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Abstract 3-Rigidity and Bivariate C_2^1-Splines I: Whiteley's Maximality Conjecture

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Abstract: A conjecture of Graver from 1991 states that the generic 3-dimensional rigidity matroid is the unique maximal abstract 3-rigidity matroid with respect to the weak order on matroids. Based on a close similarity between the generic d-dimensional rigidity matroid and the generic C_{d-2}^{d-1}-cofactor matroid from approximation theory, Whiteley made an analogous conjecture in 1996 that the generic C_{d-2}^{d-1}-cofactor matroid is the unique maximal abstract d-rigidity matroid for all $d \geq 2$. We verify the case $d = 3$ of Whiteley's conjecture in this paper. A key step in our proof is to verify a second conjecture of Whiteley that the 'double V-replacement operation' preserves independence in the generic C_2^1-cofactor matroid.

Key words and phrases: graph rigidity, abstract rigidity matroid, bivariate spline, cofactor matroid

1 Introduction

The statics of skeletal structures strongly depends on their underlying graphs. This connection has been studied since the seminal work of James Clerk Maxwell [12] in 1864, and combinatorial rigidity theory is now a fundamental topic in structural rigidity [7, 21].

We will consider a *d-dimensional (bar-joint) framework*, which is a pair $(G, \boldsymbol{p})$ consisting of a finite graph $G = (V, E)$ and a map $\boldsymbol{p} : V \to \mathbb{R}^d$. Asimow and Roth [1] and Gluck [5] observed that the properties of rigidity and infinitesimal rigidity coincide when $\boldsymbol{p}$ is generic (i.e., the set of coordinates in $\boldsymbol{p}$ is algebraically independent over the rational field), and are completely determined by G and d. This fact enables us to define the generic d-dimensional rigidity matroid $\mathcal{R}_d(G)$ on the edge set of G, whose rank characterizes the rigidity of any generic framework $(G, \boldsymbol{p})$. (See, e.g., [7, 16, 21] for more details.)

The generic 1-dimensional rigidity matroid $\mathcal{R}_1(G)$ is equal to the graphic matroid of G, and a celebrated theorem of Pollaczek-Geiringer [15] and Laman [10] can be used to obtain a concise combinatorial formula for the rank function of the generic 2-dimensional rigidity matroid $\mathcal{R}_2(G)$, see [11]. Finding a combinatorial description for $\mathcal{R}_3(G)$ remains a fundamental open problem in the field which has inspired a significant amount of research.

In 1991, Graver [6] suggested the approach of abstracting representative properties of the d-dimensional rigidity matroid and recasting rigidity as a matroid property. Suppose that $(G, \boldsymbol{p})$, $(G_1, \boldsymbol{p}_1)$ and $(G_2, \boldsymbol{p}_2)$ are generic d-dimensional frameworks such that $G = G_1 \cup G_2$ and $\boldsymbol{p}_i$ is the restriction of $\boldsymbol{p}$ to $V(G_i)$ for $i = 1, 2$. It is intuitively clear that:

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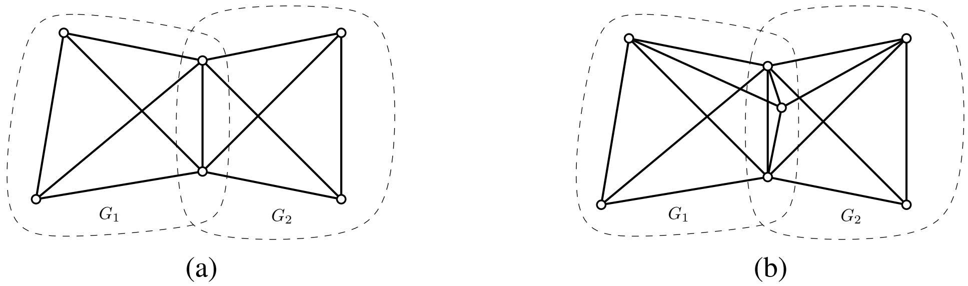


Figure 1: Examples in $d = 3$: (a) When $|V(G_1) \cap V(G_2)| = 2$, $(G_1, \boldsymbol{p}_1)$ and $(G_2, \boldsymbol{p}_2)$ can rotate about the line through the points of $V(G_1) \cap V(G_2)$, and adding any edge between $V(G_1) \setminus V(G_2)$ and $V(G_2) \setminus V(G_1)$ will restrict this motion. (b) Rigid frameworks $(G_1, \boldsymbol{p}_1)$ and $(G_2, \boldsymbol{p}_2)$ are glued together along three vertices, and the resulting framework is rigid.

- if $|V(G_1) \cap V(G_2)| \leq d - 1$ then adding any edge between $V(G_1) \setminus V(G_2)$ and $V(G_2) \setminus V(G_1)$ will restrict the 'flexibility' of $(G, \boldsymbol{p})$, see Figure 1(a);
- if $|V(G_1) \cap V(G_2)| \geq d$ and $(G_1, \boldsymbol{p}_1)$ and $(G_2, \boldsymbol{p}_2)$ are both rigid then $(G, \boldsymbol{p})$ will be rigid, see Figure 1(b).

Graver [6] observed that these physical properties can be described in terms of matroid closure, and proposed to investigate all matroids which satisfy these properties. More precisely, he defined a matroid M on the edge set $K(V)$ of the complete graph with vertex set V to be an *abstract d-rigidity matroid* if the following two properties hold:

(R1) If $E_1, E_2 \subseteq K(V)$ with $|V(E_1) \cap V(E_2)| \leq d - 1$, then $\mathrm{cl}_M(E_1 \cup E_2) \subseteq K(V(E_1)) \cup K(V(E_2))$;

(R2) If $E_1, E_2 \subseteq K(V)$ with $\mathrm{cl}_M(E_1) = K(V(E_1)), \mathrm{cl}_M(E_2) = K(V(E_2))$, and $|V(E_1) \cap V(E_2)| \geq d$, then $\mathrm{cl}_M(E_1 \cup E_2) = K(V(E_1 \cup E_2))$,

where cl_M denotes the closure operator of M. The generic d-dimensional rigidity matroid for $K(V)$, $\mathcal{R}_d(V)$, is an example of an abstract d-rigidity matroid. Graver [6] conjectured that, for all $d \geq 1$, there is a unique maximal abstract d-rigidity matroid on $K(V)$ with respect to the weak order of matroids, and further that $\mathcal{R}_d(V)$ is this maximal matroid. He verified his conjecture for $d = 1, 2$ but N. J. Thurston (see, [7, page 150]) subsequently showed that $\mathcal{R}_d(V)$ is not the unique maximal abstract d-rigidity matroid when $d \geq 4$. We will verify the first part of Graver's conjecture when $d = 3$.[1] The second part remains as a long-standing open problem.

Conjecture 1.1 (Maximality conjecture)**.** *The generic 3-dimensional rigidity matroid on $K(V)$ is the unique maximal abstract 3-rigidity matroid on $K(V)$.*

Whiteley [21] found another candidate for a maximal abstract d-rigidity matroid from approximation theory. Consider a plane polygonal domain D which has been subdivided into a set of polygonal faces Δ. A *bivariate C^r_s-spline* over Δ is a function on D which is continuously differentiable r times and is given by a polynomial of degree s over each face of Δ. The set of all C^r_s-splines over Δ forms a vector space $S^r_s(\Delta)$, and determining the dimension of $S^r_s(\Delta)$ is a major question in this area. When $r = 0$ and $d = 1$, $S^0_1(\Delta)$ is the space of piecewise linear functions, and *Maxwell's reciprocal diagram* gives a concrete correspondence between $S^0_1(\Delta)$ and the self-stresses of the 1-skeleton of Δ, regarded as a 2-dimensional framework $(G, \boldsymbol{p})$. Whiteley [19–21] extended this connection further and demonstrated how the dimension of $S^r_s(\Delta)$ can be computed from the rank of the *C^r_s-cofactor matrix* $C^r_s(G, \boldsymbol{p})$, which is a variant of the incidence matrix of

[1]The first part of Graver's conjecture, that there exists a unique maximal abstract d-rigidity matroid on $K(V)$, would follow for all d as a special case of the main result in a preprint of Sitharam and Vince [17]. Unfortunately this more general result is false, see [9, 14].

G in which the entries are replaced by block matrices whose entries are polynomials in $\boldsymbol{p}$. Billera [2] used Whiteley's approach to solve a conjecture of Strang on the generic dimension of $S_s^1(\Delta)$.

The definition of $C_s^r(G,\boldsymbol{p})$ is not dependent on the fact that $(G,\boldsymbol{p})$ is a realisation of G in the plane without crossing edges and is equally valid for any 2-dimensional framework $(G,\boldsymbol{p})$. Moreover, in the case when $r = s-1$, each row of $C_s^{s-1}(G,\boldsymbol{p})$ is associated with a distinct edge of G, and hence the row matroid of $C_s^{s-1}(G,\boldsymbol{p})$ is a matroid defined on $E(G)$. We refer to this matroid as the C_s^{s-1}-*cofactor matroid* for $(G,\boldsymbol{p})$ and denote it by $\mathcal{C}_s^{s-1}(G,\boldsymbol{p})$. Since the matroids $\mathcal{C}_s^{s-1}(G,\boldsymbol{p})$ are the same for all generic $\boldsymbol{p}$, the *generic* C_s^{s-1}-*cofactor matroid* of G, denoted by $\mathcal{C}_s^{s-1}(G)$, is defined to be $\mathcal{C}_s^{s-1}(G,\boldsymbol{p})$ for any generic $\boldsymbol{p}$. (See Section 2.1 for a formal definition). Let $\mathcal{C}_{d-1}^{d-2}(V) = \mathcal{C}_{d-1}^{d-2}(K(V))$. Whiteley [21] showed that $\mathcal{C}_{d-1}^{d-2}(V)$ is an example of an abstract d-rigidity matroid and pointed out that $\mathcal{C}_{d-1}^{d-2}(V) = \mathcal{R}_d(V)$ when $d=2$. He also showed that $\mathcal{C}_{d-1}^{d-2}(V)$ is a counterexample to Graver's original maximality conjecture for all $d \geq 4$ and conjectured further that $\mathcal{C}_{d-1}^{d-2}(V)$ is the maximal abstract d-rigidity matroid for all $d \geq 2$.

In this paper we verify Whiteley's conjecture for $d=3$, and hence prove the cofactor counterpart of Conjecture 1.1:

Theorem 1.2. *The generic C_2^1-cofactor matroid $\mathcal{C}_2^1(V)$ is the unique maximal abstract 3-rigidity matroid on $K(V)$.*

The most difficult part of the proof of Theorem 1.2 is to show that a certain graph operation (called the double V-replacement operation) preserves independence in the generic C_2^1-cofactor matroid. This is related to another long-standing conjecture on the generic 3-dimensional rigidity matroid, known as the *Henneberg construction conjecture*, which asks if every base of the generic 3-dimensional rigidity matroid $\mathcal{R}_3(V)$ can be constructed from K_4 by a sequence of simple graph operations (see [4, 18] for more details). Based on a strong similarity between rigidity matroids and cofactor matroids, Whiteley [19, page 55] posed a corresponding Henneberg construction conjecture for the generic C_2^1-cofactor matroid. We will verify this conjecture.

In his survey on generic rigidity and cofactor matroids, Whiteley gave a table of properties and conjectures for the generic 3-dimensional rigidity and C_2^1-cofactor matroids [21, page 65]. The table below updates Whiteley's table with results from [8] and the current paper.

	Generic 3-dim rigidity	Generic C_2^1-cofactor
rank K_n, $n \geq 3$	$3n-6$	$3n-6$
0-extension (vertex addition)	YES	YES
1-extension (edge split)	YES	YES
abstract 3-rigidity	YES	YES
vertex splitting	YES	YES
simplicial 2-surfaces	Rigid [5]	Rigid [2]
coning	YES	YES
X-replacement	Conjectured	YES [21]
double V-replacement	Conjectured	YES (Theorem 3.3)
Dress conjecture	NO [8]	NO [8]
$K_{4,6}$	Base	Base
$K_{5,5}$	Circuit	Circuit
maximal abstract 3-rigidity matroid	Conjectured	YES (Theorem 1.2)

We will give a combinatorial characterization of the rank function of the maximal abstract 3-rigidity matroid $\mathcal{C}_2^1(V)$ in a companion paper [3] to this one, and hence solve the cofactor counterpart to the combinatorial characterization problem for 3-dimensional rigidity. Theorem 1.2 is a key ingredient of the proof in this companion paper.

This paper focuses on proving Theorem 1.2, and hence we restrict our attention to C_2^1-cofactor matroids. In Section 2, we first provide a brief introduction to C_2^1-cofactor matroids, and then give a detailed roadmap for the proof of Theorem 1.2.

2 Preliminaries

In this section we give a formal definition of C_2^1-cofactor matroids and describe their fundamental properties. We then give a roadmap for the proof of Theorem 1.2.

We use the following notation throughout this paper. For a finite set $V = \{v_1, v_2, \ldots, v_n\}$, let $K(V)$, or $K(v_1, v_2, \ldots, v_n)$, denote the edge set of the complete graph with vertex set V. For a graph $G = (V, E)$ and $v \in V$, let $N_G(v)$ be the set of neighbors of v in G and $\hat{N}_G(v) = N_G(v) \cup \{v\}$ be the closed neighborhood of v in G. For an edge set F and a vertex v, let $d_F(v)$ denote the number of edges of F incident to v. A *star on n vertices* is a tree with n vertices in which one vertex is adjacent to all other vertices.

We often regard a map $\boldsymbol{p} : V \to \mathbb{R}^k$ as a $k|V|$-dimensional vector. The inner product $\boldsymbol{p} \cdot \boldsymbol{q}$ of two maps $\boldsymbol{p}, \boldsymbol{q} : V \to \mathbb{R}^k$ is given by this identification.

For vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ in a Euclidean space, let $\langle \boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \rangle$ be their linear span.

For a set Z of real numbers, let $\mathbb{Q}(Z)$ denote the smallest subfield of $\mathbb{R}$ that contains the rationals and Z.

The closure operator and the rank function of a matroid $\mathcal{M}$ are denoted by $\mathrm{cl}_{\mathcal{M}}$ and $r_{\mathcal{M}}$, respectively. The *weak order for matroids* is a partial order over all matroids with the same groundset E, where for two matroids $\mathcal{M}_i = (E, r_{\mathcal{M}_i})$ $(i = 1, 2)$ we have $\mathcal{M}_1 \preceq \mathcal{M}_2$ if $r_{\mathcal{M}_1}(X) \leq r_{\mathcal{M}_2}(X)$ holds for all $X \subseteq E$.

2.1 The C_2^1-cofactor Matrix

Given two points $p_i = (x_i, y_i)$ and $p_j = (x_j, y_j)$ in $\mathbb{R}^2$, we define $D(p_i, p_j) \in \mathbb{R}^3$ by

$$D(p_i, p_j) = ((x_i - x_j)^2, (x_i - x_j)(y_i - y_j), (y_i - y_j)^2).$$

For a 2-dimensional framework $(G, \boldsymbol{p})$ with vertex set $V = \{v_1, v_2, \ldots, v_n\}$, we simply write $D(v_i, v_j)$ for $D(\boldsymbol{p}(v_i), \boldsymbol{p}(v_j))$ when $\boldsymbol{p}$ is clear from the context.

We define the *C_2^1-cofactor matrix* of a 2-dimensional framework $(G, \boldsymbol{p})$ to be the matrix $C_2^1(G, \boldsymbol{p})$ of size $|E| \times 3|V|$ in which each vertex is associated with a set of three consecutive columns, each edge is associated with a row, and the row associated with the edge $e = v_i v_j$ with $i < j$ is

$$\begin{array}{c} \\ e = v_i v_j \end{array} \begin{array}{c} \\ \left[\begin{array}{ccccc} 0 \cdots 0 & \overset{v_i}{D(v_i, v_j)} & 0 \cdots 0 & \overset{v_j}{-D(v_i, v_j)} & 0 \cdots 0 \end{array}\right]. \end{array}$$

For example, for $(K_4, \boldsymbol{p})$ with $\boldsymbol{p}(v_1) = (0,0)$, $\boldsymbol{p}(v_2) = (1,0)$, $\boldsymbol{p}(v_3) = (0,1)$ and $\boldsymbol{p}(v_4) = (-1,-1)$,

$$C_2^1(K_4, \boldsymbol{p}) = \begin{array}{c} \\ v_1v_2 \\ v_1v_3 \\ v_1v_4 \\ v_2v_3 \\ v_2v_4 \\ v_3v_4 \end{array} \left[\begin{array}{ccc|ccc|ccc|ccc} & v_1 & & & v_2 & & & v_3 & & & v_4 & \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 2 & 1 & 0 & 0 & 0 & -4 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 & -1 & -2 & -4 \end{array}\right].$$

(Our definition is slightly different to that given by Whiteley [21], but the two definitions are equivalent up to elementary column operations.) It is known that the space $S_2^1(\Delta)$ of bivariate C_2^1-splines over a subdivision Δ of a polygonal domain in the plane is linearly isomorphic to the left kernel of $C_2^1(G, \boldsymbol{p})$ if $(G, \boldsymbol{p})$ is the 1-skeleton of Δ. See, [21] for more details.

2.2 C_2^1-motions

As mentioned above, the left kernel of the C_2^1-cofactor matrix $C_2^1(G, \boldsymbol{p})$ has an important role in the analysis of C_2^1-splines. An analogous situation occurs in rigidity theory where the left kernel of the rigidity matrix is the space of *self-stresses* of the framework. The right kernel of the rigidity matrix plays an equally important role in rigidity theory since it is the space of *infinitesimal motions* of the framework. In order to apply techniques from rigidity theory to cofactor matrices, we will also consider the right kernel of the C_2^1-cofactor matrix.

Let $G=(V,E)$ be a graph and $\boldsymbol{p}: V \to \mathbb{R}^2$ such that $\boldsymbol{p}(v_i)=(x_i,y_i)\in\mathbb{R}^2$ for all $v_i \in V$. A *C_2^1-motion* (or simply, a *motion*) of $(G,\boldsymbol{p})$ is a map $\boldsymbol{q}: V\to\mathbb{R}^3$ satisfying

$$D(v_i,v_j)\cdot(\boldsymbol{q}(v_i)-\boldsymbol{q}(v_j))=0 \qquad (v_iv_j\in E). \tag{1}$$

The C_2^1-cofactor matrix $C_2^1(G,\boldsymbol{p})$ defined in Section 2.1 is the matrix of coefficients of this system of linear equations in the variable $\boldsymbol{q}$, and hence each C_2^1-motion $\boldsymbol{q}$ is a vector in the right kernel $Z(G,\boldsymbol{p})$ of $C_2^1(G,\boldsymbol{p})$.

Whiteley [21] showed that $Z(G,\boldsymbol{p})$ has dimension at least six when $\boldsymbol{p}(V)$ affinely spans $\mathbb{R}^2$ by showing that the C_2^1-motions $\boldsymbol{q}_i^*: V\to\mathbb{R}^3$, $1\le i\le 6$, defined by

$$\boldsymbol{q}_1^*(v_i)=(1,0,0), \qquad \boldsymbol{q}_2^*(v_i)=(0,1,0), \qquad \boldsymbol{q}_3^*(v_i)=(0,0,1), \tag{2}$$

$$\boldsymbol{q}_4^*(v_i)=(y_i,-x_i,0), \qquad \boldsymbol{q}_5^*(v_i)=(0,-y_i,x_i) \qquad \boldsymbol{q}_6^*(v_i)=(y_i^2,-2x_iy_i,x_i^2) \tag{3}$$

for each $v_i\in V$, are linearly independent vectors in $Z(G,\boldsymbol{p})$ for all G. (We have adjusted the values of the $\boldsymbol{q}_i^*$ given in [21] to fit our modified definition of $C_2^1(G,\boldsymbol{p})$.) We will refer to a C_2^1-motion that can be described as a linear combination of the $\boldsymbol{q}_i^*, 1\le i\le 6$ as a *trivial* C_2^1-motion, and let $Z_0(G,\boldsymbol{p})$ be the space of all trivial C_2^1-motions.

The following terminologies are C_2^1-cofactor analogues of standard terminologies in rigidity theory. We say that a framework $(G,\boldsymbol{p})$ is: *C_2^1-rigid* if $Z(G,\boldsymbol{p})=Z_0(G,\boldsymbol{p})$; *$C_2^1$-independent* if the rows of $C_2^1(G,\boldsymbol{p})$ are linearly independent; *minimally C_2^1-rigid* if $(G,\boldsymbol{p})$ is both C_2^1-rigid and C_2^1-independent; a *k-degree of freedom framework*, or *k-dof framework* for short, if $\dim Z(G,\boldsymbol{p})=6+k$. We will use the same terms for the graph G if $(G,\boldsymbol{p})$ has the corresponding properties for some (or equivalently, every) generic $\boldsymbol{p}$.

2.3 Generic C_2^1-cofactor Matroids

The generic *C_2^1-cofactor matroid*, $\mathcal{C}_{2,n}^1$, is the matroid on $E(K_n)$ in which independence is given by the linear independence of the rows of $C_2^1(K_n,\boldsymbol{p})$ for any generic $\boldsymbol{p}$. We will sometimes simplify $\mathcal{C}_{2,n}^1$ to $\mathcal{C}_n$ or even $\mathcal{C}$ when it is clear from the context. Note that a graph G with n vertices is minimally C_2^1-rigid (resp., C_2^1-independent) if and only if $E(G)$ is a base (resp., an independent set) of $\mathcal{C}_{2,n}^1$.

Since $\dim Z(K_n,\boldsymbol{p})\ge\dim Z_0(K_n,\boldsymbol{p})=6$ holds when $n\ge 3$ and $\boldsymbol{p}$ is generic, the rank of $\mathcal{C}_{2,n}^1$ is at most $3n-6$ for all $n\ge 3$. Whiteley [21, Corollary 11.3.15] showed that the rank of $\mathcal{C}_{2,n}^1$ is equal to $3n-6$ when $n\ge 3$, and, more significantly, that $\mathcal{C}_{2,n}^1$ is an abstract 3-rigidity matroid. Our main result, Theorem 1.2, verifies that $\mathcal{C}_{2,n}^1$ is the unique maximal abstract 3-rigidity matroid on $E(K_n)$.

2.4 Pinning

The technique of *pinning*, i.e. specifying the value of a motion at one or more vertices, is frequently used when dealing with a k-dof framework $(G,\boldsymbol{p})$ in rigidity theory, and we will also use this technique in our analysis. The following technical lemmas will be used only in Sections 5 and 6, so the reader may wish to skip this subsection and refer back to it later.

For $v_s\in V(G)$ and $t\in\{1,2,3\}$, let $\mathbf{e}_{s,t}: V\to\mathbb{R}^3$ be such that $\mathbf{e}_{s,t}(v_s)$ is the unit vector with 1 at the t-coordinate and zeros elsewhere, and $\mathbf{e}_{s,t}(v_i)=0$ for $i\neq s$.

Lemma 2.1. *Let $(G,\boldsymbol{p})$ be a k-dof framework such that $\boldsymbol{p}(v_i)=(x_i,y_i)\in\mathbb{R}^2$ for all $v_i\in V(G)$; v_a,v_b,v_c be three distinct vertices such that $(y_a-y_b)(y_a-y_c)(y_b-y_c)\neq 0$; $\boldsymbol{q}_0,\boldsymbol{q}_1,\ldots,\boldsymbol{q}_k$ be motions of $(G,\boldsymbol{p})$ such that $\boldsymbol{q}_1,\ldots,\boldsymbol{q}_k$ are linearly independent and*

$$\boldsymbol{q}_i\cdot\mathbf{e}_{s,t}=0 \textit{ for all } (s,t)\in\{(a,1),(a,2),(a,3),(b,1),(b,2),(c,1)\} \tag{4}$$

for every $i\in\{0,1,\ldots,k\}$. Then $Z(G,\boldsymbol{p})=Z_0(G,\boldsymbol{p})\oplus\langle\boldsymbol{q}_1,\ldots,\boldsymbol{q}_k\rangle$ and $\boldsymbol{q}_0\in\langle\boldsymbol{q}_1,\ldots,\boldsymbol{q}_k\rangle$.

Proof. Suppose $\sum_{i=1}^{k}\mu_i\boldsymbol{q}_i=\sum_{j=1}^{6}\lambda_j\boldsymbol{q}_j^*$ for some $\mu_i,\lambda_j\in\mathbb{R}$. Then (4) gives

$$\sum_{j=1}^{6}\lambda_j(\boldsymbol{q}_j^*\cdot\mathbf{e}_{s,t})=0 \text{ for every } (s,t)\in\{(a,1),(a,2),(a,3),(b,1),(b,2),(c,1)\}.$$

By the definition of q_i^* in (2) and (3), this system of equations can be written as

$$\begin{pmatrix}1&0&0&y_a&0&y_a^2\\0&1&0&-x_a&-y_a&-2x_ay_a\\0&0&1&0&x_a&x_a^2\\1&0&0&y_b&0&y_b^2\\0&1&0&-x_b&-y_b&-2x_by_b\\1&0&0&y_c&0&y_c^2\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\\\lambda_3\\\lambda_4\\\lambda_5\\\lambda_6\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\\0\end{pmatrix}.$$

Let Q denote the matrix of coefficients of this system. Then $\det Q=(y_a-y_b)^2(y_c-y_a)(y_b-y_c)\neq 0$. Hence Q is non-singular, and $\lambda_j=0$ for all $1\le j\le 6$. Thus $Z_0(G,\boldsymbol{p})\cap\langle\boldsymbol{q}_1,\ldots,\boldsymbol{q}_k\rangle=\{0\}$. The fact that $(G,\boldsymbol{p})$ is a k-dof framework now gives $Z(G,\boldsymbol{p})=Z_0(G,\boldsymbol{p})\oplus\langle\boldsymbol{q}_1,\ldots,\boldsymbol{q}_k\rangle$.

Since $\boldsymbol{q}_0$ is a motion, we have $\boldsymbol{q}_0=\sum_{i=1}^{k}\mu_i'\boldsymbol{q}_i+\sum_{j=1}^{6}\lambda_j'\boldsymbol{q}_j^*$ for some $\mu_i',\lambda_j'\in\mathbb{R}$. We can use (4) to obtain

$$\sum_{j=1}^{6}\lambda_j'(\boldsymbol{q}_j^*\cdot\mathbf{e}_{s,t})=0 \text{ for every } (s,t)\in\{(a,1),(a,2),(a,3),(b,1),(b,2),(c,1)\}.$$

The same argument as in the previous paragraph now gives $\lambda_j'=0$ for all $1\le j\le 6$. In other words, $\boldsymbol{q}_0$ is a linear combination of $\boldsymbol{q}_1,\ldots,\boldsymbol{q}_k$. □

Suppose v_a,v_b,v_c are distinct vertices in a framework $(G,\boldsymbol{p})$. We define the *extended C_2^1-cofactor matrix* $\tilde{C}(G,\boldsymbol{p})$ (with respect to (v_a,v_b,v_c)) to be the matrix of size $3|V|\times(|E|+6)$ obtained from $C(G,\boldsymbol{p})$ by adding the six rows, $\mathbf{e}_{a,1},\mathbf{e}_{a,2},\mathbf{e}_{a,3},\mathbf{e}_{b,1},\mathbf{e}_{b,2},\mathbf{e}_{c,1}$.

For example, for $(K_4,\boldsymbol{p})$ with $\boldsymbol{p}(v_1)=(0,0)$, $\boldsymbol{p}(v_2)=(1,0)$, $\boldsymbol{p}(v_3)=(0,1)$ and $\boldsymbol{p}(v_4)=(-1,-1)$, the extended C_2^1-cofactor matrix with respect to (v_1,v_2,v_3) is

$$\begin{array}{c}\\ \\ \\ \\ \\ \\ v_1v_2\\v_1v_3\\v_1v_4\\v_2v_3\\v_2v_4\\v_3v_4\end{array}\overset{\begin{array}{cccc}\quad v_1\quad&\quad v_2\quad&\quad v_3\quad&\quad v_4\quad\end{array}}{\left[\begin{array}{ccc|ccc|ccc|ccc}1&0&0&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0&0&0\\1&0&0&-1&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&-1&0&0&0\\1&1&1&0&0&0&0&0&0&-1&-1&-1\\0&0&0&1&-1&1&-1&1&-1&0&0&0\\0&0&0&4&2&1&0&0&0&-4&-2&-1\\0&0&0&0&0&0&1&2&4&-1&-2&-4\end{array}\right]}.$$

Lemma 2.2. *Let $(G,\boldsymbol{p})$ be a framework with $|E(G)|=3|V(G)|-(6+k)$, and denote $\boldsymbol{p}(v_i)=(x_i,y_i)$ for each $v_i\in V$. Let $\tilde{C}(G,\boldsymbol{p})$ be the extended C_2^1-cofactor matrix with respect to three vertices (v_a,v_b,v_c) and suppose that $(y_a-y_b)(y_a-y_c)(y_b-y_c)\neq 0$. Then $\tilde{C}(G,\boldsymbol{p})$ is row independent if and only if* $\dim Z(G,\boldsymbol{p})=6+k$.

Proof. If $\tilde{C}(G,\boldsymbol{p})$ is row independent, then $C(G,\boldsymbol{p})$ is row independent and hence

$$\dim Z(G,\boldsymbol{p})=3|V(G)|-|E(G)|=6+k.$$

To see the converse, suppose $\dim Z(G,\boldsymbol{p})=6+k$ but $\tilde{C}(G,\boldsymbol{p})$ is row dependent. We can choose $k+1$ linearly independent C_2^1-motions $\boldsymbol{q}_0,\boldsymbol{q}_1,\ldots,\boldsymbol{q}_k$ of $(G,\boldsymbol{p})$ from the kernel of $\tilde{C}(G,\boldsymbol{p})$. Then each $\boldsymbol{q}_i$ satisfies (4). Hence, by Lemma 2.1, $\boldsymbol{q}_0\in\langle\boldsymbol{q}_1,\ldots,\boldsymbol{q}_k\rangle$, which is a contradiction. □

2.5 Roadmap for the proof of Theorem 1.2

The proof of Theorem 1.2 is structured as follows:

$$\text{Theorem 1.2} \overset{\text{Sec 3.2}}{\Longleftarrow} \text{Theorem 3.6} \overset{\text{Sec 3.3}}{\Longleftarrow} \text{Theorem 3.3} \overset{\text{Sec 4}}{\Longleftarrow} \text{Theorem 4.6} \overset{\text{Sec 5}}{\Longleftarrow} \text{Theorem 5.8} \overset{\text{Sec 6}}{\Longleftarrow} \text{Theorem 6.1}.$$

Below we briefly explain each theorem and step.

Section 3.2: It is known that the edge set of K_5 is a circuit in any abstract 3-rigidity matroid (see, e.g., [7] or Theorem 3.4). A matroid satisfying this circuit property is called a K_5-matroid. The class of K_5-matroids is slightly larger than the set of abstract 3-rigidity matroids. Theorem 3.6 states that the generic C_2^1-cofactor matroid $\mathcal{C}_2^1(K_n)$ is the unique maximal matroid in the poset of all K_5-matroids on $E(K_n)$. Thus Theorem 3.6 is a stronger version of Theorem 1.2.

Section 3.3: Graver [6] proved that the generic 2-dimensional rigidity matroid is the unique maximal abstract 2-rigidity matroid based on that fact that any base of the generic 2-dimensional rigidity matroid can be constructed from K_3 by a sequence of two simple graph operations, called the Henneberg constructions. Following the same approach as that of Graver [6], we show that Theorem 3.6 will follow from an inductive construction of the bases of the $\mathcal{C}_2^1$-cofactor matroid. For this purpose, Whiteley [21] already verified that various graph operations preserve C_2^1-independence. However, there remained one operation, double V-replacement, that Whiteley could not show preserved independence. (See Figure 3 for an example of double V-replacement.). We confirm that it does preserve independence in Theorem 3.3. The remainder of the paper is devoted to proving this result.

Section 4: By combining existing results from [21] and a simple combinatorial argument in Section 4, we prove that Theorem 3.3 holds in all but one special case. This remaining special case is when $\mathrm{cl}(E(G-v_0)) \cap K(N_G(v_0))$ forms a star on five vertices as shown in Figure 4, where v_0 denotes the vertex created by the double V-replacement. We formulate a new result, Theorem 4.6, to deal with this remaining special case.

Section 5: In Section 5, we first show that, if the statement of Theorem 4.6 fails, then the framework created by the double V-replacement operation has a very special C_2^1-motion. Theorem 5.8 states that such a motion cannot exist in any generic 1-dof framework.

Section 6: We prove Theorem 5.8. Our inductive proof requires a more general inductive statement, Theorem 6.1, which concerns frameworks with at most two degrees of freedom. The proof is rather long so we first outline the main ideas in Section 6.2.

3 Inductive Construction and Proof of Theorem 1.2

3.1 Inductive Construction

Our proof of Theorem 1.2 uses the following 3-dimensional versions of standard graph operations from rigidity theory. Given a graph H:

- the 0-*extension* operation adds a new vertex v and three new edges from v to vertices in H;
- the 1-*extension* operation chooses an edge e of H and adds a new vertex v and four new edges from v to vertices in $H-e$ with the proviso that two of the new edges join v to the end-vertices of e;
- the *X-replacement* operation chooses two non-adjacent edges e and f of H and adds a new vertex v and five new edges from v to vertices in $H-\{e,f\}$ with the proviso that four of the new edges join v to the end-vertices of e and f;

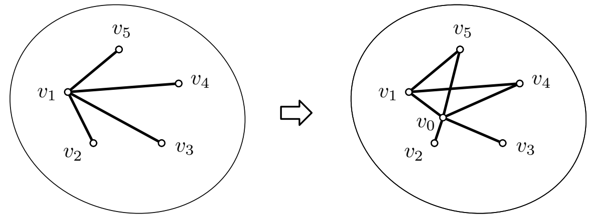


Figure 2: An extension operation which can be considered as a special case of both V-replacement and vertex splitting.

