MAGNITUDE MEETS PERSISTENCE. HOMOLOGY THEORIES FOR FILTERED SIMPLICIAL SETS

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Abstract. The Euler characteristic is an invariant of a topological space that in a precise sense captures its canonical notion of size, akin to the cardinality of a set. The Euler characteristic is closely related to the homology of a space, as it can be expressed as the alternating sum of its Betti numbers, whenever the sum is well-defined. Thus, one says that homology categorifies the Euler characteristic. In his work on the generalisation of cardinality-like invariants, Leinster introduced the magnitude of a metric space, a real number that counts the "effective number of points" of the space and has been shown to encode many invariants of metric spaces from integral geometry and geometric measure theory. In 2015, Hepworth and Willerton introduced a homology theory for metric graphs, called magnitude homology, which categorifies the magnitude of a finite metric graph. This work was subsequently generalised to enriched categories by Leinster and Shulman, and the homology theory that they introduced categorifies magnitude for arbitrary finite metric spaces. When studying a metric space, one is often only interested in the metric space up to a rescaling of the distance of the points by a non-negative real number. The magnitude function describes how the effective number of points changes as one scales the distance, and it is completely encoded by magnitude homology. When studying a finite metric space in topological data analysis using persistent homology, one approximates the space through a nested sequence of simplicial complexes so as to recover topological information about the space by studying the homology of this sequence. Here we relate magnitude homology and persistent homology as two different ways of computing homology of filtered simplicial sets.

1. Introduction

In a letter to Goldbach written in 1750, Euler [5] noted that for any polyhedron consisting of F regions, E edges and V vertices one obtains $V-E+F=2$. This sum is known as the **Euler characteristic** of the polyhedron. While one usually first encounters the Euler characteristic in relation to topological spaces, one can more generally define the Euler characteristic of an object in any symmetric monoidal category [21], and this can be thought of as its canonical size, a "dimensionless" measure. The irrelevance of topology for the notion of Euler characteristic, and how it should be thought of as an invariant giving a measure of the size or cardinality of an object was made precise among others by Schanuel [25].

In his work on the generalisation of the Euler characteristic as a cardinality-like invariant, Leinster [14] introduced an invariant for finite categories generalising

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work done by Rota on posets. The invariant introduced by Leinster generalises both the cardinality of a set, as well as the topological Euler characteristic. In subsequent work [15] Leinster generalised this invariant to enriched categories, calling it **magnitude**.

Here we are interested in the magnitude of metric spaces. In 1973 Lawvere [12] observed that every metric space is a category enriched over the monoidal category $\mathbf{[0,\infty]}^{\mathrm{op}}$ with objects non-negative real numbers, and a morphism $\epsilon' \to \epsilon$ whenever $\epsilon' \geq \epsilon$, with tensor product given by addition. Such enriched categories are called "Lawvere metric spaces", and a Lawvere metric space is the same thing as an extended quasi-pseudometric space. The magnitude of a metric space is a real number that can be thought of as measuring the "effective number of points" of the space, see [16, Proposition 2.8], and the discussion at the end of Section 7. The **magnitude function** describes how the effective number of points changes as one scales the distances of the points of the metric space by a non-negative real number.

The Euler characteristic of a topological space X is closely related to the singular homology of a space, as it can be expressed as the alternating sum of its Betti numbers

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i \beta_i(X)\,,$$

whenever the sum and the summands are finite. One then says that homology **categorifies** the Euler characteristic. Thus, a natural question to ask is whether there is a homology theory for metric spaces that categorifies in an analogous way the magnitude. Hepworth and Willerton answered this question in the affirmative for finite metric spaces associated to graphs, by introducing **magnitude homology** for graphs [6]. Their work was subsequently extended to arbitrary metric spaces by Leinster and Shulman [18], who define magnitude homology for arbitrary metric spaces as a special case of Hochschild homology for enriched categories. When the metric space is finite, this homology theory encodes the magnitude function.

Towards the end of [18] Leinster and Shulman list a series of open problems, of which the last two are as follows:

> (9) Magnitude homology only "notices" whether the triangle inequality is a strict equality or not. Is there a "blurred" version that notices "approximate equalities"?
> (10) Almost everyone who encounters both magnitude homology and persistent homology feels that there should be some relationship between them. What is it?

Here we give an answer to these questions, which we show are intertwined: we define a blurred version of magnitude homology, and show that it is the persistent homology with respect to a certain filtered simplicial set. Furthermore, we show how blurred and ordinary magnitude homology differ in the limit: blurred magnitude homology coincides with Vietoris homology, while magnitude homology is trivial. Thus, it is the blurred version of magnitude homology that is related to persistent homology. Furthermore, ordinary and blurred versions of magnitude homology are morally very different homology theories associated to filtered simplicial sets: the ordinary version forgets the information given by the filtration of the simplicial set, which is exactly the "persistent" information captured by persistent homology, and hence the blurred version of magnitude homology. In the remainder of this section we give a summary of the main notions and results presented in this manuscript.

1.1. **Persistent homology.** Persistent homology is, in an appropriate sense, the generalisation of simplicial homology of a simplicial set to filtered simplicial sets. Given a metric space (X, d), one associates to it a filtered simplicial set $S(X)$ such that $S(X)(\epsilon)$ captures topological and geometric information of X "below distance ϵ". One then considers the composite functor

$$CS(X) = \left(\mathbf{[0,\infty]} \xrightarrow{S(X)} \mathrm{sSet} \xrightarrow{\mathbb{Z}[\cdot]} \mathrm{sAb} \xrightarrow{U} \mathrm{ch}_{\mathrm{Ab}} \right) ,$$

where the functor $\mathbb{Z}[\cdot]$ is induced by the free abelian group functor, and the functor U is the functor that sends a simplicial abelian group to its unnormalised chain complex.

Definition. Let (X, d) be a metric space, and $S(X) \colon \mathbf{[0,\infty]} \longrightarrow \mathrm{sSet}$ be a functor as above. The **persistent homology of $\boldsymbol{X}$ with respect to $\boldsymbol{S}$** is the composition $H_\star(CS(X))$, where $H_\star$ is the usual homology functor for chain complexes.

1.2. **Ordinary magnitude homology.** In [18] Leinster and Shulman define magnitude homology as the homology of the geometric realisation of a certain two-sided simplicial bar construction. In online discussions [17] they had previously considered a different definition, based on the generalisation of the nerve of a category to enriched categories, called the "enriched nerve". In the special case of metric spaces, the enriched nerve is defined as follows:

Definition. Given a metric space (X, d) its **enriched nerve $\boldsymbol{N(X)}$** is a functor $\mathbf{[0,\infty]} \to \mathrm{sSet}$, where for each $\epsilon \in [0, \infty]$ the simplicial set $N(X)(\epsilon)$ has set of n-simplices given by

$$N(X)(\epsilon)_n = \{(x_0, \ldots, x_n) : \ d(x_0, x_1) + \cdots + d(x_{n-1}, x_n) \leq \epsilon\} \ .$$

One can then formulate the original definition of magnitude homology suggested by Leinster and Shulman in [17] as follows:

Magnitude homology (Definition B'). For any $\epsilon \in [0, \infty]$ let

$$A_\epsilon \colon \mathbf{[0,\infty]}^{\mathrm{op}} \to \mathrm{Ab}$$

be the coefficient functor induced by the map

$$\ell \mapsto \begin{cases} \mathbb{Z}, \text{ if } \ell = \epsilon \\ 0, \text{ otherwise} . \end{cases}$$

We consider A_ϵ as taking values in $\mathrm{ch}_{\mathrm{Ab}}$ through the canonical inclusion functor $\mathrm{Ab} \hookrightarrow \mathrm{ch}_{\mathrm{Ab}}$ that sends an abelian group to the chain complex with that abelian group concentrated in degree zero. Let

$$CN(X) \otimes_{\mathbf{[0,\infty]}} A_{-} \colon \mathbf{[0,\infty]} \to \mathrm{ch}_{\mathrm{Ab}}$$

be the functor induced by the map that sends ϵ to the coend $CN(X) \otimes_{\mathbf{[0,\infty]}} A_\epsilon$. The **magnitude homology of $\boldsymbol{X}$** is the homology of $CN(X) \otimes_{\mathbf{[0,\infty]}} A_{-}$.

In Proposition 27 we show that Definition B' of magnitude homology is equivalent to the definition based on the bar construction. Definition B' is the motivation for our definition of blurred magnitude homology.

1.3. **Blurred magnitude homology.** The functors of coefficient A_ϵ used in the definition of magnitude homology based on the enriched nerve are "degenerate" in the sense that they are supported only at points, and thus when we tensor with these functors, we forget a considerable amount of information about our original metric space. To retain more information, we can consider functors of coefficients supported at intervals instead of points, and obtain what we call blurred magnitude homology:

Definition. For any $\epsilon \in [0,\infty]$ define the coefficient functor

$$\begin{aligned} A_{[0,\epsilon]} \colon \mathbf{[0,\infty]}^{\mathrm{op}} &\to \mathrm{Ab} \\ \ell &\mapsto \begin{cases} \mathbb{Z}, \text{ if } \ell \in [0,\epsilon] \\ 0, \text{ otherwise} \end{cases} \\ (\ell' \geq \ell) &\mapsto \begin{cases} \mathrm{id}_{\mathbb{Z}},\ \ell, \ell' \in [0,\epsilon] \\ 0, \text{ otherwise} \end{cases} \end{aligned}$$

Similarly as for A_ϵ we consider $A_{[0,\epsilon]}$ as taking values in $\mathrm{ch}_{\mathrm{Ab}}$. The map that sends any ϵ to the coend $CN(X) \otimes_{\mathbf{[0,\infty]}} A_{[0,\epsilon]}$ induces a functor

$$CN(X) \otimes_{\mathbf{[0,\infty]}} A_{[0,-]} \colon \mathbf{[0,\infty]} \to \mathrm{ch}_{\mathrm{Ab}}\,.$$

The **blurred magnitude homology of *X*** is the homology of $CN(X) \otimes_{\mathbf{[0,\infty]}} A_{[0,-]}$.

And we have (see Theorem 32):

Theorem. For any metric space X, the functors $CN(X) \otimes_{[0,\infty]} A_{[0,-]}$ and $CN(X)$ are isomorphic. In particular, the blurred magnitude homology of X is the persistent homology of X with respect to the enriched nerve.

