

GEOMETRIC HARDY INEQUALITIES ON STARSHAPED SETS

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ABSTRACT. In this paper, we present the geometric Hardy inequalities on the star-shaped sets in the Carnot groups. Also, we obtain the geometric Hardy inequalities on half-spaces for general vector fields.

1. INTRODUCTION

In 1998, Danielli and Garofalo [4] firstly introduced the concept of starshapedness on the Carnot groups (see also [5]). Their paper provides the geometrical properties of starshaped and convex sets. The convexity in the Heisenberg groups was studied by many authors such as Monti and Rickly [10] who proved the geodesic convexity, or by Danielli, Garofalo, and Nhieu [3] (see also [8]) who introduced the concept of horizontal convexity (H -convexity). Bardi and Dragoni [1], [2] generalised the concept of convexity to general vector fields and introduced the notion of \mathcal{X} -convexity which is a generalisation of H -convexity. This analysis allows introducing the distance to the boundary notation for starshaped sets, so by using the distance formula one can obtain geometric Hardy type inequalities.

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The main aim of this paper is to obtain the geometric Hardy inequalities on star-shaped sets in the Carnot groups. Moreover, we present the geometric Hardy inequalities on the half-spaces for general vector fields.

We organise the paper in the following way:

- Sec. 1: We give a brief overview of the sub-Riemannian manifolds, Grushin plane, Carnot groups, Heisenberg groups, and Engel groups.
- Sec. 2: We obtain the geometric Hardy inequalities on the starshaped sets in the Carnot groups and provide some examples.
- Sec. 3: We obtain the geometric Hardy inequalities on the half-spaces for general vector fields and provide some examples.
- Sec. 4: We give the proofs of main results.

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1.1. Sub-Riemannian manifolds. Let M be a smooth manifold of dimension n with a family of vector fields $\{X_k\}_{k=1}^N$, $n \geq N$, defined on M satisfying the Hörmander rank condition. Then they induce a sub-Riemannian metric $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ on the associated space $\mathcal{H}_x = \text{span}(X_1(x), \dots, X_N(x))$. The triple $(M, \mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ is a so-called sub-Riemannian manifold (with sub-Riemannian geometry). Note that, unlike for Carnot groups, in general, it is not possible to define dilations, translations, the homogeneous norm and the distance on sub-Riemannian manifolds.

Let us denote the operator of the sum of squares of vector fields by

$$\mathcal{L} := \sum_{k=1}^N X_k^2. \quad (1.1)$$

These operators have been studied by many authors, for instance, it is well-known since Hörmander's pioneering work [9] that if the commutators of the vector fields $\{X_k\}_{k=1}^N$ generate the Lie algebra, the operator \mathcal{L} is locally hypoelliptic. The p -version of the sum of squares of vector fields can be given by the formula

$$\mathcal{L}_p f := \nabla_X \cdot (|\nabla_X f|^{p-2} \nabla_X f), \quad (1.2)$$

where

$$\nabla_X := (X_1, \dots, X_N).$$

1.2. Grushin plane. One of the important examples of a sub-Riemannian manifold is the Grushin plane. The Grushin plane is the space \mathbb{R}^2 with vector fields

$$X_1 = \frac{\partial}{\partial x_1}, \quad \text{and} \quad X_2 = x_1 \frac{\partial}{\partial x_2},$$

for $x := (x_1, x_2) \in \mathbb{R}^2$.

1.3. Carnot groups. Let $\mathbb{G} = (\mathbb{R}^n, \circ, \delta_\lambda)$ be a stratified Lie group (or a homogeneous Carnot group or just a Carnot group), with the dilation structure δ_λ and Jacobian generators X_1, \dots, X_N , so that N is the dimension of the first stratum of \mathbb{G} . Let us denote by Q the homogeneous dimension of \mathbb{G} . We refer to the recent books [7] and [15] for extensive discussions of stratified Lie groups and their properties.