- the *V-replacement* operation chooses two adjacent edges e and f of H and adds a new vertex v and five new edges from v to vertices in $H-\{e,f\}$ with the proviso that three of the new edges join v to the end-vertices of e and f;
- the *vertex splitting* operation chooses a vertex u of H and pairwise disjoint sets U_1,U_2,U_3 with $U_1\cup U_2\cup U_3 = N_H(u)$ and $|U_2|=2$, deletes all edges from u to U_3, and then adds a new vertex v and $|U_3|+3$ new edges from v to each vertex in $U_2\cup U_3\cup\{u\}$.

Whiteley [21, Lemmas 10.1.5 and 10.2.1, Theorems 10.2.7 and 10.3.1] showed that all but one of these operations preserve C_2^1-independence:

Lemma 3.1. *The* 0*-extension,* 1*-extension, X-replacement, and vertex splitting operations preserve* C_2^1*-independence.*

In general V-replacement may not preserve C_2^1-independence, but there is an important special case when it does.

Lemma 3.2. *Suppose that* v_1,v_2,v_3,v_4,v_5 *are vertices of a* C_2^1*-independent graph* $H=(V,E)$*,* $e=v_1v_2, f=v_1v_3$ *are edges of* H *and* v_1v_4,v_1v_5 *belong to the closure of* $E-e-f$ *in the* C_2^1*-cofactor matroid on* $K(V)$*. Then the graph* G *obtained from* $H-e-f$ *by adding a new vertex* v *and new edges* vv_1,vv_2,vv_3,vv_4,vv_5 *is* C_2^1*-independent.*

Proof. Let cl(.) denote the closure operator in the C_2^1-cofactor matroid on $K(V)$. Since $v_1v_4, v_1v_5\in \mathrm{cl}(E-e-f)$, we may choose a base B of $\mathrm{cl}(E-e-f)$ with $v_1v_4,v_1v_5\in B$. Then the graph $H'=(B,B+e+f)$ is C_2^1-independent and we can now apply the vertex splitting operation at v_1 to deduce that the graph G' obtained from $H'-e-f$ by adding a new vertex v and new edges vv_1,vv_2,vv_3,vv_4,vv_5 is C_2^1-independent, see Figure 2. This and the fact that $\mathrm{cl}(B)=\mathrm{cl}(E-e-f)$ imply that G is C_2^1-independent. □

Lemma 3.2 will be used several times in our proofs

Whiteley [21] conjectured that another special case of V-replacement preserves C_2^1-independence: if $H\cup\{e_1,e_2\}$ and $H\cup\{e_1',e_2'\}$ are both C_2^1-independent for two pairs of adjacent edges e_1,e_2 and e_1',e_2' with the property that the common endvertex of e_1,e_2 is distinct from that of e_1',e_2', then the graph G obtained by adding a new vertex v_0 of degree five to H in such a way that the endvertices of e_1,e_2,e_1',e_2' are all neighbours of v_0, is C_2^1-independent. See Figure 3. This operation is referred to as *double V-replacement* since G can be constructed from both $G-v_0+e_1+e_2$ and $G-v_0+e_1'+e_2'$ by a V-replacement. We will verify Whiteley's conjecture:

Theorem 3.3. *The double V-replacement operation preserves* C_2^1*-independence.*

Lemma 3.1 and Theorem 3.3 imply that we can construct all minimally C_2^1-rigid graphs from copies of K_4 with a 'Henneberg tree construction' using the 0-, 1-, X- and double V-extension operations. We refer the reader to [18, 21] for more details.

The proof of Theorem 3.3 will be spread over Sections 4, 5 and 6 of this paper. To motivate this rather long and technical proof we show in the remainder of this section how (a generalisation of) our main result, Theorem 1.2, can be deduced easily from Lemma 3.1 and Theorem 3.3.

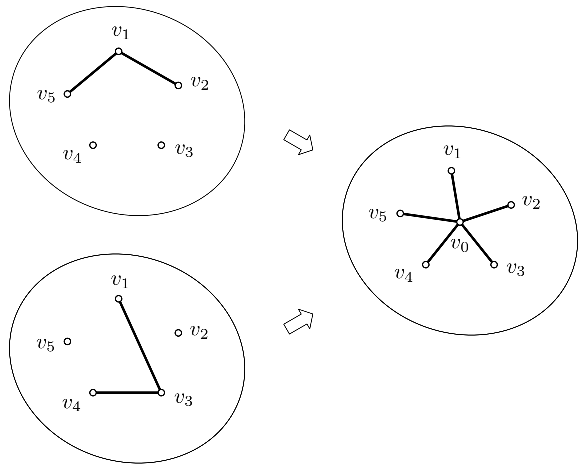


Figure 3: An example of double V-replacement.

3.2 K_t-matroids

Let n and t be positive integers with $t \leq n$. A matroid $\mathcal{M}$ on the edge set of a complete graph K_n is said to be a *K_t-matroid* if the edge set of every copy of K_t in K_n is a circuit in $\mathcal{M}$. The following simple characterization of abstract d-rigidity matroids due to Nguyen [13, Theorem 2.2] implies that every abstract d-rigidity matroid is a K_{d+2}-matroid.

Theorem 3.4. *Let n,d be positive integers with $n \geq d+2$ and $\mathcal{M}$ be a matroid on $E(K_n)$. Then $\mathcal{M}$ is an abstract d-rigidity matroid if and only if $\mathcal{M}$ is a K_{d+2}-matroid with rank $dn - \binom{d+1}{2}$.*

We will prove a stronger version of Theorem 1.2 by showing that the C_2^1-cofactor matroid is the unique maximal matroid in the family of K_5-matroids. (This strengthening will be needed in our companion paper [3].)

Our next lemma shows that $dn - \binom{d+1}{2}$ is an upper bound on the rank of any K_{d+2}-matroid on $E(K_n)$ when n is sufficiently large.

Lemma 3.5. *Suppose that $\mathcal{M}$ is a K_{d+2}-matroid on $E(K_n)$ for $n \geq d+2$. Then $\mathcal{M}$ has rank at most $dn - \binom{d+1}{2}$.*

Proof. We proceed by induction on n. The claim is trivial when $n = d+2$.

Suppose $n > d+2$, and denote the vertex set of K_n by $\{v_1, \ldots, v_n\}$. Let K be the edge set of the complete graph on $\{v_1, \ldots, v_{n-1}\}$. Since $\mathcal{M}|_K$ is a K_{d+2}-matroid, the rank of K in $\mathcal{M}$ is at most $d(n-1) - \binom{d+1}{2}$ by induction. Let $K' = K + \{v_1v_n, v_2v_n, \ldots, v_dv_n\}$. We show K' spans $E(K_n)$.

For each $i = d+1, \ldots, n-1$, let C_i be the edge set of the complete graph on $\{v_1, v_2, \ldots, v_{d-1}, v_d, v_n, v_i\}$. Note that K' contains all edges of C_i except v_iv_n. Since $\mathcal{M}$ is a K_{d+2}-matroid, v_iv_n is spanned by K'. Hence K' spans $E(K_n)$, and the rank of $\mathcal{M}$ is at most $|K'| = |K| + d = dn - \binom{d+1}{2}$. □

Theorem 3.4 and Lemma 3.5 imply that an abstract d-rigidity matroid is a K_{d+2}-matroid which attains the maximum possible rank.

3.3 Maximality of the generic C_2^1-cofactor matroid

We show that $\mathcal{C}_{2,n}^1$ is the unique maximal element, with respect to the weak order, in the set of all K_5-matroids on $E(K_n)$. Since every abstract 3-rigidity matroid is a K_5-matroid, this immediately implies Theorem 1.2.

Theorem 3.6. *The generic C_2^1-cofactor matroid $\mathcal{C}_{2,n}^1$ is the unique maximal K_5-matroid on $E(K_n)$ for all $n \geq 1$.*

Proof. The theorem is trivially true when $n \leq 5$ so we may assume that $n \geq 6$. Let $\mathcal{M}$ be an arbitrary K_5-matroid on $E(K_n)$ and $F \subseteq E(K_n)$ be an independent set in $\mathcal{M}$. We will show that F is independent in $\mathcal{C}^1_{2,n}$ by induction on $|F|$. We will abuse notation throughout this proof and use the same letter for both a subgraph of K_n and its edge set.

Since $\mathcal{M}$ is a K_5-matroid, Lemma 3.5 implies that $|F| = r(F) \leq 3|V(F)| - 6$, and hence there exists a vertex v_0 of degree at most five in F. Let $N_F(v_0) = \{v_1, \ldots, v_k\}$ (where k is the degree of v_0) and let K be the complete graph on $N_F(v_0)$.

Suppose $d_F(v_0) \leq 3$. Since $F - v_0$ is independent in $\mathcal{M}$, $F - v_0$ is independent in $\mathcal{C}^1_{2,n}$ by induction. Then F is independent in $\mathcal{C}^1_{2,n}$ by the 0-extension property in Lemma 3.1. Hence we may suppose $d_F(v_0) \in \{4,5\}$.

Claim 3.7. $\mathrm{cl}_{\mathcal{M}}(F - v_0) \cap K$ *does not contain a copy of* K_4.

Proof. Suppose, for a contradiction, that $\mathrm{cl}_{\mathcal{M}}(F - v_0)$ contains a copy of K_4. We may assume that $\mathrm{cl}_{\mathcal{M}}(F - v_0)$ contains $K(v_1, v_2, v_3, v_4)$. Let $e_i = v_0 v_i$ for $1 \leq i \leq 4$. Since $\mathcal{M}$ is a K_5-matroid, $e_4 \in \mathrm{cl}_{\mathcal{M}}(F - v_0 + e_1 + e_2 + e_3)$. This contradicts the fact that F is independent in $\mathcal{M}$. □

Suppose that $d_F(v_0) = 4$. Claim 3.7 implies that $F - v_0 + v_1 v_2$ is independent in $\mathcal{M}$ for two non-adjacent neighbors $v_1 v_2$ of v_0 in F. We can now apply induction to deduce that $F - v_0 + v_1 v_2$ is independent in $\mathcal{C}^1_{2,n}$ and then use the 1-extension property in Lemma 3.1 to deduce that F is independent in $\mathcal{C}^1_{2,n}$. Hence we may assume that $d_F(v_0) = 5$.

Claim 3.8. $K \not\subseteq \mathrm{cl}_{\mathcal{M}}(F - v_0 + e)$ *for all* $e \in K$.

Proof. Suppose, for a contradiction, that $K \subseteq \mathrm{cl}_{\mathcal{M}}(F - v_0 + e)$ for some $e \in K$. Then $F - v_0 + e$ is independent in $\mathcal{M}$ by Claim 3.7. Combining this with the fact $\mathcal{M}$ is a K_5-matroid, we obtain $r_{\mathcal{M}}(F) \leq r_{\mathcal{M}}(F - v_0 + e) + 3$. On the other hand, the fact that F is independent in $\mathcal{M}$ tells us that $r_{\mathcal{M}}(F) = r_{\mathcal{M}}(F - v_0) + 5 = r_{\mathcal{M}}(F - v_0 + e) + 4$. This is a contradiction. □

Claims 3.7 and 3.8 imply that we can choose $e_1, e_2 \in K \setminus F$ such that $(F - v_0) + e_1 + e_2$ is independent in $\mathcal{M}$. The induction hypothesis now tells us that $(F - v_0) + e_1 + e_2$ is independent in $\mathcal{C}^1_{2,n}$. If e_1, e_2 have no common endvertices, then F can be obtained from $F - v_0 + e_1 + e_2$ by X-replacement and F would be independent in $\mathcal{C}^1_{2,n}$ by Lemma 3.1, a contradiction. Hence we may suppose that v_1 is a common endvertex of e_1, e_2. Claim 3.7 now tells us that $F - v_0 + e'_1$ is independent in $\mathcal{M}$ for some $e'_1 \in K(v_2, v_3, v_4, v_5) \setminus F$, and Claim 3.8 in turn gives an edge $e'_2 \in K \setminus F$ such that $F - v_0 + e'_1 + e'_2$ is independent in $\mathcal{M}$. Then $F - v_0 + e'_1 + e'_2$ is independent in $\mathcal{C}^1_{2,n}$ by induction. Since F can be obtained from $F - v_0 + e_1 + e_2$ and $F - v_0 + e'_1 + e'_2$ by a double V-replacement, Theorem 3.3 tells us that F is independent in $\mathcal{C}^1_{2,n}$. This contradiction completes the proof. □

4 Double V-replacement

The remainder of the paper is dedicated to proving Theorem 3.3. Since we will only be concerned with the C^1_2-cofactor matroid we will often suppress specific mention of this matroid. In particular we will say that a set of edges $F \subseteq E(K_n)$ is *independent* if it is independent in $\mathcal{C}^1_{2,n}$ and use $\mathrm{cl}(F)$ to denote the closure of F in $\mathcal{C}^1_{2,n}$. We will continue to use the terms C^1_2-independent and C^1_2-rigid for graphs.

In this section, we will formulate a special case of Theorem 3.3, Theorem 4.6 below, and then show that the general result will follow from this special case. The proof of Theorem 4.6 is delayed until Sections 5 and 6.

We first need to establish a structural result for C^1_2-independent graphs.

Lemma 4.1. *Let* $H = (V, E)$ *be a* C^1_2*-independent graph,* $U = \{v_1, \ldots, v_5\}$ *be a set of five vertices in G, and* $K = K(U)$*. Then at least one of the following holds:*

(i) $\mathrm{cl}(E) \cap K$ *contains a copy of* K_4*.*

(ii) $K \subseteq \mathrm{cl}(E+e)$ *for some edge* $e \in K$.

(iii) *there are two non-adjacent edges* e_1 *and* e_2 *in* K *such that* $H+e_1+e_2$ *is* C_2^1*-independent.*

(iv) *there are two adjacent edges* e_1 *and* e_2 *in* K *such that* $H+e_1+e_2$ *is* C_2^1*-independent and the common end-vertex of* e_1 *and* e_2 *is incident to two edges in* $\mathrm{cl}(E)\cap K$.

(v) $\mathrm{cl}(E)\cap K$ *forms a star on five vertices, and* $\mathrm{cl}(E+e)\cap K$ *contains no copy of* K_4 *for all* $e \in K$.

Proof. We assume that (i) - (iv) do not hold and prove that (v) must hold.

Since (iii) does not hold, we have:

$$\text{for all } e = v_iv_j \in K\setminus \mathrm{cl}(E),\ \mathrm{cl}(E+e) \text{ contains a triangle on } U\setminus\{v_i,v_j\}. \tag{5}$$

Claim 4.2. *For all* $e \in K$, $\mathrm{cl}(E+e)\cap K$ *contains no copy of* K_4.

Proof. Since (i) does not occur, the claim follows if $e \in \mathrm{cl}(E)$. Thus we suppose that $e \notin \mathrm{cl}(E)$, and hence $E+e$ is independent.

Relabeling if necessary, we may suppose $e = v_1v_2$. By (5), $\mathrm{cl}(E+e)$ contains the triangle $K(v_3,v_4,v_5)$ on $\{v_3,v_4,v_5\}$. Since (ii) does not hold, $\mathrm{cl}(E+e)\cap K$ does not contain two copies of K_4 (since the union of two distinct copies of K_4 on U would form a C_2^1-rigid graph on U.) We consider two cases.

Case 1: Suppose that $\mathrm{cl}(E+e)\cap K$ contains a K_4 which includes $e = v_1v_2$. By symmetry we may assume that this is a K_4 on $\{v_1,v_2,v_3,v_4\}$. Since $\mathrm{cl}(E+e)\cap K$ contains $K(v_3,v_4,v_5)$ and contains at most one K_4, $v_1v_5, v_2v_5 \notin \mathrm{cl}(E+e)$. We will show that:

$$\{v_2v_3, v_2v_4\} \subset \mathrm{cl}(E). \tag{6}$$

Suppose $v_2v_3 \notin \mathrm{cl}(E)$. Since $v_2v_3 \in \mathrm{cl}(E+e)\setminus \mathrm{cl}(E)$, $E+v_2v_3$ is independent and $\mathrm{cl}(E+e) = \mathrm{cl}(E+v_2v_3)$. Since $v_1v_5 \notin \mathrm{cl}(E+e)$, this in turn implies $v_1v_5 \notin \mathrm{cl}(E+v_2v_3)$. Hence $E+v_2v_3+v_1v_5$ is independent, contradicting that (iii) does not hold. Thus $v_2v_3 \in \mathrm{cl}(E)$. The same argument for v_2v_4 implies (6).

Since $v_2v_5 \notin \mathrm{cl}(E+e)$, $E+v_1v_2+v_2v_5$ is independent. Combined with (6), this contradicts the assumption that (iv) does not hold.

Case 2: Suppose that $\mathrm{cl}(E+e)\cap K$ contains a K_4 which avoids $e = v_1v_2$. By symmetry we may assume that this is a K_4 on $\{v_1,v_3,v_4,v_5\}$. Since $\mathrm{cl}(E+e)\cap K$ contains at most one K_4, there is at most one edge from v_2 to $\{v_3,v_4,v_5\}$ in $\mathrm{cl}(E+e)\cap K$ and we may assume by symmetry that $v_2v_4, v_2v_5 \notin \mathrm{cl}(E+e)$. We will show that this case cannot occur by showing that

$$\text{all edges on } \{v_1,v_3,v_4,v_5\} \text{ are in } \mathrm{cl}(E) \tag{7}$$

and hence that (i) holds.

Suppose $v_1v_3 \notin \mathrm{cl}(E)$. Then, since $v_1v_3 \in \mathrm{cl}(E+e)\setminus \mathrm{cl}(E)$, $E+v_1v_3$ is independent and $\mathrm{cl}(E+v_1v_2) = \mathrm{cl}(E+v_1v_3)$. Since $v_2v_4 \notin \mathrm{cl}(E+v_1v_2)$, we have $v_2v_4 \notin \mathrm{cl}(E+v_1v_3)$. Hence $E+v_1v_3+v_2v_4$ is independent and (iii) holds. This contradiction implies that $v_1v_3 \in \mathrm{cl}(E)$. The same argument holds for all the other edges on $\{v_1,v_3,v_4,v_5\}$ except v_4v_5. If $v_4v_5 \notin \mathrm{cl}(E)$ then, since $v_4v_5 \in \mathrm{cl}(E+v_1v_2)\setminus \mathrm{cl}(E)$, $E+v_4v_5$ is independent and $\mathrm{cl}(E+v_1v_2) = \mathrm{cl}(E+v_4v_5)$. This in turn implies $v_2v_4 \notin \mathrm{cl}(E+v_4v_5)$, and $E+v_2v_4+v_4v_5$ is independent. This and the fact $v_1v_4, v_3v_4 \in \mathrm{cl}(E)$ contradicts the assumption that (iv) does not hold. Hence $v_4v_5 \notin \mathrm{cl}(E)$ and (7) holds. □

Claim 4.2 implies that the second part of condition (v) in the statement holds. It remains to show that $\mathrm{cl}(E)\cap K$ forms a star on U. This will follow by combining Claims 4.4 and 4.5, below. We first derive an auxiliary claim.

Claim 4.3. *Suppose that* $E+v_1v_2$ *is independent. Then* $\mathrm{cl}(E)$ *has at least two edges on* $\{v_3,v_4,v_5\}$.

Proof. Recall that $\mathrm{cl}(E+v_1v_2)$ contains a complete graph on $\{v_3,v_4,v_5\}$ by (5).

We first consider the complete graph on $\{v_1,v_2,v_3,v_4\}$. By Claim 4.2, we may assume without loss of generality that $v_1v_3 \notin \mathrm{cl}(E+v_1v_2)$. If $v_4v_5 \notin \mathrm{cl}(E)$, then $v_4v_5 \in \mathrm{cl}(E+v_1v_2)\setminus \mathrm{cl}(E)$, and so $\mathrm{cl}(E+v_4v_5) = \mathrm{cl}(E+v_1v_2)$. Hence $v_1v_3 \notin \mathrm{cl}(E+v_4v_5)$. This gives a contradiction as $E+v_4v_5+v_1v_3$ would be independent and (iii) would hold. Thus we obtain $v_4v_5 \in \mathrm{cl}(E)$.

We next consider the complete graph on $\{v_1,v_2,v_4,v_5\}$. By Claim 4.2, we may assume without loss of generality that either $v_1v_4 \notin \mathrm{cl}(E+v_1v_2)$ or $v_2v_4 \notin \mathrm{cl}(E+v_1v_2)$. If $v_3v_5 \notin \mathrm{cl}(E)$, then $v_3v_5 \in \mathrm{cl}(E+v_1v_2)\setminus \mathrm{cl}(E)$, and so $\mathrm{cl}(E+v_1v_2) = \mathrm{cl}(E+v_3v_5)$. Hence $v_1v_4 \notin \mathrm{cl}(E+v_3v_5)$ or $v_2v_4 \notin \mathrm{cl}(E+v_3v_5)$ respectively. This gives a contradiction as (iii) would hold. Thus we obtain $v_3v_5 \in \mathrm{cl}(E)$. □

Claim 4.4. *For all i with $1 \le i \le 5$, $d_{\mathrm{cl}(E)\cap K}(v_i) \ge 1$. And if $d_{\mathrm{cl}(E)\cap K}(v_i) = 1$, then the vertex v_j adjacent to v_i in $\mathrm{cl}(E)\cap K$ satisfies $d_{\mathrm{cl}(E)\cap K}(v_j) = 4$.*

Proof. Suppose that $d_{\mathrm{cl}(E)\cap K}(v_5) = 0$. Since $\mathrm{cl}(E)$ has no copy of K_4, we may assume $v_1v_2 \notin \mathrm{cl}(E)$. Then $E+v_1v_2$ is independent, and $\mathrm{cl}(E)$ has at most one edge on $\{v_3,v_4,v_5\}$ since $d_{\mathrm{cl}(E)\cap K}(v_5) = 0$. This contradicts Claim 4.3.

Suppose that $d_{\mathrm{cl}(E)\cap K}(v_5) = 1$. We may assume without loss of generality that v_5 is adjacent to v_1 in $\mathrm{cl}(E)$. Suppose further that $\mathrm{cl}(E)$ does not contain v_1v_2. Then $E+v_1v_2$ is independent. On the other hand, $\mathrm{cl}(E)$ has at most one edge on $\{v_3,v_4,v_5\}$ because v_5 is incident only to v_1 in $\mathrm{cl}(E)\cap K$. This again contradicts Claim 4.3. □

Claim 4.5. *$\mathrm{cl}(E)\cap K$ is cycle free.*

Proof. Suppose that $\mathrm{cl}(E)\cap K$ contains a cycle of length five. By Claim 4.2 we may choose two chords e_1, e_2 of this cycle such that $E+e_1+e_2$ is independent. This would contradict the assumption that (iii) and (iv) do not hold since either e_1 and e_2 are non-adjacent or their common end-vertex has degree two in $\mathrm{cl}(E)\cap K$.

Suppose that $\mathrm{cl}(E)\cap K$ contains a cycle of length four, say $v_1v_2v_3v_4v_1$. Since $\mathrm{cl}(E)\cap K$ has no K_4, we may suppose that $v_1v_3 \notin \mathrm{cl}(E)$. Then Claim 4.2 implies that $E+v_1v_3+v_2v_4$ is independent and (iii) holds, a contradiction.

Suppose that $\mathrm{cl}(E)\cap K$ contains a triangle, say $K(v_1,v_2,v_3)$. Claim 4.4 and the fact that $\mathrm{cl}(E)\cap K$ contains no cycle of length four tells us that there is exactly one edge in $\mathrm{cl}(E)\cap K$ from each of v_4, v_5 to $\{v_1,v_2,v_3\}$, and that both of these edges have a common end-vertex, say v_1. Hence $E+v_2v_4$ is independent. We can now use (5) to deduce that $v_3v_5 \in \mathrm{cl}(E+v_2v_4)$. Claim 4.2 now implies that $v_2v_5 \notin \mathrm{cl}(E+v_2v_4)$ and hence $E+v_2v_4+v_2v_5$ is independent. This gives a contradiction since v_2 has degree two in $\mathrm{cl}(E)\cap K$ and hence (iv) holds. □

Claims 4.4 and 4.5 imply that $\mathrm{cl}(E)\cap K$ forms a star on $\{v_1,v_2,\ldots,v_5\}$ and hence (v) holds. □

Let $G=(V,E)$ be a graph. A vertex v_0 in G is said to be of *type* $(\star)$ if

- v_0 has degree five with $N_G(v_0) = \{v_1,v_2,v_3,v_4,v_5\}$,
- $G-v_0+v_1v_2+v_1v_3$ and $G-v_0+v_1v_3+v_3v_4$ are both C_2^1-rigid, and
- $\mathrm{cl}(E(G-v_0))\cap K(N_G(v_0))$ forms a star on five vertices centered on v_5,

See Figure 4.

The following theorem implies that double V-replacement preserves minimal C_2^1-rigidity in the special case when the new vertex is of type $(\star)$.

Theorem 4.6. *Let $G=(V,E)$ be a graph with $|E| = 3|V|-6$ and suppose that G has a vertex v_0 of type $(\star)$. Then G is minimally C_2^1-rigid.*

As noted above, we delay the proof of this theorem to Sections 5 and 6 and instead use it to verify Theorem 3.3.

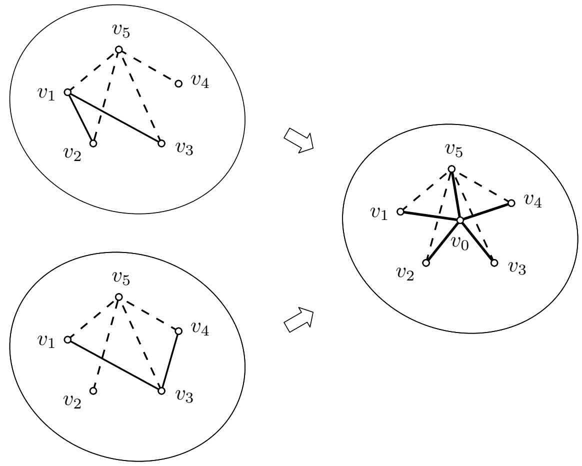


Figure 4: The double V-replacement that creates a vertex v_0 of type $(\star)$. The dashed lines represent the edges in $\mathrm{cl}(E(G-v_0)) \cap K(N_G(v_0))$, which form a star on five vertices centered on v_5.

Proof of Theorem 3.3 Let $G=(V,E)$ be a graph that has a vertex v_0 of degree five, and let $\{e_1,e_2\}$ and $\{e'_1,e'_2\}$ be two pairs of adjacent edges on $N_G(v_0)$. Suppose that $G-v_0+e_1+e_2$ and $G-v_0+e'_1+e'_2$ are both C_2^1-independent, and the common endvertex of e_1 and e_2 is distinct from that of e'_1 and e'_2. Let $N_G(v_0)=\{v_1,\dots,v_5\}$. We need to show that G is C_2^1-independent.

We apply Lemma 4.1 to $H:=G-v_0$ and $U:=N_G(v_0)$. (Note that H is C_2^1-independent.) Neither (i) nor (ii) of Lemma 4.1 hold since otherwise $G-v_0+e_1+e_2$ or $G-v_0+e'_1+e'_2$ would be dependent. If (iii) holds, then we can construct G from a C_2^1-independent graph by X-replacement, and hence G is C_2^1-independent by Lemma 3.1. Similarly, if (iv) holds, then G can be constructed from a C_2^1-independent graph by a V-replacement satisfying the hypotheses of Lemma 3.2 so is C_2^1-independent. Hence we may assume (v) of Lemma 4.1 holds, and, without loss of generality, that $\mathrm{cl}(E(H)) \cap K(U)$ forms a star whose center is v_5. Then $E(H)+v_1v_2$ and $E(H)+v_3v_4$ are both independent. If $v_3v_4 \notin \mathrm{cl}(E(H)+v_1v_2)$, then G can be obtained from the C_2^1-independent graph $H+v_1v_2+v_3v_4$ by X-replacement. Hence we may assume that $v_3v_4 \in \mathrm{cl}(E(H)+v_1v_2)$, and thus $\mathrm{cl}(E(H)+v_1v_2)=\mathrm{cl}(E(H)+v_3v_4)$ by the independence of $H+v_3v_4$. Since (v) of Lemma 4.1 holds, we may further suppose that $v_1v_3 \notin \mathrm{cl}(E(H)+v_1v_2)$. Therefore both $H+v_1v_2+v_1v_3$ and $H+v_3v_4+v_1v_3$ are C_2^1-independent.

Let $k=3|V|-6-|E|$. We complete the proof by induction on k. If $k=0$ then $H+v_1v_2+v_1v_3$ and $H+v_3v_4+v_1v_3$ are both minimally C_2^1-rigid, and v_0 is of type $(\star)$. Theorem 4.6 now implies that G is minimally C_2^1-rigid.

For the induction step, we assume that $k>0$. Since $\mathrm{cl}(E(H)+v_1v_2)=\mathrm{cl}(E(H)+v_3v_4)$, we have $\mathrm{cl}(E(H)+v_1v_2+v_1v_3)=\mathrm{cl}(E(H)+v_3v_4+v_1v_3)$. As $H+v_1v_2+v_1v_3$ is a k-dof graph with $k>0$, there is an edge $f \in K(V-v_0)$ such that $H+v_1v_2+v_1v_3+f$ and $H+v_3v_4+v_1v_3+f$ are both C_2^1-independent. Applying the induction hypothesis to $H+f$, we find that $G+f$ is C_2^1-independent. Therefore G is C_2^1-independent. □

5 Bad C_2^1-Motions

In this section, we show that if Theorem 4.6 does not hold for some graph G, then every generic realization of G in $\mathbb{R}^2$ has a special kind of motion, and derive some properties of such 'bad' motions. We use these properties in Section 6, to prove that no generic framework can have a bad motion.

The following notation will be used throughout the remainder of this paper. Suppose $p_i=(x_i,y_i)$ is a point in the plane for $i=0,1,2,3$. For any three of these points, we let

$$\Delta(p_i,p_j,p_k)=\begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{vmatrix},$$

which is twice the signed area of the triangle defined by the three points. The formula for the determinant of a Vandermonde matrix implies that

$$\begin{vmatrix} D(p_0,p_1) \\ D(p_0,p_2) \\ D(p_0,p_3) \end{vmatrix} = -\begin{vmatrix} x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}\begin{vmatrix} x_0 & y_0 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\begin{vmatrix} x_0 & y_0 & 1 \\ x_3 & y_3 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} \tag{8}$$

$$= -\Delta(p_0,p_1,p_2)\Delta(p_0,p_2,p_3)\Delta(p_0,p_3,p_1). \tag{9}$$

See, e.g., [19, page 15] for further details.

5.1 Bad local behavior

The following key lemma states that if Theorem 4.6 does not hold for some vertex v_0 of type $(\star)$ in a graph G, then every generic framework $(G,\boldsymbol{p})$ has a non-trivial C^1_2-motion $\boldsymbol{q}$ with the property that each coordinate of $\boldsymbol{q}(v_i)$, for $v_i \in \hat{N}_G(v_0)$, can be expressed as a polynomial in the coordinates of the points $\boldsymbol{p}(v_j)$, for $v_j \in \hat{N}_G(v_0)$. We note that the precise formulae of Lemma 5.1 are not important in the remaining proof, and only the fact that each coordinate is written as a polynomial in the coordinates of $\boldsymbol{p}(v_0),\boldsymbol{p}(v_1),\dots,\boldsymbol{p}(v_5)$ will be important.

Lemma 5.1. *Let $(G,\boldsymbol{p})$ be a generic framework, v_0 be a vertex of type $(\star)$ in G with $N_G(v_0)=\{v_1,\dots,v_5\}$, $H=G-v_0$ and $\boldsymbol{p}|_H$ be the restriction of $\boldsymbol{p}$ to H. Suppose that G is not C^1_2-rigid. Then there exist non-trivial C^1_2-motions $\boldsymbol{q}_1$ of $(H+v_1v_2,\boldsymbol{p}|_H)$ and $\boldsymbol{q}_2$ of $(H+v_1v_3,\boldsymbol{p}|_H)$ satisfying*

$$\begin{aligned}
&\boldsymbol{q}_1(v_1)=\boldsymbol{q}_1(v_2)=\boldsymbol{q}_1(v_5)=0,\\
&\boldsymbol{q}_1(v_3)=\Delta(\boldsymbol{p}(v_1),\boldsymbol{p}(v_2),\boldsymbol{p}(v_3))D(v_3,v_4)\times D(v_3,v_5),\\
&\boldsymbol{q}_1(v_4)=\Delta(\boldsymbol{p}(v_1),\boldsymbol{p}(v_2),\boldsymbol{p}(v_4))D(v_3,v_4)\times D(v_4,v_5),
\end{aligned}$$

and

$$\begin{aligned}
&\boldsymbol{q}_2(v_1)=\boldsymbol{q}_2(v_3)=\boldsymbol{q}_2(v_5)=0,\\
&\boldsymbol{q}_2(v_2)=\Delta(\boldsymbol{p}(v_1),\boldsymbol{p}(v_3),\boldsymbol{p}(v_2))D(v_2,v_4)\times D(v_2,v_5),\\
&\boldsymbol{q}_2(v_4)=\Delta(\boldsymbol{p}(v_1),\boldsymbol{p}(v_3),\boldsymbol{p}(v_4))D(v_2,v_4)\times D(v_4,v_5),
\end{aligned}$$

where $\times$ denotes the cross product of two vectors. Moreover, G has a non-trivial motion $\boldsymbol{q}$ such that $\boldsymbol{q}|_{V-v_0}=\alpha\boldsymbol{q}_1+\beta\boldsymbol{q}_2$, where

$$\alpha=\begin{vmatrix} D(v_0,v_2) \\ D(v_2,v_4) \\ D(v_2,v_5) \end{vmatrix}\begin{vmatrix} D(v_0,v_3) \\ D(v_0,v_1) \\ D(v_0,v_5) \end{vmatrix}, \quad \beta=-\begin{vmatrix} D(v_0,v_3) \\ D(v_3,v_4) \\ D(v_3,v_5) \end{vmatrix}\begin{vmatrix} D(v_0,v_2) \\ D(v_0,v_1) \\ D(v_0,v_5) \end{vmatrix},$$

and $\boldsymbol{q}(v_0)$ is given by

$$\boldsymbol{q}(v_0)=\Delta(\boldsymbol{p}(v_1),\boldsymbol{p}(v_2)\boldsymbol{p}(v_3))\begin{vmatrix} D(v_0,v_3) \\ D(v_3,v_4) \\ D(v_3,v_5) \end{vmatrix}\begin{vmatrix} D(v_0,v_2) \\ D(v_2,v_4) \\ D(v_2,v_5) \end{vmatrix} D(v_0,v_1)\times D(v_0,v_5).$$

Proof. For simplicity, put $D_{i,j}=D(v_i,v_j)$ and $\Delta_{i,j,k}=\Delta(\boldsymbol{p}(v_i),\boldsymbol{p}(v_j),\boldsymbol{p}(v_k))$.

