Quantitative Unique Continuation and Applications

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Quantitative unique continuation and applications

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Zusammenfassung

Diese Arbeit behandelt quantitative eindeutige Fortsetzungsprinzipien (engl.: "quantitative unique continuation principles" bzw. "UCPs") und deren Anwendungen. Für geeignete Mengen $\omega \subset \Omega \subset \mathbb{R}^d$ beweisen wir in Abschnitt 3 Ungleichungen der Form

$$\int_\Omega |f(x)|^2 \mathrm{d}x \leq C \int_\omega |f(x)|^2 \mathrm{d}x,$$

für alle f aus einem Spektralraum kompakter Energie des Schrödingeroperators $-\Delta + V$ mit beschränktem Potential V auf $L^2(\Omega)$. Die Abhängigkeit der Konstante C von dem spektralen Teilraum, von V und von der Geometrie von ω und Ω wird explizit angegeben. Wir verbessern existierende UCPs, da wir alle spektralen Teilräume endlicher Energie sowie Schrödingeroperatoren auf unbeschränkten Teilmengen von $\mathbb{R}^d$ betrachten. Die Konstante C ist skalenfrei und uniform über eine große Klasse von Geometrien (Ω, ω). Die Frage der Optimalität ihrer Abhängigkeit von der Energie und von V wird erörtert. Zudem beweisen wir eine Verallgemeinerung von spektralen Teilräumen hin zum Definitionsbereich gewisser Funktionen des Schrödingeroperators.

Als erste Anwendung zeigen wir in Abschnitt 4 untere Schranken an die Veränderung des Spektrums – insbesondere des essentiellen Spektrums – von Schrödingeroperatoren unter gewissen nicht-negativen Störungen. Zu diesem Zweck beweisen wir abstrakte Sätze über die Störung von Spektra selbstadjungierter Operatoren.

Die zweite Anwendung in Abschnitt 5 betrifft zufällige Schrödingeroperatoren. Es werden für neue Klassen zufälliger Schrödingeroperatoren Wegner-Abschätzungen hergeleitet. Dies ist ein wichtiger Schritt um Anderson-Lokalisierung für diese Modelle zu beweisen. Ein Beispiel ist das "random breather model", bei welchem das zufällige Potential aus Indikatorfunktionen von Bällen mit zufälligen Radii besteht. Darüber hinaus beweisen wir Wegner-Abschätzungen für das sogenannte "crooked magnetic alloy-type model" und für das "Landau-breather model".

Die letzte Anwendung, in Abschnitt 6, handelt von Kontrolltheorie für Gleichungen vom Wärmeleitungstyp mit innerer Kontrolle. Zunächst wird im abstrakten Rahmen Null-Kontrollierbarkeit bestimmter Cauchy-Probleme mit expliziter Schranke an die Kontrollkosten bewiesen. Unsere Abschätzung an die Kosten ist – unseres Wissens nach – die derzeit beste. Wir kombinieren diese dann mit quantitativen UCPs und erhalten explizite Abschätzungen an die Kontrollkosten von Gleichungen vom Wärmeleitungstyp auf beschränkten und unbeschränkten Gebieten zu allen Zeiten. Dieses quantitative Resultat ist sogar für die klassische Wärmeleitungsgleichung neu und ermöglicht es, das asymptotische Verhalten der Kontrollkosten im Homogenisierungs- und im komplementären Regime zu studieren.

Abstract

This thesis treats quantitative unique continuation principles (UCPs) and their applications. For suitable $\omega \subset \Omega \subset \mathbb{R}^d$ we prove in Section 3 inequalities of the form

$$\int_\Omega |f(x)|^2 \mathrm{d}x \leq C \int_\omega |f(x)|^2 \mathrm{d}x,$$

for all f in a compact energy spectral subspace of the Schrödinger operator $-\Delta + V$ in $L^2(\Omega)$ with bounded potential V. The dependence of the constant C on the spectral subspace, on V, and on the geometry of ω and Ω is explicit. We improve existing UCPs by treating all finite energy spectral subspaces and Schrödinger operators on unbounded subsets of $\mathbb{R}^d$. The constant C is scale-free and uniform over a large class of geometries (Ω, ω) and the optimality of its dependece on the energy and on V is discussed.

As a first application we establish in Section 4 lower bounds on the movement of spectra – in particular of the essential spectrum – of Schrödinger operators under particular non-negative perturbations. For that purpose, abstract results on perturbations of spectra of self-adjoint operators are proved.

The second application in Section 5 is about random Schrödinger operators. We prove a Wegner estimate, an important step in proving Anderson localization, for new classes of random Schrödinger operators. A particular example is the random breather model, where the random potential consists of characteristic functions of balls with random radii. Furthermore, we prove Wegner estimates for so-called crooked magnetic alloy-type operators with bounded magnetic potential and for the Landau-breather model.

The last application in Section 6 concerns control theory for equations of heat-type with interior control. First, in an abstract framework, null-controllability of some Cauchy problems with explicit estimates on the control cost at all times is proved. Our estimate on the control cost is – to our knowledge – best with respect to the existing literature. Then, combining this with quantitative UCPs, we obtain explicit estimates on the control cost of heat-type equations on bounded and unbounded domains at all times. This result in this quantitative form is new even for the classic heat equation and enables to study asymptotics of the control cost in homogenization and the complementary regime.

Preface

This thesis consists of two introductory sections and four main section devoted to results and proofs. Section 1 presents the topics and main results in an abbreviated form while Section 2 introduces standard notation and recalls some facts from the spectral theory of self-adjoint operators which will be used frequently. Section 3 is devoted to the name-giving quantitative unique continuation principles. Sections 4, 5, and 6 contain applications of these unique continuation principles to the perturbation of spectra, random Schrödinger operators, and control theory. Each of the Sections 3 to 6 can be read independently. Parts of this thesis are based on and coincide with the following publications and preprints by the author, obtained partially in collaborations with **Ivica Nakić**, **Albrecht Seelmann**, **Martin Tautenhahn**, and **Ivan Veselić**:

[NTTV15] I. Nakić, M. Täufer, M. Tautenhahn, and I. Veselić. Scale-free uncertainty principles and Wegner estimates for random breather potentials. *C. R. Math.*, 353(10):919–923, 2015.

[NTTV18a] I. Nakić, M. Täufer, M. Tautenhahn, and I. Veselić. Scale-free unique continuation principle, eigenvalue lifting and Wegner estimates for random Schrödinger operators. *Anal. PDE*, 11(4):1049–1081, 2018.

[NTTV18b] I. Nakić, M. Täufer, M. Tautenhahn, and I. Veselić. Unique continuation and lifting of spectral band edges of Schrödinger operators on unbounded domains (With an Appendix by Albrecht Seelmann). arXiv:1804.07816 [math.SP], 2018.

[Täu17] M. Täufer. Laplace-eigenfunctions on the torus with high vanishing order. arXiv:1710.09328 [math.AP], 2017.

[TT17] M. Täufer and M. Tautenhahn. Scale-free and quantitative unique continuation for infinite dimensional spectral subspaces of Schrödinger operators. *Commun. Pur. Appl. Anal.*, 16(5):1719–1730, 2017.

[TT18] M. Täufer and M. Tautenhahn. Wegner Estimate and Disorder Dependence for Alloy-Type Hamiltonians with Bounded Magnetic Potential. *Ann. Henri Poincaré*, 19(4):1151–1165, 2018.

[TTV16] M. Täufer, M. Tautenhahn, and I. Veselić. Harmonic analysis and random Schrödinger operators. In *Spectral theory and mathematical*

physics, volume 254 of *Oper. Theory Adv. Appl.*, pages 223–255. Birkhäuser/Springer, [Cham], 2016.

[TV15] M. Täufer and I. Veselić. Conditional Wegner estimate for the standard random breather potential. *J. Stat. Phys.*, 161(4):902–914, 2015. arXiv:1509.03507.

[TV16a] M. Täufer and I. Veselić. Wegner estimate for Landau-breather Hamiltonians. *J. Math. Phys.*, 57(7):072102, 8, 2016.

We will indicate in the text at the beginnings of the respective sections or subsections whether they contain material from these publications.

Contents

List of Figures

All figures have been generated in LaTeX using PGF/TikZ.

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My coauthors Ivica Nakić, Martin Tautenhahn and Albrecht Seelmann with whom I have worked on projects related to this thesis also deserve a special place here. I vividly remember the numerous times we spent discussing, struggling and developing ideas. Every hour was worth it!

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1. Introduction

The main object we study in this work are scale-free quantitative unique continuation principles (UCPs) for spectral subspaces of Schrödinger operators and their applications to three different fields: Perturbation of spectra, random Schrödinger operators, and control theory for heat-type equations.

Let us start by explaining unique continuation principles which will be treated in detail in Section 3. Our results presented in this section have partially already been published in [NTTV15, NTTV18a, TT17, Täu17, NTTV18b]. The quantitative unique continuation principles we prove are inequalities of the form

$$\int_\Gamma |\phi(x)|^2 \mathrm{d}x \leq C_{\mathrm{ucp}} \int_{S_{\delta,Z}(\Gamma)} |\phi(x)|^2 \mathrm{d}x \tag{1}$$

where $\Gamma \subset \mathbb{R}^d$ is a generalized rectangle in $\mathbb{R}^d$ (e.g. a hypercube, infinite strip, half-space, $\mathbb{R}^d$ itself, etc.), $S_{\delta,Z}(\Gamma)$ is a subset of Γ, consisting of a union of small δ-balls which are arranged on the centers of a so-called equidistributed sequence Z, see Definition 3.26 below for details, and ϕ is a function in the spectral subspace of a Schrödinger operator $\operatorname{Ran} P_{-\Delta+V}(E)$ where $-\Delta + V$ for some real-valued $V \in L^\infty$, denotes the Schrödinger operator in $L^2(\Gamma)$ with potential V and self-adjoint boundary conditions. Inequality (1) implies in particular that any such function which vanishes on $S_{\delta,Z}(\Gamma)$ must be zero on all of Γ. This means that functions in the function space $\operatorname{Ran} P_{-\Delta+V}(E)$ are determined only by their values in $S_{\delta,Z}(\Gamma)$ and justifies the term *unique continuation principle.* However, the statement above is stronger since it also quantifies how much a function can deviate from zero by means of the norm of its restriction on $S_{\delta,Z}(\Gamma)$. Therefore, it is called a *quantitative unique continuation principle.* In addition to that, we also have an explicit expression for the constant C_{ucp}: it is of the form

$$C_{\mathrm{ucp}} = \delta^{-C(1+\|V\|_\infty^{2/3}+\sqrt{\max\{E,0\}})}.$$

Let us underline some important features and improvements we achieved compared to existing results: Our estimates hold uniformly for all functions in spectral subspaces $\operatorname{Ran} P_{-\Delta+V}(E)$ and not only for single eigenfunctions or eigenfunctions in a small energy window. While previous research usually focused on operators on bounded subsets of $\mathbb{R}^d$, we also treat Schrödinger operators on unbounded domains. In this case, there might not exist an orthonormal basis of eigenfunctions any more and the space $\operatorname{Ran} P_{-\Delta+V}(E)$ might be infinite dimensional. The constant C_{ucp} does neither depend on Γ nor on the particular choice of the equidistributed sequence whence we call it *scale-free*. Furthermore, C_{ucp} depends on V only via its L^∞ norm. Another feature is that the dependence of C_{ucp} on the energy E is explicit and optimal. We discuss this as well as the issue of optimality of the dependence of C_{ucp} on $\|V\|_\infty$ in Subsection 3.3. Finally, we also included a generalization of the statement from spectral subspaces to the domain of exponential functions of (a square root of) the Schrödinger operator $\mathcal{D}\left(\exp(\kappa\sqrt{\max\{-\Delta+V,0\}})\right) \subset L^2(\mathbb{R}^d)$ in the sense of spectral calculus. Our results are in particular motivated by earlier research from [RMV13, Kle13] and improve results therein. In Subsection 3.1, we present a more comprehensive history of results which have led to our work. Our proofs rely on particular Carleman estimates with explicit weight functions and a technique that we call "ghost dimension" and which allows to naturally treat spectral subspaces instead of single eigenfunctions. In order to deal with operators that do not have an orthonormal basis of eigenfunctions, we resort to abstract spectral calculus as a natural generalization of sums of eigenfunctions that have previously played an important role in unique continuation for spectral subspaces. The section is concluded by a brief survey in Subsection 3.5 of existing unique continuation principles which are partially diametrical to our results. These are on the one hand scale-free unique continuation principles for some magnetic Schrödinger operators and on the other hand the Logvinenko-Sereda theorem from Fourier analysis.

Section 4 is devoted to perturbations of the spectrum of self-adjoint operators. It is based on the preprint [NTTV18b]. We first prove abstract results on perturbations of spectra which we believe are interesting in their own right and then combine them with the unique continuation principle from Section 3. Let us first explain the abstract results: We choose a self-adjoint operator A in a Hilbert space and a bounded, symmetric perturbation B. If A is lower semibounded with purely discrete spectrum and if we denote the eigenvalues of an operator T by $\lambda_1(T) \leq \lambda_2(T) \leq \dots$, counting multiplicities, then it is well-known that $|\lambda_k(A+B) - \lambda_k(A)| \leq \|B\|$. Now, if furthermore, B is assumed to be positive on a spectral subspace with respect to $A+B$ (but not necessarily positive everywhere), then one can prove estimates of the form

$$\lambda_k(A+B) \geq \lambda_k(A) + C$$

for appropriate $k \in \mathbb{N}$ and a certain $C > 0$, see Subsection 4.1.1 for more details. This leads to *lower Lipschitz bounds* on the function $t \mapsto \lambda_k(A+tB)$. While in case of lower semibounded operators with purely discrete spectrum this seems to be relatively well-known, the case where the operator A has nonempty essential spectrum is much less understood. In fact, in this case, the situation becomes much richer. One can again study the movement of isolated eigenvalues under perturbations, but the more interesting and new aspect is now the essential spectrum. The connected components of the essential spectrum of A have upper and lower edges. Again, it can be seen that these edges are locally stable under bounded perturbations $A+tB$ and that they can be parametrized by locally Lipschitz continuous functions. Now, assuming that the operator B is positive on certain spectral subspaces, we provide in Subsections 4.1.2 and 4.1.3, lower Lipschitz bounds on the movement of such spectral edges. The tools we use are special min-max principles for eigenvalues in spectral gaps as well as some results from subspace perturbation theory. Unique continuation principles as in Ineq. (1) can be understood as positivity of non-negative, but not positive definite perturbations of a Schrödinger operator on spectral subspaces. Thus, combining the abstract results with our unique continuation principles from Section 3, we conclude lower bounds on the movement of components of the spectrum and in particular of edges of the essential spectrum of Schrödinger operators. These new lower bounds complement the more classic upper bounds in terms of the norm of the perturbation.

Section 5 is devoted to random Schrödinger operators. It presents contents of [TV15, NTTV15, TV16a, NTTV18a, TT18]. Random Schrödinger operators are random families of operators $H_\omega = H_0 + V_\omega$, $\omega \in \Omega$, where H_0 is a deterministic Schrödinger operator (mostly the negative Laplacian $-\Delta$) and $\{V_\omega\}_{\omega\in\Omega}$ is a random potential parametrized by a probability space $(\Omega, \mathbb{P})$. An key phenomenon is *Anderson localization*, that is the emergence of pure point spectrum with exponentially decaying eigenfunctions at some energies due to randomness. In Physics, Anderson localization is interpreted as an explanation how randomness leads to bad transport of charge and as an explanation for electric resistance. A key ingredient in many proofs of Anderson localization are *Wegner estimates*. These are upper bounds on the expected number of eigenvalues in small energy intervals of finite volume restriction of H_ω and serve as a non-resonance condition in proofs of localization, see Subsection 5.1 for more details. Most results on random Schrödinger operators focus on one particular model of randomness, namely the *alloy-type model* or *continuum Anderson model* where the random potential is generated by a periodic arrangement of copies of a bump functions u which are linearly coupled to a sequence of independent and identically distributed random variables at every site $j \in \mathbb{Z}^d$

$$V_\omega(x) = \sum_{j\in\mathbb{Z}^d} \omega_j u(x-j), \quad x \in \mathbb{R}^d.$$

Historically, in the first proofs of Wegner estimates, periodicity and the linear coupling have played important roles. Recently, unique continuation principles have turned out to be useful tools in order to relax the assumption of periodicity or ergodicity of the random potential V_ω, cf. [RMV13, Kle13]. We use the unique continuation principle from Section 3 to obtain Wegner estimates for a new class of models which we call *random Schrödinger operators monotone in the randomness*. This class contains the classic alloy-type model as well as non-ergodic variants, but also new models such as the *(standard) random breather model* where the random potential is generated by characteristic functions of balls of random radii

$$V_\omega(x) = \sum_{j \in \mathbb{Z}^d} \mathbf{1}_{B(j,\omega_j)}(x).$$

The unique continuation principles from Section 3 were the key new ingredient, in order to pass from linear coupling of the random variables to models with a non-linear dependence on the random variables $\{\omega_j\}_{j \in \mathbb{Z}^d}$. Moreover, we consider in Subsection 5.4 random operators where the background operator H_0 is a magnetic Schrödinger operator $(-\mathrm{i}\nabla + A_0)^2$. In case of a bounded magnetic potential, we prove Wegner estimates for the so-called crooked magnetic alloy-type model. Furthermore, we treat so-called Landau operators (Schrödinger operators with homogeneous magnetic field in dimension $d = 2$) with random breather potential and obtain Wegner estimates, however only in the so-called small disorder regime, see Subsection 5.4.2 below for details. The methods used here are different from the ones we apply in the context of models monotone in the randomness and exploit either the linear coupling in the alloy-type model or the small disorder regime for the breather model to prove Wegner estimates. This is due to the fact that for magnetic Schrödinger operators we have less powerful UCPs at our disposal.

The last Section 6 is about control theory for equations of heat-type. The heat equation on a domain $\Gamma \subset \mathbb{R}^d$ with Dirichlet boundary conditions and interior control on $S \subset \Gamma$ is

$$\begin{cases} \frac{\partial}{\partial t} u - \Delta u = \mathbf{1}_S f, & u, f \in L^2([0,T] \times \Gamma), \\ u = 0 & \text{on } \partial\Gamma \times (0,T], \\ u(0) = u_0 & u_0 \in L^2(\Gamma) \end{cases} \tag{2}$$

where the function f is called a *control function*. It is known that under reasonable assumptions on Γ and S, for every $T > 0$, the system is *null-controllable* in time $T > 0$. This means that for every time $T > 0$ there exists a control function f such that the unique solution u of system (2) satisfies $u(T) = 0$. We are going to study the more general situation where the negative Laplacian in the diffusion term $-\Delta u$ is replaced by a Schrödinger operator $-\Delta + V$ with bounded potential V. We call this a *heat-type equation* or *heat equation with generation term*. In this context, we will provide very explicit estimates on the *control cost*, i.e. upper bounds on the norm of the required control function f in terms of the initial state $\|u_0\|$, in terms of the time T, and in terms of the geometry of Γ and S. A central role will again be played by the unique continuation principle. In fact, in Subsection 6.2 we will consider abstract control problems and, assuming a so-called *spectral inequality*, we will deduce precise estimates on the control cost. This itself is a new result since in contrast to existing results it takes all model parameters into account, improves existing bounds, works for all times $T > 0$, and avoids assumptions such as purely discrete spectrum of the corresponding operators. Going back to heat-type equations, we see that our unique continuation principles from Section 3 are such spectral inequalities. Thus, in Subsection 6.3 we deduce estimates on the control cost of the system (2) where Γ is a generalized rectangle and $S \subset \Gamma$ is a union of equidistributed δ-balls or a so-called *thick set* as in the Logvinenko-Sereda theorem. These estimates enable us to study the control cost of heat-type equation in the *homogenization regime* where the control set $S \subset \Gamma$ becomes more and more equidistributed within Γ while keeping its overall density, as well as in the *de-homogenization* or *coarsening regime* where fluctuations on finite scales in the density of the control set S become larger while the overall density in $\mathbb{R}^d$ remains constant. To our knowledge, this is the first time that such limits have been studied in the context of the control cost. Among other, we deduce that in the homogenization limit (homogenization of the control set), the effect of any bounded potential V will be reduced to its effect on the ground state energy of $-\Delta + V$, i.e. to the effect of a constant potential.

2. Preliminaries

This section introduces frequently used notation and recalls some facts on the spectral theory of self-adjoint operators which will be used throughout Sections 3 to 6.

2.1. Notation and definitions

For $t \in \mathbb{R}$, we write t_+ for its positive part, i.e. $t_+ := \max\{t, 0\}$. Equally, for a real-valued function f, we write $f_+(x) := (f(x))_+$.

We use the Euclidean norm, the 1-norm, and the supremum norm on $\mathbb{R}^d$ or $\mathbb{C}^d$, defined as

$$|x| := \left(\sum_{i=1}^{d} |x_i|^2\right)^{1/2}, \quad |x|_1 := \sum_{i=1}^{d} |x_i|, \quad \text{and} \quad |x|_\infty := \max_{i=1}^{d} |x_i|.$$

Given $x \in \mathbb{R}^d$ and $r > 0$, we denote by $B(x, r) := \{y \in \mathbb{R}^d \colon |y - x| < r\}$ the open ball of radius r, centered at x, and for $L > 0$, we denote by $\Lambda_L(x) := \{y \in \mathbb{R}^d \colon |y - x|_\infty < L/2\}$ the open d-dimensional cube of side length L, centered at x. If $x = 0$, we use the shorthands $B(r)$ and Λ_L. Given $U \subset \mathbb{R}^d$ and $M > 0$, we write $MU := \{x \in \mathbb{R}^d \colon x/M \in U\}$ for its dilation by the factor M. We also denote by $\sharp U$ the cardinality of U and, if $U \subset \mathbb{R}^d$ is measurable, we write $|U|$ for its Lebesgue measure.

If $U \subset \mathbb{R}^d$ is open, we write $\mathcal{M}(U, \mathbb{R})$ and $\mathcal{M}(U, \mathbb{C})$ for the set of $\mathbb{R}$- or $\mathbb{C}$-valued, measurable functions on U, respectively, and set

$$L^2(U) = \left\{\phi \in \mathcal{M}(U, \mathbb{C}) \colon \|\phi\|_{L^2(U)} < \infty\right\},$$

where $\|\phi\|_{L^2(U)} = (\int_U |\phi|^2)^{1/2}$.

If a function $g \colon \mathbb{R} \to \mathbb{R}$ or $\mathbb{C}$ is (once, twice, etc.) differentiable , we will write g', g'', etc. for its first, second, etc. (classic) derivative. By $\partial_i := \partial/\partial_{x_i}$, we denote the weak partial derivative with respect to the i-th coordinate and if $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ is a multiindex, we write ∂^α for the weak derivative $\partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$. Sometimes, if t is a variable, we will also write $\partial_t := \partial/\partial_t$ for the derivative with respect to t. The k-th order Sobolev space is

$$H^k(U) = \left\{\phi \in L^2(U) \colon \partial^\alpha \phi \in L^2(U) \text{ for all multiindices } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha|_1 \leq k\right\}$$

and for $\phi \in H^1(U)$, we define its 1-Sobolev norm

$$\|\phi\|_{H^1(U)} := \left(\int_U |\phi|^2 + |\nabla\phi|^2\right)^{1/2}.$$

If $V \subset U$ and $\phi\mid_V \in L^2(V)$, we will write

$$\|\phi\|_{L^2(V)} := \left(\int_V |\phi|^2\right)^{1/2}$$

and accordingly, if $\phi\mid_V$ and $\nabla\phi\mid_V$ are in $L^2(V)$

$$\|\phi\|_{H^1(V)} := \left(\int_V |\phi|^2 + |\nabla\phi|^2\right)^{1/2}.$$

Furthermore, we will use

$$L^\infty(U) := \{\phi \in \mathcal{M}(U) \colon \|\phi\|_\infty < \infty\}$$

where

$$\|\phi\|_\infty = \operatorname*{esssup}_{x\in U} |\phi(x)| := \sup\left\{t \in \mathbb{R} \colon \lambda^d\{x \in \mathbb{R}^d \colon |\phi(x)| \geq t\} > 0\right\}$$

is the supremum norm and λ^d is the d-dimensional Lebesgue measure. Sometimes, we write $L^\infty := L^\infty(\mathbb{R}^d)$. By $L_c^\infty(U)$ we denote the space of bounded functions with compact support.

We use the symbol $\langle\cdot,\cdot\rangle$ for the scalar product in a Hilbert space. This could be for instance $\mathbb{R}^d$, $\mathbb{C}^d$, or $L^2(U)$. If a Hilbert space is complex, then we stick to the mathematical physics convention stating that a scalar product is antilinear in the first and linear in the second entry.

The following definition will be central in our main results:

Definition 2.1. Let $M > 0$ and $\delta \in (0, M/2)$. We call a sequence $Z = \{z_j\}_{j\in M\mathbb{Z}^d}$ (M,δ)-*equidistributed*, if for every $j \in M\mathbb{Z}^d$ we have

$$B(z_j, \delta) \subset \Lambda_M(j).$$

The cubes $\{\Lambda_M(j)\}_{j\in M\mathbb{Z}^d}$ are called *elementary cells* of the lattice $M\mathbb{Z}^d$. For such a (M,δ)-equidistributed sequence $Z = \{z_j\}_{j\in M\mathbb{Z}^d}$ and an open subset $A \subset \mathbb{R}^d$, we set

$$S_{\delta,Z}(A) = \bigcup_{j\in M\mathbb{Z}^d \colon \Lambda_M(j)\subset A} B(z_j, \delta).$$

Definition 2.2. A *generalized rectangle* $\Gamma \subset \mathbb{R}^d$ is a set of the form

$$\Gamma = \bigtimes_{i=1}^{d} (a_i, b_i)$$

where $a_i, b_i \in \mathbb{R} \cup \pm\infty$ and $a_i < b_i$ for $i \in \{1, \ldots, d\}$.

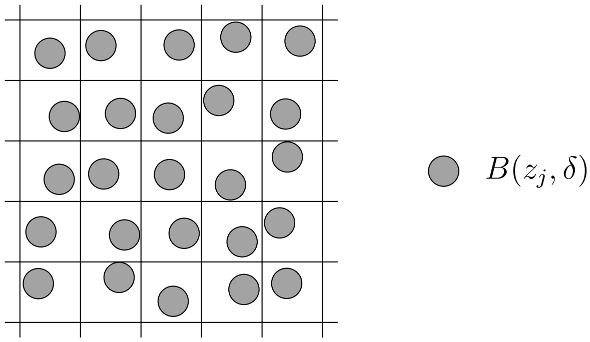

Figure 1: Example of an equidistributed arrangement $S_{\delta,Z}$ of δ-balls

Definition 2.3. For a generalized rectangle $\Gamma \subset \mathbb{R}^d$, we denote by $\Delta = \partial_1^2 + \ldots + \partial_d^2$ the *Laplace operator* or *Laplacian* in $L^2(\Gamma)$ with one of the following choices of self-adjoint boundary conditions

- Either *Dirichlet boundary conditions*, i.e. $\mathcal{D}(-\Delta) = \{\phi \in H^2(\Gamma) \colon \phi = 0 \text{ on } \partial\Gamma\}$,
- or *Neumann boundary conditions*, i.e. we define

$$\mathcal{D}(-\Delta) = \{\phi \in H^2(\Gamma) \colon \partial_\nu \phi = 0 \text{ on } \partial\Gamma\}$$

where ∂_ν is the outer normal derivative, i.e. $\partial_\nu \phi = \nu \cdot \nabla \phi$ where ν is the outer normal unit vector of Γ.

If for all $i \in \{1, \ldots, d\}$, we have $a_i > -\infty$ if and only if $b_i < \infty$ we can also prescribe

- *periodic boundary conditions*, i.e.

$$\mathcal{D}(-\Delta) = \left\{ \phi \in H^2(\Gamma) \colon \begin{array}{r} \phi(x_1, \ldots, a_i, \ldots, x_d) = \phi(x_1, \ldots, b_i, \ldots, x_d), \\ \partial_i \phi(x_1, \ldots, a_i, \ldots, x_d) = \partial_i \phi(x_1, \ldots, b_i, \ldots, x_d) \\ \text{for all } i \in \{1, \ldots, d\} \text{ with } a_i < \infty \text{ and } b_i < \infty. \end{array} \right\}$$

These identities are to be understood in trace sense, cf. [Eva98]. Since in our main results in Section 3, the choice of boundary conditions plays no role, we refrain from defining symbols for the Laplace operator with different boundary conditions here. Occasionally, in applications and when citing research by other authors, we will have to restrict to particular choices of boundary conditions and we will indicate whenever we do so.

Given a real-valued $V \in L^\infty(\Gamma)$ and a generalized rectangle $\Gamma \subset \mathbb{R}^d$, the operator of multiplication by V is a bounded perturbation of $-\Delta$ on $L^2(\Gamma)$. Hence, the operator

$$H := -\Delta + V$$

is a lower semibounded, self-adjoint operator in $L^2(\Gamma)$ with the same domain as $-\Delta$.

Remark 2.4. In the literature, in particular in the context of random Schrödinger operators, restrictions of a Schrödinger operator to subdomains of $\mathbb{R}^d$ (with self-adjoint boundary conditions) are often written with a reference to the domain. For instance, the notation H_L is often used to denote the restriction of a Schrödinger operator H from $L^2(\mathbb{R}^d)$ to $L^2(\Lambda_L)$ with appropriate boundary conditions. In order to keep the notation as lean as possible we will refrain from doing so and adopt instead the following convention: As soon as a generalized rectangle $\Gamma \subset \mathbb{R}^d$ and a real-valued function $V \in L^\infty(\Gamma)$ are specified, then H will denote the operator $-\Delta + V$ in $L^2(\Gamma)$ with Dirichlet, Neumann, or (if this is possible) periodic boundary conditions. Unless stated otherwise, statements hold for all these boundary conditions. Occasionally, when there is risk of confusion or when we cite research by other authors, it will be necessary to make exceptions from this convention. We will indicate whenever we do so.

2.2. Spectral calculus

We now recall some classic facts on spectral calculus of self-adjoint operators which can be found, e.g. in [Sch12]. Let $\mathcal{H}$ be a Hilbert space and denote by $\mathcal{B}(\mathbb{R})$ the Borel-σ-Algebra on $\mathbb{R}$.

Definition 2.5 (cf. [Sch12, Chapter 4.2.1])**.** A *spectral measure* on $\mathcal{B}(\mathbb{R})$ is a map P from $\mathcal{B}(\mathbb{R})$ into the orthogonal projections on $\mathcal{H}$ such that

(i) $P(\mathbb{R}) = \mathrm{Id}$, where Id denotes the identity operator, and

(ii) P is countably additive, i.e. $P(\cup_{n=1}^\infty A_n) = \sum_{n=1}^\infty P(A_n)$ for all mutually disjoint sequences $(A_n)_{n\in\mathbb{N}} \subset \mathcal{B}(\mathbb{R})$.

Since $\{(-\infty, E]\}_{E\in\mathbb{R}}$ is an intersection stable generator of $\mathcal{B}(\mathbb{R})$ it is sufficient to describe the spectral measures on the sets $(-\infty, E]$ and with some abuse of notation, we write

$$P(E) := P((-\infty, E]).$$

The family $\{P(E) \colon E \in \mathbb{R}\}$ is called the *resolution of identity*, corresponding to the spectral measure P.

For a spectral measure P and $\phi \in \mathcal{H}$, one defines the (real-valued) measure $\langle \phi, P(\cdot)\phi\rangle$. Furthermore, one can define for any $f \in \mathcal{M}(\mathbb{R}, \mathbb{R})$ the spectral integral

$$\int_{\mathbb{R}} f(\lambda) \mathrm{d}P(\lambda).$$

This is a possibly unbounded, self-adjoint operator in $\mathcal{H}$ with domain

$$\left\{\phi \in \mathcal{H}\colon \int_{\mathbb{R}} |f(\lambda)|^2 \mathrm{d}\left\langle \phi, P(\lambda)\phi \right\rangle < \infty \right\},$$

see [Sch12, Chapter 4.3] for more details. The spectral theorem establishes a connection between self-adjoint operators and spectral measures.

Theorem 2.6 (Spectral theorem, see [Sch12, Theorem 5.1])**.** *Let A be a self-adjoint operator in $\mathcal{H}$. Then there is a unique spectral measure $P = P_A$ on $\mathcal{B}(\mathbb{R})$ (and a unique resolution of identity) such that*

$$A = \int_{\mathbb{R}} \lambda \ \mathrm{d}P_A(\lambda)$$

where the integral is understood in the strong sense.

We can now define functions of an operator: For a self-adjoint operator A in $\mathcal{H}$ and $f \in \mathcal{M}(\mathbb{R}, \mathbb{R})$, we write

$$f(A) := \int_{\mathbb{R}} f(\lambda) \mathrm{d}P_A(\lambda).$$

Note that

$$\mathcal{D}(f(A)) = \left\{\phi \in \mathcal{H}\colon \int_{\mathbb{R}} |f(\lambda)|^2 \mathrm{d}\left\langle \phi, P_A(\lambda)\phi \right\rangle < \infty \right\}.$$

If $f(\lambda) = a_0 + \lambda a_1 + \ldots + \lambda^k a_k$ is a real-valued polynomial, and A is a bounded operator, then $f(A)$ coincides with the usual, algebraic polynomial, i.e.

$$f(A) = a_0 \mathrm{Id} + a_1 A + \ldots + a_k A^k.$$

Furthermore, for any measurable $U \subset \mathbb{R}$, we have

$$\mathbf{1}_U(A) = P_A(U).$$

We also define for a self-adjoint operator A its positive part A_+ as $A_+ := \max\{A, 0\}$ in the sense of spectral calculus.

2.3. The spectrum and spectral subsets

The *spectrum* $\sigma(A)$ of an operator A is the set of all points $\lambda \in \mathbb{C}$ such that the operator $A - \lambda \cdot \mathrm{Id}$ has no bounded inverse. Its complement is called the *resolvent set* and it is easy to see that the resolvent set is open whence the spectrum is closed. For self-adjoint operators it is well-known that the spectrum is a subset of $\mathbb{R}$ and we have, cf. [Tes09, Theorem 3.8]

$$\sigma(A) = \{\lambda \in \mathbb{R}\colon P_A([\lambda - \epsilon, \lambda + \epsilon]) \neq 0 \text{ for all } \epsilon > 0\}$$

There are several ways to decompose the spectrum of a self-adjoint operator.

Definition 2.7 (cf. [Sch12, Chapter 8.4])**.** Let A be a self-adjoint operator. The *essential spectrum* is the set

$$\sigma_{\text{ess}}(A) = \{\lambda \in \mathbb{R}\colon \dim \operatorname{Ran} P_A([\lambda - \epsilon, \lambda + \epsilon]) = \infty \text{ for all } \epsilon > 0\}$$

where Ran denotes the range of an operator, and the *discrete spectrum* is

$$\sigma_{\text{d}}(A) = \sigma(A) \backslash \sigma_{\text{ess}}(A).$$

Note that $\sigma_{\text{ess}}(A)$ is closed while $\sigma_{\text{d}}(A)$ coincides with the set of all isolated eigenvalues of finite multiplicity and does not need to be closed.

Of course, since $\sigma(A) = \sigma_{\text{ess}}(A) \cup \sigma_{\text{d}}(A)$ is a decomposition of $\sigma(A)$ into disjoint components, one could define the corresponding spectral projectors $P_A(\sigma_{\text{ess}}(A))$, and $P_A(\sigma_{\text{d}}(A))$. This would yield an orthogonal decomposition

$$\mathcal{H} = \operatorname{Ran} P_A(\sigma_{\text{ess}}(A)) \oplus \operatorname{Ran} P_A(\sigma_{\text{d}}(A)).$$

However, if one wants to describe the dynamics of the (time dependent) Schrödinger equation $i\partial_t \phi = A\phi$, there is a more suitable decomposition of $\mathcal{H}$. We present it, following the exhibition in [Tes09, Chapter 3]. First, we recall the following definition from measure theory:

Definition 2.8. A Borel-measure μ on $\mathbb{R}$ is called

- *absolutely continuous* with respect to the Lebesgue measure if $\mu(B) = 0$ for all $B \in \mathcal{B}(\mathbb{R})$ with Lebesgue measure zero,
- *singular* with respect to the Lebesgue measure if it is supported on a set $B \in \mathcal{B}(\mathbb{R})$ of Lebesgue measure zero,
- *pure point* if it singular and is a (countable) linear combination of Dirac-measures δ_x, where $\delta_x(B) = 1$ if $x \in B$ and 0 else,
- *singularly continuous* with respect to the Lebesgue measure if it is singular and no point $x \in \mathbb{R}$ has positive measure.

Every Borel-measure μ on $\mathbb{R}$ has a unique decomposition

$$\mathrm{d}\mu = \mathrm{d}\mu_{\text{ac}} + \mathrm{d}\mu_{\text{sc}} + \mathrm{d}\mu_{\text{pp}}$$

where μ_{ac} is absolutely continuous with respect to the Lebesgue measure, μ_{sc} is singularly continuous with respect to the Lebesgue measure, and μ_{pp} is pure point. Given ϕ, we consider the Borel-measure $\mathrm{d}\mu_\phi = \mathrm{d}\,\langle \phi, P(\cdot)\phi \rangle$ on $\mathbb{R}$.

Definition 2.9. Let A be a self-adjoint operator in the Hilbert $\mathcal{H}$. We define

$$\begin{aligned}\mathcal{H}_{\mathrm{ac}} &:= \{\phi \in \mathcal{H} \colon \mu_\phi \text{ is absolutely continuous}\},\\ \mathcal{H}_{\mathrm{sc}} &:= \{\phi \in \mathcal{H} \colon \mu_\phi \text{ is singularly continuous}\},\\ \mathcal{H}_{\mathrm{pp}} &:= \{\phi \in \mathcal{H} \colon \mu_\phi \text{ is pure point}\}.\end{aligned}$$

There is the following Lemma:

Lemma 2.10 ([Tes09, Lemma 3.19])**.** *We have*

$$\mathcal{H} = \mathcal{H}_{\mathrm{ac}} \oplus \mathcal{H}_{\mathrm{sc}} \oplus \mathcal{H}_{\mathrm{pp}}.$$

Sometimes, we will also write $\mathcal{H}_{\mathrm{c}} := \mathcal{H}_{\mathrm{ac}} \oplus \mathcal{H}_{\mathrm{sc}}$, since the decomposition

$$\mathcal{H} = \mathcal{H}_{\mathrm{c}} \oplus \mathcal{H}_{\mathrm{pp}}$$

plays an important role in the so-called *RAGE Theorem*, see Theorem 5.3 below.

We also define

$$\sigma_{\mathrm{ac}}(A) := \sigma(A\,|_{\mathcal{H}_{\mathrm{ac}}}), \quad \sigma_{\mathrm{sc}}(A) := \sigma(A\,|_{\mathcal{H}_{\mathrm{sc}}}), \quad \sigma_{\mathrm{pp}}(A) := \sigma(A\,|_{\mathcal{H}_{\mathrm{pp}}}).$$

Note that $\sigma_{\mathrm{ac}}(A)$, $\sigma_{\mathrm{sc}}(A)$, and $\sigma_{\mathrm{pp}}(A)$ are closed subsets and that they are not necessarily disjoint. Furthermore, $\sigma_{\mathrm{pp}}(A) = \overline{\{\text{Eigenvalues of } A\}}$, cf. the discussion in [RS80, Chapter VII.2], where we emphasize that our definition of $\sigma_{\mathrm{pp}}(A)$ and the one given therein differ.

3. Quantitative and scale-free unique continuation

This section contains material from [NTTV15, NTTV18a, TT17, Täu17, NTTV18b]. It is structured as follows: We start with an introduction to unique continuation in Subsection 3.1. Then, our main results of this section are presented in Subsection 3.2. We discuss these results and their optimality in Subsection 3.3. After the proofs in Subsection 3.4, we provide in Subsection 3.5 some overview over other results on scale-free quantitative unique continuation which will be used in later sections.

3.1. Unique continuation

Unique continuation is a manifestation of the fact that some function spaces have a rigidity property: if a function in these spaces vanishes on a non-empty, open subset, then it must vanish everywhere. Examples of such function spaces include holomorphic functions on $\mathbb{C}$, harmonic functions on $\mathbb{R}^d$ and – more generally – solutions of certain partial differential expressions. While properties such as holomorphy and analyticity have been known for centuries, an important milestone in the history of unique continuation was the work of Carleman [Car39] who introduced inequalites which are refered to as Carleman estimates. Originally, Carleman treated unique continuation for a system of differential equations of order one in two dimensions. This was generalized to the Laplacian in [Mül54] and subsequently to a larger class of elliptic differential operators. We refer to [Hör69] for an introduction on classic aspects regarding unique continuation.

Let us illustrate by means of an example how a Carleman estimate leads to unique continuation. In [KRS87], see also [Ken87], the following Carleman estimate is proved:

Proposition 3.1 ([KRS87, Theorem 3.1]). *Let $d \geq 3$, $p = 2d/(d+2)$, and $q = 2d/(d-2)$. Then there is a constant $C > 0$ such that for all $\nu \in \mathbb{R}^d$, all $\lambda \in \mathbb{R}$ and all functions u with $\mathrm{e}^{\lambda\langle\nu,\cdot\rangle} u \in H^{2,p}(\mathbb{R}^d)$ we have*

$$\|\mathrm{e}^{\lambda\langle\nu,\cdot\rangle} u\|_{L^q(\mathbb{R}^d)} \leq C \|\mathrm{e}^{\lambda\langle\nu,\cdot\rangle} \Delta u\|_{L^p(\mathbb{R}^d)}$$

As in [KRS87], Proposition 3.1 implies the following unique continuation result:

Proposition 3.2 (Unique continuation for a half-space, [KRS87, Corollary 3.1]). *Let $d \geq 3$, $p = 2d/(d+2)$ and $V \in L^{d/2}(\mathbb{R}^d)$. Then every $u \in H^{2,p}(\mathbb{R}^d)$ satisfying $|\Delta u| \leq |Vu|$ which vanishes in a half-space must vanish everywhere.*

For the sake of a complete exposition, let us reproduce here the proof from [KRS87].

Proof of Proposition 3.2. By translation and rotation we may assume that u vanishes in the half-space $\{x \in \mathbb{R}^d : x_1 < 0\}$. It suffices to show that there exists $\rho > 0$ such that u also vanishes in the strip $S_\rho := \{x \in \mathbb{R}^d : 0 \leq x_1 < \rho\}$, since then, by iterating this argument, we find $u \equiv 0$. For that purpose, we choose $\rho > 0$ sufficiently small such that $\|V\|_{L^{d/2}(S_\rho)} \leq 1/(2C)$ where C is the constant from Proposition 3.1. We estimate, using Proposition 3.1, the inequality $|\Delta u| \leq |Vu|$, and Hölder's inequality for every $\lambda > 0$

$$\begin{aligned}
\|\mathrm{e}^{-\lambda x_1} u\|_{L^q(S_\rho)} &\leq C \|\mathrm{e}^{-\lambda x_1} \Delta u\|_{L^p(\mathbb{R}^d)} \\
&\leq C \|\mathrm{e}^{-\lambda x_1} V u\|_{L^p(S_\rho)} + C \|\mathrm{e}^{-\lambda x_1} \Delta u\|_{L^p(\mathbb{R}^d \setminus S_\rho)} \\
&\leq C \|V\|_{L^{d/2}(S_\rho)} \cdot \|\mathrm{e}^{-\lambda x_1} u\|_{L^q(S_\rho)} + C \|\mathrm{e}^{-\lambda x_1} \Delta u\|_{L^p(\mathbb{R}^d \setminus S_\rho)} \\
&\leq \frac{1}{2} \|\mathrm{e}^{-\lambda x_1} u\|_{L^q(S_\rho)} + C \mathrm{e}^{-\lambda \rho} \|\Delta u\|_{L^p(\mathbb{R}^d)}.
\end{aligned}$$

Bringing the term with the $1/2$ factor to the other side and multiplying by $\mathrm{e}^{\lambda \rho}$, we find

$$\|\mathrm{e}^{\lambda(\rho - x_1)} u\|_{L^q(S_\rho)} \leq 2C \|\Delta u\|_{L^p(\mathbb{R}^d)} \quad \text{for all} \quad \lambda > 0.$$

This can only hold if $u \equiv 0$ in S_ρ. □

The above proof contains two important aspects of the application of Carleman estimates which we will encounter again in the proofs below: The first one is the *free parameter* λ in the Carleman estimate. It was essential for the concluding argument that in the Carleman estimate the parameter $\lambda \in (0, \infty)$ could be sent to infinity. In the proofs below, we will optimize over this parameter in order to obtain more refined statements. The second theme is the *choice of the geometry*, more precisely of the width ρ. It allowed to make a term sufficiently small such that it could be absorbed on the left hand side of an inequality. We will encounter similar arguments when we choose the geometry in Lemmas 3.30 and 3.31 below.

Over the years, Carleman estimates and unique continuation have led to numerous applications for partial differential equations. Among others, they have been used for uniqueness in the context of the Cauchy problem [Cal58, Hör58], absence of positive eigenvalues for Schrödinger operators, [JK85], inverse problems [IY98], boundary value problems [Tat96], control theory for the heat equation [LR95, FI96], the wave equation [TY02], and non-linear systems, see e.g. [LT97].

There are also slightly different notions of unique continuation such as *strong unique continuation*, see [Esc00], [KT02] and the surveys [Wol93, TTV16] for more on this matter.

Furthermore, recently, there has been a focus on *quantitative unique continuation.* In quantitative unique continuation, one attempts to establish inequalities of the type

$$\|f\|_{L^2(V)} \leq C\|f\|_{L^2(U)} \quad \text{for all} \quad f \in \mathcal{F}$$

with some constant C where $U \subset V \subset \mathbb{R}^d$ and $\mathcal{F} \subset L^2(V)$ is a function space. Originally also motivated by control theory, quantitative unique continuation has been introduced to random Schrödinger operators by Bourgain and Kenig [BK05].

Let us now present a sequence of results which can be considered as predecessors to our results in Subsection 3.2 below. Since the early 2000s, there has been an increasing interest in so-called multi-scale variants of quantitative unique continuation, that are unique continuation estimates which hold with a uniform constant in a variety of geometric configurations ("scales"). This development has been mainly driven by applications in the context of random Schrödinger operators, see Section 5 below for more on these applications. As a starting point, let us cite the following result from [CHK03] which has been used in proofs of so-called Wegner estimates in [CHK03, CHK07].

Proposition 3.3 (cf. [CHK03, Section 4])**.** *Let $V_0 \colon \mathbb{R}^d \to \mathbb{R}$ be $\mathbb{Z}^d$-periodic such that the operator $H = -\Delta + V_0$ has the unique continuation property in the sense that every function ϕ satisfying $H\phi = E\phi$ for some $E \in \mathbb{R}$ and which vanishes on a non-empty open set must vanish everywhere. Let $V \colon \mathbb{R}^d \to \mathbb{R}$ be a bounded, $\mathbb{Z}^d$-periodic, nonnegative function which is strictly positive on some open set. For $L \in \mathbb{N}_{\mathrm{odd}}$ denote by H_L the restriction of H onto Λ_L with periodic boundary conditions and by V_L the restriction of V to Λ_L. Then, for every bounded interval $I \subset \mathbb{R}$, there is a constant $C = C(I, V, V_0)$, such that for every $L \in \mathbb{N}_{\mathrm{odd}}$, and every $\phi \in \operatorname{Ran} P_{H_L}(I)$, we have*

$$\int_{\Lambda_L} |\phi|^2 \leq C \int_{\Lambda_L} V_L |\phi|^2.$$

Proposition 3.3 is called *scale-free* because the constant C does not depend on the scale L. If we choose V as the characteristic function of a $\mathbb{Z}^d$-periodic arrangement $S_{\delta,\mathrm{per}}$ of δ-balls, then the statement of Proposition 3.3 will read

$$\|\phi\|^2_{L^2(\Lambda_L)} \leq C\|\phi\|^2_{L^2(S_{\delta,\mathrm{per}})} \quad \text{for all} \quad \phi \in \operatorname{Ran} P_{H_L}(I).$$

The statement has been generalized to some examples of magnetic Schrödinger operators, cf. [CHK03, CHK07]. The proof of Proposition 3.3 in [CHK03] uses Floquet theory and a compactness argument in the dual torus or the Brillouin zone to turn the qualitative statement of unique continuation into a quantitative one. While this method has the advantage that it immediately allows for spectral subspaces $\operatorname{Ran} P_H(I)$ rather than single solutions of an eigenvalue equation, and that it allows for a large class of background potentials V_0 (it is only required that a unique continuation property for $-\Delta + V_0$ holds), it suffers from the restriction that only periodic V and V_0 are allowed. Furthermore, due to the compactness argument, the resulting constant $C = C(I, V, V_0)$ is non-explicit in terms of I, V, and V_0. We also emphasize that in the proof of Proposition 3.3, the underlying tools which imply unique continuation have somewhat been hidden in the machinery of the proof: It suffices to have a qualitative statement on unique continuation for distributional solutions of the eigenvalue equation $H\phi = E\phi$. In summary, Proposition 3.3 smartly uses periodicity to boost (classic, qualitative) unique continuation properties for single eigenfunctions and turns them into a quantitative, scale-free unique continuation estimate – however without providing explicit information on the constant involved.

In [BK05], Bourgain and Kenig used unique continuation in a more direct manner. Their goal was to solve the longstanding open problem of Anderson localization for the Bernoulli-Anderson model. As an important step in their proof they established lower bounds on the vanishing rate of solutions of the stationary Schrödinger equation:

Proposition 3.4 (Special case of [BK05, Lemma 3.10])**.** *Assume $\Delta u = Vu$ in $\mathbb{R}^d$ and*

$$u(0) = 1, \quad \|u\|_\infty \leq C, \quad \|V\|_\infty \leq C.$$

Let $x_0 \in \mathbb{R}^d$ with $|x_0| = R > 1$. Then, there is a constant $c' > 0$, such that

$$\max_{|x-x_0|\leq 1} |u(x)| > c' \exp(-c'(\ln R) R^{4/3}).$$

Proposition 3.4 can be interpreted as a quantitative lower bound on the vanishing rate of eigenfunctions with respect to maxima over balls. Before discussing Proposition 3.4 in more detail, we also cite the following Proposition 3.5 which has been used used by Germinet and Klein in [GK13] and by Bourgain and Klein in [BK13]. It uses similar techniques as Proposition 3.4 and can be considered as an L^2-variant thereof.

Proposition 3.5 ([BK13, Theorem 3.4], see also [GK13, Theorem A.1])**.** *Let $\Omega \subset \mathbb{R}^d$ be open and $\|V\|_\infty \leq K < \infty$. Let $\psi \in H^2(\Omega)$ be real-valued, and let $\xi \in L^2(\Omega)$ be defined by*

$$-\Delta\psi + V\psi = \xi,$$

Let $\Theta \subset \Omega$ be a bounded, measurable set where $\|\psi\|_{L^2(\Theta)} > 0$. Set

$$Q(x, \Theta) := \sup_{y\in\Theta} |y - x| \quad \text{for} \quad x \in \Omega.$$

Consider $x_0 \in \Omega \backslash \overline{\Theta}$ *such that*

$$Q = Q(x_0, \Theta) \geq 1 \quad \textit{and} \quad B(x_0, 6Q + 2) \subset \Omega.$$

Then, given

$$0 < \delta \leq \min \left\{ \operatorname{dist}(x_0, \Theta), \frac{1}{24} \right\},$$

we have

$$\left(\frac{\delta}{Q}\right)^{m(1+K^{2/3})\left(Q^{4/3}+\log \frac{\|\psi\|^2_{L^2(\Omega)}}{\|\psi\|^2_{L^2(\Theta)}}\right)} \|\psi\|^2_{L^2(\Theta)} \leq \|\psi\|^2_{L^2(B(x_0,\delta))} + \delta^2 \|\xi\|^2_{L^2(\Omega)} \tag{3}$$

More precisely, in [GK13], the results of [BK05] were generalized to Anderson Hamiltonians with singular random potentials while in [BK13], continuity of the so-called density of states for Schrödinger operators with bounded potentials in dimensions one to three was proved. For more details on the density of states, we refer again to Section 5 below.

Propositions 3.4 and 3.5 both rely on the following Carleman estimate:

Proposition 3.6 ([BK05, Lemma 3.15], see also [EV03])**.** *There are constants* C_1*,* C_2*,* C_3*, depending only on the dimension, and an increasing function* $w = w(|x|) > 0$ *satisfying for* $0 < |x| < 10$

$$\frac{1}{C_1} < \frac{w(|x|)}{|x|} < C_1$$

such that for all real-valued $f \in C_0^\infty(B(0, 10)\backslash\{0\})$*,* $\alpha > C_2$ *we have*

$$\alpha^3 \int w^{-1-2\alpha} f^2 \leq C_3 \int w^{2-2\alpha} (\Delta f)^2.$$

Similar to the Carleman estimate, Proposition 3.1 above, Proposition 3.6 estimates f from above by Δf – but with radially symmetric weight functions instead of $\mathrm{e}^{\lambda\langle\mu,\cdot\rangle}$ and in terms of L^2-norms instead of L^p and L^q norms. In the proof of the main result of this section, Theorem 3.17 below, we will apply a slightly stronger variant of this Carleman estimate which carries an additional gradient term on the left hand side and an additional scaling parameter ρ.