1.4. **Blurred vs. ordinary magnitude homology.** The characterisation of ordinary magnitude homology as given in Proposition 27 was our starting point for the definition of blurred magnitude homology, and it also allows us to relate ordinary and blurred magnitude homology as two different choices of coefficients: in ordinary magnitude homology we take coefficient functors supported at points, while in blurred magnitude homology we take coefficient functors supported at intervals. In Section 1.4.1 we explain how one can make this comparison more precise, while in Section 1.4.2 we give a way of relating ordinary and blurred magnitude homology by taking the categorical limit.

1.4.1. *Persistent vs. graded objects.* Given a monoidal poset $(P, \leq, +)$, that is, a poset together with a compatible structure of monoid, there is a natural way to assign a monoidal category $\mathbf{P}$ to it, in which objects are given by elements of P, the tensor product is given by $+$, and there is a morphism $p \to p'$ whenever $p \leq p'$. Similarly, given a set S there is a natural way to associate a discrete category S to it, with objects given by the elements of S and no morphisms apart from the identity morphisms.

For a monoidal poset $(P, \leq, +)$ and a category C we say that a functor

$$F \colon \mathbf{P} \to C$$

is a **P-persistent object of *C***. Given a set S we call a functor $S \to C$ an ***S*-graded object of *C***. For example, when $(P, \leq, +) = ([0,\infty], \leq, +)$, $S = [0,\infty]$

and $C = \mathrm{Vect}_{\mathbb{F}}$ is the category of vector spaces over a field $\mathbb{F}$, we have that $[\mathbf{0},\boldsymbol{\infty}]$-persistent vector spaces are what are usually called persistence modules in the topological data analysis literature (see Example 14), while $[0,\infty]$-graded vector spaces correspond to the usual definition of graded vector spaces.

As we explain in Remark 18, if C has a zero object, any P-graded object can be identified in a canonical way with a $\mathbf{P}$-persistent object. Thus, blurred magnitude homology is the homology of a bona fide $[\mathbf{0},\boldsymbol{\infty}]$-persistent chain complex of abelian groups, while ordinary magnitude homology is the homology of a $[0,\infty]$-graded chain complex, which we can think of as a degenerate $[\mathbf{0},\boldsymbol{\infty}]$-persistent chain complex.

1.4.2. *Limit homology.* Finally, we show how ordinary and blurred version of magnitude homology — which coincides with the persistent homology taken with respect to the enriched nerve — differ in the limit: while the limit of blurred magnitude homology coincides with Vietoris homology, the limit of ordinary magnitude homology is trivial. In what we call "limit homology", we take the categorical limit of $H_\star(CS(X))\colon [\mathbf{0},\boldsymbol{\infty}] \to \mathrm{Ab}$. Such a homology theory was first introduced by Vietoris in [27] to define a homology theory for compact metric spaces. To do this, Vietoris introduced what is now called the Vietoris–Rips complex at scale ϵ: this is the simplicial complex $V(X)(\epsilon)$ whose n-simplices are the unordered $(n+1)$-tuples of points $\{x_0,\dots,x_n\}$ of X obeying

$$\forall i,j \quad d(x_i,x_j) \le \epsilon.$$

Vietoris defined the homology of a compact metric space (X,d) to be the limit

$$\mathcal{H}_\star(X) := \varinjlim_{\epsilon} H_\star(CV(X)(\epsilon))\,.$$

Vietoris's motivation was to prove what is now called the "Vietoris mapping theorem", a result that relates the homology groups of two spaces using properties of a map between them. While there has been some work done on Vietoris homology (see, e.g., [7, 23]), the theory has not been as widely studied as other homology theories. A limit homology theory that plays a fundamental role in algebraic topology is Čech homology: given a space X and a cover $\mathcal{U}$ of X, one considers the simplicial homology $H_\star(CN(\mathcal{U}))$ of the nerve of $\mathcal{U}$. If $\mathcal{V}$ is a cover of X that refines $\mathcal{U}$, then there is a homomorphism $H_\star(CN(\mathcal{V})) \to H_\star(CN(\mathcal{U}))$. The Čech homology of X is the inverse limit of the inverse system obtained by considering all open covers of X. The difference between Vietoris and Čech homology is immaterial for compact metric spaces, as for such spaces the homology theories are canonically isomorphic, see [13]. We have (see Corollary 36):

Corollary. Let k be a non-negative integer, and let X be a metric space. Then:

$$\varinjlim_{\epsilon} H_k(CN(X) \otimes_{[\mathbf{0},\boldsymbol{\infty}]} A_{[0,\epsilon]}) \cong \mathcal{H}_k(X)\,,$$

and

$$\varinjlim_{\epsilon} H_k(CN(X) \otimes_{[\mathbf{0},\boldsymbol{\infty}]} A_{\epsilon}) \cong 0\,.$$

That is, under the limit, blurred magnitude homology is Vietoris homology, while ordinary magnitude homology is trivial.

We note that while in limit homology and persistent homology one works with simplicial complexes, the definition of magnitude homology is based on simplicial sets. Simplicial complexes present advantages from the computational point of view, as a simplex can be uniquely specified by listing its vertices, but from the theoretical point of view simplicial sets are better suited. In Section 2 we will explain how to a given simplicial complex one can assign a simplicial set such that their geometric realisations are homeomorphic. This assignment is not functorial, however this will not be a problem for our purposes.

1.5. **Structure of the paper.** The paper is structured as follows:

- We cover preliminaries about simplicial complexes and simplicial sets in Section 2; enriched categories and Lawvere metric spaces in Section 3; filtered simplicial sets in Section 4; persistent as well as graded objects in Section 5; and coends in Section 6.
- In Section 7 we recall the definition of magnitude for enriched categories, and briefly discuss the special case of the magnitude of metric spaces as well as the magnitude function.
- In Section 8 we give the definition of magnitude homology for metric spaces as a special case of Hochschild homology following [18] (see Definition A in Section 8.1), and then introduce an alternative definition based on the enriched nerve (Definition B' in Section 8.2), and show that they are equivalent in Proposition 27.
- In Section 9 we give a general definition of persistent homology, while in Section 10 we introduce blurred magnitude homology, taking as starting point the alternative definition of magnitude homology (Definition B' in Section 8.2), and show that it is the persistent homology taken with respect to the enriched nerve. We show how blurred and ordinary magnitude homology differ in the limit in Section 11.

2. Simplicial complexes and simplicial sets

The number of researchers who have a working knowledge of both simplicial complexes and simplicial sets is arguably small, therefore here we recall the basic notions and definitions. Simplicial complexes and simplicial sets can be seen as combinatorial versions of topological spaces; they are related to topological spaces by the geometric realisation. We first recall the definitions of simplicial complexes, simplicial sets and the corresponding geometric realisations. We then discuss how one can assign a simplicial set to a simplicial complex in such a way that the corresponding geometric realisations are homeomorphic.

Definition 1. A **simplicial complex** is a tuple $K = (V, \Sigma)$ where V is a set, and Σ is a set of non-empty finite subsets of V such that:

(i) for all $v \in V$ we have that $\{v\} \in \Sigma$
(ii) Σ is closed with respect to taking subsets.

The elements of Σ with cardinality $n+1$ are called ***n*-simplices** of K. The elements of V are called **vertices** of K. Given two simplicial complexes $K = (V, \Sigma)$ and $K' = (V', \Sigma')$, a **simplicial map** $K \to K'$ is a map $f\colon V \to V'$ such that for all $\sigma \in \Sigma$ we have $f(\sigma) \in \Sigma'$.

Remark 2. We note that if one wants the 0-simplices to coincide with the vertices of a simplicial complex, then condition (i) in Definition 1 cannot be dispensed of; while condition (ii) implies that all vertices contained in simplices are in Σ, condition (i) guarantees that these are the only vertices. Often in the topological data analysis literature one finds a definition of simplicial complex as a variant of Definition 1 in which condition (i) is omitted, and in such a definition one thus allows vertices that are not 0-simplices. One could give a definition equivalent to Definition 1 by only requiring closure under taking subsets as follows: let Σ be a family of non-empty finite sets closed under taking subsets, and let $V(\Sigma) = \bigcup \Sigma$. Then $(V(\Sigma), \Sigma)$ is a simplicial complex according to Definition 1.

To define simplicial sets, we first need to introduce the "simplex category" Δ. Consider the category with objects finite non-empty totally ordered sets, and morphisms given by order preserving maps. The skeleton of this category is denoted by Δ and called **simplex category**. In other words, Δ has objects given by a totally ordered set $[n] = \{0, 1, \ldots, n\}$ for every natural number n, and morphisms order-preserving maps.

Definition 3. Denote by Set the category with objects sets and morphisms maps of sets. A **simplicial set** is a functor $S \colon \Delta^{\mathrm{op}} \to \mathrm{Set}$. The elements of $S(n)$ are called ***n*-simplices**.

Explicitly, one can show that a simplicial set is a collection of sets $\{S_n\}_{n \in \mathbb{N}}$ together with so-called **face** maps

$$d_i \colon S_n \to S_{n-1}$$

and **degeneracy** maps

$$s_i \colon S_n \to S_{n+1}$$

for all $0 \leq i \leq n$, that satisfy certain compatibility conditions, see [4, Def. 1.1].

