The Characteristic Initial Value Problem in General Relativity

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Doctor of Philosophy

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Declaration

I, PENG ZHAO, confirm that the research included within this thesis is my own work or that where it has been carried out in collaboration with, or supported by others, that this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below.

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Details of collaboration and publications: parts of this work have been completed in collaboration with David Hilditch and Juan A. Valiente Kroon, and are published in the following papers:

- Revisiting the characteristic initial value problem for the vacuum einstein field equations. To appear in GRG
- Improved existence for the characteristic initial value problem with the conformal Einstein field equations. Gen.Rel.Grav. 52 (2020) 9, 85.

- The conformal Einstein field equations and the local extension of future null infinity. In Preparation

Abstract

This thesis discusses several questions related to the local existence of the characteristic initial value problem (CIVP) in general relativity (GR)

First, we study the CIVP of vacuum Einstein field equations by using Newman-Penrose (NP) formalism. Working in a gauge suggested by Stewart, and following the strategy taken in the work of Luk, we demonstrate local existence of solutions in a neighbourhood of the set on which data are given. These data are given on intersecting null hypersurfaces. Existence near their intersection is achieved by combining the observation that the field equations are symmetric hyperbolic in this gauge with the results of Rendall. To obtain existence all the way along the null-hypersurfaces themselves, a bootstrap argument involving the NP variables is performed.

Second, applying the same strategy, we analyze the asymptotic CIVP for the conformal Einstein field equations (CEFE) and demonstrate the local existence of solutions in a neighbourhood of the set on which the data are given. In particular, we obtain existence of solutions along a narrow rectangle along null infinity which, in turn, corresponds to an infinite domain in the asymptotic region of the physical spacetime. This result generalises work by Kánnár on the local existence of solutions to the CIVP by means of Rendalls reduction strategy.

In the last part of the thesis, we make use of a CIVP for the CEFE to provide an alternative proof of local extension of null infinity given by Li and Zhu see [1]. This proof builds on the framework developed in first two parts of the thesis.

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Chapter 1

Introduction

General relativity (GR) is the most successful theory of gravitation in physics. It is based on the equivalence principle and Mach's principle. Einstein explained gravity in terms of the curvature of spacetime manifold with a Lorentzian metric – the famous Einstein field equations (EFE). This theory is so beautiful and the simple elegant equations can offer us not only many incredible predictions, like black holes, gravitational waves and singularity but also promote the developments of Riemann geometry.

One important aspect of a physical theory is its Cauchy / initial value problem (IVP). That is because a well-posed IVP is related to the predictability of the theory. Another reason why the IVP of GR is of interest is that many physical concepts like the total energy, the total momentum and positivity of total energy are also related to a well-posed IVP of GR. Mathematically, the well-posed IVP in GR is the study of EFE as a second order hyperbolic system of partial differential equations (PDEs). In 1952, Choquet-Bruhat [2] showed that any initial data set $(\Sigma, \boldsymbol{h}, \boldsymbol{K})$ satisfies constraint on a 3-manifold has a future development $(\mathcal{M}, \boldsymbol{g})$. Here $\mathcal{M}$ is a 4-dimensional manifold with a Lorentzian metric $\boldsymbol{g}$ satisfying the vacuum EFE. And further in 1969 Choquet-Bruhat and Geroch showed that there is a unique maximal future development with such initial data set. So GR is a "good" theory.

The existence and stability problem are related to the global properties of solutions to EFE. So Penrose in the 1960s introduced the conformal transformation which preserve the global causal structure. With this idea, Friedrich introduced the conformal Einstein field equations (CEFE) in the study of asymptotic characteristic initial value problem for EFE where data are given on part of past null infinity. CEFE are such powerful that conformal geometry can be used to analyze the existence and stability of many asymptotically simple spacetimes. For example, the global existence and stability of de Sitter-like spacetimes and the semiglobal existence and stability of Minkowski-like spacetimes—see [3, 4], the local existence of anti-de Sitter-like spacetimes—see [5, 6], matter model — see [7, 8] and black hole model — see [9].

Rather than specifying data just on a spacelike hypersurface as in the IVP however, we can consider additionally the initial boundary value problem. In this setup we might have, for example, a compact spatial domain and then choose suitable boundary conditions on a timelike worldtube at the boundary of that domain. A third possibility, that we consider in the present work, is the characteristic initial value problem (CIVP). Here data are specified on characteristic surfaces of the equations under consideration. In the context of GR these surfaces are null hypersurfaces.

In GR the CIVP has a long history which dates back at least to the pioneering work by Bondi and collaborators on gravitational waves —see [10, 11]. The analysis in this work is based on the observation that in coordinates (*Bondi coordinates*) adapted to the geometry of outgoing light cones, the EFE give rise to a hierarchy of equations which can be formally solved in sequence if certain pieces of data are provided. These ideas were formalised in subsequent work by Sachs —see [12]. The CIVP was reconsidered by Newman & Penrose in their more geometric reformulation of the original analysis of gravitational radiation by Bondi and collaborators — see [13], which also contains the original formulation of the frame formulation of the EFE known as the *Newman-Penrose (NP) formalism.* The work by Newman & Penrose identifies particular components of the Weyl tensor (expressed in terms of a null frame) as the key pieces of free data to be specified on the characteristic hypersurfaces. The CIVP setup also underlies subsequent work by Penrose on the properties of massless spin fields and his approach of *exact sequences of fields* — see [14]. The common theme in this early work on the CIVP in GR is that is mainly concerned with the structural (i.e. algebraic) properties of the system of equations and does not systematically address the issue of existence and uniqueness of solutions.

Pioneering work on technical issues concerning the existence and uniqueness of solutions to the characteristic problem for the EFE can be found in the analysis of Müller zu Hagen and Seifert [15]. These ideas were brought to fruition in the work of Friedrich —see [16]. There, it was shown that the formulation of the characteristic problem by Newman & Penrose implies a symmetric hyperbolic evolution system for which known techniques from the theory of PDEs can be applied. In particular, Friedrich shows the local existence of solution near the intersection of the characteristic hypersurfaces under the assumption of analyticity of the freely specifiable data. This method was extended in subsequent work to characteristic problems for a conformal representation of the Einstein field equations (*the conformal Einstein field equations*) —see [17, 18]. Among other things, this work demonstrates the mathematical consistency of the work on the nature of gravitational waves by Bondi and collaborators and Newman & Penrose. The formulation of the CIVP for the Einstein equations using the NP formalism was further developed as a possible pathway towards numerical simulations of the Einstein field equations [19] —see also [20] for an alternative formulation for numerics using the Bondi approach to the characteristic problem, and also influenced work on the nature and classification of caustics in Relativity [21].

A major milestone in the analysis of the problem came with the influential work by Rendall on the *reduction* of the CIVP to a standard IVP [22], whose well-posedness is guaranteed by the classical results of Choquet-Bruhat [2]. In particular this reduction provides an improved version of the local existence theorem for the CIVP for the EFEs which only requires a finite level of differentiability of the initial data. Rendall's method was subsequently used to obtain a smooth data version Friedrich local existence result for the asymptotic CIVP for the CEFE. Ideas arising from the CIVP underline and permeate the fundamental work by Christodoulou & Klainermann and on the non-linear stability for the Einstein field equations [23, 24]. In particular, Christodoulou & Klainermann make use of a null frame formalism related to that of Newman & Penrose. Moreover, their analysis systematically exploits the nonlinear structure of the Einstein field equations when expressed in terms of such a null frame.

The structural properties identified in the analysis by Christodoulou & Klainermann paved the way for an improved local existence result for the CIVP for the

Einstein equations. Working in a gauge adopted from Christodoulou's work on the formation of black holes [25], which explicitly employs double-null coordinates, such an improved result has been given by Luk [26]. This work guarantees an existence domain no longer restricted to a neighbourhood of the intersection of the initial null hypersurfaces but that stretches along them. Recently, Luk's analysis has been extended so that the existence interval extends arbitrarily along the null hypersurfaces and, thus, the solution contains a piece of infinity —see [1]. An alternative approach to an improve local existence result for the CIVP has been pursued by Chruściel and collaborators —see [27–29] This approach makes use of second order evolution equations for which well developed theory of the CIVP exists —see e.g. [30, 31].

1.1 Main results of the thesis

In this thesis, we discuss three problems involving the CIVP in GR. The first one is about the EFE. We answer two follow-up questions for which the work of Rendall [22] and Luk [26] are most relevant. One question is how do the aforementioned results look when expressed in the language of the Newman-Penrose formalism? Following long-term existence results in harmonic gauge [32], it is apparent that a variety of formulations of GR exhibit desirable structure in their nonlinearities. Another is that we are therefore curious as to the robustness of this 'null-structure' under changes of gauge. We hence give a formulation of the CIVP heavily influenced by that of Stewart [33], and demonstrate for that formulation local existence in a full neighbourhood of the initial null surfaces. In first instance, the argument here provided gives an improved local existence result along one of the initial hypersurfaces. This argument can be adapted, *mutatis mutandi*, to obtain improved local existence along the other initial hypersurface —see Figure 1.1, (b). For conciseness, we restrict our discussion to the neighbourhood of only one of the hypersurfaces. A tertiary aim in translating to the NP formalism is to allow for the arguments and methods employed with Christodoulou's formulation to be reformed for application elsewhere. Our interest in understanding the structural properties of the NP field equations is what drives us to consider the approach to an improved local existence result for the CIVP pursued by Luk rather than the one followed by Chruściel and collaborators.

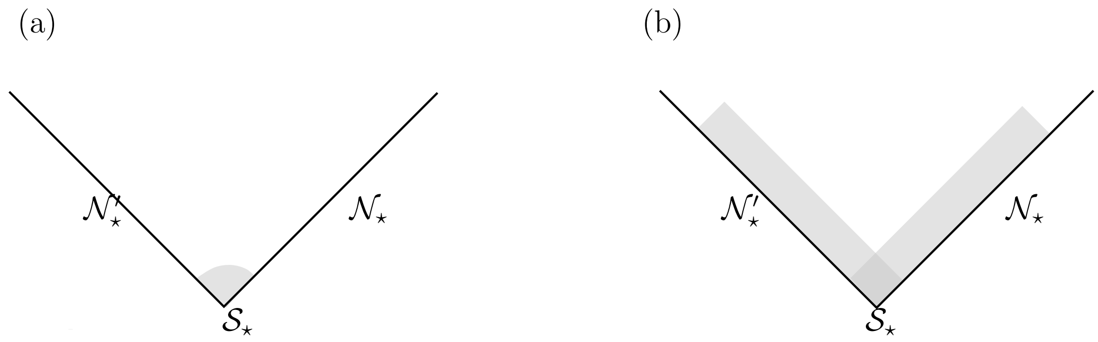

Figure 1.1: Comparison of the existence domains for the characteristic problem: (a) existence domain using Rendall's strategy based on the reduction to a standard Cauchy problem; (b) existence domain using Luk's strategy —in principle, the long side of the rectangles extends for as much as one has control on the initial data.

The second problem we consider is the *asymptotic characteristic initial value problem*, a CIVP for Friedrich's CEFE in which one of the null initial hypersurfaces is a portion of past null infinity, and show that Luk's strategy can also be adapted to this setting. Accordingly, we obtain a domain of existence of the solution to the CEFE on a narrow rectangle having a portion of null infinity as one of its long sides —see Figure 1.2. In doing so we improve Kánnár's local existence result for the asymptotic CIVP in which existence of a solution is only guaranteed in a neighbourhood of the intersection of the initial null hypersurfaces —see [34] and also [35], Chapter 18. Expressed in terms of a solution to the EFE the improved rectangular existence domain corresponds, in fact, to an infinite domain. Kánnár's result is, in turn, an extension to the setting of smooth (i.e. C^∞) functions of Friedrich's seminal analysis of the CIVP for the Einstein and conformal Einstein field equations in the analytic setting —see [16–18].

The third problem is studying the question of *the local extendibility of null infinity* by using CIVP of CEFE. To this end, initial data is prescribed on two future oriented hull hypersurfaces intersecting a 2-dimensonal surface with the topology of the 2-sphere $\mathbb{S}^2$. These null hypersurfaces are assumed to intersect future null infinity, $\mathscr{I}^+$. The question to be adressed is through this initial value problem is whether it is possible to recover a portion of future null infinity lying in the causal future of the initial hypersurfaces. Observe that in the future null infinity version of the asymptotic characteristic initial value problem analysed in Section 4, the solution

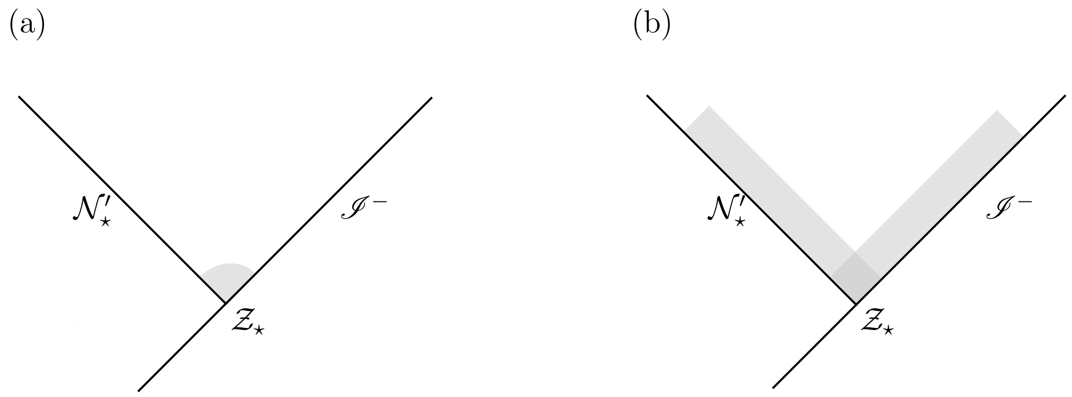

Figure 1.2: Comparison of the existence domains for the characteristic problem: (a) existence domain using Rendall's strategy based on the reduction to a standard Cauchy problem; (b) existence domain using Luk's strategy —in principle, the long side of the rectangles extends for as much as one has control on the initial data.

constructed is located in the causal past of the initial hypersurfaces —see Figure 5.1. The question of the local extendibility of null infinity through a chracteristic initial value problem has been studied by Li & Zhu in [1] directly through the Einstein field equations. In this approach, in order to encode the asymptotic behaviour of the various field at infinity it is necessary to make use of weighted functional spaces and norms. Moreover, it is necessary to consider the existence of solutions to the field equations of a domain with an infinite extent. Accordingly, this study requires a delicate and lengthy analysis. By contrast, in this section we make use of what we believe is the natural setting to address the local extendibility of null infinity: *the use of a conformal representation of the spacetime and the conformal Einstein field equations* —see e.g. [35].

1.2 Notations and Conventions

We take $\{_{a},_{b},_{c},\ldots\}$ to denote abstract tensor indices whereas $\{_{\mu},_{\nu},_{\lambda},\ldots\}$ will be used as spacetime coordinate indices with the values $0,\ldots,3$. Our conventions for

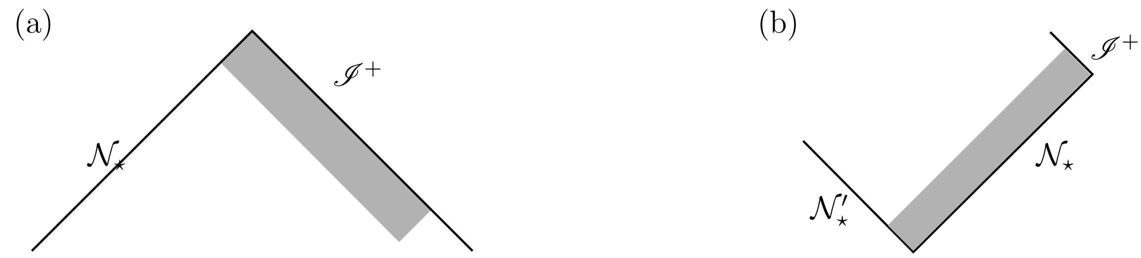

Figure 1.3: Comparison between the asymptotic characteristic problem (a) and the standard characteristic problem (b) for the conformal Einstein field equations. In the future null infinity version of the asymptotic characteristic initial value problem initial data is prescribed on future null infinity and on an outgoing lightcone $\mathcal{N}_\star$. The optimal existence result allows to recover a narrow causal diamond along null infinity. The length of this rectangle is limited by the portion of $\mathscr{I}^+$ on which one has cntrol of the initial data. Observe that region of existence of solutions lies in the causal past of the null hypersurfaces and that the existence of, at least a portion of null infinity is *a priori* assumed. In the characteristic problem considered in this article the initial data is prescribed on two standard null hypersurfaces $\mathcal{N}_\star$ and $\mathcal{N}'_\star$ with at least one of them ($\mathcal{N}_\star$) intersecting the conformal boundary. The improved existence result allows then to recover a narrow rectangle whose long side lies on $\mathcal{N}_\star$ and the short one gives a portion of future null infinity. Observe that the region of existence is on the causal future of the initial hypersurfaces and that, *a priori* only the existence of a cut of null infinity is assumed.

the connections is torsion-free and for the curvature tensors are fixed by the relation

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)v^c = R^c{}_{dab}v^d. \tag{1.1}$$

We make systematic use of the NP formalism as described, for example, in [33, 36]. In particular, the signature of Lorentzian metrics is $(+---)$. In this thesis, we restrict our attention to *the 4-dimensional vacuum case with vanishing Cosmological constant.*

Chapter 2

Mathematical Preliminaries

2.1 The Einstein Field Equations

Given a spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$, the well-known vacuum Einstein field equations (EFE) are

$$\tilde{R}_{ab} = 0. \tag{2.1}$$

Here $\tilde{R}_{ab}$ is the Ricci curvature. In order to analyse the properties of EFE we use frame formalisms. The advantages are, first, that it is simpler to handle scalars than tensors and, second, that it leads to spinorial counterpart directly. Here scalars mean the components which depend on the local choice of frame.

Let $\{\tilde{\boldsymbol{e}}_{\boldsymbol{a}}\}$ denote a frame and $\{\tilde{\boldsymbol{\omega}}^{\boldsymbol{b}}\}$ denote its dual coframe basis. Here the bold indices $\boldsymbol{a}$, $\boldsymbol{b}$, ... ranking over 0, 1, 2, 3. Namely, one has that $\langle \tilde{\boldsymbol{\omega}}^{\boldsymbol{b}}, \tilde{\boldsymbol{e}}_{\boldsymbol{a}} \rangle = \delta_{\boldsymbol{a}}{}^{\boldsymbol{b}}$. The connection coefficients of $\tilde{\boldsymbol{\nabla}}$ with respect to the frame $\{\tilde{\boldsymbol{e}}_{\boldsymbol{a}}\}$ are denoted by $\tilde{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$ and are defined by

$$\tilde{\boldsymbol{\nabla}}_{\tilde{\boldsymbol{e}}_{\boldsymbol{a}}} \tilde{\boldsymbol{e}}_{\boldsymbol{c}} = \tilde{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} \tilde{\boldsymbol{e}}_{\boldsymbol{b}}.$$

Using the torsion-free condition

$$[\tilde{\boldsymbol{e}}_{\boldsymbol{a}}, \tilde{\boldsymbol{e}}_{\boldsymbol{b}}] - (\tilde{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} - \tilde{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{a}}{}_{\boldsymbol{c}}) \tilde{\boldsymbol{e}}_{\boldsymbol{c}} = 0 \tag{2.2}$$

one obtain the first equation of the frame formalisms, the commutator relationship.

The second equation is known as the (Cartan) structure equations.

$$\tilde{P}_{\boldsymbol{abcd}} - \tilde{\rho}_{\boldsymbol{abcd}} = 0. \tag{2.3}$$

Here $\tilde{P}^{\boldsymbol{d}}{}_{\boldsymbol{cab}}$ denote the definition of Riemann curvature

$$\begin{aligned}\tilde{P}^{\boldsymbol{d}}{}_{\boldsymbol{cab}} = \tilde{\boldsymbol{e}}_{\boldsymbol{a}}(\tilde{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}}) - \tilde{\boldsymbol{e}}_{\boldsymbol{b}}(\tilde{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}}) + \tilde{\Gamma}_{\sigma}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}}(\tilde{\Gamma}_{\boldsymbol{b}}{}^{\sigma}{}_{\boldsymbol{a}} - \tilde{\Gamma}_{\boldsymbol{a}}{}^{\sigma}{}_{\boldsymbol{b}}) \\ + \tilde{\Gamma}_{\boldsymbol{b}}{}^{\sigma}{}_{\boldsymbol{c}}\tilde{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{d}}{}_{\sigma} - \tilde{\Gamma}_{\boldsymbol{a}}{}^{\sigma}{}_{\boldsymbol{c}}\tilde{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{d}}{}_{\sigma}.\end{aligned} \tag{2.4}$$

We call $\tilde{P}_{\boldsymbol{dcab}}$ the geometric curvature. And tensor $\tilde{\rho}_{\boldsymbol{dcab}}$ is known as algebraic curvature which is led to by irreducible decomposition of the Riemann curvature

$$\tilde{\rho}_{\boldsymbol{abcd}} = \tilde{C}_{\boldsymbol{abcd}} + 2\tilde{R}_{[\boldsymbol{a}[\boldsymbol{c}}\tilde{g}_{\boldsymbol{d}]\boldsymbol{b}]} + \frac{1}{3}\tilde{R}\tilde{g}_{[\boldsymbol{a}[\boldsymbol{c}}\tilde{g}_{\boldsymbol{d}]\boldsymbol{b}]}. \tag{2.5}$$

The third equation is the Bianchi identity

$$\tilde{\nabla}_{[a}\tilde{R}_{bc]de} = 0. \tag{2.6}$$

In summary, in the frame formalism a solution of the Einstein field is a collection $(\tilde{\boldsymbol{e}}_{\boldsymbol{a}}, \tilde{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}, \tilde{R}^{\boldsymbol{d}}{}_{\boldsymbol{cab}})$ satisfying Equations (2.2), (2.3) and (2.6).

2.2 The Conformal Einstein Field Equations

In this section we provide a brief introduction to Friedrich's conformal Einstein field equations —see e.g. [17, 37]. The reader interested in full details, derivation and discussion is refered to [35], Chapter 8.

Assume one has two spacetimes $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$ and $(\mathcal{M}, g_{ab})$ which are related to each other by a conformal transformation

$$g_{ab} = \Xi^2 \tilde{g}_{ab}, \tag{2.7}$$

where Ξ is the conformal factor which is formally regular up to the conformal boundary —i.e. the points for which $\Xi = 0$. We call $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$ and $(\mathcal{M}, g_{ab})$ the physical spacetime and the unphysical spacetime respectively. The sets of points of the conformal boundary giving rise to a hypersurface —i.e. the points for which $\mathbf{d}\Xi \neq 0$— are called null infinity, $\mathscr{I}$. We assume, for simplicity, that $\mathscr{I} = \partial\mathcal{M}$. Null infinity can be shown to correspond to the endpoints of null geodesics and, thus, it consists of two disconnected components —past and future null infinity, $\mathscr{I}^-$ and $\mathscr{I}^+$. For concreteness we restrict our discussion to a neighbourhood of $\mathscr{I}^-$. In what follows let ∇_a denote the Levi-Civita connection of the metric g_{ab} and let R_{ab}, $C^a{}_{bcd}$ be its Ricci and Weyl tensors, respectively.

From the conformal transformation (2.7) and the Einstein field equation for $\tilde{g}_{ab}$ one can obtain the following relationship between physical and unphysical Ricci tensor

$$R_{ab} - \frac{1}{2}Rg_{ab} = -2\Xi^{-1}(\nabla_a\nabla_b\Xi - \nabla_c\nabla^c\Xi g_{ab}) - 3\Xi^{-2}\nabla_c\Xi\nabla^c\Xi g_{ab} \tag{2.8}$$

which can be regarded as an Einstein field equation for the unphysical metric g_{ab} with energy momentum tensor $-2\Xi^{-1}(\nabla_a\nabla_b\Xi - \nabla_c\nabla^c\Xi g_{ab}) - 3\Xi^{-2}\nabla_c\Xi\nabla^c\Xi g_{ab}$. The term Ξ^{-1} becomes singular at the point for which $\Xi = 0$ which means equation (2.8) is singular at the conformal boundary.

In order to explore the conformal boundary as well as avoid the singular behaviour in equation (2.8), one can define some new fields

$$\begin{aligned}
&L_{ab} \equiv \frac{1}{2}R_{ab} - \frac{1}{12}Rg_{ab}, \\
&d^a{}_{bcd} \equiv \Xi^{-1}C^a{}_{bcd}, \\
&s \equiv \frac{1}{4}\nabla^a\nabla_a\Xi + \frac{1}{24}R\Xi,
\end{aligned}$$

—the Schouten tensor, the rescaled Weyl tensor and the Friedrich scalar. In terms of the latter the conformal vacuum Einstein field equations (CEFE) with vanishing Cosmological constant are given by:

$$\nabla_a\nabla_b\Xi = -\Xi L_{ab} + sg_{ab}, \tag{2.9a}$$

$$\nabla_a s = -L_{ac}\nabla^c\Xi, \tag{2.9b}$$

$$\nabla_c L_{db} - \nabla_d L_{cb} = \nabla_a\Xi d^a{}_{bcd}, \tag{2.9c}$$

$$\nabla_a d^a{}_{bcd} = 0, \tag{2.9d}$$

$$6\Xi s - 3\nabla_c\Xi\nabla^c\Xi = 0. \tag{2.9e}$$

One important property of these equations is that they are regular at the conformal boundary where $\Xi = 0$. Furthermore, a solution to the conformal vacuum Einstein field equations implies a solution to the Einstein field equations at the point $\Xi \neq 0$. To be specific, the relation between the conformal Einstein field equations (2.9a)-(2.9e) and the Einstein field equations is expressed in the following result:

Proposition 1 (***solutions of the conformal vacuum Einstein field equations as solutions to the vacuum Einstein field equations***)**.** *Let*

$$(g_{ab},\ \Xi,\ s,\ L_{ab},\ d^a{}_{bcd})$$

denote a solution to equations (2.9a)-(2.9d) *such that* $\Xi \neq 0$ *on an open set* $\mathcal{U} \subset \mathcal{M}$*. If, in addition, equation* (2.9e) *is satisfied at a point* $p \in \mathcal{U}$*, then the metric* $\tilde{g}_{ab} = \Xi^{-2}g_{ab}$ *is a solution to the vacuum Einstein field equations on* $\mathcal{U}$*.*

More details can be found in Chapter 8 in [35].

Given a basis $\{\boldsymbol{e_a}\}$ and in order to use the frame formalism to analyze CEFE, one need to supplement (2.9a)-(2.9e) with the structure equations

$$\begin{aligned}
&[\boldsymbol{e_a}, \boldsymbol{e_b}] - (\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} - \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{a}}{}_{\boldsymbol{c}})\boldsymbol{e_c} = 0,\\
&P^{\boldsymbol{d}}{}_{\boldsymbol{cab}} - \rho^{\boldsymbol{d}}{}_{\boldsymbol{cab}} = 0.
\end{aligned}$$

Here $P^{\boldsymbol{d}}{}_{\boldsymbol{cab}}$ is the geometric curvature of g_{ab} and $\rho^{\boldsymbol{d}}{}_{\boldsymbol{cab}}$ is algebraic curvature which is defined by

$$\rho^{\boldsymbol{d}}{}_{\boldsymbol{cab}} \equiv C^{\boldsymbol{d}}{}_{\boldsymbol{cab}} + \left(\delta^{\boldsymbol{d}}{}_{[\boldsymbol{a}}L_{\boldsymbol{b}]\boldsymbol{c}} - g_{\boldsymbol{c}[\boldsymbol{a}}L_{\boldsymbol{b}]}{}^{\boldsymbol{d}}\right).$$

All the components of tensor are with respect to basis $\{\boldsymbol{e_a}\}$.

Together with these structure equations, one obtain the frame formalism for CEFE:

$$[\boldsymbol{e_a}, \boldsymbol{e_b}] - (\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} - \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{a}}{}_{\boldsymbol{c}})\boldsymbol{e_c} = 0, \tag{2.10a}$$

$$P^{\boldsymbol{d}}{}_{\boldsymbol{cab}} - \rho^{\boldsymbol{d}}{}_{\boldsymbol{cab}} = 0, \tag{2.10b}$$
$$\nabla_{\boldsymbol{a}}\nabla_{\boldsymbol{b}}\Xi = -\Xi L_{\boldsymbol{ab}} + s g_{\boldsymbol{ab}}, \tag{2.10c}$$
$$\nabla_{\boldsymbol{a}} s = -L_{\boldsymbol{ac}}\nabla^{\boldsymbol{c}}\Xi, \tag{2.10d}$$
$$\nabla_{\boldsymbol{c}} L_{\boldsymbol{db}} - \nabla_{\boldsymbol{d}} L_{\boldsymbol{cb}} = \nabla_{\boldsymbol{a}}\Xi d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}, \tag{2.10e}$$
$$\nabla_{\boldsymbol{a}} d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} = 0, \tag{2.10f}$$
$$6\Xi s - 3\nabla_{\boldsymbol{a}}\Xi\nabla^{\boldsymbol{a}}\Xi = 0. \tag{2.10g}$$

Accordingly, a solution to the frame conformal Einstein field equations is a collection $(\boldsymbol{e_a}, \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}, \Xi, s, L_{\boldsymbol{ab}}, d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}})$ satisfying Equations (2.10a)-(2.10g).

2.3 Newman-Penrose(NP) formalism

The Newman-Penrose formalism is a type of frame formalism which is deeply related to spinors. So we provide some basic knowledge of spinors. We start by introducing a spin basis $\{o, \iota\}$ on a neighbourhood of spacetime $\mathcal{M}$ equipped with a symplectic linear structure

$$[.,.] : \mathcal{C}_p \times \mathcal{C}_p \to \mathbb{C}$$

at point p. This structure can be represented by a antisymmetric valence 2 spinor ϵ_{AB} and spin basis $\{o, \iota\}$ satisfies conditions

$$\epsilon_{AB} o^A \iota^B = 1,$$
$$\epsilon_{AB} o^A o^B = \epsilon_{AB} \iota^A \iota^B = 0.$$

The set of Hermitian spinors $\xi^{AA'}$ is a four dimensional real vector space and can describe the tangent space. $'$ means the complex conjugate. The correspondence between the metric and ϵ_{AB} is

$$g_{ab} = \epsilon_{AB}\epsilon_{A'B'}.$$

Therefore one can introduce the NP null tetrad $\{l^a,\ n^a,\ m^a,\ \bar{m}^a\}$ by

$$l^a = o^A o^{A'},\quad n^a = \iota^A \iota^{A'},\quad m^a = o^A \iota^{A'},\quad \bar{m}^a = \iota^A o^{A'},$$

$$l_a = o_A o_{A'}, \quad n_a = \iota_A \iota_{A'}, \quad m_a = o_A \iota_{A'}, \quad \bar{m}_a = \iota_A o_{A'}.$$

Then one obtain following contractions

$$\begin{aligned} &g_{ab} l^a m^b = g_{ab} l^a \bar{m}^b = g_{ab} n^a m^b = g_{ab} n^a \bar{m}^b = 0, \\ &g_{ab} l^a n^b = -g_{ab} m^a \bar{m}^b = 1. \end{aligned}$$

From the definitions $\boldsymbol{l}$ and $\boldsymbol{n}$ are real while $\boldsymbol{m}$ and $\bar{\boldsymbol{m}}$ are complex conjugate of each other. In order to introduce the spinorial counterpart of connection, one need a more systematic manner to write the spinor basis. We make use of notation

$$\epsilon_{\boldsymbol{0}}{}^A = o^A, \quad \epsilon_{\boldsymbol{1}}{}^A = \iota^A.$$

In the following, we will use bold indices $\boldsymbol{A}$, $\boldsymbol{B}$ which rank over 0 and 1 to express the component with respect to spinor basis. Hence one can define the spinorial counterpart of connection coefficients $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$ by spinor conponents

$$\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{BB'}}{}_{\boldsymbol{CC'}} \equiv \omega^{\boldsymbol{BB'}}{}_{BB'} \nabla_{\boldsymbol{AA'}} e_{\boldsymbol{CC'}}{}^{BB'}$$

where $\nabla_{\boldsymbol{AA'}} \equiv e_{\boldsymbol{AA'}}{}^{AA'} \nabla_{AA'}$ denote the directional covariant derivative in the direction of $e_{\boldsymbol{AA'}}{}^{AA'}$. Using that

$$\omega^{\boldsymbol{BB'}}{}_{BB'} = \epsilon^{\boldsymbol{B}}{}_B \bar{\epsilon}^{\boldsymbol{B'}}{}_{B'}, \quad e_{\boldsymbol{CC'}}{}^{CC'} = \epsilon_{\boldsymbol{C}}{}^C \bar{\epsilon}_{\boldsymbol{C'}}{}^{C'},$$

one can obtain

$$\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{BB'}}{}_{\boldsymbol{CC'}} = \Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}} \delta_{\boldsymbol{C'}}{}^{\boldsymbol{B'}} + \bar{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{B'}}{}_{\boldsymbol{C'}} \delta_{\boldsymbol{C}}{}^{\boldsymbol{B}} \tag{2.11}$$

where $\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}}$ are called the spin connection coefficients defined by

$$\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}} \equiv \epsilon^{\boldsymbol{B}}{}_B \nabla_{\boldsymbol{AA'}} \epsilon_{\boldsymbol{C}}{}^B. \tag{2.12}$$

In the NP formalism the directional derivatives along directions are denoted by

$$D \equiv \nabla_{00'} = l^a \nabla_a, \quad \Delta \equiv \nabla_{11'} = n^a \nabla_a, \quad \delta \equiv \nabla_{01'} = m^a \nabla_a, \quad \bar{\delta} \equiv \nabla_{10'} = \bar{m}^a \nabla_a.$$

Now, one can construct the scalar coefficient $\alpha^A \nabla \beta_A$ where α, β are o or ι and ∇ is one of D, Δ, δ and $\bar{\delta}$. Therefore one has 12 complex rotation coefficients

$$\kappa \equiv o^A D o_A, \quad \epsilon \equiv o^A D \iota_A, \quad \pi \equiv \iota^A D \iota_A, \tag{2.13a}$$

$$\tau \equiv o^A \Delta o_A, \quad \gamma \equiv o^A \Delta \iota_A, \quad \nu \equiv \iota^A \Delta \iota_A, \tag{2.13b}$$

$$\sigma \equiv o^A \delta o_A, \quad \beta \equiv o^A \delta \iota_A, \quad \mu \equiv \iota^A \delta \iota_A, \tag{2.13c}$$

$$\rho \equiv o^A \bar{\delta} o_A, \quad \alpha \equiv o^A \bar{\delta} \iota_A, \quad \lambda \equiv \iota^A \bar{\delta} \iota_A. \tag{2.13d}$$

The definitions of these coefficients with NP frame are presented in Appendix 1.

2.3.1 NP field equations

In this part we focus on the fields on the physical spactime $\tilde{\mathcal{M}}$. For convenience, we remove the tilde $\tilde{}$ in the discussion. One can start from the commutator relation (2.2) by writing its spinorial form

$$[\boldsymbol{e}_{\boldsymbol{AA'}}, \boldsymbol{e}_{\boldsymbol{BB'}}] - (\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{BB'}} - \Gamma_{\boldsymbol{BB'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{AA'}}) \boldsymbol{e}_{\boldsymbol{CC'}} = 0. \tag{2.14}$$

Then applying the decomposition (2.11) and the definition of the rotation coefficients (2.13a) one can obtain the NP commutators

$$(\Delta D - D\Delta)\psi = \big((\gamma + \bar{\gamma})D + (\epsilon + \bar{\epsilon})\Delta - (\bar{\tau} + \pi)\delta - (\tau + \bar{\pi})\bar{\delta}\big)\psi, \tag{2.15a}$$

$$(\delta D - D\delta)\psi = \big((\bar{\alpha} + \beta - \bar{\pi})D + \kappa\Delta - (\bar{\rho} + \epsilon - \bar{\epsilon})\delta - \sigma\bar{\delta}\big)\psi, \tag{2.15b}$$

$$(\delta\Delta - \Delta\delta)\psi = \big(- \bar{\nu}D + (\tau - \bar{\alpha} - \beta)\Delta + (\mu - \gamma + \bar{\gamma})\delta + \bar{\lambda}\bar{\delta}\big)\psi, \tag{2.15c}$$

$$(\bar{\delta}\delta - \delta\bar{\delta})\psi = \big((\bar{\mu} - \mu)D + (\bar{\rho} - \rho)\Delta + (\alpha - \bar{\beta})\delta - (\bar{\alpha} - \beta)\bar{\delta}\big)\psi. \tag{2.15d}$$

where ψ is any scalar field.

Next we focus on the structure equation (2.3). From the definition of the geometric curvature (2.4), one can obtain its spinorial counterpart:

$$\begin{aligned} P^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'AA'BB'}} =& \boldsymbol{e}_{\boldsymbol{AA'}}(\Gamma_{\boldsymbol{BB'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'}}) - \boldsymbol{e}_{\boldsymbol{BB'}}(\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'}}) \\ &+ \Gamma_{\boldsymbol{FF'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'}}\Gamma_{\boldsymbol{BB'}}{}^{\boldsymbol{FF'}}{}_{\boldsymbol{AA'}} - \Gamma_{\boldsymbol{FF'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'}}\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{FF'}}{}_{\boldsymbol{BB'}} \\ &+ \Gamma_{\boldsymbol{BB'}}{}^{\boldsymbol{FF'}}{}_{\boldsymbol{DD'}}\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{FF'}}{}_{\boldsymbol{DD'}} - \Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{FF'}}{}_{\boldsymbol{DD'}}\Gamma_{\boldsymbol{BB'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{FF'}}. \end{aligned} \tag{2.16}$$

For the algebraic curvature, since we know that the Weyl curvature can be expressed in terms of a valence 4 totally symmetric spinor Ψ_{ABCD} (Weyl spinor), one has that

$$C_{abcd} = \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \Psi_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD},$$

and the trace-free Ricci tensor

$$\Phi_{ab} \equiv R_{ab} - \frac{1}{4}Rg_{ab}$$

can be written in terms of a spinor $\Phi_{ABC'D'}$:

$$\Phi_{ab} = R_{AA'BB'} - \frac{1}{4}R\epsilon_{AB}\epsilon_{A'B'} = 2\Phi_{ABA'B'},$$

so that one obtains that

$$\begin{aligned}\rho_{AA'BB'CC'DD'} &= \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \Psi_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} + \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD}\\ &\quad + \bar{\Phi}_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'} + 2\Lambda(\epsilon_{AC}\epsilon_{BD}\epsilon_{A'C'}\epsilon_{B'D'} - \epsilon_{AD}\epsilon_{BC}\epsilon_{A'D'}\epsilon_{B'C'})\end{aligned}$$

where $\Lambda = -\frac{1}{24}R$. Using the spin basis $\{o, \iota\}$, one can obtain the components of the curvature spinors:

$$\begin{gathered}\Psi_0 = \Psi_{ABCD}o^A o^B o^C o^D, \quad \Psi_1 = \Psi_{ABCD}o^A o^B o^C \iota^D, \quad \Psi_2 = \Psi_{ABCD}o^A o^B \iota^C \iota^D,\\ \Psi_3 = \Psi_{ABCD}o^A \iota^B \iota^C \iota^D, \quad \Psi_4 = \Psi_{ABCD}\iota^A \iota^B \iota^C \iota^D,\end{gathered} \tag{2.17}$$

and

$$\begin{gathered}\Phi_{00} = \Phi_{ABA'B'}o^A o^B \bar{o}^{A'} \bar{o}^{B'}, \quad \Phi_{01} = \Phi_{ABA'B'}o^A o^B \bar{o}^{A'} \bar{\iota}^{B'},\\ \Phi_{02} = \Phi_{ABA'B'}o^A o^B \bar{\iota}^{A'} \bar{\iota}^{B'}, \quad \Phi_{10} = \Phi_{ABA'B'}o^A \iota^B \bar{o}^{A'} \bar{o}^{B'},\\ \Phi_{11} = \Phi_{ABA'B'}o^A \iota^B \bar{o}^{A'} \bar{\iota}^{B'}, \quad \Phi_{12} = \Phi_{ABA'B'}o^A \iota^B \bar{\iota}^{A'} \bar{\iota}^{B'},\\ \Phi_{20} = \Phi_{ABA'B'}\iota^A \iota^B \bar{o}^{A'} \bar{o}^{B'}, \quad \Phi_{21} = \Phi_{ABA'B'}\iota^A \iota^B \bar{o}^{A'} \bar{\iota}^{B'},\\ \Phi_{22} = \Phi_{ABA'B'}\iota^A \iota^B \bar{\iota}^{A'} \bar{\iota}^{B'}.\end{gathered} \tag{2.18}$$

Using the above expressions one can obtain the components from of each equation (2.3) –the so-called NP structure equations.

The spinorial Bianchi identity $\tilde{\nabla}_{[a}\tilde{R}_{bc]de} = 0$ can be expressed as

$$\tilde{\nabla}_{B'}{}^{A}\Psi_{ABCD} = \tilde{\nabla}_{B'}{}^{A}\tilde{\Phi}_{CDA'B'} - 2\tilde{\epsilon}_{B(C}\tilde{\nabla}_{D)B'}\tilde{\Lambda}.$$

Under the vacuum condition, namely $\Phi_{ABC'D'} = 0$ and $\Lambda = 0$, the Weyl spinor satisfies

$$\tilde{\nabla}_{B'}{}^{A}\Psi_{ABCD} = 0.$$

The components of the Bianchi identities, together with the NP structure equations, can be found in Appendix 6.2 and 6.2.

2.3.2 The NP formalism of CEFE

This subsection presents the NP formalism of CEFE. Now we focus on the fields on unphysical spacetime $\mathcal{M}$. From the definition of rescaled Weyl tensor d_{abcd} we define its spinorial counterpart given by the 4 valence total symmetric spinor-rescaled Weyl spinor ϕ_{ABCD}

$$\phi_{ABCD} = \Xi^{-1}\Psi_{ABCD}.$$

The spinor ϕ_{ABCD} is related to the rescaled Weyl tensor via correspondance

$$d_{abcd} = \phi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \phi_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD}.$$

Using the spin basis $\{o, \iota\}$, one can denote the components of rescaled Weyl tensor as follows:

$$\begin{gathered}\phi_0 = \phi_{ABCD}o^A o^B o^C o^D, \quad \phi_1 = \phi_{ABCD}o^A o^B o^C \iota^D, \quad \phi_2 = \phi_{ABCD}o^A o^B \iota^C \iota^D, \\ \phi_3 = \phi_{ABCD}o^A \iota^B \iota^C \iota^D, \quad \phi_4 = \phi_{ABCD}\iota^A \iota^B \iota^C \iota^D. \end{gathered} \tag{2.19}$$

The spinorial counterpart of the Schouten tensor is

$$L_{ab} = -\Lambda\epsilon_{AB}\epsilon_{A'B'} + \Phi_{ABA'B'}.$$

Then the spinorial counterpart of frame conformal Einstein field equations (2.10a)-(2.10g) are

$$[\boldsymbol{e}_{\boldsymbol{AA'}}, \boldsymbol{e}_{\boldsymbol{BB'}}] - (\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{BB'}} - \Gamma_{\boldsymbol{BB'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{AA'}})\boldsymbol{e}_{\boldsymbol{CC'}} = 0, \tag{2.20a}$$

$$P_{AA'BB'CC'DD'} - \boldsymbol{c}_{AA'BB'CC'DD'} = 0, \tag{2.20b}$$

$$\nabla_{BB'}\nabla_{AA'}\Xi = -\Xi\Phi_{ABA'B'} + s\epsilon_{AB}\bar{\epsilon}_{A'B'} + \Xi\Lambda\epsilon_{AB}\bar{\epsilon}_{A'B'}, \tag{2.20c}$$

$$\nabla_{AA'}s = \Lambda\nabla_{AA'}\Xi - \Phi_{ABA'B'}\nabla^{BB'}\Xi, \tag{2.20d}$$

$$\begin{aligned}&\nabla_{AA'}\Phi_{BCB'C'} - \nabla_{BB'}\Phi_{ACA'C'} = \epsilon_{BC}\bar{\epsilon}_{B'C'}\nabla_{AA'}\Lambda - \epsilon_{AC}\bar{\epsilon}_{A'C'}\nabla_{BB'}\Lambda\\ &-\bar{\phi}_{A'B'C'D'}\epsilon_{AB}\nabla_{C}{}^{D'}\Xi - \phi_{ABCD}\bar{\epsilon}_{A'B'}\nabla^{D}{}_{C'}\Xi,\end{aligned} \tag{2.20e}$$

$$\nabla_{DC'}\phi_{ABC}{}^{D} = 0, \tag{2.20f}$$

$$\lambda = 6\Xi s - 3\nabla_{AA'}\Xi\nabla^{AA'}\Xi. \tag{2.20g}$$

The components equations of CEFE can be found in Appendix 6.2

Chapter 3

Revisiting the characteristic initial value problem for the vacuum Einstein field equations

In this chapter, using the NP formalism we study the characteristic initial value problem in vacuum General Relativity. We work in a gauge suggested by Stewart, and following the strategy taken in the work of Luk, demonstrate local existence of solutions in a neighbourhood of the set on which data are given. These data are given on intersecting null hypersurfaces. Existence near their intersection is achieved by combining the observation that the field equations are symmetric hyperbolic in this gauge with the results of Rendall. To obtain existence all the way along the null-hypersurfaces themselves, a bootstrap argument involving the NP variables is performed.

3.1 The geometry of the problem

Let $(\mathcal{M}, \boldsymbol{g})$ denote a vacuum spacetime satisfying $R_{ab} = 0$, where $\mathcal{M}$ is a 4-dimensional manifold with boundary and an edge. The boundary consists of two null hypersurface: $\mathcal{N}_\star$, the outgoing null hypersurface; $\mathcal{N}'_\star$, the incoming null hypersurface with non-empty intersection $\mathcal{S}_\star \equiv \mathcal{N}_\star \cap \mathcal{N}'_\star$. For concreteness we will assume that $\mathcal{S}_\star \approx \mathbb{S}^2$.

Given a neighbourhood $\mathcal{U}$ of $\mathcal{S}_\star$, one can introduce coordinates $x = (x^\mu)$ with $x^0 = v$ and $x^1 = u$ such that, at least in a neighbourhood of $\mathcal{S}_\star$ one can write

$$\mathcal{N}_\star = \{p \in \mathcal{U} \mid u(p) = 0\}, \quad \mathcal{N}'_\star = \{p \in \mathcal{U} \mid v(p) = 0\}.$$

Given suitable data on $(\mathcal{N}_\star \cup \mathcal{N}'_\star) \cap \mathcal{U}$ we are interested in making statements about the existence and uniqueness of solutions to the vacuum Einstein field equations of the aforementioned type on some open set

$$\mathcal{V} \subset \{p \in \mathcal{U} \mid u(p) \geq 0, v(p) \geq 0\}$$

which we identify with a subset of the future domain of dependence, $D^+(\mathcal{N}_\star \cup \mathcal{N}'_\star)$, of $\mathcal{N}_\star \cup \mathcal{N}'_\star$.

3.1.1 Construction of the gauge: Stewart's approach

We will ultimately be concerned with existence and uniqueness of solutions, but, as is common in such constructions, it is useful to start by assuming existence in order to give a concrete PDE formulation of the problem. In this section we thus briefly review the gauge choice. In the rest of this article we will call this construction *Stewart's gauge*.

3.1.1.1 Coordinates

In the following it will be convenient to regard the 2-dimensional surface $\mathcal{S}_\star$ as a submanifold of a spacelike hypersurface S. The subsequent discussion will be restricted to the future of S. As $\mathcal{S}_\star \approx \mathbb{S}^2$, one has that $\mathcal{S}_\star$ divides S in two regions —the interior of $\mathcal{S}_\star$ and the exterior of $\mathcal{S}_\star$. Now, consider a foliation of S by 2-dimensional surfaces with the topology of $\mathbb{S}^2$ which includes $\mathcal{S}_\star$. At each of the 2-dimensional surfaces we assume there pass two null hypersurfaces. Further, we assume that:

i). one of these hypersurfaces has the property that the projection of the tangent vectors of their generators at $\mathcal{S}_\star$ point *outwards* —we call these null hypersurfaces *outgoing light cones*;

ii). one of these hypersurfaces has the property that the projection of the tangent vectors of their generators at $S_\star$ point *inwards* —we call these null hypersurfaces *ingoing light cones*.

Thus, as least close to S one obtains a 1-parameter family of outgoing null hypersurface $\mathcal{N}_u$ and a 1-parameter family of ingoing null hypersurface $\mathcal{N}'_v$. One can then define scalar fields u and v by the requirements, respectively, that u is constant on each of the $\mathcal{N}_u$ and v is constant on each $\mathcal{N}'_v$. In particular, we assume that $\mathcal{N}_0 = \mathcal{N}_\star$ and $\mathcal{N}'_0 = \mathcal{N}'_\star$. Following standard usage, we call u a *retarded time* and v an *advanced time*. We use the notation $\mathcal{N}_u(v_1, v_2)$ to denote the part of the hypersurface $\mathcal{N}_u$ with $v_1 \leq v \leq v_2$. Likewise $\mathcal{N}'_v(u_1, u_2)$ has a similar definition. We denote the sphere intersected by $\mathcal{N}_u$ and $\mathcal{N}'_v$ by $\mathcal{S}_{u,v}$. We define the region

$$\bigcup_{0\leq v'\leq v, 0\leq u'\leq u} \mathcal{S}_{u',v'} \tag{3.3}$$

as $\mathcal{D}_{u,v}$. We also define the time function

$$t \equiv u + v, \tag{3.4}$$

and the *truncated causal diamond*,

$$\mathcal{D}^{\tilde{t}}_{u,v} \equiv \mathcal{D}_{u,v} \cap \{t \leq \tilde{t}\}, \tag{3.5}$$

which will be used frequently throughout our arguments.

The scalar fields u and v introduced in the previous paragraph will be used as coordinates in a neighbourhood of $\mathcal{S}_\star$. To complete the coordinate system, consider arbitrary coordinates $(x^{\mathcal{A}})$ in a coordinate patch U on $\mathcal{S}_\star$, with the index $\mathcal{A}$ taking the values 2, 3. These coordinates are then propagated into $\mathcal{N}_\star$ by requiring them to be constant along the generators of $\mathcal{N}_\star$. Once coordinates have been defined on $\mathcal{N}_\star$, one can propagate them into $\mathcal{V}$ by requiring them to be constant along the generators of each $\mathcal{N}'_v$. In this manner, for each coordinate patch U one obtains a coordinate system $(x^\mu) = (v,\ u,\ x^{\mathcal{A}})$ on D_U in $\mathcal{V}$. Here area D_U is defined by the image of first generating U along v and then generating long u.

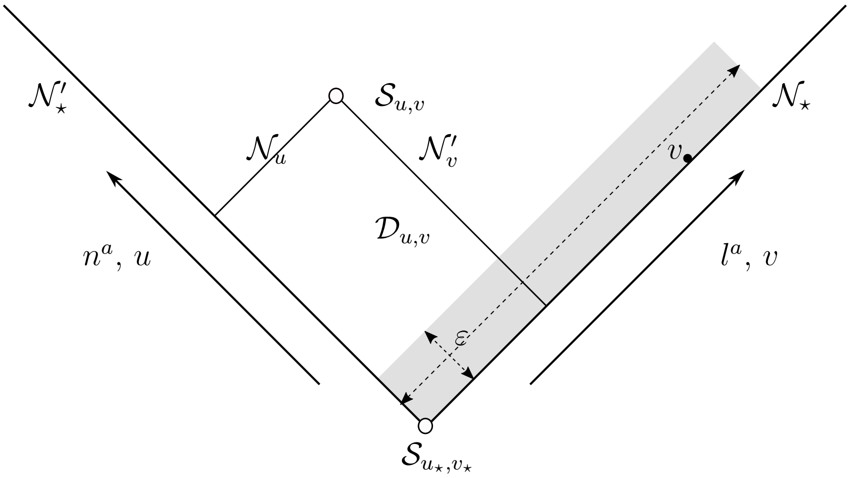

Figure 3.1: Setup for Stewart's gauge. The construction makes use of a double null foliation of the future domain of dependence of the initial hypersurface $\mathcal{N}_\star \cup \mathcal{N}'_\star$. The coordinates and NP null tetrad are adapted to this geometric setting. The analysis in this article is focused on the arbitrarily thin grey rectangular domain along the hypersurface $\mathcal{N}_\star$. The argument can be adapted, in a suitable manner, to a similar rectangle along $\mathcal{N}'_\star$. See the main text for the definitions of the various regions and objects.

3.1.1.2 The NP frame

To construct a null NP tetrad we choose vector fields l^a and n^a to be tangent to the generators of $\mathcal{N}_u$ and $\mathcal{N}'_v$ respectively. Further we require them to be normalised according to

$$g_{ab}l^a n^b = 1.$$

The latter normalisation condition is preserved under the boost,

$$l^a \mapsto \varsigma l^a, \quad n^a \mapsto \varsigma^{-1} n^a, \quad \varsigma \in \mathbb{R}.$$

This freedom can be used to set

$$n_a = \nabla_a v.$$

This requirement still leaves some freedom left as one can choose a relabelling of the form $v \mapsto V(v)$. Next, we choose the complex vector fields m^a and $\bar{m}^a$ so that they are tangent to the surfaces $\mathcal{S}_{u,v}$ and satisfy the conditions

$$g_{ab}m^a\bar{m}^b = -1, \quad g_{ab}m^a m^b = 0.$$

There is still the freedom to perform a spin

$$m^a \mapsto e^{i\theta}m^a, \quad \theta \in \mathbb{R}$$

at each point.

Remark 1. Now, observing that, by construction, on the generators of each null hypersurface $\mathcal{N}_v^\star$ only the coordinate u varies, one has that

$$n^\mu \boldsymbol{\partial}_\mu = Q\boldsymbol{\partial}_u,$$

where Q is a real function of the position. Furthermore, since the vector l^a is tangent to the generators of each $\mathcal{N}_u$ and $l^a n_a = l^a \nabla_a v = 1$, one has that

$$l^\mu \boldsymbol{\partial}_\mu = \boldsymbol{\partial}_v + C^\mathcal{A}\boldsymbol{\partial}_\mathcal{A},$$

where, again, the components $C^\mathcal{A}$ are real functions of the position. By construction, the coordinates $(x^\mathcal{A})$ do not vary along the generators of $\mathcal{N}_\star$-that is, one has that $l^a\nabla_a x^\mathcal{A} = 0$. Accordingly, one has that

$$C^\mathcal{A} = 0 \quad \text{on} \quad \mathcal{N}_\star.$$

Finally, because m^a and $\bar{m}^a$ span the tangent space of each surface $\mathcal{S}_{u,v}$, hence in these coordinate system one has that

$$m^\mu \boldsymbol{\partial}_\mu = P^\mathcal{A}\boldsymbol{\partial}_\mathcal{A},$$

where the coefficients $P^\mathcal{A}$ are complex functions.

Summarising, we make the following choice:

Assumption 1 (***Stewart's choice of the components of the frame***)**.** On a local coordinate patch D_U of $\mathcal{V}$ one can find a NP frame $\{l^a,\ n^a,\ m^a,\ \bar{m}^a\}$ of the form:

$$\boldsymbol{l} = \boldsymbol{\partial}_v + C^{\mathcal{A}}\boldsymbol{\partial}_{\mathcal{A}}, \qquad \boldsymbol{n} = Q\boldsymbol{\partial}_u, \qquad \boldsymbol{m} = P^{\mathcal{A}}\boldsymbol{\partial}_{\mathcal{A}}.$$

Remark 2. *The coordinate system we use is the same with the choice by Luk as long as we replace v to $\underline{u}$. But the null frame choice is different and up to a rescaling. To be specific, Luk chooses $e_3 = \Omega^{-1}\partial_u$ and $e_4 = \Omega^{-1}(\partial_{\underline{u}} + b^A\partial_A)$ with a normalization $g(e_3, e_4) = -2$ as null directions to decompose the equations. In his analysis, null vector $\underline{L} = \Omega e_3$ and $L = \Omega e_4$ are also needed.*

Remark 3. In view of the normalisation condition $g_{ab}m^a\bar{m}^b = -1$, there are only 3 real functions involved in the $P^{\mathcal{A}}$'s. Thus, Q, $C^{\mathcal{A}}$ together with $P^{\mathcal{A}}$ give six scalar fields describing the metric. Thus the components $(g^{\mu\nu})$ of the contravariant form of the metric $\boldsymbol{g}$ are of the form

$$(g^{\mu\nu}) = \begin{pmatrix} 0 & Q & 0 \\ Q & 0 & QC^{\mathcal{A}} \\ 0 & QC^{\mathcal{A}} & \sigma^{\mathcal{AB}} \end{pmatrix},$$

where

$$\sigma^{\mathcal{AB}} \equiv -(P^{\mathcal{A}}\bar{P}^{\mathcal{B}} + \bar{P}^{\mathcal{A}}P^{\mathcal{B}}).$$

Here and in what follows $\boldsymbol{\sigma}$ is the induced metric on $\mathcal{S}_{u,v}$, and has contravariant components $\sigma^{\mathcal{AB}}$ defined in the standard manner. Note that care is needed to distinguish σ, the NP connection coefficient, from this quantity. From the expression, we can compute that $l_\mu dx^\mu = Q^{-1}du$, $\sigma_{\mathcal{AB}}P^{\mathcal{A}}P^{\mathcal{B}} = 0$, $\sigma_{\mathcal{AB}}P^{\mathcal{A}}\bar{P}^{\mathcal{B}} = -1$ and $-\partial_{\mathcal{A}}C^{\mathcal{A}} = \bar{m}_{\mathcal{A}}\delta C^{\mathcal{A}} + m_{\mathcal{A}}\bar{\delta}C^{\mathcal{A}}$ directly.

Remark 4. On $\mathcal{N}'_\star$ one has that $\boldsymbol{n} = Q\boldsymbol{\partial}_u$. As the coordinates $(x^{\mathcal{A}})$ are constant along the generators of $\mathcal{N}_\star$ and $\mathcal{N}'_\star$, it follows that on $\mathcal{N}'_\star$ the coefficient Q is only a function of u. Thus, without loss of generality one can parameterise u so as to set $Q = 1$ on $\mathcal{N}'_\star$.

3.1.2 Analysis of the NP commutators

In this subsection we analyse some simple consequences of the NP frame of Assumption 1 and the NP commutator equations (2.15a)-(2.15d). In particular, we exploit the fact that given a choice of NP frame, the evaluation of the NP commutators on the coordinates gives rise to two different types of equations, namely i). conditions on the spin connection coefficients, and ii). equations for the coefficients of the frame. In what follows we analyse these two classes of equations. For future use observe that from the definition of the NP frame $\{l^a,\ n^a,\ m^a,\ \bar{m}^a\}$ in Assumption 1 it readily follows that,

$$Dv = 1, \qquad \Delta v = 0, \qquad \delta v = 0, \qquad \bar{\delta} v = 0, \tag{3.8a}$$

$$Du = 0, \qquad \Delta u = Q, \qquad \delta u = 0, \qquad \bar{\delta} u = 0, \tag{3.8b}$$

$$Dx^{\mathcal{A}} = C^{\mathcal{A}}, \qquad \Delta x^{\mathcal{A}} = 0, \qquad \delta x^{\mathcal{A}} = P^{\mathcal{A}}, \qquad \bar{\delta} x^{\mathcal{A}} = \bar{P}^{\mathcal{A}}. \tag{3.8c}$$

3.1.2.1 Spin connection coefficients

Direct inspection of the NP commutators (2.15a)-(2.15d) applied to the coordinates $(v,\ u,\ x^2,\ x^3)$ taking into account (3.8a)-(3.8c) yields on $\mathcal{V}$ the conditions,

$$\kappa = \nu = 0, \quad \gamma + \bar{\gamma} = 0, \quad \rho = \bar{\rho}, \quad \mu = \bar{\mu}, \quad \pi = \alpha + \bar{\beta}.$$

We will see that these gauge conditions can be refined still further.

Fixing the rotation freedom. The set up of frame vectors under Assumption 1 allows the freedom of a rotation

$$m^a \mapsto m'^a = e^{i\theta} m^a.$$

The latter, in turn, implies the transformation

$$\gamma - \bar{\gamma} \mapsto \gamma' - \bar{\gamma}' = \gamma - \bar{\gamma} - i\Delta\theta.$$

Accordingly, by requiring θ to satisfy the equation

$$\Delta\theta = i(\bar{\gamma} - \gamma) \tag{3.9}$$

it is always possible to assume that $\bar{\gamma}-\gamma=0$, which, together with the condition $\gamma+\bar{\gamma}=0$ allows us to set $\gamma=0$ on $\mathcal{V}$. A similar computation shows that

$$\epsilon-\bar{\epsilon}\mapsto\epsilon'-\bar{\epsilon}'=\epsilon-\bar{\epsilon}+iD\theta.$$

This equation can be used to set $\epsilon-\bar{\epsilon}=0$ on $\mathcal{N}_\star$. Also, after solving this equation, the result θ on $\mathcal{N}_\star$ can be the initial value of equation (3.9). The value of Q on $\mathcal{N}_\star$ can be propagated from $\mathcal{S}_\star$ using the transport equation,

$$DQ=-(\epsilon+\bar{\epsilon})Q=-2\epsilon Q$$

that is,

$$\partial_v Q=-2\epsilon Q.$$

Summarising, we have the following gauge restriction, which we employ exclusively in what follows:

Lemma 1 (***properties of the connection coefficients in Stewart's gauge***)**.** *The NP frame of Assumption 1 can be chosen such that*

$$\kappa=\nu=\gamma=0, \tag{3.10a}$$
$$\rho=\bar{\rho},\quad \mu=\bar{\mu}, \tag{3.10b}$$
$$\pi=\alpha+\bar{\beta} \tag{3.10c}$$

on $\mathcal{V}$ and, furthermore, with

$$\epsilon-\bar{\epsilon}=0\quad on\quad \mathcal{V}\cap\mathcal{N}_\star.$$

3.1.2.2 Equations for the frame coefficients

Taking into account the conditions on the spin connection coefficients given by (3.10a)-(3.10c), it follows that the remaining commutators yield the equations

$$\Delta C^{\mathcal{A}}=-(\bar{\tau}+\pi)P^{\mathcal{A}}-(\tau+\bar{\pi})\bar{P}^{\mathcal{A}}, \tag{3.11a}$$

$$\Delta P^{\mathcal{A}} = -\mu P^{\mathcal{A}} - \bar{\lambda}\bar{P}^{\mathcal{A}}, \tag{3.11b}$$

$$DP^{\mathcal{A}} - \delta C^{\mathcal{A}} = (\rho + \epsilon - \bar{\epsilon})P^{\mathcal{A}} + \sigma\bar{P}^{\mathcal{A}}, \tag{3.11c}$$

$$DQ = -(\epsilon + \bar{\epsilon})Q, \tag{3.11d}$$

$$\bar{\delta}P^{\mathcal{A}} - \delta\bar{P}^{\mathcal{A}} = (\alpha - \bar{\beta})P^{\mathcal{A}} - (\bar{\alpha} - \beta)\bar{P}^{\mathcal{A}}, \tag{3.11e}$$

$$\delta Q = (\tau - \bar{\pi})Q. \tag{3.11f}$$

Remark 5. Equations (3.11a)-(3.11b) allow us to evolve the frame coefficients $C^{\mathcal{A}}$ and $P^{\mathcal{A}}$ off of the null hypersurface $\mathcal{N}'_\star$. Equations (3.11c)-(3.11d) allow evolution of the coefficients Q and $P^{\mathcal{A}}$ along the null generators of $\mathcal{N}_\star$. Finally (3.11e)-(3.11f) provide constraints for Q and $P^{\mathcal{A}}$ on the spheres $\mathcal{S}_{u,v}$.

3.2 The formulation of the CIVP

In this section we analyse general aspects of the CIVP for the vacuum Einstein field equations on the null hypersurfaces $\mathcal{N}_\star$ and $\mathcal{N}'_\star$. The hierarchical structure allows the identification of the basic reduced initial data set $r_\star$ from which the full initial data on $\mathcal{N}_\star \cup \mathcal{N}'_\star$ can be computed.

Lemma 2 (***freely specifiable data for the CIVP***)**.** *Working in the gauge given by Assumption 1 and Lemma 1, initial data for the vacuum Einstein field equations on $\mathcal{N}_\star \cup \mathcal{N}'_\star$ can be computed (near $\mathcal{S}_\star$) from the reduced data set $\mathbf{r}_\star$ consisting of:*

$$\begin{aligned}
&\Psi_0,\ \epsilon + \bar{\epsilon} \quad on \quad \mathcal{N}_\star, \\
&\Psi_4 \quad on \quad \mathcal{N}'_\star, \\
&\lambda, \quad \sigma, \quad \mu, \quad \rho, \quad \pi, \quad P^{\mathcal{A}} \quad on \quad \mathcal{S}_\star.
\end{aligned}$$

Proof. The proof follows by inspection of the various intrinsic equations on $\mathcal{N}_\star$, $\mathcal{N}'_\star$ and $\mathcal{S}_\star$.

Data on $\mathcal{S}_\star$. Since $P^{\mathcal{A}}$ are given, the operators δ and $\bar{\delta}$ are well defined on $\mathcal{S}_\star$ and intrinsic to this 2-dimensional hypersurface. From the definition of the connection coefficients α and β it follows that the inner connection of $\mathcal{S}_\star$ is described by the combination $\alpha - \bar{\beta}$. This is readily computable from the data $P^{\mathcal{A}}$ on $\mathcal{S}_\star$. Thus, using $\alpha + \bar{\beta} = \pi$, one can compute α and β. Noting that $Q = 1$ on $\mathcal{S}_\star \subset \mathcal{N}'_\star$, we obtain that $\pi = \bar{\tau}$ from (3.11f). Then we obtain all the values of connection coefficients on $\mathcal{S}_\star$. Thus, the constraint equations (3q), (3j), (3n) of the structure equations can be used to compute the value of Ψ_1, Ψ_2, Ψ_3 on $\mathcal{S}_\star$. With that, all initial data for the connection coefficients and Weyl curvature on $\mathcal{S}_\star$ have been obtained.