The sub-Laplacian on \mathbb{G} is given by

$$\mathcal{L} = \sum_{k=1}^N X_k^2. \quad (1.3)$$

We also recall that the standard Lebesgue measure dx on \mathbb{R}^n is the Haar measure for \mathbb{G} (see, e.g. [7, Proposition 1.6.6]). Each left invariant vector field X_k has an explicit form and satisfies the divergence theorem, see e.g. [7] for the derivation of exact formula: more precisely, we can express

$$X_k = \frac{\partial}{\partial x'_k} + \sum_{l=2}^r \sum_{m=1}^{N_l} a_{k,m}^{(l)}(x', \dots, x^{(l-1)}) \frac{\partial}{\partial x_m^{(l)}}, \quad (1.4)$$

with $x = (x', x^{(2)}, \dots, x^{(r)})$, where r is the step of \mathbb{G} and $x^{(l)} = (x_1^{(l)}, \dots, x_{N_l}^{(l)})$ are the variables in the l^{th} stratum, see also [7, Section 3.1.5] for a general presentation. The

horizontal divergence is defined by

$$\operatorname{div}_H f := \nabla_H \cdot f,$$

where

$$\nabla_H := (X_1, \dots, X_N)$$

is the horizontal gradient. The p -sub-Laplacian has the form

$$\mathcal{L}_p f = \nabla_H \cdot (|\nabla_H f|^{p-2} \nabla_H f). \quad (1.5)$$

1.4. Heisenberg groups. Let \mathbb{H}_1 be the Heisenberg group, that is, the set \mathbb{R}^3 equipped with the group law

$$x \circ x' := (x_1 + x'_1, x_2 + x'_2, x_3 + x'_3 + 2(x'_1 x_2 - x_1 x'_2)),$$

where $x := (x_1, x_2, x_3) \in \mathbb{R}^3$, and $x^{-1} = -x$ is the inverse element of x with respect to the group law. The dilation operation on the Heisenberg group with respect to the group law has the form

$$\delta_\lambda(x) := (\lambda x_1, \lambda x_2, \lambda^2 x_3) \text{ for } \lambda > 0.$$

The Lie algebra \mathfrak{h} of the left-invariant vector fields on the Heisenberg group \mathbb{H}_1 is spanned by

$$\begin{aligned} X_1 &:= \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_3}, \\ X_2 &:= \frac{\partial}{\partial x_2} - 2x_1 \frac{\partial}{\partial x_3}, \end{aligned}$$

with their (non-zero) commutator

$$[X_1, X_2] = -4 \frac{\partial}{\partial x_3}.$$

The horizontal gradient on \mathbb{H}_1 is given by

$$\nabla_H := (X_1, X_2),$$

so the sub-Laplacian on \mathbb{H}_1 is given by

$$\mathcal{L} := X_1^2 + X_2^2.$$

The Heisenberg group is the most common example of a step 2 stratified group (Carnot group).

1.5. Engel groups. Let \mathbb{E} be the Engel group, that is, the set \mathbb{R}^4 equipped with the group law

$$x \circ x' := (x_1 + x'_1, x_2 + x'_2, x_3 + x'_3 + P_3, x_4 + x'_4 + P_4),$$

where

$$\begin{aligned} P_3 &= \frac{1}{2}(x_1 x'_2 - x_2 x'_1), \\ P_4 &= \frac{1}{2}(x_1 x'_3 - x_3 x'_1) + \frac{1}{12}(x_1^2 x'_2 - x_1 x'_1 (x_2 + x'_2) + x_2 x_1'^2). \end{aligned}$$

Here $x := (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. The vector fields have the following form

$$\begin{aligned} X_1 &:= \frac{\partial}{\partial x_1} - \frac{x_2}{2} \frac{\partial}{\partial x_3} - \left(\frac{x_3}{2} + \frac{x_1 x_2}{12} \right) \frac{\partial}{\partial x_4}, \\ X_2 &:= \frac{\partial}{\partial x_2} + \frac{x_1}{2} \frac{\partial}{\partial x_3} + \frac{x_1^2}{12} \frac{\partial}{\partial x_4}, \\ X_3 &:= [X_1, X_2] = \frac{\partial}{\partial x_3} + \frac{x_1}{2} \frac{\partial}{\partial x_4}, \\ X_4 &:= [X_1, X_3] = \frac{\partial}{\partial x_4}. \end{aligned}$$

The Engel group is a well-known example of a step 3 stratified group (Carnot group).

2. HARDY INEQUALITIES ON STARSHAPED SETS

In order to present the results on the starshaped domains, let us recall the definition of starshaped sets in a Carnot group $\mathbb{G} = (\mathbb{R}^n, \circ, \delta_t)$ and related arguments.

Definition 2.1 (Starshapedness [4]). Let $\Omega \subset \mathbb{G}$ be a C^1 domain containing the identity e . Then Ω is starshaped with respect to e if for every $x \in \partial\Omega$ one has

$$\langle Z(x), n(x) \rangle \geq 0, \quad (2.1)$$

where n is the Riemannian outer normal to $\partial\Omega$.

When the strict inequality holds, then Ω is said to be strictly starshaped with respect to e .