The geometric realisation functor gives a canonical way to associate a topological space to a simplicial complex or set. For this, one first chooses a topological model for n-simplices, namely the standard n-simplex Δ^n:

Definition 4. The **standard *n*-simplex** is the subset of Euclidean space

$$\Delta^n = \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=0}^{n} x_i = 1\,, \text{ and } 0 \leq x_i \leq 1 \text{ for all } i \right\} .$$

There are **boundary** maps

$$\delta_i \colon \Delta^n \to \Delta^{n-1} \colon (x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_i + x_{i+1}, \ldots, x_n)$$

and **face** maps

$$\sigma_i \colon \Delta^n \to \Delta^{n+1} \colon (x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_{i-1}, 0, x_i, \ldots, x_n)\,.$$

Then, to define the geometric realisation one proceeds to glue together standard simplices using the face and boundary maps:

Definition 5. Given a simplicial complex $K = (V, \Sigma)$, its **geometric realisation** $|\boldsymbol{K}|$ is the quotient space

$$\bigcup_{\sigma \in \Sigma} \Delta^{|\sigma|-1} \times \{\sigma\} / \sim$$

where $\bigcup_{\sigma\in\Sigma} \Delta^{|\sigma|-1} \times \{\sigma\}$ is endowed with the disjoint union space topology, while the equivalence relation $\sim$ is the transitive closure of the following relation

$$\left\{ \Big((x,\sigma), (\delta_i(x),\tau) \Big) : x \in \Delta_{|\sigma|-1},\ \text{and}\ \ \sigma \subseteq \tau \text{ with } |\tau| = |\sigma|+1 \right\}.$$

Similarly, given a simplicial set $S\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$, its **geometric realisation $|S|$** is the quotient space

$$\bigcup_{n\in\mathbb{N}} \Delta^n \times S_n/\sim$$

where the equivalence relation $\sim$ is the transitive closure of the union of the relations

$$\left\{ \Big((x, d_i(\sigma)), (\sigma_i(x), \sigma) \Big) : x \in \Delta^n \text{ and } \sigma \in S_{n+1} \right\},\ \text{and}$$

$$\left\{ \Big((x, s_i(\sigma)), (\delta_i(x), \sigma) \Big) : x \in \Delta^n \text{ and } \sigma \in S_{n-1} \right\}.$$

Now, given a simplicial complex $K = (V, \Sigma)$, we assign to it a simplicial set so that its geometric realisation is homeomorphic to that of K.

Definition 6. Let $K = (V, \Sigma)$ be a simplicial complex. Choose a total order on V. Define

$$K_n^{\mathrm{sim}} = \{(x_0, \dots, x_n) \mid \{x_0, \dots, x_n\} \in \Sigma \text{ and } x_0 \leq \dots \leq x_n\},$$

and for $0 \leq i \leq n$ let

$$d_i\colon K_n^{\mathrm{sim}} \to K_{n-1}^{\mathrm{sim}}\colon (x_0, \dots, x_n) \mapsto (x_0, \dots, \hat{x_i}, \dots, x_n),$$

where $\hat{x_i}$ means that that entry is missing, and let

$$s_i\colon K_n^{\mathrm{sim}} \to K_{n+1}^{\mathrm{sim}}\colon (x_0, \dots, x_n) \mapsto (x_0, \dots, x_i, x_i, \dots, x_n).$$

It is then easy to show that $\{K_n^{\mathrm{sim}}\}_{n\in\mathbb{N}}$ together with the maps d_i and s_i is a simplicial set. We denote this simplicial set by K^{sim}. Furthermore, we have:

Lemma 7. The geometric realisations of K^{sim} and K are homeomorphic.

Proof. This is easy to see, since the non-degenerate simplices are in bijection, and all degenerate simplices are in the image of some non-degenerate simplex. See also [4]. ☐

The assignment $K \mapsto K^{\mathrm{sim}}$ is not functorial, since it depends on the choice of a total order on V. One could assign a simplicial set to a simplicial complex in a functorial way, so that their geometric realisations are homotopy equivalent rather than homeomorphic, however this is at the cost of adding many more simplices. For our purposes the non-functorial assignment $K \mapsto K^{\mathrm{sim}}$ suffices.

3. Enriched categories and Lawvere metric spaces

An ordinary (small) category C is given by a set of objects, and for every pair of objects x, y a set of morphisms $C(x, y)$, together with composition maps

$$C(x, y) \times C(y, z) \to C(x, z)$$

and maps assigning to every object x its identity morphism

$$\{\star\} \to C(x, x),$$

such that the composition of morphisms is associative and the identity morphism for every object is the neutral element for this composition. Let $\boldsymbol{\mathcal{V}}$ be a monoidal category with tensor product $\otimes_{\boldsymbol{\mathcal{V}}}$ and unit $1_{\boldsymbol{\mathcal{V}}}$. A (small) **category enriched over $\boldsymbol{\mathcal{V}}$** (or **$\boldsymbol{\mathcal{V}}$-category**) is a generalisation of an ordinary category: we still have a set of objects, but now for every pair of objects x, y we are given an object $C(x, y)$ in $\boldsymbol{\mathcal{V}}$, together with composition and identity assigning morphisms in $\boldsymbol{\mathcal{V}}$, namely

$$C(x, y) \otimes_{\boldsymbol{\mathcal{V}}} C(y, z) \to C(x, z)$$

and

$$1_{\boldsymbol{\mathcal{V}}} \to C(x, x)\,,$$

which satisfy associativity and unitality conditions. When $\boldsymbol{\mathcal{V}}$ is the category of sets, a category enriched over $\boldsymbol{\mathcal{V}}$ is an ordinary category. We note that while an enriched category is in general not a category, it has an "underlying" category, see [10] for details.

In [12] Lawvere observed that any metric space is an enriched category:

Definition 8. Let $[\mathbf{0}, \boldsymbol{\infty}]^{\mathrm{op}}$ denote the symmetric monoidal category with objects given by the extended non-negative real numbers (that is, elements of $[0, \infty]$), exactly one morphism $\epsilon' \to \epsilon$ if $\epsilon' \geq \epsilon$, tensor product given by addition, and unit by 0. A **Lawvere metric space** is a small category enriched over $[\mathbf{0}, \boldsymbol{\infty}]^{\mathrm{op}}$.

In other words, a Lawvere metric space is given by a set X, together with for all $x, y \in X$ a number $X(x, y) \in [0, \infty]$, and for all $x, y, z \in X$ a morphism

$$X(x, y) + X(y, z) \to X(x, z) \tag{1}$$

as well as a morphism

$$0 \to X(x, x)\,. \tag{2}$$

Equation (1) is the triangle inequality, while Equation (2) implies that $X(x, x) = 0$. Thus, a Lawvere metric space is the same thing as an extended (since we are allowing infinite distances) quasi-pseudometric space (as distances are not necessarily symmetric, and we allow distinct elements to have zero distance).

4. Filtered simplicial sets

Given a metric space (X, d) we are interested in associating to it filtered simplicial sets, namely functors $S(X)\colon [\mathbf{0}, \boldsymbol{\infty}] \to \mathrm{sSet}$. Two main examples that we consider in this paper are the enriched nerve and the Vietoris–Rips simplicial set. We next recall their definitions.

Definition 9. Let (X, d) be a metric space. The **enriched nerve** of X is the functor $N(X)\colon [\mathbf{0}, \boldsymbol{\infty}] \longrightarrow \mathrm{sSet}$ such that for any $\epsilon \in [0, \infty]$ the simplicial set $N(X)(\epsilon)$ has set of n-simplices given by

$$N(X)(\epsilon)_n = \left\{ (x_0, \ldots, x_n) \mid x_i \in X, \text{ and } \sum_{i=0}^{n-1} d(x_i, x_{i+1}) \leq \epsilon) \right\}$$

and the obvious degeneracy and face maps. Further, for any $\epsilon \leq \epsilon'$ the simplicial maps $N(X)(\epsilon \leq \epsilon')\colon N(X)(\epsilon) \to N(X)(\epsilon')$ are the canonical inclusion maps.

When adding up pairwise lengths of an ordered tuple, we will often talk about the "length" of the tuple:

Definition 10. Let (X, d) be a metric space. The **length** of an ordered tuple $(x_0, \dots, x_n)$ of elements of X is $\sum_{i=0}^{n-1} d(x_i, x_{i+1})$.

Definition 11. Let (X, d) be a metric space. The **Vietoris–Rips simplicial set** of X is the functor $V^{\mathrm{sim}}(X)\colon \mathbf{[0,\infty]} \longrightarrow \mathrm{sSet}$ with set of n-simplices given by

$$V^{\mathrm{sim}}(X)(\epsilon)_n = \big\{(x_0, \dots, x_n) \mid d(x_i, x_j) \leq \epsilon \text{ for all } i, j \in \{0, \dots, n\}\big\}$$

and the obvious degeneracy and face maps. Furthermore, for any $\epsilon \leq \epsilon'$ the simplicial maps $V^{\mathrm{sim}}(X)(\epsilon \leq \epsilon')\colon V^{\mathrm{sim}}(X)(\epsilon) \to V^{\mathrm{sim}}(X)(\epsilon')$ are the canonical inclusion maps.

We note that we are indeed interested in studying simplicial sets filtered by the monoidal category $\mathbf{[0,\infty]}$, and not merely by the category associated to the poset $([0,\infty], \leq)$. Firstly, the enriched nerve is the generalisation of the nerve of a category to the enriched setting, and it can be defined, using the Yoneda embedding, as a simplicial object in the category of presheaves $\mathrm{Set}^{\mathbf{[0,\infty]}}$, see Section 4.1. Secondly, as we will explain in the next section, a fundamental observation in persistent homology is that functors $\mathbf{[0,\infty]} \to \mathbb{K}\mathrm{Vect}$ can be identified with graded modules over a certain monoid ring, and implicit in this identification is the fact that the poset has a monoid structure compatible with the order. The monoidal structure is also crucial for the study of questions related to stability in persistent homology, see [1].

4.1. **The nerve of an enriched category.** We recall the construction of the nerve for enriched categories, and, in particular, for metric spaces. The author learned about this construction from John Baez, and the following discussion is due to him.

Given an ordinary category C, the **nerve** $N(C)$ is a simplicial set whose n-simplices are composable n-tuples of morphisms in C:

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} x_{n-1} \xrightarrow{f_n} x_n.$$

In other words, the set of n-simplices of the nerve is a disjoint union of products:

$$N(C)_n = \bigsqcup_{x_0,\dots,x_n \in \mathrm{ob}C} C(x_0, x_1) \times \cdots \times C(x_{n-1}, x_n). \tag{3}$$

The face maps in $N(C)$ are defined using composition, while the degeneracy maps are defined using identity morphisms.

To generalise this concept to categories enriched over an arbitrary monoidal category $\boldsymbol{\mathcal{V}}$ we proceed as follows. The product of sets in Equation (3) should be replaced by the tensor product $\otimes_{\boldsymbol{\mathcal{V}}}$ in $\boldsymbol{\mathcal{V}}$. The disjoint union of sets is a special case of a coproduct. While $\boldsymbol{\mathcal{V}}$ may not have coproducts, the category of presheaves on $\boldsymbol{\mathcal{V}}$, denoted $\widehat{\boldsymbol{\mathcal{V}}}$ does. The objects of this category are functors $F\colon \boldsymbol{\mathcal{V}}^{\mathrm{op}} \to \mathrm{Set}$, called **presheaves** on $\boldsymbol{\mathcal{V}}$. The morphisms are natural transformations.