There is an interesting aspect regarding Proposition 3.6. In [BK13], continuity of the density of states of Schrödinger operators is proved in dimensions only up to three. The technical obstacle preventing a similar result in higher dimensions is the factor $Q^{4/3}$ in the exponent of Ineq. (3) which itself is due to the Carleman estimate, Proposition 3.6. As discussed in [BK13], if this was Q^β for $\beta > 1$, they would obtain continuity of the density of states in dimensions $d < \beta/(\beta - 1)$. In other words, if one managed to prove a variant of Proposition 3.5 with $Q^{4/3}$ replaced by Q^{β} for all $\beta > 1$, then continuity of the density of states in all dimensions would follow. Even though Bourgain and Klein claim in the introduction of [BK13] that it is reasonable to assume that analogs of Proposition 3.5 should hold for all $\beta > 1$, this turns out to be not an easy task. In fact it seems impossible to achieve this by merely improving the Carleman estimate Proposition 3.6 since it is known that such an improved Carleman estimate would lead to yield lower bounds on the vanishing rate of solutions of $\Delta u = Vu$ at infinity which contradict a counterexample by Meshkov [Mes92], see also the discussion in Section 3 of [BK05] and our discussion in Subsection 3.3.2 below. There are still possible loopholes around this since Meshkov's counterexample relies on a complex-valued V, but it illustrates the difficulty here. We stress these points since in our main results, Theorems 3.9, 3.13, and 3.17 below, a similar effect occurs which yields a term $\|V\|_\infty^{2/3}$ in the unique continuation constant.

In 2013, Rojas-Molina and Veselić applied Proposition 3.6 to remedy the restriction of Proposition 3.3 to periodic situations, however only for single solutions of an eigenvalue inequality. Recall the definition of $(1, \delta)$-equidistributed sequences in Section 2.1. Then Rojas-Molina and Veselić proved:

Proposition 3.7 ([RMV13, Theorem 2.1])**.** *There is a constant C, depending only on the dimension, such that for all $\delta \in (0, 1/2)$, all $(1, \delta)$-equidistributed sequences Z, all $L \in \mathbb{N}_{\mathrm{odd}}$, all $K_V \geq 0$, all measurable and bounded $V : \mathbb{R}^d \to [-K_V, K_V]$, and all real-valued ψ which are in the intersections of the domains of the Laplacian with Dirichlet and periodic boundary conditions on Λ_L, and which satisfy*

$$|\Delta \psi| \leq |V \psi| \tag{4}$$

we have

$$\|\psi\|^2_{L^2(\Lambda_L)} \leq \left(\frac{\delta}{C} \right)^{-C(1 + K_V^{2/3})} \|\psi\|^2_{L^2(S_{\delta,Z} \cap \Lambda_L)}. \tag{5}$$

Proposition 3.7 does not require periodicity as in Proposition 3.3 above any more and the constant carries an explicit dependence on δ and K_V. However, in contrast to Proposition 3.3 which yields a statement on spectral subspaces, Ineq. (4) is more restrictive. If ψ is an eigenfunction of a Schrödinger operator $H = -\Delta + V_0$ to the eigenvalue $\lambda \in \mathbb{R}$, then we have indeed

$$|\Delta \psi| \leq |(V_0 + \lambda) \psi|$$

whence Ineq. (5) holds with $K_V = |E| + \|V_0\|_\infty$. However, if one would like to go beyond single eigenfunctions and for instance linear combinations of eigenfunctions in some energy interval, i.e. spectral subspaces, then Proposition 3.7 is not sufficient. We also emphasize that the constant in (5) is polynomial in δ and carries the term $K_V^{2/3}$ in the exponent which is related to the factor $Q^{4/3}$ in Proposition 3.5, originating from the Carleman estimate Proposition 3.6 as discussed above. As already mentioned, we will encounter this polynomial dependence on δ and the 2/3 in the exponent later, see Subsection 3.3.2 below. While allowing to pass beyond periodic situations which was crucial for the application to non-ergodic operators as in [RMV13], Proposition 3.7 is not necessarily an improvement compared to Proposition 3.3 since the statement is essentially restricted to eigenfunctions. Therefore, in [RMV13], the authors asked as an open question if an analogous result also holds for finite energy spectral subspaces. In [Kle13], the restriction to single eigenfunctions was partially removed by using a perturbation argument.

Proposition 3.8 ([Kle13, Theorem 1.1])**.** *Let $V \in L^\infty(\mathbb{R}^d)$, $\delta \in (0, 1/2)$, Z a $(1, \delta)$-equidistributed sequence, $E_0 > 0$, and $L \in \mathbb{N}_{\mathrm{odd}}$ with $L \geq 72\sqrt{d}$. Set $H = -\Delta + V$ and denote the restriction of H to Λ_L with Dirichlet or periodic boundary conditions by H_L. Then there exists a constant $M > 0$, depending only on the dimension, such that for $\gamma = \gamma(d, K, \delta) > 0$ defined as*

$$\gamma^2 = \frac{1}{2}\delta^{M\left(1+K^{2/3}\right)}, \quad K = 2\|V\|_\infty + E_0$$

we have for every closed interval $I \subset (-\infty, E_0]$ of length at most γ, and every $\phi \in \operatorname{Ran} P_{H_L}(I)$ that

$$\|\psi\|^2_{L^2(\Lambda_L)} \leq \gamma^{-2}\|\psi\|^2_{L^2(S_{\delta,Z}\cap\Lambda_L)}.$$

In comparison to Proposition 3.7, Proposition 3.8 holds for all functions in certain spectral subspaces. This positively answers the questions posed in [RMV13] in the special case of small energy intervals and immediately leads to an improvement of the Wegner estimate of [RMV13] and to localization at low energies, cf. [Kle13].

However, Proposition 3.8 carries a smallness condition on the lenght γ of the energy interval in terms of the parameters δ and K. There are applications where this is still not sufficient. These applications include lower bounds of the sensitivity of the spectrum under perturbations, random Schrödinger operators, and control theory and are the subjects of Sections 4, 5, and 6, respectively. The main results of the following section remove this smallness condition on γ. In particular Theorem 3.9 below affirmatively answers the question from [RMV13] in full generality.

3.2. Main results

The following results have partly been published in joint works with Ivica Nakic, Martin Tautenhahn and Ivan Veselić, [NTTV15, NTTV18a, TT17, NTTV18b]. More precisely, Theorems 3.9 and 3.13 have been stated and proved in the joint publications [NTTV15, NTTV18a, NTTV18b] with Ivica Nakić, Martin Tautenhahn, and Ivan Veselić, and Theorem 3.16 has been published in the joint publication [TT17] with Martin Tautenhahn. Theorem 3.17 is a new result that covers these previous results.

For the sake of readability, we start by citing special instances, namely Theorems 3.9, 3.13, and 3.16, before turning to the most general result of this section, Theorem 3.17. After Theorem 3.17, we briefly explain how to deduce the other results. Theorem 3.17 itself is proved in Subsection 3.4.

Let us recall the notation introduced in Section 2.1: Given a generalized rectangle $\Gamma \subset \mathbb{R}^d$ and a bounded and real-valued potential V, the operator $H = -\Delta + V$ is the corresponding self-adjoint Schrödinger operator with potential V in $L^2(\Gamma)$ with Dirichlet, Neumann and (if possible) periodic boundary conditions.

3.2.1. Scale-free, quantitative uniqe continuation for spectral subspaces

The following theorem removes the smallness assumption on the energy intervals in [Kle13]. It has been announced in [NTTV15] and published in [NTTV18a]. It combines techniques developed in [RMV13] and [Kle13] with a technique from [LRL12] which allows to treat linear combinations of eigenfunctions by using an additional "ghost dimension", see Subsection 3.4.1 for details.

Recall that for $L > 0$, $\Lambda_L = (-L/2, L/2)$ denotes the centered hypercupe of side length L. For a (M, δ)-equidistributed sequence Z, we have introduced in Definition 3.26 the set $S_{\delta,Z}(\Lambda_L)$ as the union of δ-balls with centers at the points of Z whose elementary cells are entirely contained in Λ_L, see also Figure 2, and its characteristic function $W_{\delta,Z}(\Lambda_L)$.

Theorem 3.9 ([NTTV15, Theorem 2.1], [NTTV18a, Theorem 2.2 and Corollary 2.3])**.** *There is $C > 0$, depending only on the dimension, such that for all $M > 0$, all $\delta \in (0, M/2)$, all (M, δ)-equidistributed sequences Z, all $L \in M\mathbb{N}$, all $V \in L^\infty(\Lambda_L)$, all $E \geq 0$ and all $\phi \in \operatorname{Ran} P_H(E)$ we have*

$$\|\phi\|^2_{L^2(S_{\delta,Z}(\Lambda_L))} \geq \left(\frac{\delta}{M}\right)^{C(1+M^{4/3}\|V\|_\infty^{2/3}+ME^{1/2})} \|\phi\|^2_{L^2(\Lambda_L)} \tag{6}$$

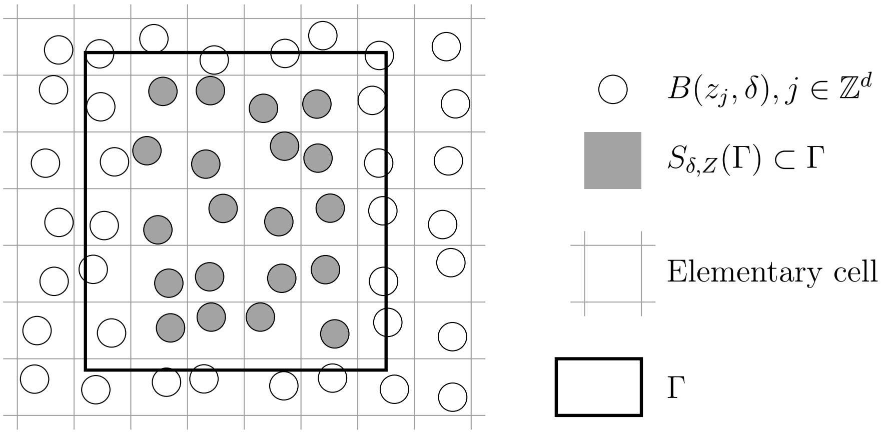

Figure 2: The set $S_{\delta,Z}(\Gamma) \subset \Gamma$ consists of all δ-balls such that the corresponding elementary cells are entirely contained in Γ.

where $\Gamma = \Lambda_L$, i.e. H is the Schrödinger operator $-\Delta + V$ in $L^2(\Lambda_L)$ with Dirichlet, Neumann or periodic boundary conditions.

The following corollary is an immediate consequence:

Corollary 3.10. *Under the assumptions of Theorem 3.9 we have*

$$P_H(E) W_{\delta,Z} P_H(E) \geq \left(\frac{\delta}{M}\right)^{C(1+M^{4/3}\|V\|_\infty^{2/3}+ME^{1/2})} P_H(E) \tag{7}$$

in quadratic form sense.

Remark 3.11. Note that since $\operatorname{Ran} P_H((-\infty, E]) \subset \operatorname{Ran} P_H((-\infty, 0])$ for all $E < 0$, the assumption $E \geq 0$ is in fact no restriction. Furthermore, given any measurable $B \subset (-\infty, E]$, we can multiply Ineq. (7) from both sides with $P_H(B)$ and immediately deduce

$$P_H(B) W_{\delta,Z} P_H(B) \geq \left(\frac{\delta}{M}\right)^{C(1+M^{4/3}\|V\|_\infty^{2/3}+ME^{1/2})} \cdot P_H(B).$$

The constant in the unique continuation principle (6) now carries an improved (compared to Propositions 3.7 and 3.8 above) dependence on the energy E: While in the latter ones, the energy enters as $E_0^{2/3}$ in the exponent, see the discussion after Proposition 3.7, it is now of order $E^{1/2}$. The reason for that is that Propositions 3.7 and 3.8 rely on techniques developed for single solutions of partial differential expressions and they include the energy E by absorbing the eigenvalue into the potential whence they end up with an order $E^{2/3}$. The order $E^{1/2}$ in the exponent is optimal, see Subsection 3.3 below for a discussion.

However, one can use this trick to further optimize the constant in Theorem 3.9: Replacing the potential V by $V - \lambda_0$ for some $\lambda_0 \in \mathbb{R}$ corresponds to an energy shift. In fact, we have

$$P_{-\Delta+V}(E) = P_{-\Delta+V-\lambda_0}(E - \lambda_0)$$

which allows to minimize over an additional parameter $\lambda \in \mathbb{R}$. This leads to the following corollary.

Corollary 3.12. *There is $C > 0$, depending only on the dimension, such that for all $M > 0$, all $\delta \in (0, M/2)$, all (M, δ)-equidistributed sequences Z, all $L \in M\mathbb{N}$, all $V \in L^\infty(\Lambda_L)$, all $E \in \mathbb{R}$, all $\lambda_0 \in \mathbb{R}$, and all $\phi \in \operatorname{Ran} P_H(E)$ we have*

$$\|\phi\|^2_{L^2(S_{\delta,Z}(\Lambda_L))} \geq \left(\frac{\delta}{M}\right)^{C(1+M^{4/3}\|V-\lambda_0\|_\infty^{2/3}+M(E-\lambda_0)_+^{1/2})} \|\phi\|^2_{L^2(\Lambda_L)}.$$

In the application to control theory in Section 6 below, it will be convenient to make the choice $\lambda = \inf \sigma(H)$.

Theorem 3.9 has led to some applications: a Wegner estimate for the so-called standard random breather model [TV15, NTTV15, NTTV18a] and estimates on the control cost for the heat equation in a multi-scale setting [NTTV18a]. The Wegner estimate will be discussed in Section 5 below while the control cost estimate will be analyzed in an even more general situation in Section 6.

Finally let us stress that for every $L > 0$, the operator $H = -\Delta + V$ in $L^2(\Lambda_L)$ is lower semibounded with purely discrete spectrum. Thus, all $\phi \in \operatorname{Ran} P_H(E)$ turn out to be finite linear combinations of eigenfunctions. This expansion into eigenfunctions has been used in [NTTV18a].

3.2.2. Generalization to unbounded domains

If one wants to extend Theorem 3.9 to more general geometric situations such as operators living on the whole space $\mathbb{R}^d$, one is confronted with the difficulty that these operators will – in general – exhibit continuous spectrum. Of course, one can deduce statements on single eigenfunctions, cf. [TV16b] where an analog of Proposition 3.7 for eigenfunctions of the full-space operator is deduced. However, eigenfunctions, if any exist at all, might span only a subspace. Therefore, a concrete description of spectral subspaces $P_H(E)$ might become more challenging since functions in these spaces are not finite linear combinations of eigenfunctions any more. However, the spectral calculus provides appropriate tools to also treat unbounded situations. This is done by the next theorem which is contained in the joint work [NTTV18b] with Ivica Nakić, Albrecht Seelmann, Martin Tautenhahn, and Ivan Veselić.

Theorem 3.13 (Infinite volume, finite energy [NTTV18b]). *There is $C > 0$, depending only on the dimension, such that for all $M > 0$, all $\delta \in (0, M/2)$, all (M, δ)-equidistributed sequences Z, all generalized rectangles Γ containing at least one elementary cell of $M\mathbb{Z}^d$, all $V \in L^\infty(\Gamma)$, all $E \geq 0$ and all $\phi \in \operatorname{Ran} P_H(E)$ we have*

$$\|\phi\|^2_{L^2(S_{\delta,Z}(\Gamma))} \geq \left(\frac{\delta}{M}\right)^{C(1+M^{4/3}\|V\|_\infty^{2/3}+ME^{1/2})} \|\phi\|^2_{L^2(\Gamma)}.$$

Again, this theorem can be reformulated in terms of quadratic forms:

Corollary 3.14. *Under the assumptions of Theorem 3.13 we have*

$$P_H(E) W_{\delta,Z} P_H(E) \geq \left(\frac{\delta}{M}\right)^{C(1+M^{4/3}\|V\|_\infty^{2/3}+ME^{1/2})} P_H(E)$$

in quadratic form sense.

Obviously Remark 3.11 holds analogously here and we find the analogon of Corollary 3.12:

Corollary 3.15. *There is $C > 0$, depending only on the dimension, such that for all $M > 0$, all $\delta \in (0, M/2)$, all (M, δ)-equidistributed sequences Z, all generalized rectangles Γ containing at least one elementary cell of $M\mathbb{Z}^d$, all $V \in L^\infty(\Gamma)$, all $E \geq 0$, all $\lambda \in \mathbb{R}$, and all $\phi \in \operatorname{Ran} P_H(E)$ we have*

$$\|\phi\|^2_{L^2(S_{\delta,Z}(\Gamma))} \geq \left(\frac{\delta}{M}\right)^{C(1+M^{4/3}\|V-\lambda_0\|_\infty^{2/3}+M(E-\lambda_0)_+^{1/2})} \|\phi\|^2_{L^2(\Gamma)}.$$

3.2.3. Beyond finite energy

Besides extending it to infinite domains, there is another natural way to extend Theorem 3.9 to infinite dimensional subspaces. In [JL99, Theorem 14.10], and [LRL12, Proposition 5.6], the observation was made that certain infinite series of eigenfunctions with rapidly decaying coefficients have a unique continuation property. Building upon this observation, in [TT17], the following multi-scale, quantitative, series-of-eigenfunctions analog of Theorem 3.9 was proved.

Theorem 3.16 (Finite volume, infinite energy, [TT17, Theorem 2.2]). *There is $C > 0$, depending only on the dimension, such that for all $\kappa > 0$, all $M \in (0, \kappa/(18\mathrm{e}\sqrt{d})]$, all $\delta \in (0, M/2)$, all (M, δ)-equidistributed sequences Z, all $L \in M\mathbb{N}$, all $V \in L^\infty(\Lambda_L)$, all $D \geq 1$, and all $\phi \in L^2(\Lambda_L)$ satisfying*

$$\sum_{k\in\mathbb{N}} \exp\left(\kappa\sqrt{(\lambda_k(H))_+}\right) |\alpha_k|^2 \leq D \sum_{k\in\mathbb{N}} |\alpha_k|^2 = D\|\phi\|^2_{L^2(\Lambda_L)},$$

where $\Gamma = \Lambda_L$ and $\lambda_k(H)$ denotes the k-th eigenvalue of the operator H in $L^2(\Lambda_L)$, enumerated increasingly and counting multiplicities with normalized eigenfunction ψ_k and $\alpha_k = \langle \psi_k, \phi \rangle$, we have

$$\|\phi\|^2_{L^2(S_{\delta,Z}(\Lambda_L))} \geq \left(\frac{\delta}{M}\right)^{C\left(1+M^{4/3}\|V\|_\infty^{2/3}+\ln D\right)} \|\phi\|^2_{L^2(\Lambda_L)}.$$

We will comment on the condition $M \in (0, \kappa/(18\mathrm{e}\sqrt{d})]$ later in Subsection 3.3.3 when discussing optimality of the results. Theorem 3.16 again relies on the fact that on finite volume boxes, the operator H has purely discrete spectrum. Note that the inequality

$$\sum_{k\in\mathbb{N}} \exp\left(\kappa\sqrt{(\lambda_k(H))_+}\right)|\alpha_k|^2 \leq D \sum_{k\in\mathbb{N}} |\alpha_k|^2 < \infty$$

in Theorem 3.16 is in particular satisfied for some D if

$$\phi \in \mathcal{D}\left(\mathrm{e}^{\frac{\kappa}{2}\sqrt{H_+}}\right),$$

see Subsection 2.2 on spectral calculus above. Thus we see immediately that the following Theorem 3.17 is a generalization of Theorem 3.16. It is the most general statement of this section and we explain below how it implies Theorems 3.9 and 3.13.

Theorem 3.17. *There is $C > 0$, depending only on the dimension, such that for all $\kappa > 0$, all $M \in (0, \kappa/(18\mathrm{e}\sqrt{d})]$, all $\delta \in (0, M/2)$, all (M, δ)-equidistributed sequences Z, all generalized rectangles Γ that contain at least one elementary cell of $M\mathbb{Z}^d$, all $V \in L^\infty(\Gamma)$ and all*

$$0 \neq \phi \in \mathcal{D}\left(\mathrm{e}^{\frac{\kappa}{2}\sqrt{H_+}}\right),$$

we have

$$\|\phi\|_{L^2(S_{\delta,Z}(\Gamma))} \geq \left(\frac{\delta}{M}\right)^{C(1+M^{4/3}\|V\|_\infty^{2/3}+\ln D(\phi))} \|\phi\|_{L^2(\Gamma)}$$

where

$$D(\phi) = \frac{\|\mathrm{e}^{\frac{\kappa}{2}\sqrt{H_+}}\phi\|^2_{L^2(\Gamma)}}{\|\phi\|^2_{L^2(\Gamma)}}.$$

We already explained above that Theorem 3.16 is a special case of Theorem 3.17. Let us now explain how Theorems 3.9 and 3.13 follow from Theorem 3.17: Let $M > 0$ and set $\kappa := 18\mathrm{e}\sqrt{d}M$. Then, for every $E \in \mathbb{R}$, we have

$$\operatorname{Ran} P_H(E) \subset \mathcal{D}\left(\mathrm{e}^{\frac{\kappa}{2}\sqrt{H_+}}\right),$$

see Subsection 2.2, and

$$\ln\left(\frac{\|\mathrm{e}^{\frac{\kappa}{2}\sqrt{H_+}}\phi\|^2_{L^2(\Gamma)}}{\|\phi\|^2_{L^2(\Gamma)}}\right) \leq \ln\left(\frac{\mathrm{e}^{\kappa\sqrt{E}}\|\phi\|^2_{L^2(\Gamma)}}{\|\phi\|^2_{L^2(\Gamma)}}\right) \leq 18\mathrm{e}\sqrt{d}M\sqrt{E}.$$

Hence, Theorems 3.9 or 3.13, respectively, follow with C replaced by $18\mathrm{e}\sqrt{d}C$.

3.3. Discussion on optimality

Before turning to the proof of Theorem 3.17 in Section 3.4, let us make some remarks on optimality of the above results. Parts of this subsection are based on parts of [Täu17, TT17].

3.3.1. Dependence on energy

We first argue that the bounds in Theorems 3.9, and 3.13 are sharp in terms of the energy dependence $\exp(\sqrt{E})$. For that purpose, we cite [LRL12, Proposition 5.5], see also [JL99, Proposition 14.9] for an earlier, but less explicit version.

Proposition 3.18 ([LRL12, Proposition 5.5])**.** *Let Ω be a bounded, open set in $\mathbb{R}^d$, denote by $-\Delta$ the negative Laplacian on Ω with Dirichlet or Neumann boundary conditions and let $\omega \subset \Omega$ be non-empty and open with $\overline{\omega} \neq \Omega$. Then there exist $C > 0$ and $E_0 > 0$, such that for all $E \geq E_0$ there exists a function $\phi \in \operatorname{Ran} P_{-\Delta}(E)$ with*

$$\|\phi\|^2_{L^2(\Omega)} \geq C\exp(CE^{1/2})\|\phi\|^2_{L^2(\omega)}. \tag{8}$$

Comparing this with our estimate in Theorem 3.9

$$\|\phi\|^2_{L^2(S_{\delta,Z}(\Lambda_L))} \geq \left(\frac{\delta}{M}\right)^{C(1+M^{4/3}\|V\|_\infty^{2/3}+M\sqrt{E})}\|\phi\|^2_{L^2(\Lambda_L)} \geq c\exp(cE^{1/2})\|\phi\|^2_{L^2(\Lambda_L)},$$

and choosing $\Omega = \Lambda_L$, $\omega = S_{\delta,Z}(\Lambda_L)$, we see that the order $\exp(\sqrt{E})$ is sharp and cannot be improved.

However, there exist other approaches to weaken or even get rid of the E-dependence in the bound

$$\left(\frac{\delta}{M}\right)^{C(1+M^{4/3}\|V\|_\infty^{2/3}+ME^{1/2})}$$

For instance, in [EV16] it has been suggested to consider spectral projectors corresponding to closeby eigenvalues, i.e. $\phi \in \operatorname{Ran} P_H([E-w,E])$ for some $w > 0$ instead of $\phi \in \operatorname{Ran} P_H(E)$. In [EV16], the authors provide some indications to possibly prove the following: Fix L, δ, and a function $w(E)$, tending sufficiently fast to 0 as $E \to \infty$. Then for every $E \geq 0$, we have

$$\|\phi\|^2_{L^2(S_{\delta,Z}(\Lambda_L))} \geq C\|\phi\|^2_{L^2(\Lambda_L)} \quad \text{for all } \phi \in \operatorname{Ran} P_H([E - w(E), E]) \tag{9}$$

with a uniform $C > 0$. There are even hints that in some situations and for small dimensions one might be able to choose $w(E)$ constant, see [EV16] and the references therein. However, the L-dependence and – more crucially – the δ-dependence are non-explicit in this reasoning. One question is whether one can still recover a polynomial dependence in δ as in Theorems 3.9 and 3.13 since this polynomial dependence turned out to be crucial in our application to Wegner estimates in Section 5. This encouraged Egidi and Veselić to ask a question, a special case of which we cite here:

Question (Special case of [EV16, Question 3])**.** *Let $L > 0$, $V \in L^\infty(\Lambda_L)$, and fix $w \in (0, \infty)$. Denote by H the operator $-\Delta + V$ in $L^2(\Lambda_L)$ with periodic boundary conditions. Is there a constant M, which may depend on w and V, such that for all $E \in \mathbb{R}$, all $0 < \delta < L/2$, and all $f \in \operatorname{Ran} P_H[E - w, E]$, the estimate*

$$\int_{B(\delta)} |f|^2 \geq \delta^M \int_{\Lambda_L} |f|^2.$$

holds true?

In [Täu17], we showed that in dimension $d \geq 2$, the answer to this question in no.

Theorem 3.19. *The answer to Question 3 in [EV16] is no in dimension $d \geq 2$.*

The proof of Theorem 3.19 can be found in Appendix A.1. It relies on the observation that the spectral subspaces $\operatorname{Ran} P_{-\Delta}([E - w, E])$ can have arbitrarily high dimension which allows to construct counterexamples. In fact, a similar idea an be found in [Kir87]. In dimensions $d \geq 3$, this high dimensionality of subspaces can be deduced from Weyl asymptotics. However, in order to also cover the case $d = 2$, we pursue a different approach and use a number theoretic argument to find eigenvalues of high multiplicity. As a byproduct of identifying high dimensional eigenvalues, it follows that even if the subspaces $\operatorname{Ran} P_{-\Delta+V}([E - w, E])$ are replaced by $\operatorname{Ran} P_{-\Delta+V}([E - w(E), E])$ with $w(E) \to 0$ as $E \to \infty$, the answer will still be no. In conclusion, we see that a relaxation of the assumption $\phi \in \operatorname{Ran} P_H(E)$ of Theorems 3.9 and 3.13 in the spirit of (9) is not possible without loosing the polynomial δ-dependence.

We now turn to Theorems 3.16, and 3.17. They establish that all functions in the domain of operators of the form $\exp(\sqrt{H_+})$, H being a Schrödinger operator, have a quantifiable rigidity. An interesting question is whether one can still expect a result as in Theorems 3.16 or 3.17 if the condition $\phi \in \mathcal{D}(\mathrm{e}^{\kappa\sqrt{H_+}})$ is replaced by $\phi \in \mathcal{D}(H_+^n)$ for some $n \in \mathbb{N}$, i.e. if we replace exponential integrability of the spectral measure $\mathrm{d}\langle \phi, P_H(\cdot), \phi \rangle$ by polynomial integrability. We now show that this is also not possible. The following lemma shows that in this situation, every $\phi \in C_0^\infty(\Lambda_L)$ is in some $\mathcal{D}(H_+^n)$.

Lemma 3.20 ([TT17, Lemma 4.1])**.** *Let $L > 0$, $V \equiv 0$, $\kappa > 0$, $\phi \in C_0^\infty(\Lambda_L)$. Denote by λ_k and e_k, $k \in \mathbb{N}$ the eigenvalues and corresponding normalized eigenfunctions of $-\Delta$ on $L^2(\Lambda_L)$ and write $\phi = \sum_k \alpha_l e_k$. Then there is $C = C(\phi, L, \kappa) > 0$ such that*

$$\sum_{k\in\mathbb{N}} |\lambda_k|^\kappa |\alpha_k|^2 < C.$$

The proof of Lemma 3.20 can be found in Appendix A.2. Choosing $\phi \in C_0^\infty(\Lambda_L)$ non-zero and vanishing on $S_{\delta,Z}(\Lambda_L)$, we see that the function ϕ satisfies

$$\sum_{k\in\mathbb{N}} |\lambda_k|^\kappa |\alpha_k|^2 \leq D \|\phi\|^2_{L^2(\Lambda_L)}$$

with $D := C/\|\phi\|^2_{L^2(\Lambda_L)}$, but not $\|\phi\|^2_{L^2(S_{\delta,Z}(\Lambda_L))} \geq C_{\mathrm{sfuc}} \|\phi\|^2_{L^2(\Lambda_L)}$. Therefore:

Corollary 3.21. *Under the assumptions of Theorems 3.16 and 3.17, one cannot replace the condition $\phi \in \mathcal{D}(\mathrm{e}^{\kappa\sqrt{H_+}})$ by $\phi \in \mathcal{D}(H_+^n)$ for any $n \in \mathbb{N}$.*

3.3.2. The term $\|V\|_\infty^{2/3}$

We now discuss whether the term $\|V\|_\infty^{2/3}$, appearing in the exponent in Theorems 3.9, 3.13, 3.16, and 3.17 can be improved. One can at best hope to reduce this to $\|V\|_\infty^{1/2}$, since if the exponent was smaller than 1/2, then constant potentials (which correspond to an energy shift) would lead to a contradiction to the optimal energy dependence $\exp(\sqrt{E})$ discussed in the previous subsection.

Let us discuss where the term $\|V\|_\infty^{2/3}$ comes from. Looking at the proof of Theorem 3.17, more precisely at (45), we see that the $\|V\|_\infty^{2/3}$ contribution stems from the fact that the parameters α_1 and β_2 are chosen of order $\|V\|_\infty^{2/3}$. From now on, we focus on the parameter α_1 which is due to the Carleman estimate in Proposition 3.28. The parameter β_2 originates from another Carleman estimate, Proposition 3.27, and leads to an analogous discussion. The reason for choosing α_1 of order $\|V\|_\infty^{2/3}$ is that in the proof of Lemma 3.31 a term with a factor $\|V\|_\infty^2$ has to be absorbed on the left hand side of (24). Since the corresponding term on the left hand side of (24) has an α^3 prefactor, stemming from Proposition 3.28, the choice $\alpha \geq \alpha_1 \sim \|V\|_\infty^{2/3}$ is required. In summary, we can say that the exponent 2/3 corresponds to the relation between the prefactor α^3 on the left hand side of the Carleman estimate, Proposition 3.27, and the fact that $|\Delta\Phi|^2 = |V\Phi|^2 \sim \|V\|_\infty^2$.

So, one thinkable venue how to improve the term $\|V\|_\infty^{2/3}$ would be attempting to increase the parameter α^3 on the left hand side of Proposition 3.28 to α^θ, $\theta \in (3, 4]$. In this case, the term $\|V\|_\infty^{2/3}$ in the exponent of our main results would improve to $\|V\|_\infty^{2/\theta}$.

Unfortunately, this would be a futile task. In fact, an improved Carleman estimate would also lead to an improvement of [BK05, Lemma 3.10] which would itself contradict a counterexample by Meshkov [Mes92]. In fact, one can prove the following meta-theorem:

Proposition 3.22. *Assume that an improved variant of the the Carleman estimate Proposition 3.27 holds where α^3 on the left hand side has been replaced by α^θ for some $\theta > 0$. Then, for all $u, V \in L^\infty(\mathbb{R}^d)$ satisfying $\Delta u = Vu$ in $\mathbb{R}^d$, and $u(0) = 1$, all $R > 1$, and all $x_0 \in \mathbb{R}^d$ with $|x_0| = R$ we would have*

$$\max_{|x-x_0|\leq 1} |u(x)| > C \exp(-C \log(R) R^{4/\theta}), \quad \textit{for some} \quad C > 0. \tag{10}$$

Proof. One can follow literally the proof of [BK05, Lemma 3.10] with the obvious modification that $\alpha \sim 4/3$ needs to be replaced by $\alpha \sim R^{4/\theta}$. □

However, in [Mes92, Section 2], Meshkov gave an example of functions $u, V \in L^\infty(\mathbb{R}^2, \mathbb{C}) \cap L^2(\mathbb{R}^2, \mathbb{C})$ which satisfy $u(0) \neq 0$, $\Delta u = Vu$ and

$$\max_{|x|\geq R} |u(x)| \leq C \exp(-C R^{4/3}). \tag{11}$$

Comparing (10) and (11), we deduce the following corollary:

Corollary 3.23. *An improved variant of the Carleman estimate, Proposition 3.27, where the exponent α^3 on the left hand side is replaced by α^θ for some $\theta > 3$ cannot hold.*

This illustrates the limitations of the Carleman approach and explains why new ideas will be required in hope of improving the term $\|V\|_\infty^{2/3}$. There is one silver lining here: Meshkov's examples relies on a *complex-valued* potential V. Since we have a real-valued potential V, in our situation the problem enjoys additional structure which might be exploited in the future.

3.3.3. Relation between κ and M

In Theorems 3.16 and 3.17, the parameters κ (decay rate of prefactors of high energy modes) and M (grid size) must satisfy $M/\kappa \leq 18\mathrm{e}\sqrt{d}$. At first sight, this fits the intuition of uncertainty principles: delocalization in momentum space (large κ) corresponds to localization in position space, i.e. a fine grid (small M) is required in order to obtain an estimate as in Theorems 3.16 and 3.17. It also seems that this condition on M and κ occurs naturally when using Carleman estimates. In fact, a similar assumption is required in an analog result for solutions of variable coefficient second order elliptic operators with Lipschitz continuous coefficients, see [BTV15]. There, the Lipschitz constant assumes (on a technical level) the role of $1/\kappa$ from our setting. Thus, our condition corresponds into a smallness condition on the Lipschitz constant in the main result of [BTV15]. Of course, the factor $18\mathrm{e}\sqrt{d}$ seems somewhat arbitrary and it could be slightly improved by optimizing the covering arguments in the proof of Proposition 3.33 below. Since we do not believe that this will yield anything optimal, we refrained from doing so.

However, instead of considering a *quantitative* tweaking of the factor $18\mathrm{e}\sqrt{d}$, one could ask if the relation between κ and M can be *qualitatively* improved, i.e. if a quantitative unique continuation principle as in Theorems 3.16 and 3.17 holds for every pair (κ, M). An indication for this is Proposition 5.6 in [RL12] where the following statement is proved in the special case $V \equiv 0$: *Let $\omega \subset \Lambda_L$ be open and $\kappa > 0$. Then for all functions $u = \sum_{k\in\mathbb{N}} \alpha_k \phi_k$ with $|\alpha_k| \leq \exp(-\kappa\sqrt{E_k})$, $k \in \mathbb{N}$, we have $u \equiv 0$ if $u|_\omega \equiv 0$.* Even though it would be possible without effort to turn this qualitative into a quantitative statement of the form

$$\|u\|^2_{L^2(\omega)} \geq C\|u\|^2_{L^2(\Lambda_L)},$$

the method in [RL12] does not provide any control over the constant C in terms of δ, L, and κ, which is of special interest. Thus, the question, if the relation $M/\kappa \leq 18\mathrm{e}\sqrt{d}$ in Theorems 3.16, and 3.17 can be dropped, remains open.

3.4. Proof of Theorem 3.17

We will first prove Theorem 3.17 in the special case where $M = 1$ and where ϕ is subject to an energy cutoff, i.e.

$$\phi \in \operatorname{Ran} P_H(E) \cap \mathcal{D}\left(\mathrm{e}^{\kappa\sqrt{H_+}}\right) = \operatorname{Ran} P_H(E)$$

for some $E \geq 0$. In Section 3.4.5 we will deduce the general case.

3.4.1. Ghost dimension and the function Φ

In this section, we take a function $\phi \in \operatorname{Ran} P_H(E)$ and turn it into a V-harmonic function Φ by extending it to an additional dimension. This is a technique, inspired by [RL12, Theorem 5.4], see also [JL99], and allows to directly apply the Carleman formalism to the function Φ. We will define the function Φ via spectral calculus whence it is convenient to first describe the construction in an abstract setting.

Let $\mathcal{H}$ be a Hilbert space and A a self-adjoint operator on $\mathcal{H}$ with domain $\mathcal{D}(A)$. We define a family of operators $(\mathcal{F}_t)_{t\in\mathbb{R}}$ on $\mathcal{H}$ as

$$\mathcal{F}_t = \int_{-\infty}^{\infty} s_t(\lambda) \mathrm{d}P_A(\lambda) \quad \text{where} \quad s_t(\lambda) = \begin{cases} \sinh(\sqrt{\lambda}t)/\sqrt{\lambda}, & \lambda > 0, \\ t, & \lambda = 0, \\ \sin(\sqrt{-\lambda}t)/\sqrt{-\lambda}, & \lambda < 0. \end{cases}$$

The operators $\mathcal{F}_t$ are self-adjoint operators with $\operatorname{Ran} P_A([a,b]) \subset \mathcal{D}(\mathcal{F}_t)$ for $-\infty < a < b < \infty$. For $\phi \in P_A([a,b])$ we define the function $\Phi\colon \mathbb{R} \to \mathcal{H}$ as

$$\Phi(t) = \mathcal{F}_t \phi.$$

Fix $T > 0$ and define the operator $\hat{A}$ in $L^2((-T,T);\mathcal{H})$ on $\mathcal{D}(\hat{A}) = \{\Psi\colon t \mapsto A(\Psi(t)) - (\partial_t^2\Psi)(t) \in L^2((-T,T);\mathcal{H})\}$ by

$$(\hat{A}\Phi)(t) = A(\Phi(t)) - (\partial_t^2\Phi)(t).$$

where $\partial_t^2\Phi$ denotes the second $\mathcal{H}$-derivative with respect to t.

Lemma 3.24. *For all $a, b \in \mathbb{R}$ with $a < b$ and all $\phi \in \operatorname{Ran} P_A([a,b])$ we have:*

(i) The map $\mathbb{R} \ni t \mapsto \Phi(t) \in \mathcal{H}$ is infinitely $\mathcal{H}$-differentiable. In particular,

$$(\partial_t\Phi)(0) = \phi. \tag{12}$$

(ii) For all $T > 0$ we have $\Phi \in \mathcal{D}(\hat{A})$ and

$$\hat{A}\Phi = 0. \tag{13}$$

Proof of Lemma 3.24. We aim to identify the $L^2(\mathbb{R}^d)$-derivative and calculate by a standard application of the dominated convergence theorem

$$\lim_{h\to 0}\left\| \int_a^b \left(\frac{s_{t+h}(\lambda) - s_t(\lambda)}{h} - \partial_t s_t(\lambda) \right) \mathrm{d}P_A(\lambda)\phi \right\|_{\mathcal{H}}^2 = \lim_{h\to 0} \int_a^b \left| \frac{s_{t+h}(\lambda) - s_t(\lambda)}{h} - \partial_t s_t(\lambda) \right|^2 \mathrm{d}\langle \phi, P_A(\lambda)\phi\rangle = 0.$$

The calculation for higher derivatives is analogous and we find for $k \in \mathbb{N}_0$ that

$$(\partial_t^k \Phi)(t) = \left(\int_a^b \partial_t^k s_t(\lambda) \mathrm{d}P_A(\lambda) \right) \phi \in \mathcal{H}.$$

Since A is self-adjoint and $\mathcal{F}_t P_A([a,b])$ is bounded, the operator $A\mathcal{F}_t P_A([a,b])$ is closed. For part (ii) we infer from [Sch12, Theorem 5.9] that

$$A\mathcal{F}_t P_A([a,b])\phi = \overline{A\mathcal{F}_t P_A([a,b])}\phi = \left(\int_a^b \lambda s_t(\lambda) \mathrm{d}P_A(\lambda) \right) \phi \in \mathcal{H},$$

which implies that $\phi \in \mathcal{D}(A\mathcal{F}_t P_A([a,b]))$. Hence $\mathcal{F}_t P_A([a,b])\phi = \Phi(t) \in \mathcal{D}(A)$. We then calculate using $\lambda s_t(\lambda) - \partial_t^2 s_t(\lambda) = 0$

$$\int_{-T}^T \|A(\Phi(t)) - (\partial_t^2 \Phi)(t)\|_{\mathcal{H}}^2 \mathrm{d}t = \int_{-T}^T \int_a^b |\lambda s_t(\lambda) - \partial_t^2 s_t(\lambda)|^2 \mathrm{d}\|P_A(\lambda)\phi\|^2 \mathrm{d}t = 0 \quad \square$$

Remark 3.25. We will apply Lemma 3.24 with $\mathcal{H} = L^2(\Gamma)$, $A = H$, $a = -\|V\|_\infty$, and $b = E \geq 0$. Then $L^2((-T,T);\mathcal{H}) = L^2(\Gamma \times (-T,T))$, i.e. Φ can be understood as a map from $\Gamma \times (-T,T)$ to $\mathbb{C}$ and we have

$$\hat{A} = -\Delta + V \quad \text{on} \quad L^2(\Gamma \times (-T,T))$$

with corresponding boundary conditions on $(\partial\Gamma) \times (-T,T)$ where we extended V constantly to the extra dimension, i.e. $V(x,t) = V(x)$ for all $t \in (-T,T)$. Since $\Phi \in \mathcal{D}(\hat{A})$ by Lemma 3.24, we find $\Phi \in H^1(\Gamma \times (-T,T))$.

In case where $\Gamma \neq \mathbb{R}^d$, we define extensions of V and Φ from $\Gamma \times (-T,T)$ to the supersets $\mathbb{R}^d \times (-T,T)$, and $\Gamma^{(k)} \times (-T,T)$, $k \in \mathbb{N}$ where

$$\Gamma^{(k)} := \bigtimes_{i=1}^d (a_i - k \cdot (b_i - a_i), b_i + k \cdot (b_i - a_i)).$$

We first extend V and Φ to $\mathbb{R}^d \times (-T,T)$ and then possibly restrict to $\Gamma^{(k)} \times (-T,T)$. The way these extensions are defined depends on the type of boundary conditions:

- In case of *Dirichlet boundary conditions*, V is extended by symmetric and Φ by antisymmetric reflections on the boundary surfaces $\{x_i = a_i\}$ and $\{x_i = b_i\}$ whenever a_i or b_i are finite.

- In case of *Neumann boundary conditions*, both V and Φ are extended by symmetric reflections on the boundary surfaces $\{x_i = a_i\}$ and $\{x_i = b_i\}$ whenever a_i or b_i are finite.

- In case of *periodic boundary conditions*, both V and Φ are extended periodically in every direction where both a_i and b_i are finite.

Iterating this procedure yields extensions to $\mathbb{R}^d \times (-T,T)$ which we then restrict to $\Gamma^{(k)} \times (-T,T)$. We will use the same symbol for the original as well as the extended V and Φ. By construction, we have

$$(-\Delta + V)\Phi = 0 \quad \text{on} \quad \Gamma^{(k)} \times (-T,T). \tag{14}$$

The extended Φ is in $H^2(\Gamma^{(k)} \times (-T,T))$ and satisfies the corresponding boundary conditions on $\partial\Gamma^{(k)}$.

Definition 3.26. Given a $(1,\delta)$-equidistributed sequence Z and a generalized rectangle Γ which contains at least one elementary cell of $\mathbb{Z}^d$, let

$$\hat{S}_{\delta,Z}(\Gamma) = \cup_{j \in \mathbb{Z}^d \colon \Lambda_1(j) \subset \Gamma} \{z_j\}.$$

Obviously,

$$S_{\delta,Z}(\Gamma) = \cup_{z \in \hat{S}_{\delta,Z}(\Gamma)} B(z,\delta).$$

3.4.2. Carleman and doubling estimates

In this section, we state two Carleman estimates, Propositions 3.27 and 3.28, and deduce local doubling estimates for the function Φ. They will ultimately play different roles in the proof of Theorem 3.17: While Proposition 3.28 will perform the actual unique continuation step for the function Φ, Proposition 3.27 contains a boundary term which is going to serve as a ticket back to $\mathbb{R}^d$ and back to the original function ϕ by exploiting the relation $\partial_{d+1}\Phi \mid_{t=0} = \phi$.

Since we aim for a quantitative result, we need to keep control over the model parameters. Therefore, it is paramount to have explicit expressions for the corresponding weight functions and carefully keep track of the geometry of their level sets. This requires particular choices of two Carleman estimates. Let us now fix some notation: We denote by $\mathbb{R}^{d+1}_+ := \{(x,t) \in \mathbb{R}^{d+1} \colon t \geq 0\}$ the $d+1$-dimensional half-space and by $B_r^+ := \{(x,t) \in \mathbb{R}^{d+1}_+ \colon |(x,t)| < r\}$ the $d+1$-dimensional half-ball. For functions $F \in C^\infty(\mathbb{R}^{d+1}_+)$ we use the notation $F_0 = F|_{t=0}$.

In the appendix of [LR95], Lebeau and Robbiano state a Carleman estimate for complex-valued functions with support in B_r^+ by using a real-valued weight function $u \in C^\infty(\mathbb{R}^{d+1})$ satisfying the two conditions

$$\forall (x,t) \in B_r^+ \colon \quad (\partial_{d+1} u)(x,t) \neq 0, \tag{15}$$

and for all $\xi \in \mathbb{R}^{d+1}$ and $(x,t) \in B_r^+$ the

$$\left.\begin{aligned} 2\langle \xi, \nabla u\rangle &= 0 \\ |\xi|^2 &= |\nabla u|^2 \end{aligned}\right\} \quad \Rightarrow \quad \sum_{j,k=1}^{d+1} (\partial_{jk} u)\big(\xi_j \xi_k + (\partial_j u)(\partial_k u)\big) > 0. \tag{16}$$

As in [JL99] we choose $r < 2 - \sqrt{2}$ and the special weight function $u : \mathbb{R}^{d+1} \to \mathbb{R}$,

$$u(x,t) = -t + \frac{t^2}{2} - \frac{|x|^2}{4}. \tag{17}$$

Note that $u(x,t) \leq 0$ for all $(x,t) \in B_2^+$. This function u indeed satisfies the assumptions (15) and (16). Condition (15) is trivial for $r < 1$. In order to show the implication (16) we show

$$|\xi|^2 = |\nabla u|^2 \;\Rightarrow\; \sum_{j,k=1}^{d+1} (\partial_{jk} u)(\xi_j \xi_k + (\partial_j u)(\partial_k u)) > 0. \tag{18}$$

We use the hypothesis of (18) and calculate

$$\begin{aligned} \sum_{j,k=1}^{d+1} \partial_{jk} u(\xi_j \xi_k + \partial_j u \partial_k u) &= -\frac{1}{2} \sum_{i=1}^{d} \xi_i^2 + \xi_{d+1}^2 - \frac{1}{8}|x|^2 + (t-1)^2 \\ &= \frac{3}{2}\xi_{d+1}^2 - \frac{1}{4}|x|^2 + \frac{1}{2}(t-1)^2. \end{aligned}$$

Since $|x|^2 \leq r^2$ and $(t-1)^2 \geq (1-r)^2$, assumption (18) is satisfied if $r < 2 - \sqrt{2}$. Now let

$$\begin{aligned} &C_{c,0}^\infty(B_r^+) = \big\{F : \mathbb{R}_+^{d+1} \to \mathbb{C} \colon F \equiv 0 \text{ on } \{t = 0\}, \\ &\quad \exists \phi \in C^\infty(\mathbb{R}^{d+1}) \text{ with } \operatorname{supp}\phi \subset \{(x,t) \in \mathbb{R}^{d+1} \colon |(x,t)| < r\} \text{ and } F \equiv \phi \text{ on } \mathbb{R}_+^{d+1}\big\}. \end{aligned}$$

Hence, using that $F \equiv 0$ on $\{t = 0\}$ we deduce the following Carleman estimate with a boundary term as a corollary of Proposition 1 in the appendix of [LR95].

Proposition 3.27 (Simplified version of [LR95], Proposition 1 in the Appendix)**.** *Let $u \in C^\infty(\mathbb{R}^{d+1}, \mathbb{R})$ be as in Eq. (17) and $\rho \in (0, 2-\sqrt{2})$. Then there are constants $\beta_0, C_1 \geq 1$ such that for all $\beta \geq \beta_0$, and all $F \in C_{c,0}^\infty(B_\rho^+)$ we have*

$$\int_{\mathbb{R}^{d+1}} \mathrm{e}^{2\beta u} \left(\beta |\nabla F|^2 + \beta^3 |F|^2\right) \leq C_1 \left(\int_{\mathbb{R}^{d+1}} \mathrm{e}^{2\beta u} |\Delta F|^2 + \beta \int_{\mathbb{R}^d} \mathrm{e}^{2\beta u_0} |(\partial_{d+1} F)_0|^2 \right).$$

We will need another Carleman estimate with a weight function the level sets of which can be explicitly controlled.

Proposition 3.28 ([EV03, BK05, NRT15, Dav14, KT16])**.** *Let $\rho > 0$ and define $v\colon \mathbb{R}^d \to \mathbb{R}, v(x) = \frac{|x|}{\rho}\int_0^{|x|/\rho} \frac{1-e^{-t}}{t}\,dt$. In particular,*

$$\forall x \in B(\rho)\colon \frac{|x|}{\rho \mathrm{e}} \leq v(x) \leq \frac{|x|}{\rho}.$$

Then there are constants $\alpha_0, C_2 \geq 1$, depending only on the dimension, such that for all $\alpha \geq \alpha_0$, and all $G \in H^2(\mathbb{R}^{d+1})$ with support in $B(\rho) \setminus \{0\}$ we have

$$\int_{\mathbb{R}^{d+1}} \left(\alpha \rho^2 v^{1-2\alpha} |\nabla G|^2 + \alpha^3 v^{-1-2\alpha} |G|^2\right) \mathrm{d}x \leq C_2 \rho^4 \int_{\mathbb{R}^{d+1}} v^{2-2\alpha} \left|\Delta G\right|^2 \mathrm{d}x.$$

While this Carleman estimate can essentially be found in [EV03] and [BK05], the variants stated therein are not quite sufficient for our purpose. The estimate in [EV03] lacks a quantitative statement about the admissible functions G, while in [BK05] the gradient term on the left hand side is missing. Even though it would be possible to modify the proofs of [EV03, BK05] without employing new techniques, we cite [NRT15] for a complete proof. We also mention that this estimate is stated without proof in [KT16, Lemma 2.1] and that also [Dav14] contains a Carleman estimate which is less explicit than Proposition 3.28, but would still be sufficient for our purpose.

Remark 3.29. Note that it would be possible to prove Theorem 3.17 solely relying on Proposition 3.27. However, the geometry of the level sets of the weight function in Proposition 3.27 is more complicated (hyperbolas) than in Proposition 3.28 (balls). Thus, using only Proposition 3.27 would unnecessarily complicate later steps of the proof without improving the final result.

We now use the Carleman inequalities, Propositions 3.27, and 3.28 to deduce two doubling estimates for the function Φ in a very precise geometric setting. To do so, we introduce some more notation. For $\delta \in (0, 1/2)$, we set

$$\begin{aligned} u_1 &= -\delta^2/16, & u_2 &= -\delta^2/8, & u_3 &= -\delta^2/4, \\ r_1 &= \frac{1}{2} - \frac{1}{8}\sqrt{16 - \delta^2}, & r_2 &= 1, & r_3 &= 8.5\mathrm{e}\sqrt{d}, \\ R_1 &= 1 - \frac{1}{4}\sqrt{16 - \delta^2}, & R_2 &= 8\sqrt{d}, & R_3 &= 9\mathrm{e}\sqrt{d}, \end{aligned}$$

and define for $i \in \{1, 2, 3\}$ the sets

$$\mathcal{U}_i := \left\{(x,t) \in \mathbb{R}^{d+1} \colon u(x,t) > u_i, t \in [0,1]\right\} \subset \mathbb{R}^{d+1}_+$$

and

$$\mathcal{V}_i := B(R_i) \setminus \overline{B(r_i)} \subset \mathbb{R}^{d+1}.$$

The sets $\mathcal{U}_i$ are cusps of hyperbolas while the sets $\mathcal{V}_i$ are annuli. Furthermore, we have

$$\mathcal{U}_1 \subset \mathcal{U}_2 \subset \mathcal{U}_3 \subset B_\delta^+ \subset \mathbb{R}_+^{d+1}.$$

Lemma 3.30. *For all $\delta \in (0, 1/2)$, all $V \in L^\infty(\mathbb{R}^d)$, all $\Phi \in H^2(\mathcal{U}_3)$ with $\Delta\Phi(x,t) = V(x)\Phi(x,t)$, we have*

$$\mathrm{e}^{2\beta u_1} \|\Phi\|_{H^1(\mathcal{U}_1)}^2 \leq 24 C_1 \Theta_1^2 \mathrm{e}^{2\beta u_2} \|\Phi\|_{H^1(\mathcal{U}_3)}^2 + 2C_1 \|(\partial_{d+1}\Phi)_0\|_{L^2(B(0,\delta))}^2$$

for all $\beta \geq \beta_1$ where β_1 is given in Eq. (21), C_1 is the constant from Proposition 3.27 and Θ_1 is given in Eq. (19).