Here the vector fields Z are the infinitesimal generator of this group automorphism. This vector fields Z takes the form

$$Z = \sum_{i=1}^N x'_i \frac{\partial}{\partial x'_i} + 2 \sum_{l=1}^{N_2} x_{2,l} \frac{\partial}{\partial x_{2,l}} + \cdots + r \sum_{l=1}^{N_r} x_{r,l} \frac{\partial}{\partial x_{r,l}}. \quad (2.2)$$

Then for $x' \in \mathbb{R}^N$ and $x^{(i)} \in \mathbb{R}^{N_i}$ with $i = 2, \dots, r$ we have

$$Z(x) = (x', 2x^{(2)}, \dots, rx^{(r)}), \quad (2.3)$$

and

$$\begin{aligned} \langle Z(x), n(x) \rangle &= x' n' + 2x^{(2)} n^{(2)} + \dots + rx^{(r)} n^{(r)} \\ &= x'_1 n'_1 + \dots + x'_N n'_N + 2(x_{2,1} n_{2,1} + \dots + x_{2,N_2} n_{2,N_2}) \\ &\quad + \dots + r(x_{r,1} n_{r,1} + \dots + x_{r,N_r} n_{r,N_r}), \end{aligned}$$

since $n(x) := (n', n^{(2)}, \dots, n^{(r)})$ with $n' \in \mathbb{R}^N$ and $n^{(i)} \in \mathbb{R}^{N_i}$, $i = 2, \dots, r$.

Based on the above arguments now we present the geometric Hardy inequalities on the starshaped sets for the sub-Laplacians.

Theorem 1. *Let Ω be a starshaped set on a Carnot group. Then for every $\gamma \in \mathbb{R}$ and $p > 1$ we have the following Hardy inequality*

$$\begin{aligned} \int_{\Omega} |\nabla_H f(x)|^p dx &\geq - (p-1)(|\gamma|^{\frac{p}{p-1}} + \gamma) \int_{\Omega} \frac{|\nabla_H \langle Z(x), n(x) \rangle|^p}{|\langle Z(x), n(x) \rangle|^p} |f(x)|^p dx \\ &\quad + \gamma \int_{\Omega} \frac{\mathcal{L}_p(\langle Z(x), n(x) \rangle)}{|\langle Z(x), n(x) \rangle|^{p-1}} |f(x)|^p dx, \end{aligned} \quad (2.4)$$

for every function $f \in C_0^\infty(\Omega)$.

Corollary 2. *Let \mathbb{H}^* be a starshaped set on the Heisenberg group \mathbb{H}_1 . Then for $p > 1$, we have the following Hardy inequality*

$$\int_{\mathbb{H}^*} |\nabla_H f(x)|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^*} \frac{|(n_1 + 4x_2n_3, n_2 - 4x_1n_3)|^p}{|x_1n_1 + x_2n_2 + 2x_3n_3|^p} |f(x)|^p dx, \quad (2.5)$$

for every function $f \in C_0^\infty(\mathbb{H}^*)$.

Remark 3. Note that in the case

$$\mathbb{H}^* := \{\langle Z(x), n(x) \rangle > 0, \forall x \in \partial\mathbb{H}^*, Z(x) := (x_1, x_2, 2x_3)\} = \{x \in \mathbb{H}_1 \cong \mathbb{R}^3 : x_3 > 0\}$$

with $n(x) := (0, 0, 1)$, and $p = 2$, we have the inequality

$$\int_{\mathbb{H}^*} |\nabla_H f(x)|^2 dx \geq \int_{\mathbb{H}^*} \frac{|x_1|^2 + |x_2|^2}{|x_3|^2} |f(x)|^2 dx.$$

Proof of Corollary 2. We begin the proof of Corollary 2 by a simple computation such as

$$\begin{aligned} \langle Z(x), n(x) \rangle &= x_1n_1 + x_2n_2 + 2x_3n_3, \\ \nabla_H \langle Z(x), n(x) \rangle &= (n_1 + 4x_2n_3, n_2 - 4x_1n_3), \\ |\nabla_H \langle Z(x), n(x) \rangle|^p &= ((n_1 + 4x_2n_3)^2 + (n_2 - 4x_1n_3)^2)^{p/2}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}_p \langle Z(x), n(x) \rangle &= \nabla_H \cdot (|\nabla_H \langle Z(x), n(x) \rangle|^{p-2} \nabla_H \langle Z(x), n(x) \rangle) \\ &= X_1(|\nabla_H \langle Z(x), n(x) \rangle|^{p-2} (n_1 + 4x_2n_3)) \\ &\quad + X_2(|\nabla_H \langle Z(x), n(x) \rangle|^{p-2} (n_2 - 4x_1n_3)) \\ &= -4(p-2)|\nabla_H \langle Z(x), n(x) \rangle|^{p-4} (n_1 + 4x_2n_3)(n_2 - 4x_1n_3)n_3 \\ &\quad + 4(p-2)|\nabla_H \langle Z(x), n(x) \rangle|^{p-4} (n_2 - 4x_1n_3)(n_1 + 2x_4n_3)n_3 \\ &= 0. \end{aligned}$$