The category $\widehat{\boldsymbol{\mathcal{V}}}$ contains $\boldsymbol{\mathcal{V}}$ as a subcategory via the Yoneda embedding

$$Y\colon \boldsymbol{\mathcal{V}} \to \widehat{\boldsymbol{\mathcal{V}}}$$

which sends each object $\epsilon \in \mathrm{ob}\boldsymbol{\mathcal{V}}$ to the so-called representable presheaf

$$\boldsymbol{\mathcal{V}}(-, \epsilon)\colon \boldsymbol{\mathcal{V}}^{\mathrm{op}} \to \mathrm{Set}.$$

The coproducts in $\widehat{\boldsymbol{\mathcal{V}}}$ are computed objectwise: if $\{F_j\}_{j\in J}$ is a collection of presheaves on $\boldsymbol{\mathcal{V}}$, their coproduct is given by

$$\left(\bigsqcup_{j\in J} F_j\right)(\epsilon) = \bigsqcup_{j\in J} F_j(\epsilon)$$

for all $\epsilon \in \mathrm{ob}\boldsymbol{\mathcal{V}}$, see [20, Sec. V.3] for more details. Now we can generalise the nerve to a category enriched over $\boldsymbol{\mathcal{V}}$:

Definition 12. Let C be a $\boldsymbol{\mathcal{V}}$-category. The **enriched nerve of $\boldsymbol{C}$** is the functor

$$N(C)\colon \boldsymbol{\mathcal{V}}^{\mathrm{op}} \to \mathrm{sSet}$$

where for each $\epsilon \in \mathrm{ob}\boldsymbol{\mathcal{V}}$ the set of n-simplices is given by

$$N(C)(\epsilon)_n = \bigsqcup_{x_0,\dots,x_n\in \mathrm{ob}C} \boldsymbol{\mathcal{V}}(\epsilon, C(x_0,x_1)\otimes_{\boldsymbol{\mathcal{V}}}\cdots\otimes_{\boldsymbol{\mathcal{V}}} C(x_{n-1},x_n))\,.$$

The maps

$$d_i\colon N(C)(\epsilon)_n \to N(C)(\epsilon)_{n-1} \qquad i = 0,\dots,n$$

are defined using composition morphisms in C, while the degeneracy maps

$$s_i\colon N(C)(\epsilon)_n \to N(C)(\epsilon)_{n+1} \qquad i = 0,\dots,n$$

are defined using identity-assigning morphisms, all in a manner closely mimicking the usual nerve.

When $\boldsymbol{\mathcal{V}} = [\mathbf{0},\boldsymbol{\infty}]^{\mathrm{op}}$ and X is a $\boldsymbol{\mathcal{V}}$-category, the set

$$\boldsymbol{\mathcal{V}}\big(\epsilon, X(x_0,x_1)\otimes_{\boldsymbol{\mathcal{V}}}\cdots\otimes_{\boldsymbol{\mathcal{V}}} X(x_{n-1},x_n)\big) = \boldsymbol{\mathcal{V}}\big(\epsilon, d(x_0,x_1)+\cdots+d(x_{n-1},x_n)\big)$$

is a singleton if

$$\epsilon \geq d(x_0,x_1)+\cdots+d(x_{n-1},x_n)$$

and empty otherwise. Thus, we have a canonical isomorphism

$$N(X)_n(\epsilon) \cong \Big\{(x_0,\dots,x_n) \mid\ d(x_0,x_1)+\cdots+d(x_{n-1},x_n) \leq \epsilon\Big\}.$$

We can take this isomorphic set as the set of n-simplices in the enriched nerve $N(X)$.

5. Persistent vs. graded objects

Our aim is to study the homology of filtered simplicial sets such as those introduced in Section 4, and we are thus interested in functors $[\mathbf{0},\boldsymbol{\infty}] \to \mathrm{ch}_{\mathrm{Ab}}$. Since such functors are the central object of study in persistent homology, we introduce the following definition:

Definition 13. Let C be a small category, and $(P,\leq,+)$ a monoidal poset, that is, a poset together with a monoid structure compatible with the order. We identify $(P,\leq,+)$ with the symmetric monoidal category $\boldsymbol{P}$ with objects given by the elements of P, exactly one morphism $p' \to p$ if $p' \leq p$, and tensor product given by $+$. A functor $\boldsymbol{P} \to C$ is called a **$\boldsymbol{P}$-persistent** element of the set of objects of C.

Example 14. Consider $(\mathbb{N}, \leq, +)$ where $\leq$ and $+$ are the usual order and addition on the natural numbers. Further, let $C = \mathbb{K}\mathrm{Vect}$ be the category of vector spaces over a field $\mathbb{K}$ together with $\mathbb{K}$-linear maps. There is an isomorphism of categories between the functor category of $\boldsymbol{\mathbb{N}}$-persistent vector spaces over $\mathbb{K}$ and the category of $\mathbb{N}$-graded modules over the polynomial ring $\mathbb{K}[x]$. Similarly, when we consider the monoidal poset $([0,\infty], \leq, +)$ where $\leq$ and $+$ are the usual order and addition on real numbers, there is an isomorphism of categories between the functor category of $\boldsymbol{[0,\infty]}$-persistent vector spaces and the category of modules graded by $([0,\infty], \leq +)$ over the monoid ring $\mathbb{K}[([0,\infty], +)]$. Furthermore, finitely generated modules correspond to persistent vector spaces of "finitely presented type" [3]. This is known as the Correspondence Theorem in the persistent homology literature, and $\boldsymbol{\mathbb{N}}$-, as well as $\boldsymbol{[0,\infty]}$-persistent vector spaces are usually called **persistence modules**.

We will see that in magnitude homology one "forgets" the information given by the inclusion maps in the filtration of a simplicial sets, and thus the chain complexes that one ends up with are more properly *graded* objects, rather than persistent objects.

Definition 15. Let C be a small category, and I a set, which we identify with the discrete category I with objects given by the elements of I and no morphisms apart from the identity morphisms. A functor $I \to C$ is called an ***I*-graded** element of the set of objects of C.

If C has all coproducts, one can characterise such functors as follows:

Proposition 16. Let C be a small category with all coproducts, and let I be a set. There is an isomorphism of categories between the functor category of I-graded objects of C and the category with objects pairs $(c, \{c_i\}_{i\in I})$ such that c is isomorphic to the coproduct of $\{c_i\}_{i\in I}$, and morphisms $(c, \{c_i\}_{i\in I}) \to (c', \{c'_i\}_{i\in I})$ given by $\{f_i\}_{i\in I}$ where for each $i \in I$ we have that $f_i \colon c_i \to c'_i$ is a morphism in C.

Example 17. When $I = [0,\infty]$, we have that a $[0,\infty]$-graded chain complex of abelian groups can be identified with a chain complex of $[0,\infty]$-graded abelian groups, because coproducts of chain complexes are computed componentwise.

Thus, while a $[0,\infty]$-graded vector space over $\mathbb{K}$ is simply a vector space V together with a direct sum decomposition $V = \oplus_{l\in[0,\infty]} V_l$, we have that a $\boldsymbol{[0,\infty]}$-persistent vector space over $\mathbb{K}$ is a $[0,\infty]$-graded vector space together with an action of the monoid ring $\mathbb{K}[([0,\infty], +)]$, which corresponds to the information given by the non-trivial maps $\epsilon \longrightarrow \epsilon'$ whenever $\epsilon \leq \epsilon'$.

Remark 18. Given a monoidal poset $(P, \leq, +)$, and a category C with zero morphisms, we can identify any P-graded object in C with a $\boldsymbol{P}$-persistent object in a canonical way. Namely, consider the full subcategory of the functor category $\mathrm{Fun}(\boldsymbol{P}, C)$, given by all functors that send every morphism to the zero morphism in C. Then this category is easily seen to be isomorphic to the category of P-graded objects of C, that is, the functor category $\mathrm{Fun}(P, C)$.

6. Coends

One of the main ingredients in the definition of blurred magnitude homology that we will give in Section 10 is the coend, a construction that is ubiquitous in category theory. For ease of reference we briefly recall its definition here.

Intuitively, given a bivariate functor with mixed variance $F\colon D^{\mathrm{op}} \times D \to C$, its coend is an object in C that identifies the "left action" of F with the "right action" of F; for instance, the tensor product of a left module with a right module over a ring is an example of coend, see [20, Section IX.6].

While one can define a coend in this general setting, we will make use of the following characterisation of coends in the case that D is cocomplete and C small.

Definition 19. Suppose that D is a cocomplete category, and C is a small category. Given a functor $F\colon C^{\mathrm{op}} \times C \to D$, its coend is the coequaliser of the diagram

$$\coprod_{f\colon c \to c'} F(c', c) \rightrightarrows \coprod_{c \in C} F(c, c)\,,$$

where the two parallel morphisms are the unique morphisms induced by the morphisms $F(f, 1_c)\colon F(c', c) \to F(c, c)$, and $F(1_{c'}, f)\colon F(c', c) \to F(c', c')$, respectively. Given two functors $L\colon C^{\mathrm{op}} \to D$ and $R\colon C \to D$, we denote the coend of $L \times R$ by $L \otimes_D R$.

For more details on coends we refer the reader to [20, Section IX.6], as well as the survey [19].

7. Magnitude

We first recall the definition of magnitude for arbitrary enriched categories loosely following [15, 18], and then briefly summarise some properties of the magnitude function of a metric space.

Let $\boldsymbol{\mathcal{V}}$ be a symmetric monoidal category with tensor product $\otimes_{\boldsymbol{\mathcal{V}}}$ and unit $1_{\boldsymbol{\mathcal{V}}}$. We denote the set of objects of $\boldsymbol{\mathcal{V}}$ by $\mathrm{ob}\boldsymbol{\mathcal{V}}$.