Since v_0 is of type $(\star)$ in G, $(H+v_1v_2+v_1v_3,\boldsymbol{p}|_H)$ and $(H+v_1v_3+v_3v_4,\boldsymbol{p}|_H)$ are both C^1_2-rigid. Also $\mathrm{cl}(E(H))$ forms a star on $N_G(v_0)$ centered at v_5. We first prove a result concerning the C^1_2-motions of $(H,\boldsymbol{p}|_H)$.

Claim 5.2. *There exist non-trivial C^1_2-motions $\boldsymbol{q}_1$ of $(H+v_1v_2,\boldsymbol{p}|_H)$ and $\boldsymbol{q}_2$ of $(H+v_1v_3,\boldsymbol{p}|_H)$ such that:*
(a) $\boldsymbol{q}_1(v_1)=\boldsymbol{q}_1(v_2)=\boldsymbol{q}_1(v_5)=0$, $D_{3,4}\cdot[\boldsymbol{q}_1(v_3)-\boldsymbol{q}_1(v_4)]=0$, and $D_{1,3}\cdot\boldsymbol{q}_1(v_3)\neq 0$;
(b) $\boldsymbol{q}_2(v_1)=\boldsymbol{q}_2(v_3)=\boldsymbol{q}_2(v_5)=0$, $D_{2,4}\cdot[\boldsymbol{q}_2(v_2)-\boldsymbol{q}_2(v_4)]=0$, and $D_{1,2}\cdot\boldsymbol{q}_2(v_2)\neq 0$;
(c) $Z(H,\boldsymbol{p}|_H)=Z_0(H,\boldsymbol{p}|_H)\oplus\langle\boldsymbol{q}_1,\boldsymbol{q}_2\rangle$.

Proof. Since $\mathrm{cl}(E(H))$ forms a star on $N_G(v_0)$ centered at v_5, $\mathrm{cl}(E(H)+v_1v_2)$ contains the triangle $K(v_1,v_2,v_5)$. Since G contains a 1-extension of $H+v_1v_2$ and is not C_2^1-rigid, $H+v_1v_2$ is not C_2^1-rigid. Hence we can choose a non-trivial motion $\boldsymbol{q}_1$ of $(H+v_1v_2,\boldsymbol{p}|_H)$ such that $\boldsymbol{q}_1(v_1)=\boldsymbol{q}_1(v_2)=\boldsymbol{q}_1(v_5)=0$ (by pinning the triangle on $\{v_1,v_2,v_5\}$). The fact that $(H+v_1v_2+v_1v_3,\boldsymbol{p}|_H)$ is C_2^1-rigid implies that $D_{1,3}\cdot[\boldsymbol{q}_1(v_3)-\boldsymbol{q}_1(v_1)]=D_{1,3}\cdot\boldsymbol{q}_1(v_3)\neq 0$. If $D_{3,4}\cdot[\boldsymbol{q}_1(v_3)-\boldsymbol{q}_1(v_4)]\neq 0$ then $H+v_1v_2+v_3v_4$ would be C_2^1-rigid, and G would also be C_2^1-rigid since it can be obtained from $H+v_1v_2+v_3v_4$ by an X-replacement. Thus $D_{3,4}\cdot[\boldsymbol{q}_1(v_3)-\boldsymbol{q}_1(v_4)]=0$. This completes the proof of (a). Part (b) can be proved analogously.

The facts that $D_{1,2}\cdot[\boldsymbol{q}_1(v_1)-\boldsymbol{q}_1(v_2)]=0\neq D_{1,2}\cdot[\boldsymbol{q}_2(v_1)-\boldsymbol{q}_2(v_2)]$ and $D_{1,3}\cdot[\boldsymbol{q}_1(v_1)-\boldsymbol{q}_1(v_3)]\neq 0=D_{1,3}\cdot[\boldsymbol{q}_2(v_1)-\boldsymbol{q}_2(v_3)]$ imply that $\boldsymbol{q}_1,\boldsymbol{q}_2$ are linearly independent and can be extended to a base of $Z(H,\boldsymbol{p}|_H)$ by adding the vectors in a base of $Z_0(H,\boldsymbol{p}|_H)$. Hence $Z(H,\boldsymbol{p}|_H)=Z_0(H,\boldsymbol{p}|_H)\oplus\langle\boldsymbol{q}_1,\boldsymbol{q}_2\rangle$. □

Claim 5.3. *There exists a non-trivial C_2^1-motion $\boldsymbol{q}$ of $(G,\boldsymbol{p})$ such that $\boldsymbol{q}|_H=\alpha\boldsymbol{q}_1+\beta\boldsymbol{q}_2$ for some $\alpha,\beta\in\mathbb{R}$ with $\beta\neq 0$.*

Proof. Since G is not C_2^1-rigid, we may choose a non-trivial C_2^1-motion $\boldsymbol{q}$ of $(G,\boldsymbol{p})$. Since $d_G(v_0)=5$, $\boldsymbol{q}|_H$ is a non-trivial C_2^1-motion of $(H,\boldsymbol{p}|_H)$. By Claim 5.2(c), $\boldsymbol{q}|_H=\boldsymbol{q}_0+\alpha\boldsymbol{q}_1+\beta\boldsymbol{q}_2$ for some $\boldsymbol{q}_0\in Z_0(H,\boldsymbol{p}|_H)$ and $\alpha,\beta\in\mathbb{R}$. We may choose $\tilde{\boldsymbol{q}}\in Z_0(G,\boldsymbol{p})$ such that $\tilde{\boldsymbol{q}}|_H=\boldsymbol{q}_0$. Then $\hat{\boldsymbol{q}}=\boldsymbol{q}-\tilde{\boldsymbol{q}}$ is a non-trivial C_2^1-motion of $(G,\boldsymbol{p})$ and $\hat{\boldsymbol{q}}|_H=\alpha\boldsymbol{q}_1+\beta\boldsymbol{q}_2$. Since $(G+v_1v_2,\boldsymbol{p})$ can be obtained from $(H+v_1v_2+v_1v_3,\boldsymbol{p}|_H)$ by first performing a 1-extension operation and then adding the edge v_0v_5, $(G+v_1v_2,\boldsymbol{p})$ is C_2^1-rigid. This tells us that $D_{1,2}\cdot[\hat{\boldsymbol{q}}(v_1)-\hat{\boldsymbol{q}}(v_2)]\neq 0$ and hence $\beta\neq 0$. □

Let α,β be as described in Claim 5.3, and put $\gamma=\alpha/\beta$. For all $1\le i\le 5$, G contains the edge v_0v_i, and so $D_{0,i}\cdot[\boldsymbol{q}(v_0)-\boldsymbol{q}(v_i)]=0$. Since $\boldsymbol{q}(v_i)=\alpha\boldsymbol{q}_1(v_i)+\beta\boldsymbol{q}_2(v_i)$ for $1\le i\le 5$, we obtain the system of equations

$$D_{0,i}\cdot[\boldsymbol{q}(v_0)/\beta-\gamma\boldsymbol{q}_1(v_i)]=D_{0,i}\cdot\boldsymbol{q}_2(v_i)\text{ for }1\le i\le 5.$$

Putting $\boldsymbol{q}(v_0)/\beta=(a_0,b_0,c_0)$, and using the facts that $\boldsymbol{q}_1$ is zero on v_1,v_2,v_5 and $\boldsymbol{q}_2$ is zero on v_1,v_3,v_5, we may rewrite this system as the matrix equation

$$\begin{pmatrix} D_{0,1} & 0\\ D_{0,2} & 0\\ D_{0,3} & -D_{0,3}\cdot\boldsymbol{q}_1(v_3)\\ D_{0,4} & -D_{0,4}\cdot\boldsymbol{q}_1(v_4)\\ D_{0,5} & 0\end{pmatrix}\begin{pmatrix}a_0\\ b_0\\ c_0\\ \gamma\end{pmatrix}=\begin{pmatrix}0\\ D_{0,2}\cdot\boldsymbol{q}_2(v_2)\\ 0\\ D_{0,4}\cdot\boldsymbol{q}_2(v_4)\\ 0\end{pmatrix}. \tag{10}$$

Since this equation has a solution for a_0,b_0,c_0,γ we have

$$\begin{vmatrix} D_{0,1} & 0 & 0\\ D_{0,2} & 0 & D_{0,2}\cdot\boldsymbol{q}_2(v_2)\\ D_{0,3} & -D_{0,3}\cdot\boldsymbol{q}_1(v_3) & 0\\ D_{0,4} & -D_{0,4}\cdot\boldsymbol{q}_1(v_4) & D_{0,4}\cdot\boldsymbol{q}_2(v_4)\\ D_{0,5} & 0 & 0\end{vmatrix}=0. \tag{11}$$

Let $\boldsymbol{p}(v_0)=(x_0,y_0)$. The left hand side of (11) is a polynomial in x_0,y_0 with coefficients in $\mathbb{Q}(\boldsymbol{p}|_H,\boldsymbol{q}_1,\boldsymbol{q}_2)$. Since it is equal to zero for all generic values of x_0,y_0, each coefficient is zero. This implies that (11) will hold for *all* values of x_0 and y_0, including non-generic choices of $\boldsymbol{p}(v_0)$.

Claim 5.4. $D_{3,4}\cdot\boldsymbol{q}_1(v_3)=0$.

Proof. We choose $\boldsymbol{p}(v_0)$ to be a point on the interior of the line segment joining $\boldsymbol{p}(v_3)$ and $\boldsymbol{p}(v_4)$. Then $D_{0,3}$ and $D_{0,4}$ are scalar multiples of $D_{3,4}$. We can now use elementary row and column operations to convert (11) to

$$\begin{vmatrix} D_{0,1} & 0 & 0 \\ D_{0,2} & 0 & D_{0,2}\cdot \boldsymbol{q}_2(v_2) \\ D_{3,4} & D_{3,4}\cdot \boldsymbol{q}_1(v_3) & 0 \\ 0 & D_{3,4}\cdot(\boldsymbol{q}_1(v_4)-\boldsymbol{q}_1(v_3)) & D_{3,4}\cdot \boldsymbol{q}_2(v_4) \\ D_{0,5} & 0 & 0 \end{vmatrix} = 0. \tag{12}$$

Since $D_{3,4}\cdot(\boldsymbol{q}_1(v_4)-\boldsymbol{q}_1(v_3)) = 0$ by Claim 5.2(a), this gives

$$\begin{vmatrix} D_{0,1} & 0 & 0 \\ D_{0,2} & 0 & D_{0,2}\cdot \boldsymbol{q}_2(v_2) \\ D_{3,4} & D_{3,4}\cdot \boldsymbol{q}_1(v_3) & 0 \\ 0 & 0 & D_{3,4}\cdot \boldsymbol{q}_2(v_4) \\ D_{0,5} & 0 & 0 \end{vmatrix} = 0. \tag{13}$$

Hence

$$\begin{vmatrix} D_{0,1} \\ D_{0,2} \\ D_{0,5} \end{vmatrix} (D_{3,4}\cdot \boldsymbol{q}_1(v_3))\,(D_{3,4}\cdot \boldsymbol{q}_2(v_4)) = 0. \tag{14}$$

As long as $\boldsymbol{p}(v_0)$ is generic on the line through $\boldsymbol{p}(v_3)$ and $\boldsymbol{p}(v_4)$, the first term is non-zero. Since $\boldsymbol{q}_2$ is a non-trivial C_2^1-motion of $(H+v_1v_3, \boldsymbol{p}|_H)$ and $(H+v_1v_3+v_3v_4, \boldsymbol{p}|_H)$ is C_2^1-rigid, we also have $D_{3,4}\cdot \boldsymbol{q}_2(v_4) = D_{3,4}\cdot(\boldsymbol{q}_2(v_4)-\boldsymbol{q}_2(v_3)) \neq 0$. Hence (14) gives $D_{3,4}\cdot \boldsymbol{q}_1(v_3) = 0$. □

Since the edge v_3v_5 exists in $\mathrm{cl}(E(H))$ and $\boldsymbol{q}_1(v_5) = 0$, we also have

$$D_{3,5}\cdot \boldsymbol{q}_1(v_3) = 0. \tag{15}$$

Claim 5.4 and (15) imply that $\boldsymbol{q}_1(v_3)$ is a vector in the orthogonal complement of the span of $D_{3,4}$ and $D_{3,5}$. Hence, scaling $\boldsymbol{q}_1$ appropriately, we may assume that

$$\boldsymbol{q}_1(v_3) = \Delta_{1,2,3} D_{3,4}\times D_{3,5}. \tag{16}$$

By an analogous argument on the line through $\boldsymbol{p}(v_2)$ and $\boldsymbol{p}(v_4)$, we obtain

$$\boldsymbol{q}_2(v_2) = \Delta_{1,3,2} D_{2,4}\times D_{2,5}. \tag{17}$$

We next calculate $\boldsymbol{q}_1(v_4)$. Since the edge v_4v_5 exists in $\mathrm{cl}(E(H))$ and $\boldsymbol{q}_1(v_5) = 0$, we have $D_{4,5}\cdot \boldsymbol{q}_1(v_4) = 0$. Also the edge v_3v_4 exists in $\mathrm{cl}(E(H)+v_1v_2)$, since otherwise we could perform an X-replacement on $H+v_1v_2+v_3v_4$ and deduce that G is C_2^1-rigid. Hence $D_{3,4}\cdot(\boldsymbol{q}_1(v_4)-\boldsymbol{q}_1(v_3)) = 0$. Since $D_{3,4}\cdot \boldsymbol{q}_1(v_3) = 0$ by Claim 5.4, we have $D_{3,4}\cdot \boldsymbol{q}_1(v_4) = 0$. Thus $\boldsymbol{q}_1(v_4)$ is in the orthogonal complement of the span of $D_{4,5}$ and $D_{3,4}$, and $\boldsymbol{q}_1(v_4) = sD_{3,4}\times D_{4,5}$ for some $s\in\mathbb{R}$. The next claim determines the value of s:

Claim 5.5. $s = \Delta_{1,2,4}$.

Proof. We put $\boldsymbol{p}(v_0)$ on the interior of the line segment joining $\boldsymbol{p}(v_1)$ and $\boldsymbol{p}(v_2)$. Then $D_{0,1}$ is a scalar multiple of $D_{0,2}$. With this choice of $\boldsymbol{p}(v_0)$, we may expand the determinant in (11) to obtain

$$\begin{vmatrix} D_{0,1} \\ D_{0,5} \\ D_{0,3} \end{vmatrix} (D_{0,4}\cdot \boldsymbol{q}_1(v_4))(D_{0,2}\cdot \boldsymbol{q}_2(v_2)) - \begin{vmatrix} D_{0,1} \\ D_{0,5} \\ D_{0,4} \end{vmatrix} (D_{0,3}\cdot \boldsymbol{q}_1(v_3))(D_{0,2}\cdot \boldsymbol{q}_2(v_2)) = 0. \tag{18}$$

We may use the fact that $D_{1,2}\cdot \boldsymbol{q}_2(v_2) \neq 0$ by Claim 5.2(b) to deduce that $D_{0,2}\cdot \boldsymbol{q}_2(v_2) \neq 0$ and then rewrite (18) as

$$\begin{vmatrix} D_{0,1} \\ D_{0,5} \\ D_{0,3} \end{vmatrix} (D_{0,4}\cdot \boldsymbol{q}_1(v_4)) - \begin{vmatrix} D_{0,1} \\ D_{0,5} \\ D_{0,4} \end{vmatrix} (D_{0,3}\cdot \boldsymbol{q}_1(v_3)) = 0. \tag{19}$$

Using $\boldsymbol{q}_1(v_4) = sD_{3,4} \times D_{4,5}$, and substituting $\boldsymbol{q}_1(v_3)$ with (16), we obtain

$$s = \frac{\begin{vmatrix} D_{0,1} \\ D_{0,5} \\ D_{0,4} \end{vmatrix} \begin{vmatrix} D_{0,3} \\ D_{3,4} \\ D_{3,5} \end{vmatrix}}{\begin{vmatrix} D_{0,1} \\ D_{0,5} \\ D_{0,3} \end{vmatrix} \begin{vmatrix} D_{0,4} \\ D_{3,4} \\ D_{4,5} \end{vmatrix}} \Delta_{1,2,3}. \tag{20}$$

By using the identity (9) and the fact $D_{i,j} = D_{j,i}$, (20) becomes

$$s = \frac{\Delta_{0,1,5}\Delta_{0,5,4}\Delta_{0,4,1}\Delta_{3,0,4}\Delta_{3,4,5}\Delta_{3,5,0}}{\Delta_{0,1,5}\Delta_{0,5,3}\Delta_{0,3,1}\Delta_{4,0,3}\Delta_{4,3,5}\Delta_{4,5,0}} \Delta_{1,2,3} = \frac{\Delta_{0,4,1}}{\Delta_{0,3,1}} \Delta_{1,2,3}. \tag{21}$$

Finally since $\boldsymbol{p}(v_0)$ is on the interior of the line segment joining $\boldsymbol{p}(v_1)$ and $\boldsymbol{p}(v_2)$, we get

$$s = \frac{\Delta_{0,4,1}}{\Delta_{0,3,1}} \Delta_{1,2,3} = \frac{\Delta_{1,2,4}}{\Delta_{1,2,3}} \Delta_{1,2,3} = \Delta_{1,2,4}.$$ □

This completes the derivation of the formula for $\boldsymbol{q}_1$ in the statement of the lemma. The formula for $\boldsymbol{q}_2$ can be obtained using a symmetric argument by changing the role of v_2 and v_3.

It remains to compute $\boldsymbol{q}(v_0)$. By (10), $\boldsymbol{q}(v_0)/\beta$ is in the orthogonal complement of the span of $D_{0,1}$ and $D_{0,5}$. Hence $\boldsymbol{q}(v_0)/\beta = kD_{0,1} \times D_{0,5}$ for some $k \in \mathbb{R}$. By the second equation in (10), we also have $D_{0,2} \cdot \boldsymbol{q}_2(v_2) = D_{0,2} \cdot (\boldsymbol{q}(v_0)/\beta) = kD_{0,2} \cdot (D_{0,1} \times D_{0,5})$. Combining this with (17), we get

$$k = \Delta_{1,3,2} \frac{D_{0,2} \cdot (D_{2,4} \times D_{2,5})}{D_{0,2} \cdot (D_{0,1} \times D_{0,5})}. \tag{22}$$

By the third equation in (10), $(D_{0,3} \cdot \boldsymbol{q}_1(v_3))\gamma = D_{0,3} \cdot \boldsymbol{q}(v_0)/\beta = kD_{0,3} \cdot (D_{0,1} \times D_{0,5})$. Hence by (22) and (16) we get

$$\begin{aligned} \frac{\alpha}{\beta} = \gamma = \frac{kD_{0,3} \cdot (D_{0,1} \times D_{0,5})}{D_{0,3} \cdot \boldsymbol{q}_1(v_3)} &= \frac{\Delta_{1,3,2} D_{02} \cdot (D_{24} \times D_{25}) D_{03} \cdot (D_{01} \times D_{05})}{\Delta_{1,2,3} D_{03} \cdot (D_{34} \times D_{35}) D_{02} \cdot (D_{01} \times D_{05})} \\ &= -\frac{\begin{vmatrix} D_{0,2} \\ D_{2,4} \\ D_{2,5} \end{vmatrix} \begin{vmatrix} D_{0,3} \\ D_{0,1} \\ D_{0,5} \end{vmatrix}}{\begin{vmatrix} D_{0,3} \\ D_{3,4} \\ D_{3,5} \end{vmatrix} \begin{vmatrix} D_{0,2} \\ D_{0,1} \\ D_{0,5} \end{vmatrix}}. \end{aligned}$$

By scaling $\boldsymbol{q}$ appropriately, we may take

$$\alpha = \begin{vmatrix} D_{0,2} \\ D_{2,4} \\ D_{2,5} \end{vmatrix} \begin{vmatrix} D_{0,3} \\ D_{0,1} \\ D_{0,5} \end{vmatrix} \text{ and } \beta = -\begin{vmatrix} D_{0,3} \\ D_{3,4} \\ D_{3,5} \end{vmatrix} \begin{vmatrix} D_{0,2} \\ D_{0,1} \\ D_{0,5} \end{vmatrix}.$$

Since $\boldsymbol{q}(v_0) = \beta k D_{0,1} \times D_{0,5}$, we can now use (22) to obtain the formula for $\boldsymbol{q}(v_0)$ given in the statement of the lemma. □

5.2 Bad motions and proof of Theorem 4.6

Let $(G, \boldsymbol{p})$ be a generic framework and v_0 be a vertex of type $(\star)$ in G with $N_G(v_0) = \{v_1, \ldots, v_5\}$. Lemma 5.1 shows that, if G is not C_2^1-rigid, then $(G, \boldsymbol{p})$ is a 1-dof framework and has a non-trivial motion $\boldsymbol{q}$ with the property that the coordinates of $\boldsymbol{q}$ at each vertex of $\hat{N}_G(v_0)$ are three polynomials in the coordinates of

$\boldsymbol{p}(\hat{N}_G(v_0))$ with coefficients in $\mathbb{Q}$. This seems unlikely as the graph on $N_G(v_0)$ induced by $\mathrm{cl}(E(G-v_0))$ is a star on the five vertices (since v_0 is of type $(\star)$), which has a large degree of freedom as a subframework. Our goal is to give a rigorous proof that this cannot happen. The following concepts will be useful for this purpose.

Let $G=(V,E)$ be a graph with $V=\{v_1,v_2,\dots,v_n\}$. For $U\subseteq V$, let

$$b:U\to\mathbb{Q}[X_1,Y_1,X_2,Y_2,\dots,X_n,Y_n]^3$$

be a map which associates a 3-tuple b_i of polynomials in $2n$ variables to each $v_i\in U$.

For a given framework $(G,\boldsymbol{p})$ with $\boldsymbol{p}(v_i)=(x_i,y_i)$ for all $1\le i\le n$, the substitution of (X_i,Y_i) with (x_i,y_i) in b_i for all $1\le i\le n$ gives a vector $b_i(\boldsymbol{p})$ in $\mathbb{R}^3$ for each $v_i\in U$. We say that a C_2^1-motion $\boldsymbol{q}$ of $(G,\boldsymbol{p})$ is a *b-motion* if $\boldsymbol{q}(v_i)=b_i(\boldsymbol{p})$ for all $v_i\in U$.

The following lemma states that the property of having a b-motion is stable over a large set S of realisations. (In particular, it will imply that this property is generic). Since we will need the realisations in S to satisfy the hypotheses of Lemma 2.2 it will be convenient to say that the realisation $\boldsymbol{p}$ is *non-degenerate on* U if there exist three distinct vertices in U which have distinct 'y-coordinates' in $(G,\boldsymbol{p})$.

Lemma 5.6. *Let $G=(V,E)$ be a C_2^1-independent graph with $V=\{v_1,\dots,v_n\}$, $U\subseteq V$, $F\subseteq K(U)$, and $S=\{\boldsymbol{p}: (G+F,\boldsymbol{p})$ is minimally C_2^1-rigid and $\boldsymbol{p}$ is non-degenerate on $U\}$. Suppose that $b:U\to\mathbb{Q}[X_1,Y_1,\dots,X_n,Y_n]^3$ and $(G,\boldsymbol{p})$ has a b-motion for some generic $\boldsymbol{p}$. Then $(G,\boldsymbol{p})$ has a b-motion for all $\boldsymbol{p}\in S$.*

Lemma 5.6 can be understood as follows. Since each entry of $C(G,\boldsymbol{p})$ is a polynomial function of the coordinates of $\boldsymbol{p}$, we can choose a base of the space of C_2^1-motions of $(G,\boldsymbol{p})$ such that each entry of the vectors in the base is a polynomial function of the coordinates of $\boldsymbol{p}$. By the assumption of the lemma, $(G,\boldsymbol{p})$ has a b-motion for some generic $\boldsymbol{p}$, which is spanned by this base. Since this holds for some generic $\boldsymbol{p}$ and b is a polynomial function, it will hold for all generic $\boldsymbol{p}$. The statement of Lemma 5.6 follows for all generic $\boldsymbol{p}$. We need a slightly more involved argument to verify the statement for all $\boldsymbol{p}\in S$. This is given in Appendix A.

Suppose v_0 is a vertex of type $(\star)$ in a generic framework $(G,\boldsymbol{p})$ and $(G,\boldsymbol{p})$ is not C_2^1-rigid. Then Lemma 5.1 tells us that $(G,\boldsymbol{p})$ has a non-trivial motion $\boldsymbol{q}$ with the property that the value of $\boldsymbol{q}$ at each vertex of $\hat{N}_G(v_0)$ can be described by three polynomials in the coordinates of $\boldsymbol{p}(\hat{N}_G(v_0))$ with coefficients in $\mathbb{Q}$. We will see in Lemma 5.7 below that $\boldsymbol{q}$ also has the property that the graph on $N_G(v_0)$ with edge set $\{v_iv_j : D(v_i,v_j)\cdot(\boldsymbol{q}(v_i)-\boldsymbol{q}(v_j))=0\}$ is a star. We will concentrate on b-motions which share these properties.

Formally, let $(G,\boldsymbol{p})$ be a framework and v_0 be a vertex of degree five in G with $N_G(v_0)=\{v_1,v_2,\dots,v_5\}$. We say that a C_2^1-motion $\boldsymbol{q}$ of $(G,\boldsymbol{p})$ is *bad at* v_0 if $\boldsymbol{q}$ is a b-motion for some $b:\hat{N}_G(v_0)\to\mathbb{Q}[X_0,Y_0,X_1,Y_1,\dots,X_5,Y_5]^3$ for which

$$\text{the graph on } N_G(v_0) \text{ with edge set } \{v_iv_j : D_{i,j}\cdot(b(v_i)-b(v_j))=0\} \text{ is a star,} \tag{23}$$

where $D_{i,j}=((X_i-X_j)^2,(X_i-X_j)(Y_i-Y_j),(Y_i-Y_j)^2)$. (Here $D_{i,j}\cdot(b(v_i)-b(v_j))=0$ means that polynomial $D_{i,j}\cdot(b(v_i)-b(v_j))$ is identically zero.)

Our next result verifies that the motion defined in Lemma 5.1 is indeed a bad motion.

Lemma 5.7. *Let $(G,\boldsymbol{p})$ be a generic framework and v_0 be a vertex of type $(\star)$ with $N_G(v_0)=\{v_1,\dots,v_5\}$. Suppose that G is not C_2^1-rigid. Then $(G,\boldsymbol{p})$ has a bad motion at v_0.*

Proof. Motivated by Lemma 5.1, we define $b : \hat{N}_G(v_0) \to \mathbb{Q}[X_0, Y_0, X_1, \ldots, Y_5]^3$ by

$$b(v_0) = \Delta_{1,2,3} \begin{vmatrix} D_{0,3} \\ D_{3,4} \\ D_{3,5} \end{vmatrix} \begin{vmatrix} D_{0,2} \\ D_{2,4} \\ D_{2,5} \end{vmatrix} D_{0,1} \times D_{0,5}, \tag{24}$$

$$b(v_1) = 0, \tag{25}$$

$$b(v_2) = \beta \Delta_{1,3,2} D_{2,4} \times D_{2,5}, \tag{26}$$

$$b(v_3) = \alpha \Delta_{1,2,3} D_{3,4} \times D_{3,5}, \tag{27}$$

$$b(v_4) = \alpha \Delta_{1,2,4} D_{3,4} \times D_{4,5} + \beta \Delta_{1,3,4} D_{2,4} \times D_{4,5}, \tag{28}$$

$$b(v_5) = 0; \tag{29}$$

where $\Delta_{i,j,k} = \begin{vmatrix} X_i & Y_i & 1 \\ X_j & Y_j & 1 \\ X_k & Y_k & 1 \end{vmatrix}$, $\alpha = \begin{vmatrix} D_{0,2} \\ D_{2,4} \\ D_{2,5} \end{vmatrix} \begin{vmatrix} D_{0,3} \\ D_{0,1} \\ D_{0,5} \end{vmatrix}$, $\beta = -\begin{vmatrix} D_{0,3} \\ D_{3,4} \\ D_{3,5} \end{vmatrix} \begin{vmatrix} D_{0,2} \\ D_{0,1} \\ D_{0,5} \end{vmatrix}$ are regarded as polynomials in $X_0, Y_0, X_1, \ldots, Y_5$. Then, by Lemma 5.1, $(G, \boldsymbol{p})$ has a b-motion at v_0.

It remains to show that b satisfies (23) symbolically. This can be done by a hand computation using the explicit formulae, see Appendix B for the details. □

Our next result asserts that no generic framework with one degree of freedom can have a bad motion.

Theorem 5.8. *Let $(G, \boldsymbol{p})$ be a generic 1-dof framework and v_0 be a vertex of degree five. Then $(G, \boldsymbol{p})$ has no bad motion at v_0.*

We will delay the proof of Theorem 5.8 until Section 6. Instead we show how it can be used to deduce Theorem 4.6.

Proof of Theorem 4.6. Suppose the theorem is false. Let G be a counterexample and $\boldsymbol{p}$ be a generic realisation of G. Then Lemma 5.7 implies that $(G, \boldsymbol{p})$ has a bad motion at v_0 and this contradicts Theorem 5.8. (Note that $(G, \boldsymbol{p})$ is a 1-dof framework since v_0 is type $(\star)$.) □

5.3 Projective transformations

Recall that each point $p = (x, y) \in \mathbb{R}^2$ can be associated to a point $p^\uparrow = [x, y, 1]^T$ in 2-dimensional real projective space. A map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a *projective transformation* if there exists a 3×3 non-singular matrix M such that, for all $p \in \mathbb{R}^2$, $f(p)^\uparrow = \lambda_p M p^\uparrow$ for some non-zero $\lambda_p \in \mathbb{R}$. We say that a framework $(G, \boldsymbol{p}')$ is a *projective image* of a framework $(G, \boldsymbol{p})$ if there is a projective transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\boldsymbol{p}'(v_i) = f(\boldsymbol{p}(v_i))$ for all $v_i \in V(G)$. Whiteley [21, Theorem 11.3.2.] showed that C_2^1-rigidity is invariant under projective transformations. We shall use the following lemma, which implies that a certain projective transformation preserves the existence of a bad motion, to simplify our calculations in the proof of Theorem 5.8. Its statement implicitly uses the fact that, for any two ordered sets S and T of four points in general position in $\mathbb{R}^2$, there is a unique projective transformation that maps S onto T.

Lemma 5.9. *Let $(G, \boldsymbol{p})$ be a generic framework, v_0 be a vertex of degree five with $N_G(v_0) = \{v_1, \ldots, v_5\}$ and $(G, \boldsymbol{p}')$ be a projective image of $(G, \boldsymbol{p})$ such that $\boldsymbol{p}'(v_1) = (1, 0)$, $\boldsymbol{p}'(v_2) = (0, 0)$, $\boldsymbol{p}'(v_3) = (0, 1)$, and $\boldsymbol{p}'(v_4) = (1, 1)$. If $(G, \boldsymbol{p}')$ has a bad motion at v_0, then $(G, \boldsymbol{p})$ has a bad motion at v_0.*

Lemma 5.9 will follow from the fact that the entries in the matrix which defines the projective transformation given in the lemma are determined by the coordinates of $\boldsymbol{p}(v_i)$ for $1 \le i \le 4$. However, a rigorous proof seems to require a precise understanding of how the projective transformation changes the motion space of the framework. Whiteley's proof of projective invariance does not provide this as it uses a self-stress argument. We develop a new proof technique for the projective invariance of C_2^1-rigidity in Appendix C, and then use it to derive Lemma 5.9.

6 Proof of Theorem 5.8

6.1 Locally k-dof parts

Let $G=(V,E)$ be a graph. A set $X \subseteq V$ is said to be a *locally C_2^1-rigid part* in G if every edge in $K(X)$ is in $\mathrm{cl}(E(G))$. The set X is said to be a *locally k-dof part* in G if there is a set F of k edges in $K(V)$ such that X is a locally C_2^1-rigid part in $G+F$, but no smaller edge set has this property.

Our inductive proof of Theorem 5.8 requires the following more general inductive hypothesis.

Theorem 6.1. *Let $(G,\boldsymbol{p})$ be a generic framework and v_0 be a vertex of degree five such that $\hat{N}_G(v_0)$ is a locally k-dof part in G with $k=1$ or $k=2$. Then $(G,\boldsymbol{p})$ has no bad motion at v_0.*

We give an outline of the proof of Theorem 6.1 in Section 6.2, and delay the full proof to Section 6.3.

6.2 Outline of the proof of Theorem 6.1

Theorem 6.1 is proved by contradiction. We choose a counterexample $(G,\boldsymbol{p})$ with $G=(V,E)$ and $|V|$ as small as possible and, subject to this condition, k as small as possible. The proof proceeds as follows.

Section 6.3.1. The first step is to prove that G is a 2-dof graph (Claim 6.6) and G has a vertex u_0 of degree five with $u_0 \notin \hat{N}_G(v_0)$ (Claim 6.9).

Sections 6.3.2 and 6.3.3. Let $G_0=(V_0,E_0)=G-u_0$. We prove statement (46): $\mathrm{cl}(E_0)\cap K(N_G(u_0))$ is a star on five vertices. The proof of (45) proceeds as in that of Lemma 4.1, but to apply a similar argument we need to prepare several preliminary claims in Section 6.3.2.