Plugging the above expressions into inequality (2.4) and maximising with respect to γ , we arrive at inequality (2.5) which proves Corollary 2. \square

Corollary 4. *Let \mathbb{E}^* be a starshaped set on the Engel group \mathbb{E} . Then for every function $f \in C_0^\infty(\mathbb{E}^*)$, $\gamma \in \mathbb{R}$ and $p = 2$, we have*

$$\begin{aligned} \int_{\mathbb{E}^*} |\nabla_H f(x)|^2 dx &\geq -(|\gamma|^2 + \gamma) \int_{\mathbb{E}^*} \frac{|\nabla_H \langle Z(x), n(x) \rangle|^2}{\langle Z(x), n(x) \rangle^2} |f(x)|^2 dx \\ &\quad + \frac{\gamma}{2} \int_{\mathbb{E}^*} \frac{x_2 n_4}{\langle Z(x), n(x) \rangle} |f(x)|^2 dx. \end{aligned} \quad (2.6)$$

Proof of Corollary 4. We begin the proof of Corollary 4 by a simple computation such as

$$\begin{aligned} \langle Z(x), n(x) \rangle &= x_1 n_1 + x_2 n_2 + 2x_3 n_3 + 3x_4 n_4, \\ \nabla_H \langle Z(x), n(x) \rangle &= \left(n_1 - x_2 n_3 - \frac{3x_3 n_4}{2} - \frac{x_1 x_2 n_4}{4}, n_2 + x_1 n_3 + \frac{x_1^2 n_4}{4} \right), \\ |\nabla_H \langle Z(x), n(x) \rangle|^2 &= \left(n_1 - x_2 n_3 - \frac{3x_3 n_4}{2} - \frac{x_1 x_2 n_4}{4} \right)^2 + \left(n_2 + x_1 n_3 + \frac{x_1^2 n_4}{4} \right)^2, \end{aligned}$$

and

$$\begin{aligned} \mathcal{L} \langle Z(x), n(x) \rangle &= \nabla_H \cdot \nabla_H \langle Z(x), n(x) \rangle \\ &= X_1 \left(n_1 - x_2 n_3 - \frac{3x_3 n_4}{2} - \frac{x_1 x_2 n_4}{4} \right) + X_2 \left(n_2 + x_1 n_3 + \frac{x_1^2 n_4}{4} \right) \\ &= \frac{x_2 n_4}{2}. \end{aligned}$$

Plugging the above expressions into inequality (2.4)

$$\begin{aligned} \int_{\mathbb{E}^*} |\nabla_H f(x)|^2 dx &\geq -(|\gamma|^2 + \gamma) \int_{\mathbb{E}^*} \frac{|\nabla_H \langle Z(x), n(x) \rangle|^2}{\langle Z(x), n(x) \rangle^2} |f(x)|^2 dx \\ &\quad + \frac{\gamma}{2} \int_{\mathbb{E}^*} \frac{x_2 n_4}{\langle Z(x), n(x) \rangle} |f(x)|^2 dx, \end{aligned}$$

which proves Corollary 4. □

3. HARDY INEQUALITIES ON HALF-SPACES FOR GENERAL VECTOR FIELDS

Let us define the half-space of a sub-Riemannian manifold by

$$\Omega^+ := \{x \in \mathbb{R}^n : \langle x, n(x) \rangle > d\},$$

where $n(x) \in \mathbb{R}^n$ is the Riemannian outer unit normal to $\partial\Omega^+$ and $d \in \mathbb{R}$. The Euclidean distance to the boundary $\partial\Omega^+$ is denoted by $\text{dist}(x, \partial\Omega^+)$ and defined by

$$\text{dist}(x, \partial\Omega^+) := \langle x, n(x) \rangle - d.$$

Then we have:

Theorem 5. *Let M be a sub-Riemannian manifold, let $\Omega^+ \subset M$ be a half-space and let X_1, \dots, X_N be the general vector fields. Then for every $\gamma \in \mathbb{R}$ and $p > 1$ we*

have the following Hardy inequality

$$\begin{aligned} \int_{\Omega^+} |\nabla_X f(x)|^p dx &\geq - (p-1)(|\gamma|^{\frac{p}{p-1}} + \gamma) \int_{\Omega^+} \frac{|\nabla_X \text{dist}(x, \partial\Omega^+)|^p}{\text{dist}(x, \partial\Omega^+)^p} |f(x)|^p dx \\ &\quad + \gamma \int_{\Omega^+} \frac{\mathcal{L}_p(\text{dist}(x, \partial\Omega^+))}{\text{dist}(x, \partial\Omega^+)^{p-1}} |f(x)|^p dx, \end{aligned} \quad (3.1)$$

for every function $f \in C_0^\infty(\Omega^+)$.

Note that inequality (3.1) was obtained in the Carnot groups by the authors in [12], but here we extend it to general sub-Riemannian manifolds.

Let us give examples for the Heisenberg group (step 2), the Engel group (step 3), and the Grushin plane which does not have a group structure, but serves as an important example of the sub-Riemannian geometry.

Corollary 6. *Let Ω^+ be a half-space in the Grushin plane G . Then for every function $f \in C_0^\infty(\Omega^+)$ and $p > 1$, we have the following Hardy inequality*

$$\begin{aligned} \int_{\Omega^+} |\nabla_X f(x)|^p dx &\geq - (p-1)(|\gamma|^{\frac{p}{p-1}} + \gamma) \int_{\Omega^+} \frac{(n_1^2 + x_1^2 n_2^2)^{p/2}}{(x_1 n_1 + x_2 n_2 - d)^p} |f(x)|^p dx \\ &\quad + (p-2)\gamma \int_{\Omega^+} \frac{|\nabla_X \text{dist}(x, \partial\Omega^+)|^{p-4} n_1 n_2^2 x_1}{(x_1 n_1 + x_2 n_2 - d)^{p-1}} |f(x)|^p dx. \end{aligned} \quad (3.2)$$

If one of the cases $n(x) = (1, 0)$ or $n(x) = (0, 1)$ holds, then we have

$$\int_{\Omega^+} |\nabla_X f(x)|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_{\Omega^+} \frac{(n_1^2 + x_1^2 n_2^2)^{p/2}}{(x_1 n_1 + x_2 n_2 - d)^p} |f(x)|^p dx, \quad (3.3)$$

where $\text{dist}(x, \partial\Omega^+) = \langle x, n(x) \rangle - d$ and $d \in \mathbb{R}$.

Remark 7. Note that, with ∇_X the Grushin gradient,

- If $\Omega^+ = \{x \in \mathbb{R}^2 : x_1 > d\}$ with $n(x) = (1, 0)$, then we have

$$\int_{\Omega^+} |\nabla_X f(x)|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_{\Omega^+} \frac{|f(x)|^p}{|x_1 - d|^p} dx.$$

- If $\Omega^+ := \{x \in \mathbb{R}^2 : x_2 > d\}$ with $n(x) = (0, 1)$, then we have

$$\int_{\Omega^+} |\nabla_X f(x)|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_{\Omega^+} \frac{|x_1|^p}{|x_2 - d|^p} |f(x)|^p dx.$$

Proof of Corollary 6. We begin the proof of Corollary 6 by a simple computation such as

$$\begin{aligned} \text{dist}(x, \partial\Omega^+) &= x_1 n_1 + x_2 n_2 - d, \\ \nabla_X \text{dist}(x, \partial\Omega^+) &= (n_1, x_1 n_2), \\ |\nabla_X \text{dist}(x, \partial\Omega^+)|^p &= (n_1^2 + x_1^2 n_2^2)^{p/2}, \end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_p \text{dist}(x, \partial\Omega^+) &= \nabla_X \cdot (|\nabla_X \text{dist}(x, \partial\Omega^+)|^{p-2} \nabla_X \text{dist}(x, \partial\Omega^+)) \\ &= \frac{\partial}{\partial x_1} ((n_1^2 + x_1^2 n_2^2)^{\frac{p-2}{2}} n_1) + x_1 \frac{\partial}{\partial x_2} ((n_1^2 + x_1^2 n_2^2)^{\frac{p-2}{2}} x_1 n_2) \\ &= (p-2) |\nabla_X \text{dist}(x, \partial\Omega^+)|^{p-4} n_1 n_2^2 x_1.\end{aligned}$$