Data on $\mathcal{N}'_\star$. On the incoming null hypersurface $\mathcal{N}'_\star$ we can obtain that $Q = 1$ leads to $\tau = \bar{\pi}$ from equation (3.11f) and $\Delta = \partial_u$. Making use of the structure equations (3g) and (3o), which can be reduced by the gauge condition, namely

$$\frac{\partial \mu}{\partial u} = -\lambda\bar{\lambda} - \mu^2,$$
$$\frac{\partial \lambda}{\partial u} = -\Psi_4 - 2\lambda\mu,$$

we can obtain the value of μ and λ on $\mathcal{N}'_\star$. Then the frame coefficients $P^{\mathcal{A}}$ on $\mathcal{N}'_\star$ are computed using equation (3.11b) which takes the form

$$\frac{\partial P^{\mathcal{A}}}{\partial u} = -\mu P^{\mathcal{A}} - \bar{\lambda}\bar{P}^{\mathcal{A}}.$$

Thus we can compute the δ-direction derivative on $\mathcal{N}'_\star$. Solving the structure equations (3d), (3k) with the Bianchi identity equation (4d), namely

$$-\frac{\partial \alpha}{\partial u} = \Psi_3 + \beta\lambda + \alpha\bar{\mu} + \lambda\tau,$$
$$-\frac{\partial \beta}{\partial u} = \alpha\bar{\lambda} + \beta\mu + \mu\tau,$$
$$\frac{\partial \Psi_3}{\partial u} - P^{\mathcal{A}}\frac{\partial \Psi_4}{\partial x^{\mathcal{A}}} = (4\beta - \tau)\Psi_4 - 4\mu\Psi_3,$$

together we can compute the value of α, β and Ψ_3 on $\mathcal{N}'_\star$. Then equation (3.11a)

$$\frac{\partial C^{\mathcal{A}}}{\partial u} = -(\bar{\tau} + \pi)P^{\mathcal{A}} - (\tau + \bar{\pi})\bar{P}^{\mathcal{A}}$$

reveals the value of the frame coefficients $C^{\mathcal{A}}$ on $\mathcal{N}'_\star$. With the above information at hand one can use equations (3a), (3i), (3r) and (4e):

$$\frac{\partial \epsilon}{\partial u} = -\Psi_2 - \beta\pi - \alpha\bar{\pi} - \alpha\tau - \pi\tau - \beta\bar{\tau},$$

$$
\begin{aligned}
P^{\mathcal{A}}\frac{\partial \tau}{\partial x^{\mathcal{A}}} - \frac{\partial \sigma}{\partial u} &= \bar{\lambda}\rho + \mu\sigma - \bar{\alpha}\tau + \beta\tau + \tau^2, \\
\bar{P}^{\mathcal{A}}\frac{\partial \tau}{\partial x^{\mathcal{A}}} - \frac{\partial \rho}{\partial u} &= \Psi_2 + \bar{\mu}\rho + \lambda\sigma + \alpha\tau - \bar{\beta}\tau + \tau\bar{\tau}, \\
\frac{\partial \Psi_2}{\partial u} - P^{\mathcal{A}}\frac{\partial \Psi_3}{\partial x^{\mathcal{A}}} &= \sigma\Psi_4 + 2(\beta - \tau)\Psi_3 - 3\mu\Psi_2
\end{aligned}
$$

to compute the value of ϵ, σ, ρ and Ψ_2 on $\mathcal{N}'_\star$. The Bianchi identity equation (4h)

$$
\frac{\partial \Psi_1}{\partial u} - P^{\mathcal{A}}\frac{\partial \Psi_2}{\partial x^{\mathcal{A}}} = -2\mu\Psi_1 - 3\tau\Psi_2 + 2\sigma\Psi_3,
$$

provides the value of Ψ_1 on $\mathcal{N}'_\star$. With the results above, we can then compute the value of Ψ_0 from equation (4b)

$$
\frac{\partial \Psi_0}{\partial u} - P^{\mathcal{A}}\frac{\partial \Psi_1}{\partial x^{\mathcal{A}}} = -\mu\Psi_0 - 2(2\tau + \beta)\Psi_1 + 3\sigma\Psi_2.
$$

Data on $\mathcal{N}_\star$. From equation (3.11d) one has that $\partial_v Q = -(\epsilon+\bar{\epsilon})Q$ so that, using the value of Q at $\mathcal{S}_\star$ one can compute the value of Q on $\mathcal{N}_\star$. The structure equations (3f) and (3m) give

$$
\begin{aligned}
\frac{\partial \sigma}{\partial v} &= \Psi_0 + 3\epsilon\sigma - \bar{\epsilon}\sigma + 2\rho\sigma, \\
\frac{\partial \rho}{\partial v} &= 2\epsilon\rho + \rho^2 + \sigma\bar{\sigma}.
\end{aligned}
$$

Solving these last equations one can obtain the value of σ and ρ on $\mathcal{N}_\star$. Then the value of $P^{\mathcal{A}}$ on $\mathcal{N}_\star$ can be computed using equation (3.11c) which in the present setting takes the form

$$
\frac{\partial P^{\mathcal{A}}}{\partial v} = \rho P^{\mathcal{A}} + \sigma\bar{P}^{\mathcal{A}}.
$$

Then the structure equations (3e), (3l) and the Bianchi identity (4a), namely,

$$
\begin{aligned}
P^{\mathcal{A}}\frac{\partial \epsilon}{\partial x^{\mathcal{A}}} - \frac{\partial \beta}{\partial v} &= -\Psi_1 + \bar{\alpha}\epsilon + \beta\bar{\epsilon} - \epsilon\bar{\pi} - \beta\rho - \alpha\sigma - \pi\sigma, \\
\bar{P}^{\mathcal{A}}\frac{\partial \epsilon}{\partial x^{\mathcal{A}}} - \frac{\partial \alpha}{\partial v} &= 2\alpha\epsilon + \bar{\beta}\epsilon - \alpha\bar{\epsilon} - \epsilon\pi - \alpha\rho - \pi\rho - \beta\bar{\sigma},
\end{aligned}
$$

$$\bar{P}^{\mathcal{A}}\frac{\partial \Psi_0}{\partial x^{\mathcal{A}}} - \frac{\partial \Psi_1}{\partial v} = (4\alpha - \pi)\Psi_0 - 2(2\rho + \epsilon)\Psi_1,$$

provide us the value of α, β and Ψ_1 on $\mathcal{N}_\star$. Next, the structure equation (3b) which takes the form

$$\frac{\partial \tau}{\partial v} = \Psi_1 + \bar{\pi}\rho + \pi\sigma + \epsilon\tau - \bar{\epsilon}\tau + \rho\tau + \sigma\bar{\tau}$$

gives us the value of τ on $\mathcal{N}_\star$. Similarly, the structure equations (3h), (3p) and the Bianchi identity equation (4e)

$$\begin{aligned}
P^{\mathcal{A}}\frac{\partial \pi}{\partial x^{\mathcal{A}}} - \frac{\partial \mu}{\partial v} &= -\Psi_2 + \epsilon\mu + \bar{\epsilon}\mu + \bar{\alpha}\pi - \beta\pi - \pi\bar{\pi} - \mu\rho - \lambda\sigma,\\
\bar{P}^{\mathcal{A}}\frac{\partial \pi}{\partial x^{\mathcal{A}}} - \frac{\partial \lambda}{\partial v} &= 3\epsilon\lambda - \bar{\epsilon}\lambda - \alpha\pi + \bar{\beta}\pi - \pi^2 - \lambda\rho - \mu\bar{\sigma},\\
\frac{\partial \Psi_2}{\partial v} - \bar{P}^{\mathcal{A}}\frac{\partial \Psi_1}{\partial x^{\mathcal{A}}} &= -\lambda\Psi_0 + 2(\pi - \alpha)\Psi_1 + 3\rho\Psi_2,
\end{aligned}$$

give us the value of μ, λ and Ψ_2 on $\mathcal{N}_\star$. Next, the Bianchi identity equations (4g) and (4c)

$$\begin{aligned}
\frac{\partial \Psi_3}{\partial v} - \bar{P}^{\mathcal{A}}\frac{\partial \Psi_2}{\partial x^{\mathcal{A}}} &= 2(\rho - \epsilon)\Psi_3 + 3\pi\Psi_2 - 2\lambda\Psi_1,\\
\bar{P}^{\mathcal{A}}\frac{\partial \Psi_3}{\partial x^{\mathcal{A}}} - \frac{\partial \Psi_4}{\partial v} &= (4\epsilon - \rho)\Psi_4 - 2(2\pi + \alpha)\Psi_3 + 3\lambda\Psi_2,
\end{aligned}$$

show us the value of Ψ_3 and Ψ_4 on $\mathcal{N}_\star$. Finally, we have obtained all the initial values on $\mathcal{N}_\star \cup \mathcal{N}'_\star$ from the reduced data set $r_\star$. □

3.3 Rendall's local existence theory

In order to apply the basic local existence theory for the CIVP as formulated by Rendall [22] (see also Section 12.5 of [35]), one has to extract a suitable symmetric hyperbolic evolution system from the Einstein field equations. The gauge introduced in Section 3.1.1 allows us to perform this reduction.

3.3.1 Construction of the reduced evolution system

In the following it will be convenient to group the components of the frame in the vector valued function

$$\boldsymbol{e}^t \equiv (C^{\mathcal{A}},\ P^{\mathcal{A}},\ Q),$$

the spin connection coefficients not fixed by the gauge in

$$\boldsymbol{\Gamma}^t \equiv (\epsilon,\ \pi,\ \beta,\ \mu,\ \alpha,\ \lambda,\ \tau,\ \sigma,\ \rho),$$

and the independent components of the Weyl spinor as

$$\boldsymbol{\Psi}^t \equiv (\Psi_0,\ \Psi_1,\ \Psi_2, \Psi_3,\ \Psi_4),$$

where superscript-t denotes the operation of taking the transpose of a column vector.

A suitable symmetric hyperbolic system for the the frame components and the spin coefficients can be obtained from equations (3.11a), (3.11b), (3.11d) and (3a), (3b), (3c), (3d), (3f), (3g), (3k), (3m), (3o), respectively. These can be written in the schematic form

$$\begin{aligned}\boldsymbol{\mathcal{D}}_1 \boldsymbol{e} &= \boldsymbol{B}_1(\boldsymbol{\Gamma}, \boldsymbol{e})\boldsymbol{e},\\ \boldsymbol{\mathcal{D}}_2 \boldsymbol{\Gamma} &= \boldsymbol{B}_2(\boldsymbol{\Gamma}, \boldsymbol{\Psi})\boldsymbol{\Gamma},\end{aligned}$$

where $\boldsymbol{\mathcal{D}}_1$ and $\boldsymbol{\mathcal{D}}_2$ are matrix operators given by,

$$\begin{aligned}\boldsymbol{\mathcal{D}}_1 &= \operatorname{diag}(\Delta,\ \Delta,\ D),\\ \boldsymbol{\mathcal{D}}_2 &= \operatorname{diag}(\Delta,\ \Delta,\ \Delta,\ \Delta,\ \Delta,\ \Delta, D,\ D,\ D),\end{aligned}$$

and $\boldsymbol{B}_1$, $\boldsymbol{B}_2$ are smooth matrix-valued functions of their arguments whose explicit form will not be required in the subsequent analysis in this section.

The Bianchi identity equations (4a)-(4h) can be reorganised as

$$\boldsymbol{\mathcal{D}}_3 \boldsymbol{\Psi} = \boldsymbol{B}_3 \boldsymbol{\Psi} \tag{3.12}$$

where

$$\mathcal{D}_3 = \begin{pmatrix} \Delta & -\delta & 0 & 0 & 0 \\ -\bar{\delta} & D+\Delta & -\delta & 0 & 0 \\ 0 & -\bar{\delta} & D+\Delta & -\delta & 0 \\ 0 & 0 & -\bar{\delta} & D+\Delta & -\delta \\ 0 & 0 & 0 & -\bar{\delta} & D \end{pmatrix}$$

and $\boldsymbol{B_3} = \boldsymbol{B_3}(\boldsymbol{\Gamma})$. Writing

$$\boldsymbol{\mathcal{D}}_3 = \boldsymbol{A}_3^{\mu}\partial_\mu$$

one has that

$$\begin{aligned} \boldsymbol{A}_3^{v} &= \mathrm{diag}(0,\ 1,\ 1,\ 1,\ 1), \\ \boldsymbol{A}_3^{u} &= \mathrm{diag}(Q,\ Q,\ Q,\ Q,\ 0), \end{aligned}$$

and

$$\boldsymbol{A}_3^{\mathcal{A}} = \begin{pmatrix} 0 & -P^{\mathcal{A}} & 0 & 0 & 0 \\ -\bar{P}^{\mathcal{A}} & C^{\mathcal{A}} & -P^{\mathcal{A}} & 0 & 0 \\ 0 & -\bar{P}^{\mathcal{A}} & C^{\mathcal{A}} & -P^{\mathcal{A}} & 0 \\ 0 & 0 & -\bar{P}^{\mathcal{A}} & C^{\mathcal{A}} & -P^{\mathcal{A}} \\ 0 & 0 & 0 & -\bar{P}^{\mathcal{A}} & C^{\mathcal{A}} \end{pmatrix}.$$

The evolution system (3.12) for the components of the Weyl tensor are obtained through the combinations (4b), (4h)-(4a), (4e)+(4f), (4d)+(4g) and $-$(4c) respectively. It can be readily verified that the matrices $\boldsymbol{A}_3^{\mu}$ are Hermitian. Moreover,

$$\boldsymbol{A}_3^{\mu}(l_\mu + n_\mu) = \mathrm{diag}(1,\ 2,\ 2,\ 2,\ 1)$$

is clearly positive definite. We can summarise the above discussion with:

Lemma 3 (***the evolution system***)**.** *The evolution system*

$$\boldsymbol{\mathcal{D}}_1\boldsymbol{e} = \boldsymbol{B}_1\boldsymbol{e}, \tag{3.13a}$$

$$\boldsymbol{\mathcal{D}}_2\boldsymbol{\Gamma} = \boldsymbol{B}_2\boldsymbol{\Gamma}, \tag{3.13b}$$

$$\boldsymbol{\mathcal{D}}_3\boldsymbol{\Psi} = \boldsymbol{B}_3\boldsymbol{\Psi}, \tag{3.13c}$$

implied by the NP field equations written in Stewart's gauge (see Section 3.1.1) is symmetric hyperbolic with respect to the direction given by $\tau^a = l^a + n^a$.

Remark 6. In the following, making use of the standard terminology, we call the evolution system the reduced Einstein field equations.

Remark 7. *The symmetric hyperbolicity of the reduced equations* (3.13a)-(3.13c) *is the key structural property which allows us to employ Rendall's local existence strategy —see the discussion in Section 3.3.2 below.*

As the hyperbolic reduction leading to the previous result makes use of a subset of the NP equations, it is also key to have a *propagation of the constraints* result for the discarded equations. Making use of analysis similar to the one discussed in Section 12.5 of [35] one obtains the following:

Proposition 2 (***propagation of the constraints***)**.** *A solution of the reduced vacuum Einstein field equations* (3.13a)-(3.13c) *on a neighbourhood $\mathcal{V}$ of $\mathcal{S}_\star$ on $J^+(\mathcal{S}_\star)$, the causal future of $\mathcal{S}_\star$, that coincides with initial data on $\mathcal{N}'_\star \cup \mathcal{N}_\star$ satisfying the vacuum Einstein equations is a solution to the vacuum Einstein field equations on $\mathcal{V}$.*

Remark 8. A consequence of the propagation of the constraints, once local existence has been established, is that we may use any combination of the NP field equations in their gauge simplified form in the required subsequent analysis. For example, from this point on we have $\pi = \alpha + \bar{\beta}$, and hence discard π or view it as a shorthand in what follows.

3.3.2 Computation of the formal derivatives on $\mathcal{N}'_\star \cup \mathcal{N}_\star$

As already mentioned, Rendall's approach to the local existence of solutions to the characteristic problem for symmetric hyperbolic systems makes use of an auxiliary Cauchy problem on a spacelike hypersurface

$$\mathcal{S}_\star \equiv \{p \in \mathbb{R} \times \mathbb{R} \times \mathbb{S}^2 \mid v(p) + u(p) = 0\}.$$

The formulation of this problem crucially depends on Whitney's extension theorem. To apply this extension theorem it is necessary to be able to evaluate all derivatives (interior and transverse) of the initial data on $\mathcal{N}'_\star \cup \mathcal{N}_\star$. A discussion of the ideas behind Rendall's approach can be found in Section 12.5 of [35]. For completeness, a formulation of Rendall's result is given below:

Theorem 1 (***local existence for the CIVP, Rendall***). *Let $\mathcal{N}_\star$ and $\mathcal{N}'_\star$ denote two characteristic hypersurfaces for the symmetric hyperbolic system*

$$\mathbf{A}^\mu(x, \mathbf{u})\partial_\mu \mathbf{u} = \mathbf{B}(x, \mathbf{u})$$

with smooth, freely specifiable data on $\mathcal{N}_\star$ and $\mathcal{N}_\star$ such that all (formal) derivatives of $\mathbf{u}$ on $\mathcal{N}_\star \cup \mathcal{N}'_\star$ to any desired order can be computed in a neighbourhood $\mathcal{W} \subset \mathcal{N}_\star \cup \mathcal{N}'_\star$ of $\mathcal{N}_\star \cap \mathcal{N}'_\star$. Then there exists a unique solution $\mathbf{u}$ to the CIVP in a neighbourhood $\mathcal{V}$ of $\mathcal{N}_\star \cap \mathcal{N}'_\star$ with $u \geq 0$, $v \geq 0$.

An important property of the NP equations in Stewart's gauge is that they allow the computation of the (formal) derivatives of all the fields to any order from the reduced data $\mathbf{r}_\star$ provided in Lemma 2. This property is discussed in the next paragraphs.

Computation of formal derivatives on $\mathcal{N}_\star$. To compute the formal derivatives on $\mathcal{N}_\star$ one first observes that the partial derivatives ∂_v, ∂_2, ∂_3 are interior whereas ∂_u is transverse. In this case, direct inspection shows that except for

$$\partial_u Q, \qquad \partial_u \tau, \qquad \partial_u \Psi_4,$$

all ∂_u-derivatives of the unknowns in the vectors $\boldsymbol{e}$, $\boldsymbol{\Gamma}$, $\boldsymbol{\Psi}$ can be computed using the structure equations (3.11a), (3.11b), the NP Ricci identities (3a), (3c), (3d), (3g), (3i), (3k), (3o), (3r), and the Bianchi identities (4b), (4d), (4f) and (4h).

To obtain these exceptional cases one first applies $Q\boldsymbol{\partial}_u$ to both sides of equations (3.11d), (3b) and (4c) to obtain

$$\begin{aligned}
&Q\partial_v(\partial_u Q) = -Q^2\partial_u(\epsilon + \bar{\epsilon}) - Q(\epsilon + \bar{\epsilon})\partial_u Q, \\
&Q\partial_v(\partial_u \tau) = L(\partial_u \tau), \\
&Q\partial_v(\partial_u \Psi_4) - Q\partial_u \bar{P}^{\mathcal{A}}\partial_{\mathcal{A}}\Psi_3 - Q\bar{P}^{\mathcal{A}}\partial_u\partial_{\mathcal{A}}\Psi_3 = M(\partial_u \Psi_4),
\end{aligned}$$

where L, M are smooth functions of $\{\boldsymbol{e}, \boldsymbol{\Gamma}, \boldsymbol{\Psi}\}$ and their $\boldsymbol{n}$-direction derivatives. One can regard the above equations as first order linear ordinary differential equations for $\partial_u Q, \partial_u \tau, \partial_u \Psi_4$ along the generators of $\mathcal{N}_\star$. Since we have all the initial values of the components of $\{\boldsymbol{e}, \boldsymbol{\Gamma}, \boldsymbol{\Psi}\}$ on $\mathcal{N}'_\star \cup \mathcal{N}_\star$, we can compute the initial value of $\partial_u Q,\ \partial_u \tau,\ \partial_u \Psi_4$ on $\mathcal{S}_\star$. The general results for the existence theorem of ordinary differential equations ensures that the above equation system can be solved in a neighbourhood of $\mathcal{S}_\star$. *In the following, we assume that the initial data provided is such that it yields a uniform existence domain for the solutions to the transport equations —this is a major assumption on the initial data in this construction.* Accordingly, all the first transverse derivatives on $\mathcal{N}_\star$ can be explicitly computed. The higher order ∂_u-derivatives can be computed in a similar way. Throughout it is assumed that the neighbourhood on which this construction can be done in uniform for any order of the derivatives.

Computation of formal derivatives on $\mathcal{N}'_\star$. The analysis of the formal derivatives on $\mathcal{N}'_\star$ is almost the mirror image of that on $\mathcal{N}_\star$. In this case $\partial_u, \partial_2, \partial_3$ are interior while ∂_v is transverse. Accordingly, except for

$$\partial_v C^A, \qquad \partial_v \epsilon, \qquad \partial_v \Psi_0,$$

all ∂_v-derivatives of the components of $\{\boldsymbol{e}, \boldsymbol{\Gamma}, \boldsymbol{\Psi}\}$ can be computed using the structure equations (3.11c)-(3.11d), the Ricci identities and the Bianchi identity. Applying the directional derivative $D = \partial_v + C^A \partial_A$ to both sides of equations (3.11a), (3a) and (4b) one obtains equations which can be regarded as first order linear ordinary differential equations for $\partial_v C^A, \partial_v \epsilon, \partial_v \Psi_0$. The solutions to these equations can be obtained from the initial values prescribed on $\mathcal{S}_\star$. Thus, all transverse derivatives can be computed in a neighbourhood of $\mathcal{S}_\star$ on $\mathcal{N}'_\star$. A similar procedure applies to higher order derivatives.

The analysis described in the previous paragraph proves the following lemma:

Lemma 4 (***computation of formal derivatives***)**.** *Any arbitrary formal derivatives of the unknown functions $\{\boldsymbol{e}, \boldsymbol{\Gamma}, \boldsymbol{\Psi}\}$ on $\mathcal{N}'_\star \cup \mathcal{N}_\star$ can be computed from the prescribed initial data $\mathbf{r}_\star$ for the reduced vacuum Einstein field equations on $\mathcal{N}'_\star \cap \mathcal{N}_\star$.*

Combining the analysis above and applying Rendall's reduction strategy for the

CIVP for symmetric hyperbolic systems (see e.g. Section 12.5 of [35]) one obtains the following local existence result in a neighbourhood of $\mathcal{S}_\star = \mathcal{N}'_\star \cup \mathcal{N}_\star$:

Theorem 2 (***existence and uniqueness to the characteristic problem*).** *Given a smooth reduced initial data set $\boldsymbol{r}_\star$ for the vacuum Einstein field equations on $\mathcal{N}'_\star \cup \mathcal{N}_\star$, there exists a unique smooth solution of the vacuum Einstein field equations in a neighbourhood $\mathcal{V}$ of $\mathcal{S}_\star$ on $J^+(\mathcal{S}_\star)$ which implies the prescribed initial data on $\mathcal{N}'_\star \cup \mathcal{N}_\star$.*

Remark 9. The proof of the above result has two distinct parts. In a first stage one uses Rendall's reduction procedure to show the existence of a solution in a neighbourhood of $\mathcal{V}$. In a second stage one shows that this solution to the reduced equations implies, in fact, a solution to the full Einstein field equations. This part of the argument relies on the propagation of the constraints as given in Proposition 2.

3.4 Setting-up Luk's strategy

In this section we begin the implementation of Luk's strategy to obtain an improved existence interval for the solutions to the CIVP for the NP field equations in Stewart's gauge.

3.4.1 Outline and main strategy

As the argument leading to the improved existence result for the CIVP is lengthy, we provide here a summary of the role of the various lemmas and propositions and a discussion of how they fit into the overall analysis. The whole scheme is based on the use of sequentially more sophisticated *a priori* estimates of an arbitrary solution that, ultimately, arrives at a contradiction giving us the desired result. This priori estimate is made in the rectangular neighbourhood along $\mathcal{N}_\star$ and hence we name the outgoing direction, $\boldsymbol{l}$–the long direction and the ingoing direction, $\boldsymbol{n}$–the short direction.

Step 0. Estimates for the components of the frame. The basic step in the construction is to obtain estimates on the components of the frame. This can be done by assuming control on the L^∞-norm on the spheres $\mathcal{S}_{u,v}$ of a number of spin connection coefficients by a constant Δ_Γ. A peculiarity of the analysis is that one needs to introduce a certain derivative (to be denoted by χ) of the components of the frame as an unknown to quick-start the argument —this quantity, which is at the level of the spin connection coefficients, does not arise in the original NP formalism. The key result in this step is Lemma 5 and 6 in which the frame coefficients Q and $P^{\mathcal{A}}$ are controlled by their initial data and Lemma 7 in which the frame coefficients $C^{\mathcal{A}}$ are controlled along the *short direction*.

The bounds on the components of the frame allow us to control in a systematic and streamlined manner the solutions to transport equations along null directions in terms of integral quantities over the spheres $\mathcal{S}_{u,v}$. The technical results required to this end are presented in Lemmas 8 and 9. From these, more specific results valid for L^p and L^∞ norms are given in Propositions 3, 4, 6 and 7. Within our geometric setup and gauge these results are fairly general and are used repeatedly in the subsequent steps of the procedure.

Step 1. Estimates for the connection coefficients. With the general technology to study transport equations along the generators of light cones has been established, one can proceed to control the spin connection coefficients. The key idea of this analysis is the integration of the transport equations implied by the Ricci identities. In a first step, in Proposition 8, assuming control on the supremum norm of the third angular derivatives of the NP connection coefficient τ and on the components of the curvature one obtains control on the supremum norm of the various connection coefficients and τ itself. This result is used in turn in Proposition 9 to obtain control on the L^4-norms of the connection coefficients and the L^2-norm of their derivatives in Proposition 10.

Step 2. First estimate for the curvature. A first estimate for the components of the Weyl tensor is given in Proposition 11. In this result one assumes control of the components of the Weyl tensor along the light cones and of the L^2-norm of the third angular derivatives of the connection coefficient τ on the spheres to obtain control of the components of the Weyl tensor on the spheres.

The results of the steps 1 and 2 are conveniently summarised in Proposition 12 in which an assumed control on the components of the curvature along light cones and of the L^2-norm of the third angular derivatives of τ is used to obtain control on the spheres $\mathcal{S}_{u,v}$ of various norms of the connection and its derivatives and of the components of the curvature.

Step 3. Improved estimate for the connection. In the next step one obtains an improved estimate for the connection in which the third angular derivatives of the connection, including τ, are controlled assuming control only on the curvature along the light cones. This result is given in Proposition 13.

Step 4. Main estimates for the curvature. At this point we are in a position to run the central part of the argument, which depends crucially on the particular structure of the Bianchi identities. General inequalities for integrals of the various components of the Weyl tensor implied by the Bianchi identities are given in Propositions 14, 15 and 16 and 17. The whole argument is wrapped up in Proposition 18 in which, under the boundedness of the connection and the curvature on the initial null hypersurfaces one obtains control of the curvature on later null hypersurfaces. This is the crucial estimate which allows us to close the lengthy boostrap argument.

Final step. Last slice argument. The control of various norms of the connection and curvature obtained in the previous steps do not provide, by themselves, the improved existence result. For this, we make use of a *last slice argument* in which one argues by contradiction under the assumption that the solution to the evolution equations breaks down at some point. The estimates of the previous steps show that this assumption leads to a contradiction.

3.4.2 Definitions and conventions

In this section we set up the conventions for the various norms that will be used in the subsequent analysis.

Integration. In the following let ϕ denote a scalar field. Let U be a coordinate patch on $\mathcal{S}_\star$ and $U_{0,v}$ be a coordinate patch on $\mathcal{S}_{0,v}$ defined by generating U along $\boldsymbol{l}$ direction. Then define $U_{u,v}$ be a coordinate patch on $\mathcal{S}_{u,v}$ by generating U along $\boldsymbol{n}$ direction. Let $D_U \equiv \cup_{0\leq u\leq \epsilon, 0\leq v\leq I} U_{u,v}$. Let f^*_U be a partition of unity in U and then again generate along $\boldsymbol{l}$ and $\boldsymbol{n}$ to D_U where we denote by f_U. Functions f_U can be the partition of unity on $\mathcal{S}_{u,v}$. Now one has that $Df_U = 0 = \Delta f_U$. For conciseness, we will use the notation

$$\int_{\mathcal{S}_{u,v}} \phi \equiv \sum_U \int_{\mathcal{S}_{u,v}} \phi f_U \mathrm{d}\boldsymbol{\sigma}$$

to denote integration on the spheres $\mathcal{S}_{u,v}$ of constant u and v. In the previous expression $\mathrm{d}\boldsymbol{\sigma} \equiv \sqrt{|\det \boldsymbol{\sigma}|}\mathrm{d}x^2\mathrm{d}x^3$ denotes the volume element of the induced metric $\boldsymbol{\sigma}$ on $\mathcal{S}_{u,v}$. On the truncated causal diamonds $\mathcal{D}^t_{u,v}$ we define integration using the volume form of the spacetime metric,

$$\begin{aligned}\int_{\mathcal{D}^t_{u,v}} \phi &\equiv \sum_U \int_0^u \int_0^{\tilde{v}} \int_{\mathcal{S}_{u',v'}} \phi f_U \sqrt{|\det \boldsymbol{g}|}\mathrm{d}x^2\mathrm{d}x^3\mathrm{d}v'\mathrm{d}u' \\ &= \sum_U \int_0^u \int_0^{\tilde{v}} \int_{\mathcal{S}_{u',v'}} Q^{-1}\phi f_U \sqrt{|\det \boldsymbol{\sigma}|}\mathrm{d}x^2\mathrm{d}x^3\mathrm{d}v'\mathrm{d}u',\end{aligned}$$

with $\tilde{v} \equiv \min(v, t-u)$. We will denote integration over the complete causal diamond in the obvious manner by the natural omission of the superscript t on $\mathcal{D}^t_{u,v}$. As there are no canonical volume forms on the null hypersurfaces $\mathcal{N}_u$ and $\mathcal{N}'_v$ we define, for convenience the following:

$$\begin{aligned}\int_{\mathcal{N}_u(0,v)} \phi &\equiv \sum_U \int_0^v \int_{S_{u,v'}} \phi f_U \sqrt{|\det \boldsymbol{\sigma}|}\mathrm{d}x^2\mathrm{d}x^3\mathrm{d}v', \\ \int_{\mathcal{N}'_v(0,u)} \phi &\equiv \sum_U \int_0^u \int_{S_{u',v}} \phi f_U \sqrt{|\det \boldsymbol{\sigma}|}\mathrm{d}x^2\mathrm{d}x^3\mathrm{d}u'.\end{aligned}$$

We will often use the notation

$$\int_{\mathcal{N}^t_u} \phi \equiv \int_{\mathcal{N}_u(I^t)} \phi, \qquad \int_{\mathcal{N}'^t_v} \phi \equiv \int_{\mathcal{N}'_v[0,\varepsilon]^t} \phi$$

where $I^t \equiv [0, \min(v_\bullet, t-u)]$, with $v_\bullet \in \mathbb{R}^+$, denotes the *truncated long integration interval*. Similarly, the interval $[0,\varepsilon]^t \equiv [0, \min(\varepsilon, t-v)]$ will be called the *truncated short integration interval*. Dropping the superscript t we define the full long and short integration intervals, I and $[0,\varepsilon]$ respectively, and the norms on the full outgoing and incoming slices in the natural way.

Norms. Keeping the above conventions for integration in mind, we can now define the various norms to be used in our analysis. As before, let ϕ define a scalar field. For $1 \leq p < \infty$ we define the L^p-norms

$$||\phi||_{L^p(\mathcal{S}_{u,v})} \equiv \left(\int_{\mathcal{S}_{u,v}} |\phi|^p\right)^{1/p}, \quad ||\phi||_{L^p(\mathcal{N}_u^t)} \equiv \left(\int_{\mathcal{N}_u^t} |\phi|^p\right)^{1/p}, \quad ||\phi||_{L^p(\mathcal{N}_v'^t)} \equiv \left(\int_{\mathcal{N}_v'^t} |\phi|^p\right)^{1/p}.$$

The L^∞-norm is defined by

$$||\phi||_{L^\infty(\mathcal{S}_{u,v})} \equiv \sup_{\mathcal{S}_{u,v}} |\phi|.$$

For a tensor field $\phi_{a_1\ldots a_p}$ on the 2-sphere, we define

$$||\phi||_{L^p(\mathcal{S}_{u,v})} \equiv \left(\int_{\mathcal{S}_{u,v}} \langle\phi,\phi\rangle_{\boldsymbol{\sigma}}^{p/2}\right)^{1/p}, \; ||\phi||_{L^p(\mathcal{N}_u^t)} \equiv \left(\int_{\mathcal{N}_u^t} \langle\phi,\phi\rangle_{\boldsymbol{\sigma}}^{p/2}\right)^{1/p},$$
$$||\phi||_{L^p(\mathcal{N}_v'^t)} \equiv \left(\int_{\mathcal{N}_v'^t} \langle\phi,\phi\rangle_{\boldsymbol{\sigma}}^{p/2}\right)^{1/p},$$

where $\langle\phi,\phi\rangle_{\boldsymbol{\sigma}} \equiv \sigma^{a_1b_1}\ldots\sigma^{a_pb_p}\bar{\phi}_{a_1,\ldots,a_p}\phi_{b_1,\ldots,b_p}$. As in the definition of the integrals, suppresion of the label t denotes taking the norms over the full long and short integration intervals.

Remark 10. *By assumption, $\mathcal{S}_{u,v}$ is topological 2-sphere. So one has at least two coordinate patches U_1 and U_2 in an atlas: $U_1 \cap U_2 \neq \emptyset$, $U_1 \cup U_2 = \mathcal{S}_{u,v}$. Let f_1 and f_2 be the partition of unity for U_1 and U_2 such that: $f_1 + f_2 = 1$ and $\mathrm{supp} f_i \subset U_i$. Here $\mathrm{supp} f_i$ is the support of f_i which is compact. On each of U_1 and U_2 consider pairs $\{\boldsymbol{m}_1, \bar{\boldsymbol{m}}_1\}$, $\{\boldsymbol{m}_2, \bar{\boldsymbol{m}}_2\}$ spaning $\mathcal{TU}_1$ and $\mathcal{TU}_2$. On $U_1 \cap U_2$ these two frame are related via a rotation:*

$$\boldsymbol{m}_1 = e^{i\theta}\boldsymbol{m}_2 \qquad (\bar{\boldsymbol{m}}_1 = e^{-i\theta}\bar{\boldsymbol{m}}_2)$$

For a spin-weighted scalar ϕ over $\mathcal{S}_{u,v}$ (i.e. $\phi \to e^{is\theta}\phi$ for some s if $\boldsymbol{m} \to e^{i\theta}\boldsymbol{m}$) one has that $|\phi|^2 = \phi\bar{\phi}$ is defined unambiguously over the whole $\mathcal{S}_{u,v}$:

$$|\phi|^2 = \phi\bar{\phi} \to e^{is\theta}e^{-is\theta}\phi\bar{\phi} = |\phi|^2.$$

Accordingly, the L^p-type norm over $\mathcal{S}_{u,v}$ is unambiguously defined if ϕ has definite spin-weight. Thus the strategy in the argument is always working with spin-weight scalars. For example the NP scalars κ, τ, σ, ρ, ν, π, μ and λ, the Weyl curvature Ψ_i and the Ricci curvature Φ_{ij}. The most notorious exception is give by α and β. Although the gauge could set $\pi = \alpha + \bar{\beta}$ with π having spin-weight, $\varpi = \beta - \bar{\alpha}$ which is the connection on $\mathcal{S}_{u,v}$ has no definite spin weight. Let ϖ_i be the value of ϖ on patch U_i with respect to a fixed basis $\{\boldsymbol{m}_i, \bar{\boldsymbol{m}}_i\}$. One can define a frame-dependent scalar $\varpi \equiv \varpi_1 f_1 + \varpi_2 f_2$ over $\mathcal{S}_{u,v}$ globally. Then its norm can be defined in standard way introduced above. The discussion for ϵ is the same.

Integration by parts. In the following we denote by $\nabla\!\!\!\!/$ the covariant derivative of the induced metric $\boldsymbol{\sigma}$ on the spheres $\mathcal{S}_{u,v}$ of constant u and v. Similarly, $\Delta\!\!\!\!/$ will denote the associated Laplacian. As these spheres have no boundary we have

$$
\begin{aligned}
||\nabla\!\!\!\!/\phi||^2_{L^2(\mathcal{S}_{u,v})} &= \int_{\mathcal{S}_{u,v}} \sigma^{ab}\nabla\!\!\!\!/_a\phi\nabla\!\!\!\!/_b\bar{\phi} = \int_{\mathcal{S}_{u,v}} \nabla\!\!\!\!/_a(\sigma^{ab}\phi\nabla\!\!\!\!/_b\bar{\phi}) - \int_{\mathcal{S}_{u,v}} \phi\Delta\!\!\!\!/\bar{\phi}, \\
&= -\int_{\mathcal{S}_{u,v}} \phi\Delta\!\!\!\!/\bar{\phi} \leq 2\left(\int_{\mathcal{S}_{u,v}} |\phi|^2\right)^{1/2}\left(\int_{\mathcal{S}_{u,v}} |\nabla\!\!\!\!/^2\phi|^2\right)^{1/2}
\end{aligned}
$$

where in the last step inequality (13) in Appendix 6.2 has been used. Integrating over $\langle\phi,\pi\rangle_{\boldsymbol{\sigma}}$ over two-spheres naturally defines an inner product, so we similarly obtain,

$$
\begin{aligned}
||\nabla\!\!\!\!/\phi||_{L^2(\mathcal{S}_{u,v})} &\leq ||\phi||_{L^2(\mathcal{S}_{u,v})} + ||\nabla\!\!\!\!/^2\phi||_{L^2(\mathcal{S}_{u,v})}, \\
||\nabla\!\!\!\!/^2\phi||_{L^2(\mathcal{S}_{u,v})} &\leq ||\nabla\!\!\!\!/\phi||_{L^2(\mathcal{S}_{u,v})} + ||\nabla\!\!\!\!/^3\phi||_{L^2(\mathcal{S}_{u,v})}.
\end{aligned}
$$

3.4.3 Estimates for the components of the frame

As a preliminary step we now show that, assuming the components of the connection coefficients are controlled by a basic boostrap assumption, it is possible to estimate the components of the NP frame in terms of the size of its initial data on $\mathcal{N}_\star \cup \mathcal{N}'_\star$. The key observation in the argument is that the structure equations provide Δ-equations for all the components of the frame. Given our particular choice of gauge, these equations are essentially ordinary differential equations with respect to the coordinate u. In fact as the structure equations form a neat hierarchy, they can be integrated sequentially. The quantity,

$$\Delta_{e_\star} \equiv \sup_{\mathcal{N}_\star, \mathcal{N}'_\star} \big(|Q|, |Q^{-1}|, |C^{\mathcal{A}}|, |P^{\mathcal{A}}| \big) \tag{3.14}$$

will be used to measure of the size of the initial data of the coefficients of the frame. *Throughout, given that the procedure has only a finite number of steps we denote all constants depending on the initial data generically by $C(\Delta_{e_\star})$ —the latter corresponds to the largest constant arising in the various steps.* For convenience in the subsequent discussion let

$$\chi \equiv \Delta \log Q.$$

The scalar χ, being a derivative of a component of the frame is at the same level of the connection coefficients. It provides a component of the connection which does not arise in the original NP formalism, but is needed to obtain a complete set of Δ equations for the frame. A direct computation using the definition of $\chi = \Delta \log Q$ and the NP Ricci identities yields

$$D\chi = \Psi_2 + \bar{\Psi}_2 + 2\alpha\tau + 2\bar{\beta}\tau + 2\bar{\alpha}\bar{\tau} + 2\beta\bar{\tau} + 2\tau\bar{\tau} - (\epsilon + \bar{\epsilon})\chi. \tag{3.15}$$

The initial data of χ on $\mathcal{N}'_\star$ is 0 due to the gauge choice that $Q = 1$ on $\mathcal{N}'_\star$. On $\mathcal{N}_\star$, making use of the information of α, β, τ, ϵ and Ψ_2 obtained in Lemma 2, one can compute the value of χ with equation (3.15). It will also be convenient to define,

$$\varpi \equiv \beta - \bar{\alpha}$$

corresponding to the only independent component of the connection on the spheres $\mathcal{S}_{u,v}$. As mentioned above, the proof is based on demonstrating a priori estimates for an arbitrary solution and consequently demonstrating that any such solution must extend to a neighborhood of $\mathcal{N}_\star \cup \mathcal{N}'_\star$. We therefore now introduce the following, which will be initially guaranteed on a sufficiently small diamond by Theorem 1, and will be employed in most of what follows:

Assumption 2 (***assumption to control the coefficients of the frame***)**.** *As-*

sume that we have a solution to the vacuum EFEs in Stewart's gauge satisfying,

$$||\{\mu, \lambda, \alpha, \beta, \tau, \chi\}||_{L^\infty(\mathcal{S}_{u,v})} \leq \Delta_\Gamma,$$

on a truncated causal diamond $\mathcal{D}^t_{u,v_\bullet}$*, where* Δ_Γ *is some constant.*

Lemma 5 (***control on the scalar field Q***)**.** *Under Assumption 2, if* $\varepsilon > 0$ *is sufficiently small, there exists a constant* C *depending on the size of the initial data such that*

$$Q^{-1}, Q \leq C(\Delta_{e_\star}),$$

on $\mathcal{D}^t_{u,v_\bullet}$.

Proof. Work under Assumption 2. Integrating the definition of $\chi = \Delta \log Q$ in the short (i.e. u) direction along an incoming null geodesic one readily finds that,

$$|Q - Q_\star| = \left| \int_0^\varepsilon \chi \mathrm{d}u \right| \leq \int_0^\varepsilon |\chi| \mathrm{d}u \leq \int_0^\varepsilon \Delta_\Gamma \mathrm{d}u = \Delta_\Gamma \varepsilon$$

for any v. It follows that

$$||Q - Q_\star||_{L^\infty(\mathcal{S}_{u,v})} \leq \Delta_\Gamma \varepsilon.$$

Hence, one can find a constant C depending on the initial data such that

$$Q^{-1}, Q \leq C(\Delta_{e_\star}).$$

□

Lemma 6 (***control on the components of the frame***)**.** *We require that* $P^\mathcal{A}$ *are bounded on* $U_{0,v}$ *such that* $\boldsymbol{\sigma}^{\mathcal{AB}}$ *is invertible and bounded above and below. Here* $U_{0,v}$ *is coordinate patch on* $\mathcal{S}_{0,v}$ *generated along* $\boldsymbol{l}$ *from coordinate patch* U *on* $\mathcal{S}_\star$*. Under Assumption 2, if* $\varepsilon > 0$ *is sufficiently small, there exists a constant* C*,* c *depending on the size of the initial data such that on* D_U *such that*

$$|\sigma^{\mathcal{AB}}|, \, |\sigma_{\mathcal{AB}}| \leq C(\Delta_{e_\star}), \qquad c(\Delta_{e_\star}) \leq |\det \boldsymbol{\sigma}| \leq C(\Delta_{e_\star}).$$

and on $\mathcal{D}^t_{u,v_\bullet}$

$$\sup_{u,v} |\mathrm{Area}(\mathcal{S}_{u,v}) - \mathrm{Area}(\mathcal{S}_{0,v})| \leq C(\Delta_{e_\star})\Delta_\Gamma \varepsilon,$$

Proof. We can integrate the components $P^{\mathcal{A}}$ in the short direction using equation (3.11b). It follows then that

$$\begin{aligned}
\partial_u |P^{\mathcal{A}}|^2 &= \partial_u (P^{\mathcal{A}} \bar{P}^{\mathcal{A}}) = P^{\mathcal{A}} \partial_u \bar{P}^{\mathcal{A}} + \bar{P}^{\mathcal{A}} \partial_u P^{\mathcal{A}} \\
&= -Q^{-1} \left(P^{\mathcal{A}} (\bar{\mu} \bar{P}^{\mathcal{A}} + \lambda P^{\mathcal{A}}) + \bar{P}^{\mathcal{A}} (\mu P^{\mathcal{A}} + \bar{\lambda} \bar{P}^{\mathcal{A}}) \right) \\
&= -Q^{-1} \left(\bar{\mu} |P^{\mathcal{A}}|^2 + \lambda (P^{\mathcal{A}})^2 + \mu |P^{\mathcal{A}}|^2 + \bar{\lambda} (\bar{P}^{\mathcal{A}})^2 \right) \\
&\leq Q^{-1} (\mu + \bar{\mu} + \lambda + \bar{\lambda}) |P^{\mathcal{A}}|^2.
\end{aligned}$$

In the previous chain of inequalities it is understood that there is no summation on the repeated indices $\mathcal{A}$. From the last inequality one readily concludes that

$$\partial_u \ln |P^{\mathcal{A}}|^2 \leq 4 Q^{-1} \Delta_\Gamma$$

so that

$$|P^{\mathcal{A}}|^2 \leq |P^{\mathcal{A}}_\star|^2 \exp(4C(\Delta_{e_\star})\Delta_\Gamma \varepsilon).$$

As ε is arbitrary, we can choose it so that

$$|P^{\mathcal{A}}| \leq C(\Delta_{e_\star}), \text{ for any } u \text{ and fixed } v.$$

Then follows from the relation

$$\sigma^{\mathcal{AB}} = -P^{\mathcal{A}} \bar{P}^{\mathcal{B}} - P^{\mathcal{B}} \bar{P}^{\mathcal{A}},$$

one can control the components of the induced metric on the coordinate patch $U_{u,v}$ of $\mathcal{S}_{u,v}$:

$$|\sigma^{\mathcal{AB}}|, \, |\sigma_{\mathcal{AB}}| \leq C(\Delta_{e_\star}), \qquad c(\Delta_{e_\star}) \leq |\det \boldsymbol{\sigma}| \leq C(\Delta_{e_\star}).$$

Here $c(\Delta_{e_\star})$ and $C(\Delta_{e_\star})$ are non-negative constants depend on $\Delta_{e_\star}$. Moreover, the boundedness of $|\det \boldsymbol{\sigma}|$ on each coordinate patch leads to that

$$\sup_{u,v} |Area(\mathcal{S}_{u,v}) - Area(\mathcal{S}_{0,v})| \leq C(\Delta_{e_\star})\Delta_\Gamma \varepsilon,$$

on $\mathcal{D}^t_{u,v_\bullet}$. Consequently the area of $\mathcal{S}_{u,v}$ is bounded above by a constant depending in initial data in the same region, for ε sufficiently small. $\square$

One can now use equation (3.11a) to integrate the coefficients $C^{\mathcal{A}}$. By a procedure similar to that used in the previous steps one has,

$$\begin{aligned}
|C^{\mathcal{A}} - C^{\mathcal{A}}_\star| &= \left| \int_0^\epsilon Q^{-1} \left((\bar{\tau} + \pi)P^{\mathcal{A}} + (\tau + \bar{\pi})\bar{P}^{\mathcal{A}} \right) \mathrm{d}u \right| \\
&\leq C(\Delta_{e_\star}) \int_0^\epsilon |(\bar{\tau} + \pi)P^{\mathcal{A}} + (\tau + \bar{\pi})\bar{P}^{\mathcal{A}}| \mathrm{d}u \\
&\leq 2C(\Delta_{e_\star}) \int_0^\epsilon |\bar{\tau} + \pi||P^{\mathcal{A}}| \mathrm{d}u \leq 2C(\Delta_{e_\star})^2 \Delta_\Gamma \varepsilon.
\end{aligned}$$

Here π should be viewed as a shorthand for $\pi = \alpha + \bar{\beta}$. Since $C^{\mathcal{A}}_\star = 0$ on $\mathcal{N}_\star$, we arrive at:

Lemma 7 (***control on the components of the frame. II***). *Under Assumption 2, if $\varepsilon > 0$ is sufficiently small, then there is a constant $C(\Delta_{e_\star}, \Delta_\Gamma)$ depending only on the initial data such that on coordinate patch D_U, choosing ε suitably, one has $|C^{\mathcal{A}}| \leq C(\Delta_{e_\star}, \Delta_\Gamma)\varepsilon$ on $\mathcal{D}^t_{u,v_\bullet}$.*

3.4.4 General estimates for transport equations

The purpose of this section is to develop a general set of tools that allow us to obtain estimates from the transport equations on hypersurfaces of constant u or v. The prototype of these transport equations are the NP Ricci identities (3a)-(3r). The results of this section do not depend on Assumption 2 unless explicitly stated.

Derivatives of integrals over $\mathcal{S}_{u,v}$. We are mostly interested on integral estimates over the spheres $\mathcal{S}_{u,v}$ and how they evolve along null directions. In the following we will systematically need to compute derivatives of integrals over $\mathcal{S}_{u,v}$ with respect to the advanced and retarded null coordinates. The key observation in this respect is the following:

Lemma 8 (***computing derivatives of integrals over $\mathcal{S}_{u,v}$***)**.** *Given a scalar ϕ one has that*

$$\frac{\mathrm{d}}{\mathrm{d}v}\int_{\mathcal{S}_{u,v}}\phi=\int_{\mathcal{S}_{u,v}}\left(D\phi-2\rho\phi\right), \tag{3.16a}$$

$$\frac{\mathrm{d}}{\mathrm{d}u}\int_{\mathcal{S}_{u,v}}\phi=\int_{\mathcal{S}_{u,v}}Q^{-1}\left(\Delta\phi+2\mu\phi\right), \tag{3.16b}$$

Proof. The proof follows a direct computation. More precisely, one has that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}v}\int_{\mathcal{S}_{u,v}}\phi &= \int_{\mathcal{S}_{u,v}}\frac{\partial}{\partial v}(\phi\sqrt{|\det\boldsymbol{\sigma}|})\mathrm{d}x^2\mathrm{d}x^3\\
&=\int_{\mathcal{S}_{u,v}}\Big(D(\phi\sqrt{|\det\boldsymbol{\sigma}|})-C^{\mathcal{A}}\partial_{\mathcal{A}}(\phi\sqrt{\det\boldsymbol{\sigma}}\Big))\mathrm{d}x^2\mathrm{d}x^3\\
&=\int_{\mathcal{S}_{u,v}}\Big(D\phi\sqrt{|\det\boldsymbol{\sigma}|}+\phi D\sqrt{|\det\boldsymbol{\sigma}|}-C^{\mathcal{A}}\partial_{\mathcal{A}}(\phi\sqrt{|\det\boldsymbol{\sigma}|})\Big)\,\mathrm{d}x^2\mathrm{d}x^3.
\end{aligned}$$

For the second term in the integrand, $\phi D\sqrt{|\det\boldsymbol{\sigma}|}$, we find that

$$\begin{aligned}
D\sqrt{|\det\boldsymbol{\sigma}|}&=\frac{1}{2\sqrt{|\det\boldsymbol{\sigma}|}}D\det\boldsymbol{\sigma}=\frac{|\det\boldsymbol{\sigma}|}{2\sqrt{|\det\boldsymbol{\sigma}|}}\sigma^{\mathcal{AB}}D\sigma_{\mathcal{AB}}=-\frac{\sqrt{|\det\boldsymbol{\sigma}|}}{2}\sigma_{\mathcal{AB}}D\sigma^{\mathcal{AB}}\\
&=\sqrt{|\det\boldsymbol{\sigma}|}\sigma_{\mathcal{AB}}\left(\bar{P}^{\mathcal{B}}DP^{\mathcal{A}}+P^{\mathcal{A}}D\bar{P}^{\mathcal{B}}\right)\\
&=\sqrt{|\det\boldsymbol{\sigma}|}\left(\sigma_{\mathcal{AB}}\bar{P}^{\mathcal{B}}\delta C^{\mathcal{A}}+\sigma_{\mathcal{AB}}P^{\mathcal{A}}\bar{\delta}C^{\mathcal{B}}-2\rho+\sigma\sigma_{\mathcal{AB}}\bar{P}^{\mathcal{A}}\bar{P}^{\mathcal{B}}+\bar{\sigma}\sigma_{\mathcal{AB}}P^{\mathcal{A}}P^{\mathcal{B}}\right)\\
&=\sqrt{|\det\boldsymbol{\sigma}|}\left(\bar{m}_{\mathcal{A}}\delta C^{\mathcal{A}}+m_{\mathcal{A}}\bar{\delta}C^{\mathcal{A}}-2\rho\right)=-\sqrt{|\det\boldsymbol{\sigma}|}\left(\partial_{\mathcal{A}}C^{\mathcal{A}}+2\rho\right),
\end{aligned}$$

where we have used Remark 1 and the structure equation (3.11c). For the third term in the integral one has that

$$\begin{aligned}
&\int_{\mathcal{S}_{u,v}}C^{\mathcal{A}}\partial_{\mathcal{A}}\left(\phi\sqrt{|\det\boldsymbol{\sigma}|}\right)\mathrm{d}x^2\mathrm{d}x^3\\
&=\int_{\mathcal{S}_{u,v}}\partial_{\mathcal{A}}\left(C^{\mathcal{A}}\phi\sqrt{|\det\boldsymbol{\sigma}|}\right)\mathrm{d}x^2\mathrm{d}x^3-\int_{\mathcal{S}_{u,v}}\phi\partial_{\mathcal{A}}C^{\mathcal{A}}\sqrt{|\det\boldsymbol{\sigma}|}\mathrm{d}x^2\mathrm{d}x^3\\
&=-\int_{\mathcal{S}_{u,v}}\phi\partial_{\mathcal{A}}C^{\mathcal{A}}\sqrt{|\det\boldsymbol{\sigma}|}\mathrm{d}x^2\mathrm{d}x^3+\int_{\mathcal{S}_{u,v}}\nabla_{\mathcal{A}}\left(C^{\mathcal{A}}\phi\sqrt{|\det\boldsymbol{\sigma}|}\right)
\end{aligned}$$

$$= -\int_{\mathcal{S}_{u,v}} \phi \partial_{\mathcal{A}} C^{\mathcal{A}} \sqrt{|\det \boldsymbol{\sigma}|} \mathrm{d}x^2 \mathrm{d}x^3,$$

where for the last equality we have use Stokes' theorem and the fact that sphere has no boundary. Combining the above observations one finds that

$$\frac{\mathrm{d}}{\mathrm{d}v} \int_{\mathcal{S}_{u,v}} \phi = \int_{\mathcal{S}_{u,v}} (D\phi - 2\rho\phi) \sqrt{|\det \boldsymbol{\sigma}|} \mathrm{d}x^2 \mathrm{d}x^3.$$

To compute the derivative with respect to u, we first consider

$$\begin{aligned}
\Delta \sqrt{|\det \boldsymbol{\sigma}|} &= -\frac{1}{2} \sqrt{|\det \boldsymbol{\sigma}|} \sigma_{\mathcal{A}\mathcal{B}} \Delta \sigma^{\mathcal{A}\mathcal{B}} = \frac{1}{2} \sqrt{|\det \boldsymbol{\sigma}|} \sigma_{\mathcal{A}\mathcal{B}} \left(\bar{P}^{\mathcal{B}} \Delta P^{\mathcal{A}} + P^{\mathcal{A}} \Delta \bar{P}^{\mathcal{B}} \right) \\
&= \frac{1}{2} \sqrt{|\det \boldsymbol{\sigma}|} \sigma_{\mathcal{A}\mathcal{B}} \left(\bar{P}^{\mathcal{B}} (-\mu P^{\mathcal{A}} - \bar{\lambda} \bar{P}^{\mathcal{A}}) + P^{\mathcal{A}} (-\bar{\mu} \bar{P}^{\mathcal{B}} - \lambda P^{\mathcal{B}}) \right) \\
&= \frac{1}{2} (\mu + \bar{\mu}) \sqrt{|\det \boldsymbol{\sigma}|} = \mu \sqrt{|\det \boldsymbol{\sigma}|}.
\end{aligned}$$

From the above identity one readily obtains

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}u} \int_{\mathcal{S}_{u,v}} \phi &= \int_{\mathcal{S}_{u,v}} \frac{\partial}{\partial u} \left(\phi \sqrt{|\det \boldsymbol{\sigma}|} \right) \mathrm{d}x^2 \mathrm{d}x^3 \\
&= \int_{\mathcal{S}_{u,v}} Q^{-1} \left(\sqrt{|\det \boldsymbol{\sigma}|} \Delta \phi + \phi \Delta \sqrt{|\det \boldsymbol{\sigma}|} \right) \mathrm{d}x^2 \mathrm{d}x^3 \\
&= \int_{\mathcal{S}_{u,v}} Q^{-1} (\Delta \phi + 2\mu\phi) \sqrt{|\det \boldsymbol{\sigma}|} \mathrm{d}x^2 \mathrm{d}x^3,
\end{aligned}$$

as required. □

Remark 11. *We follow the strategy from Chapter 3.1 of book [38] and provide a geometric version of proof. For each $\mathcal{S}_{u,v}$, there are two null directions $\boldsymbol{l}$ and $\boldsymbol{n}$, which are the direction of the generator on $\mathcal{N}_u$ and $\mathcal{N}_v$. For the convenience one define new null vector field $\hat{\boldsymbol{l}}$ ($\hat{\boldsymbol{n}}$) whose the 1-parameter family of the diffeomorphisms $\{\phi_t\}$ ($\{\underline{\phi}_t\}$) maps the $\mathcal{S}_{u,v}$ entirely to $\mathcal{S}_{u,v'}$ ($\mathcal{S}_{u',v}$). Then vector field $\hat{\boldsymbol{l}}$ ($\hat{\boldsymbol{n}}$) is said to be equivariant relative to the foliation. From the definition, a curve on $\mathcal{S}_{u,v}$ can be mapped to a curve on $\mathcal{S}_{u,v'}$ by ϕ_t, hence the push forward of vector $V|_{\mathcal{S}_{u,v}}$, $\phi_{t\star}V$ is tangent on $\mathcal{S}_{u,v'}$. From the Lemma 3.1.2 of book [38], this requirement leads to $\hat{\boldsymbol{l}}(u) = 0$, $\hat{\boldsymbol{l}}(v) = 1$ and $\hat{\boldsymbol{n}}(u) = 1$,$\hat{\boldsymbol{n}}(v) = 0$. Therefore $\hat{\boldsymbol{l}}$ must be proportional to $\boldsymbol{l}$ and $\hat{\boldsymbol{n}}$ is proportional to $\boldsymbol{n}$. Then one can compute and obtain that $\hat{\boldsymbol{l}} = \boldsymbol{l}$ and $\hat{\boldsymbol{n}} = Q^{-1}\boldsymbol{n}$.*

With these preparations in hand, precisely one has that

$$\frac{\mathrm{d}}{\mathrm{d}v}\int_{\mathcal{S}_{u,v}} f\varepsilon_{ab} = \lim_{\Delta v\to 0}\frac{1}{\Delta v}\left(\int_{\mathcal{S}_{u,v+\Delta v}} f\varepsilon_{ab} - \int_{\mathcal{S}_{u,v}} f\varepsilon_{ab}\right)$$

Here ε_{ab} is the volume element on $\mathcal{S}$. The diffeomorphism $\phi_{\Delta v}$ generated by vector field $\hat{\boldsymbol{l}}$ maps $\mathcal{S}_{u,v}$ to $\mathcal{S}_{u,v+\Delta v}$. Then one has

$$\int_{\mathcal{S}_{u,v+\Delta v}} f\varepsilon_{ab} = \int_{\mathcal{S}_{u,v}} \phi^{\star}_{\Delta v}(f\varepsilon_{ab}).$$

Hence one obtain that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}v}\int_{\mathcal{S}_{u,v}} f\varepsilon_{ab} &= \lim_{\Delta v\to 0}\frac{1}{\Delta v}\int_{\mathcal{S}_{u,v}}\left(\phi^{\star}_{\Delta v}(f\varepsilon_{ab}) - f\varepsilon_{ab}\right)\\ &= \int_{\mathcal{S}_{u,v}} \mathscr{L}_{\hat{\boldsymbol{l}}}(f\varepsilon_{ab}).\end{aligned}$$

Here $\mathscr{L}_{\hat{\boldsymbol{l}}}$ is the Lie derivative with respect to $\hat{\boldsymbol{l}}$. One can compute that

$$\mathscr{L}_{\hat{\boldsymbol{l}}}(f\varepsilon_{ab}) = \mathscr{L}_{\boldsymbol{l}}(f\varepsilon_{ab}) = \mathscr{L}_{\boldsymbol{l}}(f)\varepsilon_{ab} + f\mathscr{L}_{\boldsymbol{l}}(\varepsilon_{ab}).$$

The first term on the right hand side is $Df\varepsilon_{ab}$ and the second term is a 2-form on $\mathcal{S}_{u,v}$ which is proportional to ε_{ab}. Namely one has a scalar h such that

$$\mathscr{L}_{\boldsymbol{l}}(\varepsilon_{ab}) = h\varepsilon_{ab}$$

Contract with ε^{ab} one can fix that $h = \frac{1}{2}\sigma^{ab}\nabla_a l_b = -(\rho+\bar{\rho})$. Collect the results above and consider the gauge choice $\bar{\rho} = \rho$ one has

$$\frac{\mathrm{d}}{\mathrm{d}v}\int_{\mathcal{S}_{u,v}} \phi = \int_{\mathcal{S}_{u,v}} \left(D\phi - 2\rho\phi\right).$$

The derivative of integral with respect to u can be obtained in the same way.

Integrals over $\mathcal{D}_{u,v}$. The construction of energy-type estimates for the components of the Weyl tensor require further integral identities. These integrals allow us to write the integral over the diamond $\mathcal{D}_{u,v}$ of the D and Δ-derivatives of the components of the Weyl tensor in terms of integrals on the light cones and an integral over the bulk diamond of the (undifferentiated) components.

Lemma 9 (***integral over causal diamonds of derivatives of a scalar***)**.** *Let f be a scalar field in the causal diamond $\mathcal{D}_{u,v}$. One has then that*

$$
\begin{aligned}
\int_{\mathcal{D}_{u,v}} Df &= \int_{\mathcal{N}'_v(0,u)} Q^{-1}f - \int_{\mathcal{N}'_0(0,u)} Q^{-1}f + \int_{\mathcal{D}_{u,v}} (2\rho + \epsilon + \bar{\epsilon})f, \\
\int_{\mathcal{D}_{u,v}} \Delta f &= \int_{\mathcal{N}_u(0,v)} f - \int_{\mathcal{N}_0(0,v)} f - \int_{\mathcal{D}_{u,v}} 2\mu f.
\end{aligned}
$$

Proof. The proof of the identities follows by integration by parts. For the long direction we have, by definition, that

$$
\int_{\mathcal{D}_{u,v}} Df = \int_0^u \int_0^v \int_{\mathcal{S}_{u',v'}} Q^{-1}(\partial_v f + C^{\mathcal{A}}\partial_{\mathcal{A}} f)\sqrt{|\det \boldsymbol{\sigma}|}\mathrm{d}x^2\mathrm{d}x^3\mathrm{d}u'\mathrm{d}v'.
$$

Now, on the one hand, integrating by parts with respect to v one has that,

$$
\begin{aligned}
&\int_0^u \int_0^v \int_{\mathcal{S}_{u',v'}} Q^{-1}\partial_v f \sqrt{|\det \boldsymbol{\sigma}|}\mathrm{d}x^2\mathrm{d}x^3\mathrm{d}v'\mathrm{d}u' \\
&\quad = \int_0^u \int_0^v \int_{\mathcal{S}_{u',v'}} \partial_v (Q^{-1} f \sqrt{|\det \boldsymbol{\sigma}|})\mathrm{d}x^2\mathrm{d}x^3\mathrm{d}v'\mathrm{d}u' \\
&\qquad - \int_0^u \int_0^v \int_{\mathcal{S}_{u',v'}} f\partial_v (Q^{-1} \sqrt{|\det \boldsymbol{\sigma}|})\mathrm{d}x^2\mathrm{d}x^3\mathrm{d}v'\mathrm{d}u', \\
&\quad = \int_{\mathcal{N}'_v(0,u)} Q^{-1}f - \int_{\mathcal{N}'_0(0,u)} Q^{-1}f \\
&\qquad - \int_0^u \int_0^v \int_{\mathcal{S}_{u',v'}} (f\partial_v Q^{-1}\sqrt{|\det \boldsymbol{\sigma}|} + Q^{-1}f\partial_v \sqrt{|\det \boldsymbol{\sigma}|})\mathrm{d}x^2\mathrm{d}x^3\mathrm{d}v'\mathrm{d}u'.
\end{aligned}
$$

On the other hand, integration by parts respect to the angular coordinates gives

$$
\begin{aligned}
&\int_0^u \int_0^v \int_{\mathcal{S}_{u',v'}} Q^{-1}C^{\mathcal{A}}\partial_{\mathcal{A}} f \sqrt{|\det \boldsymbol{\sigma}|}\mathrm{d}x^2\mathrm{d}x^3\mathrm{d}v'\mathrm{d}u' \\
&\quad = -\int_0^u \int_0^v \int_{\mathcal{S}_{u',v'}} f\partial_{\mathcal{A}} (Q^{-1}C^{\mathcal{A}} \sqrt{|\det \boldsymbol{\sigma}|})\mathrm{d}x^2\mathrm{d}x^3\mathrm{d}v'\mathrm{d}u',
\end{aligned}
$$

$$
= -\int_0^u \int_0^v \int_{\mathcal{S}_{u',v'}} (f\sqrt{|\det \boldsymbol{\sigma}|} C^{\mathcal{A}} \partial_{\mathcal{A}} Q^{-1} + Q^{-1} f \sqrt{|\det \boldsymbol{\sigma}|} \partial_{\mathcal{A}} C^{\mathcal{A}}
+ Q^{-1} f C^{\mathcal{A}} \partial_{\mathcal{A}} \sqrt{|\det \boldsymbol{\sigma}|}) \mathrm{d}x^2 \mathrm{d}x^3 \mathrm{d}v' \mathrm{d}u'.
$$

Thus, we have

$$
\int_{\mathcal{D}_{u,v}} Df = \int_{\mathcal{N}'_v(0,u)} Q^{-1} f - \int_{\mathcal{N}'_0(0,u)} Q^{-1} f
- \int_0^u \int_0^v \int_{S_{u',v'}} (\sqrt{|\det \boldsymbol{\sigma}|} f Q^{-2} DQ + Q^{-1} f D\sqrt{|\det \boldsymbol{\sigma}|} + Q^{-1} f \sqrt{|\det \boldsymbol{\sigma}|} \partial_{\mathcal{A}} C^{\mathcal{A}}) \mathrm{d}x^2 \mathrm{d}x^3 \mathrm{d}v' \mathrm{d}u'.
$$

Finally, making use of the expressions for DQ from equation (3.11d) and $D\sqrt{|\det \boldsymbol{\sigma}|}$ from Proposition 8, respectively, one obtains the desired identity.

To demonstrate the identity along the short direction one proceeds in a similar fashion. □

Corollary 1. *If $f = f_1 f_2$, then*

$$
\int_{\mathcal{D}_{u,v}} f_1 D f_2 + \int_{\mathcal{D}_{u,v}} f_2 D f_1 = \int_{\mathcal{N}'_v(0,u)} Q^{-1} f_1 f_2 - \int_{\mathcal{N}'_0(0,u)} Q^{-1} f_1 f_2 + \int_{\mathcal{D}_{u,v}} (2\rho + \epsilon + \bar{\epsilon}) f_1 f_2,
$$

$$
\int_{\mathcal{D}_{u,v}} f_1 \Delta f_2 + \int_{\mathcal{D}_{u,v}} f_2 \Delta f_1 = \int_{\mathcal{N}_u(0,v)} f_1 f_2 - \int_{\mathcal{N}_0(0,v)} f_1 f_2 - \int_{\mathcal{D}_{u,v}} 2\mu f_1 f_2.
$$

Basic L^p estimates. The first step in the analysis is the construction of L^p estimates. These estimates require *a priori* control of the NP spin connection coefficients ρ and μ. The reason for their special treatment can be traced back to their appearance in Lemma 8. Proceeding in this way we obtain the following:

Proposition 3 (***control of the L^p-norm with transport equations***)**.** *Work under Assumption 2. Assume furthermore on $\mathcal{D}^t_{u,v_\bullet}$ that*

$$
\sup_{u,v} ||\{\rho, \mu\}||_{L^\infty(\mathcal{S}_{u,v})} \leq \mathcal{O}.
$$

Then there exists $\varepsilon_\star = \varepsilon_\star(\Delta_{e_\star}, \mathcal{O})$ such that for all $\varepsilon \leq \varepsilon_\star$ and for every $1 \leq p < \infty$, *we have the estimates:*

$$||\phi||_{L^p(\mathcal{S}_{u,v})} \leq C(I,\mathcal{O})\left(||\phi||_{L^p(\mathcal{S}_{u,0})} + \int_0^v ||D\phi||_{L^p(\mathcal{S}_{u,v'})}\mathrm{d}v'\right),$$
$$||\phi||_{L^p(\mathcal{S}_{u,v})} \leq 2\left(||\phi||_{L^p(\mathcal{S}_{0,v})} + C(\Delta_{e_\star},\mathcal{O})\int_0^u ||\Delta\phi||_{L^p(\mathcal{S}_{u',v})}\mathrm{d}u'\right),$$

where, as elsewhere, I denotes the long direction interval.

