrand considered in the definition of the singular density function (Definition 5.1.5) is space-independent we simply get

$$\Big\|\frac{1}{T-s}\Big\|_{\alpha,\beta,\mathcal{P}(p,t,a_1\, r_{\mathrm{Rm}})} = \left(\int_{t-a_1^2 r_{\mathrm{Rm}}^2}^{t+a_1^2 r_{\mathrm{Rm}}^2} \Big(\frac{1}{T-s}\Big)^{\beta} \mu_{g(s)}^{\beta/\alpha}\Big(B_{g(t)}(p, a_1\, r_{\mathrm{Rm}}(p,t))\Big)\, ds\right)^{1/\beta} \tag{5.3.7}$$

if $(t-a_1^2 r_{\mathrm{Rm}}^2, t+a_1^2 r_{\mathrm{Rm}}^2) \subseteq (0,T)$. We proceed with the proof.

Proof of Theorem 5.1.6. According to Perelman's noncollapsing theorem [65], there exists a constant $\kappa = \kappa(n, g(0), T) > 0$ such that the Ricci flow in consideration is κ-noncollapsed at scale $\varrho = \sqrt{T}$. Hence, using (5.3.4) and (5.3.6), we obtain

$$c(n,\kappa) a_1^n r_{\mathrm{Rm}}^n(p,t) \leq \mu_{g(s)}\big(B_{g(t)}(p, a_1\, r_{\mathrm{Rm}}(p,t))\big) \leq C(n) a_1^n r_{\mathrm{Rm}}^n(p,t), \tag{5.3.8}$$

for some constants $c(n,\kappa) > 0$ and $C(n) < \infty$, and where the lower bound requires $r_{\mathrm{Rm}} < \varrho$.

As in the proof of the previous theorem, if $p \in \Sigma$ we know that $r_{\mathrm{Rm}}^2(p,t) \leq (T-t)$ from Corollary 5.2.4 and therefore (5.3.8) holds for sufficiently large t and we also have (5.3.2) and (5.3.3). From (5.3.2) we obtain in particular also

$$\frac{2}{3(T-t)} \leq \frac{1}{T-s} \leq \frac{2}{T-t}, \qquad \forall s \in (t - a_1^2 r_{\mathrm{Rm}}^2(p,t), t + a_1^2 r_{\mathrm{Rm}}^2(p,t)). \tag{5.3.9}$$

Using (5.3.7), we can therefore estimate

$$\begin{aligned}
\Big\|\frac{1}{T-s}\Big\|_{\alpha,\beta,\mathcal{P}(p,t,a_1\, r_{\mathrm{Rm}})} &\leq \left(\int_{t-a_1^2 r_{\mathrm{Rm}}^2}^{t+a_1^2 r_{\mathrm{Rm}}^2} \Big(\frac{1}{T-s}\Big)^{\beta} \Big(C(n) a_1^n r_{\mathrm{Rm}}^n(p,t)\Big)^{\beta/\alpha} ds\right)^{\frac{1}{\beta}} \\
&\leq C(n)^{1/\alpha} (a_1\, r_{\mathrm{Rm}}(p,t))^{n/\alpha} \left(\int_{t-a_1^2 r_{\mathrm{Rm}}^2}^{t+a_1^2 r_{\mathrm{Rm}}^2} \Big(\frac{1}{T-s}\Big)^{\beta} ds\right)^{\frac{1}{\beta}} \\
&\leq C(n)^{1/\alpha} (a_1\, r_{\mathrm{Rm}}(p,t))^{n/\alpha} (2 a_1^2 r_{\mathrm{Rm}}^2(p,t))^{1/\beta} \frac{2}{T-t}.
\end{aligned}$$

Using that for an optimal pair (α,β) we have $\frac{n}{\alpha} + \frac{2}{\beta} = 2$, we thus obtain

$$\Big\|\frac{1}{T-s}\Big\|_{\alpha,\beta,\mathcal{P}(p,t,a_1\, r_{\mathrm{Rm}})} \leq 4C(n)^{1/\alpha} a_1^2 \,\frac{r_{\mathrm{Rm}}^2(p,t)}{T-t}.$$

It is straightforward now to deduce from Theorem 5.1.4 that if $p \in \Sigma_I$, this is uniformly bounded from above by $4C(n)^{1/\alpha} a_1^2 (\widetilde{c}_I)^{-1}$ for every t close enough to T, whereas if $p \in \Sigma_{II}$ this gives $\Theta(p) = 0$. Analogously, the lower bounds in (5.3.8) and (5.3.9) yield

$$\begin{aligned}\Big\|\frac{1}{T-s}\Big\|_{\alpha,\beta,\mathcal{P}(p,t,a_1\, r_{\mathrm{Rm}})} &\geq \left(\int_{t-a_1^2 r_{\mathrm{Rm}}^2}^{t+a_1^2 r_{\mathrm{Rm}}^2} \Big(\frac{1}{T-s}\Big)^{\beta}\Big(c(n,\kappa) a_1^n r_{\mathrm{Rm}}^n(p,t)\Big)^{\beta/\alpha} ds\right)^{\frac{1}{\beta}} \\ &\geq c(n,\kappa)^{1/\alpha}(a_1\, r_{\mathrm{Rm}}(p,t))^{n/\alpha}(2a_1^2 r_{\mathrm{Rm}}^2(p,t))^{1/\beta}\frac{2}{3(T-t)} \\ &\geq \frac{2}{3} c(n,\kappa)^{1/\alpha} a_1^2\, \frac{r_{\mathrm{Rm}}^2(p,t)}{T-t}.\end{aligned}$$

If $p \in \Sigma_I$, then $\Theta(p)$ is bounded away from 0 as $\liminf_{t\nearrow T} \frac{r_{\mathrm{Rm}}^2(p,t)}{T-t} \geq (\widetilde{C}_I)^{-1}$ by Theorem 5.1.4.

It remains to show that $\Theta(p) = \infty$ for regular points. For $p \in \mathfrak{Reg}$, Theorem 5.1.4 ensures the existence of a constant $C > 0$ such that $r_{\mathrm{Rm}}^{-2}(p,t) \leq C^2$ for every $t \in [0,T]$. Thus for every $t \geq T - a_1^2/C^2$ we have $t + a_1^2 r_{\mathrm{Rm}}^2 > T$ and therefore

$$\mathcal{P}(p,t,a_1\, r_{\mathrm{Rm}}(p,t)) \supseteq B_{g(t)}\Big(p, \frac{a_1}{C}\Big) \times (t,T). \tag{5.3.10}$$

We can appeal to Perelman's noncollapsing theorem [65] to obtain the existence of a constant $\kappa = \kappa(n, g(0), T, C) > 0$ such that the Ricci flow in consideration is κ-noncollapsed at scale $\varrho = C^{-1}$. Therefore, since by (5.3.10)

$$|\mathrm{Rm}| \leq r_{\mathrm{Rm}}^{-2}(p,t) \leq \frac{C^2}{a_1^2} \qquad \text{on} \qquad B_{g(t)}\Big(p, \frac{a_1}{C}\Big) \times (t,T),$$

we have for every $t \geq T - \frac{a_1^2}{C^2}$ and every $s \in (t,T)$ the uniform bound

$$\mu_{g(s)}\Big(B_{g(t)}\Big(p, \frac{a_1}{C}\Big)\Big) \geq c(n,\kappa)\frac{a_1^n}{C^n} > 0.$$

Because $(T-s)^{-\beta} \notin L^1\big(t,T\big)$ for every $t \geq T - \frac{a_1^2}{C^2}$, we obtain

$$\begin{aligned}\Big\|\frac{1}{T-s}\Big\|_{\alpha,\beta,\mathcal{P}(p,t,a_1\, r_{\mathrm{Rm}})} &\geq \Big\|\frac{1}{T-s}\Big\|_{\alpha,\beta,B_{g(t)}(p,\frac{a_1}{C})\times(t,T)} \\ &\geq \Big(c(n,\kappa)\frac{a_1^n}{C^n}\Big)^{1/\alpha}\left(\int_t^T \frac{1}{(T-s)^{\beta}} ds\right)^{\frac{1}{\beta}} = +\infty,\end{aligned}$$

from which we clearly see $\Theta(p) = +\infty$. □

5.4 The Ricci Singular Sets

We begin this section with the definition of two different versions of Ricci scale, analogous to the definitions of the Riemann scales given in the introduction, Definition 5.1.3. Note that we always mark the fixed time-slice scales with a tilde in this Chapter to distinguish them from the forwards-backwards scales that are mainly used in the local singularity analysis.

Definition 5.4.1 (Ricci Scale). *Let $(M, g(t))$ be a Ricci flow defined on $[0,T)$ and consider a space-time point $(p,t) \in M \times [0,T)$.*

i) We define the Ricci scale $r_{\mathrm{Ric}}(p,t)$ *at (p,t) by*

$$r_{\mathrm{Ric}}(p,t) := \sup\{r > 0 \mid |\mathrm{Ric}| < a_0(n) r^{-2} \text{ on } \mathcal{P}(p,t,r)\}, \tag{5.4.1}$$

where $a_0(n) = \sqrt{n}(n-1)$ as before, or equivalently by

$$r_{\mathrm{Ric}}(p,t) := \sup\{r > 0 \mid -(n-1)r^{-2} g < \mathrm{Ric} < (n-1) r^{-2} g \text{ on } \mathcal{P}(p,t,r)\} \tag{5.4.2}$$

If $(M, g(t))$ is Ricci-flat for every $t \in [0,T)$, we set $r_{\mathrm{Ric}}(p,t) = +\infty$. Moreover, by slight abuse of notation, we may sometimes write $\mathcal{P}(p,t,r_{\mathrm{Ric}})$ for $\mathcal{P}(p,t,r_{\mathrm{Ric}}(p,t))$.

ii) The time-slice Ricci scale *at (p,t) is given by $\widetilde{r}_{\mathrm{Ric}}(p,t) = +\infty$ if the flow is Ricci flat, otherwise we set*

$$\widetilde{r}_{\mathrm{Ric}}(p,t) := \sup\{r > 0 \mid |\mathrm{Ric}| < a_0 r^{-2} \text{ on } B_{g(t)}(p,r)\}, \tag{5.4.3}$$

where $a_0 = a_0(n) := \sqrt{n}(n-1)$.