Proof. We choose a cutoff function $\chi \in C^\infty(\mathbb{R}^{d+1}, [0,1])$ which is symmetric with respect to the $d+1$-st coordinate, $\operatorname{supp} \chi \cap \mathbb{R}_+^{d+1} \subset \overline{\mathcal{U}_3}$, $\chi(x) = 1$ if $x \in \mathcal{U}_2$ and

$$\max\{\|\Delta\chi\|_\infty, \||\nabla\chi|\|_\infty\} \leq \frac{\tilde{\Theta}_1}{\delta^4} =: \Theta_1, \tag{19}$$

where $\tilde{\Theta}_1 = \tilde{\Theta}_1(d)$ depends only on the dimension. This is due to the fact that the distance of $\mathcal{U}_2$ and $\mathbb{R}_+^{d+1} \setminus \mathcal{U}_3$ is bounded from below by $\delta^2/16$, see Appendix B. We want to apply Proposition 3.27 with $F = \chi\Phi$. However, the restricion of $\chi\Phi$ onto $\mathbb{R}_+^{d+1}$ is not in $C_{c,0}^\infty(B(\rho))$ but merely in $H^2(\mathbb{R}_+^{d+1})$ and we need to approximate it appropriately.

Since $\chi\Phi \in H^2(\mathbb{R}^d)$ there is a sequence $(F_n)_{n\in\mathbb{N}}$ of $C_0^\infty(\mathbb{R}^{d+1})$-functions such that $F_n \to \chi\Phi$, $\nabla F_n \to \nabla(\chi\Phi)$, $\Delta F_n \to \Delta(\chi\Phi)$ in $L^2(\mathbb{R}^{d+1})$ and $(\partial_{d+1}F_n)_0 \to \partial_{d+1}(\chi\Phi)_0$ in $L^2(\mathbb{R}^d)$, cf. [Eva98, Chapter 5.3 for approximation and Chapter 5.5 for convergence of the traces on $\mathbb{R}^d$]. The function $\chi\Phi$ is symmetric with respect to the $d+1$-st coordinate whence we may assume that the F_n are also symmetric (else replace $F_n(\cdot, x_{d+1})$ by the symmetrized $(F_n(\cdot, x_{d+1}) + F_n(\cdot, -x_{d+1}))/2$). We may also assume that $\operatorname{supp} F_n \subset B(\delta)$ whence the restrictions of F_n onto $\mathbb{R}_+^{d+1}$ are in $C_{c,0}^\infty(B_\delta^+)$. Thus, we can apply Proposition 3.27 with $F = F_n$ and $\rho = 1/2$, take the limit $n \to \infty$ and obtain for all $\beta \geq \beta_0 \geq 1$

$$\begin{aligned} \int_{\mathcal{U}_3} \mathrm{e}^{2\beta u} \left(\beta |\nabla(\chi\Phi)|^2 + \beta^3 |\chi\Phi|^2\right) \leq C_1 \int_{\mathcal{U}_3} \mathrm{e}^{2\beta u} |\Delta(\chi\Phi)|^2 + \\ + \beta C_1 \int_{B(\delta)} \mathrm{e}^{2\beta u_0} |(\partial_{d+1}(\chi\Phi))_0|^2. \end{aligned} \tag{20}$$

Note that β_0 and C_1 only depend on the dimension. For the first summand on the right hand side we have the upper bound

$$\begin{aligned}\int_{\mathcal{U}_3} e^{2\beta u}|\Delta(\chi\Phi)|^2 &\leq 3\int_{\mathcal{U}_3} e^{2\beta u}\left(4|\nabla\chi|^2|\nabla\Phi|^2 + |\Delta\chi|^2|\Phi|^2 + |\Delta\Phi|^2|\chi|^2\right)\\ &\leq 3e^{2\beta u_2}\int_{\mathcal{U}_3\setminus\mathcal{U}_2}\left(4\Theta_1^2|\nabla\Phi|^2 + \Theta_1^2|\Phi|^2\right) + \int_{\mathcal{U}_3} 3e^{2\beta u}|V\Phi\chi|^2\\ &\leq 12\Theta_1^2 e^{2\beta u_2}\|\Phi\|_{H^1(\mathcal{U}_3)}^2 + 3\|V\|_\infty^2\int_{\mathcal{U}_3} e^{2\beta u}|\chi\Phi|^2.\end{aligned}$$

The second summand is bounded from above by $\beta C_1\int_{B(\delta)}|(\partial_{d+1}\Phi)_0|^2$, since $\Phi = 0$ and $u \leq 0$ on $\{x_{d+1} = 0\}$. Hence,

$$\begin{aligned}\beta\int_{\mathcal{U}_3} e^{2\beta u}|\nabla(\chi\Phi)|^2 + (\beta^3 - 3\|V\|_\infty^2 C_1)\int_{\mathcal{U}_3} e^{2\beta u}|\chi\Phi|^2&\\ \leq 12C_1\Theta_1^2 e^{2\beta u_2}\|\Phi\|_{H^1(\mathcal{U}_3)}^2 + C_1\beta\|(\partial_{d+1}\Phi)_0\|_{L^2(B(\delta))}^2.&\end{aligned}$$

Additionally to $\beta \geq \beta_0$ we choose $\beta \geq (6\|V\|_\infty^2 C_1)^{1/3} =: \tilde{\beta}_0$. This ensures that for all

$$\beta \geq \beta_1 := \max\{\beta_0, \tilde{\beta}_0\} \tag{21}$$

we have

$$\frac{1}{2}\int_{\mathcal{U}_3} e^{2\beta u}\left(\beta|\nabla(\chi\Phi)|^2 + \beta^3|\chi\Phi|^2\right) \leq 12C_1\Theta_1^2 e^{2\beta u_2}\|\Phi\|_{H^1(\mathcal{U}_3)}^2 + C_1\beta\|(\partial_{d+1}\Phi)_0\|_{L^2(B(\delta))}^2.$$

Since $\beta \geq 1$, $\mathcal{U}_3 \supset \mathcal{U}_1$, $\chi = 1$ and $e^{2\beta u} \geq e^{2\beta u_1}$ on $\mathcal{U}_1$, we obtain

$$e^{2\beta u_1}\|\Phi\|_{H^1(\mathcal{U}_1)}^2 \leq 24C_1\Theta_1^2 e^{2\beta u_2}\|\Phi\|_{H^1(\mathcal{U}_3)}^2 + 2C_1\|(\partial_{d+1}\Phi)_0\|_{L^2(B(\delta))}^2.$$

□

Lemma 3.31. *For all $\delta \in (0, 1/2)$, all $V \in L^\infty(\mathbb{R}^d)$, all $\Phi \in H^2(\mathcal{U}_3)$ with $\Delta\Phi(x,t) = V(x)\Phi(x,t)$, we have*

$$\|\Phi\|_{H^1(\mathcal{V}_2)} \leq 24C_2R_3^3\left[\Theta_2^2\left(\frac{eR_2}{r_1}\right)^{2\alpha-2}\|\Phi\|_{H^1(\mathcal{V}_1)}^2 + \Theta_3^2\left(\frac{eR_2}{r_3}\right)^{2\alpha-2}\|\Phi\|_{H^1(\mathcal{V}_3)}^2\right]$$

for all $\alpha \geq \alpha_1$ where α_1 is given in eq. (25)*, C_2 is the constant in Proposition 3.28 and Θ_1, Θ_2 are given in equations* (19) *and* (22)*.*

Proof. We choose a cutoff function $\chi \in C_c^\infty(\mathbb{R}^{d+1}, [0,1])$ with $\operatorname{supp}\chi \subset B(R_3)\setminus\overline{B(r_1)}$, $\chi(x) = 1$ if $x \in B(r_3)\setminus\overline{B(R_1)}$,

$$\max\{\|\Delta\chi\|_{\infty,\mathcal{V}_1}, \||\nabla\chi|\|_{\infty,\mathcal{V}_1}\} \leq \frac{\tilde{\Theta}_2}{\delta^4} =: \Theta_2 \tag{22}$$

and

$$\max\{\|\Delta\chi\|_{\infty,\mathcal{V}_3}, \||\nabla\chi|\|_{\infty,\mathcal{V}_3}\} \leq \Theta_3, \tag{23}$$

where $\tilde{\Theta}_2$ depends only on the dimension and Θ_3 is an absolute constant, see Appendix B. We set $G = \chi\Phi$. We apply Proposition 3.28 with $\rho = R_3$ to the function G and obtain for all $\alpha \geq \alpha_0 \geq 1$

$$\int_{B(R_3)} \left(\alpha R_3^2 v^{1-2\alpha}|\nabla G|^2 + \alpha^3 v^{-1-2\alpha}|G|^2\right) \mathrm{d}x \leq C_2 R_3^4 \int_{B(R_3)} v^{2-2\alpha}|\Delta G|^2 \mathrm{d}x.$$

Since $v \leq 1$ on $B(R_3)$ we can replace the exponent of the weight function v at all three places by $2-2\alpha$, i.e.

$$\int_{B(R_3)} \left(\alpha R_3^2 v^{2-2\alpha}|\nabla G|^2 + \alpha^3 v^{2-2\alpha}|G|^2\right) \mathrm{d}x \leq C_2 R_3^4 \int_{B(R_3)} v^{2-2\alpha}|\Delta G|^2 \mathrm{d}x =: I. \tag{24}$$

For the right hand side we use

$$\Delta G = 2(\nabla\chi)(\nabla\Phi) + (\Delta\chi)\Phi + (\Delta\Phi)\chi$$

and $\Delta\Phi = V\Phi$ to obtain

$$I \leq 3C_2 R_3^4 \int_{B(R_3)} v^{2-2\alpha} \left(4|(\nabla\chi)(\nabla\Phi)|^2 + |(\Delta\chi)\Phi|^2 + \|V\|_\infty^2 |\chi\Phi|^2\right) \mathrm{d}x =: I_1 + I_2 + I_3.$$

If we choose α sufficiently large, i.e.

$$\alpha \geq \left(6C_2 R_3^4 \|V\|_\infty^2\right)^{1/3} =: \tilde{\alpha}_0,$$

we can subsume the term I_3 into the left hand side of Ineq. (24). We obtain for all

$$\alpha \geq \alpha_1 := \max\{\alpha_0, \tilde{\alpha}_0\} \tag{25}$$

the estimate

$$\int_{B(R_3)} \left(\alpha R_3^2 v^{2-2\alpha}|\nabla G|^2 + \frac{\alpha^3}{2} v^{2-2\alpha}|G|^2\right) \mathrm{d}x \leq I_1 + I_2.$$

For the "new" left hand side we have the lower bound

$$I_1 + I_2 \geq \int_{B(R_3)} \left(\alpha R_3^2 v^{2-2\alpha}|\nabla G|^2 + \frac{\alpha^3}{2} v^{2-2\alpha}|G|^2\right) \mathrm{d}x \geq \frac{1}{2}\left(\frac{R_3}{R_2}\right)^{2\alpha-2} \|\Phi\|_{H^1(\mathcal{V}_2)}^2.$$

For I_1 and I_2 we have the estimates

$$I_1 \leq 3C_2 R_3^4 \left[4\Theta_2^2 \left(\frac{\mathrm{e}R_3}{r_1}\right)^{2\alpha-2} \int_{\mathcal{V}_1} |\nabla\Phi|^2 + 4\Theta_3^2 \left(\frac{\mathrm{e}R_3}{r_3}\right)^{2\alpha-2} \int_{\mathcal{V}_3} |\nabla\Phi|^2\right]$$

and

$$I_2 \leq 3C_2 R_3^4 \left[\Theta_2^2 \left(\frac{\mathrm{e}R_3}{r_1} \right)^{2\alpha-2} \int_{\mathcal{V}_1} |\Phi|^2 + \Theta_3^2 \left(\frac{\mathrm{e}R_3}{r_3} \right)^{2\alpha-2} \int_{\mathcal{V}_3} |\Phi|^2 \right].$$

Putting everything together, the Carleman estimate from Proposition 3.28 implies for $\alpha \geq \alpha_1$

$$\|\Phi\|_{H^1(\mathcal{V}_2)}^2 \leq 24 C_2 R_3^4 \left[\Theta_2^2 \left(\frac{\mathrm{e}R_2}{r_1} \right)^{2\alpha-2} \|\Phi\|_{H^1(\mathcal{V}_1)}^2 + \Theta_3^2 \left(\frac{\mathrm{e}R_2}{r_3} \right)^{2\alpha-2} \|\Phi\|_{H^1(\mathcal{V}_3)}^2 \right]. \quad (26)$$

□

3.4.3. Interpolation inequalities

We will now sum shifted variants of the inequalities in Lemma 3.30 and 3.31 and turn the sum on the right hand side into a product by an appropriate choice of the parameters α and β. Before doing so, let us introduce the following notation. For $x \in \mathbb{R}^d$ and $i \in \{1,2,3\}$ we define

$$\mathcal{U}_i(x) = \{(y,t) \in \mathbb{R}^{d+1} \colon (y-x,t) \in \mathcal{U}_i\} \text{ and } \mathcal{V}_i(x) = \{(y,t) \in \mathbb{R}^{d+1} \colon (y-x,t) \in \mathcal{V}_i\}$$

as well as

$$\mathcal{W}_i(\Gamma, Z) := \cup_{z \in \hat{S}_{\delta,Z}(\Gamma)} \mathcal{U}_i(z).$$

Furthermore, we define the set

$$\mathcal{X}_1 := \Gamma \times (-1,1) \subset \mathbb{R}^{d+1},$$

let $\lceil R_3 \rceil$ denote the least integer larger or equal than R_3 and

$$\tilde{\mathcal{X}}_{R_3} := \Gamma^{\lceil R_3 \rceil} \times (-R_3, R_3) \subset \mathbb{R}^{d+1}.$$

Proposition 3.32. *For all $\delta \in (0,1/2)$, all $(1,\delta)$-equidistributed sequences Z, all measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}$, all $\phi \in L^2(\Gamma)$, satisfying $\phi \in \operatorname{Ran} P_H(-\infty, E])$ for some $E \in \mathbb{R}$ we have*

$$\|\Phi\|_{H^1(\mathcal{W}_1(\Gamma,Z))} \leq D_1 \|\phi\|_{L^2(S_{\delta,Z}(\Gamma))}^{1/2} \|\Phi\|_{H^1(\mathcal{W}_3(\Gamma,Z))}^{1/2},$$

where D_1 is given in Eq. (31).

Proof. We apply Lemma 3.30 to the translates $\mathcal{U}_i(z)$ where $z \in \hat{S}_{\delta,Z}(\Gamma)$ and sum up:

$$\mathrm{e}^{2\beta u_1} \sum_{z \in \hat{S}_{\delta,Z}(\Gamma)} \|\Phi\|_{H^1(\mathcal{U}_1(z))}^2 \leq 24 C_1 \Theta_1^2 \mathrm{e}^{2\beta u_2} \sum_{z \in \hat{S}_{\delta,Z}(\Gamma)} \|\Phi\|_{H^1(\mathcal{U}_3(z))}^2 + \\ + 2C_1 \sum_{z \in \hat{S}_{\delta,Z}(\Gamma)} \|(\partial_{d+1}\Phi)_0\|_{L^2(B(z,\delta))}^2.$$

Since $\mathcal{U}_i(z) \cap \mathcal{U}_i(z') = \emptyset$ for $z, z' \in z \in \hat{S}_{\delta,Z}(\Gamma)$ with $z \neq z'$ and by the definition of $\mathcal{W}_i(\Gamma, Z)$ and $S_{\delta,Z}(\Gamma)$, we have for all $\beta \geq \beta_1$

$$\|\Phi\|^2_{H^1(\mathcal{W}_1(Z,\Gamma))} \leq \tilde{D}_1 \|\Phi\|^2_{H^1(\mathcal{W}_3(Z,\Gamma))} + \hat{D}_1 \|(\partial_{d+1}\Phi)_0\|^2_{L^2(S_{\delta,Z}(\Gamma))},$$

where

$$\tilde{D}_1(\beta) = 24 C_1 \Theta_1^2 \mathrm{e}^{2\beta(u_2 - u_1)} \quad \text{and} \quad \hat{D}_1(\beta) = 2 C_1 \mathrm{e}^{-2\beta u_1}. \tag{27}$$

We choose β such that

$$\mathrm{e}^{\beta} = \left[\frac{1}{12\Theta_1^2} \frac{\|(\partial_{d+1}\Phi)_0\|^2_{L^2(\mathcal{W}_{\delta,Z}(\Gamma)}}{\|\Phi\|^2_{H^1(\mathcal{W}_3(Z,\Gamma))}} \right]^{\frac{1}{2u_2}}. \tag{28}$$

Now we distinguish two cases. If $\beta \geq \beta_1$ we obtain by using $u_1 = 2u_2$

$$\|\Phi\|^2_{H^1(\mathcal{W}_1(Z,\Gamma))} \leq 8\sqrt{3} C_1 \Theta_1 \|\Phi\|_{H^1(\mathcal{W}_3(Z,\Gamma))} \|(\partial_{d+1}\Phi)_0\|_{L^2(S_{\delta,Z}(\Gamma))}. \tag{29}$$

If $\beta < \beta_1$ we use Lemma 5.2 of [LRL12]. In particular, one concludes from Eq. (28) that

$$\|\Phi\|^2_{H^1(\mathcal{W}_3(Z,\Gamma))} < \frac{1}{12\Theta_1^2} \mathrm{e}^{-2\beta_1 u_2} \|(\partial_{d+1}\Phi)_0\|^2_{L^2(S_{\delta,Z}(L))}.$$

This gives us in case $\beta < \beta_1$

$$\|\Phi\|^2_{H^1(\mathcal{W}_1(Z,\Gamma))} \leq \|\Phi\|^2_{H^1(\mathcal{W}_3(Z,\Gamma))} < \frac{\mathrm{e}^{-\beta_1 u_2}}{\sqrt{12}\Theta_1} \|\Phi\|_{H^1(\mathcal{W}_3(Z,\Gamma))} \|(\partial_{d+1}\Phi)_0\|_{L^2(S_{\delta,Z}(L))}. \tag{30}$$

Using $(\partial_{d+1}\Phi)_0 = \phi$ and setting

$$D_1^2 = \max\left\{ 8\sqrt{3} C_1 \Theta_1, \frac{\mathrm{e}^{-\beta_1 u_2}}{\Theta_1 \sqrt{12}} \right\}, \tag{31}$$

we conclude the statement of the proposition from Ineqs. (29) and (30). □

Proposition 3.33. *For all $\delta \in (0, 1/2)$, all $(1, \delta)$-equidistributed sequences Z, all measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}$, all $\phi \in L^2(\Gamma)$ satisfying $\phi \in \operatorname{Ran}(P_H((-\infty, E])$ for some $E \in \mathbb{R}$ we have*

$$\|\Phi\|_{H^1(\mathcal{X}_1)} \leq D_2 \|\Phi\|^{\gamma}_{H^1(\mathcal{W}_1(Z,\Gamma))} \|\Phi\|^{1-\gamma}_{H^1(\tilde{\mathcal{X}}_{R_3})},$$

where γ and D_2 are given in Eq. (41) *and* (42).

Proof of Proposition 3.33. Let us start by emphasizing that we now work with the extended function $\Phi : \mathbb{R}^{d+1} \to \mathbb{C}$. We apply Lemma 3.31 to translates $\mathcal{V}_i(z)$ and sum over $z \in \hat{S}_{\delta,Z}(\Gamma)$ to obtain

$$\sum_{z \in \hat{S}_{\delta,Z}(\Gamma)} \|\Phi\|^2_{H^1(\mathcal{V}_2(z))} \leq 24 C_2 R_3^4 \left[\Theta_2^2 \left(\frac{\mathrm{e}R_2}{r_1} \right)^{2\alpha-2} \sum_{z \in \hat{S}_{\delta,Z}(\Gamma)} \|\Phi\|^2_{H^1(\mathcal{V}_1(z))} \right.$$
$$\left. + \Theta_3^2 \left(\frac{\mathrm{e}R_2}{r_3} \right)^{2\alpha-2} \sum_{z \in \hat{S}_{\delta,Z}(\Gamma)} \|\Phi\|^2_{H^1(\mathcal{V}_3(z))} \right]. \tag{32}$$

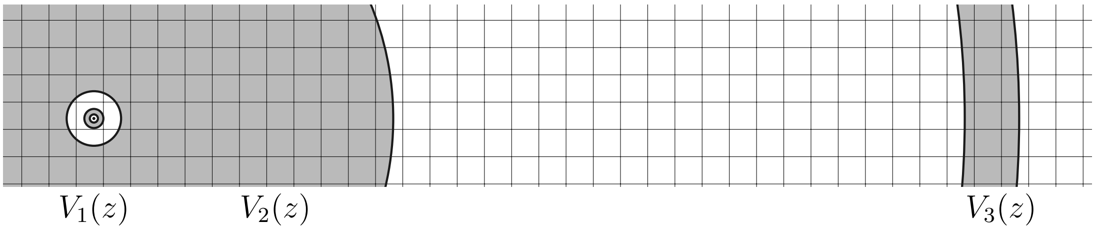

Figure 3: Sketch of the annuli $\mathcal{V}_1(z)$, $\mathcal{V}_2(z)$, $\mathcal{V}_3(z)$ in dimension $d = 2$

We now further estimate the three sums in Ineq. (32). First, we note that since $R_1 < \delta$, the sets $\mathcal{V}_1(z)$, $z \in \hat{S}_{\delta,Z}(\Gamma)$, are mutually disjoint. Furthermore we have $\mathcal{V}_1(z) \cap \mathbb{R}^{d+1}_+ \subset \mathcal{U}_1(z)$, hence $\cup_{z\in\hat{S}_{\delta,Z}(\Gamma)} \mathcal{V}_1(z_j) \cap \mathbb{R}^{d+1}_+ \subset \mathcal{W}_{1,Z}$. Together with the antisymmetry of Ψ in the $(d+1)$-coordinate, this yields

$$\sum_{z\in\hat{S}_{\delta,Z}(\Gamma)} \|\Psi\|^2_{H^1(\mathcal{V}_1(z))} = \|\Psi\|^2_{H^1(\bigcup_{z\in\hat{S}_{\delta,Z}(\Gamma)} \mathcal{V}_1(z))}$$

$$= 2\|\Psi\|^2_{H^1(\bigcup_{z\in\hat{S}_{\delta,Z}(\Gamma)} \mathcal{V}_1(z)\cap\mathbb{R}^{d+1}_+)} \leq 2\|\Psi\|^2_{H^1(\mathcal{W}_{1,Z})}.$$

Moreover, for every $(x,t) \in \tilde{\mathcal{X}}_{R_3} = \Gamma^{\lceil R_3 \rceil} \times (-R_3, R_3)$, there are at most $(2R_3 + 2)^d$ points $z \in \hat{S}_{\delta,Z}(\Gamma)$ such that $(x,t) \in \mathcal{V}_3(z)$. Thus, we have

$$\sum_{z\in\hat{S}_{\delta,Z}(\Gamma)} \|\Psi\|^2_{H^1(\mathcal{V}_3(z))} \leq (2R_3 + 2)^d \|\Psi\|^2_{H^1(\tilde{X}_{R_3})} =: K_d \|\Psi\|^2_{H^1(\tilde{X}_{R_3})}.$$

Finally, we claim that

$$\sum_{z\in\hat{S}_{\delta,Z}(\Gamma)} \|\Psi\|^2_{H^1(V_2(z))} \geq \|\Psi\|^2_{H^1(X_1)}, \tag{33}$$

Since the sets V_2 are rather large (they are annuli with outer radius $R_2 = 8\sqrt{d}$ and inner radius 1), and the points $z \in \hat{S}_{\delta,Z}(\Gamma)$ are equidistributed, Ineq. (33) is quite credible but in particular if Γ is "small", it turns out to be a bit technical to provide a rigorous proof. Note that if the boundaries of Γ coincided with the boundaries of the underlying lattice $\mathbb{Z}^d$ as in [NTTV18a], then Ineq. (33) would follow almost immediately. Also, if we had chosen a larger R_2, the following argument would be substantially simpler. However, this would have forced us to also choose the parameter R_3 larger which itself would have led to a more restrictive assumption on κ and M in Theorems 3.16, and 3.17.

Ineq. (33) is best seen by distinguishing cases:

- Let us first assume that Γ is sufficiently large, i.e. there there is at least one direction $k \in \{1, \dots, d\}$ such that $b_k - a_k \geq 5$. We will then show that for every $x \in \Gamma$, there is $z \in \hat{S}_{\delta,Z}(\Gamma)$ such that

$$1 < |x - z| < 7\sqrt{d}. \tag{34}$$

Then, since $V_2(z) = B(z, 8\sqrt{d}) \backslash \overline{B(z,1)}$, it will follow that $\{x\} \times (-1,1) \subset \mathcal{V}_2(z)$. Consequently, identifying such a point $z \in \hat{S}_{\delta,Z}(\Gamma)$ for every $x \in \Gamma$, we find $\mathcal{X}_1 = \Gamma \times (-1,1) \subset \bigcup_{z \in \hat{S}_{\delta,Z}(\Gamma)} \mathcal{V}_2(z)$. To find such a $z \in \hat{S}_{\delta,Z}$, we proceed as follows: The point x is contained in an elementary cell $\Lambda_1 + j_0$, $j_0 \in \mathbb{Z}^d$, of the lattice $\mathbb{Z}^d$. Note that this elementary cell does not need to be contained entirely in Γ. However, since Γ contains an elementary cell of $\mathbb{Z}^d$ and has side length at least 5 in one dimension, the set $\Lambda_5(j_0) \backslash \Lambda_3(j_0)$ (which is a union of $(5^d - 3^d)$ elementary cells) contains at least one elementary cell $\Lambda_1(j)$, $j \in \mathbb{Z}^d$, which is entirely in Γ. The corresponding $z = z_j \in \hat{S}_{\delta,Z}$ then satisfies (34).

- In the other case, we have $b_k - a_k < 5$ for all $k \in \{1, \ldots, d\}$. Now, the set $\cup_{j \in \hat{S}_{\delta,Z}(\Gamma)} \mathcal{V}_2(z_j)$ does not necessarily cover $\mathcal{X}_1 = \Gamma \times (-1,1)$ but we claim that it will cover a translate $\tilde{\Gamma} \times (-1,1)$. Ineq. (33) then follows from the symmetry properties of the extension of Φ since $\|\Phi\|_{H^1(\tilde{\Gamma} \times (-1,1))} = \|\Phi\|_{H^1(\Gamma \times (-1,1))}$. Let us first consider this in dimension $d = 1$, i.e. $\Gamma = (a,b)$, where $b - a < 5$
  - If $3 \leq b - a < 5$, then there is a point $z_1 \in \hat{S}_{\delta,Z}(\Gamma) \cap (b-3, b-1)$ and a point $z_2 \in \hat{S}_{\delta,Z}(\Gamma) \cap (b-2, b)$, since both intervals have side length 2. Note that z_1 and z_2 might coincide. For all $x \in \tilde{\Gamma} = (b, b + (b-a))$ we then have

$$\text{either} \quad 1 < |x - z_1| \leq 7 \quad \text{or} \quad 1 < |x - z_2| \leq 7$$

    whence for all $x \in \tilde{\Gamma} = (b, b+(b-a))$ and all $t \in (-1,1)$, there is $j \in \{1,2\}$ such that

$$1 < |(x,t) - (z_j, 0)| \leq \sqrt{7^2 + 1} < 8, \quad \text{i.e. } (x,t) \in V_2(z_j).$$

  - If $1 \leq b - a < 3$, then there is a point $z \in \hat{S}_{\delta,Z}(\Gamma)$ which has distance at most 1.5 to either a or b. Without loss of generality let $|z - b| \leq 1.5$. We then choose $\tilde{\Gamma} := (b + (b-a), b + 2(b-a))$ and for every $x \in \tilde{\Gamma}$ and every $t \in (-1,1)$ we have

$$1 < |(z,0) - (x,t)| \leq \sqrt{7.5^2 + 1} < 8, \quad \text{i.e. } (x,t) \in V_2(z_j).$$

  In either case, this implies $\cup_{j \in \hat{S}_{\delta,Z}(\Gamma)} \mathcal{V}_2(z_j) \subset \tilde{\Gamma} \times (-1,1)$ and we found (33) for $d = 1$. Let us now assume $d \geq 2$. We distinguish again two subcases:
  - Let us first assume that there is $z \in \hat{S}_{\delta,Z}$ with distance larger than 1 to the boundary $\partial\Gamma$ of Γ. We then choose $\tilde{\Gamma}$ to be a neighboring translate

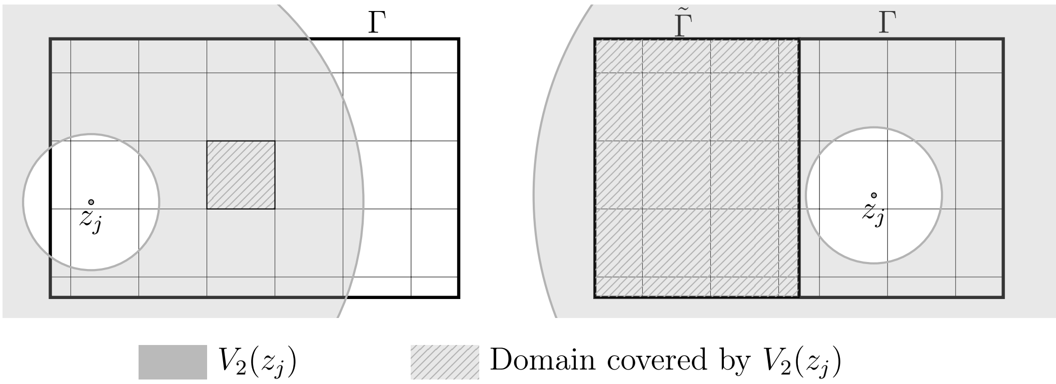

Figure 4: The annuli $V_2(z_j)$ either each already cover another elementary cell in Γ, such that all of Γ will be covered, or Γ is so small that an annulus $V_2(z_j)$ already covers an entire translated copy $\tilde{\Gamma}$. In this figure, the outer radii of $V_2(z_j)$ are not drawn in scale.

of Γ, touching Γ on a boundary surface and find for all $x \in \tilde{\Gamma}$ and all $t \in (-1, 1)$

$$1 < \operatorname{dist}((z,0),(x,t)) = \sqrt{|z-x|^2 + t^2} \leq \sqrt{5^2(d-1) + (4+5)^2 + 1} \\ = \sqrt{25d + 57} < 8\sqrt{d}$$

where the last inequality is easily verified by an elementary calculation using $d \geq 2$.

– If this is not the case, i.e. every point $z \in \hat{S}_{\delta,Z}(\Gamma)$ has distance less than 1 to the boundary of Γ, then in at least one direction $k \in \{1, \ldots, d\}$ we must have $b_k - a_k \leq 4$. Without loss of generality, let this be the x_1 direction. We choose $z \in \hat{S}_{\delta,Z}(\Gamma)$ and $\tilde{\Gamma}$ as a next-to-one neighboring copy of Γ in x_1-direction, i.e. $\tilde{\Gamma} = \Gamma + 2(b-a)e_1$ where e_k is a unit vector in x_1 direction. For every point $x \in \tilde{\Gamma}$, there is a point $z \in \hat{S}_{\delta,Z}(\Gamma)$ such that $z + 2(b-a)e_1 \in \tilde{\Gamma}$ has distance at most 2 in every coordinate direction to x. Therefore, we find for all $x \in \tilde{\Gamma}$ and all $t \in (-1, 1)$

$$1 < \operatorname{dist}((z,0),(x,t)) \leq \sqrt{2^2(d-1) + (2+8)^2 + 1} \\ = \sqrt{4d + 97} < 8\sqrt{d}$$

where the last step follows again by an elementary calculation and the fact that $d \geq 2$.

We thus showed Ineq. (33). Going back to Ineq. (32), we therefore established

$$\|\Phi\|^2_{H^1(\mathcal{X}_1)} \leq \tilde{D}_2(\alpha)\|\Phi\|^2_{H^1(\mathcal{W}_1(Z,\Gamma))} + \hat{D}_2(\alpha)\|\Phi\|_{H^1(\tilde{\mathcal{X}}_{R_3})}, \tag{35}$$

where

$$\tilde{D}_2(\alpha) = 48C_2R_3^4\Theta_2^2\left(\frac{\mathrm{e}R_2}{r_1}\right)^{2\alpha-2} \text{ and } \hat{D}_2(\alpha) = 24C_2R_3^4\Theta_3^2K_d\left(\frac{\mathrm{e}R_2}{r_3}\right)^{2\alpha-2}. \tag{36}$$

If we let $c_1 = 48C_2\Theta_2^2R_3^4r_1^2/(\mathrm{e}R_2)^2$, $c_2 = 24C_2\Theta_3^2K_dR_3^4r_3^2/(\mathrm{e}R_2)^2$,

$$p^+ = 2\ln\left(\frac{\mathrm{e}R_2}{r_1}\right) > 0 \quad \text{and} \quad p^- = 2\ln\left(\frac{\mathrm{e}R_2}{r_3}\right) < 0,$$

then Ineq. (35) reads

$$\frac{1}{5^d}\|\Phi\|^2_{H^1(\mathcal{X}_1)} \leq c_1\mathrm{e}^{p^+\alpha}\|\Phi\|^2_{H^1(\mathcal{W}_1(Z,\Gamma))} + c_2\mathrm{e}^{p^-\alpha}\|\Phi\|^2_{H^1(\tilde{\mathcal{X}}_{R_3})}. \tag{37}$$

We choose α such that

$$\mathrm{e}^\alpha = \left(\frac{c_2}{c_1}\frac{\|\Phi\|^2_{H^1(\tilde{\mathcal{X}}_{R_3})}}{\|\Phi\|^2_{H^1(\mathcal{W}_1(Z,\Gamma))}}\right)^{\frac{1}{p^+-p^-}}. \tag{38}$$

If $\alpha \geq \alpha_1$ we obtain from Ineq. (37) that

$$\frac{1}{5^d}\|\Phi\|^2_{H^1(\mathcal{X}_1)} \leq 2c_1^\gamma c_2^{1-\gamma}\|\Phi\|^{2\gamma}_{H^1(\mathcal{W}_1(Z,\Gamma))}\|\Phi\|^{2-2\gamma}_{H^1(\tilde{\mathcal{X}}_{R_3})}, \quad \text{where} \quad \gamma = \frac{-p^-}{p^+-p^-}. \tag{39}$$

If $\alpha < \alpha_1$, we proceed as in the last part of the proof of Proposition 3.32, i.e. we conclude from Eq. (38) that

$$\|\Phi\|^2_{H^1(\tilde{\mathcal{X}}_{R_3})} < \frac{c_1}{c_2}\mathrm{e}^{\alpha_1(p^+-p^-)}\|\Phi\|^2_{H^1(\mathcal{W}_1(Z,\Gamma))}$$

and thus

$$\|\Phi\|^2_{H^1(\mathcal{X}_1)} \leq \|\Phi\|^{2\frac{p^+-p^-}{p^+-p^-}}_{H^1(\tilde{\mathcal{X}}_{R_3})} < \|\Phi\|^{2(1-\gamma)}_{H^1(\tilde{\mathcal{X}}_{R_3})}\left(\frac{c_1}{c_2}\mathrm{e}^{\alpha_1(p^+-p^-)}\right)^\gamma\|\Phi\|^{2\gamma}_{H^1(\mathcal{W}_1(Z,\Gamma))}. \tag{40}$$

We calculate

$$\gamma = \frac{\ln 2}{\ln(r_3/r_1)}, \tag{41}$$

set

$$D_2^2 = \max\left\{5^d 192\cdot 9^4C_2\Theta_3^2K_d\mathrm{e}^4d^2\left(\frac{2\Theta_2^2r_1^2}{\Theta_3^2K_dr_3^2}\right)^\gamma, \left(\frac{2\Theta_2^2}{\Theta_3^2K_d}\left(\frac{r_3}{r_1}\right)^{2(\alpha_1-1)}\right)^\gamma\right\} \tag{42}$$

and conclude the statement of the proposition from Ineqs. (39) and (40). □

3.4.4. Proof of Theorem 3.17 in the special case

In this section, we use the interpolation inequalities, Proposition 3.32, and 3.33 to prove Theorem 3.17 in the special case where $M = 1$ and $\phi \in \operatorname{Ran} P_H(E)$ for some $E \geq 0$. We shall need one more ingredient: The following proposition uses the special structure of ϕ and Φ to show that the norms of ϕ and Φ are comparable.

Proposition 3.34. *For all $E \in \mathbb{R}$, $\phi \in \operatorname{Ran} P_E(H)$ and $\tau > 0$ we have*

$$\frac{\tau}{2}\|\phi\|^2_{L^2(\Gamma)} \leq \|\Phi\|^2_{H^1(\Gamma\times(-\tau,\tau))} \leq 2\tau(1 + (1 + \|V\|_\infty)\tau^2)\|\mathrm{e}^{2\tau\sqrt{H_+}}\phi\|^2_{L^2(\Gamma)}.$$

Proof. For the function $\Phi : \Gamma \times \mathbb{R} \to \mathbb{C}$ we have for $\tau > 0$

$$\|\Phi\|^2_{H^1(\Gamma\times[-\tau,\tau])} = \int_{-\tau}^{\tau}\int_{\Phi}\left(|\partial_{d+1}\Phi|^2 + |\nabla_d\Phi|^2 + |\Phi|^2\right)\mathrm{d}x.$$

By Green's theorem and Lemma 3.24 we have

$$\int_\Gamma |\nabla_d\Phi|^2\mathrm{d}x' = -\int_\Gamma(\Delta_d\Phi)\overline{\Phi}\mathrm{d}x' - \int_\Gamma V|F|^2\mathrm{d}x' + \int_\Gamma(\partial_t^2\Phi)\overline{\Phi}\mathrm{d}x'$$

for all $t \in \mathbb{R}$. First we estimate

$$\begin{aligned}
\|\Phi\|^2_{H^1(\Gamma\times(-\tau,\tau))} &= \int_{-\tau}^{\tau}\int_\Gamma\left(|\partial_{d+1}\Phi|^2 - V|\Phi|^2 + (\partial_t^2\Phi)\overline{\Phi} + |\Phi|^2\right)\mathrm{d}x \\
&\leq \int_{-\tau}^{\tau}\int_\Gamma\left(|\partial_{d+1}\Phi|^2 + (\partial_t^2\Phi)\overline{\Phi} + (1 + \|V\|_\infty)|\Phi|^2\right)\mathrm{d}x \\
&= 2\int_{-\infty}^{E} I(\lambda)\mathrm{d}\|P_H(\lambda)\phi\|^2_{L^2(\Gamma)},
\end{aligned}$$

where

$$\begin{aligned}
I(\lambda) &:= \int_0^\tau\left((1 + \|V\|_\infty)s_\lambda(t)^2 + s'_\lambda(t)^2 + s''_\lambda(t)s_\lambda(t)\right)\mathrm{d}t \\
&= (1 + \|V\|_\infty)\int_0^\tau s_\lambda(t)^2\mathrm{d}t + s'_\lambda(t)s_\lambda(t).
\end{aligned}$$

In particular, the above integral is finite since I is bounded on $(-\infty, E]$. For $\lambda \leq 0$, we estimate using $s_\lambda(t) \leq t$ and $s'_\lambda(t)s_\lambda(t) \leq t$ for $t > 0$

$$I(\lambda) \leq (1 + \|V\|_\infty)\tau^3/3 + \tau \leq ((1 + \|V\|_\infty)\tau^3 + \tau)\mathrm{e}^{2\tau\sqrt{\lambda}}.$$

For $\lambda > 0$ we use $\sinh(\sqrt{\lambda}t)/\sqrt{\lambda} \leq t\cosh(\sqrt{\lambda}t)$ for $t > 0$ and $\cosh(\sqrt{\lambda}t)^2 \leq \mathrm{e}^{2\sqrt{\lambda}t}$ to obtain

$$\begin{aligned}
I(\lambda) &= (1 + \|V\|_\infty)\int_0^\tau\frac{\sinh^2(\sqrt{\lambda}t)}{\lambda}\mathrm{d}t + \sinh(\sqrt{\lambda}\tau)\cosh(\sqrt{\lambda}\tau)/\sqrt{\lambda} \\
&\leq ((1 + \|V\|_\infty)\tau^3\cosh^2(\sqrt{\lambda}\tau) + \tau\cosh^2(\sqrt{\lambda}\tau)) \leq ((1 + \|V\|_\infty)\tau^3 + \tau)\mathrm{e}^{2\tau\sqrt{\lambda}}.
\end{aligned}$$

This shows the upper bound. For the lower bound we drop the gradient term and obtain

$$\|\Phi\|^2_{H^1(\Gamma\times(-\tau,\tau))} \geq \int_{-\tau}^{\tau}\int_\Gamma \left(|\partial_{d+1}\Phi|^2 + |\Phi|^2\right) \mathrm{d}x\mathrm{d}t = 2\int_{-\infty}^{E} I(\lambda)\|P_H(\lambda)\phi\|^2_{L^2(\Gamma)},$$

where

$$\tilde I(\lambda) := \int_0^\tau \left(s_\lambda(t)^2 + s'_\lambda(t)^2\right) \mathrm{d}t.$$

If $\lambda = 0$, the lower bound $\tilde I(\lambda) \geq \tau$ follows immediately. Else, we have $s_\lambda(t)^2 \geq \sin^2(\sqrt{|\lambda|}t)/|\lambda|$ and $s'_\lambda(t)^2 \geq \cos(|\lambda|t)$ whence

$$\tilde I(\lambda) \geq \int_0^\tau \frac{\sin^2(|\lambda|t)}{|\lambda|^2} + \cos^2(|\lambda|t)\mathrm{d}t \geq \int_0^\tau \cos^2(|\lambda|t)\mathrm{d}t = \frac{\tau}{2} + \frac{\sin(2|\lambda|\tau)}{4|\lambda|}.$$

Now, if $2|\lambda|\tau < \pi$, the sin term is positive and we drop it to find $\tilde I(\lambda) \geq \tau/2$. If $2|\lambda|\tau \geq \pi$, we have $\sin(2|\lambda|\tau) \geq -1$ and estimate

$$\tilde I(\lambda) \geq \frac{\tau}{2} - \frac{1}{4|\lambda|} = \frac{\tau}{2} - \frac{\pi}{4\pi|\lambda|} \geq \frac{\tau}{2} - \frac{T}{2\pi} \geq \frac{\tau}{4}. \qquad \square$$

We are now ready to prove Theorem 3.17 in case $M = 1$ and ϕ is in some $\operatorname{Ran} P_H(E)$.

We have

$$\|\Phi\|^2_{H^1(\tilde{\mathcal X}_{R_3})} \leq \lceil R_3\rceil^d \|\Phi\|^2_{H_1(\Gamma\times(-R_3,R_3)}$$

whence we find by Proposition 3.34

$$\begin{aligned}\frac{\|\Phi\|^2_{H^1(\tilde{\mathcal X}_{R_3})}}{\|\Phi\|^2_{H^1(\mathcal X_1)}} &\leq \lceil R_3\rceil^d 4R_3(1+(1+\|V\|_\infty R_3^2)\frac{\|\mathrm{e}^{2R_3\sqrt{H_+}}\phi\|^2_{L^2(\Gamma)}}{\|\phi\|^2_{L^2(\Gamma)}}\\ &= \lceil R_3\rceil^d 4R_3(1+(1+\|V\|_\infty R_3^2)\hat D(\phi) =: D_3^2.\end{aligned}$$

Using this estimate, Propositions 3.33 and 3.32, and $\mathcal W_3(\Gamma, Z) \subset \tilde{\mathcal X}_{R_3}$, we find

$$\begin{aligned}\|\Phi\|_{H^1(\tilde{\mathcal X}_{R_3})} &\leq D_3\|\Phi\|^2_{H^1(\mathcal X_1)} \leq D_2D_3\|\Phi\|^\gamma_{H^1(\mathcal W_1(Z,\Gamma))}\|\Phi\|^{1-\gamma}_{H^1(\tilde{\mathcal X}_{R_3})}\\ &\leq D_1^\gamma D_2D_3\|\phi\|^{\gamma/2}_{L^2(S_{\delta,Z}(\Gamma))}\|\Phi\|^{\gamma/2}_{H^1(\mathcal W_3(\Gamma,Z))}\|\Phi\|^{1-\gamma}_{H^1(\tilde{\mathcal X}_{R_3})}\\ &\leq D_1^\gamma D_2D_3\|\phi\|^{\gamma/2}_{L^2(S_{\delta,Z}(\Gamma))}\|\Phi\|^{1-\gamma/2}_{H^1(\tilde{\mathcal X}_{R_3})}.\end{aligned}$$

Dividing by $\|\Phi\|^{1-\gamma/2}_{H^1(\tilde{\mathcal X}_{R_3})}$ and raising to the power $2/\gamma$ yields

$$\|\Phi\|_{H^1(\tilde{\mathcal X}_{R_3})} \leq (D_1^\gamma D_2D_3)^{2/\gamma}\|\phi\|_{L^2(S_{\delta,Z}(\Gamma))}.$$

By Proposition 3.34, we have

$$\|\phi\|^2_{L^2(\Gamma)} \leq \frac{2}{R_3}\|\Phi\|^2_{H^1(\Gamma\times(-R_3,R_3))} \leq \frac{2}{R_3}\|\Phi\|^2_{H^1(\tilde{\mathcal{X}}_{R_3})}. \tag{43}$$

This finally yields

$$\|\phi\|_{L^2(\Gamma)} \leq \sqrt{\frac{2}{R_3}}(D_1^\gamma D_2 D_3)^{2/\gamma}\|\phi\|_{L^2(S_{\delta,Z}(\Gamma))} = \sqrt{\frac{2}{R_3}}D_1^2 D_2^{2/\gamma} D_3^{2/\gamma}\|\phi\|_{L^2(S_{\delta,Z}(\Gamma))}. \tag{44}$$

It remains to see that there is $C > 0$ such that

$$\sqrt{\frac{2}{R_3}}D_1^2 D_2^{2/\gamma} D_3^{2/\gamma} \leq \delta^{-C/2(1+\|V\|_\infty + \ln \hat{D}(\phi))}.$$

Recall that

$$\begin{aligned}
D_1^2 &= \max\left\{8\sqrt{3}C_1\Theta_1, \frac{\mathrm{e}^{-\beta_1 u_2}}{\Theta_1\sqrt{12}}\right\},\\
D_2^2 &= \max\left\{5^d 192\cdot 9^4 C_2\Theta_3^2 K_d \mathrm{e}^4 d^2\left(\frac{2\Theta_2^2 r_1^2}{\Theta_3^2 K_d r_3^2}\right)^\gamma, \left(\frac{2\Theta_2^2}{\Theta_3^2 K_d}\left(\frac{r_3}{r_1}\right)^{2(\alpha_1-1)}\right)^\gamma\right\},\\
D_3^2 &= \lceil R_3\rceil^d 4R_3(1+(1+\|V\|_\infty R_3^2)\hat{D}(\phi),\\
\gamma &= \frac{\ln 2}{\ln(r_3/r_1)},
\end{aligned}$$

as well as

$$\begin{aligned}
\Theta_1 &= \frac{\tilde{\Theta}_1}{\delta^4},\\
\Theta_2 &= \frac{\tilde{\Theta}_2}{\delta^4},\\
\alpha_1 &\geq \left(6C_2R_3^4\|V\|^2_\infty\right)^{1/3} = \tilde{\alpha}_0\\
\beta_1 &\geq (6\|V\|^2_\infty C_1)^{1/3} = \tilde{\beta}_0,\\
u_2 &= -\delta^2/8\\
\frac{r_1}{r_3} &= \frac{\frac{1}{2}-\frac{1}{8}\sqrt{16-\delta^2}}{8.5\mathrm{e}\sqrt{d}} \in \left[\frac{\delta^2}{64}\frac{1}{8.5\mathrm{e}\sqrt{d}}, \frac{\delta}{64}\frac{1}{8.5\mathrm{e}\sqrt{d}}\right] \subset (0,1).
\end{aligned} \tag{45}$$

where $C_1, C_2, R_3, \tilde{\Theta}_1, \tilde{\Theta}_2, \Theta_3, K_d$ are constants that depend only on the dimension. In the following, we will denote by $\tilde{C}_i$ positive constants, depending only on the dimension. We easily deduce

$$2/\gamma \leq -\tilde{C}_1 \ln \delta$$

and

$$D_1^2 \leq \tilde{C}_2 \left(\delta^{-4} + \delta^4 \mathrm{e}^{\tilde{C}_3 \|V\|_\infty^{2/3} \delta^2}\right) \leq \tilde{C}_2 \left(\delta^{-4} + \delta^{-\tilde{C}_3 \|V\|_\infty^{2/3}}\right) \leq \tilde{C}_4 \delta^{-\tilde{C}_5 (1+\|V\|_\infty^{2/3})}, \quad (46)$$

$$D_2^{2/\gamma} \leq \tilde{C}_6 \left(\tilde{C}_7^{2/\gamma} \delta^4 + \delta^{-8(\alpha_1 - 1)}\right) \leq \tilde{C}_6 \left(\tilde{C}_7^{-\tilde{C}_1 \ln \delta} + \delta^{-\tilde{C}_8 \|V\|_\infty^{2/3}}\right) \quad (47)$$
$$= \tilde{C}_6 \left(\delta^{-\tilde{C}_1 \ln \tilde{C}_7} + \delta^{-\tilde{C}_8 \|V\|_\infty^{2/3}}\right) \leq \tilde{C}_9 \left(\delta^{-\tilde{C}_{10}(1+\|V\|_\infty^{2/3})}\right),$$

$$D_3^{2/\gamma} \leq \left(\tilde{C}_{11} (1 + \|V\|_\infty) \hat{D}(\phi)\right)^{-\tilde{C}_1 \ln \delta} = \delta^{-\tilde{C}_1 \ln(\tilde{C}_{12}(1+\|V\|_\infty)\hat{D}(\phi))} \quad (48)$$
$$\leq \delta^{-\tilde{C}_{12}(1+\|V\|_\infty^{2/3} + \ln \hat{D}(\phi))}$$

where we used $\delta < 1/2$, the identity $a^{b \ln \delta} = \delta^{b \ln a}$ and $\ln(1 + \|V\|_\infty) \leq \|V\|_\infty^{2/3}$. Combining ineq. (44) with (46), (47), (48), and using

$$t = 2^{\log_2(t)} \leq \delta^{-\log_2(t)}.$$

for all $t > 0$, we find

$$\|\phi\|_{L^2(\Gamma)}^2 \leq \tilde{C}_{13} \delta^{-\tilde{C}_{14}(1+\|V\|_\infty^{2/3} + \ln \hat{D}(\phi))} \|\phi\|_{S_{\delta,Z}(\Gamma)}^2 \leq \delta^{-C(1+\|V\|_\infty^{2/3} + \ln \hat{D}(\phi))} \|\phi\|_{S_{\delta,Z}(\Gamma)}^2.$$

This concludes the proof of Theorem 3.17 in the special case where $M = 1$ and where ϕ is subject to some energy cutoff.

3.4.5. Limiting and scaling argument

We now deduce the general statement of Theorem 3.17. Let us first explain how to relax the condition $\phi \in \operatorname{Ran} P_H(E)$ to $\phi \in \mathcal{D}(\mathrm{e}^{\frac{\kappa}{2}\sqrt{d}\sqrt{H_+}})$. For this purpose, we need the following lemma.

Lemma 3.35. *Assume* $\phi \in \mathcal{D}\left(\mathrm{e}^{18\mathrm{e}\sqrt{d}\sqrt{H_+}}\right)$ *Then, for all* $E \in \mathbb{R}$

$$\frac{\|\mathrm{e}^{18\mathrm{e}\sqrt{d}\sqrt{H_+}} P_H(E)\phi\|_{L^2(\Gamma)}^2}{\|P_H(E)\phi\|_{L^2(\Gamma)}^2} \leq \frac{\|\mathrm{e}^{18\mathrm{e}\sqrt{d}\sqrt{H_+}} \phi\|_{L^2(\Gamma)}^2}{\|\phi\|_{L^2(\Gamma)}^2} \quad (49)$$

where we use the convention $0/0 = 0$.