Plugging the above expressions into inequality (3.1) we arrive at

$$\begin{aligned}\int_{\Omega^+} |\nabla_X f(x)|^p dx &\geq -(p-1) (|\gamma|^{\frac{p}{p-1}} + \gamma) \int_{\Omega^+} \frac{(n_1^2 + x_1^2 n_2^2)^{p/2}}{(x_1 n_1 + x_2 n_2 - d)^p} |f(x)|^p dx \\ &\quad + (p-2) \gamma \int_{\Omega^+} \frac{|\nabla_X \text{dist}(x, \partial\Omega^+)|^{p-4} n_1 n_2^2 x_1}{(x_1 n_1 + x_2 n_2 - d)^{p-1}} |f(x)|^p dx,\end{aligned}$$

which proves inequality (3.2). If one of the cases $n(x) = (1, 0)$ or $n(x) = (0, 1)$ holds, then the last term of the above inequality vanishes, so that we get

$$\int_{\Omega^+} |\nabla_X f(x)|^p dx \geq -(p-1) (|\gamma|^{\frac{p}{p-1}} + \gamma) \int_{\Omega^+} \frac{(n_1^2 + x_1^2 n_2^2)^{p/2}}{(x_1 n_1 + x_2 n_2 - d)^p} |f(x)|^p dx. \quad (3.4)$$

Then, we maximise above inequality by differentiating with respect to γ , so that we have

$$\frac{p}{p-1} |\gamma|^{\frac{1}{p-1}} + 1 = 0,$$

which leads to

$$\gamma = - \left(\frac{p-1}{p} \right)^{p-1}.$$

By putting the value of γ into inequality (3.4), we obtain inequality (3.3). \square

Corollary 8. *Let Ω^+ be a half-space on the Heisenberg group. Then for every function $f \in C_0^\infty(\Omega^+)$ and $p > 1$, we have*

$$\int_{\Omega^+} |\nabla_H f(x)|^p dx \geq \left(\frac{p-1}{p} \right)^p \int_{\Omega^+} \frac{|(n_1 + 2x_2 n_3, n_2 - 2x_1 n_3)|^p}{\text{dist}(x, \partial\Omega^+)^p} |f(x)|^p dx, \quad (3.5)$$

where $\text{dist}(x, \partial\Omega^+) = \langle x, n(x) \rangle - d$ and $d \in \mathbb{R}$.

Remark 3.1. Note that if we choose $n(x) = (0, 0, 1)$, $p = 2$ and $d = 0$ in inequality (3.5), then we get

$$\int_{\Omega^+} |\nabla_H f(x)|^2 dx \geq \int_{\Omega^+} \frac{|x_1|^2 + |x_2|^2}{|x_3|^2} |f(x)|^2 dx. \quad (3.6)$$

The Hardy inequality of the form (3.6) in the half-space on the Heisenberg group was shown by Luan and Young [16].

Proof of Corollary 8. We begin the proof of Corollary 8 by a simple computation such as

$$\begin{aligned}\text{dist}(x, \partial\Omega^+) &= x_1 n_1 + x_2 n_2 + x_3 n_3 - d, \\ \nabla_X \text{dist}(x, \partial\Omega^+) &= (n_1 + 2x_2 n_3, n_2 - 2x_1 n_3), \\ |\nabla_X \text{dist}(x, \partial\Omega^+)|^p &= ((n_1 + 2x_2 n_3)^2 + (n_2 - 2x_1 n_3)^2)^{p/2}.\end{aligned}$$

Then we compute

$$\begin{aligned}
 \mathcal{L}_p \text{dist}(x, \partial\Omega^+) &= \nabla_H \cdot (|\nabla_H \text{dist}(x, \partial\Omega^+)|^{p-2} \nabla_H \text{dist}(x, \partial\Omega^+)) \\
 &= X_1((n_1 + 2x_2n_3)^2 + (n_2 - 2x_1n_3)^2)^{\frac{p-2}{2}}(n_1 + 2x_2n_3) \\
 &\quad + X_2((n_1 + 2x_2n_3)^2 + (n_2 - 2x_1n_3)^2)^{\frac{p-2}{2}}(n_2 - 2x_1n_3) \\
 &= -2(p-2)|\nabla_H \text{dist}(x, \partial\Omega^+)|^{p-4}(n_1 + 2x_2n_3)(n_2 - 2x_1n_3)n_3 \\
 &\quad + 2(p-2)|\nabla_H \text{dist}(x, \partial\Omega^+)|^{p-4}(n_2 - 2x_1n_3)(n_1 + 2x_2n_3)n_3 \\
 &= 0.
 \end{aligned}$$