Remark 5.4.2. *It might maybe seem more natural to define the Ricci scale as*

$$\sup\{r > 0 \mid |\mathrm{Ric}| < r^{-2} \text{ on } \mathcal{P}(p,t,r)\} \tag{5.4.4}$$

but our normalisation is more convenient for the purpose of this Chapter for two main reasons. Firstly, because $|\mathrm{Rm}| < r^{-2}$ implies

$$-(n-1)r^{-2} g < \mathrm{Ric} < (n-1) r^{-2} g, \tag{5.4.5}$$

and thus $|\mathrm{Ric}| < a_0 r^{-2}$, we get the simple relation

$$r_{\mathrm{Ric}}(p,t) \geq r_{\mathrm{Rm}}(p,t) \tag{5.4.6}$$

for any space-time point (p,t). Secondly, it is in fact exactly the property (5.4.5) which is used in a variety of proofs in the local singularity analysis and hence this normalisation allows a unified approach. (On the other hand, using (5.4.4) would allow to work with constants that do not depend *on the dimension n, which could have advantages in other contexts.)*

Remark 5.4.3. *From the Pseudolocality Proposition 3.2 in [7] we see that $\widetilde{r}_{\mathrm{Rm}}$ and r_{Rm} are comparable for bounded scalar curvature Ricci flows. It is not clear whether a similar relation also holds for the Ricci scales, apart from the obvious estimate $\widetilde{r}_{\mathrm{Ric}}(p,t) \geq r_{\mathrm{Ric}}(p,t)$ that simply follows from the fact that the definition of $r_{\mathrm{Ric}}(p,t)$ requires a bound on a larger set.*

We first show that like the Riemann scale, also the Ricci scale is Lipschitz continuous in space and Hölder continuous in time, yielding in particular the following result.

Theorem 5.4.4 (Lipschitz-Hölder Continuity of Ricci Scale)**.** *Suppose $(M, g(t))$ is a complete Ricci flow defined on $[0,T)$. Then for any couple of space-time points (p,t) and (q,s) we have*

$$|r_{\mathrm{Ric}}(p,t) - r_{\mathrm{Ric}}(q,s)| \leq \min\{d_{g(t)}(p,q), d_{g(s)}(p,q)\} + C_1|t-s|^{\frac{1}{2}}, \tag{5.4.7}$$

where $C_1 = 2\sqrt[4]{2/3}\sqrt{n-1}$ as in Theorem 5.2.2.

Proof. We first note that if $(M, g(t))$ is a complete Ricci flow defined on $[0,T)$, then for any point $(p,t) \in M \times [0,T)$ we have the inclusion

$$B_{g(t)}(p, r_{\mathrm{Ric}}(p,t)) \supseteq B_{g(s)}(p, r_{\mathrm{Ric}}(p,t) - C_1^2 r_{\mathrm{Ric}}^{-1}(p,t)|s-t|) \tag{5.4.8}$$

for all $s \in (t - r_{\mathrm{Ric}}^2(p,t), t + r_{\mathrm{Ric}}^2(p,t))$. The proof of this inclusion is exactly the same as the one of Lemma 5.2.1, since it relies only on Ricci curvature bounds of the type (5.4.5).

We can now deduce the Lipschitz-Hölder continuity exactly as in the proof of Theorem 5.2.2. In fact, the only ingredient of the proof that does not rely on elementary estimates like the triangle inequality, is the use of (5.4.8) established above. □

A first corollary of this theorem gives upper and lower bounds for r_{Ric}^{-2} on a parabolic cylinder in terms of its value at the center of this cylinder. The proof is exactly the same as for the Riemann scale, exploiting the continuity from Theorem 5.4.4 above.

Corollary 5.4.5 (Local Ricci Harnack-Type Inequality)**.** *Suppose $(M, g(t))$ is a complete Ricci flow maximally defined on $[0,T)$, and let $(p,t) \in M \times (0,T)$. Then for $a_1 = \frac{1}{2(1+C_1)} \in (0,1)$ as in Corollary 5.2.3, we have that*

$$\frac{1}{4} r_{\mathrm{Ric}}^{-2}(p,t) \leq r_{\mathrm{Ric}}^{-2}(\cdot,\cdot) \leq 4 r_{\mathrm{Ric}}^{-2}(p,t) \quad \text{on } \mathcal{P}(p,t,a_1 r_{\mathrm{Ric}}(p,t)). \tag{5.4.9}$$

Similar to Definition 5.1.1, we can now define the following concepts of different types of *Ricci singular sets*, depending only on the Ricci curvature rather than the full Riemannian curvature tensor.

Definition 5.4.6 (Ricci Singular Points)**.** *Let $(M, g(t))$ be a Ricci flow maximally defined on the time interval $[0, T)$, $T < \infty$ and assume that $(M, g(t))$ has bounded Ricci curvature for all $t \in [0, T)$. For any fixed $t \in [0, T)$, we consider again the parabolically rescaled Ricci flow $\widetilde{g}_t(s)$ defined for $s \in [-\frac{t}{T-t}, 1)$ as $\widetilde{g}_t(s) := (T-t)^{-1} g(t + (T-t)s)$.*

i) We say that a point $p \in M$ is a Ricci singular point *if for any neighbourhood U of p, the Ricci curvature becomes unbounded on U as t approaches T. The* Ricci singular set Σ^{Ric} *is the set of all such points and its complement is the* Ricci regular set $\mathfrak{Reg}^{\mathrm{Ric}}$*.*

ii) We say that a point $p \in M$ is a Type I Ricci singular point *if there exist constants $c_I, C_I, r_I > 0$ such that we have*

$$a_0 c_I < \limsup_{t \nearrow T} \sup_{B_{\widetilde{g}_t(0)}(p, r_I) \times (-r_I^2, r_I^2)} |\mathrm{Ric}_{\widetilde{g}_t}|_{\widetilde{g}_t} \leq a_0 C_I. \tag{5.4.10}$$

We denote the set of such points by Σ_I^{Ric} and call it the Type I Ricci singular set*.*

iii) We say that a point p is a Type II Ricci singular point *if for any $r > 0$ we have*

$$\limsup_{t \nearrow T} \sup_{B_{\widetilde{g}_t(0)}(p, r) \times (-r^2, r^2)} |\mathrm{Ric}_{\widetilde{g}_t}|_{\widetilde{g}_t} = \infty. \tag{5.4.11}$$

We denote the set of such points by $\Sigma_{II}^{\mathrm{Ric}}$ and call it the Type II Ricci singular set*.*

We note that the upper bound in (5.4.10) is directly implied by the upper bound in (5.1.1). Obviously this is *not* true for the respective lower bounds.

We immediately obtain the following alternative characterisations of the different Ricci singular sets which should be compared to their Riemann counterparts in Theorem 5.1.4.

Theorem 5.4.7 (Alternative Characterisation of Ricci Singular Sets)**.** *Let $(M, g(t))$ be a Ricci flow on a manifold M of dimension n, maximally defined on $[0, T)$, $T < +\infty$ and with bounded Ricci curvature time-slices. Let Σ^{Ric}, Σ_I^{Ric}, and $\Sigma_{II}^{\mathrm{Ric}}$ be given by Definition 5.4.6. Then*

i) $p \in \Sigma^{\mathrm{Ric}}$ if and only if $\limsup_{t \nearrow T} r_{\mathrm{Ric}}^{-2}(p, t) = \infty$.

ii) $p \in \Sigma_I^{\mathrm{Ric}}$ if and only if for some $0 < \widetilde{c}_I, \widetilde{C}_I$ we have $\widetilde{c}_I < \limsup_{t \nearrow T} (T-t) r_{\mathrm{Ric}}^{-2}(p, t) \leq \widetilde{C}_I$.

iii) $p \in \Sigma_{II}^{\mathrm{Ric}}$ if and only if $\limsup_{t \nearrow T} (T-t) r_{\mathrm{Ric}}^{-2}(p, t) = \infty$.

Proof. The first part of the Theorem follows the same way as for the Riemann scale. The main point why this goes through is again that the proof relies on various applications of distance distortion estimates that only depend on the Ricci curvature bounds and not full Riemann bounds. The other parts of the proof can also be adopted almost verbatim, including the choices of all the constants – for instance (5.4.10) implies

$$0 < \widetilde{c}_I := \min\{r_I^{-2}, c_I\} < (T - t_i) r_{\mathrm{Ric}}^{-2}(p, t_i) \leq \max\{r_I^{-2}, C_I\} =: \widetilde{C}_I. \tag{5.4.12}$$

This works because our normalisations in (5.4.1) and (5.4.10) agree. □

This then implies a non-oscillation result for the Ricci curvature along a Ricci flow, stating that it cannot oscillate between a Type I rate and a lower rate arbitrarily close to the singular time. Once again, the proof is exactly the same as for the Riemann scale.

Corollary 5.4.8 (Type I non-Oscillation of Ricci scale)**.** *Suppose $(M, g(t))$ is a Ricci flow maximally defined on a finite time interval $[0,T)$, with either bounded Ricci curvature or complete time slices. Then $p \in \Sigma^{\mathrm{Ric}}$ if and only if*

$$r_{\mathrm{Ric}}^{-2}(p,t) > \frac{1}{T-t}, \quad \forall t \in [0,T). \tag{5.4.13}$$

Remark 5.4.9. *This result may not yet be an improvement over Theorem 1 in [84], but it is its combination with Theorem 5.1.8 that gives a clear improvement from a global gap to a local one.*

As for the Riemann scale, combining Theorem 5.4.7 with Corollary 5.4.8, we obtain the following decomposition of the Ricci singular set.