Proof. Noting that

$$\|\mathrm{e}^{18\mathrm{e}\sqrt{d}\sqrt{H_+}} P_H(E)\phi\|_{L^2(\Gamma)}^2 = \int_{-\infty}^{E} \mathrm{e}^{18\mathrm{e}\sqrt{d}\cdot\sqrt{E_+}} \mathrm{d}\|P_H(E)\phi\|_{L^2(\Gamma)}^2$$

and

$$\|P_H(E)\phi\|_{L^2(\Gamma)}^2 = \int_{-\infty}^{E} \mathrm{d}\|P_H(E)\phi\|_{L^2(\Gamma)}^2,$$

we need to see that the map

$$\mathbb{R} \ni E \mapsto \frac{\int_{-\infty}^{E} \mathrm{e}^{18\mathrm{e}\sqrt{d}\cdot\sqrt{E_+}} \mathrm{d}\|P_H(E)\phi\|_{L^2(\Gamma)}^2}{\int_{-\infty}^{E} \mathrm{d}\|P_H(E)\phi\|_{L^2(\Gamma)}^2}$$

is nondecreasing. This follows from

$$\begin{aligned}
&\frac{\int_{-\infty}^{E+\epsilon} s(E)\mathrm{d}\mu(E)}{\int_{-\infty}^{E+\epsilon} \mathrm{d}\mu(E)} - \frac{\int_{-\infty}^{E} s(E)\mathrm{d}\mu(E)}{\int_{-\infty}^{E} \mathrm{d}\mu(E)} \\
&= \frac{\int_{E}^{E+\epsilon} s(E)\mathrm{d}\mu(E) \int_{-\infty}^{E} \mathrm{d}\mu(E) - \int_{E}^{E+\epsilon} \mathrm{d}\mu(E) \int_{-\infty}^{E} s(E)\mathrm{d}\mu(E)}{\mu((-\infty, E+\epsilon]) \cdot \mu((-\infty, E])} \\
&\geq \mu([E, E+\epsilon]) \frac{\int_{-\infty}^{E} s(E) - s(E)\mathrm{d}\mu(E)}{\mu((-\infty, E+\epsilon]) \cdot \mu((-\infty, E])} \geq 0
\end{aligned}$$

for every finite Borel measure μ, every non-decreasing function $s : \mathbb{R} \to [0, \infty)$, every $E \in \mathbb{R}$ and every $\epsilon > 0$. □

We are now ready to prove Theorem 3.17 in case $M = 1$, but with the relaxed assumption $\phi \in \mathcal{D}(\mathrm{e}^{18\mathrm{e}\sqrt{d}\sqrt{H_+}})$ instead of $\phi \in \operatorname{Ran} P_H(E)$.

Proof. Assume that

$$0 \neq \phi \in \mathcal{D}\left(\mathrm{e}^{18\mathrm{e}\sqrt{d}\sqrt{H_+}}\right).$$

Since $\phi \neq 0$, there is $E_0 \in \mathbb{R}$ such that $\|P_H(E)\phi\|_{L^2(\Gamma)} \neq 0$ for all $E \geq E_0$ and we have by Lemma 3.35

$$\hat{D}(P_H(E)\phi) = \frac{\|\mathrm{e}^{18\mathrm{e}\sqrt{d}\sqrt{H_+}} P_H(E)\phi\|_{L^2(\Gamma)}^2}{\|P_H(E)\phi\|_{L^2(\Gamma)}^2} \leq \frac{\|\mathrm{e}^{18\mathrm{e}\sqrt{d}\sqrt{H_+}} \phi\|_{L^2(\Gamma)}^2}{\|\phi\|_{L^2(\Gamma)}^2} = \hat{D}(\phi).$$

By the finite energy case of Theorem 3.17, which we proved above, and using $\delta < 1$, we find

$$\begin{aligned}
\|P_H(E)\phi\|_{L^2(S_{\delta,Z}(\Gamma))} &\geq \delta^{C(1+\|V\|_\infty^{2/3}+\ln \hat{D}(P_H(E)\phi))} \|P_H(E)\phi\|_{L^2(\Gamma)} \\
&\geq \delta^{C(1+\|V\|_\infty^{2/3}+\ln \hat{D}(P_H(E)\phi))} \|P_H(E)\phi\|_{L^2(\Gamma)} \\
&\geq \delta^{C(1+\|V\|_\infty^{2/3}+\ln \hat{D}(\phi))} \|P_H(E)\phi\|_{L^2(\Gamma)}.
\end{aligned}$$

Since

$$\lim_{E\to\infty} \|P_H(E)\phi\|_{S_{\delta,Z}(\Gamma)}^2 = \|\phi\|_{S_{\delta,Z}(\Gamma)}^2, \quad \text{and} \quad \lim_{E\to\infty} \|P_H(E)\phi\|_{L^2(\Gamma)}^2 = \|\phi\|_{L^2(\Gamma)}^2,$$

Theorem 3.17 (in case $M = 1$) follows by taking the limit $E \to \infty$. □

It remains to perform a scaling argument and drop the condition $M = 1$.

Proof. Let $\kappa, M, \delta, Z, \Gamma, V, E$ and ϕ be as in Theorem 3.17. We define $\tilde{V} : M^{-1}\Gamma \to \mathbb{R}$ by $\tilde{V}(x) = M^2 V(Mx)$, the operator $\tilde{H} = -\Delta + \tilde{V}$ in $L^2(M^{-1}\Gamma)$ with its resolution of identity $\{P_{\tilde{H}}(E) \colon E \in \mathbb{R}\}$, the bounded operator $S : L^2(\Gamma) \to L^2(M^{-1}\Gamma)$ by $(Sf)(x) = f(Mx)$, and $\tilde{\phi} \in L^2(M^{-1}\Gamma)$ by $\tilde{\phi} = S\phi$. By a straightforward calculation one obtains $\tilde{H} = M^2 SHS^{-1}$. We also define the map $\mathbb{R} \ni E \mapsto \hat{P}(E) := SP_H(E)S^{-1}$. Then $\{\hat{P}(E) \colon E \in \mathbb{R}\}$ is the resolution of identity corresponding to the operator $SHS^{-1} = M^{-2}\tilde{H}$. This follows by verifying the defining properties of the resolution of identity, cf. [Sch12, Chapters 4 and 5], using in particular that $\{P_H(E) \colon E \in \mathbb{R}\}$ is a resolution of the identity of H, and the formula $S^* = M^{-d}S^{-1}$. Now,

$$\begin{aligned}\|P_H(E)\phi\|^2_{L^2(\Gamma)} &= \left\langle \tilde{\phi}, (S^{-1})^* P_H(E) S^{-1}\tilde{\phi} \right\rangle = M^d \left\langle \tilde{\phi}, SP_H(E)S^{-1}\tilde{\phi} \right\rangle \\ &= M^d \left\langle \tilde{\phi}, P_{\tilde{H}}(E/M^2)\tilde{\phi} \right\rangle = M^d \|P_{\tilde{H}}(E/M^2)\tilde{\phi}\|^2_{L^2(M^{-1}\Gamma)}.\end{aligned}$$

Using this, the transformation formula for spectral measures, cf. [Sch12, Prop. 4.24], with $E \mapsto E/M^2$, and $\kappa \geq M \cdot 18\mathrm{e}\sqrt{d}$, we obtain

$$\begin{aligned}\|\mathrm{e}^{18\mathrm{e}\sqrt{d}\sqrt{\tilde{H}_+}}\tilde{\phi}\|^2_{L^2(M^{-1}\Gamma)} &= \int_{-\infty}^{\infty} \mathrm{e}^{18\mathrm{e}\sqrt{d}\sqrt{E_+}} \mathrm{d}\|P_{\tilde{H}}(E)\tilde{\phi}\|^2_{L^2(M^{-1}\Gamma)} \\ &= \int_{-\infty}^{\infty} \mathrm{e}^{M\cdot 18\mathrm{e}\sqrt{d}\sqrt{E_+}} \mathrm{d}\|P_{\tilde{H}}(E/M^2)\tilde{\phi}\|^2_{L^2(M^{-1}\Gamma)} \\ &= M^{-d} \int_{-\infty}^{\infty} \mathrm{e}^{M\cdot 18\mathrm{e}\sqrt{d}\sqrt{E_+}} \mathrm{d}\|P_H(E)\phi\|^2_{L^2(\Gamma)} \\ &\leq M^{-d} \int_{-\infty}^{\infty} \mathrm{e}^{\kappa\sqrt{E_+}} \mathrm{d}\|P_H(E)\phi\|^2_{L^2(\Gamma)} \\ &= M^{-d}\|\mathrm{e}^{\kappa\sqrt{H_+}}\phi\|^2_{L^2(\Gamma)} < \infty.\end{aligned}$$

This establishes $\tilde{\phi} \in \mathcal{D}(\mathrm{e}^{18\mathrm{e}\sqrt{d}\sqrt{\tilde{H}_+}})$. Applying the Theorem in case $M = 1$ with δ/M, $M^{-1}Z$, $M^{-1}\Gamma$, $\tilde{V}$ and $\tilde{\phi}$ leads to

$$\begin{aligned}\|\phi\|^2_{L^2(S_{\delta,Z}(\Gamma))} &= M^d \|\tilde{\phi}\|^2_{L^2(M^{-1}\cdot S_{\delta,Z}(\Gamma))} \\ &\geq M^d \left(\frac{\delta}{M}\right)^{C(1+\|\tilde{V}\|_\infty^{2/3} + \ln \hat{D}(\tilde{\phi}))} \|\tilde{\phi}\|^2_{L^2(M^{-1}\Gamma)} \\ &\geq \left(\frac{\delta}{M}\right)^{C(1+M^{4/3}\|V\|_\infty^{2/3} + \ln D(\phi))} \|\phi\|^2_{L^2(\Gamma)}\end{aligned}$$

where we used $\delta/M < 1/2$ and

$$\hat{D}(\tilde{\phi}) = \frac{\|\mathrm{e}^{18\mathrm{e}\sqrt{d}\sqrt{\tilde{H}_+}}\tilde{\phi}\|^2_{L^2(M^{-1}\Gamma)}}{\|\tilde{\phi}\|^2_{L^2(M^{-1}\Gamma)}} \leq \frac{\|\mathrm{e}^{\kappa\sqrt{H_+}}\phi\|^2_{L^2(\Gamma)}}{\|\phi\|^2_{L^2(\Gamma)}} = D(\phi).$$

This concludes the proof of Theorem 3.17. □

3.5. Related results on scale-free unique continuation

Let us conclude this section by citing some results which are neither implied by our main results nor stronger, but will be used in later sections.

We start with a generalization of Proposition 3.7 (the UCP for functions satisfying an eigenvalue inequality from [RMV13]), and Proposition 3.8 (the unique continuation principle on boxes Λ_L for spectral intervals with a smallness condition on the interval length from [Kle13]) to some operators with magnetic field. It is proved in [BTV15]. In fact, the results therein are stated in even greater generality, namely for some second order elliptic partial differential expressions with variable coefficients. However, we are only going to need the special case in which the leading term is the pure Laplacian. Let

$$\mathcal{H}u := -\Delta + b^{\mathrm{T}}\nabla u + cu$$

with $b \in L^\infty(\mathbb{R}^d;\mathbb{C}^d)$ and $c \in L^\infty(\mathbb{R}^d;\mathbb{C})$. For $L > 0$ we denote by $\mathcal{D}(\Delta_L)$ the domain of the Laplace operator in $L^2(\Lambda_L)$ subject to Dirichlet boundary conditions. For $L > 0$ we define the differential operator $H_L : \mathcal{D}(\Delta_L) \to L^2(\Lambda_L)$ by $H_L\psi = \mathcal{H}\psi$. If b and c satisfy

$$b = \mathrm{i}\tilde{b} \quad \text{and} \quad c = \tilde{c} + \mathrm{i}\,\mathrm{div}\,\tilde{b}/2 \tag{50}$$

for some $\tilde{b} \in L^\infty(\mathbb{R}^d,\mathbb{R}^d)$ and $\tilde{c} \in L^\infty(\mathbb{R}^d)$, then H_L is a self-adjoint operator in $L^2(\Lambda_L)$. The following proposition is a special case of Theorems 13 and 14 in [BTV15].

Proposition 3.36. *Let the above assumptions and in particular* (50) *be satisfied. Then for all* $L \in \mathbb{N}_{\mathrm{odd}}$*, all measurable and bounded* $V : \Lambda_L \to \mathbb{R}$*, all* $\psi \in \mathcal{D}(\Delta_L)$ *and* $\zeta \in L^2(\Lambda_L)$ *satisfying* $|\mathcal{H}\psi| \leq |V\psi| + |\zeta|$*, all* $\delta \in (0, 1/2)$*, all* $(1,\delta)$*-equidistributed sequences* Z*, and all* $\psi \in \mathrm{Ran}\, P_{H_L}([E-\gamma, E+\gamma])$ *with*

$$\gamma^2 = \delta^{N_1\left(1+|E|^{2/3}+\|b\|_\infty^2+\|c\|_\infty^{2/3}\right)}$$

we have

$$\|\psi\|^2_{L^2(S_{\delta,Z}(\Lambda_L))} \geq \gamma^2 \|\psi\|^2_{L^2(\Lambda_L)}.$$

Here $N_1 \geq 1$ *is a constant depending only on the dimension.*

Remark 3.37. The careful reader might notice that by extracting the statement of [BTV15, Theorems 13 and 14], one ends up with a term $\gamma/2$ instead of γ^2 in the last inequality. However, since $\gamma < 1/2$, the version we state is an immediate consequence. We stated it in this way since it makes notation more transparent in later applications.

Recall that for the union $S_{\delta,Z}(\Lambda_L)$ of δ-balls within Λ_L, distributed according to a equidistributed sequence, we had defined $W_{\delta,Z}(\Lambda_L)$ as the operator of multiplication by the characteristic function $\mathbf{1}_{S_{\delta,Z}(\Lambda_L)}$. In Subsection 5.4.1, we will apply Proposition 3.36 in the form of the following quadratic form corollary:

Proposition 3.38. *Let* (50) *be satisfied,* $E_0 \in \mathbb{R}$, $\delta \in (0, 1/2)$, *and*

$$\gamma^2 = \delta^{N_1\left(1+|E_0|^{2/3}+\|b\|_\infty^2+\|c\|_\infty^{2/3}\right)}.$$

Then for all $I \subset (-\infty, E_0]$ *with* $|I| \leq 2\gamma$, *and all* $(1, \delta)$*-equidistributed sequences* Z, *we have*

$$P_{H_L}(I) W_{\delta,Z}(\Lambda_L) P_{H_L}(I) \geq \gamma^2 P_{H_L}(I).$$

Beyond results relying on Floquet theory or Carleman estimates there are scale-free quantitative unique continuation results for a particular operator, namely for the *Landau Hamiltonian.* This is the self-adjoint operator $H_B = (-\mathrm{i}\nabla - A)^2$ with $A = (B/2)(x_2, -x_1)$ in $L^2(\mathbb{R}^2)$ where $B > 0$ is called the *magnetic field strength.*

Since $b := -2\mathrm{i}A$ and $c := (-\mathrm{i}(\nabla A) + A^2)$ are bounded on every cube Λ_L and since one can rewrite

$$\begin{aligned}(-\mathrm{i}\nabla - A)^2\phi &= (-\Delta - \mathrm{i}\nabla \cdot A - \mathrm{i}A \cdot \nabla + A^2)\phi \\ &= -\Delta\phi - 2\mathrm{i}A \cdot (\nabla\phi) + (-\mathrm{i}(\nabla A) + A^2)\phi\end{aligned}$$

this would put us into the situation of Proposition 3.36. However, since $\|A\,|_{\Lambda_L}\|_\infty$ grows linearly with L, also $\|b\|_\infty$ and $\|c\|_\infty$ would grow whence the length γ of the energy interval in Proposition 3.36 would exponentially tend to 0 as $L \to \infty$. This is not good enough for our purposes.

Thus, we proceed differently. We define a scale $L_B > 0$ by choosing

$$K_B := 2\lceil\sqrt{B/(4\pi)}\rceil, \quad L_B = K_B\sqrt{4\pi/B}, \quad \text{and} \quad \mathbb{N}_B = L_B\mathbb{N}$$

where $\lceil t\rceil$ denotes the least integer larger or equal than t. Physically, this means that cubes of side length $L \in \mathbb{N}_B$ have integer magnetic flux. Now, for $L \in \mathbb{N}_B$, the spectrum of the Landau Hamiltonian on Λ_L with periodic boundary conditions consists of an increasing sequence of isolated eigenvalues of finite multiplicity at the Landau levels $B(2n-1)$, $n = 1, 2, \ldots$, see for instance [GKS07, Section 5]. We denote the spectral projector onto the n-th Landau level by $\Pi_{n,L}$. Furthermore, for $L \in \mathbb{N}_B$ and $x \in \mathbb{R}^2/L\mathbb{Z}^2$, we write $\hat{\mathbf{1}}_{x,L}$ for the characteristic function of the cube with side length L, centered at x on the torus $\mathbb{R}^2/L\mathbb{Z}^2$. The following proposition is [GKS07, Lemma 5.3] which is an adaptation of [CHKR04, Lemma 2] which in turn uses results from [RW02b, RW02a].

Proposition 3.39. *Fix $B > 0$, $n \in \mathbb{N}$, $R > r > 0$ and $\eta > 0$. If $\kappa > 1$ and $L \in \mathbb{N}_B$ are such that $L > 2(L_B + \kappa R)$ then for all $x \in \Lambda_L$ we have*

$$\Pi_{n,L}\hat{\mathbf{1}}_{x,r}\Pi_{n,L} \geq C_0\Pi_{n,L}\left(\hat{\mathbf{1}}_{x,R} - \eta\hat{\mathbf{1}}_{x,\kappa R}\right)\Pi_{n,L} + \Pi_{n,L}\tilde{\mathcal{E}}_x\Pi_{n,L}$$

where $C_0 = C_0(n, B, r, R, \eta) > 0$ is a constant and the symmetric error operator

$$\tilde{\mathcal{E}}_x = \tilde{\mathcal{E}}_x(n, L, B, r, R, \eta)$$

satisfies

$$\|\tilde{\mathcal{E}}_x\| \leq C_{n,B,r,R,\eta}\mathrm{e}^{-m_{n,B}L}$$

for some constants $C_{n,B,r,R,\eta} > 0$ and $m_{n,B} > 0$ not depending on x.

In Subsection 5.4.2, we will use this in the form of the following lemma:

Lemma 3.40 ([TV16a])**.** *Fix $B > 0$, and $n \in \mathbb{N}$. Then there exist $C_1 = C_1(n, B, r) = C_0(n, B, r, 4, 1/162)/4$, and $L_0 = L_0(n, B, r) > 0$ such that for all $L \in \mathbb{N}_B$ with $L \geq L_0$, all $r \in (0, 1/2)$ and all r-equidistributed sequences we have*

$$\Pi_{n,L}W_{\delta,Z}(\Lambda_L)\Pi_{n,L} \geq C_1\Pi_{n,L}.$$

Remark 3.41. This bound can also be found in [RM12, Lemma 5.3], where it is used to prove a Wegner estimate for the Delone-alloy-type model with the method from [CHK07].

Proof. We choose a large $L \in \mathbb{N}_B$ to be determined later and apply Proposition 3.39 with r, $R = 4$, $\kappa = 2$ and $\eta = 1/162$. Recall that $r < 1/2 < R$. We estimate

$$\begin{aligned}\Pi_{n,L}W_{\delta,Z}(\Lambda_L)\Pi_{n,L} &\geq \sum_{j\in\mathbb{Z}^2 : B(x_j,r)\subset\Lambda_L} \Pi_{n,L}\hat{\mathbf{1}}_{x_j,r}\Pi_{n,L} \\ &\geq C_0 \sum_{j\in\mathbb{Z}^2 : B(x_j,r)\subset\Lambda_L} \left(\Pi_{n,L}(\hat{\mathbf{1}}_{x_j,4} - \eta\hat{\mathbf{1}}_{x_j,8})\Pi_{n,L} + \Pi_{n,L}\tilde{\mathcal{E}}_{x_j}\Pi_{n,L}\right).\end{aligned}$$

Since $x_j \in \Lambda_1(j)$ for all $j \in \mathbb{Z}^2$, we have for $L \geq 3$ that for every $x \in \Lambda_L$ there is $j \in \mathbb{Z}^2$ with $B(x_j, r) \subset \Lambda_L$ such that $x \in \Lambda_4(x_j)$. Therefore,

$$\bigcup_{j\in\mathbb{Z}^2 : B(x_j,r)\subset\Lambda_L} \hat{\Lambda}_4(x_j) \supset \mathbb{R}^2/L\mathbb{Z}^2, \quad \text{(the } L-\text{torus)}.$$

Furthermore, given $x \in \Lambda_L$, there are at most 81 elementary cells $\Lambda_1(j)$ of $\mathbb{Z}^2$ in which $\hat{\mathbf{1}}_{x_j,8}$ can be non-zero. Hence, we can bound the sum from below by

$$C_0\left(\Pi_{n,L} - \eta 81\Pi_{n,L}\right) + \sum_{j\in\mathbb{Z}^2 : B(x_j,r)\subset\Lambda_L} \Pi_{n,L}\tilde{\mathcal{E}}_{x_j}\Pi_{n,L} = \frac{C_0}{2}\Pi_{n,L} + \Pi_{n,L}\mathcal{E}_L\Pi_{n,L} \quad (51)$$

with a symmetric error operator $\mathcal{E}_L$ satisfying

$$\|\mathcal{E}_L\| \leq C_{n,B} \left(\frac{L}{L_B}\right)^2 \mathrm{e}^{-m_{n,B}L}.$$

This implies that there is $\tilde{L}_0 > 0$ such that for all $L \in \mathbb{N}_B$ with $L \geq \tilde{L}_0$ we have $\|\mathcal{E}_L\| \leq C_0/4$ whence in particular $\mathcal{E}_L \geq -C_0/4 \cdot \mathrm{Id}$ in quadratic form sense. Since we used $L \geq 3$ and since inequality (51) requires $L \geq 2(L_B + 8)$, we need $L \geq L_0 := \max\{\tilde{L}_0, 2(L_B + 8), 3\}$ to deduce the estimate

$$\Pi_{n,L} W_{\delta,Z}(\Lambda_L) \Pi_{n,L} \geq \frac{C_0}{4} \Pi_{n,L}. \qquad \square$$

The last type of unique continuation principle we want to state is from the realm of Fourier analysis: the so-called Logvinenko-Sereda theorem. It has been proved on $\mathbb{R}$ by Logvinenko and Sereda in [LS74], and independently by Kacnel'son in [Kac73], improved to $\mathbb{R}^d$ in [Kov00], and recently, a scale-free variant on finite cubes with periodic boundary conditions (or tori) has been proved in [EV16]. We also refer to the recent [BPJ18] for a related result. It is an analog of Theorems 3.9 and 3.13 which relies on tools from Fourier analysis. While it has the drawback that it is a priori restricted to the pure Laplacian, its advantage over the results in Sections 3.2, and 3.5 is that it allows for a more general geometry, namely *thick sets* instead of equidistributed union of δ-balls as in Definition 3.26.

Definition 3.42. Let $a = (a_1, \dots, a_d) \in \mathbb{R}^d$ with $a_j > 0$ for all $j = 1, \dots, d$, and $\gamma \in (0, 1]$. A measurable subset $S \subset \mathbb{R}^d$ is (γ, a)-*thick* if for every parallelepiped $P = [x_1 - a_1/2, x_1 + a_1/2] \times \ldots \times [x_d - a_d/2, x_d + a_d/2] \subset \mathbb{R}^d$ we have

$$|S \cap P| \geq \gamma |P|.$$

We can now state the multi-dimensional Logvinenko-Sereda theorem. It holds for L^p spaces. However, in Section 6, we will only use the L^2 variant.

Proposition 3.43 ([Kov00])**.** *Let $f \in L^p(\mathbb{R}^d)$ with $p \in [1, \infty]$, $S \subset \mathbb{R}^d$ be a (γ, a)-thick set, and assume that the Fourier transform $\hat{f}$ of f satisfies*

$$\operatorname{supp} \hat{f} \subset \bigtimes_{j=1}^{d} \left[x_j - \frac{b_j}{2}, x_j + \frac{b_j}{2}\right] \quad \textit{for some} \quad x_j \in \mathbb{R} \textit{ and } b_j > 0, \quad j \in \{1, \dots, d\}.$$

Then

$$\|f\|_{L^p(S)} \geq \left(\frac{\gamma}{C^d}\right)^{C(d + \langle a, b \rangle)} \|f\|_{L^p(\mathbb{R}^d)}.$$

Here C is a constant, and $\langle a, b \rangle$ denotes the standard Euclidean scalar product of $a, b \in \mathbb{R}^d$.

There is an analog variant on $L^p(\Lambda_L)$ or more precisely on L^2 of the torus $\mathbb{T}_L^d := \mathbb{R}^d/(2\pi L\mathbb{Z}^d)$. For $f \in L^p(\mathbb{T}_L^d)$, its Fourier coefficients $\hat{f}(k)$, $k \in \left(\frac{1}{L}\mathbb{Z}\right)^d$, are given by

$$\hat{f}\left(\frac{k_1}{L}, \dots, \frac{k_d}{L}\right) = \frac{1}{(2\pi L)^d} \int_{\mathbb{T}_L^d} f(x) \mathrm{e}^{-\frac{\mathrm{i}}{L} x \cdot k} \mathrm{d}x$$

if they exist. We say that $\hat{f}$ has support in $B \subset \mathbb{R}^d$, if $\hat{f}(\xi) = 0$ for all $\xi \in \frac{1}{L}\mathbb{Z}^d \backslash B$. It is important to note that if L varies, then also the set on which the sequence of Fourier coefficients of $f \in L^2(\mathbb{T}_L^d)$ is supported, will change.

Proposition 3.44 ([EV16]). *Let $L > 0$, $\mathbb{T}_L^d = [0, 2\pi L]^d$, $f \in L^p(\mathbb{T}_L^d)$ with $p \in [1, \infty]$,*

$$\operatorname{supp} \hat{f} \subset \bigtimes_{j=1}^{d} \left[x_j - \frac{b_j}{2}, x_j + \frac{b_j}{2}\right] \quad \textit{for some} \quad x_j \in \mathbb{R} \textit{ and } b_j > 0, \quad j \in \{1, \dots, d\},$$

and $S \subset \mathbb{R}^d$ be a (γ, a)-thick set with $0 < a_j \leq 2\pi L$ for $j \in \{1, \dots, d\}$. Then

$$\|f\|_{L^p(S \cap \mathbb{T}_L^d)} \geq \left(\frac{\gamma}{C^d}\right)^{C a \cdot b + \frac{6d+1}{p}} \|f\|_{L^p(\mathbb{T}_L^d)}$$

Here C is a constant, and $a \cdot b$ denotes the standard Euclidean scalar product of $a, b \in \mathbb{R}^d$.

An important observation is that the Fourier transform or the sequence of Fourier coefficients are related to the Laplacian in $L^2(\mathbb{R}^d)$ or on $L^2(\mathbb{T}_L^d)$ with periodic boundary conditions, respectively. In fact, for any Borel-set $B \subset \mathbb{R}$, the orthogonal projector $P_{-\Delta}(B))$ is the orthogonal projector onto the closed subspace

$$\left\{ f \in L^2(\mathbb{R}^d) \text{ or } L^2(\mathbb{T}_L^d) \colon \operatorname{supp} \hat{f} \subset \{\xi \in \mathbb{R}^d \colon |\xi|^2 \in B\} \right\} \subset L^2(\mathbb{R}^d) \text{ or } L^2(\mathbb{T}_L^d).$$

This yields the following corollary, see also [EV18, Section 5]:

Corollary 3.45. *Let $d \in \mathbb{N}$, $\Gamma = \mathbb{T}_{L/(2\pi)}^d = \Lambda_L$ for some $L > 0$ or $\Gamma = \mathbb{R}^d$, and denote by Δ the Laplacian on Γ, where we impose periodic boundary conditions if $\Gamma = \Lambda_L$. Let $S \subset \mathbb{R}^d$ be a (γ, a)-thick set with $a_j \leq L$ for $j \in \{1, \dots, d\}$, and let $E \geq 0$. Then, for all $f \in \operatorname{Ran} P_{-\Delta}(E)$, we have*

$$\|f\|_{L^2(\Gamma \cap S)} \geq \left(\frac{\gamma}{C^d}\right)^{3d + \frac{1}{2} + |a|\sqrt{E}} \|f\|_{L^2(\Gamma)}.$$

where $C > 0$ is a universal constant.

Let us note that by passing to a cube of side length $2L$, it is rather straightforward to pass from periodic boundary conditions to Dirichlet and Neumann boundary conditions. We recover the statement of Theorems 3.9, and 3.13, however only in cases $\Gamma = \mathbb{R}^d$ or $\Gamma = \Lambda_L$ and $V \equiv 0$. The advantage of Corollary 3.45 is that it requires a less restrictive geometric setting S (thick sets) than our results, which require subsets of unions of equidistributed δ-balls. Recent results on necessary and sufficient conditions for null-controllability of the heat equation on $\mathbb{R}^d$ [WWZZ17, EV18] indicate that thick sets should be the optimal geometric configuration to expect here.

4. Application to perturbation of spectral edges

This section is based on the joint preprint [NTTV18b] with A. Seelmann, I. Nakić, M. Tautenhahn, and I. Veselić. The purpose of this section is to apply the unique continuation principles from the previous section to establish lower bounds on the sensitivity of the spectrum of self-adjoint operators with respect to certain, non-negative perturbations. For that goal, we start with some abstract results on perturbation of spectra of self-adjoint operators in Section 4.1 before combining them with our unique continuation estimates in Section 4.2 below.

4.1. Perturbation of spectra of self-adjoint operators

Throughout this section all operators are defined on the same Hilbert space $\mathcal{H}$ and we recall that the domain of an operator A is denoted by $\mathcal{D}(A)$. If A is a self-adjoint operator, B a bounded, symmetric perturbation, and $\lambda = \lambda(A)$ an isolated (single) eigenvalue of A, then by a standard Neumann series argument, there is a well-defined, real analytic map $t \mapsto \lambda(A + tB)$ in a neighborhood of $t = 0$ which describes the movement of the isolated eigenvalue under the perturbation, see for instance [Ves08b] and [See14b] for some background.

Similarly, we can consider edges of connected components of the essential spectrum or briefly spectral edges. The following lemma formulates the fact that such spectral edges are locally stable under perturbations and establishes that there exists a Lipschitz continuous function $t \mapsto f(t)$ describing the spectral edge of $A + tB$.

Lemma 4.1. *Let A be self-adjoint, $B \neq 0$ bounded, symmetric and non–negative. Let $a \in \sigma_{\mathrm{ess}}(A)$ and let $b > a$ be such that $(a, b) \cap \sigma_{\mathrm{ess}}(A) = \emptyset$. Let $t_0 = (b - a)/\|B\|$. Then $f\colon (-t_0, t_0) \to \mathbb{R}$, $f(t) = \sup(\sigma_{\mathrm{ess}}(A + tB) \cap (-\infty, b - t_-\|B\|))$ satisfies for all $t \in (-t_0, t_0)$,*

(a) $f(0) = a$,

(b) $f(t) \in \sigma_{\mathrm{ess}}(A + tB)$,

(c) $(f(t), b - t_-\|B\|) \cap \sigma_{\mathrm{ess}}(A + tB) = \emptyset$, *where* $t_- = \max\{0, -t\}$,

(d) f is Lipschitz continuous with Lipschitz coefficient $\|B\|$.

In fact, Lemma 4.1 describes *left edges* of components of the essential spectrum. One can also consider *right edges*, i.e. the analogous situation where $(a, b) \cap \sigma_{\mathrm{ess}}(A) = \emptyset$ and $b \in \sigma_{\mathrm{ess}}(A)$, cf. Cor. 4.18. Lemma 4.1 holds as well in case of indefinite B, provided one replaces t_0 by $(b - a)/(2\|B\|)$ and (c) by $(f(t), b - |t|\|B\|) \cap \sigma_{\mathrm{ess}}(A + tB) = \emptyset$.

Lemma 4.1 follows immediately from

Lemma 4.2. *Let A be self-adjoint and B bounded and non-negative. If $(a,b) \subset \mathbb{R}$, then*

$$(a,b) \cap \sigma_{\mathrm{ess}}(A) = \emptyset \quad \Rightarrow \quad (a + \|B\|, b) \cap \sigma_{\mathrm{ess}}(A+B) = \emptyset, \tag{52}$$

and

$$(a,b) \cap \sigma_{\mathrm{ess}}(A) \neq \emptyset \quad \Rightarrow \quad (a, b + \|B\|) \cap \sigma_{\mathrm{ess}}(A+B) \neq \emptyset. \tag{53}$$

Here we use the convention that $(c,d) = \emptyset$ if $c \geq d$.

Remark 4.3. Since the essential spectrum is closed, the statements of the Lemma 4.2 also hold if the corresponding intervals are replaced by closed or semi-closed intervals. Furthermore, analogous statements for negative perturbations $-B$ follow by applying the contraposition of (52) and (53).

Proof of Lemma 4.2. Let us assume $(a,b) \cap \sigma_{\mathrm{ess}}(A) = \emptyset$. Then for every $\epsilon > 0$, A has at most finitely many eigenvalues with finite multiplicity in $(a+\epsilon, b-\epsilon)$. Thus, there is a finite rank perturbation T such that $(a+\epsilon, b-\epsilon) \cap \sigma(A+T) = \emptyset$. From Proposition 2.1 in [See17] we infer that $(a + \epsilon + \|B\|, b - \epsilon) \cap \sigma(A + T + B) = \emptyset$. Since finite rank perturbations leave the essential spectrum unchanged, we obtain

$$(a + \epsilon + \|B\|, b - \epsilon) \cap \sigma_{\mathrm{ess}}(A+B) = \emptyset \quad \text{for all} \quad \epsilon > 0.$$

This shows (52). The relation (53) is equivalent to (52) by contraposition. □

Lemma 4.1 yields an *upper bound* on the maximal movement of spectral edges, namely the norm of the perturbation. Our goal is now to complement this by *lower bounds* on the lifting of spectral edges and eigenvalues in gaps of the essential spectrum. All these results will hold under an abstract positivity condition on the perturbation.

4.1.1. Below the essential spectrum

In order to explain the underlying mechanism we first consider the infimum of the essential spectrum and eigenvalues below it. For a lower semibounded self-adjoint operator A, we set $\lambda_\infty(A) = \inf \sigma_{\mathrm{ess}}(A)$. Note that $\lambda_\infty(A) = \infty$ if $\sigma_{\mathrm{ess}}(A) = \emptyset$. Moreover, we denote by $\{\lambda_k(A)\}_{k \in \mathbb{N}}$ the sequence of eigenvalues of A below $\sigma_{\mathrm{ess}}(A)$, enumerated non-decreasingly and counting multiplicities. If there are exactly $N \in \mathbb{N}_0$ eigenvalues in $(-\infty, \lambda_\infty(A))$ then we set $\lambda_k(A) = \lambda_\infty(A)$ for all $k \in \mathbb{N}$ with $k > N$. If A has purely discrete spectrum then the sequence $\lambda_k(A)$, $k \in \mathbb{N}$, is the non-decreasing sequence of eigenvalues of A.

Lemma 4.4. *Let A be self-adjoint and lower semibounded, B bounded and symmetric, $E \in \mathbb{R}$, $\kappa \in \mathbb{R}$, and*

$$\forall x \in \operatorname{Ran}(P_{A+B}(E))\colon \quad \langle x, Bx\rangle \geq \kappa \|x\|^2.$$

Then for all $k \in \mathbb{N} \cup \{\infty\}$ such that $\lambda_k(A+B) < E$, we have

$$\lambda_k(A+B) \geq \lambda_k(A) + \kappa.$$

Proof. Let $\epsilon_0 = E - \lambda_k(A+B) > 0$. Then we have by assumption

$$\begin{aligned}\lambda_k(A+B) &= \inf_{0<\epsilon\leq\epsilon_0} \sup_{\substack{x\in\operatorname{Ran} P_{A+B}(\lambda_k(A+B)+\epsilon)\\ \|x\|=1}} (\langle x, Ax\rangle + \langle x, Bx\rangle)\\ &\geq \inf_{0<\epsilon\leq\epsilon_0} \sup_{\substack{x\in\operatorname{Ran} P_{A+B}(\lambda_k(A+B)+\epsilon)\\ \|x\|=1}} \langle x, Ax\rangle + \kappa.\end{aligned}$$

Since $\dim \operatorname{Ran} P_{A+B}(\lambda_k(A+B)+\epsilon) \geq k$ for $\epsilon > 0$, we have by the standard variational principle

$$\begin{aligned}\sup_{\substack{x\in\operatorname{Ran} P_{A+B}(\lambda_k(A+B)+\epsilon)\\ \|x\|=1}} \langle x, Ax\rangle &= \sup_{\substack{\mathcal{L}\subset\operatorname{Ran} P_{A+B}(\lambda_k(A+B)+\epsilon)\\ \dim\mathcal{L}=k}} \sup_{\substack{x\in\mathcal{L}\\ \|x\|=1}} \langle x, Ax\rangle\\ &\geq \inf_{\substack{\mathcal{L}\subset\operatorname{Ran} P_{A+B}(\lambda_k(A+B)+\epsilon)\\ \dim\mathcal{L}=k}} \sup_{\substack{x\in\mathcal{L}\\ \|x\|=1}} \langle x, Ax\rangle\\ &\geq \inf_{\substack{\mathcal{L}\subset\mathcal{D}(A)\\ \dim\mathcal{L}=k}} \sup_{\substack{x\in\mathcal{L}\\ \|x\|=1}} \langle x, Ax\rangle = \lambda_k(A).\end{aligned}$$

□

4.1.2. Ordering from right to left in gaps

If one considers eigenvalues in gaps of the essential spectrum, the situation becomes more intricate. In every such gap, A may have finitely or infinitely many eigenvalues with possible accumulation points at the upper and lower edge of the gap. Therefore, the notion of the k-th eigenvalue in a gap might become meaningless and we introduce a different concept in order to count eigenvalues by fixing a reference point $\gamma \in \mathbb{R}$ and counting eigenvalues below or above γ. We start by considering eigenvalues below a reference point.

For a self-adjoint (not necessarily semibounded) operator A, and $\gamma \in \rho(A) \cap \mathbb{R}$, we set $\lambda^{\leftarrow}_{\infty,\gamma}(A) = \sup\{\lambda < \gamma\colon \lambda \in \sigma_{\mathrm{ess}}(A)\}$. If there is no essential spectrum below γ this means $\lambda^{\leftarrow}_{\infty,\gamma}(A) = -\infty$, otherwise $\lambda^{\leftarrow}_{\infty,\gamma}(A)$ is the right edge of the component of $\sigma_{\mathrm{ess}}(A)$ below γ. Moreover, we denote by $\{\lambda^{\leftarrow}_{k,\gamma}(A)\}_{k\in\mathbb{N}}$ the sequence of eigenvalues of A in $(\lambda^{\leftarrow}_{\infty,\gamma}(A), \gamma)$, enumerated non-increasingly and counting multiplicities.

If there are infinitely many eigenvalues inside $(\lambda^{\leftarrow}_{\infty,\gamma}(A),\gamma)$, then it follows that $\lambda^{\leftarrow}_{\infty,\gamma}(A) = \inf_k \lambda^{\leftarrow}_{k,\gamma}(A) = \lim_k \lambda^{\leftarrow}_{k,\gamma}(A)$. If there are exactly $N \in \mathbb{N}_0$ eigenvalues inside $(\lambda^{\leftarrow}_{\infty,\gamma}(A),\gamma)$ we set $\lambda^{\leftarrow}_{k,\gamma}(A) = \lambda^{\leftarrow}_{\infty,\gamma}(A)$ for all $k \in \mathbb{N}$ with $k > N$. Moreover, we define

$$\mathcal{M}^-_\gamma = \{\mathcal{M} \colon \mathcal{M} \text{ is a maximal } (A-\gamma)\text{-non-positive subspace of } \mathcal{D}(A)\}.$$

Here, a subspace $\mathcal{M} \subset \mathcal{H}$ is called *A-non-positive* if $\langle x, Ax\rangle \le 0$ for all $x \in \mathcal{M}$ and $\mathcal{M}$ is called *maximal* if there is no A-non-positive subspace which properly contains $\mathcal{M}$.

For $\gamma \in \rho(A) \cap \mathbb{R}$ we define

$$\operatorname{dist}^{\leftarrow}(\gamma,\sigma(A)) = \operatorname{dist}(\gamma,\sigma(A)\cap(-\infty,\gamma]) = \gamma - \sup\left(\sigma(A)\cap(-\infty,\gamma]\right), \text{ and}$$
$$\operatorname{dist}^{\rightarrow}(\gamma,\sigma(A)) = \operatorname{dist}(\gamma,\sigma(A)\cap[\gamma,\infty)) = \inf(\sigma(A)\cap[\gamma,\infty)) - \gamma.$$

Note that $\operatorname{dist}^{\leftarrow}(\gamma,\sigma(A)) = \infty$ if $\gamma < \inf\sigma(A)$, and that $\operatorname{dist}^{\rightarrow}(\gamma,\sigma(A)) = \infty$ if $\gamma > \sup\sigma(A)$.

Lemma 4.5. *Let A be self-adjoint, $\gamma \in \rho(A)\cap\mathbb{R}$, B bounded and symmetric, satisfying either*

(i) $\|B\| < \frac{1}{2}\operatorname{dist}(\gamma,\sigma(A))$, *or*

(ii) $0 \le B < \operatorname{dist}(\gamma,\sigma(A))$.

Then $\operatorname{Ran} P_{A+B}(\gamma) \cap \mathcal{D}(A)$ *is a maximal $(A-\gamma)$-non-positive subspace of $\mathcal{D}(A)$.*

Proof. For all $x \in \operatorname{Ran} P_{A+B}(\gamma)\cap\mathcal{D}(A)$ with $\|x\| = 1$ we have

$$\langle x,(A-\gamma)x\rangle = \langle x,(A+B-\gamma)x\rangle - \langle x,Bx\rangle \le -\operatorname{dist}^{\leftarrow}(\gamma,\sigma(A+B)) - \langle x,Bx\rangle.$$

If (ii) is satisfied, this is clearly negative. If (i) is satisfied, we use $\operatorname{dist}^{\leftarrow}(\gamma,\sigma(A+B)) \ge \operatorname{dist}(\gamma,\sigma(A)) - \|B\|$ and conclude

$$\langle x,(A-\gamma)x\rangle < -\operatorname{dist}^{\leftarrow}(\gamma,\sigma(A)) + \|B\| - \langle x,Bx\rangle \le -\operatorname{dist}^{\leftarrow}(\gamma,\sigma(A)) + 2\|B\| < 0.$$

Hence, $\operatorname{Ran} P_{A+B}(\gamma)\cap\mathcal{D}(A)$ is an $(A-\gamma)$-non-positive subspace of $\mathcal{D}(A)$.

Let us assume it is not maximal. Then we can choose $x \in \operatorname{Ran} P_{A+B}(\gamma)^\perp \cap \mathcal{D}(A)$ satisfying $\|x\| = 1$ and $\langle x,(A-\gamma)x\rangle \le 0$. In case (i), we use $\|B\| < \operatorname{dist}(\gamma,\sigma(A))/2$, to obtain

$$\langle x,(A+B-\gamma)x\rangle \ge \operatorname{dist}(\gamma,\sigma(A+B)) \ge \operatorname{dist}(\gamma,\sigma(A)) - \|B\| \ge \frac{1}{2}\operatorname{dist}(\gamma,\sigma(A)).$$

This leads to the contradiction

$$0 \geq \langle x, (A-\gamma)x\rangle = \langle x, (A+B-\gamma)x\rangle - \langle x, Bx\rangle \geq \frac{1}{2}\operatorname{dist}(\gamma, \sigma(A)) - \|B\| > 0.$$

In case (ii), we use $\langle x, (A+B-\gamma)x\rangle \geq \operatorname{dist}^{\rightarrow}(\gamma, \sigma(A))$, and find the contradiction

$$0 \geq \langle x, (A-\gamma)x\rangle = \langle x, (A+B-\gamma)x\rangle - \langle x, Bx\rangle \geq \operatorname{dist}^{\rightarrow}(\gamma, \sigma(A)) - \|B\| > 0. \quad \square$$

We are again going to apply a variational principle for eigenvalues, however this time, we shall need a more sophisticated variant, suitable for eigenvalues in spectral gaps. The following proposition is a reformulation of Theorem 3.1 in [LS16], obtained by replacing T by $-T$, and by working with operator domains instead of quadratic form domains.

Proposition 4.6. *Let A be self-adjoint. For all $\gamma \in \rho(A) \cap \mathbb{R}$ and $k \in \mathbb{N}$ we have*

$$\lambda^{\leftarrow}_{k,\gamma}(a) = \inf_{\mathcal{M} \in \mathcal{M}^{-}_{\gamma}} \inf_{\substack{\mathcal{L} \subset \mathcal{M} \\ \dim \mathcal{L} = k-1}} \sup_{\substack{x \in \mathcal{M} \\ x \perp \mathcal{L} \\ \|x\|=1}} \langle x, Ax\rangle. \tag{54}$$

Remark 4.7. Let us briefly comment on the (trivial) situation where $\gamma < \inf \sigma(A)$ and explain that formula (54) still makes sense. In this case, by definition we have $\lambda^{\leftarrow}_{k,\gamma}(A) = -\infty$. Since we also have $\mathcal{M}^{-}_{\gamma} = \{0\}$, the last supremum on the right hand side of Eq. (54) is taken over the empty set. Hence, the right hand side of Eq. (54) is as well minus infinity as it should be.

With Lemma 4.5 and Proposition 4.6 at our disposal we are prepared to prove

Theorem 4.8. *Let A be self-adjoint, $\gamma \in \rho(A) \cap \mathbb{R}$, $\kappa \in \mathbb{R}$, B bounded and symmetric, satisfying either*

(i) $\|B\| < \frac{1}{2}\operatorname{dist}(\gamma, \sigma(A))$, *or*

(ii) $0 \leq B < \operatorname{dist}(\gamma, \sigma(A))$.

Assume further

$$\forall x \in \operatorname{Ran}(P_{A+B}(\gamma)): \quad \langle x, Bx\rangle \geq \kappa \|x\|^2.$$

Then for all $k \in \mathbb{N} \cup \{\infty\}$, we have

$$\lambda^{\leftarrow}_{k,\gamma}(A+B) \geq \lambda^{\leftarrow}_{k,\gamma}(A) + \kappa.$$

Proof. First consider $k \in \mathbb{N}$. Since $\|B\| < \operatorname{dist}(\gamma, \sigma(A))$ we have $\gamma \in \rho(A+B)$. We apply the standard variational principle to $-(A+B)|_{\operatorname{Ran} P_{A+B}(\gamma)}$ and obtain

$$\begin{aligned}\lambda^{\leftarrow}_{k,\gamma}(A+B) &= \inf_{\substack{\mathcal{L}\subset \operatorname{Ran} P_{A+B}(\gamma)\cap\mathcal{D}(A)\\ \dim\mathcal{L}=k-1}} \sup_{\substack{x\in \operatorname{Ran} P_{A+B}(\gamma)\cap\mathcal{D}(A)\\ x\perp\mathcal{L}\\ \|x\|=1}} \langle x, (A+B)x\rangle \\ &\geq \inf_{\substack{\mathcal{L}\subset \operatorname{Ran} P_{A+B}(\gamma)\cap\mathcal{D}(A)\\ \dim\mathcal{L}=k-1}} \sup_{\substack{x\in \operatorname{Ran} P_{A+B}(\gamma)\cap\mathcal{D}(A)\\ x\perp\mathcal{L}\\ \|x\|=1}} \langle x, Ax\rangle + \kappa.\end{aligned}$$

By Lemma 4.5, the subspace $\operatorname{Ran} P_{A+B}(\gamma)\cap\mathcal{D}(A)$ is a maximal $(A-\gamma)$-non-positive subspace of $\mathcal{D}(A)$. Hence,

$$\lambda^{\leftarrow}_{k,\gamma}(A+B) \geq \inf_{\mathcal{M}\in\mathcal{M}^-_\gamma} \inf_{\substack{\mathcal{L}\subset\mathcal{M}\\ \dim\mathcal{L}=k-1}} \sup_{\substack{x\in\mathcal{M}\\ x\perp\mathcal{L}\\ \|x\|=1}} \langle x, Ax\rangle + \kappa.$$

For $k \in \mathbb{N}$, the statement of the theorem follows from Theorem 4.6.

If there are infinitely many eigenvalues $\lambda^{\leftarrow}_{k,\gamma}(A+B)$ in $(\lambda^{\leftarrow}_{\infty,\gamma}(A+B), \gamma)$ then

$$\lambda^{\leftarrow}_{\infty,\gamma}(A+B) = \inf_{k\in\mathbb{N}} \lambda^{\leftarrow}_{k,\gamma}(A+B) \geq \inf_{k\in\mathbb{N}} \lambda^{\leftarrow}_{k,\gamma}(A) + \kappa = \lambda^{\leftarrow}_{\infty,\gamma}(A) + \kappa. \qquad \square$$

In case of a non-negative perturbation, one would actually expect such lifting estimates as long as the norm of $B \geq 0$ is smaller than the distance between the reference point $\gamma \in \rho(A) \cap \mathbb{R}$ and the closest spectral value below γ. This is the statement of the following theorem which, however, requires a stronger assumption on the positivity of B.

Theorem 4.9. *Let A be self-adjoint, $\gamma \in \rho(A)\cap\mathbb{R}$, $\kappa \in \mathbb{R}$, B bounded and symmetric satisfying $0 \leq B < \operatorname{dist}^{\leftarrow}(\gamma, \sigma(A))$. Let n be the smallest integer larger than $\|B\|/\operatorname{dist}^{\rightarrow}(\gamma, \sigma(A))$, and assume that*

$$\forall x \in \bigcup_{j=1}^{n} \operatorname{Ran}(P_{A+jB/n}(\gamma))\colon \quad \langle x, Bx\rangle \geq \kappa\|x\|^2.$$

Then for all $k \in \mathbb{N}\cup\{\infty\}$, we have

$$\lambda^{\leftarrow}_{k,\gamma}(A+B) \geq \lambda^{\leftarrow}_{k,\gamma}(A) + \kappa.$$

Proof. First assume additionally that $B < \operatorname{dist}^{\rightarrow}(\gamma, \sigma(A))$. Then $0 \leq B < \operatorname{dist}(\gamma, \sigma(A))$ and the statement follows from Theorem 4.8.

Now we drop the assumption $B < \operatorname{dist}^{\rightarrow}(\gamma, \sigma(A))$, and consider the case $k \in \mathbb{N}\cup\{\infty\}$ and $0 \leq B < \operatorname{dist}^{\leftarrow}(\gamma, \sigma(A))$. Recall that $n \in \mathbb{N}$ satisfies $0 \leq B/n < \operatorname{dist}^{\rightarrow}(\gamma, \sigma(A))$. It follows that

$$0 \leq B/n < \operatorname{dist}(\gamma, \sigma(A+jB/n)) \quad \text{for } j \in \{0, \ldots, n-1\},$$

see [See17, Proposition 2.1]. We now apply the result obtained above iteratively for $j \in \{0, \dots, n-1\}$ to the operator $A + jB/n$ instead of A, with the perturbation B replaced by B/n, and with κ replaced by κ/n. □

We now turn to $\sigma_{\text{ess}}(A)$ itself. If we are only interested in a lifting estimate for edges of $\sigma_{\text{ess}}(A)$ the location of eigenvalues within the gap of $\sigma_{\text{ess}}(A)$ should be irrelevant. Theorem 4.10 below makes this precise.

Theorem 4.10. *Let A be self-adjoint, $(a,b) \cap \sigma_{\text{ess}}(A) = \emptyset$, $a \in \sigma_{\text{ess}}(A)$, $\kappa \geq 0$, and B bounded and symmetric satisfying $0 \leq B < b - a$. Assume that*

$$\forall x \in \bigcup_{t \in [0,1]} \operatorname{Ran} P_{A+tB}(b) \colon \quad \langle x, Bx \rangle \geq \kappa \|x\|^2.$$

Then

$$[a + \kappa, b) \cap \sigma_{\text{ess}}(A + B) \neq \emptyset.$$

Proof. Define $\epsilon = b - a - \|B\| > 0$. We define a sequence of disjoint intervals

$$I_k = \left(b - \epsilon + \frac{\epsilon}{2^k}, b - \epsilon + \frac{3\epsilon}{2^{k+1}} \right), \quad k \in \mathbb{N}.$$

Note that by (52) in Lemma 4.2, for all $t \in [0,1]$ we have $\sigma_{\text{ess}}(A+tB) \cap (b-\epsilon, b) = \emptyset$.

Choose $\gamma_1 \in \rho(A) \cap I_1$ and $s_1 = \min\{\operatorname{dist}^{\leftarrow}(\gamma_1, \sigma(A))/\|B\|, 1\}$. We will now recursively define sequences

$$(\gamma_n)_{n \in \mathbb{N}} \subset \bigcup_{k \in \mathbb{N}} I_k \quad \text{and} \quad (s_n)_{n \in \mathbb{N}} \subset [0,1].$$

For that purpose, we will denote by $k_n \in \mathbb{N}$, the (unique) index such that $\gamma_n \in I_{k_n}$. If $s_n = 1$, we set $s_m = 1$ and $\gamma_m = \gamma_n$ for all $m > n$. Else, given $n \in \mathbb{N}$, γ_n and $s_n < 1$, we choose

$$\begin{cases} \gamma_{n+1} = \gamma_n & \text{if} \quad \sigma(A + s_n B) \cap [\sup I_{k_n+1}, \gamma_n) = \emptyset, \\ \gamma_{n+1} \in I_{k_n+1} \cap \rho(A + s_n B) & \text{else,} \end{cases}$$

and set $s_{n+1} = \min\{s_n + \operatorname{dist}^{\leftarrow}(\gamma_{n+1}, \sigma(A + s_n B))/\|B\|, 1\}$. The sequence $(s_n)_{n \in \mathbb{N}}$ is monotone increasing and bounded by 1.