Definition 20. Let $(\mathbb{K}, \cdot, +, 1, 0)$ be a semiring. A **size** is a function $\#\colon \mathrm{ob}(\boldsymbol{\mathcal{V}}) \to \mathbb{K}$ such that

(i) if $x \cong y$ then $\#x = \#y$
(ii) $\#(1_{\boldsymbol{\mathcal{V}}}) = 1$ and $\#(x \otimes_{\boldsymbol{\mathcal{V}}} y) = \#(x) \cdot \#(y)$.

Lemma 21. If $\boldsymbol{\mathcal{V}}$ is essentially small (i.e., equivalent to a small category), there is a size function $\#\colon \mathrm{ob}(\boldsymbol{\mathcal{V}}) \to \mathbb{K}$ that is universal, in the sense that any other size function $\#'\colon \mathrm{ob}(\boldsymbol{\mathcal{V}}) \to \mathbb{K}'$ factors through it.

Proof. First note that the set of isomorphism classes of $\boldsymbol{\mathcal{V}}$ has the structure of a monoid, where $[x] \cdot [y] = [x \otimes_{\boldsymbol{\mathcal{V}}} y]$, and $1 \cdot [x] = [1_{\boldsymbol{\mathcal{V}}} \otimes_{\boldsymbol{\mathcal{V}}} x]$.

Now, for any monoid $(M, \cdot, 1_M)$, there is a canonical way to associate a semiring $F(M)$ to it: the functor F is the left-adjoint functor to the forgetful functor that sends a semiring $(R, +, 0, \cdot, 1, +)$ to the monoid $(R, \cdot, 1)$, thus forgetting the monoid structure $(R, +, 0)$. The semiring $F(M)$ is thus given by formal $\mathbb{N}$-linear combinations of elements of M, subject to the relation $m^0 \sim 1_M$ for all $m \in M$, and the relations that make the binary operation $\cdot$ of M distributive over taking linear combinations.

The claim then follows by taking $\mathbb{K}$ to be the free semiring $F(\mathrm{ob}\boldsymbol{\mathcal{V}}/\cong)$ associated to the monoid of isomorphism classes of $\mathrm{ob}\boldsymbol{\mathcal{V}}$.

□

The previous lemma motivates the following definition:

Definition 22. Let $\mathcal{V}$ be an essentially small symmetric monoidal category. The **canonical size function** of $\mathcal{V}$ is the function

$$\begin{aligned} \#\colon \mathrm{ob}\mathcal{V} &\to F(\mathrm{ob}\mathcal{V}/\cong) \\ x &\mapsto [x] \end{aligned}$$

that sends every object to its isomorphism class.

Example 23. For $\mathcal{V} = \mathbf{[0, \infty]}$ the elements of the monoid $F(\mathrm{ob}\mathcal{V}/\cong)$ are formal $\mathbb{N}$-linear combinations of numbers in $[0, \infty]$, and we have $[l_1] \cdot [l_2] = [l_1 + l_2]$. We denote this semiring by $\mathbb{N}([0, \infty])$.

Definition 24. Let X be a finite $\mathcal{V}$-category, together with a size $\#\colon \mathrm{ob}\mathcal{V} \to \mathbb{K}$, and put a total order on the objects of X. The **zeta function** of X is the matrix Z_X indexed by the objects of X with

$$Z_X(x, y) = \#(X(x, y)) .$$

One says that X has **Möbius inversion with respect to #** if Z_X is invertible over $\mathbb{K}$. In this case, the **magnitude of X with respect to #** is the sum of all entries of Z_X^{-1}, and is denoted by $|X|$.

For finite metric spaces and real-valued size functions $\#\colon [0, \infty] \to \mathbb{R}$, if one requires the function to be measurable, then one invariably must have $\#(\ell) = q^\ell$ for some non-negative real constant q [15]. As Leinster observes in [15], one can consider the different choices of a constant $q \in (0, 1)$ as rescaling the distance between the points of the metric space by a non-negative real number t, since for $-t = log(q)$ we have that $e^{-t\ell} = q^l$. This, and the observation that in many applications one is interested in a metric space only up to a rescaling of the distance between the points, lead Leinster to associate to a metric space the 1-parameter family of magnitudes obtained by rescaling the distance between its points:

Definition 25. Let (X, d) be a metric space. For any $t \in (0, \infty)$ let $tX := (X, td)$ be the metric space obtained from (X, d) by rescaling the distance between its points by a non-negative real number t. Furthermore, let $0X$ be the one-point metric space. The **magnitude function** of (X, d) is the partially defined function

$$\begin{aligned} f\colon [0, \infty] &\to \mathbb{R} \\ t &\mapsto |tX| , \end{aligned}$$

where $|X|$ is the magnitude of (X, d) with respect to the size function

$$\begin{aligned} [0, \infty] &\to \mathbb{R} \\ \ell &\mapsto e^{-\ell} . \end{aligned}$$

A crucial observation is that the magnitude function of a metric space is completely encoded in the magnitude with respect to its canonical size function. By the universal property of the canonical size function we have obtain the following commutative diagram for any $t \in [0, \infty]$

$$\begin{array}{ccc} & \ell \longmapsto & [\ell] \\ \ell & [0,\infty] \longrightarrow & \mathbb{N}[[0,\infty]] \\ \big\downarrow & \big\downarrow & \swarrow \\ e^{-t\ell} & \mathbb{R} & \end{array}$$

Thus, when seeking to categorify the magnitude function of a finite metric space, it is enough to categorify the magnitude with respect to its canonical size function.

For arbitrary finite metric spaces the magnitude function can be ill-behaved: it might only be partially defined, take on negative values, or the magnitude of a subspace might be greater than the magnitude of the whole space, see, e.g., [15, Figure 3]. However, for a certain class of nice spaces — so-called positive-definite spaces — the magnitude function is well-defined and has many of the properties that one would expect a cardinality-like invariant to have [15, Section 2.4]. In particular, all finite subsets of Euclidean space are positive-definite spaces. For such spaces one has:

- The magnitude function is well-defined [15, Lemma 2.4.2 (i)].
- $1 \leq f(t)$ for all t, see [15, Corollary 2.4.5].
- $\lim_{t\to\infty} f(t) = \#X$, where $\#X$ here denotes the cardinality of the set X (this holds for arbitrary finite metric spaces) [15, Proposition 2.2.6].

These properties also support the interpretation of the magnitude as giving the "effective number of points" of a metric space, as observed in [16, Proposition 2.8].

8. Magnitude homology

Hepworth and Willerton introduced magnitude homology for graphs in [6] as the categorification of the magnitude of a finite metric space associated to a graph. Subsequently, Leinster and Shulman generalised magnitude homology to arbitrary finite metric spaces [18]. Here we first briefly recall the definition of magnitude homology as given in [18], and then we give an alternative equivalent definition that will serve as the starting point to relate persistent homology to magnitude homology.

8.1. Magnitude homology for arbitrary finite metric spaces.

Instead of with $\mathbf{[0,\infty]}$, Leinster and Shulman choose to work with the category $\mathbf{[0,\infty)}$ with set of objects given by the non-negative real numbers $[0,\infty)$, with exactly one morphism $\epsilon \longrightarrow \epsilon'$ whenever $\epsilon \leq \epsilon'$, tensor product given by addition, and unit by 0. See [18, Section 2] for an explanation. Here we will adopt the same choice. In this setting we have that a $\mathbf{[0,\infty)}^{\mathrm{op}}$-category is a quasi-pseudometric space. Leinster and Shulman then give the following definition:

Magnitude homology (Definition A). [18, Section 5]: Let (X,d) be a finite quasi-pseudometric space. The **magnitude homology** of X is the homology of the chain complex $M(X)$ of $[0,\infty)$-graded abelian groups defined as follows:

$$M(X)_n = \bigoplus_{l\in[0,\infty)} \mathbb{Z}\left[\left\{(x_0,\ldots,x_n) \mid \sum_{i=0}^{n} d(x_i,x_{i+1}) = l\right\}\right]. \tag{4}$$

Thus, in degree n it is the free $[0,\infty)$-graded abelian group, which in degree l is generated by the ordered tuples $(x_0,\dots,x_n)$ of length exactly l. Furthermore, the boundary map $d_n\colon M(X)_n \to M(X)_{n-1}$ is given by the alternating sum of maps d_n^i, defined as follows for all $1 \le i \le n-1$:

$$d_n^i((x_0,\dots,x_n)) = \begin{cases} (x_0,\dots,x_{i-1},x_{i+1},\dots x_n), & \text{if } d(x_{i-1},x_i)+d(x_i,x_{i+1}) = d(x_{i-1},x_{i+1}) \\ 0, & \text{otherwise} \end{cases}$$

while for $i=0$ we have

$$d_n^0((x_0,\dots,x_n)) = \begin{cases} (x_1,x_2,\dots x_n), & \text{if } d(x_0,x_1)=0 \\ 0, & \text{otherwise} \end{cases}$$

and similarly for $i=n$.

The assignment $X \mapsto H_\star(M(X))$ induces a functor from the category with objects $\mathbf{[0,\infty)}$-categories and morphisms $\mathbf{[0,\infty)}$-functors to the category of $[0,\infty)$-graded abelian groups [18, Theorem 4.4]. Futhermore, for finite quasi-pseudometric spaces X, the magnitude homology of X categorifies the magnitude of X with respect to the canonical size function, see [18, Theorem 5.11, Corollary 6.28].

8.2. **Magnitude homology: an alternative viewpoint.** In online discussions [17] Leinster and Shulman initially gave a different definition of magnitude homology. Here we recall this definition (Definition B), and prove that an adaptation of it (Definition B') agrees with the definition given in the previous section (Definition A). We will use Definition B' of magnitude homology to relate magnitude homology to persistent homology.

Denote by Ab the category of abelian groups with monoidal structure given by the tensor product of abelian groups, which we denote by $\boxtimes$; this induces a monoidal structure on the category of chain complexes over Ab, which we again denote by $\boxtimes$. Given a $\mathbf{[0,\infty)}^{\mathrm{op}}$-category X, Leinster and Shulman consider the following functor

$$CN(X) = \left(\mathbf{[0,\infty)} \xrightarrow{N(X)} \mathrm{sSet} \xrightarrow{\mathbb{Z}[\cdot]} \mathrm{sAb} \xrightarrow{U} \mathrm{ch}_{\mathrm{Ab}}\right), \tag{5}$$

where the functor $\mathbb{Z}[\cdot]$ is induced by the free abelian group functor, and the functor U is the functor that sends a simplicial abelian group to its unnormalised chain complex. They then introduce functors of coefficients $A\colon \mathbf{[0,\infty)}^{\mathrm{op}} \longrightarrow \mathrm{Ab}$, where one views A as taking values in $\mathrm{ch}_{\mathrm{Ab}}$ through the canonical inclusion $\mathrm{Ab} \hookrightarrow \mathrm{ch}_{\mathrm{Ab}}$, and give the following definition:

Magnitude homology (Definition B). The **magnitude homology of *X* with coefficients in *A*** is the homology of the chain complex given by the coend $CN(X) \otimes_{\mathbf{[0,\infty)}} A$.