Section 6.3.4. The final goal of the proof is to contradict the minimal choice of $(G,\boldsymbol{p})$ by showing that $(G_0+e_1+e_2,\boldsymbol{p}|_{V_0})$ is a 2-dof framework with a bad motion at v_0 for two edges $e_1,e_2\in K(N_G(u_0))$. To do this we prove the existence of distinct edges $f_1,f_2\in K(N_G(v_0))$ such that $G+f_1+f_2$ is C_2^1-rigid and distinct edges $e_{j,1},e_{j,2}\in K(N_G(u_0))$ for $j=1,2,3$ such that $(G_0+f_i+e_{j,1}+e_{j,2},\boldsymbol{p})$ has a non-trivial motion $\boldsymbol{q}_j^i$ for all $i=1,2$ and $j=1,2,3$ so that $Z(G_0+f_i,\boldsymbol{p}|_{V_0})=Z_0(G_0+f_i,\boldsymbol{p}|_{V_0})\oplus\langle \boldsymbol{q}_1^i,\boldsymbol{q}_2^i,\boldsymbol{q}_3^i\rangle$.

Section 6.3.5. Let $\boldsymbol{q}^i$ be a nontrivial motion of $(G+f_i,\boldsymbol{p})$ for $i=1,2$. Since $G+f_1+f_2$ is C_2^1-rigid, we have $Z(G,\boldsymbol{p})=Z_0(G,\boldsymbol{p})\oplus\langle \boldsymbol{q}^1,\boldsymbol{q}^2\rangle$. By $Z(G_0+f_i,\boldsymbol{p}|_{V_0})=Z_0(G_0+f_i,\boldsymbol{p}|_{V_0})\oplus\langle \boldsymbol{q}_1^i,\boldsymbol{q}_2^i,\boldsymbol{q}_3^i\rangle$, we may suppose (by subtracting a suitable trivial motion) that, for both $i=1,2$, $\boldsymbol{q}^i|_{V_0}=\sum_{j=1}^3\alpha_j^i\boldsymbol{q}_j^i$ for some scalars α_j^i.

Since $\mathrm{cl}(E_0)\cap K(N_G(u_0))$ forms a star on five vertices, we can use the proof technique of Lemma 5.1 to derive an explicit formula for the scalars α_j^i.

Section 6.3.6. The original framework $(G,\boldsymbol{p})$ has a bad motion $\boldsymbol{q}_{\mathrm{bad}}$ at v_0, and its restriction to V_0 is a motion of $(G_0,\boldsymbol{p}|_{V_0})$. Since $Z(G,\boldsymbol{p})=Z_0(G,\boldsymbol{p})\oplus\langle\boldsymbol{q}^1,\boldsymbol{q}^2\rangle$, we have

$$\boldsymbol{q}_{\mathrm{bad}}|_{\hat{N}(v_0)}\in\left\langle \sum_{j=1}^3\alpha_j^1\,\boldsymbol{q}_j^1|_{\hat{N}(v_0)},\sum_{j=1}^3\alpha_j^2\,\boldsymbol{q}_j^2|_{\hat{N}(v_0)}\right\rangle.$$

We can use the explicit formulae for the scalars α_j^i derived in Section 6.3.5 and the facts that the entries of $\boldsymbol{q}_{\mathrm{bad}}|_{\hat{N}(v_0)}$ are contained in $\mathbb{Q}(\boldsymbol{p}(V_0))$ and $p(u_0)$ is generic over $\mathbb{Q}(\boldsymbol{p}(V_0))$ to show that

$$\boldsymbol{q}_{\mathrm{bad}}|_{\hat{N}(v_0)}\in\left\langle \boldsymbol{q}_j^1|_{\hat{N}(v_0)},\boldsymbol{q}_j^2|_{\hat{N}(v_0)}\right\rangle$$

for some $j\in\{1,2,3\}$. Since any linear combination of $\boldsymbol{q}_j^1$ and $\boldsymbol{q}_j^2$ is a motion of $(G_0+e_{j,1}+e_{j,2},\boldsymbol{p}|_{V_0})$, this implies that $(G_0+e_{j,1}+e_{j,2},\boldsymbol{p}|_{V_0})$ has a bad motion at v_0. This contradiction to the minimal choice of $(G,\boldsymbol{p})$ completes the proof.

6.3 Proof of Theorem 6.1

We assume the statement of Theorem 6.1 is false and choose a counterexample $(G,\boldsymbol{p})$ with $G=(V,E)$ and $v_0\in V$ such that the G is minimal under the lexicographic ordering given by $(|V|,k,\text{dof }G,|E|)$. Then $(G,\boldsymbol{p})$ has a bad motion at v_0, $d_G(v_0)=5$ and $\hat{N}_G(v_0)$ is a locally k-dof part in G.

6.3.1 Preliminary results on the structure of G

The first step of the proof is to show that G is a C_2^1-independent, 2-dof graph (statement (31) and Claims 6.2 and 6.6) and G has a vertex u_0 of degree five with $u_0\notin\hat{N}_G(v_0)$ (Claim 6.9).

Let $N_G(v_0)=\{v_1,\dots,v_5\}$ and denote for simplicity $K_v=K(N_G(v_0))$. Since $(G,\boldsymbol{p})$ is a counterexample, $(G,\boldsymbol{p})$ has a b-motion $\boldsymbol{q}_{\text{bad}}$ for some $b:\hat{N}_G(v_0)\to\mathbb{Q}[X_0,Y_0,\dots,Y_5]$ which satisfies (23). Relabeling $N_G(v_0)$ if necessary, we may suppose that, for all $v_i,v_j\in N_G(v_0)$ with $i<j$,

$$D(v_i,v_j)\cdot(\boldsymbol{q}_{\text{bad}}(v_i)-\boldsymbol{q}_{\text{bad}}(v_j))=0 \text{ holds if and only if } j=5. \tag{30}$$

By (30), $v_iv_j\notin\text{cl}(E)$ for all $1\le i<j\le 4$. If $v_iv_5\notin\text{cl}(E)$ for some $1\le i\le 4$, then again by (30), $(G+v_iv_5,\boldsymbol{p})$ would be a $(k-1)$-dof framework having the bad motion $\boldsymbol{q}_{\text{bad}}$, which would contradict either the fact that $(G+v_iv_5,\boldsymbol{p})$ is C_2^1-rigid (when $k=1$) or the minimality of k (when $k=2$). Hence, for all $v_i,v_j\in N_G(v_0)$, we have $v_iv_j\in\text{cl}(E)$ if and only if $j=5$. Then $\text{cl}(E)\cap K_v$ induces a star on five vertices so is C_2^1-independent. This implies that we may choose a base E' of $\text{cl}(E)$ in $\mathcal{C}_2^1$ with $\text{cl}(E)\cap K_v\subseteq E'$ and $N_E(v_0)=N_{E'}(v_0)$. Let G' be the subgraph of $K(V)$ induced by E'. Then $(G,\boldsymbol{p})$ and $(G',\boldsymbol{p})$ have the same space of motions. The minimality of $|E|$ now implies that $|E|=|E'|$ and G is C_2^1-independent. Replacing G by G' if necessary, we may assume that

$$G \text{ is } C_2^1\text{-independent and } E\cap K_v=\text{cl}(E)\cap K_v \text{ is a star on five vertices centred on } v_5. \tag{31}$$

Claim 6.2. *G is a k-dof graph.*

Proof. Suppose that G has k' degrees of freedom. We show that there exists a set of $(k'-k)$ edges D of $K(V-v_0)$ such that $G+D$ is a k-dof graph, $X:=\hat{N}_G(v_0)$ is a locally k-dof part in $G+D$ and, for any generic framework $(G,\boldsymbol{p})$ and any motion $\boldsymbol{q}$ of $(G,\boldsymbol{p})$, $(G+D,\boldsymbol{p})$ has a motion $\boldsymbol{q}'$ satisfying $\boldsymbol{q}'(v)=\boldsymbol{q}(v)$ for all $v\in X$.

Let F be a set of k edges in $K(V)$ such that X is a locally C_2^1-rigid part in $G+F$, and D be a minimal set of edges in $K(V-v_0)$ such that $G+F+D$ is C_2^1-rigid. (Note that D exists since $d_G(v_0)=5$.) For each $e\in F\cup D$, $G+F+D-e$ is a 1-dof graph and we can choose a non-trivial C_2^1-motion $\boldsymbol{q}_e$ of $(G+F+D-e,\boldsymbol{p})$. Then $B_1:=\{\boldsymbol{q}_i^*:1\le i\le 6\}\cup\{\boldsymbol{q}_e:e\in F\cup D\}$ is a base for $Z(G,\boldsymbol{p})$ and $B_2:=\{\boldsymbol{q}_i^*:1\le i\le 6\}\cup\{\boldsymbol{q}_e:e\in F\}$ is a base for $Z(G+D,\boldsymbol{p})$, where $\{q_i^*:1\le i\le 6\}$ is the basis for $Z_0(G,p)$ given in Section 2.2. The facts that the motions in B_1 are linearly independent and that, for each $e\in F$, we have $D(u,v)\cdot(\boldsymbol{q}_e(u)-\boldsymbol{q}_e(v))\neq 0$ for some $u,v\in X$ now imply that we must add at least k edges to $G+D$ to make X locally C_2^1-rigid. Hence X is a locally k-dof part in $G+D$.

Let $\boldsymbol{q}$ be a motion of $(G,\boldsymbol{p})$. Since B_1 is a base for $Z(G,\boldsymbol{p})$, we have $\boldsymbol{q}=\sum_{i=1}^6\lambda_i\boldsymbol{q}_i^*+\sum_{e\in F\cup D}\mu_e\boldsymbol{q}_e$ for some $\lambda_i,\mu_e\in\mathbb{R}$. Since X is a locally C_2^1-rigid part of $G+F$, $\boldsymbol{q}_e|_X$ is a trivial motion of $(K(X),\boldsymbol{p}|_X)$ for all $e\in D$. Hence $\boldsymbol{q}_e|_X=\boldsymbol{q}_e^*|_X$ for some trivial motion $\boldsymbol{q}_e^*$ of $(G,\boldsymbol{p})$. Let $\boldsymbol{q}'=\sum_{i=1}^6\lambda_i\boldsymbol{q}_i^*+\sum_{e\in F}\mu_e\boldsymbol{q}_e+\sum_{e\in D}\mu_e\boldsymbol{q}_e^*$. Then $\boldsymbol{q}'\in Z(G+D,\boldsymbol{p})$. Since $\boldsymbol{q}_e|_X=\boldsymbol{q}_e^*|_X$ for $e\in D$, we also have $\boldsymbol{q}'|_X=\boldsymbol{q}|_X$.

Hence $(G+D,\boldsymbol{p})$ is a counterexample. The minimality of dof G now implies that $k'=k$ and $D=\emptyset$. Hence G is a k-dof graph. □

As G is a k-dof graph for some $1\le k\le 2$, (31) implies that $|V|\ge 7$.

Claim 6.3. *G has a vertex u_0 of degree at most five with $u_0\notin\hat{N}_G(v_0)$.*

Proof. Put $G-\hat{N}_G(v_0)=G'=(V',E')$ and let F be the set of edges of G from $\hat{N}_G(v_0)$ to V'. Since $|V|\geq 7$, we have $|V'|\geq 1$. By (31) we have

$$9+|F|+|E'|=|E|=3|V|-6-k=3(|V'|+6)-6-k. \tag{32}$$

Suppose every vertex in V' has degree at least six in G. Then

$$\sum_{v'\in V'} d_G(v')=|F|+2|E'|\geq 6|V'|. \tag{33}$$

We can now use (32) and (33) to deduce that $|E'|\geq 3|V'|-3+k\geq 3|V'|-2$. This contradicts the fact that G is C_2^1-independent. □

Let $u_0\in V\setminus \hat{N}_G(v_0)$ be a vertex of degree at most five in G, $K_u=K(N_G(u_0))$ and put $G-u_0=G_0=(V_0,E_0)$. Let $\mathcal{C}_{n-1}$ be the generic C_2^1-cofactor matroid on $K(V_0)$ and let $\mathcal{C}_{n-1}/\mathrm{cl}(E_0)$ be the matroid obtained from $\mathcal{C}_{n-1}$ by contracting $\mathrm{cl}(E_0)$.

Claim 6.4. *$F_v:=\{v_1v_2,v_1v_3,v_1v_4\}$ is a base of $K_v\setminus \mathrm{cl}(E_0)$ in $\mathcal{C}_{n-1}/\mathrm{cl}(E_0)$ (and hence $K_v\setminus\mathrm{cl}(E_0)$ has rank three in $\mathcal{C}_{n-1}/\mathrm{cl}(E_0)$).*

Proof. By (31) the subgraph of $G+F_v$ induced by $\hat{N}_G(v_0)$ is C_2^1-rigid. Hence F_v spans $K_v\setminus\mathrm{cl}(E_0)$ in $\mathcal{C}_{n-1}/\mathrm{cl}(E_0)$. Choose a base F' of F_v in $\mathcal{C}_{n-1}/\mathrm{cl}(E_0)$. Since the closure of F' contains F_v, $\mathrm{cl}(E_0\cup F')$ contains a C_2^1-rigid subgraph on $\hat{N}_G(v_0)$. Hence $\hat{N}_G(v_0)$ is a locally t-dof part in G_0 with $t=|F'|$. Since $\boldsymbol{q}_{\mathrm{bad}}|_{V_0}$ is a bad motion of $(G_0,\boldsymbol{p}|_{V_0})$ at v_0 and G_0 is not a counterexample by the minimality of $|V|$, we must have $t=|F'|=3$ and $F'=F_v$. □

Claim 6.5. $d_G(u_0)\in\{4,5\}$.

Proof. If $d_G(u_0)\leq 3$, then G_0 would be a t-dof graph for some $t\leq 2$. This would contradict Claim 6.4. □

Claim 6.6. $k=2$.

Proof. Suppose for a contradiction that $k=1$. If $d_G(u_0)=4$ then G_0 would be a t-dof graph for some $t\leq 2$ and this would contradict Claim 6.4. Hence we may suppose $d_G(u_0)=5$. Let $N_G(u_0)=\{u_1,u_2,\dots,u_5\}$. We will obtain a contradiction by showing that $(G_0+u_iu_j,\boldsymbol{p}|_{V_0})$ has a bad motion at v_0 for some $1\leq i<j\leq 5$.

Since G is a 1-dof graph, G_0 is a 3-dof graph and hence, by Claim 6.4, G_0+F_v is minimally C_2^1-rigid. As $G+F_v-u_0u_4-u_0u_5$ is obtained from G_0+F_v by a 0-extension, Lemma 3.1 now implies that $G+F_v-u_0u_4-u_0u_5$ is minimally C_2^1-rigid. Since G is C_2^1-independent, $G+F-u_0u_5$ is minimally C_2^1-rigid for some $F\subset F_v$ with $|F|=2$. We may now deduce that $G_0+F+u_iu_j$ is minimally C_2^1-rigid for some $1\leq i<j\leq 4$, and hence that $G_0+u_iu_j$ is a 2-dof graph. It remains to show that there is a bad motion for $(G_0+u_iu_j,\boldsymbol{p}|_{V_0})$ at v_0.

Let $G'=G-u_0u_5$. Observe that G' is obtained from $G_0+u_iu_j$ by a 1-extension. We shall consider the following non-generic realisation $\boldsymbol{p}'$ of G', which was used in the proof of [21, Theorem 10.2.1] to show that 1-extension preserves independence. For $w\in V$, let

$$\boldsymbol{p}'(w)=\begin{cases}\boldsymbol{p}(w) & (\text{if } w\in V\setminus\{u_0\})\\ \text{the mid-point of the line segment joining } \boldsymbol{p}(u_i) \text{ and } \boldsymbol{p}(u_j) & (\text{if } w=u_0).\end{cases}$$

See Figure 5.

We now use Lemma 5.6 to show that $(G',\boldsymbol{p}')$ has a bad motion at v_0. Let

$$S=\left\{\bar{\boldsymbol{p}}:(G'+F,\bar{\boldsymbol{p}}) \text{ is minimally } C_2^1\text{-rigid and } \bar{\boldsymbol{p}} \text{ is non-degenerate on } \hat{N}_{G'}(v_0)\right\}.$$

We may use the minimal C_2^1-rigidity of $(G_0+F+u_iu_j,\boldsymbol{p}|_{V_0})$, and the fact that $\{u_0,u_i,u_j\}$ is collinear in $(G'+F+u_iu_j,\boldsymbol{p}')$ to deduce that $(G'+F,\boldsymbol{p}')$ is minimally C_2^1-rigid, see the proof of [21, Theorem 10.2.1]

(a) (b)

Figure 5: (a) $(G', \boldsymbol{p}')$ and (b) $(G_0 + u_iu_j, \boldsymbol{p}_{|V_0})$.

for more details. Since $\boldsymbol{p}$ is generic, $\boldsymbol{p}'$ is non-degenerate on $\hat{N}_{G'}(v_0)$ and hence $\boldsymbol{p}' \in S$. Since $\boldsymbol{q}_{\text{bad}}$ is a bad motion of $(G', \boldsymbol{p})$ at v_0, Lemma 5.6 now implies that $(G', \boldsymbol{p}')$ has a bad motion $\boldsymbol{q}'_{\text{bad}}$ at v_0. The definition of $\boldsymbol{p}'$ implies that $\boldsymbol{q}'_{\text{bad}}|_{V_0}$ is a motion of $(G_0 + u_iu_j, \boldsymbol{p}|_{V_0})$. Since $(G_0 + u_iu_j, \boldsymbol{p}|_{V_0})$ is a generic 2-dof framework, $\boldsymbol{q}'_{\text{bad}}|_{V_0}$ is bad at v_0 and this contradicts the minimality of G. □

Recall that $F_v = \{v_1v_2, v_1v_3, v_1v_4\}$. We will denote the set of all pairs of edges in F_v by $\binom{F_v}{2}$. We say that $F \in \binom{F_v}{2}$ is *good* if $G + F$ is C_2^1-independent (and hence minimally C_2^1-rigid by $k = 2$). We say that an edge $f \in F_v$ is *very good* if $\{f, f'\}$ is good for any $f' \in F_v \setminus \{f\}$.

Claim 6.7. *There is at least one very good edge in F_v. (Equivalently, at most one pair in $\binom{F_v}{2}$ is not good).*

Proof. Let $F_v = \{f_1, f_2, f_3\}$, and suppose that $\{f_1, f_2\}$ and $\{f_1, f_3\}$ are not good. By (31), $f_1 \notin \text{cl}(E)$. Hence $G + f_1$ is C_2^1-independent and $f_2, f_3 \in \text{cl}(E + f_1)$. This in turn implies that $K_v \subseteq \text{cl}(E + f_1)$ and hence $\hat{N}_G(v_0)$ is a locally 1-dof part in G. This is a contradiction as $\hat{N}_G(v_0)$ is a locally k-dof part with $k = 2$ by Claim 6.6. □

Claim 6.8. *Let $F \in \binom{F_v}{2}$ be a good pair. Then* $\text{cl}(E_0 \cup F) \cap K_u$ *does not contain a copy of* K_4.

Proof. Suppose that $\text{cl}(E_0 \cup F) \cap K_u$ contains a copy of K_4, say on $\{u_1, \dots, u_4\} \subseteq N_G(u_0)$. Then $u_0u_4 \in \text{cl}((E_0 \cup F) + u_0u_1 + u_0u_2 + u_0u_3)$. This in turn implies that $E \cup F$ is dependent and contradicts the hypothesis that F is good. □

Claim 6.9. $d_G(u_0) = 5$.

Proof. Assume this is false. Then by Claim 6.5, $d_G(u_0) = 4$. Choose a good pair $F \in \binom{F_v}{2}$. By Claim 6.8, there is an edge $e = u_iu_j \in K_u$ such that $G_0 + F + u_iu_j$ is C_2^1-independent. Since $d_G(u_0) = 4$, it is minimally C_2^1-rigid. We can now proceed as in the last paragraph of the proof of Claim 6.6 (with G' replaced by G) to show that $(G_0 + u_iu_j, \boldsymbol{p}|_{V_0})$ has a bad motion at v_0 and hence contradict the minimality of $|V|$. □

6.3.2 Preliminary results on $\text{cl}(E_0) \cap K_u$

Our next goal is to show that $\text{cl}(E_0) \cap K_u$ is a star on five vertices (statement (45) in the next section). In this section we verify some preliminary claims. Let $N_G(u_0) = \{u_1, u_2, \dots, u_5\}$.

Claim 6.10. *$G_0 + F + e_1 + e_2$ is C_2^1-dependent for all $F \in \binom{F_v}{2}$ and all pairs of non-adjacent edges $e_1, e_2 \in K_u \setminus E_0$.*

Proof. Suppose that $G_0 + F + e_1 + e_2$ is C_2^1-independent (and hence minimally C_2^1-rigid) for $e_1 = u_1u_2$ and $e_2 = u_3u_4$. We will contradict the minimality of G by showing that $(G_0 + e_1 + e_2, \boldsymbol{p}|_{V_0})$ has a bad motion at v_0.

Observe that $G + F$ is obtained from $G_0 + F + e_1 + e_2$ by an X-replacement. We shall consider the following non-generic realisation $\boldsymbol{p}'$ of G, which was used in the proof of [21, Theorem 10.3.1] to show that X-replacement preserves C_2^1-independence. For $w \in V$, let $\boldsymbol{p}'(w) = \boldsymbol{p}(w)$ for $w \in V_0$ and $\boldsymbol{p}'(u_0)$ be the point of intersection of the line through $\boldsymbol{p}(u_1)$ and $\boldsymbol{p}(u_2)$, and the line through $\boldsymbol{p}(u_3)$ and $\boldsymbol{p}(u_4)$. We first use Lemma 5.6 to show that $(G,\boldsymbol{p}')$ has a bad motion at v_0. To do this, let

$$S = \left\{ \bar{\boldsymbol{p}} : (G+F,\bar{\boldsymbol{p}}) \text{ is minimally } C_2^1\text{-rigid and } \bar{\boldsymbol{p}} \text{ is non-degenerate on } \hat{N}_G(v_0) \right\}.$$

We may use the minimal C_2^1-rigidity of $(G_0+F+e_1+e_2,\boldsymbol{p}|_{V_0})$, and the fact the vertex sets $\{u_0,u_1,u_2\}$ and $\{u_0,u_3,u_4\}$ are collinear in $(G+F+e_1+e_2,\boldsymbol{p}')$ to deduce that $(G+F,\boldsymbol{p}')$ is minimally C_2^1-rigid, see the proof of [21, Theorem 10.3.1] for more details. Since $\boldsymbol{p}$ is generic, $\boldsymbol{p}'$ is non-degenerate on $\hat{N}_G(v_0)$ and hence $\boldsymbol{p}' \in S$. Since $(G,\boldsymbol{p})$ has a bad motion at v_0, Lemma 5.6 now implies that $(G,\boldsymbol{p}')$ has a bad motion $\boldsymbol{q}'_{\text{bad}}$ at v_0. The definition of $\boldsymbol{p}'$ implies that $\boldsymbol{q}'_{\text{bad}}|_{V_0}$ is a motion of $(G_0+e_1+e_2,\boldsymbol{p}|_{V_0})$. Since $(G_0+e_1+e_2,\boldsymbol{p}|_{V_0})$ is a generic 2-dof framework, $\boldsymbol{q}'_{\text{bad}}|_{V_0}$ is bad at v_0 and this contradicts the minimality of G. □

Claim 6.10 implies the following.

Claim 6.11. *Let $F \in \binom{F_v}{2}$ and let $e = u_iu_j \in K_u \setminus E_0$ such that G_0+F+e is C_2^1-independent. Then $\mathrm{cl}((E_0 \cup F)+e)$ contains a triangle on $N_G(u_0)\setminus\{u_i,u_j\}$.*

Proof. If not, then we can choose an edge e' on $N_G(u_0)\setminus\{u_i,u_j\}$ such that $G_0+F+e+e'$ is C_2^1-independent. This contradicts Claim 6.10. □

Claim 6.12. *$G_0+F+e_1+e_2$ is C_2^1-dependent for all $F \in \binom{F_v}{2}$ and all pairs of adjacent edges $e_1,e_2 \in K_u \setminus E_0$ whose common end-vertex has degree two in $\mathrm{cl}(E_0)\cap K_u$.*

Proof. Suppose that $G_0+F+e_1+e_2$ is C_2^1-independent (and hence minimally C_2^1-rigid). Our aim is to contradict the minimality of G by showing that $(G_0+e_1+e_2,\boldsymbol{p}|_{V_0})$ has a bad motion at v_0.

Let u_1 be the common endvertex of e_1 and e_2. Since u_1 has degree two in $\mathrm{cl}(E_0)\cap K_u$ we may choose a base E_0' of $\mathrm{cl}(E_0)$ such that u_1 has degree two in $E_0'\cap K_u$ and v_0 is incident with the same set of edges in E_0 and E_0'. Let G_0' be the graph induced by E_0' and G' be the graph obtained from G_0' by adding u and the edges of G incident to u. Then $G_0'+F+e_1+e_2$ is C_2^1-independent and hence $G'+F$ is C_2^1-independent by Lemma 3.2. In addition, the fact that $\mathrm{cl}(E_0)=\mathrm{cl}(E_0')$ implies that (G,p) and (G',p) will have the same space of motions. Relabelling G' by G, we may assume that u_1 has degree two in $E_0\cap K_u$ and $G+F$ is obtained from $G_0+F+e_1+e_2$ by a vertex splitting.

Consider the realisation $\boldsymbol{p}_0$ of G defined by $\boldsymbol{p}_0(w)=\boldsymbol{p}(w)$ for $w\in V_0$ and $\boldsymbol{p}_0(u_0)=\boldsymbol{p}(u_1)$. Let $C^*(G+F,\boldsymbol{p}_0)$ be obtained from the cofactor matrix $C(G+F,\boldsymbol{p}_0)$ by replacing the zero-row indexed by the edge u_0u_1 with a row of the form

$$\begin{array}{c} \\ e{=}u_0u_1 \end{array}\begin{array}{c} \quad\quad\quad\quad u_0 \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad u_1 \\ \left[\; 0\ldots0 \quad D(\boldsymbol{p}(u_0),\boldsymbol{p}(u_1)) \quad 0\ldots0 \quad -D(\boldsymbol{p}(u_0),\boldsymbol{p}(u_1)) \quad 0\ldots0 \;\right]. \end{array}$$

The argument used to show that vertex splitting preserves C_2^1-independence in [21, Theorem 10.2.7] implies that $C^*(G+F,\boldsymbol{p}_0)$ is row independent. If $C^*(G+F,\boldsymbol{p}_0)$ were the C_2^1-cofactor matrix of $(G+F,\boldsymbol{p}_0)$ then we could find a bad motion for $(G_0+e_1+e_2,\boldsymbol{p}|_{V_0})$ by applying Lemma 5.6 to $(G+F,\boldsymbol{p}_0)$ as in the last paragraph of Claim 6.10, and this would contradict the minimality of G. Since $C^*(G+F,\boldsymbol{p}_0)$ is not a C_2^1-cofactor matrix we instead apply the proof technique of Lemma 5.6 directly to $C^*(G+F,\boldsymbol{p}_0)$ to show that

$$z|_{V_0} \text{ is a bad motion of } (G_0+e_1+e_2,\boldsymbol{p}|_{V_0}) \text{ for some } z\in\ker C^*(G,\boldsymbol{p}_0) \tag{34}$$

and obtain the same contradiction. We give the proof of (34) immediately after the proof of Lemma 5.6 in Appendix A. □

6.3.3 The structure of $\mathrm{cl}(E_0)\cap K_u$

We show that $\mathrm{cl}(E_0)\cap K_u$ is a star on $N_G(u_0)$. The proof proceeds as in that of Lemma 4.1.

Claim 6.13. *For every good pair $F\in\binom{F_v}{2}$ and every $e\in K_u$ such that $(E_0\cup F)+e$ is independent,* $\mathrm{cl}((E_0\cup F)+e)\cap K_u$ *does not contain a C_2^1-rigid subgraph spanning $N_G(u_0)$.*

Proof. Suppose $\mathrm{cl}((E_0\cup F)+e)\cap K_u$ contains a C_2^1-rigid subgraph spanning $N_G(u_0)$. Then $K_u\subseteq \mathrm{cl}((E_0\cup F)+e)$ and hence $u_0u_4,u_0u_5\in \mathrm{cl}((E_0\cup F)+e+u_0u_1+u_0u_2+u_0u_3)$. Since G is C_2^1-independent, we can use two base exchanges to show that $\mathrm{cl}((E_0\cup F)+e+u_0u_1+u_0u_2+u_0u_3)=\mathrm{cl}(E+e')$ for some $e'\in F+e$. Then $F\subseteq \mathrm{cl}(E+e')$. This contradicts the hypothesis that F is good. □

For $F\subseteq K(V_0)\setminus \mathrm{cl}(E_0)$, we use $[F]$ to denote the closure of F in $\mathcal{C}_{n-1}/\mathrm{cl}(E_0)$. Thus an edge $e\in K(V_0)\setminus \mathrm{cl}(E_0)$ belongs to $[F]$ if and only if we have $e\in C\subseteq (E_0\cup F)+e$ for some circuit C of $\mathcal{C}_{n-1}$. If F is a singleton $\{f\}$, then we denote $[\{f\}]$ by $[f]$.

We say that a set $F\subset F_v$ is *influential* if $[F]\cap K_u\neq\emptyset$. A set $F+e$ with $F\in\binom{F_v}{2}$ and $e\in K_u\setminus \mathrm{cl}(E_0)$ is said to be a *stabilizer* if $\hat{N}(v_0)$ is a locally C_2^1-rigid part of G_0+F+e. By Claim 6.4, $F+e$ is a stabilizer if and only if $\mathrm{cl}((E_0\cup F)+e)=\mathrm{cl}(E_0\cup F_v)$. This fact will be used frequently.

Claim 6.14. *Let $F\in\binom{F_v}{2}$ be a good pair and $e\in K_u\setminus \mathrm{cl}(E_0)$ such that $(E_0\cup F)+e$ is independent. Suppose that either F is not influential or $F+e$ is a stabilizer. Then* $\mathrm{cl}((E_0\cup F)+e)\cap K_u$ *contains no copy of K_4.*

Proof. Without loss of generality, let $e=u_1u_2$.

Suppose that $\mathrm{cl}((E_0\cup F)+e)\cap K_u$ contains a copy of K_4. If $\mathrm{cl}((E_0\cup F)+e)\cap K_u$ contains two distinct copies of K_4, then their union would be a C_2^1-rigid subgraph of K_u. This would contradict Claim 6.13. Hence $\mathrm{cl}((E_0\cup F)+e)\cap K_u$ contains exactly one copy of K_4. We consider two cases depending on the position of the copy of K_4.

Case 1: The copy of K_4 in $\mathrm{cl}((E_0\cup F)+e)\cap K_u$ contains $e=u_1u_2$.
By symmetry we may assume that this is a K_4 on $\{u_1,u_2,u_3,u_4\}$. By Claim 6.11, $\mathrm{cl}((E_0\cup F)+e)\cap K_u$ also contains $\{u_3u_5,u_4u_5\}$. Hence, by Claim 6.13,

$$u_1u_5,u_2u_5\notin \mathrm{cl}((E_0\cup F)+e)\cap K_u.$$

We next show that:

$$u_1u_3,u_1u_4\in \mathrm{cl}(E_0\cup F). \tag{35}$$

To see this, suppose $u_1u_3\notin \mathrm{cl}(E_0\cup F)$. Since $u_1u_3\in \mathrm{cl}((E_0\cup F)+e)$, we have that $(E_0\cup F)+u_1u_3$ is independent and $u_2u_5\notin \mathrm{cl}((E_0\cup F)+e)=\mathrm{cl}((E_0\cup F)+u_1u_3)$. Hence $(E_0\cup F)+u_1u_3+u_2u_5$ is independent. This contradicts Claim 6.10 so $u_1u_3\in \mathrm{cl}(E_0\cup F)$. The same argument for u_1u_4 gives (35).

Since $(E_0\cup F)+e+u_1u_5$ is independent, Claim 6.12 implies that $\{u_1u_3,u_1u_4\}\not\subseteq \mathrm{cl}(E_0)$. By symmetry, we may suppose that $u_1u_3\notin \mathrm{cl}(E_0)$, i.e., $u_1u_3\in \mathrm{cl}((E_0\cup F))\setminus \mathrm{cl}(E_0)=[F]$. This implies that F is influential, and hence (by an hypothesis of the claim) $F+e$ is a stabilizer. Thus, for the unique edge $f_e\in F_v\setminus F$, we have $\mathrm{cl}((E_0\cup F)+e)=\mathrm{cl}((E_0\cup F)+f_e)$. Since $u_1u_3\in[F]$, we also have $u_2u_5\notin \mathrm{cl}((E_0\cup F)+e)=\mathrm{cl}((E_0\cup F)+f_e)=\mathrm{cl}(E_0+f+u_1u_3+f_e)$ for some edge $f\in F$. Then $(E_0\cup\{f,f_e\})+u_1u_3+u_2u_5$ is independent. This contradicts Claim 6.10.

Case 2: $\mathrm{cl}((E_0\cup F)+e)\cap K_u$ contains a copy of K_4 which avoids $e=u_1u_2$.
By symmetry we may assume that this is a K_4 on $\{u_1,u_3,u_4,u_5\}$. Since there is no other copy of K_4 in $\mathrm{cl}((E_0\cup F)+e)\cap K_u$, we may also assume

$$u_2u_3,u_2u_4\notin \mathrm{cl}((E_0\cup F)+e).$$

We proceed in a similar way to Case 1. We first show:

$$\{u_1u_3,u_1u_4,u_1u_5,u_3u_5,u_4u_5\}\subseteq \mathrm{cl}(E_0\cup F). \tag{36}$$

To see this, suppose $u_1u_3 \notin \mathrm{cl}(E_0 \cup F)$. Since $u_1u_3 \in \mathrm{cl}((E_0 \cup F) + e)$, $(E_0 \cup F) + u_1u_3$ is independent and $u_2u_4 \notin \mathrm{cl}((E_0 \cup F) + e) = \mathrm{cl}((E_0 \cup F) + u_1u_3)$. Hence $(E_0 \cup F) + u_1u_3 + u_2u_4$ is independent. This contradicts Claim 6.10. The same proof can be applied to the other edges to give (36).