Assume that $\lim_n s_n = s < 1$. If there is $n_0 \in \mathbb{N}$ such that $\gamma_n = \gamma_{n_0}$ for all $n \geq n_0$, then for all $n \in \mathbb{N}$, $s_{n+1} - s_n$ is bounded from below by the distance between I_{n_0} and I_{n_0+1} which is a fixed positive number for all $n \geq n_0$. Hence, the sequence $(\gamma_n)_{n \in \mathbb{N}}$ cannot become stationary. This implies that A has infinitely many eigenvalues in

$(b-\epsilon-\|B\|s,\gamma_1)$. This is a contradiction, since $\sigma_{\mathrm{ess}}(A)\cap[b-\epsilon-\|B\|s,\gamma_1]=\emptyset$. This shows $\lim_n s_n = 1$.

By Lemma 4.2, we have $\lambda^{\leftarrow}_{\infty,\gamma}(A+tB)=\lambda^{\leftarrow}_{\infty,\tilde\gamma}(A+tB)$ for all $t\in[0,1]$ and $\gamma,\tilde\gamma\in\rho(A+tB)\cap(b-\epsilon,b)$. Given $n\in\mathbb{N}$, we apply Theorem 4.9 with $\gamma=\gamma_n$, A replaced by $A+s_nB$ and B replaced by $(s_{n+1}-s_n)B$ and obtain

$$\begin{aligned}\lambda^{\leftarrow}_{\infty,\gamma_n}(A+s_nB) &= \lambda^{\leftarrow}_{\infty,\gamma_n}(A+s_{n-1}B+(s_n-s_{n-1})B)\\ &\geq \lambda^{\leftarrow}_{\infty,\gamma_n}(A+s_{n-1}B)+(s_n-s_{n-1})\kappa\\ &= \lambda^{\leftarrow}_{\infty,\gamma_{n-1}}(A+s_{n-1}B)+(s_n-s_{n-1})\kappa.\end{aligned}$$

Iteratively, we obtain for all $n\in\mathbb{N}$ that $\lambda^{\leftarrow}_{\infty,\gamma_n}(A+s_nB)\geq\lambda^{\leftarrow}_{\infty,\gamma_1}(A)+s_n\kappa=a+s_n\kappa$, which implies

$$[a+s_n\kappa,a+s_n\|B\|]\cap\sigma_{\mathrm{ess}}(A+s_nB)\neq\emptyset.$$

Using (53) from Lemma 4.2 and $0\leq(1-s_n)B\leq B$ this yields in particular

$$(a-\delta+s_n\kappa,a+\|B\|+\delta)\cap\sigma_{\mathrm{ess}}(A+B)\neq\emptyset,\quad\text{for all }\delta>0$$

and since $\sigma_{\mathrm{ess}}(A+B)$ is closed, we find

$$[a+s_n\kappa,b)\cap\sigma_{\mathrm{ess}}(A+B)\neq\emptyset.$$

The statement follows by taking the supremum over $n\in\mathbb{N}$. □

Finally, combining the last theorem with Lemma 4.1, we arrive at two sided Lipschitz estimates on the sensitivity of upper edges of the essential spectrum.

Corollary 4.11. *Let A be self-adjoint, $B\neq0$ bounded, symmetric and non–negative, $a\in\sigma_{\mathrm{ess}}(A)$, $b>a$ such that $(a,b)\cap\sigma_{\mathrm{ess}}(A)=\emptyset$, and $t_0=(b-a)/\|B\|$. Assume that*

$$\forall x\in\bigcup_{t\in[0,1]}\operatorname{Ran}P_{A+tB}(b)\colon\quad\langle x,Bx\rangle\geq\kappa\|x\|^2.$$

Then $f(t)=\sup\left(\sigma_{\mathrm{ess}}(A+tB)\cap(-\infty,b)\right)$ satisfies

$$\kappa\epsilon\leq f(t+\epsilon)-f(t)\leq\|B\|\epsilon\quad\textit{for all } t\in[0,t_0)\textit{ and }\epsilon\in[0,t_0-t).$$

4.1.3. Ordering from left to right in gaps

We now turn to eigenvalues above a reference point and to lower edges of components of the essential spectrum.

Let A be self-adjoint. For $\gamma\in\rho(A)\cap\mathbb{R}$ we set $\lambda^{\rightarrow}_{\infty,\gamma}(A)=\inf\{\lambda>\gamma\colon\lambda\in\sigma_{\mathrm{ess}}(A)\}$. If there is no essential spectrum above γ this means $\lambda^{\rightarrow}_{\infty,\gamma}(A)=\infty$, otherwise $\lambda^{\rightarrow}_{\infty,\gamma}(A)$

is the lower edge of the component of $\sigma_{\mathrm{ess}}(A)$ above γ. Moreover, we denote by $\{\lambda^{\rightarrow}_{k,\gamma}(A)\}_{k\in\mathbb{N}}$ the sequence of eigenvalues of A in $(\gamma, \lambda^{\rightarrow}_{\infty,\gamma}(A))$, enumerated non-decreasingly and counting multiplicities. If there are infinitely many eigenvalues inside $(\gamma, \lambda^{\rightarrow}_{\infty,\gamma}(A))$ it follows that $\lambda^{\rightarrow}_{\infty,\gamma}(A) = \sup_k \lambda^{\rightarrow}_{k,\gamma}(A) = \lim_k \lambda^{\rightarrow}_{k,\gamma}(A)$. If there are exactly $N \in \mathbb{N}_0$ eigenvalues inside $(\gamma, \lambda^{\rightarrow}_{\infty,\gamma}(A))$ then we set $\lambda^{\rightarrow}_{k,\gamma}(A) = \lambda^{\rightarrow}_{\infty,\gamma}(A)$ for all $k \in \mathbb{N}$ with $k > N$.

The two main results of this subsection are the following theorems, dealing with indefinite and non-negative perturbations, respectively.

Theorem 4.12. *Let A be self-adjoint, B bounded and symmetric, $\gamma \in \rho(A) \cap \mathbb{R}$, $\|B\| \leq \operatorname{dist}(\gamma, \sigma(A))/2$, $\kappa \in \mathbb{R}$, $E > \gamma$, and*

$$\forall x \in \operatorname{Ran}(P_{A+B}(E)) \colon \quad \langle x, Bx\rangle \geq \kappa \|x\|^2.$$

Then for all $k \in \mathbb{N} \cup \{\infty\}$ with $\lambda^{\rightarrow}_{k,\gamma}(A+B) < E$ we have

$$\lambda^{\rightarrow}_{k,\gamma}(A+B) \geq \lambda^{\rightarrow}_{k,\gamma}(A) + \kappa.$$

If the perturbation B is non-negative, only the distance between γ and the closest spectral value below γ should matter which allows to relax the condition on the norm of B. This is the statement of the following theorem:

Theorem 4.13. *Let A be self-adjoint, $\gamma \in \rho(A) \cap \mathbb{R}$, $\kappa \in \mathbb{R}$, B bounded and symmetric satisfying $0 \leq B < \operatorname{dist}^{\leftarrow}(\gamma, \sigma(A))$, $E > \gamma$, and*

$$\forall x \in \operatorname{Ran}(P_{A+B}(E)) \colon \quad \langle x, Bx\rangle \geq \kappa \|x\|^2.$$

Then for all $k \in \mathbb{N} \cup \{\infty\}$ with $\lambda^{\rightarrow}_{k,\gamma}(A+B) < E$ we have

$$\lambda^{\rightarrow}_{k,\gamma}(A+B) \geq \lambda^{\rightarrow}_{k,\gamma}(A) + \kappa.$$

Now we turn to the proofs of the two Theorems. As in the previous subsection, we will apply a variational principle for eigenvalues. This time, we will employ a variant, proved by Albrecht Seelmann in the Appendix of [NTTV18b].

Proposition 4.14 ([NTTV18b, Theorem 3.16 and Theorem A.2])**.** *Let A be self-adjoint, B be bounded and symmetric, and $\gamma \in \mathbb{R}$. If*

(i) $\langle x, (A-\gamma)x\rangle \leq 0$ *for all* $x \in \operatorname{Ran} P_{A+B}(\gamma) \cap \mathcal{D}(A)$ *and*

(ii) $\|P^{\perp}_{A+B}(\gamma) P_A(\gamma)\| < 1$,

then

$$\lambda_k(A|_{\operatorname{Ran} P_A^\perp(\gamma)}) = \inf_{\substack{\mathcal{L} \subset \operatorname{Ran} P_{A+B}^\perp(\gamma) \cap \mathcal{D}(A) \\ \dim(\mathcal{L})=k}} \sup_{\substack{x \in \mathcal{L} \oplus \operatorname{Ran} P_{A+B}(\gamma) \cap \mathcal{D}(A) \\ \|x\|=1}} \langle x, Ax \rangle$$

for all $k \in \mathbb{N}$ *with* $k \leq \dim(\operatorname{Ran} P_{A+B}^\perp(\gamma))$.

Remark 4.15. One might wonder whether instead of Proposition 4.14 one could appropriately adapt Proposition 4.6 above. One would naturally try to transform one variational principle into the other, e.g. by using the sign-flip $A - \gamma \mapsto -A + \gamma$. However, this does not work directly, because the assumption

$$B \geq \kappa > 0 \quad \text{on} \quad \operatorname{Ran} P_{A+B}(E)$$

is not symmetric under the sign flip.

There also exist other variational principles which might be used instead of Proposition 4.14, e.g. the variational principles in [GLS99] or [MM15], which are formulated for quadratic forms. However, we use the above variant since at least a direct adaptation of the mentioned results does not seem to yield the optimal critical value for the norm of the perturbation $\|B\|$. Furthermore, the hypotheses of Proposition 4.14 are particularly easy to verify. In fact, in our application the criteria are ensured by the Davis-Kahan $\sin 2\Theta$ theorem on the perturbation of spectral projections. This opens up an interesting perspective on a connection between perturbations of spectral subspaces in the context of $\sin\Theta$ theorems, and perturbation theory for eigenvalues, which to our knowledge has not yet been exploited.

Before turning to the proofs of Theorem 4.12 and 4.13, we isolate the following step which is used is both proofs:

Lemma 4.16. *Let* A *be self-adjoint,* B *bounded and symmetric,* $\gamma \in \rho(A+B) \cap \mathbb{R}$, $E > \gamma$, $\kappa \in \mathbb{R}$, *and let*

$$\forall x \in \operatorname{Ran}(P_{A+B}(E)) \colon \quad \langle x, Bx \rangle \geq \kappa \|x\|^2.$$

Then for all $k \in \mathbb{N}$ *with* $\lambda_{k,\gamma}^{\rightarrow}(A+B) < E$ *we have*

$$\lambda_{k,\gamma}^{\rightarrow}(A+B) \geq \inf_{\substack{\mathcal{L} \subset \operatorname{Ran} P_{A+B}^\perp(\gamma) \cap \mathcal{D}(A) \\ \dim \mathcal{L}=k}} \sup_{\substack{x \in \mathcal{L} \oplus (\operatorname{Ran} P_{A+B}(\gamma) \cap \mathcal{D}(A)) \\ \|x\|=1}} \langle x, Ax \rangle + \kappa. \tag{55}$$

Proof. By assumption, we have for $\epsilon_0 = E - \lambda^{\to}_{k,\gamma}(A+B)$

$$\lambda^{\to}_{k,\gamma}(A+B) = \inf_{0<\epsilon<\epsilon_0} \sup_{\substack{x\in \operatorname{Ran} P_{A+B}(\lambda^{\to}_{k,\gamma}(A+B)+\epsilon)\cap\mathcal{D}(A) \\ \|x\|=1}} \langle x, (A+B)x\rangle$$
$$\geq \inf_{0<\epsilon<\epsilon_0} \sup_{\substack{x\in \operatorname{Ran} P_{A+B}(\lambda^{\to}_{k,\gamma}(A+B)+\epsilon)\cap\mathcal{D}(A) \\ \|x\|=1}} \langle x, Ax\rangle + \kappa.$$

For an self-adjoint operator A we use the notation $P_A(E_1,E_2]) := \mathbf{1}_{(E_1,E_2]}(A)$. Since

$$\operatorname{Ran} P_{A+B}((\gamma, \lambda^{\to}_{k,\gamma}(A+B)+\epsilon]) \cap \mathcal{D}(A)$$

is a subspace of $\operatorname{Ran} P^{\perp}_{A+B}(\gamma)\cap\mathcal{D}(A)$ and has dimension at least k for all $\epsilon > 0$, we further estimate

$$\sup_{\substack{x\in \operatorname{Ran} P_{A+B}(\lambda^{\to}_{k,\gamma}(A+B)+\epsilon)\cap\mathcal{D}(A) \\ \|x\|=1}} \langle x, Ax\rangle$$
$$= \sup_{\substack{\mathcal{L}\subset \operatorname{Ran} P_{A+B}((\gamma,\lambda^{\to}_{k,\gamma}(A+B)+\epsilon])\cap\mathcal{D}(A) \\ \dim\mathcal{L}\geq k}} \sup_{\substack{x\in\mathcal{L}\oplus(\operatorname{Ran} P_{A+B}(\gamma)\cap\mathcal{D}(A)) \\ \|x\|=1}} \langle x, Ax\rangle$$
$$\geq \sup_{\substack{\mathcal{L}\subset \operatorname{Ran} P_{A+B}((\gamma,\lambda^{\to}_{k,\gamma}(A+B)+\epsilon)])\cap\mathcal{D}(A) \\ \dim\mathcal{L}= k}} \sup_{\substack{x\in\mathcal{L}\oplus(\operatorname{Ran} P_{A+B}(\gamma)\cap\mathcal{D}(A)) \\ \|x\|=1}} \langle x, Ax\rangle$$
$$\geq \inf_{\substack{\mathcal{L}\subset \operatorname{Ran} P_{A+B}((\gamma,\lambda^{\to}_{k,\gamma}(A+B)+\epsilon])\cap\mathcal{D}(A) \\ \dim\mathcal{L}= k}} \sup_{\substack{x\in\mathcal{L}\oplus(\operatorname{Ran} P_{A+B}(\gamma)\cap\mathcal{D}(A)) \\ \|x\|=1}} \langle x, Ax\rangle$$
$$\geq \inf_{\substack{\mathcal{L}\subset \operatorname{Ran} P^{\perp}_{A+B}(\gamma)\cap\mathcal{D}(A) \\ \dim\mathcal{L}= k}} \sup_{\substack{x\in\mathcal{L}\oplus\operatorname{Ran} P_{A+B}(\gamma)\cap\mathcal{D}(A) \\ \|x\|=1}} \langle x, Ax\rangle. \qquad \square$$

We are now ready to prove Theorems 4.12 and 4.13

Proof of Theorem 4.12. We first consider the case $k \in \mathbb{N}$. Since $\|B\| < \operatorname{dist}(\gamma, \sigma(A))$ we have $\gamma \in \rho(A+B)$. We can thus apply Lemma 4.16 and obtain Ineq. (55) for all $k \in \mathbb{N}$ with $\lambda^{\to}_{k,\gamma}(A+B) < E$. Since $\lambda^{\to}_{k,\gamma}(A) = \lambda_k(A|_{\operatorname{Ran} P^{\perp}_A(\gamma)})$, it now suffices to check the assumptions of Proposition 4.14. For all normalized $x \in \operatorname{Ran} P_{A+B}(\gamma)\cap\mathcal{D}(A)$ we have by the assumption $\|B\| \leq \operatorname{dist}(\gamma,\sigma(A))/2$ that

$$\langle x, (A+B)x\rangle \leq \sup(\sigma(A+B)\cap(-\infty,\gamma]) \leq \gamma - \operatorname{dist}(\gamma,\sigma(A+B))$$
$$\leq \gamma - \operatorname{dist}(\gamma,\sigma(A)) + \|B\| \leq \gamma - \|B\| \leq \gamma + \langle x, Bx\rangle\,.$$

This shows that assumption (i) of Proposition 4.14 is satisfied. It remains to check assumption (ii) of Proposition 4.14. This is a consequence of the Davis-Kahan $\sin 2\Theta$

theorem [DK70, Theorem 8.2]. We apply a version given in [See14a, Remark 2.9], and obtain

$$\begin{aligned}\|P_{A+B}^{\perp}(\gamma)\cdot P_A(\gamma)\| &\le \|P_{A+B}(\gamma)-P_A(\gamma)\|\\ &\le \sin\left(\frac{1}{2}\arcsin\left(2\frac{\|B\|}{\operatorname{dist}(\gamma,\sigma(A))}\right)\right)\le\frac{1}{\sqrt{2}}<1.\end{aligned}$$

The case $k=\infty$ follows by taking the supremum. □

Proof of Theorem 4.13. First we consider claim (1) and the case $k\in\mathbb{N}$. Since $0\le B<\operatorname{dist}^{\leftarrow}(\gamma,\sigma(A))$ we have $\gamma\in\rho(A+B)$, see [See17, Proposition 2.1]. Applying Lemma 4.16, we arrive at Ineq. (55) for all $k\in\mathbb{N}$ with $\lambda_{k,\gamma}^{\to}(A+B)<E$. Since B is non-negative, assumption (i) of Theorem 4.14 is satisfied. Assumption (ii) of the same theorem follows in a similar way as in in the proof of Theorem 4.12, but using the $\sin 2\Theta$ theorem for non-negative perturbations in [See17, Theorem 1]. This shows the statement for $k\in\mathbb{N}$. □

By mimicking the proof of Theorem 4.10, we also find the following theorem.

Theorem 4.17. *Let A be self-adjoint, $(a,b)\cap\sigma_{\mathrm{ess}}(A)=\emptyset$, $b\in\sigma_{\mathrm{ess}}(A)$, $\kappa\ge 0$, and B an operator satisfying $0\le B<b-a$. Assume that*

$$\forall x\in\bigcup_{t\in[0,1]}\operatorname{Ran}P_{A+tB}(b+\|B\|):\quad\langle x,Bx\rangle\ge\kappa\|x\|^2.$$

Then

$$[b,b+\kappa)\cap\sigma_{\mathrm{ess}}(A+B)=\emptyset.$$

This allows us to describe the movement of an lower edge of a component of the essential spectrum.

Corollary 4.18. *Let A be self-adjoint, $B\neq 0$ bounded, symmetric and non–negative, $b\in\sigma_{\mathrm{ess}}(A)$, $b>a$ such that $(a,b)\cap\sigma_{\mathrm{ess}}(A)=\emptyset$, and $t_0=(b-a)/\|B\|$. Assume that*

$$\forall x\in\bigcup_{t\in[0,1]}\operatorname{Ran}P_{A+tB}(b+\|B\|):\quad\langle x,Bx\rangle\ge\kappa\|x\|^2.$$

Then $f(t)=\inf\left(\sigma_{\mathrm{ess}}(A+tB)\cap(a+t\|B\|,\infty)\right)$ satisfies

$$\kappa\epsilon\le f(t+\epsilon)-f(t)\le\|B\|\epsilon\quad\textit{for all } t\in[0,t_0)\textit{ and }\epsilon\in[0,t_0-t).$$

4.2. Perturbation of spectral band edges and eigenvalues of Schrödinger operators

We now consider again consider Schrödinger operators $H = -\Delta + V$ in $L^2(\Gamma)$ with Dirichlet, Neumann or periodic boundary conditions as in Section 2.1, where $\Gamma \subset \mathbb{R}^d$ is a generalized rectangle containing at least one elementary cell of $M\mathbb{Z}^d$. We perturb this operator by a non-negative potential W which is strictly positive on the equidistributed set $S_{\delta,Z} \subset \Gamma$. Since the perturbation W is positive only on a subset, it is not immediately clear if the spectrum will be lifted at all. However, combining the results from the previous Section 4.1 with the unique continuation principle Theorem 3.13 above, we obtain lower bounds on the lifting of eigenvalues and of the edges of the essential spectrum, see Theorem 4.19.

Moreover, we consider the family of operators $H + tW$ with coupling constant $t \in \mathbb{R}$. As seen in Lemma 4.1 one can locally parametrize the edges of the essential spectrum of $H + tW$ as a function $t \mapsto f(t)$. In Corollary 4.20 we conclude that $t \mapsto f(t)$ is strictly monotone, and provide upper and lower bounds in terms of linear functions of t.

Theorem 4.19. *Let $G > 0$, $\Gamma \subset \mathbb{R}^d$ be G-admissible, $\delta \in (0, G/2)$, Z be a (G,δ)-equidistributed sequence, $V \in L^\infty(\Gamma)$ be real-valued, and $W \in L^\infty(\Gamma)$ be real-valued such that*

$$W \geq \vartheta \mathbf{1}_{S_{\delta,Z}}$$

for some $\vartheta > 0$. Moreover, for $s \in \mathbb{R}$ and with N as in Theorem 3.13 we set

$$\kappa(s) = \vartheta \left(\frac{\delta}{G}\right)^{N\left(1+G^{4/3}(\|V\|_\infty+\|W\|_\infty)^{2/3}+G\sqrt{s_+}\right)}.$$

(a) ***Lifting of spectral values not exceeding*** $\inf \sigma_{\mathrm{ess}}(H)$***.*** *Let $E \in \mathbb{R}$. Then for all $k \in \mathbb{N} \cup \{\infty\}$ such that $\lambda_k(H + W) < E$, we have*

$$\lambda_k(H + W) \geq \lambda_k(H) + \kappa(E).$$

(b) ***Lifting of eigenvalues (counted decreasingly) in gaps of*** $\sigma_{\mathrm{ess}}(H)$***.*** *Let $\gamma \in \rho(H) \cap \mathbb{R}$ and $\|W\|_\infty < \operatorname{dist}^{\leftarrow}(\gamma, \sigma(H))$. Then for all $k \in \mathbb{N} \cup \{\infty\}$*

$$\lambda^{\leftarrow}_{k,\gamma}(H + W) \geq \lambda^{\leftarrow}_{k,\gamma}(H) + \kappa(\gamma).$$

(c) ***Lifting of eigenvalues (counted increasingly) in gaps of*** $\sigma_{\mathrm{ess}}(H)$***.*** *Let $\gamma \in \rho(H) \cap \mathbb{R}$, $\|W\|_\infty < \operatorname{dist}^{\leftarrow}(\gamma, \sigma(H))$, and $E > \gamma$. Then for all $k \in \mathbb{N} \cup \{\infty\}$ such that $\lambda^{\rightarrow}_{k,\gamma}(H + W) < E$, we have*

$$\lambda^{\rightarrow}_{k,\gamma}(H + W) \geq \lambda^{\rightarrow}_{k,\gamma}(H) + \kappa(E)$$

(d) ***Lifting of a lower edge of a gap in*** $\sigma_{\mathrm{ess}}(H)$. *Let* $(a,b) \cap \sigma_{\mathrm{ess}}(H) = \emptyset$, $a \in \sigma_{\mathrm{ess}}(A)$ *and assume that* $\|W\|_\infty < b - a$. *Then*

$$[a + \kappa(b), b) \cap \sigma_{\mathrm{ess}}(H + W) \neq \emptyset.$$

(e) ***Lifting of an upper edge of a gap in*** $\sigma_{\mathrm{ess}}(H)$. *Let* $(a,b) \cap \sigma_{\mathrm{ess}}(H) = \emptyset$, $b \in \sigma_{\mathrm{ess}}(A)$ *and assume that* $\|W\|_\infty < b - a$. *Then*

$$[b, b + \kappa(b + \|W\|_\infty)) \cap \sigma_{\mathrm{ess}}(H + W) = \emptyset.$$

The parameter κ neither depends on the set Γ nor on the choice of the equidistributed sequence and depends on the potentials V and W only by their L^∞-norm. We also note that analogously to the argument leading to Corollary 3.12 above, we can introduce a spectral shift parameter λ and replace κ by the optimized

$$\tilde{\kappa}(s) = \vartheta \sup_{\lambda \in \mathbb{R}} \left(\frac{\delta}{G}\right)^{N\left(1 + G^{4/3}(\|V-\lambda\|_\infty + \|W\|_\infty)^{2/3} + G\sqrt{(s-\lambda)_+}\right)} \geq \kappa(s).$$

Proof. By Theorem 3.13, for all $t \in [0,1]$, $s \in \mathbb{R}$ and $x \in \operatorname{Ran} P_{H+tW}(s)$, we have

$$\begin{aligned}\langle x, Wx \rangle \geq \vartheta \left\langle x, \mathbf{1}_{S_{\delta,Z}} x \right\rangle &\geq \vartheta \sup_{\lambda \in \mathbb{R}} \left(\frac{\delta}{G}\right)^{N\left(1 + G^{4/3}\|V + tW - \lambda\|_\infty^{2/3} + G\sqrt{(s-\lambda)_+}\right)} \|x\|^2 \\ &\geq \vartheta \sup_{\lambda \in \mathbb{R}} \left(\frac{\delta}{G}\right)^{N\left(1 + G^{4/3}(\|V-\lambda\|_\infty + \|W\|_\infty)^{2/3} + G\sqrt{(s-\lambda)_+}\right)} \|x\|^2.\end{aligned}$$

Then the statements (a)–(e) follow by applying Lemma 4.4 and Theorems 4.9, 4.13, 4.10, 4.17, respectively. □

In the language of Lemma 4.1 we obtain the following corollary.

Corollary 4.20. *Under the assumptions of Theorem 4.19, let* $a, b \in \sigma_{\mathrm{ess}}(H)$, *and* $b > a$ *such that* $(a,b) \cap \sigma_{\mathrm{ess}}(H) = \emptyset$, *and define* $t_0 = (b-a)/\|W\|_\infty$. *Then,* $f_\pm : (-t_0, t_0) \to \mathbb{R}$,

$$f_-(t) = \sup\left(\sigma_{\mathrm{ess}}(H + tW) \cap (-\infty, b - t_-\|W\|_\infty)\right),$$
$$f_+(t) = \inf\left(\sigma_{\mathrm{ess}}(H + tW) \cap (a + t_+\|W\|_\infty, \infty)\right),$$

are Lipschitz continuous with Lipschitz constant $\|W\|_\infty$ *and satisfy*

$$\begin{aligned} t\kappa \leq f_-(t+\epsilon) - f_-(t) &\leq t\|W\| && \text{for all } t \in (0, t_0) && \text{and } \epsilon \in [0, t_0 - t), \\ t\|W\| \leq f_-(t+\epsilon) - f_-(t) &\leq t\kappa && \text{for all } t \in (-t_0, 0) && \text{and } \epsilon \in [0, t_0 + t), \\ t\kappa \leq f_+(t+\epsilon) - f_+(t) &\leq t\|W\| && \text{for all } t \in (0, t_0) && \text{and } \epsilon \in [0, t_0 - t), \\ b + t\|W\| \leq f_+(t+\epsilon) - f_+(t) &\leq b + t\kappa && \text{for all } t \in (-t_0, 0) && \text{and } \epsilon \in [0, t_0 + t), \end{aligned}$$

where

$$\kappa = \vartheta \sup_{\lambda \in \mathbb{R}} \left(\frac{\delta}{G} \right)^{N\left(1+G^{4/3}(\|V-\lambda\|_\infty + t_0\|W\|_\infty)^{2/3} + G\sqrt{(2b-a-\lambda)_+}\right)},$$

and, for real $t \in \mathbb{R}$, *we set* $t_+ = \max\{0, t\}$ *and* $t_- = \max\{0, -t\}$.

5. Application to random Schrödinger operators

This section contains parts of published work by the author. More precisely, Subsections 5.2 and 5.3 are based on the author's publications [TV15, NTTV15, NTTV18a] while Subsection 5.4 contains material from [TV15, TV16a, TT18].

We start with an introduction to random Schrödinger operators in Subsection 5.1. After that, in Subsection 5.2 we introduce a class of random Schrödinger operators which we call *operators monotone in the randomness* and state a Wegner estimate which we prove in Subsection 5.3. Subsection 5.4 is devoted to Wegner estimates for some random Schrödinger operators with magnetic field.

5.1. Some basics on random Schrödinger operators

The topic of random Schrödinger operators and Anderson localization goes back to the work of physicist Philipp Warren Anderson [And58] from 1958 who was awarded the Nobel Prize for Physics in 1977 for his work on localization. He argued that in the presence of disorder lattice operators modeling the dynamics of electrons in solids exhibit absence of diffusion of mass or charge. This is in stark contrast to the situation in the corresponding periodic systems where typically transport occurs. Since then, the notion of *Anderson localization* has been the subject of intensive study in Physics as well as in Mathematics. In fact, it seems almost impossible to provide a complete list of relevant references. Therefore, we will focus on important milestones and on results which are relevant for our purpose of providing an introduction to the field.

Mathematically, the notion of localization can be interpreted from a spectral or a dynamical point of view. In order to properly define the notion of localization, it is convenient to start by properly defining random operators and by introducing ergodic random operators.

Definition 5.1 (cf. [KM82, CL90]). A *random operator* in $L^2(\mathbb{R}^d)$ is a measurable map

$$\Omega \ni \omega \mapsto H_\omega$$

where all H_ω are self-adjoint operators in $L^2(\mathbb{R}^d)$. By measurable, we mean that the mappings $\omega \mapsto f(H_\omega)$ are weakly measurable for all bounded, real-valued Borel-measurable functions f. If there is no risk of confusion, we will simply write H_ω for the random family $\{H_\omega\}_{\omega\in\Omega}$.
A family of measure-preserving transformations $\{\tau_i\}_{i\in I}$ on Ω is *ergodic* if all events which are invariant under this family are of probability zero or one.

We call the family H_ω a $\mathbb{Z}^d$-*ergodic random operator* if there is an ergodic family of measure preserving transformations $\{\tau_y\}_{y\in\mathbb{Z}^d}$ on Ω such that

$$U(y)H_\omega U^*(y) = H_{\tau_y(\omega)} \quad \text{for all} \quad y \in \mathbb{Z}^d. \tag{56}$$

where $U(y)\colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ denotes the unitary operator of translation by y.

In general, ergodicity of a measure preserving family of transformation is not easy to verify. However, if a measure preserving family of transformations satisfies identity (56), and if restrictions of the random operator H_ω on sufficiently far apart open domains $B_1, B_2 \subset \mathbb{R}^d$ are independent, then the family $\{\tau_y\}_{y\in\mathbb{Z}^d}$ is ergodic by Kolmogorov's zero–one law, cf. [Kle14, Chapter 2.3]. This property is also called *independence at distance* and holds for all examples of random Schrödinger operators which we treat below. We note however, that below, we will occasionally treat non-ergodic models of random Schrödinger operators such as the crooked alloy-type model.

An important consequence of ergodicity is that the spectrum is almost surely deterministic, i.e. there is a set $\Sigma \subset \mathbb{R}$ such that $\sigma(H_\omega) = \Sigma$ with probability one. Furthermore, the decomposition of the spectrum into pure point, absolutely continuous and singular continuous spectrum is deterministic, i.e. there are sets $\Sigma_{\mathrm{pp}}, \Sigma_{\mathrm{ac}}, \Sigma_{\mathrm{sc}} \subset \mathbb{R}$ such that with probability one, we have $\sigma_{\mathrm{pp}}(H_\omega) = \Sigma_{\mathrm{pp}}$, $\sigma_{\mathrm{ac}}(H_\omega) = \Sigma_{\mathrm{ac}}$, and $\sigma_{\mathrm{sc}}(H_\omega) = \Sigma_{\mathrm{sc}}$, cf. [KM82, CL90].

Obviously, the same statements hold for $\mathbb{Z}^d$-periodic operators such as the pure negative Laplacian with periodic potential (this can be considered as "trivial" ergodic randomness). In this case, the operator typically exhibits purely absolutely continuous spectrum and ballistic transport. Anderson's main contribution can be summarized by the statement that when passing from periodic to random ergodic operators, localization typically occurs.

Let us start with a weak form of localization:

Definition 5.2. An ergodic random operator exhibits *spectral localization* in an interval $I \subset \mathbb{R}$ if it has only pure point spectrum in I. This means that $\Sigma_{\mathrm{ac}} \cap I = \Sigma_{\mathrm{sc}} \cap I = \emptyset$, and $\Sigma \cap I = \Sigma_{\mathrm{pp}} \cap I \neq \emptyset$.

The reason why pure point spectrum is refered to as spectral localization is due to the so-called *RAGE theorem*, named after Ruelle, Amrein, Georgescu and Enss [Rue69, AG74, Ens78]. Let us recall that the time-dependent Schrödinger equation is

$$\mathrm{i}\frac{\partial}{\partial t}\phi = A\phi \tag{57}$$

where A is a self-adjoint operator on a Hilbert space $\mathcal{H}$. If we are given an initial state $\phi(0) \in \mathcal{H}$, then the solution of (57) is given by

$$\phi(t) = \mathrm{e}^{-\mathrm{it}A}\phi(0), \quad t \in \mathbb{R}.$$

The group of unitary operators $\mathrm{e}^{-\mathrm{i}tA}$ thus determines the time evolution of the system. The RAGE theorem describes the effect of the time evolution on initial states, depending on which spectral subspace they are from:

Theorem 5.3 (RAGE Theorem)**.** *Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$ and let K_n, $n \in \mathbb{N}$, be a sequence of relative A-compact operators which converges strongly to the identity. Then for the decomposition $\mathcal{H} = \mathcal{H}_{\mathrm{c}} \oplus \mathcal{H}_{\mathrm{pp}}$ of $\mathcal{H}$ it holds that*

$$\mathcal{H}_{\mathrm{pp}} = \left\{ \psi \in \mathcal{H}\colon \lim_{n\to\infty} \sup_{t\geq 0} \|(\mathrm{Id} - K_n)\mathrm{e}^{-itA}\psi\| = 0 \right\},$$

$$\mathcal{H}_{\mathrm{c}} = \left\{ \psi \in \mathcal{H}\colon \lim_{n\to\infty} \lim_{T\to\infty} \frac{1}{T} \int_0^T \|K_n \mathrm{e}^{-itA}\psi\| \mathrm{d}t = 0 \right\}.$$

If A is a Schrödinger operator in $L^2(\mathbb{R}^d)$ (for simplicity with bounded potential), then one may choose K_R as the operator of multiplication by the characteristic functions of a ball with radius R for $R \in \mathbb{N}$. In this case, the statement of the RAGE theorem becomes:

$$\psi \in \mathcal{H}_{\mathrm{pp}} \quad \Leftrightarrow \quad \lim_{R\to\infty} \sup_{t\geq 0} \int_{|x|\geq R} |\mathrm{e}^{-itA}\psi(x)|^2 \mathrm{d}x = 0,$$

$$\psi \in \mathcal{H}_{\mathrm{c}} \quad \Leftrightarrow \quad \lim_{R\to\infty} \lim_{T\to\infty} \frac{1}{T} \int_0^T \int_{|x|<R} |\mathrm{e}^{-itA}\psi(x)|^2 \mathrm{d}x\mathrm{d}t = 0.$$

This can be interpreted as follows: if an ergodic, random operator H_ω in $L^2(\mathbb{R}^d)$ has only pure point spectrum in an interval I with probability one, then the mass (or the electric charge) of every normalized initial state ψ with energies in I will almost surely stay essentially locally confined under the time evolution $\mathrm{e}^{-\mathrm{i}tH_\omega}$ generated by the random operator, uniformly in time. This justifies the notion of spectral localization.

The term *Anderson localization* has been originally used in the mathematical community to describe spectral localization with exponentially decaying eigenfunctions, but nowadays it is also used collectively for stronger versions of localization. In fact, there is a whole hierarchy of notions of localization, cf. [Kle08] for a more detailed overview. Let us merely cite the following variants:

Definition 5.4. An ergodic random operator H_ω exhibits

- *dynamical localization* in an interval $I \subset \mathbb{R}$ if $\Sigma \cap I \neq \emptyset$ and for $\mathbb{P}$-almost every $\omega \in \Omega$, every compact interval $J \subset I$ and every $\psi \in L^2(\mathbb{R}^d)$ with compact support, we have

$$\sup_{t\in\mathbb{R}} \left\|(1+|x|^2)^{n/2} P_{H_\omega}(J) \mathrm{e}^{-itH_\omega}\psi\right\|^2 < \infty \quad \text{for all } n \geq 0.$$

- *strong dynamical localization* in an interval $I \subset \mathbb{R}$ if $\Sigma \cap I \neq \emptyset$ and for every compact interval $J \subset I$ and every $\psi \in L^2(\mathbb{R}^d)$ with compact support, we have

$$\mathbb{E}\left[\sup_{t\in\mathbb{R}} \left\|(1+|x|^2)^{n/2} P_{H_\omega}(J) \mathrm{e}^{-itH_\omega}\psi\right\|^2\right] < \infty \quad \text{for all } n \geq 0.$$

Strong dynamical localization implies dynamical localization which implies spectral localization, but the reverse implication is not true in general.

There is a variety of models in the continuum for which localization in its various manifestations has been established. The first such model would be the natural generalization of the lattice analogs to the continuum: the *alloy-type model* or *continuum Anderson model*. Some more complicated models include the random displacement model [KLNS12], the Poisson potential [GHK07], and the Bernoulli-Anderson model [BK05]. Another research direction is when the potential is assumed to be generated by a random Gaussian field [Uek04, FHLM97, FLM00, Ves11]. Further examples where localization has been proved include Schrödiner operators with random δ-interaction [BTV18] as well as acoustic and Maxwell operators in random media [FK94, CHT99, KK01, KK04].

The first mathematically sound proof of localization was given in [GMP77] for dimension one. However, the techniques used therein are restricted to the one-dimensional situation. Methods of wider applicability are the *multi-scale analysis* [FS83, FMSS85, Dre87, vDK89] and the *fractional moment method* [AM93, Aiz94, AFHS01, AEN$^+$06, BNSS06].

In the following we focus on the multi-scale-analysis. It has originally been developed for operators on lattice graphs in [FS83], but was soon generalized to the continuum in [HM84]. Furthermore, the first proofs of localization via multi-scale analysis only yielded spectral localization with exponentially decaying eigenfunctions, but later it has been improved to stronger forms of localization [GDB98, DS01, GK01].

The multi-scale method is an inductive process over growing boxes which establishes that almost surely generalized eigenfunctions of the operator H_ω in an energy region turn out to be $L^2(\mathbb{R}^d)$-eigenfunctions. Apart from some standard analytical and probabilistic tools, it requires two important ingredients: the *initial length scale* *estimate*, which serves as an induction anchor and the *Wegner estimate* for the induction step.

Wegner estimates are bounds of the form

$$\mathbb{P}\left(\sigma(H_{\omega,L}) \cap I \neq \emptyset\right) \leq C|I|^{\alpha} L^{\beta d}, \tag{58}$$

where $H_{\omega,L}$ is the restriction of the random operator H_ω to Λ_L with Dirichlet, Neumann, or periodic boundary conditions, I is a bounded interval, $C > 0$ a constant depending only on the maximum of I, $0 < \alpha \leq 1$, and $\beta \geq 1$. We will deduce the slightly stronger variant

$$\mathbb{E}\left[\operatorname{Tr}\left[P_{H_{\omega,L}}(I)\right]\right] \leq C|I|^{\alpha} L^{d} \tag{59}$$

where $\mathbb{E}$ denotes the expectation with respect to the probability measure $\mathbb{P}$. Clearly, since $\operatorname{Tr}\left[P_{H_{\omega,L}}(I)\right]$ is the number of eigenvalues of $H_{\omega,L}$ in I, Ineq. (59) is stronger than Ineq. (58) by Chebyshev's inequality. We also note that in (59), the factor L^d is optimal as can be seen by the Weyl asymptotics for eigenvalues. Wegner estimates with such an *optimal volume dependence* are helpful when studying another object of interest in the context of random operators: the *Integrated Density of States* (IDS) N. For a lower semibounded Schrödinger operator H on $L^2(\mathbb{R}^d)$, its IDS is defined as

$$N(E) := \lim_{L\to\infty} \frac{\operatorname{Tr} P_{H_L}(E)}{L^d} = \lim_{L\to\infty} \frac{\#\{\text{No. of eigenvalues of } H_L \text{ below } E\}}{L^d}, \quad E \in \mathbb{R},$$

if this limit exists, where H_L denotes the restriction of H to Λ_L with Dirichlet, Neumann, or periodic boundary conditions. The IDS (if it exists) is left continuous and monotone increasing. Ergodic random Schrödinger operators H_ω have an almost sure IDS [Pas71] which means that there is a function $N \colon \mathbb{R} \to [0,\infty)$ such that for almost every $\omega \in \Omega$, the IDS of H_ω exists and is equal to N. We also refer to [Ves08a] for more on the IDS.

Recently, there has also been interest in non-ergodic random Schrödinger operators. Even though the lack of ergodicity makes the question of spectral localization and the existence of the IDS more delicate [GMRM15], there have been recent results [RM12, RMV13, Kle13, GMRM15, TT18] on localization and on the IDS. Scale-free quantitative unique continuation has played an important role in this context.

Our main results in this section are Wegner estimates for several models in Theorems 5.6, 5.14, 5.16, and 5.24. More precisely, Theorems 5.6 and 5.24 treat random Schrödinger operators where a sequence of elementary random variables generates a random potential in a non-linear way: the so-called *random breather model*. In fact, Theorem 5.6 treats a more general situation, namely *random Schrödinger operators monotone in the randomness*. Theorems 5.14, 5.16 and 5.24 are concerned with magnetic Schrödinger operators. This means that the negative Laplacian $-\Delta$ is replaced by a magnetic Schrödinger operator $(-\mathrm{i}\nabla + A_0)^2$. We also note that Theorems 5.6, 5.14, 5.16 describe models which typically are non-ergodic.

Beyond Wegner estimates, we also comment on the IDS, on initial scale estimates, and on localization.

5.2. A Wegner estimate for operators monotone in the randomness

This subsection presents results and proofs from [TV15, NTTV15, NTTV18a]. We establish Wegner estimates for a new class of models of random Schrödinger operators which we call *random Schrödinger operators monotone in the randomness*, see Definition 5.5. Special cases of this class include the the classic alloy-type model [HM84], the Delone-alloy-type model [RM12], and the crooked Anderson Hamiltonians [Kle13]. However, it also contains new example such as the *standard random breather model*. For this model, we also obtain localization by combining it with recent results of Schumacher and Veselić [SV17] and applying the multi-scale analysis.

Let us now define the model:

Definition 5.5. Let $\{u_j\}_{j\in\mathbb{Z}^d}$ be a sequence of measurable mappings $u_j\colon [0,1]\times\mathbb{R}^d \to \mathbb{R}$ such that the following properties hold:

(a) There is $u_{\max}$ such that for all $j \in \mathbb{Z}^d$ and all $t \in [0,1]$, we have $\|u_j(t,\cdot)\|_\infty \leq u_{\max}$.

(b) There is $M > 0$ such that $\operatorname{supp} u_j \subset (-M/2, M/2)^d$ for all $j \in \mathbb{Z}^d$.

(c) There are $\alpha_1, \beta_1 > 0$ and $\alpha_2, \beta_2 \geq 0$ such that for all $j \in \mathbb{Z}^d$, and all $0 \leq s < t \leq 1$, there exists a point $x_0 = x_0(j,t,s) \in \mathbb{R}^d$ such that

$$u_j(t,\cdot) - u_j(s,\cdot) \geq \alpha_1 (t-s)^{\alpha_2} \cdot \mathbf{1}_{B(x_0, \beta_1 (t-s)^{\beta_2})}(\cdot) \geq 0. \tag{60}$$

Now let $0 \leq \omega_- < \omega_+ < 1$ and let $\{\omega_j\}_{j\in\mathbb{Z}^d}$ be an independent sequence of random variables on the probability space

$$(\Omega, \mathcal{A}, \mathbb{P}) = \left(\bigtimes_{j\in\mathbb{Z}^d} \mathbb{R}, \quad \bigotimes_{j\in\mathbb{Z}^d} \mathcal{B}(\mathbb{R}), \quad \bigotimes_{j\in\mathbb{Z}^d} \mu_j \right)$$

where the μ_j are probability measures with support in $[\omega_-, \omega_+]$ and with uniformly bounded Lebesgue densities ν_j. This means that the independent random variables ω_j

is distributed according to the probability measure μ_j. Furthermore let $V_0 \in L^\infty(\mathbb{R}^d)$. We then define the random potential

$$V_\omega(x) = V_0 + \sum_{j \in \mathbb{Z}^d} u_j(\omega_j, x - Mj), \quad \omega \in \Omega,$$

and the random Schrödinger operator

$$H_\omega = -\Delta + V_\omega = -\Delta + V_0 + \sum_{j \in \mathbb{Z}^d} u_j(\omega_j, \cdot - Mj), \quad \omega \in \Omega.$$

We call this a *random Schrödinger operator monotone in the randomness*. Finally, for $L > 0$, we define $H_{\omega,L}$ as the restriction of H_ω to $L^2(\Lambda_L)$, where $\Lambda_L = (-L/2, L/2)^d$, with Dirichlet boundary conditions.

The main result is the following Wegner estimate.

Theorem 5.6. *Let H_ω be a random Schrödinger operator monotone in the randomness and let $E_0 \in \mathbb{R}$. Then there are constants $C > 0$, $\kappa \in [0, 1)$, and $\epsilon_0 > 0$, such that for all $L \geq M$, all $0 < \epsilon \leq \epsilon_0$, and all $E \in \mathbb{R}$ such that $[E - \epsilon, E + \epsilon] \subset (-\infty, E_0]$, we have*

$$\mathbb{E}\left[\operatorname{Tr}\left[P_{H_{\omega,L}}(I)\right]\right] \leq C\epsilon^\kappa |\ln \epsilon|^d L^d.$$

In particular, the constants only depend on d, E_0, M, $u_{\max}$, α_1, α_2, β_1, β_2, ω_+, $\sup_{j \in \mathbb{Z}^d} \|\nu_j\|_\infty$, and on $\|V_0\|_\infty$. Furthermore, their dependence on $\|V_0\|_\infty$ is monotone.

An easy consequence is local Hölder continuity of the IDS:

Corollary 5.7. *Let $\{H_\omega\}_{\omega \in \Omega}$ be a random Schrödinger operator monotone in the randomness and assume that its IDS $N : \mathbb{R} \to [0, \infty)$ exists. Then, for every $E_0 \in \mathbb{R}$, the function N is $\tilde{\kappa}$-Hölder continuous in $(-\infty, E_0]$ for every exponent $\tilde{\kappa} < \kappa$, where $\kappa \in (0, 1]$ is as in Theorem 5.6.*

Let us now provide some examples of random Schrödinger operators monotone in the randomness as in Definition 5.5.

Standard random breather model: Let μ be the uniform distribution on the interval $[0, 1/4]$, and let $u_t(x) = \mathbf{1}_{B(0,t)}$, $j \in \mathbb{Z}^d$. Then $V_\omega = \sum_{j \in \mathbb{Z}^d} \mathbf{1}_{B(j,\omega_j)}$ is the characteristic function of a disjoint union of balls with random radii. This model has been introduced in [TV15].

General random breather models Let $0 \leq u \in L_0^\infty(\mathbb{R}^d)$ and define $u_t(x) = u(x/t)$ for $t > 0$ and $u_0 \equiv 0$, and assume that the family $\{u_t : t \in [0, 1]\}$ satisfies

(60). Then $V_\omega(x) = \sum_{j\in\mathbb{Z}^d} u_{\omega_j}(x-j)$ is a sum of random dilations of a single-site potential u at each lattice site $j \in \mathbb{Z}^d$. Natural examples of functions u, satisfying (60) include

- characteristic functions of bounded convex sets,
- the hat-potential $(1-|x|)\mathbf{1}_{\{|x|<1\}}$,
- or the bump function $\exp\left(1/(|x|^2-1)\right)\mathbf{1}_{\{|x|<1\}}$,

see also Appendix C for details.

Alloy-type model Random Schrödinger operators monotone in the randomness also naturally treat the classic alloy-type or continuum Anderson model. Let $0 \leq u \in L_0^\infty(\mathbb{R}^d)$, $u \geq \alpha > 0$ on some open set and let $u_t(x) := tu(x)$. Then $V_\omega(x) = \sum_{j\in\mathbb{Z}^d} \omega_j u(x-j)$ is a sum of copies of u at all lattice sites $j \in \mathbb{Z}^d$, multiplied with ω_j.

Crooked Anderson Hamiltonians and Delone-alloy-type model Also non-ergodic systems such as crooked Anderson Hamiltonians [Kle13] and the Delone-alloy-type model [RM12] are also a special case. However, while Theorem 5.6 does not have an optimal (i.e. linear) dependence of the upper bound in terms of the length of the energy interval, there are stronger results for this model, cf. [Kle13], which do have an upper bound optimal in the length of the energy interval.

Remark 5.8 (Initial scale estimate and localization). For these models, as soon as the Wegner estimate is complemented by an appropriate initial scale estimate, we can perform the bootstrap multi-scale analysis and obtain localization (spectral localization with exponentially decaying eigenfunctions, dynamical, and strong dynamical localization), cf. [GK01, GK03]. We emphasize that this procedure also works for the non-ergodic operators considered above, cf. [RM12]. In [NTTV18a, Theorem 2.10], an initial scale for operators monotone in the randomness is proved under the additional assumptions that the probability distributions μ_j are sufficiently thin near the bottom of their support. Hence, localization at low energies immediately follows in this case, albeit under a technical condition on the probability measure.

Initial scale estimates can also be inferred from so-called *Lifshitz tail estimates* on the asymptotic behavior of the IDS near the bottom of the almost sure spectrum. For the standard random breather model, Lifshitz tails are proved (without the technical condition on the thinness of the probability distribution near the infimum of its support) in [SV17]. Therefore, localization near the bottom of the almost sure spectrum follows for this model.