Proof. Making use of the definition of $||\phi||_{L^p(\mathcal{S}_{u,v})}$ and the identity in Lemma 8, we have

$$\begin{aligned}
||\phi||^p_{L^p(\mathcal{S}_{u,v})} &= ||\phi||^p_{L^p(\mathcal{S}_{u,0})} + \int_0^v \frac{\mathrm{d}}{\mathrm{d}v}||\phi||^p_{L^p(\mathcal{S}_{u,v'})}\mathrm{d}v' \\
&= ||\phi||^p_{L^p(\mathcal{S}_{u,0})} + \int_0^v \left(\frac{\mathrm{d}}{\mathrm{d}v}\int_{\mathcal{S}_{u,v'}} |\phi|^p\right)\mathrm{d}v' \\
&= ||\phi||^p_{L^p(\mathcal{S}_{u,0})} + \int_0^v \left(\int_{\mathcal{S}_{u,v'}} (D|\phi|^p - 2\rho|\phi|^p)\right)\mathrm{d}v'.
\end{aligned}$$

Now, Young's inequality gives

$$D|\phi|^p = p|\phi|^{p-1}D|\phi| \leq p\left(\frac{(|\phi|^{p-1})^{\frac{p}{p-1}}}{p/(p-1)} + \frac{(D|\phi|)^p}{p}\right) = (p-1)|\phi|^p + (D|\phi|)^p.$$

Thus, we have that

$$\begin{aligned}
||\phi||^p_{L^p(\mathcal{S}_{u,v})} &\leq ||\phi||^p_{L^p(\mathcal{S}_{u,0})} + \int_0^v \left(\int_{\mathcal{S}_{u,v'}} (D|\phi|)^p + (p-1-2\rho)|\phi|^p\right)\mathrm{d}v' \\
&\leq ||\phi||^p_{L^p(\mathcal{S}_{u,0})} + \int_0^v \left(\int_{\mathcal{S}_{u,v'}} (D|\phi|)^p + C_1(\mathcal{O})|\phi|^p\right)\mathrm{d}v' \\
&\leq ||\phi||^p_{L^p(\mathcal{S}_{u,0})} + \int_0^v ||D\phi||^p_{L^p(\mathcal{S}_{u,v'})}\mathrm{d}v' + C_1(\mathcal{O})\int_0^v ||\phi||^p_{L^p(\mathcal{S}_{u,v'})}\mathrm{d}v'.
\end{aligned}$$

Now, making use of Grönwall's inequality, we obtain

$$||\phi||^p_{L^p(\mathcal{S}_{u,v})} \leq C(I,\mathcal{O})\left(||\phi||^p_{L^p(\mathcal{S}_{u,0})} + \int_0^v ||D\phi||^p_{L^p(\mathcal{S}_{u,v'})}\mathrm{d}v'\right)$$

$$
\begin{aligned}
&\leq C(I,\mathcal{O})\left(||\phi||^p_{L^p(\mathcal{S}_{u,0})} + \left(\int_0^v ||D\phi||_{L^p(\mathcal{S}_{u,v'})}\mathrm{d}v'\right)^p\right)\\
&\leq C(I,\mathcal{O})\left(||\phi||_{L^p(\mathcal{S}_{u,0})} + \int_0^v ||D\phi||_{L^p(\mathcal{S}_{u,v'})}\mathrm{d}v'\right)^p,
\end{aligned}
$$

so that, in fact, one has

$$
||\phi||_{L^p(\mathcal{S}_{u,v})} \leq C(I,\mathcal{O})\left(||\phi||_{L^p(\mathcal{S}_{u,0})} + \int_0^v ||D\phi||_{L^p(\mathcal{S}_{u,v'})}\mathrm{d}v'\right).
$$

Now, for the integration in the short direction $0 \leq u \leq \varepsilon$, using the assumption that $\sup_{u,v}||\mu||_{L^\infty(\mathcal{S}_{u,v})} \leq \mathcal{O}$, a similar argument as before, and now using Lemma 5, allows us to show that

$$
||\phi||^p_{L^p(\mathcal{S}_{u,v})} \leq ||\phi||^p_{L^p(\mathcal{S}_{0,v})} + C(\Delta_{e_\star})\left(C(\mathcal{O})\int_0^u ||\phi||^p_{L^p(\mathcal{S}_{u',v})}\mathrm{d}u' + \int_0^u ||\Delta\phi||^p_{L^p(\mathcal{S}_{u',v})}\mathrm{d}u'\right),
$$

so that one has

$$
||\phi||_{L^p(\mathcal{S}_{u,v})} \leq ||\phi||_{L^p(\mathcal{S}_{0,v})} + C(\Delta_{e_\star},\mathcal{O})\left(\int_0^u ||\phi||_{L^p(\mathcal{S}_{u',v})}\mathrm{d}u' + \int_0^u ||\Delta\phi||_{L^p(\mathcal{S}_{u',v})}\mathrm{d}u'\right).
$$

Then, using Grönwall's inequality one is led to

$$
||\phi||_{L^p(\mathcal{S}_{u,v})} \leq \exp(C(\Delta_{e_\star},\mathcal{O})\varepsilon)\left(||\phi||_{L^p(\mathcal{S}_{0,v})} + C(\Delta_{e_\star},\mathcal{O})\int_0^u ||\Delta\phi||_{L^p(\mathcal{S}_{u',v})}\mathrm{d}u'\right).
$$

From, the latter choosing $\varepsilon > 0$ small enough one concludes that

$$
||\phi||_{L^p(\mathcal{S}_{u,v})} \leq 2\left(||\phi||_{L^p(\mathcal{S}_{0,v})} + C(\Delta_{e_\star},\mathcal{O})\int_0^u ||\Delta\phi||_{L^p(\mathcal{S}_{u',v})}\mathrm{d}u'\right).
$$

□

As a particular example of the previous discussion consider $\phi = \delta f$, with $p = 2$. In this case one has

$$
||\delta f||_{L^2(\mathcal{S}_{u,v})} \leq C(I,\mathcal{O})\left(||\delta f||_{L^2(\mathcal{S}_{u,0})} + \int_0^v \left(\int_{\mathcal{S}_{u,v'}} D|\delta f|^2\right)^{1/2}\mathrm{d}v'\right).
$$

If $p=4$ one has that

$$||\delta f||_{L^4(\mathcal{S}_{u,v})} \leq C(I,\mathcal{O}) \left(||\delta f||_{L^4(\mathcal{S}_{u,0})} + \int_0^v \left(\int_{\mathcal{S}_{u,v'}} D|\delta f|^2 \right)^{1/4} \mathrm{d}v' \right).$$

For the short direction one readily obtains analogous expressions.

Basic L^∞ estimates. Our analysis will also require estimates on the L^∞ norm of various scalars. The first result in this direction is the following:

Proposition 4 (***supremum norm of solutions to transport equations*****).** *Work under Assumption 2. There exists $\varepsilon_\star$ such that for all $\varepsilon \leq \varepsilon_\star$, we have*

$$||\phi||_{L^\infty(\mathcal{S}_{u,v})} \leq ||\phi||_{L^\infty(\mathcal{S}_{u,0})} + \int_0^v ||D\phi||_{L^\infty(\mathcal{S}_{u,v'})}\mathrm{d}v',$$
$$||\phi||_{L^\infty(\mathcal{S}_{u,v})} \leq ||\phi||_{L^\infty(\mathcal{S}_{0,v})} + C(\Delta_{e_\star}) \int_0^u ||\Delta\phi||_{L^\infty(\mathcal{S}_{u',v})}\mathrm{d}u',$$

on $\mathcal{D}^t_{u,v_\bullet}$.

Proof. Given a fixed point $(u,0,x^{\mathcal{A}})$ on $\mathcal{N}'_\star$, and then integrating out along integral curves of l^a, conveniently parametrizing with v, gives

$$\phi_{\mathcal{S}_{u,v}} - \phi_{\mathcal{S}_{u,0}} = \int_0^v \frac{d\phi}{dv}\mathrm{d}v' = \int_0^v D\phi\mathrm{d}v'.$$

Fixing u, varying the angular point x^A on $\mathcal{N}'_\star$ arbitrarily, and taking the supremum we obtain the inequality of the of the proposition. The proof of the second inequality is similar. □

More advanced L^p-estimates. Finally, we discuss the construction of more refined L^p-estimates. As in the case of the basic L^p-estimates, these estimates require some *a priori* control on the L^∞-norm of the the NP spin connection coefficients ρ and μ. More precisely, one has the following:

Proposition 5 (L^4***-norm of solutions to transport equations*****).** *Work under Assumption 2. Assume, as in Proposition 3, furthermore that*

$$\sup_{u,v} ||\{\rho,\mu\}||_{L^\infty(\mathcal{S}_{u,v})} \leq \mathcal{O}.$$

on $\mathcal{D}^t_{u,v_\bullet}$. Then there exists $\varepsilon_\star = \varepsilon_\star(\Delta_{e_\star}, \mathcal{O})$ such that for all $\varepsilon \leq \varepsilon_\star$ we have the estimates:

$$||\phi||_{L^4(\mathcal{S}_{u,v})} \leq C(\Delta_{e_\star}, \mathcal{O}) \left(||\phi||_{L^4(\mathcal{S}_{u,0})} + ||D\phi||^{1/2}_{L^2(\mathcal{N}_u(0,v))} \left(||\phi||^2_{L^2(\mathcal{N}_u(0,v))} + ||\nabla\!\!\!\!/\, \phi||^2_{L^2(\mathcal{N}_u(0,v))} \right)^{1/4} \right),$$

$$||\phi||_{L^4(\mathcal{S}_{u,v})} \leq 2 \left(||\phi||_{L^4(\mathcal{S}_{0,v})} + C(\Delta_{e_\star}) ||\Delta\phi||^{1/2}_{L^2(\mathcal{N}'_v(0,u))} \left(||\phi||^2_{L^2(\mathcal{N}'_v(0,u))} + ||\nabla\!\!\!\!/\, \phi||^2_{L^2(\mathcal{N}'_v(0,u))} \right)^{1/4} \right),$$

on $\mathcal{D}^t_{u,v_\bullet}$.

Proof. The proof proceeds by direct computation. We first obtain the estimate on the long direction. Following arguments similar to those used in Proposition 3, we find that

$$\begin{aligned} ||\phi||^4_{L^4(\mathcal{S}_{u,v})} &= ||\phi||^4_{L^4(\mathcal{S}_{u,0})} + \int_0^v \left(\int_{S_{u,v'}} D|\phi|^4 - 2\rho|\phi|^4 \right) \mathrm{d}v' \\ &\leq ||\phi||^4_{L^4(\mathcal{S}_{u,0})} + 2\,\mathcal{O} \int_0^v ||\phi||^4_{L^4(\mathcal{S}_{u,v'})} \mathrm{d}v' + 4 \left(\int_{\mathcal{N}_u(0,v)} |\phi|^6 \right)^{1/2} \left(\int_{\mathcal{N}_u(0,v)} |D\phi|^2 \right)^{1/2}. \end{aligned} \tag{3.17}$$

Now, for small enough ε, using the Nirenberg-Sobolev inequality (see Appendix 6.2) we estimate:

$$\begin{aligned} \int_{\mathcal{N}_u(0,v)} |\phi|^6 &= \int_0^v \int_{\mathcal{S}_{u,v'}} |\phi|^6 \mathrm{d}v' = \int_0^v |||\phi|^3||^2_{L^2(\mathcal{S}_{u,v'})} \mathrm{d}v' \\ &\leq C(\Delta_{e_\star}) \int_0^v \left(|||\phi|^3||_{L^1(\mathcal{S}_{u,v'})} + ||\nabla\!\!\!\!/\, |\phi|^3||_{L^1(\mathcal{S}_{u,v'})} \right)^2 \mathrm{d}v' \\ &\leq C(\Delta_{e_\star}) \int_0^v \left(|||\phi|^2||_{L^2(\mathcal{S}_{u,v'})} ||\phi||_{L^2(\mathcal{S}_{u,v'})} + |||\phi|^2||_{L^2(\mathcal{S}_{u,v'})} ||\nabla\!\!\!\!/\, \phi||_{L^2(\mathcal{S}_{u,v'})} \right)^2 \mathrm{d}v' \\ &\leq C(\Delta_{e_\star}) \int_0^v ||\phi||^4_{L^4(\mathcal{S}_{u,v'})} \left(||\phi||_{L^2(\mathcal{S}_{u,v'})} + ||\nabla\!\!\!\!/\, \phi||_{L^2(\mathcal{S}_{u,v'})} \right)^2 \mathrm{d}v' \\ &\leq 2C(\Delta_{e_\star}) \left(\sup_{u,v} ||\phi||^4_{L^4(\mathcal{S}_{u,v})} \right) \int_0^v \left(||\phi||^2_{L^2(\mathcal{S}_{u,v'})} + ||\nabla\!\!\!\!/\, \phi||^2_{L^2(\mathcal{S}_{u,v'})} \right) \mathrm{d}v' \\ &\leq C(\Delta_{e_\star}) \left(\sup_{u,v} ||\phi||^4_{L^4(\mathcal{S}_{u,v})} \right) \left(||\phi||^2_{L^2(\mathcal{N}_u(0,v))} + ||\nabla\!\!\!\!/\, \phi||^2_{L^2(\mathcal{N}_u(0,v))} \right), \end{aligned}$$

where to pass from the second to the third line we have made use of Hölder's inequality and, to pass from the third to fourth we have extracted common factors. Making use of the above estimate in inequality (3.17), we have that

$$
\begin{aligned}
||\phi||^4_{L^4(\mathcal{S}_{u,v})} \leq{}& ||\phi||^4_{L^4(\mathcal{S}_{u,0})} + 2\,\mathcal{O}\int_0^v ||\phi||^4_{L^4(\mathcal{S}_{u,v'})}\mathrm{d}v' \\
&+ C(\Delta_{e_\star})\left(\sup_{u,v}||\phi||^2_{L^4(\mathcal{S}_{u,v})}\right)\left(||\phi||^2_{L^2(\mathcal{N}_u(0,v))} + ||\nabla\!\!\!/\,\phi||^2_{L^2(\mathcal{N}_u(0,v))}\right)^{1/2}||D\phi||_{L^2(\mathcal{N}_u(0,v))} \\
\leq{}& ||\phi||^4_{L^4(\mathcal{S}_{u,0})} + 2\,\mathcal{O}\int_0^v ||\phi||^4_{L^4(\mathcal{S}_{u,v'})}\mathrm{d}v' + C(\Delta_{e_\star})\delta\left(\sup_{u,v}||\phi||^4_{L^4(\mathcal{S}_{u,v})}\right) \\
&+ \frac{C(\Delta_{e_\star})}{4\delta}\left(||\phi||^2_{L^2(\mathcal{N}_u(0,v))} + ||\nabla\!\!\!/\,\phi||^2_{L^2(\mathcal{N}_u(0,v))}\right)||D\phi||^2_{L^2(\mathcal{N}_u(0,v))},
\end{aligned}
$$

for some $\delta > 0$. Now, choosing δ sufficiently small and making use of Grönwall's inequality, one finally obtains that

$$
||\phi||^4_{L^4(\mathcal{S}_{u,v})} \leq C(\Delta_{e_\star},\mathcal{O})\left(||\phi||^4_{L^4(\mathcal{S}_{u,0})} + ||D\phi||^2_{L^2(\mathcal{N}_u(0,v))}\left(||\phi||^2_{L^2(\mathcal{N}_u(0,v))} + ||\nabla\!\!\!/\,\phi||^2_{L^2(\mathcal{N}_u(0,v))}\right)\right).
$$

The proof of the estimate along the short direction is similar. In this case we can choose $\varepsilon > 0$ sufficiently small to make the overall constant equal to, say, 2. □

3.4.5 Sobolev inequalities

In the last step in our preparatory work, we now obtain Sobolev-type inequalities on the spheres $\mathcal{S}_{u,v}$ —i.e. estimates of the L^p-norms of a scalar in terms of its L^2-norms and those of its derivatives. The key tool in this analysis is the *isoperimetric Sobolev inequality on* $\mathcal{S}_{u,v}$ —see [24]:

Theorem 3 (***isoperimetric Sobolev inequality on*** $\mathcal{S}_{u,v}$**).** *Let ϕ denote an integrable function and with integrable first derivatives on $\mathcal{S}_{u,v}$. Then we have that*

$$
\int_{\mathcal{S}_{u,v}}|\phi-\bar{\phi}|^2 \leq \mathcal{I}(\mathcal{S}_{u,v})\left(\int_{\mathcal{S}_{u,v}}|\nabla\!\!\!/\,\phi|\right)^2, \tag{3.18}
$$

where $\bar{\phi}$ denotes the average of ϕ over $\mathcal{S}_{u,v}$ and $\mathcal{I}(\mathcal{S}_{u,v})$ is the isoperimetric constant.

Remark 12. The isoperimetric inequality can be shown to be controlled by the area of the 2-dimensional surfaces $\mathcal{S}_{u,v}$ —see e.g. [24]. Thus, if one has control over the area of the surface (as it is, in principle, in our setup), one has also control over the isoperimetric constant.

Using this we can prove the following result concerning Sobolev-type inequalities:

Proposition 6 (***Sobolev-type inequality. I***). *Work under Assumption 2. Let ϕ be a scalar field on $\mathcal{S}_{u,v}$ which is square-integrable with square-integrable first covariant derivatives. Then for each $2 < p < \infty$, $\phi \in L^p(\mathcal{S}_{u,v})$, there exists $\varepsilon_\star = \varepsilon_\star(\Delta_{e_\star}, \Delta_\Gamma)$ such that as long as $\varepsilon \leq \varepsilon_\star$, we have*

$$||\phi||_{L^p(\mathcal{S}_{u,v})} \leq G_p(\boldsymbol{\sigma}) \left(||\phi||_{L^2(\mathcal{S}_{u,v})} + ||\nabla\!\!\!\!/\,\phi||_{L^2(\mathcal{S}_{u,v})}\right)$$

where $G_p(\boldsymbol{\sigma})$ is a constant also depends on the isoperimetric constant $\mathcal{I}(\mathcal{S}_{u,v})$ and p, but is controlled by some $C(\Delta_{e_\star})$, $\nabla\!\!\!\!/$ is the induced connection on $\mathcal{S}_{u,v}$ which is associated with the metric $\boldsymbol{\sigma}$.

Proof. We make use of the following result which can be found in Lemma 5.1 in Chapter 5.2 of [25]:

$$(\mathrm{Area}(\mathcal{S}_{u,v}))^{-1/p} \, ||\phi||_{L^p(\mathcal{S}_{u,v})} \leq C_p \sqrt{\mathcal{I}'(\mathcal{S}_{u,v})} \left((\mathrm{Area}(\mathcal{S}_{u,v}))^{-1/2} \, ||\phi||_{L^2(\mathcal{S}_{u,v})} + ||\nabla\!\!\!\!/\,\phi||_{L^2(\mathcal{S}_{u,v})} \right), \tag{3.19}$$

where C_p is a numerical constant depending only on p,

$$\mathcal{I}'(\mathcal{S}_{u,v}) = \max\{1, \mathcal{I}(\mathcal{S}_{u,v})\},$$

where as above $\mathcal{I}(\mathcal{S}_{u,v})$ is the isoperimetric constant of $\mathcal{S}_{u,v}$. Now, under Assumption 2 we have that the area of $\mathcal{S}_{u,v}$ is finite in the tilted rectangle. Accordingly, inequality (3.19) can be adapted to our particular setting. □

Consequently we have the following two results:

Proposition 7 (***Sobolev-type inequality. II***). *Work under Assumption 2. There exists $\varepsilon_\star = \varepsilon_\star(\Delta_{e_\star}, \Delta_\Gamma)$ such that as long as $\varepsilon \leq \varepsilon_\star$, we have*

$$||\phi||_{L^\infty(\mathcal{S}_{u,v})} \leq G_p(\boldsymbol{\sigma}) \left(||\phi||_{L^p(\mathcal{S}_{u,v})} + ||\nabla\!\!\!\!/\,\phi||_{L^p(\mathcal{S}_{u,v})}\right),$$

with $2 < p < \infty$ *and* $G_p(\boldsymbol{\sigma}) \leq C(\Delta_{e_\star})$ *as above.*

Corollary 2 (***Sobolev-type inequality. III***)**.** *Work under Assumption 2. There exists* $\varepsilon_\star = \varepsilon_\star(\Delta_{e_\star}, \Delta_\Gamma)$ *such that as long as* $\varepsilon \leq \varepsilon_\star$*, we have*

$$||\phi||_{L^4(\mathcal{S}_{u,v})} \leq G(\boldsymbol{\sigma}) \left(||\phi||_{L^2(\mathcal{S}_{u,v})} + ||\nabla\!\!\!\!/\, \phi||_{L^2(\mathcal{S}_{u,v})} \right),$$
$$||\phi||_{L^\infty(\mathcal{S}_{u,v})} \leq G(\boldsymbol{\sigma}) \left(||\phi||_{L^2(\mathcal{S}_{u,v})} + ||\nabla\!\!\!\!/\, \phi||_{L^2(\mathcal{S}_{u,v})} + ||\nabla\!\!\!\!/^2 \phi||_{L^2(\mathcal{S}_{u,v})} \right),$$

again with $G(\boldsymbol{\sigma}) \leq C(\Delta_{e_\star})$*.*

3.5 Main estimates

In this section we provide a discussion of the construction of the main estimates required to obtain the improved existence result for the CIVP. The arguments rely heavily on the preparatory work carried out in the previous section.

3.5.1 Norms for the initial data

The boostrap argument requires assumptions on the size of the initial data. Following Luk [26], we define the following:

i). Norm for the initial value of the connection coefficients, given by

$$\Delta_{\Gamma_\star} \equiv \sup_{\mathcal{S}_{u,v} \subset \mathcal{N}_\star, \mathcal{N}'_\star} \sup_{\Gamma \in \{\mu,\lambda,\rho,\sigma,\alpha,\beta,\tau,\epsilon\}} \max\{1, \sum_{i=0}^{1} ||\nabla\!\!\!\!/^i \Gamma||_{L^\infty(\mathcal{S}_{u,v})}, \sum_{i=0}^{2} ||\nabla\!\!\!\!/^i \Gamma||_{L^4(\mathcal{S}_{u,v})},$$
$$\sum_{i=0}^{3} ||\nabla\!\!\!\!/^i \Gamma||_{L^2(\mathcal{S}_{u,v})}\}.$$

ii). Norm for the initial value of the components of the Weyl tensor, given by

$$\Delta_{\Psi_\star} \equiv \sup_{\mathcal{S}_{u,v} \subset \mathcal{N}_\star, \mathcal{N}'_\star} \sup_{\Psi \in \{\Psi_0,\Psi_1,\Psi_2,\Psi_3,\Psi_4\}} \max\{1, \sum_{i=0}^{1} ||\nabla\!\!\!\!/^i \Psi||_{L^4(\mathcal{S}_{u,v})}, \sum_{i=0}^{2} ||\nabla\!\!\!\!/^i \Psi||_{L^2(\mathcal{S}_{u,v})}\}$$
$$+ \sum_{i=0}^{3} \sup_{\Psi \in \{\Psi_0,\Psi_1,\Psi_2,\Psi_3\}} ||\nabla\!\!\!\!/^i \Psi||_{L^2(\mathcal{N}_\star)} + \sup_{\Psi \in \{\Psi_1,\Psi_2,\Psi_3,\Psi_4\}} ||\nabla\!\!\!\!/^i \Psi||_{L^2(\mathcal{N}'_\star)}.$$

iii). Norm for the components of the Weyl tensor at later null hypersurfaces, given by

$$\Delta_{\Psi} \equiv \sum_{i=0}^{3} \sup_{\Psi\in\{\Psi_0,\Psi_1,\Psi_2,\Psi_3\}} \sup_{u} ||\nabla^i \Psi||_{L^2(\mathcal{N}_u^t)} + \sup_{\Psi\in\{\Psi_1,\Psi_2,\Psi_3,\Psi_4\}} \sup_{v} ||\nabla^i \Psi||_{L^2(\mathcal{N}_v'^t)}$$

where the suprema in u and v are taken over $\mathcal{D}^t_{u,v_\bullet}$.

iv). Sup over the L^2-norm of the components of the Weyl tensor at spheres of constant u, v, given by,

$$\Delta_{\Psi}(\mathcal{S}) = \sum_{i=0}^{2} \sup_{u,v} ||\nabla^i(\Psi_0, \Psi_1, \Psi_2, \Psi_3)||_{L^2(\mathcal{S}_{u,v})},$$

with the supremum taken over $\mathcal{D}^t_{u,v_\bullet}$, and in which u will be taken sufficiently small to apply our estimates.

Remark 13. There is no appearance of χ in $\Delta_{\Gamma_\star}$ because initial data for χ used in the following calculations are required only on $\mathcal{N}'_\star$ where χ is zero.

Remark 14. In addition to the above norms, we recall that the norm $\Delta_{e_\star}$, as defined in equation (3.14) has been used to control the initial value of the components of the frame.

Remark 15. Observe that the above expressions do not include any norm for the components of the connection coefficients away from the initial null hypersurfaces. Instead such norms will be controlled by local bootstrap arguments within the proof.

Remark 16. Throughout the proof besides keeping track of $\Delta_{\Psi_\star}$ and $\Delta_{\Psi_\star}(\mathcal{S})$, to assist in future generalization, we trace also the dependence of our various constants on $I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi_\star}$. Note that because of the way that we setup our frame none of the constants so far depend upon I.

3.5.2 Estimates for the connection coefficients

In this section we show how to construct estimates on the coefficients of the connection. The strategy is an application of the tools developed in Section 3.4.4 to estimate the solutions of generic transport equations along null hypersurfaces. In this approach, as a bootstrap, control is assumed of the curvature (components of the Weyl tensor) on the double foliation of null hypersurfaces and on the 2-spheres of constant u and v through the norms Δ_Ψ and $\Delta_\Psi(\mathcal{S})$.

In a first step we obtain basic control of the L^∞-norm of the connection coefficients by assuming finiteness of Δ_Ψ and $\Delta_\Psi(\mathcal{S})$ and of third derivatives of the NP coefficient τ in terms of the L^2-norm on the 2-spheres $\mathcal{S}_{u,v}$.

Proposition 8 (***control on the supremum norm of the connection coefficients***)**.** *Assume that we have a solution of the vacuum EFEs in Stewart's gauge in a region $\mathcal{D}^t_{u,v_\bullet}$ with*

$$\sup_{u,v} ||\{\mu,\lambda,\alpha,\beta,\epsilon,\rho,\sigma,\tau,\chi\}||_{L^\infty(S_{u,v})} \leq \Delta_\Gamma\,,$$

for some positive Δ_Γ. Assume also

$$\sup_{u,v} ||\nabla\!\!\!\!/^2\tau||_{L^2(S_{u,v})} < \infty, \qquad \sup_{u,v} ||\nabla\!\!\!\!/^3\tau||_{L^2(S_{u,v})} < \infty, \qquad \Delta_\Psi(\mathcal{S}) < \infty, \qquad \Delta_\Psi < \infty,$$

on the same domain. Then there exists

$$\varepsilon_\star = \varepsilon_\star(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\sup_{u,v}||\nabla\!\!\!\!/^2\tau||_{L^2(\mathcal{S}_{u,v})},\sup_{u,v}||\nabla\!\!\!\!/^3\tau||_{L^2(\mathcal{S}_{u,v})},\Delta_\Psi),$$

such that when $\varepsilon \leq \varepsilon_\star$, we have

$$\begin{aligned}
&\sup_{u,v} ||\{\tau,\chi\}||_{L^\infty(\mathcal{S}_{u,v})} \leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_\Psi(\mathcal{S})),\\
&\sup_{u,v} ||\{\mu,\lambda,\alpha,\beta,\epsilon,\rho,\sigma\}||_{L^\infty(\mathcal{S}_{u,v})} \leq 3\Delta_{\Gamma_\star},
\end{aligned}$$

on $\mathcal{D}^t_{u,v_\bullet}$.

Remark 17. Observe that in the above proposition, as well as in several of the following ones, the NP spin connection coefficient τ is singled out as it requires additional hypotheses.

Remark 18. The first assumption here covers Assumption 2, which allows us to employ Lemma 5, Lemma 6 Lemma 7, Proposition 4 and the Sobolev inequalities of

Propositions 6, 7 and Corollary 2. It also permits the use of Propositions 3 and 5.

Proof.

Basic bootstrap assumption. We start by making the bootstrap assumption

$$\sup_{u,v} ||(\{\mu, \lambda, \alpha, \beta, \epsilon, \rho, \sigma\}||_{L^\infty(\mathcal{S}_{u,v})} \leq 4\Delta_{\Gamma_\star}.$$

Estimate for τ. As first step we prove that

$$||\tau||_{L^\infty(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_\Psi(\mathcal{S})).$$

For this, we make use of the D-equation (3b) for the NP coefficient τ:

$$D\tau = (\epsilon - \bar{\epsilon} + \rho)\tau + \sigma\bar{\tau} + \bar{\pi}\rho + \pi\sigma + \Psi_1.$$

Making use of the Sobolev inequality in Proposition 7, we readily obtain from our assumptions that for ε sufficiently small,

$$||\Psi_0,\ \Psi_1,\ \Psi_2,\ \Psi_3,\ \Psi_4||_{L^\infty(\mathcal{S}_{u,v})} \leq \Delta_\Psi(\mathcal{S}) < \infty.$$

Moreover, the inequalities in Proposition 4 show that

$$\begin{aligned}
||\tau||_{L^\infty(\mathcal{S}_{u,v})} &\leq ||\tau||_{L^\infty(\mathcal{S}_{u,0})} + \int_0^v ||D\tau||_{L^\infty(\mathcal{S}_{u,v'})}\mathrm{d}v' \\
&\leq ||\tau||_{L^\infty(\mathcal{S}_{u,0})} + \int_0^v ||\bar{\pi}\rho + \pi\sigma + \Psi_1||_{L^\infty(\mathcal{S}_{u,v'})}\mathrm{d}v' \\
&\qquad + \int_0^v |\epsilon - \bar{\epsilon} + \rho|\, ||\tau||_{L^\infty(\mathcal{S}_{u,v'})}\mathrm{d}v' + \int_0^v |\sigma|\, ||\bar{\tau}||_{L^\infty(\mathcal{S}_{u,v'})}\mathrm{d}v' \\
&\leq \Delta_{\Gamma_\star} + (32\Delta_{\Gamma_\star}^2 + \Delta_\Psi(\mathcal{S}))v_\bullet + 16\Delta_{\Gamma_\star}\int_0^v ||\tau||_{L^\infty(\mathcal{S}_{u,v'})}\mathrm{d}v'.
\end{aligned}$$

Using Grönwall's inequality in the previous expression one then concludes that

$$||\tau||_{L^\infty(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_\Psi(\mathcal{S})).$$

Estimate for χ. To obtain the estimate for χ we proceed in a similar manner. We use the D-transport equation equation (3.15) for χ to obtain

$$\begin{aligned}||\chi||_{L^\infty(\mathcal{S}_{u,v})} &\leq ||\chi||_{L^\infty(\mathcal{S}_{u,0})} + \int_0^v ||D\chi||_{L^\infty(\mathcal{S}_{u,v'})}\mathrm{d}v' \\ &\leq (2\Delta_\Psi(\mathcal{S}) + c\Delta_{\Gamma_\star} + C)v_\bullet + 2\Delta_{\Gamma_\star}\int_0^v ||\chi||_{L^\infty(\mathcal{S}_{u,v'})}\mathrm{d}v',\end{aligned}$$

where c is a positive constant and the constant C is related to the constant appearing in the estimate for τ. From the latter, Grönwall's inequality readily yields

$$||\chi||_{L^\infty(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_\Psi(\mathcal{S})).$$

Estimates for μ and λ. To obtain estimates of the NP coefficients μ and λ we make use of the Δ-transport equations (3g) and (3o):

$$\begin{aligned}\Delta\mu &= -\mu^2 - \lambda\bar{\lambda}, \\ \Delta\lambda &= -2\mu\lambda - \Psi_4.\end{aligned}$$

These are Riccati-type equations and, thus, they can only be naively integrated for a small distance in the u direction —i.e. $u \in [0, \varepsilon]$. Now, making use of the inequalities in Proposition 4 we find that

$$||\mu||_{L^\infty(\mathcal{S}_{u,v})} \leq ||\mu||_{L^\infty(\mathcal{S}_{0,v})} + C(\Delta_{e_\star})\int_0^\varepsilon ||\Delta\mu||_{L^\infty(\mathcal{S}_{u',v})}\mathrm{d}u'.$$

Accordingly, one concludes that

$$\begin{aligned}||\mu||_{L^\infty(\mathcal{S}_{u,v})} &\leq ||\mu||_{L^\infty(\mathcal{S}_{0,v})} + C(\Delta_{e_\star})\int_0^\varepsilon ||\mu^2 + \lambda\bar{\lambda}||_{L^\infty(\mathcal{S}_{u',v})}\mathrm{d}u' \\ &\leq ||\mu||_{L^\infty(\mathcal{S}_{0,v})} + 32C(\Delta_{e_\star})\int_0^\varepsilon \Delta^2_{\Gamma_0}\mathrm{d}u' \\ &\leq \Delta_{\Gamma_\star} + 32C(\Delta_{e_\star})\Delta^2_{\Gamma_\star}\varepsilon.\end{aligned}$$

For λ one obtains that

$$||\lambda||_{L^\infty(\mathcal{S}_{u,v})} \leq \Delta_{\Gamma_\star} + 32C(\Delta_{e_\star})\Delta^2_{\Gamma_\star}\varepsilon + C(\Delta_{e_\star})\int_0^u ||\Psi_4||_{L^\infty(\mathcal{S}_{u',v})}du'$$

$$\leq \Delta_{\Gamma_\star} + C(\Delta_{e_\star}, \Delta_{\Gamma_\star})\varepsilon + C(\Delta_{e_\star}) \int_0^u \sum_{i=0}^{2} ||\nabla\!\!\!/^i \Psi_4||_{L^2(\mathcal{S}_{u',v})} \mathrm{d}u',$$

where in the second inequality we have made use of the Sobolev embedding property —see corollary 2. Now, using Hölder's inequality, we can transform the estimate of Ψ_4 from one on sphere $\mathcal{S}_{u,v}$ to one on a null hypersurface. More precisely, one has that

$$\begin{aligned}\int_0^u ||\nabla\!\!\!/^i \Psi_4||_{L^2(\mathcal{S}_{u',v})} \mathrm{d}u' = \int_0^u \left(\int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/^i \Psi_4|^2 \right)^{1/2} \mathrm{d}u' \leq \left(\int_0^u \int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/^i \Psi_4|^2 \mathrm{d}u' \right)^{1/2} \left(\int_0^u 1 \mathrm{d}u' \right)^{1/2} \\ \leq C\varepsilon^{1/2} ||\nabla\!\!\!/^i \Psi_4||_{L^2(\mathcal{N}'_v(0,u))}.\end{aligned}$$

Hence, we conclude that

$$||\lambda||_{L^\infty(\mathcal{S}_{u,v})} \leq \Delta_{\Gamma_\star} + C(\Delta_{e_\star}, \Delta_{\Gamma_\star})\varepsilon + C\Delta_\Psi \varepsilon^{1/2}.$$

Together, the estimates for μ and λ show that the maximum of these functions will not be too far away from their initial value for ε sufficiently small.

Estimates for α, β and ϵ. Estimates α, β and ϵ can be obtained by a similar method —i.e. integration along the *short* direction. In this case the relevant Δ-transport equations are given by the structure equations (3k), (3d) and (3a),

$$\begin{aligned}\Delta\alpha &= -\mu\alpha - \lambda\beta - \lambda\tau - \Psi_3, \\ \Delta\beta &= -\bar{\lambda}\alpha - \mu\beta - \tau\mu, \\ \Delta\epsilon &= -\alpha\bar{\pi} - \beta\pi - \alpha\tau - \beta\bar{\tau} - \pi\tau - \Psi_2,\end{aligned}$$

where it is recalled that in the present gauge one has that $\pi = \alpha + \bar{\beta}$ —see Lemma 1, equation (3.10c). The details are omitted.

Estimates for ρ and σ. In this case the relevant Δ-transport equations are the structure equations (3i) and (3r):

$$\begin{aligned}\Delta\rho &= \bar{\delta}\tau - \mu\rho - \lambda\sigma - \alpha\tau + \bar{\beta}\tau - \tau\bar{\tau} - \Psi_2, \\ \Delta\sigma &= \delta\tau - \bar{\lambda}\rho - \mu\sigma + \bar{\alpha}\tau - \beta\tau - \tau^2.\end{aligned}$$

Observe that these equations contain the derivatives $\delta\tau$ and $\bar{\delta}\tau$. To control these terms from our hypotheses, we make use of the Sobolev inequalities in corollary 2 which, together with integration by parts on $\mathcal{S}_{u,v}$ allows us to show that,

$$
\begin{aligned}
||\nabla\!\!\!/\tau||_{L^\infty(\mathcal{S}_{u,v})} &\leq C(\Delta_{e_\star})\sum_{i=1}^{3}||\nabla\!\!\!/^i\tau||_{L^2(\mathcal{S}_{u,v})} \\
&\leq C(\Delta_{e_\star})\big(||\tau||_{L^2(\mathcal{S}_{u,v})}+||\nabla\!\!\!/^2\tau||_{L^2(\mathcal{S}_{u,v})}+||\nabla\!\!\!/^3\tau||_{L^2(\mathcal{S}_{u,v})}\big).
\end{aligned}
$$

It follows then from the Hölder inequality

$$
||\tau||_{L^2(\mathcal{S}_{u,v})} \leq ||\tau||_{L^\infty(\mathcal{S}_{u,v})}\,\mathrm{Area}(\mathcal{S}_{u,v})^{1/2}
$$

and the boundedness assumptions on $||\nabla\!\!\!/^i\tau||_{L^2(\mathcal{S}_{u,v})}$ for $i=2,3$, that

$$
||\nabla\!\!\!/\tau||_{L^\infty(\mathcal{S}_{u,v})} < \infty.
$$

From this observation, an argument similar to the one used for μ and λ yields the required estimates.

Concluding the argument. From the estimates for the NP connection coefficients constructed above it follows that one can choose

$$
\varepsilon = \varepsilon(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\sup_{u,v}||\nabla\!\!\!/^2\tau||_{L^2(\mathcal{S}_{u,v})},\sup_{u,v}||\nabla\!\!\!/^3\tau||_{L^2(\mathcal{S}_{u,v})},\Delta_\Psi(\mathcal{S}),\Delta_\Psi)
$$

sufficiently small so that

$$
\sup_{u,v}||\{\mu,\lambda,\alpha,\beta,\epsilon,\rho,\sigma\}||_{L^\infty(\mathcal{S}_{u,v})} \leq 3\Delta_{\Gamma_\star}.
$$

Accordingly, we have improved our initial bootstrap assumption. As this is our first such improvement we give an overview of the technique. Recall that to complete a bootstrap argument we need first, to verify that the hypothesis, in our case that $\sup_{u,v}||\{\mu,\lambda,\alpha,\beta,\epsilon,\rho,\sigma\}||_{L^\infty(\mathcal{S}_{u,v})} \leq 4\Delta_{\Gamma_\star}$ holds over the region of interest, is satisfied. We then need to demonstrate, as in the previous argument, that the hypothesis can be improved for ε sufficiently small. Obviously if the conclusion $\sup_{u,v}||\{\mu,\lambda,\alpha,\beta,\epsilon,\rho,\sigma\}||_{L^\infty(\mathcal{S}_{u,v})} \leq 3\Delta_{\Gamma_\star}$ holds at some point then our hypothesis holds in a neighborhood of that point. Since the interval $[0,\varepsilon]$ is connected and the set on which our desired conclusion holds is open, closed and non-empty, it follows that the desired conclusion holds for $u \in [0,\varepsilon]$. In the argument above we have shown that we can improve the hypothesis from a bound $4\Delta_{\Gamma_\star}$ to $3\Delta_{\Gamma_\star}$. Evidently the same arguments could be used to improve from $(N+1)\Delta_{\Gamma_\star}$ to $N\Delta_{\Gamma_\star}$ for any natural number $N \geq 3$. Given our initial assumption that $||\{\mu,\lambda,\alpha,\beta,\epsilon,\rho,\sigma\}||_{L^\infty(\mathcal{S}_{u,v})} \leq \Delta_\Gamma$ we can therefore choose N so that $\Delta_\Gamma \leq N\Delta_{\Gamma_\star}$ and iterate from N down to 4 to guarantee that our hypothesis is indeed satisfied in some truncated diamond, demonstrating the statement. $\square$

The existence proof also requires control over the L^4-norms of the δ and $\bar{\delta}$ derivatives of the NP spin connection coefficients. This is provided by the following:

Proposition 9 (***control on the*** L^4***-norm of the connection coefficients***)**.** *Make the same assumptions as in Proposition 8, and additionally assume that,*

$$\sup_{u,v} ||\nabla\!\!\!/ \{\mu,\lambda,\alpha,\beta,\epsilon,\rho,\sigma\}||_{L^4(\mathcal{S}_{u,v})} \leq \Delta_\Gamma,$$

in the truncated diamond $\mathcal{D}^t_{u,v_\bullet}$. Then there exists,

$$\varepsilon_\star = \varepsilon_\star(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \sup_{u,v} ||\nabla\!\!\!/^2\tau||_{L^2(\mathcal{S}_{u,v})}, \sup_{u,v} ||\nabla\!\!\!/^3\tau||_{L^2(\mathcal{S}_{u,v})}, \Delta_\Psi(\mathcal{S}), \Delta_\Psi),$$

such that when $\varepsilon \leq \varepsilon_\star$, we have,

$$\sup_{u,v} ||\nabla\!\!\!/ \{\tau,\chi\}||_{L^4(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_\Psi(\mathcal{S})),$$
$$\sup_{u,v} ||\nabla\!\!\!/ \{\mu,\lambda,\alpha,\beta,\epsilon,\rho,\sigma\}||_{L^4(\mathcal{S}_{u,v})} \leq 3\Delta_{\Gamma_\star},$$

on $\mathcal{D}^t_{u,v_\bullet}$.

Proof.

Basic bootstrap assumption. In order to run the argument we make the following bootstrap assumption:

$$\sup_{u,v} ||\nabla\!\!\!/ \{\mu,\lambda,\alpha,\beta,\epsilon,\rho,\sigma\}||_{L^4(\mathcal{S}_{u,v})} \leq 4\Delta_{\Gamma_\star}.$$

Estimates for $\nabla\!\!\!/\tau$. First we make use of the boundedness of the L^2-norm of τ and its angular derivatives up to third order to estimate the L^4-norm of the first order angular derivatives of τ. For this, we apply δ to the D-transport equation for τ —equation (3b). After making use of the commutators of directional covariant derivatives one arrives at the equations

$$
\begin{aligned}
D\delta\tau &= (\rho+\bar\rho+2\epsilon-2\bar\epsilon)\delta\tau+\sigma\bar\delta\tau+\sigma\delta\bar\tau+\delta(\epsilon-\bar\epsilon+\rho)\tau \\
&\qquad +\bar\tau\delta\sigma+\rho\delta\bar\pi+\bar\pi\delta\rho+\sigma\delta\pi+\pi\delta\sigma+\delta\Psi_1, &\text{(3.20a)}\\
D\bar\delta\tau &= 2\rho\bar\delta\tau+\sigma\bar\delta\bar\tau+\bar\sigma\delta\tau+\tau\bar\delta(\epsilon-\bar\epsilon+\rho)+\bar\tau\bar\delta\sigma \\
&\qquad +\rho\bar\delta\bar\pi+\bar\pi\delta\sigma\rho+\sigma\bar\delta\pi+\pi\bar\delta\sigma+\bar\delta\Psi_1. &\text{(3.20b)}
\end{aligned}
$$

The above equation contains terms of the form $\Gamma\nabla\!\!\!/\Gamma$ —i.e. products of connection coefficients and their derivatives. In the following the L^4-norm of these products will be split using the Hölder inequality as follows:

$$
||\Gamma\nabla\!\!\!/\Gamma||_{L^4(\mathcal{S}_{u,v})}\leq||\Gamma||_{L^\infty(\mathcal{S}_{u,v})}||\nabla\!\!\!/\Gamma||_{L^4(\mathcal{S}_{u,v})}.
$$

Observe that from Proposition 8 it follows that terms of the type $||\Gamma||_{L^\infty(\mathcal{S}_{u,v})}$ are bounded.

Now, making use of the Sobolev inequality in Proposition 6, we obtain that

$$
\sum_{j=0}^{1}||\nabla\!\!\!/^j\Psi_i||_{L^4(\mathcal{S}_{u,v})}\leq\Delta_\Psi(\mathcal{S})<\infty,\qquad i=0,1,2,3.
$$

Combining this with the inequality in the long direction shown in Proposition 3 we find that

$$
\begin{aligned}
&||\delta\tau||_{L^4(\mathcal{S}_{u,v})}+||\bar\delta\tau||_{L^4(\mathcal{S}_{u,v})}\\
&\qquad\leq C(I,\Delta_{\Gamma_\star})\Big(||\delta\tau||_{L^4(\mathcal{S}_{u,0})}+||\bar\delta\tau||_{L^4(\mathcal{S}_{u,0})}+\int_0^v||D\delta\tau||_{L^4(\mathcal{S}_{u,v'})}+||D\bar\delta\tau||_{L^4(\mathcal{S}_{u,v'})}\mathrm{d}v'\Big).
\end{aligned}
$$

Substituting the expressions for $D\delta\tau$ and $D\bar\delta\tau$ given by equations (3.20a)-(3.20b) one concludes that

$$\begin{aligned}
&||\delta\tau||_{L^4(\mathcal{S}_{u,v})} + ||\bar{\delta}\tau||_{L^4(\mathcal{S}_{u,v})} \\
&\qquad \leq C_1(I, \Delta_{\Gamma_\star}, \Delta_\Psi(\mathcal{S})) + C_2(I, \Delta_{\Gamma_\star}) \int_0^v (||\delta\tau||_{L^4(\mathcal{S}_{u,v'})} + ||\bar{\delta}\tau||_{L^4(\mathcal{S}_{u,v'})}) \mathrm{d}v'.
\end{aligned}$$

Thus, using Grönwall's inequality it follows that

$$||\delta\tau||_{L^4(\mathcal{S}_{u,v})} + ||\bar{\delta}\tau||_{L^4(\mathcal{S}_{u,v})} \leq C(I, \Delta_{\Gamma_\star}, \Delta_\Psi(\mathcal{S})).$$

Consequently, one has

$$||\nabla\!\!\!/\,\tau||_{L^4(\mathcal{S}_{u,v})} \leq C(I, \Delta_{\Gamma_\star}, \Delta_\Psi(\mathcal{S}))$$

as required.

Estimates for $\nabla\!\!\!/\,\chi$. From equation (3.15) one can readily compute that

$$D\delta\chi = (\bar{\rho} - 2\bar{\epsilon})\delta\chi + \sigma\bar{\delta}\chi + \delta(\Psi_2 + \bar{\Psi}_2) + \Gamma\delta\Gamma - \chi\delta(\epsilon + \bar{\epsilon}),$$

where Γ represents a combination of connection coefficients whose particular form is not essential. A similar equation for $D\bar{\delta}\chi$ can be computed. Using the same strategy used for $\nabla\!\!\!/\,\chi$ one concludes from the above equations that,

$$||\delta\chi||_{L^4(\mathcal{S}_{u,v})} + ||\bar{\delta}\chi||_{L^4(\mathcal{S}_{u,v})} \leq C(I, \Delta_{\Gamma_\star}, \Delta_\Psi(\mathcal{S})).$$

In other words, we find that

$$||\nabla\!\!\!/\,\chi||_{L^4(\mathcal{S}_{u,v})} \leq C(I, \Delta_{\Gamma_\star}, \Delta_\Psi(\mathcal{S})).$$

Estimates for the remaining connection coefficients. In order to obtain equations for $\delta\mu$ and $\delta\lambda$, we apply the Δ-directional derivative on both sides of equations (3g) and (3o). This gives,

$$\begin{aligned}
\Delta\delta\mu &= (\tau - \bar{\alpha} - \beta)(\mu^2 + \lambda\bar{\lambda}) - 3\mu\delta\mu - \bar{\lambda}\bar{\delta}\mu - \lambda\delta\bar{\lambda} - \bar{\lambda}\delta\lambda, \\
\Delta\delta\lambda &= (\tau - \bar{\alpha} - \beta)(2\mu\lambda + \Psi_4) - 3\mu\delta\lambda - \bar{\lambda}\bar{\delta}\lambda - 2\lambda\delta\mu - \delta\Psi_4.
\end{aligned}$$

A direct computation using Proposition 3 shows that there exists an $\varepsilon_\star$ such that

$$||\nabla\!\!\!/\{\mu,\lambda\}||_{L^4(\mathcal{S}_{u,v})}\leq 3\Delta_{\Gamma_\star}$$

if $\varepsilon \leq \varepsilon_\star$. The details of this computation can be found in Appendix 6.2. We can estimate $\delta\alpha$, $\delta\beta$ and $\delta\epsilon$ by the same method. Since, by our bootstrap assumption $\sup_{u,v}||\nabla\!\!\!/^3\tau||_{L^2(\mathcal{S}_{u,v})}<\infty$, it follows from the Sobolev inequalities in Corollary 2 that $||\nabla\!\!\!/^i\tau||_{L^4(\mathcal{S}_{u,v})}$ for $i\leq 2$ are finite. Using this information we can estimate $\delta\rho$ and $\delta\sigma$ applying the δ-directional derivative to equations (3i) and (3r).

Concluding the argument. From the previous estimates it follows that we can find an $\varepsilon_\star$ depending on I, $\Delta_{e_\star}$,$\Delta_{\Gamma_\star}$, $\sup_{u,v}||\nabla\!\!\!/^2\tau||_{L^2(\mathcal{S}_{u,v})}$, $\sup_{u,v}||\nabla\!\!\!/^3\tau||_{L^2(\mathcal{S}_{u,v})}$, $\Delta_\Psi(\mathcal{S})$, and Δ_Ψ, such that

$$\sup_{u,v}||\nabla\!\!\!/\{\mu,\lambda,\alpha,\beta,\epsilon,\rho,\sigma\}||_{L^4(\mathcal{S}_{u,v})}\leq 3\Delta_{\Gamma_\star}.$$

The bootstrap can hence be closed as in Proposition 8. □

In a similar vein, the next proposition shows how to obtain control on the L^2-norms of the NP connection coefficients and their first and second derivatives.

Proposition 10 (***control on the L^2-norm of the connection coefficients***). *Assume that we have a solution of the vacuum EFEs in Stewart's gauge in a region $\mathcal{D}^t_{u,v_\bullet}$ with*

$$\begin{aligned}\sup_{u,v}||\{\mu,\lambda,\alpha,\beta,\epsilon,\rho,\sigma,\tau,\chi\}||_{L^\infty(S_{u,v})}&\leq\Delta_\Gamma\,,\\ \sup_{u,v}||\nabla\!\!\!/\{\mu,\lambda,\alpha,\beta,\epsilon,\rho,\sigma\}||_{L^4(\mathcal{S}_{u,v})}&\leq\Delta_\Gamma,\\ \sup_{u,v}||\nabla\!\!\!/^2\{\mu,\lambda,\alpha,\beta,\epsilon,\rho,\sigma,\tau\}||_{L^2(\mathcal{S}_{u,v})}&\leq\Delta_\Gamma,\end{aligned}$$

for some positive Δ_Γ. Assume also

$$\sup_{u,v}||\nabla\!\!\!/^3\tau||_{L^2(S_{u,v})}<\infty,\qquad \Delta_\Psi(\mathcal{S})<\infty,\qquad \Delta_\Psi<\infty,$$

on the same domain. We have that there exists

$$\varepsilon_\star = \varepsilon_\star(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \sup_{u,v} ||\nabla\!\!\!\!/^3\tau||_{L^2(\mathcal{S}_{u,v})}, \Delta_\Psi(\mathcal{S}), \Delta_\Psi),$$

such that when $\varepsilon \leq \varepsilon_\star$*, we have that*

$$\sup_{u,v} ||\nabla\!\!\!\!/^2\{\tau,\chi\}||_{L^2(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_\Psi(\mathcal{S})),$$
$$\sup_{u,v} ||\nabla\!\!\!\!/^2\{\mu,\lambda,\alpha,\beta,\epsilon,\rho,\sigma\}||_{L^2(\mathcal{S}_{u,v})} \leq 3\Delta_{\Gamma_\star}.$$

Proof.

Basic bootstrap assumption. Examining the above hypotheses we first observe that both Propositions 8 and 9 are applicable. We start then with the following basic bootstrap assumption:

$$\sup_{u,v} ||\nabla\!\!\!\!/^2\{\mu,\lambda,\alpha,\beta,\epsilon,\rho,\sigma\}||_{L^2(\mathcal{S}_{u,v})} \leq 4\Delta_{\Gamma_\star}.$$

Estimates for $||\nabla\!\!\!\!/^2\tau||_{L^2(\mathcal{S}_{u,v})}$ and $||\nabla\!\!\!\!/^2\chi||_{L^2(\mathcal{S}_{u,v})}$. Starting from equation (3.20a), applying the δ-directional derivative and using the commutators one obtains a D-transport equation of the form

$$D\delta^2\tau = \Gamma\delta^2\tau + \Gamma\delta^2\bar{\tau} + \Gamma\bar{\delta}\delta\tau + \Gamma\delta\bar{\delta}\tau + \delta^2\Psi_1 + \Gamma_1\delta^2\Gamma_1 + \delta\Gamma_1\delta\Gamma_1,$$

where Γ depends linearly on ϵ, ρ, σ, while Γ_1 depends linearly on τ, α, β, ϵ, ρ, σ. Similar computations lead to equations for $D\bar{\delta}\tau$ and $D\delta\bar{\delta}\tau$. The term $\delta\Gamma_1\delta\Gamma_1$ is dealt with using the Hölder inequality to obtain

$$||\delta\Gamma_1\delta\Gamma_1||_{L^2(\mathcal{S}_{u,v})} \leq ||\delta\Gamma_1||_{L^4(\mathcal{S}_{u,v})}||\delta\Gamma_1||_{L^4(\mathcal{S}_{u,v})}.$$

Using Proposition 9, it follows then that the left-hand side of the inequality is finite.

Now, the inequality in the long direction of Proposition 3 and the equation for $D\delta\tau$ show that,

$$||\delta^2\tau||_{L^2(\mathcal{S}_{u,v})} \leq C(I, \Delta_{\Gamma_\star}) \left(||\delta^2\tau||_{L^2(\mathcal{S}_{u,0})} + \int_0^v ||D\delta^2\tau||_{L^2(\mathcal{S}_{u,v'})}\mathrm{d}v' \right),$$

$$\leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_\Psi(\mathcal{S})) + C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}) \int_0^v ||\nabla\!\!\!/^2 \tau||_{L^2(\mathcal{S}_{u,v'})} \mathrm{d}v'.$$

Similar estimates can be obtained for $\bar{\delta}^2\tau$, $\delta\bar{\delta}\tau$ and $\bar{\delta}\delta\tau$.

Recalling the result in Lemma 6 that the area of $\mathcal{S}_{u,v}$ is bounded one can estimate the norm $||\delta\tau||_{L^2(\mathcal{S}_{u,v})}$ by observing that

$$||\delta\tau||_{L^2(\mathcal{S}_{u,v})} \leq C(\Delta_{e_\star}, \Delta_{\Gamma_\star}) ||\delta\tau||_{L^4(\mathcal{S}_{u,v})}.$$

Hence, using Proposition 9 it follows that $||\delta\tau||_{L^2(\mathcal{S}_{u,v})}$ is also finite. Now, from inequality (10) we then obtain that

$$||\nabla\!\!\!/^2\tau||_{L^2(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_\Psi(\mathcal{S})) + C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}) \int_0^v ||\nabla\!\!\!/^2 \tau||_{L^2(\mathcal{S}_{u,v'})} \mathrm{d}v'.$$

so that using Grönwall's inequality one concludes that

$$||\nabla\!\!\!/^2\tau||_{L^2(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_\Psi(\mathcal{S})).$$

Estimates for $||\nabla\!\!\!/^2\chi||_{L^2(\mathcal{S}_{u,v})}$. An analysis analogous to that for τ, readily shows that $||\nabla\!\!\!/^2\chi||_{L^2(\mathcal{S}_{u,v})}$ is bounded.

Estimates for the the remaining spin connection coefficients. The remaining connection coefficients can be estimated using the same ideas as in Proposition 8 — namely, we first compute equations for $\Delta\delta^2\Gamma$ and $\Delta\bar{\delta}\delta\Gamma$ using the NP Ricci identities and the commutators for covariant directional derivatives. In a second step we make use of the short direction inequality of Proposition 3. It then follows that one can choose ε small enough so that,

$$\sup_{u,v} ||\nabla\!\!\!/^2\{\mu, \lambda, \alpha, \beta, \epsilon, \rho, \sigma\}||_{L^2(\mathcal{S}_{u,v})} \leq 3\Delta_{\Gamma_\star},$$

for,

$$\varepsilon \leq \varepsilon_\star(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \sup_{u,v} ||\nabla\!\!\!/^3\tau||_{L^2(\mathcal{S}_{u,v})}, \Delta_\Psi(\mathcal{S}), \Delta_\Psi).$$

Details of the generic calculations involved in these last steps are discussed in Appendix 6.2. $\square$

3.5.3 A first estimate for the curvature

Having obtained estimates for the NP spin connection coefficients, we are now in the position to obtain a first estimate for the curvature. The proposition of this section provides for bounds the components of the Weyl tensor of the spheres $\mathcal{S}_{u,v}$ assuming, as a bootstrap, their boundedness on the null hypersurfaces and boundedness on τ and its derivatives.

Proposition 11 (***basic control of the curvature***)**.** *Assume that we are given a solution to the vacuum EFEs in Stewart's gauge satisfying the assumptions of Proposition 10. Then there exists*

$$\varepsilon_\star = \varepsilon_\star(\Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi_\star}, \sup_{u,v} ||\nabla\!\!\!/^3\tau||_{L^2(\mathcal{S}_{u,v})}, \Delta_\Psi)$$

such that for $\varepsilon \leq \varepsilon_\star$, one has

$$\Delta_\Psi(\mathcal{S}) \leq C(\Delta_{\Psi_\star}),$$

on $\mathcal{D}^t_{u,v_\bullet}$.

Proof.

Boostrap assumption. In this proof we start with the following bootstrap assumption:

$$\sup_{u,v} ||\nabla\!\!\!/^i\{\Psi_0, \Psi_1, \Psi_2, \Psi_3\}||_{L^2(\mathcal{S}_{u,v})} \leq 4\Delta_{\Psi_\star}, \qquad i = 0, ..., 2,$$

which we then aim to improve.

L^2-norm of the components $\{\Psi_0, \Psi_1, \Psi_2, \Psi_3\}$. Estimates for the L^2-norms of the components $\{\Psi_0, \Psi_1, \Psi_2, \Psi_3\}$ can be obtained from the Δ-Bianchi identity equations (3b), (3h), (3f) and (3d) which are then integrated along the short direction. As an example of the procedure we consider here the coefficient Ψ_2. From Proposition 3 it follows that

$$
\begin{aligned}
||\Psi_2||_{L^2(\mathcal{S}_{u,v})} &\leq 2\left(||\Psi_2||_{L^2(\mathcal{S}_{0,v})} + C(\Delta_{e_\star},\Delta_{\Gamma_\star})\int_0^u ||\Delta\Psi_2||_{L^2(\mathcal{S}_{u',v})}\mathrm{d}u'\right)\\
&\leq 2\Big(\Delta_{\Psi_\star} + C(\Delta_{e_\star},\Delta_{\Gamma_\star})\int_0^u ||\bar{\eth}\Psi_3||_{L^2(\mathcal{S}_{u',v})} + ||3\mu\Psi_2||_{L^2(\mathcal{S}_{u',v})}\\
&\qquad + ||2(\beta-\tau)\Psi_3||_{L^2(\mathcal{S}_{u',v})} + ||\sigma\Psi_4||_{L^2(\mathcal{S}_{u',v})}\mathrm{d}u'\Big)\\
&\leq 2\left(\Delta_{\Psi_\star} + C(\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Psi_\star})\varepsilon + C(\Delta_{e_\star},\Delta_{\Gamma_\star})\Delta_\Psi\varepsilon^{1/2}\right.\\
&\qquad \left.+C(\Delta_{e_\star},\Delta_{\Gamma_\star})||\Psi_4||_{L^2(\mathcal{N}'_v(0,u))}\varepsilon^{1/2}\right)\\
&\leq 2\Delta_{\Psi_\star} + C(\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Psi_\star})\varepsilon + C(\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Psi_\star})\Delta_\Psi\varepsilon^{1/2},
\end{aligned}
$$

In passing from the second to the third inequality we have used that the term

$$
\int_0^u ||\bar{\eth}\Psi_3||_{L^2(\mathcal{S}_{u',v})}\mathrm{du}'
$$

is, in fact, an statement on the light cone and, hence, it is controlled by the definition of Δ_Ψ. Moreover, we have also used Hölder's inequality in the form

$$
\int_0^u ||\Psi_4||_{L^2(\mathcal{S}_{u',v})}\mathrm{d}u' \leq C\varepsilon^{1/2}||\Psi_4||_{L^2(\mathcal{N}_{v'}(0,u))}.
$$

The analysis for the coefficients Ψ_0,Ψ_1,Ψ_3 is similar. Consequently, we can find $\varepsilon_\star$ depending on the initial data, Δ_Ψ and I such that for $\varepsilon\leq\varepsilon_\star$, we have

$$
\sup_{u,v}||\{\Psi_0,\Psi_1,\Psi_2,\Psi_3\}||_{L^2(\mathcal{S}_{u,v})} \leq 3\Delta_{\Psi_\star}.
$$

Estimates for $||\bar{\eth}\{\Psi_0,\Psi_1,\Psi_2,\Psi_3\}||_{L^2(\mathcal{S}_{u,v})}$. Again, we focus our discussion on the analysis of the coefficient Ψ_2. From Proposition 3 we find that

$$
\begin{aligned}
||\bar{\eth}\Psi_2||_{L^2(\mathcal{S}_{u,v})} &\leq 2\left(||\bar{\eth}\Psi_2||_{L^2(\mathcal{S}_{0,v})} + C(\Delta_{e_\star},\Delta_{\Gamma_\star})\int_0^u\left(\int_{\mathcal{S}_{u',v}}\Delta\left\langle\bar{\eth}\Psi_2,\bar{\eth}\Psi_2\right\rangle_{\boldsymbol{\sigma}}\right)^{1/2}\mathrm{d}u'\right)\\
&\leq 2\Delta_{\Psi_\star} + C(\Delta_{e_\star},\Delta_{\Gamma_\star})\int_0^u\left(\int_{\mathcal{S}_{u',v}}|\bar{\eth}\Psi_2|(|\Delta\delta\Psi_2| + |\Delta\bar{\delta}\Psi_2|)\right)^{1/2}\mathrm{d}u'.
\end{aligned}
$$

Now, using the expression for $\Delta\delta\Psi_2$ and $\Delta\bar{\delta}\Psi_2$ obtained from using the commutators on the Δ-Bianchi equation for Ψ_2, and schematically denoting arbitrary connection coefficients by Γ, one obtains that

$$
\begin{aligned}
&\int_0^u \left(\int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/\Psi_2|(|\Delta\delta\Psi_2| + |\Delta\bar{\delta}\Psi_2|) \right)^{1/2} \mathrm{d}u' \\
&\quad \leq \int_0^u \Big(\int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/\Psi_2||\Gamma|^2|\Psi_{2,3}| \Big)^{1/2} \mathrm{d}u' + \int_0^u \Big(\int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/\Psi_2||\Gamma|^2|\Psi_4| \Big)^{1/2} \mathrm{d}u' \\
&\quad + \int_0^u \Big(\int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/\Psi_2||\Gamma||\nabla\!\!\!/\Psi_{2,3}| \Big)^{1/2} \mathrm{d}u' + \int_0^u \Big(\int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/\Psi_2||\Gamma||\nabla\!\!\!/\Psi_4| \Big)^{1/2} \mathrm{d}u' \\
&\quad + \int_0^u \Big(\int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/\Psi_2||\nabla\!\!\!/\Gamma||\Psi_{2,3}| \Big)^{1/2} \mathrm{d}u' + \int_0^u \Big(\int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/\Psi_2||\nabla\!\!\!/\Gamma||\Psi_4| \Big)^{1/2} \mathrm{d}u' \\
&\quad + \int_0^u \Big(\int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/\Psi_2||\nabla\!\!\!/^2\Psi_3| \Big)^{1/2} \mathrm{d}u'.
\end{aligned} \tag{3.21}
$$

In the first and third terms of the right-hand side or the above inequality we can separate the L^∞-norm of the connection coefficients. Thus, using the bootstrap assumption with Proposition 8, we find that

$$
\begin{aligned}
&\int_0^u \Big(\int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/\Psi_2||\Gamma|^{2-i}|\nabla\!\!\!/^i\Psi_{2,3}| \Big)^{1/2} \mathrm{d}u' \\
&\leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi_\star}) \int_0^u ||\nabla\!\!\!/\Psi_2||^{1/2}_{L^2(\mathcal{S}_{u',v})} ||\nabla\!\!\!/^i\Psi_{2,3}||^{1/2}_{L^2(\mathcal{S}_{u',v})} \mathrm{d}u',
\end{aligned}
$$

for $i = 0, 1$. Accordingly, using the bootstrap assumption once again, we conclude that,

$$
\int_0^u \Big(\int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/^i\Psi_2||\Gamma|^{2-i}|\nabla\!\!\!/\Psi_{2,3}| \Big)^{1/2} \mathrm{d}u' \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi_\star})\varepsilon,
$$

for $i = 0, 1$. The second and fourth term in the right-hand side of inequality (3.21) can be handled in an analogous manner. Since we do not have control on the the $L^2(\mathcal{S}_{u,v})$ norm of Ψ_4, we transform the $L^2(\mathcal{S}_{u,v})$ norm to a norm over the light cone. More precisely, one has that using Hölder's inequality

$$\int_0^u \Big(\int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/ \Psi_2||\Gamma|^{2-i}|\nabla\!\!\!/^i\Psi_4|\Big)^{1/2}\mathrm{d}u' \leq \int_0^u ||\nabla\!\!\!/ \Psi_2||^{1/2}_{L^2(\mathcal{S}_{u',v})}||\nabla\!\!\!/^i\Psi_4||^{1/2}_{L^2(\mathcal{S}_{u',v})}\mathrm{d}u'$$
$$\leq C(\Delta_{\Psi_\star})||\nabla\!\!\!/^i\Psi_4||^{1/2}_{L^2(\mathcal{N}'_v(0,u))}\varepsilon^{3/4}, \; i=0,1.$$

Hence, we conclude that

$$\int_0^u \Big(\int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/ \Psi_2||\Gamma|^2|\Psi_4|\Big)^{1/2}\mathrm{d}u', \quad \int_0^u \Big(\int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/ \Psi_2||\Gamma||\nabla\!\!\!/ \Psi_4|\Big)^{1/2}\mathrm{d}u' \leq C(\Delta_{\Psi_\star},\Delta_\Psi)\varepsilon^{3/4}.$$

Now, for the fifth term in inequality (3.21) one has that

$$\int_0^u \Big(\int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/ \Psi_2||\nabla\!\!\!/ \Gamma||\Psi_{2,3}|\Big)^{1/2}\mathrm{d}u' \leq \int_0^u \left(||\Psi_{2,3}||_{L^\infty(\mathcal{S}_{u,v})}||\nabla\!\!\!/ \Psi_2||_{L^2(\mathcal{S}_{u,v})}||\nabla\!\!\!/ \Gamma||_{L^2(\mathcal{S}_{u,v})}\right)^{1/2}\mathrm{d}u',$$

where the first term in the integral in the right-hand side can be controlled by the bootstrap assumption and Sobolev embedding (Corollary 2). The third term can be controlled by the $L^4(\mathcal{S}_{u,v})$ norm as given by Proposition 9, again in combination with the bootstrap assumption. One then concludes that,

$$\int_0^u \Big(\int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/ \Psi_2||\nabla\!\!\!/ \Gamma||\Psi_{2,3}|\Big)^{1/2}\mathrm{d}u' \leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Psi_\star})\sum_{i=0}^{2}\int_0^u ||\nabla\!\!\!/^i\Psi_{2,3}||^{1/2}_{L^2(\mathcal{S}_{u',v})}\mathrm{d}u'$$
$$\leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Psi_\star},\Delta_\Psi)\varepsilon^{3/4}.$$

The sixth term in inequality (3.21) can also be dealt with by transforming the norms of the coefficients of the Weyl tensor on $\mathcal{S}_{u,v}$ to norms on the light cone. More precisely, one has that

$$\int_0^u \Big(\int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/ \Psi_2||\nabla\!\!\!/ \Gamma||\Psi_4|\Big)^{1/2}\mathrm{d}u' \leq \int_0^u \left(||\Psi_4||_{L^\infty(\mathcal{S}_{u,v})}||\nabla\!\!\!/ \Psi_2||_{L^2(\mathcal{S}_{u,v})}||\nabla\!\!\!/ \Gamma||_{L^2(\mathcal{S}_{u,v})}\right)^{1/2}\mathrm{d}u'$$
$$\leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Psi_\star})\left(\int_0^u\sum_{i=0}^{2}||\nabla\!\!\!/^i\Psi_4||_{L^2(\mathcal{S}_{u,v})}\mathrm{d}u'\right)^{1/2}$$
$$\leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Psi_\star})\Big(\sum_{i=0}^{2}||\nabla\!\!\!/^i\Psi_4||_{L^2(\mathcal{N}'_v(0,u))}\Big) \leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Psi_\star},\Delta_\Psi)\varepsilon^{3/4}.$$

Finally, the last integral in the right-hand side of inequality (3.21) can be separated into two L^2-norms. The estimate of $\nabla\!\!\!/^2\Psi_3$ can, in turn, be transformed to an estimate on the light cone and, hence, it can be controlled by the definition of Δ_Ψ.