We next show that:

$$\{u_1u_3, u_1u_4, u_1u_5, u_3u_5, u_4u_5\} \subseteq \mathrm{cl}(E_0). \tag{37}$$

To see this, suppose $u_1u_3 \notin \mathrm{cl}(E_0)$. Then $u_1u_3 \in \mathrm{cl}((E_0 \cup F)) \setminus \mathrm{cl}(E_0) = [F]$. This implies that F is influential, and hence (by an hypothesis of the claim) $F + e$ is a stabilizer. Thus, for the unique edge $f_e \in F_v \setminus F$, $\mathrm{cl}((E_0 \cup F) + e) = \mathrm{cl}((E_0 \cup F) + f_e)$ holds, and we have $u_2u_4 \notin \mathrm{cl}((E_0 \cup F) + e) = \mathrm{cl}((E_0 \cup F) + f_e) = \mathrm{cl}(E_0 + f + u_1u_3 + f_e)$ for some edge $f \in F$. Hence $(E_0 \cup \{f, f_e\}) + u_1u_3 + u_2u_4$ is independent. This contradicts Claim 6.10. The same argument can be applied to the other edges to give (37).

Suppose that $u_3u_4 \notin \mathrm{cl}(E_0 \cup F)$. As $u_3u_4 \in \mathrm{cl}((E_0 \cup F) + e)$, $(E_0 \cup F) + u_3u_4$ is independent and $u_2u_4 \notin \mathrm{cl}((E_0 \cup F) + e) = \mathrm{cl}((E_0 \cup F) + u_3u_4)$. Hence $(E_0 \cup F) + u_3u_4 + u_2u_4$ is independent. Since u_4 has degree two in $\mathrm{cl}(E_0) \cap K_u$ by (37), this contradicts Claim 6.12. Hence $u_3u_4 \in \mathrm{cl}(E_0 \cup F)$. This and (36) imply that $\mathrm{cl}(E_0 \cup F)$ contains a copy of K_4 on $\{u_1, u_3, u_4, u_5\}$, contradicting Claim 6.8. □

Claim 6.15. *Let $F \in \binom{F_v}{2}$ be a good pair and $e = u_iu_j \in K_u$ be such that $(E_0 \cup F) + e$ is independent. Suppose that either F is not influential or $F + e$ is a stabilizer. Then $\mathrm{cl}(E_0)$ contains at least two edges of the triangle on $N_G(u_0) \setminus \{u_i, u_j\}$.*

Proof. Without loss of generality suppose $e = u_1u_2$. By Claim 6.11, $\mathrm{cl}(E_0 + F + e)$ contains a triangle on $\{u_3, u_4, u_5\}$.

We first consider the complete graph on $\{u_1, u_2, u_3, u_4\}$. By Claim 6.14 we may assume without loss of generality that $\mathrm{cl}((E_0 \cup F) + e)$ does not contain u_1u_3. If $u_4u_5 \in \mathrm{cl}((E_0 \cup F) + e) \setminus \mathrm{cl}(E_0 \cup F)$, then $(E_0 \cup F) + u_4u_5$ is independent and $\mathrm{cl}((E_0 \cup F) + u_4u_5) = \mathrm{cl}((E_0 \cup F) + e)$. This would imply that $(E_0 \cup F) + u_1u_3 + u_4u_5$ is independent and contradict Claim 6.10. Hence $u_4u_5 \in \mathrm{cl}(E_0 \cup F)$.

Suppose next that $u_4u_5 \in \mathrm{cl}(E_0 \cup F) \setminus \mathrm{cl}(E_0)$, i.e., $u_4u_5 \in [F]$. Then F is influential and so, by the hypothesis, $F + e$ is a stabilizer. Hence $\mathrm{cl}((E_0 \cup F) + e) = \mathrm{cl}(E_0 \cup F_v) = \mathrm{cl}(E_0 + f_1 + f_2 + u_4u_5)$ holds for some edges $f_1, f_2 \in F_v$. Then $E_0 + f_1 + f_2 + u_4u_5 + u_1u_3$ is independent and we again contradict Claim 6.10. Hence $u_4u_5 \in \mathrm{cl}(E_0)$.

By applying the same argument to the complete graph on $\{u_1, u_2, u_4, u_5\}$, we also get $u_3u_4 \in \mathrm{cl}(E_0)$ or $u_3u_5 \in \mathrm{cl}(E_0)$. (Specifically, we have $u_3u_4 \in \mathrm{cl}(E_0)$ when $u_1u_5 \notin \mathrm{cl}(E_0 + f + e)$ or $u_2u_5 \notin \mathrm{cl}((E_0 \cup F) + e)$ holds, and we have $u_3u_5 \in \mathrm{cl}(E_0)$ when $u_1u_4 \notin \mathrm{cl}((E_0 \cup F) + e)$ or $u_2u_4 \notin \mathrm{cl}((E_0 \cup F) + e)$ holds.) □

Recall that an edge $f \in F_v$ is said to be very good if $\{f, f'\}$ is a good pair for every $f' \in F_v \setminus \{f\}$. In order to apply Claim 6.13 for a given edge $e \in K_u$, we need to find a good pair $F \subset F_v$ such that $(E_0 \cup F) + e$ is independent. The following claim shows we can do this under a mild assumption.

Claim 6.16. *Let $e \in K_u \setminus \mathrm{cl}(E_0)$, and suppose that $e \notin [f]$ for some very good edge $f \in F_v$. Then there is a good pair $F \in \binom{F_v}{2}$ such that $(E_0 \cup F) + e$ is independent.*

Proof. Let $F_v = \{f_1, f_2, f_3\}$, and suppose that an edge $e \in K_u \setminus \mathrm{cl}(E_0)$ satisfies $e \notin [f_1]$ for a very good edge f_1. Since $e \notin [f_1]$, $E_0 + f_1 + e$ is independent. As f_1 is very good, $\{f_1, f_2\}$ and $\{f_1, f_3\}$ are both good. Hence, if no good pair has the desired property, then $\{f_2, f_3\} \subset \mathrm{cl}(E_0 + f_1 + e)$. By Claim 6.4, $E_0 + f_1 + f_2$ is independent. Since $f_2 \in \mathrm{cl}(E_0 + f_1 + e)$, this gives $\mathrm{cl}(E_0 + f_1 + e) = \mathrm{cl}(E_0 + f_1 + f_2)$, and $f_3 \in \mathrm{cl}(E_0 + f_1 + f_2)$ follows since $f_3 \in \mathrm{cl}(E_0 + f_1 + e)$. This would contradict Claim 6.4. □

Our next claim extends Claim 6.14. (Note that F is not required to be a good pair in the following claim.)

Claim 6.17. $\mathrm{cl}(E_0 \cup F_v) \cap K_u$ *has no copy of K_4 and, for any stabilizer $F + e$,* $\mathrm{cl}((E_0 \cup F) + e) \cap K_u$ *has no copy of K_4.*

Proof. Suppose that a copy of K_4 exists in $\mathrm{cl}(E_0 \cup F_v) \cap K_u$. By Claim 6.7 we can choose a very good edge f from F_v. Then Claim 6.8 implies that $\mathrm{cl}(E_0 + f) \cap K_u$ has no copy of K_4. Hence there is an edge e in the copy of K_4 with $e \notin \mathrm{cl}(E_0 + f)$. By Claim 6.16, there exists a good pair F of edges in F_v such that $(E_0 \cup F) + e$ is independent. Since $e \in \mathrm{cl}(E_0 \cup F_v)$ and $(E_0 \cup F) + e$ is independent, we have $F_v \subset \mathrm{cl}(E_0 \cup F_v) = \mathrm{cl}((E_0 \cup F) + e)$. (Note that $E_0 \cup F_v$ is independent by Claim 6.4.) Now $F_v \subset \mathrm{cl}((E_0 \cup F) + e)$ implies that $F + e$ is a stabilizer while $\mathrm{cl}(E_0 \cup F_v) = \mathrm{cl}((E_0 \cup F) + e)$ implies that $\mathrm{cl}((E_0 \cup F) + e)$ has a copy of K_4. This contradicts Claim 6.14, and completes the proof of the first part of the claim. The second part now follows from the fact that $\mathrm{cl}(E_0 \cup F_v) = \mathrm{cl}((E_0 \cup F) + e)$ when $F + e$ is a stabilizer. □

Claim 6.18. *Let $e = u_i u_j \in K_u \setminus \mathrm{cl}(E_0)$ be such that $e \notin [f]$ for some very good edge $f \in F_v$. Then $\mathrm{cl}(E_0)$ contains at least two edges of the triangle on $N_G(u_0) \setminus \{u_i, u_j\}$.*

Proof. We may assume, without loss of generality, that $e = u_1 u_2$. Suppose that two edges on $\{u_3, u_4, u_5\}$ are missing in $\mathrm{cl}(E_0)$. By symmetry, we may assume that $u_3 u_5$ and $u_4 u_5$ are missing.

We first show that for any $F \in \binom{F_v}{2}$ with $(E_0 \cup F) + e$ independent, we have

$$\{u_1u_2, u_1u_3, u_1u_4, u_2u_3, u_2u_4, u_3u_4, u_3u_5, u_4u_5\} \subset \mathrm{cl}((E_0 \cup F) + e). \tag{38}$$

To see this, choose an $F \in \binom{F_v}{2}$ such that $(E_0 \cup F) + e$ is independent. By Claim 6.11, $\mathrm{cl}((E_0 \cup F) + e)$ contains a triangle on $\{u_3, u_4, u_5\}$. To show that $\{u_1u_2, u_1u_3, u_1u_4, u_2u_3, u_2u_4\}$ is contained in $\mathrm{cl}((E_0 \cup F) + e)$, we first prove:

$$\begin{array}{c}\text{if } u_k u_l \notin \mathrm{cl}(E_0) \text{ for some } k, l \in \{3,4,5\}, \text{ then} \\ \mathrm{cl}((E_0 \cup F) + e) \text{ contains a triangle on } N(u_0) \setminus \{u_k, u_l\}.\end{array} \tag{39}$$

This follows from Claim 6.11 if $u_k u_l \notin \mathrm{cl}(E_0 \cup F)$. Hence we assume $u_k u_l \in \mathrm{cl}(E_0 \cup F) \setminus \mathrm{cl}(E_0) = [F]$. By Claim 6.4, the unique edge f' in $F_v \setminus F$ satisfies $f' \notin [F]$. Also, since $u_k u_l \in [F]$, we can choose $f'' \in F$ such that $\mathrm{cl}(E_0 \cup F) = \mathrm{cl}(E_0 + f'' + u_k u_l)$. Since $f' \notin \mathrm{cl}(E_0 \cup F)$, this implies that $(E_0 \cup \{f', f''\}) + u_k u_l$ is independent. If $e = u_1 u_2 \notin \mathrm{cl}((E_0 \cup \{f', f''\}) + u_k u_l)$, then $(E_0 \cup \{f', f''\}) + u_k u_l + e$ would be independent, which contradicts Claim 6.10. Hence $e \in \mathrm{cl}((E_0 \cup \{f', f''\}) + u_k u_l)$. As $[\{f'', u_k u_l\}] = [F]$ and $(E_0 \cup F) + e$ is independent, we have $\mathrm{cl}((E_0 \cup \{f', f''\}) + u_k u_l) = \mathrm{cl}((E_0 \cup F) + e)$. By Claim 6.11, $\mathrm{cl}((E_0 \cup \{f', f''\}) + u_k u_l)$ contains a triangle on $N_G(u_0) \setminus \{u_k, u_l\}$. This in turn implies (39), since $\mathrm{cl}((E_0 \cup \{f', f''\}) + u_k u_l) = \mathrm{cl}((E_0 \cup F) + e)$.

The assumption that $u_3 u_5 \notin \mathrm{cl}(E_0)$ and (39) imply that $u_1u_2, u_1u_4, u_2u_4 \in \mathrm{cl}((E_0 \cup F) + e)$. Similarly, $u_4 u_5 \notin \mathrm{cl}(E_0)$ and (39) imply that $u_1u_3, u_2u_3 \in \mathrm{cl}((E_0 \cup F) + e)$. Hence (38) holds. In particular, $\mathrm{cl}((E_0 \cup F) + e)$ contains a copy of K_4, and Claim 6.17 now gives:

$$\text{for any } F \in \binom{F_v}{2} \text{ such that } (E_0 \cup F) + e \text{ is independent, } F + e \text{ is not a stabilizer.} \tag{40}$$

Let $F_v = \{f_1, f_2, f_3\}$. By Claim 6.16 there is a good pair F such that $(E_0 \cup F) + e$ is independent. We may assume without loss of generality that $F = \{f_1, f_2\}$. We next prove:

$$E_0 + f_1 + f_3 + e \text{ and } E_0 + f_2 + f_3 + e \text{ are both independent.} \tag{41}$$

Suppose $E_0 + f_1 + f_3 + e$ is dependent. Then, since $E_0 + f_1 + e$ is independent, $f_3 \in \mathrm{cl}(E_0 + f_1 + e)$. This in turn implies that $f_1 + f_2 + e$ is a stabilizer, contradicting (40). Hence $E_0 + f_1 + f_3 + e$ is independent. A similar argument for $E_0 + f_2 + f_3 + e$ gives (41).

We next show that:

$$S := \{u_1u_2, u_1u_3, u_1u_4, u_2u_3, u_2u_4, u_3u_4, u_3u_5, u_4u_5\} \subset \mathrm{cl}(E_0 + e). \tag{42}$$

To see this suppose $u_i u_j \notin \mathrm{cl}(E_0 + e)$ for some edge $u_i u_j \in S$. Since $u_i u_j \in \mathrm{cl}(E_0 + f_1 + f_2 + e)$ by (38) but $u_i u_j \notin \mathrm{cl}(E_0 + e)$, we may assume without loss of generality that $\mathrm{cl}(E_0 + f_1 + f_2 + e) = \mathrm{cl}(E_0 + f_1 + e + u_i u_j)$.

Then $f_2 \in \mathrm{cl}(E_0 + f_1 + e + u_iu_j)$. We also have $u_iu_j \in \mathrm{cl}(E_0 + f_1 + f_3 + e)$ by (38) and (41). This gives $\mathrm{cl}(E_0 + f_1 + f_3 + e) = \mathrm{cl}(E_0 + f_1 + f_3 + e + u_iu_j)$. Since $f_2 \in \mathrm{cl}(E_0 + f_1 + f_3 + e + u_iu_j)$, this in turn implies that $f_1 + f_3 + e$ is a stabilizer, contradicting (40). Thus (42) holds.

Since F is a good pair, Claim 6.13 implies that $\mathrm{cl}((E_0 \cup F) + e) \cap K_u$ cannot contain a C_2^1-rigid graph which spans $N_G(u_0)$. Combining this fact with (42), we have

$$\mathrm{cl}((E_0 \cup F) + e) \cap K_u = \mathrm{cl}(E_0 + e) \cap K_u = S. \tag{43}$$

In particular, u_1u_5 and u_2u_5 are missing in $\mathrm{cl}((E_0 \cup F) + e)$. If $u_1u_3, u_1u_4 \in \mathrm{cl}(E_0)$, then the fact that $(E_0 \cup F) + e + u_1u_5$ is independent would contradict Claim 6.12. Thus we may assume $u_1u_3 \notin \mathrm{cl}(E_0)$. Then $u_1u_3 \in [e]$ by (43). This implies that $\mathrm{cl}((E_0 \cup F) + e) = \mathrm{cl}((E_0 \cup F) + u_1u_3)$, and hence $(E_0 \cup F) + u_1u_3 + u_2u_5$ is independent as $u_2u_5 \notin \mathrm{cl}((E_0 \cup F) + e)$. This contradicts Claim 6.10. □

Claim 6.19. *For all i with $1 \le i \le 5$, $d_{\mathrm{cl}(E_0)\cap K_u}(u_i) \ge 1$. And if $d_{\mathrm{cl}(E_0)\cap K_u}(u_i) = 1$, then the vertex u_j adjacent to u_i in $\mathrm{cl}(E_0) \cap K_u$ satisfies $d_{\mathrm{cl}(E_0)\cap K_u}(u_j) = 4$.*

Proof. Let $F_v = \{f_1, f_2, f_3\}$. By Claim 6.7, we may assume that f_1 is a very good edge.

Suppose that $d_{\mathrm{cl}(E_0)\cap K_u}(u_5) = 0$. By Claim 6.8, $\mathrm{cl}(E_0 + f_1) \cap K_u$ has no copy of K_4. Hence we can choose an edge e on $\{u_1, u_2, u_3, u_4\}$ such that $e \notin \mathrm{cl}(E_0 + f_1)$. We can then apply Claim 6.18 to e to deduce that $\mathrm{cl}(E_0)$ has an edge incident to u_5. This gives a contradiction and completes the proof of first part of the claim.

To prove the second part we assume that $d_{\mathrm{cl}(E_0)\cap K_u}(u_5) = 1$ and that u_5 is adjacent to u_1 in $\mathrm{cl}(E_0)$. We will show that $u_1u_2, u_1u_3, u_1u_4 \in \mathrm{cl}(E_0)$. Suppose, for a contradiction, that $u_1u_2 \notin \mathrm{cl}(E_0)$. If $u_1u_2 \notin [f_1]$, then Claim 6.18 would imply that u_5 has degree at least two in $\mathrm{cl}(E_0)$, a contradiction. Hence $u_1u_2 \in [f_1]$ which gives $u_1u_2 \in \mathrm{cl}(E_0 + f_1)$. We aslo have $u_1u_3, u_1u_4 \in \mathrm{cl}(E_0 + f_1)$ since, if $u_1u_i \notin \mathrm{cl}(E_0)$ for some $i = 1, 2$, then we can use the same argument as for u_1u_2 to deduce that $u_1u_i \in [f_1]$. The facts that $u_1u_2 \in [f_1]$ and F_v is independent in $\mathcal{C}_{n-1}/\mathrm{cl}(E_0)$ imply that $E_0 + f_2 + f_3 + u_1u_2$ is independent and $\mathrm{cl}(E_0 \cup F_v) = \mathrm{cl}(E_0 + f_2 + f_3 + u_1u_2)$. Since $u_3u_4, u_4u_5, u_5u_3 \in \mathrm{cl}(E_0 + f_2 + f_3 + u_1u_2)$ by Claim 6.11, $\mathrm{cl}(E_0 \cup F_v)$ contains a copy of K_4 on $\{u_1, u_3, u_4, u_5\}$. This contradicts Claim 6.17. □

Claim 6.20. *$\mathrm{cl}(E_0) \cap K_u$ is cycle free.*

Proof. We choose $F \in \binom{F_v}{2}$ and $e \in K_u$ as follows.

- When $[F_v] \cap K_u \neq \emptyset$, we first choose $e \in [F_v] \cap K_u$ and then choose F such that $F + e$ is independent. Such a choice of F is possible by Claim 6.4 and will ensure that $F + e$ is a stabilizer.

- When $[F_v] \cap K_u = \emptyset$, we first choose F to be good and then choose $e \in K_u \setminus \mathrm{cl}(E_0)$. This choice will ensure that $(E_0 \cup F) + e$ is independent and F is good but not influential.

This choice of F and e and Claims 6.14 and 6.17 imply that

$$\mathrm{cl}((E_0 \cup F) + e) \cap K_u \text{ contains no copy of } K_4. \tag{44}$$

Suppose that $\mathrm{cl}(E_0) \cap K_u$ contains a cycle of length five, say $u_1u_2u_3u_4u_5u_1$. We may assume without loss of generality that $e = u_1u_3$. By (44), either $u_2u_4 \notin \mathrm{cl}((E_0 \cup F) + e)$ or $u_1u_4 \notin \mathrm{cl}((E_0 \cup F) + e)$ holds. Both alternatives lead to a contradiction by applying Claim 6.10 and 6.12, respectively.

Suppose that $\mathrm{cl}(E_0) \cap K_u$ contains a cycle of length four, say $u_1u_2u_3u_4u_1$. If e is a chord of the cycle, then the other chord is missing in $\mathrm{cl}((E_0 \cup F) + e)$ by (44), and we can apply Claim 6.10 to get a contradiction. Hence e is not a chord of the cycle. Since at least one diagonal of the cycle is missing in $\mathrm{cl}((E_0 \cup F) + e)$ by (44), we can again apply Claim 6.10 or Claim 6.12 to get a contradiction.

It remains to consider the case when $\mathrm{cl}(E_0) \cap K_u$ contains a cycle of length three, say $u_1u_2u_3u_1$. Claim 6.19 and (44) tell us that there is exactly one edge in $\mathrm{cl}(E_0) \cap K_u$ from each of u_4, u_5 to $\{u_1, u_2, u_3\}$, and that both of these edges have a common end-vertex, say u_1. By symmetry, we may assume that either $e = u_2u_4$ or

$e = u_4u_5$ holds. If $e = u_2u_4$, then $u_3u_5 \in \mathrm{cl}((E_0 \cup F) + e)$ by Claim 6.11, and $u_2u_5 \notin \mathrm{cl}((E_0 \cup F) + e)$ by (44). Then $(E_0 \cup F) + e + u_2u_5$ is independent, contradicting Claim 6.12. Hence $e = u_4u_5$.

Recall our selection of e at the beginning of the proof. If $[F_v] \cap K_u = \emptyset$, then e could have been chosen to be any edge in $K_u \setminus \mathrm{cl}(E_0)$. In particular we could choose $e = u_2u_4$, and obtain the contradiction in the previous paragraph. Hence $[F_v] \cap K_u \neq \emptyset$. Since e can be any edge in $[F_v] \cap K_u$, we must have $[F_v] \cap K_u = \{u_4u_5\}$.

Choose any good pair $F' \in \binom{F_v}{2}$, and consider $F' + u_2u_4$. Since $[F_v] \cap K_u = \{u_4u_5\}$, $F' + u_2u_4$ is independent. Hence by Claims 6.10 and 6.12, $u_2u_5, u_3u_5 \in \mathrm{cl}((E_0 \cup F') + u_2u_4)$. Since $u_3u_5 \notin \mathrm{cl}(E_0 \cup F')$, we have $\mathrm{cl}((E_0 \cup F') + u_2u_4) = \mathrm{cl}((E_0 \cup F') + u_3u_5)$. Then by Claim 6.12, $\mathrm{cl}((E_0 \cup F') + u_3u_5)$ also contains u_3u_4. Summarizing, we have shown that $\mathrm{cl}((E_0 \cup F') + u_2u_4)$ contains a C_2^1-rigid graph which spans $N_G(u_0)$. This contradicts Claim 6.13. □

Combining Claim 6.19 and 6.20, we conclude that $\mathrm{cl}(E_0) \cap K_u$ is a star on five vertices. Without loss of generality, we may assume that

$$\mathrm{cl}(E_0) \cap K_u \text{ is a star centered at } u_5. \tag{45}$$

6.3.4 Further structural properties of G

We let K be the edge set of the complete graph on $\{u_1, u_2, u_3, u_4\}$ and, for each edge $e \in K$, we refer to the unique edge of K which is not adjacent to e as the *opposite edge to* e and denote it by $\tilde{e}$. For $e \in K$ and $f \in F_v$, we say that *e is f-coupled* if $\tilde{e} \in [\{e, f\}]$.

Claim 6.21. *Let $e \in [F_v] \cap K$. Then $\tilde{e} \in [F_v]$.*

Proof. Since $e \in [F_v]$, $e \notin \mathrm{cl}(E_0)$ and we can choose $F \in \binom{F_v}{2}$ such that $(E_0 \cup F) + e$ is independent. Together with Claim 6.11, this implies that $\tilde{e} \in \mathrm{cl}((E_0 \cup F) + e) = \mathrm{cl}(E_0 \cup F_v)$. Since $\tilde{e} \notin \mathrm{cl}(E_0)$ by (45), we have $\tilde{e} \in [F_v]$. □

Claim 6.22. *Suppose e is not f-coupled for some $e \in K$ and $f \in F_v$. Then $e, \tilde{e} \in [F_v]$.*

Proof. Let $F_v = \{f_1, f_2, f_3\}$ where $f = f_1$.

Suppose $e \notin [F_v]$. Then, by Lemma 6.4, $E_0 + f_i + f_j + e$ is independent for all distinct $i, j \in \{1, 2, 3\}$. Hence, for each $i \in \{2, 3\}$, Claim 6.11 implies $\tilde{e} \in [\{f_1, f_i, e\}]$. Since $\tilde{e} \notin [\{f_1, e\}]$ (as e is not f_1-coupled), we have $f_i \in \mathrm{cl}(E_0 + f_1 + f_i + e) = \mathrm{cl}(E_0 + f_1 + e + \tilde{e})$ for $i \in \{2, 3\}$. As $E_0 \cup F_v$ is independent, this in turn implies that $\mathrm{cl}(E_0 + f_1 + e + \tilde{e}) = \mathrm{cl}(E_0 \cup F_v)$, and contradicts our initial assumption that $e \notin [F_v]$.

Hence $e \in [F_v]$. Claim 6.21 now implies that $\tilde{e} \in [F_v]$. □

Claim 6.23. *Suppose $E_0 + f + e + e'$ is independent for some $f \in F_v$ and distinct $e, e' \in [F_v] \cap K$. Then $\mathrm{cl}(E_0 + f + e + e') = \mathrm{cl}(E_0 + K_v)$.*

Proof. Recall that, by Claim 6.4, F_v is a base of $K_v \setminus \mathrm{cl}(E_0)$ in $\mathcal{C}_{n-1}/\mathrm{cl}(E_0)$. Since $E_0 + f + e + e'$ is independent and $f, e, e' \in [F_v]$, $\{f, e, e'\}$ spans F_v in $\mathcal{C}_{n-1}/\mathrm{cl}(E_0)$. This implies that $\mathrm{cl}(E_0 + f + e + e') = \mathrm{cl}(E_0 \cup F_v) = \mathrm{cl}(E_0 \cup K_v)$. □

We will assume henceforth that $F = \{f_1, f_2\} \in \binom{F_v}{2}$ is a good pair. This is justified by Claim 6.7. We next categorize $G_0 + f_i$ according to the properties of $\mathrm{cl}(E_0 + f_i)$. We say that $G_0 + f_i$ is:

Type 0 if $K_u \subseteq \mathrm{cl}(E_0 + f_i + e_1 + e_2)$ for some two edges $e_1, e_2 \in K_u$.

Type A if the graphs $A_1^i := G_0 + f_i + u_1u_2 + u_2u_3$, $A_2^i := G_0 + f_i + u_2u_3 + u_3u_1$, and $A_3^i := G_0 + f_i + u_3u_1 + u_1u_2$ have the following properties.

(1) Each A_j^i is C_2^1-independent and has one degree of freedom.

(2) Each $\mathrm{cl}(A_j^i) \cap K_u$ is as shown in Figure 6.

(3) For some (or equivalently, every) non-trivial motion $\boldsymbol{q}^i_j$ of $(A^i_j,\boldsymbol{p})$, $1\le j\le 3$, we have $Z(G_0+f_i,\boldsymbol{p}) = Z_0(G_0+f_i,\boldsymbol{p})\oplus\langle \boldsymbol{q}^i_1,\boldsymbol{q}^i_2,\boldsymbol{q}^i_3\rangle$.

Type B if the graphs $B^i_1 := G_0+f_i+u_1u_2+u_3u_4$, $B^i_2 := G_0+f_i+u_2u_3+u_3u_1$, and $B^i_3 := G_0+f_i+u_3u_1+u_1u_2$ have the following properties.

(1) Each B^i_j is C^1_2-independent and has one degree of freedom.

(2) Each $\mathrm{cl}(B^i_j)\cap K_u$ is as shown in Figure 7.

(3) For some (or equivalently, every) non-trivial motion $\boldsymbol{q}^i_j$ of $(B^i_j,\boldsymbol{p})$, $1\le j\le 3$, we have $Z(G_0+f_i,\boldsymbol{p}) = Z_0(G_0+f_i,\boldsymbol{p})\oplus\langle \boldsymbol{q}^i_1,\boldsymbol{q}^i_2,\boldsymbol{q}^i_3\rangle$.

(4) $K_v\subset \mathrm{cl}(B^i_1)$ and $\{u_1u_2,u_3u_4\}\subset [F_v]$.

Type C if the graphs $C^i_1 := G_0+f_i+u_1u_2+u_2u_3$, $C^i_2 := G_0+f_i+u_2u_3+u_3u_1$, and $C^i_3 := G_0+f_i+u_3u_1+u_1u_2$ have the following properties.

(1) Each C^i_j is C^1_2-independent and has one degree of freedom.

(2) Each $\mathrm{cl}(C^i_j)\cap K_u$ is as shown in Figure 8.

(3) For some (or equivalently, every) non-trivial motion $\boldsymbol{q}^i_j$ of $(C^i_j,\boldsymbol{p})$, $1\le j\le 3$, we have $Z(G_0+f_i,\boldsymbol{p}) = Z_0(G_0+f_i,\boldsymbol{p})\oplus\langle \boldsymbol{q}^i_1,\boldsymbol{q}^i_2,\boldsymbol{q}^i_3\rangle$.

(4) $K_v\subset \mathrm{cl}(C^i_1)$.

Type D if the graphs $D^i_1 := G_0+f_i+u_1u_2+u_3u_4$, $D^i_2 := G_0+f_i+u_1u_3+u_3u_4$, and $D^i_3 := G_0+f_i+u_3u_1+u_1u_2$ have the following properties.

(1) Each D^i_j is C^1_2-independent and has one degree of freedom.

(2) Each $\mathrm{cl}(D^i_j)\cap K_u$ is as shown in Figure 9.

(3) For some (or equivalently, every) non-trivial motion $\boldsymbol{q}^i_j$ of $(D^i_j,\boldsymbol{p})$, $1\le j\le 3$, we have $Z(G_0+f_i,\boldsymbol{p}) = Z_0(G_0+f_i,\boldsymbol{p})\oplus\langle \boldsymbol{q}^i_1,\boldsymbol{q}^i_2,\boldsymbol{q}^i_3\rangle$.

(4) $K_v\subset \mathrm{cl}(D^i_1)$.

In addition, for X,Y in $\{$A,B,C,D$\}$, we say that (G_0,F) is *type* XY if G_0+f_1 is type X and G_0+f_2 is type Y.

Claim 6.24. *Suppose $F=\{f_1,f_2\}$ is a good pair and $f_i\in F$. Then, up to a possible relabeling of u_1,u_2,u_3,u_4, G_0+f_i is type 0, A, B, C, or D, and, if neither G_0+f_1 nor G_0+f_2 is type 0, then (G_0,F) is type AA, AB, BA, BB, CC or DD.*

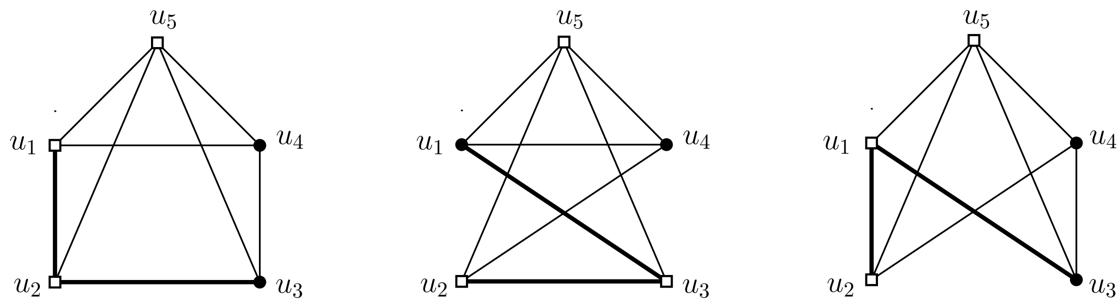
Proof. Since the claim holds trivially if G_0+f_i is type 0 for some $i\in\{1,2\}$, we may assume that this is not the case. Recall that K is the edge set of the complete graph on $\{u_1,u_2,u_3,u_4\}$. By Claim 6.21, $|[F_v]\cap K|$ is even. We will consider the different alternatives for $|[F_v]\cap K|$.

Suppose that $|[F_v]\cap K|=6$. Since F is a good pair, Claim 6.8 implies there is an edge $e\in K$ such that $(E_0\cup F)+e$ is independent. Since $e\in K=[F_v]\cap K$, this and (45) give $K_u\subset \mathrm{cl}(E_0\cup F_v)=\mathrm{cl}((E_0\cup F)+e)$. This contradicts Claim 6.13. Hence $|[F_v]\cap K|\in\{0,2,4\}$, and we split the proof accordingly.

Case 1: $|[F_v]\cap K|=0$. We show G_0+f_i is type A and hence (G_0,F) is type AA. Note that Claim 6.22 and the fact that $[F_v]\cap K=\emptyset$ imply that $\tilde{e}\in[\{f_i,e\}]$ for all $e\in K$, so every $e\in K$ is f_i-coupled.

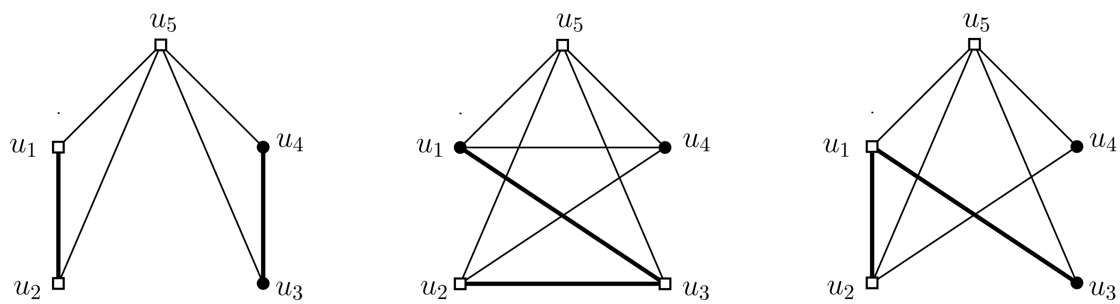
Let $H^i := G_0+f_i+u_1u_2+u_2u_3+u_3u_1$. If H^i is C^1_2-dependent then, by symmetry, we may assume that $u_3u_1\in \mathrm{cl}(E_0+f_i+u_1u_2+u_2u_3)$. Since each $e\in K$ is f_i-coupled, this gives $K\subset \mathrm{cl}(E_0+f_i+u_1u_2+u_2u_3)$, and contradicts our initial assumption that G_0+f_i is not type 0. Hence H^i is C^1_2-independent.