5.3. Proof of Theorem 5.6

We define $\delta_0 = 1 - \omega_+ > 0$. Given $\omega = (\omega_j)_{j\in\mathbb{Z}^d} \in [\omega_-, \omega_+]^{\mathbb{Z}^d}$ and $0 \le \delta \le \delta_0$, we set $\omega + \delta = \{\omega_j + \delta\}_{j\in\mathbb{Z}^d} \in [0,1]^{\mathbb{Z}^d}$. Let us first note that from Definition 5.5 it follows for all $\omega \in [\omega_-, \omega_+]^{\mathbb{Z}^d}$, and all $\delta \le \delta_0$ that

$$0 \le \alpha_1 \delta^{\alpha_2} \cdot \mathbf{1}_S \le V_{\omega+\delta} - V_\omega \le u_{\max}$$

for some set $S = S_{\beta_1\delta^{\beta_2},Z}$ which is a union of $(\beta_1\delta^{\beta_2})$-balls, distributed according to a $(M, \beta_1\delta^{\beta_2})$-equidistributed sequence $Z = Z(\omega, \delta)$. Furthermore, it holds that

$$\|V_\omega\|_\infty \le \|V_0\|_\infty + u_{\max}.$$

Fix $L \ge M$ and let $k \in \mathbb{N}$ such that $\lambda_k(H_{\omega,L}) \le E_0$. Then,

$$\lambda_k(H_{\omega+\delta,L}) \le \lambda_k(H_{\omega,L}) + u_{\max},$$

and by the eigenvalue lifting estimate, Theorem 4.19 a), we have

$$\begin{aligned}\lambda_k(H_{\omega+\delta,L}) &\ge \lambda_k(H_{\omega,L}) + \alpha_1\delta^{\alpha_2}\left(\frac{\beta_1\delta_2^{\beta}}{M}\right)^{N(1+M^{4/3}(\|V_0\|_\infty+u_{\max})^{2/3}+M\sqrt{|E_0+u_{\max}|})}\\ &\ge \lambda_k(H_{\omega,L}) + \delta^\kappa\end{aligned}$$

for a constant $\kappa > 0$. We emphasize that κ depends on V_0 only via $\|V_0\|_\infty$ and the dependence is monotone. We choose $\tilde\epsilon_0 = \delta_0^\kappa/4$. This implies for all $0 < \epsilon \le \tilde\epsilon_0$

$$\lambda_k(H_{\omega+\delta,L}) \ge \lambda_k(H_{\omega,L}) + 4\epsilon \quad \text{where} \quad \delta = (4\epsilon)^{1/\kappa}. \tag{61}$$

Let now $\rho \in C^\infty(\mathbb{R}, [-1,0])$ be a smooth, non-decreasing function with $\rho = -1$ on $(-\infty; -\epsilon]$ and $\rho = 0$ on $[\epsilon; \infty)$. We can assume $\|\rho'\|_\infty \le 1/\epsilon$. It holds that

$$\mathbf{1}_{[E-\epsilon;E+\epsilon]}(x) \le \rho(x - E + 2\epsilon) - \rho(x - E - 2\epsilon) = \rho(x - E - 2\epsilon + 4\epsilon) - \rho(x - E - 2\epsilon)$$

for all $x \in \mathbb{R}$ and together with (61) this implies

$$\begin{aligned}\mathbb{E}\left[\mathrm{Tr}\left[P_{H_{\omega,L}}([E-\epsilon;E+\epsilon])\right]\right] &\le \mathbb{E}\left[\mathrm{Tr}\left[\rho(H_{\omega,L} - E - 2\epsilon + 4\epsilon) - \rho(H_{\omega,L} - E - 2\epsilon)\right]\right]\\ &\le \mathbb{E}\left[\mathrm{Tr}\left[\rho\left(H_{\omega+\delta,L} - E - 2\epsilon\right) - \rho\left(H_{\omega,L} - E - 2\epsilon\right)\right]\right].\end{aligned} \tag{62}$$

Now let $\tilde\Lambda_L := \{j \in \mathbb{Z}^d : \Lambda_M + j \cap \Lambda_L \neq \emptyset\}$ be the set of indices which can influence the potential within Λ_L. Note that $|\tilde\Lambda_L| \le (L + 2M)^d \le 3^d L^d$. We enumerate the points in $\tilde\Lambda_L$ by $k : \{1, \ldots, |\tilde\Lambda_L|\} \to \mathbb{Z}^d$, $n \mapsto k(n)$. The upper bound in (62) will be expanded in a telescopic sum by changing the $|\tilde{\Lambda}_L|$ indices from ω_j to $\omega_j + \delta$ successively. In order to do so, we introduce the following notation: Given $\omega \in [\omega_-, \omega_+]^{\mathbb{Z}^d}$, $n \in \{1, \dots, |\tilde{\Lambda}_L|\}$, $\delta \in [0, \delta_{\max}]$ and $t \in [\omega_-, \omega_+]$, we define $\tilde{\omega}^{(n,\delta)}(t) \in [\omega_-, 1]^{\mathbb{Z}^d}$ inductively by defining

$$\left(\tilde{\omega}^{(1,\delta)}(t)\right)_j := \begin{cases} t & \text{if } j = k(1), \\ \omega_j & \text{else,} \end{cases}$$

and for $j \in \{2, \dots, |\tilde{\Lambda}_L|\}$

$$\left(\tilde{\omega}^{(n,\delta)}(t)\right)_j : = \begin{cases} t & \text{if } j = k(n), \\ \left(\tilde{\omega}^{(n-1,\delta)}(\omega_j + \delta)\right)_j & \text{else.} \end{cases}$$

The function $\tilde{\omega}^{(n,\delta)} : [\omega_-, 1] \to [\omega_-, 1]^{\mathbb{Z}^d}$ is the rank-one perturbation of ω in the $k(n)$-th coordinate with the additional requirement that all sites $k(1), \dots, k(n-1)$ have already been increased by δ. We also define

$$\Theta_n(t) := \operatorname{Tr}\left[\rho\left(H_{\tilde{\omega}^{(n,\delta)}(t),L} - E - 2\epsilon\right)\right], \text{ for } n = 1, \dots, |\tilde{\Lambda}_L|.$$

Note that

$$\begin{aligned} \Theta_1(\omega_{k(1)}) &= \operatorname{Tr}\left[\rho\left(H_{\omega,L} - E - 2\epsilon\right)\right], \\ \Theta_n(\omega_{k(n)}) &= \Theta_{n-1}(\omega_{k(n-1)} + \delta) \quad \text{for } n = 2, \dots, |\tilde{\Lambda}_L| \quad \text{and} \\ \Theta_{|\tilde{\Lambda}_L|}(\omega_{k(|\tilde{\Lambda}_L|)} + \delta) &= \operatorname{Tr}\left[\rho\left(H_{\omega+\delta,L} - E - 2\epsilon\right)\right]. \end{aligned}$$

Hence the upper bound in (62) is

$$\begin{aligned} &\mathbb{E}\left[\operatorname{Tr}\left[\rho(H_{\omega+\delta,L} - E - 2\epsilon)\right] - \operatorname{Tr}\left[\rho(H_{\omega,L} - E - 2\epsilon\right]\right] \\ &\quad = \mathbb{E}\left[\Theta_{|\tilde{\Lambda}_L|}(\omega_{k(|\tilde{\Lambda}_L|)} + \delta) - \Theta_1(\omega_{k(1)})\right] = \sum_{n=1}^{|\tilde{\Lambda}_L|} \mathbb{E}\left[\Theta_n(\omega_{k(n)} + \delta) - \Theta_n(\omega_{k(n)})\right]. \end{aligned}$$

Using the product structure of the probability space, we apply Fubini's Theorem to each summand and obtain

$$\mathbb{E}\left[\Theta_n(\omega_{k(n)} + \delta) - \Theta_n(\omega_{k(n)})\right] = \mathbb{E}\left[\int_{\omega_-}^{\omega_+} \Theta_n(\omega_{k(n)} + \delta) - \Theta_n(\omega_{k(n)}) \mathrm{d}\mu(\omega_{k(n)})\right].$$

Note that $\Theta_n : [\omega_-, 1] \to \mathbb{R}$ is monotone and bounded. We will further estimate each summand by means of the following Lemma.

Lemma 5.9. *Let $-\infty < \omega_- < \omega_+ \leq +\infty$. Assume that μ is a probability distribution with bounded density ν_μ and support in the interval $[\omega_-, \omega_+]$ and let Θ be a non-decreasing, bounded function. Then for all $\delta > 0$*

$$\int_{\mathbb{R}} [\Theta(\lambda + \delta) - \Theta(\lambda)] \,\mathrm{d}\mu(\lambda) \leq \|\nu_\mu\|_\infty \cdot \delta \,[\Theta(\omega_+ + \delta) - \Theta(\omega_-)] \,.$$

Proof of Lemma 5.9. We calculate

$$\begin{aligned}
&\int_{\mathbb{R}} [\Theta(\lambda + \delta) - \Theta(\lambda)] \,\mathrm{d}\mu(\lambda) \\
\leq &\|\nu_\mu\|_\infty \int_{\omega_-}^{\omega_+} [\Theta(\lambda + \delta) - \Theta(\lambda)] \,\mathrm{d}\lambda = \|\nu_\mu\|_\infty \left[\int_{\omega_- + \delta}^{\omega_+ + \delta} \Theta(\lambda)\mathrm{d}\lambda - \int_{\omega_-}^{\omega_+} \Theta(\lambda)\mathrm{d}\lambda \right] \\
= &\|\nu_\mu\|_\infty \left[\int_{\omega_+}^{\omega_+ + \delta} \Theta(\lambda)\mathrm{d}\lambda - \int_{\omega_-}^{\omega_- + \delta} \Theta(\lambda)\mathrm{d}\lambda \right] \leq \|\nu_\mu\|_\infty \cdot \delta \,[\Theta(\omega_+ + \delta) - \Theta(\omega_-)] \,. \square
\end{aligned}$$

Thus, we find for all $n = 1, \dots, |\tilde{\Lambda}_L|$

$$\int_{\omega_-}^{\omega_+} \left[\Theta_n(\omega_{k(n)} + \delta) - \Theta_n(\omega_{k(n)}) \mathrm{d}\mu(\omega_{k(n)}) \right] \leq \|\nu_j\|_\infty \cdot \delta \,[\Theta_n(\omega_+ + \delta) - \Theta_n(\omega_-)] \,.$$

It now suffices to estimate the differences $\Theta_n(\omega_+ + \delta) - \Theta_n(\omega_-)$. This will be done by the following result. Its proof relies on the so-called spectral shift function, see, e.g., Theorem 2 in [HKN$^+$06] and [TV15, Proposition 3.4] where some more details are provided in a special case.

Proposition 5.10. *Let $H_0 := -\Delta + A$ be a Schrödinger operator with a bounded potential $A \geq 0$, and let $H_1 := H_0 + B$ for some bounded potential $B \geq 0$ with compact support. Denote the corresponding Dirichlet restrictions to Λ by H_0^Λ and H_1^Λ, respectively. There are constants K_1, K_2 depending only on d and monotonously on $\operatorname{diam} \operatorname{supp} B$ such that for any smooth, bounded function $g : \mathbb{R} \to \mathbb{R}$ with compact support in $(-\infty, E_0]$ and the property that $g(H_1^\Lambda) - g(H_0^\Lambda)$ is trace class we have*

$$\operatorname{Tr} \left[g(H_1^\Lambda) - g(H_0^\Lambda) \right] \leq K_1 \mathrm{e}^{E_0} + K_2 \left(\ln(1 + \|g'\|_\infty)^d \right) \|g'\|_1.$$

where $\|g'\|_1 = \int_{\mathbb{R}} |g'|$.

Proposition 5.10 implies

Lemma 5.11. *Let $0 < \epsilon \leq \tilde{\epsilon}_0$. Then $\Theta_n(\omega_+ + \delta) - \Theta_n(\omega_-) \leq (K_1 \mathrm{e}^{E_0} + 2^d K_2) |\ln \epsilon|^d$, where K_1, K_2 are as in Proposition 5.10 and thus only depend on d and on M.*

Proof of Lemma 5.11. Let $g(\cdot) := \rho(\cdot - E - 2\epsilon)$. By our choice of ρ, g has support in $(-\infty, E_0]$, $\|g'\|_\infty \leq 1/\epsilon$ and $\|g'\|_1 = 1$. We define the operators

$$H_0^\Lambda := H\left(\tilde{\omega}^{(n,\delta)}(\omega_-), L\right) \quad \text{and} \quad H_1^\Lambda := H\left(\tilde{\omega}^{(n,\delta)}(\omega_+ + \delta), L\right).$$

They are lower semibounded operators with purely discrete spectrum and since g has support in $(-\infty, E_0]$, the difference $g(H_1^\Lambda) - g(H_0^\Lambda)$ is trace class. By the previous proposition

$$\Theta_n(\omega_+ + \delta) - \Theta_n(\omega_-) = \operatorname{Tr}\left[g(H_1^\Lambda) - g(H_0^\Lambda)\right] \leq K_1 \mathrm{e}^{E_0} + K_2 \left(\ln(1 + 1/\epsilon)\right)^d.$$

To conclude, we set $\epsilon_0 = \min\{\tilde{\epsilon}_0, 1/2\}$ which implies in particular $\epsilon \leq \epsilon_0 \leq \frac{1}{2}$. Thus $\ln(1 + 1/\epsilon) \leq 2|\ln \epsilon|$ and $1 \leq |\ln \epsilon| \leq |\ln \epsilon|^d$. □

Putting everything together and recalling $\delta = (4\epsilon)^{1/\kappa}$ we find

$$\begin{aligned}\mathbb{E}\left[\operatorname{Tr}\left[P_{H_{\omega,L}}([E-\epsilon, E+\epsilon])\right]\right] &\leq \left(K_1 \mathrm{e}^{E_0} + 2^d K_2\right) \left(\sup_{j \in \mathbb{Z}^d} \|\nu_j\|_\infty\right) \delta\, |\ln \epsilon|^d\, |\tilde{\Lambda}_L| \\ &\leq \left(K_1 \mathrm{e}^{E_0} + 2^d K_2\right) \left(\sup_{j \in \mathbb{Z}^d} \|\nu_j\|_\infty\right) (4\epsilon)^{1/\kappa}\, |\ln \epsilon|^d\, 3^d L^d.\end{aligned}$$

This shows the statement of Theorem 5.6. □

5.4. What to do in the presence of magnetic fields

This subsection contains proofs and theorems from [TV15, TV16a, TT18].

In Theorem 5.6 we used the unique continuation principles from Section 3 to deduce a Wegner estimate. However, so far, we have only discussed Schrödinger operators where the non-random part was the negative Laplacian. In this subsection, we consider the situation where the Laplacian is replaced by the magnetic Schrödinger operator $(-\mathrm{i}\nabla + A)^2$, A being a vector field. In this situation, A is called a *magnetic vector potential.* It can be considered as a 1-form and the physically relevant *magnetic field* B can be considered as a 2-form, obtained via $B = \mathrm{d}A$, where d denotes the exterior derivative [AHS78]. Note that in a d-dimensional space, 2-forms are isomorphic to $\binom{d}{2} = d(d-1)/2$-dimensional vector valued functions, i.e. it is only in dimension $d = 3$ that the dimension of the magnetic field and the space dimension coincide. In this case, we have in particular $B = \operatorname{Curl} A$. In dimension $d = 2$, the magnetic field is a scalar-valued function, while for $d \geq 4$, magnetic fields are described as vector fields on a higher dimensional space. A simple generalization of Theorem 5.6 is the following statement, a special case of which can be found in [TV15, Theorem 1.3].

Theorem 5.12. *Let $A \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ and define the operator $H_0 := (-\mathrm{i}\nabla + A)^2$. Assume that for all $L \in \mathbb{N}$, all $E \geq 0$, all $\delta \in (0, 1/2)$, all $V \in L^\infty(\mathbb{R}^d)$, and all $(1, \delta)$-equidistributed sequences Z we have the unique continuation principle*

$$P_{H_0+V|_{\Lambda_L}}(E) \geq \delta^C P_{H_0+V|_{\Lambda_L}}(E) W_{\delta,Z}(\Lambda_L) P_{H_0+V|_{\Lambda_L}}(E) \tag{63}$$

in quadratic form sense, where the constant $C = C(\|V\|_\infty, E)$ depends in a monotone way on $\|V\|_\infty$ and on E. Then the Wegner estimate of Theorem 5.6 holds with $-\Delta$ replaced by $(-\mathrm{i}\nabla + A)^2$.

Proof. Let $E_0 \in \mathbb{R}$. By Lemma 4.4, the unique continuation principle implies for all $L \in \mathbb{N}$, all $\omega \in \Omega$, all sufficiently small $\delta > 0$, and all $k \in \mathbb{N}$ such that $\lambda_k(H_{\omega+\delta,L}) \leq E_0$ that

$$\lambda_k(H_{\omega+\delta,L}) \geq \lambda_k(H_{\omega,L}) + \delta^C.$$

The rest of the proof is completely analogous to the proof of Theorem 5.6 above, see also [TV15], where it is elaborated for the special case of the standard random breather model. The only issue to address is that Proposition 5.10 has to be replaced by an appropriate variant for magnetic Schrödinger operators. Such estimates are known in the literature, see for instance [TV15, Corollary 2.5]. □

There is one problem with Theorem 5.12: it outsources one main difficulty since the proof of Ineq. (63) requires a magnetic analogon of Theorem 3.9.

As discussed in Subsection 3.5, such magnetic analogs seem to be in reach in the next years by applying appropriate Carleman estimates. However, the Carleman formalism is exponentially sensitive to the infinity norm of the parameters of the operator involved and thus the magnetic UCPs are expected to carry a term proportional to $\exp(\sup_{x \in \Lambda_L} |A(x)|)$. This dependence would then be inherited by the upper bound in the Wegner estimate. But for the multi-scale analysis, this would not be sufficient. Thus, the applicability of Theorem 5.12 would be restricted to bounded (or strictly speaking to logarithimically growing) magnetic potentials A which already excludes the physically most relevant case of a homogeneous magnetic field since in this case, the potential grows of order L.

For the rest of this section, we focus on results that can be achieved with existing tools. First, we use Proposition 3.38, the magnetic variant of Proposition 3.8 which has recently been proved in [BTV15] to study so-called crooked magnetic alloy-type potentials with bounded magnetic fields. We also comment on the disorder dependence of the appearing constants which might hint to the limitations of the multi-scale analysis. Then, we consider the special case of the Landau Hamiltonian (homogeneous magnetic field in two dimensions). Wegner estimates and localization for the Landau Hamiltonian with (ergodic and non-ergodic) alloy-type potentials has been established in [CHKR04, CHK07, RM12]. However, due to the structure of the spectrum of the Landau Hamiltonian it is even possible to obtain Wegner estimates for some random breather models in the small disorder regime, see Theorem 5.24. We emphasize that this trick is not possible when the background operator is the non-magnetic pure Laplacian.

5.4.1. Wegner estimate and disorder dependence for alloy-type models with bounded magnetic potential

This subsection is based on the work [TT18]. Let us define the class of random Schrödinger operators studied in this subsection. It is a generalization of the models studied in [Kle13].

Definition 5.13. Let H_0 be a magnetic Schrödinger operator of the form

$$H_0 = (-\mathrm{i}\nabla + A_0)^2 + V_0$$

in $L^2(\mathbb{R}^d)$ with a real-valued and bounded electric potential $V_0 \in L^\infty(\mathbb{R}^d)$, and a bounded magnetic vector potential $A_0 \in L^\infty(\mathbb{R}^d, \mathbb{R}^d)$ such that $\operatorname{div}(A_0)$ is bounded and $\inf \sigma(H_0) = 0$. Note that we can rewrite H_0 to $H_0 = -\Delta + b_0^T \nabla + c_0$ where

$$b_0(x) = -2\mathrm{i}A_0(x) \quad \text{and} \quad c_0(x) = V_0(x) + |A_0(x)|^2 - \mathrm{i}\operatorname{div}(A_0)(x), \tag{64}$$

for bounded functions $b\colon \mathbb{R}^d \to \mathbb{C}^d$, and $c\colon \mathbb{R}^d \to \mathbb{C}$. Now let $\{\omega_j\}_{j\in\mathbb{Z}^d}$ be an independent sequence of random variables on the probability space

$$(\Omega, \mathcal{A}, \mathbb{P}) = \left(\bigtimes_{j\in\mathcal{D}} \mathbb{R}, \quad \bigotimes_{j\in\mathcal{D}} \mathcal{B}(\mathbb{R}), \quad \bigotimes_{j\in\mathcal{D}} \mu_j \right)$$

where μ_j, $j \in \mathbb{Z}^d$, are probability measures on $\mathbb{R}$ with $\operatorname{supp} \mu_j \subset [0, \omega_+]$ for some $\omega_+ > 0$ and all $j \in \mathbb{Z}^d$. We write $S_\mu(t) := \sup_{a\in\mathbb{R}} \mu([a, a+t])$ for the concentration function of a probability measure μ and set for $t \geq 0$

$$S_L(t) := \sup_{j\in\Lambda_L\cap\mathbb{Z}^d} S_{\mu_j}(t).$$

Furthermore, we assume that the μ_j are non-singular which means that for all $L > 0$ we have $S_L(t) \to 0$ as $t \to 0$.

Let now $\delta_- \in (0, 1/2)$ and $Z = (z_j)_{j\in\mathbb{Z}^d} \subset \mathbb{R}^d$ such that

$$\forall j \in \mathbb{Z}^d\colon \quad B(\delta_-, z_j) \subset \Lambda_1(j).$$

For each $\omega \in \Omega$, the *crooked alloy-type potential* $V_\omega : \mathbb{R}^d \to \mathbb{R}$ is defined by

$$V_\omega(x) = \sum_{j\in\mathbb{Z}^d} \omega_j u_j(x - z_j),$$

where the single-site potentials $(u_j)_{j\in\mathbb{Z}^d}$, are measurable and real-valued functions on $\mathbb{R}^d$ satisfying

$$u_- \mathbf{1}_{B(\delta_-)} \leq u_j \leq \mathbf{1}_{\Lambda_{\delta_+}(0)}$$

for some $u_- \in (0, 1]$ and $\delta_+ > 0$. For each $\omega \in \Omega$ and $\lambda > 0$ we define the self-adjoint operator

$$H_\omega = H_0 + \lambda V_\omega$$

in $L^2(\mathbb{R}^d)$, and call the family of operators $(H_\omega)_{\omega\in\Omega}$ the *magnetic crooked alloy-type model.* For $L > 0$ we denote by $H_{\omega,L}$ the restrictions of H_ω to $L^2(\Lambda_L)$ subject to Dirichlet boundary conditions.

For later use, we also define the non-negative functions

$$U(x) := \sum_{j\in\mathbb{Z}^d} u_j(x - z_j) \quad \text{and} \quad W(x) := \sum_{j\in\mathbb{Z}^d} \mathbf{1}_{B(\delta_-,z_j)},$$

as well as their restrictions U_L and W_L to Λ_L for $L > 0$.

Theorem 5.14. *Let $\{H_\omega\}_{\omega\in\Omega}$ be a crooked magnetic alloy-type Hamiltonian as in Definition 5.13. Let $E_0 > 0$ and set*

$$\gamma = \frac{1}{2}\delta_-^{N_1\left(1+|E_0|^{2/3}+\|b_0\|_\infty^2+\left(\|c_0\|_\infty+\lambda\omega_+(2+\delta_+)^d\right)^{2/3}\right)} \tag{65}$$

where $N_1 > 0$ is the constant from Theorem 3.38 and b_0, c_0 are as in Eq. (64). Then there is a constant $C_2 = C_2(d, \delta_+, \|V_0\|_\infty)$ such that for any closed interval $I \subset (-\infty, E_0]$ with $|I| \leq 2\gamma$, any $\lambda > 0$, and any $L \in \mathbb{N}_{\mathrm{odd}}$ with $L \geq 2 + \delta_+$, we have

$$\mathbb{E}\left[\operatorname{Tr} P_{H_{\omega,L}}(I)\right] \leq C_2 \left(u_-^{-2}\gamma^{-4}(1 + E_0)\right)^{2^{1+\frac{\log d}{\log 2}}} S_L(\lambda^{-1}|I|)|\Lambda_L|.$$

Theorem 5.14 is an adaptation of [Kle13, Theorem 1.5] to the magnetic setting. Hence, as in [Kle13], we shall need the following lemma for its proof.

Lemma 5.15. *Let $I \subset (-\infty, E_0]$ be a closed interval and $L \in \mathbb{N}_{\mathrm{odd}}$, $L \geq 2 + \delta_+$. Suppose that there is $\kappa > 0$ such that*

$$P_{H_{\omega,L}}(I) U_L P_{H_{\omega,L}}(I) \geq \kappa P_{H_{\omega,L}}(I) \quad \textit{with probability one.}$$

Then there is a constant

$$C_4 = C_4(d, \delta_+, \|V_0\|_\infty)$$

such that

$$\mathbb{E}\big(\operatorname{Tr} P_{H_{\omega,L}}(L)\big) \leq C_4 \left(\kappa^{-2}(1+E_0)\right)^{2^{1+\frac{\log d}{\log 2}}} S_L(\lambda^{-1}|I|)|\Lambda_L|.$$

Proof of Lemma 5.15. One can follow verbatim the proof of Lemma 3.1 in [Kle13] which partially relies on a result from [CHK07]. The latter one applies to the class of magnetic Schrödinger operators as considered here as well. The only issue to address is the dependence of C_4 on the various parameters. In Eqs. (3.8) and (3.17) of [Kle13], constants from Combes-Thomas estimates (for non-magnetic Schrödinger operators) which depend only on d, δ_+ and on $\|V_0\|_\infty$ enter the final constant C_4. Combes-Thomas estimates for magnetic Schrödinger operators do not depend on the magnetic potential A_0, see Theorem 4.6 of [She14]. Therefore the constant C_4 will not depend on the magnetic potential A_0. □

Proof of Theorem 5.14. We follow the proof of Theorem 1.5 in [Kle13]. Given $E_0 > 0$, define γ as in Eq. (65). Proposition 3.38 yields for all Λ_L with $L \in \mathbb{N}_{\mathrm{odd}}$, all intervals $I \subset (-\infty, E_0]$ with $|I| \leq 2\gamma$ and almost all $\omega \in \Omega$ the estimate

$$P_{H_{\omega,L}}(I) \leq \gamma^{-2} P_{H_{\omega,L}}(I) W_{\delta_-,Z}(\Lambda_L) P_{H_{\omega,L}}(I) \leq u_-^{-1}\gamma^{-2} P_{H_{\omega,L}}(I) U_L P_{H_{0,L}}(I).$$

The statement now follows from Lemma 5.15. □

An interesting aspect of Wegner estimates as in Theorem 5.14 is the behavior of the constant in the *large disorder regime* which to the author's knowledge has first been discussed in [Kle13]. In order to discuss the disorder dependence, we introduce some notation: For $t \geq 0$ we define

$$H_0(t) := H_0 + t \sum_{j \in \mathbb{Z}^d} u_j(\cdot - z_j) = H_0 + tU(\cdot),$$

and set

$$E_0(t) := \inf \sigma(H_0(t)) \quad \text{and} \quad E_0(\infty) := \lim_{t\to\infty} E_0(t) = \sup\left\{E_0(t) : t \geq 0\right\}.$$

The next result is a similar Wegner estimate as in Theorem 5.14, but it will be motivated by the discussion on the disorder dependence, cf. Remark 5.19 below.

Theorem 5.16. *We have $E_0(\infty) > 0$. Let $E_1 \in (0, E_0(\infty))$ and set*

$$\kappa_0 = \sup_{s>0: E_0(s) \geq E_1} \frac{E_0(s) - E_1}{s} > 0.$$

Then for any Borel set $B \subset (-\infty, E_1]$, any $\lambda > 0$, any $L \in \mathbb{N}_{\mathrm{odd}}$, and almost all $\omega \in \Omega$ we have

$$P_{H_{\omega,L}}(B)\Big(\sum_{j\in\Lambda_L\cap\mathbb{Z}^d} u_j(\cdot - z_j)\Big)P_{H_{\omega,L}}(B) \geq \kappa_0 P_{H_{\omega,L}}(B). \tag{66}$$

Moreover, for any closed interval $I \subset (-\infty, E_1]$, any $\lambda > 0$, and for any $L \in \mathbb{N}_{\mathrm{odd}}$ with $L \geq 2 + \delta_+$, we have

$$\mathbb{E}\left[\operatorname{Tr} P_{H_{\omega,L}}(I)\right] \leq C_3 \left(\kappa_0^{-2}(1+E_1)\right)^{2^{1+\frac{\log d}{\log 2}}} S_L(\lambda^{-1}|I|)|\Lambda_L|, \tag{67}$$

where $C_3 > 0$ is a constant depending on d, δ_+, $\|V_0\|_\infty$, $\|b_0\|_\infty$, and $\|c_0\|_\infty$.

For the proof of Theorem 5.16, we shall need an abstract uncertainty relation for Schrödinger operators at the bottom of the spectrum which has been developed in [BLS11]. The following lemma is a slight generalization thereof, see Lemma 4.1 of [Kle13].

Lemma 5.17. *Let H be a self-adjoint operator on a Hilbert space $\mathcal{H}$, bounded from below, and let $Y \geq 0$ be a bounded operator on $\mathcal{H}$. Let $H(t) = H + tY$ for $t \geq 0$, and set $E(t) = \inf\sigma(H(t))$ and $E(\infty) = \lim_{t\to\infty} E(t) = \sup_{t\geq 0} E(t)$. Suppose that $E(\infty) > E(0)$. For $E_1 \in (E(0), E(\infty))$ let*

$$\kappa = \kappa(H, Y, E_1) = \sup_{s>0:\ E(s)>E_1} \frac{E(s) - E_1}{s} > 0.$$

Then for all bounded operators $V \geq 0$ on $\mathcal{H}$ and Borel sets $B \subset (-\infty, E_1]$ we have

$$P_{H+V}(B) Y P_{H+V}(B) \geq \kappa P_{H+V}(B).$$

We recall that $H_0(t) = H_0 + t\sum_{j\in\mathbb{Z}^d} u_j(\cdot - z_j)$ for $t \geq 0$, $E_0(t) = \inf\sigma(H_0(t))$, and $E_0(\infty) = \lim_{t\to\infty} E_0(t) = \sup_{t\geq 0} E_0(t)$.

Lemma 5.18. *We have $E_0(\infty) > 0$.*

Proof. By monotonicity of $t \mapsto E_0(t)$, it suffices to show $E_0(t_0) > 0$ for some $t_0 > 0$. Now, for all $t \geq 0$, we have $E_0(t) = \lim_{L\to\infty} E_{0,L}(t)$ where $E_{0,L}$ denotes the restriction of $H_0 + tU$ to Λ_L with Dirichlet boundary conditions. This can for instance be seen by using a Weyl sequence argument. Furthermore, since $E_0(0) = 0$ by assumption it now suffices to establish

$$E_{0,L}(t_0) \geq E_{0,L}(0) + t_0 u_- \delta_-^{N_1\left(2+\|b_0\|_\infty^2+\|c_0+t_0U\|_\infty^{2/3}\right)}$$

for all sufficiently large $L \in \mathbb{N}_{\mathrm{odd}}$. For that purpose, we choose $t_0 > 0$ such that defining

$$\gamma^2 = \gamma(t_0)^2 := \delta_-^{N_1\left(2+\|b_0\|_\infty^2+\|c_0+t_0U\|_\infty^{2/3}\right)},$$

which is of the form of the term in Proposition 3.36 with $E = 1$, we have $\gamma(t_0) \geq t_0\|U\|_\infty$. This is possible since $\gamma(t_0)$ converges to a positive constant as $t_0 \searrow 0$.

We then have $E_{0,L}(t_0) \in [E_{0,L}(0)-\gamma, E_{0,L}(0)+\gamma]$. Choosing L sufficiently large and using $\gamma \leq \delta_- \leq 1/2$, we furthermore may assume that $E_{0,L}(0) \leq 1$. Proposition 3.36 then implies for all $\psi \in \operatorname{Ran} P_{H_{0,L}(t_0)}(E_{0,L}(t_0))$ that

$$\begin{aligned}\langle \psi, t_0 U \psi\rangle &\geq t_0 u_- \langle \psi, W\psi\rangle \\ &= t_0 u_- \|\psi\|_{L^2(S_{\delta_-,Z(L)})} \geq t_0 u_- \gamma^2 \|\psi(t)\|^2_{L^2(\Lambda_L)}.\end{aligned}$$

From Lemma 4.4 we conclude $E_{0,L}(t_0) \geq E_{0,L}(0) + t_0 u_- \gamma(t_0)^2$. □

Proof of Theorem 5.16. By Lemma 5.18 we have $E_0(\infty) > 0$ and hence $\kappa_0 > 0$. For $L \in \mathbb{N}_{\mathrm{odd}}$ we denote by $H_{0,L}(t)$ the restriction of $H_0(t)$ to $L^2(\Lambda_L)$ subject to Dirichlet boundary conditions with domain $\mathcal{D}(\Delta_L)$, and set $E_{0,L}(t) = \inf \sigma(H_{0,L}(t))$. Using $0 \leq E_0(t) \leq E_{0,L}(t)$ we obtain

$$\kappa_{0,L} := \sup_{s>0: E_{0,L}(s) \geq E_1} \frac{E_{0,L}(s) - E_1}{s} \geq \kappa_0 = \sup_{s>0: E_0(s)\geq E_0(1)} \frac{E_0(s) - E_1}{s} > 0.$$

Hence, the assumptions of Lemma 5.17 are satisfied with $H = H_{0,L}$ and $Y = \sum_{j\in\mathbb{Z}^d} u_j(\cdot - z_j)$, and we obtain Ineq. (66). Ineq. (67) now follows from Ineq. (66) and Lemma 5.15. □

Remark 5.19. The Wegner estimates in Theorems 5.14 and 5.16 can be used as ingredients for the multi-scale analysis [FS83, vDK89, GK01, GK03, GK06]. Let us emphasize that the multi-scale analysis requires that the concentration functions S_L, $L \in \mathbb{N}$, are sufficiently regular, e.g. with a uniformly bounded density, or uniformly Hölder continuous, cf. the just mentioned references. The multi-scale analysis is an induction argument to establish localization in its various manifestations (spectral, dynamical, etc.). Hence, if the concentration functions S_L are sufficiently regular, Theorems 5.14 and 5.16 will imply localization at energies where an appropriate initial scale estimate is satisfied, see [GK01, GK03] for the bootstrap multi-scale analysis, and [RM12] for the adaptation to the non-ergodic setting. If the constant in the Wegner estimate becomes small at large disorder, the initial scale estimate will follow from the Wegner estimate at sufficiently large disorder as observed in [vDK89], see also [Kir08, Kle13].

The upper bound in Theorem 5.14 grows as the disorder λ increases. This is not sufficient to deduce an initial length-scale estimate and localization at large disorder. In contrast to that, the upper bound in Theorem 5.16 converges to 0 as the disorder parameter λ tends to ∞. Hence, an initial scale estimate and localization at large disorder follow, albeit only for energies below $E_0(\infty)$. Note that if a covering condition

$$U(\cdot) = \sum_{j \in \mathbb{Z}^d} u_j(\cdot - z_j) \geq \epsilon > 0$$

is satisfied, then $E_0(\infty) = \infty$ and the statement of Theorem 5.16 holds at all energies, see [CH94] which is formulated in the special case of vanishing magnetic field. Thus, in case of a covering condition, for every energy $E \geq 0$, we find a disorder strength $\lambda_0 = \lambda_0(E)$ such that for disorder $\lambda \geq \lambda_0$, we have localization in a neighborhood of E. In contrast, $E(\infty)$ might be finite if we do not assume a covering condition.

Corollary 5.20. *Let $0 \leq E_0 < E_0(\infty)$. Then there is $\lambda_0 > 0$, such that for all disorder strengths $\lambda \geq \lambda_0$, we have localization (spectral localization with exponentially decaying eigenfunctions and strong dynamical localization) in $\Sigma \cap (-\infty, E_0]$.*

Remark 5.21. Since Wegner estimates with a disorder dependence as in Theorem 5.16 provide a relatively simple path to localization at large disorder, it is natural to ask if such a disorder dependence can be expected at all energies, even if no covering condition is assumed. However, so far one was not able to prove such a Wegner estimate for alloy-type models with and without magnetic field above the threshold $E(\infty)$, cf. [Sto10, BLS11, Kle13].

Our next theorem shows that this is indeed not possible. A disorder dependence as in Theorem 5.16 holds if and only if we consider energy intervals below $E_0(\infty)$. In particular this shows that at high energies and at high disorder there is a fundamental difference between alloy-type models with and without a covering condition. This is a new result, even in the special case of vanishing magnetic potential ($A_0 = 0$) and ergodic potential (V_0 periodic, $z_j = j$, $u_j = u_0$, and $\mu_j = \mu_0$).

Theorem 5.22. *Let $E_2 \in \mathbb{R}$. The following are equivalent:*

(i) $E_2 \leq E_0(\infty)$.

(ii) For all sufficiently large $L > 0$, and all closed intervals $I \subset (-\infty, E_2]$, we have

$$\mathbb{E}\left(\operatorname{Tr} P_{H_{\omega,L}}(I)\right) \to 0 \quad as \quad \lambda \to \infty. \tag{68}$$

Proof. The implication (i) $\Rightarrow$ (ii) is the statement of Theorem 5.16. In order to show the converse, we prove the contraposition: Let $E_2 > E_0(\infty)$, and $I = (-\infty, E_2]$. Note that for almost all $\omega \in \Omega$, we have $H_{\omega,L} \leq H_{\eta,L}$, where $\eta \in \Omega$, $\eta_k = \omega_+$ for all $k \in \mathbb{Z}^d$, hence

$$\mathbb{E}\left(\operatorname{Tr} P_{H_{\omega,L}}(I)\right) \geq \operatorname{Tr} P_{H_{\eta,L}}(I).$$

Since

$$\lim_{t\to\infty} \lim_{L\to\infty} E_{0,L}(t) < E_2,$$

there are $L_0 > 0$ and $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ we have

$$1 \leq \operatorname{Tr} P_{H_{\eta,L_0}}(I) \leq \mathbb{E}\left(\operatorname{Tr} P_{H_{\omega,L_0}}(I)\right).$$

Hence, (68) cannot hold. □

5.4.2. Wegner estimate for Landau-breather models at small disorder

This subsection is based on [TV16a]. The previous Subsection 5.4.1 treated magnetic Schrödinger operators with bounded vector potential. However, this does by far not cover all magnetic fields. The most important example are magnetic Schrödinger operators with *homogeneous magnetic field.* In dimension $d = 2$, the magnetic Schrödinger operator with homogeneous magnetic field is called the *Landau Hamiltonian.* Fortunately, for this operator, some explicit calculations are possible which allows to deduce unique continuation principles as in Lemma 3.40. We use them to prove Wegner estimates for the Landau-breather model, albeit only for small disorder. See Remark 5.28 below on a discussion why our result is restricted to this regime.

Definition 5.23. The Landau Hamiltonian is the self-adjoint operator $H_B = (-\mathrm{i}\nabla - A)^2$ with the vector field $A = (B/2)\left(\begin{smallmatrix} x_2 \\ -x_1 \end{smallmatrix}\right)$ on $L^2(\mathbb{R}^2)$ where $B > 0$ is the magnetic field strength. We define a scale $L_B > 0$ such that corresponding squares have integer flux by letting

$$K_B := 2\lceil \sqrt{B/(4\pi)} \rceil, \quad L_B = K_B\sqrt{4\pi/B}, \quad \text{and} \quad \mathbb{N}_B = L_B\mathbb{N}$$

where $\lceil a \rceil$ denotes the least integer larger or equal than a. Now let $\{\omega_j\}_{j\in\mathbb{Z}^d}$ be an independent sequence of random variables on the probability space

$$(\Omega, \mathcal{A}, \mathbb{P}) = \left(\bigtimes_{j\in\mathcal{D}} \mathbb{R}, \quad \bigotimes_{j\in\mathcal{D}} \mathcal{B}(\mathbb{R}), \quad \bigotimes_{j\in\mathcal{D}} \mu \right)$$

where μ is a probability measure with support in $[\omega_-, \omega_+] \subset [0, 1/2)$ and with uniformly bounded Lebesgue density ν_μ. For the single-site potential $u : \mathbb{R}^2 \to [0, \infty)$ we make the following assumptions:

(i) u is measurable, bounded and compactly supported.

(ii) For every $t \in [\omega_-, \omega_+]$, the map $x \mapsto \partial_t u(x/t)$ exists for almost every $x \in \mathbb{R}^2$.

(iii) There is $C_u > 0$ such that for every $t \in [\omega_-, \omega_+]$ we find $x_0 = x_0(t) \in \Lambda_1$ with

$$\frac{\partial}{\partial t} u\left(\frac{x}{t}\right) \geq C_u \mathbf{1}_{B(x_0(t),r)}(x) \text{ for almost every } x \in \mathbb{R}^2. \tag{69}$$

Now, we define the random breather potential as

$$V_\omega(x) := \sum_{j \in \mathbb{Z}^2} u_{\omega_j}(x - j) \quad \text{where} \quad u_t(x) := u\left(\frac{x}{t}\right).$$

The *Landau-breather Hamiltonian* is the random operator $H_B^\omega = H_B + \lambda V_\omega, \quad \omega \in \Omega$, where $\lambda > 0$ is the *disorder parameter*. Let $H_{B,L}$ and $H_{B,L}^\omega$ be restrictions of the operators H_B and H_B^ω to $L^2(\Lambda_L)$ with periodic boundary conditions, respectively.

In Remark 5.26 we comment on these assumptions and provide some explicit examples.

Theorem 5.24 (Wegner estimate for the Landau-breather model at small disorder)**.** *Assume that u satisfies the hypotheses (i)–(iii) above and let $B > 0$, $E_0 \in \mathbb{R}$, $\theta \in (0, 1)$. Then there is $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$ we find $C = C(B, E_0, \theta) > 0$ and $L_0 \in \mathbb{N}_B$ such that for all intervals $I \subset (-\infty, E_0]$ with $|I| \leq B/2$ and all $L \in \mathbb{N}_B$ with $L \geq L_0$ we have*

$$\mathbb{E}\left[\operatorname{Tr}\left(P_{H_{B,L}^\omega}(I)\right)\right] \leq C|I|^\theta |\Lambda_L|.$$

The critical disorder parameter λ_0 only depends on B, E_0 and u and is explicitly given in inequality (75)*.*

Since the Landau-breather Hamiltonian is ergodic with respect to magnetic shifts, it has an almost sure IDS $N(\cdot)$. We have the following corollary:

Corollary 5.25. *Fix $B > 0$, $E_0 \in \mathbb{R}$ and $\theta \in (0, 1)$. Then, for all disorder parameters $0 < \lambda \leq \lambda_0$ the IDS $N \colon \mathbb{R} \to [0, \infty)$ of the Landau-breather Hamiltonian satisfies*

$$N(E) - N(E - \epsilon) \leq C \cdot |\epsilon|^\theta \quad \textit{for all } \epsilon > 0,\ E \leq E_0.$$

Thus, the IDS of H_B^ω is locally Hölder continuous in $(-\infty, E_0]$ with respect to any Hölder exponent $\theta \in (0,1)$. In the context of the Landau Hamiltonian this is somewhat more remarkable than for – say – random Schrödinger operators where the background operator is the negative Laplacian. While $-\Delta$ has a continuous IDS, the IDS of the Landau Hamiltonian is a step function with jumps at the Landau levels $2(n-1)B$, $n \in \mathbb{N}$, cf. [Nak01, HLMW01, RW02a], and Corollary 5.25 states that arbitrarily small random perturbations make the IDS continuous.

Remark 5.26. Note that in condition (69) the radius r and the constant C_u need to be t-independent, while we can allow x_0 to vary with t. Condition (69) translates into

$$-x/t^2 \cdot (\nabla u)(x/t) \geq C_u \mathbf{1}_{B(x_0,r)}(x) \quad \text{for all } x \in \mathbb{R}^2,\ t \in [\omega_-, \omega_+]$$

or equivalently

$$-y \cdot \nabla u(y) \geq C_u t \mathbf{1}_{B(x_0,r)}(ty) = C_u t \mathbf{1}_{\frac{1}{t}B(x_0,r)}(y) \quad \text{for all } y, t$$

and can be compared to the condition on the breather potential

$$u \in C^1(\mathbb{R}^2 \setminus \{0\}), \quad -x \cdot \nabla u \geq \epsilon_0 u \text{ for all } x \in \mathbb{R}^2 \setminus \{0\}$$

with fixed $\epsilon_0 > 0$ in [CHN01] which implies a singularity at the origin which we do not have.

Let us give two examples of single-site potentials u satisfying our assumptions.

- The smooth function

$$u(x) = \exp\left(-\frac{1}{1-|x|^2}\right) \mathbf{1}_{|x|<1},$$

  since, using $\omega_- \leq t \leq 1/2$

$$\begin{aligned} -x/t^2 \cdot (\nabla u)(x/t) &= x/t^2 \cdot \exp\left(-\frac{1}{1-|x/t|^2}\right) \frac{2(x/t^2)}{(1-|x/t|^2)^2} \cdot \mathbf{1}_{|x|<t} \\ &\geq 2\mathrm{e}^{-4/3}|x|^2 \cdot \mathbf{1}_{|x|<t/2} \geq \frac{|x|^2}{2} \mathbf{1}_{\omega_-/4 \leq |x| < \omega_-/2} \geq \frac{\omega_-^2}{32} \mathbf{1}_{B(x_0, \omega_-/8)} \end{aligned}$$

  for every point x_0 with $|x_0| = 3\omega_-/8$.

- The hat potential $u(x) = \mathbf{1}_{|x|<1}(1-|x|)$, since

$$-x/t^2 \cdot (\nabla u)(x/t) = \frac{|x|}{t^3} \mathbf{1}_{|x|<t} \geq \frac{\mathbf{1}_{t/2 \leq |x| < t}}{2t^2} \geq \frac{1}{2\omega_+^2} \mathbf{1}_{B(x_0, \omega_-/4)} \geq 2\mathbf{1}_{B(x_0,\omega_-/4)}$$

  for every point x_0 with $|x_0| = 3t/4$.

Unfortunately, our assumptions are not satisfied for the *standard random breather potential* from Subsection 3.2.

We start the proof of Theorem 5.24 with the following abstract theorem:

Theorem 5.27. *Let H be a lower semibounded self-adjoint operator with purely discrete spectrum, V a bounded symmetric operator, and $I \subset J \subset \mathbb{R}$ two intervals. We assume that there are $C_2 > 0$ and a positive, symmetric operator W such that*

$$P_H(J) W P_H(J) \geq C_2 P_H(J). \tag{70}$$

Then, for $\|V\|_\infty < \operatorname{dist}(I, J^c)\sqrt{C_2/(C_2 + 1 + \|W\|_\infty)}$ there is C_3 depending only on C_2, $\operatorname{dist}(I, J^c)$ and on $\|V\|_\infty$ such that

$$\operatorname{Tr}\left[P_{H+V}(I)\right] \leq C_3 \operatorname{Tr}(P_{H+V}(I)(W + W^2)) \tag{71}$$

More precisely, we have

$$C_3 = \frac{\operatorname{dist}(I, J^c)^2}{C_2 \operatorname{dist}(I, J^c)^2 - \|V\|_\infty^2 (C_2 + 1 + \|W\|_\infty)}.$$

Proof. We decompose

$$\operatorname{Tr}(P_{H+V}(I)) = \operatorname{Tr}(P_{H+V}(I) P_H(J)) + \operatorname{Tr}(P_{H+V}(I) P_H(J^c)(H)). \tag{72}$$

We estimate the term in (72) containing $P_H(J^c)$ by expanding the trace in eigenfunctions ϕ_j in the range of $P_{H+V}(I)$.

From the eigenvalue equation $(H + V - E_j)\phi_j = 0$ we deduce

$$-(H - E_j)^{-1} P_H(J^c) V \phi_j = P_H(J^c)\phi_j.$$

This yields

$$\operatorname{Tr}\left(P_{H+V}(I) P_H(J^c)\right) = \sum_j \langle \phi_j, P_H(J^c)\phi_j \rangle = \tag{73}$$

$$\sum_j \left\langle \phi_j, \left(V \frac{P_H(J^c)}{(H - E_j)^2} V \right) \phi_j \right\rangle \leq \frac{\|V\|_\infty^2}{\operatorname{dist}(I, J^c)^2} \operatorname{Tr}\left(P_{H+V}(I)\right).$$

Now we turn to the first summand on the right hand side of (72). Using the assumption (70), we have

$$\begin{aligned}
\operatorname{Tr}(P_{H+V}(I) P_H(J)) &\leq \frac{1}{C_2} \operatorname{Tr}(P_{H+V}(I) P_H(J) W P_H(J)) \\
&= \frac{1}{C_2} [\operatorname{Tr}(P_{H+V}(I) W) + \operatorname{Tr}(P_{H+V}(I) P_H(J^c) W P_H(J^c)) \\
&\quad - 2 \operatorname{Re}(\operatorname{Tr}\left[P_{H+V}(I) P_H(J^c) W\right)] \\
&\leq \frac{1}{C_2} [\operatorname{Tr}(P_{H+V}(I) W) + \|W\|_\infty \operatorname{Tr}(P_{H+V}(I) P_H(J^c)) \\
&\quad + \operatorname{Tr}(P_{H+V}(I) P_H(J^c)) + \operatorname{Tr}(P_{H+V}(I) W^2)].
\end{aligned}$$

In the last step, we used $-\operatorname{Re}(x) \le |x|$, cyclicity of the trace, the Hölder inequality for traces, and the fact that $2ab \le a^2 + b^2$ to estimate

$$\begin{aligned}-2\operatorname{Re}(\operatorname{Tr}(P_{H+V}(I)P_H(J^c)W) &\le 2|\operatorname{Tr}(P_{H+V}(I)P_H(J^c)WP_{H+V}(I))| \\ &\le \operatorname{Tr}(P_{H+V}(I)P_H(J^c)) + \operatorname{Tr}(P_{H+V}(I)W^2).\end{aligned}$$

This simplifies to

$$\begin{aligned}&\operatorname{Tr}(P_{H+V}(I)P_H(J)) \le \\ &\quad\le \frac{1+\|W\|_\infty}{C_2}\operatorname{Tr}(P_{H+V}(I)P_H(J^c)) + \frac{1}{C_2}\operatorname{Tr}(P_{H+V}(I)(W+W^2)).\end{aligned} \tag{74}$$

Combining (72) with (73) and (74) we find

$$\begin{aligned}\operatorname{Tr}(P_{H+V}(I)) \le \left(1+\frac{1+\|W\|_\infty}{C_2}\right)&\frac{\|V\|_\infty^2}{\operatorname{dist}(I,J^c)^2}\operatorname{Tr}(P_{H+V}(I)) \\ &+\frac{1}{C_2}\operatorname{Tr}(P_{H+V}(I)(W+W^2)),\end{aligned}$$

that is

$$\begin{aligned}&\operatorname{Tr}(P_{H+V}(I)) \le \\ &\quad\le \frac{\operatorname{dist}(I,J^c)^2}{C_2\operatorname{dist}(I,J^c)^2 - \|V\|_\infty^2(C_2+1+\|W\|_\infty)}\operatorname{Tr}(P_{H+V}(I)(W+W^2)). \qquad \square\end{aligned}$$

We are now ready for the proof of Theorem 5.24.

Proof of Theorem 5.24. Given $B > 0$ and $E_0 \in \mathbb{R}$, there are finitely many Landau levels below $E_0 + B/4$. Let $L \in \mathbb{N}_B$, $L \ge L_0$ as in Lemma 3.40. Take

$$\tilde{C} := \min\{C_1(n,B,r) \text{ from Lemma 3.40} : n \in \mathbb{N} \text{ with } B(2n-1) \le E_0 + B/4\}.$$

Let $I = [I_-, I_+] \subset (-\infty, E_0]$ with $I_+ - I_- \le B/2$. For every constellation $\{\omega_j\}_{j\in\mathbb{Z}^2}$, we apply Theorem 5.27 with $V = \lambda V_\omega$ and $J = [I_- - B/4, I_+ + B/4]$ and

$$W = \sum_{j\in\mathbb{Z}^2\colon B(x_j+j,r)\subset\Lambda_L} \mathbf{1}_{B(x_0(\omega_j)+j,r)}$$

where the $x_0(\omega_j)$ are the points from (69). Note that J contains at most one Landau level. Hence, (70) holds by Lemma 3.40 with $C_2 = \tilde{C}$. We find

$$\operatorname{Tr}(P_{H+V}(I)) \le C_3 \operatorname{Tr}(P_{H^\omega_{B,L}}(I)W),$$

where

$$C_3 = \frac{(B/4)^2}{\tilde{C}(B/4)^2 - \|V_\omega\|_\infty^2\lambda^2(\tilde{C}+2)}.$$

We have $\|V_\omega\|_\infty \leq V_\infty := \lceil \max \operatorname{supp} u \rceil^2 \|u\|_\infty$. For

$$\lambda \leq \lambda_0 := B\sqrt{\tilde{C}/(32V_\infty^2(\tilde{C}+2))}, \tag{75}$$

it holds that $C_3 \leq 2/\tilde{C}$. We have estimated so far

$$\operatorname{Tr}(P_{H+V}(I)) \leq \frac{2}{\tilde{C}} \operatorname{Tr}(P_{H^\omega_{B,L}}(I)W).$$

Now we take a monotone decreasing function $f \in C^1(\mathbb{R})$ with $f \equiv 1$ on $(-\infty, I_- - B/4]$ and $f \equiv 0$ on $[I_+ + B/4, \infty)$ such that $-C_4|I|f'(x) \geq \mathbf{1}_I(x)$ for some $C_4 > 0$. Then

$$\begin{aligned}\operatorname{Tr}(P_{H^\omega_{B,L}}(I)W) &\leq -C_4|I| \operatorname{Tr}(f'(H^\omega_{B,L})W) \\ &\leq -C_u^{-1}C_4|I| \sum_{j\in\mathbb{Z}^2\cap\Lambda_L} \operatorname{Tr}(f'(H^\omega_{B,L})\frac{\partial}{\partial\omega_j}u_{\omega_j}(x-j)) \\ &= -C_u^{-1}C_4|I| \sum_{j\in\mathbb{Z}^2\cap\Lambda_L} \operatorname{Tr}(\frac{\partial}{\partial\omega_j}f(H^\omega_L)).\end{aligned}$$

We take the expectation and obtain

$$\mathbb{E}\left[\operatorname{Tr}(P_{H^\omega_{B,L}}(I)W)\right] \leq -C_u^{-1}C_4|I| \sum_{j\in\mathbb{Z}^2\cap\Lambda_L} \mathbb{E}\left[\frac{\partial}{\partial\omega_j}\operatorname{Tr}(f(H^\omega_{B,L}))\right].$$

We evaluate the expectation in every summand with respect to the random variable ω_j

$$\begin{aligned}0 \leq -\mathbb{E}\left[\frac{\partial}{\partial\omega_j}\operatorname{Tr}(f(H^\omega_{B,L}))\right] &= -\mathbb{E}\left[\int_{\omega_-}^{\omega_+}\frac{\partial}{\partial\omega_j}\operatorname{Tr}(f(H^\omega_{B,L}))\mathrm{d}\omega_j\right] \\ \leq \|\nu_\mu\|_\infty \mathbb{E}\left[\left|\operatorname{Tr}\left(f(H^\omega_{B,L}\mid_{\omega_j=\omega_+}) - f(H^\omega_{B,L}\mid_{\omega_j=\omega_-})\right)\right|\right].\end{aligned}$$

Analogously to [CHK03], Appendix A we find for every $\theta \in (0,1)$ a constant $C_\theta > 0$ such that

$$\left|\operatorname{Tr}\left(f(H^\omega_{B,L}\mid_{\omega_j=\omega_+}) - f(H^\omega_{B,L}\mid_{\omega_j=\omega_-})\right)\right| \leq C_\theta|I|^{\theta-1}.$$

All together we found

$$\operatorname{Tr}\left(P_{H^\omega_{B,L}}(I)\right) \leq \frac{4}{C_u\tilde{C}}C_4|I|C_\theta|I|^{\theta-1}\#\{\Lambda_L\cap\mathbb{Z}^2\} = C|I|^\theta|\Lambda_L|. \qquad \square$$

Remark 5.28. Let us explain why our result is restricted to the small coupling regime and discuss possible approaches how to remove the condition on the disorder strength λ. One candidate for replacing the smallness condition on $\|V\|_\infty$ in Theorem 5.27, which is necessary to ensure positivity of $C_2 \operatorname{dist}(I, J^c)^2 - \|V\|_\infty^2(C_2 + 1 + \|W\|_\infty)$, would be a largeness condition on $\operatorname{dist}(I, J^c)$. In the application (i.e. in the proof of Theorem 5.24) an upper bound on $\operatorname{dist}(I, J^c)$ is due to the fact that J should contain at most one Landau level. Therefore, it would be desirable to improve Lemma 3.40 to something like

$$\left(\sum_{k=1}^{n} \Pi_{n,L}\right) W_{\delta,Z}(\Lambda_L) \left(\sum_{k=1}^{n} \Pi_{n,L}\right) \geq C \left(\sum_{k=1}^{n} \Pi_{n,L}\right),$$

where $C = C(n, B, r)$ behaves asymptotically for large n as

$$C(n, B, r) \gg n^{-2}.$$

6. Application to control theory for heat-type equations

In this section, we apply the scale-free, quantitative unique continuation principle from Theorem 3.13 and the Logvinenko-Sereda theorem to controlled heat-type equations with interior control in a multi-scale setting on bounded and unbounded domains.