Collecting all the estimates for the various terms in inequality (3.21) we conclude that,

$$||\nabla\!\!\!/\Psi_2||_{L^2(\mathcal{S}_{u,v})} \leq 2\Delta_{\Psi_\star} + C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Psi_\star})\varepsilon + C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Psi_\star},\Delta_\Psi)\Delta_\Psi\varepsilon^{3/4}.$$

The latter inequality implies that we can improve the bootstrap assumption by choosing ε small enough. A similar strategy allows us to estimate $\nabla\!\!\!/\{\Psi_0,\Psi_1,\Psi_3\}$. Therefore we have that

$$\sup_{u,v}||\nabla\!\!\!/\{\Psi_0,\Psi_1,\Psi_2,\Psi_3\}||_{L^2(S_{u,v})} \leq 3\Delta_{\Psi_\star}.$$

Estimates for $||\nabla\!\!\!/^2\{\Psi_0,\Psi_1,\Psi_2,\Psi_3\}||_{L^2(\mathcal{S}_{u,v})}$. As before, we focus the discussion on $||\nabla\!\!\!/^2\Psi_2||_{L^2(\mathcal{S}_{u,v})}$. The estimate along the short direction in Proposition 3 shows that

$$\begin{aligned}||\nabla\!\!\!/^2\Psi_2||_{L^2(\mathcal{S}_{u,v})} &\leq 2\left(||\nabla\!\!\!/^2\Psi_2||_{L^2(\mathcal{S}_{0,v})} + C(\Delta_{e_\star},\Delta_{\Gamma_\star})\int_0^u\left(\int_{\mathcal{S}_{u',v}}\Delta\left\langle\nabla\!\!\!/^2\Psi_2,\nabla\!\!\!/^2\Psi_2\right\rangle_{\boldsymbol{\sigma}}\right)^{1/2}\mathrm{d}u'\right)\\ &\leq 2\Delta_{\Psi_\star} + C(\Delta_{e_\star},\Delta_{\Gamma_\star})\int_0^u\left(\int_{\mathcal{S}_{u',v}}|\nabla\!\!\!/^2\Psi_2|(|\Delta T|)\right)^{1/2}\mathrm{d}u',\end{aligned} \tag{3.22}$$

where T denotes an expression involving products of connection coefficients, their derivatives and components of the Weyl tensor and their derivatives. In particular, one has that

$$\begin{aligned}&\int_{\mathcal{S}_{u',v}}|\nabla\!\!\!/^2\Psi_2|(|\Delta T|)\\ &\leq \int_{S_{u',v}}|\nabla\!\!\!/^2\Psi_2||\Psi\nabla\!\!\!/^2\Gamma + \Gamma\nabla\!\!\!/^2\Psi + \nabla\!\!\!/\Psi\nabla\!\!\!/\Gamma + \Gamma^2\nabla\!\!\!/\Psi + \Gamma\Psi\nabla\!\!\!/\Psi + \Gamma^3\Psi + \Psi_3\nabla\!\!\!/\Psi_2 + \nabla\!\!\!/^3\Psi_3|.\end{aligned}$$

We can then proceed with a strategy similar to that used in the analysis of the estimates for the first order derivatives of the components of the Weyl tensor. In particular, we use Hölder's inequality to split products and then apply the Sobolev embedding theorem as necessary. The estimates on the sphere for the terms $\nabla^i\Psi_4$ and $\nabla^3\Psi_3$ are transformed into estimates on the light cone. Hence the integral on the right-hand-side of inequality (3.22) can be made as small as necessary by choosing a suitable ε. Ultimately, we conclude that

$$\sup_{u,v}||\nabla^2\{\Psi_0,\Psi_1,\Psi_2,\Psi_3\}||_{L^2(\mathcal{S}_{u,v})}\leq 3\Delta_{\Psi_\star}.$$

Concluding the argument. Collecting all the estimates in the previous steps one obtains the statement

$$\sup_{u,v}||\nabla^i\{\Psi_0,\Psi_1,\Psi_2,\Psi_3\}||_{L^2(\mathcal{S}_{u,v})}\leq 3\Delta_{\Psi_\star},\qquad i=0,...,2,$$

which improves the starting bootstrap assumption. □

Applying the standard embedding of L^p into L^q for $p\leq q$, we can summarise the results of Propositions 8, 9, 10 and 11 in the following proposition:

Proposition 12 (***summary of the basic estimates for the NP quantities*).** *Suppose we are given a solution to the vacuum EFE's in Stewart's gauge emanating from data for the CIVP as prepared in Lemma 2, satisfying*

$$\sup_{u,v}||\{\mu,\lambda,\alpha,\beta,\epsilon,\rho,\sigma,\tau,\chi\}||_{L^\infty(\mathcal{S}_{u,v})}<\infty\,,\quad \sup_{u,v}||\nabla\{\mu,\lambda,\alpha,\beta,\epsilon,\rho,\sigma\}||_{L^4(\mathcal{S}_{u,v})}<\infty\,,$$
$$\sup_{u,v}||\nabla^2\{\mu,\lambda,\alpha,\beta,\epsilon,\rho,\sigma,\tau\}||_{L^2(\mathcal{S}_{u,v})}<\infty\,,\quad \sup_{u,v}||\nabla^3\tau||_{L^2(\mathcal{S}_{u,v})}<\infty\,,$$
$$\Delta_\Psi(\mathcal{S})<\infty\,,\quad \Delta_\Psi<\infty\,,$$

on some truncated causal diamond $\mathcal{D}^t_{u,v_\bullet}$. Then there exists,

$$\varepsilon_\star=\varepsilon_\star(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Psi_\star},\sup_{u,v}||\nabla^3\tau||_{L^2(\mathcal{S}_{u,v})},\Delta_\Psi)\,,$$

such that for $\varepsilon\leq\varepsilon_\star$, we have

$$||\Gamma||_{L^\infty(\mathcal{S}_{u,v})}\leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Psi_\star}),\qquad \sum_{i=0}^{1}||\nabla^i\Gamma||_{L^4(\mathcal{S}_{u,v})}\leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Psi_\star}),$$

$$\sum_{i=0}^{2} ||\nabla\!\!\!/^i\Gamma||_{L^2(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi_\star}), \qquad \Delta_\Psi(\mathcal{S}) \leq C(\Delta_{\Psi_\star}),$$

on $\mathcal{D}^t_{u,v_\bullet}$, with Γ standing for an arbitrary connection coefficient.

3.5.4 Estimates on the third derivatives of connection coefficients

We are now in the position to obtain estimates for the NP spin connection coefficients which only require assumptions on the curvature on the light cone. More precisely, one has the following:

Proposition 13 (***further control on the L^2-norm of the connection coefficients***)**.** *Assume, as in the previous proposition, that we are given a solution to the vacuum EFE's in Stewart's gauge emanating from data for the CIVP as prepared in Lemma 2. Suppose that,*

$$\sup_{u,v} ||\{\mu, \lambda, \alpha, \beta, \epsilon, \rho, \sigma, \tau, \chi\}||_{L^\infty(\mathcal{S}_{u,v})} < \infty\,, \quad \sup_{u,v} ||\nabla\!\!\!/\{\mu, \lambda, \alpha, \beta, \epsilon, \rho, \sigma\}||_{L^4(\mathcal{S}_{u,v})} < \infty\,,$$
$$\sup_{u,v} ||\nabla\!\!\!/^2\{\mu, \lambda, \alpha, \beta, \epsilon, \rho, \sigma, \tau\}||_{L^2(\mathcal{S}_{u,v})} < \infty\,, \quad \Delta_\Psi(\mathcal{S}) < \infty\,, \quad \Delta_\Psi < \infty\,,$$

and furthermore that,

$$\sup_{u,v} ||\nabla\!\!\!/^3\{\mu, \lambda, \alpha, \beta, \epsilon, \tau\}||_{L^2(\mathcal{S}_{u,v})} < \infty\,,$$

on $\mathcal{D}^t_{u,v_\bullet}$. Then there exists $\varepsilon_\star = \varepsilon_\star(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi_\star}, \Delta_\Psi)$ such that for $\varepsilon \leq \varepsilon_\star$, we have

$$\sup_{u,v} ||\nabla\!\!\!/^3\{\mu, \lambda, \alpha, \beta, \epsilon\}||_{L^2(\mathcal{S}_{u,v})} \leq 3\Delta_{\Gamma_\star},$$
$$\sup_{u,v} ||\nabla\!\!\!/^3\{\rho, \sigma\}||_{L^2(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi_\star}),$$
$$\sup_{u,v} ||\nabla\!\!\!/^3\{\tau, \chi\}||_{L^2(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi_\star}, \Delta_\Psi).$$

Proof.

Bootstrap assumption. In order to start the proof we place bootstrap assumptions on $\mu, \lambda, \alpha, \beta$ and ϵ, and name the bound on τ as follows,

$$\sup_{u,v} ||\nabla\!\!\!/^3\{\mu, \lambda, \alpha, \beta, \epsilon\}||_{L^2(\mathcal{S}_{u,v})} \leq 4\Delta_{\Gamma_\star}, \qquad \sup_{u,v} ||\nabla\!\!\!/^3\tau||_{L^2(\mathcal{S}_{u,v})} \leq \Delta_\tau.$$

Estimates for ρ and σ. We first estimate the spin connection coefficients ρ and σ using the long direction transport equations (3m) and (3f) as this allows to avoid higher derivatives on the sphere that arise in the short direction equations. Using the expression for $||\nabla\!\!\!/^3 f||_{L^2(\mathcal{S}_{u,v})}$ for an arbitrary scalar f given in Appendix 6.2, we will discuss four typical terms. The first is $\delta^3\rho$. Making use of the commutators of directional covariant derivatives, we can compute the long direction derivative of any third derivatives of ρ on the sphere —for example, one has that,

$$\begin{aligned} D\delta^3\rho = {} & \Gamma^5 + \Gamma^3\delta\Gamma + \Gamma(\delta\Gamma)^2 + \Gamma^2\delta^2\Gamma + \delta\Gamma\delta^2\Gamma + \rho\delta^3(\epsilon + \bar{\epsilon}) \\ & + (4\epsilon - 2\bar{\epsilon} + 5\rho)\delta^3\rho + \sigma\delta^3\bar{\sigma} + \bar{\sigma}\delta^3\sigma + \sigma\delta^2\bar{\delta}\rho, \end{aligned}$$

where here Γ represents linear combinations of the coefficients ϵ, ρ and σ, whose precise form is not crucial for the discussion. The L^2-norm of the term $\delta\Gamma\delta^2\Gamma$ can be split as

$$||\delta\Gamma\delta^2\Gamma||_{L^2(\mathcal{S}_{u,v})} \leq ||\nabla\!\!\!/\Gamma||_{L^4(\mathcal{S}_{u,v})} ||\nabla\!\!\!/^2\Gamma||_{L^4(\mathcal{S}_{u,v})}.$$

The first term on the right-hand side of the inequality can be controlled using the results of Proposition 9. The second term can be controlled using the Sobolev inequality,

$$||\nabla\!\!\!/^2\Gamma||_{L^4(\mathcal{S}_{u,v})} \leq C(\Delta_{e_\star}) \left(||\nabla\!\!\!/^2\Gamma||_{L^2(\mathcal{S}_{u,v})} + ||\nabla\!\!\!/^3\Gamma||_{L^2(\mathcal{S}_{u,v})} \right).$$

Proceeding in a similar way with the other terms in the equation for $D\delta^3\rho$ and the using the long direction inequality in Proposition 3 leads to

$$\begin{aligned} ||\delta^3\rho||_{L^2(\mathcal{S}_{u,v})} \leq {} & C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi_\star}) \\ & + C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi_\star}) \int_0^v \left(||\nabla\!\!\!/^3\rho||_{L^2(\mathcal{S}_{u,v'})} + ||\nabla\!\!\!/^3\sigma||_{L^2(\mathcal{S}_{u,v'})} \right) \mathrm{d}v'. \end{aligned}$$

The second representative term in the expansion of $||\nabla\!\!\!/^3\rho||_{L^2(\mathcal{S}_{u,v})}$ is $||\varpi\delta^2\rho||_{L^2(S_{\mathcal{S}_{u,v}})}$ (recall that $\varpi \equiv \beta - \bar{\alpha}$). One has

$$
\begin{aligned}
D(\varpi\delta^2\rho) &= D\varpi(\delta^2\rho) + \varpi D\delta^2\rho \\
&\qquad = (\Psi_1 + \Gamma^2 + \delta\epsilon - \delta\bar{\epsilon})\delta^2\rho + \Gamma^5 + \Gamma^3\delta\Gamma + \varpi(\delta\Gamma)^2 + \Gamma^2\delta^2\Gamma,
\end{aligned}
$$

from which we can conclude that

$$
||\varpi\delta^2\rho||_{L^2(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi_\star}) + C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi_\star}) \int_0^v ||\nabla\!\!\!/^3\rho||_{L^2(\mathcal{S}_{u,v'})}\mathrm{d}v',
$$

by Sobolev embedding as before. The third representative term is $||\delta\varpi\delta\rho||_{L^2(\mathcal{S}_{u,v})}$ for which we have

$$
D(\delta\varpi\delta\rho) = -\Psi_1\bar{\pi}\delta\rho + \Gamma^3\delta\rho + \delta\Psi_1\delta\rho + \Gamma(\delta\Gamma)^2 + \delta^2(\epsilon - \bar{\epsilon})\delta\rho,
$$

so that

$$
||\delta\varpi\delta\rho||_{L^2(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi_\star}) + C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi_\star}) \int_0^v ||\nabla\!\!\!/^3\rho||_{L^2(\mathcal{S}_{u,v'})}\mathrm{d}v'.
$$

The fourth representative term is $\varpi^2\delta\rho$ for which we can compute

$$
D(\varpi^2\delta\rho) = 2\varpi\Psi_1\delta\rho + \Gamma^3\delta\Gamma + \Gamma(\delta\Gamma)^2 + \Gamma^5.
$$

Consequently, one finds that

$$
||\varpi^2\delta\rho||_{L^2(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi_\star}).
$$

Combining all the expressions arising in the expansion of $\nabla\!\!\!/^3\rho$ one then concludes,

$$
\begin{aligned}
||\nabla\!\!\!/^3\rho||_{L^2(\mathcal{S}_{u,v})} &\leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi_\star}) \\
&\qquad + C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi_\star}) \int_0^v \Big(||\nabla\!\!\!/^3\rho||_{L^2(\mathcal{S}_{u,v'})} + ||\nabla\!\!\!/^3\sigma||_{L^2(\mathcal{S}_{u,v'})}\Big)\,\mathrm{d}v',
\end{aligned}
$$

and Grönwall's inequality finally gives

$$||\nabla\!\!\!/^3\rho||_{L^2(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi_\star}) + C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi_\star}) \int_0^v ||\nabla\!\!\!/^3\sigma||_{L^2(\mathcal{S}_{u,v'})}\mathrm{d}v'.$$

In order to estimate $||\nabla\!\!\!/^3\sigma||_{L^2(\mathcal{S}_{u,v})}$, we make use, again, of the general expressions contained in Appendix 6.2. The structure of equation $D\delta^3\sigma$ is similar to that of ρ. Then one can also calculate and obtain the inequality for $\nabla\!\!\!/^3\sigma$ like

$$||\nabla\!\!\!/^3\sigma||_{L^2(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi_\star}) + C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi_\star}) \int_0^v ||\nabla\!\!\!/^3\rho||_{L^2(\mathcal{S}_{u,v'})}\mathrm{d}v'.$$

Combine with the equality of ρ, one can obtain that

$$\begin{aligned}
&||\nabla\!\!\!/^3\sigma||_{L^2(\mathcal{S}_{u,v})} + ||\nabla\!\!\!/^3\rho||_{L^2(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi_\star}) \\
&+ C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi_\star}) \int_0^v \left(||\nabla\!\!\!/^3\rho||_{L^2(\mathcal{S}_{u,v'})} + ||\nabla\!\!\!/^3\sigma||_{L^2(\mathcal{S}_{u,v})}\right) \mathrm{d}v'.
\end{aligned}$$

So the Grönwall's estimate gives us

$$||\nabla\!\!\!/^3\sigma||_{L^2(\mathcal{S}_{u,v})} + ||\nabla\!\!\!/^3\rho||_{L^2(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi_\star}).$$

These inequalities above in turn imply that

$$\begin{aligned}
||\nabla\!\!\!/^3\rho||_{L^2(\mathcal{S}_{u,v})} &\leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi_\star}), \\
||\nabla\!\!\!/^3\sigma||_{L^2(\mathcal{S}_{u,v})} &\leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi_\star}).
\end{aligned}$$

Estimates for τ and χ. Making use of the structure equation (3b) and the commutators we obtain

$$\begin{aligned}
D\delta^3\tau = {}& \delta^3\Psi_1 + \Gamma\delta^3\Gamma_1 + \Gamma\delta^3\tau + \Gamma\delta^2\Psi_1 + \delta\Gamma\delta^2\Gamma + \Gamma^2\delta^2\Gamma \\
&+ \Gamma^2\delta\Psi_1 + \delta\Gamma\delta\Psi_1 + \Gamma^3\delta\Gamma + \Gamma(\delta\Gamma)^2,
\end{aligned}$$

where Γ_1 contains combinations of ϵ, α, β, ρ and σ. Thus, using the main bootstrap assumption and the definition of Δ_Ψ we obtain that

$$||\nabla\!\!\!/^3\tau||_{L^2(\mathcal{S}_{u,v})} \leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Psi_\star},\Delta_\Psi) + C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Psi_\star})\int_0^v ||\nabla\!\!\!/^3\tau||_{L^2(\mathcal{S}_{u,v'})}\mathrm{d}v'.$$

Accordingly, using Grönwall's inequality one arrives to

$$||\nabla\!\!\!/^3\tau||_{L^2(\mathcal{S}_{u,v})} \leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Psi_\star},\Delta_\Psi).$$

The construction of an estimate for χ is similar. In this case we obtain that

$$||\nabla\!\!\!/^3\chi||_{L^2(\mathcal{S}_{u,v})} \leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Psi_\star},\Delta_\Psi).$$

Estimates for the remaining connection coefficients. In order to provide estimates for

$$||\nabla\!\!\!/^3\{\mu,\lambda,\alpha,\beta,\epsilon\}||_{L^2(\mathcal{S}_{u,v})},$$

we make use of the transport equations along the short direction. The proofs for the various coefficients are similar so for brevity we discuss only the argument for ϵ. In this case one can readily compute that

$$\begin{aligned}\Delta\delta^3\epsilon = &-\delta^3\Psi_2 + \Gamma\delta^3\Gamma_1 + \Gamma\delta^3\epsilon + \Psi_1\delta^2\Gamma + \delta\Gamma\delta^2\Gamma + \Gamma^2\delta^2\Gamma \\ &+ \Gamma\delta^2\Psi_2 + \Gamma^2\delta\Psi_2 + \Gamma^3\delta\Gamma + \Gamma(\delta\Gamma)^2 + \Gamma^3\Psi_2 + \Gamma^5,\end{aligned}$$

where the coefficients Γ_1 do not contain ϵ. Making use of the short direction inequality of Proposition 3 we obtain that

$$\begin{aligned}||\nabla\!\!\!/^3\epsilon||_{L^2(\mathcal{S}_{u,v})} \leq\ & 2||\nabla\!\!\!/^3\epsilon||_{L^2(\mathcal{S}_{0,v})} + C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Psi_\star})\Delta_\Psi\varepsilon^{1/2} \\ &+ C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Psi_\star})\int_0^u ||\nabla\!\!\!/^3\epsilon||_{L^2(\mathcal{S}_{u',v})}\mathrm{d}u'.\end{aligned}$$

In particular, we can choose the range of integration sufficiently small so that

$$||\nabla\!\!\!/^3\epsilon||_{L^2(\mathcal{S}_{u,v})} \leq 3\Delta_{\Gamma_\star}.$$

The argument for $||\nabla^3\{\mu,\lambda,\alpha,\beta\}||_{L^2(\mathcal{S}_{u,v})}$ is the same.

Concluding the argument. An inspection of the estimates obtained in the previous paragraphs shows that we have improved the initial bootstrap assumption. This concludes the proof of the proposition. □

3.5.5 Main estimates for the curvature

We are now in the position to obtain the main estimates for the components of the Weyl tensor. We start with an estimate on a given pair of null hypersurfaces in terms of their value at hypersurfaces in the past.

Proposition 14 (***basic control of components of the Weyl tensor on the light cones in terms of its values on causal diamonds***). *Suppose that we are given a solution to the vacuum EFEs in Stewart's gauge and that $\mathcal{D}_{u,v}$ is contained in the existence area. The following L^2 estimates for the Weyl curvature hold:*

$$\sum_{i=0,1,2}\int_{\mathcal{N}_u(0,v)}|\Psi_i|^2+\sum_{j=1,2,3}\int_{\mathcal{N}'_v(0,u)}Q^{-1}|\Psi_j|^2$$
$$\leq\sum_{i=0,1,2}\int_{\mathcal{N}_0(0,v)}|\Psi_i|^2+\sum_{j=1,2,3}\int_{\mathcal{N}'_0(0,u)}Q^{-1}|\Psi_j|^2+\int_{\mathcal{D}_{u,v}}|\Psi_H\Psi\Gamma+cc|,$$

where Ψ contains Ψ_k, $k=0,...,4$, Ψ_H denotes the components Ψ_k, $k=0,...,3$, "cc" denotes the complex conjugate of the last term on the right-hand side and Γ stands for arbitrary connection coefficients from the collection $\{\mu,\lambda,\alpha,\beta,\epsilon,\rho,\sigma,\tau\}$.

Proof. Assuming, as always that the vacuum field equations of GR are satisfied, we start considering the Bianchi identities (4b) and (4a) written schematically as

$$\Delta\Psi_0=\delta\Psi_1+\Gamma\Psi,$$
$$D\Psi_1=\bar{\delta}\Psi_0+\Gamma\Psi.$$

Then, integration by parts one obtains (again, using schematic notation) that

$$\int_{\mathcal{D}_{u,v}}\bar{\Psi}_0\Delta\Psi_0=\int_{\mathcal{D}_{u,v}}\bar{\Psi}_0\delta\Psi_1+\int_{\mathcal{D}_{u,v}}\bar{\Psi}_0\Gamma\Psi$$

$$
\begin{aligned}
&= -\int_{\mathcal{D}_{u,v}} \Psi_1 \delta \bar{\Psi}_0 - \int_{\mathcal{D}_{u,v}} \Psi_1 \bar{\Psi}_0 \varpi + \int \bar{\Psi}_0 \Gamma \Psi \\
&= -\int_{\mathcal{D}_{u,v}} \Psi_1 D \bar{\Psi}_1 + \int_{\mathcal{D}_{u,v}} \{\bar{\Psi}_0, \Psi_1\} \Gamma \Psi.
\end{aligned}
$$

Hence, using the identities in Lemma 9, we conclude that

$$
\begin{aligned}
\int_{\mathcal{N}_u(0,v)} |\Psi_0|^2 + \int_{\mathcal{N}'_v(0,u)} Q^{-1} |\Psi_1|^2 \le \int_{\mathcal{N}_0(0,v)} |\Psi_0|^2 + \int_{\mathcal{N}'_0(0,u)} Q^{-1} |\Psi_1|^2 \\
+ \int_{\mathcal{D}_{u,v}} (|\{\Psi_0, \Psi_1\} \Psi \Gamma + \mathrm{cc}|),
\end{aligned}
$$

where in the previous expression Ψ contains $\Psi_{0,1,2}$. Analogous inequalities can be obtained for the pairs $\Delta\Psi_1$, $D\Psi_2$, and $\Delta\Psi_2$, $D\Psi_3$. □

Similar estimates can be obtained for the first angular derivatives of the components of the Weyl tensor.

Proposition 15 (***control of the first angular derivatives of the components of the Weyl tensor***)**.** *Again let $\mathcal{D}_{u,v}$ be contained in the existence area, then we have that*

$$
\begin{aligned}
\sum_{i=0,1,2} \int_{\mathcal{N}_u(0,v)} |\nabla\!\!\!/ \Psi_i|^2 + \sum_{j=1,2,3} \int_{\mathcal{N}'_v(0,u)} Q^{-1} |\nabla\!\!\!/ \Psi_j|^2 \\
\le \sum_{i=0,1,2} \int_{\mathcal{N}_0(0,v)} |\nabla\!\!\!/ \Psi_i|^2 + \sum_{j=1,2,3} \int_{\mathcal{N}'_0(0,u)} Q^{-1} |\nabla\!\!\!/ \Psi_j|^2 \\
+ \int_{\mathcal{D}_{u,v}} |\nabla\!\!\!/ \Psi_H| (|\Psi \Gamma^2| + |\Gamma \nabla\!\!\!/ \Psi| + |\Psi \nabla\!\!\!/ \Gamma|),
\end{aligned}
$$

where Ψ contains Ψ_k, $k = 0, ..., 4$, and Ψ_H contains Ψ_k, $k = 0, ..., 3$, and again Γ stands for some combination of the connection coefficients $\{\mu, \lambda, \alpha, \beta, \epsilon, \rho, \sigma, \tau\}$.

Proof. Again, we make use of integration by parts. Consider for example

$$
\begin{aligned}
\int_{\mathcal{D}_{u,v}} \bar{\delta} \bar{\Psi}_0 \Delta \delta \Psi_0 &= \int_{\mathcal{D}_{u,v}} \bar{\delta} \bar{\Psi}_0 \delta^2 \Psi_1 + \int_{\mathcal{D}_{u,v}} \bar{\delta} \bar{\Psi}_0 (\Gamma^2 \Psi_i + \Gamma \delta \Psi_i + \Psi_i \delta \Gamma) \\
&= -\int_{\mathcal{D}_{u,v}} \delta \bar{\delta} \bar{\Psi}_0 \delta \Psi_1 + \int_{\mathcal{D}_{u,v}} \bar{\delta} \bar{\Psi}_0 (\Gamma^2 \Psi_i + \Gamma \delta \Psi_i + \Psi_i \delta \Gamma)
\end{aligned}
$$

$$= -\int_{\mathcal{D}_{u,v}} \delta\Psi_1 D\bar{\delta}\bar{\Psi}_1 + \int_{\mathcal{D}_{u,v}} (\bar{\delta}\bar{\Psi}_0, \delta\Psi_1)(\Gamma^2\Psi_i + \Gamma\delta\Psi_i + \Psi_i\delta\Gamma)$$

with $i = 0,\ 1,\ 2$. A similar expression can be obtained for the combination

$$\int_{\mathcal{D}_{u,v}} \delta\bar{\Psi}_0 \Delta\bar{\delta}\Psi_0 + \int_{\mathcal{D}_{u,v}} \bar{\delta}\Psi_1 D\delta\bar{\Psi}_1.$$

Thus, using Lemma 9 can conclude that

$$\begin{aligned}\int_{\mathcal{N}_u(0,v)} |\nabla\!\!\!/\,\Psi_0|^2 + \int_{\mathcal{N}'_v(0,u)} Q^{-1}|\nabla\!\!\!/\,\Psi_1|^2 \leq \int_{\mathcal{N}_0(0,v)} |\nabla\!\!\!/\,\Psi_0|^2 + \int_{\mathcal{N}'_0(0,v)} Q^{-1}|\nabla\!\!\!/\,\Psi_1|^2 \\ + \int_{\mathcal{D}_{u,v}} |\nabla\!\!\!/\,\{\Psi_0, \Psi_1\}|(|\Psi\Gamma^2| + |\Gamma\nabla\!\!\!/\,\Psi| + |\Psi\nabla\!\!\!/\,\Gamma|),\end{aligned}$$

where Ψ contains the components Ψ_0, Ψ_1 and Ψ_2. A similar computation for the other pairs of components renders the desired result. $\square$

The previous result can be extended to include higher order derivatives. More precisely:

Proposition 16 (***control of the higher angular derivatives of the components of the Weyl tensor***)**.** *Let $\mathcal{D}_{u,v}$ again be contained in the existence area. Given a non-negative integer m, one has*

$$\begin{aligned}\sum_{i=0,1,2} \int_{\mathcal{N}_u(0,v)} |\nabla\!\!\!/^m\Psi_i|^2 + \sum_{j=1,2,3} \int_{\mathcal{N}'_v(0,u)} Q^{-1}|\nabla\!\!\!/^m\Psi_j|^2 \\ \leq \sum_{i=0,1,2} \int_{\mathcal{N}_0(0,v)} |\nabla\!\!\!/^m\Psi_i|^2 + \sum_{j=1,2,3} \int_{\mathcal{N}'_0(0,v)} Q^{-1}|\nabla\!\!\!/^m\Psi_j|^2 \\ + \int_{\mathcal{D}_{u,v}} |\nabla\!\!\!/^m\Psi_H| \sum_{i_1+i_2+i_3+i_4=m} |\nabla\!\!\!/^{i_1}\Gamma^{i_2}||\nabla\!\!\!/^{i_3}\Gamma||\nabla\!\!\!/^{i_4}\Psi|.\end{aligned}$$

where Ψ contains the components Ψ_k, $k = 0, ..., 4$, and Ψ_H contains the components Ψ_k, $k = 0, ..., 3$. Again Γ stands for some combination of the connection coefficients $\{\mu, \lambda, \alpha, \beta, \epsilon, \rho, \sigma, \tau\}$.

To wrap up the argument we also need estimates on the components Ψ_3 and Ψ_4.

These follow from the Bianchi identities

$$\begin{aligned}\Delta\Psi_3 - \delta\Psi_4 &= 4\Psi_4\beta - \Psi_4\tau - 4\Psi_3\mu,\\ D\Psi_4 - \bar{\delta}\Psi_3 &= \Psi_4(\rho - 4\epsilon) + 2\Psi_3(3\alpha + 2\beta) - 3\Psi_2\lambda.\end{aligned} \tag{3.23}$$

Using a similar approach to the one used in the previous propositions one can prove the following:

Proposition 17 (***control of the higher angular derivatives of the "bad" components of the Weyl tensor***)**.** *Let $\mathcal{D}_{u,v}$ be contained in the existence area. Given a non-negative integer m, one has that*

$$\begin{aligned}&\int_{\mathcal{N}_u(0,v)} |\nabla\!\!\!/^m\Psi_3|^2 + \int_{\mathcal{N}'_v(0,u)} Q^{-1}|\nabla\!\!\!/^m\Psi_4|^2\\ &\quad\le \int_{\mathcal{N}_0(0,v)} |\nabla\!\!\!/^m\Psi_3|^2 + \int_{\mathcal{N}'_0(0,u)} Q^{-1}|\nabla\!\!\!/^m\Psi_4|^2\\ &\qquad + \int_{\mathcal{D}_{u,v}} |\nabla\!\!\!/^m\Psi_4| \sum_{i_1+i_2+i_3+i_4=m} |\nabla\!\!\!/^{i_1}\Gamma'^{i_2}||\nabla\!\!\!/^{i_3}\Gamma'||\nabla\!\!\!/^{i_4}\Psi_4|\\ &\qquad + \int_{\mathcal{D}_{u,v}} |\nabla\!\!\!/^m\Psi_3| \sum_{i_1+i_2+i_3+i_4=m} |\nabla\!\!\!/^{i_1}\Gamma^{i_2}||\nabla\!\!\!/^{i_3}\Gamma||\nabla\!\!\!/^{i_4}\Psi|\\ &\qquad + \int_{\mathcal{D}_{u,v}} |\nabla\!\!\!/^m\Psi_4| \sum_{i_1+i_2+i_3+i_4=m} |\nabla\!\!\!/^{i_1}\Gamma^{i_2}||\nabla\!\!\!/^{i_3}\Gamma||\nabla\!\!\!/^{i_4}\Psi'_H|,\end{aligned}$$

where Ψ contains the components Ψ_3 and Ψ_4, while Ψ'_H contains the components Ψ_2 and Ψ_3. Here Γ stands for some combination of the connection coefficients $\{\mu, \lambda, \alpha, \beta, \epsilon, \rho, \tau, \sigma\}$. Because neither the coefficient of Ψ_4 on the right hand side of (3.23) *nor the NP $\delta\bar{\delta}$-commutator* (2.15d) *contain τ, χ terms, neither does Γ'.*

Propositions 14-17 clearly make no use of the estimates demonstrated in the previous sections. Finally, we therefore conclude this section with the main estimate for the components of the Weyl tensor employing our earlier work. This proposition makes only assumptions on the initial data.

Proposition 18 (***control of the components of the Weyl tensor in terms of the initial data***)**.** *Suppose we are given a solution to the vacuum EFE's in*

Stewart's gauge emanating from data for the CIVP as prepared in Lemma 2, satisfying

$$\Delta_{e_\star},\ \Delta_{\Gamma_\star},\ \Delta_{\Psi_\star} < \infty,$$

with the solution itself satisfying

$$\begin{aligned}
&\sup_{u,v} ||\{\mu,\lambda,\alpha,\beta,\epsilon,\rho,\sigma,\tau,\chi\}||_{L^\infty(\mathcal{S}_{u,v})} < \infty\,, \quad \sup_{u,v} ||\nabla\!\!\!/ \{\mu,\lambda,\alpha,\beta,\epsilon,\rho,\sigma\}||_{L^4(\mathcal{S}_{u,v})} < \infty\,,\\
&\sup_{u,v} ||\nabla\!\!\!/^2 \{\mu,\lambda,\alpha,\beta,\epsilon,\rho,\sigma,\tau\}||_{L^2(\mathcal{S}_{u,v})} < \infty\,, \quad \sup_{u,v} ||\nabla\!\!\!/^3 \{\mu,\lambda,\alpha,\beta,\epsilon,\tau\}||_{L^2(\mathcal{S}_{u,v})} < \infty\,,\\
&\Delta_\Psi(\mathcal{S}) < \infty\,, \quad \Delta_\Psi < \infty\,,
\end{aligned}$$

on some truncated causal diamond $\mathcal{D}^t_{u,v_\bullet}$. *Then there exists* $\varepsilon_\star = \varepsilon_\star(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi_\star})$ *such that for* $\varepsilon_\star \leq \varepsilon$ *we have*

$$\Delta_\Psi \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi_\star}).$$

Proof. The aim in this proof is to control the terms involving integrals on the diamond $\mathcal{D}_{u,v}$ arising in Propositions 16 and 17 for $m \leq 3$. Starting with Proposition 16 one has that the relevant integral is given by

$$\int_{\mathcal{D}_{u,v}} |\nabla\!\!\!/^m \Psi_H| \sum_{i_1+i_2+i_3+i_4=m} |\nabla\!\!\!/^{i_1}\Gamma^{i_2}||\nabla\!\!\!/^{i_3}\Gamma||\nabla\!\!\!/^{i_4}\Psi|, \tag{3.24}$$

for (u,v) in $\mathcal{D}^t_{\varepsilon,v_\bullet}$. On the one hand, for the first factor in this integral, given that $\Psi_H \in \{\Psi_0, \Psi_1, \Psi_2, \Psi_3\}$ can be controlled in $L^2(\mathcal{N}_u(0,v))$, one readily obtains

$$||\nabla\!\!\!/^m \Psi_H||_{L^2(\mathcal{D}_{u,v})} = \left(\int_0^u \int_0^v \int_{\mathcal{S}_{u',v'}} |\nabla\!\!\!/^m \Psi_H|^2 \mathrm{d}v' \mathrm{d}u'\right)^{1/2} \leq C\Delta_\Psi \varepsilon^{1/2},$$

On the other, for the factors contains Ψ_4, one only has control on $\mathcal{N}'_v(0,u)$ —that is,

$$||\nabla\!\!\!/^m \Psi||_{L^2(\mathcal{D}_{u,v})} \leq C\Delta_\Psi.$$

It then follows that the integral (3.24) can be estimated as,

$$\int_{\mathcal{D}_{u,v}} |\nabla\!\!\!/^m \Psi_H| \sum_{i_1+i_2+i_3+i_4=m} |\nabla\!\!\!/^{i_1}\Gamma^{i_2}||\nabla\!\!\!/^{i_3}\Gamma||\nabla\!\!\!/^{i_4}\Psi|$$
$$\leq C\varepsilon^{1/2}\Delta_\Psi \sum_{i_1+i_2+i_3+i_4\leq 3} ||\nabla\!\!\!/^{i_1}\Gamma^{i_2}\nabla\!\!\!/^{i_3}\Gamma\nabla\!\!\!/^{i_4}\Psi||_{L^2(\mathcal{D}_{u,v})}. \tag{3.25}$$

In particular, for $m = 0$, the right-hand side of the above inequality gives

$$C\varepsilon^{1/2}\Delta_\Psi||\Gamma\Psi||_{L^2(\mathcal{D}_{u,v})} \leq C\epsilon^{1/2}\Delta_\Psi||\Gamma||_{L^\infty(S)}||\Psi||_{L^2(\mathcal{D}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi_\star}, \Delta_\Psi)\varepsilon^{1/2}.$$

Next, when $m = 1$, we have that the right-hand of inequality (3.25) gives

$$C\varepsilon^{1/2}\Delta_\Psi||\Gamma^2\Psi + \Gamma|\nabla\!\!\!/\Psi| + \Psi|\nabla\!\!\!/\Gamma|||_{L^2(\mathcal{D}_{u,v})}.$$

The first two terms can be controlled like the case $m = 0$, and the third can be controlled by means of Sobolev embedding:

$$\begin{aligned}||\Psi|\nabla\!\!\!/\Gamma|||_{L^2(\mathcal{S}_{u,v})} &\leq ||\nabla\!\!\!/\Gamma||_{L^\infty(\mathcal{S}_{u,v})}||\Psi||_{L^2(\mathcal{D}_{u,v})}\\ &\leq \left(||\nabla\!\!\!/\Gamma||_{L^2(\mathcal{S}_{u,v})} + ||\nabla\!\!\!/^2\Gamma||_{L^2(\mathcal{S}_{u,v})} + ||\nabla\!\!\!/^3\Gamma||_{L^2(\mathcal{S}_{u,v})}\right)||\Psi||_{L^2(\mathcal{D}_{u,v})}.\end{aligned}$$

For the case $m = 2$, the terms on the right-hand side of inequality (3.25) give

$$C\varepsilon^{1/2}\Delta_\Psi||\Gamma|\nabla\!\!\!/^2\Psi| + \Gamma^3\Psi + \Gamma^2|\nabla\!\!\!/\Psi| + \Psi\Gamma|\nabla\!\!\!/\Gamma| + |\nabla\!\!\!/\Psi||\nabla\!\!\!/\Gamma| + \Psi|\nabla\!\!\!/^2\Gamma|||_{L^2(\mathcal{D}_{u,v})}. \tag{3.26}$$

All terms, save last one, can be controlled by analysis analogous to that used in the previous cases. To see this, we split the L^∞-norm of the connection coefficient and the L^2-normal of the curvature. The L^∞-normal can then be controlled by means of Sobolev embedding. For the last term, we have

$$\begin{aligned}\left(\int_0^u\int_0^v\int_{\mathcal{S}_{u',v'}}(\Psi|\nabla\!\!\!/^2\Gamma|)^2\mathrm{d}v'\mathrm{d}u'\right)^{1/2} &\leq \left(\int_0^u\int_0^v ||\Psi||^2_{L^\infty(\mathcal{S}_{u',v'})}||\nabla\!\!\!/^2\Gamma||^2_{L^2(\mathcal{S}_{u',v'})}\mathrm{d}v'\mathrm{d}u'\right)^{1/2}\\ &\leq \left(\sup_{\mathcal{D}_{u,v}}||\nabla\!\!\!/^2\Gamma||_{L^2(\mathcal{S}_{u',v'})}\right)\sum_{i=0}^2||\nabla\!\!\!/^i\Psi||_{L^2(\mathcal{D}_{u,v})},\end{aligned}$$

hence (3.26) under control.

Finally, when $m = 3$ the terms on the right-hand side of inequality (3.25) give

$$C\varepsilon^{1/2}\Delta_\Psi||(\Gamma|\nabla^3\Psi| + \Psi|\nabla^3\Gamma| + |\nabla\Gamma||\nabla^2\Psi| + |\nabla\Psi||\nabla^2\Gamma| + \Gamma^2|\nabla^2\Psi| + \Gamma\Psi|\nabla^2\Gamma| \\ + \Gamma|\nabla\Gamma||\nabla\Psi| + \Psi|\nabla\Gamma|^2 + \Gamma^3|\nabla\Psi| + \Psi\Gamma^2|\nabla\Gamma| + \Gamma^4\Psi)||_{L^2(\mathcal{D}_{u,v})}.$$

The various terms in this expression can be estimated in a manner analogous to the previous cases. We conclude that the integral over $\mathcal{D}_{u,v}$ can be controlled by

$$\int_{\mathcal{D}_{u,v}} |\nabla^m\Psi_H| \sum_{i_1+i_2+i_3+i_4=m} |\nabla^{i_1}\Gamma^{i_2}||\nabla^{i_3}\Gamma||\nabla^{i_4}\Psi| \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi_\star}, \Delta_\Psi)\varepsilon^{1/2}.$$

We now proceed to examine the estimate from Proposition 17. The terms in

$$\int_{\mathcal{D}_{u,v}} |\nabla^m\Psi_3| \sum_{i_1+i_2+i_3+i_4=m} |\nabla^{i_1}\Gamma^{i_2}||\nabla^{i_3}\Gamma||\nabla^{i_4}\Psi|$$

are identical to those already analysed and can be controlled by

$$C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi_\star}, \Delta_\Psi)\varepsilon^{1/2}.$$

The terms

$$\int_{\mathcal{D}_{u,v}} |\nabla^m\Psi_4| \sum_{i_1+i_2+i_3+i_4=m} |\nabla^{i_1}\Gamma^{i_2}||\nabla^{i_3}\Gamma||\nabla^{i_4}\Psi'_H|$$

can also be controlled because the components of the Weyl tensor contained in $\Psi'_H = \{\Psi_2, \Psi_3\}$ have already been shown to be controlled. The remaining terms are

$$\int_{\mathcal{D}_{u,v}} |\nabla^m\Psi_4| \sum_{i_1+i_2+i_3+i_4=m} |\nabla^{i_1}\Gamma'^{i_2}||\nabla^{i_3}(\rho+\epsilon)||\nabla^{i_4}\Psi_4|.$$

We proceed to by treating $m = 0, \ldots, 3$ individually. Notice in particular, that Γ' does contains neither τ nor χ. Crucially the weakest bounds of Proposition 12 and Proposition 13 involving Δ_Ψ are therefore not invoked in the resulting computation, and so after a lengthy analysis one concludes that these terms satisfy

$$\begin{aligned}
\int_{\mathcal{D}_{u,v}} |\nabla\!\!\!/^m \Psi_4| &\sum_{i_1+i_2+i_3+i_4=m} |\nabla\!\!\!/^{i_1}\Gamma^{i_2}||\nabla\!\!\!/^{i_3}(\rho+\epsilon)||\nabla\!\!\!/^{i_4}\Psi_4| \\
&\leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Psi_\star})\int_0^v ||\nabla\!\!\!/^m\Psi_4||_{L^2(\mathcal{N}'_{v'}(0,u))}\sum_{i=0}^m ||\nabla\!\!\!/^i\Psi_4||_{L^2(\mathcal{N}'_{v'}(0,u))}\mathrm{d}v' \\
&\leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Psi_\star})\int_0^v \sum_{i=0}^m ||\nabla\!\!\!/^i\Psi_4||^2_{L^2(\mathcal{N}'_{v'}(0,u))}\mathrm{d}v'.
\end{aligned}$$

Substituting the previous expressions into the inequality of Proposition 17 one concludes that

$$\begin{aligned}
\sum_{i=0}^3 ||\nabla\!\!\!/^i\Psi_4||^2_{L^2(\mathcal{N}'_v(0,u))} &\leq C\Delta_{\Psi_\star} + C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Psi_\star},\Delta_\Psi)\varepsilon^{1/2} \\
&\qquad + C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Psi_\star})\int_0^v\sum_{i=0}^m ||\nabla\!\!\!/^i\Psi_4||^2_{L^2(\mathcal{N}'_v(0,u))}\mathrm{d}v'.
\end{aligned}$$

Accordingly, using Grönwall's inequality and taking ε sufficiently small one finds,

$$\sum_{i=0}^3 ||\nabla\!\!\!/^i\Psi_4||^2_{L^2(\mathcal{N}'_v(0,u))} \leq C\Delta_{\Psi_\star} + C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Psi_\star},\Delta_\Psi)\varepsilon^{1/2} \leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Psi_\star}).$$

Using this estimate, it follows that

$$\Delta_\Psi \leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Psi_\star}) + C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Psi_\star},\Delta_\Psi)\varepsilon^{1/2}.$$

Taking ε small enough we have proven the proposition. □

3.6 Last slice argument and the end of the proof

In this section we make use of the estimates developed in the previous sections to show the existence of solutions to the vacuum Einstein field equations exists in the rectangular domain

$$\mathcal{D} = \{0 \leq u \leq \varepsilon,\ 0 \leq v \leq v_\bullet\}.$$

The strategy makes use of an argument by contradiction known as the *last slice argument*, in which it is assumed that the solution does not fill the whole of $\mathcal{D}$ and, accordingly, there exists a hypersurface (the last slice) which bounds the domain of existence of the solution. The estimates we have constructed in the previous sections allow then to show that, in fact, on this slice the solution and its derivatives are bounded. Thus, it is possible to make use of the standard Cauchy problem for the Einstein field equations to show that the solution extends beyond the hypersurface t^* —an observation which contradicts the original assumption.

3.6.1 Setup

In order to implement the above strategy one foliates the rectangle $\mathcal{D}$ by means of spacelike hypersurfaces. To this end recall definition (3.4) of the *time function*

$$t \equiv u + v$$

so that ∇t is timelike. Let Σ_t denote the level sets of t.

The last slice argument starts by invoking the local existence result for the CIVP based on Rendall's reduction strategy. This result ensures the existence of a solution to evolution equations in a neighbourhood $\mathcal{V}$ of $\mathcal{S}_\star$ on $J^+(\mathcal{S}_\star)$ —see Theorem 2. Within this neighbourhood there exists a truncated causal diamond on which all the bootstrap assumptions required to obtain the estimates from the previous sections hold. Thus, we know that the set on which the bootstrap hypotheses hold is non-empty, and hence render our estimates applicable. The rest of the last slice argument proceeds now to show that this basic truncated causal diamond can be progressively enlarged as long as one has control on the initial data on the null cone $\mathcal{N}_\star$ thus exhausting the domain $\mathcal{D}$.

If the solution does not exist in the whole of $\mathcal{D}$, we must have $t^* \in (0, I + \varepsilon)$ such that

$$t^* = \sup\{t : \text{ the spacetime exists in } \mathcal{D} \cap \cup_{\tau \in [0,t)} \Sigma_\tau\}.$$

Let $\boldsymbol{h}_t$ and $\boldsymbol{K}_t$ be, respectively, the induced metric and second fundamental form on Σ_t. A schematic depiction of the geometric set-up is shown in Figure 3.2.

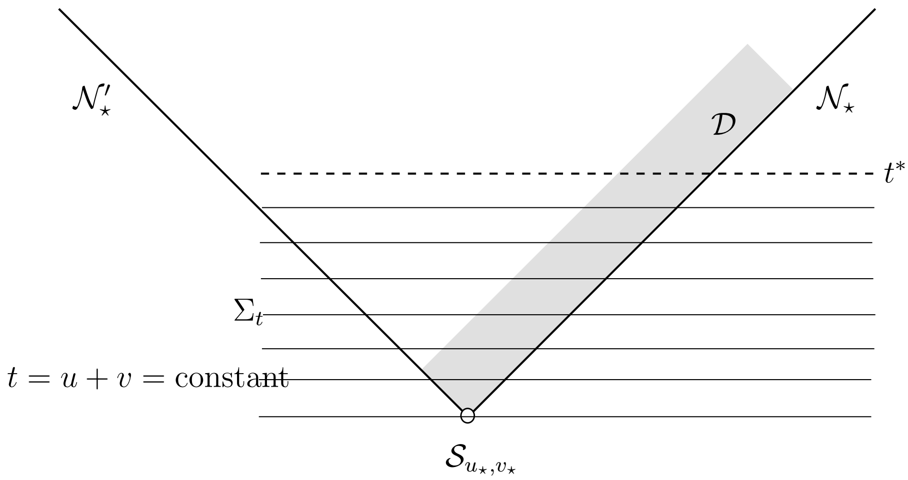

Figure 3.2: Setup for the *last slice argument*. On each slice of the family of hypersurfaces Σ_t one has a smooth initial data set $(\boldsymbol{h}_t, \boldsymbol{K}_t)$ for the vacuum Einstein field equations. The estimates of Proposition 18 then show that even on the *last slice* Σ_{t^*} one has a well initial data set. Thus, the solution can be extended beyond this slice —a contradiction!

3.6.2 Main argument

In the following we will show that the fields $\boldsymbol{h}_t$ and $\boldsymbol{K}_t$ converge in C^∞ to fields $\boldsymbol{h}_{t^*}$ and $\boldsymbol{K}_{t^*}$. Moreover, it will be shown that the pair $(\boldsymbol{h}_{t^*}, \boldsymbol{K}_{t^*})$ satisfy the Einstein constraint equations on Σ_{t^*}. In order to show this, it is necessary to show that all derivatives of $\boldsymbol{h}_t$ are bounded uniformly in $L^2(\Sigma_t)$ for all $t < t^*$. The method proceeds by induction:

Base step. The first step corresponds, in essence, to the estimates obtained in the previous sections. More precisely, we have first derived uniform estimates for the L^∞-norm of the *zeroth order derivatives* of connection on $\mathcal{D}$ —see Proposition 8. For this we needed to assume that

$$\sup_{u,v} ||\nabla\!\!\!/^2 \tau||_{L^2(\mathcal{S}_{u,v})} < \infty, \qquad \sup_{u,v} ||\nabla\!\!\!/^3 \tau||_{L^2(\mathcal{S}_{u,v})} < \infty, \qquad \Delta_\Psi(\mathcal{S}) < \infty, \qquad \Delta_\Psi < \infty \tag{3.27}$$

on the truncated causal diamond. These conditions also lead to the analysis the L^4-norms of the first order derivatives (Proposition 9) and L^2-norms of the second order

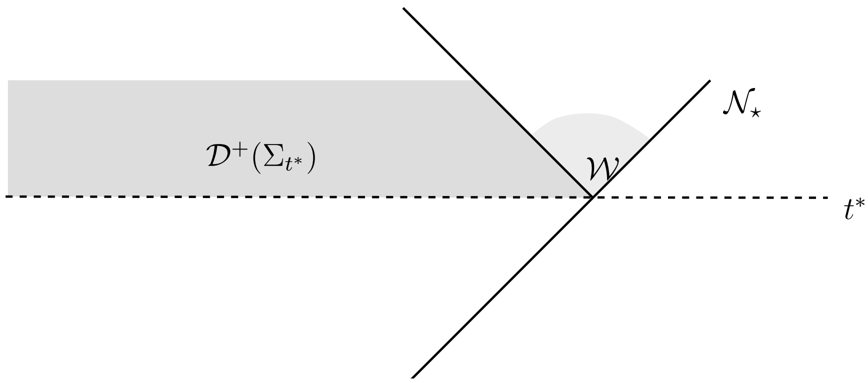

Figure 3.3: Zoom in on the hypothetical last slice. Regular Cauchy initial data on Σ_{t^*} allows to extend the solution to, at least, a slab on $\mathcal{D}^+(\Sigma_{t^*})$ making use of the standard Cauchy problem for the Einstein field equations. On the wedge $\mathcal{W}$, a solution can be recovered by appealing to Rendall's formulation of the local CIVP.

derivatives of the connection —see Proposition 10. Now, using the bootstrap assumptions, it follows that $\Delta_\Psi(\mathcal{S}) < \infty$ uniformly on $\mathcal{D}$ with bounds given in terms of the initial data —thus, this condition can be removed from the list in (3.27). Similarly, we can also drop the condition $||\nabla\!\!\!/^3\tau||_{L^2(\mathcal{S}_{u,v})} < \infty$ and estimate the L^2-norm of the third order angular derivatives of the connection. In order to do so, we make use of the D-direction (i.e. the *long direction*) equations for the NP coefficients ρ and σ, rather than the equations along the short direction as we want to avoid dealing with the higher order derivative of τ on spheres $\mathcal{S}_{u,v}$. Now, using integration by parts, one concludes that Δ_Ψ satisfies a similar uniform bound on $\mathcal{D}$. Thus, it has been shown that given some initial data on the initial light cone, it is possible to estimate the L^2-norm on the spheres $\mathcal{S}_{u,v}$ of the connection coefficients and their derivatives up to third order.

Intermediate step. The previous analysis is the *base step* of the induction. As an intermediate induction step one analyses the fourth order derivatives of the connection coefficients. To this end, we make use of the same approach used in the analysis of the third order derivatives in Proposition 13 This approach requires the control of the norms of the fourth order derivatives of the components of the Weyl tensor on the light cone. As in the case of the Base Step, the required bounds need to be uniform on the truncated causal diamond with bounds given in terms of the initial data. This control can be achieved by the using integration by parts as in the analysis of Proposition 18.

Remark 19. The reason the method to analyse the fourth order derivatives of the connection coefficients is different from that of the third and lower orders lies in the structural properties of the equations —these properties become manifest when considering higher order derivatives. In particular, one has that:

i). For zeroth-order derivatives, we cannot make use of the Codazzi equation to access the norms of ρ and σ, since the Codazzi equation is a first order equation for the derivatives of ρ and σ. Further difficulties arise from the nonlinear term ρ^2 in the D-direction equation (3m) for the coefficient ρ.

ii). For the first-order derivatives, we can readily estimate the L^2-norm of the connection. However, this is not enough for the second order derivatives. In the L^2 estimate for the second order derivatives of the connections, we need Hölder's inequality to separate products of the form $\delta\Gamma \times \delta\Gamma$. This procedure leads to estimates involving the L^4-norm.

Induction step. A procedure analogous to the one used to control the fourth order derivatives of the connection coefficients is employed to estimate the $k+1$-th order derivatives of the connection if control on the derivatives of k-th order is assumed. This calculation, requires, in particular, control of the value of such norms on the initial light cone —this control follows readily from the procedure used to evaluate the *formal derivatives* on the initial light cone —see Lemma 4.

Concluding the argument. The previous step shows that it is possible to obtain control over the L^2-norms of all angular derivatives of the connection over the rectangular domain $\mathcal{D}$. Control of the derivatives respect to the optical functions u and v can be obtained by applying, as required, the directional covariant derivatives D and Δ to the evolution equations and commuting. Since the domain is bounded, then all derivatives of $\boldsymbol{h}_t$ and $\boldsymbol{K}_t$ are bounded uniformly in $L^2(\Sigma_t)$ for $t < t^*$. Moreover, one has that the 1-parameter family of data $(\boldsymbol{h}_t, \boldsymbol{K}_t)$ converges uniformly in C^∞ to a pair $(\boldsymbol{h}_{t^*}, \boldsymbol{K}_{t^*})$. The pair $(\boldsymbol{h}_{t^*}, \boldsymbol{K}_{t^*})$ satisfies the Einstein constraint equations on the hypersurface defined $t = t^*$ —see [26]. This leads to a contradiction with the assumption of the existence of a last slice as the theory of the Cauchy problem for the Einstein field equations allows us to readily obtain a (future) development of the data set $(\boldsymbol{h}_{t^*}, \boldsymbol{K}_{t^*})$ —see Figure 3.3 Thus, no such *last slice* exists and the solution to the Einstein vacuum equations exists on the whole of the rectangular domain $\mathcal{D}$.

3.6.3 Statement of the main result

The long analysis of the preceding sections leads to the following:

Theorem 4 (***main result —improved local existence for the CIVP for the EFE***)**.** *Given regular initial data for the vacuum Einstein field equations as contructed in Lemma 2 on the null hypersurfaces $\mathcal{N}_\star \cup \mathcal{N}'_\star$ for $I \equiv \{0 \leq v \leq v_\bullet\}$, there exists $\varepsilon > 0$ such that a unique smooth solution to the vacuum Einstein field equations exists in the region where $v \in I$ and $0 \leq u \leq \varepsilon$ defined by the null coordinates (u, v). The number ε can be chosen to depend only on I, $\Delta_{e_\star}$, $\Delta_{\Gamma_\star}$ and $\Delta_{\Psi_\star}$. Furthermore, in this area one has that,*

$$
\begin{aligned}
&\sup_{u,v} \sup_{\Gamma \in \{\mu,\lambda,\rho,\sigma,\alpha,\beta,\epsilon,\tau,\chi\}} \max\left\{ \sum_{i=0}^{1} ||\nabla^i \Gamma||_{L^\infty(\mathcal{S}_{u,v})}, \sum_{i=0}^{2} ||\nabla^i \Gamma||_{L^4(\mathcal{S}_{u,v})}, \sum_{i=0}^{3} ||\nabla^i \Gamma||_{L^2(\mathcal{S}_{u,v})} \right\} \\
&+ \sum_{i=0}^{3} \sup_{\Psi \in \{\Psi_0,\Psi_1,\Psi_2,\Psi_3\}} \sup_{u} ||\nabla^i \Psi||_{L^2(\mathcal{N}_u)} + \sup_{\Psi \in \{\Psi_1,\Psi_2,\Psi_3,\Psi_4\}} \sup_{v} ||\nabla^i \Psi||_{L^2(\mathcal{N}'_v)} \\
&\leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi_\star}).
\end{aligned}
$$

Chapter 4

Improved existence for the characteristic initial value problem with the conformal Einstein field equations

In this chapter we adapt Luk's analysis of the characteristic initial value problem in General Relativity to the asymptotic characteristic problem for the conformal Einstein field equations (see 2.2) to demonstrate the local existence of solutions in a neighbourhood of the set on which the data are given. In particular, we obtain existence of solutions along a narrow rectangle along null infinity which, in turn, corresponds to an infinite domain in the asymptotic region of the physical spacetime. This result generalises work by Kánnár [34] on the local existence of solutions to the characteristic initial value problem by means of Rendall's reduction strategy. In analysing the conformal Einstein equations we make use of the Newman-Penrose formalism and a gauge due to J. Stewart.

4.1 The geometry of the problem

In this section, we will discuss the geometric and the gauge choices in the asymptotic CIVP on past null infinity.

4.1.1 Basic setting

Our basic geometric setting consists of an unphysical manifold $\mathcal{M}$ with a boundary and an edge. The boundary consists of two null hypersurfaces: $\mathscr{I}^-$, past null infinity on which $\Xi = 0$; and $\mathcal{N}'_\star$, an incoming null hypersurface with non-vacuum intersection $\mathcal{S}_\star \equiv \mathscr{I}^- \cap \mathcal{N}'_\star$. We will assume that $\mathcal{S}_\star \approx \mathbb{S}^2$. In a neighbourhood $\mathcal{U}$ of $\mathcal{S}_\star$, one can introduce coordinates $x = (x^\mu)$ with $x^0 = v$ and $x^1 = u$ such that, at least in a neighbourhood of $\mathcal{S}_\star$ one can write

$$\mathscr{I}^- = \{p \in \mathcal{U} \mid u(p) = 0\}, \qquad \mathcal{N}'_\star = \{p \in \mathcal{U} \mid v(p) = 0\}.$$

Given suitable data on $(\mathscr{I}^- \cup \mathcal{N}'_\star) \cap \mathcal{U}$ one is interested in making statements about the existence and uniqueness of solutions to the conformal Einstein field equations on some open set

$$\mathcal{V} \subset \{p \in \mathcal{U} \mid u(p) \geq 0, v(p) \geq 0\} \tag{4.1}$$

which we identify with a subset of the future domain of dependence, $D^+(\mathscr{I}^- \cup \mathcal{N}'_\star)$, of $\mathscr{I}^- \cup \mathcal{N}'_\star$.

4.1.2 Stewart's gauge

The basic geometric setting described in the previous section is supplemented by a gauge choice first introduced by Stewart [33].

4.1.2.1 Coordinates

It is convenient to regard the 2-dimensional surface $S_\star$ as a submanifold of spacelike hypersurfaces. The subsequent discussion will be restricted to the future of the hypersurface. As $S_\star \approx \mathbb{S}^2$, one has that $S_\star$ divides the spacelike hypersurface into two regions, the interior of $S_\star$ and the exterior of $S_\star$. Now, consider a foliation of the spacelike hypersurface by 2-dimensional surfaces with the topology of $\mathbb{S}^2$ which includes $S_\star$. At each of the 2-dimensional surfaces we assume there pass two null hypersurfaces. Further, we assume that one of these hypersurfaces has the property that the projection of the tangent vectors of their generators at $S_\star$

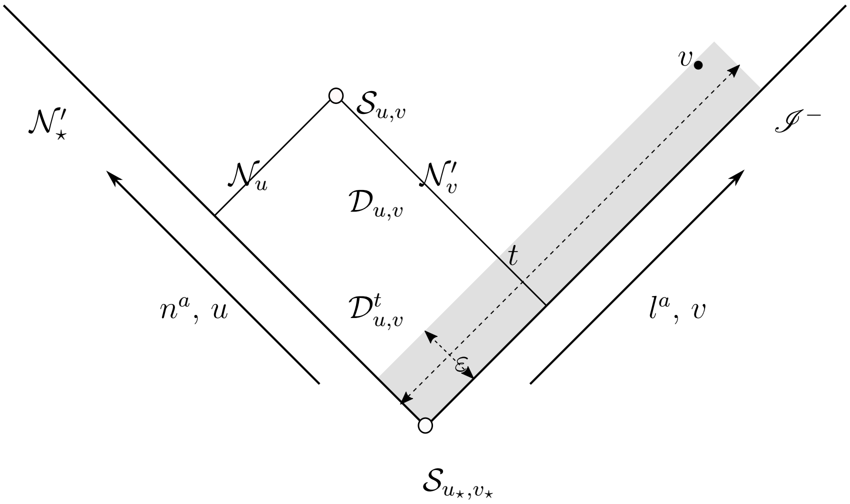

Figure 4.1: Setup for Stewart's gauge. The construction makes use of a double null foliation of the future domain of dependence of the initial hypersurface $\mathscr{I}^- \cup \mathcal{N}'_\star$. The coordinates and NP null tetrad are adapted to this geometric setting. The analysis in this article is focused on the arbitrarily thin grey rectangular domain along the hypersurface $\mathscr{I}^-$. The argument can be adapted, in a suitable manner, to a similar rectangle along $\mathcal{N}'_\star$. See the main text for the definitions of the various regions and objects.

point *outwards*. We call these null hypersurfaces *outgoing light cones*. Moreover, it is also assumed that one of these hypersurfaces has the property that the projection of the tangent vectors of their generators at $S_\star$ point *inwards*. We call these null hypersurfaces *ingoing light cones*.

Thus, at least locally, one obtains a 1-parameter family of outgoing null hypersurface $\mathcal{N}_u$ and a 1-parameter family of ingoing null hypersurface $\mathcal{N}'_v$. One can then define scalar fields u and v by the requirements, respectively, that u is constant on each of the $\mathcal{N}_u$ and v is constant on each $\mathcal{N}'_v$. In particular, we assume that $\mathcal{N}_0 = \mathscr{I}^-$ and $\mathcal{N}'_0 = \mathcal{N}'_\star$. Following standard usage, we call u a *retarded time* and v a *advanced time*. The scalar fields u and v will be used as coordinates in a neighbourhood of $\mathcal{S}_\star$. To complete the coordinate system, consider arbitrary coordinates $(x^\mathcal{A})$ in a coordinate patch U on $\mathcal{S}_\star$, with the index ${}^\mathcal{A}$ taking the values 2, 3. These coordinates are then propagated into $\mathscr{I}^-$ by requiring them to be constant along the generators of $\mathscr{I}^-$. Once coordinates have been defined on $\mathscr{I}^-$, one can propagate them into $\mathcal{V}$ by requiring them to be constant along the generators of each $\mathcal{N}'_v$. In this manner one obtains a coordinate system $(x^\mu) = (v,\ u,\ x^{\mathcal{A}})$ on D_U in $\mathcal{V}$. Here area D_U is defined by the image of first generating U along v and then generating long u.