Since each A^i_j is a subgraph of H^i, each A^i_j is a C^1_2-independent 1-dof graph. The C^1_2-independence of H^i and the fact that each $e\in K$ is f_i-coupled imply that $\mathrm{cl}(A^i_j)\cap K_u$ is as shown in Figure 6. This in turn implies that $Z(G_0+f_i,\boldsymbol{p}) = Z_0(G_0+f_i,\boldsymbol{p})\oplus\langle \boldsymbol{q}^i_1,\boldsymbol{q}^i_2,\boldsymbol{q}^i_3\rangle$ for any non-trivial motions $\boldsymbol{q}^i_j$ of $(A^i_j,\boldsymbol{p})$, $1\le j\le 3$.



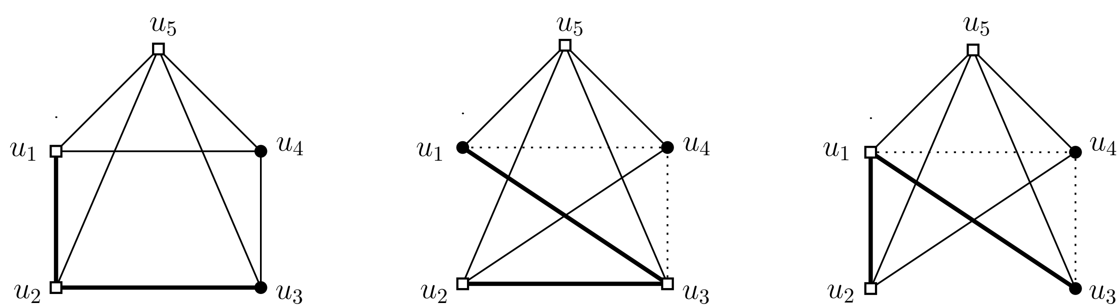
(i) $\mathrm{cl}(A_1^i) \cap K_u$ (ii) $\mathrm{cl}(A_2^i) \cap K_u$ (iii) $\mathrm{cl}(A_3^i) \cap K_u$

Figure 6: $\mathrm{cl}(A_j^i) \cap K_u$ for $1 \leq j \leq 3$.



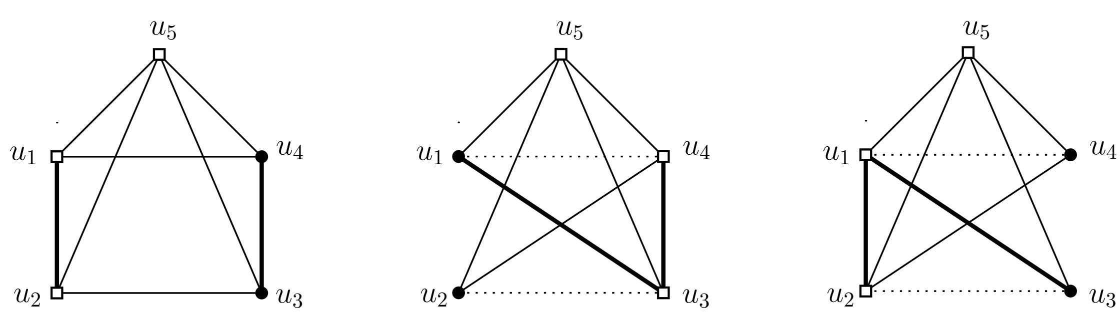
(i) $\mathrm{cl}(B_1^i) \cap K_u$ (ii) $\mathrm{cl}(B_2^i) \cap K_u$ (iii) $\mathrm{cl}(B_3^i) \cap K_u$

Figure 7: $\mathrm{cl}(B_j^i) \cap K_u$ for $1 \leq j \leq 3$.



(i) $\mathrm{cl}(C_1^i) \cap K_u$ (ii) $\mathrm{cl}(C_2^i) \cap K_u$ (iii) $\mathrm{cl}(C_3^i) \cap K_u$

Figure 8: $\mathrm{cl}(C_j^i) \cap K_u$ for $1 \leq j \leq 3$ (at most one dotted edge may exist in each graph).



(i) $\mathrm{cl}(D_1^i) \cap K_u$ (ii) $\mathrm{cl}(D_2^i) \cap K_u$ (iii) $\mathrm{cl}(D_3^i) \cap K_u$

Figure 9: $\mathrm{cl}(D_j^i) \cap K_u$ for $1 \leq j \leq 3$ (at most one dotted edge may exist in each graph).

Case 2: $|[F_v]\cap K| = 2$. By relabeling u_1,u_2,u_3,u_4 if necessary, we may assume that $[F_v]\cap K = \{u_1u_2, u_3u_4\}$. Then by Claim 6.22, each of the edges in $K\setminus[F_v]$ is f_i-coupled. We will show G_0+f_i is type A or type B, and hence (G_0,F) is type AA, AB, BA or BB.

Let $H^i := G_0+f_i+u_1u_2+u_2u_3+u_3u_1$. Since u_2u_3, u_3u_1 are f_i-coupled and G_0+f_i is not type 0, H^i is C_2^1-independent. If u_1u_2 is f_i-coupled, then we may use the same proof as in Case 1 to show that G_0+f_i is type A. Hence we may assume that $u_3u_4\notin[\{f_i,u_1u_2\}]$.

Since B_2^i and B_3^i are subgraphs of H^i, they are C_2^1-independent 1-dof graphs. The assumption that G_0+f_i is not type 0 now implies that $\mathrm{cl}(B_2^i)\cap K_u$ and $\mathrm{cl}(B_3^i)\cap K_u$ are as shown in Figure 7(ii)(iii). (The edge u_3u_4 does not exist in $\mathrm{cl}(B_2^i)\cap K_u$ since G_0+f_i is not type 0. The reason why the edge u_3u_4 does not exist in $\mathrm{cl}(B_3^i)\cap K_u$ will be clarified later.) The graph B_1^i is C_2^1-independent since $u_3u_4\notin[\{f_i,u_1u_2\}]$. Also, by Claim 6.23, $\mathrm{cl}(B_1^i)=\mathrm{cl}(E_0+f_i+u_1u_2+u_3u_4)=\mathrm{cl}(E_0+K_v)$. This gives $K_v\subset\mathrm{cl}(B_1^i)$. Since $[F_v]\cap K=\{u_1u_2,u_3u_4\}$, it also implies that $\mathrm{cl}(B_1^i)\cap K_u$ is as shown in Figure 7(i). The fact that $\mathrm{cl}(B_1^i)\cap K_u$ is as shown in Figure 7(i) now tells us that u_3u_4 does not exist in $\mathrm{cl}(B_3^i)\cap K_u$.

The assertion that $Z(G_0+f_i,\boldsymbol{p})=Z_0(G_0+f_i,\boldsymbol{p})\oplus\langle\boldsymbol{q}_1^i,\boldsymbol{q}_2^i,\boldsymbol{q}_3^i\rangle$ for any non-trivial motions $\boldsymbol{q}_j^i$ of $(B_j^i,\boldsymbol{p})$ easily follows from the fact that $\mathrm{cl}(B_j^i)\cap K_u$ is as in Figure 7.

Case 3: $|[F_v]\cap K|=4$. By relabeling u_1,u_2,u_3,u_4 if necessary, we may assume that $[F_v]\cap K=\{u_1u_2,u_2u_3,u_3u_4,u_4u_1\}$. Then by Claim 6.22, u_1u_3 and u_2u_4 are f_i-coupled. We will show that G_0+f_i is type C or type D. If $|[\{f_i,u_1u_2\}]\cap\{u_2u_3,u_3u_4,u_1u_4\}|\geq 2$ then, since u_1u_3 is coupled, $\mathrm{cl}(E_0+f_i+u_1u_2+u_1u_3)$ would contain at least five edges from K. This in turn would imply that $K_u\subset\mathrm{cl}(E_0+f_i+u_1u_2+u_1u_3)$ and contradict our initial assumption that G_0+f_i is not type 0. Hence

$$|[\{f_i,u_1u_2\}]\cap\{u_2u_3,u_3u_4,u_1u_4\}|\leq 1. \tag{46}$$

Consider the following two subcases:

Subcase 3-1: $[\{f_1,u_1u_2\}]\cap\{u_2u_3,u_1u_4\}=\{u_1u_4\}$ and $[\{f_2,u_1u_2\}]\cap\{u_2u_3,u_1u_4\}=\{u_2u_3\}$. We will show that (G_0,F) is type DD. We first show that $J^i:=G_0+f_i+u_1u_2+u_1u_3+u_3u_4$ is C_2^1-independent for both $i=1,2$. To see this, observe that $E_0+f_i+u_1u_2+u_3u_4$ is independent by (46) and the assumption of subcase 3-1. Hence, by Claim 6.23,

$$\mathrm{cl}(E_0+f_i+u_1u_2+u_3u_4)=\mathrm{cl}(E_0\cup F_v)=\mathrm{cl}(E_0\cup K_v). \tag{47}$$

Since $[F_v]\cap K=\{u_1u_2,u_2u_3,u_3u_4,u_4u_1\}$ we have $u_1u_3\notin\mathrm{cl}(E_0+f_i+u_1u_2+u_3u_4)$ and hence J^i is C_2^1-independent.

Since each D_j^i is a subgraph of J^i, each D_j^i is a C_2^1-independent 1-dof graph. We have $K_v\subset\mathrm{cl}(D_1^i)$ by (47). The assumption that G_0+f_i is not type 0 and (47) also imply that $\mathrm{cl}(D_j^i)\cap K_u$ is as shown in Figure 9. The assertion that $Z(G_0+f_i,\boldsymbol{p})=Z_0(G_0+f_i,\boldsymbol{p})\oplus\langle\boldsymbol{q}_1^i,\boldsymbol{q}_2^i,\boldsymbol{q}_3^i\rangle$ for any non-trivial motions $\boldsymbol{q}_j^i$ of $(D_j^i,\boldsymbol{p})$, $1\leq i\leq 3$, follows easily from the fact that $\mathrm{cl}(D_j^i)\cap K_u$ is as in Figure 9.

Subcase 3-2: Either $[\{f_i,u_1u_2\}]\cap\{u_2u_3,u_1u_4\}\subseteq\{u_1u_4\}$ for both $i=1,2$, or
$[\{f_i,u_1u_2\}]\cap\{u_2u_3,u_1u_4\}\subseteq\{u_2u_3\}$ for both $i=1,2$.
Relabeling if necessary, we may assume that the first alternative holds. We will show that (G_0,F) is type CC. We first show that $H^i:=G_0+u_1u_2+u_2u_3+u_3u_1$ is C_2^1-independent (for both $i=1,2$). To see this, observe that $E_0+f_i+u_1u_2+u_2u_3$ is independent since $[\{f_i,u_1u_2\}]\cap\{u_2u_3,u_1u_4\}\subseteq\{u_1u_4\}$. Hence, by Claim 6.23,

$$\mathrm{cl}(E_0+f_i+u_1u_2+u_2u_3)=\mathrm{cl}(E_0\cup F_v)=\mathrm{cl}(E_0\cup K_v). \tag{48}$$

Since $[F_v]\cap K=\{u_1u_2,u_2u_3,u_3u_4,u_4u_1\}$, we have $u_1u_3\notin\mathrm{cl}(E_0+f_i+u_1u_2+u_2u_3)$ and hence H^i is C_2^1-independent.

Since each C_j^i is a subgraph of H^i, each C_j^i is a C_2^1-independent 1-dof graph. We have $K_v\subset\mathrm{cl}(C_1^i)$ by (48). The assumption that G_0+f_i is not type 0 and (48) imply that $\mathrm{cl}(C_j^i)\cap K_u$ is as shown in Figure 8. The assertion that $Z(G_0+f_i,\boldsymbol{p})=Z_0(G_0+f_i,\boldsymbol{p})\oplus\langle\boldsymbol{q}_1^i,\boldsymbol{q}_2^i,\boldsymbol{q}_3^i\rangle$ for any non-trivial motions $\boldsymbol{q}_j^i$ of $(C_j^i,\boldsymbol{p})$ follows easily from the fact that $\mathrm{cl}(C_j^i)\cap K_u$ is as in Figure 8. □

6.3.5 The motion spaces of $(G_0+f_i, p|_{V_0})$ and $(G+f_i, p)$

We will obtain expressions for a non-trivial motion of each of the 1-dof frameworks $(A^i_j, \boldsymbol{p}|_{V_0})$, $(B^i_j, \boldsymbol{p}|_{V_0})$, $(C^i_j, \boldsymbol{p}|_{V_0})$ and $(D^i_j, \boldsymbol{p}|_{V_0})$. We first apply a projective transformation to $(G, \boldsymbol{p})$ to ensure that

$$\boldsymbol{p}(u_1) = (1,0),\ \boldsymbol{p}(u_2) = (0,0),\ \boldsymbol{p}(u_3) = (0,1),\ \boldsymbol{p}(u_4) = (1,1), \tag{49}$$

and all other points in $\boldsymbol{p}(V_0)$ are generically placed. Note that this transformation does not change the underlying C_2^1-cofactor matroid of $(G, \boldsymbol{p})$ by the projective invariance of C_2^1-rigidity (see [21] or Appendix C), and does not change the fact that $(G, \boldsymbol{p})$ has a bad motion at v_0 by Lemma 5.9.

For simplicity we use the notation

$$D_{ij} = D(\boldsymbol{p}(u_i), \boldsymbol{p}(u_j)) \quad \text{and} \quad \Delta_{ijk} = \Delta(\boldsymbol{p}(u_i), \boldsymbol{p}(u_j), \boldsymbol{p}(u_k))$$

throughout the remainder of this section.

Let $F = \{f_1, f_2\}$ be the fixed good pair, and suppose that $G_0 + f_i$ is type A for some $i \in \{1,2\}$. Since $(G, \boldsymbol{p})$ is 2-dof, $(G+f_i, \boldsymbol{p})$ is 1-dof. Let $\boldsymbol{q}^i$ be a non-trivial motion of $(G+f_i, \boldsymbol{p})$. Then $\boldsymbol{q}^i|_{V_0}$ is a motion of $(G_0+f_i, \boldsymbol{p}|_{V_0})$ and property (3) in the definition of type A implies that $\boldsymbol{q}^i|_{V_0}$ is a linear combination of $\boldsymbol{q}^i_1, \boldsymbol{q}^i_2, \boldsymbol{q}^i_3$ and a trivial motion, where $\boldsymbol{q}^i_j$ is a non-trivial motion of $(A^i_j, \boldsymbol{p}|_{V_0})$. By adding a suitable trivial motion to $\boldsymbol{q}^i$, we may assume that we have

$$\boldsymbol{q}^i|_{V_0} = \sum_{j=1}^{3} \alpha^i_j \boldsymbol{q}^i_j \tag{50}$$

for some scalars $\alpha^i_j \in \mathbb{R}$.

Claim 6.25. *Suppose that $G_0 + f_i$ is type A. Then for all $1 \le j \le 3$, $(A^i_j, \boldsymbol{p}|_{V_0})$ has a non-trivial motion $\boldsymbol{q}^i_j$ satisfying $\boldsymbol{q}^i_j(w) \in \mathbb{Q}(\boldsymbol{p}(V_0))$ for all $w \in V_0$, and*

- $\boldsymbol{q}^i_1(u_5) = \boldsymbol{q}^i_1(u_1) = \boldsymbol{q}^i_1(u_2) = 0, \boldsymbol{q}^i_1(u_3) = D_{3,2} \times D_{3,5}$, $\boldsymbol{q}^i_1(u_4) = t_1 D_{4,1} \times D_{4,5}$;
- $\boldsymbol{q}^i_2(u_5) = \boldsymbol{q}^i_2(u_2) = \boldsymbol{q}^i_2(u_3) = 0, \boldsymbol{q}^i_2(u_1) = D_{1,3} \times D_{1,5}$, $\boldsymbol{q}^i_2(u_4) = t_2 D_{4,2} \times D_{4,5}$;
- $\boldsymbol{q}^i_3(u_5) = \boldsymbol{q}^i_3(u_1) = \boldsymbol{q}^i_3(u_2) = 0, \boldsymbol{q}^i_3(u_3) = D_{3,1} \times D_{3,5}$, $\boldsymbol{q}^i_3(u_4) = t_3 D_{4,2} \times D_{4,5}$;

where $t_1 = \Delta_{523}/\Delta_{514}$, $t_2 = -\Delta_{513}/\Delta_{524}$ and $t_3 = \Delta_{513}/\Delta_{524}$.

In addition, $(G+f_i, \boldsymbol{p})$ has a non-trivial motion $\boldsymbol{q}^i$ such that $\boldsymbol{q}^i|_{V_0} = \sum_{j=1}^3 \alpha^i_j \boldsymbol{q}^i_j$ where:

$$\begin{aligned}
\alpha^i_1 &= -\Delta_{012}\Delta_{013}\Delta_{024}\Delta_{135}\Delta_{245}t_2 - \Delta_{013}^2\Delta_{024}\Delta_{135}^2 + \Delta_{013}\Delta_{023}\Delta_{024}\Delta_{135}\Delta_{245}t_3;\\
\alpha^i_2 &= \Delta_{012}\Delta_{013}\Delta_{014}\Delta_{135}\Delta_{145}t_1 - \Delta_{012}\Delta_{023}\Delta_{024}\Delta_{235}\Delta_{245}t_3;\\
\alpha^i_3 &= \Delta_{012}\Delta_{023}\Delta_{024}\Delta_{235}\Delta_{245}t_2 - \Delta_{013}\Delta_{014}\Delta_{023}\Delta_{135}\Delta_{145}t_1 + \Delta_{013}\Delta_{023}\Delta_{024}\Delta_{135}\Delta_{235}.
\end{aligned}$$

Proof. We will suppress the superscript i in $A^i_j, \boldsymbol{q}^i_j, \boldsymbol{q}^i, \alpha^i_j$ as it remains constant throughout the proof.

Consider $(A_1, \boldsymbol{p}|_{V_0})$. Since $\mathrm{cl}(A_1)$ contains the triangle on $\{u_1, u_2, u_5\}$ and $u_1u_3 \notin \mathrm{cl}(A_1)$, $(A_1, \boldsymbol{p}|_{V_0})$ has a non-trivial motion $\boldsymbol{q}_1$ such that $\boldsymbol{q}_1(u_1) = \boldsymbol{q}_1(u_2) = \boldsymbol{q}_1(u_5) = 0$ and $\boldsymbol{q}_1(u_3) \neq 0$. Since $\mathrm{cl}(A_1)$ contains u_2u_3 and u_3u_5, $\boldsymbol{q}_1(u_3) = s_1 D_{2,3} \times D_{3,5}$ for some scalar s_1. Since $\boldsymbol{q}_1(u_3) \neq 0$ we can scale $\boldsymbol{q}_1$, so that $s_1 = 1$. Since $\mathrm{cl}(A_1)$ contains u_1u_4 and u_4u_5, $\boldsymbol{q}_1(v_4) = t_1 D_{1,4} \times D_{4,5}$ for some scalar t_1. Since $u_3u_4 \in \mathrm{cl}(A_1)$, we have

$$0 = D_{34} \cdot (\boldsymbol{q}_1(u_3) - \boldsymbol{q}_1(u_4)) = \begin{vmatrix} D_{34} \\ D_{32} \\ D_{35} \end{vmatrix} - \begin{vmatrix} D_{43} \\ D_{41} \\ D_{45} \end{vmatrix} t_1 \text{ so } t_1 = \begin{vmatrix} D_{34} \\ D_{32} \\ D_{35} \end{vmatrix} \Big/ \begin{vmatrix} D_{43} \\ D_{41} \\ D_{45} \end{vmatrix}.$$

This can be simplified to $t_1 = \Delta_{523}/\Delta_{514}$ by using (9) and (49). Furthermore, since $(A_1, \boldsymbol{p}|_{V_0})$ is a 1-dof framework and $\boldsymbol{q}_1(u_3)$ takes fixed non-zero values in $\mathbb{Q}(\boldsymbol{p}(V_0))$, $\boldsymbol{q}_1$ is uniquely determined by Cramer's rule and will satisfy $\boldsymbol{q}_1(w) \in \mathbb{Q}(\boldsymbol{p}(V_0))$ for all $w \in V_0$.

We may apply the same argument, to deduce that $(A_2, \boldsymbol{p}|_{V_0})$ and $(A_3, \boldsymbol{p}|_{V_0})$ have the non-trivial motions $\boldsymbol{q}_2$ and $\boldsymbol{q}_3$ given in the claim.

We next prove the second part of the claim. The existence of a non-trivial motion $\boldsymbol{q}$ such that $\boldsymbol{q}|_{V_0} = \sum_{j=1}^{3} \alpha_j \boldsymbol{q}_j$ is established in (50). Since u_0u_k is an edge of G for all $1 \le k \le 5$, we have $D_{0,k} \cdot [\boldsymbol{q}(u_0) - \boldsymbol{q}(u_k)] = 0$ for all $1 \le k \le 5$. Hence we obtain the system of equations

$$D_{0,k} \cdot [\boldsymbol{q}(u_0) - \alpha_1 \boldsymbol{q}_1(u_k) - \alpha_2 \boldsymbol{q}_2(u_k) - \alpha_3 \boldsymbol{q}_3(u_k)] = 0 \text{ for } 1 \le k \le 5,$$

or in matrix form,

$$\begin{pmatrix} D_{0,1} \cdot \boldsymbol{q}_1(u_1) & D_{0,1} \cdot \boldsymbol{q}_2(u_1) & D_{0,1} \cdot \boldsymbol{q}_3(u_1) & D_{0,1} \\ D_{0,2} \cdot \boldsymbol{q}_1(u_2) & D_{0,2} \cdot \boldsymbol{q}_2(u_2) & D_{0,2} \cdot \boldsymbol{q}_3(u_2) & D_{0,2} \\ D_{0,3} \cdot \boldsymbol{q}_1(u_3) & D_{0,3} \cdot \boldsymbol{q}_2(u_3) & D_{0,3} \cdot \boldsymbol{q}_3(u_3) & D_{0,3} \\ D_{0,4} \cdot \boldsymbol{q}_1(u_4) & D_{0,4} \cdot \boldsymbol{q}_2(u_4) & D_{0,4} \cdot \boldsymbol{q}_3(u_4) & D_{0,4} \\ D_{0,5} \cdot \boldsymbol{q}_1(u_5) & D_{0,5} \cdot \boldsymbol{q}_2(u_5) & D_{0,5} \cdot \boldsymbol{q}_3(u_5) & D_{0,5} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ -a_0 \\ -b_0 \\ -c_0 \end{pmatrix} = 0, \tag{51}$$

where $(a_0, b_0, c_0) = \boldsymbol{q}(u_0)$. Substituting the zero values for $\boldsymbol{q}_j(u_k)$ given in the first part of the claim, we may rewrite this system as:

$$\begin{pmatrix} 0 & D_{0,1} \cdot \boldsymbol{q}_2(u_1) & 0 & D_{0,1} \\ 0 & 0 & 0 & D_{0,2} \\ D_{0,3} \cdot \boldsymbol{q}_1(u_3) & 0 & D_{0,3} \cdot \boldsymbol{q}_3(u_3) & D_{0,3} \\ D_{0,4} \cdot \boldsymbol{q}_1(u_4) & D_{0,4} \cdot \boldsymbol{q}_2(u_4) & D_{0,4} \cdot \boldsymbol{q}_3(u_4) & D_{0,4} \\ 0 & 0 & 0 & D_{0,5} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ -a_0 \\ -b_0 \\ -c_0 \end{pmatrix} = 0. \tag{52}$$

By scaling $\boldsymbol{q}$, we may suppose that $\alpha_1 + \alpha_2 + \alpha_3 = \mu$, where μ is a non-zero number which will be chosen later. Let M' be the matrix of coefficients in (52) and M denote the square matrix $\begin{pmatrix} b \\ M' \end{pmatrix}$ where $b = (1, 1, 1, 0, 0, 0)$. Then we may rewrite (52) as $M(\alpha_1, \alpha_2, \alpha_3, -a_0, -b_0, -c_0)^\top = (\mu, 0, 0, 0, 0, 0)^\top$ and Cramer's rule gives

$$\alpha_1 = \frac{\mu}{\det M} \begin{vmatrix} D_{0,1} \cdot \boldsymbol{q}_2(u_1) & 0 & D_{0,1} \\ 0 & 0 & D_{0,2} \\ 0 & D_{0,3} \cdot \boldsymbol{q}_3(u_3) & D_{0,3} \\ D_{0,4} \cdot \boldsymbol{q}_2(u_4) & D_{0,4} \cdot \boldsymbol{q}_3(u_4) & D_{0,4} \\ 0 & 0 & D_{0,5} \end{vmatrix}$$

$$\alpha_2 = -\frac{\mu}{\det M} \begin{vmatrix} 0 & 0 & D_{0,1} \\ 0 & 0 & D_{0,2} \\ D_{0,3} \cdot \boldsymbol{q}_1(u_3) & D_{0,3} \cdot \boldsymbol{q}_3(u_3) & D_{0,3} \\ D_{0,4} \cdot \boldsymbol{q}_1(u_4) & D_{0,4} \cdot \boldsymbol{q}_3(u_4) & D_{0,4} \\ 0 & 0 & D_{0,5} \end{vmatrix}$$

$$\alpha_3 = \frac{\mu}{\det M} \begin{vmatrix} 0 & D_{0,1} \cdot \boldsymbol{q}_2(u_1) & D_{0,1} \\ 0 & 0 & D_{0,2} \\ D_{0,3} \cdot \boldsymbol{q}_1(u_3) & 0 & D_{0,3} \\ D_{0,4} \cdot \boldsymbol{q}_1(u_4) & D_{0,4} \cdot \boldsymbol{q}_2(u_4) & D_{0,4} \\ 0 & 0 & D_{0,5} \end{vmatrix}$$

Note that each entry of the form $D_{0,k} \cdot \boldsymbol{q}_j(u_k)$ in the above determinants can be written as a product of areas of three triangles. For example, we have $D_{01} \cdot \boldsymbol{q}_2(u_1) = \begin{vmatrix} D_{10} \\ D_{13} \\ D_{15} \end{vmatrix} = -\Delta_{103}\Delta_{135}\Delta_{150}$, where the first equation follows from the first part of the claim and the second equation follows from the Vadermonde identity (9).

We may expand the determinant in the formula for α_1 to obtain

$$\begin{aligned}\frac{\det M}{\mu}\alpha_1 &= D_{0,1}\cdot \boldsymbol{q}_2(u_1)\left(-D_{0,3}\cdot \boldsymbol{q}_3(u_3)\begin{vmatrix} D_{0,2}\\ D_{0,4}\\ D_{0,5}\end{vmatrix} + D_{0,4}\cdot \boldsymbol{q}_3(u_4)\begin{vmatrix} D_{0,2}\\ D_{0,3}\\ D_{0,5}\end{vmatrix}\right)\\ &\quad - D_{0,4}\cdot \boldsymbol{q}_2(u_4) D_{0,3}\cdot \boldsymbol{q}_3(u_3)\begin{vmatrix} D_{0,1}\\ D_{0,2}\\ D_{0,5}\end{vmatrix}\\ &= \begin{vmatrix} D_{1,0}\\ D_{1,3}\\ D_{1,5}\end{vmatrix}\left(-\begin{vmatrix} D_{3,0}\\ D_{3,1}\\ D_{3,5}\end{vmatrix}\begin{vmatrix} D_{0,2}\\ D_{0,4}\\ D_{0,5}\end{vmatrix} + t_3\begin{vmatrix} D_{4,0}\\ D_{4,2}\\ D_{4,5}\end{vmatrix}\begin{vmatrix} D_{0,2}\\ D_{0,3}\\ D_{0,5}\end{vmatrix}\right) - t_2\begin{vmatrix} D_{4,0}\\ D_{4,2}\\ D_{4,5}\end{vmatrix}\begin{vmatrix} D_{3,0}\\ D_{3,1}\\ D_{3,5}\end{vmatrix}\begin{vmatrix} D_{0,1}\\ D_{0,2}\\ D_{0,5}\end{vmatrix}\\ &= (\Delta_{013}\Delta_{135}\Delta_{015}\Delta_{035}\Delta_{024}\Delta_{045}\Delta_{025})(-\Delta_{013}\Delta_{135} + t_3\Delta_{245}\Delta_{023} - t_2\Delta_{245}\Delta_{012}).\end{aligned}$$

The required expression for α_1 follows by putting $\mu = \det M/\Delta_{015}\Delta_{025}\Delta_{035}\Delta_{045}$.

Similar calculations give the required formulae for α_2 and α_3 (using the same constant μ). □

We next obtain an analogous claim for the case when $(G+f_i)$ is type B.

Claim 6.26. *Suppose that G_0+f_i is type B. Then for all $1\le j\le 3$, $(B^i_j, \boldsymbol{p}|_{V_0})$ has a non-trivial motion $\boldsymbol{q}^i_j$ satisfying $\boldsymbol{q}^i_j(w)\in\mathbb{Q}(\boldsymbol{p}(V_0))$ for all $w\in V_0$, and*

- $\boldsymbol{q}^i_1(u_5)=\boldsymbol{q}^i_1(u_1)=\boldsymbol{q}^i_1(u_2)=0$,
  $\boldsymbol{q}^i_1(u_3)=D_{3,1}\times D_{3,5}+s_{12}D_{3,2}\times D_{3,5}$,
  $\boldsymbol{q}^i_1(u_4)=t_{11}D_{4,1}\times D_{4,5}+t_{12}D_{4,2}\times D_{4,5}$;
- $\boldsymbol{q}^i_2(u_5)=\boldsymbol{q}^i_2(u_2)=\boldsymbol{q}^i_2(u_3)=0, \boldsymbol{q}^i_2(u_1)=D_{1,3}\times D_{1,5}$, $\boldsymbol{q}^i_2(u_4)=t_2D_{4,2}\times D_{4,5}$;
- $\boldsymbol{q}^i_3(u_5)=\boldsymbol{q}^i_3(u_1)=\boldsymbol{q}^i_3(u_2)=0, \boldsymbol{q}^i_3(u_3)=D_{1,3}\times D_{3,5}$, $\boldsymbol{q}^i_3(u_4)=t^i_3D_{4,2}\times D_{4,5}$;

for $t_2=-\Delta_{513}/\Delta_{524}$ and some non-zero constants $s_{12},t_{11},t_{12},t^i_3\in\mathbb{Q}(\boldsymbol{p}(V_0))$, and we may take s_{12},t_{11},t_{12} to be the same constants for both $\boldsymbol{q}^1_1$ and $\boldsymbol{q}^2_1$ when (G_0,F) is type BB.

In addition, $(G+f_i,\boldsymbol{p})$ has a non-trivial motion $\boldsymbol{q}^i$ such that $\boldsymbol{q}^i|_{V_0}=\sum_{j=1}^3\beta^i_j\boldsymbol{q}^i_j$ where:

$$\begin{aligned}\beta^i_1 &= -\Delta_{012}\Delta_{013}\Delta_{024}\Delta_{135}\Delta_{245}t_2 - \Delta_{013}^2\Delta_{024}\Delta_{135}^2 + \Delta_{013}\Delta_{023}\Delta_{024}\Delta_{135}\Delta_{245}t^i_3;\\ \beta^i_2 &= \Delta_{012}\Delta_{013}\Delta_{014}\Delta_{135}\Delta_{145}t_{11} - \Delta_{012}\Delta_{013}\Delta_{024}\Delta_{135}\Delta_{245}t^i_3\\ &\quad + \Delta_{012}\Delta_{013}\Delta_{024}\Delta_{135}\Delta_{245}t_{12} - \Delta_{012}\Delta_{023}\Delta_{024}\Delta_{235}\Delta_{245}s_{12}t^i_3;\\ \beta^i_3 &= \Delta_{012}\Delta_{013}\Delta_{024}\Delta_{135}\Delta_{245}t_2 + \Delta_{012}\Delta_{023}\Delta_{024}\Delta_{235}\Delta_{245}s_{12}t_2\\ &\quad + \Delta_{013}^2\Delta_{024}\Delta_{135}^2 - \Delta_{013}\Delta_{014}\Delta_{023}\Delta_{135}\Delta_{145}t_{11}\\ &\quad + \Delta_{013}\Delta_{023}\Delta_{024}\Delta_{135}\Delta_{235}s_{12} - \Delta_{013}\Delta_{023}\Delta_{024}\Delta_{135}\Delta_{245}t_{12}.\end{aligned}$$

Proof. We will suppress the superscript i in $B^i_j,\boldsymbol{q}^i_j,\beta^i_j,t^i_3$ as it remains constant throughout the proof.