We first give an introduction to the controlled heat equation in Subsection 6.1. After that, in Subsection 6.2, which is based on unpublished work with Ivica Nakić, Martin Tautenhahn, and Ivan Veselić, we prove null-controllability with explicit estimates on the control cost in a general setting: We consider abstract controlled Cauchy problems satisfying a so-called spectral inequality. We emphasize that our abstract result is interesting in its own right since it improves and unifies existing results and is applicable beyond the scope of heat-type equations. Heat-type equations however are a special case of such controlled Cauchy problems and the unique continuation principles from Section 3.2 are a special case of spectral inequalities. Thus, in Subsection 6.3, we deduce estimates on the control cost of heat-type equations and discuss its asymptotic behavior in certain regimes, namely in the *homogenization* and in the *de-homogenization* or *coarsening* regime.

6.1. A brief introduction to the controlled heat equation

In order to present the setting and to better motivate our results below, we start with an introduction on control theory for the (standard) heat equation. We emphasize that the amount of relevant literatur in this context is substantial and we do not claim to provide a full list of references, but rather focus on results relevant for our results below.

Let $\Omega \subset \mathbb{R}^d$ be a (open, connected) domain and $\omega \subset \Omega$. Given a time $T > 0$, the *controlled heat equation with interior control on* ω is

$$\begin{cases} \partial_t u - \Delta u = \mathbf{1}_\omega f, & u \in L^2([0,T] \times \Omega), \\ \mathfrak{B} u = 0, & \text{for all } t \in (0,T), \\ u(0,\cdot) = u_0, & u_0 \in L^2(\Omega), \end{cases} \tag{76}$$

where $\mathfrak{B}u = 0$ stands for boundary conditions which make the operator Δ self-adjoint in $L^2(\Omega)$. This could be for instance Dirichlet, Neumann or periodic boundary conditions. If $\Omega = \mathbb{R}^d$ then no boundary conditions are required and the condition $\mathfrak{B}u = 0$ is void. The function $f \in L^2([0,T] \times \Omega)$ is called *control function* and since its action on the system is penalized by the indicator function $\mathbf{1}_\omega$, it can be considered as a function $f \in L^2([0,T] \times \omega)$. It is easy to see that for fixed u_0 and f, the unique mild solution of the system (76) in $L^2([0,T],\Omega)$ is given by

$$u(t) = \mathrm{e}^{\Delta t} u_0 + \int_0^t \mathrm{e}^{(t-s)\Delta}(\mathbf{1}_\omega f(s))\mathrm{d}s, \quad t \in [0,T]. \tag{77}$$

The fundamental motivation in control theory revolves around the following question: Given an initial state u_0, and a target state u_T, does there exist a control function f, such that the solution (77) of system (76) satisfies $u(T) = u_T$? And if the answer is yes: What can be said about such a function f?

The first important observation is that it is convenient to restrict the attention to *null-controllability* i.e. to the target state $u_T = 0$. A formal definition of null-controllability will be given in Definition 6.3 below. The following lemma seems to be folklore, however, for convenience we provide a short proof:

Lemma 6.1 (Null-controllability implies controllability on the range of the semi-group)**.** *Assume that for every $u_0 \in L^2(\Omega)$ there exists a control function such that the solution of system* (76) *with initial state u_0 satisfies $u(T) = 0$. Then, for every pair (u_0, u_T) where $u_0 \in L^2(\Omega)$, and u_T is in the range of the operator $\mathrm{e}^{T\Delta}$ there exists a control f such that the solution of system* (76) *with initial state u_0 satisfies $u(T) = u_T$.*

Proof. By assumption, we have $u_T = \mathrm{e}^{T\Delta} v_0$ for some $v_0 \in L^2(\Omega)$. We choose a null-control $f = f_{u_0 - v_0}$, driving the initial state $u_0 - v_0$ to zero in time T. Then, by (77)

$$0 = \mathrm{e}^{T\Delta}(u_0 - v_0) + \int_0^T \mathrm{e}^{(T-s)\Delta}\mathbf{1}_\omega f(s)\mathrm{d}s$$

whence

$$u_T = \mathrm{e}^{T\Delta} v_0 = \mathrm{e}^{T\Delta} u_0 + \int_0^T \mathrm{e}^{(T-s)\Delta}\mathbf{1}_\omega f(s)\mathrm{d}s = u(T). \qquad \square$$

For bounded and connected domains Ω, it is well-established that system (76) is null-controllable in all times $T > 0$ as soon as ω is non-empty and open [FI96, LR95]. In [AE13, AEWZ14], it is proven that for bounded domains Ω, it suffices ω to have positive Lebesgue measure. If $\Omega = \mathbb{R}^d$, it has recently been proved [WWZZ17, EV18] that the system (76) is null-controllable if and and only if the set $\omega \subset \mathbb{R}^d$ is *thick*, as in Definition 3.42 in the context of the Logvinenko-Sereda theorem above. Furthermore, in [EV18], the control cost of the heat equation on cubes with thick control sets in a multiscale setting has been considered for the first time. We will study the same situation in Theorem 6.8 below.

Our results below focus on several aspects. First, instead of the pure heat equation we consider the *heat-type equation* or the *heat equation with generation term* where the generator $-\Delta$ of the free evolution semigroup in (76) has been replaced by the operator $-\Delta + V$ with $V \in L^\infty$. Even though such operators have been considered and null-controllability has been established in [FI96], the analysis – in particular in the large time regime – is more challenging since the operator $-\Delta + V$ might not be semidefinite any more which is a common assumption in control theory. A second important aspect is that we simultaneously treat bounded and unbounded domains. On unbounded domains, the operator $-\Delta + V$ will have essential spectrum and there does not necessarily exist an orthonormal basis of eigenfunctions any more – a fact which is prominently used in many proofs of null-controllability. Finally, the most important aspect of our results is that we provide explicit estimates on the so-called *control cost* in terms of all model parameters – in particular parameters describing the geometry of the sets Ω and ω – and which are to our knowledge the best with respect to the existing literature. Estimates on the control cost are upper bounds on the norm of the control function $f \in L^2([0,T] \times \omega)$, see Definition 6.3 below for a proper definition of the control cost. There exist many results in the literature which estimate the control cost and some also derive parameter dependences. A plethora of articles has studied the dependence of the control cost on the time parameter T [Güi85, FZ00, Phu04, TT07, Mil06, Mil04, Mil10, EZ11, Lis12]. Hence, today its dependence on T and $\|V\|_\infty$ in the $T \to 0$ regime is well understood [FI96, FZ00], see also [Zua07, Chapter 5]. It has emerged that the control cost of the heat equation has an $\exp(C/T)$ singularity at $T = 0$ [FZ00]. Furthermore, if $V \not\equiv 0$ and $-\Delta$ is replaced by $-\Delta + V$ for some bounded V, then at least in even space dimension, there must be a $\exp(C\|V\|_\infty^{2/3})$ contribution to the control cost [DZZ08]. Let us emphasize that many existing results focus on the small time regime and that few is known about the dependence of the control cost on the geometry.

In a number of recent publications [Mil10, Mil17, BPS18, TT11] the issue of null-controllability has been studied in an abstract setting. In these, controlled Cauchy problems are considered and under the abstract assumption of a so-called *spectral inequality*, controllability is proved. This is motivated by the aim to develop a general framework which then allows to systematically treat controllability of linear systems – of the classic heat equation as well as for instance for the fractional heat equation where the negative Laplacian $-\Delta$ is replaced by a fractal version $(-\Delta)^\alpha$. There exists a classic equivalence between *controllability* and so-called *observability estimates*, that are estimates of the form

$$\|\mathrm{e}^{T\Delta}\phi\|_{L^2(\Omega)}^2 \leq C_{\mathrm{obs}} \int_0^T \|\mathrm{e}^{t\Delta}\phi\|_{L^2(\omega)}^2 \mathrm{d}t, \quad \phi \in L^2(\Omega),$$

and the square root of the constant C_{obs} in the observability estimate is an upper bound on the control cost. Observability estimates themselves can be deduced from spectral inequalities. The unique continuation principles from Subsection 3.2 and the Logvinenko-Sereda theorem from Subsection 3.5 turn out to be special cases of spectral inequalities. Since we have the quantitative unique continuation principles from Section 3 with an explicit control on all occurring parameters at our disposal such abstract estimates on the control cost will be useful to us. However, some of the above mentioned abstract control results impose conditions such as the small time regime [Mil10], or are not sufficiently explicit in the control cost and restricted to operators with purely discrete spectrum [TT11]. In order to study phenomena such as homogenization, we will have to remove these restrictions and develop our own framework for null-controllability and explicit estimates on the control cost for abstract Cauchy problems in the following Subsection 6.2. It is inspired by and has a more explicit constant that the result in [TT11].

This general result is interesting in its own right since it can be seen to be optimal in many regards. Combining the abstract results with the unique continuation principles from Section 3, we will then show null-controllability with explicit estimates on the control cost in a multi-scale setting in Subsection 6.3 and discuss homogenization as well as de-homogenization of the control set.

6.2. Abstract observability, null-controllability and control cost

Let X and U be Hilbert spaces with inner products $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_U$ and norms $\|\cdot\|$ and $\|\cdot\|_U$, respectively. Let $A \geq 0$ be a self-adjoint operator in X with domain $\mathcal{D}(A)$.

Let $\beta \leq 0$. We define on X the inner product

$$\langle x, y \rangle_\beta = \big\langle (I + A^2)^{\beta/2} x, (I + A^2)^{\beta/2} y \big\rangle \tag{78}$$

and denote by $X_\beta \supset X$ the extrapolation space obtained as the completion of X with respect to the norm induced by the inner product (78). Clearly, $X_0 = X$. Furthermore, we denote by $X_{-\beta} := \mathcal{D}(I + A^2)^{-\beta/2} \subset X$ the interpolation space, endowed with the scalar product

$$\langle x, y \rangle_{-\beta} = \big\langle (I + A^2)^{-\beta/2} x, (I + A^2)^{-\beta/2} y \big\rangle.$$

Let $B \in \mathcal{L}(U, X_\beta)$, the space of bounded linear maps from U to X_β. The spaces $X_{-\beta} \subset X \subset X_\beta$ form a Gelfand triple and we will take the dual of B with respect to the pivot space $X = X_0$, i.e. $B^* \in \mathcal{L}(X_{-\beta}, U)$.

We study null-controllability of an abstract parabolic equation of the form

$$\frac{\partial}{\partial t} u + Au = Bf, \quad u(0) = u_0 \in X. \tag{79}$$

Remark 6.2. In our application in Subsection 6.3, we will choose $X = L^2(\Gamma)$ for a generalized rectangle $\Gamma \subset \mathbb{R}^d$, $U = L^2(S) \subset L^2(\Gamma)$ for an appropriate $S \subset \Gamma$, $A = -\Delta + V$, where $V \in L^\infty$ with self-adjoint boundary conditions on $\partial\Gamma$, and $B = \mathbf{1}_S$ the canonical injection of $L^2(S)$ into $L^2(\Gamma)$. We note that in this case, we even have $B \in \mathcal{L}(U, X)$ and thus the construction of the interpolation space X_β is unnecessary in this setting. However, for the sake of flexibility of our abstract result, we stick to this generalization. In fact, if one wants to treat the controlled heat equation on a bounded domain with boundary control, then the corresponding operator B will be an inverse boundary trace and merely relatively bounded with respect to a power of the Laplacian. In order not to exclude such systems in our abstract setting, we included the formulation with the interpolation space X_β.

Definition 6.3. We say that the system (79) is *null-controllable* in time $T > 0$ if for all $u_0 \in X$ there exists a function $f \in L^2([0,T]; U)$ such that the solution of (79) satisfies $u(T) = 0$. Moreover, we define the *control cost* as

$$\mathcal{C} = \mathcal{C}(T) = \sup_{u_0 \neq 0} \frac{\inf\{\|f\|_{L^2([0,T];U)} : \text{the solution of (79) satisfies } u(T) = 0\}}{\|u_0\|}.$$

Theorem 6.4. *Assume that there are $d_0 > 0$, $d_1 \geq 0$ and $\gamma \in (0,1)$ such that for all $\lambda > 0$ and all $\phi \in X$ we have*

$$\|P_A(\lambda)\phi\|^2 \leq d_0 \mathrm{e}^{d_1 \lambda^\gamma} \|B^* P_A(\lambda)\phi\|_U^2. \tag{80}$$

Then for all $T > 0$ and all $\phi \in X$ we have the observability estimate

$$\left\|\mathrm{e}^{-AT}\phi\right\|^2 \leq C_{\mathrm{obs}}^2 \int_0^T \left\|B^* \mathrm{e}^{-At}\phi\right\|_U^2 \mathrm{d}t, \tag{81}$$

where C_{obs} satisfies

$$C_{\mathrm{obs}}^2 = \frac{C_1 d_0}{T} K_1^{C_2} \exp\left(C_3 \left(\frac{d_1 + (-\beta)^{C_4}}{T^\gamma} \right)^{\frac{1}{1-\gamma}} \right) \quad \text{with} \quad K_1 = 2 d_0 \mathrm{e}^{-\beta} \|B\|_{\mathcal{L}(U, X_\beta)}^2 + 1.$$

Here, $C_i > 0$, $i \in \{1,2,3,4\}$, are constants depending only on γ. They are explicitely given by Eq. (93). Moreover, for all $T > 0$ the system (79) is null-controllable in time T with costs satisfying $\mathcal{C} \leq C_{\mathrm{obs}}$.

Some words should be said on well-definedness of (80) and (81) since $B^* \in \mathcal{L}(X_{-\beta}, U)$ and $X_{-\beta} \subset X$. It is straightforward to verify that $\operatorname{Ran} \mathrm{e}^{-At}$ and $\operatorname{Ran} P_A(\lambda)$ ar subsets of $X_{-\beta}$ for all $\beta \leq 0$, $t > 0$ and $\lambda \in \mathbb{R}$. Hence, for every $\phi \in X$, the expressions $B^* P_A(\lambda)\phi$ and $B^* \mathrm{e}^{-At}\phi$ are well-defined as elements in U. An analogous reasoning shows well-definedness of corresponding expressions appearing in the subsequent proof.

We also note that in the applications below, we will always work in the case where $\beta = 0$. The proof of Theorem 6.4 is based on techniques developed in the proof of [TT11, Theorem 1.2].

Proof of Theorem 6.4. Let $T > 0$. For $\phi \in X$, $t \in (0, T]$, and $\lambda > 0$ we use the notation

$$\begin{aligned} F(t) &= \left\|\mathrm{e}^{-At}\phi\right\|^2, & G(t) &= \left\|B^*\mathrm{e}^{-At}\phi\right\|_U^2, \\ F_\lambda(t) &= \left\|\mathrm{e}^{-At}P_A(\lambda)\phi\right\|^2, & G_\lambda(t) &= \left\|B^*\mathrm{e}^{-At}P_A(\lambda)\phi\right\|_U^2, \\ F_\lambda^\perp(t) &= \left\|\mathrm{e}^{-At}(\mathrm{Id} - P_A(\lambda))\phi\right\|^2, & G_\lambda^\perp(t) &= \left\|B^*\mathrm{e}^{-At}(\mathrm{Id} - P_A(\lambda))\phi\right\|_U^2. \end{aligned}$$

Since $A \geq 0$ we have $F(t_1) \geq F(t_2)$, $F_\lambda(t_1) \geq F_\lambda(t_2)$, and $F_\lambda^\perp(t_1) \geq F_\lambda^\perp(t_2)$ if $t_1 \leq t_2$ and $\lambda > 0$. By monotonicity and our assumption (80), we obtain for all $t \in (0, T]$ and all $\lambda > 0$

$$F_\lambda(t) = \frac{2}{t}\int_{t/2}^t F_\lambda(t)\mathrm{d}\tau \leq \frac{2}{t}\int_{t/2}^t F_\lambda(\tau)\mathrm{d}\tau \leq \frac{2d_0\mathrm{e}^{d_1\lambda^\gamma}}{t}\int_{t/2}^t G_\lambda(\tau)\mathrm{d}\tau. \tag{82}$$

By spectral calculus we have

$$\begin{aligned} G_\lambda^\perp(t) &\leq \|B\|_{\mathcal{L}(U,X_\beta)}^2 \|\mathrm{e}^{-At}(I - P_A(\lambda))\phi\|_{X_{-\beta}}^2 \\ &= \|B\|_{\mathcal{L}(U,X_\beta)}^2 \|(I + A^2)^{-\beta/2}\mathrm{e}^{-At}(I - P_A(\lambda))\phi\|^2 \\ &= \|B\|_{\mathcal{L}(U,X_\beta)}^2 \int_\lambda^\infty (1+\mu^2)^{-\beta}\mathrm{e}^{-2\mu t}\mathrm{d}\|P_A(\mu)\phi\|^2. \end{aligned}$$

Recall that $\beta \leq 0$. Let $\Theta > 0$ to be specified later. For $\mu, t > 0$ we estimate

$$(1+\mu^2)^{-\beta}\mathrm{e}^{-\mu t} \leq \left(1 + \left(-\frac{2\beta}{t}\right)^2\right)^{-\beta} \leq \exp\left(\frac{C_\Theta}{t^\Theta} - \beta\right), \quad C_\Theta = 2^\Theta(-\beta)^{\Theta+1}\left(\frac{2+\Theta}{\Theta}\right),$$

where the first inequality follows by maximizing with respect to μ, and the second one follows from the inequality $\ln(1+x) \leq (2/\Theta + 1)x^{\Theta/2} + 1$ for $x \geq 0$. Hence,

$$G_\lambda^\perp(t) \leq \|B\|_{\mathcal{L}(U,X_\beta)}^2 \int_\lambda^\infty \mathrm{e}^{C_\Theta/t^\Theta - \beta - \mu t}\mathrm{d}\|P_A(\mu)\phi\|^2 \leq \|B\|_{\mathcal{L}(U,X_\beta)}^2 \mathrm{e}^{C_\Theta/t^\Theta - \beta - \lambda t/2}F(t/2). \tag{83}$$

Similarly we find

$$F_\lambda^\perp(t) = \int_\lambda^\infty \mathrm{e}^{-2\mu t}\mathrm{d}\|P_A(\mu)\phi\|^2 \leq \mathrm{e}^{-3\lambda t/2}\int_\lambda^\infty \mathrm{e}^{-\mu t/2}\mathrm{d}\|P_A(\mu)\phi\|^2 \leq \mathrm{e}^{-3\lambda t/2}F(t/4).$$

From the last inequality and Ineq. (82) we obtain

$$F(t) = F_\lambda(t) + F_\lambda^\perp(t) \leq \frac{2d_0\mathrm{e}^{d_1\lambda^\gamma}}{t}\int_{t/2}^t G_\lambda(\tau)\mathrm{d}\tau + \mathrm{e}^{-3\lambda t/2}F(t/4).$$

Since $G_\lambda(t) \leq 2(G_\lambda^\perp(t) + G(t))$ and by Ineq. (83) we obtain for all $t \in (0,T]$ and all $\lambda > 0$

$$\begin{aligned} F(t) &\leq \frac{4d_0\mathrm{e}^{d_1\lambda^\gamma}}{t}\int_{t/2}^t (G_\lambda^\perp(\tau) + G(\tau))\mathrm{d}\tau + \mathrm{e}^{-3\lambda t/2}F(t/4) \\ &\leq \frac{4d_0\mathrm{e}^{d_1\lambda^\gamma}}{t}\int_{t/2}^t G(\tau)\mathrm{d}\tau + \frac{4d_0\mathrm{e}^{-\beta}\mathrm{e}^{d_1\lambda^\gamma}\|B\|^2_{\mathcal{L}(U,X_\beta)}}{t}\int_{t/2}^t \frac{F(\tau/2)}{\mathrm{e}^{\lambda\tau/2 - C_\Theta/t^\Theta}}\mathrm{d}\tau + \frac{F(t/4)}{\mathrm{e}^{3\lambda t/2}}. \end{aligned}$$

Since $F(\tau/2) \leq F(t/4)$, $\mathrm{e}^{-\lambda\tau/2} \leq \mathrm{e}^{-\lambda t/4}$, and $\mathrm{e}^{C_\Theta/\tau^\Theta} \leq \mathrm{e}^{2^\Theta C_\Theta/t^\Theta}$ for $\tau \geq t/2$, we obtain

$$\begin{aligned} F(t) &\leq \frac{4d_0\mathrm{e}^{d_1\lambda^\gamma}}{t}\int_{t/2}^t G(\tau)\mathrm{d}\tau + \mathrm{e}^{-\lambda t/4 + 2^\Theta C_\Theta/t^\Theta}\left(2d_0\mathrm{e}^{-\beta}\mathrm{e}^{d_1\lambda^\gamma}\|B\|^2_{\mathcal{L}(U,X_\beta)} + 1\right)F(t/4) \\ &\leq \frac{4d_0\mathrm{e}^{d_1\lambda^\gamma}}{t}\int_{t/2}^t G(\tau)\mathrm{d}\tau + \mathrm{e}^{-\lambda t/4 + 2^\Theta C_\Theta/t^\Theta + d_1\lambda^\gamma}\left(2d_0\mathrm{e}^{-\beta}\|B\|^2_{\mathcal{L}(U,X_\beta)} + 1\right)F(t/4). \end{aligned}$$

With the notation

$$\begin{aligned} D_1(t,\lambda) &= \frac{4d_0\mathrm{e}^{d_1\lambda^\gamma}}{t}\int_{t/2}^t G(\tau)\mathrm{d}\tau, \quad \text{and} \\ D_2(t,\lambda) &= \mathrm{e}^{-\lambda t/4 + 2^\Theta C_\Theta/t^\Theta + d_1\lambda^\gamma}\left(2d_0\mathrm{e}^{-\beta}\|B\|^2_{\mathcal{L}(U,X_\beta)} + 1\right) \end{aligned}$$

we can summarize that for all $t \in (0,T]$ we have

$$F(t) \leq D_1(t,\lambda) + D_2(t,\lambda)F(t/4). \tag{84}$$

This inequality can be iterated. For $k \in \mathbb{N}_0$ let $\lambda_k = \nu\alpha^k$ with $\nu > 0$ and $\alpha > 1$ to be specified later. In particular, applying Ineq. (84) with $t = T$ and $\lambda = \lambda_0$ at the first place, the term $F(4^{-1}T)$ on the right hand side can then be estimated by Ineq. (84) with $t = 4^{-1}T$ and $\lambda = \lambda_1$. This way, we obtain after two steps

$$\begin{aligned} F(T) &\leq D_1(T,\lambda_0) + D_2(T,\lambda_0)\left(D_1(4^{-1}T,\lambda_1) + D_2(4^{-1}T,\lambda_1)F(4^{-2}T)\right) \\ &= D_1(T,\lambda_0) + D_1(4^{-1}T,\lambda_1)D_2(T,\lambda_0) + D_2(T,\lambda_0)D_2(4^{-1}T,\lambda_1)F(4^{-2}T). \end{aligned}$$

After $N+1$ steps of this type we obtain

$$F(T) \leq D_1(T,\lambda_0)+\sum_{k=1}^{N} D_1(4^{-k}T,\lambda_k)\prod_{l=0}^{k-1} D_2(4^{-l}T,\lambda_l)+F(4^{-N-1}T)\prod_{k=0}^{N} D_2(4^{-k}T,\lambda_k). \tag{85}$$

In order to study the limit $N \to \infty$, we assume that $4^{\Theta+1} \leq \alpha$, $\alpha^\gamma \leq \alpha/4$, and $\nu T > 2^{\Theta+2} C_\Theta T^{-\Theta} + d_1 \nu^\gamma \alpha$. This ensures that the constants

$$K_1 = 2d_0 \mathrm{e}^{-\beta} \|B\|^2_{\mathcal{L}(U,X_\beta)}+1, \quad K_2 = \nu T/4 - 2^\Theta C_\Theta / T^\Theta - d_1\nu^\gamma, \quad K_3 = \frac{K_2}{\alpha/4-1} - d_1 \nu^\gamma \tag{86}$$

are positive. Then we have that

$$\begin{aligned}\prod_{k=0}^{N} D_2(4^{-k}T,\lambda_k) &= K_1^{N+1}\prod_{k=0}^{N} \mathrm{e}^{-\nu(\alpha/4)^k T/4 + 2^\Theta C_\Theta 4^{\Theta k}/T^\Theta + d_1\nu^\gamma(\alpha^\gamma)^k} \\ &\leq K_1^{N+1}\prod_{k=0}^{N} \mathrm{e}^{(\alpha/4)^k(-\nu T/4 + 2^\Theta C_\Theta/T^\Theta + d_1\nu^\gamma)} = K_1^{N+1}\prod_{k=0}^{N} \mathrm{e}^{-K_2(\alpha/4)^k}\end{aligned} \tag{87}$$

Since $K_1, K_2 > 0$ and $\alpha > 4$ this tends to zero as N tends to infinity. From Ineq. (87) and the Definitions of $D_1(4^{-k}T,\lambda_k)$ and K_3, we infer that the middle term of the right hand side of Ineq. (85) obeys the upper bound

$$\begin{aligned}&\sum_{k=1}^{N} D_1(4^{-k}T,\lambda_k)\prod_{l=0}^{k-1} D_2(4^{-l}T,\lambda_l) \\ &\qquad \leq \int_0^T G(\tau)\mathrm{d}\tau \sum_{k=1}^{N} \frac{4^{k+1}d_0 \exp(d_1\nu^\gamma(\alpha/4)^k)}{T} K_1^k \exp\left(-K_2\frac{(\alpha/4)^k-1}{\alpha/4-1}\right) \\ &\qquad = \int_0^T G(\tau)\mathrm{d}\tau \frac{4d_0}{T}\exp\left(\frac{K_2}{\alpha/4-1}\right)\sum_{k=1}^{N} (4K_1)^k \exp\left(-K_3(\alpha/4)^k\right).\end{aligned} \tag{88}$$

Letting N tend to infinity we obtain from Ineqs. (85), (87) and (88) that

$$\left\|\mathrm{e}^{-AT}\phi\right\|^2 \leq \tilde{C}^2_{\mathrm{obs}} \int_0^T \left\|B^*\mathrm{e}^{-At}\phi\right\|_U^2 \mathrm{d}t,$$

where

$$\tilde{C}^2_{\mathrm{obs}} = \frac{4d_0\mathrm{e}^{d_1\nu^\gamma}}{T} + \frac{4d_0}{T}\exp\left(\frac{K_2}{\alpha/4-1}\right)\sum_{k=1}^{\infty} (4K_1)^k \exp\left(-K_3(\alpha/4)^k\right). \tag{89}$$

We choose α and ν as in (90) and conclude the observability inequality (81) from Lemma 6.5.

It remains to establish null-controllability. This follows by the Hilbert Uniqueness Method (HUM) as in [TT11]. In fact, we define the operator $\mathcal{G} \in \mathcal{L}(L^2([0,T],U),X)$ by its adjoint

$$(\mathcal{G}^*\phi)(t) = B^* \mathrm{e}^{-At}\phi.$$

Note that $\|\mathcal{G}^*\phi\|^2 = G(t)$. Applying Ineq. (81) we obtain

$$\|\mathrm{e}^{-AT}\phi\|^2 \leq C_{\mathrm{obs}} \|\mathcal{G}^*\phi\|^2_{L^2([0,T],U)}.$$

By Lemma 2.1 from [TT11] we infer that there is $H \in \mathcal{L}(X, L^2([0,T],U))$ such that $\mathcal{G}H = -\mathrm{e}^{-AT}$. Moreover, the norm is bounded by $\|H\|_{\mathcal{L}(X,L^2([0,T],U))} \leq C_{\mathrm{obs}}$. We now choose a control function $f \in L^2([0,T],U)$ via $f(T-t) = (Hz)(t)$. Then we have for all $\phi \in X$

$$\begin{aligned}\langle \mathrm{e}^{-AT}z, \phi\rangle &= -\langle Hz, \mathcal{G}^*\phi\rangle_{L^2([0,T],U)} = -\int_0^T \langle f(T-t), B^*\mathrm{e}^{-At}\phi\rangle_U \mathrm{d}t \\ &= \int_0^T \langle \mathrm{e}^{-A(T-t)}Bf(t), \phi\rangle \mathrm{d}t.\end{aligned}$$

Hence, the solution of the system (79) with our choice of f satisfies

$$u(T) = \mathrm{e}^{-AT}z + \int_0^T \mathrm{e}^{-A(T-s)}Bu(s)\mathrm{d}s = 0. \qquad \square$$

Lemma 6.5. *Let $d_0 > 0$, $d_1 \geq 0$, $\gamma \in (0,1)$, $T > 0$,*

$$\Theta = \frac{\gamma^2}{1-\gamma}, \quad \alpha = 8 \cdot 4^{\frac{1}{1-\gamma}}, \quad \text{and} \quad \nu = \left(\frac{\alpha d_1}{T} + \frac{D}{T^{1-\gamma}} + \frac{E}{T}\right)^{\frac{1}{1-\gamma}}, \tag{90}$$

where

$$D = (3\alpha \ln(4K_1))^{1-\gamma}, \quad E = \left(\frac{8 \cdot 2^{\Theta} C_\Theta}{D}\right)^{\frac{1-\gamma}{\gamma}}, \quad C_\Theta = 2^\Theta (-\beta)^{\Theta+1}\left(\frac{2+\Theta}{\Theta}\right),$$

and $K_1 = 2d_0 \mathrm{e}^{-\beta}\|B\|^2_{\mathcal{L}(U,X_\beta)} + 1$. Then we have $4^{\Theta+1} \leq \alpha$, $\alpha^\gamma \leq \alpha/4$, and $\nu T > 2^{\Theta+2}C_\Theta T^{-\Theta} + d_1 \nu^\gamma \alpha$. Moreover, for all $T > 0$ the constant $\tilde{C}^2_{\mathrm{obs}}$ from (89) satisfies

$$\tilde{C}^2_{\mathrm{obs}} \leq \frac{C_1 d_0}{T} K_1^{C_2} \exp\left(C_3 \left(\frac{d_1 + (-\beta)^{C_4}}{T^\gamma}\right)^{\frac{1}{1-\gamma}}\right).$$

Here, $C_i > 0$, $i \in \{1,2,3,4\}$, are constants depending only on γ. They are explicitely given by Eq. (93).

Proof. It is easy to see that $4^{\Theta+1} \leq \alpha$, and $\alpha^\gamma \leq \alpha/4$. For the constant K_3 from (86) we have

$$\begin{aligned} K_3 &= \frac{\nu T/4 - 2^\Theta C_\Theta/T^\Theta - d_1\nu^\gamma\alpha/4}{(\alpha/4-1)} \\ &= \frac{\nu^\gamma}{\alpha-4}\left[\left(\frac{\alpha d_1}{T} + \frac{D}{T^{1-\gamma}} + \frac{E}{T}\right)T - \frac{4\cdot 2^\Theta C_\Theta}{T^\Theta}\left(\frac{\alpha d_1}{T} + \frac{D}{T^{1-\gamma}} + \frac{E}{T}\right)^{-\frac{\gamma}{1-\gamma}} - d_1\alpha\right] \\ &\geq \frac{\nu^\gamma}{\alpha-4}\left[DT^\gamma - \frac{4\cdot 2^\Theta C_\Theta}{T^\Theta}\frac{E^{\frac{-\gamma}{1-\gamma}}}{T^{\frac{-\gamma}{1-\gamma}}}\right] = \frac{\nu^\gamma DT^\gamma}{2(\alpha-4)}. \end{aligned}$$

This shows in particular that $\nu T > 2^{\Theta+2}C_\Theta T^{-\Theta} + d_1\nu^\gamma\alpha$. We further estimate

$$K_3 \geq \frac{\left(\frac{D}{T^{1-\gamma}}\right)^{\gamma/(1-\gamma)} DT^\gamma}{2(\alpha-4)} = \frac{D^{1/(1-\gamma)}}{2(\alpha-4)}.$$

For the constant K_2 from (86) we estimate using $\alpha \geq 8$

$$\frac{K_2}{\alpha/4-1} \leq \nu T/2 = \frac{T}{2}\left(\frac{\alpha d_1}{T} + \frac{D}{T^{1-\gamma}} + \frac{E}{T}\right)^{\frac{1}{1-\gamma}} \leq \frac{\alpha^{\frac{1}{1-\gamma}}}{2}\left(\frac{\alpha d_1 + E}{T^\gamma} + D\right)^{\frac{1}{1-\gamma}}.$$

Let us now note that for all $A > 1$, and $B > 0$ we have

$$\sum_{k=1}^\infty A^k \mathrm{e}^{-B2^k} \leq \left(\frac{2\ln A}{B\mathrm{e}\ln 2}\right)^{\frac{\ln A}{\ln 2}} \frac{1}{B}, \tag{91}$$

since

$$\sum_{k=1}^\infty \mathrm{e}^{-\frac{B}{2}2^k} \leq \sum_{k=1}^\infty \mathrm{e}^{-kB} = \frac{\mathrm{e}^{-B}}{1-\mathrm{e}^{-B}} = \frac{1}{\mathrm{e}^B - 1} \leq \frac{1}{B}$$

and

$$\sum_{k=1}^\infty A^k \mathrm{e}^{-B2^k} \leq \sup_{x\geq 1}(A^x \mathrm{e}^{-\frac{B}{2}2^x}) \sum_{k=1}^\infty \mathrm{e}^{-\frac{B}{2}2^k} = \left(\frac{2\ln A}{B\mathrm{e}\ln 2}\right)^{\frac{\ln A}{\ln 2}} \sum_{k=1}^\infty \mathrm{e}^{-\frac{B}{2}2^k}.$$

We use $\alpha \geq 8$ and apply Ineq. (91) with $A = 4K_1 > 1$, and $B = K_3$ to obtain

$$\sum_{k=1}^\infty (4K_1)^k \exp\left(-K_3(\alpha/4)^k\right) \leq \sum_{k=1}^\infty (4K_1)^k \exp\left(-K_3 2^k\right) \leq \left(\frac{2\ln(4K_1)}{K_3\mathrm{e}\ln 2}\right)^{\frac{\ln(4K_1)}{\ln 2}} \frac{1}{K_3}. \tag{92}$$

By the above estimate on K_3 and since $\alpha > 4$ we find

$$\frac{2\ln(4K_1)}{K_3\mathrm{e}\ln 2} \leq \frac{2}{\mathrm{e}\ln 2}\frac{\ln(4K_1)2(\alpha-4)}{D^{1/(1-\gamma)}} = \frac{2}{\mathrm{e}\ln 2}\frac{2(\alpha-4)}{3\alpha} \leq 1.$$

Note that the exponent $\ln(4K_1)/\ln 2$ in (92) is positive, and that $D \geq 1$. Hence, the right hand side of (92) is bounded from above by $2(\alpha - 4)$. Using this, $\alpha \geq 8$, and $d_1\nu^\gamma \leq d_1\nu^\gamma + K_3 = K_2/(\alpha/4 - 1)$, we find

$$\begin{aligned}\tilde{C}_{\mathrm{obs}}^2 &= \frac{4d_0 \mathrm{e}^{d_1\nu^\gamma}}{T} + \frac{4d_0}{T}\exp\left(\frac{K_2}{\alpha/4-1}\right)\sum_{k=1}^{\infty}(4K_1)^k\exp\left(-K_3(\alpha/4)^k\right)\\ &\leq \frac{4d_0}{T}(1+K_3^{-1})\exp\left(\frac{K_2}{\alpha/4-1}\right)\\ &\leq \frac{4d_0}{T}(1+2(\alpha-4))\exp\left(\frac{\alpha^{\frac{1}{1-\gamma}}}{2}\left(\frac{\alpha d_1+E}{T^\gamma}+D\right)^{\frac{1}{1-\gamma}}\right),\end{aligned}$$

Since $(a+b)^x \leq 2^{x-1}(a^x+b^x)$ for $x > 1$ and $a, b \geq 0$ we obtain

$$\begin{aligned}\tilde{C}_{\mathrm{obs}}^2 \leq \frac{4d_0}{T}&(1+2(\alpha-4))\,(4K_1)^{3\alpha^{\frac{2-\gamma}{1-\gamma}}2^{2\Theta+3}}\\ &\times\exp\left(\alpha^{\frac{2}{1-\gamma}}4^{\frac{\gamma+\Theta+2}{1-\gamma}}\left(\frac{\Theta+2}{\Theta}\right)^{\frac{1}{1-\gamma}}\left(\frac{d_1+(-\beta)^{\Theta+1}}{T^\gamma}\right)^{\frac{1}{1-\gamma}}\right).\end{aligned} \tag{93}$$

□

6.3. Explicit control cost of heat-type equations

Theorem 6.4 translates spectral inequalities into null-controllability of the corresponding controlled Cauchy problem with explicit estimates on the control cost in all times. In the situation where $A = -\Delta + V$, a Schrödinger operator on a generalized rectangle Γ, and B is the characteristic function of a set $S_{\delta,Z}(\Gamma)$ as in Definition 3.26, the unique continuation principles in Subsection 3.2 are precisely such spectral inequalities. Furthermore, if Γ is $\mathbb{R}^d$ or a cube Λ_L with periodic boundary conditions, and S is a thick set as in Definition 3.42, then the Logvinenko-Sereda theorem is a spectral inequality. We will now combine these ingredients.

We assume that $\Gamma \subset \mathbb{R}^d$ is a generalized rectangle as in Definition 2.2 and $S \subset \Gamma$ is a suitable control set to be specified below. For $V \in L^\infty$ we consider the controlled heat equation with heat generation term $(-V)$.

$$\frac{\partial}{\partial t}u - \Delta u + Vu = f\mathbf{1}_S, \quad u(0,\cdot) = u_0 \in L^2(\Gamma), \tag{94}$$

where, $T > 0$, $V \in L^\infty(\Gamma)$ is non-negative, and Δ stands for the self-adjoint Laplace operator in $L^2(\Gamma)$ subject to Dirichlet, Neumann or periodic boundary conditions. If $\Gamma = \mathbb{R}^d$, Δ denotes the standard Laplace operator in $L^2(\mathbb{R}^d)$ with domain $H^2(\mathbb{R}^d)$. Note that we simultaneously treat bounded and unbounded domains such as $\mathbb{R}^d$, half-spaces, infinite strips, or cubes.

We will consider two geometric situations for the control set $S \subset \Gamma$.

Situation 1: There exist $M > 0$ such that the set Γ contains an elementary cell of the lattice $M\mathbb{Z}^d$, a parameter $\delta < M/2$, and a (M, δ) equidistributed sequence Z. In this case, we set $S = S_{\delta,Z}(\Gamma)$, the union of δ-balls defined in Definition 3.26 above.

Situation 2: We set $V \equiv 0$ and let $\Gamma \in \{\Lambda_L, \mathbb{R}^d\}$ where Λ_L is equipped with periodic boundary conditions. We set $S = \Gamma \cap F$ where F is a (ρ, a)-thick set as in Definition 3.42. In case $\Gamma = \Lambda_L$, we further assume $a_j \leq L$ for all $j \in \{1, \ldots, d\}$.

The spectral inequality is then satisfied - either by the scale-free quantitative unique continuation principle from Subsection 3.2, Theorem 3.13, or by the Logvinenko-Sereda theorem in the form of Corollary 3.45, discussed in Subsection 3.5. This is summarized by the following proposition:

Proposition 6.6. *Let either Situation 1 or Situation 2 from above hold. Then for all $\lambda \geq 0$ and all $\phi \in L^2(\Gamma)$ we have*

$$\|P_{-\Delta+V}(\lambda)\phi\|_{L^2(\Gamma)}^2 \leq d_0 \mathrm{e}^{d_1 \lambda^\rho} \|\mathbf{1}_{S\cap\Gamma} P_{-\Delta+V}(\lambda)\phi\|_{L^2(\Gamma)}^2$$

where

$$d_0 = \left(\frac{M}{\delta}\right)^{C(1+M^{4/3}\|V\|_\infty^{2/3})}, \qquad d_1 = M \ln\left(\frac{M}{\delta}\right) \qquad \text{in Situation 1,}$$

$$d_0 = \left(\frac{\tilde{C}^d}{\rho}\right)^{Cd}, \qquad d_1 = 4\tilde{C}|a|_1 \ln\left(\frac{\tilde{C}^d}{\rho}\right) \qquad \text{in Situation 2.}$$

Here, $C \geq 1$ is a constant depending only on the dimension, and $\tilde{C} \geq 1$ is an absolute constant.

Note that in Situation 2, Proposition 6.6 is essentially the statement of Theorems 7 and 8 in [EV18]. We also emphasize that Theorem 6.4 requires that the operator A is non-negative. Thus, in Situation 1, in order to combine Theorem 6.4 with Proposition 6.6, we need to assume that the operator $-\Delta + V$ is non-negative, which is definitely ensured by assuming $V \geq 0$. It is rather straightforward to naively adapt Theorem 6.4 to lower semibounded operators. However, if the operator A is merely lower semibounded, then the dynamics of the system will dramatically depend on the sign of the lower bound. If the operator A satisfies $A \geq \lambda_0 > 0$, then the uncontrolled system will have a natural tendency to exponentially converge to the zero state and the cost of null-controllability in large times will be exponentially decaying. Conversely, if A has some spectrum in $(-\infty, 0)$, then the free system will have a tendency to exponentially blow up in time and the control cost is expected to be much larger (in fact, it will not converge to zero, even in the large time limit). In order to deduce optimal estimates in both these situations, a careful discussion is required which we perform below in the context of Theorems 6.11 and 6.14.

6.3.1. Nonnegative generator

Let us first dwell on the situation when $-\Delta + V$ is non-negative (or, more precisely when $V \geq 0$) without yet exploiting possible positive definiteness of $-\Delta + V$. In this situation, we can already discuss homogenization as well as de-homogenization. Theorem 6.4 and Proposition 6.6 yield:

Theorem 6.7 (Equidistributed sets)**.** *Let Situation 1 from above hold and assume additionally that* $0 \leq V$. *Then, for all* $\phi \in L^2(\Gamma)$ *and all* $T > 0$, *we have*

$$\|\mathrm{e}^{(\Delta - V)T}\phi\|^2_{L^2(\Gamma)} \leq C_{\mathrm{obs}} \int_0^T \|\mathrm{e}^{(\Delta - V)t}\phi\|^2_{L^2(S_{\delta,Z}(\Gamma))} \mathrm{d}t,$$

where

$$C_{\mathrm{obs}} = \frac{C_1}{T}\left(\frac{\delta}{M}\right)^{-C_2(1+M^{4/3}\|V\|_\infty^{2/3})} \exp\left(\frac{C_3 M^2 \ln^2(\delta/M)}{T}\right).$$

Here, C_1, C_2, *and* C_3 *are positive constants depending only on the dimension. Moreover, for all* $T > 0$ *and all* $u_0 \in L^2(\Gamma)$ *system* (94) *with* $S = S_{\delta,Z}(\Gamma)$ *is null-controllable in time* T *with cost* $\mathcal{C} \leq \sqrt{C_{\mathrm{obs}}}$.

Theorem 6.8 (Thick sets)**.** *Let Situation 2 from above hold. Then for all* $\phi \in L^2(\Gamma)$, *and all* $T > 0$ *we have*

$$\|\mathrm{e}^{\Delta T}\|^2_{L^2(\Gamma)} \leq C_{\mathrm{obs}} \int_0^T \|\mathrm{e}^{\Delta t}\phi\|^2_{L^2(S\cap\Gamma)} \mathrm{d}t,$$

where

$$C_{\mathrm{obs}} = \frac{C_1}{T}\rho^{-C_2 d} \exp\left(\frac{C_3|a|_1^2 \ln^2(C_4^d/\rho)}{T}\right)$$

Here, C_1, C_2, C_3, *and* C_4 *are absolute positive constants. Moreover, for all* $T > 0$ *and all* $u_0 \in L^2(\Gamma)$ *the system* (94) *is null-controllable in time* T *with cost* $\mathcal{C} \leq \sqrt{C_{\mathrm{obs}}}$.

Theorem 6.8 is an improvement of Thoerems 3 and 4 in [EV18] in the sense that the expression for C_{obs} is sharper. More precisely in Theorem 6.8, the argument of the exponential term in C_{obs} is proportional to $|a|_1^2$ whereas the constant in the corresponding expression in [EV18] is not. We are going to see in the discussion and the examples below that the of the exponent on $|a|_1$ as in Theorem 6.8 is natural. Let us now discuss some consequences of Theorems 6.7 and Theorem 6.8.

Example 1 (Homogenization counteracts the exponential singularity of the control cost in time $T = 0$)**.** *We consider the controlled (classic) heat equation in $\mathbb{R}^d$ or on a hypercube Λ_L with periodic boundary conditions, where the control set becomes more and more homogenized. More precisely, we fix a time $T > 0$ and a density $\rho \in (0,1)$. Given a vector a with positive entries, we choose a (ρ, a)-thick set S_a. Then, Theorem 6.8 yields null-controllability of the heat equation with control on S_a with cost satisfying*

$$\mathcal{C}^2 \leq \frac{C_1}{T}\rho^{-C_2 d}\exp\left(\frac{C_3|a|_1^2\ln^2(C_4^d/\rho)}{T}\right)$$

We could also work with Theorem 6.7, equidistributed sets and $V = 0$, but thick sets are more general, whence we stick to them here. Homogenizing now means that we let the vector a tend to zero while keeping ρ constant. This corresponds to reducing local fluctuations in the density of the control set S_a while keeping the overall density of the control. The estimate on the cost in this case tends to

$$\mathcal{C} \sim \sqrt{\frac{C_1}{\rho^{C_2 d^2}}}\frac{1}{\sqrt{T}}. \tag{95}$$

While the large time asymptotic behavior stays roughly the same, the small time behavior is improved in the $a \to 0$ limit. We conclude that homogenization counteracts the $\exp(1/T)$ *singularity of the control cost in time $T = 0$.*

Remark 6.9. Another way to interpret Example 1 is that homogenization effectively reduces the control cost of the system to the control cost of a one-dimensional system.

Let us explain this: We choose the Hilbert space $\mathbb{C}$ and study the system

$$\frac{\partial}{\partial t}u = Cf, \quad u(0) = u_0 \in \mathbb{C}, \tag{96}$$

where $u, f \in L^2([0,T])$ and C is a positive constant. We claim that for every $T > 0$, this system has control cost $\mathcal{C} = (C\sqrt{T})^{-1}$ in time $T > 0$.

This can be seen as by a simple calculation: for $u_0 \in \mathbb{C}$, and $T > 0$ the function $f(t) = -u_0/(CT)$ is a null-control and we have

$$\|f\|_{L^2([0,T])} = \frac{|u_0|}{CT}\left(\int_0^T \mathrm{d}t\right)^{1/2} = \frac{|u_0|}{C\sqrt{T}}, \quad \text{which implies} \quad \mathcal{C} \leq (C\sqrt{T})^{-1}.$$

Conversely, assume that we have $u_0 \in \mathbb{C}$ and a control $f \in L^2([0,T])$ such that the solution of (96) satisfies $u(T) = 0$. Then

$$0 = u(T) = u_0 + C\int_0^T \frac{\partial}{\partial t}u(t)\mathrm{d}t = u_0 + C\int_0^T f(t)\mathrm{d}t$$

whence by the Cauchy-Schwarz inequality

$$|u(0)| \leq \sqrt{T} C \|f\|_{L^2([0,T])}, \quad \text{which implies} \quad \mathcal{C} \geq (C\sqrt{T})^{-1}.$$

Comparing this with the homogenization limit of the control cost, obtained in (95), we conclude that in the homogenization regime, our estimate on the control cost tends to the control cost of the controlled one-dimensional ODE system (96) where the constant C is polynomial in the density ρ of the thick set.

Example 2 (Homogenization annihilates the effects of non-negative potentials on the control cost)**.** *We now assume that $V \geq 0$ is a non-negative potential and study the control cost of the heat-type equation on a generalized rectangle Γ (e.g. on $\Gamma = \mathbb{R}^d$) in the homogenization regime. For sufficiently small $M > 0$, and $\delta < M/2$, we choose a (M, δ)-equidistributed sequence Z and the set $S_{\delta,Z}(\Gamma)$ as the union of δ-balls. Then, Theorem 6.7 yields null-controllability of the system* (94) *with cost*

$$\mathcal{C}^2 \leq \frac{C_1}{T} \left(\frac{\delta}{M}\right)^{-C_2(1+M^{4/3}\|V\|_\infty^{2/3})} \exp\left(\frac{C_3 M^2 \ln^2(\delta/M)}{T}\right).$$

Homogenization now means sending M and δ to zero while keeping δ/M constant. In this limit, we have for our estimate on the control cost

$$\mathcal{C} \sim \sqrt{\frac{C_1}{T}} \left(\frac{\delta}{M}\right)^{-C_2/2}$$

which is independent of V. In particular, homogenization of the control set will make the effect of the potential on the control cost disappear. We emphasize that apart from boundedness, no regularity assumption on V has been made.

Example 3 (De-homogenization or coarsening)**.** *For this example let us go back to the situation of Example 1 above, i.e. $V = 0$, and $\Gamma = \mathbb{R}^d$. Instead of* homogenizing, *we now want to* de-homogenize *or study the* coarsening regime *of the (ρ, a)-equidistributed control set by letting a tend to infinity while keeping ρ constant. This corresponds to an increase in fluctuations of the density of the control set on all finite scales while the overall density remains constant. It is unsurprising that for fixed time T, our upper bound on the control cost will in general increase since the diffusive nature of the heat equation makes it harder for components of the control set to exert control in larger and larger areas where there is no or only little control. Let us now have a closer look at the estimate on the control cost from Theorem 6.8*

$$\mathcal{C} \leq \frac{C_1}{T} \rho^{-C_2 d^2} \exp\left(\frac{C_3 |a|_1^2 \ln^2(C_4^d/\rho)}{T}\right).$$

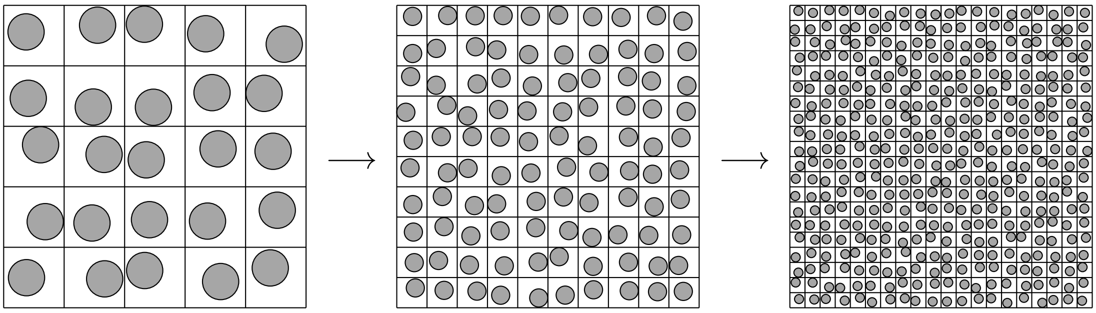

Figure 5: Homogenization limit $\delta, M \to 0$ with $\delta/M = \text{const}$

There are three model parameters here: the parameters ρ and a, describing the geometry of the control set, and the time T. Since we already chose ρ and a, the only remaining way to accommodate for the increase in our upper bound when a tends to infinity is to modify the remaining parameter T by choosing

$$T \sim |a|_1^2$$

(with a small, logarithmic correction in order to compensate for the $1/T$ term in front of the exponential). We have recovered the relation between time and space derivatives

$$\frac{\partial}{\partial t} u = \Delta u$$

from the underlying heat equation. This is an indication that our estimates on the control cost with respect to time and space parameters are close to being optimal.

6.3.2. Lower semibounded generator

So far, we have investigated the situation where V was assumed to be non-negative. However, our estimates did not distinguish between non-negative V and potentials V which are strictly positive and bounded away from zero. Furthermore, we now also want to be able to treat the case of possibly negative V.

For this purpose, let us denote $\lambda_0 := \inf \sigma(-\Delta + V)$. Depending on the sign of λ_0, there are two very different situations:

If $\lambda_0 > 0$: In this case, the semigroup $\mathrm{e}^{(\Delta - V)t}$ describing the evolution of the free system is strictly contractive and if T tends to ∞, solutions will exponentially tend to zero. Of course, one could simply resort to Theorem 6.7 above, but this would be non-optimal since it does not take the strict positivity of $-\Delta + V$ into account. In fact, a good control strategy for large times T would be to first let the system evolve without control until a time $T_0 < T$ and then use Theorem 6.4 and apply control for times $t \in (T - T_0, T)$.