Corollary 5.4.10 (Decomposition of the Ricci Singular Set)**.** *Let $(M, g(t))$ be a Ricci flow on a manifold M of dimension n, maximally defined on $[0,T)$, $T < +\infty$, and such that $(M, g(t))$ has bounded Ricci curvature for every t in $[0,T)$. Then $\Sigma^{\mathrm{Ric}} = \Sigma_I^{\mathrm{Ric}} \cup \Sigma_{II}^{\mathrm{Ric}}$.*

As an immediate consequence of Corollary 5.4.10, we also find the following result.

Corollary 5.4.11. *Let $(M, g(t))$ be a Ricci flow on a manifold M of dimension n, maximally defined on $[0,T)$, $T < +\infty$. Suppose the Ricci tensor satisfies a Type I bound. Then $\Sigma^{\mathrm{Ric}} = \Sigma_I^{\mathrm{Ric}}$.*

We can now also prove the following ε-regularity type result.

Theorem 5.4.12 (Integral Ricci Curvature Concentration)**.** *Let $\kappa > 0$ and (α, β) be an optimal pair of integrability exponents as in Definition 4.1.1. Then there exist constants $C_2 = C_2(n, \kappa, \alpha) > 0$ and $C_3 = C_3(n)$ such that the following holds. Let $(M, g(t))$ be a complete Ricci flow defined on $[0,T)$, $T < \infty$, which is κ-non-local-collapsed on a scale ϱ relative to the scalar curvature. Then for a space-time point $(p,t) \in \Sigma \times (T - \varrho^2, T)$, we have the integral bounds*

$$C_2 \leq \|r_{\mathrm{Ric}}^{-2}\|_{\alpha,\beta,\mathcal{P}(p,t,a_1 r_{\mathrm{Ric}})} \leq C_3, \tag{5.4.14}$$

where $a_1 \in (0,1)$ is the constant from Corollary 5.2.3.

Proof. We note that, in addition to elementary estimates, the proof of Theorem 5.3.1 relies on the local Harnack type inequality for the Riemann scale (which holds with the same constants also for the Ricci scale), as well as distance and volume distortion estimates and Bishop-Gromov inequality in [28] (all of which only rely on Ricci rather than full Riemann curvature bounds). Hence the proof can be adopted verbatim, simply changing every instance of r_{Rm} to r_{Ric}. We leave it to the reader to check the details. □

In the remainder of this section, we focus on a localisation of the Sesum and Wang results, proving Theorem 5.1.8. Our proof adapts some arguments from Theorem 2 in Sesum [69] as well as ideas from Proposition 5.2 in Hein-Naber [46], combined with Theorem 5.4.7 above.

Proof of Theorem 5.1.8. First of all, the inequality (5.4.6) clearly implies, together with Theorem 5.4.7, that $\Sigma^{\mathrm{Ric}} \subseteq \Sigma$, so we only need to prove the opposite inclusion. In order to do so, we argue by contradiction and assume that there exists a point $p \in \Sigma$ such that $p \notin \Sigma^{\mathrm{Ric}}$. Using again Theorem 5.4.7, there exists a constant $\delta > 0$ such that $r_{\mathrm{Ric}}(p,t) > \delta$ for every $t \in [0,T]$ while by Theorem 5.1.4 there exists a sequence of times $t_i \nearrow T$ so that $r_{\mathrm{Rm}}(p,t_i) \leq \frac{1}{i} \to 0$. For any $i \in \mathbb{N}$, let q_i be a minimiser of the function w_i defined by

$$w_i(q) = w_{(p,t_i)}(q) := \frac{r_{\mathrm{Rm}}(q,t_i)}{d_{g(t_i)}(q, \partial B_{g(t_i)}(p,\delta))}, \tag{5.4.15}$$

on the set $B_{g(t_i)}(p,\delta)$. We clearly have $w_i(q_i) \leq w_i(p) = r_{\mathrm{Rm}}(p,t_i)/\delta \leq (i\delta)^{-1}$. As a consequence of the bounded curvature of the time-slices, we must also have that for every i, $r_{\mathrm{Rm}}(q_i,t_i) > 0$, and therefore $w_i(q_i) > 0$.

Set $r_i := r_{\mathrm{Rm}}(q_i,t_i)$ and consider the sequence of pointed rescaled Ricci flows $(M, g_i(t), q_i)$ defined by $g_i(t) := r_i^{-2} g(t_i + r_i^2 t)$ on $M \times [-r_i^{-2}t_i, r_i^{-2}(T-t_i))$. We first note that by definition of q_i, we have $r_i = r_{\mathrm{Rm}}(q_i,t_i) \leq r_{\mathrm{Rm}}(p,t_i)$ and thus, by Corollary 5.2.4, $r_i^{-2}(T-t_i) \geq 1$, hence the flows $g_i(t)$ exist at least for times $t \in [-1,1)$. By definition, they satisfy $r_{\mathrm{Rm}_i}(q_i,0) = 1$ for every i and by the scaling properties of the distance

$$d_i := \frac{1}{2} d_{g_i(0)}(q_i, \partial B_{g_i(0)}(p, \delta r_i^{-1})) = \frac{1}{2w_{(p,0)}(q_i)} \geq \frac{\delta i}{2} \to +\infty.$$

Notice that by definition of d_i we obtain $B_{g_i(0)}(q_i,d_i) \subseteq B_{g_i(0)}(p,\delta r_i^{-1}) = B_{g(t_i)}(p,\delta)$. Since for every $q \in B_{g_i(0)}(q_i,d_i)$ its $g_i(0)$-distance to $\partial B_{g_i(0)}(p,\delta r_i^{-1})$ is at least d_i, we deduce from the minimising property of q_i that

$$\frac{1}{2d_i} = w_{(p,t_i)}(q_i) \leq w_{(p,t_i)}(q) \leq \frac{r_{\mathrm{Rm}_i}(q,0)}{d_i} \iff r_{\mathrm{Rm}_i}(q,0) \geq \frac{1}{2}.$$

Perelman's non local-collapsing theorem [65] applied to any of the cylinders $\mathcal{P}(q, 0, r_{\mathrm{Rm}_i}(q, 0))$ with $q \in B_{g_i(0)}(q_i, d_i)$ as before, guarantees the existence of a uniform injectivity radius lower bound. Therefore, we can apply Topping's compactness theorem [79, Theorem 1.6] to extract a pointed smooth Cheeger-Gromov limit Ricci flow $(M_\infty, g_\infty(t), q_\infty)$, defined and complete in $M_\infty \times (-\frac{1}{4}, \frac{1}{4})$. To do so, we just need to check that for every $r > 0$ there exists (in Topping's notation) some $K(r) \in \mathbb{N}$ such that for every $i \geq K(r)$ the curvature is uniformly bounded by M on $B_{g_i(0)}(q_i, r)$. But this is obviously true with $M = 4$ and $K(r)$ such that $d_i \geq r$ for $i \geq K(r)$.

This limit flow inherits several properties. First of all, $r_{\mathrm{Rm}_\infty}(q_\infty, 0) = 1$ and $|\mathrm{Rm}_\infty| \leq 4$ on $M_\infty \times (-\frac{1}{4}, \frac{1}{4})$. Secondly, we deduce from $B_{g_i(0)}(q_i, d_i) \subseteq B_{g(t_i)}(p, \delta)$ that $|\mathrm{Ric}_i| \leq a_0 r_i^2 \delta^{-2}$, so the limit flow must satisfy $\mathrm{Ric}_\infty \equiv 0$, i.e. $g_\infty(t) \equiv g_\infty$ is a static Ricci flat metric. We have therefore reconducted the study to a situation similar to the case treated by Sesum [69].

For any $r > 0$, the smooth Cheeger-Gromov convergence ensures

$$\frac{\mu_{g_\infty}(B_{g_\infty}(q_\infty, r))}{\omega_n r^n} = \lim_{i \to +\infty} \frac{\mu_{g_i(0)}(B_{g_i(0)}(q_i, r))}{\omega_n r^n} = \lim_{i \to +\infty} \frac{\mu_{g(t_i)}(B_{g(t_i)}(q_i, r r_i))}{\omega_n (r r_i)^n}.$$

Given $\delta > 0$ as above, there exists $i_0(\delta, r)$ such that $r r_i < \delta$ and $t_i + \delta^2 > T$ for every $i \geq i_0$, therefore we can appeal to the bound $|\mathrm{Ric}| \leq a_0 \delta^{-2}$ to use the multiplicative distance distortion estimates in $B_{g(t_i)}(p, r r_i) \times [t_{i_0} - \delta^2, T)$ for every $i \geq i_0$. For any $\varepsilon > 0$ and a possibly even larger $i_1(\delta, \varepsilon)$, we have the following two conditions satisfied for all $i \geq i_1$

$$(t_i - t_{i_1}) \leq a_0^{-1} \delta^2 \varepsilon,$$

and

$$\frac{\mu_{g_\infty}(B_{g_\infty}(q_\infty, r))}{\omega_n r^n} \geq \frac{\mu_{g(t_i)}(B_{g(t_i)}(q_i, r r_i))}{\omega_n (r r_i)^n} - \frac{\varepsilon}{2}.$$

Using the distortion estimate we see that

$$B_{g(t_i)}(q_i, r r_i) \supseteq B_{g(t_{i_1})}(q_i, r r_i e^{-a_0 \delta^{-2}(t_i - t_{i_1})}) \supseteq B_{g(t_{i_1})}(q_i, r r_i c_\varepsilon).$$

Here $c_\varepsilon = e^{-\varepsilon} \to 1$ as $\varepsilon \to 0$. Therefore, by the bound on the scalar curvature and the evolution equation for the volume element we obtain

$$\begin{aligned}\mu_{g(t_i)}(B_{g(t_{i_1})}(q_i, r r_i c_\varepsilon)) &\geq e^{-\sqrt{n}\, a_0 \delta^{-2}(t_i - t_{i_1})} \mu_{g(t_{i_1})}(B_{g(t_{i_1})}(q_i, r r_i c_\varepsilon)) \\ &\geq c_\varepsilon^{\sqrt{n}} \mu_{g(t_{i_1})}(B_{g(t_{i_1})}(q_i, r r_i c_\varepsilon)).\end{aligned}$$