We first consider $(B_2,\boldsymbol{p}|_{V_0})$. Since $K(u_2,u_3,u_5)\subset \mathrm{cl}(B_2)$ and $u_1u_2\notin \mathrm{cl}(B_2)$, $(B_2,\boldsymbol{p}|_{V_0})$ has a non-trivial motion $\boldsymbol{q}_2$ such that $\boldsymbol{q}_2(u_2)=\boldsymbol{q}_2(u_3)=\boldsymbol{q}_2(u_5)=0$ and $\boldsymbol{q}_2(u_1)\neq 0$. Since $\mathrm{cl}(B_2)$ contains u_1u_3 and u_1u_5, $\boldsymbol{q}_2(u_1)=s_2D_{1,3}\times D_{1,5}$ for some scalar s_2. Since $\boldsymbol{q}_2(u_1)\neq 0$ we can scale $\boldsymbol{q}_2$, so that $s_2=1$. Since $\mathrm{cl}(B_2)$ contains u_4u_2 and u_4u_5, $\boldsymbol{q}_2(v_4)=t_2D_{4,2}\times D_{4,5}$ for some scalar t_2. Since $u_1u_4\in \mathrm{cl}(B_2)$, we have $0=D_{14}\cdot(\boldsymbol{q}_2(u_1)-\boldsymbol{q}_2(u_4))$ and we can now use (9) and (49) to deduce that $t_2=-\Delta_{513}/\Delta_{524}$. Furthermore, since $(B_2,\boldsymbol{p}|_{V_0})$ is a 1-dof framework and $\boldsymbol{q}_2(u_1)$ takes fixed non-zero values in $\mathbb{Q}(\boldsymbol{p}(V_0))$, $\boldsymbol{q}_2$ is uniquely determined by Cramer's rule and will satisfy $\boldsymbol{q}_2(w)\in\mathbb{Q}(\boldsymbol{p}(V_0))$ for all $w\in V_0$.

We can apply the same argument to $(B_3,\boldsymbol{p}|_{V_0})$ using the fact that $K(u_1,u_2,u_5)\subset B_3$ to obtain the formula for $\boldsymbol{q}_3$. Note that $\boldsymbol{q}_3(u_3)\neq 0\neq \boldsymbol{q}_3(u_4)$ since $u_1u_4,u_2u_3\notin \mathrm{cl}(B_3)$ and that we cannot obtain an explicit formula for t_3 because $u_3u_4\notin \mathrm{cl}(B_3)$.

We next consider $(B_1, \boldsymbol{p}|_{V_0})$. Since $B_1 - u_3u_4 + u_3u_1 = B_3$, $\boldsymbol{q}_3$ is a motion of $(B_1 - u_3u_4, \boldsymbol{p}|_{V_0})$ satisfying $\boldsymbol{q}_3(u_5) = \boldsymbol{q}_3(u_1) = \boldsymbol{q}_3(u_2) = 0, \boldsymbol{q}_3(u_3) = D_{1,3} \times D_{3,5}, \boldsymbol{q}_3(u_4) = t_3 D_{4,2} \times D_{4,5}$ and $D_{3,4} \times (\boldsymbol{q}_3(u_3) - \boldsymbol{q}_3(u_4)) \neq 0$. We may use the symmetry between u_3 and u_4 in type B, see Figure 7, to deduce that $(B_1 - u_3u_4, \boldsymbol{p}|_{V_0})$ also has a nontrivial motion $\bar{\boldsymbol{q}}_3$ satisfying $\bar{\boldsymbol{q}}_3(u_5) = \bar{\boldsymbol{q}}_3(u_1) = \bar{\boldsymbol{q}}_3(u_2) = 0, \bar{\boldsymbol{q}}_3(u_3) = D_{2,3} \times D_{3,5}, \bar{\boldsymbol{q}}_3(u_4) = \bar{t}_3 D_{4,1} \times D_{4,5}$ and $D_{3,4} \times (\bar{\boldsymbol{q}}_3(u_3) - \bar{\boldsymbol{q}}_3(u_4)) \neq 0$. (The motion $\bar{\boldsymbol{q}}_3$ is a non-trivial motion of $\bar{B}_3 := B_1 - u_3u_4 + u_4u_1$ and we have $\mathrm{cl}(\bar{B}_3) \cap K_u = (\mathrm{cl}(B_3) \cap K_u) - u_1u_3 - u_2u_4 + u_1u_4 + u_2u_3$.)

We can now obtain the non-trivial motion $\boldsymbol{q}_1$ of $(B_1, \boldsymbol{p}|_{V_0})$ satisfying $\boldsymbol{q}_1(u_5) = \boldsymbol{q}_1(u_1) = \boldsymbol{q}_1(u_2) = 0$, $\boldsymbol{q}_1(u_3) = s_{11} D_{3,1} \times D_{3,5} + s_{12} D_{3,2} \times D_{3,5}$ and $\boldsymbol{q}_1(u_4) = t_{11} D_{4,1} \times D_{4,5} + t_{12} D_{4,2} \times D_{4,5}$ by choosing a non-trivial linear combination of $\boldsymbol{q}_3$ and $\bar{\boldsymbol{q}}_3$ which will satisfy the constraint given by the edge u_3u_4. The constants $s_{11}, s_{12}, t_{11}, t_{12}$ will all be non-zero since $u_3u_2, u_3u_1, u_4u_2, u_4u_1 \notin \mathrm{cl}(B_1)$, and so we can scale $\boldsymbol{q}_1$ to ensure that $s_{11} = 1$.

It remains to show that we can choose the same values for $s^i_{12}, t^i_{11}, t^i_{12}$ when $i = 1,2$ and $G_0 + f_1$ and $G_0 + f_2$ are both type B. Property (4) in the definition of type B gives $\{u_1u_2, u_3u_4\} \subset [F_v]$. Claim 6.23 and the independence of B_1 now imply that $\mathrm{cl}(B^1_1) = \mathrm{cl}(E_0 + K_v) = \mathrm{cl}(B^2_1)$. This in turn implies that $Z(B^1_1, \boldsymbol{p}|_{V_0}) = Z(B^2_1, \boldsymbol{p}|_{V_0})$, so we can take a common non-trivial motion that has the form stated in the claim. This completes the proof of the first part of the claim.

We next prove the second part of the claim. We can construct a non-trivial motion $\boldsymbol{q}^i$ of $(G + f_i, \boldsymbol{p})$ with $\boldsymbol{q}^i|_{V_0} = \sum_{j=1}^3 \beta^i_j \boldsymbol{q}^i_j$ for some $\beta^i_j \in \mathbb{R}$ by adding a suitable trivial motion to an arbitrary non-trivial motion of $(G + f_i, \boldsymbol{p})$, as in the derivation of (50). We need to show we can choose the β^i_j to take the values stated in the claim. The proof is identical to that of the second part of Claim 6.25 so we only give a sketch. As i is fixed, we once more suppress the superscripts on $\beta^i, \boldsymbol{q}^i$ and $\boldsymbol{q}^i_j$.

Since u_0u_k is an edge of G for all $1 \leq k \leq 5$, we have $D_{0,k} \cdot [\boldsymbol{q}(u_0) - \boldsymbol{q}(u_k)] = 0$ for all $1 \leq k \leq 5$. Hence we obtain the system of equations

$$D_{0,k} \cdot [\boldsymbol{q}(u_0) - \beta_1 \boldsymbol{q}_1(u_k) - \beta_2 \boldsymbol{q}_2(u_k) - \beta_3 \boldsymbol{q}_3(u_k)] = 0 \text{ for } 1 \leq k \leq 5.$$

Putting $(a_0, b_0, c_0) = \boldsymbol{q}(u_0)$ and substituting the zero values for $\boldsymbol{q}_j(u_k)$ given in the first part of the claim, we may rewrite this system as:

$$\begin{pmatrix} 0 & D_{0,1} \cdot \boldsymbol{q}_2(u_1) & 0 & D_{0,1} \\ 0 & 0 & 0 & D_{0,2} \\ D_{0,3} \cdot \boldsymbol{q}_1(u_3) & 0 & D_{0,3} \cdot \boldsymbol{q}_3(u_3) & D_{0,3} \\ D_{0,4} \cdot \boldsymbol{q}_1(u_4) & D_{0,4} \cdot \boldsymbol{q}_2(u_4) & D_{0,4} \cdot \boldsymbol{q}_3(u_4) & D_{0,4} \\ 0 & 0 & 0 & D_{0,5} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ -a_0 \\ -b_0 \\ -c_0 \end{pmatrix} = 0. \tag{53}$$

Each entry of the form $D_{0,k} \cdot \boldsymbol{q}_j(u_k)$ in the above matrix can be expressed as a product of areas of three triangles by using the non-zero values for $\boldsymbol{q}_j(u_k)$ given in the first part of the claim. By scaling $\boldsymbol{q}$, we may suppose that $\beta_1 + \beta_2 + \beta_3 = \mu$ for some constant μ. We can now proceed as in the proof of Claim 6.25 and obtain the stated formula for β_j by setting $\mu = \det M / \Delta_{015} \Delta_{025} \Delta_{035} \Delta_{045}$, where $M = \begin{pmatrix} b \\ M' \end{pmatrix}$, M' is the matrix of coefficients in (53) and $b = (1,1,1,0,0,0)$. □

We next state the analogous claims for the cases when $(G + f_i)$ is type C or D without proof as their proofs are identical to the proof of Claim 6.25.

Claim 6.27. *Suppose that $G_0 + f_i$ is type C. Then for all $1 \leq j \leq 3$, $(C^i_j, \boldsymbol{p}|_{V_0})$ has a non-trivial motion $\boldsymbol{q}^i_j$ satisfying $\boldsymbol{q}^i_j(w) \in \mathbb{Q}(\boldsymbol{p}(V_0))$ for all $w \in V_0$, and*

- $\boldsymbol{q}^i_1(u_5) = \boldsymbol{q}^i_1(u_1) = \boldsymbol{q}^i_1(u_2) = 0, \boldsymbol{q}^i_1(u_3) = D_{3,2} \times D_{3,5}$, $\boldsymbol{q}^i_1(u_4) = t_1 D_{4,1} \times D_{4,5}$;
- $\boldsymbol{q}^i_2(u_5) = \boldsymbol{q}^i_2(u_2) = \boldsymbol{q}^i_2(u_3) = 0, \boldsymbol{q}^i_2(u_1) = D_{1,3} \times D_{1,5}$, $\boldsymbol{q}^i_2(u_4) = t^i_2 D_{4,2} \times D_{4,5}$;

- $\boldsymbol{q}_3^i(u_5) = \boldsymbol{q}_3^i(u_1) = \boldsymbol{q}_3^i(u_2) = 0, \boldsymbol{q}_3^i(u_3) = D_{3,1} \times D_{3,5}$, $\boldsymbol{q}_3^i(u_4) = t_3^i D_{4,2} \times D_{4,5}$;

for $t_1 = \Delta_{523}/\Delta_{514}$ and some $t_2^i, t_3^i \in \mathbb{Q}(\boldsymbol{p}(V_0))$.

In addition, $(G+f_i, \boldsymbol{p})$ has a non-trivial motion $\boldsymbol{q}^i$ such that $\boldsymbol{q}^i|_{V_0} = \sum_{j=1}^3 \gamma_j^i \boldsymbol{q}_j^i$ where:

$$\gamma_2^i = \Delta_{012}\Delta_{013}\Delta_{014}\Delta_{135}\Delta_{145}t_1 - \Delta_{012}\Delta_{023}\Delta_{024}\Delta_{235}\Delta_{245}t_3^i;$$
$$\gamma_3^i = \Delta_{012}\Delta_{023}\Delta_{024}\Delta_{235}\Delta_{245}t_2^i - \Delta_{013}\Delta_{014}\Delta_{023}\Delta_{135}\Delta_{145}t_1 + \Delta_{013}\Delta_{023}\Delta_{024}\Delta_{135}\Delta_{235}.$$

Claim 6.28. *Suppose that $G_0 + f_i$ is type D. Then for all $1 \le j \le 3$, $(D_j^i, \boldsymbol{p}|_{V_0})$ has a non-trivial motion $\boldsymbol{q}_j^i$ satisfying $\boldsymbol{q}_j^i(w) \in \mathbb{Q}(\boldsymbol{p}(V_0))$ for all $w \in V_0$, and*

- $\boldsymbol{q}_1^i(u_5) = \boldsymbol{q}_1^i(u_1) = \boldsymbol{q}_1^i(u_2) = 0, \boldsymbol{q}_1^i(u_3) = D_{3,2} \times D_{3,5}$, $\boldsymbol{q}_1^i(u_4) = t_1 D_{4,1} \times D_{4,5}$;
- $\boldsymbol{q}_2^i(u_5) = \boldsymbol{q}_2^i(u_3) = \boldsymbol{q}_2^i(u_4) = 0, \boldsymbol{q}_2^i(u_1) = s_2^i D_{1,3} \times D_{1,5}$, $\boldsymbol{q}_2^i(u_2) = t_2^i D_{2,4} \times D_{2,5}$;
- $\boldsymbol{q}_3^i(u_5) = \boldsymbol{q}_3^i(u_1) = \boldsymbol{q}_3^i(u_2) = 0, \boldsymbol{q}_3^i(u_3) = s_3^i D_{3,1} \times D_{3,5}$, $\boldsymbol{q}_3^i(u_4) = t_3^i D_{4,2} \times D_{4,5}$;

for $t_1 = \Delta_{523}/\Delta_{514}$ and some $s_2^i, t_2^i, s_3^i, t_3^i \in \mathbb{Q}(\boldsymbol{p}(V_0))$ with $\Delta_{513}s_j^i - \Delta_{524}t_j^i \neq 0$ for $j = 2,3$.[2]

In addition, $(G+f_i, \boldsymbol{p})$ has a non-trivial motion $\boldsymbol{q}^i$ such that $\boldsymbol{q}^i|_{V_0} = \sum_{j=1}^3 \delta_j^i \boldsymbol{q}_j^i$ where:

$$\begin{aligned}\delta_2^i =& \Delta_{012}\Delta_{013}\Delta_{014}\Delta_{135}\Delta_{145}s_3^i t_1 - \Delta_{012}\Delta_{023}\Delta_{024}\Delta_{235}\Delta_{245}t_3^i;\\ \delta_3^i =& -\Delta_{013}\Delta_{014}\Delta_{023}\Delta_{135}\Delta_{145}s_2^i t_1 + \Delta_{013}\Delta_{014}\Delta_{024}\Delta_{145}\Delta_{245}t_1 t_2^i\\ &+ \Delta_{013}\Delta_{023}\Delta_{024}\Delta_{135}\Delta_{235}s_2^i - \Delta_{014}\Delta_{023}\Delta_{024}\Delta_{235}\Delta_{245}t_2^i.\end{aligned}$$

We will show that each of the above expressions for a motion of $(G+f_i, p)$ lead to a contradiction. We first show that the motion given by Claim 6.25 cannot occur.

Claim 6.29. *$G_0 + f_i$ cannot be of type A.*

Proof. Suppose, for a contradiction, that $G_0 + f_i$ is type A. Let $\boldsymbol{q}_j^i$ be the nontrivial motion of $(A_j^i, \boldsymbol{p}|_{V_0})$ given by Claim 6.25. We will omit the superscript i from $\boldsymbol{q}_j^i$ throughout the proof as it remains fixed. Let $\boldsymbol{p}(u_0) = (x_0, y_0)$ and $\boldsymbol{p}(u_5) = (x_5, y_5)$, and recall that the values of $\boldsymbol{p}(u_k)$, $1 \le k \le 4$, are given by (49). We will contradict the choice of G by showing that $(G+f_i, \boldsymbol{p})$ is a 1-dof framework with a bad motion at u_0.

By Claim 6.25, $(G+f_i, \boldsymbol{p})$ has a non-trivial motion $\boldsymbol{q}^i$ such that $\boldsymbol{q}^i|_{V_0} = \sum_{j=1}^3 \alpha_j \boldsymbol{q}_j$. The formulae for α_j, $j = 1,2,3$, and $\boldsymbol{q}_j(u_k)$, $1 \le k \le 5$, given in Claim 6.26 imply that each coordinate of $\boldsymbol{q}^i(u_k)$, $1 \le k \le 5$, can be expressed as a polynomial in x_0, y_0, x_5, y_5 over $\mathbb{Q}$. We show that the same is true for $\boldsymbol{q}^i(u_0)$.

Let $\boldsymbol{q}^i(u_0) = (a_0, b_0, c_0)$. We recall the the following linear system in the proof of Claim 6.25:

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & D_{0,1}\cdot\boldsymbol{q}_2(u_1) & 0 & D_{0,1} \\ 0 & 0 & 0 & D_{0,2} \\ D_{0,3}\cdot\boldsymbol{q}_1(u_3) & 0 & D_{0,3}\cdot\boldsymbol{q}_3(u_3) & D_{0,3} \\ D_{0,4}\cdot\boldsymbol{q}_1(u_4) & D_{0,4}\cdot\boldsymbol{q}_2(u_4) & D_{0,4}\cdot\boldsymbol{q}_3(u_4) & D_{0,4} \\ 0 & 0 & 0 & D_{0,5} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ -a_0 \\ -b_0 \\ -c_0 \end{pmatrix} = \begin{pmatrix} \mu \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{54}$$

where $\mu = \alpha_1 + \alpha_2 + \alpha_3 \in \mathbb{Q}(\boldsymbol{p}(\hat{N}_G(u_0)))$ takes the value given at the end of the proof of Claim 6.25 so can be considered as a rational function of x_0, y_0, x_5, y_5 over $\mathbb{Q}$. Since each entry of the matrix of coefficients in (54) is a polynomial in x_0, y_0, x_5, y_5, each of a_0, b_0, c_0 can be expressed as a rational function of x_0, y_0, x_5, y_5 over $\mathbb{Q}$. Hence, by a further scaling $\boldsymbol{q}^i$, we can suppose that each component of $\boldsymbol{q}^i(u_k)$, $0 \le k \le 5$, can be

[2]The expression for q_2^i contains two parameters s_2^i, t_2^i because one of the edges u_1u_4, u_2u_3 may exist in $\mathrm{cl}(D_2^i)$, and hence we could have $q_2^i(u_1) = 0$ or $q_2^i(u_2) = 0$. We can deduce that $\Delta_{513}s_2^i - \Delta_{524}t_2^i \neq 0$ from the fact that the edge u_1u_2 is not in $\mathrm{cl}(D_2^i)$. A similar remark holds for q_3^i.

expressed as a polynomial in x_0, y_0, x_5, y_5. This gives us a map $b : \hat{N}_G(u_0) \to \mathbb{Q}[X_0, Y_0, X_5, Y_5]^3$ such that $\boldsymbol{q}^i$ is a b-motion on $\hat{N}_G(u_0)$.

We next show that b satisfies (23) i.e. the polynomial identity given by an edge $u_k u_\ell$, $1 \le k < \ell \le 5$, is satisfied if and only if $\ell = 5$. Since x_0, y_0, x_5, y_5 are generic, this is equivalent to showing that the motion $\boldsymbol{q}^i$ satisfies the constraint corresponding to the edge $u_k u_\ell$, if and only if $\ell = 5$. Since the current motion $\boldsymbol{q}^i$, is a non-zero scalar multiple of our original motion it will suffice to show this for the original $\boldsymbol{q}^i$. (More specifically, it will suffice to show $D_{k,\ell} \cdot (\boldsymbol{q}^i(u_k) - \boldsymbol{q}^i(u_\ell)) \neq 0$ for $1 \le k < \ell \le 4$ as $\boldsymbol{q}^i$ is a motion of $(G + f_i, \boldsymbol{p})$ and $\mathrm{cl}(G + f_i)$ contains the star on $N_G(u_0)$ centered at u_5.) This follows since $\boldsymbol{q}^i|_{V_0} = \sum_{j=1}^3 \alpha_j \boldsymbol{q}_j$ and each $u_k u_\ell$ is an edge of $\mathrm{cl}(A^i_j)$ for exactly two values $1 \le j \le 3$. Thus $(G + f_i, \boldsymbol{p})$ has a bad motion at u_0.

Although $(G + f_i, \boldsymbol{p})$ is not generic as it satisfies (49), Lemma 5.9 ensures that we may use a projective transformation to construct a generic 1-dof framework $(G + f_i, \boldsymbol{p}')$ which has a bad motion at u_0. This contradicts our initial choice of $(G, \boldsymbol{p})$ to minimize $|V|$ and then k since $k = 2$ and $G + f_i$ is 1-dof. □

Our proof that the motions given by Claim 6.26, 6.27 and 6.28 cannot occur is more complicated because the expressions for the motions $\boldsymbol{q}^i_j$ in these claims contain undetermined parameters. We first give a preliminary claim which shows that specific values of these parameters cannot occur when $G_0 + f_i$ is type B.

Claim 6.30. *Suppose that $G_0 + f_i$ is type B. Let $\boldsymbol{q}^i$ and $\boldsymbol{q}^i_j$ be the non-trivial motion of $(G + f_i, \boldsymbol{p})$ and $(B^i_j, \boldsymbol{p}|_{V_0})$, respectively, given in Claim 6.26. Then*

$$(s_{12}, t_{11}, t_{12}, t_2, t^i_3) \neq \left(-\frac{\Delta_{513}}{\Delta_{523}}, \frac{\Delta_{513}}{\Delta_{514}}, -\frac{\Delta_{513}}{\Delta_{524}}, -\frac{\Delta_{513}}{\Delta_{524}}, -\frac{\Delta_{513}}{\Delta_{524}}\right).$$

Proof. Suppose that $(s_{12}, t_{11}, t_{12}, t_2, t^i_3) = \left(-\frac{\Delta_{513}}{\Delta_{523}}, \frac{\Delta_{513}}{\Delta_{514}}, -\frac{\Delta_{513}}{\Delta_{524}}, -\frac{\Delta_{513}}{\Delta_{524}}, -\frac{\Delta_{513}}{\Delta_{524}}\right)$. Then we can use the same argument as in the proof of Claim 6.29 to show that some scalar multiple of $\boldsymbol{q}^i$ is a b-motion of $(G + f_i, \boldsymbol{p})$ on $\hat{N}_G(u_0)$ for some $b : \hat{N}_G(u_0) \to \mathbb{Q}[X_0, Y_0, X_5, Y_5]^3$. In order to verify that b satisfies (23) it suffices to show that b satisfies $D_{k,\ell} \cdot (\boldsymbol{q}^i(u_k) - \boldsymbol{q}^i(u_\ell)) \neq 0$ for all $u_k u_\ell \in K(u_1, u_2, u_3, u_4)$. This inequality holds for the edges u_1u_2, u_1u_3, u_2u_4 since they are edges of $\mathrm{cl}(A^i_j)$ for exactly two values $1 \le j \le 3$. We can verify the inequality holds for the remaining three edges by a direct computation using the expressions for the motions $\boldsymbol{q}^i$ and $\boldsymbol{q}^i_j$ given in Claim 6.26. For example, the facts that u_1u_4 is in $\mathrm{cl}(A^i_2)$, $q^i_1(u_1) = 0 = q^i_3(u_1)$ and $t_{12} = t^i_3$ give

$$\begin{aligned} D_{1,4} \cdot (\boldsymbol{q}^i(u_1) - \boldsymbol{q}^i(u_4)) &= -D_{1,4} \cdot (\boldsymbol{q}^i_1(u_4) + \boldsymbol{q}^i_3(u_4)) = t_{12}(\beta_1 + \beta_3) \begin{vmatrix} D_{4,1} \\ D_{4,2} \\ D_{4,3} \end{vmatrix} \\ &= t_{12} \Delta^2_{513} x_0 (2x_0 - 1)(x_0 - 1) \begin{vmatrix} D_{4,1} \\ D_{4,2} \\ D_{4,3} \end{vmatrix} \neq 0. \end{aligned}$$

Thus $(G + f_i, \boldsymbol{p})$ has a bad motion at u_0.

We can now use Lemma 5.9 to construct a generic 1-dof framework $(G + f_i, \boldsymbol{p}')$ which has a bad motion at u_0. This contradicts our initial choice of $(G, \boldsymbol{p})$ to minimize $|V|$ and then k since $k = 2$ and $G + f_i$ is 1-dof. □

We can use Claim 6.30 to obtain the following key result for handling the case when $G_0 + f_i$ is type B.

Claim 6.31. *Suppose that $G_0 + f_i$ is type B, and, for $1 \le j \le 3$, let $\boldsymbol{q}^i_j$ be the non-trivial motion of $(B^i_j, \boldsymbol{p}|_{V_0})$ given in Claim 6.26. If $\lambda_2 \beta_2 + \lambda_3 \beta_3 = 0$ for some $\lambda_2, \lambda_3 \in \mathbb{Q}(\boldsymbol{p}(V_0))$, then $\lambda_2 = \lambda_3 = 0$.*

Proof. We will again omit the superscript i from the proof since it stays fixed. We assume that $\lambda_2 \beta_2 + \lambda_3 \beta_3 = 0$ for some $\lambda_2, \lambda_3 \in \mathbb{Q}(\boldsymbol{p}(V_0))$ which are not both zero, and show that this contradicts Claim 6.30.

Let $\boldsymbol{p}(u_i) = (x_i, y_i)$ for $0 \le i \le 5$. By Claim 6.26 we have an explicit formula for each β_i in terms of $x_0, \ldots, x_5, y_0, \ldots, y_5$. Since $V_0 = V \setminus \{u_0\}$, we may regard each β_i as a polynomial in x_0, y_0 with coefficients in $\mathbb{Q}(\boldsymbol{p}(V_0))$. If $\lambda_2\beta_2+\lambda_3\beta_3=0$ with $\lambda_i\in\mathbb{Q}(\boldsymbol{p}(V_0))$, then $\lambda_2\beta_2+\lambda_3\beta_3$ is identically zero as a polynomial in x_0,y_0. In particular, the coefficient of each monomial is zero.

Using Claim 6.26 and (49), a computation of the list of coefficients of the polynomial $\lambda_2\beta_2+\lambda_3\beta_3$ in x_0,y_0 over $\mathbb{Q}(\boldsymbol{p}(V_0))$ gives:

$$x_0^3:\ -\Delta_{513}^2\lambda_3+\Delta_{513}\Delta_{514}\lambda_3t_{11}-\Delta_{513}\Delta_{523}\lambda_3s_{12}+\Delta_{513}\Delta_{524}\lambda_3t_{12},$$

$$x_0^2y_0:\ -\Delta_{513}\Delta_{514}\lambda_2t_{11}+\Delta_{513}\Delta_{514}\lambda_3t_{11}+\Delta_{513}\Delta_{523}\lambda_3s_{12}-\Delta_{513}\Delta_{524}\lambda_2t_{12}$$
$$+\Delta_{513}\Delta_{524}\lambda_2t_3+\Delta_{523}\Delta_{524}\lambda_2s_{12}t_3,$$

$$x_0^2:\ 2\Delta_{513}^2\lambda_3-2\Delta_{513}\Delta_{514}\lambda_3t_{11}+\Delta_{513}\Delta_{523}\lambda_3s_{12}-\Delta_{513}\Delta_{524}\lambda_3t_{12},$$

$$x_0y_0^2:\ \Delta_{513}^2\lambda_3-\Delta_{513}\Delta_{514}\lambda_2t_{11}-\Delta_{513}\Delta_{524}\lambda_3t_{12}-\Delta_{523}\Delta_{524}\lambda_2s_{12}t_3,$$

$$x_0y_0:\ -\Delta_{513}^2\lambda_3+2\Delta_{513}\Delta_{514}\lambda_2t_{11}-\Delta_{513}\Delta_{514}\lambda_3t_{11}-\Delta_{513}\Delta_{523}\lambda_3s_{12}+\Delta_{513}\Delta_{524}\lambda_2t_{12}-\Delta_{513}\Delta_{524}\lambda_2t_3+\Delta_{513}\Delta_{524}\lambda_3t_{12},$$

$$x_0:\ -\Delta_{513}^2\lambda_3+\Delta_{513}\Delta_{514}\lambda_3t_{11},$$

$$y_0^3:\ \Delta_{513}\Delta_{524}\lambda_2t_{12}-\Delta_{513}\Delta_{524}\lambda_2t_3,$$

$$y_0^2:\ -\Delta_{513}^2\lambda_3+\Delta_{513}\Delta_{514}\lambda_2t_{11}-\Delta_{513}\Delta_{524}\lambda_2t_{12}+\Delta_{513}\Delta_{524}\lambda_2t_3,$$

$$y_0:\ \Delta_{513}^2\lambda_3-\Delta_{513}\Delta_{514}\lambda_2t_{11}$$

where s_{12},t_{11},t_{12},t_3 are all non-zero by Claim 6.26. Since $\lambda_2\beta_2+\lambda_3\beta_3=0$, all coefficients are zero. The hypothesis that λ_2,λ_3 are not both zero and the expression for the coefficient of y_0 give $\lambda_2\neq 0$ and $\lambda_3\neq 0$. The coefficient of x_0 now gives $t_{11}=\Delta_{513}/\Delta_{514}$, and the coefficient of y_0 now gives $\lambda_2=\lambda_3$. Then the coefficients of x_0^3 and y_0^2, give $s_{12}=\Delta_{524}t_{12}/\Delta_{523}$ and $t_3=t_{12}$. Finally, the coefficient of $x_0^2y_0$ gives $t_{12}=-\Delta_{513}/\Delta_{524}$. Hence
$$(s_{12},t_{11},t_{12},t_2,t_3)=\left(-\frac{\Delta_{513}}{\Delta_{523}},\frac{\Delta_{513}}{\Delta_{514}},-\frac{\Delta_{513}}{\Delta_{524}},-\frac{\Delta_{513}}{\Delta_{524}},-\frac{\Delta_{513}}{\Delta_{524}}\right).$$
This contradicts Claim 6.30. □

Our next result is key to handling the cases when G_0+f_i is type C or D.

Claim 6.32. *Suppose γ_j^i,δ_j^i, are as in Claims 6.27 and 6.28, respectively.*
(a) If $\lambda_2\gamma_2^i+\lambda_3\gamma_3^i=0$ for some $\lambda_2,\lambda_3\in\mathbb{Q}(\boldsymbol{p}(V_0))$, then $\lambda_2=\lambda_3=0$.
(b) If $\lambda_2\delta_2^i+\lambda_3\delta_3^i=0$ for some $\lambda_2,\lambda_3\in\mathbb{Q}(\boldsymbol{p}(V_0))$, then $\lambda_2=\lambda_3=0$.

Proof. We first prove (a). Let $\boldsymbol{p}(u_i)=(x_i,y_i)$ for $0\leq i\leq 5$. As in the proof of Claim 6.31, we regard each γ_j^i as a polynomial in x_0,y_0 with coefficients in $\mathbb{Q}(\boldsymbol{p}(V_0))$. Suppose $\sum_{2\leq j\leq 3}\lambda_j\gamma_j^i=0$ with $\lambda_j^i\in\mathbb{Q}(\boldsymbol{p}(V_0))$. Then $P:=\sum_{2\leq j\leq 3}\lambda_j\gamma_j^i$ is identically zero as a polynomial in x_0,y_0. Using Claim 6.27 and (49), we find that the coefficient of y_0^2 in P is $\Delta_{513}\Delta_{523}\lambda_2$, and hence $\lambda_2=0$. This implies that the coefficient of x_0y_0 in P is $-2\Delta_{513}\Delta_{523}\lambda_3$ and hence $\lambda_3=0$.

We next prove (b). If $\sum_{2\leq j\leq 3}\lambda_j\delta_j^i=0$ with $\lambda_j^i\in\mathbb{Q}(\boldsymbol{p}(V_0))$, then $Q:=\sum_{2\leq j\leq 3}\lambda_j\delta_j^i$ is identically zero as a polynomial in x_0,y_0. Using Claim 6.28 and (49), we find that the coefficient of x_0^2 in Q is $(-\Delta_{513}s_2^i+\Delta_{524}t_2^i)\Delta_{523}\lambda_3$. Since $\Delta_{513}s_2^i-\Delta_{524}t_2^i\neq 0$ by Claim 6.28, $\lambda_3=0$. This implies that the coefficient of $x_0^2y_0$ in Q is $(-\Delta_{513}s_3^i+\Delta_{524}t_3^i)\Delta_{523}\lambda_2$. Since $\Delta_{513}s_3^i-\Delta_{524}t_3^i\neq 0$ by Claim 6.28, $\lambda_2=0$. □

We next obtain quadratic versions of Claims 6.31 and 6.32.

Claim 6.33. *Let $\beta_j^i,\gamma_j^i,\delta_j^i$ be as defined in Claims 6.26, 6.27 and 6.28, and $\lambda_{h,j}\in\mathbb{Q}(\boldsymbol{p}(V_0))$ for all $1\leq h,j\leq 3$.*

BB: *If $\sum_{2\leq h\leq 3}\sum_{2\leq j\leq 3}\lambda_{h,j}\beta_h^1\beta_j^2=0$, then $\lambda_{k,k}=0$ for some $k\in\{2,3\}$.*

CC: *If $\sum_{2\leq h\leq 3}\sum_{2\leq j\leq 3}\lambda_{h,j}\gamma_h^1\gamma_j^2=0$, then $\lambda_{k,k}=0$ for some $k\in\{2,3\}$.*

DD: *If* $\sum_{2\le h\le 3}\sum_{2\le j\le 3}\lambda_{h,j}\delta_h^1\delta_j^2=0$*, then* $\lambda_{k,k}=0$ *for some* $k\in\{2,3\}$.

Proof. We first check case (CC). Suppose $\sum_{2\le i\le 3}\sum_{2\le j\le 3}\lambda_{i,j}\gamma_i^1\gamma_j^2=0$. We may regard this as a polynomial identity with indeterminates x_0,y_0 and hence the coefficient of each monomial must be zero. If we compute the formula for $\sum_{2\le i\le 3}\sum_{2\le j\le 3}\lambda_{i,j}\gamma_i^1\gamma_j^2$ explicitly using Claim 6.27 and (49), we find that the coefficient of y_0^4 is $\Delta_{513}^2\Delta_{523}^2\lambda_{2,2}$. Since $\Delta_{513}\neq 0$ and $\Delta_{523}\neq 0$, we obtain $\lambda_{2,2}=0$.

We next check case (DD). Suppose $\sum_{2\le i\le 3}\sum_{2\le j\le 3}\lambda_{i,j}\delta_i^1\delta_j^2=0$. If we compute the formula for $\sum_{2\le i\le 3}\sum_{2\le j\le 3}\lambda_{i,j}\delta_i^1\delta_j^2$ explicitly using Claim 6.28 and (49), we find that the coefficient of x_0^4 is $\Delta_{523}^2\lambda_{3,3}(\Delta_{524}t_2^1-T_{513}s_2^1)(\Delta_{524}t_2^2-\Delta_{513}s_2^2)$. Since $\Delta_{524}t_2^i-\Delta_{513}s_2^i\neq 0$ by Claim 6.28, we obtain $\lambda_{3,3}=0$.