We use the notation $\mathcal{N}_u(v_1, v_2)$ to denote the part of the hypersurface $\mathcal{N}_u$ with $v_1 \leq v \leq v_2$. Likewise $\mathcal{N}'_v(u_1, u_2)$ has a similar definition. We denote the sphere intersected by $\mathcal{N}_u$ and $\mathcal{N}'_v$ by $\mathcal{S}_{u,v}$. We define the region

$$\bigcup_{0\leq v'\leq v, 0\leq u'\leq u} \mathcal{S}_{u',v'}$$

as $\mathcal{D}_{u,v}$. We also define the time function $t \equiv u + v$, and the *truncated causal diamond*,

$$\mathcal{D}^{\tilde{t}}_{u,v} \equiv \mathcal{D}_{u,v} \cap \{t \leq \tilde{t}\}.$$

Remark 20. It is observed that while the null coordinte u has a compact range, this is, in principle, not the case for v.

4.1.2.2 The NP frame

A null Newman-Penrose (NP) tetrad is constructed by choosing vector fields l^a and n^a tangent to the generators of $\mathcal{N}_u$ and $\mathcal{N}'_v$ respectively. Following the standard conventions we make use of the normalisation $g_{ab}l^a n^b = 1$ is preserved under boost transformations. This freedom can be used to set $n_a = \nabla_a v$. This requirement still leaves some freedom left as one can choose a relabelling of the form $v \mapsto V(v)$. Next, we choose the vector fields m^a and $\bar{m}^a$ so that they are tangent to the surfaces $\mathcal{S}_{u,v} \equiv \mathcal{N}_u \cap \mathcal{N}'_v$ and satisfy the conditions $g_{ab}m^a\bar{m}^b = -1$, $g_{ab}m^a m^b = 0$. This leaves the freedom to perform a spin transformation at each point.

Now, observing that, by construction, on the generators of each null hypersurface $\mathcal{N}^\star_v$ only the coordinate u varies, one has that

$$n^\mu \boldsymbol{\partial}_\mu = Q\boldsymbol{\partial}_u,$$

where Q is a real function of the position. Further, since the vector l^a is tangent to the generators of each $\mathcal{N}_u$ and $l^a n_a = l^a \nabla_a v = 1$, one has that

$$l^\mu \boldsymbol{\partial}_\mu = \boldsymbol{\partial}_v + C^{\mathcal{A}} \boldsymbol{\partial}_{\mathcal{A}},$$

where, again, the components $C^{\mathcal{A}}$ are real functions of the position. By construction, the coordinates $(x^{\mathcal{A}})$ do not vary along the generators of $\mathscr{I}^-$, that is, one has that $l^a \nabla_a x^{\mathcal{A}} = 0$. Accordingly, one has

$$C^{\mathcal{A}} = 0 \quad \text{on} \quad \mathscr{I}^-.$$

Finally, since m^a and $\bar{m}^a$ span the tangent space of each surface $\mathcal{S}_{u,v}$ one has that

$$m^\mu \boldsymbol{\partial}_\mu = P^{\mathcal{A}} \boldsymbol{\partial}_{\mathcal{A}},$$

where the coefficients $P^{\mathcal{A}}$ are complex functions.

Summarising, we make the following assumption:

Assumption 3. On a local coordinate patch D_U of $\mathcal{V}$ one can find a Newman-Penrose frame $\{l^a,\ n^a,\ m^a,\ \bar{m}^a\}$ of the form:

$$\boldsymbol{l} = \boldsymbol{\partial}_v + C^{\mathcal{A}} \boldsymbol{\partial}_{\mathcal{A}}, \tag{4.2a}$$

$$\boldsymbol{n} = Q \boldsymbol{\partial}_u, \tag{4.2b}$$

$$\boldsymbol{m} = P^{\mathcal{A}} \boldsymbol{\partial}_{\mathcal{A}}. \tag{4.2c}$$

Remark 21. In view of the normalisation condition $g_{ab} m^a \bar{m}^b = -1$, there are only 3 independent real functions in the coefficients $P^{\mathcal{A}}$. Thus, Q, $C^{\mathcal{A}}$ together with $P^{\mathcal{A}}$ give six scalar fields describing the metric. The components $(g^{\mu\nu})$ of the contravariant form of the metric g_{ab} are of the form

$$(g^{\mu\nu}) = \begin{pmatrix} 0 & Q & 0 \\ Q & 0 & QC^{\mathcal{A}} \\ 0 & QC^{\mathcal{A}} & \sigma^{\mathcal{AB}} \end{pmatrix},$$

where

$$\sigma^{\mathcal{AB}} \equiv -(P^{\mathcal{A}}\bar{P}^{\mathcal{B}} + \bar{P}^{\mathcal{A}}P^{\mathcal{B}})$$

is the (contravariant) induced metric on $\mathcal{S}_{u,v}$.

On $\mathcal{N}'_\star$ one has that $\boldsymbol{n} = Q\boldsymbol{\partial}_u$. As the coordinates $(x^{\mathcal{A}})$ are constant along the generators of $\mathscr{I}^-$ and $\mathcal{N}'_\star$, it follows that on $\mathcal{N}'_\star$ the coefficient Q is only a function of u. Thus, without loss of generality one can reparameterise u so as to set $Q = 1$ on $\mathcal{N}'_\star$.

4.1.2.3 Spin connection coefficients

Direct inspection of the NP commutators (2.15a)-(2.15d) applied to the coordinates $(v,\ u,\ x^2,\ x^3)$ taking into account together with the remaining gauge freedom in the vector frame of Assumption 4.8 leads to the following:

Lemma 10. *The NP frame of Assumption 4.8 can be chosen such that*

$$\kappa = \nu = \gamma = 0, \tag{4.3a}$$

$$\rho = \bar{\rho}, \quad \mu = \bar{\mu}, \tag{4.3b}$$

$$\pi = \alpha + \bar{\beta}, \tag{4.3c}$$

on $\mathcal{V}$ and, furthermore, with

$$\epsilon - \bar{\epsilon} = 0 \quad \text{on} \quad \mathcal{V} \cap \mathscr{I}^-.$$

Proof. The proof of this result is analogous to that of Lemma 1 in Chapter 3. □

4.1.2.4 Equations for the frame coefficients

Taking into account the conditions of the spin connection coefficients given by (4.3a)-(4.3c), the remaining commutators yield the equations

$$\Delta C^{\mathcal{A}} = -(\bar{\tau} + \pi)P^{\mathcal{A}} - (\tau + \bar{\pi})\bar{P}^{\mathcal{A}}, \tag{4.4a}$$

$$\Delta P^{\mathcal{A}} = -\mu P^{\mathcal{A}} - \bar{\lambda}\bar{P}^{\mathcal{A}}, \tag{4.4b}$$

$$
\begin{aligned}
&DP^{\mathcal{A}} - \delta C^{\mathcal{A}} = (\bar{\rho} + \epsilon - \bar{\epsilon})P^{\mathcal{A}} + \sigma \bar{P}^{\mathcal{A}}, && \text{(4.4c)}\\
&DQ = -(\epsilon + \bar{\epsilon})Q, && \text{(4.4d)}\\
&\bar{\delta} P^{\mathcal{A}} - \delta \bar{P}^{\mathcal{A}} = (\alpha - \bar{\beta})P^{\mathcal{A}} - (\bar{\alpha} - \beta)\bar{P}^{\mathcal{A}}, && \text{(4.4e)}\\
&\delta Q = (\tau - \bar{\pi})Q. && \text{(4.4f)}
\end{aligned}
$$

Equations (4.4a)-(4.4b) allow the evolution of the frame coefficients $C^{\mathcal{A}}$ and $P^{\mathcal{A}}$ off of the null hypersurface $\mathcal{N}'_\star$. Equations (4.4c)-(4.4d) allow to evolve the coefficients Q and $P^{\mathcal{A}}$ to be evolved along the null generators of $\mathscr{I}^-$. Finally, (4.4e)-(4.4f) provide constraints for Q and $P^{\mathcal{A}}$ on the spheres $\mathcal{S}_{u,v}$.

4.1.2.5 The conformal gauge conditions

The conformal Einstein field equations have an in-built conformal freedom which can be exploited to simplify the geometric setting of the problem. This freedom allows us, in particular, to select in an indirect manner the conformal factor via a *conformal gauge source function*. More precisely, one has the following:

Lemma 11 (***conformal gauge conditions***)**.** *Let $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ denote an asymptotically simple spacetime satisfying Ric$[\tilde{\boldsymbol{g}}] = 0$ and let $(\mathcal{M}, \boldsymbol{g}, \Xi)$ with $\boldsymbol{g} = \Xi^2 \tilde{\boldsymbol{g}}$ be a conformal extension for which the condition $\Xi = 0$ describes past null infinity $\mathscr{I}^-$. Given the previous NP frame* (4.2a)-(1)*, the conformal factor Ξ can be chosen so that given a null hypersurface $\mathcal{N}'_\star$ intersecting $\mathscr{I}^-$ on $\mathcal{S}_\star \approx \mathbb{S}^2$ one has*

$$
\Lambda = -\frac{1}{24} R(x), \qquad \textit{in a neighourhood} \quad \mathcal{V} \quad \textit{of} \quad \mathcal{S}_\star \quad \textit{on} \quad J^+(\mathcal{S}_\star)
$$

where $R(x)$ is an arbitrary function of the coordinates. Moreover, one has the additional gauge conditions

$$
\begin{aligned}
&\Sigma_2 = 1, \quad \mu = \rho = 0 \qquad \textit{on} \quad \mathcal{S}_\star,\\
&\Phi_{22} = 0 \qquad \textit{on} \quad \mathcal{N}'_\star,\\
&\Phi_{00} = 0 \qquad \textit{on} \quad \mathscr{I}^-.
\end{aligned}
$$

Proof. The definition of asymptotically simple spacetime follows from Def 7.1. and the proof of the above result is analogous to that of Lemma 18.2 in [35]. □

4.2 The formulation of the CIVP

In this section we analyse general aspects of the asymptotic CIVP for the conformal vacuum Einstein field equations with data on the null hypersurface $\mathscr{I}^-$ and $\mathcal{N}'_\star$. A key feature of the setting is the existence of a hierarchical structure in the reduced conformal equations which allows to identify the basic reduced initial data set from which the full initial data on $\mathscr{I}^- \cup \mathcal{N}'_\star$ for the conformal Einstein field equations can be computed.

4.2.1 Specifiable free data

The following result identifies a possible choice of freely specifiable initial data for the asymptotic CIVP:

Lemma 12 (***freely specifiable data for the characteristic problem***)**.** *Assume that the gauge condition given by Lemma 10 and Lemma 11 is satisfied in a neighbourhood $\mathcal{V}$ of $\mathcal{S}_\star$. Initial data for the conformal Einstein field equations on $\mathscr{I}^- \cup \mathcal{N}'_\star$ can be computed from the reduced data set $\mathbf{r}_\star$ consisting of:*

$$
\begin{aligned}
&\phi_0, \ \epsilon + \bar{\epsilon} \quad on \quad \mathscr{I}^-, \\
&\phi_4 \ \ on \ \ \mathcal{N}'_\star, \\
&\lambda, \quad \phi_2 + \bar{\phi}_2, \quad \Phi_{20}, \quad \phi_3, \quad P^{\mathcal{A}}, \quad on \quad \mathcal{S}_\star.
\end{aligned}
$$

Remark 22. An alternative, less symmetric, reduced initial data set is given by:

$$
\begin{aligned}
&\lambda, \ \epsilon + \bar{\epsilon} \quad \text{on} \quad \mathscr{I}^-, \\
&\phi_4 \ \ \text{on} \ \ \mathcal{N}'_\star, \\
&\phi_3, \quad \phi_2 + \bar{\phi}_2, \quad P^{\mathcal{A}}, \quad \text{on} \quad \mathcal{S}_\star.
\end{aligned}
$$

Proof. The proof of this result follows from a lengthy but straightforward computation on the same lines of Lemma 2 in Paper I. See also Section 18.2 in [35] and [34]. □

Remark 23. From the smoothness of the freely specifiable component ϕ_4 on the incoming null hypersurface $\mathcal{N}'_\star$ and, in particular at the intersection with $\mathscr{I}^-$ it follows that that the resulting spacetime will satisfy the *peeling behaviour* near past null infinity —see e.g. [35], Chapter 10. A reformulation of our characteristic problem to future null infinity, for which now ϕ_0 is freely specifiable data along an outgoing null hypersurface, gives rise *mutatis mutandi* to solutions with peeling at $\mathscr{I}^+$.

4.2.2 Basic local existence

To apply the theory of CIVP, as discussed say in Section 12.5 of [35], one has to extract a suitable symmetric hyperbolic evolution system out of the conformal field equations and the structure equations. The gauge introduced in Section 3.1.1 allows us to perform this reduction.

4.2.2.1 The reduced conformal field equations

In what follows, we group the four directional derivatives of the conformal factor and s as

$$\mathbf{\Sigma}^t \equiv (\Sigma_1,\ \Sigma_2,\ \Sigma_3,\ \Sigma_4,\ s),$$

the components of the frame as

$$\boldsymbol{e}^t \equiv (C^{\mathcal{A}},\ P^{\mathcal{A}},\ Q),$$

the spin connection coefficients not fixed by the gauge as

$$\mathbf{\Gamma}^t \equiv (\epsilon,\ \pi,\ \beta,\ \mu,\ \alpha,\ \lambda,\ \tau,\ \sigma,\ \rho),$$

the independent components of the rescaled Weyl spinor as

$$\boldsymbol{\phi}^t \equiv (\phi_0,\ \phi_1,\ \phi_2,\ \phi_3,\ \phi_4),$$

and finally those of tracefree Ricci spinor as

$$\mathbf{\Phi}^t \equiv (\Phi_{00},\ \Phi_{01},\ \Phi_{11},\ \Phi_{02}, \Phi_{12},\ \Phi_{22}),$$

where ^{t} denotes the operation of taking the transpose of a column vector.

A suitable symmetric hyperbolic systems for the four directional derivatives of the conformal factor, the frame components and the spin coefficients can be obtained from equations (5d)-(5f), (5f)*, (6b), (4.4a), (4.4b), (4.4d) and (3a)-(3d), (3f), (3g), (3k), (3m), (3o), respectively. Here * means the complex conjugate of the equation. These can be written in the schematic form,

$$\begin{aligned}
\boldsymbol{\mathcal{D}}_0\boldsymbol{\Sigma} &= \boldsymbol{B}_0(\boldsymbol{\Sigma}, \boldsymbol{\Gamma}, s)\boldsymbol{\Sigma},\\
\boldsymbol{\mathcal{D}}_1\boldsymbol{e} &= \boldsymbol{B}_1(\boldsymbol{\Gamma}, \boldsymbol{e})\boldsymbol{e},\\
\boldsymbol{\mathcal{D}}_2\boldsymbol{\Gamma} &= \boldsymbol{B}_2(\boldsymbol{\Gamma}, \boldsymbol{\phi}, \boldsymbol{\Phi})\boldsymbol{\Gamma},
\end{aligned}$$

where $\mathcal{D}_0$, $\mathcal{D}_1$ and $\mathcal{D}_2$ are given by

$$\begin{aligned}
\boldsymbol{\mathcal{D}}_0 &\equiv \mathrm{diag}(\Delta,\ \Delta,\ \Delta,\ \Delta,\ \Delta),\\
\boldsymbol{\mathcal{D}}_1 &\equiv \mathrm{diag}(\Delta,\ \Delta,\ D),\\
\boldsymbol{\mathcal{D}}_2 &\equiv \mathrm{diag}(\Delta,\ \Delta,\ \Delta,\ \Delta,\ \Delta,\ \Delta, D,\ D,\ D),
\end{aligned}$$

and $\boldsymbol{B}_0$, $\boldsymbol{B}_1$, $\boldsymbol{B}_2$ are smooth matrix-valued functions of their arguments, whose explicit form will not be required.

The components of the third conformal field equation (2.9c), equations (7a)-(7m) can be reorganised as

$$\boldsymbol{\mathcal{D}}_3\boldsymbol{\Phi} = \boldsymbol{B}_3\boldsymbol{\Phi}$$

where

$$\mathcal{D}_3 = \begin{pmatrix}
\Delta & -\bar{\delta} & 0 & 0 & 0 & 0\\
-\delta & D+2\Delta & -\delta & -\bar{\delta} & 0 & 0\\
0 & -\bar{\delta} & D+\Delta & 0 & -\bar{\delta} & 0\\
0 & -\delta & 0 & D+\Delta & -\delta & 0\\
0 & 0 & -\delta & -\bar{\delta} & 2D+\Delta & -\delta\\
0 & 0 & 0 & 0 & -\bar{\delta} & D
\end{pmatrix}$$

with $\boldsymbol{B_3} = \boldsymbol{B_3}(\Phi,\ \phi,\ \Gamma,\ \Sigma_i)$. Writing

$$\boldsymbol{\mathcal{D}}_3 = \boldsymbol{A}_3^{\mu}\partial_{\mu}$$

one has that

$$\begin{aligned}
\boldsymbol{A}_3^{v} &= \mathrm{diag}(0,\ 1,\ 1,\ 1,\ 2,\ 1),\\
\boldsymbol{A}_3^{u} &= \mathrm{diag}(Q,\ 2Q,\ Q,\ Q,\ Q,\ 0).
\end{aligned}$$

and

$$\boldsymbol{A}_3^{\mathcal{A}} = \begin{pmatrix}
0 & -\bar{P}^{\mathcal{A}} & 0 & 0 & 0 & 0\\
-P^{\mathcal{A}} & C^{\mathcal{A}} & -P^{\mathcal{A}} & -\bar{P}^{\mathcal{A}} & 0 & 0\\
0 & -\bar{P}^{\mathcal{A}} & C^{\mathcal{A}} & 0 & -\bar{P}^{\mathcal{A}} & 0\\
0 & -P^{\mathcal{A}} & 0 & C^{\mathcal{A}} & -P^{\mathcal{A}} & 0\\
0 & 0 & -P^{\mathcal{A}} & -\bar{P}^{\mathcal{A}} & 2C^{\mathcal{A}} & -P^{\mathcal{A}}\\
0 & 0 & 0 & 0 & -\bar{P}^{\mathcal{A}} & C^{\mathcal{A}}
\end{pmatrix}.$$

To be specific, the equations above are obtained through the combinations (7a)+(7k), (7j)+2(7b)+(7l), (7d)*+(7h)*, (7c)+(7i), (7e)+2(7g)+(7l) and (7f)+(7m), respectively. It can be readily verified that the matrices $\boldsymbol{A}_3^{\mu}$ are Hermitian. Moreover,

$$\boldsymbol{A}_3^{\mu}(l_{\mu} + n_{\mu}) = \mathrm{diag}(1,\ 3,\ 2,\ 2,\ 3,\ 1)$$

is likewise clearly positive definite.

The components of the fourth conformal equation (2.9d), (8a)-(8h), can be grouped as

$$\boldsymbol{\mathcal{D}}_4\boldsymbol{\phi} = \boldsymbol{B}_4\boldsymbol{\phi}$$

where

$$
\boldsymbol{\mathcal{D}}_4 = \begin{pmatrix} \Delta & -\delta & 0 & 0 & 0 \\ -\bar{\delta} & D+\Delta & -\delta & 0 & 0 \\ 0 & -\bar{\delta} & D+\Delta & -\delta & 0 \\ 0 & 0 & -\bar{\delta} & D+\Delta & -\delta \\ 0 & 0 & 0 & -\bar{\delta} & D \end{pmatrix},
$$

and $\boldsymbol{B}_4 = \boldsymbol{B}_4(\boldsymbol{\phi},\ \boldsymbol{\Gamma})$. Again, writing

$$
\boldsymbol{\mathcal{D}}_4 = \boldsymbol{A}_4^\mu \partial_\mu,
$$

one has that

$$
\begin{aligned}
\boldsymbol{A}_4^v &= \mathrm{diag}(0,\ 1,\ 1,\ 1,\ 1), \\
\boldsymbol{A}_4^u &= \mathrm{diag}(Q,\ Q,\ Q,\ Q,\ 0),
\end{aligned}
$$

and

$$
\boldsymbol{A}_4^{\mathcal{A}} = \begin{pmatrix} 0 & -P^{\mathcal{A}} & 0 & 0 & 0 \\ -\bar{P}^{\mathcal{A}} & C^{\mathcal{A}} & -P^{\mathcal{A}} & 0 & 0 \\ 0 & -\bar{P}^{\mathcal{A}} & C^{\mathcal{A}} & -P^{\mathcal{A}} & 0 \\ 0 & 0 & -\bar{P}^{\mathcal{A}} & C^{\mathcal{A}} & -P^{\mathcal{A}} \\ 0 & 0 & 0 & -\bar{P}^{\mathcal{A}} & C^{\mathcal{A}} \end{pmatrix}.
$$

Specifically, the above matricial expressions are obtained from the combinations (8a), (8b)+(8e), (8c)+(8f), (8d)+(8g) and (8h). Again, the matrices $\boldsymbol{A}_4^\mu$ can be seen to be Hermitian and, moreover, one has that

$$
\boldsymbol{A}_4^\mu (l_\mu + n_\mu) = \mathrm{diag}(1,\ 2,\ 2,\ 2,\ 1)
$$

is clearly positive definite.

We can summarise the above discussion as:

Lemma 13. *The evolution system*

$$\boldsymbol{\mathcal{D}}_0\boldsymbol{\Sigma} = \boldsymbol{B}_0\boldsymbol{\Sigma} \tag{4.6a}$$

$$\boldsymbol{\mathcal{D}}_1\boldsymbol{e} = \boldsymbol{B}_1\boldsymbol{e}, \tag{4.6b}$$

$$\boldsymbol{\mathcal{D}}_2\boldsymbol{\Gamma} = \boldsymbol{B}_2\boldsymbol{\Gamma}, \tag{4.6c}$$

$$\boldsymbol{\mathcal{D}}_3\boldsymbol{\Phi} = \boldsymbol{B}_3\boldsymbol{\Phi}, \tag{4.6d}$$

$$\boldsymbol{\mathcal{D}}_4\boldsymbol{\phi} = \boldsymbol{B}_4\boldsymbol{\phi}, \tag{4.6e}$$

is symmetric hyperbolic with respect to the direction given by

$$\tau^a = l^a + n^a.$$

4.2.2.2 Computation of the formal derivatives on $\mathcal{N}'_\star \cup \mathscr{I}^-$ and propagation of the constraints

As discussed in Section 12.5 of [35], the existence and uniqueness of solutions to a CIVP can be obtained via an auxiliary Cauchy problem on the spacelike hypersurface

$$S \equiv \{p \in \mathbb{R} \times \mathbb{R} \times \mathbb{S}^2 \mid v(p) + u(p) = 0\}.$$

The formulation of this problem depends crucially on Whitney's extension theorem, which requires being able to evaluate all derivatives (interior and transverse) of initial data on $\mathcal{N}'_\star \cup \mathscr{I}^-$. One has the following:

Lemma 14 (***computation of formal derivatives***)**.** *Any arbitrary formal derivatives of the unknown functions* $\{\boldsymbol{\Sigma},\ \boldsymbol{e},\ \boldsymbol{\Gamma},\ \boldsymbol{\Phi},\ \boldsymbol{\phi}\}$ *on* $\mathcal{N}'_\star \cup \mathscr{I}^-$ *can be computed from the prescribed initial data* $\boldsymbol{r}_\star$ *for the reduced conformal field equations on* $\mathcal{N}'_\star \cup \mathscr{I}^-$*.*

Proof. The statement follows from a careful inspection of the conformal field equations in the present gauge, see Section 18.2 in [35] and [34] for more details. □

Moreover, using arguments similar to those discussed in [35], Section 12.5, one can establish the following result concerning the relation between the reduced equations and the full conformal vacuum Einstein field equations:

Proposition 19 (***propagation of the constraints***)**.** *A solution of the reduced conformal field equations* (4.6a)-(4.6e) *on a neighbourhood* $\mathcal{V}$ *of* $\mathcal{S}_\star$ *on* $J^+(\mathcal{S}_\star)$ *that coincides with initial data on* $\mathcal{N}'_\star \cup \mathscr{I}^-$ *satisfying the conformal equations gives rise to a solution of the conformal Einstein field equations* (2.9a)-(2.9e) *on* $\mathcal{V}$.

In addition, one has that:

Corollary 3 (***preservation of the conformal gauge***)**.** *Let* $\boldsymbol{u}$ *denote a solution to the characteristic problem for the conformal field equations on a neighbourhood* $\mathcal{V}$ *of* $\mathcal{S}_\star$ *on* $J^+(\mathcal{S}_\star)$ *which satisfies the gauge conditions given in Lemmas 1 and 2. Then the metric* $\boldsymbol{g}$ *constructed from the components of the solution* $\boldsymbol{u}$ *satisfies the conformal vacuum Einstein field equations in a gauge for which* $R[\boldsymbol{g}] = R(x)$.

4.2.2.3 Summary

Combining the analysis above and applying the theory of the CIVP for the symmetric hyperbolic systems of Section 12.5 of [35], one obtains the following existence result:

Theorem 5 (***existence and uniqueness to the standard asymptotic characteristic problem***)**.** *Given a smooth reduced initial data set* $\mathbf{r}_\star$ *for the conformal Einstein field equations on* $\mathcal{N}'_\star \cup \mathscr{I}^-$, *there exists a unique smooth solution of the conformal field equations in a neighbourhood* $\mathcal{V}$ *of* $\mathcal{S}_\star$ *on* $J^+(\mathcal{S}_\star)$ *which implies the prescribed initial data on* $\mathcal{N}'_\star \cup \mathscr{I}^-$. *Moreover, this solution to the conformal Einstein field equations implies, in turn, a solution to the vacuum Einstein field equations in a neighbourhood of past null infinity.*

Remark 24. Although the region $\mathcal{V}$ is, in the unphysical picture, finite, from the physical point of view, it corresponds to an infinite domain of the asymptotic region near past null infinity.

4.3 Improved existence result

In this section we provide the basic setting for the improved local existence result for the asymptotic CIVP for the conformal Einstein field equations using Luk's method.

Our analysis builds on the general formalism developed in chapter 3. Similarly, the long direction mean $\boldsymbol{l}$-direction and the short direction means $\boldsymbol{n}$. Hence the Gronwell's type estimate Prop. 3 can be applied in the following analysis.

The main difference between the present analysis and that of chapter 3 is that when dealing with the conformal Einstein field equations one has more unknown equations to take care of. Specifically, we now have the conformal factor, its derivatives and the components of the tracefree Ricci tensor.

4.3.1 Estimates for the components of the frames and the conformal factor

A first step in the analysis in chapter 3 was the construction of basic estimates for the components of the frame in terms of the initial conditions. A similar step is required for the conformal Einstein field equations. The main difference in this case is that one also needs to obtain some basic control on the conformal factors and its derivatives. These estimates are constructed presently.

4.3.1.1 Definitions

Following chapter 3, in the following it will be convenient to define the following norm measuring the size of the initial value of the components of the frame:

$$\Delta_{e_\star} \equiv \sup_{\mathscr{I}^-, \mathcal{N}'_\star} \left(|Q|, |Q^{-1}|, |C^{\mathcal{A}}|, |P^{\mathcal{A}}| \right).$$

Moreover, we define a scalar

$$\chi \equiv \Delta \log Q,$$

and with the NP Ricci identities we obtain that

$$D\chi = 2\Phi_{11} + \Psi_2 + \bar{\Psi}_2 + 2\alpha\tau + 2\bar{\beta}\tau + 2\bar{\alpha}\bar{\tau} + 2\beta\bar{\tau} + 2\tau\bar{\tau} - (\epsilon + \bar{\epsilon})\chi. \qquad (4.7)$$

Now, as a consequence of the gauge choice $Q = 1$ on $\mathcal{N}'_\star$, the initial data for χ on $\mathcal{N}'_\star$ is 0. For convenience we also define

$$\varpi \equiv \beta - \bar{\alpha}$$

corresponding to the only independent component of the connection on the spheres $\mathcal{S}_{u,v}$.

4.3.1.2 The estimates

Following the main strategy in 3, we construct estimates for the components of the frame and the conformal factor through the analysis of Δ-equations under the following bootstrap assumption:

Assumption 4 (***assumption to control the coefficients of the frame and the conformal factor***)**.** *Assume that we have a solution to the vacuum conformal Einstein field equations in Stewart's gauge satisfying*

$$||\{\chi, \mu, \lambda, \alpha, \beta, \tau, \Sigma_2\}||_{L^\infty(\mathcal{S}_{u,v})} \leq \Delta_\Gamma$$

on a truncated causal diamond $\mathcal{D}^t_{u,v_\bullet}$, where Δ_Γ is some (possibly large) constant.

The construction of the estimates proceeds along the following steps:

Step 1. We integrate $\chi = \Delta \log Q = \partial_u Q$ in the short direction so as to obtain

$$|Q - Q_\star| = |\int_0^\varepsilon \chi \mathrm{d}u| \leq \int_0^\varepsilon |\chi| \mathrm{d}u \leq \int_0^\varepsilon \Delta_\Gamma \mathrm{d}u = \Delta_\Gamma \varepsilon$$

for any v. Then we have

$$||Q - Q_\star||_{L^\infty(\mathcal{S}_{u,v})} \leq \Delta_\Gamma \varepsilon.$$

So there is a constant C depending on the initial data such that

$$Q^{-1}, \quad Q \leq C(\Delta_{e_\star}).$$

Step 2.: To estimate the conformal factor Ξ, we integrate Ξ along the short direction

$$|\Xi| = |\int_0^{\varepsilon} Q^{-1}\Sigma_2 \mathrm{d}u| \leq C(\Delta_\Gamma)\varepsilon.$$

Accordingly, we have the following lemma:

Lemma 15 (***control of conformal factor***)**.** *Under Assumption 4.8, if $\varepsilon > 0$ is sufficiently small, there exists a constant C depending on the size of the initial data such that*

$$||\Xi||_{L^\infty(\mathcal{S}_{u,v})} \leq C(\Delta_\Gamma)\varepsilon$$

on $\mathcal{D}^t_{u,v_\bullet}$.

Step 3. Integrating $P^{\mathcal{A}}$ in the short direction using equation (4.4b) one readily obtains the following lemma:

Lemma 16 (***control on the components of the frame, I***)**.** *We require that $P^{\mathcal{A}}$ are bounded on $U_{0,v}$ such that $\boldsymbol{\sigma}^{\mathcal{AB}}$ is invertible and bounded above and below. Here $U_{0,v}$ is coordinate patch on $\mathcal{S}_{0,v}$ generated along $\boldsymbol{l}$ from coordinate patch U on $\mathcal{S}_\star$. Under Assumption 4.8, if $\varepsilon > 0$ is sufficiently small, there exists a constant C depending on the size of the initial data such that*

$$|\{P^{\mathcal{A}}, (P^{\mathcal{A}})^{-1}\}| \leq C(\Delta_{e_\star}),$$

on coordinate patch D_U of $\mathcal{D}^t_{u,v_\bullet}$. Moreover, since

$$\sigma^{\mathcal{AB}} = -P^{\mathcal{A}}\bar{P}^{\mathcal{B}} - P^{\mathcal{B}}\bar{P}^{\mathcal{A}},$$

we also obtain that

$$\begin{aligned}&|\sigma^{\mathcal{AB}}|, |\sigma_{\mathcal{AB}}| \leq C(\Delta_{e_\star}),\\ &c(\Delta_{e_\star}) \leq \det\sigma \leq C(\Delta_{e_\star}).\end{aligned}$$

Thus, for any vector v^a on $\mathcal{S}_{u,v}$, we have that the norms

$$\int_{\mathcal{S}_{u,v}} (\sigma_{\mathcal{AB}} v^{\mathcal{A}} v^{\mathcal{B}})^{p/2}, \qquad \textit{and} \qquad \int_{\mathcal{S}_{u,v}} ((v^1)^2 + (v^2)^2)^{p/2},$$

are equivalent. Finally, one also has

$$\sup_{u,v} |\mathrm{Area}(\mathcal{S}_{u,v}) - \mathrm{Area}(\mathcal{S}_{0,v})| \leq C\Delta_{\Gamma}\varepsilon.$$

Step 4. Integrating $C^{\mathcal{A}}$ in the short direction using equation (4.4a) yields the lemma

Lemma 17 (***control on the components of the frame, II***)**.** *Choosing ε suitably, since $C^{\mathcal{A}}_{\star} = 0$ on $\mathscr{I}^-$ one has that*

$$|C^{\mathcal{A}}| \leq C\Delta_{\Gamma}\varepsilon$$

on a coordinate patch of $\mathcal{D}^t_{u,v_\bullet}$.

4.4 Main estimates

In this section we discuss the construction of the main estimates to obtain the improved existence results for the asymptotic CIVP for the conformal Einstein field equations. The strategy of the arguments resemble that in Einstein field equations. As many of the ideas and techniques are similar to those in chapter 3, as elsewhere, in this section we focus our attention on the particular aspects of arising from the use of the conformal Einstein equations.

4.4.1 Norms

The argument in this and subsequent sections relies on the use of a number of tailor-made norms. We define the following:

(i) Norm for the initial value of the connection coefficients, given by

$$\Delta_{\Gamma_\star} \equiv \sup_{\mathcal{S}_{u,v}\subset\mathscr{I}^-,\mathcal{N}'_\star} \sup_{\Gamma\in\{\mu,\lambda,\rho,\sigma,\alpha,\beta,\tau,\epsilon\}} \max\{1, \sum_{i=0}^{1} ||\nabla\!\!\!/^i\Gamma||_{L^\infty(\mathcal{S}_{u,v})}, \sum_{i=0}^{2} ||\nabla\!\!\!/^i\Gamma||_{L^4(\mathcal{S}_{u,v})},$$

$$\sum_{i=0}^{3}||\nabla\!\!\!/^i\Gamma||_{L^2(\mathcal{S}_{u,v})}\}.$$

(ii) Norm for the initial value of the derivative of conformal factor Σ_a, given by

$$\Delta_{\Sigma_\star} \equiv \sup_{\mathcal{S}_{u,v}\subset\mathscr{I}^-} \max\{1, \sum_{i=0}^{1}||\nabla\!\!\!/^i\Sigma_2||_{L^\infty(\mathcal{S}_{u,v})}, \sum_{i=0}^{2}||\nabla\!\!\!/^i\Sigma_2||_{L^4(\mathcal{S}_{u,v})}, \sum_{i=0}^{3}||\nabla\!\!\!/^i\Sigma_2||_{L^2(\mathcal{S}_{u,v})}\}.$$

(iii) Norm for the initial value of the Ricci curvature components, given by

$$\Delta_{\Phi_\star} \equiv \sup_{\mathcal{S}_{u,v}\subset\mathscr{I}^-,\mathcal{N}'_\star} \sup_{\Phi\in\{\Phi_{00},\Phi_{01},\Phi_{02},\Phi_{11},\Phi_{12}\}} \max\{1, \sum_{i=0}^{1}||\nabla\!\!\!/^i\Phi||_{L^4(\mathcal{S}_{u,v})}, \sum_{i=0}^{2}||\nabla\!\!\!/^i\Phi||_{L^2(\mathcal{S}_{u,v})}\}$$
$$+\sum_{i=0}^{3} \sup_{\Phi\in\{\Phi_{00},\Phi_{01},\Phi_{02},\Phi_{11},\Phi_{12}\}} ||\nabla\!\!\!/^i\Phi||_{L^2(\mathscr{I}^-)} + \sup_{\Phi\in\{\Phi_{01},\Phi_{02},\Phi_{11},\Phi_{12},\Phi_{22}\}} ||\nabla\!\!\!/^i\Phi||_{L^2(\mathcal{N}'_\star)}.$$

(iv) Norm for the initial value of the rescaled Weyl curvature components, given by

$$\Delta_{\phi_\star} \equiv \sup_{\mathcal{S}_{u,v}\subset\mathscr{I}^-,\mathcal{N}'_\star} \sup_{\phi\in\{\phi_0,\phi_1,\phi_2,\phi_3,\phi_4\}} \max\{1, \sum_{i=0}^{1}||\nabla\!\!\!/^i\phi||_{L^4(\mathcal{S}_{u,v})}, \sum_{i=0}^{2}||\nabla\!\!\!/^i\phi||_{L^2(\mathcal{S}_{u,v})}\}$$
$$+\sum_{i=0}^{3} \sup_{\phi\in\{\phi_0,\phi_1,\phi_2,\phi_3\}} ||\nabla\!\!\!/^i\phi||_{L^2(\mathscr{I}^-)} + \sup_{\phi\in\{\phi_1,\phi_2,\phi_3,\phi_4\}} ||\nabla\!\!\!/^i\phi||_{L^2(\mathcal{N}'_\star)}.$$

(v) Norm for the components of the Ricci curvature components at later null hypersurfaces, given by

$$\Delta_\Phi \equiv \sum_{i=0}^{3} \sup_{\Phi\in\{\Phi_{00},\Phi_{01},\Phi_{02},\Phi_{11},\Phi_{12}\}} ||\nabla\!\!\!/^i\Phi||_{L^2(\mathcal{N}_u^t)} + \sup_{\Phi\in\{\Phi_{01},\Phi_{02},\Phi_{11},\Phi_{12},\Phi_{22}\}} ||\nabla\!\!\!/^i\Phi||_{L^2(\mathcal{N}'^t_v)},$$

where the suprema in u and v are taken over $\mathcal{D}^t_{u,v_\bullet}$.

(vi) Supremum-type norm over the L^2-norm of the components of the Ricci curvature at spheres of constant u, v, given by

$$\Delta_{\Phi}(\mathcal{S}) \equiv \sum_{i=0}^{2} \sup_{u,v} ||\nabla^i \{\Phi_{00}, \Phi_{01}, \Phi_{02}, \Phi_{11}, \Phi_{12}\}||_{L^2(\mathcal{S}_{u,v})},$$

where the supremum is taken over $\mathcal{D}^t_{u,v_\bullet}$.

(vii) Norm for the components of the Weyl tensor at later null hypersurfaces, given by

$$\Delta_{\phi} \equiv \sum_{i=0}^{3} \sup_{\phi\in\{\phi_0,\phi_1,\phi_2,\phi_3\}} ||\nabla^i \phi||_{L^2(\mathcal{N}^t_u)} + \sup_{\phi\in\{\phi_1,\phi_2,\phi_3,\phi_4\}} ||\nabla^i \phi||_{L^2(\mathcal{N}'^t_v)},$$

where the suprema in u and v are taken over $\mathcal{D}^t_{u,v_\bullet}$.

(viii) Supremum-type norm over the L^2-norm of the components of the rescaled Weyl curvature at spheres of constant u, v, given by,

$$\Delta_{\phi}(\mathcal{S}) \equiv \sum_{i=0}^{2} \sup_{u,v} ||\nabla^i \{\phi_0, \phi_1, \phi_2, \phi_3\}||_{L^2(\mathcal{S}_{u,v})},$$

with the supremum taken over $\mathcal{D}^t_{u,v_\bullet}$ and in which u will be taken sufficiently small to apply our estimates.

4.4.2 Estimates for the connection coefficients and the derivative of conformal factor

In this subsection, we prove estimates for connection coefficients and derivatives of the conformal factor. We assume first that the norms of curvature are bounded and prove that the short range ε can be chosen such that connection coefficients and the derivative of conformal factor can be controlled by initial data and $\Delta_{\Phi}(\mathcal{S})$. This can be achieved by considering the transport equations. For the connection coefficients τ and χ, we only have their long direction D equations. However, the fact that there is no quadratic term in τ or χ themselves allows us to regard these as linear equations for τ and χ. Then the Grönwall-type inequalities will show us that these two connection coefficients are bounded. Accordingly, except for τ and χ, we can analyse the Δ-equations for the connection coefficients and the derivatives of conformal. The small range of ε does not let them drift too far from their initial data on $\mathscr{I}^-$. Consequently, we find that although Σ_1, Σ_3 and Σ_4 are all small, the component Σ_1 has a different power of ε than Σ_3 and Σ_4 in our estimates.

Proposition 20 (***control on the supremum norm of the connection coefficients and the derivatives of the conformal factor***)**.** *Assume that we have a solution of the vacuum conformal Einstein field equations in Stewart's gauge in a region* $\mathcal{D}^t_{u,v_\bullet}$ *with*

$$\sup_{u,v} ||\{\mu, \lambda, \alpha, \beta, \epsilon, \rho, \sigma, \tau, \chi, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4\}||_{L^\infty(\mathcal{S}_{u,v})} \leq \Delta_{\Gamma,\Sigma}\,,$$

for some positive $\Delta_{\Gamma,\Sigma}$*. Assume also that*

$$\Delta_\Phi(\mathcal{S}) < \infty, \quad \Delta_\Phi < \infty, \quad \Delta_\phi(\mathcal{S}) < \infty, \quad \Delta_\phi < \infty,$$
$$\sup_{u,v} ||\nabla\!\!\!/^i \tau||_{L^2(\mathcal{S}_{u,v})} < \infty, \quad i = 2, 3,$$

on the same domain. Then there exists

$$\varepsilon_\star = \varepsilon_\star(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \sup_{u,v} ||\nabla\!\!\!/^2 \tau||_{L^2(\mathcal{S}_{u,v})}, \sup_{u,v} ||\nabla\!\!\!/^3 \tau||_{L^2(\mathcal{S}_{u,v})}, \Delta_{\Sigma_\star}, \Delta_\phi, \Delta_\Phi),$$

such that when $\varepsilon \leq \varepsilon_\star$*, we have*

$$\sup_{u,v} ||\{\mu, \lambda, \rho, \sigma, \alpha, \beta, \epsilon\}||_{L^\infty(\mathcal{S}_{u,v})} \leq 3\Delta_{\Gamma_\star},$$
$$\sup_{u,v} ||\{\tau, \chi\}||_{L^\infty(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_\Phi(\mathcal{S})),$$
$$\sup_{u,v} ||\Sigma_2||_{L^\infty(\mathcal{S}_{u,v})} \leq 3\Delta_{\Sigma_\star},$$
$$\sup_{u,v} ||\{\Sigma_1, \Sigma_3, \Sigma_4\}||_{L^\infty(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_\Phi(\mathcal{S}))\varepsilon,$$
$$\sup_{u,v} ||s||_{L^\infty(\mathcal{S}_{u,v})} \leq C(\Delta_{e_\star}, \Delta_{\Sigma_\star}, \Delta_\Phi)\varepsilon^{1/2},$$

on $\mathcal{D}^t_{u,v_\bullet}$*.*

Proof.

Basic bootstrap assumption. Place the following bootstrap assumptions:

$$\sup_{u,v} ||\{\mu, \lambda, \rho, \sigma, \alpha, \beta, \epsilon\}||_{L^\infty(\mathcal{S}_{u,v})} \leq 4\Delta_{\Gamma_\star},$$
$$\sup_{u,v} ||\{\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4\}||_{L^\infty(\mathcal{S}_{u,v})} \leq 4\Delta_{\Sigma_\star}.$$

Estimate for τ. First we prove that $||\tau||_{L^\infty(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_\Phi(\mathcal{S}))$. We make use of the D-direction equation of τ, (3b),

$$D\tau = (\epsilon - \bar{\epsilon} + \rho)\tau + \sigma\bar{\tau} + \bar{\pi}\rho + \pi\sigma + \Xi\phi_1 + \Phi_{01}. \tag{4.8}$$

The above equation crucially contains no τ^2 terms. Making use of the Sobolev inequality in the Proposition 7, we obtain that

$$||\phi_i||_{L^\infty(\mathcal{S}_{u,v})} \leq \Delta_\phi(\mathcal{S}) < \infty, \qquad i = 0, 1, 2, 3,$$
$$||\Phi_H||_{L^\infty(\mathcal{S}_{u,v})} \leq \Delta_\Phi(\mathcal{S}) < \infty,$$

where $\Phi_H = \{\Phi_{00}, \Phi_{01}, \Phi_{02}, \Phi_{11}, \Phi_{12}\}$. Then the inequalities in Proposition 4 show that

$$\begin{aligned} ||\tau||_{L^\infty(\mathcal{S}_{u,v})} &\leq ||\tau||_{L^\infty(\mathcal{S}_{u,0})} + \int_0^v ||D\tau||_{L^\infty(\mathcal{S}_{u,v'})} \mathrm{d}v' \\ &\leq \Delta_{\Gamma_\star} + C(\Delta_{\Gamma_\star}, \Delta_{e_\star}, \Delta_\Phi(\mathcal{S}))v_\bullet + C(I, \Delta_{\Sigma_\star}, \Delta_{e_\star}, \Delta_\phi(\mathcal{S}))\varepsilon \\ &\quad + C(\Delta_{\Gamma_\star}) \int_0^v ||\tau||_{L^\infty(\mathcal{S}_{u,v'})} \mathrm{d}v'. \end{aligned}$$

Now, choosing ε sufficiently small, it follows from Grönwall's inequality that

$$||\tau||_{L^\infty(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_\Phi(\mathcal{S})).$$

Estimate for χ In order to estimate χ, we use the D-direction equation (4.7) for χ. A similar analysis as before yields

$$||\chi||_{L^\infty(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_\Phi(\mathcal{S})).$$

Estimates for μ, λ, α, β and ϵ. To estimate the coefficients μ and λ, we consider equations (3g) and (3o):

$$
\begin{aligned}
\Delta\mu &= -\mu^2 - \lambda\bar{\lambda} - \Phi_{22}, \\
\Delta\lambda &= -2\mu\lambda - \Xi\phi_4.
\end{aligned}
$$

Making use of the inequalities in Proposition 4 for the short direction, we obtain that

$$
\begin{aligned}
||\mu||_{L^\infty(\mathcal{S}_{u,v})} &\leq ||\mu||_{L^\infty(\mathcal{S}_{0,v})} + C(\Delta_{e_\star}) \int_0^\varepsilon ||\Delta\mu||_{L^\infty(\mathcal{S}_{u',v})} \mathrm{d}u' \\
&\leq \Delta_{\Gamma_\star} + C(\Delta_{e_\star}, \Delta_{\Gamma_\star})\varepsilon + C(\Delta_{e_\star}) \int_0^u ||\Phi_{22}||_{L^\infty(\mathcal{S}_{u',v})} \mathrm{d}u'.
\end{aligned}
$$

From the Sobolev and Hölder inequalities, we further find that

$$
\begin{aligned}
\int_0^u ||\Phi_{22}||_{L^\infty(\mathcal{S}_{u',v})} \mathrm{d}u' &\leq C(\Delta_{e_\star}) \int_0^u \sum_{i=0}^{2} ||\nabla^i \Phi_{22}||_{L^2(\mathcal{S}_{u',v})} \mathrm{d}u' \\
&= \sum_{i=0}^{2} C(\Delta_{e_\star}) \int_0^u \left(\int_{\mathcal{S}} |\nabla^i \Phi_{22}|^2 \right)^{1/2} \mathrm{d}u' \\
&\leq \left(\sum_{i=0}^{2} C(\Delta_{e_\star}) \int_0^u \int_{\mathcal{S}} |\nabla^i \Phi_{22}|^2 \mathrm{d}u' \right)^{1/2} \left(\int_0^u 1 \mathrm{d}u' \right)^{1/2} \\
&\leq C(\Delta_{e_\star}) \varepsilon^{1/2} ||\nabla^i \Phi_{22}||_{L^2(\mathcal{N}'_v(0,u))}.
\end{aligned}
$$

Hence we obtain that

$$
||\mu||_{L^\infty(\mathcal{S}_{u,v})} \leq \Delta_{\Gamma_\star} + C(\Delta_{e_\star}, \Delta_{\Gamma_\star})\varepsilon + C\varepsilon^{1/2}\Delta_\Phi.
$$

For the connection coefficient λ, a similar computation yields

$$
\begin{aligned}
||\lambda||_{L^\infty(\mathcal{S}_{u,v})} &\leq \Delta_{\Gamma_\star} + C(\Delta_{e_\star}, \Delta_{\Gamma_\star})\varepsilon + C(\Delta_{e_\star}) \int_0^u ||\Xi\phi_4||_{L^\infty(\mathcal{S}_{u',v})} du', \\
&\leq \Delta_{\Gamma_\star} + C(\Delta_{e_\star}, \Delta_{\Gamma_\star})\varepsilon + C\varepsilon^{3/2}\Delta_\phi.
\end{aligned}
$$

With the same method, we can estimate α, β and ϵ by using their short direction structure equations (3k), (3d) and (3a):

$$
\begin{aligned}
\Delta\alpha &= -\mu\alpha - \lambda\beta - \lambda\tau - \Xi\phi_3,\\
\Delta\beta &= -\bar{\lambda}\alpha - \mu\beta - \tau\mu - \Phi_{12},\\
\Delta\epsilon &= -\alpha\bar{\pi} - \beta\pi - \alpha\tau - \beta\bar{\tau} - \pi\tau - \Xi\phi_2 - \Phi_{11},
\end{aligned}
$$

The details are omitted.

Estimates for ρ and σ. In this case, the relevant Δ-transport equations are the structure equations (3i) and (3r):

$$
\begin{aligned}
\Delta\rho &= \bar{\delta}\tau - \mu\rho - \lambda\sigma - \alpha\tau + \bar{\beta}\tau - \tau\bar{\tau} - \Xi\phi_2,\\
\Delta\sigma &= \delta\tau - \bar{\lambda}\rho - \mu\sigma + \bar{\alpha}\tau - \beta\tau - \tau^2 - \Phi_{02}.
\end{aligned}
$$

In order to estimate $\delta\tau$ and $\bar{\delta}\tau$, we make use of the Sobolev inequalities in Corollary 2 and partial integration on $\mathcal{S}_{u,v}$ to obtain

$$
\begin{aligned}
||\nabla\!\!\!/\,\tau||_{L^\infty(\mathcal{S}_{u,v})} &\leq C(\Delta_{e_\star})\sum_{i=1}^{3}||\nabla\!\!\!/^{\,i}\tau||_{L^2(\mathcal{S}_{u,v})}\\
&\leq C(\Delta_{e_\star})\big(||\tau||_{L^2(\mathcal{S}_{u,v})} + ||\nabla\!\!\!/^{\,2}\tau||_{L^2(\mathcal{S}_{u,v})} + ||\nabla\!\!\!/^{\,3}\tau||_{L^2(\mathcal{S}_{u,v})}\big).
\end{aligned}
$$

Then the Hölder inequality

$$
||\tau||_{L^2(\mathcal{S}_{u,v})} \leq ||\tau||_{L^\infty(\mathcal{S}_{u,v})}\mathrm{Area}(\mathcal{S})^{1/2}
$$

and the assumptions

$$
\sup_{u,v}||\nabla\!\!\!/^{\,2}\tau||_{L^2(\mathcal{S}_{u,v})}, \qquad \sup_{u,v}||\nabla\!\!\!/^{\,3}\tau||_{L^2(\mathcal{S}_{u,v})} < \infty
$$

show us that $||\nabla\!\!\!/\,\tau||_{L^\infty(\mathcal{S}_{u,v})}$ is bounded. So we can estimate the $||\nabla\!\!\!/\,\tau||_{L^\infty(\mathcal{S}_{u,v})}$ term in the short direction using equations (3i) and (3r) for σ and ρ, respectively.

Estimate for s. Before estimating the derivatives of the conformal factor, we first analyse the Friedrich scalar s. Making use of the conformal Einstein field equations (6b),

$$\Delta s = -\Sigma_1\Phi_{22} - \Sigma_2\Phi_{11} + \Sigma_3\Phi_{21} + \Sigma_4\Phi_{12}$$

and the initial value $s|_{\mathscr{I}^-} = 0$, we readily have that

$$\begin{aligned}||s||_{L^\infty(\mathcal{S}_{u,v})} &\leq C(\Delta_{e_\star})\int_0^u ||\Sigma_2\Phi_{11} - \Sigma_4\Phi_{12} - \Sigma_3\Phi_{21} + \Sigma_1\Phi_{22}||_{L^\infty(\mathcal{S}_{u',v})}\mathrm{d}u',\\ &\leq C(\Delta_{e_\star}, \Delta_{\Sigma_\star}, \Delta_\Phi(\mathcal{S}))\varepsilon + C(\Delta_{e_\star}, \Delta_{\Sigma_\star}, \Delta_\Phi)\varepsilon^{1/2}.\end{aligned}$$

Estimate for Σ_2. Making use of the conformal Einstein field equation (5e)

$$\Delta\Sigma_2 = -\Xi\Phi_{22},$$

we have that

$$||\Sigma_2||_{L^\infty(\mathcal{S}_{u,v})} \leq \Delta_{\Sigma_\star} + C(\Delta_{e_\star})\int_0^u ||\Xi\Phi_{22}||_{L^\infty(\mathcal{S}_{u',v})}\mathrm{d}u' \leq \Delta_{\Sigma_\star} + C(\Delta_{e_\star}, \Delta_{\Sigma_\star}, \Delta_\Phi)\varepsilon^{3/2}.$$

Thus, we can choose $\varepsilon_\star$ sufficiently small such that $||\Sigma_2||_{L^\infty(\mathcal{S}_{u,v})}$ remains close to its initial value.

Estimate for Σ_1. Next, equation (5d)

$$\Delta\Sigma_1 = -\Sigma_4\tau - \Sigma_3\bar{\tau} + s - \Xi\Phi_{11}$$

and the initial value $\Sigma_1|_{\mathscr{I}^-} = 0$, gives that

$$\begin{aligned}||\Sigma_1||_{L^\infty(\mathcal{S}_{u,v})} &\leq C(\Delta_{e_\star})\int_0^u ||-\Sigma_4\tau - \Sigma_3\bar{\tau} + s - \Xi\Phi_{11}||_{L^\infty(S_{u',v})}\mathrm{d}u'\\ &\leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_\Phi(\mathcal{S}))\varepsilon + C(\Delta_{e_\star}, \Delta_{\Sigma_\star}, \Delta_\Phi)\varepsilon^{3/2}\\ &+ C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_\Phi(\mathcal{S}))\varepsilon^2.\end{aligned}$$

Estimates for Σ_3 and Σ_4. Equation (5f)

$$\Delta\Sigma_3 = -\Sigma_2\tau - \Xi\Phi_{12}$$

readily gives that

$$\begin{aligned}||\Sigma_3||_{L^\infty(\mathcal{S}_{u,v})} &\leq C(\Delta_{e_\star})\int_0^u ||\Sigma_2\tau + \Xi\Phi_{12}||_{L^\infty(\mathcal{S}_{u',v})}\mathrm{d}u',\\ &\leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Sigma_\star},\Delta_\Phi(\mathcal{S}))\varepsilon + C(\Delta_{e_\star},\Delta_{\Sigma_\star},\Delta_\Phi(\mathcal{S}))\varepsilon^2.\end{aligned}$$

The method is the same for Σ_4.

Concluding the argument. From the estimates for the NP connection coefficients and $\Sigma_{AA'}$ constructed above it follows that one can choose

$$\varepsilon_\star = \varepsilon_\star(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\sup_{u,v}||\nabla\!\!\!\!/^2\tau||_{L^2(\mathcal{S}_{u,v})},\sup_{u,v}||\nabla\!\!\!\!/^3\tau||_{L^2(\mathcal{S}_{u,v})},\Delta_{\Sigma_\star},\Delta_\phi,\Delta_\Phi,\Delta_\phi,\Delta_\Phi(\mathcal{S})),$$

sufficiently small so that

$$\begin{aligned}&\sup_{u,v}||\{\mu,\lambda,\alpha,\beta,\epsilon,\rho,\sigma\}||_{L^\infty(\mathcal{S}_{u,v})} \leq 3\Delta_{\Gamma_\star},\\ &\sup_{u,v}||\Sigma_2||_{L^\infty(\mathcal{S}_{u,v})} \leq 3\Delta_{\Sigma_\star},\\ &\sup_{u,v}||\{\Sigma_1,\Sigma_3,\Sigma_4\}||_{L^\infty(\mathcal{S}_{u,v})} \leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Sigma_\star},\Delta_\Phi(\mathcal{S}))\varepsilon.\end{aligned}$$

Accordingly, we have improved our initial bootstrap assumption. □

Now we use the similar method to analyse the L^4 estimate of the connection coefficients and the derivative of conformal factor.

Proposition 21 (***control on the L^4-norm of the connection coefficients and the derivatives of the conformal factor***)**.** *With the same assumptions in Proposition 20, and additionally assuming that*

$$\sup_{u,v}||\nabla\!\!\!\!/\{\mu,\lambda,\alpha,\beta,\epsilon,\rho,\sigma,\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4\}||_{L^4(\mathcal{S}_{u,v})} \leq \Delta_{\Gamma,\Sigma},$$

in the truncated diamond $\mathcal{D}^t_{u,v_\bullet}$, we find that there exists

$$\varepsilon_\star = \varepsilon_\star(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\sup_{u,v}||\nabla\!\!\!\!/^2\tau||_{L^2(\mathcal{S}_{u,v})},\sup_{u,v}||\nabla\!\!\!\!/^3\tau||_{L^2(\mathcal{S}_{u,v})},\Delta_{\Sigma_\star},\Delta_\phi,\Delta_\Phi,\Delta_\phi(\mathcal{S}),\Delta_\Phi(\mathcal{S})),$$

such that when $\varepsilon \leq \varepsilon_\star$, we have we have

$$\begin{aligned}
&\sup_{u,v} ||\nabla\{\tau,\chi\}||_{L^4(\mathcal{S}_{u,v})} \leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Phi}(\mathcal{S})),\\
&\sup_{u,v} ||\nabla\{\mu,\lambda,\rho,\sigma,\alpha,\beta,\epsilon\}||_{L^4(\mathcal{S}_{u,v})} \leq 3\Delta_{\Gamma_\star},\\
&\sup_{u,v} ||\nabla\Sigma_2||_{L^4(\mathcal{S}_{u,v})} \leq 3\Delta_{\Sigma_\star},\\
&\sup_{u,v} ||\nabla\{\Sigma_1,\Sigma_3,\Sigma_4\}||_{L^4(\mathcal{S}_{u,v})} \leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Sigma_\star},\Delta_{\Phi}(\mathcal{S}))\varepsilon.
\end{aligned}$$

on $\mathcal{D}^t_{u,v_\bullet}$.

Proof.

Basic bootstrap assumption. We make bootstrap assumptions

$$\begin{aligned}
&\sup_{u,v} ||\nabla(\mu,\lambda,\rho,\sigma,\alpha,\beta,\epsilon)||_{L^4(\mathcal{S}_{u,v})} \leq 4\Delta_{\Gamma_\star}\\
&\sup_{u,v} ||\nabla\{\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4\}||_{L^4(\mathcal{S}_{u,v})} \leq 4\Delta_{\Sigma_\star}.
\end{aligned}$$

Estimates for $\nabla\tau$. First, we estimate the $L^4(\mathcal{S}_{u,v})$ norm of $\nabla\tau$. Apply the δ-derivative to the D-direction equation of τ and the commutator of directional covariant derivatives we obtain

$$\begin{aligned}
D\delta\tau = (\rho+\bar{\rho}+2\epsilon-2\bar{\epsilon})\delta\tau + \sigma\bar{\delta}\tau + \sigma\delta\bar{\tau} + \delta(\epsilon-\bar{\epsilon}+\rho)\tau + \bar{\tau}\delta\sigma + \rho\delta\bar{\pi}\\
+ \bar{\pi}\delta\rho + \sigma\delta\pi + \pi\delta\sigma + \Gamma^3 + \Sigma_3\phi_1 + \Xi\delta\phi_1 + \Xi\phi_1\Gamma + \delta\Phi_{01} + \Phi_{01}\Gamma.
\end{aligned}$$

In order to estimate the terms in $||\Gamma\nabla\Gamma||_{L^4(\mathcal{S}_{u,v})}$, we use the Hölder inequality and split it as

$$||\Gamma\nabla\Gamma||_{L^4(\mathcal{S}_{u,v})} \leq ||\Gamma||_{L^\infty(\mathcal{S}_{u,v})}||\nabla\Gamma||_{L^4(\mathcal{S}_{u,v})}.$$

Now, Proposition 20 shows that terms of the form $||\Gamma||_{L^\infty(\mathcal{S}_{u,v})}$ are, in fact, bounded. Making use of the Sobolev inequality in Proposition 6 and the long direction inequality in Proposition 5, we find that

$$||\delta\tau||_{L^4(\mathcal{S}_{u,v})} + ||\bar{\delta}\tau||_{L^4(\mathcal{S}_{u,v})}$$

$$
\begin{aligned}
&\leq C\big(||\delta\tau||_{L^4(S_{u,0})} + ||\bar{\delta}\tau||_{L^4(S_{u,0})} + \int_0^v ||D\delta\tau||_{L^4(S_{u,v'})} + ||D\bar{\delta}\tau||_{L^4(S_{u,v'})}\mathrm{d}v'\big) \\
&\leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_\Phi(\mathcal{S})) + C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_\Phi(\mathcal{S}), \Delta_\phi(\mathcal{S}))\varepsilon \\
&\quad + C(\Delta_{\Gamma_\star}) \int_0^v \big(||\delta\tau||_{L^4(\mathcal{S}_{u,v'})} + ||\bar{\delta}\tau||_{L^4(\mathcal{S}_{u,v'})}\big)\mathrm{d}v'.
\end{aligned}
$$

Thus Grönwall's inequality gives

$$
||\delta\tau||_{L^4(\mathcal{S}_{u,v})} + ||\bar{\delta}\tau||_{L^4(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_\Phi(\mathcal{S})) + C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_\Phi(\mathcal{S}), \Delta_\phi(\mathcal{S}))\varepsilon.
$$

Accordingly, for a small range ε, we obtain that

$$
||\nabla\!\!\!/\,\tau||_{L^4(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_\Phi(\mathcal{S})).
$$

Estimates for $\nabla\!\!\!/\,\chi$. A direct computation shows that

$$
D\delta\chi = (\bar{\rho} - 2\bar{\epsilon})\delta\chi + \sigma\bar{\delta}\chi + \Gamma\delta\Gamma - \chi\delta(\epsilon + \bar{\epsilon}) + \Sigma_3(\phi_2 + \bar{\phi}_2) + \Xi\delta(\phi_2 + \bar{\phi}_2) + \delta\Phi_{11},
$$

where Γ represents a combination of the connection coefficients whose particular form is not required. A similar equation can be obtained for $D\bar{\delta}\chi$. Using the same method as for the coefficient τ, we obtain that $||\nabla\!\!\!/\,\chi||_{L^4(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_\Phi(\mathcal{S}))$.

Estimates for $\nabla\!\!\!/\,\{\mu, \lambda, \rho, \sigma, \alpha, \beta, \epsilon\}$. Applying the operator Δ to equations (3g) and (3o) we find that

$$
\begin{aligned}
\Delta\delta\mu &= (\tau - \bar{\alpha} - \beta)(\mu^2 + \lambda\bar{\lambda}) - 3\mu\delta\mu - \bar{\lambda}\bar{\delta}\mu - \lambda\delta\bar{\lambda} - \bar{\lambda}\delta\lambda - \delta\Phi_{22}, \\
\Delta\delta\lambda &= (\tau - \bar{\alpha} - \beta)(2\mu\lambda + \Xi\phi_4) - 3\mu\delta\lambda - \bar{\lambda}\bar{\delta}\lambda - 2\lambda\delta\mu - \Sigma_3\phi_4 - \Xi\delta\phi_4.
\end{aligned}
$$

Now, a direct computation applying Proposition 3 shows that we can find an $\varepsilon_\star$ such that when $\varepsilon \leq \varepsilon_\star$, we have

$$
||\nabla\!\!\!/\,\{\mu, \lambda\}||_{L^4(\mathcal{S}_{u,v})} \leq 3\Delta_{\Gamma_\star}.
$$

We can estimate $\delta\alpha$, $\delta\beta$ and $\delta\epsilon$ by using the same method. Since we are using the assumption $\sup_{u,v} ||\nabla\!\!\!/^3\tau||_{L^2(\mathcal{S}_{u,v})} < \infty$ in the truncated causal diamond, the Sobolev inequalities of Corollary 2 show that $||\nabla\!\!\!/^2\tau||_{L^4(\mathcal{S}_{u,v})}$ is finite. Proceeding in a similar way we can estimate $\delta\sigma$ and $\delta\rho$ by applying δ to equations (3i) and (3r).

Estimate for $\nabla\!\!\!/\,\Sigma_2$. Applying δ to the short direction equation (5e) for Σ_2 and using the commutators we find that

$$\begin{aligned}\Delta\delta\Sigma_2 &= -\Xi\delta\Phi_{22} - \Sigma_3\Phi_{22} + \Xi\Phi_{22}(\tau-\bar{\pi}) + \Xi\Phi_{21}\bar{\lambda} + \Xi\Phi_{12}\mu, \\ &+ \Sigma_2(\pi\bar{\lambda} + \bar{\pi}\mu) - \Sigma_3(\lambda\bar{\lambda} + \mu^2) - 2\Sigma_4\bar{\lambda}\bar{\mu}.\end{aligned}$$

Similar arguments to the ones used for the connection coefficients show that

$$||\nabla\!\!\!/\,\Sigma_2||_{L^4(\mathcal{S}_{u,v})} \leq 2\Delta_{\Sigma_\star} + C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Sigma_\star})\varepsilon + C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Sigma_\star},\Delta_{\Phi}(\mathcal{S}))\varepsilon^2 + o(\varepsilon^2).$$

Accordingly, the $\varepsilon_\star$ can be chosen sufficiently small to ensure that $||\nabla\!\!\!/\,\Sigma_2||_{L^4(\mathcal{S}_{u,v})}$ is no more than $3\Delta_{\Sigma_\star}$.

Estimate for $\nabla\!\!\!/\,\Sigma_1$. Making use of the equation for $\Delta\delta\Sigma_1$:

$$\begin{aligned}\Delta\delta\Sigma_1 &= -\Sigma_1\Phi_{12} - \Sigma_2\Phi_{01} + \Sigma_4\Phi_{02} + s(\bar{\pi}-\tau) + \Xi\Phi_{11}(\tau-\bar{\pi}) + \Sigma_4\tau(\tau-\bar{\pi}) \\ &+ \Sigma_3\bar{\tau}(\tau-\bar{\pi}) - \mu\delta\Sigma_1 - \bar{\tau}\delta\Sigma_3 - \tau\delta\Sigma_4 - \Xi\delta\Phi_{11} - \Sigma_4\delta\tau - \Sigma_3\delta\bar{\tau} - \bar{\lambda}\bar{\delta}\Sigma_1,\end{aligned}$$

it follows from the bootstrap assumption, that

$$||\nabla\!\!\!/\,\Sigma_1||_{L^4(\mathcal{S}_{u,v})} \leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Sigma_\star},\Delta_{\Phi}(\mathcal{S}))\varepsilon + o(\varepsilon).$$

Estimate for $\nabla\!\!\!/\,\Sigma_{3,4}$. A direct computation yields the equation

$$\Delta\delta\Sigma_3 = -\Sigma_3\Phi_{12} + \Xi\Phi_{12}(\tau-\bar{\pi}) + \Sigma_2\tau(\tau-\bar{\pi}) - \tau\delta\Sigma_2 - \Sigma_2\delta\tau - \mu\delta\Sigma_3 - \Xi\delta\Phi_{12} - \bar{\lambda}\bar{\delta}\Sigma_3.$$

accordingly, one can readily find that

$$||\nabla\!\!\!/\,\Sigma_3||_{L^4(\mathcal{S}_{u,v})} \leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Sigma_\star},\Delta_{\Phi}(\mathcal{S}))\varepsilon + o(\varepsilon).$$

A similar result holds for $||\nabla\!\!\!/\,\Sigma_4||_{L^4(S)}$. It follows from the previous discussion that when ε is suitably small, we can improve the bootstrap assumption.