If $\lambda_0 < 0$: In this case, the semigroup $e^{(\Delta - V)t}$ will not be contractive. Initial states from the negative spectral subspace of $-\Delta + V$ will grow exponentially under the time evolution. Thus, there is a conflict of interest between "quickly killing the expanding part", which is expensive (remember the $\exp(1/T)$ singularity in the control cost in time $T = 0$) and "taking as much time as possible" to reduce the cost for killing the positive energy part of the system.

Before generalizing Theorem 6.7 to these situations, let us start by providing some *lower bounds* on the control cost. They shall serve as benchmarks for us.

Lemma 6.10. *Let $V \in L^\infty$, let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain and let $\omega \subset \Omega$ such that the system*

$$\frac{\partial}{\partial t} u - \Delta u + Vu = \mathbf{1}_\omega f, \quad u(0) = u_0 \in L^2(\Omega), \tag{97}$$

is null-controllable in every time $T > 0$ where we put Dirichlet or – if the geometry allows for it – Neumann or periodic boundary conditions on $\partial\Omega$. Denote $\lambda_0 := \inf \sigma(-\Delta + V)$. Then, for every $T > 0$, the control cost of system (97) *in time T is at least*

- $\sqrt{|2\lambda_0|}$, *if $\lambda_0 < 0$,*
- $\sqrt{2\lambda_0} \exp(-\lambda_0 T)$, *if $\lambda_0 > 0$,*
- $\sqrt{T}^{-1}$, *if $\lambda_0 = 0$.*

Proof. The operator $\Delta + V$ in $L^2(\Omega)$ has eigenvalues $\{\lambda_k\}_{k \in \mathbb{N}}$, where $\lambda_1 = \lambda_0$, with corresponding normalized eigenfunctions $\{\phi_k\}_{k \in \mathbb{N}}$. We choose $u_0 := \phi_1$ as initial state and pick for $T > 0$ a control function $f \in L^2([0,T] \times \Omega)$ such that the solution of (97) satisfies $u(T) = 0$. We want to lower bound $\|f\|_{L^2([0,T]\times\omega)} = \|\mathbf{1}_\omega f\|_{L^2([0,T]\times\Omega)}$. For that purpose, we expand the function $\mathbf{1}_\omega f$ in the eigenfunction basis of $-\Delta + V$, i.e. $(\mathbf{1}_\omega f)(t) = \sum_{k=1}^\infty \alpha_k(t)\phi_k$ for $t \in [0,T]$ with $\alpha_k(\cdot) \in L^2([0,T])$ for all $k \in \mathbb{N}$. Then, by (77), we have

$$0 = \exp(-\lambda_0 T)\phi_1 + \int_0^T \sum_{k=1}^\infty \exp(-\lambda_k(T-s)\alpha_k(s)\phi_k \mathrm{d}s.$$

Projecting onto ϕ_1, and using the Cauchy Schwarz inequality in $L^2([0,T])$ yields for $\lambda_0 \neq 0$

$$\begin{aligned}\exp(-\lambda_0 T) &= -\int_0^T \exp(-\lambda_0(T-s))\alpha_1(s)\mathrm{d}s \\ &\leq \left(\int_0^T \exp(-2\lambda_0(T-s))\mathrm{d}s\right)^{1/2} \cdot \left(\int_0^T |\alpha_1(s)|^2 \mathrm{d}s\right)^{1/2} \\ &= \left(-\frac{\mathrm{e}^{-2\lambda_0 T}-1}{2\lambda_0}\right)^{1/2} \cdot \left(\int_0^T |\alpha_1(s)|^2 \mathrm{d}s\right)^{1/2} \\ &\leq \left(-\frac{\mathrm{e}^{-2\lambda_0 T}-1}{2\lambda_0}\right)^{1/2} \cdot \|f\|_{L^2([0,T]\times\Omega)}.\end{aligned}$$

Rearranging, we find

$$\|f\|^2_{L^2([0,T]\times\Omega)} \geq \frac{-2\lambda_0 \exp(-2\lambda_0 T)}{\exp(-2\lambda_0 T)-1} = \frac{-2\lambda_0}{1-\exp(2\lambda_0 T)}.$$

If $\lambda_0 < 0$, this is bounded away from zero, uniformly in T, and if $\lambda_0 > 0$, the cost cannot converge to zero faster than $\sim \exp(-2\lambda_0 T)$. The statement for $\lambda_0 = 0$ follows by a completely analogous calculation. □

In particular, we see that if $\lambda_0 < 0$, the control cost is strictly bounded away from zero in all times – in contrast to the situation $\lambda_0 \geq 0$, where the control cost vanishes in the large time limit. However, it remains unclear whether the lower bound $\sqrt{|2\lambda_0|}$ is optimal.

Equipped with the lower bounds of Lemma 6.10, we can now discuss upper bounds on the control cost. We start with the case $\lambda_0 \geq 0$.

Theorem 6.11 (Control cost if $\lambda_0 \geq 0$)**.** *Let $M > 0$, $\Gamma \subset \mathbb{R}^d$ a generalized rectangle, containing at least one elementary cell of the lattice $M\mathbb{Z}^d$, $\delta \in (0, M/2)$, Z a (M,δ)-equidistributed sequence Z, and $V \in L^\infty(\Gamma)$ such that* $\inf \sigma(-\Delta + V) = \lambda_0 \geq 0$*. Then, for all $\phi \in L^2(\Gamma)$, and all $T > 0$, the system* (94) *is null-controllable in time T with cost satisfying*

$$\mathcal{C}^2 \leq \inf_{T_0 \in (0,T)} \frac{C_1}{T-T_0} \left(\frac{\delta}{M}\right)^{-C_2(1+M^{4/3}\|V\|_\infty^{2/3})} \exp\left(\frac{C_3 M^2 \ln^2(\delta/M)}{T-T_0} - 2\lambda_0 T_0\right).$$

Proof. By contractivity of the semigroup we have for all $u_0 \in L^2(\Gamma)$ that

$$\|\mathrm{e}^{(\Delta-V)T_0}\phi\|_{L^2(\Gamma)} \leq \mathrm{e}^{-2\lambda_0 T_0}\|\phi\|_{L^2(\Gamma)}.$$

Applying no control in $[0, T_0]$ implies that $\|u(T_0)\|^2 \leq \mathrm{e}^{-2\lambda_0 T_0}\|u_0\|^2$. Applying then Theorem 6.4 and Proposition 6.6 with the initial state $u(T_0)$ in the remaining time interval of length $T - T_0$ yields the result. □

Remark 6.12. If $\lambda_0 > 0$, one could now optimize over T_0. We refrain from doing so and merely note that for large times T, the choice $T_0 = T - 1$ shows that

$$\mathcal{C} \leq C_1 \left(\frac{\delta}{M}\right)^{-C_2(1+M^{4/3}\|V\|_\infty^{2/3})} \exp\bigl(C_3 M^2 \ln^2(\delta/M) + 2\lambda_0 - 2\lambda_0 T\bigr) \sim \exp(-2\lambda_0 T).$$

This means that in the large time limit, the upper bound coincides (up to a T-independent factor) with the lower bound of Lemma 6.10.

In the situation where $\lambda_0 < 0$, we have to go one step further back, namely to the level of the observability estimate. The following proposition generalizes the observability estimate from Theorem 6.4:

Proposition 6.13. *In the situation of Theorem 6.4 assume that the operator A is only lower semibounded by $\lambda_0 < 0$ and that for all $\lambda \geq \lambda_0$ and all $\phi \in X$ we have the spectral inequality*

$$\|P_A(\lambda)\phi\|^2 \leq d_0 \mathrm{e}^{d_1(\lambda-\lambda_0)^\gamma} \|B^* P_A(\lambda)\phi\|_U^2. \tag{98}$$

Then for all $T > 0$ and all $\phi \in X$ we have the observability estimate

$$\begin{aligned}\left\|\mathrm{e}^{-AT}\phi\right\|^2 &\leq C_{\mathrm{obs}} \int_0^T \mathrm{e}^{-2\lambda_0(T-t)} \left\|B^*\mathrm{e}^{-At}\phi\right\|_U^2 \mathrm{d}t \\ &\leq C_{\mathrm{obs}} \mathrm{e}^{-2\lambda_0 T} \int_0^T \left\|B^*\mathrm{e}^{-At}\phi\right\|_U^2 \mathrm{d}t\end{aligned}$$

with C_{obs} as in Theorem 6.4.

Proof. Clearly, $P_A(\lambda) = P_{A-\lambda_0}(\lambda - \lambda_0)$. Hence, we have for all $\mu \geq 0$ that

$$\|P_{A-\lambda_0}(\mu)\phi\|^2 \leq d_0 \mathrm{e}^{d_1\mu^\gamma} \|B^* P_{A-\lambda_0}(\mu)\phi\|_U^2$$

and since the operator $A - \mu$ is non-negative, we obtain from Theorem 6.4 the observability estimate

$$\left\|\mathrm{e}^{-(A-\lambda_0)T}\phi\right\|^2 \leq C_{\mathrm{obs}} \int_0^T \left\|B^*\mathrm{e}^{-(A-\lambda_0)t}\phi\right\|_U^2 \mathrm{d}t = C_{\mathrm{obs}} \int_0^T \mathrm{e}^{2\lambda_0 t} \left\|B^*\mathrm{e}^{-At}\phi\right\|_U^2 \mathrm{d}t$$

Dividing by $\mathrm{e}^{2\lambda_0 T}$ yields the result. □

One could now proceed as in Theorem 6.4. Combining the previous proposition with Corollary 3.15, it would follow that in every time $T > 0$ the system (94) is null-controllable with cost satisfying

$$\mathcal{C}^2 \leq \frac{C_1}{T} \left(\frac{\delta}{M}\right)^{-C_2(1+M^{4/3}\|V-\lambda_0\|_\infty^{2/3})} \exp\left(\frac{C_3 M^2 \ln^2(\delta/M)}{T} - 2\lambda_0 T\right). \tag{99}$$

Uncommented, this would be a rather non-optimal statement: Since $\lambda_0 < 0$, the upper bound increases exponentially with large times. However, the control cost must be nonincreasing in time: If there exists a control function that drives the system to zero in time T, then this function, continued by zero, will also work for all larger times.

Therefore, we arrive at the following theorem:

Theorem 6.14 (Control cost if $\lambda_0 < 0$). *Let $M > 0$, $\Gamma \subset \mathbb{R}^d$ a generalized rectangle, containing at least one elementary cell of the lattice $M\mathbb{Z}^d$, $\delta \in (0, M/2)$, Z a (M,δ)-equidistributed sequence Z, and $V \in L^\infty(\Gamma)$ such that* $\inf \sigma(-\Delta + V) = \lambda_0 < 0$. *Then, for all $\phi \in L^2(\Gamma)$, and all $T > 0$, the system* (94) *is null-controllable in time T with control cost satisfying*

$$\mathcal{C}^2 \leq \min_{T' \in (0,T]} \frac{C_1}{T'} \left(\frac{\delta}{M} \right)^{-C_2(1+M^{4/3}\|V-\lambda_0\|_\infty^{2/3})} \exp\left(\frac{C_3 M^2 \ln^2(\delta/M)}{T'} - 2\lambda_0 T' \right). \quad (100)$$

Proof. Let $T > 0$ and $T' \in (0, T]$. Corollary 3.15 implies the spectral inequality

$$\|\phi\|_{L^2(\Gamma)}^2 \leq d_0 \mathrm{e}^{d_1(\lambda-\lambda_0)^{1/2}} \|\phi\|_{L^2(S_{\delta,Z}(\Gamma))}^2$$

for all $\lambda \geq \lambda_0$ and all $\phi \in \operatorname{Ran} P_{-\Delta+V}(\lambda)$ where

$$d_0 = \left(\frac{M}{\delta} \right)^{C(1+M^{4/3}\|V-\lambda_0\|_\infty^{2/3})}, \quad \text{and} \quad d_1 = M \ln\left(\frac{M}{\delta} \right).$$

We combine this spectral inequality with Proposition 6.13 and use the HUM method as at the end of Theorem 6.4 to deduce null-controllability in time T' with control cost satisfying

$$\mathcal{C}^2 \leq \frac{C_1}{T'} \left(\frac{\delta}{M} \right)^{-C_2(1+M^{4/3}\|V-\lambda_0\|_\infty^{2/3})} \exp\left(\frac{C_3 M^2 \ln^2(\delta/M)}{T'} - 2\lambda_0 T' \right).$$

Since any control function null-controlling the system in time T' can be extended by zero in $(T', T]$ and hence null-controls the system in time $T \geq T'$, we obtain null-controllability in time T with the same cost. □

In order to discuss this bound, let us define the following quantity:

Definition 6.15. The *control cost in infinite time* $\mathcal{C}_\infty$ is

$$\mathcal{C}_\infty := \lim_{T\to\infty} \{\text{Control cost } \mathcal{C} \text{ in time } T\} = \inf_{T>0} \{\text{Control cost } \mathcal{C} \text{ in time } T\}.$$

From Theorems 6.7 and 6.11 it follows that if $\lambda_0 \geq 0$, the quantity $\mathcal{C}_\infty$ is zero and thus trivial. In case where $\lambda_0 < 0$, Lemma 6.10 shows that the control cost in this case is strictly positive and at least $\sqrt{|2\lambda_0|}$. Thus, in this case, it is worthwhile to study upper bounds on $\mathcal{C}_\infty$. By elementary calculus, it is easy to see that the term

$$\frac{C_1}{T'}\left(\frac{\delta}{M}\right)^{-C_2(1+M^{4/3}\|V-\lambda_0\|_\infty^{2/3})}\exp\left(\frac{C_3M^2\ln^2(\delta/M)}{T'}-2\lambda_0 T'\right),$$

on the left hand side of (100) in Theorem 6.14 takes its overall minimum over $T' \in (0,\infty)$ at

$$T' = \frac{1+\sqrt{1-8\lambda_0 C_3 M^2\ln^2(\delta/M)}}{-4\lambda_0}$$

whence we find

$$\begin{aligned}-2\lambda_0 \leq \mathcal{C}_\infty^2 \leq&\\ &\leq -4\lambda_0 C_1\left(\frac{\delta}{M}\right)^{-C_2(1+M^{4/3}\|V-\lambda_0\|_\infty^{2/3})}\frac{\exp\left(1+\sqrt{1-8\lambda_0 C_3M^2\ln^2(\delta/M)}\right)}{1+\sqrt{1-8\lambda_0 C_3M^2\ln^2(\delta/M)}}\end{aligned}$$

in the situation of Theorem 6.14. Comparing the λ_0-dependence, we see that for $\lambda_0 \to -\infty$, there is still a discrepancy between the upper and the lower bound.

We conclude this section by discussing homogenization and de-homogenization, analogous to Examples 1 to 3.

Example 4 (Homogenization for all λ_0). *Let us assume that $V \not\equiv 0$ and that $\Gamma \subset \mathbb{R}^d$ is a generalized rectangle. We study the heat-type equation on Γ (where we put Dirichlet, Neumann or periodic boundary conditions if Γ has non-empty boundary). For sufficiently small M and $\delta < M/2$, we pick a (M,δ)-equidistributed sequence and choose $S = S_{\delta,Z}(\Gamma)$ as union of δ-balls. Then, Theorems 6.11 and 6.14 show null-controllability of the system 94 with control cost*

$$\mathcal{C}^2 \leq \begin{cases}\inf\limits_{T_0\in(0,T)}\frac{C_1}{T-T_0}\left(\frac{\delta}{M}\right)^{-C_2(1+M^{4/3}\|V\|_\infty^{2/3})}\exp\left(\frac{C_3M^2\ln^2(\delta/M)}{T-T_0}-2\lambda_0T_0\right) & \text{if } \lambda_0 \geq 0,\\ \min\limits_{T'\in(0,T]}\frac{C_1}{T'}\left(\frac{\delta}{M}\right)^{-C_2(1+M^{4/3}\|V-\lambda_0\|_\infty^{2/3})}\exp\left(\frac{C_3M^2\ln^2(\delta/M)}{T'}-2\lambda_0T'\right) & \text{if } \lambda_0 < 0.\end{cases}$$

Considering again the homogenization limit, i.e. letting M and δ tend to 0 while keeping δ/M constant, we see that as in Example 1, homogenization will send the term which is responsible for the exponential singularity of the control cost in $T = 0$ to zero.

Furthermore, similarly to Example 2, we see that homogenization reduces the effect of the potential V up to the shift of the ground state energy of the system, caused by

V. In other words, by homogenizing the control set, we can reduce the effect of the potential on $\mathcal{C}$ to the effect of a constant potential.

A. Proof of Theorem 3.19 and Lemma 3.20

In this appendix, we prove Theorem 3.19, and Lemma 3.20. We use statements and proofs from [Täu17] and [TT17]. For our arguments below, it is convenient to explicitly know the eigenvalues and eigenfunctions of the negative Laplacian $-\Delta_L$ in $L^2(\Lambda_L)$. Depending on the boundary conditions we choose a index set $\mathcal{I} = \mathbb{N}$ in case of Dirichlet boundary conditions, $\mathcal{I} = \mathbb{N}_0$ in case of Neumann boundary conditions, and $\mathcal{I} = 2\mathbb{Z}$ in case of periodic boundary conditions. Then, the eigenvalues of $-\Delta_L$ are given by

$$\lambda_y = \left(\frac{\pi}{L}\right)^2 |y|_2^2, \quad y \in \mathcal{I}^d, \tag{101}$$

with corresponding normalized eigenfunctions

$$e_y(x) = \begin{cases} \|e_y\|^{-1} \prod_{l=1}^{d} \sin\left(\frac{\pi y_l}{L}(x_l + L/2)\right) & \text{in case of Dirichlet b.c.,} \\ \|e_y\|^{-1} \prod_{l=1}^{d} \cos\left(\frac{\pi y_l}{L}(x_l + L/2)\right) & \text{in case of Neumann b.c.,} \\ \|e_y\|^{-1} \exp\left(\frac{\mathrm{i}\pi}{L} y \cdot x\right) & \text{in case of periodic b.c..} \end{cases} \tag{102}$$

The normalization constants $\|e_y\|^{-1}$ can be easily calculated, though we will not need them. Moreover, there exists a bijection $p : \mathbb{N} \to \mathcal{I}^d$ such that

$$\lambda_{p(k)}, \quad k \in \mathbb{N},$$

is the k-th eigenvalue of $-\Delta_L$ enumerated in increasing order counting multiplicities. This bijection is unique up to permutations of sites $y \in \mathcal{I}^d$ with the same Euclidean norm $|y|$.

A.1. Proof of Theorem 3.19

We will consider the case $L = 2\pi$ and $V = 0$ and periodic boundary conditions. We say that a function $f : \Lambda_L \to \mathbb{C}$ vanishes to order $N > 0$ at $x_0 \in \Lambda_L$ if

$$\limsup_{\delta \to 0} \frac{\sup_{y \in B(x_0,\delta)} |f(x)|}{\delta^N} < \infty.$$

Theorem A.1. *Let $d \geq 2$. For every $N > 0$ there is a nonzero function $f \in L^2(\Lambda_L)$ with $-\Delta f = \lambda f$ for some $\lambda > 0$ such that f vanishes to order at least N at 0.*

Proof. For $\lambda \geq 0$ let $\mathcal{I}_\lambda^d := \{k \in \mathbb{Z}^d \colon |k|^2 = \lambda\}$ and recall (101), and (102). A function $f \in L^2(\mathbb{T}^d)$ is an eigenfunction to the eigenvalue $\lambda \geq 0$ if and only if f is of the form

$$f = \sum_{\substack{k/2 \in \mathcal{I}_\lambda^d, \\ k \in \mathcal{I}}} \mu_k e_k = \sum_{k \in \mathcal{I}_\lambda^d} \mu_k \exp(ik\cdot), \quad \mu_k \in \mathbb{C}.$$

We expand the functions $x \mapsto \exp(ikx)$ in a Taylor series around 0

$$\exp(ikx)(x) = \sum_{\alpha \in \mathbb{N}_0^d} \frac{1}{\alpha!}(D^\alpha \exp(ik\cdot))(0) \cdot x^\alpha = \sum_{\alpha \in \mathbb{N}_0^d} \frac{(ik)^\alpha}{\alpha!} x^\alpha$$

where we used multindex notation, i.e. given $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}_0^d$, we write $\alpha! := \alpha_1! \cdot \ldots \cdot \alpha_d!$, $D^\alpha := \partial_{x_1}^{\alpha_1} \ldots \partial_{x_d}^{\alpha_d}$, $k^\alpha := k_1^{\alpha_1} \cdot \ldots \cdot k_d^{\alpha_d}$, $|\alpha|_1 := \alpha_1 + \ldots + \alpha_d$. Then, f can be expressed as

$$f(x) = \sum_{k \in \mathcal{I}_\lambda^d} \mu_k \exp(ikx) = \sum_{\alpha \in \mathbb{N}_0^d} \frac{(ix)^\alpha}{\alpha!} \left(\sum_{k \in \mathcal{I}_\lambda^d} \mu_k k^\alpha \right).$$

Since the Taylor series is locally absolutely convergent, f vanishes to order N at 0 if for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha|_1 \leq N$, we have

$$\sum_{k \in \mathcal{I}_\lambda^d} \mu_k k^\alpha = 0.$$

This is a system of finitely many linear equations, indexed by α, with variables $\{\mu_k\}_{k \in I_\lambda}$. More precisely, we have

$$\sharp\{\text{equations}\} = \sharp\{\alpha \in \mathbb{N}_0^d \colon |\alpha|_1 \leq N\} =: C(N), \quad \textit{fixed}, \text{ once we chose } N,$$
$$\sharp\{\text{variables}\} = \sharp\{k \in \mathbb{Z}^d \colon |k|^2 = \lambda\}.$$

If $\sharp\{\text{variables}\} > \sharp\{\text{equations}\}$ then there will be a non-trivial solution $\{\mu_k\}_{k \in I_\lambda}$. This will yield a function f which vanishes to order N at 0. Since f is a nontrivial linear combination of orthogonal ψ_k, it is non-zero in $L^2(\Lambda_L)$ sense.

Thus, it remains to show that for every $C \in \mathbb{N}$, there is $\lambda \geq 0$ such that $\sharp\{k \in \mathbb{Z}^d \colon |k|^2 = \lambda\} \geq C$. Clearly, it suffices to establish this in dimension $d = 2$ and in this case, the task boils down to finding $\lambda \geq 0$ such that the number of all pairs $(x, y) \in \mathbb{Z}^2$ satisfying $x^2 + y^2 = \lambda$ exceeds C. For $\lambda \in \mathbb{N}$, this number is explicitly given by the so-called sum-of-squares theorem, sometimes also referred to as Gauss's formula, which can be found in [Gau01]. See also [Fri82, Chapter 1] for a more modern reference. The sum-of-squares theorem states that for $\lambda \in \mathbb{N}$ with prime factor decomposition

$$\lambda = p_1^{a_1} \cdot \ldots \cdot p_k^{a_k} \cdot q_1^{b_1} \cdot \ldots \cdot q_l^{b_l} \cdot 2^c,$$

where p_i are primes of the form $4k+1$ and q_i are primes of the form $4k+3$, the number of pairs $(x,y) \in \mathbb{Z}^2$ with $x^2+y^2=\lambda$ is

$$\begin{cases} 4 \cdot (1+a_1) \cdot \ldots \cdot (1+a_n) & \text{if all } b_i \text{ are even,} \\ 0 & \text{else.} \end{cases}$$

Choosing e.g. $\lambda = 5^C$, this implies

$$\sharp\{(x,y) \in \mathbb{Z}^2 \colon x^2+y^2=\lambda\} = 4 \cdot (1+C) \geq C. \qquad \square$$

An analogous argument works if the Laplacian on the torus is replaced by the Laplacian on a hypercube with Dirichlet or Neumann boundary conditions.

The above notion of vanishing to order N at a point x_0 can be understood as vanishing with respect to the sup norm. The UCPs we are interested in are then bounds on the vanishing order with respect to the L^2 norm, i.e. estimates of the form

$$\int_{B(\delta)} |f|^2 \geq \delta^M \int_{\Lambda_L} |f|^2, \quad 0 < \delta \leq \pi. \tag{103}$$

The following lemma clarifies the connection between functions of high vanishing order and counterexamples to (103).

Lemma A.2. *If $0 \neq f \in L^2(\Lambda_L)$ vanishes of order N at 0, then* (103) *cannot hold with $M = 2N$.*

Proof. We estimate by Hölder's inequality

$$\lim_{\delta \to 0} \frac{\int_{B(\delta)} |f|^2}{\delta^{2N}} \leq \lim_{\delta \to 0} \operatorname{Vol}(B(\delta)) \cdot \left(\frac{\sup_{B(\delta)} |f|}{\delta^N} \right)^2 = 0$$

where we used that the second term on the right hand side remains bounded as $\delta \to 0$. Thus, Ineq. (103) cannot hold for $M = 2N$. $\square$

Proof of Theorem 3.19. It suffices to consider $V = 0$, i.e. the case of the pure Laplacian. By the above lemma, it suffices to find for every $N \geq 0$ a function $f \in \operatorname{Ran} P_{-\Delta}([E_0 - \omega, E_0])$ for some $E_0 \in \mathbb{R}$ that vanishes to order N at 0. Eigenfunctions are definitely in some $\operatorname{Ran} P_{-\Delta}([E_0 - \omega, E_0])$ and by Theorem A.1, we find an eigenfunction f, vanishing to order N at 0. $\square$

A.2. Proof of Lemma 3.20

Proof of Lemma 3.20. Since the eigenfunctions and eigenvalues of $-\Delta$ on Λ_L are explicitly given in (101) and (102), we can replace the sum on the left hand side by

$$\sum_{k \in \mathbb{N}} |E_k|^\kappa |\alpha_k|^2 = \sum_{y \in \mathcal{I}^d} \left(\frac{\pi}{L} \right)^{2\kappa} |y|_2^{2\kappa} |\langle e_y, \phi \rangle|^2 \leq \left(\frac{\pi}{L} \right)^{2\kappa} \sum_{y \in \mathcal{I}^d} |y|_2^{2N} |\langle e_y, \phi \rangle|^2$$

where $N \in 2\mathbb{N}$ is the least even integer larger than κ. For the eigenfunctions, see Eq. (102), we have $\partial_i^N e_y = -(\pi/L)^N |y_i|^N e_y$ for $i \in \{1, \cdots, d\}$. We calculate using integration by parts

$$\sum_{y \in \mathcal{I}^d} |y|_2^{2N} |\langle e_y, \phi \rangle|^2 \leq N \sum_{i=1}^d \sum_{y \in \mathcal{I}^d} |y_i|^{2N} |\langle e_y, \phi \rangle|^2 = N \left(\frac{L}{\pi}\right)^{2N} \sum_{i=1}^d \sum_{y \in \mathcal{I}^d} |\langle \partial_i^N e_y, \phi \rangle|^2$$
$$= N \left(\frac{L}{\pi}\right)^{2N} \sum_{i=1}^d \sum_{y \in \mathcal{I}^d} |\langle e_y, \partial_i^N \phi \rangle|^2 = N \left(\frac{L}{\pi}\right)^{2N} \sum_{i=1}^d \|\partial_i^N \phi\|_{\Lambda_L}^2. \qquad \square$$

B. Constants in the proof of Theorem 3.17

In this appendix, we provide some technical aspects which have been omitted in the proof of Theorem 3.17. This appendix coincides with [NTTV16, Appendix B], the preprint of [NTTV18a]

B.1. Cutoff functions

Let $f, \psi : \mathbb{R} \to [0,1]$ be given by

$$f(x) = \begin{cases} \mathrm{e}^{-1/x} & x > 0, \\ 0 & x \leq 0, \end{cases} \quad \text{and} \quad \psi(x) = \frac{f(x)}{f(x) + f(1-x)}.$$

Note that the function ψ is $C^\infty(\mathbb{R})$ and satisfies

$$\sup_{x \in \mathbb{R}} \psi'(x) \leq 2 =: C', \quad \sup_{x \in \mathbb{R}} \psi''(x) \leq 10 =: C'', \quad \text{and} \quad \psi(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x \geq 1. \end{cases}$$

For $\epsilon > 0$ we define $\psi_\epsilon : \mathbb{R} \to [0,1]$ by

$$\psi_\epsilon(x) = \psi(x/\epsilon).$$

Let now $A \subset \mathbb{R}^{d+1}$ and $h_A : \mathbb{R}^{d+1} \to \mathbb{R}$ with $h_A(x) \geq \operatorname{dist}(x, A)$ if $x \notin A$ and $h_A(x) \leq 0$ if $x \in A$. For $\epsilon > 0$ we define $\chi : \mathbb{R}^{d+1} \to [0,1]$ by

$$\chi_{A,\epsilon}(x) = \psi_\epsilon\big(\epsilon - h_A(x)\big).$$

Of course, $h_A(x) := \operatorname{dist}(x, A)$ is a possible choice, but in applications we will require h_A to have certain additional properties. By construction we have (cf. Fig. 6)

$$\chi_{A,\epsilon}(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } \operatorname{dist}(x, A) \geq \epsilon. \end{cases}$$

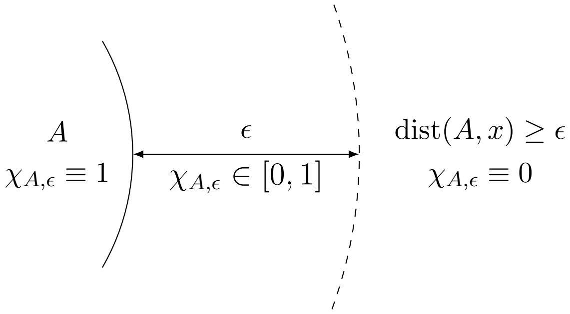

Figure 6: Cutoff function $\chi_{A,\epsilon}$

B.1.1. The constants Θ_2 and Θ_3

We want to construct a cutoff function $\chi \in C_c^\infty(\mathbb{R}^{d+1}; [0,1])$ with $\operatorname{supp}\chi \subset B(R_3)\setminus\{0\}$ and $\chi(x) = 1$ if $x \in B(r_3) \setminus \overline{B(R_1)}$. We set $\tilde{A} = B(r_3)$, $2\tilde{\epsilon} = R_3 - r_3$, $h_{\tilde{A}}(x) = |x| - r_3$ and define

$$\tilde{\chi}(x) = \chi_{\tilde{A},\tilde{\epsilon}}(x).$$

Note that

$$\tilde{\chi}(x) = \begin{cases} 1 & \text{if } x \in B(r_3), \\ 0 & \text{if } x \notin B((r_3+R_3)/2). \end{cases}$$

For the partial derivatives we calculate

$$(\partial_i\tilde{\chi})(x) = -\frac{1}{\tilde{\epsilon}}\psi'(1 - h_{\tilde{A}}(x)/\tilde{\epsilon})\frac{x_i}{|x|},$$
$$(\partial_i^2\tilde{\chi})(x) = \frac{1}{\tilde{\epsilon}^2}\psi''(1 - h_{\tilde{A}}(x)/\tilde{\epsilon})\frac{x_i^2}{|x|^2} - \frac{1}{\tilde{\epsilon}}\psi'(1 - h_{\tilde{A}}(x)/\tilde{\epsilon})\left(\frac{1}{|x|} - \frac{x_i^2}{|x|^3}\right).$$

Hence, using $\Delta\tilde{\chi}(x) = 0$ if $x \notin B(R_3) \setminus B(r_3)$ and $2\tilde{\epsilon} = R_3 - r_3 = 3\mathrm{e}\sqrt{d}$, we obtain

$$\|\nabla\tilde{\chi}\|_\infty \leq \frac{C'}{\tilde{\epsilon}} = \frac{4}{R_3 - r_3} = \frac{4}{3\mathrm{e}\sqrt{d}} \leq 1,$$
$$\|\Delta\tilde{\chi}\|_\infty \leq \frac{C''}{\tilde{\epsilon}^2} + \frac{C'}{\tilde{\epsilon}}\frac{d}{r_3} \leq \frac{80 + 4d}{18\mathrm{e}^2 d} \leq \frac{84}{18\mathrm{e}^2} \leq 1.$$

Analogously we find a function $\hat{\chi}$ with values in $[0,1]$, $\hat{\chi}(x) = 0$ if $x \in B(r_1)$, $\hat{\chi}(x) = 1$ if $x \notin B(R_1)$ and, using $R_1 - r_1 = r_1 \geq \delta^2/64$,

$$\|\nabla\hat{\chi}\|_\infty \leq \frac{C'}{R_1 - r_1} \leq \frac{128}{\delta^2},$$
$$\|\Delta\hat{\chi}\|_\infty \leq \frac{C''}{(R_1 - r_1)^2} + \frac{C'}{(R_1 - r_1)}\frac{d}{r_1} \leq \frac{10 \cdot 64^2}{\delta^4} + \frac{2d64^2}{\delta^4} \leq \frac{12d64^2}{\delta^4}.$$

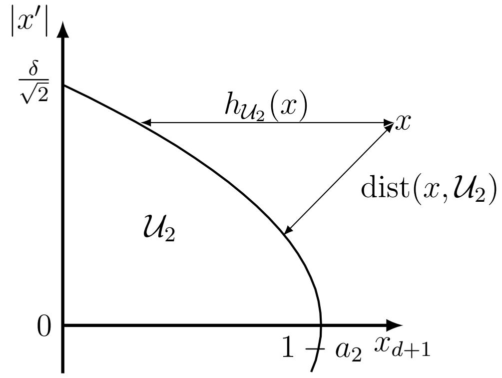

Figure 7: Distance to $\mathcal{U}_2$

Our cutoff function $\chi \in C_c^\infty(\mathbb{R}^{d+1};[0,1])$ with $\operatorname{supp}\chi \subset B(R_3)\setminus\{0\}$ and $\chi(x)=1$ if $x \in B(r_3)\setminus\overline{B(R_1)}$ can be defined by

$$\chi(x) = \begin{cases} \chi(x) = \hat{\chi}(x) & \text{if } x \in B(R_1)\setminus\overline{B(r_1)}, \\ \chi(x) = 1 & \text{if } x \in B(r_3)\setminus\overline{B(R_1)}, \\ \chi(x) = \tilde{\chi}(x) & \text{if } x \in B(R_3)\setminus\overline{B(r_3)}, \end{cases}$$

and has the properties (recall $\mathcal{V}_i = B(R_i)\setminus\overline{B(r_i)}$)

$$\max\{\|\Delta\chi\|_{\infty,\mathcal{V}_1}, \||\nabla\chi|\|_{\infty,\mathcal{V}_1}\} \le \frac{12d64^2}{\delta^4} =: \frac{\tilde{\Theta}_2}{\delta^4} =: \Theta_2$$

and

$$\max\{\|\Delta\chi\|_{\infty,\mathcal{V}_3}, \||\nabla\chi|\|_{\infty,\mathcal{V}_3}\} \le \frac{4}{3\mathrm{e}} =: \Theta_3.$$

B.1.2. The constant Θ_1

We choose $A =_2$, $\epsilon = \delta^2/16$ and

$$h_{\mathcal{U}_2}(x) = x_{d+1} - 1 + \sqrt{a_2^2 + \frac{|x'|^2}{2}}.$$

Obviously, $h_{\mathcal{U}_2}(x) \ge \mathrm{dist}(x,\mathcal{U}_2)$ if $x \notin \mathcal{U}_2$ and $h_{\mathcal{U}_2}(x) \le 0$ if $x \in \mathcal{U}_2$, cf. Fig. 7. Since the distance between the sets $\mathcal{U}_2$ and $\mathbb{R}^{d+1}_+ \setminus \mathcal{U}_3$ is bounded from below by $\delta^2/16$, see Appendix B.1.3, we find that

$$\chi_{\mathcal{U}_2,\epsilon}(x) = \begin{cases} 1 & \text{if } x \in \mathcal{U}_2, \\ 0 & \text{if } x \in \mathbb{R}^{d+1}_+ \setminus \mathcal{U}_3. \end{cases}$$

For the partial derivatives we calculate for $x \in \mathcal{U}_3 \setminus \mathcal{U}_2$

$$(\partial_i\chi)(x) = -\frac{1}{\epsilon}\psi'(1 - h_{\mathcal{U}_2}(x)/\epsilon) \begin{cases} \frac{x_i}{2}\left(a_2^2 + \frac{|x'|^2}{2}\right)^{-1/2} & \text{if } i \in \{1,\dots,d\}, \\ 1 & \text{if } i = d+1, \end{cases}$$

and find by using $|x'|^2 \le 1/4$ for $x \in \mathcal{U}_3 \setminus \mathcal{U}_2$ and $a_2^2 \in [15/16, 1]$

$$\|\nabla\chi_{\mathcal{U},\epsilon}\|_\infty^2 \le \frac{16}{466}\left(\frac{C'}{\epsilon}\right)^2, \quad \text{hence,} \quad \|\nabla\chi_{\mathcal{U},\epsilon}\|_\infty \le \frac{6}{\delta^2}.$$

For the second partial derivatives we calculate for $i \in \{1, \dots, d\}$

$$\begin{aligned}(\partial_i^2\chi)(x) = {}& \frac{1}{\epsilon^2}\psi''(1 - h_{\mathcal{U}}(x)/\epsilon)\frac{x_i^2}{4}\left(a_2^2 + \frac{|x'|^2}{2}\right)^{-1} \\ & - \frac{1}{\epsilon}\psi'(1 - h_{\mathcal{U}}(x)/\epsilon)\left[\frac{1}{2}\left(a_2^2 + \frac{|x'|^2}{2}\right)^{-1/2} - \frac{x_i^2}{4}\left(a_2^2 + \frac{|x'|^2}{2}\right)^{-3/2}\right],\end{aligned}$$

and $\partial_{d+1}^2\chi(x) = (1/\epsilon^2)\psi''(1 - h_{\mathcal{U}}(x)/\epsilon)$. Hence, using $|x'|^2 \le 1/4$ for $x \in \mathcal{U}_3 \setminus \mathcal{U}_2$ and $a_2^2 \in [15/16, 1]$

$$\|\Delta\chi\|_\infty \le \frac{C''}{\epsilon^2}\frac{237}{233} + \frac{C'}{2\epsilon a_2}(d + 8/233) \le \frac{16^2 \cdot 11d}{\delta^4} =: \frac{\tilde{\Theta}_1}{\delta^4} =: \Theta_1.$$

B.1.3. Distance of $\mathcal{U}_2$ and $\mathbb{R}_+^{d+1} \setminus \mathcal{U}_3$

The distance between the sets $\mathcal{U}_2$ and $\mathbb{R}_+^{d+1} \setminus \mathcal{U}_3$ is given by the distance between the two hyperbolas

$$h_i\colon \frac{(x-1)^2}{a_i^2} - \frac{y^2}{b_i^2} = 1, \quad i \in \{2, 3\}$$

in $\{(x, y) \in \mathbb{R}^2 \colon x, y \ge 0\}$, where a_i and b_i are given by

$$a_2^2 = 1 - \frac{\delta^2}{4}, \quad a_3^2 = 1 - \frac{\delta^2}{2} \quad \text{and} \quad b_i^2 = 2a_i^2.$$

See Fig. 8 for an illustration. By symmetry we can consider the case $y \ge 0$ only. First we show that in order to estimate the distance between h_2 and h_3 from below, it is sufficient to consider the distance between the intersection point of h_2 with the x-axis and h_3. For every point (x, y) on h_2, we define the distance $a(y)$ between h_2 and h_3 in x-direction and the distance $b(x)$ in y-direction. This gives rise to a rectangular triangle with catheti of length a and b. Due to concavity and monotonicity of h_2 and h_3, considered as functions of x, a lower bound for the distance of (x, y) to h_3 is given by the height of this rectangular triangle, given by

$$h(x) := \frac{a(x)b(x)}{\sqrt{a^2(x) + b^2(x)}}.$$

By a straightforward calculation, we see that $b(x)$ is strictly increasing as a function of x while $a(y)$ is strictly decreasing as a function of y. Thus, taking the triangle at

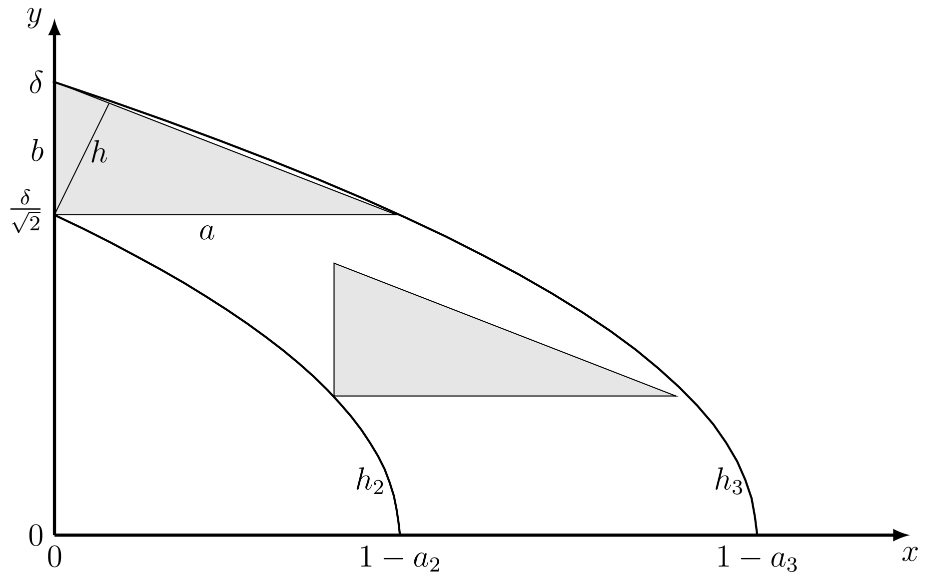

Figure 8: Distance between the hyperbolas h_2 and h_3

the point $(0, \delta/\sqrt{2})$ and moving it along h_2, the triangle will always stay below h_3, see Fig. 8. Hence, h evaluated at the point $(0, \delta/\sqrt{2})$ is a lower bound for $\operatorname{dist}(h_2, h_3)$. We have

$$a(\delta/\sqrt{2}) = 1 - \sqrt{1 - \frac{\delta^2}{4}} \quad \text{and} \quad b(0) = \left(1 - \frac{1}{\sqrt{2}}\right)\delta.$$

Hence,

$$\operatorname{dist}(h_2, h_3) \geq \frac{\left(1 - \frac{1}{\sqrt{2}}\right)\delta\left(1 - \sqrt{1 - \frac{\delta^2}{4}}\right)}{\sqrt{\left(1 - \frac{1}{\sqrt{2}}\right)^2\delta^2 + \left(1 - \sqrt{1 - \frac{\delta^2}{4}}\right)^2}}.$$

We use $\delta^2/8 \leq 1 - \sqrt{1 - \delta^2/4} \leq \delta/2$ and obtain the bound

$$\operatorname{dist}(h_2, h_3) \geq \frac{\left(1 - \frac{1}{\sqrt{2}}\right)\delta^2}{8\sqrt{\left(1 - \frac{1}{\sqrt{2}}\right)^2 + 1/4}} > \frac{\delta^2}{16}.$$

C. On single-site potentials for the breather model

This appendix coincides with [NTTV18a, Appendix A.1]. We discuss here our conditions on the single-site potential in the random breather model in Subsection 5.2. Recall that the ω_j were supported in $[\omega_-, \omega_+] \subset [0, 1)$ whence we consider $t \in [\omega_-, \omega_+]$ and $\delta \in [0, 1 - \omega_+]$.

Definition C.1. We say that a family $\{u_t\}_{t\in[0,1]}$ of measurable functions $u_t : \mathbb{R}^d \to \mathbb{R}$ satisfies condition

(A) if the u_t are uniformly bounded, have uniform compact support and if there are $\alpha_1, \beta_1 > 0$ and $\alpha_2, \beta_2 \geq 0$ such that for all $t \in [\omega_-, \omega_+]$, $\delta \leq 1 - \omega_+$ there is $x_0 = x_0(t, \delta) \in \mathbb{R}^d$ with

$$u_{t+\delta} - u_t \geq \alpha_1 \delta^{\alpha_2} \mathbf{1}_{B(x_0, \beta_1 \delta^{\beta_2})}. \tag{104}$$

(B) if u_t is the dilation of a function u by t, defined as $u_t(x) := u(x/t)$ for $t > 0$ and $u_0 \equiv 0$, where u is the characteristic function of a bounded convex set K with $0 \in \overline{K}$.

(C) if u_t is the dilation of a measurable function u which is positive, radially symmetric, compactly supported, bounded with decreasing radial part $r_u : [0, \infty) \to [0, \infty)$ and such there is a point $\tilde{x} > 0$ where r_u is differentiable, $r_u'(\tilde{x}) < 0$ and $r_u(\tilde{x}) > 0$.

(D) if u_t is the dilation of a measurable function u which is positive, radially symmetric, radially decreasing, compactly supported, bounded and which has a discontinuity away from 0.

(E) if u_t is the dilation of a measurable function which is non-positive, radially symmetric, radially increasing, compactly supported, bounded, and such there is a point $\tilde{x} > 0$ where the radial part r_u is differentiable, $r_u'(\tilde{x}) > 0$ and $r_u(\tilde{x}) < 0$.

Remark C.2. Condition (A) is the abstract assumption on random Schrödinger operators monotone in the randomness in Definition 5.5. Conditions (B) to (E) are relatively easy to verify for specific examples of single-site potentials. In particular, (C) holds for many natural choices of single-site potentials such as the smooth function $\mathbf{1}_{|x|<1} \exp(1/(|x|^2 - 1))$ or the hat-potential $\mathbf{1}_{|x|<1}(1 - |x|)$. Furthermore, we note that if we have families $\{u_t\}_{t\in[0,1]}$ and $\{v_t\}_{t\in[0,1]}$ where u_t satisfies (A) and $v_{t+\delta} - v_t \geq 0$ for all $t \in [\omega_-, \omega_+]$ and $\delta \in (0, 1 - \omega_+]$, then the family $\{u_t + v_t\}_{t\in[0,1]}$ also satisfies (A).

Lemma C.3. *We have that each of the assumptions (B) to (E) implies (A).*

Proof. Assume (B). We will show (A) with $\alpha_1 = 1$, $\alpha_2 = 0$, $\beta_2 = 1$ and $\beta_1 = c$, and hence it is enough to show the existence of a $c\delta$-ball in $K_{t+\delta} \backslash K_t$.

For $K \subset \mathbb{R}^d$ and $t > 0$ we define $K_t := \{x \in \mathbb{R}^d : x/t \in K\}$ and $K_0 := \emptyset$. Without loss of generality let $x := (1, 0, ..., 0)$ be a point in $\overline{K}$ which maximizes $|x|$ over $\overline{K}$. For $\lambda \in \mathbb{R}$ define the half-space $H_\lambda := \{x \in \mathbb{R}^d : x_1 \leq \lambda\}$, where x_1 stands for the first coordinate of x. By scaling, the existence of a $c\delta$-ball in $K_{t+\delta} \backslash K_t$ is equivalent to the existence of a $c\delta/(t+\delta)$-ball in $K \backslash K_{t/(t+\delta)}$. By maximality of $(1, 0, ..., 0)$, we have $K \subset H_1$ and hence $K_{t/(t+\delta)} \subset H_{t/(t+\delta)}$. Thus, it is sufficient to find a $c\delta/(t+\delta)$-ball in $K \backslash H_{t/(t+\delta)}$. By convexity of K, the set $\{z \in K : z_1 = 1/2\}$ is non-empty and since K is open, we find $z_0 \in K$ with $z_1 = 1/2$ and $0 < c < 1/2$ such that $B(z_0, c) \subset K$. We define for $\lambda \in [0, 1)$ the set $X(\lambda) \subset \mathbb{R}^d$ as $X(\lambda) := B(z_0 + \lambda((1, 0, ..., 0) - z_0), c\cdot(1-\lambda))$. By convexity and the fact that $(1, 0, ..., 0) \in \overline{K}$, we have $X(\lambda) \subset K$. In fact, let $\{x_n\}_{n\in\mathbb{N}} \subset K$ be a sequence with $x_n \to (1, 0, ..., 0)$. We define open sets $X_n(\lambda)$ by replacing $(1, 0, ..., 0)$ by x_n in the definition of $X(\lambda)$. By convexity of K, every X_n is a subset of K whence $\bigcup_{n\in\mathbb{N}} X_n(\lambda) \subset K$. Furthermore we have $X(\lambda) \subset \bigcup_{n\in\mathbb{N}} X_n(\lambda)$. Thus $X(\lambda) \subset K$. We now choose $\lambda := t/(t + \delta)$. Then $X(\lambda) \cap H_\lambda = \emptyset$. Noting that $c(1 - \lambda) = c\delta/(t + \delta)$, we see that $X(\lambda)$ is the desired $c\delta/(t + \delta)$-ball.

Now we assume (C). Let $r_u'(\tilde{x}) = -C_1$. Then there is $\tilde{\epsilon} > 0$ such that

$$r_u(\tilde{x} + \epsilon) - r_u(\tilde{x}) \in \left[-2\epsilon C_1, \frac{-\epsilon}{2} C_1\right] \quad \text{for all } |\epsilon| < \tilde{\epsilon}. \tag{105}$$

It is sufficient to prove the following: There are $C_2, C_3 > 0$ such that for every $0 \leq t \leq \omega_+$ and every $0 < \delta \leq 1 - \omega_+$ there is $\hat{x} = \hat{x}(t, \delta)$ such that

$$r_u\left(\frac{\hat{x} + C_2\delta}{t + \delta}\right) - r_u\left(\frac{\hat{x}}{t}\right) \geq C_3\delta. \tag{106}$$

Indeed, by monotonicity of r_u, (106) implies that for every $x \in [\hat{x}, \hat{x} + C_2\delta]$ we have

$$r_u\left(\frac{x}{t + \delta}\right) - r_u\left(\frac{x}{t}\right) \geq r_u\left(\frac{\hat{x} + C_2\delta}{t + \delta}\right) - r_u\left(\frac{\hat{x}}{t}\right) \geq C_3\delta$$

whence (A) holds with $x_0 := (\hat{x} + C_2\delta/2)e_1$, $\alpha_1 = C_3$, $\beta_1 = C_2/2$, $\alpha_2 = \beta_2 = 1$.

In order to see (106), let $\hat{x} = (t + \delta)\tilde{x}$. We choose $\kappa \in (0, 1/4)$ and assume that $\tilde{x} - 4\kappa\tilde{\epsilon} > 0$ (this is no restriction since (105) also holds for smaller $\tilde{\epsilon}$). Furthermore, we define $C_2 := \kappa\tilde{\epsilon}$. Now we distinguish two cases. If $\tilde{x}\delta/t \leq \tilde{\epsilon}$, then (105) implies

$$\begin{aligned} r_u\left(\frac{\hat{x} + C_2\delta}{t + \delta}\right) - r_u\left(\frac{\hat{x}}{t}\right) &= r_u\left(\tilde{x} + \kappa\frac{\tilde{\epsilon}\delta}{t + \delta}\right) - r_u(\tilde{x}) + r_u(\tilde{x}) - r_u\left(\tilde{x} + \tilde{x}\frac{\delta}{t}\right) \\ &\geq -2\kappa C_1\frac{\tilde{\epsilon}\delta}{t + \delta} + C_1\frac{\tilde{x}\delta}{2t} \geq \delta\frac{C_1}{2}\frac{\tilde{x} - 4\kappa\tilde{\epsilon}}{t + \delta}. \end{aligned}$$

If $\tilde{x}\delta/t > \tilde{\epsilon}$, we use $r_u(\tilde{x}) - r_u(\tilde{x} + \tilde{x}\delta/t) \geq r_u(\tilde{x}) - r_u(\tilde{x} + \tilde{\epsilon})$ and (105) to obtain

$$r_u\left(\frac{\hat{x} + C_2\delta}{t + \delta}\right) - r_u\left(\frac{\hat{x}}{t}\right) \geq -2\kappa C_1\frac{\tilde{\epsilon}\delta}{t + \delta} + C_1\frac{\tilde{\epsilon}}{2} = \frac{C_1\tilde{\epsilon}}{2}\left(1 - \frac{4\kappa\delta}{t + \delta}\right) \geq \frac{C_1\tilde{\epsilon}}{2}(1 - 4\kappa).$$

Hence

$$r_u\left(\frac{\hat{x}+C_2\delta}{t+\delta}\right)-r_u\left(\frac{\hat{x}}{t}\right)\geq C_3\delta,\ \text{where } C_3:=\min\left\{\frac{C_1(\tilde{x}-4\kappa\tilde{\epsilon})}{2},\frac{C_1\tilde{\epsilon}(1-4\kappa)}{2(1-\omega_+)}\right\}>0.$$

The fact that (D) implies (A) is a consequence of (B). In fact, a functions u as in (D) can be decomposed $u=v+w$ where v is (a multiple of) a characteristic function of a ball, centered at the origin, and w is positive, radially symmetric and decreasing. Indeed, let x_0 be the point of discontinuity with the smallest norm. Then we can take $v=(u(x_0-)-u(x_0+))\mathbf{1}_{B(0,|x_0|)}$.

The function v satisfies (A) by (B) (since balls are convex) and we have $w_{t+\delta}-w_t\geq 0$. By Remark C.2, the family $\{u_t\}_{t\in[0,1]}=\{v_t+w_t\}_{t\in[0,1]}$ also satisfies (A). The case (E) is an adaptation of (C). □

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