One can describe this chain complex as follows for a particular choice of coefficient functor.

Lemma 26. For any $\epsilon \in [0, \infty)$ define the following functor of coefficients

$$\begin{aligned} A_\epsilon \colon \mathbf{[0,\infty)}^{\mathrm{op}} &\to \mathrm{Ab} \\ l &\mapsto \begin{cases} \mathbb{Z}, \text{ if } l = \epsilon \\ 0, \text{ otherwise} \end{cases} \\ (\ell \geq \ell') &\mapsto \begin{cases} \mathrm{id}_{\mathbb{Z}}, \text{ if } \ell = \ell' = \epsilon \\ 0, \text{ otherwise}\,. \end{cases} \end{aligned}$$

We consider A_ϵ as taking values in $\mathrm{ch}_{\mathrm{Ab}}$ through the canonical inclusion functor $\mathrm{Ab} \hookrightarrow \mathrm{ch}_{\mathrm{Ab}}$.

Then, the chain complex $CN(X) \otimes_{\mathbf{[0,\infty)}} A_\epsilon$ is given in degree n by the free abelian group on the tuples $(x_0, \ldots, x_n)$ that have length exactly ϵ. The boundary maps

$$d_n \colon \left(CN(X) \otimes_{\mathbf{[0,\infty)}} A_\epsilon\right)_n \to \left(CN(X) \otimes_{\mathbf{[0,\infty)}} A_\epsilon\right)_{n-1}$$

are alternating sums of maps d_n^i which can be described as follows, for $0 < i < n$:

$$d_n^i((x_0, \ldots, x_n)) = \begin{cases} (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), & \text{if } d(x_{i-1}, x_{i+1}) = d(x_{i-1}, x_i) + d(x_i, x_{i+1}) \\ 0, & \text{otherwise} \end{cases}$$

while for $i = 0$ we have

$$\begin{aligned} d_n^0 \colon \left(CN(X) \otimes_{\mathbf{[0,\infty)}} A_\epsilon\right)_n &\to \left(CN(X) \otimes_{\mathbf{[0,\infty)}} A_\epsilon\right)_{n-1} \\ (x_0, \ldots, x_n) &\mapsto \begin{cases} (x_1, x_2, \ldots, x_n), & \text{if } d(x_0, x_1) = 0 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

and similarly for $i = n$.

Proof. [1] The coend $CN(X) \otimes_{\mathbf{[0,\infty)}} A_\epsilon$ is the coequaliser of the following diagram:

$$\bigoplus_{\ell \leq \ell'} CN(X)(\ell) \boxtimes A_\epsilon(\ell') \; \underset{\bigoplus_{\ell \leq \ell'} 1 \boxtimes A_\epsilon(\ell \leq \ell')}{\overset{\bigoplus_{\ell \leq \ell'} CN(X)(\ell \leq \ell') \boxtimes 1}{\rightrightarrows}} \; \bigoplus_{\ell \in [0,\infty)} CN(X)(\ell) \boxtimes A_\epsilon(\ell)\,.$$

Thus, it is the coproduct over $\ell \in [0, \infty)$ of the chain complexes $CN(X)(\ell) \boxtimes A_\epsilon(\ell)$, modulo the relations given by equating the two parallel morphisms on the left hand side. By definition of A_ϵ, the morphisms are both trivial if $\ell' \neq \epsilon$, thus we assume that $\ell' = \epsilon$. The two morphisms are identical if $\ell = \epsilon$, thus we assume that $\ell \neq \epsilon$, and so we have $\epsilon > \ell$. Thus, the bottom parallel morphism is zero, while the top parallel morphism is

$$CN(X)(\ell) \longrightarrow CN(X)(\epsilon)\,.$$

Furthermore, we have

$$\bigoplus_{\ell \in [0,\infty)} CN(X)(\ell) \boxtimes A_\epsilon(\ell) \cong CN(X)(\epsilon)$$

[1] We note that parts of this proof were given by Shulman in an online comment [26].

since tensoring with $A_\epsilon(\ell)$ makes all summands vanish, except if $\ell = \epsilon$. Thus, in degree n the chain complex $CN(X) \otimes_{\mathbf{[0,\infty)}} A_\epsilon$ is the free abelian group on the tuples $(x_0, \dots, x_n)$ that have length exactly ϵ.

Now, denote by $D(\epsilon)$ the subcomplex of $CN(X)(\epsilon)$ whose n-chains are the n-tuples with length strictly less than ϵ, so that $CN(X) \otimes_{\mathbf{[0,\infty)}} A_\epsilon \cong CN(X)(\epsilon)/D(\epsilon)$ by the previous discussion. Note that the boundary map on the quotient chain complex $CN(X)(\epsilon)/D(\epsilon)$ is the alternating sum of maps

$$d_n^i \colon CN(X)(\epsilon)_n/D(\epsilon)_n \to CN(X)(\epsilon)_{n-1}/D(\epsilon)_{n-1}$$

which send $c + D(\epsilon)_n$ to $d_n^{i,C}(c) + D(\epsilon)_{n-1}$, where $d_n^{i,C} \colon CN(X)(\epsilon)_n \to CN(X)(\epsilon)_{n-1}$ is the map induced by the ith face map. Thus $d_n^i(x_0, \dots, x_n)$ is the map induced by the ith face maps if and only if by deleting the ith entry the length of the tuple is unchanged, and is the zero map otherwise.

□

Our aim is to relate Definition B with Definition A. For any $\epsilon \leq \epsilon'$ define the natural transformation $\iota \colon A_\epsilon \Rightarrow A_{\epsilon'}$ where ι_ℓ is the identity if $\ell = \epsilon = \epsilon'$, and the zero map otherwise. This induces a chain map $CN(X) \otimes_{\mathbf{[0,\infty)}} A_\epsilon \to CN(X) \otimes_{\mathbf{[0,\infty)}} A_{\epsilon'}$, and we thus have a functor $CN(X) \otimes_{\mathbf{[0,\infty)}} A_- \colon \mathbf{[0,\infty)} \to \mathrm{ch_{Ab}}$ that assigns to a number ϵ the chain complex $CN(X) \otimes_{\mathbf{[0,\infty)}} A_\epsilon$. The functor $CN(X) \otimes_{\mathbf{[0,\infty)}} A_-$ factors through the forgetful functor $\mathbf{[0,\infty)} \to [0,\infty)$, where $[0,\infty)$ is the discrete category with objects given by non-negative real numbers. Recall that a chain complex of $[0,\infty)$-graded abelian groups can be identified with an $[0,\infty)$-graded chain complex (see Example 17). Thus, in particular, we can identify the chain complex $M(X)$ with a functor $\mathbf{[0,\infty)} \to \mathrm{ch_{Ab}}$ that coincides with $M(X)$ on the set of objects, and sends every morphism to the trivial chain map (see Remark 18). We have:

Proposition 27. The functors $CN(X) \otimes_{\mathbf{[0,\infty)}} A_-$ and $M(X)$ are isomorphic. In particular, the magnitude homology of X (Definition A) is isomorphic to the homology of $CN(X) \otimes_{\mathbf{[0,\infty)}} A_-$.

Proof. By Lemma 26 and (4) the chain complexes $M(X)_\epsilon$ and $CN(X) \otimes_{\mathbf{[0,\infty)}} A_\epsilon$ are canonically isomorphic. Next, for $\epsilon \leq \epsilon'$, consider the following diagram:

$$\begin{array}{ccc} M(X)_\epsilon & \longrightarrow & M(X)_{\epsilon'} \\ \downarrow & & \downarrow \\ CN(X) \otimes_{\mathbf{[0,\infty)}} A_\epsilon & \longrightarrow & CN(X) \otimes_{\mathbf{[0,\infty)}} A_{\epsilon'} \end{array}$$

where the vertical arrows are the canonical isomorphisms, while the horizontal arrows are zero maps. Thus every such square commutes, so the canonical isomorphisms assemble into a natural isomorphism between $CN(X) \otimes_{\mathbf{[0,\infty)}} A_-$ and $M(X)$. □

For ease of reference, we state here the equivalent definition of magnitude homology, as given by Proposition 27:

Magnitude homology (Definition B'). The **magnitude homology of X** is the homology of the $[0,\infty)$-graded chain complex $CN(X) \otimes_{\mathbf{[0,\infty)}} A_-$.

9. Persistent homology

Persistent homology is, in an appropriate sense, the generalisation of simplicial homology of a simplicial set to persistent simplicial sets. Given a metric space (X, d), we seek to study its geometric and topological properties by associating to it $\mathbf{[0,\infty)}$-persistent simplicial sets $S(X)$. We then consider the functor

$$CS(X) = \left(\mathbf{[0,\infty)} \xrightarrow{S(X)} \mathrm{sSet} \xrightarrow{\mathbb{Z}[\cdot]} \mathrm{sAb} \xrightarrow{U} \mathrm{ch_{Ab}} \right),$$

where $\mathbb{Z}[\cdot]$ and U are defined as in (5).

Let $\mathbb{F}$ be a field. Consider the constant functor of coefficients

$$A\colon \mathbf{[0,\infty)} \to \mathrm{ch_{Ab}}$$

that sends ℓ to the chain complex with a copy of $\mathbb{F}$ concentrated in degree zero, and further sends $\ell \leq \ell'$ to the identity chain map if $\ell = \ell'$ and to the zero chain map otherwise.

The composite $H_\star(CS(X) \boxtimes A)$, where $H_\star\colon \mathrm{ch_{Ab}} \to \mathrm{Ab}$ is the usual homology functor, is usually called the "persistent homology of X (with respect to S) with coefficients in $\mathbb{F}$." Using this coefficient functor has the advantage that, under appropriate finiteness conditions, isomorphism classes of such functors can be completely characterised by a collection of intervals, called the **barcode**, see e.g., [22, Theorem 1.9].