Concluding the argument. From the analysis above, it follows we can choose

$$\varepsilon_\star = \varepsilon_\star(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \sup_{u,v} ||\nabla^2 \tau||_{L^2(\mathcal{S}_{u,v})}, \sup_{u,v} ||\nabla^3 \tau||_{L^2(\mathcal{S}_{u,v})}, \Delta_{\Sigma_\star}, \Delta_\phi, \Delta_\Phi, \Delta_\phi(\mathcal{S}), \Delta_\Phi(\mathcal{S})),$$

sufficiently small so that

$$\begin{aligned}
&\sup_{u,v} ||\nabla\{\mu, \lambda, \rho, \sigma, \alpha, \beta, \epsilon\}||_{L^4(\mathcal{S}_{u,v})} \leq 3\Delta_{\Gamma_\star},\\
&\sup_{u,v} ||\nabla \Sigma_2||_{L^4(\mathcal{S}_{u,v})} \leq 3\Delta_{\Sigma_\star},\\
&\sup_{u,v} ||\nabla\{\Sigma_1, \Sigma_3, \Sigma_4\}||_{L^4(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_\Phi(\mathcal{S}))\varepsilon.
\end{aligned}$$

The above estimates improve the bootstrap assumption. □

The discussion of this section is concluded with L^2-estimates for the connection coefficients and the derivative of conformal factor.

Proposition 22 (***control on the L^2-norm of the connection coefficients and the derivatives of the conformal factor***)**.** *Assume that we have a solution of the vacuum conformal Einstein field equations in Stewart's gauge in a region $\mathcal{D}^t_{u,v_\bullet}$ with*

$$\begin{aligned}
&\sup_{u,v} ||\{\mu, \lambda, \alpha, \beta, \epsilon, \rho, \sigma, \tau, \chi, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4\}||_{L^\infty(\mathcal{S}_{u,v})} \leq \Delta_{\Gamma,\Sigma}\,,\\
&\sup_{u,v} ||\nabla\{\mu, \lambda, \alpha, \beta, \epsilon, \rho, \sigma, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4\}||_{L^4(\mathcal{S}_{u,v})} \leq \Delta_{\Gamma,\Sigma},\\
&\sup_{u,v} ||\nabla^2\{\mu, \lambda, \alpha, \beta, \epsilon, \rho, \sigma, \tau, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4\}||_{L^2(\mathcal{S}_{u,v})} \leq \Delta_{\Gamma,\Sigma},
\end{aligned}$$

for some positive $\Delta_{\Gamma,\Sigma}$. Assume also

$$\sup_{u,v} ||\nabla^3 \tau||_{L^2(\mathcal{S}_{u,v})} < \infty, \qquad \Delta_\Phi(\mathcal{S}) < \infty, \qquad \Delta_\Phi < \infty, \qquad \Delta_\phi(\mathcal{S}) < \infty, \qquad \Delta_\phi < \infty$$

on the same domain. We have that there exists

$$\varepsilon_\star = \varepsilon_\star(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star} \sup_{u,v} ||\nabla^3 \tau||_{L^2(\mathcal{S}_{u,v})}, \Delta_\phi, \Delta_\Phi, \Delta_\phi(\mathcal{S}), \Delta_\Phi(\mathcal{S})),$$

such that when $\varepsilon \leq \varepsilon_\star$, we have that

$$
\begin{aligned}
&\sup_{u,v} ||\nabla\!\!\!/^2\{\tau,\chi\}||_{L^2(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_\Phi(\mathcal{S})),\\
&\sup_{u,v} ||\nabla\!\!\!/^2\{\mu,\lambda,\alpha,\beta,\epsilon,\rho,\sigma\}||_{L^2(\mathcal{S}_{u,v})} \leq 3\Delta_{\Gamma_\star},\\
&\sup_{u,v} ||\nabla\!\!\!/^2\Sigma_2||_{L^4(\mathcal{S}_{u,v})} \leq 3\Delta_{\Sigma_\star},\\
&\sup_{u,v} ||\nabla\!\!\!/^2\{\Sigma_1,\Sigma_3,\Sigma_4\}||_{L^4(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_\Phi(\mathcal{S}))\varepsilon.
\end{aligned}
$$

Proof.

Basic bootstrap assumption. We make following bootstrap assumptions:

$$
\begin{aligned}
&\sup_{u,v} ||\nabla\!\!\!/^2\{\mu,\lambda,\rho,\sigma,\alpha,\beta,\epsilon\}||_{L^2(\mathcal{S}_{u,v})} \leq 4\Delta_{\Gamma_\star},\\
&\sup_{u,v} ||\nabla\!\!\!/^2\{\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4\}||_{L^2(\mathcal{S}_{u,v})} \leq 4\Delta_{\Sigma_\star}.
\end{aligned}
$$

Estimates for $||\nabla\!\!\!/^2\tau||_{L^2(\mathcal{S}_{u,v})}$ and $||\nabla\!\!\!/^2\chi||_{L^2(\mathcal{S}_{u,v})}$. Applying the operator δ to the equation for $D\delta\tau$ and using the commutators, one obtains following the D-direction equation of $\delta^2\tau$:

$$
\begin{aligned}
D\delta^2\tau =\, &\Gamma\delta^2\tau + \Gamma\delta^2\bar{\tau} + \Gamma\bar{\delta}\delta\tau + \Gamma\delta\bar{\delta}\tau + \Gamma_1^4 + \Gamma_1\delta^2\Gamma_1\\
&+ \delta\Gamma_1\delta\Gamma_1 + \Gamma_1^2\delta\Gamma_1 + \delta^2\Phi_{01} + \Gamma_1\delta\Phi_{01} + \Phi_{01}\delta\Gamma_1 + \Phi_{01}\Gamma_1^2\\
&+ \delta\Sigma_3\phi_1 + 2\Sigma_3\delta\phi_1 + \Xi\delta\phi_1 + \Xi\phi_1\Gamma^2 + \Xi\Gamma_1\delta\phi_1 + \Xi\phi_1\delta\Gamma_1 + \Xi\delta^2\phi_1 + \Sigma_3\phi_1\Gamma_1,
\end{aligned}
$$

where Γ contains a combination of the coefficients ρ,σ,ϵ, Γ_1 contains a combination of $\tau,\alpha,\beta,\sigma,\epsilon,\rho$. A similar computation renders equations for $D\bar{\delta}\tau$, $D\delta\bar{\delta}\tau$. Terms of the form $\delta\Gamma_1\delta\Gamma_1$ can be handled using the Hölder inequality

$$
||\delta\Gamma_1\delta\Gamma_1||_{L^2(\mathcal{S}_{u,v})} \leq ||\delta\Gamma_1||_{L^4(\mathcal{S}_{u,v})}||\delta\Gamma_1||_{L^4(\mathcal{S}_{u,v})},
$$

where Proposition 21 shows that the bound is finite. The analysis for the term $\delta\Sigma_3\phi_1$ is the same. More precisely, one has that

$$
||\delta\Sigma_3\phi_1||_{L^2(\mathcal{S}_{u,v})} \leq ||\delta\Sigma_3||_{L^4(\mathcal{S}_{u,v})}||\phi_1||_{L^4(\mathcal{S}_{u,v})}
$$

$$\leq C(\Delta_{e_\star})||\delta\Sigma_3||_{L^4(\mathcal{S}_{u,v})}\big(||\phi_1||_{L^2(\mathcal{S}_{u,v})} + ||\nabla\!\!\!/\phi_1||_{L^2(\mathcal{S}_{u,v})}\big).$$

Similar arguments can be employed in the rest of the terms for the equation for $D\delta^2\tau$ so that with the long direction inequality in Proposition 3 we obtain

$$\begin{aligned}
||\delta^2\tau||_{L^2(\mathcal{S}_{u,v})} &\leq C(I,\Delta_{\Gamma_\star})\left(||\delta^2\tau||_{L^2(\mathcal{S}_{u,0})} + \int_0^v ||D\delta^2\tau||_{L^2(\mathcal{S}_{u,v'})}\mathrm{d}v'\right),\\
&\leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_\Phi(\mathcal{S})) + C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Sigma_\star},\Delta_\Phi(\mathcal{S}),\Delta_\phi(\mathcal{S}))\varepsilon\\
&\quad + C(I,\Delta_{e_\star},\Delta_{\Gamma_\star})\int_0^v ||\nabla\!\!\!/^2\tau||_{L^2(\mathcal{S}_{u,v'})}\mathrm{d}v'.
\end{aligned}$$

Similar estimates can be obtained for $\bar{\delta}^2\tau$, $\delta\bar{\delta}\tau$ and $\bar{\delta}\delta\tau$. To estimate $||\delta\tau||_{L^2(\mathcal{S}_{u,v})}$ we can make use of the fact that the area of $\mathcal{S}_{u,v}$ is bounded so that

$$||\delta\tau||_{L^2(\mathcal{S}_{u,v})} \leq C(\Delta_{e_\star},\Delta_{\Gamma_\star})||\delta\tau||_{L^4(\mathcal{S}_{u,v})},$$

hence, Proposition 21 shows us that this is also finite. From inequality (32) of Paper I we get

$$\begin{aligned}
||\nabla\!\!\!/^2\tau||_{L^2(\mathcal{S}_{u,v})} &\leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_\Phi(\mathcal{S})) + C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Gamma_\star},\Delta_\Phi(\mathcal{S}),\Delta_\phi(\mathcal{S}))\varepsilon\\
&\quad + C(I,\Delta_{e_\star},\Delta_{\Gamma_\star})\int_0^v ||\nabla\!\!\!/^2\tau||_{L^2(\mathcal{S}_{u,v'})}\mathrm{d}v',
\end{aligned}$$

so that using Grönwall's inequality we conclude that

$$||\nabla\!\!\!/^2\tau||_{L^2(\mathcal{S}_{u,v})} \leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_\Phi(\mathcal{S})) + C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_\phi(\mathcal{S}))\varepsilon.$$

Hence, one finds that $||\nabla\!\!\!/^2\tau||_{L^2(\mathcal{S}_{u,v})}$ is bounded by a constant $C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_\Phi(\mathcal{S}))$. Using the same analysis, we can conclude that $||\nabla\!\!\!/^2\chi||_{L^2(\mathcal{S}_{u,v})}$ is bounded.

Estimates for the the remaining spin connection coefficients. Estimates for the remaining connection coefficients can be obtained by the same methods as in Proposition 21 namely, first we compute equations for $\Delta\delta^2\Gamma$ and $\Delta\bar{\delta}\delta\Gamma$, and make use of the short direction inequality in Proposition 3 to find that

$$||\nabla\!\!\!/^2\{\mu,\lambda,\alpha,\beta,\epsilon,\sigma,\rho\}||_{L^2(\mathcal{S}_{u,v})} \leq 3\Delta_{\Gamma_\star}$$

for sufficiently small ε.

Estimates for $\not\nabla^2\Sigma_2$. A direct calculation shows that

$$\begin{aligned}\Delta\delta^2\Sigma_2 &= \Gamma\delta^2\Sigma_2 + \delta\Gamma\delta\Sigma_2 + \Gamma^2\delta\Sigma_2 + \Xi\delta^2\Phi_{22} + \Xi\Phi_{22}\delta\Gamma \\ &\quad + \Phi_{22}\delta\Sigma_3 + \Sigma_3\delta\Phi_{22} + \Xi\Gamma\delta\Phi_{22} + \Sigma_3\Phi_{22}\Gamma + \Xi\Phi_{22}\Gamma^2.\end{aligned}$$

The other short direction equation for the remaining second order spherical derivatives of Σ_2 have the same structure. From these equations we obtain that

$$||\not\nabla^2\Sigma_2||_{L^2(\mathcal{S}_{u,v})} \leq 2\Delta_{\Sigma_\star} + C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_\Phi(\mathcal{S}))\varepsilon + o(\varepsilon).$$

The term $o(\varepsilon)$ arises from the presence of $\delta^i\Xi$, $i = 0, 1, 2$.

Estimates for $\not\nabla^2\Sigma_1$ and $\not\nabla^2\Sigma_{3,4}$. Again, a direct computation yields the equation

$$\begin{aligned}\Delta\delta^2\Sigma_1 &= \Gamma\delta^2\Sigma_1 + \Gamma\delta^2\Sigma' + \Sigma\Gamma\delta\Gamma + \delta\Sigma\delta\Gamma + \Sigma\delta^2\Gamma + \Sigma\Gamma^3 + \Gamma^2\delta\Sigma + \Gamma\Sigma\Phi, \\ &\quad + \Gamma^2\Xi\Phi + s\Gamma^2 + s\delta\Gamma + \Phi\delta\Sigma + \Sigma\delta\Phi + \Xi\Gamma\delta\Phi + \Xi\Phi\delta\Gamma + \Xi\delta^2\Phi,\end{aligned}$$

where Γ contains τ, Σ contains Σ_2, and Σ' does not contain Σ_1, while Φ does not contain Φ_{22}. Making use of the same arguments as for Σ_2, we obtain that

$$||\not\nabla^2\Sigma_1||_{L^2(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_\Phi(\mathcal{S}))\varepsilon + o(\varepsilon).$$

Similar arguments give

$$||\not\nabla^2\Sigma_3||_{L^2(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_\Phi(\mathcal{S}))\varepsilon + o(\varepsilon).$$

Concluding the argument. From the analysis in the previous paragraphs it follows that we can choose

$$\varepsilon_\star = \varepsilon_\star(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \sup_{u,v}||\not\nabla^3\tau||_{L^2(\mathcal{S}_{u,v})}, \Delta_{\Sigma_\star}, \Delta_\phi, \Delta_\Phi, \Delta_\phi(\mathcal{S}), \Delta_\Phi(\mathcal{S})),$$

sufficiently small so that

$$\sup_{u,v}||\not\nabla^2\{\mu, \lambda, \rho, \sigma, \alpha, \beta, \epsilon\}||_{L^2(\mathcal{S}_{u,v})} \leq 3\Delta_{\Gamma_\star},$$

$$
\begin{aligned}
&\sup_{u,v} ||\nabla\!\!\!\!/^2 \Sigma_2||_{L^2(\mathcal{S}_{u,v})} \leq 3\Delta_{\Sigma_\star}, \\
&\sup_{u,v} ||\nabla\!\!\!\!/^2 \{\Sigma_1, \Sigma_3, \Sigma_4\}||_{L^2(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_\Phi(\mathcal{S}))\varepsilon.
\end{aligned}
$$

The above estimates improve the bootstrap assumptions. □

4.4.3 First estimates for the curvature

Building on the L^p-estimates for the connection coefficients and the derivative of the conformal factor obtained in the previous section, we now show that the norms $\Delta_\Phi(\mathcal{S})$ and $\Delta_\phi(\mathcal{S})$ are bounded by the initial data. This is achieved in the next two propositions.

Proposition 23 (***basic control of the Ricci curvature***)**.** *Assume that we are given a solution to the vacuum CEFEs in Stewart's gauge satisfying the assumptions of Proposition 22. Then there exists*

$$
\varepsilon_\star = \varepsilon_\star(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_{\Phi_\star}, \Delta_\Phi, \Delta_\phi, \sup_{u,v} ||\nabla\!\!\!\!/^3 \tau||_{L^2(\mathcal{S}_{u,v})})
$$

such that for $\varepsilon \leq \varepsilon_\star$, we have

$$
\Delta_\Phi(\mathcal{S}) < 3\Delta_{\Phi_\star}.
$$

on $\mathcal{D}^t_{u,v_\bullet}$.

Proof.

Bootstrap assumption. We make the following bootstrap assumption:

$$
\sup_{u,v} ||\nabla\!\!\!\!/^i \{\Phi_{00}, \Phi_{01}, \Phi_{02}, \Phi_{11}, \Phi_{12}\}||_{L^2(\mathcal{S}_{u,v})} \leq 4\Delta_{\Phi_\star}, \qquad i = 0, ..., 2.
$$

L^2-norm of the components $\{\Phi_{00}, \Phi_{01}, \Phi_{02}, \Phi_{11}, \Phi_{12}\}$. We focus on the $L^2(\mathcal{S})$ norm of $\{\Phi_{00}, \Phi_{01}, \Phi_{02}, \Phi_{11}, \Phi_{12}\}$. We will use the short direction equations (7a)-(7e) to estimate these components. We take Φ_{11} as an example. The relevant equation is in this case given by

$$\Delta\Phi_{11} = \delta\Phi_{21} + 2\beta\Phi_{21} - \bar{\lambda}\Phi_{20} - 2\mu\Phi_{11} + \bar{\rho}\Phi_{22} - \tau\Phi_{21} - \bar{\tau}\Phi_{21} + \Sigma_2\bar{\phi}_2 - \Sigma_4\bar{\phi}_3. \tag{4.9}$$

It follows then that

$$\begin{aligned}||\Phi_{11}||_{L^2(\mathcal{S}_{u,v})} &\leq 2\left(||\Phi_{11}||_{L^2(\mathcal{S}_{0,v})} + C(\Delta_{e_\star}, \Delta_{\Gamma_\star})\int_0^u ||\Delta\Phi_{11}||_{L^2(\mathcal{S}_{u',v})}\mathrm{d}u'\right), \\ &\leq 2\Delta_{\Phi_\star} + C(\Delta_{e_\star}, \Delta_{\Gamma_\star})\int_0^u \Big(||\delta\Phi_{21}||_{L^2(\mathcal{S}_{u',v})} + ||\bar{\rho}\Phi_{22}||_{L^2(\mathcal{S}_{u',v})} + ||\Sigma_2\bar{\phi}_2||_{L^2(\mathcal{S}_{u',v})}, \\ &\quad + ||2\beta\Phi_{21} + \bar{\lambda}\Phi_{20} + 2\mu\Phi_{11} + \tau\Phi_{21} + \bar{\tau}\Phi_{21}||_{L^2(\mathcal{S}_{u',v})} + ||\Sigma_4\bar{\phi}_3||_{L^2(\mathcal{S}_{u',v})}\Big)\mathrm{d}u'.\end{aligned}$$

Using the Hölder inequality, the first three terms can be transformed to a norm on the light cone. More precisely, one has

$$\begin{aligned}\int_0^u ||\delta\Phi_{21}||_{L^2(\mathcal{S}_{u',v})}du' &= \int_0^u \left(\int_{\mathcal{S}_{u',v}} |\delta\Phi_{21}|^2\right)^{1/2}\mathrm{d}u' \leq \left(\int_0^u\int_{\mathcal{S}_{u',v}} |\delta\Phi_{21}|^2\right)^{1/2}\left(\int_0^u 1\right)^{1/2} \\ &\leq \varepsilon^{1/2}||\delta\Phi_{21}||_{L^2(\mathcal{N}'_v(0,u))} \leq \Delta_\Phi\varepsilon^{1/2}.\end{aligned}$$

Similarly, one has that

$$\int_0^u ||\bar{\rho}\Phi_{22}||_{L^2(\mathcal{S}_{u',v})}du' \leq C(\Delta_{\Gamma_\star}, \Delta_\Phi)\varepsilon^{1/2}, \qquad \int_0^u ||\Sigma_2\bar{\phi}_2||_{L^2(\mathcal{S}_{u',v})}du' \leq C(\Delta_{\Sigma_\star}, \Delta_\phi)\varepsilon^{1/2}.$$

The (large) fourth term can be estimated as follows:

$$\int_0^u ||\Gamma\Phi||_{L^2(\mathcal{S}_{u',v})}\mathrm{d}u' \leq \int_0^u ||\Gamma||_{L^\infty(\mathcal{S}_{u',v})}||\Phi||_{L^2(\mathcal{S}_{u',v})}\mathrm{d}u' \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Phi_\star})\varepsilon.$$

For the last term we have that

$$\begin{aligned}\int_0^u ||\Sigma_4\bar{\phi}_3||_{L^2(\mathcal{S}_{u',v})}du' &\leq \int_0^u ||\Sigma_4||_{L^\infty(\mathcal{S}_{u',v})}||\bar{\phi}_3||_{L^2(\mathcal{S}_{u',v})}\mathrm{d}u' \\ &\leq C\varepsilon||\phi_3||_{L^2(\mathcal{N}'_v(0,u))}\varepsilon^{1/2} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_\phi)\varepsilon^{3/2}.\end{aligned}$$

Hence, we find that

$$||\Phi_{11}||_{L^2(\mathcal{S}_{u,v})} \leq 2\Delta_{\Phi_\star} + C(\Delta_{e_\star}, \Delta_{\Sigma_\star}, \Delta_{\Gamma_\star}, \Delta_{\Phi}, \Delta_{\phi})\varepsilon^{1/2} + C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Phi_\star})\varepsilon$$
$$+ C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_{\phi})\varepsilon^{3/2}.$$

Accordingly, $\varepsilon_\star$ can be chosen sufficiently small so that $||\Phi_{11}||_{L^2(\mathcal{S}_{u,v})}$ is less than $3\Delta_{\Phi_\star}$, and similarly for the remaining terms. Consequently, we have improved the bootstrap assumption and finished Step 1, that is, we have

$$\sup_{u,v} ||(\Phi_{00}, \Phi_{01}, \Phi_{02}, \Phi_{11}, \Phi_{12})||_{L^2(\mathcal{S}_{u,v})} \leq 3\Delta_{\Phi_\star}.$$

Estimates for $||\nabla\!\!\!/\{\Phi_{00}, \Phi_{01}, \Phi_{02}, \Phi_{11}, \Phi_{12}\}||_{L^2(\mathcal{S}_{u,v})}$**.** We now focus on the $L^2(\mathcal{S}_{u,v})$-norm of the first derivative of the Ricci curvature. We take $\nabla\!\!\!/\Phi_{11}$ as an example. Using the results of Proposition 3 we readily have

$$||\nabla\!\!\!/\Phi_{11}||_{L^2(\mathcal{S}_{u,v})} \leq 2\left(||\nabla\!\!\!/\Phi_{11}||_{L^2(\mathcal{S}_{0,v})} + C(\Delta_{e_\star}, \Delta_{\Gamma_\star}) \int_0^u \left(\int_{\mathcal{S}_{u',v}} \Delta \left\langle \nabla\!\!\!/\Phi_{11}, \nabla\!\!\!/\Phi_{11} \right\rangle_\sigma \right)^{1/2} \mathrm{d}u' \right),$$
$$\leq 2\Delta_{\Phi_\star} + C(\Delta_{e_\star}, \Delta_{\Gamma_\star}) \int_0^u \left(\int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/\Phi_{11}|(|\Delta\delta\Phi_{11}| + |\Delta\bar{\delta}\Phi_{11}|) \right)^{1/2} \mathrm{d}u',$$

while the short direction equation for $\delta\Phi_{11}$ is given by

$$\Delta\delta\Phi_{11} = \delta^2\Phi_{21} + \Sigma_2\bar{\phi}_2(\bar{\pi} - \tau) + \bar{\phi}_2\delta\Sigma_2 + \Sigma_2\delta\bar{\phi}_2 + \Sigma_4\bar{\phi}_3(\tau - \bar{\pi}) - \bar{\phi}_3\delta\Sigma_4 + \Sigma_4\delta\bar{\phi}_3$$
$$+ \Phi_{22}\bar{\rho}(\bar{\pi} - \tau) + \bar{\rho}\delta\Phi_{22} + \Phi_{22}\delta\bar{\rho} + \Phi\Gamma^2 + \Gamma\delta\Phi + \Phi\delta\Gamma.$$

Here the letter Φ is used to denote $\{\Phi_{20}, \Phi_{21}, \Phi_{11}\}$. The first term on the right hand side of the previous equation, $\delta^2\Phi_{21}$, can be controlled by

$$\int_0^u \left(\int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/\Phi_{11}||\nabla\!\!\!/^2\Phi_{21}| \right)^{1/2} \mathrm{d}u' \leq \int_0^u \left(\int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/\Phi_{11}|^2 \right)^{1/4} \left(\int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/^2\Phi_{21}|^2 \right)^{1/4} \mathrm{d}u'$$
$$\leq \sup_{u,v} ||\nabla\!\!\!/\Phi_{11}||^{1/2}_{L^2(\mathcal{S}_{u,v})} ||\nabla\!\!\!/^2\Phi_{21}||^{1/2}_{L^2(\mathcal{N}'_v(0,u))} \varepsilon^{3/4}$$
$$\leq C(\Delta_{\Phi_\star}, \Delta_{\Phi})\varepsilon^{3/4}.$$

In the case of the terms

$$\Sigma_2\bar{\phi}_2(\bar{\pi}-\tau)+\bar{\phi}_2\delta\Sigma_2+\Sigma_2\delta\bar{\phi}_2+\Phi_{22}\bar{\rho}(\bar{\pi}-\tau)+\bar{\rho}\delta\Phi_{22}+\Phi_{22}\delta\bar{\rho},$$

the use of the estimates of the curvature of the light cone (rather than on the sphere) gives a contribution with the same power of ε. Furthermore, the terms

$$\Sigma_4\bar{\phi}_3(\tau-\bar{\pi}), \qquad \text{and} \qquad \Sigma_4\delta\bar{\phi}_3$$

contribute with a power $\varepsilon^{5/4}$ since $||\Sigma_4||_{L^\infty(\mathcal{S}_{u,v})}$ is controlled by ε in Proposition 20. For the term $\bar{\phi}_3\delta\Sigma_4$ we have that

$$\begin{aligned}
&\int_0^u\left(\int_{\mathcal{S}_{u',v}}|\nabla\!\!\!\!/\,\Phi_{11}||\bar{\phi}_3\nabla\!\!\!\!/\,\Sigma_4|\right)^{1/2}\mathrm{d}u' \leq \sup_{u,v}||\nabla\!\!\!\!/\,\Phi_{11}||^{1/2}_{L^2(\mathcal{S}_{u,v})}||\nabla\!\!\!\!/\,\Sigma_4||^{1/2}_{L^2(\mathcal{S}_{u,v})}\int_0^u||\phi_3||^{1/2}_{L^\infty(\mathcal{S}_{u',v})}\mathrm{d}u'\\
&\leq C(\Delta_{e_\star})\sup_{u,v}||\nabla\!\!\!\!/\,\Phi_{11}||^{1/2}_{L^2(\mathcal{S}_{u,v})}||\nabla\!\!\!\!/\,\Sigma_4||^{1/2}_{L^4(\mathcal{S}_{u,v})}\sum_{i=0}^{2}||\nabla\!\!\!\!/^{\,i}\phi_3||^{1/2}_{L^2(\mathcal{N}'_v(0,u))}\varepsilon^{3/4}\\
&\leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Sigma_\star},\Delta_{\Phi_\star},\Delta_\phi)\varepsilon^{5/4}.
\end{aligned}$$

Here we have used the Sobolev inequality and Proposition 20. Next, the term $\Phi\Gamma^2$ gives us

$$\begin{aligned}
\int_0^u\left(\int_{\mathcal{S}_{u',v}}|\nabla\!\!\!\!/\,\Phi_{11}||\Phi\Gamma^2|\right)^{1/2}\mathrm{d}u' &\leq \sum_{i=0}^{2}C(\Delta_{e_\star})\sup_{u,v}||\Gamma||_{L^\infty(\mathcal{S}_{u,v})}||\nabla\!\!\!\!/\,\Phi_{11}||^{1/2}_{L^2(\mathcal{S}_{u,v})}||\nabla\!\!\!\!/^{\,i}\Phi||^{1/2}_{L^2(\mathcal{S}_{u,v})}\varepsilon^{3/4}\\
&\leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Phi_\star})\varepsilon^{3/4}.
\end{aligned}$$

Terms $\Gamma\delta\Phi$ and $\Phi\delta\Gamma$ give a similar contribution. Putting everything together we find that

$$||\nabla\!\!\!\!/\,\Phi_{11}||_{L^2(\mathcal{S}_{u,v})}\leq 2\Delta_{\Phi_\star}+C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Phi_\star})\varepsilon^{3/4}+C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Sigma_\star},\Delta_{\Phi_\star},\Delta_\phi)\varepsilon^{5/4},$$

so that it is possible to choose a suitably small $\varepsilon_\star$ to improve the bootstrap assumption.

Estimates for $||\nabla\!\!\!\!/^{\,2}\{\Phi_{00},\Phi_{01},\Phi_{02},\Phi_{11},\Phi_{12}\}||_{L^2(\mathcal{S}_{u,v})}$. We present the analysis of $\nabla\!\!\!\!/^{\,2}\Phi_{11}$

as an example. The relevant short direction equation is

$$\begin{aligned}\Delta\delta^2\Phi_{11} &= \delta^3\Phi_{21} + \Phi\delta^2\Gamma + \Gamma\delta^2\Phi + \delta\Phi\delta\Gamma + \Phi\Gamma\delta\Gamma + \Gamma^2\delta\Phi \\ &\quad + \Phi\Gamma^3 + \Sigma\delta\phi + \phi\delta^2\Sigma + \delta\Sigma\delta\phi + \Sigma\Gamma\delta\phi + \phi\Gamma\delta\Sigma + \Sigma\phi\Gamma^2.\end{aligned}$$

Then, making use of the short direction Grönwall-type estimate one obtains

$$\begin{aligned}||\nabla\!\!\!/^2\Phi_{11}||_{L^2(\mathcal{S}_{u,v})} &\leq 2\left(||\nabla\!\!\!/^2\Phi_{11}||_{L^2(\mathcal{S}_{0,v})} + C(\Delta_{e_\star},\Delta_{\Gamma_\star})\int_0^u\left(\int_{\mathcal{S}_{u',v}}\Delta\left\langle\nabla\!\!\!/^2\Phi_{11},\nabla\!\!\!/^2\Phi_{11}\right\rangle_\sigma\right)^{1/2}\mathrm{d}u'\right) \\ &\leq 2\Delta_{\Phi_\star} + C(\Delta_{e_\star},\Delta_{\Gamma_\star})\int_0^u\left(\int_{\mathcal{S}_{u',v}}|\nabla\!\!\!/^2\Phi_{11}|(|\Delta T_1| + |\Delta T_2|)\right)^{1/2}\mathrm{d}u',\end{aligned}$$

where

$$T_1 \equiv \bar{\delta}\bar{\delta}\Phi_{11} + (\bar{\beta}-\alpha)\bar{\delta}\Phi_{11}, \qquad T_2 \equiv \bar{\delta}\delta\Phi_{11} + (\alpha-\bar{\beta})\delta\Phi_{11}.$$

Since Φ contains only the components $\{\Phi_{11},\Phi_{20},\Phi_{21},\Phi_{22}\}$, we can analyse terms which contain Φ in a similar way. Namely, we make use of the Hölder inequality to separate the product terms, and then we make use of the Sobolev embedding theorem. When we encounter the terms $\nabla\!\!\!/^i\Phi_{22}$ and $\nabla\!\!\!/^3\Phi_{21}$, we can make use of the estimate on the light cone. Finally, a quick inspection of the remaining terms reveals that only those related to Σ_2 contribute to the integration. For example, the term $\Sigma_2\delta\phi$ gives

$$\begin{aligned}\int_0^u\left(\int_{\mathcal{S}_{u',v}}|\nabla\!\!\!/^2\Phi_{11}||\Sigma_2\delta\phi|\right)^{1/2}\mathrm{d}u' &\leq \int_0^u ||\nabla\!\!\!/^2\Phi_{11}||^{1/2}_{L^2(\mathcal{S}_{u',v})}||\Sigma_2||^{1/2}_{L^\infty(\mathcal{S}_{u',v})}||\nabla\!\!\!/\phi||^{1/2}_{L^2(\mathcal{S}_{u',v})}\mathrm{d}u' \\ &\leq \sup_{u,v}||\nabla\!\!\!/^2\Phi_{11}||^{1/2}_{L^2(\mathcal{S}_{u,v})}||\Sigma_2||^{1/2}_{L^\infty(\mathcal{S}_{u,v})}||\nabla\!\!\!/\phi||^{1/2}_{L^2(\mathcal{N}'_v(0,u))}\varepsilon^{3/4} \\ &\leq C(\Delta_{\Sigma_\star},\Delta_{\Phi_\star},\Delta_\phi)\varepsilon^{3/4}.\end{aligned}$$

Similarly, the Hölder and the Sobolev inequalities allow us to analyse other terms which also controlled by ε. Putting everything together one finds that

$$\sup_{u,v} ||\nabla\!\!\!/^2\{\Phi_{00}, \Phi_{01}, \Phi_{02}, \Phi_{11}, \Phi_{12}\}||_{L^2(\mathcal{S}_{u,v})} \leq 3\Delta_{\Phi_\star}.$$

Concluding the argument. From the estimates obtained in the previous paragraphs one concludes that

$$\sup_{u,v} ||\nabla\!\!\!/^i\{\Phi_{00}, \Phi_{01}, \Phi_{02}, \Phi_{11}, \Phi_{12}\}||_{L^2(\mathcal{S}_{u,v})} \leq 3\Delta_{\Phi_\star}, \qquad i = 0, \ldots, 2.$$

Hence, we have improved the starting bootstrap assumption. □

Using a similar method, we can obtain the following result:

Proposition 24. *Assume that we are given a solution to the vacuum CEFEs in Stewart's gauge satisfying the same assumptions of Proposition 22. Then there exists*

$$\varepsilon_\star = \varepsilon_\star(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_{\Phi_\star}, \Delta_\Phi, \Delta_\phi, \sup_{u,v} ||\nabla\!\!\!/^3 \tau||_{L^2(\mathcal{S}_{u,v})})$$

such that for $\varepsilon \leq \varepsilon_\star$, we have

$$\Delta_\phi(\mathcal{S}) < 3\Delta_{\phi_\star}.$$

In order to estimate the curvature, we need $L^2(\mathcal{S}_{u,v})$-estimates of the connection coefficients and the derivatives of the conformal factor up to third order. These estimates can be obtained, except for ρ and σ, by a method similar to the one used in the previous proof. For these coefficients, instead of considering their n-direction equations, we make use of their long direction equations and the Codazzi equation to obtain the required estimates.

Proposition 25 (***further control on the L^2-norm of the connection coefficients***)**.** *Assume again that we have a solution of the vacuum CEFEs in Stewart's gauge in a region $\mathcal{D}^t_{u,v_\bullet}$ with*

$$\sup_{u,v} ||\{\mu, \lambda, \alpha, \beta, \epsilon, \rho, \sigma, \tau, \chi, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4\}||_{L^\infty(\mathcal{S}_{u,v})} \leq \infty,$$

$$\sup_{u,v} ||\nabla\!\!\!/\{\mu, \lambda, \alpha, \beta, \epsilon, \rho, \sigma, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4\}||_{L^4(\mathcal{S}_{u,v})} \leq \infty,$$

$$\sup_{u,v} ||\nabla\!\!\!/^2\{\mu,\lambda,\alpha,\beta,\epsilon,\rho,\sigma,\tau,\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4\}||_{L^2(\mathcal{S}_{u,v})} \leq \infty,$$
$$\Delta_\Phi(\mathcal{S}) < \infty, \qquad \Delta_\Phi < \infty, \qquad \Delta_\phi(\mathcal{S}) < \infty, \qquad \Delta_\phi < \infty$$

for some positive $\Delta_{\Gamma,\Sigma}$ and furthermore that

$$\sup_{u,v} ||\nabla\!\!\!/^3\{\mu,\lambda,\alpha,\beta,\epsilon,\tau,\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4\}||_{L^2(\mathcal{S}_{u,v})} < \infty$$

on $\mathcal{D}^t_{u,v_\bullet}$. Then there exists $\varepsilon_\star = \varepsilon_\star(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Sigma_\star},\Delta_{\Phi_\star},\Delta_\Phi,\Delta_\phi)$ such that for $\varepsilon \leq \varepsilon_\star$, we have

$$\begin{aligned}
&\sup_{u,v} ||\nabla\!\!\!/^3\{\mu,\lambda,\alpha,\beta,\epsilon\}||_{L^2(\mathcal{S}_{u,v})} \leq 3\Delta_{\Gamma_\star},\\
&\sup_{u,v} ||\nabla\!\!\!/^3\{\rho,\sigma\}||_{L^2(\mathcal{S}_{u,v})} \leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Phi_\star},\Delta_\Phi),\\
&\sup_{u,v} ||\nabla\!\!\!/^3\{\tau,\chi\}||_{L^2(\mathcal{S}_{u,v})} \leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Phi_\star},\Delta_\Phi),\\
&\sup_{u,v} ||\nabla\!\!\!/^3\Sigma_2||_{L^2(\mathcal{S}_{u,v})} \leq 3\Delta_{\Sigma_\star},\\
&\sup_{u,v} ||\nabla\!\!\!/^3\{\Sigma_1,\Sigma_3,\Sigma_4\}||_{L^2(\mathcal{S}_{u,v})} \leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Sigma_\star},\Delta_{\Phi_\star})\varepsilon.
\end{aligned}$$

Proof.

Bootstrap assumption. We make the following bootstrap assumption to start the proof:

$$\begin{aligned}
&\sup_{u,v} ||\nabla\!\!\!/^3\{\mu,\lambda,\alpha,\beta,\epsilon\}||_{L^2(\mathcal{S}_{u,v})} \leq 4\Delta_{\Gamma_\star},\\
&\sup_{u,v} ||\nabla\!\!\!/^3\tau||_{L^2(\mathcal{S}_{u,v})} \leq \Delta_\tau,\\
&\sup_{u,v} ||\nabla\!\!\!/^3\{\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4\}||_{L^2(\mathcal{S}_{u,v})} \leq 4\Delta_{\Sigma_\star},
\end{aligned}$$

where Δ_τ is a constant whose value will be fixed later.

Estimates for ρ and σ. We first estimate ρ and σ using the long direction equations (3m) and (3f) as we want to avoid the higher derivatives on sphere in the short direction equations. From the full expression of $||\nabla\!\!\!/^3\rho||_{L^2(\mathcal{S}_{u,v})}$ (see Appendix C in Paper I), we will analyse four typical terms namely, $\delta^3\rho$, $\xi\delta^2\rho$, $\delta\xi\delta\rho$ and $\xi^2\delta\rho$. For the term $\delta^3\rho$, we have

$$D\delta^3\rho = \Gamma^5 + \Gamma^3\delta\Gamma + \Gamma(\delta\Gamma)^2 + \Gamma^2\delta^2\Gamma + \delta\Gamma\delta^2\Gamma + \rho\delta^3(\epsilon+\bar{\epsilon})$$
$$+ (4\epsilon - 2\bar{\epsilon} + 5\rho)\delta^3\rho + \sigma\delta^3\bar{\sigma} + \bar{\sigma}\delta^3\sigma + \sigma\delta^2\bar{\delta}\rho + \delta^3\Phi_{00}.$$

The term $\delta\Gamma\delta^2\Gamma$ can be estimated as

$$\begin{aligned}||\delta\Gamma\delta^2\Gamma||_{L^2(\mathcal{S}_{u,v})} &\leq ||\nabla\!\!\!/\Gamma||_{L^4(\mathcal{S}_{u,v})}||\nabla\!\!\!/^2\Gamma||_{L^4(\mathcal{S}_{u,v})} \\ &\leq C(\Delta_{e_\star})||\nabla\!\!\!/\Gamma||_{L^4(\mathcal{S}_{u,v})}\left(||\nabla\!\!\!/^2\Gamma||_{L^2(\mathcal{S}_{u,v})} + ||\nabla\!\!\!/^3\Gamma||_{L^2(\mathcal{S}_{u,v})}\right),\end{aligned}$$

where Γ contains ϵ, ρ and σ. Then, making use of the norm of Φ_{00} on the long light cone, we find that

$$\int_0^v ||\delta^3\Phi_{00}||_{L^2(\mathcal{S}_{u,v'})}\mathrm{d}v' \leq \left(\int_0^v\int_{\mathcal{S}_{u,v'}}|\delta^3\Phi_{00}|^2\mathrm{d}v'\right)^{1/2}\left(\int_0^v 1\mathrm{d}v'\right)^{1/2} \leq C(I)||\nabla\!\!\!/^3\Phi_{00}||_{L^2(\mathcal{N}_u(0,v))}.$$

Hence, the long direction of inequality in Proposition 3 yields

$$||\delta^3\rho||_{L^2(\mathcal{S}_{u,v})} \leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Phi_\star},\Delta_\Phi) + C(I,\Delta_{\Gamma_\star})\int_0^v\left(||\nabla\!\!\!/^3\rho||_{L^2(\mathcal{S}_{u,v'})} + ||\nabla\!\!\!/^3\sigma||_{L^2(\mathcal{S}_{u,v'})}\right)\mathrm{d}v'.$$

For the term $\varpi\delta^2\rho$, we readily find that

$$||\varpi\delta^2\rho||_{L^2(\mathcal{S}_{u,v})} \leq ||\varpi||_{L^\infty(\mathcal{S}_{u,v})}||\nabla\!\!\!/^2\rho||_{L^2(\mathcal{S}_{u,v})} \leq C(\Delta_{\Gamma_\star}).$$

Similar estimates can be found for $\delta\varpi\delta\rho$ and $\varpi^2\delta\rho$. Hence, we conclude that

$$||\nabla\!\!\!/^3\rho||_{L^2(\mathcal{S}_{u,v})} \leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Phi_\star},\Delta_\Phi) + C(I,\Delta_{\Gamma_\star})\int_0^v\left(||\nabla\!\!\!/^3\rho||_{L^2(\mathcal{S}_{u,v'})} + ||\nabla\!\!\!/^3\sigma||_{L^2(\mathcal{S}_{u,v'})}\right)\mathrm{d}v'.$$

From here, using Grönwall's inequality one finds that

$$||\nabla\!\!\!/^3\rho||_{L^2(\mathcal{S}_{u,v})} \leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Phi_\star},\Delta_\Phi) + C(I,\Delta_{\Gamma_\star})\int_0^v ||\nabla\!\!\!/^3\sigma||_{L^2(\mathcal{S}_{u,v'})}\mathrm{d}v'.$$

Similarly, one can have the estimate for σ

$$||\nabla\!\!\!/^3\sigma||_{L^2(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Phi_\star}, \Delta_\Phi) + C(I, \Delta_{\Gamma_\star}) \int_0^v ||\nabla\!\!\!/^3\rho||_{L^2(\mathcal{S}_{u,v'})} \mathrm{d}v'.$$

Combine these two inequality above one have

$$\begin{aligned} ||\nabla\!\!\!/^3\sigma||_{L^2(\mathcal{S}_{u,v})} + ||\nabla\!\!\!/^3\rho||_{L^2(\mathcal{S}_{u,v})} &\leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Phi_\star}, \Delta_\Phi) \\ + C(I, \Delta_{\Gamma_\star}) \int_0^v &\left(||\nabla\!\!\!/^3\rho||_{L^2(\mathcal{S}_{u,v'})} + ||\nabla\!\!\!/^3\sigma||_{L^2(\mathcal{S}_{u,v})} \right) \mathrm{d}v'. \end{aligned}$$

Then the Grönwall's inequality gives us

$$||\nabla\!\!\!/^3\sigma||_{L^2(\mathcal{S}_{u,v})} + ||\nabla\!\!\!/^3\rho||_{L^2(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Phi_\star}, \Delta_\Phi).$$

Finally we obtain

$$\begin{aligned} ||\nabla\!\!\!/^3\rho||_{L^2(\mathcal{S}_{u,v})} &\leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Phi_\star}, \Delta_\Phi), \\ ||\nabla\!\!\!/^3\sigma||_{L^2(\mathcal{S}_{u,v})} &\leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Phi_\star}, \Delta_\Phi). \end{aligned}$$

Estimates for τ and χ. The Δ-equation for $\nabla\!\!\!/^3\tau$ can be obtained from the structure equation (3b) and the commutator relationship. More precisely, one has that

$$\begin{aligned} D\delta^3\tau = \delta^3(\Xi\phi_1) + \delta^3\Phi_{01} + \Gamma\delta^3\Gamma_1 + \Gamma\delta^3\tau + \Gamma\delta^2\Psi_1 \\ + \delta\Gamma\delta^2\Gamma + \Gamma^2\delta^2\Gamma + \Gamma^3\delta\Gamma + \Gamma(\delta\Gamma)^2, \end{aligned}$$

where Γ_1 contains $\epsilon, \alpha, \beta, \rho$ and σ. Then, using the bootstrap assumption and the definition of Δ_Ψ, we obtain

$$\begin{aligned} ||\nabla\!\!\!/^3\tau||_{L^2(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Phi_\star}, \Delta_\Phi) + C(I, \Delta_{e_\star}, \Delta_{\Sigma_\star}, \Delta_{\Gamma_\star}, \Delta_{\Phi_\star}, \Delta_\phi)\varepsilon \\ + C(I, \Delta_{\Gamma_\star}) \int_0^v ||\nabla\!\!\!/^3\tau||_{L^2(\mathcal{S}_{u,v'})} \mathrm{d}v', \end{aligned}$$

so that using Grönwall's inequality we conclude that

$$||\nabla\!\!\!/^3\tau||_{L^2(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Phi_\star}, \Delta_\Phi) + C(\Delta_{e_\star}, \Delta_{\Sigma_\star}, \Delta_{\Gamma_\star}, \Delta_{\Phi_\star}, \Delta_\phi)\varepsilon.$$

We can then choose the constant Δ_τ larger than the right side above so as to improve the bootstrap assumption. The estimate of χ is similar:

$$||\nabla\!\!\!/^3\chi||_{L^2(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Phi_\star}, \Delta_\Phi) + C(\Delta_{e_\star}, \Delta_{\Sigma_\star}, \Delta_{\Gamma_\star}, \Delta_{\Phi_\star}, \Delta_\phi)\varepsilon.$$

Estimates for the the remaining spin connection coefficients. To obtain the estimates for

$$||\nabla\!\!\!/^3\{\mu, \lambda, \alpha, \beta, \epsilon\}||_{L^2(\mathcal{S}_{u,v})},$$

we make use of their short direction equations. Since the proof are similar, we only show the details of ϵ as a representative example. In this case the relevant equation is

$$\Delta\delta^3\epsilon = -\delta^3(\Xi\phi_2 + \Phi_{12}) + \Gamma\delta^3\Gamma_1 + \Gamma\delta^3\epsilon + \delta\Gamma\delta^2\Gamma + \Gamma^2\delta^2\Gamma + \Gamma^3\delta\Gamma + \Gamma(\delta\Gamma)^2 + \Gamma^5,$$

where Γ_1 does not contain ϵ. We can then make use of the short inequality in Proposition 3 and obtain that

$$||\nabla\!\!\!/^3\epsilon||_{L^2(\mathcal{S}_{u,v})} \leq 2\Delta_{\Gamma_\star} + C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Phi_\star}, \Delta_\Phi)\varepsilon^{3/4} + o(\varepsilon^{3/4}).$$

Choosing the integral range sufficiently small we conclude that

$$||\nabla\!\!\!/^3\epsilon||_{L^2(\mathcal{S}_{u,v})} \leq 3\Delta_{\Gamma_\star}.$$

The estimates of $||\nabla\!\!\!/^3\{\mu, \lambda, \alpha, \beta\}||_{L^2(\mathcal{S}_{u,v})}$ are are similar. Hence, we have improved the bootstrap assumption for the connection coefficients.

Estimates for $\nabla\!\!\!/^3\Sigma_2$. The short direction equation for $\delta^3\Sigma_2$ can be analysed by the same method. Starting from

$$\begin{aligned}\Delta\delta^3\Sigma_2 = {} & \Gamma^3\delta\Sigma_2 + \Gamma\delta\Sigma_2\delta\Gamma + \Gamma^2\delta^2\Sigma_2 + \delta\Gamma\delta^2\Sigma_2 + \delta\Sigma_2\delta^2\Gamma + \Gamma\delta^3\Sigma_2 \\ & + \sum_{i_1+\ldots+i_4=3} \delta^{i_1}\Xi\delta^{i_2}\Gamma^{i_3}\delta^{i_4}\Phi_{22},\end{aligned}$$

where Γ contains τ and it is observed that the terms in the summation will contribute higher order of ε in the integration. Then applying Proposition 23 we find that

$$||\nabla\!\!\!/^3\Sigma_2||_{L^2(\mathcal{S}_{u,v})} \leq 2\Delta_{\Sigma_\star} + C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_{\Phi_\star})\varepsilon + o(\varepsilon),$$

where the term $o(\varepsilon)$ arises from the summation.

Estimates for $\nabla\!\!\!/^3\Sigma_1$. In this case one has that the Δ-equation for $\Delta\delta^3\Sigma_1$ is of the form

$$\begin{aligned}\Delta\delta^3\Sigma_1 = {} & \Sigma_2\Phi\Gamma^2 + \Phi\Gamma\nabla\!\!\!/\Sigma_2 + \nabla\!\!\!/\Sigma_2\nabla\!\!\!/\Phi + \Sigma_2\Phi\nabla\!\!\!/\Gamma + \Phi\nabla\!\!\!/^2\Sigma_2 \\ & + s\Gamma^3 + s\Gamma\delta\Gamma + s\delta^2\Gamma + \sum_{i_1+\ldots+i_4=3} \delta^{i_1}\Xi\delta^{i_2}\Gamma^{i_3}\delta^{i_4}\Phi,\end{aligned}$$

here the first line on the right hand side contains the leading order contribution, and Φ does not contain Φ_{22}. From this equation one readily obtains that

$$||\nabla\!\!\!/^3\Sigma_1||_{L^2(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_{\Phi_\star})\varepsilon + o(\varepsilon).$$

Estimates for $\nabla\!\!\!/^3\Sigma_{3,4}$. In this case the term contributing to the leading order of the estimate of $||\nabla\!\!\!/^3\Sigma_3||_{L^2(\mathcal{S}_{u,v})}$ is

$$\begin{aligned}\Delta\delta^3\Sigma_3 = {} & \Sigma_2\Gamma^4 + \Gamma\nabla\!\!\!/\Sigma_2 + \Sigma_2\Gamma^2\nabla\!\!\!/\Gamma + \Sigma_2(\nabla\!\!\!/\Gamma)^2 + \Gamma^2\nabla\!\!\!/^2\Sigma_2 + \Sigma_2\Gamma\nabla\!\!\!/^2\Gamma + \Sigma_2\nabla\!\!\!/^3\Gamma + \Gamma\nabla\!\!\!/^3\Sigma_2 \\ & + \sum_{i_1+\ldots+i_4=3} \delta^{i_1}\Xi\delta^{i_2}\Gamma^{i_3}\delta^{i_4}\Phi,\end{aligned}$$

again here the first line of the right hand side offers the leading contribution, and gives

$$||\nabla\!\!\!/^3\Sigma_3||_{L^2(\mathcal{S}_{u,v})} \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_{\Phi_\star})\varepsilon + o(\varepsilon).$$

Concluding the argument. From the analysis above, it follows that we can choose

$$\varepsilon_\star = \varepsilon_\star(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_{\phi_\star}, \Delta_{\Phi_\star}, \Delta_\phi, \Delta_\Phi),$$

sufficiently small so that

$$
\begin{aligned}
&\sup_{u,v} ||\nabla\!\!\!/^3\{\mu,\lambda,\alpha,\beta,\epsilon\}||_{L^2(\mathcal{S}_{u,v})} \leq 3\Delta_{\Gamma_\star},\\
&\sup_{u,v} ||\nabla\!\!\!/^3\{\rho,\sigma\}||_{L^2(\mathcal{S}_{u,v})} \leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Phi_\star},\Delta_{\Phi}),\\
&\sup_{u,v} ||\nabla\!\!\!/^3\{\tau,\chi\}||_{L^2(\mathcal{S}_{u,v})} \leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Phi_\star},\Delta_{\Phi}),\\
&\sup_{u,v} ||\nabla\!\!\!/^3\Sigma_2||_{L^2(\mathcal{S}_{u,v})} \leq 3\Delta_{\Sigma_\star},\\
&\sup_{u,v} ||\nabla\!\!\!/^3\{\Sigma_1,\Sigma_3,\Sigma_4\}||_{L^2(\mathcal{S}_{u,v})} \leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Sigma_\star},\Delta_{\Phi_\star})\varepsilon,
\end{aligned}
$$

on $\mathcal{D}^t_{u,v_\bullet}$. □

4.4.4 The energy estimates for the curvature

In this subsection, we show how to obtain the main energy estimates for the components of the Ricci and rescaled Weyl curvature.

4.4.4.1 Analysis of the rescaled Weyl tensor

We begin by introducing some integral identities which follow from using integration by parts in the conformal equations satisfied by the components of the rescaled Weyl tensor, equations (8a)-(8h). The proof of these results follows the same arguments used for the components of the Weyl tensor in Paper I as the (vacuum) Bianchi identities have an identical structure to that of the equations for the rescaled Weyl tensor and are thus omitted.

Proposition 26 (***control of the angular derivatives of the components of the rescaled Weyl tensor***). *Suppose that we are given a solution to the CEFEs in Stewart's gauge and that $\mathcal{D}_{u,v}$ is contained in the existence area. The following L^2 estimates for the components of the rescaled Weyl curvature hold. First,*

$$
\begin{aligned}
&\sum_{i=0,1,2}\int_{\mathcal{N}_u(0,v)}|\phi_i|^2+\sum_{j=1,2,3}\int_{\mathcal{N}'_v(0,u)}Q^{-1}|\phi_j|^2\\
&\qquad\leq \sum_{i=0,1,2}\int_{\mathcal{N}_0(0,v)}|\phi_i|^2+\sum_{j=1,2,3}\int_{\mathcal{N}'_0(0,u)}Q^{-1}|\phi_j|^2+\int_{\mathcal{D}_{u,v}}\phi_H\phi\Gamma,
\end{aligned}
$$

then

$$\begin{aligned}&\sum_{i=0,1,2}\int_{\mathcal{N}_u(0,v)}|\nabla\phi_i|^2+\sum_{j=1,2,3}\int_{\mathcal{N}'_v(0,u)}Q^{-1}|\nabla\phi_j|^2\\&\leq\sum_{i=0,1,2}\int_{\mathcal{N}_0(0,v)}|\nabla\phi_i|^2+\sum_{j=1,2,3}\int_{\mathcal{N}'_0(0,u)}Q^{-1}|\nabla\phi_j|^2\\&+\int_{\mathcal{D}_{u,v}}|\nabla\phi_H|(\phi\Gamma^2+\Gamma|\nabla\phi|+\phi|\nabla\Gamma|),\end{aligned}$$

next

$$\begin{aligned}&\sum_{i=0,1,2}\int_{\mathcal{N}_u(0,v)}|\nabla^2\phi_i|^2+\sum_{j=1,2,3}\int_{\mathcal{N}'_v(0,u)}Q^{-1}|\nabla^2\phi_j|^2\\&\leq\sum_{i=0,1,2}\int_{\mathcal{N}_0(0,v)}|\nabla^2\phi_i|^2+\sum_{j=1,2,3}\int_{\mathcal{N}'_0(0,u)}Q^{-1}|\nabla^2\phi_j|^2\\&\qquad+\int_{\mathcal{D}_{u,v}}|\nabla^2\phi_H|(\Gamma|\nabla^2\phi|+\phi|\nabla^2\Gamma|+|\nabla\phi||\nabla\Gamma|+\Gamma^2|\nabla\phi|+\phi\Gamma|\nabla\Gamma|+\Gamma^3\phi),\end{aligned}$$

and finally

$$\begin{aligned}&\sum_{i=0,1,2}\int_{\mathcal{N}_u(0,v)}|\nabla^3\phi_i|^2+\sum_{j=1,2,3}\int_{\mathcal{N}'_v(0,u)}Q^{-1}|\nabla^3\phi_j|^2\\&\leq\sum_{i=0,1,2}\int_{\mathcal{N}_0(0,v)}|\nabla^3\phi_i|^2+\sum_{j=1,2,3}\int_{\mathcal{N}'_0(0,u)}Q^{-1}|\nabla^3\phi_j|^2\\&\quad+\int_{\mathcal{D}_{u,v}}|\nabla^3\phi_H|\big(\Gamma|\nabla^3\phi|+\phi|\nabla^3\Gamma|+|\nabla\Gamma||\nabla^2\phi|+|\nabla\phi||\nabla^2\Gamma|+\Gamma^2|\nabla^2\phi|+\Gamma\phi|\nabla^2\Gamma|\\&\qquad+\Gamma|\nabla\Gamma||\nabla\phi|+\phi|\nabla\Gamma|^2+\Gamma^3|\nabla\phi|+\phi\Gamma^2|\nabla\Gamma|+\Gamma^4\phi\big),\end{aligned}$$

where Γ *stands for arbitrary connection coefficients from the collection* $\{\mu,\lambda,\alpha,\beta,\epsilon,\rho,\sigma,\tau\}$.

To summarise, the previous results can be given a more general formulation:

Proposition 27. *Suppose that we are given a solution to the CEFEs in Stewart's gauge and that* $\mathcal{D}_{u,v}$ *is contained in the existence area. Then we have that*

$$\sum_{i=0,1,2}\int_{\mathcal{N}_u(0,v)}|\nabla^m\phi_i|^2+\sum_{j=1,2,3}\int_{\mathcal{N}'_v(0,u)}Q^{-1}|\nabla^m\phi_j|^2\leq\sum_{i=0,1,2}\int_{\mathcal{N}_0(0,v)}|\nabla^m\phi_i|^2$$

$$+\sum_{j=1,2,3}\int_{\mathcal{N}'_0(0,u)} Q^{-1}|\nabla\!\!\!/^m\phi_j|^2+\int_{\mathcal{D}_{u,v}}|\nabla\!\!\!/^m\phi_H|\sum_{i_1+i_2+i_3+i_4=m}|\nabla\!\!\!/^{i_1}\Gamma^{i_2}||\nabla\!\!\!/^{i_3}\Gamma||\nabla\!\!\!/^{i_4}\phi|,$$

where ϕ contains ϕ_k, $k=0,...,4$, ϕ_H contains ϕ_k, $k=0,...,3$.

In addition, we have the following proposition:

Proposition 28 (***control of the angular derivatives of the "bad" components of the rescaled Weyl tensor***)**.** *Suppose that we are given a solution to the CEFEs in Stewart's gauge and that $\mathcal{D}_{u,v}$ is contained in the existence area. Then we have that*

$$\int_{\mathcal{N}_u(0,v)}|\nabla\!\!\!/^m\phi_3|^2+\int_{\mathcal{N}'_v(0,u)} Q^{-1}|\nabla\!\!\!/^m\phi_4|^2\leq\int_{\mathcal{N}_0(0,v)}|\nabla\!\!\!/^m\phi_3|^2+\int_{\mathcal{N}'_0(0,u)} Q^{-1}|\nabla\!\!\!/^m\phi_4|^2$$
$$+\int_{\mathcal{D}_{u,v}}|\nabla\!\!\!/^m\phi_4|\sum_{i_1+i_2+i_3+i_4=m}|\nabla\!\!\!/^{i_1}\Gamma^{i_2}||\nabla\!\!\!/^{i_3}(\rho+\epsilon)||\nabla\!\!\!/^{i_4}\phi_4|$$
$$+\int_{\mathcal{D}_{u,v}}|\nabla\!\!\!/^m\phi_3|\sum_{i_1+i_2+i_3+i_4=m}|\nabla\!\!\!/^{i_1}\Gamma^{i_2}||\nabla\!\!\!/^{i_3}\Gamma||\nabla\!\!\!/^{i_4}\phi|$$
$$+\int_{\mathcal{D}_{u,v}}|\nabla\!\!\!/^m\phi_4|\sum_{i_1+i_2+i_3+i_4=m}|\nabla\!\!\!/^{i_1}\Gamma^{i_2}||\nabla\!\!\!/^{i_3}\Gamma||\nabla\!\!\!/^{i_4}\phi'_H|,$$

where ϕ contains ϕ_3 and ϕ_4, ϕ'_H contains ϕ_2 and ϕ_3.

4.4.4.2 Analysis of the Ricci curvature

In order to estimate the L^2-norms of the components of the Ricci tensor we need inequalities analogous to the ones used for the rescaled Weyl tensor. In order to obtain these, we first we need to regroup the conformal equations for the Ricci tensor shown in Appendix 6.2. More precisely, we pair the components Φ_{01} and Φ_{11} by analysing equations (7b) and (7h); pair the components Φ_{02} and Φ_{12} by analysing (7c) and (7g)+(7l); pair the components Φ_{11} and Φ_{12} by analysing (7d) and (7g); pair the components Φ_{01} and Φ_{02} by analysing (7b)+(7l) and (7h). Making use of this strategy one obtains the following:

Proposition 29. *Suppose that we are given a solution to the CEFEs in Stewart's* *gauge and that $\mathcal{D}_{u,v}$ is contained in the existence area. Then we have that*

$$\sum_{\Phi_i\in\Phi_L}\int_{\mathcal{N}_u(0,v)}|\Phi_i|^2+\sum_{\Phi_j\in\Phi_S}\int_{\mathcal{N}'_v(0,u)}Q^{-1}|\Phi_j|^2\leq\sum_{\Phi_i\in\Phi_L}\int_{\mathcal{N}_0(0,v)}|\Phi_i|^2+\sum_{\Phi_j\in\Phi_S}\int_{\mathcal{N}'_0(0,u)}Q^{-1}|\Phi_j|^2$$
$$+\int_{\mathcal{D}_{u,v}}\Phi_H\Gamma\Phi+\int_{\mathcal{D}_{u,v}}\Phi_H\Sigma\phi,$$

where $\Phi_L =\equiv \{\Phi_{00},\Phi_{01},\Phi_{02},\Phi_{11}\}$, $\Phi_S\equiv\{\Phi_{01},\Phi_{02},\Phi_{11},\Phi_{12}\}$, $\Phi\equiv\{\Phi_{00},\Phi_{01},\Phi_{02},\Phi_{11},\Phi_{12},\Phi_{22}\}$ *and* $\Phi_H\equiv\{\Phi_{00},\Phi_{01},\Phi_{02},\Phi_{11},\Phi_{12}\}$.