We have arrived at

$$\frac{\mu_{g_\infty}(B_{g_\infty}(q_\infty, r))}{\omega_n r^n} \geq c_\varepsilon^{\sqrt{n}} \frac{\mu_{g(t_{i_1})}(B_{g(t_{i_1})}(q_i, c_\varepsilon r r_i))}{\omega_n (r r_i)^n} - \frac{\varepsilon}{2}.$$

The key feature of this inequality is that the right hand side has a controllable dependence on i. Arguing as in [69], using the fact that we have full control on the geometry of $g(t_{i_1})$ by the bounded curvature assumption and Shi's estimates as well as the fact that $r_i \to 0$, we obtain

$$\mu_{g(t_{i_1})}(B_{g(t_{i_1})}(q_i, c_\varepsilon r r_i)) \geq \omega_n (c_\varepsilon r r_i)^n \Big(1 - \frac{\varepsilon}{2}\Big)$$

for sufficiently large $i \geq i_2(\varepsilon, g_{i_1}, n)$. In particular, sending i to infinity, we obtain

$$\frac{\mu_{g_\infty}(B_{g_\infty}(q_\infty, r))}{\omega_n r^n} \geq \Big(1 - \frac{\varepsilon}{2}\Big) c_\varepsilon^{n+\sqrt{n}} - \frac{\varepsilon}{2}. \tag{5.4.16}$$

We can now let ε go to 0 to find

$$\frac{\mu_{g_\infty}(B_{g_\infty}(q_\infty, r))}{\omega_n r^n} \geq 1. \tag{5.4.17}$$

On the other hand, since g_∞ is Ricci flat, by Bishop-Gromov inequality (see [28]) we obtain

$$\frac{\mu_{g_\infty}(B_{g_\infty}(q_\infty, r))}{\omega_n r^n} \leq 1, \tag{5.4.18}$$

and hence we are in the equality case where $B_{g_\infty}(q_\infty, r)$ has exactly Euclidean volume growth. By the rigidity statement in Bishop-Gromov's inequality, we deduce that g_∞ is flat, in contradiction with $r_{\mathrm{Rm}_\infty}(q_\infty) = 1$. This means that $\Sigma \subseteq \Sigma^{\mathrm{Ric}}$ as claimed. □

It is now easy to prove Corollary 5.1.9.

Proof. Fix a point $p \in \Sigma^{\mathrm{Ric}}_{II}$. Then by Theorem 5.4.7 we know that

$$\limsup_{t \nearrow T} (T-t) r^{-2}_{\mathrm{Ric}}(p,t) = +\infty, \tag{5.4.19}$$

and hence by the inequality $r^{-2}_{\mathrm{Rm}}(p,t) \geq r^{-2}_{\mathrm{Ric}}(p,t)$ also

$$\limsup_{t \nearrow T} (T-t) r^{-2}_{\mathrm{Rm}}(p,t) = +\infty. \tag{5.4.20}$$

Theorem 5.1.4 *iii*) guarantees that $p \in \Sigma_{II}$. The other inclusion is easily deduced from this one, Theorem 5.1.2, Corollary 5.4.10, and Theorem 5.1.8. □

Arguing as we did in Proposition 5.2.10 the reader can easily verify the following result.

Proposition 5.4.13 (Characterisation of Ricci Singular Set using Fixed Time Slice Scale)**.** *Let $(M, g(t))$ be a Ricci flow on $[0,T)$, $T < \infty$, such that $(M, g(t))$ is complete and has bounded Ricci curvature for every $t \in [0,T)$. Then $p \in \Sigma^{\mathrm{Ric}}$ if and only if* $\liminf_{t \nearrow T} \widetilde{r}_{\mathrm{Ric}}(p,t) = 0$.

Together with our Theorem 5.1.8 and Proposition 5.2.10 , we deduce the following corollary.

Corollary 5.4.14. *Let $(M, g(t))$ be a Ricci flow defined on $[0, T)$, $T < \infty$, such that $(M, g(t))$ is complete and has bounded geometry for every $t \in [0, T)$. Then for any point $p \in M$, we have the equivalence* $\liminf_{t \nearrow T} \widetilde{r}_{\mathrm{Rm}}(p, t) = 0$ *if and only if* $\liminf_{t \nearrow T} \widetilde{r}_{\mathrm{Ric}}(p, t) = 0$.

5.5 Applications to Bounded Scalar Curvature Ricci Flows

The aim of this section is to discuss applications of our local theory to Ricci flows with bounded scalar curvature. We first recall several results from Bamler [7] and Bamler-Zhang [11].

We first point out that in our discussion about noncollapsing relative to the scalar curvature (see Definition 2.4.9 and the paragraph below), we worked with an initial injectivity radius bound and an implicit initial Ricci curvature lower bound (on which Aubin's result relies [5]). Alternatively, we could have assumed a lower bound on Perelman's entropy of the initial metric, $\nu_0 := \nu[g(0), 2T] > -A$ to obtain these noncollapsing results. This is the condition present in Bamler's work [7], but we will rewrite the results we need from his paper using injectivity radius bounds instead; we start by recalling his extremely powerful estimate of the volume of sublevel sets of $\widetilde{r}_{\mathrm{Rm}}$ for Ricci flows with bounded scalar curvature.

Theorem 5.5.1 (Proposition 6.4 of [7]). *For any natural number $n \in \mathbb{N}$, and reals $T < +\infty$, $R_0 > 0$, $i_0 > 0$, $k_0 > 0$ and $d' \in (0, 4)$ there exist constant $E' = E'(n, R_0, i_0, k_0, d')$ and $t_0 = t_0(n, R_0, T) < T$ such that the following statement holds. Let $(M^n, g(t))$ be a closed Ricci flow maximally defined on $[0, T)$ and satisfying $\mathrm{inj}(M, g(0)) > i_0$ and $\mathrm{Ric}_{g(0)} > -(n-1)k_0 g(0)$. Assume that the scalar curvature is uniformly bounded, $|\mathrm{R}| \leq n(n-1)R_0$. Then for any $p \in M$, $t \in [t_0, T)$ and $r, s \in (0, 1)$, we have*

$$\mu_{g(t)}(\{p \mid \widetilde{r}_{\mathrm{Rm}}(p, t) < sr\} \cap B_{g(t)}(p, r)) < E' s^{d'} r^n. \tag{5.5.1}$$

Bamler uses this estimate to prove a codimension four estimate of the singular set (after passing to a weak limit). We will use it for our codimension eight result, Theorem 5.1.12. Let us remark that in its original wording, Proposition 6.4 of [7] requires $|\mathrm{R}| \leq 1$ on a time interval $[-2, 0]$ (yielding $t_0 = -1$), but the result as stated above can be obtained from this by parabolic rescaling. The proof of the above theorem is extremely involved and occupies most of [7]. It relies on a detailed analysis on the geometry of Perelman's reduced length, which allows the author to improve further and further the bound on the volume of high curvature regions. Luckily, we do not need to modify any of these estimates but manage to use Theorem 5.5.1 as a "black box".

The theorem has the following corollary.

Corollary 5.5.2 (Theorem 1.7 of [7])**.** *For any $n \in \mathbb{N}$, $T < +\infty$, $R_0 > 0$, $i_0 > 0$, $k_0 > 0$ and $\varepsilon > 0$, there exist constants $F = F(n, R_0, i_0, k_0, \varepsilon)$ and and $t_0 = t_0(n, R_0, T) < T$ as in the theorem above, such that the following statement holds. Let $(M^n, g(t))$ be a closed Ricci flow maximally defined on $[0, T)$ and satisfying $\mathrm{inj}(M, g(0)) > i_0$ and $\mathrm{Ric}_{g(0)} > -(n-1)k_0 g(0)$. Assume that the scalar curvature is uniformly bounded, $|\mathrm{R}| \le n(n-1)R_0$. Then for every $p \in M$, $r' \in (0,1)$ and $t \in [t_0, T)$, we have*

$$\int_{B_{g(t)}(p,r')} \widetilde{r}_{\mathrm{Rm}}^{-\alpha}(\cdot, t)\, d\mu_{g(t)} \le F(r')^{n-4+2\varepsilon}, \tag{5.5.2}$$

where $\alpha := 4 - 2\varepsilon$.

As this result follows directly from Theorem 5.5.1, we give a quick sketch here.

Proof. For $\alpha = 4 - 2\varepsilon$, set $d' := 4 - \varepsilon$ and pick E' and t_0 as in Theorem 5.5.1. Similar to the noncollapsing volume bound, the assumptions on our flow ensure a noninflation result, meaning that for some constant $K_1 = K_1(n, i_0, k_0, R_0)$ we have

$$\mu_{g(t)}(B_{g(t)}(p, r)) \le K_1 r^n \tag{5.5.3}$$

for all $t \in [t_0, T)$, see e.g. [92]. The dependency of the constant K_1 on the initial metric $g(0)$ is not explicitely related to i_0 and k_0 in [92], but it is shown that it depends only on the initial $\mathcal{F}$-entropy and initial Sobolev constants bounds, which can be both bounded in terms of i_0 and k_0 as remarked by the same author in [94], see the Remark after Theorem 1.1 there.