More generally, we give the following definition:

Definition 28. Let (X, d) be a metric space, let $S(X)$ be a $\mathbf{[0,\infty)}$-persistent simplicial set, and $A\colon \mathbf{[0,\infty)} \to \mathrm{ch_{Ab}}$ a functor. The **persistent homology of X with respect to S and with coefficients in A** is the composition $H_\star(CS(X) \boxtimes A)$. When A is the unit for $\boxtimes$ we call the homology of $CS(X) \boxtimes A$ the **persistent homology of X (with respect to S)**.

For arbitrary coefficient functors one in general no longer has a barcode. However, such functors of coefficients might be interesting for applications, as they might allow to capture more refined information, for instance different torsion or orientability phenomena over different filtration scales, which might be detected by taking, e.g., coefficients over $\mathbb{F}_2$ (the field with two elements) over a certain interval $I \subset [0,\infty)$, and coefficients over $\mathbb{F}_3$ over a different disjoint interval $J \subset [0,\infty)$. More complicated coefficient functors might allow for an even more refined analysis.

10. Magnitude meets persistence

In the final section of [18] Leinster and Shulman list a series of open problems; the last two of these are (from [18, Section 8]):

(9) Magnitude homology only "notices" whether the triangle inequality is a strict equality or not. Is there a "blurred" version that notices "approximate equalities"?

(10) Almost everyone who encounters both magnitude homology and persistent homology feels that there should be some relationship between them. What is it?

In this section we attempt a first answer to these questions, which we believe are intertwined: it is the blurred version of magnitude homology that is related to persistent homology. Indeed, as is apparent from Proposition 27, the magnitude homology of a metric space X is a homology theory that in a certain sense forgets the maps induced on the homology groups by the inclusions of simplicial sets $N(X)(\epsilon) \to NX(\epsilon')$, whenever $\epsilon \leq \epsilon'$, whereas the "persistence" in persistent homology is exactly the information given by such maps. Thus, morally, these are very different homology theories.

Our starting point is Definition B' of magnitude, which we adapt to coefficient functors not supported at points, but on intervals.

Definition 29. For any $\epsilon \in [0, \infty]$ define the functor of coefficients

$$\begin{aligned} A_{[0,\epsilon]} \colon \mathbf{[0,\infty)}^{\mathrm{op}} &\to \mathrm{Ab} \\ \ell &\mapsto \begin{cases} \mathbb{Z}, \text{ if } \ell \in [0,\epsilon] \\ 0, \text{ otherwise} \end{cases} \\ (\ell \geq \ell') &\mapsto \begin{cases} \mathrm{id}_{\mathbb{Z}}, \ \ell, \ell' \in [0,\epsilon] \\ 0, \text{ otherwise }. \end{cases} \end{aligned}$$

We consider $A_{[0,\epsilon]}$ as taking values in $\mathrm{ch}_{\mathrm{Ab}}$ through the canonical inclusion functor $\mathrm{Ab} \hookrightarrow \mathrm{ch}_{\mathrm{Ab}}$.

Now, for any $\epsilon \leq \epsilon'$ we consider the natural transformation $\iota \colon A_{[0,\epsilon]} \Rightarrow A_{[0,\epsilon']}$ where ι_ℓ is the identity if $\ell \in [0,\epsilon]$, and the zero map otherwise. This natural transformation induces a chain map $CN(X) \otimes_{\mathbf{[0,\infty)}} A_{[0,\epsilon]} \to CN(X) \otimes_{\mathbf{[0,\infty)}} A_{[0,\epsilon']}$, and we thus have a functor $CN(X) \otimes_{\mathbf{[0,\infty)}} A_{[0,-]} \colon \mathbf{[0,\infty)} \to ch_{\mathrm{Ab}}$ that assigns to a number ϵ the chain complex $CN(X) \otimes_{\mathbf{[0,\infty)}} A_{[0,\epsilon]}$. Explicitly, we can describe the chain complexes $CN(X) \otimes_{\mathbf{[0,\infty)}} A_{[0,\epsilon]}$ as follows:

Lemma 30. For any $\epsilon \in [0, \infty)$ we have

$$CN(X) \otimes_{\mathbf{[0,\infty)}} {A_{[0,\epsilon]}}_n = \mathbb{Z}\left[\left\{(x_0, \ldots, x_n) \mid \sum_{i=0}^{n} d(x_i, x_{i+1}) \leq \epsilon\right\}\right]$$

with boundary maps given by alternating sums of maps induced by the face maps.

Proof. The chain complex $CN(X) \otimes_{\mathbf{[0,\infty)}} A_{[0,\epsilon]}$ is the coequaliser of the following diagram:

$$\bigoplus_{\ell \leq \ell'} CN(X)(\ell) \boxtimes A_{[0,\epsilon]}(\ell') \underset{\bigoplus_{\ell \leq \ell'} 1 \boxtimes A_{[0,\epsilon]}(\ell \leq \ell')}{\overset{\bigoplus_{\ell \leq \ell'} CN(X)(\ell \leq \ell') \boxtimes 1}{\rightrightarrows}} \bigoplus_{\ell \in [0,\infty)} CN(X)(\ell) \boxtimes A_{[0,\epsilon]}(\ell)\,.$$

First, note that

$$\bigoplus_{\ell \in [0,\infty)} CN(X)(\ell) \boxtimes A_{[0,\epsilon]}(\ell) \cong \bigoplus_{\ell \in [0,\epsilon]} CN(X)(\ell)\,,$$

as the summands vanish if $\ell > \epsilon$, by definition of $A_{[0,\epsilon]}$. We next see what relations are given by the two parallel morphisms in the diagram. For $\ell' > \epsilon$ we have that the morphisms are both zero, so we assume that $\ell' \in [0, \epsilon]$. Furthermore, the morphisms are identical if $\ell = \ell'$, so we assume that $\ell \neq \ell'$, and thus have $0 \leq \ell < \ell' \leq \epsilon$. Thus, the top horizontal morphism is $CN(X)(\ell) \to CN(X)(\ell')$, while the bottom morphism is $CN(X)(\ell) \to CN(X)(\ell)$. By equating these morphisms in $\oplus_{\ell\in[0,\epsilon]} CN(X)(\ell)$ we are thus identifying the summand $CN(X)(\ell)$ with the image of the inclusion of $CN(X)(\ell)$ in $CN(X)(\ell')$. We thus obtain

$$CN(X) \otimes_{\mathbf{[0,\infty)}} A_{[0,\epsilon]} \cong CN(X)(\epsilon)\,.$$

Similarly, the relations given by the pair of parallel morphisms tell us that the boundary maps on the quotient chain complex are the boundary maps of the chain complex $CN(X)(\epsilon)$, thus alternating sums of maps induced by face maps. □

Definition 31. Let (X, d) be a metric space. The **blurred magnitude homology** of X is the homology of $CN(X) \otimes_{\mathbf{[0,\infty)}} A_{[0,-]})$.

We have:

Theorem 32. The functors $CN(X) \otimes_{\mathbf{[0,\infty)}} A_{[0,-]}$ and $CN(X)$ are isomorphic. In particular, the blurred magnitude homology of X is isomorphic to the persistent homology of X with respect to the enriched nerve.

Proof. By Lemma 30 we know that there is an isomorphism between $CN(X) \otimes_{\mathbf{[0,\infty)}} A_{[0,\epsilon]}$ and $CN(X)(\epsilon)$ for any $\epsilon \in [0,\infty)$. Next, for any $\epsilon \leq \epsilon'$, the square

$$\begin{array}{ccc} CN(X) \otimes_{\mathbf{[0,\infty)}} A_{[0,\epsilon]} & \longrightarrow & CN(X) \otimes_{\mathbf{[0,\infty)}} A_{[0,\epsilon']} \\ \downarrow & & \downarrow \\ CN(X)(\epsilon) & \longrightarrow & CN(X)(\epsilon') \end{array}$$

where the vertical maps are the isomorphisms, commutes, as the horizontal morphisms are inclusions.

□

11. Limit homology

In [27] Vietoris introduced what is now called the Vietoris–Rips complex, as a way to define a homology theory for compact metric spaces[2]. One starts by considering the homology of the $\mathbf{[0,\infty)}$-persistent chain complex

$$CV^{\mathrm{sim}}(X) = \left(\mathbf{[0,\infty)} \overset{V^{\mathrm{sim}}(X)}{\longrightarrow} \mathrm{sSet} \longrightarrow \mathrm{sAb} \longrightarrow \mathrm{ch}_{\mathrm{Ab}}\right)$$

where $V^{\mathrm{sim}}(X)$ is the Vietoris–Rips simplicial set associated to the metric space X, see Definition 11. Vietoris defined the homology of X (for a compact metric space X) to be the limit

$$\mathcal{H}_\star(X) := \varinjlim_{\epsilon} H_\star(CV^{\mathrm{sim}}(X)(\epsilon))\,. \tag{6}$$

[2]We note that while Vietoris introduced what is called the "Vietoris–Rips simplicial complex" (at level ϵ) $V(X)(\epsilon)$, here we discuss this homology theory using the simplicial set $V^{\mathrm{sim}}(X)(\epsilon)$ associated to it, see Lemma 7 and Definition 11.

In later work, Hausmann [7] proposed a cohomological counterpart of the homology theory introduced by Vietoris, by considering the colimit of the functor that one obtains by taking simplicial cohomology of the Vietoris–Rips simplicial sets:

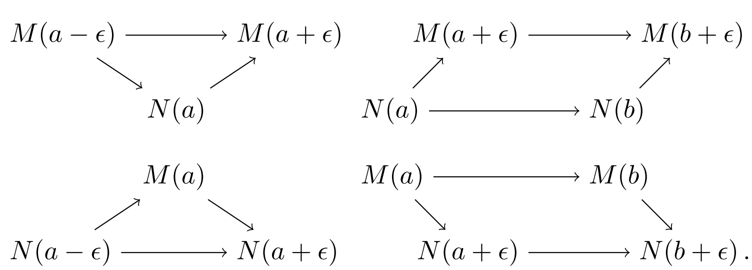
$$\mathcal{H}^\star(X) := \varinjlim_{\epsilon} H^\star(CV^{\mathrm{sim}}(X)(\epsilon))\,.$$

He called this cohomology theory "metric cohomology", and not Vietoris cohomology, because the adjective Vietoris had already been used to designate a cohomology theory which is in general not isomorphic to the cohomological counterpart of the homology theory introduced by Vietoris [7]. The denomination "Vietoris–Rips" for the complex introduced by Vietoris is also due to Hausmann, as the complex introduced by Vietoris was in the meantime known as Rips complex [24].