Proof. For simplicity, we demonstrate the argument with the conformal equations (7a) and (7j) written in the form

$$\Delta\Phi_{00}=\delta\Phi_{10}+\Gamma\Phi+\Sigma\phi,$$
$$D\Phi_{01}=\delta\Phi_{00}+\Gamma\Phi+\Sigma\phi.$$

Integrating by parts we have that

$$\int_{\mathcal{N}_u(0,v)}|\Phi_{00}|^2+\int_{\mathcal{N}'_v(0,u)}Q^{-1}|\Phi_{01}|^2\leq\int_{\mathcal{N}_0(0,v)}|\Phi_{00}|^2+\int_{\mathcal{N}'_0(0,u)}Q^{-1}|\Phi_{01}|^2$$
$$+\int_{\mathcal{D}_{u,v}}(\Phi_{00},\Phi_{01})\Gamma\Phi+\int_{\mathcal{D}_{u,v}}(\Phi_{00},\Phi_{01})\Sigma\phi.$$

A similar argument applies to the pairs Φ_{01} and Φ_{11}, Φ_{02} and Φ_{12}, Φ_{11} and Φ_{12}, Φ_{01} and Φ_{02}. Putting everything together we obtain the required result. □

Now, applying the angular derivatives to the conformal equations we obtain the following statement:

Proposition 30. *Suppose that we are given a solution to the CEFEs in Stewart's gauge and that $\mathcal{D}_{u,v}$ is contained in the existence area. Then we have first that*

$$\sum_{\Phi_i\in\Phi_L}\int_{\mathcal{N}_u(0,v)}|\nabla\!\!\!/\Phi_i|^2+\sum_{\Phi_j\in\Phi_S}\int_{\mathcal{N}'_v(0,u)}Q^{-1}|\nabla\!\!\!/\Phi_j|^2$$
$$\leq\sum_{\Phi_i\in\Phi_L}\int_{\mathcal{N}_0(0,v)}|\nabla\!\!\!/\Phi_i|^2+\sum_{\Phi_j\in\Phi_S}\int_{\mathcal{N}'_0(0,u)}Q^{-1}|\nabla\!\!\!/\Phi_j|^2$$

$$+\int_{\mathcal{D}_{u,v}} |\nabla\!\!\!/\,\Phi_H|\big(\Phi\Gamma^2 + \Gamma|\nabla\!\!\!/\,\Phi| + \Phi|\nabla\!\!\!/\,\Gamma|\big) + \int_{\mathcal{D}_{u,v}} |\nabla\!\!\!/\,\Phi_H|\big(\Sigma\phi\Gamma + \phi|\nabla\!\!\!/\,\Sigma| + \Sigma|\nabla\!\!\!/\,\phi|\big),$$

and also,

$$\begin{aligned}
&\sum_{\Phi_i\in\Phi_L}\int_{\mathcal{N}_u(0,v)} |\nabla\!\!\!/^2\Phi_i|^2 + \sum_{\Phi_j\in\Phi_S}\int_{\mathcal{N}'_v(0,u)} Q^{-1}|\nabla\!\!\!/^2\Phi_j|^2 \\
&\leq \sum_{\Phi_i\in\Phi_L}\int_{\mathcal{N}_0(0,v)} |\nabla\!\!\!/^2\Phi_i|^2 + \sum_{\Phi_j\in\Phi_S}\int_{\mathcal{N}'_0(0,u)} Q^{-1}|\nabla\!\!\!/^2\Phi_j|^2 \\
&\quad+\int_{\mathcal{D}_{u,v}} |\nabla\!\!\!/^2\Phi_H|\big(\Gamma|\nabla\!\!\!/^2\Phi| + \Phi|\nabla\!\!\!/^2\Gamma| + |\nabla\!\!\!/\,\Phi||\nabla\!\!\!/\,\Gamma| + \Gamma^2|\nabla\!\!\!/\,\Phi| + \Phi\Gamma|\nabla\!\!\!/\,\Gamma| + \Gamma^3\Phi\big) \\
&\quad+\int_{\mathcal{D}_{u,v}} |\nabla\!\!\!/^2\Phi_H|\big(\Sigma\phi\Gamma^2 + \Gamma\phi|\nabla\!\!\!/\,\Sigma| + \Gamma\Sigma|\nabla\!\!\!/\,\phi| + \Sigma\phi|\nabla\!\!\!/\,\Gamma| + |\nabla\!\!\!/\,\phi||\nabla\!\!\!/\,\Sigma| + \phi|\nabla\!\!\!/^2\Sigma| + \Sigma|\nabla\!\!\!/^2\phi|\big),
\end{aligned}$$

and finally,

$$\begin{aligned}
&\sum_{\Phi_i\in\Phi_L}\int_{\mathcal{N}_u(0,v)} |\nabla\!\!\!/^3\Phi_i|^2 + \sum_{\Phi_j\in\Phi_S}\int_{\mathcal{N}'_v(0,u)} Q^{-1}|\nabla\!\!\!/^3\Phi_j|^2 \\
&\leq \sum_{\Phi_i\in\Phi_L}\int_{\mathcal{N}_0(0,v)} |\nabla\!\!\!/^3\Phi_i|^2 + \sum_{\Phi_j\in\Phi_S}\int_{\mathcal{N}'_0(0,u)} Q^{-1}|\nabla\!\!\!/^3\Phi_j|^2 \\
&\quad+\int_{\mathcal{D}_{u,v}} |\nabla\!\!\!/^3\Phi_H|\big(\Gamma|\nabla\!\!\!/^3\Phi| + \Phi|\nabla\!\!\!/^3\Gamma| + |\nabla\!\!\!/\,\Gamma||\nabla\!\!\!/^2\Phi| + |\nabla\!\!\!/\,\Phi||\nabla\!\!\!/^2\Gamma| + \Gamma^2|\nabla\!\!\!/^2\Phi| + \Gamma\Phi|\nabla\!\!\!/^2\Gamma| \\
&\qquad\quad+ \Gamma|\nabla\!\!\!/\,\Gamma||\nabla\!\!\!/\,\Phi| + \Phi|\nabla\!\!\!/\,\Gamma|^2 + \Gamma^3|\nabla\!\!\!/\,\Phi| + \Phi\Gamma^2|\nabla\!\!\!/\,\Gamma| + \Gamma^4\Phi\big) \\
&\quad+\int_{\mathcal{D}_{u,v}} |\nabla\!\!\!/^3\Phi_H|\big(\Sigma|\nabla\!\!\!/^3\phi| + \phi|\nabla\!\!\!/^3\Sigma| + |\nabla\!\!\!/^2\phi||\nabla\!\!\!/\,\Sigma| + |\nabla\!\!\!/^2\Sigma||\nabla\!\!\!/\,\phi| + \Sigma\Gamma|\nabla\!\!\!/^2\phi| + \Sigma\phi|\nabla\!\!\!/^2\Gamma| \\
&\qquad\quad+ \phi\Gamma|\nabla\!\!\!/^2\Sigma| + \Gamma|\nabla\!\!\!/\,\Sigma||\nabla\!\!\!/\,\phi| + \Sigma|\nabla\!\!\!/\,\phi||\nabla\!\!\!/\,\Gamma| + \phi|\nabla\!\!\!/\,\Sigma||\nabla\!\!\!/\,\Gamma| \\
&\qquad\quad+ \phi\Gamma^2|\nabla\!\!\!/\,\Sigma| + \Sigma\Gamma|\nabla\!\!\!/\,\phi| + \Sigma\phi\Gamma|\nabla\!\!\!/\,\Gamma| + \Sigma\phi\Gamma^3\big).
\end{aligned}$$

As before, we can summarise the previous estimates in the following more concise statement:

Proposition 31 (***control of the higher angular derivatives of the components of the Ricci tensor***)**.** *Suppose that we are given a solution to the CEFEs in Stewart's gauge and that $\mathcal{D}_{u,v}$ is contained in the existence area. Then we have* *that*

$$\sum_{\Phi_i\in\Phi_L}\int_{\mathcal{N}_u(0,v)}|\nabla\!\!\!/^m\Phi_i|^2+\sum_{\Phi_j\in\Phi_S}\int_{\mathcal{N}'_v(0,u)}Q^{-1}|\nabla\!\!\!/^m\Phi_j|^2$$
$$\leq\sum_{\Phi_i\in\Phi_L}\int_{\mathcal{N}_0(0,v)}|\nabla\!\!\!/^m\Phi_i|^2+\sum_{\Phi_j\in\Phi_S}\int_{\mathcal{N}'_0(0,u)}Q^{-1}|\nabla\!\!\!/^m\Phi_j|^2$$
$$+\int_{\mathcal{D}_{u,v}}|\nabla\!\!\!/^m\Phi_H|\sum_{i_1+i_2+i_3+i_4=m}(|\nabla\!\!\!/^{i_1}\Gamma^{i_2}||\nabla\!\!\!/^{i_3}\Gamma||\nabla\!\!\!/^{i_4}\Phi|+|\nabla\!\!\!/^{i_1}\Gamma^{i_2}||\nabla\!\!\!/^{i_3}\Sigma||\nabla\!\!\!/^{i_4}\phi|),$$

where $m=0,\ 1,\ 2,\ 3.$

Using equations (7e) and (7f) we can obtain a similar control over the components Φ_{12} and Φ_{22}. More precisely, one has that:

Proposition 32 (***control of the higher angular derivatives of the "bad" components of the Ricci tensor***)**.** *Suppose that we are given a solution to the CEFEs in Stewart's gauge and that $\mathcal{D}_{u,v}$ is contained in the existence area. Then we have that*

$$\int_{\mathcal{N}_u(0,v)}|\nabla\!\!\!/^m\Phi_{12}|^2+\int_{\mathcal{N}'_v(0,u)}Q^{-1}|\nabla\!\!\!/^m\Phi_{22}|^2\leq\int_{\mathcal{N}_0(0,v)}|\nabla\!\!\!/^m\Phi_{12}|^2+\int_{\mathcal{N}'_0(0,u)}Q^{-1}|\nabla\!\!\!/^m\Phi_{22}|^2$$
$$+\int_{\mathcal{D}_{u,v}}|\nabla\!\!\!/^m\Phi_{22}|\sum_{i_1+i_2+i_3+i_4=m}|\nabla\!\!\!/^{i_1}\Gamma'^{i_2}||\nabla\!\!\!/^{i_3}\Gamma'||\nabla\!\!\!/^{i_4}\Phi_{22}|$$
$$+\int_{\mathcal{D}_{u,v}}|\nabla\!\!\!/^m\Phi_{12}|\sum_{i_1+i_2+i_3+i_4=m}|\nabla\!\!\!/^{i_1}\Gamma^{i_2}||\nabla\!\!\!/^{i_3}\Gamma||\nabla\!\!\!/^{i_4}\Phi|$$
$$+\int_{\mathcal{D}_{u,v}}|\nabla\!\!\!/^m\Phi_{22}|\sum_{i_1+i_2+i_3+i_4=m}|\nabla\!\!\!/^{i_1}\Gamma^{i_2}||\nabla\!\!\!/^{i_3}\Gamma||\nabla\!\!\!/^{i_4}\Phi'_H|$$
$$+\int_{\mathcal{D}_{u,v}}\left(|\nabla\!\!\!/^m\Phi_{12}||\nabla\!\!\!/^{i_1}\Gamma^{i_2}||\nabla\!\!\!/^{i_3}\Sigma||\nabla\!\!\!/^{i_4}\phi|+|\nabla\!\!\!/^m\Phi_{22}||\nabla\!\!\!/^{i_1}\Gamma^{i_2}||\nabla\!\!\!/^{i_3}\Sigma||\nabla\!\!\!/^{i_4}\phi'_H|\right),$$

where Γ' does not contain τ and χ, Φ does not contain Φ_{00}, Φ'_H does not contains Φ_{22} and Φ_{00}, ϕ contains ϕ_3 and ϕ_4, ϕ'_H contains ϕ_2 and ϕ_3.

Making use of the previous estimates for the Ricci tensor, we can show their boundedness in the truncated diamonds:

Proposition 33 (***control of the components of the Ricci tensor in terms of the initial data***). *Suppose we are given a solution to the vacuum CEFE's in Stewart's gauge arising from data for the CIVP satisfying*

$$\Delta_{e_\star},\ \Delta_{\Gamma_\star},\ \Delta_{\Sigma_\star},\ \Delta_{\Phi_\star}\ \Delta_{\phi_\star} < \infty,$$

with the solution itself satisfying

$$\sup_{u,v} ||\{\mu, \lambda, \alpha, \beta, \epsilon, \rho, \sigma, \tau, \chi, \Sigma_i\}||_{L^\infty(\mathcal{S}_{u,v})} < \infty\,, \quad \sup_{u,v} ||\nabla\!\!\!\!/\,\{\mu, \lambda, \alpha, \beta, \epsilon, \rho, \sigma, \Sigma_i\}||_{L^4(\mathcal{S}_{u,v})} < \infty\,,$$
$$\sup_{u,v} ||\nabla\!\!\!\!/^2\{\mu, \lambda, \alpha, \beta, \epsilon, \rho, \sigma, \tau, \Sigma_i\}||_{L^2(\mathcal{S}_{u,v})} < \infty\,, \quad \sup_{u,v} ||\nabla\!\!\!\!/^3\{\mu, \lambda, \alpha, \beta, \epsilon, \tau, \Sigma_i\}||_{L^2(\mathcal{S}_{u,v})} < \infty\,,$$
$$\Delta_\Phi(\mathcal{S}) < \infty\,, \quad \Delta_\Phi < \infty\,, \quad \Delta_\phi(\mathcal{S}) < \infty\,, \quad \Delta_\phi < \infty,$$

on some truncated causal diamond $\mathcal{D}^t_{u,v_\bullet}$. *Then there exists* $\varepsilon_\star = \varepsilon_\star(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_{\Phi_\star}, \Delta_\phi)$ *such that for* $\varepsilon_\star \leq \varepsilon$ *we have*

$$\Delta_\Phi < C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Phi_\star}).$$

Proof. We need to control the integration in $\mathcal{D}_{u,v}$ in Propositions 31 and 32. Firstly, we focus on Proposition 31. We need control

$$\int_{\mathcal{D}_{u,v}} |\nabla\!\!\!\!/^m\Phi_H| \sum_{i_1+i_2+i_3+i_4=m} (|\nabla\!\!\!\!/^{i_1}\Gamma^{i_2}||\nabla\!\!\!\!/^{i_3}\Gamma||\nabla\!\!\!\!/^{i_4}\Phi| + |\nabla\!\!\!\!/^{i_1}\Gamma^{i_2}||\nabla\!\!\!\!/^{i_3}\Sigma||\nabla\!\!\!\!/^{i_4}\phi|),$$

where $\Phi_H = \{\Phi_{00}, \Phi_{01}, \Phi_{02}, \Phi_{11}, \Phi_{12}\}$. We can separate $|\nabla\!\!\!\!/^m\Phi_H|$ and the summation using the Hölder inequality. In turn, the term $|\nabla\!\!\!\!/^m\Phi_H|$ can be controlled as follows:

$$||\nabla\!\!\!\!/^m\Phi_H||_{L^2(\mathcal{D}_{u,v})} = \left(\int_0^u\int_0^v\int_S |\nabla\!\!\!\!/^m\Phi_H|^2\right)^{1/2} \leq C\Delta_\Phi\varepsilon^{1/2}.$$

We observe that as Φ contains Φ_{22}, we can only control it on $\mathcal{N}'_v$. Accordingly, we have that

$$||\nabla\!\!\!\!/^m\Phi||_{L^2(\mathcal{D}_{u,v})} \leq C\Delta_\Phi.$$

Next, we need to analyse the L^2-norm of the summation. Observing that the first term of the summation has a structure similar to that of the Weyl tensor Ψ in vacuum Einstein case, we readily obtain that this term is controlled by

$$C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Phi_\star}, \Delta_\Phi)\varepsilon^{1/2}.$$

The second term in the summation can be shown to be less than

$$C\Delta_\Phi \varepsilon^{1/2} \sum_{i_1+i_2+i_3+i_4=m} ||\nabla^{i_1}\Gamma^{i_2}\nabla^{i_3}\Sigma\nabla^{i_4}\phi||_{L^2(\mathcal{D}_{u,v})}.$$

Every time we encounter the components ϕ_0 to ϕ_3 and their derivatives, we can control them through the L^2-norm on the long light cone $\mathcal{N}_u$. Moreover, by analogy to Φ_{22}, we control ϕ_4 and its derivatives on the short light cone $\mathcal{N}'_v$. Hence following the same procedure we can obtain that this norm is less than

$$C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_{\Phi_\star}, \Delta_\phi, \Delta_\Phi)\varepsilon^{3/2}.$$

In the next step, we consider the terms on the right hand side of the estimate in Proposition 32. The terms

$$\int_{\mathcal{D}_{u,v}} |\nabla^m \Phi_{12}| \sum_{i_1+i_2+i_3+i_4=m} |\nabla^{i_1}\Gamma^{i_2}||\nabla^{i_3}\Gamma||\nabla^{i_4}\Phi|$$

can be controlled in the same manner as it was done in Proposition 31 and are bounded by

$$C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Phi_\star}, \Delta_\Phi)\varepsilon^{1/2}.$$

Next, the terms

$$\int_{\mathcal{D}_{u,v}} |\nabla^m \Phi_{22}| \sum_{i_1+i_2+i_3+i_4=m} |\nabla^{i_1}\Gamma^{i_2}||\nabla^{i_3}\Gamma||\nabla^{i_4}\Phi'_H|$$

can also be controlled because it does not contains the term $(\Phi_{22})^2$. Moreover, the terms

$$\sum_{i_1+i_2+i_3+i_4=m} \int_{\mathcal{D}_{u,v}} |\nabla\!\!\!/^m \Phi_{12}||\nabla\!\!\!/^{i_1} \Gamma^{i_2}||\nabla\!\!\!/^{i_3} \Sigma||\nabla\!\!\!/^{i_4} \phi|$$

can be controlled by

$$C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_{\Phi_\star}, \Delta_\phi, \Delta_\Phi)\varepsilon^{3/2}.$$

In the case of the term

$$\int_{\mathcal{D}_{u,v}} |\nabla\!\!\!/^m \Phi_{22}| \sum_{i_1+i_2+i_3+i_4=m} |\nabla\!\!\!/^{i_1} \Gamma'^{i_2}||\nabla\!\!\!/^{i_3} \Gamma'||\nabla\!\!\!/^{i_4} \Phi_{22}|,$$

we readily found that it is bounded by

$$\begin{aligned} &C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Phi_\star}) \int_0^v ||\nabla\!\!\!/^m \Phi_{22}||_{L^2(\mathcal{N}'_v(0,u))} \sum_{i=0}^m ||\nabla\!\!\!/^i \Phi_{22}||_{L^2(\mathcal{N}'_v(0,u))} \\ &\leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Phi_\star}) \int_0^v \sum_{i=0}^m ||\nabla\!\!\!/^i \Phi_{22}||^2_{L^2(\mathcal{N}'_v(0,u))}. \end{aligned}$$

Similarly, we also have that

$$\begin{aligned} &\sum_{i_1+i_2+i_3+i_4=m} \int_{\mathcal{D}_{u,v}} |\nabla\!\!\!/^m \Phi_{22} \nabla\!\!\!/^{i_1} \Gamma^{i_2} \nabla\!\!\!/^{i_3} \Sigma \nabla\!\!\!/^{i_4} \phi'_H| \\ &\leq \varepsilon^{3/2} C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_{\Phi_\star}, \Delta_\phi) \int_0^v ||\nabla\!\!\!/^m \Phi_{22}||^2_{L^2(\mathcal{N}'_v(0,u))}. \end{aligned}$$

Putting together the above estimates in the inequality of Proposition 32 we have that

$$\begin{aligned} &\sum_{i=0}^3 ||\nabla\!\!\!/^i \Phi_{22}||^2_{L^2(\mathcal{N}'_v)} \leq C\Delta_{\Phi_\star} + C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_\Phi)\varepsilon^{1/2} \\ &+ C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_{\Phi_\star}, \Delta_\phi, \Delta_\Phi)\varepsilon^{3/2} + (C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Phi_\star}) \\ &+ C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_{\Phi_\star}, \Delta_\phi)\varepsilon^{3/2}) \int_0^v \sum_{i=0}^m ||\nabla\!\!\!/^i \Phi_{22}||^2_{L^2(\mathcal{N}'_v(0,u))}. \end{aligned}$$

Thus, applying the Grönwall's inequality, we obtain that

$$\Delta_\Phi \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Phi_\star}) + C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_{\Phi_\star}, \Delta_\Phi)\varepsilon^{1/2} + o(\epsilon^{1/2}).$$

Finally, taking ε small enough we prove the proposition.

□

The final ingredient in our analysis is the following proposition whose proof is analogous to that of Proposition 17 in Paper I:

Proposition 34 (***control of the components of the Rescaled Weyl tensor in terms of the initial data***)**.** *With the same assumptions in proposition 38 on some truncated causal diamond $\mathcal{D}^t_{u,v_\bullet}$. Then there exists $\varepsilon_\star = \varepsilon_\star(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_{\Phi_\star}, \Delta_{\phi_\star})$ such that for $\varepsilon_\star \leq \varepsilon$ we have*

$$\Delta_\phi \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Phi_\star}, \Delta_{\phi_\star}).$$

4.5 Concluding the argument

The estimates obtained in the previous sections can be used in a *last slice argument* to obtain our main result. The proof is completely analogous to that given in Section 7 in Paper I and is thus omitted.

Theorem 6. *Given smooth initial data on $\mathscr{I}^- \cup \mathcal{N}'_\star$ for $0 \leq v \leq I$ as constructed in Lemma 20, there exists ε such that an unique smooth solution to the vacuum conformal Einstein field equations exists in the region where $0 \leq v \leq I$ and $0 \leq u \leq \varepsilon$ under the coordinate system and ε can be chosen to depend only on $\Delta_{e_\star}$, $\Delta_{\Gamma_\star}$, $\Delta_{\Sigma_\star}$, $\Delta_{\Phi_\star}$ and $\Delta_{\phi_\star}$. Moreover, in this area,*

$$\begin{aligned}
&\sup_{u,v} \sup_{\Gamma\in\{\chi,\mu,\lambda,\rho,\sigma,\alpha,\beta,\tau,\epsilon\}} \max\{\sum_0^1 ||\nabla^i\Gamma||_{L^\infty(\mathcal{S}_{u,v})}, \sum_{i=0}^2 ||\nabla^i\Gamma||_{L^4(\mathcal{S}_{u,v})}, \sum_{i=0}^3 ||\nabla^i\Gamma||_{L^2(\mathcal{S}_{u,v})}\} \\
&+ \sup_{u,v} \max\{\sum_{i=0}^1 ||\nabla^i\Sigma_2||_{L^\infty(\mathcal{S}_{u,v})}, \sum_{i=0}^2 ||\nabla^i\Sigma_2||_{L^4(\mathcal{S}_{u,v})}, \sum_{i=0}^3 ||\nabla^i\Sigma_2||_{L^2(\mathcal{S}_{u,v})}\} \\
&+ \Delta_\Phi + \Delta_\phi \leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_{\Phi_\star}, \Delta_{\phi_\star})
\end{aligned}$$

and

$$\sup_{u,v}\max\{\sum_{i=0}^{1}||\nabla\!\!\!/^{i}\Sigma_{1,3,4}||_{L^\infty(\mathcal{S}_{u,v})},\sum_{i=0}^{2}||\nabla\!\!\!/^{i}\Sigma_{1,3,4}||_{L^4(\mathcal{S}_{u,v})},\sum_{i=0}^{3}||\nabla\!\!\!/^{i}\Sigma_{1,3,4}||_{L^2(\mathcal{S}_{u,v})}\}$$
$$\leq C(I,\Delta_{e_\star},\Delta_{\Gamma_\star},\Delta_{\Sigma_\star},\Delta_{\Phi_\star})\varepsilon.$$

Chapter 5

The conformal Einstein field equations and the local extension of future null infinity

In this chapter we make use of an optimal existence result for the characteristic initial value problem for the conformal Einstein equations to show that given initial data on two null hypersurfaces $\mathcal{N}_\star$ and $\mathcal{N}'_\star$ such that the conformal factor (but not its gradient) vanishes on a section of $\mathcal{N}_\star$ one recovers a portion of null infinity.

The problem. In this chapter we study the question of *the local extendibility of null infinity*. To this end, initial data is prescribed on two future oriented null hypersurfaces intersecting a 2-dimensonal surface with the topology of the 2-sphere $\mathbb{S}^2$. These null hypersurfaces are assumed to intersect future null infinity, $\mathscr{I}^+$. The question to be adressed is through this initial value problem is whether it is possible to recover a portion of future null infinity lying in the causal future of the initial hypersurfaces. Observe that in the future null infinity version of the asymptotic characteristic initial value problem analysed in Chapter 4, the solution constructed is located in the causal past of the initial hypersurfaces —see Figure 5.1. The question of the local extendibility of null infinity through a chracteristic initial value problem has been studied by Li & Zhu in [1] directly through the Einstein field equations. In this approach, in order to encode the asymptotic behaviour of the various field at infinity it is necessary to make use of weighted functional spaces and norms. Moreover, it is necessary to consider the existence of solutions to the field equations of a domain with an infinite extent. Accordingly, this study requires a delicate and lengthy analysis. By contrast, in this chapter we make use of what we believe is the natural setting to address the local extendibility of null infinity: *the use of a conformal representation of the spacetime and the conformal Einstein field equations* —see e.g. [35].

Conformal methods. The use of conformal methods in the study of the local extendibility of (future) null infinity allows to transform question of existence of solutions to hyperbolic evolution equations on an infinite domain into the study of solutions on a finite region. Moroever, the asymptotic decay of the various fields fields is conveniently encoded through regularity of the fields. Accordingly, it is possible to work with standard (unweighted) functional spaces and norms. Luk's strategy to analyse the characteristic initial value problem allows to ensure the existence of solutions on *causal diamonds* having a *long* and a *short* direction —see Figure 5.1. Existence on the long direction is ensured as long as one has control on the initial data. On the other hand, the extent of the short direction is restricted by the potential appearance of *singularities in finite time* due to the presence of *Riccati-type* equations in the evolution system. In the present problem the conformal framework provides a natural causal diamond with one of its sides lying on one of the null initial hypersurfaces, $\mathcal{N}_\star$, and a short side covering a portion of null infinity. Although from the point of view of the conformal representation this domain has a finite size, in the physical spacetime it actually represents an infinite domain contained between two *parallel* null hypersurfaces. The main result of this chapter is that *it is possible to ensure the existence of solutions to the conformal Einstein field equations on the causal diamond with sides on $\mathcal{N}_\star$ and $\mathscr{I}^+$*. Thus, it is possible to recover a portion of null infinity to the future of $\mathcal{N}_\star$ —i.e. we have extended $\mathscr{I}^+$.

The hyperboloidal initial value problem. Historically, the first resolution of the local extendibility of null infinity has been given by Friedrich's in his analysis of the hyperboloidal initial value problem for the conformal Einstein field equations —see [3, 39], also [4, 35]. In this case, initial data is prescribed on a spatial hypersurface $\mathcal{H}_\star$ which becomes asymptotically null near the conformal boundary. Due to the formal regularity of the conformal Einstein field equations at the conformal

Figure 5.1: Comparison between the asymptotic characteristic problem (a) and the standard characteristic problem (b) for the conformal Einstein field equations. In the future null infinity version of the asymptotic characteristic initial value problem initial data is prescribed on future null infinity and on an outgoing lightcone $\mathcal{N}_\star$. The optimal existence result allows to recover a narrow causal diamond along null infinity. The length of this rectangle is limited by the portion of $\mathscr{I}^+$ on which one has cntrol of the initial data. Observe that region of existence of solutions lies in the causal past of the null hypersurfaces and that the existence of, at least a portion of null infinity is *a priori* assumed. In the characteristic problem considered in this article the initial data is prescribed on two standard null hypersurfaces $\mathcal{N}_\star$ and $\mathcal{N}'_\star$ with at least one of them ($\mathcal{N}_\star$) intersecting the conformal boundary. The improved existence result allows then to recover a narrow rectangle whose long side lies on $\mathcal{N}_\star$ and the short one gives a portion of future null infinity. Observe that the region of existence is on the causal future of the initial hypersurfaces and that, *a priori* only the existence of a cut of null infinity is assumed.

boundary, the standard local existence theory for symmetric hyperbolic systems allows to recover a slab of spacetime in the causal future of $\mathcal{H}_\star$ which covers a portion of null infinity —see Figure 5.2. A particular drawback of this approach, in constrast with the characteristic initial value problem, is the increased complexity in solving the constraint equations on $\mathcal{H}_\star$ and obtaining conditions ensuring peeling (see below) —see [40–42]. In view of the later and the historical and practical relevance of the characteristic initial value problem we believe it is of interest to discuss the extendibility of null infinity form this alternative point of view.

Peeling. The problem here considered is closely related to one of the central issues on the study of the asymptotics of the gravitational field: *peeling*. As part of the formulation of the characteristic initial value problem here considered it is necessary to prescribe the value of one of the components of the Weyl tensor (ϕ_0) on one of

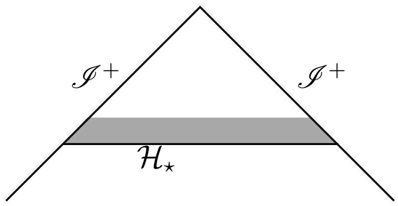

Figure 5.2:

the null hypersurfaces. This component is usually loosely interpreted as describing some sort of incoming radiation —see [43]. For simplicity, in the present analysis it is assumed that *the component* ϕ_0 *is smooth at null infinity*. It follows that on the portion of future null infinity recovered by the optimal local existence result for the characteristic initial value problem *the Weyl tensor satisfies the peeling behaviour*. If a finite regularity is assumed below a certain threshold, then the assumptions of the *peeling theorem* are no longer satisfied.

Differences with the asymptotic characteristic problem. The characteristic initial value problem considered in this article differs from the one in Chapter 4 in that in the former reference one of the initial hypersurfaces coincides with the conformal boundary. This leads to a number of simplifications in the gauge and equations. In the present case, both initial null hypersurfaces lie in the physical spacetime —except for their intersections with null infinity. Thus, one has to deal with a somewhat more general set up. Nevertheless, a careful inspection of the analysis of Chapter 4 shows that all the main assertions and estimates hold in the present situation. Roughly speaking these estimates control the size of the L^2-norm of the fields appearing in the conformal Einstein field equations in terms of the size of the initial data. Thus, if the data is finite, so will also the solutions to the conformal Einstein field equations. The existence of solutions on the causal diamond containing a portion of null infinity then follows from a *last slice argument* in which the basic existence domain arising from the use of *Rendall's reduction strategy* [22] is progressively extended.

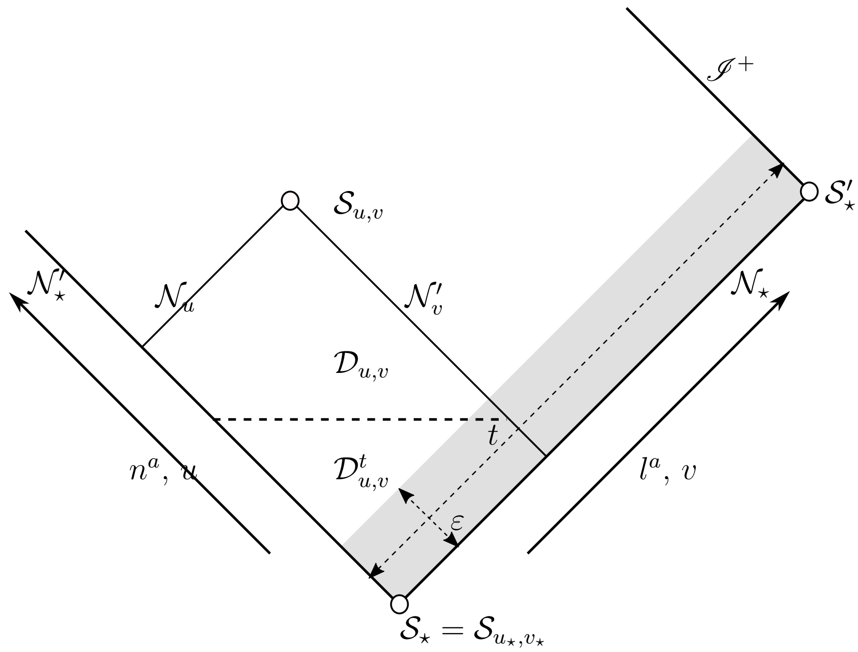

Figure 5.3: Geometric setup for the analysis of the local extension of future null infinity. The construction makes use of a double null foliation of the domain of dependence of the initial hypersurface $\mathcal{N}'_\star \cup \mathcal{N}_\star$. The null hypersurface $\mathcal{N}_\star$ terminates at the conformal boundary where $\Xi = 0$. Our construction allows us to recover a portion of length ϵ on $\mathscr{I}^+$. The coordinates and null NP tetrad are adapted to this geometric setting. The analysis is focused on the thing grey rectangular domain along $\mathcal{N}_\star$. The conformal Einstein field equation allows to treat this problem on an infinite domain in terms of a problem in unphysical space on a finite domain.

5.1 The geometry of the problem

In this section we discuss the geometric setting of the local extension of future null infinity. This is very similar to the one used in Chapter Revisiting the characteristic initial value problem for the vacuum Einstein field equations and 4 and makes use of a gauge which we will call *Stewart's gauge*. The reader is refered to [44, 45] for further details and discussion —see also [19, 33].

5.1.1 Basic geometric setting

In our basic setting, the unphysical manifold $\mathcal{M}$ has a boundary and two edges. The boundary consists of three null hypersurfaces: the outgoing null hypersurface $\mathcal{N}_\star$; the incoming null hypersurface $\mathcal{N}'_\star$ with non-vacuum intersection $\mathcal{S}_\star \equiv \mathcal{N}_\star \cap \mathcal{N}'_\star$; future null infinity $\mathscr{I}^+$ intersecting with $\mathcal{N}_\star$ at the corner $\mathcal{S}'_\star$. For concreteness, we will assume that $\mathcal{S}_\star, \mathcal{S}'_\star \approx \mathbb{S}^2$. See Figure 5.3 for further details.

One can introduce coordinates $x = (x^\mu)$ in a neighbourhood $\mathcal{U}$ of $\mathcal{Z}_\star$ with $x^0 = v$ and $x^1 = u$ such that, at least in a neighbourhood of $\mathcal{S}_\star$ one can write

$$\mathcal{N}_\star = \{p \in \mathcal{U} \mid u(p) = 0\}, \qquad \mathcal{N}'_\star = \{p \in \mathcal{U} \mid v(p) = 0\}.$$

Given suitable data on $(\mathcal{N}_\star \cap \mathcal{N}'_\star) \cap \mathcal{U}$ we are interested in making statements about the existence and uniqueness of solutions to the CEFE on some open set

$$\mathcal{V} \subset \{p \in \mathcal{U} \mid u(p) \geq 0, v(p) \geq 0\}$$

which we identify with a subset of the future domain of dependence, $D^+(\mathcal{N}_\star \cup \mathcal{N}'_\star)$ of $\mathcal{N}_\star \cup \mathcal{N}'_\star$. Moreover, we want to show that the existence region can be extended along $\mathcal{N}_\star$ to reach the conformal conformal boundary —this *improved existence domain* corresponds to the grey rectangle in Figure 5.3.

5.1.2 Stewart's Gauge

Following the discussion of Chapter 3 and 4, in the following we assume that the future of $\mathcal{S}_\star$ can be foliated by a family of null hypersurfaces: $\mathcal{N}_u$ (the *outgoing null hypersurfaces*) and $\mathcal{N}'_v$ (the *ingoing null hypersurfaces*). The scalars u and v satisfy

$$\boldsymbol{g}^\sharp(\mathbf{d}u, \mathbf{d}u) = \boldsymbol{g}^\sharp(\mathbf{d}v, \mathbf{d}v) = 0.$$

In particular, we assume that $\mathcal{N}_0 = \mathcal{N}_\star$ and $\mathcal{N}'_0 = \mathcal{N}'_\star$. Following standard usage, we call u an *retarded time* and v a *advanced time* and use these two scalar fields u and v as coordinates in a neighbourhood of $\mathcal{S}_\star$. To complete the coordinate system, consider arbitrary coordinates $(x^\mathcal{A})$ on $\mathcal{D}_\star$, with the index $^\mathcal{A}$ taking the values 2, 3. These coordinates are then propagated into $\mathcal{N}_\star$ by requiring them to be constant along the generators of $\mathcal{N}_\star$. Once coordinates have been defined on $\mathcal{N}_\star$, one can propagate them into $\mathcal{V}$ by requiring them to be constant along the generators of each $\mathcal{N}'_v$. In this manner one obtains a coordinate system $(x^\mu) = (u,\, v,\, x^{\mathcal{A}})$ in $\mathcal{V}$. Moreover, we define $\mathcal{S}_{u,v} \equiv \mathcal{N}_u \cap \mathcal{N}'_v \approx \mathbb{S}^2$. Our analysis will be mostly carried out in *causal diamonds* of the form

$$\mathcal{D}_{u',v'} \equiv \{0 \leq v \leq v',\ 0 \leq u \leq u'\} = \cup_{0\leq v \leq v', 0 \leq u \leq u'} \mathcal{S}_{u,v}.$$

By means of the time function $t \equiv u + v$ one can readily define the *truncated causal diamond*

$$\mathcal{D}^{\tilde{t}}_{u',v'} \equiv \mathcal{D}_{u',v'} \cap \{t \leq \tilde{t}\}.$$

The above coordinate construction is complemented by a Newman-Penrose (NP) null tetrad $\{\boldsymbol{l},\, \boldsymbol{n},\, \boldsymbol{m},\, \bar{\boldsymbol{m}}\}$ with the vectors $\boldsymbol{l}$ and $\boldsymbol{n}$ tangent to the generators of the null hypersurfaces $\mathcal{N}_u$ and $\mathcal{N}'_v$ respectively. Following the same discussion of Chapter 3 and 4 we make

Gauge choice 1 (***Stewart's choice of the components of the frame*****).** On $\mathcal{V}$ we consider a NP frame of the form

$$\boldsymbol{l} = \boldsymbol{\partial}_v + C^{\mathcal{A}}\boldsymbol{\partial}_{\mathcal{A}}, \quad \boldsymbol{n} = Q\boldsymbol{\partial}_u, \quad \boldsymbol{m} = P^{\mathcal{A}}\boldsymbol{\partial}_{\mathcal{A}},$$

where $C^{\mathcal{A}} = 0$ on $\mathcal{N}_\star$, $\boldsymbol{m}$ and $\bar{\boldsymbol{m}}$ span the tangent space of $\mathcal{S}_{u,v}$. On $\mathcal{N}'_\star$ one has that $\boldsymbol{n} = Q\boldsymbol{\partial}_u$. As the coordinates $(x^{\mathcal{A}})$ are constant along the generators of $\mathcal{N}_\star$ and $\mathcal{N}'_\star$, it follows that on $\mathcal{N}'_\star$ the coefficient Q is only a function of u. Thus, without loss of generality one can reparametrise u so as to set $Q = 1$ on $\mathcal{N}'_\star$.

Direct inspection of the NP commutators applied to the coordinates $(u,\, v,\, x^{\mathcal{A}})$ leads to the following:

Lemma 18 (***conditions on the connection coefficients***)**.** *The NP frame of the Gauge Choice 1 can be chosen such that*

$$\kappa = \nu = \gamma = 0, \tag{5.1a}$$

$$\rho = \bar{\rho}, \quad \mu = \bar{\mu}, \tag{5.1b}$$

$$\pi = \alpha + \bar{\beta} \tag{5.1c}$$

on $\mathcal{V}$ and, furthermore, with

$$\epsilon - \bar{\epsilon} = 0 \quad on \quad \mathcal{V} \cap \mathcal{N}_\star.$$

Remark 25. Additional commutator relations can be used to obtain equations for the frame coefficients Q, $P^{\mathcal{A}}$ and $C^{\mathcal{A}}$ —see equations (4.4a)-(4.4f) in Chapter 4.

In addition to the *coordinate* and *frame gauge freedom* we also need to fix the *conformal gauge freedom.* This is done in the following lemma whose proof follows the same scheme as Lemma 11 in Chapter 4:

Lemma 19 (***conformal gauge conditions for characteristic problem***)**.** *Let $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ denote a vacuum asymptotically simple spacetime and let $(\mathcal{M}, \boldsymbol{g}, \Xi)$ with $\boldsymbol{g} = \Xi^2\tilde{\boldsymbol{g}}$ a conformal extension. Given the NP frame of the Gauge Choice 1, the conformal factor Ξ can be chosen so that*

$$R[\boldsymbol{g}] = R(x), \qquad \textit{in a neighourhood} \quad \mathcal{V} \quad \textit{of} \quad \mathcal{S}_\star \quad \textit{on} \quad J^+(\mathcal{S}_\star)$$

where $R(x)$ is an arbitrary function of the coordinates. Moreover, one has the additional gauge conditions

$$\begin{aligned} &\Sigma_2 = 1, \quad \mu = \rho = 0 \quad \textit{on} \ \ \mathcal{S}_\star, \\ &\Phi_{22} = 0 \quad \textit{on} \ \ \mathcal{N}'_\star, \\ &\Phi_{00} = 0 \quad \textit{on} \ \ \mathcal{N}_\star. \end{aligned}$$

5.2 The formulation of the characteristic initial value problem

This section provides a brief discussion of the basic set up and local existence theory of the characteristic initial value problem for the conformal Einstein field equations with data on the null hypersurfaces $\mathcal{N}_\star$ and $\mathcal{N}'_\star$ using Rendall's reduction strategy [22] —see also Section 12.5 of [35]. The analysis is completely analogous to the one carried out in Chapter 3 in which the initial value problem for the vacuum Einstein field equations was considered —note, by contrast, the conceptual difference with Chapter 4 in which an asymptotic characteristic problem was considered.

5.2.1 Specifiable free data

In order to obtain a solution in the domain $J^+(\mathcal{S}_\star)$, we need to provide initial data for the evolution equations on $\mathcal{N}_\star \cup \mathcal{N}'_\star$. In particular, we need to know the value of the derivatives of conformal factor $\{\Sigma_1,\ \Sigma_2,\ \Sigma_3,\ \Sigma_4\}$, the components of frame $\{C^{\mathcal{A}},\ P^{\mathcal{A}}, Q\}$, the spin connection coefficients $\{\epsilon,\ \pi,\ \beta,\ \mu,\ \alpha,\ \lambda,\ \tau,\ \sigma,\ \rho\}$, the rescaled Weyl tensor $\{\phi_0,\ \phi_1,\ \phi_2, \phi_3,\ \phi_4\}$ and the Ricci tensor $\{\Phi_{00},\ \Phi_{01},\ \Phi_{11},\ \Phi_{02},\ \Phi_{12},\ \Phi_{22}\}$ on the initial hypersurfaces. However, as a consequence of the constraints implied by the CEFE, this data cannot be freely specified. As in the case of the discussion in Chapter 3 and 4, The hierarchical structure of the CEFE allows to identify the basic reduced initial data set r_* from which the full initial data on $\mathcal{N}_\star \cup \mathcal{N}'_\star$ for the conformal Einstein field equations can be computed. The following lemma shows us the freely specifiable data for our characteristic problem.

Lemma 20 (***freely specifiable data for the characteristic problem***)**.** *Assume that the Gauge Choice 1 and the gauge conditions implied by Lemmas 1 and 19 are satisfied in a neighbourhood $\mathcal{V}$ of $\mathcal{S}_\star$. Initial data for the conformal Einstein field equations on $\mathcal{N}_\star \cup \mathcal{N}'_\star$ can be computed from the reduced data set $\mathbf{r}_\star$ consisting of:*

$$
\begin{aligned}
&\phi_0,\ \Xi, \quad \textit{on} \quad \mathcal{N}_\star,\\
&\phi_4 \quad \textit{on} \quad \mathcal{N}'_\star,\\
&\lambda, \quad \phi_2+\bar{\phi}_2, \quad \Phi_{20}, \quad \phi_3, \quad P^{\mathcal{A}}, \quad \textit{on} \quad \mathcal{S}_\star.
\end{aligned}
$$

The proof of this result is completely analogous to that of Lemma 20 in Chapter 4—see also 2 in Chapter 3.

In the problem under consideration we require that $\mathcal{N}_\star$ has a finite range $v \in [0, v^+]$ and extends to the conformal boundary $\mathscr{I}^+$ —i.e. future null infinity. This idea can be encoded in the following requirements on Ξ:

$$
\Xi > 0 \qquad \text{on} \quad \mathcal{N}_\star/\mathscr{I}^+
$$

$$\Xi = 0 \qquad \text{on} \quad \mathcal{S}_{0,v^+}.$$

In addition, it is also necessary to ensure that one remains on $\mathscr{I}^+$ if we move away from $\mathcal{S}_{0,v^+}$ along the direction given by $\boldsymbol{n}$. This is ensured by the following Lemma.

Lemma 21 (***Conditions for*** Ξ ***on conformal boundary*** $\mathscr{I}^+$)**.** *Under the same assumptions in Lemma 20, and with a conformal factor satisfying*

$$\Xi = 0 \qquad on \quad S_{0,v^+},$$

we have

$$\Xi = 0, \quad \mathbf{d}\Xi \neq 0, \qquad on \quad \mathcal{N}'_{v^+}.$$

Proof. From the definition of Σ_a and the conformal equation (2.9a) it follows that in our gauge one has

$$\begin{aligned} \Delta\Xi &= \Sigma_2, \\ \Delta\Sigma_2 &= -\Xi\Phi_{22}, \end{aligned}$$

along $\mathcal{N}'_{v_\star}$. Combining these equations we find that

$$\Delta^2\Xi = -\Xi\Phi_{22},$$

so that $\Xi = 0$ is a solution such that $\Xi|_{S_{0,v_\star}}$. The theory of ordinary differential equation shows that this is the unique solution. □

5.2.2 The reduced conformal field equations

In Chapter 4 it has been discussed how the CEFE expressed in Stewart's gauge imply a symmetric hyperbolic evolution system. More precisely, letting

$$\begin{aligned} \boldsymbol{\Sigma}^t &\equiv (\Xi, \Sigma_1,\ \Sigma_2,\ \Sigma_3,\ \Sigma_4,\ s), \\ \boldsymbol{e}^t &\equiv (C^{\mathcal{A}},\ P^{\mathcal{A}},\ Q), \\ \boldsymbol{\Gamma}^t &\equiv (\epsilon,\ \pi,\ \beta,\ \mu,\ \alpha,\ \lambda,\ \tau,\ \sigma,\ \rho), \\ \boldsymbol{\phi}^t &\equiv (\phi_0,\ \phi_1,\ \phi_2,\ \phi_3,\ \phi_4), \end{aligned}$$

$$\mathbf{\Phi}^t \equiv (\Phi_{00},\ \Phi_{01},\ \Phi_{11},\ \Phi_{02}, \Phi_{12},\ \Phi_{22}),$$

it can be shown that

$$\boldsymbol{\mathcal{D}}^\mu(x, \boldsymbol{u})\partial_\mu \boldsymbol{u} = \boldsymbol{B}(x, \boldsymbol{u})\boldsymbol{u} \tag{5.2}$$

with

$$\boldsymbol{u} = (\boldsymbol{e}^t, \mathbf{\Sigma}^t, \mathbf{\Gamma}^t, \mathbf{\Phi}^t, \boldsymbol{\phi}^t)^t$$

is a symmetric hyperbolic system with respect to the direction given by

$$\tau^a = l^a + n^a.$$

In particular, $\boldsymbol{\mathcal{D}}^\mu(x, \boldsymbol{u})$ are Hermitian matrices and $\boldsymbol{B}(x, \boldsymbol{u})$ is smooth matrix-valued functions of their arguments whose explicit form will not be required in the subsequent discussion in this section. We call the evolution system (5.2) the *reduced conformal Einstein field equations.*

Remark 26. The propagation of the constraint equations implied by the CEFE on the initial hypersurface $\mathcal{N}_\star \cup \mathcal{N}'_\star$ can be addressed along the same lines of the analysis in Section 12.5 of [35]. It follows from the latter that a solution of the reduced conformal field equations on a neighbourhood $\mathcal{V}$ of $\mathcal{S}_\star$ on $J^+(\mathcal{S}_\star)$ that coincides with initial data on $\mathcal{N}'_\star \cup \mathcal{N}_\star$ satisfying the conformal equations is, in fact, a solution to the conformal Einstein field equations on $\mathcal{V}$.

Rendall's approach to the existence and uniqueness of solutions of CIVP can be obtained via an auxiliary Cauchy initial value problem on a spacelike hypersurface S_* denoted by $\{p \in \mathbb{R} \times \mathbb{R} \times \mathbb{S}^2 \mid v(p) + u(p) = 0\}$. The formulation of this problem crucially depends on Whitney's extension theorem which requires being able to evaluate all derivatives (interior and transverse) of initial data on $\mathcal{N}'_\star \cup \mathcal{N}_\star$. A key property of the NP equations in Stewart's gauge is that any arbitrary formal derivatives of the unknown functions $\{\mathbf{\Sigma},\ \boldsymbol{e},\ \mathbf{\Gamma},\ \mathbf{\Phi},\ \boldsymbol{\phi}\}$ on $\mathcal{N}'_\star \cup \mathcal{N}_\star$ can be computed from the prescribed initial data $\boldsymbol{r}_\star$ for the reduced conformal field equations on $\mathcal{N}'_\star \cup \mathcal{N}_\star$. This observation allows to make use of Whitney's extension theorem. More details can be found in Chapter 3 and Chapter 4.

Combining the previous analysis and applying the theory of CIVP for the symmetric hyperbolic systems of Section 12.5 of [35], one obtains the following existence result:

Theorem 7 (***existence and uniqueness to the standard asymptotic characteristic problem***). *Given an smooth reduced initial data set $\boldsymbol{r}_\star$ on $\mathcal{N}'_\star \cup \mathcal{N}_\star$, there exists a unique smooth solutionto the CEFE in a neighbourhood $\mathcal{V}$ of $\mathcal{Z}_\star$ on $J^+(\mathcal{S}_\star)$ which implies the prescribed initial data on $\mathcal{N}'_\star \cup \mathcal{N}_\star$. Moreover, this solution to the conformal Einstein field equations implied, in turn, a solution to the vacuum Einstein field equations in a neighbourhood of null infinity.*

5.3 Basic set up for the improved existence result

As already discussed, in this article we address the question of the local extendibility of null infinity by means of an improved existence result for the CIVP for the CEFE. In this section we briefly review the basic technical tools for this construction.

5.3.1 Norms

In the following we make use the same conventions for the norms of functions as in Chapter 4 —see Section 4.3.

5.3.2 Estimates for the frame and the conformal factor

The first step in the analysis of the improved existence result is to obtain control on the coefficients of the frame and the conformal factor. The asymptotic characteristic initial value problem considered in Chapter 4 leads to some non-generic simplifications which do not arise when one of the initial null hypersurfaces is not the conformal boundary. Nevertheless, the basic analysis follows through.

In the following we make use of

$$\Delta_{e_\star,\Xi_\star} \equiv \max\{ \sup_{\mathcal{N}_*,\mathcal{N}'_*} \left(|Q|, |Q^{-1}|, |C^{\mathcal{A}}|, |P^{\mathcal{A}}|\right), \sup_{\mathcal{N}_*}(\Xi)\}$$

to measure the size of the initial data of frame and the conformal factor. In addition, for convenience we define the scalar

$$\chi \equiv \Delta \log Q,$$

which, being a derivative of a component of the frame, is at the same level of the connection coefficients. A direct computation using the definition of $\chi = \Delta \log Q$ and the NP Ricci identities yields

$$D\chi = \Psi_2 + \bar{\Psi}_2 + 2\alpha\tau + 2\bar{\beta}\tau + 2\bar{\alpha}\bar{\tau} + 2\beta\bar{\tau} + 2\tau\bar{\tau} - (\epsilon + \bar{\epsilon})\chi. \tag{5.3}$$

In view of the gauge choice $Q = 1$ on $\mathcal{N}'_\star$ it follows that $\chi = 0$ on $\mathcal{N}'_\star$. We also define

$$\varpi \equiv \beta - \bar{\alpha}$$

corresponding to the only independent component of the connection on the spheres $\mathcal{S}_{u,v}$.

In order to start the analysis we make the following:

Assumption 5 (***assumption to control the coefficients of the frame and conformal factor***)**.** *Assume that we have a solution to the vacuum CEFEs in Stewart's gauge satisfying,*

$$||\{\mu, \lambda, \alpha, \beta, \tau, \chi, \Sigma_2\}||_{L^\infty(\mathcal{S}_{u,v})} \leq \Delta_\Gamma,$$

on a truncated causal diamond $\mathcal{D}^t_{u,v_\bullet}$, where Δ_Γ is some constant.

This assumption is initially guaranteed on a sufficiently small diamond. With the above assumption and the definition of χ, Σ_2 and making use of the equations for the frame coefficients implied by the NP commutators we obtain the following basic estimates for metric and conformal factor:

Lemma 22 (***control on the metric and conformal factor***)**.** *Given sufficiently small $\varepsilon > 0$ there exist constants C_1, C_2 and C_3 depending $\Delta_{e_\star,\Xi_\star}$ and Δ_Γ such that on $\mathcal{D}^t_{u,v_\bullet}$*

$$\begin{aligned}
&||Q, Q^{-1}||_{L^\infty(\mathcal{S}_{u,v})}, |P^{\mathcal{A}}|, |(P^{\mathcal{A}})^{-1}| \leq C_1(\Delta_{e_\star,\Xi_\star}),\\
&|C^{\mathcal{A}}| \leq C_2(\Delta_{e_\star,\Xi_\star}, \Delta_\Gamma)\varepsilon,
\end{aligned}$$

$$||\Xi||_{L^\infty(\mathcal{S}_{u,v})} \leq C_3(\Delta_{e_\star,\Xi_\star}).$$

Moreover one has

$$|\sigma^{\mathcal{AB}}|,\ |\sigma_{\mathcal{AB}}| \leq C(\Delta_{e_\star,\Xi_\star}), \qquad c(\Delta_{e_\star}) \leq |\det \boldsymbol{\sigma}| \leq C(\Delta_{e_\star,\Xi_\star}),$$

and finally

$$\sup_{u,v} |\mathrm{Area}(\mathcal{S}_{u,v}) - \mathrm{Area}(\mathcal{S}_{0,v})| \leq C(\Delta_{e_\star,\Xi_\star})\Delta_\Gamma \varepsilon.$$

5.4 Main analysis

In this section we present the main analysis of this article leading, ultimately, to an improved existence result for the characteristic initial value problem for the conformal Einstein equations. The strategy followed is very similar to that in Chapter 4. In view of this, most of the proofs of the various lemmas and propositions are omitted and we focus our attention at the points where there may be differences in the analysis of Chapter 4.

5.4.1 Norms and statement of the main result

As in Chapter 4, we make use of a number of tailor-made norms to control the various assumptions and conclusions of the bootstrap argument underpinning our analysis.

(i) Norm for the initial value of the connection coefficients, given by

$$\Delta_{\Gamma_\star} \equiv \sup_{\mathcal{S}_{u,v}\subset \mathcal{N}_\star,\mathcal{N}'_\star} \sup_{\Gamma\in\{\mu,\lambda,\rho,\sigma,\alpha,\beta,\tau,\epsilon\}} \max\{1, \sum_{i=0}^{1} ||\nabla\!\!\!\!/^{\,i}\Gamma||_{L^\infty(\mathcal{S}_{u,v})}, \sum_{i=0}^{2} ||\nabla\!\!\!\!/^{\,i}\Gamma||_{L^4(\mathcal{S}_{u,v})},$$
$$\sum_{i=0}^{3} ||\nabla\!\!\!\!/^{\,i}\Gamma||_{L^2(\mathcal{S}_{u,v})}\}.$$

(ii) Norm for the initial value of the derivative of conformal factor Σ_a, given by

$$\Delta_{\Sigma_\star} \equiv \sup_{\mathcal{S}_{u,v}\subset\mathcal{N}_\star} \sup_{j=1,\ldots,4} \max\{1, \sum_{i=0}^{1} ||\nabla\!\!\!/^i \Sigma_j||_{L^\infty(\mathcal{S}_{u,v})}, \sum_{i=0}^{2} ||\nabla\!\!\!/^i \Sigma_j||_{L^4(\mathcal{S}_{u,v})}, \sum_{i=0}^{3} ||\nabla\!\!\!/^i \Sigma_j||_{L^2(\mathcal{S}_{u,v})}\}.$$

(iii) Norm for the initial value of the components of the Ricci curvature given by

$$\Delta_{\Phi_\star} \equiv \sup_{\mathcal{S}_{u,v}\subset\mathcal{N}_\star,\mathcal{N}'_\star} \sup_{\Phi\in\{\Phi_{00},\Phi_{01},\Phi_{02},\Phi_{11},\Phi_{12}\}} \max\{1, \sum_{i=0}^{1} ||\nabla\!\!\!/^i \Phi||_{L^4(\mathcal{S}_{u,v})}, \sum_{i=0}^{2} ||\nabla\!\!\!/^i \Phi||_{L^2(\mathcal{S}_{u,v})}\}$$
$$+ \sum_{i=0}^{3} \sup_{\Phi\in\{\Phi_{00},\Phi_{01},\Phi_{02},\Phi_{11},\Phi_{12}\}} ||\nabla\!\!\!/^i \Phi||_{L^2(\mathcal{N}_\star)} + \sup_{\Phi\in\{\Phi_{01},\Phi_{02},\Phi_{11},\Phi_{12},\Phi_{22}\}} ||\nabla\!\!\!/^i \Phi||_{L^2(\mathcal{N}'_\star)}.$$

(iv) Norm for the initial value of the components of the rescaled Weyl curvature , given by

$$\Delta_{\phi_\star} \equiv \sup_{\mathcal{S}_{u,v}\subset\mathcal{N}_\star,\mathcal{N}'_\star} \sup_{\phi\in\{\phi_0,\phi_1,\phi_2,\phi_3,\phi_4\}} \max\{1, \sum_{i=0}^{1} ||\nabla\!\!\!/^i \phi||_{L^4(\mathcal{S}_{u,v})}, \sum_{i=0}^{2} ||\nabla\!\!\!/^i \phi||_{L^2(\mathcal{S}_{u,v})}\}$$
$$+ \sum_{i=0}^{3} \sup_{\phi\in\{\phi_0,\phi_1,\phi_2,\phi_3\}} ||\nabla\!\!\!/^i \phi||_{L^2(\mathcal{N}_\star)} + \sup_{\phi\in\{\phi_1,\phi_2,\phi_3,\phi_4\}} ||\nabla\!\!\!/^i \phi||_{L^2(\mathcal{N}'_\star)}.$$

(v) Norm for the components of the Ricci curvature components at later null hypersurfaces, given by

$$\Delta_\Phi \equiv \sum_{i=0}^{3} \sup_{\Phi\in\{\Phi_{00},\Phi_{01},\Phi_{02},\Phi_{11},\Phi_{12}\}} ||\nabla\!\!\!/^i \Phi||_{L^2(\mathcal{N}_u^t)} + \sup_{\Phi\in\{\Phi_{01},\Phi_{02},\Phi_{11},\Phi_{12},\Phi_{22}\}} ||\nabla\!\!\!/^i \Phi||_{L^2(\mathcal{N}'^t_v)},$$

where the suprema in u and v are taken over $\mathcal{D}^t_{u,v_\bullet}$.

(vi) Supremum-type norm over the L^2-norm of the components of the Ricci curvature at spheres of constant u, v, given by

$$\Delta_\Phi(\mathcal{S}) \equiv \sum_{i=0}^{2} \sup_{u,v} ||\nabla\!\!\!/^i \{\Phi_{00}, \Phi_{01}, \Phi_{02}, \Phi_{11}, \Phi_{12}\}||_{L^2(\mathcal{S}_{u,v})},$$

where the supremum is taken over $\mathcal{D}^t_{u,v_\bullet}$.

(vii) Norm for the components of the Weyl tensor at later null hypersurfaces, given by

$$\Delta_\phi \equiv \sum_{i=0}^{3} \sup_{\phi\in\{\phi_0,\phi_1,\phi_2,\phi_3\}} ||\nabla\!\!\!/^i \phi||_{L^2(\mathcal{N}_u^t)} + \sup_{\phi\in\{\phi_1,\phi_2,\phi_3,\phi_4\}} ||\nabla\!\!\!/^i \phi||_{L^2(\mathcal{N}_v'^{\,t})},$$

where the suprema in u and v are taken over $\mathcal{D}^t_{u,v_\bullet}$.

(viii) Supremum-type norm over the L^2-norm of the components of the rescaled Weyl curvature at spheres of constant u, v, given by,

$$\Delta_\phi(\mathcal{S}) \equiv \sum_{i=0}^{2} \sup_{u,v} ||\nabla\!\!\!/^i \{\phi_0,\phi_1,\phi_2,\phi_3\}||_{L^2(\mathcal{S}_{u,v})},$$

with the supremum taken over $\mathcal{D}^t_{u,v_\bullet}$ and in which u will be taken sufficiently small to apply our estimates.

The main result of this article can be expressed, in terms of the above norms, as:

Theorem 8 (***local extension of null infinity***)**.** *Given regular initial data for the conformal Einstein field equations on $\mathcal{N}_\star \cup \mathcal{N}'_\star$ such that $\Xi|_{v=v_\bullet}$ for some $v_\bullet \in [0,\infty)$, there exists $\varepsilon > 0$ such that an unique smooth solution to the vacuum conformal Einstein field equations exists in the region*

$$\mathcal{D} \equiv \{0 \le u \le \varepsilon,\ 0 \le v \le v_\bullet\}$$

and such that $\varepsilon_\star$ can be chosen to depend only on $\Delta_{e_\star,\Xi_\star}$, $\Delta_{\Gamma_\star}$, $\Delta_{\Sigma_\star}$, $\Delta_{\Phi_\star}$ and $\Delta_{\phi_\star}$. The set defined by the condition $v = v_\bullet$ can be identified with a portion of future null infinity $\mathscr{I}^+$. Furthermore, on $\mathcal{D}$ one has that

$$\begin{aligned}
&\sup_{u,v} \sup_{\Gamma\in\{\mu,\lambda,\rho,\sigma,\alpha,\beta,\tau,\epsilon,\chi\}} \max\{\sum_{0}^{1} ||\nabla\!\!\!/^i\Gamma||_{L^\infty(S)}, \sum_{i=0}^{2} ||\nabla\!\!\!/^i\Gamma||_{L^4(S)}, \sum_{i=0}^{3} ||\nabla\!\!\!/^i\Gamma||_{L^2(S)}\} \\
&\quad + \sup_{u,v} \sup_{j=1,\ldots,4} \{\sum_{i=0}^{1} ||\nabla\!\!\!/^i\Sigma_j||_{L^\infty(S)}, \sum_{i=0}^{2} ||\nabla\!\!\!/^i\Sigma_j||_{L^4(S)}, \sum_{i=0}^{3} ||\nabla\!\!\!/^i\Sigma_j||_{L^2(S)}\} + \Delta_\Phi + \Delta_\phi \\
&\qquad \le C(I, \Delta_{e_\star,\Xi_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_{\Phi_\star}, \Delta_{\phi_\star}).
\end{aligned}$$

The proof of the above result is based on a lengthy bootstrap argument. All the main ingredients for it have already been developed in Chapter 3 and Chapter 4. The main task in this chapter is to verify that the arguments follow through in the slightly different setting of the problem of the local extension of null infinity. The various steps in the proof are as follows:

(0) Construct L^∞ estimates for the components of the frame and the conformal factor and its derivatives on the spheres $\mathcal{S}_{u,v}$ in terms of initial data and the length ε of the short direction of integration. These bounds, in turn, allow to control in a systematic manner the solutions of the transport equations implied by the CEFE along null directions.

(i) Construction of L^∞, L^2 and L^4 estimates for the connection coefficients over the spheres $\mathcal{S}_{u,v}$. These estimates require the assumption that the components of the curvature are bounded.

(ii) Show that the components of the curvature are bounded in the L^2 norm on the spheres $\mathcal{S}_{u,v}$. These bounds are given in terms of the initial conditions an the value of the curvature of the light cones $\mathcal{N}_u$ and $\mathcal{N}'_v$.

(iii) Show that the norms of the curvature on the light cones can be bounded in terms of the initial data.

(iv) Last slice argument. Make use of the estimates obtained in the previous steps to show that the solution to the evolution equations exists close to $\mathcal{N}_\star$ as long as one has control of the data on this initial hypersurface.

5.4.2 Estimates for the connection coefficients and the derivative of conformal factor

In this section we provide a discussion of the first step of our bootstrap argument and provide estimates for connection coefficients and the derivative of conformal factor. In order to prove these estimates it is assumed that the norms of the components of the curvature spinors are bounded. It follows then that the short range ε can be chosen such that connection coefficients and the derivative of conformal factor can be controlled by the norm of the initial data and the norm $\Delta_\Phi(\mathcal{S})$. The main tool in this estimation are the transport equations satisfied by the various fields. Most of the connection coefficients satisfy transport equations in both the D and Δ directions. Only for the connection coefficients τ and χ, we only have their long direction D equations. Crucially, however, these equations do not contain quadratic terms and can basically be regarded as linear equations.

The first step in the argumentation to control the supemum norm of the connection coefficients and the derivatives of the conformal factor —cf. Proposition 20 in Chapter 4. The assumptions in this estimate are that there exists a positive constant $\Delta_{\Gamma,\Sigma}$ $\mathcal{D}_{u,v_\bullet}$ such that

$$\sup_{u,v} ||\{\mu, \lambda, \alpha, \beta, \epsilon, \rho, \sigma, \tau, \chi, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4\}||_{L^\infty(\mathcal{S}_{u,v})} \leq \Delta_{\Gamma,\Sigma}\,,$$

in a causal diamond and that, moreover,

$$\sup_{u,v} ||\nabla\!\!\!/^3\tau||_{L^2(\mathcal{S}_{u,v})} < \infty, \quad \Delta_\Phi(\mathcal{S}) < \infty, \quad \Delta_\Phi < \infty, \quad \Delta_\phi(\mathcal{S}) < \infty, \quad \Delta_\phi < \infty.$$

Next, one constructs L^4-estimates of the connection coefficients and the derivative of conformal factor —cf. Proposition 21 in Chapter 4. These estimates are needed to make use of the Gagliardo-Nirenberg inequality in dealing with the non-linearities of the evolution equations when constructing L^2-estimates. This step requires the further assumption that

$$\sup_{u,v} ||\nabla\!\!\!/\{\mu, \lambda, \alpha, \beta, \epsilon, \rho, \sigma, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4\}||_{L^4(\mathcal{S}_{u,v})} \leq \Delta_{\Gamma,\Sigma}.$$

The last step in this process is a L^2-estimate for the connection coefficients and the derivative of conformal factor —cf. Proposition 22 in Chapter 4— which is obtained without the need of any further assumptions.

In order to estimate the components of the curvature, we need L^2-estimates of the connection coefficients and derivatives of the conformal factor up to third order. This can be achieved by a method similar to the one used to estimate the undifferentiated fields —cf. Proposition 25 in Chapter 4. The analysis described in the previous paragraphs can be summarised as follows:

Proposition 35 (***estimates for the L^∞, L^4 and L^2 norms of the connection coefficients and the derivatives of the conformal factor to second derivative***)**.** *Assume*

$$\Delta_\Phi < \infty, \qquad \Delta_\phi < \infty,$$

in the truncated diamond $\mathcal{D}^t_{u,v_\bullet}$. Then there exists

$$\varepsilon_\star = \varepsilon_\star(I, \Delta_{e_\star,\Xi_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_\Phi, \Delta_\phi, \sup_{u,v} ||\nabla\!\!\!/^3\tau||_{L^2(\mathcal{S}_{u,v})})$$

such that for $\varepsilon \leq \varepsilon_\star$, we have

$$\begin{aligned}
&\sup_{u,v} \sup_{\Gamma\in\{\mu,\lambda,\alpha,\beta,\epsilon,\rho,\sigma,\tau,\chi\}} \left(||\Gamma||_{L^\infty(\mathcal{S}_{u,v})} + \sum_{i=0}^{1} ||\nabla\!\!\!/^i\Gamma||_{L^4(\mathcal{S}_{u,v})} + \sum_{i=0}^{2} ||\nabla\!\!\!/^i\Gamma||_{L^2(\mathcal{S}_{u,v})} \right) \\
&\leq C(I, \Delta_{e_\star,\Xi_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_\Phi(\mathcal{S}), \Delta_\phi(\mathcal{S})), \\
&\sup_{u,v} \sup_{j=1,\ldots,4} \left(||\Sigma_j||_{L^\infty(\mathcal{S}_{u,v})} + \sum_{i=0}^{1} ||\nabla\!\!\!/^i\Sigma_j||_{L^4(\mathcal{S}_{u,v})} + \sum_{i=0}^{2} ||\nabla\!\!\!/^i\Sigma_j||_{L^2(\mathcal{S}_{u,v})} \right) \leq C(\Delta_{e_\star,\Xi_\star}, \Delta_{\Sigma_\star}),
\end{aligned}$$

in the truncated diamond $\mathcal{D}^t_{u,v_\bullet}$.

Armed with L^2-estimates for the connection coefficients and the derivative of conformal factor up to the second order, it is now possible to show that the norms $\Delta_\Phi(\mathcal{S})$ and $\Delta_\phi(\mathcal{S})$ are finite —see Proposition 23 in Chapter 4. More precisely, one has that

Proposition 36 (***boundedness of the components of the curvature***)**.** *Assume that*

$$\Delta_\Phi < \infty, \qquad \Delta_\phi < \infty, \qquad \sup_{u,v} ||\nabla\!\!\!/^3\tau||_{L^2(\mathcal{S}_{u,v})} < \infty$$

in the truncated diamond $\mathcal{D}^t_{u,v_\bullet}$. Then there exists

$$\varepsilon_\star = \varepsilon_\star(I, \Delta_{e_\star,\Xi_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_{\Phi_\star}, \Delta_{\phi_\star}, \Delta_\Phi, \Delta_\phi, \sup_{u,v} ||\nabla\!\!\!/^3\tau||_{L^2(\mathcal{S}_{u,v})})$$

such that for $\varepsilon \leq \varepsilon_\star$, we have

$$\Delta_\Phi(\mathcal{S}) < \infty, \qquad \Delta_\phi(\mathcal{S}) < \infty.$$

With the results above, we gather all the estimates for connection coefficients and derivative of conformal factor

Proposition 37 (***estimates for the L^∞, L^4 and L^2 norms of the connection coefficients and the derivatives of the metric***)**.** *Assume*

$$\Delta_\Phi < \infty, \qquad \Delta_\phi < \infty,$$

in the truncated diamond $\mathcal{D}^t_{u,v_\bullet}$. Then there exists

$$\varepsilon_\star = \varepsilon_\star(I, \Delta_{e_\star,\Xi_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_{\Phi_\star}, \Delta_{\phi_\star}, \Delta_\Phi, \Delta_\phi)$$

such that for $\varepsilon \leq \varepsilon_\star$, we have

$$\begin{aligned}
&\sup_{u,v} \sup_{\Gamma \in \{\mu,\lambda,\alpha,\beta,\epsilon\}} \left(\sum_{i=0}^{1} ||\nabla\!\!\!/^i \Gamma||_{L^\infty(\mathcal{S}_{u,v})} + \sum_{i=0}^{2} ||\nabla\!\!\!/^i \Gamma||_{L^4(\mathcal{S}_{u,v})} + \sum_{i=0}^{3} ||\nabla\!\!\!/^i \Gamma||_{L^2(\mathcal{S}_{u,v})} \right) \\
&\leq C(\Delta_{e_\star,\Xi_\star}, \Delta_{\Gamma_\star}), \\
&\sup_{u,v} \left(||\{\rho,\sigma\}||_{L^\infty(\mathcal{S}_{u,v})} + \sum_{i=0}^{1} ||\nabla\!\!\!/^i \{\rho,\sigma\}||_{L^4(\mathcal{S}_{u,v})} + \sum_{i=0}^{2} ||\nabla\!\!\!/^i \{\rho,\sigma\}||_{L^2(\mathcal{S}_{u,v})} \right) \\
&\leq C(\Delta_{e_\star,\Xi_\star}, \Delta_{\Gamma_\star}), \\
&\sup_{u,v} \left(||\{\tau,\chi\}||_{L^\infty(\mathcal{S}_{u,v})} + \sum_{i=0}^{1} ||\nabla\!\!\!/^i \{\tau,\chi\}||_{L^4(\mathcal{S}_{u,v})} + \sum_{i=0}^{2} ||\nabla\!\!\!/^i \{\tau,\chi\}||_{L^2(\mathcal{S}_{u,v})} \right) \\
&\leq C(I, \Delta_{e_\star,\Xi_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_{\Phi_\star}, \Delta_{\phi_\star}), \\
&\sup_{u,v} \left(||\nabla\!\!\!/ \{\rho,\sigma\}||_{L^\infty(\mathcal{S}_{u,v})} + ||\nabla\!\!\!/^2 \{\rho,\sigma\}||_{L^4(\mathcal{S}_{u,v})} + ||\nabla\!\!\!/^3 \{\rho,\sigma\}||_{L^2(\mathcal{S}_{u,v})} \right) \\
&\leq C(I, \Delta_{e_\star,\Xi_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_\Phi, \Delta_\phi), \\
&\sup_{u,v} \left(||\nabla\!\!\!/ \{\tau,\chi\}||_{L^\infty(\mathcal{S}_{u,v})} + ||\nabla\!\!\!/^2 \{\tau,\chi\}||_{L^4(\mathcal{S}_{u,v})} + ||\nabla\!\!\!/^3 \{\tau,\chi\}||_{L^2(\mathcal{S}_{u,v})} \right) \\
&\leq C(I, \Delta_{e_\star,\Xi_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_\Phi, \Delta_\phi), \\
&\sup_{u,v} \sup_{j=1,\ldots,4} \left(\sum_{i=0}^{1} ||\nabla\!\!\!/^i \Sigma_j||_{L^\infty(\mathcal{S}_{u,v})} + \sum_{i=0}^{2} ||\nabla\!\!\!/^i \Sigma_j||_{L^4(\mathcal{S}_{u,v})} + \sum_{i=0}^{3} ||\nabla\!\!\!/^i \Sigma_j||_{L^2(\mathcal{S}_{u,v})} \right)
\end{aligned}$$

$$\leq C(\Delta_{e_\star,\Xi_\star},\Delta_{\Sigma_\star}),$$

in the truncated diamond $\mathcal{D}^t_{u,v_\bullet}$.