We can therefore estimate

$$\begin{aligned}
I(r') &:= \int_{B_{g(t)}(p,r')} \widetilde{r}_{\mathrm{Rm}}^{-\alpha}(q,t)\, d\mu_{g(t)}(q) \\
&\le \int_{B_{g(t)}(p,r')} \left(\int_0^{(r')^{-\alpha}} 1\, dy + \int_{(r')^{-\alpha}}^{\infty} \chi_{\{y < \widetilde{r}_{\mathrm{Rm}}^{-\alpha}(q,t)\}}(q,y)\, dy \right) d\mu_{g(t)}(q) \\
&= (r')^{-\alpha} \mu_{g(t)}(B_{g(t)}(p,r')) + \int_{(r')^{-\alpha}}^{\infty} \mu_{g(t)}(\{q \mid \widetilde{r}_{\mathrm{Rm}}(q,t) < y^{-1/\alpha}\} \cap B_{g(t)}(p,r'))\, dy \\
&\le K_1 (r')^{n-4+2\varepsilon} + \int_{(r')^{-\alpha}}^{\infty} E' y^{-d'/\alpha} (r')^{-d'} (r')^n\, dy \\
&\le K_1 (r')^{n-4+2\varepsilon} + E'(r')^{n-4+\varepsilon} \int_{(r')^{-\alpha}}^{\infty} y^{-d'/\alpha} dy \\
&= F(r')^{n-4+2\varepsilon},
\end{aligned}$$

where $F := K_1 + E'\varepsilon/\alpha$. Here, we have used (5.5.1) with $s := y^{-1/\alpha}/r'$ and the fact that the very last integral is equal to $(r')^{\varepsilon} \cdot \varepsilon/\alpha$ for the choices of d' and α as above. □

As a last ingredient for our results, we recall the following lemma, essentially due to Wang [84], though our version resembles more the one from Bamler and Zhang [11], clarifying in which terms the square root of the Riemann curvature controls the Ricci curvature on a flow with bounded scalar curvature.

Lemma 5.5.3 (Lemma 6.1 in [11]). *For any $n \in \mathbb{N}$, $R_0 > 0$, i_0 and k_0 there exists a constant $C_4(n, R_0, i_0, k_0)$ such that the following holds. Let $(M^n, g(t))$ be a Ricci flow defined on $[0,T)$, $T < \infty$ and satisfying $\operatorname{inj}(M, g(0)) > i_0$ and $\operatorname{Ric}_{g(0)} > -(n-1)k_0 g(0)$. Suppose that for some $(p_0, t_0) \in M \times (0,T)$ and $r_0 \in (0, \min\{R_0^{-1/2}, \sqrt{t_0}, 1\})$ we have $B_{g(t_0)}(p_0, r_0) \subset\subset M$. If on the backward cylinder $\mathcal{P}^-(p_0, t_0, r_0) := B_{g(t_0)}(p_0, r_0) \times (t_0 - r_0^2, t_0]$ we have $|\mathrm{R}| \leq n(n-1)R_0$ and $|\mathrm{Rm}| \leq r_0^{-2}$, then we also get $|\operatorname{Ric}(p_0, t_0)| \leq C_4 a_0 r_0^{-1}$.*

As this result is essential for our applications of the local theory to bounded scalar curvature Ricci flows, we recall the proof here, closely following the one given in [11].

Proof. By parabolically rescaling the flow with factor r_0^{-2} we may assume $|\mathrm{Rm}| \leq 1$ on the cylinder $\mathcal{P}^-(p_0, t_0, 1)$ and $|\mathrm{R}| \leq A := n(n-1)R_0 r_0^2$. We need to show the existence of a constant $D(n, R_0, i_0, k_0)$ such that $|\operatorname{Ric}(p_0, t_0)| \leq D\sqrt{A}$, yielding the claim for $C_4 = D\sqrt{R_0}/\sqrt{n-1}$.

Using Shi's estimates we see that for universal constants $\{C_m\}_{m \in \mathbb{N}}$ we have

$$|\nabla^m \mathrm{Rm}\,| \leq C_m \quad \text{on } \mathcal{P}^-\big(x_0, t_0, \tfrac{1}{2}\big). \tag{5.5.4}$$

Notice that by our assumption, we have that $\mathcal{P}^-(x_0, t_0, a_1 r_0) \subset\subset M \times (0,T)$, where a_1 is the constant defined in Section 5.2. Arguing as in [11] we deduce the existence of a universal constant $b_1 \leq \min\{a_1, 1/2\}$ such that the map $\exp_{p_0} \colon B(0, b_1) \subset \mathbb{R}^n \to M$ given by the $g(t_0)-$exponential map centered at p_0 is injective, and the pull-back Ricci flow $\widetilde{g}(t) := \exp_{p_0}^* g(t)$ defined on the cylinder $B(0, b_1) \times [t_0 - b_1^2, t_0]$ inherits smooth bounds by (5.5.4), with the metrics $\widetilde{g}(t)$ being 2-Lipschitz equivalent to the Euclidean metric for every $t \in [t_0 - b_1^2, t_0]$. Note that the bounds on the curvatures are preserved by the pull-back (up to universal constants). From now on we will focus on the metrics $\widetilde{g}(t)$, and drop the tilde to simplify the notation. Fix a cut-off function $\phi \in C_0^\infty(B(0, b_1))$ such that $\phi \in [0,1]$ and $\phi \equiv 1$ on $B(0, \frac{b_1}{2})$. This can be done in such a way that $|\partial\phi|, |\partial^2\phi| \leq E_1$ for some universal constant E_1. Therefore, for some other universal E_2 we have $|\Delta_{g(t)}\phi| \leq E_2$ on $B(0, b_1) \times [t_0 - b_1^2, t_0]$. Testing the evolution equation for the scalar curvature against ϕ, and integrating by parts we obtain

$$\begin{aligned}
&\left|\partial_t \int_{B(0,b_1)} \mathrm{R}(\cdot,t)\phi\, d\mu_{g(t)} - \int_{B(0,b_1)} 2|\operatorname{Ric}(\cdot,t)|^2 \phi\, d\mu_{g(t)}\right| \\
&\qquad = \left| - \int_{B(0,b_1)} \mathrm{R}(\cdot,t)^2 \phi\, d\mu_{g(t)} + \int_{B(0,b_1)} \Delta_{g(t)} \mathrm{R}(\cdot,t)\phi\, d\mu_{g(t)}\right| \\
&\qquad = \left| - \int_{B(0,b_1)} \mathrm{R}(\cdot,t)^2 \phi\, d\mu_{g(t)} + \int_{B(0,b_1)} \Delta_{g(t)}\phi\, \mathrm{R}(\cdot,t) d\mu_{g(t)}\right|.
\end{aligned}$$

The inequality $|\mathrm{R}| \leq A \leq n(n-1)R_0$ implies for every $t \in [t_0 - b_1^2, t_0]$ the non-inflating property $\mu_{g(t)}(B(0,b_1)) \leq K_1 b_1^n := E_3$ for some $K_1 = K_1(n, i_0, k_0, R_0)$ by [92]. Moreover, by the discussion above, the Laplace term is controlled, so we get the upper bound for the right-hand side

$$\begin{aligned}\Big| - \int_{B(0,b_1)} \mathrm{R}(\cdot,t)^2 \phi \, d\mu_{g(t)} + \int_{B(0,b_1)} \Delta_{g(t)} \phi \, \mathrm{R}(\cdot,t) d\mu_{g(t)} \Big| \\ \leq E_3 A^2 + E_3 E_2 A \leq (n(n-1)R_0 E_3 + E_2 E_3) A =: E_4 A.\end{aligned}$$

The constant E_4 depends only on n, i_0, k_0 and R_0. Integrating this inequality in time we deduce

$$\begin{aligned}\|\mathrm{Ric}\|^2_{2,\mathcal{P}^-(0,b_1/2)} &\leq \int_{t_0-b_1^2}^{t_0} \int_{B(0,b_1)} |\mathrm{Ric}(\cdot,t)|^2 \phi \, d\mu_{g(t)} dt \\ &\leq \int_{B(0,b_1)} |\mathrm{R}(\cdot,t_0)| \phi \, d\mu_{g(t_0)} + \int_{B(0,b_1)} |\mathrm{R}(\cdot,t_0 - b_1^2)| \phi \, d\mu_{g(t_0 - b_1^2)} + b_1^2 E_4 A \\ &\leq 2E_3 A + b_1^2 E_4 A =: E_5 A.\end{aligned}$$

Recall that the Ricci tensor solves the parabolic system

$$(\partial_t + \Delta_{g(t)} - 2\,\mathrm{Rm})\,\mathrm{Ric} = 0, \tag{5.5.5}$$

which can be interpreted as linear in Ric, with coefficients universally bounded in every C^m norm. Thus, the standard parabolic theory ensures the existence of a universal constant E_6 such that, once we set $b_2 := b_1/4$, we have

$$|\mathrm{Ric}(0,t_0)| \leq \|\mathrm{Ric}\|_{\infty,\mathcal{P}^-(0,b_2)} \leq E_6 \|\mathrm{Ric}\|_{2,\mathcal{P}^-(0,b_1/2)} \leq E_6 \sqrt{E_5 A} =: D\sqrt{A}, \tag{5.5.6}$$

concluding the proof. □

The proof above shows that, for a universal constant $b_2 \in (0,1)$, one can extend the Ricci bounds to the backward cylinder $\mathcal{P}^-(p_0, t_0, b_2 r_0)$. A backward-forward analogue of this result would be more in line with the results in the rest of this Chapter and it is indeed also possible. Nevertheless, in each case, the bound obtained is not strong enough to get the desired estimate $\widetilde{r}^2_{\mathrm{Ric}} \gtrsim \widetilde{r}_{\mathrm{Rm}}$, because this would require a Ricci bound on the (bigger) scale $\sqrt{r_0}$. This issue comes directly from the proof as given above, since one can appeal to linear parabolic regularity theory only on the scale r_0. A proof of a global relation $\widetilde{r}^2_{\mathrm{Ric}} \gtrsim \widetilde{r}_{\mathrm{Rm}}$ would have extremely interesting consequences, for example it would imply $\Sigma_I = \emptyset$, but without extra assumptions, it currently seems out of reach.