Plugging the above expressions into inequality (3.1), we arrive at

$$\int_{\Omega^+} |\nabla_H f(x)|^p dx \geq -(p-1)(|\gamma|^{\frac{p}{p-1}} + \gamma) \int_{\Omega^+} \frac{|(n_1 + 2x_2n_3, n_2 - 2x_1n_3)|^p}{(x_1n_1 + x_2n_2 + x_3n_3 - d)^p} |f(x)|^p dx, \quad (3.7)$$

which can be maximised by differentiating with respect to γ , then we have

$$\frac{p}{p-1} |\gamma|^{\frac{1}{p-1}} + 1 = 0,$$

that leads to

$$\gamma = - \left(\frac{p-1}{p} \right)^{p-1}.$$

By putting the value of γ into inequality (3.7), we obtain inequality

$$\int_{\Omega^+} |\nabla_H f(x)|^p dx \geq \left(\frac{p-1}{p} \right)^p \int_{\Omega^+} \frac{|(n_1 + 2x_2n_3, n_2 - 2x_1n_3)|^p}{\text{dist}(x, \partial\Omega^+)^p} |f(x)|^p dx,$$

which proves Corollary 8. \square

Corollary 9. *Let Ω^+ be a half-space on the Engel group \mathbb{E} . Then for every function $f \in C_0^\infty(\Omega^+)$, $\gamma \in \mathbb{R}$ and $p = 2$, we have*

$$\begin{aligned}
 \int_{\Omega^+} |\nabla_H f(x)|^2 dx &\geq -(|\gamma|^2 + \gamma) \int_{\Omega^+} \frac{|\nabla_H \text{dist}(x, \partial\Omega^+)|^2}{\text{dist}(x, \partial\Omega^+)^2} |f(x)|^2 dx \\
 &\quad + \frac{\gamma}{6} \int_{\Omega^+} \frac{x_2n_4}{\text{dist}(x, \partial\Omega^+)} |f(x)|^2 dx,
 \end{aligned} \quad (3.8)$$

where $\text{dist}(x, \partial\Omega^+) = \langle x, n(x) \rangle - d$ and $d \in \mathbb{R}$.

Proof of Corollary 9. We begin the proof of Corollary 9 by a simple computation such as

$$\begin{aligned}
 \text{dist}(x, \partial\Omega^+) &= x_1n_1 + x_2n_2 + x_3n_3 + x_4n_4 - d, \\
 \nabla_H \text{dist}(x, \partial\Omega^+) &= \left(n_1 - \frac{x_2n_3}{2} - \frac{x_3n_4}{2} - \frac{x_1x_2n_4}{12}, n_2 + \frac{x_1n_3}{2} + \frac{x_1^2n_4}{12} \right), \\
 |\nabla_H \text{dist}(x, \partial\Omega^+)|^2 &= \left(n_1 - \frac{x_2n_3}{2} - \frac{x_3n_4}{2} - \frac{x_1x_2n_4}{12} \right)^2 + \left(n_2 + \frac{x_1n_3}{2} + \frac{x_1^2n_4}{12} \right)^2,
 \end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}(\text{dist}(x, \partial\Omega^+)) &= \nabla_H \cdot \nabla_H \text{dist}(x, \partial\Omega^+) \\
&= \nabla_H \cdot \left(n_1 - \frac{x_2 n_3}{2} - \frac{x_3 n_4}{2} - \frac{x_1 x_2 n_4}{12}, n_2 + \frac{x_1 n_3}{2} + \frac{x_1^2 n_4}{12} \right) \\
&= X_1 \left(n_1 - \frac{x_2 n_3}{2} - \frac{x_3 n_4}{2} - \frac{x_1 x_2 n_4}{12} \right) + X_2 \left(n_2 + \frac{x_1 n_3}{2} + \frac{x_1^2 n_4}{12} \right) \\
&= \frac{x_2 n_4}{6}.
\end{aligned}$$

Plugging the above expressions into inequality (3.1), we get

$$\begin{aligned}
\int_{\Omega^+} |\nabla_H f(x)|^2 dx &\geq -(|\gamma|^2 + \gamma) \int_{\Omega^+} \frac{|\nabla_H \text{dist}(x, \partial\Omega^+)|^2}{\text{dist}(x, \partial\Omega^+)^2} |f(x)|^2 dx \\
&\quad + \frac{\gamma}{6} \int_{\Omega^+} \frac{x_2 n_4}{\text{dist}(x, \partial\Omega^+)} |f(x)|^2 dx,
\end{aligned}$$

which proves Corollary 9. \square

4. PROOF OF MAIN RESULTS

The approach to prove the main results is based on the works [11] and [12] (see, also [13]-[14]). For a vector field $g \in C^\infty(\Omega)$ we compute