5.4.3 The energy estimates for the curvature

The next step in the boostrap argument leading to the optimal local existence result is to make use of the estimates provided by Proposition 35 to obtain sharper energy estimates for the components of the Ricci and rescaled Weyl curvature spinors. The hierarchical structure of the CEFE allows to proceed with this estimation in a two-step process: first one looks at the components of the Weyl tensor –cf. Propositions 26 and 28 of Chapter 4. In the second step one estmates the components of the Ricci tensor —cf. Propositions 31 and 32. For both the rescaled Weyl tensor and the Ricci tensor the analysis of most of the components is straightforwrad. Only certain *bad* components require extra consideration —the components ϕ_3 and ϕ_4 of the Weyl tensor and the components Φ_{12} and Φ_{22} of the Ricci tensor. The final result of this analysis is the following Proposition estimating the components of the curvature in terms of the initial data. The key ingredient in this proposition is the assumption that the curvature is assumed to be bounded.

Proposition 38 (***control of the components of the curvature in terms of the initial data***)**.** *Suppose we are given a solution to the vacuum CEFE's in Stewart's gauge arising from data for the CIVP satisfying*

$$\Delta_{e_\star,\Xi_\star},\ \Delta_{\Gamma_\star},\ \Delta_{\Sigma_\star},\ \Delta_{\Phi_\star}\ \Delta_{\phi_\star}<\infty,$$

with the solution itself satisfying

$$\sup_{u,v}||\{\mu,\lambda,\alpha,\beta,\epsilon,\rho,\sigma,\tau,\chi,\Sigma_i\}||_{L^\infty(\mathcal{S}_{u,v})}<\infty\,,\quad \sup_{u,v}||\nabla\!\!\!\!/\{\mu,\lambda,\alpha,\beta,\epsilon,\rho,\sigma,\Sigma_i\}||_{L^4(\mathcal{S}_{u,v})}<\infty\,,$$
$$\sup_{u,v}||\nabla\!\!\!\!/^2\{\mu,\lambda,\alpha,\beta,\epsilon,\rho,\sigma,\tau,\Sigma_i\}||_{L^2(\mathcal{S}_{u,v})}<\infty\,,\quad \sup_{u,v}||\nabla\!\!\!\!/^3\{\mu,\lambda,\alpha,\beta,\epsilon,\tau,\Sigma_i\}||_{L^2(\mathcal{S}_{u,v})}<\infty\,,$$
$$\Delta_\Phi(\mathcal{S})<\infty\,,\quad \Delta_\Phi<\infty\,,\quad \Delta_\phi(\mathcal{S})<\infty\,,\quad \Delta_\phi<\infty,$$

on some truncated causal diamond $\mathcal{D}^t_{u,v_\bullet}$. *Then there exists* $\varepsilon_\star=\varepsilon_\star(I,\Delta_{e_\star,\Xi_\star},\Delta_{\Gamma_\star},\Delta_{\Sigma_\star},\Delta_{\Phi_\star},\Delta_{\phi_\star})$

such that for $\varepsilon_\star \leq \varepsilon$ we have

$$\Delta_\Phi < C_1(I, \Delta_{e_\star,\Xi_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_{\Phi_\star}, \Delta_{\phi_\star}),$$
$$\Delta_\phi \leq C_2(I, \Delta_{e_\star,\Xi_\star}, \Delta_{\Gamma_\star}, \Delta_{\Sigma_\star}, \Delta_{\Phi_\star}, \Delta_{\phi_\star}).$$

5.4.4 Last slice argument

The estimates discussed in the previous subsections can be used, in turn, to show that the solution to the conformal Einstein field equations exist on a rectangular domain of the form

$$\mathcal{D} \equiv \{0 \leq u \leq \varepsilon,\ 0 \leq v \leq v_\bullet\},$$

with $v_\bullet$ such that $\Xi|_{v=v_\bullet} = 0$. Accordingly, the set $\{v = v_\bullet\} \cap \mathcal{D}$ corresponds to a portion of future null infinity $\mathscr{I}^+$. The strategy to show this result is similar to the one used in Chapter 3 and Chapter 4 and is based on a *last slice argument.* In this scheme one argues by contradiction and assumes that the solution does not fill the whole domain $\mathcal{D}$. Accordingly, there must exist a hypersurface (the last slice) which bounds the domain of existence of the solution. The estimates constructed in the previous subsections allow then to show that in this last slice the solution and its derivatives are bounded so that it is possible to formulate a (standard) initial value problem for the conformal Einstein field equations to show that the solution extends beyond the last slice —thus resulting in a contradiction.

As the workings of the last slice argument have been discussed in detail in Chapter 3 —see Section 3.6 of this reference— here we focus on the modifications that need to be taken into account due to the peculiarities of the problem at hand. As the main purpose of the present analysis is to ensure that one recovers a portion of future null infinity, in order to ensure existence of the solution to the CEFE on the domain $\mathcal{D}$ one actually needs to show existence in a slightly larger domain. This is because the existence domains are given in terms of open sets. As the CEFE are regular at the sets where $\Xi = 0$, one can consider an initial hypersurface $\mathcal{N}_\star$ which extends beyond $\mathscr{I}^+$. The basic initial data on $\mathcal{N}_\star$ as described in Proposition 20 can be extended in an arbitrary, but controlled, manner beyond the intersection of null infinity with $\mathscr{I}^+$ up to, say $v_\bullet + \frac{1}{10}$, in such a way that it coincides with the original data for $v \in [0, v_\bullet]$ —see Figure 5.4. In particular we require that the extension is

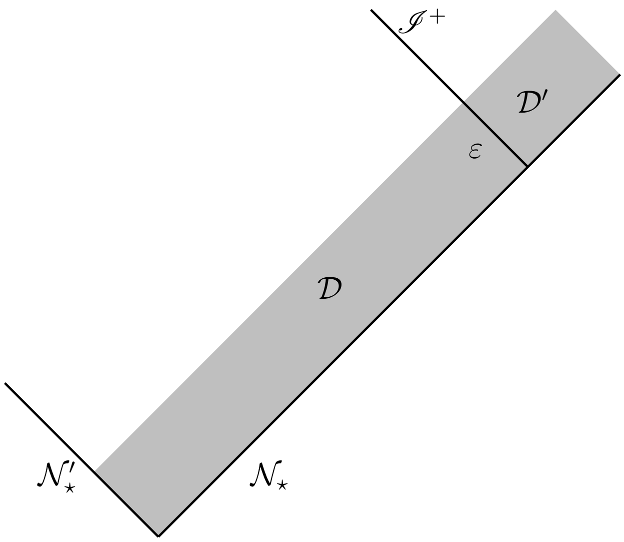

Figure 5.4: Extension of the initial data on $\mathcal{N}_\star$. By causality the choice of the extension of the data beyond $\mathscr{I}^+$ does not influence the solution of the causal domain $\mathcal{D}$.

such that the norms $\Delta_{e_\star,\Xi_\star}$, $\Delta_{\Gamma_\star}$, $\Delta_{\Sigma_\star}$, $\Delta_{\Phi_\star}$ and $\Delta_{\phi_\star}$ which have a contribution along $\mathcal{N}_\star$ are finite. Using this extended data on $\mathcal{N}_\star$ together with the data on $\mathcal{N}'_\star$ and $\mathcal{S}_\star$ one can compute the full initial data set for the conformal evolution equations. The last slice argument as discussed in Chapter 3 and Chapter 4 can then be used to ensure existence on

$$\mathcal{D}' \equiv \{0 \leq u \leq \varepsilon,\ 0 \leq v \leq v_\bullet + \tfrac{1}{10}\} \supset \mathcal{D}.$$

As a consequence of Lemma 21, one has that the set defined by the condition $v = v_\bullet$ is a null hypersurface and, accordingly, our domain of existence contains a portion of $\mathscr{I}^+$. Finally, observe that by causality the solution on $\mathcal{D}$ is independent of the choice of extended data on $\{u = 0,\ v \in (v_\bullet, v_\bullet + \tfrac{1}{10}\}$ —$\mathscr{I}^+$ is the Cauchy horizon of the data on $\{v = 0,\ u \in [0,\varepsilon\} \cup \{u = 0,\ v \in [0, v_\bullet]\}$.

Chapter 6

Outlook and Conclusions

6.1 Summary

Chapters 3, 4 and 5 present the main themes of this thesis—the local existence problem of CIVP of EFE and CEFE and applications. In Chapter 3, following Luk's strategy, we use NP language to study the existence of CIVP for EFE under Stewart's gauge. Adapting the same strategy, in Chapter 4, we analyze the CIVP of CEFE and demonstrate the local existence of solutions in a narrow rectangle along null infinity. Based on the aforementioned result, in Chapter 5, we make use of CEFE to show the existence of CIVP on the local extension of the future null infinity.

6.2 Future Directions

Naturally, the first direction is the CIVP of CEFE coupled with a matter field, like the electromagnetic field, conformally invariant scalar field and radiation perfect fluids. These fields possess the property of a trace-free energy-momentum tensor which leads to a simple transformation law for the equations for the matter models. We hope to explore the equations of these matter models and find a similar hierarchy structure to discuss the existence problem in the rectangular area. The second direction is to use the tool of CEFE to explore the perturbation problem of the Kerr horizon. For this purpose, we shall discuss the Killing spinor data [46] in an unphysical spacetime. Finally, one could try to adapt the methods of An [47] which are also based on the characteristic problem to the CEFE to study the formation of trapped surfaces.

Appendix A

Field equations in NP formalism

In this appendix, we provide the NP equations for structure equations, the Bianchi identities and the conformal Einstein equations. All the fields are defined in unphysical spacetime $\mathcal{M}$.

Given the NP frame $\{l^a,\ n^a,\ m^a,\ \bar{m}^a\}$, the following is the definition the complex spin connection coefficients and curvature with NP frame:

$$\kappa = -m^a l^b \nabla_b l_a, \quad \rho = -m^a \bar{m}^b \nabla_b l_a, \quad \sigma = -m^a m^b \nabla_b l_a, \quad \tau = -m^a n^b \nabla_b l_a, \tag{1a}$$

$$\nu = \bar{m}^a n^b \nabla_b n_a, \quad \mu = \bar{m}^a m^b \nabla_b n_a, \quad \lambda = \bar{m}^a \bar{m}^b \nabla_b n_a, \quad \pi = \bar{m}^a l^b \nabla_b n_a, \tag{1b}$$

$$\alpha = \frac{1}{2}(l^a \bar{m}^b \nabla_b n_a - m^a \bar{m}^b \nabla_b \bar{m}_a), \quad \beta = \frac{1}{2}(\bar{m}^a m^b \nabla_b m_a - n^a m^b \nabla_b l_a), \tag{1c}$$

$$\epsilon = \frac{1}{2}(\bar{m}^a l^b \nabla_b m_a - n^a l^b \nabla_b l_a), \quad \gamma = \frac{1}{2}(l^a n^b \nabla_b n_a - m^a n^b \nabla_b \bar{m}_a), \tag{1d}$$

and

$$\Psi_0 = C_{abcd} l^a m^b l^c m^d, \quad \Psi_1 = C_{abcd} l^a n^b l^c m^d, \quad \Psi_2 = \frac{1}{2} C_{abcd} l^a n^b (l^c n^d - m^c \bar{m}^d) \tag{2a}$$

$$\Psi_3 = C_{abcd} n^a l^b n^c \bar{m}^d, \quad \Psi_4 = C_{abcd} n^a \bar{m}^b n^c \bar{m}^d, \tag{2b}$$

$$\Phi_{00} = \frac{1}{2} R_{\{ab\}} l^a l^b, \quad \Phi_{01} = \frac{1}{2} R_{\{ab\}} l^a m^b, \quad \Phi_{02} = \frac{1}{2} R_{\{ab\}} m^a m^b, \tag{2c}$$

$$\Phi_{11} = \frac{1}{4} R_{\{ab\}} (l^a n^b + m^a \bar{m}^b), \quad \Phi_{12} = \frac{1}{2} R_{\{ab\}} n^a m^b, \tag{2d}$$

$$\Phi_{22} = \frac{1}{2} R_{\{ab\}} n^a n^b, \quad \Lambda = -\frac{R}{24}, \tag{2e}$$

The structure equation

$$\Delta\epsilon - D\gamma = \Lambda - \Phi_{11} - \Psi_2 + \epsilon(2\gamma + \bar{\gamma}) + \gamma\bar{\epsilon} + \kappa\nu - \beta\pi - \alpha\bar{\pi} - \alpha\tau - \pi\tau - \beta\bar{\tau}, \quad (3a)$$

$$\Delta\kappa - D\tau = -\Phi_{01} - \Psi_1 + 3\gamma\kappa + \bar{\gamma}\kappa - \bar{\pi}\rho - \pi\sigma - \epsilon\tau + \bar{\epsilon}\tau - \rho\tau - \sigma\bar{\tau}, \quad (3b)$$

$$\Delta\pi - D\nu = -\Phi_{21} - \Psi_3 + 3\epsilon\nu + \bar{\epsilon}\nu - \gamma\pi + \bar{\gamma}\pi - \mu\pi - \lambda\bar{\pi} - \lambda\tau - \mu\bar{\tau}, \quad (3c)$$

$$\delta\gamma - \Delta\beta = \Phi_{12} - \bar{\alpha}\gamma - 2\beta\gamma + \beta\bar{\gamma} + \alpha\bar{\lambda} + \beta\mu - \epsilon\bar{\nu} - \nu\sigma + \gamma\tau + \mu\tau, \quad (3d)$$

$$\delta\epsilon - D\beta = -\Psi_1 + \bar{\alpha}\epsilon + \beta\bar{\epsilon} + \gamma\kappa + \kappa\mu - \epsilon\bar{\pi} - \beta\bar{\rho} - \alpha\sigma - \pi\sigma, \quad (3e)$$

$$\delta\kappa - D\sigma = -\Psi_0 + \bar{\alpha}\kappa + 3\beta\kappa - \kappa\bar{\pi} - 3\epsilon\sigma + \bar{\epsilon}\sigma - \rho\sigma - \bar{\rho}\sigma + \kappa\tau, \quad (3f)$$

$$\delta\nu - \Delta\mu = \Phi_{22} + \lambda\bar{\lambda} + \gamma\mu + \bar{\gamma}\mu + \mu^2 - \bar{\alpha}\nu - 3\beta\nu - \bar{\nu}\pi + \nu\tau, \quad (3g)$$

$$\delta\pi - D\mu = -2\Lambda - \Psi_2 + \epsilon\mu + \bar{\epsilon}\mu + \kappa\nu + \bar{\alpha}\pi - \beta\pi - \pi\bar{\pi} - \mu\bar{\rho} - \lambda\sigma, \quad (3h)$$

$$\delta\tau - \Delta\sigma = \Phi_{02} - \kappa\bar{\nu} + \bar{\lambda}\rho - 3\gamma\sigma + \bar{\gamma}\sigma + \mu\sigma - \bar{\alpha}\tau + \beta\tau + \tau^2, \quad (3i)$$

$$\bar{\delta}\beta - \delta\alpha = -\Lambda - \Phi_{11} + \Psi_2 - \alpha\bar{\alpha} + 2\alpha\beta - \beta\bar{\beta} - \epsilon\mu + \epsilon\bar{\mu} - \gamma\rho - \mu\rho + \gamma\bar{\rho} + \lambda\sigma, \quad (3j)$$

$$\bar{\delta}\gamma - \Delta\alpha = \Psi_3 - \bar{\beta}\gamma - \alpha\bar{\gamma} + \beta\lambda + \alpha\bar{\mu} - \epsilon\nu - \nu\rho + \lambda\tau + \gamma\bar{\tau}, \quad (3k)$$

$$\bar{\delta}\epsilon - D\alpha = -\Phi_{10} + 2\alpha\epsilon + \bar{\beta}\epsilon - \alpha\bar{\epsilon} + \gamma\bar{\kappa} + \kappa\lambda - \epsilon\pi - \alpha\rho - \pi\rho - \beta\bar{\sigma}, \quad (3l)$$

$$\bar{\delta}\kappa - D\rho = -\Phi_{00} + 3\alpha\kappa + \bar{\beta}\kappa - \kappa\pi - \epsilon\rho - \bar{\epsilon}\rho - \rho^2 - \sigma\bar{\sigma} + \bar{\kappa}\tau, \quad (3m)$$

$$\bar{\delta}\mu - \delta\lambda = -\Phi_{21} + \Psi_3 - \bar{\alpha}\lambda + 3\beta\lambda - \alpha\mu - \bar{\beta}\mu - \mu\pi + \bar{\mu}\pi - \nu\rho + \nu\bar{\rho}, \quad (3n)$$

$$\bar{\delta}\nu - \Delta\lambda = \Psi_4 + 3\gamma\lambda - \bar{\gamma}\lambda + \lambda\mu + \lambda\bar{\mu} - 3\alpha\nu - \bar{\beta}\nu - \nu\pi + \nu\bar{\tau}, \quad (3o)$$

$$\bar{\delta}\pi - D\lambda = -\Phi_{20} + 3\epsilon\lambda - \bar{\epsilon}\lambda + \bar{\kappa}\nu - \alpha\pi + \bar{\beta}\pi - \pi^2 - \lambda\rho - \mu\bar{\sigma}, \quad (3p)$$

$$\bar{\delta}\sigma - \delta\rho = -\Phi_{01} + \Psi_1 - \kappa\mu + \kappa\bar{\mu} - \bar{\alpha}\rho - \beta\rho + 3\alpha\sigma - \bar{\beta}\sigma - \rho\tau - \bar{\rho}\tau, \quad (3q)$$

$$\bar{\delta}\tau - \Delta\rho = 2\Lambda + \Psi_2 - \kappa\nu - \gamma\rho - \bar{\gamma}\rho + \bar{\mu}\rho + \lambda\sigma + \alpha\tau - \bar{\beta}\tau + \tau\bar{\tau}. \quad (3r)$$

The Bianchi identities

$$\begin{aligned}\bar{\delta}\Psi_0 - D\Psi_1 + D\Phi_{01} - \delta\Phi_{00} &= (4\alpha - \pi)\Psi_0 - 2(2\rho + \epsilon)\Psi_1 + 3\kappa\Psi_2 \\ &+ (\bar{\pi} - 2\bar{\alpha} - 2\beta)\Phi_{00} + 2(\epsilon + \bar{\rho})\Phi_{01} + 2\sigma\Phi_{10} - 2\kappa\Phi_{11} - \bar{\kappa}\Phi_{02}, \quad (4a)\end{aligned}$$

$$\begin{aligned}\Delta\Psi_0 - \delta\Psi_1 + D\Phi_{02} - \delta\Phi_{01} &= (4\gamma - \mu)\Psi_0 - 2(2\tau + \beta)\Psi_1 + 3\sigma\Psi_2 - \bar{\lambda}\Phi_{00} \\ &+ 2(\bar{\pi} - \beta)\Phi_{01} + 2\sigma\Phi_{11} + (\bar{\rho} + 2\epsilon - 2\bar{\epsilon})\Phi_{02} - 2\kappa\Phi_{12}, \quad (4b)\end{aligned}$$

$$\bar{\delta}\Psi_3 - D\Psi_4 + \bar{\delta}\Phi_{21} - \Delta\Phi_{20} = (4\epsilon - \rho)\Psi_4 - 2(2\pi + \alpha)\Psi_3 + 3\lambda\Psi_2 + 2\lambda\Phi_{11}$$

$$- 2\nu\Phi_{10} - \bar{\sigma}\Phi_{22} + (2\gamma - 2\bar{\gamma} + \bar{\mu})\Phi_{20} + 2(\bar{\tau} - \alpha)\Phi_{21}, \tag{4c}$$

$$\Delta\Psi_3 - \delta\Psi_4 + \bar{\delta}\Phi_{22} - \Delta\Phi_{21} = (4\beta - \tau)\Psi_4 - 2(2\mu + \gamma)\Psi_3 + 3\nu\Psi_2 + 2\lambda\Phi_{12}$$

$$- 2\nu\Phi_{11} - \bar{\nu}\Phi_{20} + (\bar{\tau} - 2\bar{\beta} - 2\alpha)\Phi_{22} + 2(\gamma + \bar{\mu})\Phi_{21}, \tag{4d}$$

$$D\Psi_2 - \bar{\delta}\Psi_1 + \Delta\Phi_{00} - \bar{\delta}\Phi_{01} + 2D\Lambda = -\lambda\Psi_0 + 2(\pi - \alpha)\Psi_1 + 3\rho\Psi_2 - 2\kappa\Psi_3$$

$$- 2\tau\Phi_{10} + 2\rho\Phi_{11} + \bar{\sigma}\Phi_{02} + (2\gamma + 2\bar{\gamma} - \bar{\mu})\Phi_{00} - 2(\bar{\tau} + \alpha)\Phi_{01}, \tag{4e}$$

$$\Delta\Psi_2 - \delta\Psi_3 + D\Phi_{22} - \delta\Phi_{21} + 2\Delta\Lambda = \sigma\Psi_4 + 2(\beta - \tau)\Psi_3 - 3\mu\Psi_2 + 2\nu\Psi_1$$

$$+ 2\pi\Phi_{12} - 2\mu\Phi_{11} - \bar{\lambda}\Phi_{20} + (\bar{\rho} - 2\epsilon - 2\bar{\epsilon})\Phi_{22} + 2(\bar{\pi} + \beta)\Phi_{21}, \tag{4f}$$

$$D\Psi_3 - \bar{\delta}\Psi_2 - D\Phi_{21} + \delta\Phi_{20} - 2\bar{\delta}\Lambda = -\kappa\Psi_4 + 2(\rho - \epsilon)\Psi_3 + 3\pi\Psi_2 - 2\lambda\Psi_1$$

$$- 2\pi\Phi_{11} + 2\mu\Phi_{10} + \bar{\kappa}\Phi_{22} + (2\bar{\alpha} - 2\beta - \bar{\pi})\Phi_{20} - 2(\bar{\rho} - \epsilon)\Phi_{21}, \tag{4g}$$

$$\Delta\Psi_1 - \delta\Psi_2 - \Delta\Phi_{01} + \bar{\delta}\Phi_{02} - 2\delta\Lambda = \nu\Psi_0 + 2(\gamma - \mu)\Psi_1 - 3\tau\Psi_2 + 2\sigma\Psi_3$$

$$+ 2\tau\Phi_{11} - 2\rho\Phi_{12} - \bar{\nu}\Phi_{00} + (\bar{\tau} - 2\bar{\beta} + 2\alpha)\Phi_{02} + 2(\bar{\mu} - \gamma)\Phi_{01}, \tag{4h}$$

$$D\Phi_{11} - \delta\Phi_{10} - \bar{\delta}\Phi_{01} + \Delta\Phi_{00} + 3D\Lambda = (2\gamma - \mu + 2\bar{\gamma} - \bar{\mu})\Phi_{00} + (\pi - 2\alpha - 2\bar{\tau})\Phi_{01}$$

$$+ \bar{\sigma}\Phi_{02} + \sigma\Phi_{20} + (\bar{\pi} - 2\bar{\alpha} - 2\tau)\Phi_{10} + 2(\rho + \bar{\rho})\Phi_{11} - \bar{\kappa}\Phi_{12} - \kappa\Phi_{21}, \tag{4i}$$

$$D\Phi_{12} - \delta\Phi_{11} - \bar{\delta}\Phi_{02} + \Delta\Phi_{01} + 3\delta\Lambda = (-2\alpha + 2\bar{\beta} + \pi - \bar{\tau})\Phi_{02} + (\bar{\rho} + 2\rho - 2\bar{\epsilon})\Phi_{12}$$

$$+ \bar{\nu}\Phi_{00} - \bar{\lambda}\Phi_{10} + 2(\bar{\pi} - \tau)\Phi_{11} + (2\gamma - 2\bar{\mu} - \mu)\Phi_{01} + \sigma\Phi_{21} - \kappa\Phi_{22}, \tag{4j}$$

$$D\Phi_{22} - \delta\Phi_{21} - \bar{\delta}\Phi_{12} + \Delta\Phi_{11} + 3\Delta\Lambda = (\rho + \bar{\rho} - 2\epsilon - 2\bar{\epsilon})\Phi_{22} + (2\bar{\beta} + 2\pi - \bar{\tau})\Phi_{12}$$

$$+ \nu\Phi_{01} + \bar{\nu}\Phi_{10} + (2\beta + 2\bar{\pi} - \tau)\Phi_{21} - 2(\mu + \bar{\mu})\Phi_{11} - \bar{\lambda}\Phi_{20} - \lambda\Phi_{02}. \tag{4k}$$

Conformal vacuum Einstein field equations

CFE1

The spinorial counterpart of the first equation (11)

$$\nabla_{BB'}\nabla_{AA'}\Xi = -\Xi\Phi_{ABA'B'} + s\epsilon_{AB}\bar{\epsilon}_{A'B'} + \Xi\Lambda\epsilon_{AB}\bar{\epsilon}_{A'B'}.$$

When we decompose it by the Newman-Penrose null tetrad, we have:

$$-\Sigma_1(\epsilon + \bar{\epsilon}) + \Sigma_4\kappa + \Sigma_3\bar{\kappa} + D\Sigma_1 = -\Xi\Phi_{00}, \tag{5a}$$

$$\Sigma_2(\epsilon + \bar{\epsilon}) - \Sigma_3\pi - \Sigma_4\bar{\pi} + D\Sigma_2 = s + \Xi\Lambda - \Xi\Phi_{11}, \tag{5b}$$

$$-\Sigma_3(\epsilon - \bar{\epsilon}) + \Sigma_2\kappa - \Sigma_1\bar{\pi} + D\Sigma_3 = -\Xi\Phi_{01}, \tag{5c}$$

$$-\Sigma_1(\gamma+\bar{\gamma})+\Sigma_4\tau+\Sigma_3\bar{\tau}+\Delta\Sigma_1 = s+\Xi\Lambda-\Xi\Phi_{11}, \tag{5d}$$
$$\Sigma_2(\gamma+\bar{\gamma})-\Sigma_3\nu-\Sigma_4\bar{\nu}+\Delta\Sigma_2 = -\Xi\Phi_{22}, \tag{5e}$$
$$-\Sigma_3(\gamma-\bar{\gamma})-\Sigma_1\bar{\nu}+\Sigma_2\tau+\Delta\Sigma_3 = -\Xi\Phi_{12}, \tag{5f}$$
$$-\Sigma_1(\bar{\alpha}+\beta)+\Sigma_3\bar{\rho}+\Sigma_4\sigma+\delta\Sigma_1 = -\Xi\Phi_{01}, \tag{5g}$$
$$\Sigma_2(\bar{\alpha}+\beta)-\Sigma_4\bar{\lambda}-\Sigma_3\mu+\delta\Sigma_2 = -\Xi\Phi_{12}, \tag{5h}$$
$$-\Sigma_3(-\bar{\alpha}+\beta)-\Sigma_1\bar{\lambda}+\Sigma_2\sigma+\delta\Sigma_3 = -\Xi\Phi_{02}, \tag{5i}$$
$$\Sigma_4(-\bar{\alpha}+\beta)-\Sigma_1\mu+\Sigma_2\bar{\rho}+\delta\Sigma_4 = -s-\Xi\Lambda-\Xi\Phi_{11}. \tag{5j}$$

CFE2

The spinorial counterpart of the second equation (6) is

$$\nabla_{AA'}s = \Lambda\nabla_{AA'}\Xi - \Phi_{ABA'B'}\nabla^{BB'}\Xi,$$

we can decompose it by Newman-Penrose null tetrad:

$$-Ds = -\Sigma_1\Lambda+\Sigma_2\Phi_{00}-\Sigma_4\Phi_{01}-\Sigma_3\Phi_{10}+\Sigma_1\Phi_{11}, \tag{6a}$$
$$-\Delta s = -\Sigma_2\Lambda+\Sigma_2\Phi_{11}-\Sigma_4\Phi_{12}-\Sigma_3\Phi_{21}+\Sigma_1\Phi_{22}, \tag{6b}$$
$$-\delta s = -\Sigma_3\Lambda+\Sigma_2\Phi_{01}-\Sigma_4\Phi_{02}-\Sigma_3\Phi_{11}+\Sigma_1\Phi_{12}. \tag{6c}$$

CFE3

The spinorial counterpart of the third equation (2.20e) is

$$\begin{aligned}\nabla_{AA'}\Phi_{BCB'C'}-\nabla_{BB'}\Phi_{ACA'C'} &= \epsilon_{BC}\bar{\epsilon}_{B'C'}\nabla_{AA'}\Lambda-\epsilon_{AC}\bar{\epsilon}_{A'C'}\nabla_{BB'}\Lambda\\ &-\bar{\phi}_{A'B'C'D'}\epsilon_{AB}\nabla_{C}{}^{D'}\Xi-\phi_{ABCD}\bar{\epsilon}_{A'B'}\nabla^{D}{}_{C'}\Xi.\end{aligned}$$

The components of this equation with respect to the null tetrad are:

$$\begin{aligned}\Delta\Phi_{00}-\delta\Phi_{10}+2D\Lambda &= \Phi_{00}(2\gamma+2\bar{\gamma}-\mu)-2\Phi_{10}(\bar{\alpha}+\tau)-2\Phi_{01}\bar{\tau}+2\Phi_{11}\bar{\rho}+\Phi_{20}\sigma\\ &+\Sigma_3\bar{\phi}_1-\Sigma_1\bar{\phi}_2\end{aligned} \tag{7a}$$
$$\begin{aligned}\Delta\Phi_{01}-\delta\Phi_{11}+\delta\Lambda &= \Phi_{01}(2\gamma-\mu)+\Phi_{00}\bar{\nu}+\Phi_{12}\bar{\rho}+\Phi_{21}\sigma-\Phi_{10}\bar{\lambda}-2\Phi_{11}\tau-\Phi_{02}\bar{\tau}\\ &+\Sigma_3\bar{\phi}_2-\Sigma_1\bar{\phi}_1,\end{aligned} \tag{7b}$$

$$\Delta\Phi_{02} - \delta\Phi_{12} = \Phi_{02}(2\gamma - 2\bar{\gamma} - \mu) + 2\Phi_{12}(\bar{\alpha} - \tau) + 2\Phi_{01}\bar{\nu} + \Phi_{22}\sigma - 2\Phi_{11}\bar{\lambda}$$
$$+\Sigma_3\bar{\phi}_3 - \Sigma_1\bar{\phi}_4, \qquad (7c)$$

$$\Delta\Phi_{11} - \delta\Phi_{21} + \Delta\Lambda = \Phi_{01}\nu + \Phi_{10}\bar{\nu} + \Phi_{21}(2\beta - \tau) + \Phi_{22}\bar{\rho} - \Phi_{20}\bar{\lambda} - 2\Phi_{11}\mu - \Phi_{12}\bar{\tau}$$
$$+\Sigma_2\bar{\phi}_2 - \Sigma_4\bar{\phi}_3, \qquad (7d)$$

$$\Delta\Phi_{12} - \delta\Phi_{22} = \Phi_{22}(2\bar{\alpha} + 2\beta - \tau) + \Phi_{02}\nu + 2\Phi_{11}\bar{\nu} - 2\Phi_{12}(\bar{\gamma} + \mu) - 2\Phi_{21}\bar{\lambda}$$
$$+\Sigma_2\bar{\phi}_3 - \Sigma_4\bar{\phi}_4, \qquad (7e)$$

$$D\Phi_{22} - \delta\Phi_{21} + 2\Delta\Lambda = \Phi_{22}(\bar{\rho} - 2\epsilon - 2\bar{\epsilon}) + 2\Phi_{21}(\beta + \bar{\pi}) + 2\Phi_{12}\pi - \Phi_{20}\bar{\lambda} - 2\Phi_{11}\mu$$
$$+\Sigma_3\phi_3 - \Sigma_2\phi_2 \qquad (7f)$$

$$D\Phi_{12} - \delta\Phi_{11} + \delta\Lambda = \Phi_{02}\pi + 2\Phi_{11}\bar{\pi} + \Phi_{12}(\bar{\rho} - 2\bar{\epsilon}) + \Phi_{21}\sigma - \Phi_{22}\kappa - \Phi_{10}\bar{\lambda} - \Phi_{01}\mu$$
$$-\Sigma_2\phi_1 + \Sigma_3\phi_2, \qquad (7g)$$

$$D\Phi_{11} - \delta\Phi_{10} + D\Lambda = \Phi_{01}\pi + \Phi_{10}(\bar{\pi} - 2\bar{\alpha}) + 2\Phi_{11}\bar{\rho} + \Phi_{20}\sigma - \Phi_{21}\kappa - \Phi_{12}\bar{\kappa} - \Phi_{00}\mu$$
$$-\Sigma_4\phi_1 + \Sigma_1\phi_2, \qquad (7h)$$

$$D\Phi_{02} - \delta\Phi_{01} = \Phi_{02}(2\epsilon - 2\bar{\epsilon} + \bar{\rho}) + 2\Phi_{01}(\bar{\pi} - \beta) + 2\Phi_{11}\sigma - 2\Phi_{12}\kappa - \Phi_{00}\bar{\lambda}$$
$$-\Sigma_2\phi_0 + \Sigma_3\phi_1, \qquad (7i)$$

$$D\Phi_{01} - \delta\Phi_{00} = 2\Phi_{01}(\epsilon + \bar{\rho}) + \Phi_{00}\bar{\pi} + 2\Phi_{10}\sigma - 2\Phi_{00}(\bar{\alpha} + \beta) - 2\Phi_{11}\kappa - \Phi_{02}\bar{\kappa}$$
$$-\Sigma_4\phi_0 + \Sigma_1\phi_1, \qquad (7j)$$

$$\delta\Phi_{10} - \bar{\delta}\Phi_{01} = \Phi_{00}(\mu - \bar{\mu}) + 2\Phi_{11}(\rho - \bar{\rho}) + 2\Phi_{10}\bar{\alpha} + \Phi_{02}\bar{\sigma} - 2\Phi_{01}\alpha - \Phi_{20}\sigma$$
$$+\Sigma_4\phi_1 - \Sigma_3\bar{\phi}_1 - \Sigma_1\phi_2 + \Sigma_1\bar{\phi}_2, \qquad (7k)$$

$$\delta\Phi_{11} - \bar{\delta}\Phi_{02} + \delta\Lambda = 2\Phi_{02}(\bar{\beta} - \alpha) + \Phi_{01}(\mu - 2\bar{\mu}) + \Phi_{12}(2\rho - \bar{\rho}) + \Phi_{10}\bar{\lambda} - \Phi_{21}\sigma$$
$$+\Sigma_2\phi_1 - \Sigma_3(\phi_2 + \bar{\phi}_2) + \Sigma_1\bar{\phi}_3, \qquad (7l)$$

$$\delta\Phi_{21} - \bar{\delta}\Phi_{12} = 2\Phi_{11}(\mu - \bar{\mu}) + \Phi_{22}(\rho - \bar{\rho}) + 2\Phi_{12}\bar{\beta} + \Phi_{20}\bar{\lambda} - 2\Phi_{21}\beta - \Phi_{02}\lambda$$
$$+\Sigma_2(\phi_2 - \bar{\phi}_2) - \Sigma_3\phi_3 + \Sigma_4\bar{\phi}_3. \qquad (7m)$$

CFE4

The spinorial counterpart of the forth equation (2.20f) is

$$\nabla_{DC'}\phi_{ABC}{}^{D} = 0.$$

Decomposing it by the null tetrad, we obtain:

$$\Delta\phi_0 - \delta\phi_1 = -2\phi_1(\beta + 2\tau) + \phi_0(4\gamma - \mu) + 3\phi_2\sigma, \tag{8a}$$

$$\Delta\phi_1 - \delta\phi_2 = 2\phi_1(\gamma - \mu) + \phi_0\nu + 2\phi_3\sigma - 3\phi_2\tau, \tag{8b}$$

$$\Delta\phi_2 - \delta\phi_3 = 2\phi_3(\beta - \tau) - 3\phi_2\mu + 2\phi_1\nu + \phi_4\sigma, \tag{8c}$$

$$\Delta\phi_3 - \delta\phi_4 = \phi_4(4\beta - \tau) + 3\phi_2\nu - 2\phi_3(\gamma + 2\mu), \tag{8d}$$

$$D\phi_1 - \bar{\delta}\phi_0 = \phi_0(\pi - 4\alpha) + 2\phi_1(\epsilon + 2\rho) - 3\phi_2\kappa, \tag{8e}$$

$$D\phi_2 - \bar{\delta}\phi_1 = 2\phi_1(\pi - \alpha) - \phi_0\lambda + 3\phi_2\rho - 2\phi_3\kappa, \tag{8f}$$

$$D\phi_3 - \bar{\delta}\phi_2 = 2\phi_3(\rho - \epsilon) - 2\phi_1\lambda + 3\phi_2\pi - \phi_4\kappa, \tag{8g}$$

$$D\phi_4 - \bar{\delta}\phi_3 = \phi_4(\rho - 4\epsilon) + 2\phi_3(\alpha + 2\pi) - 3\phi_2\lambda. \tag{8h}$$

CFE5

The spinorial counterpart of the fifth equation (9a) is:

$$\lambda = 6\Xi s - 3\nabla_{AA'}\Xi\nabla^{AA'}\Xi,$$

and its components equation is:

$$\lambda = -6\Sigma_1\Sigma_2 + 6\Sigma_3\Sigma_4 + 6\Xi s \tag{9a}$$

Appendix B

Inequalities

In this appendix, as a quick reference, we list the key inequalities which are used routinely in our analysis. These inequalities are standard and proofs can be found, e.g. in [48].

Cauchy-Schwarz inequality. If $u_1, ..., u_n \in \mathbb{C}$ and $v_1, ..., v_n \in \mathbb{C}$, we have

$$|u_1 v_1 + ... + u_n v_n|^2 \leq (|u_1|^2 + ... + |u_n|^2)(|v_1|^2 + ... + |v_n|^2).$$

Grönwall's inequality. If $\beta(t)$ is a non-negative continuous function and $u(t)$ satisfies

$$u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)\mathrm{d}s, \quad \forall t \in [a,\ b],$$

then

$$u(t) \leq \alpha(t) + \int_a^t \alpha(s)\beta(s)\exp\left(\int_s^t \beta(r)\mathrm{d}r\right)\mathrm{d}s, \quad t \in [a,\ b].$$

In addition, if the function α is non-decreasing, then

$$u(t) \leq \alpha(t)\exp\left(\int_a^t \beta(s)\mathrm{d}s\right), \quad t \in [a,\ b].$$

Moreover, if $\beta \equiv C$ where C is a positive constant, then

$$u(t) \leq C(b-a)\alpha(t).$$

Young's inequality. If a and b are non negative real numbers and p and q are positive real numbers such that $1/p + 1/q = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

The equality holds if and only if $a^p = b^q$. Moreover, if a and b are non negative real numbers and $p \geq 1$, then

$$a^p + b^p \leq (a+b)^p.$$

Finally, if $f(x)$ is non-negative continuous function and $p \geq 1$, then

$$\int_K f^p \leq \left(\int_K f \right)^p,$$

where K is a compact set.

Generalised Hölder's inequality. Let K be a measurable space. Assume $f \in L^p(K)$ and $g \in L^q(K)$ with $1 \leq p, q \leq \infty$ and $1/r = 1/p + 1/q \leq 1$, then

$$||fg||_{L^r(K)} \leq ||f||_{L^p(K)} ||g||_{L^q(K)}.$$

Gagliardo-Nirenberg-Sobolev inequality. Let U be a bounded, open subset of $\mathbb{R}^n$, and assume ∂U is C^1. Let $1 \leq p < n$, and suppose that $u \in W^{1,p}(U)$. Then $u \in L^{p*}(U)$, with the estimate,

$$||u||_{L^{p*}(U)} \leq C ||u||_{W^{1,p}(U)}$$

the constant C depending only on p, n and U and $1/p + 1/p^* = 1/n$.

Appendix C

Angular derivatives of a scalar function

In our analysis we make repeated use of properties of the angular derivatives of a scalar field over the 2-spheres $\mathcal{S}_{u,v}$ of constant u, v. In the following let $f : \mathcal{S}_{u,v} \to \mathbb{C}$ denote a sufficiently smooth complex scalar field.

Definitions and basic inequalities

In terms of the NP vectors m^a and $\bar{m}^a$ one has that

$$\nabla\!\!\!/_a f = -m_a \bar{m}^b \nabla\!\!\!/_b f - \bar{m}_a m^b \nabla\!\!\!/_b f = -m_a \bar{\delta} f - \bar{m}_a \delta f.$$

Moreover, we have that

$$|\nabla\!\!\!/ f|^2 \equiv -\sigma^{ab} \overline{\nabla\!\!\!/_a f} \nabla\!\!\!/_b f = \bar{\delta} f \delta \bar{f} + \bar{\delta} \bar{f} \delta f.$$

A direct computation shows that,

$$\begin{aligned}
||\nabla\!\!\!/ f||_{L^p(\mathcal{S}_{u,v})} &= \left(\int_{\mathcal{S}_{u,v}} |\bar{\delta} f \delta \bar{f} + \bar{\delta} \bar{f} \delta f|^{p/2} \right)^{1/p} = |||\delta f|^2 + |\bar{\delta} f|^2||^{1/2}_{L^{p/2}(\mathcal{S}_{u,v})} \\
&\leq \left(|||\delta f|^2||_{L^{p/2}(\mathcal{S}_{u,v})} + |||\bar{\delta} f|^2||_{L^{p/2}(\mathcal{S}_{u,v})} \right)^{1/2} \\
&\leq |||\delta f|^2||^{1/2}_{L^{p/2}(\mathcal{S}_{u,v})} + |||\bar{\delta} f|^2||^{1/2}_{L^{p/2}(\mathcal{S}_{u,v})} \\
&= ||\delta f||_{L^p(\mathcal{S}_{u,v})} + ||\bar{\delta} f||_{L^p(\mathcal{S}_{u,v})}.
\end{aligned}$$

Conversely, we have

$$||\delta f||_{L^p(\mathcal{S}_{u,v})}, \quad ||\bar{\delta} f||_{L^p(\mathcal{S}_{u,v})} \leq \left(\int_{\mathcal{S}_{u,v}} |\bar{\delta} f \delta \bar{f} + \bar{\delta} \bar{f} \delta f|^{p/2} \right)^{1/p} \leq ||\nabla\!\!\!/ f||_{L^p(\mathcal{S}_{u,v})}.$$

Thus, we can estimate $\nabla\!\!\!/ f$ in terms of δf and $\bar{\delta} f$ and vice versa. *This observation is used repeatedly in the main text.*

The Hessian

The Hessian $\nabla\!\!\!/_a \nabla\!\!\!/_b f$ of the scalar function f can be expanded in terms of NP objects as

$$\begin{aligned} \nabla\!\!\!/_a \nabla\!\!\!/_b f = & \left(\bar{\delta}\bar{\delta} f + (\bar{\beta} - \alpha)\bar{\delta} f\right) m_a m_b + \left(\delta\delta f + (\beta - \bar{\alpha})\delta f\right) \bar{m}_a \bar{m}_b \\ & + \left(\bar{\delta}\delta f + (\alpha - \bar{\beta})\delta f\right) m_a \bar{m}_b + \left(\delta\bar{\delta} f + (\bar{\alpha} - \beta)\bar{\delta} f\right) \bar{m}_a m_b, \end{aligned}$$

where we have made use of the expansion

$$\nabla\!\!\!/_a m_b = (\alpha - \bar{\beta}) m_a m_b + (\beta - \bar{\alpha}) \bar{m}_a m_b.$$

Defining, for convenience, the scalars

$$\begin{aligned} T_1 &\equiv \bar{\delta}\bar{\delta} f + (\bar{\beta} - \alpha)\bar{\delta} f, \qquad & T_2 &\equiv \bar{\delta}\delta f + (\alpha - \bar{\beta})\delta f, \\ T_3 &\equiv \delta\bar{\delta} f + (\bar{\alpha} - \beta)\bar{\delta} f, \qquad & T_4 &\equiv \delta\delta f + (\beta - \bar{\alpha})\delta f, \end{aligned}$$

one can then write

$$|\nabla\!\!\!/^2 f|^2 \equiv \sigma^{ab}\sigma^{cd}\overline{\nabla\!\!\!/_a \nabla\!\!\!/_c f}\nabla\!\!\!/_b \nabla\!\!\!/_d f = |T_1|^2 + |T_2|^2 + |T_3|^2 + |T_4|^2.$$

Making use of the above decomposition we then have that

$$\begin{aligned} ||\nabla\!\!\!/^2 f||_{L^p(\mathcal{S}_{u,v})} &= \left(\int_{\mathcal{S}_{u,v}} (|T_1|^2 + |T_2|^2 + |T_3|^2 + |T_4|^2)^{p/2} \right)^{1/p} \leq \textstyle\sum_{i=1}^4 ||T_i||_{L^p(\mathcal{S}_{u,v})} \\ &\leq ||\delta^2 f||_{L^p(\mathcal{S}_{u,v})} + ||\bar{\delta}^2 f||_{L^p(\mathcal{S}_{u,v})} + ||\delta\bar{\delta} f||_{L^p(\mathcal{S}_{u,v})} + ||\bar{\delta}\delta f||_{L^p(\mathcal{S}_{u,v})} \\ &\qquad + 4\Delta_\Gamma (||\delta f||_{L^p(\mathcal{S}_{u,v})} + ||\bar{\delta} f||_{L^p(\mathcal{S}_{u,v})}), \end{aligned} \tag{10}$$

where Δ_Γ is defined as in the main text. Also, observe that $||\not\nabla^2 f||_{L^p(\mathcal{S}_{u,v})}$ is not smaller than any of the individual terms in the right side of the first inequality (10).

A final observation following the *irreducible decomposition*

$$\not\nabla_a \not\nabla_b f = \not\nabla_{\{a} \not\nabla_{b\}} f + \frac{1}{2}\sigma_{ab} \not\Delta f + \not\nabla_{[a} \not\nabla_{b]} f \tag{11}$$

of the Hessian, where the curly brackets denote the symmetric-tracefree part with respect to the metric σ_{ab}, is that

$$|\not\nabla_a \not\nabla_b f|^2 = |\not\nabla_{\{a} \not\nabla_{b\}} f|^2 + \frac{1}{2}|\not\Delta f|^2 + |\not\nabla_{[a} \not\nabla_{b]} f|^2, \tag{12}$$

so that

$$|\not\Delta f|^2 \leq 2|\not\nabla_a \not\nabla_b f|^2 \tag{13}$$

Third derivatives of a scalar field

As in the main text denote by $\varpi \equiv \beta - \bar{\alpha}$ the simple independent component of the connection of the 2-sphere $\mathcal{S}_{u,v}$. It follows from the from the structure equation (3j) and its complex conjugate, that the Gaussian curvature curvature

$$K \equiv 2\varpi\bar{\varpi} + 2\delta\bar{\varpi} + 2\bar{\delta}\varpi$$

satisfies the relation

$$K = \sigma\lambda + \bar{\sigma}\bar{\lambda} - \rho\mu - \bar{\rho}\bar{\mu} + \Psi_2 + \bar{\Psi}_2,$$

see [36] for details.

Now, the third order covariant derivative of f on $\mathcal{S}_{u,v}$ can be expanded as

$$\begin{aligned}\not\nabla_a \not\nabla_b \not\nabla_c f = {} & M_1 m_a m_b m_c + M_5 \bar{m}_a \bar{m}_b \bar{m}_c + M_2 \bar{m}_a m_b m_c + M_6 m_a \bar{m}_b \bar{m}_c \\ & + M_3 m_a m_b \bar{m}_c + M_7 \bar{m}_a \bar{m}_b m_c + M_4 \bar{m}_a m_b \bar{m}_c + M_8 m_a \bar{m}_b m_c,\end{aligned}$$

where,

$$
\begin{aligned}
M_1 &\equiv -(\bar{\delta}^3 f + 3\bar{\varpi}\bar{\delta}^2 f + \bar{\delta}\bar{\varpi}\bar{\delta} f + 2\bar{\varpi}^2\bar{\delta} f), \\
M_2 &\equiv -\delta\bar{\delta}^2 f - \bar{\varpi}\delta\bar{\delta} f + 2\varpi\bar{\delta}^2 f - \delta\bar{\varpi}\bar{\delta} f + 2\varpi\bar{\varpi}\bar{\delta} f, \\
M_3 &\equiv -\bar{\delta}^2\delta f + \bar{\varpi}\bar{\delta}\delta f + \bar{\delta}\bar{\varpi}\delta f, \\
M_4 &\equiv -\delta\bar{\delta}\delta f + \bar{\varpi}\delta^2 f + \delta\bar{\varpi}\delta f, \\
M_5 &\equiv -(\delta^3 f + 3\varpi\delta^2 f + \delta\varpi\delta f + 2\varpi^2\delta f), \\
M_6 &\equiv -\bar{\delta}\delta^2 f - \varpi\bar{\delta}\delta f + 2\bar{\varpi}\delta^2 f - \bar{\delta}\varpi\delta f + 2\varpi\bar{\varpi}\delta f, \\
M_7 &\equiv -\delta^2\bar{\delta} f + \varpi\delta\bar{\delta} f + \delta\varpi\bar{\delta} f, \\
M_8 &\equiv -\bar{\delta}\delta\bar{\delta} f + \varpi\bar{\delta}^2 f + \bar{\delta}\varpi\bar{\delta} f.
\end{aligned}
$$

It follows then that,

$$
|\not\nabla^3 f|^2 = \sum_{i=1}^{8} |M_i|^2.
$$

From the above expression one finds that

$$
\begin{aligned}
||\not\nabla^3 f||_{L^p(\mathcal{S}_{u,v})} \leq\ & ||\delta^3 f||_{L^p(\mathcal{S}_{u,v})} + ||\bar{\delta}^3 f||_{L^p(\mathcal{S}_{u,v})} + ||\delta^2\bar{\delta} f||_{L^p(\mathcal{S}_{u,v})} + ||\delta\bar{\delta}^2 f||_{L^p(\mathcal{S}_{u,v})} \\
&+ ||\bar{\delta}^2\delta f||_{L^p(\mathcal{S}_{u,v})} + ||\bar{\delta}\delta^2 f||_{L^p(\mathcal{S}_{u,v})} + ||\delta\bar{\delta}\delta f||_{L^p(\mathcal{S}_{u,v})} + ||\bar{\delta}\delta\bar{\delta} f||_{L^p(\mathcal{S}_{u,v})} \\
&+ 3||\bar{\varpi}\bar{\delta}^2 f||_{L^p(\mathcal{S}_{u,v})} + ||\bar{\varpi}\delta\bar{\delta} f||_{L^p(\mathcal{S}_{u,v})} + 2||\varpi\bar{\delta}^2 f||_{L^p(\mathcal{S}_{u,v})} + ||\bar{\varpi}\bar{\delta}\delta f||_{L^p(\mathcal{S}_{u,v})} \\
&+ ||\bar{\varpi}\delta^2 f||_{L^p(\mathcal{S}_{u,v})} + 3||\varpi\delta^2 f||_{L^p(\mathcal{S}_{u,v})} + ||\varpi\bar{\delta}\delta f||_{L^p(\mathcal{S}_{u,v})} + 2||\bar{\varpi}\delta^2 f||_{L^p(\mathcal{S}_{u,v})} \\
&+ ||\varpi\delta\bar{\delta} f||_{L^p(\mathcal{S}_{u,v})} + ||\varpi\bar{\delta}^2 f||_{L^p(\mathcal{S}_{u,v})} + ||\bar{\delta}\bar{\varpi}\bar{\delta} f||_{L^p(\mathcal{S}_{u,v})} + ||\delta\bar{\varpi}\bar{\delta} f||_{L^p(\mathcal{S}_{u,v})} \\
&+ ||\bar{\delta}\bar{\varpi}\delta f||_{L^p(\mathcal{S}_{u,v})} + ||\delta\bar{\varpi}\delta f||_{L^p(\mathcal{S}_{u,v})} + ||\delta\varpi\delta f||_{L^p(\mathcal{S}_{u,v})} + ||\bar{\delta}\varpi\delta f||_{L^p(\mathcal{S}_{u,v})} \\
&+ ||\delta\varpi\bar{\delta} f||_{L^p(\mathcal{S}_{u,v})} + ||\bar{\delta}\varpi\bar{\delta} f||_{L^p(\mathcal{S}_{u,v})} + 2||\bar{\varpi}^2\bar{\delta} f||_{L^p(\mathcal{S}_{u,v})} + 2||\varpi\bar{\varpi}\bar{\delta} f||_{L^p(\mathcal{S}_{u,v})} \\
&+ 2||\varpi^2\delta f||_{L^p(\mathcal{S}_{u,v})} + 2||\varpi\bar{\varpi}^2\delta f||_{L^p(\mathcal{S}_{u,v})}.
\end{aligned}
$$

The above expression contains four representative terms, namely $||\delta^3 f||_{L^p(\mathcal{S}_{u,v})}$, $||\varpi\delta^2 f||_{L^p(\mathcal{S}_{u,v})}$, $||\delta\varpi\delta f||_{L^p(\mathcal{S}_{u,v})}$ and $||\varpi^2\delta f||_{L^p(\mathcal{S}_{u,v})}$ which will be used to illustrate the analysis in the main text.

Appendix D

Details in Propositions 9 and 10

In this appendix we provide further details regarding the lengthy computations arising in the analysis of Propositions 9 and 10.

Estimates on the L^4-norm of connection coefficients

In the following we consider, for conciseness, the NP spin connection coefficient λ. Making use of Proposition 3 to estimate $||\lambda||_{L^4(S)}$ one finds that

$$||\not{\nabla}\lambda||_{L^4(\mathcal{S}_{u,v})} \leq 2\left(||\not{\nabla}\lambda||_{L^4(\mathcal{S}_{0,v})} + C(\Delta_{e_\star}, \Delta_{\Gamma_\star}) \int_0^u \left(\int_{\mathcal{S}_{u',v}} \Delta \langle \not{\nabla}\lambda, \not{\nabla}\lambda \rangle^2_{\boldsymbol{\sigma}} \right)^{1/4} \mathrm{d}u' \right).$$

One can then estimate

$$\begin{aligned}
\int_{\mathcal{S}_{u',v}} |\Delta \langle \not{\nabla}\lambda, \not{\nabla}\lambda \rangle^2_{\boldsymbol{\sigma}}| &= \int_{\mathcal{S}_{u',v}} |\not{\nabla}\lambda|^2 |\Delta(\bar{\delta}\lambda\delta\bar{\lambda} + \bar{\delta}\bar{\lambda}\delta\lambda)| \\
&= \int_{\mathcal{S}_{u',v}} |\not{\nabla}\lambda|^2 |(\Delta\delta\lambda)\bar{\delta}\bar{\lambda} + \delta\lambda\Delta\bar{\delta}\bar{\lambda} + \delta\bar{\lambda}\Delta\bar{\delta}\lambda + \bar{\delta}\lambda\Delta\delta\bar{\lambda}| \\
&\leq \int_{\mathcal{S}_{u',v}} |\not{\nabla}\lambda|^2 \sqrt{2|\delta\lambda|^2 + 2|\bar{\delta}\lambda|^2}\sqrt{2|\Delta\delta\lambda|^2 + 2|\Delta\bar{\delta}\lambda|^2} \\
&\leq 2\int_{\mathcal{S}_{u',v}} |\not{\nabla}\lambda|^3 \left(|\Delta\delta\lambda| + |\Delta\bar{\delta}\lambda| \right),
\end{aligned}$$

where we have made use of the Cauchy-Schwarz inequality in the first inequality. Now, making use of the expressions for $\Delta\delta\lambda$ and $\Delta\bar{\delta}\lambda$ one further finds that,

$$
\begin{aligned}
\int_{\mathcal{S}_{u',v}} |\Delta \langle \nabla\lambda, \nabla\lambda\rangle^2_{\boldsymbol{\sigma}}| &\leq 2\int_{\mathcal{S}_{u',v}} |\nabla\lambda|^3 \left(|\Gamma|^3 + |\Gamma||\Psi_4| + |\Gamma'||\nabla\lambda| + 4|\lambda||\nabla\mu| + |\nabla\Psi_4|\right) \\
&\leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi}(\mathcal{S}))\left(\int_{\mathcal{S}_{u',v}} |\nabla\lambda|^3 + ||\Psi_4||_{L^\infty(\mathcal{S}_{u',v})}\int_{\mathcal{S}_{u',v}} |\nabla\lambda|^3\right) \\
&\quad + C(\Delta_{\Gamma_\star})\int_{\mathcal{S}_{u',v}} |\nabla\lambda|^4 + C(\Delta_{\Gamma_\star})\int_{\mathcal{S}_{u',v}} |\nabla\lambda|^3|\nabla\mu| + 2||\nabla\Psi_4||_{L^\infty(\mathcal{S}_{u',v})}\int_{\mathcal{S}_{u',v}} |\nabla\lambda|^3 \\
&\leq C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi}(\mathcal{S}))\mathrm{Area}(\mathcal{S}_{u',v})^{1/4}||\nabla\lambda||^3_{L^4(\mathcal{S}_{u',v})}\left(1 + \Big(\sum_{i=0}^{2} ||\nabla^i\Psi_4||_{L^2(\mathcal{S}_{u',v})}\Big)\right) \\
&\quad + C(\Delta_{\Gamma_\star})||\nabla\lambda||^3_{L^4(\mathcal{S}_{u',v})}||\nabla\mu||_{L^4(\mathcal{S}_{u',v})} + C(\Delta_{\Gamma_\star})||\nabla\lambda||^4_{L^4(\mathcal{S}_{u',v})} \\
&\quad + C(\Delta_{\Gamma_\star})\mathrm{Area}(\mathcal{S}_{u',v})^{1/4}||\nabla\lambda||^3_{L^4(\mathcal{S}_{u',v})}\Big(\sum_{i=1}^{3} ||\nabla^i\Psi_4||_{L^2(\mathcal{S}_{u',v})}\Big),
\end{aligned}
$$

where in the previous chain of inequalities we have made use of Hölder's inequality and the Sobolev's embedding. Moreover, here Γ represents a linear combination of the NP spin connection coefficients $\tau, \alpha, \beta, \mu, \lambda$ whereas Γ' contains no τ term, which allows the use of sharper estimates. Both Γ and Γ' are controlled in $L^\infty(\mathcal{S}_{u',v})$ as a result of Proposition 8.

Making use of the latter estimate and of the bootstrap assumption in Proposition 9, one readily obtains that

$$
||\nabla\lambda||_{L^4(\mathcal{S}_{u,v})} \leq 2\Delta_{\Gamma_\star} + C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi}(\mathcal{S}))\varepsilon + C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\Psi}(\mathcal{S}))\Delta_{\Psi}\varepsilon^{7/8},
$$

where it has been used that

$$
\int_0^u \left(\int_{\mathcal{S}_{u',v}} |\Psi_4|^2\right)^{1/8} \mathrm{d}u' \leq \left(\int_0^u \int_{\mathcal{S}_{u',v}} |\Psi_4|^2 \mathrm{d}u'\right)^{1/8} \left(\int_0^u 1\mathrm{d}u'\right)^{7/8} \leq \varepsilon^{7/8}||\Psi_4||^{1/4}_{L^2(\mathcal{N}'_v(0,u))}.
$$

Thus, we can choose a suitable $\varepsilon > 0$ such that $||\nabla\lambda||_{L^4(\mathcal{S}_{u,v})} \leq 3\Delta_{\Gamma_\star}$. This improves the starting bootstrap assumption.

Estimates on $||\nabla\!\!\!/^2\lambda||_{L^2(\mathcal{S}_{u,v})}$

In this case we start from

$$\begin{aligned}\int_{\mathcal{S}_{u,v}} |\Delta \left\langle \nabla\!\!\!/^2\lambda, \nabla\!\!\!/^2\lambda \right\rangle_{\boldsymbol{\sigma}}| &= \int_{\mathcal{S}_{u,v}} 2|\Delta(T_1\bar{T}_1 + T_2\bar{T}_2 + T_3\bar{T}_3 + T_4\bar{T}_4)| \\ &\leq 2\sqrt{2}\int_{\mathcal{S}_{u,v}} |\nabla\!\!\!/^2\lambda| \left(|\Delta T_1| + |\Delta T_2| + |\Delta T_3| + |\Delta T_4|\right).\end{aligned}$$

we can then further expand to obtain (in schematic notation for simplicity) that

$$\begin{aligned}\int_{\mathcal{S}_{u',v}} |\Delta \left\langle \nabla\!\!\!/^2\lambda, \nabla\!\!\!/^2\lambda \right\rangle_{\boldsymbol{\sigma}}| \leq{}& 2\sqrt{2}\int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/^2\lambda|(|\Gamma'||\nabla\!\!\!/^2\lambda| + |\Gamma'||\nabla\!\!\!/^2\Gamma| + |\nabla\!\!\!/^2\Psi_4| + |\nabla\!\!\!/\Gamma||\nabla\!\!\!/\Gamma| \\ &+ |\Gamma^2||\nabla\!\!\!/\Gamma| + |\Psi_4||\nabla\!\!\!/\Gamma| + |\Psi_3||\nabla\!\!\!/\lambda| + |\Gamma||\nabla\!\!\!/\Psi_4| + |\Psi_4||\Gamma^2| + |\Gamma^4|) \\ \leq{}& C(\Delta_{\Gamma_\star})\int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/^2\lambda|^2 + C(\Delta_{\Gamma_\star})\int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/^2\lambda||\nabla\!\!\!/^2\Gamma| + \int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/^2\lambda||\nabla\!\!\!/^2\Psi_4| \\ &+ \int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/^2\lambda||\nabla\!\!\!/\Gamma||\nabla\!\!\!/\Gamma| + C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_\Psi(\mathcal{S}))\int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/^2\lambda||\nabla\!\!\!/\Gamma| \\ &+ C(\Delta_\Psi(\mathcal{S}))\int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/^2\lambda||\nabla\!\!\!/\Gamma| + C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_\Psi(\mathcal{S}))\int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/^2\lambda||\nabla\!\!\!/\Psi_4| \\ &+ C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_\Psi(\mathcal{S}))\int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/^2\lambda| + \int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/^2\lambda||\nabla\!\!\!/\Gamma||\Psi_4| \\ &+ C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_\Psi(\mathcal{S}))\int_{\mathcal{S}_{u',v}} |\nabla\!\!\!/^2\lambda||\Psi_4| \\ \leq{}& C(\Delta_{\Gamma_\star})||\nabla\!\!\!/^2\lambda||^2_{L^2(\mathcal{S}_{u',v})} + C(\Delta_{\Gamma_\star})||\nabla\!\!\!/^2\lambda||_{L^2(\mathcal{S}_{u',v})}||\nabla\!\!\!/^2\Gamma||_{L^2(\mathcal{S}_{u',v})} \\ &+ ||\nabla\!\!\!/^2\lambda||_{L^2(\mathcal{S}_{u',v})}||\nabla\!\!\!/^2\Psi_4||_{L^2(\mathcal{S}_{u',v})} + ||\nabla\!\!\!/^2\lambda||_{L^2(\mathcal{S}_{u',v})}||\nabla\!\!\!/\Gamma||^2_{L^4(\mathcal{S}_{u',v})} \\ &+ C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_\Psi(\mathcal{S}))||\nabla\!\!\!/^2\lambda||_{L^2(\mathcal{S}_{u',v})}||\nabla\!\!\!/\Gamma||_{L^4(\mathcal{S}_{u',v})} \\ &+ C(\Delta_{e_\star}, \Delta_{\Gamma_\star})||\Psi_4||_{L^\infty(\mathcal{S}_{u',v})}||\nabla\!\!\!/^2\lambda||_{L^2(\mathcal{S}_{u',v})} \\ &+ C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_\Psi(\mathcal{S}))||\nabla\!\!\!/^2\lambda||_{L^2(\mathcal{S}_{u',v})}||\nabla\!\!\!/\Psi_4||_{L^2(\mathcal{S}_{u',v})} \\ &+ C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_\Psi(\mathcal{S}))||\nabla\!\!\!/^2\lambda||_{L^2(\mathcal{S}_{u',v})} \\ &+ C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_\Psi(\mathcal{S}))||\nabla\!\!\!/^2\lambda||_{L^2(\mathcal{S}_{u',v})}||\nabla\!\!\!/\Gamma||_{L^4(\mathcal{S}_{u',v})}||\Psi_4||_{L^\infty(\mathcal{S}_{u',v})} \\ \leq{}& C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_\Psi(\mathcal{S}))(1 + ||\Psi_4||_{L^2(\mathcal{S}_{u',v})} + ||\nabla\!\!\!/\Psi_4||_{L^2(\mathcal{S}_{u',v})} + ||\nabla\!\!\!/^2\Psi_4||_{L^2(\mathcal{S}_{u',v})}).\end{aligned}$$

In the previous chain of inequalities we have made repeated use of our bootstrap assumption, the results in Proposition 5 and of Hölder's inequality. Finally, combining with the short direction estimate in Proposition 3 we conclude that

$$||\nabla\!\!\!/^2\lambda||_{L^2(\mathcal{S}_{u,v})} \leq 2\Delta_{\Gamma_\star} + C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_\Psi(\mathcal{S}))\varepsilon + C(I, \Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_\Psi(\mathcal{S}))\Delta_\Psi\varepsilon^{3/4}.$$

The factor $\varepsilon^{3/4}$ results from the transferring of the 2-sphere estimate of Ψ_4 to the light cone.

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