Here, we overcome this difficulty by considering only points where the Ricci curvature is almost maximal or well behaved points as defined in the introduction, so that any bound on the Ricci curvature that we find extends naturally to a bound on the Ricci scale.

Theorem 5.5.4 (Quadratic Scale Comparison at Certain Points). *For any $n \in \mathbb{N}$, $T < +\infty$, $R_0 > 0$, $i_0 > 0$ and $k_0 > 0$ there exists a constant $a_3 = a_3(n, R_0, i_0, k_0, T)$ such that the following holds. Let $(M, g(t))$ be a Ricci flow defined on a finite time interval $[0, T)$, satisfying $\operatorname{inj}(M, g(0)) > i_0$, $\operatorname{Ric}_{g(0)} \geq -(n-1)k_0 g(0)$ and with complete and bounded curvature time slices. Assume that $|\mathrm{R}| \leq n(n-1)R_0$ on $M \times [0, T)$. Then we deduce the two properties below.*

i) Let $(p_0, t_0) \in M \times (0, T)$ be such that $\frac{1}{2}\sup_M |\mathrm{Ric}|(\cdot, t_0) \leq |\mathrm{Ric}|(p_0, t_0) =: a_0 r_0^{-2}$. Then if $r_0 \leq \min\{R_0^{-1/2}, \sqrt{t_0}, 1\}$, we have $\widetilde{r}^2_{\mathrm{Ric}}(p_0, t_0) \geq a_3 \widetilde{r}_{\mathrm{Rm}}(p_0, t_0)$.

ii) Let $(p_0, t_0) \in G_{\delta, t_1}$ for some $\delta \in (0, 1)$, where $G_\delta = G_{\delta, t_1}$ denotes the set of well behaved points for some $t_1 \leq t_0$, see (5.1.12). Then if $r_{\mathrm{Ric}}(p_0, t_0) \leq \min\{R_0^{-1/2}, \sqrt{t_0}, 1\}$, we have $r^2_{\mathrm{Ric}}(p_0, t_0) \geq \delta a_3 r_{\mathrm{Rm}}(p_0, t_0)$.

Proof. The proof is essentially the same for both cases.

i) Fix any such space-time point (p_0, t_0) with almost maximal Ricci curvature. From the definition of the time-slice Ricci scale, it is clear that $a_0 \widetilde{r}^{-2}_{\mathrm{Ric}}(p_0, t_0) \geq |\mathrm{Ric}|(p_0, t_0) = a_0 r_0^{-2}$. Moreover, since p_0 has almost maximal Ricci curvature, $|\mathrm{Ric}|(\cdot, t_0) \leq 2|\mathrm{Ric}|(p_0, t_0)$ everywhere and thus $a_0 \widetilde{r}^{-2}_{\mathrm{Ric}}(p_0, t_0) \leq 2|\mathrm{Ric}|(p_0, t_0) = 2a_0 r_0^{-2}$. Together, we have

$$r_0^{-2} \leq \widetilde{r}^{-2}_{\mathrm{Ric}}(p_0, t_0) \leq 2r_0^{-2}. \tag{5.5.7}$$

By definition of the time-slice Riemann scale, we can bound $|\mathrm{Rm}| \leq \widetilde{r}^{-2}_{\mathrm{Rm}}(p_0, t_0)$ on the ball $B_{g(t_0)}(p_0, \widetilde{r}_{\mathrm{Rm}}(p_0, t_0))$. The Backward Pseudolocality Theorem of Bamler-Zhang (Theorem 1.4 in [11]) applied to the scale $r_1 := \widetilde{r}_{\mathrm{Rm}}(p_0, t_0) \leq \widetilde{r}_{\mathrm{Ric}}(p_0, t_0) \leq r_0 \leq \sqrt{t_0}$ yields the existence of constants ε and K (without loss of generality $K \leq \varepsilon^{-2}$) depending only on n, T, i_0 and k_0 such that we have $|\mathrm{Rm}| \leq K r_1^{-2} \leq (\varepsilon r_1)^{-2}$ on the backwards parabolic cylinder $\mathcal{P}^-(p_0, t_0, \varepsilon r_1)$. Using Lemma 5.5.3, we can estimate $a_0 r_0^{-2} = |\mathrm{Ric}|(p_0, t_0) \leq C_4 a_0 (\varepsilon r_1)^{-1}$ for some constant $C_4 = C_4(n, R_0, i_0, k_0)$, or equivalently

$$\widetilde{r}^{-2}_{\mathrm{Ric}}(p_0, t_0) \leq 2r_0^{-2} \leq 2C_4(\varepsilon r_1)^{-1} = \frac{2C_4}{\varepsilon} \widetilde{r}^{-1}_{\mathrm{Rm}}(p_0, t_0).$$

The conclusion now follows for $a_3 \leq \frac{\varepsilon}{2C_4}$.

ii) Fix a well behaved point $(p_0, t_0) \in G_\delta$. Denoting again $|\mathrm{Ric}|(p_0, t_0) =: a_0 r_0^{-2}$, we obtain the following analogue of equation (5.5.7)

$$r_0^{-2} \leq r^{-2}_{\mathrm{Ric}}(p_0, t_0) \leq \delta^{-1} r_0^{-2}. \tag{5.5.8}$$

Since $r_{\mathrm{Rm}}(p_0, t_0) \leq r_{\mathrm{Ric}}(p_0, t_0) \leq \min\{R_0^{-1/2}, \sqrt{t_0}, 1\}$, and we have, in particular, the bound $|\mathrm{Rm}| \leq r^{-2}_{\mathrm{Rm}}(p_0, t_0)$ on the (backwards) parabolic cylinder $\mathcal{P}^-(p_0, t_0, r_{\mathrm{Rm}})$, we can apply again

Lemma 5.5.3 to conclude $a_0 r_0^{-2} = |\mathrm{Ric}|(p_0,t_0) \leq C_4 a_0 (r_{\mathrm{Rm}}(p_0,t_0))^{-1}$. This means that

$$r_{\mathrm{Ric}}^{-2}(p_0,t_0) \leq \delta^{-1} r_0^{-2} \leq \delta^{-1} C_4 r_{\mathrm{Rm}}^{-1}(p_0,t_0)),$$

and hence the conclusion for $a_3 \leq \frac{1}{C_4}$. □

Next, we prove the following integral Ricci curvature concentration result, which is the time-slice analogue of Theorem 5.4.12.

Lemma 5.5.5 (Time-Slice Ricci Scale Integral Concentration). *Let $(M, g(t))$ be an n-dimensional complete Ricci flow defined on a finite time interval $[0,T)$ with complete and bounded curvature time-slices, as well as $\mathrm{inj}(M, g(0)) > i_0$ and $\mathrm{Ric}_{g(0)} \geq -(n-1)k_0 g(0)$ for some $i_0, k_0 > 0$. Assume there exists $R_0 < \infty$ such that $|\mathrm{R}| \leq n(n-1)R_0$ on $M \times [0,T)$, and let $\alpha \geq \frac{n}{2}$. Then for any $\delta \in (0, \frac{1}{2})$, there exists $C_5 = C_5(n, R_0, T, i_0, k_0, \alpha, \delta) > 0$ such that for any $(p_0,t_0) \in M \times [0,T)$ with $\widetilde{r}_{\mathrm{Ric}}(p_0,t_0) \leq \min\{R_0^{-1/2}, \sqrt{T}\}$, we can bound*

$$\int_{B_{g(t_0)}(p_0, \delta \widetilde{r}_{\mathrm{Ric}}(p_0,t_0))} \widetilde{r}_{\mathrm{Ric}}^{-2\alpha} \, d\mu_{g(t_0)} \geq C_5 > 0. \tag{5.5.9}$$

Proof. We first note that while the fixed time-slice scales might not satisfy nice continuity properties in time, their spatial continuity is well understood. In fact, for any $t \in [0,T)$, the functions $\widetilde{r}_{\mathrm{Rm}}(\cdot,t)$ and $\widetilde{r}_{\mathrm{Ric}}(\cdot,t)$ are 1-Lipschitz continuous with respect to the metric $d_{g(t)}$. This is proven in the exact same way as the corresponding results for the parabolic scales $r_{\mathrm{Rm}}(\cdot,t)$ and $r_{\mathrm{Ric}}(\cdot,t)$ in Theorem 5.2.2 and Theorem 5.4.4.

A straightforward consequence of the Lipschitz continuity is the following local Harnack type inequality: if $(p_0,t_0) \in M \times [0,T)$ is a point in a Ricci flow (say not identically Ricci flat), then

$$\tfrac{1}{2}\,\widetilde{r}_{\mathrm{Ric}}(p_0,t_0) \leq \widetilde{r}_{\mathrm{Ric}} \leq \tfrac{3}{2}\,\widetilde{r}_{\mathrm{Ric}}(p_0,t_0), \quad \text{on } B_{g(t_0)}(p_0, \tfrac{1}{2}\,\widetilde{r}_{\mathrm{Ric}}(p_0,t_0)). \tag{5.5.10}$$

To prove the integral concentration of the time-slice Ricci scale near singular points let then (p_0,t_0) be a point satisfying the assumption of the lemma and set $r_0 := \widetilde{r}_{\mathrm{Ric}}(p_0,t_0)$. From the discussion following Definition 2.4.9, we have that $\mu_{g(t_0)}(B_{g(t_0)}(p_0,\delta r_0)) \geq \kappa_1 \delta^n r_0^n$, where $\kappa_1 = \kappa_1(n, i_0, k_0, T, R_0)$. Moreover, we can use (5.5.10) to get $\widetilde{r}_{\mathrm{Ric}} \leq \frac{3}{2} r_0$ on $B_{g(t_0)}(p_0,\delta r_0)$. Then we easily compute

$$\int_{B_{g(t_0)}(p_0,\delta r_0)} \widetilde{r}_{\mathrm{Ric}}^{-2\alpha} \, d\mu_{g(t_0)} \geq \kappa_1 \delta^n r_0^n (\tfrac{3}{2}\, r_0)^{-2\alpha} \geq \kappa_1 \delta^n (\tfrac{2}{3})^{2\alpha} \min\left\{ R_0^{\frac{2\alpha-n}{2}}, 1 \right\} =: C_5 > 0.$$

In the second inequality we have used $n - 2\alpha \leq 0$. □

We are now ready to prove the non-existence of well-behaved singularities in dimensions less than eight.