$$\begin{aligned}
\int_{\Omega} \text{div}_X g |f(x)|^p dx &= -p \int_{\Omega} |f(x)|^{p-1} \langle g, \nabla_X f(x) \rangle dx \\
&\leq p \left(\int_{\Omega} |\nabla_X f(x)|^p dx \right)^{\frac{1}{p}} \left(\int_{\Omega} |g|^{\frac{p}{p-1}} |f(x)|^p dx \right)^{\frac{p-1}{p}} \\
&\leq \int_{\Omega} |\nabla_H f(x)|^p dx + (p-1) \int_{\Omega} |g|^{\frac{p}{p-1}} |f(x)|^p dx.
\end{aligned}$$

Here we have first used the divergence theorem, then we applied the Hölder inequality and the Young inequality. By rearranging the above expression, we arrive at

$$\int_{\Omega} |\nabla_X f(x)|^p dx \geq \int_{\Omega} (\text{div}_X g - (p-1)|g|^{\frac{p}{p-1}}) |f(x)|^p dx. \quad (4.1)$$

A suitable choice of the vector field g in each special case is a key argument of our proofs.

Proof of Theorem 1. Let us set

$$g = \gamma \frac{|\nabla_H \langle Z(x), n(x) \rangle|^{p-2}}{|\langle Z(x), n(x) \rangle|^{p-1}} \nabla_H \langle Z(x), n(x) \rangle,$$

so that we have

$$|g|^{\frac{p}{p-1}} = |\gamma|^{\frac{p}{p-1}} \frac{|\nabla_H \langle Z(x), n(x) \rangle|^p}{|\langle Z(x), n(x) \rangle|^p}, \quad (4.2)$$

and

$$\text{div}_H g = \gamma \frac{\mathcal{L}_p(\langle Z(x), n(x) \rangle)}{|\langle Z(x), n(x) \rangle|^{p-1}} - \gamma(p-1) \frac{|\nabla_H \langle Z(x), n(x) \rangle|^p}{|\langle Z(x), n(x) \rangle|^p}. \quad (4.3)$$

Plugging the above expressions (4.2) and (4.3) into inequality (4.1), we get

$$\begin{aligned} \int_{\Omega} |\nabla_H f(x)|^p dx &\geq - (p-1)(|\gamma|^{\frac{p}{p-1}} + \gamma) \int_{\Omega} \frac{|\nabla_H \langle Z(x), n(x) \rangle|^p}{|\langle Z(x), n(x) \rangle|^p} |f(x)|^p dx \\ &\quad + \gamma \int_{\Omega} \frac{\mathcal{L}_p(\langle Z(x), n(x) \rangle)}{|\langle Z(x), n(x) \rangle|^{p-1}} |f(x)|^p dx, \end{aligned}$$

which proves inequality (2.4). \square

Proof of Theorem 5. Let us take

$$g = \gamma \frac{|\nabla_X \text{dist}(x, \partial\Omega^+)|^{p-2}}{\text{dist}(x, \partial\Omega^+)^{p-1}} \nabla_X \text{dist}(x, \partial\Omega^+), \quad (4.4)$$

so that we have

$$|g|^{\frac{p}{p-1}} = |\gamma|^{\frac{p}{p-1}} \frac{|\nabla_X \text{dist}(x, \partial\Omega^+)|^p}{\text{dist}(x, \partial\Omega^+)^p}, \quad (4.5)$$

and

$$\text{div}_X g = \gamma \frac{\mathcal{L}_p \text{dist}(x, \partial\Omega^+)}{\text{dist}(x, \partial\Omega^+)^{p-1}} - \gamma(p-1) \frac{|\nabla_X \text{dist}(x, \partial\Omega^+)|^p}{\text{dist}(x, \partial\Omega^+)^p}. \quad (4.6)$$

Combining expressions (4.5) and (4.6) with inequality (4.1), we obtain

$$\begin{aligned} \int_{\Omega} |\nabla_X f(x)|^p dx &\geq - (p-1)(|\gamma|^{\frac{p}{p-1}} + \gamma) \int_{\Omega} \frac{|\nabla_X \text{dist}(x, \partial\Omega^+)|^p}{\text{dist}(x, \partial\Omega^+)^p} |f(x)|^p dx \\ &\quad + \gamma \int_{\Omega} \frac{\mathcal{L}_p \text{dist}(x, \partial\Omega^+)}{\text{dist}(x, \partial\Omega^+)^{p-1}} |f(x)|^p dx, \end{aligned}$$

which proves inequality (3.1). \square

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