Instead of the Vietoris–Rips simplicial set, we can consider the enriched nerve associated to a metric space X, and take the limit of resulting homology functor:

$$\varinjlim_{\epsilon} H_\star(CN(X)(\epsilon))\,. \tag{7}$$

In the following we relate the limits (6) and (7). Let C be any category, and $(P, \leq, +)$ a monoidal poset. The category with objects given by $\boldsymbol{P}$-persistent objects of C, and morphisms given by natural transformations between them, can be endowed with an extended pseudo-distance, called **interleaving distance** [1]. The interleaving distance was first introduced in [2] for the monoidal poset $(\mathbb{R}, \leq, +)$. The central notion is that of interleaving: for $\epsilon \geq 0$ two functors $M, N\colon \mathbb{R} \to C$ are **ϵ-interleaved** if there are collections of morphisms $\{\phi_\epsilon\colon M(a) \to N(a+\epsilon) \mid a \in \mathbb{R}\}$ and $\{\psi_\epsilon\colon N(a) \to M(a+\epsilon) \mid a \in \mathbb{R}\}$ such that all diagrams of the following form commute:



Two functors that are ϵ-interleaved have bounded interleaving distance [2, Theorem 4.4].

In many examples of filtered spaces that one considers in topological data analysis, what one obtains is not an interleaving of the corresponding homologies, but rather what is called an "approximation". Two functors $M, N\colon \mathbb{R} \to C$ are ***c*-approximations** of each other if there are collections of morphisms $\{\phi_c\colon M(a) \to N(ca) \mid a \in \mathbb{R}\}$ and $\{\psi_c\colon N(a) \to M(ca) \mid a \in \mathbb{R}\}$ such that all diagrams of the following form commute:

$$
\begin{array}{ccc}
M(a) & \longrightarrow & M(c^2a) \\
 & \searrow \quad \nearrow & \\
 & N(ca) &
\end{array}
\qquad
\begin{array}{cccc}
 & M(ca) & \longrightarrow & M(cb) \\
 \nearrow & & \nearrow & \\
N(a) & \longrightarrow & N(b) &
\end{array}
$$

$$
\begin{array}{ccc}
 & M(ca) & \\
 & \nearrow \quad \searrow & \\
N(a) & \longrightarrow & N(c^2a)
\end{array}
\qquad
\begin{array}{cccc}
M(a) & \longrightarrow & M(b) & \\
 & \searrow & & \searrow \\
 & N(ca) & \longrightarrow & N(cb)\,.
\end{array}
$$

It shouldn't then be too surprising that functors that are c-approximations of each other have bounded interleaving distance in the log scale [11].

For ease of reference, we state the definition of c-approximations for $\mathbf{[0,\infty)}$-persistent objects:

Definition 33. Let C be a category, and let $M, N\colon \mathbf{[0,\infty)} \longrightarrow C$ be two functors. For any $c \geq 1$ denote by $D_c\colon \mathbf{[0,\infty)} \to \mathbf{[0,\infty)}$ the functor that sends a to ca. Furthermore, denote by $\eta_c\colon id_{\mathbf{[0,\infty)}} \Rightarrow D_c$ the natural transformation given by $\eta_c(a)\colon a \to ca$. A ***c*-approximation** of M and N is a pair of natural transformations

$$\phi\colon M \Rightarrow ND_c$$

and

$$\psi\colon N \Rightarrow MD_c$$

such that $(\psi D_c)\phi = M\eta_{c^2}$ and $(\phi D_c)\psi = N\eta_{c^2}$.

Lemma 34. Let $M, N\colon \mathbf{[0,\infty)} \longrightarrow \mathrm{Ab}$ be two $\mathbf{[0,\infty)}$-persistent abelian groups. If there exists a c-approximation between M and N, then

$$\varinjlim_{\epsilon} M(\epsilon) \cong \varinjlim_{\epsilon} N(\epsilon)\,.$$

Proof. Let $\phi\colon M \Rightarrow ND_c$ and $\psi\colon N \Rightarrow MD_c$ be the natural transformations which are part of the data of the c-approximation. These induce homomorphisms

$$\bar{\phi}\colon \varinjlim_{\epsilon} M(\epsilon) \longrightarrow \varinjlim_{\epsilon} N(\epsilon)$$

and

$$\bar{\psi}\colon \varinjlim_{\epsilon} N(\epsilon) \longrightarrow \varinjlim_{\epsilon} M(\epsilon)$$

which are inverse to each other. □

Theorem 35. For all $k = 0, 1, 2, \ldots$ there is an isomorphism

$$\varinjlim_{\epsilon} H_k(CN(X)(\epsilon)) \cong \mathcal{H}_k(X)\,.$$

Proof. While there is an inclusion map $N(X)(\epsilon) \to V^{\mathrm{sim}}(X)(\epsilon)$ for all $\epsilon \in [0,\infty]$, in general there is an inclusion $V^{\mathrm{sim}}(X)(\epsilon) \to N(X)(c\epsilon)$ only for $c = \infty$, since a p-simplex in $V^{\mathrm{sim}}(X)(\epsilon)$ may have length equal to $p\epsilon$. On the other hand, when computing simplicial homology in dimension k, we only need to consider simplices up to dimension $k+1$, and we will therefore consider the truncations of the simplicial sets $N(X)(\epsilon)_{\leq k}$, given by precomposing $N(X)(\epsilon)\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ with the inclusion $\Delta^{\mathrm{op}}_{\leq k} \to \Delta^{\mathrm{op}}$, and similarly for $V^{\mathrm{sim}}(X)(\epsilon)$.

The inclusion $N(X)(\epsilon) \to V^{\mathrm{sim}}(X)(\epsilon)$ induces an inclusion

$$\phi_c(\epsilon)\colon N(X)_{\leq k+1}(\epsilon) \to V^{\mathrm{sim}}(X)_{\leq k+1}(c\epsilon)$$

for any $c \geq 1$. If σ is a p-simplex in $V^{\mathrm{sim}}(X)(\epsilon)$, then its length is bounded by $p\epsilon$, and thus there is an inclusion map

$$\psi_{k+1}(\epsilon)\colon V^{\mathrm{sim}}(X)_{\leq k+1}(\epsilon) \to N(X)_{\leq k+1}((k+1)\epsilon)\,.$$

The collection of maps

$$\left\{\phi_{k+1}(\epsilon)\colon N(X)_{\leq k+1}(\epsilon) \to V^{\mathrm{sim}}(X)_{\leq k+1}((k+1)\epsilon) \mid \epsilon \in [0,\infty)\right\}$$

and

$$\left\{\psi_{k+1}(\epsilon)\colon V^{\mathrm{sim}}(X)_{\leq k+1}(\epsilon) \to N(X)_{\leq k+1}((k+1)\epsilon) \mid \epsilon \in [0,\infty)\right\}$$

are easily seen to satisfy the properties of a $k+1$-approximation, as all maps involved are inclusions. Applying homology we obtain a $k+1$-approximation between the functors $H_k(CV^{\mathrm{sim}}(X))$ and $H_k(CN(X))$. We can now use Lemma 34 and obtain an isomorphism

$$\varinjlim_{\epsilon} H_k(CN(X)_{\leq k+1}(\epsilon)) \cong \varinjlim_{\epsilon} H_k(CV^{\mathrm{sim}}(X)_{\leq k+1}(\epsilon))\,.$$

Since $H_k(CN(X)_{\leq k+1}(\epsilon))$ is equal to $H_k(CN(X)(\epsilon))$ for all k, and similarly for the truncation of the Vietoris–Rips simplicial set, we obtain the claim.

□

Corollary 36. Let k be a non-negative integer, and let X be a metric space with $\mathcal{H}_k(X) \ncong 0$. Then

$$\varinjlim_{\epsilon} H_k(CN(X) \otimes_{\mathbf{[0,\infty)}} A_\epsilon) \ncong \varinjlim_{\epsilon} H_k(CN(X) \otimes_{\mathbf{[0,\infty)}} A_{[0,\epsilon]})\,.$$

That is, under the limit, the kth ordinary and blurred magnitude homology of X are not isomorphic. In particular, for any finite metric space the limits differ for $k = 0$.

Proof. First, note that

$$\varinjlim_{\epsilon} H_k(CN(X) \otimes_{\mathbf{[0,\infty)}} A_\epsilon) \cong 0$$

since for any $\epsilon \leq \epsilon'$ we have that $CN(X) \otimes_{\mathbf{[0,\infty)}} A_\epsilon \to CN(X) \otimes_{\mathbf{[0,\infty)}} A_{\epsilon'}$ is the zero chain map. By Lemma 30 we know that $CN(X) \otimes_{\mathbf{[0,\infty)}} A_{[0,\epsilon]} \cong CN(X)(\epsilon)$, and further by Theorem 35 we have that $\varinjlim_{\epsilon} H_k(CN(X)(\epsilon))$ is the Vietoris homology of X. □

12. Conclusion

In this manuscript we relate persistent homology to magnitude homology as two different ways of computing homology of filtered simplicial sets. We give an answer to two of the open problems formulated by Leinster and Shulman in [18], and listed on Page 2 of this manuscript, which we show are intertwined. We define a blurred version of magnitude homology and show that it is the persistent homology taken with respect to a certain filtered simplicial set. Furthermore, we show how blurred and ordinary magnitude homology differ in the limit: blurred magnitude homology coincides with Vietoris homology, while magnitude homology is trivial. Based on this comparison we conjecture the existence of an Euler characteristic for persistent homology and the magnitude (function) that it would encode. Whether this would coincide with the magnitude function given by magnitude homology would be a further open question; if yes, the behaviour in the limit of magnitude homology, as well as the findings [8, 9], clarify that the right way to categorify the magnitude is given by the blurred version of magnitude homology. If not, then this invariant should give much richer information about the geometry of a metric space than the magnitude.

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