Proof of Theorem 5.1.10. Without loss of generality $n \geq 4$. Fix any $t_1 \in (0,T)$ and $\delta \in (0,1)$. Assume by contradiction that there exists a blow-up sequence (p_i, t_i) with $r_i := r_{\mathrm{Ric}}(p_i,t_i) \to 0$ and which is δ-well behaved. Since we have assumed that M is closed, the initial metric satisfies $\mathrm{inj}(M, g(0)) > i_0 > 0$ and $\mathrm{Ric}_{g(0)} \geq -(n-1)k_0 g(0)$, for some $i_0 > 0$, $k_0 > 0$. By the Pseudolocality Proposition 3.2 in [7] we see that $\widetilde{r}_{\mathrm{Rm}}$ and r_{Rm} are comparable for bounded scalar curvature Ricci flows, that is, there exists a constant $C = C(n, R_0, i_0, k_0, T)$ such that $C\widetilde{r}_{\mathrm{Rm}} \leq r_{\mathrm{Rm}}$ for all points (p,t) with t large enough. Together with Theorem 5.5.4, this implies in particular that $\widetilde{r}^2_{\mathrm{Ric}} \geq r^2_{\mathrm{Ric}} \geq \delta a_3 r_{\mathrm{Rm}} \geq C\delta a_3 \widetilde{r}_{\mathrm{Rm}}$ on the ball $B_{g(t_i)}(p_i, \sqrt{\delta}\widetilde{r}_{\mathrm{Ric}}(p_i,t_i))$ for i large enough so that $t_i \geq t_1$.

We can then apply Corollary 5.5.2 with $\varepsilon < 1/8$, we obtain for every i and every $r'_i \in (0,1)$

$$\int_{B_{g(t_i)}(p_i, r'_i)} \widetilde{r}^{-\alpha}_{\mathrm{Rm}} \, d\mu_{g(t_i)} \leq F(r'_i)^{n-4+2\varepsilon}, \tag{5.5.11}$$

where $\alpha := 4 - 2\varepsilon$ and $F = F(n, R_0, i_0, k_0, \varepsilon)$. In order to apply Lemma 5.5.5, notice that $\alpha \geq \frac{n}{2}$ if and only if $n < 8$. Choosing $r'_i := \sqrt{\delta} r_i$ we can employ Lemma 5.5.5, and together with the inequality $\widetilde{r}^2_{\mathrm{Ric}} \geq C\delta a_3 \widetilde{r}_{\mathrm{Rm}}$, we see that

$$\begin{aligned} 0 < C_5 &\leq \int_{B_{g(t_i)}(p_i, r'_i)} \widetilde{r}^{-2\alpha}_{\mathrm{Ric}} \, d\mu_{g(t_i)} \\ &\leq (C\delta a_3)^{-\alpha} \int_{B_{g(t_i)}(p_i, r'_i)} \widetilde{r}^{-\alpha}_{\mathrm{Rm}} \, d\mu_{g(t_i)} \\ &\leq (C\delta a_3)^{-\alpha} F(\sqrt{\delta} r_i)^{n-4+2\varepsilon}. \end{aligned}$$

It is sufficient to let i go to infinity to get the desired contradiction.

To prove the second statement in the theorem, assume that the flow cannot be extended past time T and pick a point $p \in \Sigma$. Then by Theorem 5.1.8 as well as Proposition 5.4.13, we see that for every $t_i \nearrow T$, the sequence (p, t_i) is a blow-up sequence, which is δ-well behaved because $M = G_\delta$, in contradiction with what we proved above. □

We have seen in Theorem 5.5.4 that a point p almost maximising the Ricci curvature is well behaved. By Shi's estimates, also points in a sufficiently small neighbourhood of p are almost maximising, namely on a scale comparable to its Riemann scale. It would be interesting to see which scale one can reach, considering that any improvement of this scale might be used (with the same argument as above) to exclude the singularity formation in low dimensions. Under the additional assumption of an injectivity radius bound as in (5.1.14), we are able to reach a sufficiently large scale, as the following proof shows.

Proof of Corollary 5.1.11. We can again assume without loss of generality that $n \geq 4$. Suppose towards a contradiction that the flow cannot be extended past time $T < \infty$. Then by Sesum's result in [69], there exists a sequence (p_i, t_i) with $t_i \nearrow T$ such that $|\mathrm{Ric}|(p_i, t_i) = \sup_{M \times [0,t_i]} |\mathrm{Ric}| \to \infty$. Set $r_i := \widetilde{r}_{\mathrm{Ric}}(p_i, t_i)$. Notice that $r_i \to 0$.

For i large enough, due to the injectivity radius bound (5.1.14), we can apply Lemma 1 of Chen [22] (with $T = t_i$) to obtain the existence of some $\beta = \beta(\alpha) > 0$ such that $|\nabla \mathrm{Ric}| \leq \beta r_i^{-3}$. Setting $\delta = \frac{1}{2\beta}$, this yields

$$|\mathrm{Ric}|(q, t_i) \geq \frac{1}{2} |\mathrm{Ric}|(p_i, t_i) = \frac{1}{2} \sup_M |\mathrm{Ric}|(\cdot, t_i)$$

for all $q \in B_{g(t_i)}(p_i, \delta r_i)$. For sufficiently large i so that $r_i \leq \min\{R_0^{-1/2}, \sqrt{t_i}, 1\}$, we can thus apply Part $i)$ of Theorem 5.5.4 to obtain $\widetilde{r}_{\mathrm{Ric}}^2(q, t_i) \geq a_3 \widetilde{r}_{\mathrm{Rm}}(q, t_i)$ on $B_{g(t_i)}(p_i, \delta r_i)$. We can therefore argue as in the proof of Theorem 5.1.10 to obtain the desired contradiction. □

We finish this Chapter with a proof of Theorem 5.1.12. It heavily relies on our localised version of Sesum's result from Theorem 5.1.8, as well as the fact that the Ricci curvature blows-up at least at a Type I rate near any singular point.

Proof of Theorem 5.1.12. First of all, by Theorem 5.1.8, $\Sigma = \Sigma^{\mathrm{Ric}}$. By Corollay 5.4.8, we furthermore know that $p \in \Sigma = \Sigma^{\mathrm{Ric}}$ if and only if $r_{\mathrm{Ric}}(p, t) < \sqrt{T - t}$ for all t. In particular, for any $t \in [0, T)$ we know that $\Sigma \subseteq \{p \mid r_{\mathrm{Ric}}(p, t) < \sqrt{T - t}\}$. We therefore have the inclusion

$$\Sigma_\delta = \Sigma \cap G_\delta \subseteq \{p \mid r_{\mathrm{Ric}}(p, t) < \sqrt{T - t}\} \cap G_\delta \tag{5.5.12}$$

for every $t \in [0, T)$. Pick t_1 sufficiently large, such that $\sqrt{T - t} \leq \min\{R_0^{-1/2}, \sqrt{t}, 1\}$ for all times $t \in [t_1, T)$. We can then appeal to Part $ii)$ of Theorem 5.5.4 to get the inclusions

$$\{p \mid r_{\mathrm{Ric}}(p, t) < \sqrt{T - t}\} \cap G_\delta \subseteq \{p \mid r_{\mathrm{Rm}}(p, t) < (a_3 \delta)^{-1} (T - t)\} \cap G_\delta.$$

By the Pseudolocality Proposition 3.2 in [7] we see that $\widetilde{r}_{\mathrm{Rm}}$ and r_{Rm} are comparable for bounded scalar curvature Ricci flows, that is, there exists a constant $C = C(n, R_0, i_0, k_0, T)$ such that, for all points (p, t) with $t \in [t_1, T)$, we have $C \widetilde{r}_{\mathrm{Rm}} \leq r_{\mathrm{Rm}}$. Hence, we see that

$$\{p \mid r_{\mathrm{Ric}}(p, t) < \sqrt{T - t}\} \cap G_\delta \subseteq \{p \mid \widetilde{r}_{\mathrm{Rm}}(p, t) < (C a_3 \delta)^{-1} (T - t)\} \cap G_\delta \tag{5.5.13}$$

for every $t \in [t_1, T)$. Increasing t_1 possibly even further, we can then also ensure that $t_1 \geq t_0$ (where t_0 is the constant from Theorem 5.5.1) as well as $s := 2(C a_3 \delta)^{-1} (T - t) < 1$. This allows us to apply Theorem 5.5.1 with this $s < 1$ and $r := \frac{1}{2} < 1$, yielding for any exponent $d' \in (0, 4)$ a constant E' such that we obtain the upper bound for every $t \in [t_1, T)$

$$\mu_{g(t)}(\Sigma_\delta \cap B_{g(t)}(p, \tfrac{1}{2})) \le \mu_{g(t)}(\{p \mid \widetilde{r}_{\mathrm{Rm}}(p,t) < sr\} \cap B_{g(t)}(p,r)) < E' s^{d'} r^n < E(\sqrt{T-t})^{2d'}.$$

Here we have set $E := E' 2^{d'} (Ca_3\delta)^{-d'}$. We note that $d = 2d' \in (0, 8)$. This concludes the proof. □

Chapter 6

Future work

In this final chapter, we briefly outline a plan for future developments of the local singularity analysis we have introduced and give some ideas of possible applications. The results presented here are part of a joint project by the author and his supervisor R. Buzano. We aim to localise the results of Mantegazza-Müller in [61], that is to generalise Theorems 1.2.4 and 1.2.5 through local preservation of Perelman's entropy. However, we will need to focus our attention to Type I singular points, obtaining interesting results such as Theorem 6.1.1 below for instance. In order to state our main theorems, let us recall a few definitions and results about the scalar heat equation under the Ricci flow.

Consider a fixed background Ricci flow defined on $M \times [0,T)$ with complete and bounded geometry time-slices. We use $K(p,t;q,s)$ to denote the *heat kernel* or *fundamental solution* based at the space-time point (q,s), that is the minimal solution of

$$\begin{aligned} (\partial_t - \triangle_{p,t})K(p,t;q,s) &= 0, \\ \lim_{t \searrow s} K(\cdot,t;q,s) &= \delta_q, \end{aligned} \tag{6.1.1}$$

for $p,q \in M$ and $0 \leq s < t < T$. Here, $\triangle_{p,t}$ denotes the Laplacian in the p-variable with respect to the metric $g(t)$. Notice that as a function of (q,s), with (p,t) fixed, $K(p,t;q,s)$ is the fundamental solution of the *conjugate* heat equation, i.e.

$$\begin{aligned} (-\partial_s - \triangle_{q,s} + R_{g(s)})K(p,t;q,s) &= 0, \\ \lim_{s \nearrow t} K(p,t;\cdot,s) &= \delta_p, \end{aligned} \tag{6.1.2}$$

where $R_{g(s)}$ is the scalar curvature of $(M,g(s))$. Due to this fact, we interpret a bound on $K(p,t;q,s)$ involving the distance with respect to $g(t)$ as a heat kernel bound, and instead a bound involving the distance with respect to $g(s)$ as a conjugate heat kernel bound.

Let us remark that by parabolic regularity theory we have smooth local bounds for the conjugate heat kernel on any compact subset K of $M \times (0,T)$, independent of the base point (p,t) with $t > \max\{t' \mid (q,t') \in K \text{ for some } q \in M\}$, and hence we can define a conjugate heat kernel *based at a singular time-slice* point (p,T) via a limiting procedure. In [61], the authors highlighted the importance of this kernel in the Ricci flow context; assuming a global Type I bound for the curvature, they showed Gaussian upper and lower bounds for $K(p,T;\cdot,\cdot)$ depending only on the global Type I constant. Then, they implemented a study of Perelman's entropy functional W computed exactly at these functions, in the sense we describe below. Given a metric g, a function $f : M \to \mathbb{R}$ and a scale τ, Perelman's entropy functional W is defined by

$$W(g,f,\tau) := \int_M \left(\tau(\mathrm{R}_g + |\nabla f|_g^2) + f - n\right) \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}} d\mu_g, \tag{6.1.3}$$

under the condition

$$\int_M \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}} d\mu_g = 1. \tag{6.1.4}$$

For any fixed s, consider a function $f_{p,T}$ such that $K(p,T;\cdot,s) := \frac{e^{-f_{p,T}(\cdot,s)}}{(4\pi(T-s))^{\frac{n}{2}}}$. The authors consider the quantity $\theta(p,s) := W(g(s), f_{p,T}(\cdot,s), T-s)$. They show that this function is monotone in s, and the (continuous) density function $\theta(p) := \lim_{s \nearrow T} \theta(p,s)$ associated to it remains preserved under the blow-up procedure in a suitable sense. This fact allows them to reprove Theorems 1.2.4 and 1.2.5 for Type I Ricci flows. The global Gaussian bounds for K mentioned above are used in order to prevent the entropy θ from being lost at infinity in the blow-up procedure. Their analysis has been later extended by Hallgren to Ricci flows under a global Type I bound on the scalar curvature in [41]. A key ingredient in his analysis consists once again in Gaussian upper and lower bounds for the conjugate heat kernels under this global assumption, which he achieves to combine with the weak compactness results from [7].

In general however, one does *not* expect global Gaussian (conjugate) heat kernel bounds around the points where the δ-functions form (except possibly for times s and t very close together) for two reasons. Firstly, the concentration might happen around other points. For example, if $(M, g(t))$ is a Bryant soliton and p is its tip, then despite the property $\lim_{s \nearrow t} K(p,t;\cdot,s) = \delta_p$, the conjugate heat kernel at time $s < t$ is *not* concentrated at the tip but rather near points q with approximate distance $|t-s|$ to the tip. A similar behaviour is also expected for conjugate heat kernels based at the tip of a degenerate neckpinch. This is beautifully explained in Bamler's work [8], where, as a way to overcome this issue, he develops Gaussian-type upper bounds that are "off-centre" (that is, they are centred around so-called H_n-centres rather than around the point p). Nevertheless, by work of Buzano and Yudowitz [15], one does expect a Gaussian-type behaviour around the points where the δ-functions form *if distances remain comparable*, as is for example the case when the Ricci curvature is controlled. This suggests that a natural scale on which such bounds could hold is the Ricci scale, and without further assumptions one may not in principle extend these.

Secondly, the bounds might deteriorate due to collapsedness on large scales. We know for example by Li-Yau [56] that for (static) manifolds with non-negative Ricci curvature we have

$$K(p,t;q,s) \sim \frac{1}{\mathrm{Vol}_g(B_g(p,\sqrt{t-s}))} e^{-\frac{d_g(p,q)^2}{D(t-s)}}. \tag{6.1.5}$$

Here $D > 4$ can be taken arbitrarily close to 4. The *upper* bound discovered by Cao-Zhang [18] for Ricci flows with $\mathrm{Ric} \geq 0$ takes the same form. While for s close to t we have the asymptotic behaviour $\mathrm{Vol}_g(B_g(x,\sqrt{t-s})) \sim (t-s)^{n/2}$ and therefore recover a Gaussian type upper bound, we may have $\mathrm{Vol}_g(B_g(x,\sqrt{t-s})) \ll (t-s)^{n/2}$ for $(t-s)$ large, so that a Gaussian decay cannot be expected. Luckily, this second issue is taken care of by the fact that the flow is non-collapsed on the Ricci scale thanks to Perelman's non-collapsing Theorem [65], indicating again that the Ricci scale is a natural choice for proving Gaussian bounds, unless further assumptions are made.

In our work, we confirm this heuristic showing local Gaussian bounds for both the heat and conjugate heat kernels on a Ricci scale around their base point. In the case of the heat kernel, we can extend this Gaussian bound *globally in space* in full generality. In particular, for a possibly short time around the base time, an exponential quadratic decay in space of the heat kernel is established. However, these bounds are not good enough to perform a local singularity analysis, because the interval of times over which they hold may collapse as the base time t approaches the singular time T, as well as the distance $d_{g(t)}$ may become degenerate in this limit.

Therefore we move to a deeper study for the conjugate heat kernel. In order to extend the Gaussian bounds *globally in space and time*, and hence deduce bounds for $K(p,T;\cdot,\cdot)$, we need to assume two conditions: the base point must be a Type I singular point or a regular point, and the Ricci tensor must be globally bounded from below at a Type I rate. The first assumptions allows us to extend the bounds globally in time, whereas the second one in space. Combining these two hypothesis we deduce the following theorem.

Theorem 6.1.1 (Type I Singular Points are Modelled on Shrinkers)**.** *Suppose $(M, g(t))$ is a closed Ricci flow such that there exists a constant K with $\mathrm{Ric} \geq -\frac{K}{T-t}$ on $M \times [0,T)$. Let $p \in \Sigma_I$ be a Type I singular point. Then for any sequence of times $t_j \nearrow T$ there exist sequences of parameters $\lambda_j \nearrow +\infty$ and open sets $U_j \subset M$ such that, once set $g_j(s) := \lambda_j g(t_j + s/\lambda_j)$, the pointed Ricci flows (U_j, g_j, p) (sub-)converge to a Ricci flow $(U_\infty, g_\infty, p_\infty)$ (defined on a non empty interval of times $I \ni 0$) in the pointed smooth Cheeger-Gromov sense. Moreover, the limit g_∞ verifies*

$$\mathrm{Ric}_{g_\infty}(s) + \nabla^2 f_\infty(s) = \frac{g_\infty(s)}{2(\ell - s)}, \tag{6.1.6}$$

for a suitable function $f_\infty : U_\infty \times I \to \mathbb{R}$ and a value $\ell = \lim_{j\to\infty} \lambda_j (T - t_j)$.

We believe the limit flow g_∞ cannot be flat, but we currently do not have any proof of this result. Under this claimed non-flatness, we would deduce as a corollary of this theorem the following equivalence result generalising Theorem 1.2.5 to a great extent.

Conjecture 6.1.2. *Suppose* $(M, g(t))$ *is a closed Ricci flow such that there exists a constant* K *with* $\mathrm{Ric} \geq -\frac{K}{T-t}$ *on* $M \times [0, T)$*. Then at any Type I singular point* $p \in \Sigma_I$ *we must have* $\mathrm{R}(p, t) \geq \frac{L}{T-t}$ *for some constant* $L > 0$*.*

Notice that the lower bound assumed on the Ricci tensor is guaranteed for any Type I flow, and for such flows we also have $\Sigma = \Sigma_I$, so we would generalise Theorem 1.2.5. Using the global Gaussian bounds proved by Bamler-Zhang in [11] in the bounded scalar curvature case (and later generalised by Hallgren as said above), we could show an even stronger result in this particular case.

Conjecture 6.1.3. *Suppose* $(M, g(t))$ *is a closed Ricci flow defined on* $[0, T)$*, with bounded scalar curvature all along the flow. Then the Type I singular set is empty* $\Sigma_I = \emptyset$ *and hence* $\Sigma = \Sigma_{II}$*.*

The proofs of both these corollaries would come from a contradiction argument. On one hand, the local limit flow we get at a Type I singular point would be a non-flat shrinker thanks to Theorem 6.1.1 above. On the other hand, in both cases we could obtain by contradiction a shrinker centered at a point with zero scalar curvature. A rigidity result by Pigola-Rimoldi-Setti would then ensure that the limit should in fact be Ricci flat, hence a static Ricci flow. Finally, this property would imply curvature bounds up to the final time, which would translate in the point being regular, a contradiction!

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