Case (BB) can be verified in the same way by using (49), Claim 6.26, and Claim 6.30. The main steps are as follows. We suppose that $P:=\sum_{2\le h\le 3}\sum_{2\le j\le 3}\lambda_{h,j}\beta_h^1\beta_j^2=0$ and $\lambda_{k,k}\neq 0$ for $k\in\{2,3\}$. The coefficient of x_0^2 in P implies that $t_{11}=\Delta_{513}/\Delta_{514}$, and the coefficient of y_0^6 implies that either $t_{12}=t_3^1$ or $t_{12}=t_3^2$. By symmetry we may assume that $t_{12}=t_3^1$. Then the coefficient of x_0^4 gives $s_{12}=t_3^1\Delta_{524}/\Delta_{523}$, and the coefficient of y_0^2 gives $\lambda_{3,2}=\lambda_{2,2}+\lambda_{3,3}-\lambda_{2,3}$. The coefficient of $x_0y_0^2$ now implies that either $\lambda_{3,3}=\lambda_{2,3}$ or $t_3^1=t_3^2$. If $t_3^1\neq t_3^2$ then we have $\lambda_{3,3}=\lambda_{2,3}$, the coefficient of x_0y^5 now implies that $t_3^1=-\Delta_{513}/\Delta_{524}$ and the resulting values of t_{12},s_{12} contradict Claim 6.30. Hence $t_3^1=t_3^2$. The coefficient of $x_0y_0^3$ and the fact that $t_3^1\neq-\Delta_{513}/\Delta_{524}$ by Claim 6.30 then give $\lambda_{2,2}=\lambda_{3,3}$. The coefficient of $x_0^2y_0^4$ now gives $t_3^1=-\Delta_{513}/\Delta_{524}$ and this again contradicts Claim 6.30. □

We close this section with an observation on the type 0 case.

Claim 6.34. *Suppose that* G_0+f_i *is type 0. Then* $(G+f_i,\boldsymbol{p})$ *has a non-trivial motion* $\boldsymbol{q}^i$ *such that* $\boldsymbol{q}^i|_{V_0}$ *is a non-trivial motion of* $(G_0+f_i+K_u,\boldsymbol{p}|_{V_0})$*,* $\boldsymbol{q}^i(v_0)=\boldsymbol{q}^i(v_1)=\boldsymbol{q}^i(v_5)=0$*, and* $\boldsymbol{q}^i(w)\in\mathbb{Q}(\boldsymbol{p}(V_0))$ *for all* $w\in V_0$.

Proof. Since G_0+f_i is type 0, there are two edges $e_1,e_2\in K$ such that $K_u\subseteq \mathrm{cl}(E_0+f_i+e_1+e_2)$. Since $G+f_i$ is C_2^1-independent, we have $\mathrm{cl}(G+f_i+e_1+e_2-u_0u_4-u_0u_5)=\mathrm{cl}(G+f_i)$ and hence $K_u\subseteq\mathrm{cl}(G+f_i)$. This implies that every non-trivial motion of $(G_0+f_i+e_1+e_2,\boldsymbol{p}|_{V_0})$ is extendable to a non-trivial motion $\boldsymbol{q}^i$ of $(G+f_i,\boldsymbol{p})$. Since $K(v_0,v_1,v_5)\subset G$ by (31), we may combine $\boldsymbol{q}^i$ with a suitable trivial motion to ensure that $\boldsymbol{q}^i(v_0)=\boldsymbol{q}^i(v_1)=\boldsymbol{q}^i(v_5)=0$. By scaling, we may also assume that some component of $\boldsymbol{q}_i|_{V_0}$ is equal to one. Since $(G_0+f_i+e_1+e_2,\boldsymbol{p}|_{V_0})$ is a 1-dof framework, we may now use Cramer's rule to deduce that $\boldsymbol{q}^i(w)\in\mathbb{Q}(\boldsymbol{p}(V_0))$ for all $w\in V_0$. □

6.3.6 Completing the proof of Theorem 6.1

We are now ready to complete the proof of Theorem 6.1. Recall that $(G,\boldsymbol{p})$ is a minimal counterexample to Theorem 6.1, and has a motion $\boldsymbol{q}_{\rm bad}$ which is bad at v_0, and $F=\{f_1,f_2\}\in\binom{F_v}{2}$ is a good pair. Since $K(v_0,v_1,v_5)\subset G$ by (31), we may combine $\boldsymbol{q}_{\rm bad}$ with a suitable trivial motion $\boldsymbol{t}$ so that the resulting motion

$$\hat{\boldsymbol{q}}_{\rm bad}:=\boldsymbol{q}_{\rm bad}+\boldsymbol{t} \tag{55}$$

satisfies $\hat{\boldsymbol{q}}_{\rm bad}(v_0)=\hat{\boldsymbol{q}}_{\rm bad}(v_1)=\hat{\boldsymbol{q}}_{\rm bad}(v_5)=0$. Similarly, if G_0+f_i is type X for some X in {B,C,D}, we may combine the motion $\boldsymbol{q}_j^i$ of $(X_j^i,\boldsymbol{p}|_{V_0})$ given in Claims 6.25-6.28 with a suitable trivial motion $\boldsymbol{t}_j^i$ so that the resulting motion

$$\hat{\boldsymbol{q}}_j^i:=\boldsymbol{q}_j^i+\boldsymbol{t}_j^i \tag{56}$$

satisfies $\hat{\boldsymbol{q}}_j^i(v_0)=\hat{\boldsymbol{q}}_j^i(v_1)=\hat{\boldsymbol{q}}_j^i(v_5)=0$. Since all entries of $\boldsymbol{q}_{\rm bad}(v_k)$ and $\boldsymbol{q}_j^i(v_k)$ are contained in $\mathbb{Q}(\boldsymbol{p}(V_0))$ for each $v_k\in\hat{N}_G(v_0)$, we may deduce that all entries of $\hat{\boldsymbol{q}}_{\rm bad}(v_k)$ and $\hat{\boldsymbol{q}}_j^i(v_k)$ are contained in $\mathbb{Q}(\boldsymbol{p}(V_0))$ for each $v_k\in\hat{N}_G(v_0)$.

We will use the following parameterization of the motion space of $(G,\boldsymbol{p})$ over $\hat{N}_G(v_0)$. Let G_v be the subgraph of G induced by $\hat{N}_G(v_0)$. Then (31) implies that G_v is isomorphic to the graph obtained from a complete bipartite graph $K_{2,4}$ by adding an edge between the two vertices in the 2-set $\{v_0,v_5\}$ of the bipartition. See Figure 10. Let Z' be the space of all motions $\boldsymbol{q}'$ of $(G_v,\boldsymbol{p}|_{\hat{N}(v_0)})$ with $\boldsymbol{q}'(v_0)=\boldsymbol{q}'(v_1)=\boldsymbol{q}'(v_5)=0$. As

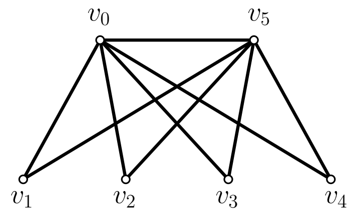


Figure 10: G_v.

G_v is a 3-dof graph, Z' is a 3-dimensional linear space. Since $u_0 \notin V(G_v)$ we may choose a base B for Z' in which the coordinates of each vector lie in $\mathbb{Q}(\boldsymbol{p}(V_0))$, and let $\psi : Z' \to \mathbb{R}^3$ be the linear bijection which maps B onto the standard base of $\mathbb{R}^3$. Note that ψ can be represented by a matrix over $\mathbb{Q}(\boldsymbol{p}(V_0))$.

Using Claims 6.24 and 6.29, and relabeling u_1, u_2, u_3, u_4 and f_1, f_2 if necessary, we may assume that each $G_0 + f_i$ is type 0, B, C or D, and that (G_0, F) is type BB, CC or DD if neither $G_0 + f_1$ nor $G_0 + f_2$ is type 0. The proof is completed by considering the following three cases, depending on the type of each $G + f_i$.

Case 1: Neither $G_0 + f_1$ nor $G_0 + f_2$ is type 0.
Let $\boldsymbol{q}^i$ be the non-trivial motion of $(G + f_i, \boldsymbol{p})$ given in Claims 6.26-6.28. Then $\boldsymbol{q}^i|_{V_0} = \sum_{j=1}^3 \chi_j^i \boldsymbol{q}_j^i$ for some appropriate $\chi \in \{\beta, \gamma, \delta\}$ and we put

$$\hat{\boldsymbol{q}}^i = \boldsymbol{q}^i + \sum_{j=1}^3 \chi_j^i \boldsymbol{t}_j^i \tag{57}$$

where $\boldsymbol{t}_j^i$ is the trivial motion given in (56) (and we are abusing notation by using the same symbol $\boldsymbol{t}_j^i$ for a trivial motion of $(G + f_i, \boldsymbol{p})$ and its restriction to V_0). Then, by $\boldsymbol{q}^i|_{V_0} = \sum_{j=1}^3 \chi_j^i \boldsymbol{q}_j^i$ and (56),

$$\hat{\boldsymbol{q}}^i|_{V_0} = \boldsymbol{q}^i|_{V_0} + \sum_{j=1}^3 \chi_j^i \boldsymbol{t}_j^i = \sum_{j=1}^3 \chi_j^i \boldsymbol{q}_j^i + \sum_{j=1}^3 \chi_j^i \boldsymbol{t}_j^i = \sum_{j=1}^3 \chi_j^i (\boldsymbol{q}_j^i + \boldsymbol{t}_j^i) = \sum_{j=1}^3 \chi_j^i \hat{\boldsymbol{q}}_j^i.$$

As (f_1, f_2) is a good pair, $Z(G, \boldsymbol{p}) = Z_0(G, \boldsymbol{p}) \oplus \langle \hat{\boldsymbol{q}}^1, \hat{\boldsymbol{q}}^2 \rangle$ holds. Hence, by Lemma 2.1,

$$\hat{\boldsymbol{q}}_{\text{bad}} \in \langle \hat{\boldsymbol{q}}^1, \hat{\boldsymbol{q}}^2 \rangle. \tag{58}$$

Suppose (G_0, F) is type BB. Since $\hat{\boldsymbol{q}}_{\text{bad}} \in \langle \hat{\boldsymbol{q}}^1, \hat{\boldsymbol{q}}^2 \rangle$ by (57), we have

$$\hat{\boldsymbol{q}}_{\text{bad}}|_{V_0} \in \left\langle \sum_{j=1}^3 \beta_j^1 \hat{\boldsymbol{q}}_j^1|_{V_0}, \sum_{j=1}^3 \beta_j^2 \hat{\boldsymbol{q}}_j^2|_{V_0} \right\rangle. \tag{59}$$

We also have $K_v \subseteq \text{cl}(B_1^i)$ by Property (4) of the definition of type B, so $\hat{\boldsymbol{q}}_1^i|_{\hat{N}(v_0)} = 0$ for both $i = 1, 2$. Hence the restriction of (59) to $\hat{N}(v_0)$ gives

$$\hat{\boldsymbol{q}}_{\text{bad}}|_{\hat{N}(v_0)} \in \left\langle \sum_{j=2}^3 \beta_j^1 \hat{\boldsymbol{q}}_j^1|_{\hat{N}(v_0)}, \sum_{j=2}^3 \beta_j^2 \hat{\boldsymbol{q}}_j^2|_{\hat{N}(v_0)} \right\rangle. \tag{60}$$

Let k and k_j^i denote the images of $\hat{\boldsymbol{q}}_{\text{bad}}|_{\hat{N}(v_0)}$ and $\hat{\boldsymbol{q}}_j^i|_{\hat{N}(v_0)}$, respectively, under ψ. Then (60) implies that

$$\det \left(k \;\middle|\; \sum_{j=2}^3 \beta_j^1 k_j^1 \;\middle|\; \sum_{j=2}^3 \beta_j^2 k_j^2 \right) = 0,$$

and the bilinearity of the determinant gives

$$\sum_{i=2}^3 \sum_{j=2}^3 \det \left(k \;\middle|\; k_i^1 \;\middle|\; k_j^2 \right) \beta_i^1 \beta_j^2 = 0.$$

Claim 6.33 now implies that

$$\det\left(\begin{array}{c|c|c} k & k_j^1 & k_j^2 \end{array}\right) = 0$$

for some $j \in \{2,3\}$. We may assume without loss of generality that $j = 2$. Then k is a linear combination of k_2^1 and k_2^2 so $\hat{\boldsymbol{q}}_{\text{bad}}|_{\hat{N}(v_0)} = a\hat{\boldsymbol{q}}_2^1|_{\hat{N}(v_0)} + b\hat{\boldsymbol{q}}_2^2|_{\hat{N}(v_0)}$ for some $a, b \in \mathbb{R}$. Let

$$\bar{\boldsymbol{q}} := a\hat{\boldsymbol{q}}_2^1 + b\hat{\boldsymbol{q}}_2^2.$$

Since $\hat{\boldsymbol{q}}_2^i$ is a non-trivial motion of $(B_2^i, \boldsymbol{p}|_{V_0})$ and $B_2^i = G_0 + f_i + u_1u_3 + u_2u_3$, $\bar{\boldsymbol{q}}$ is a motion of $(G_0 + u_1u_3 + u_2u_3, \boldsymbol{p}|_{V_0})$. Let

$$\tilde{\boldsymbol{q}} := \bar{\boldsymbol{q}} - \boldsymbol{t},$$

where $\boldsymbol{t}$ is the trivial motion given in (55). Then $\tilde{\boldsymbol{q}}$ is a motion of the 2-dof framework $(G_0 + u_1u_3 + u_2u_3, \boldsymbol{p}|_{V_0})$ and

$$\tilde{\boldsymbol{q}}|_{\hat{N}(v_0)} = \bar{\boldsymbol{q}}|_{\hat{N}(v_0)} - \boldsymbol{t}|_{\hat{N}(v_0)} = \hat{\boldsymbol{q}}_{\text{bad}}|_{\hat{N}(v_0)} - \boldsymbol{t}|_{\hat{N}(v_0)} = \boldsymbol{q}_{\text{bad}}|_{\hat{N}(v_0)}.$$

Hence $\tilde{\boldsymbol{q}}$ is bad a bad motion at v_0 for the framework $(G_0 + u_1u_3 + u_2u_3, \boldsymbol{p}|_{V_0})$. This contradicts the fact that $(G, \boldsymbol{p})$ is a minimal counterexample.

The remaining cases when (G_0, F) is type CC or DD can be solved similarly.

Case 2: $G_0 + f_1$ is type 0 and $G_0 + f_2$ is not.
We only give a proof for the case when $G_0 + f_2$ is type B since the other cases can be solved in an identical manner.

By Claim 6.34, $(G + f_1, \boldsymbol{p})$ has a non-trivial motion $\boldsymbol{q}^1$ such that $\boldsymbol{q}^1|_{V_0}$ is a motion of $(G_0 + f_1 + K_u, \boldsymbol{p}|_{V_0})$, $\boldsymbol{q}^1(v_0) = \boldsymbol{q}^1(v_1) = \boldsymbol{q}^1(v_5) = 0$ and $\boldsymbol{q}^1(w) \in \mathbb{Q}(\boldsymbol{p}(V_0))$ for all $w \in V_0$.

Let $\hat{\boldsymbol{q}}^2 = \boldsymbol{q}^2 + \sum_{j=1}^3 \beta_j^2 \boldsymbol{t}_j^2$ be as defined in (57). Since (f_1, f_2) is a good pair, $Z(G, \boldsymbol{p}) = Z_0(G, \boldsymbol{p}) \oplus \langle \boldsymbol{q}^1, \hat{\boldsymbol{q}}^2 \rangle$ holds. Hence, by Lemma 2.1, $\hat{\boldsymbol{q}}_{\text{bad}} \in \langle \boldsymbol{q}^1, \hat{\boldsymbol{q}}^2 \rangle$. Since $K_v \subseteq \text{cl}(B_1^2)$ by Property (4) of the definition of type B, this gives

$$\hat{\boldsymbol{q}}_{\text{bad}}|_{\hat{N}(v_0)} \in \left\langle \boldsymbol{q}^1|_{\hat{N}(v_0)}, \sum_{j=2}^{3} \beta_j^2 \hat{\boldsymbol{q}}_j^2|_{\hat{N}(v_0)} \right\rangle. \tag{61}$$

As in the previous case, this implies that

$$\det\left(\begin{array}{c|c|c} k & k^1 & \sum_{j=2}^3 \beta_j^2 k_j^2 \end{array}\right) = 0,$$

where k, k^1, k_j^2 are the image of $\hat{\boldsymbol{q}}_{\text{bad}}|_{\hat{N}(v_0)}$, $\boldsymbol{q}^1|_{\hat{N}(v_0)}$, $\hat{\boldsymbol{q}}_j^2|_{\hat{N}(v_0)}$, respectively, under ψ. The bilinearity of the determinant now gives

$$\sum_{2 \le j \le 3} \det\left(\begin{array}{c|c|c} k & k^1 & k_j^2 \end{array}\right) \beta_j^2 = 0,$$

and Claim 6.31 implies that

$$\det\left(\begin{array}{c|c|c} k & k^1 & k_j^2 \end{array}\right) = 0$$

for each $j \in \{2,3\}$. In particular, we have that k is a linear combination of k^1 and k_2^2 so $\hat{\boldsymbol{q}}_{\text{bad}}|_{\hat{N}(v_0)} = a\boldsymbol{q}^1|_{\hat{N}(v_0)} + b\hat{\boldsymbol{q}}_2^2|_{\hat{N}(v_0)}$ for some $a, b \in \mathbb{R}$. Let $\bar{\boldsymbol{q}} := a\boldsymbol{q}^1|_{V_0} + b\hat{\boldsymbol{q}}_2^2$. Since $\boldsymbol{q}^1|_{V_0}$ is a motion of $(G_0 + f_1 + K_u, \boldsymbol{p}|_{V_0})$ and $\hat{\boldsymbol{q}}_2^2$ is a motion of $(B_2^2, \boldsymbol{p}|_{V_0})$ where $B_2^2 = G_0 + f_2 + u_2u_3 + u_3u_1$, $\bar{\boldsymbol{q}}$ is a motion of $(G_0 + u_2u_3 + u_3u_1, \boldsymbol{p}|_{V_0})$.

Let $\tilde{\boldsymbol{q}} := \bar{\boldsymbol{q}} - \boldsymbol{t}$, where $\boldsymbol{t}$ is the trivial motion given in (55). Then $\tilde{\boldsymbol{q}}$ is a motion of the 2-dof framework $(G_0 + u_2u_3 + u_3u_1, \boldsymbol{p}|_{V_0})$ and

$$\tilde{\boldsymbol{q}}|_{\hat{N}(v_0)} = \bar{\boldsymbol{q}}|_{\hat{N}(v_0)} - \boldsymbol{t}|_{\hat{N}(v_0)} = \hat{\boldsymbol{q}}_{\text{bad}}|_{\hat{N}(v_0)} - \boldsymbol{t}|_{\hat{N}(v_0)} = \boldsymbol{q}_{\text{bad}}|_{\hat{N}(v_0)}.$$

Hence $\tilde{\boldsymbol{q}}$ is bad at v_0. This contradicts the fact that $(G, \boldsymbol{p})$ is a minimal counterexample.

Case 3: Both $G_0 + f_1$ and $G_0 + f_2$ are type 0.
By Claim 6.34, $(G + f_i, \boldsymbol{p})$ has a motion $\boldsymbol{q}^i$ such that $\boldsymbol{q}^i|_{V_0}$ is a motion of $(G_0 + f_i + K_u, \boldsymbol{p}|_{V_0})$, $\boldsymbol{q}^i(v_0) = \boldsymbol{q}^i(v_1) = \boldsymbol{q}^i(v_5) = 0$ and $\boldsymbol{q}^i(w) \in \mathbb{Q}(\boldsymbol{p}(V_0))$ for all $w \in V_0$.

As (f_1, f_2) is a good pair, $Z(G, \boldsymbol{p}) = Z_0(G, \boldsymbol{p}) \oplus \langle \boldsymbol{q}^1, \boldsymbol{q}^2 \rangle$ holds. Hence, by Lemma 2.1, $\hat{\boldsymbol{q}}_{\text{bad}} \in \langle \boldsymbol{q}^1, \boldsymbol{q}^2 \rangle$. This gives

$$\hat{\boldsymbol{q}}_{\text{bad}}|_{\hat{N}(v_0)} \in \left\langle \boldsymbol{q}^1|_{\hat{N}(v_0)}, \boldsymbol{q}^2|_{\hat{N}(v_0)} \right\rangle. \tag{62}$$

and hence

$$\det \left(\begin{array}{c|c|c} k & k^1 & k^2 \end{array} \right) = 0,$$

where k, k^1, k^2 are the image of $\hat{\boldsymbol{q}}_{\text{bad}}|_{\hat{N}(v_0)}$, $\boldsymbol{q}^1|_{\hat{N}(v_0)}$, $\boldsymbol{q}^2|_{\hat{N}(v_0)}$respectively, under ψ. This in turn implies that $\hat{\boldsymbol{q}}_{\text{bad}}|_{\hat{N}(v_0)}$ is a linear combination of $\boldsymbol{q}^1|_{\hat{N}(v_0)}$ and $\boldsymbol{q}^2|_{\hat{N}(v_0)}$. We can now use a similar argument to the previous case to deduce that $(G_0 + K_u, \boldsymbol{p}|_{V_0})$ has a bad motion at v_0. Since $(G_0 + K_u, \boldsymbol{p}|_{V_0})$ is a k-dof framework for some $k = 1, 2$, this contradicts the minimality of $(G, \boldsymbol{p})$.

This completes the proof of Theorem 6.1.

Appendix

A Proof of Lemma 5.6 and statement (34)

We deduce both Lemma 5.6 and statement (34) from Lemma A.1 below, which is a general result on matrices with polynomial entries.

Given a matrix M with entries in $\mathbb{R}[X_1, X_2, \ldots, X_m]$ and $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$ we use $M(x)$ to denote the real matrix obtained by substituting the values $X_i = x_i$ into M. We assume that the rows and columns of M are indexed by two disjoint sets R and C respectively. For $D \subseteq R$ and $U \subseteq C$, we use M_D to denote the submatrix of M with rows indexed by D, and $M_{D,U}$ to denote the submatrix of M_D with columns indexed by U. We also set $\overline{D} = R \setminus D$ and $\overline{U} = C \setminus U$. For $x \in \mathbb{R}^m$, we consider the components of each vector $z(x) \in \ker M(x)$ to be indexed by C and use $z_i(x)$ to denote the i'th component of $z(x)$ for $i \in C$.

Lemma A.1. *Let $\mathbb{F}$ be a subfield of $\mathbb{R}$, M be a square matrix with entries in $\mathbb{F}[X_1, \ldots, X_m]$, and $T = \{x \in \mathbb{R}^m : M(x)$ is non-singular$\}$. Let $D \subseteq R$ and $U \subseteq C$ such that every entry of $M_{D,\overline{U}}$ is zero, and let $b : U \to \mathbb{F}[X_1, X_2, \ldots, X_m]$. Suppose that, for some $\tilde{x} \in T$ which is generic over $\mathbb{F}$, there is a vector $z(\tilde{x}) \in \ker M_{\overline{D}}(\tilde{x})$ such that $z_i(\tilde{x}) = b_i(\tilde{x})$ for all $i \in U$. Then, for all $x \in T$, there is a vector $z(x) \in \ker M_{\overline{D}}(x)$ such that $z_i(x) = b_i(x)$ for all $i \in U$.*

Proof. Choose any $x \in T$. As $M(x)$ is non-singular, it is row independent and hence $M_{R-d}(x)$ is row independent for all $d \in D$. Since each entry of $M_{R-d}(x)$ is a polynomial function of the coordinates of x, Cramer's rule implies that we can choose a non-zero $z_d \in \ker M_{R-d}(x)$ in such a way that each of its coordinates is a polynomial function of the coordinates of x. If for some $d_1 \in D$, the vector z_{d_1} is spanned by $\{z_d : d \in D - d_1\}$ then we would have $\ker M_{\overline{D}}(x) = \ker M_{\overline{D-d_1}}(x)$ and this would contradict the row independence of $M(x)$. Thus $\{z_d : d \in D\}$ is linearly independent, and so $\dim \ker M_{\overline{D}}(x) \geq |D|$. The independence of $M(x)$ implies this holds with equality. Hence $\{z_d : d \in D\}$ is a base for $\ker M_{\overline{D}}(x)$ and each $z \in \ker M_{\overline{D}}(x)$ can be expressed as $z = \sum_{d \in D} \lambda_d z_d$ for some $\lambda_d \in \mathbb{R}$.

Let $z_d|_U$ denote the restriction of z_d to U for each $d \in D$. We will show that

$$\{z_d|_U : d \in D\} \text{ is linearly independent.} \tag{63}$$

Suppose to the contrary that, for some some $d_1 \in D$, $z_{d_1}|_U$ is spanned by $\{z_d|_U : d \in D - d_1\}$. Then the hypothesis that every entry of $M_{D,\overline{U}}$ is zero implies that we again have $\ker M_{\overline{D}}(x) = \ker M_{\overline{D-d_1}}(x)$ and contradict the row independence of $M(x)$. Hence (63) holds.

We next consider the matrix equation $A_x\lambda = b_x$ with variable λ, where A_x is the $|U| \times |D|$ matrix with columns $z_d|_U$ for $d \in D$, $\lambda = (\lambda_1, \dots, \lambda_{|D|})^\top$, and $b_x \in \mathbb{R}^{|U|}$ is the column vector with entries given by the coordinates of $b_i(x)$ for $i \in U$. By (63), $\operatorname{rank} A_x = |D|$ for all $x \in T$.

Since, for some generic $\tilde{x} \in T$, there is a vector $z(\tilde{x}) \in \ker M_{\overline{D}}(\tilde{x})$ such that $z_i(\tilde{x}) = b_i(\tilde{x})$ for all $i \in U$, the equation $A_{\tilde{x}}\lambda = b_{\tilde{x}}$ has a solution, and hence $\operatorname{rank}(A_{\tilde{x}}, b_{\tilde{x}}) = \operatorname{rank} A_{\tilde{x}} = |D|$ for this generic $\tilde{x}$. Since each entry in (A_x, b_x) is a polynomial function of the coordinates of x, this implies that $\operatorname{rank}(A_x, b_x) \le |D|$ for all $x \in T$. On the other hand we have seen that $\operatorname{rank} A_x = |D|$ for all $x \in T$. Hence $\operatorname{rank}(A_x, b_x) = \operatorname{rank} A_x$ holds for all $x \in T$, and as a result, the equation $A_x\lambda = b_x$ has a solution for all $x \in T$. This solution will give us a vector $z(x) \in \ker M_{\overline{D}}(x)$ such that $z_i(x) = b_i(x)$ for all $i \in U$ and all $x \in T$. □

We are now ready to prove Lemma 5.6. For convenience we repeat the statement.

Lemma 5.6. *Let $G = (V, E)$ be a C_2^1-independent graph with $V = \{v_1, \dots, v_n\}$, $U \subseteq V$, $F \subseteq K(U)$, and $S = \{\boldsymbol{p} : (G+F, \boldsymbol{p})$ is minimally C_2^1-rigid and $\boldsymbol{p}$ is non-degenerate on $U\}$. Suppose that $b : U \to \mathbb{Q}[X_1, Y_1, \dots, X_n, Y_n]^3$ and $(G, \boldsymbol{p})$ has a b-motion for some generic $\boldsymbol{p}$. Then $(G, \boldsymbol{p})$ has a b-motion for all $\boldsymbol{p} \in S$.*

Proof. Choose $\boldsymbol{p}_1 \in S$ and let $\boldsymbol{p}_1(v_i) = (x_i, y_i)$ for all $v_i \in V$. Since $\boldsymbol{p}_1$ is non-degenerate on U, we can choose three distinct vertices $v_a, v_b, v_c \in U$ which have distinct 'y-coordinates' in $(G, \boldsymbol{p}_1)$. Let $\tilde{C}(G+F, \boldsymbol{p}_1)$ be the extended C_2^1-cofactor matrix with respect to v_a, v_b, v_c and let M be the matrix obtained from $\tilde{C}(G+F, \boldsymbol{p}_1)$ by replacing each of the coordinates x_i, y_i in the definition of $\tilde{C}(G+F, \boldsymbol{p}_1)$ by indeterminates X_i, Y_i for all $v_i \in V$. Then each entry in M belongs to $\mathbb{Q}[X_1, Y_1, \dots, X_n, Y_n]$. We will deduce that $(G, \boldsymbol{p}_1)$ has a b-motion at v_0 by applying Lemma A.1 to M.

Let $T = \{\boldsymbol{p} : M(\boldsymbol{p}) \text{ is non-singular}\}$ and let S_1 be the set of all $\boldsymbol{p} \in S$ such that $\boldsymbol{p}(v_a)$, $\boldsymbol{p}(v_b)$ and $\boldsymbol{p}(v_c)$ have distinct y-coordinates. Lemma 2.2 implies that $\tilde{C}(G+F, \boldsymbol{p})$ is non-singular for all $\boldsymbol{p} \in S_1$ so $S_1 \subseteq T$. Let D be the set of rows of M indexed by

$$F \cup \{\boldsymbol{e}_{s,t} : (s,t) \in \{(a,1), (a,2), (a,3), (b,1), (b,2), (c,1)\},$$

using the row labelling defined in Section 2.4. Then every entry of $M_{D,\overline{U}}$ is zero, and $M_{\overline{D}}(\boldsymbol{p}) = C(G, \boldsymbol{p})$ for all realisations $\boldsymbol{p}$. Since $(G, \boldsymbol{p})$ has a b-motion for some generic $\boldsymbol{p}$, Lemma A.1 tells us that $(G, \boldsymbol{p})$ has a b-motion for all $\boldsymbol{p} \in T$. In particular $(G, \boldsymbol{p}_1)$ has a b-motion. □

Proof of statement (34). Since q_{bad} is a bad motion of $(G, \boldsymbol{p})$ at v_0, there exists $b : \hat{N}(v_0) \to \mathbb{Q}[X_0, Y_0, \dots, X_5, Y_5]^3$ which satisfies (23) and has $q_{\text{bad}}(v) = b(v)$ for all $v \in \hat{N}_G(v_0)$.

Consider the family of realisations $\boldsymbol{p}_t$ of G parametrized by $t \in \mathbb{R}$ defined by $\boldsymbol{p}_t(w) = \boldsymbol{p}(w)$ for $w \in V_0$ and $\boldsymbol{p}_t(u_0) = t\boldsymbol{p}(u_0) + (1-t)\boldsymbol{p}(u_1)$. (This family was used in the proof of [21, Theorem 10.2.7] to show that vertex-splitting preserves C_2^1-independence.) Since $\boldsymbol{p}$ is generic and $\boldsymbol{p}_t|_{V_0} = \boldsymbol{p}|_{V_0}$, we can choose three distinct vertices $v_a, v_b, v_c \in \hat{N}_G(v_0)$ which have distinct 'y-coordinates' in $(G, \boldsymbol{p}_t)$ for all $t \in \mathbb{R}$. Let $\tilde{C}(G+F, \boldsymbol{p}_t)$ be the extended C_2^1-cofactor matrix for $(G+F, \boldsymbol{p}_t)$ with respect to v_a, v_b, v_c, and let $C^*(G+F, \boldsymbol{p}_t)$ be obtained from $\tilde{C}(G+F, \boldsymbol{p}_t)$ by replacing the row indexed by the edge u_0u_1 with a row of the form

$$\begin{array}{c} \\ e=u_0u_1 \end{array} \begin{array}{c} \begin{array}{ccccc} & u_0 & & u_1 & \end{array} \\ \left[\begin{array}{ccccc} 0\dots0 & D(\boldsymbol{p}(u_0), \boldsymbol{p}(u_1)) & 0\dots0 & -D(\boldsymbol{p}(u_0), \boldsymbol{p}(u_1)) & 0\dots0 \end{array}\right]. \end{array}$$

Since $D(\boldsymbol{p}_t(u_0),\boldsymbol{p}_t(u_1)) = t^2 D(\boldsymbol{p}(u_0),\boldsymbol{p}_t(u_1))$, $\ker\tilde{C}(G,\boldsymbol{p}_t) = \ker C^*(G,\boldsymbol{p}_t)$ for $t \neq 0$. Moreover, when $t = 0$, the argument for vertex splitting in [21, Theorem 10.2.7] can be applied directly to show that $C^*(G+F,\boldsymbol{p}_0)$ is non-singular.[3]

Observe that $\boldsymbol{p}_t$ is generic over $\mathbb{Q}$ whenever t is taken to be generic over $\mathbb{Q}(\boldsymbol{p}(V))$. Together with Lemma 5.6 and the facts that $\boldsymbol{p}$ is generic over $\mathbb{Q}$ and $(G,\boldsymbol{p})$ has a b-motion, this implies that $(G,\boldsymbol{p}_t)$ has a b-motion for any t which is generic over $\mathbb{Q}(\boldsymbol{p}(V))$.

We next consider $M := C^*(G+F,\boldsymbol{p}_t)$ to be a matrix with entries in $\mathbb{F}[t]$ where $\mathbb{F} = \mathbb{Q}(p(V))$ and t is an indeterminate. Let $T = \{t \in \mathbb{R} : M(t) \text{ is non-singular}\}$. Then $0 \in T$ and $t \in T$ for any generic t over $\mathbb{Q}(\boldsymbol{p}(V))$. Let $U = \hat{N}_G(v_0)$ and D be the set of rows of M indexed by $F \cup \{\boldsymbol{e}_{h,k} : (h,k) \in \{(a,1),(a,2),(a,3),(b,1),(b,2),(c,1)\}$, using the row labelling defined in Section 2.4. Then every entry of $M_{D,\overline{U}}$ is zero, and $M_{\overline{D}}(t) = C^*(G,\boldsymbol{p}_t)$ for all $t \in \mathbb{R}$. Since $(G,\boldsymbol{p}_t)$ has a b-motion and $\boldsymbol{p}_t \in T$ for any t which is generic over $\mathbb{Q}(\boldsymbol{p}(V))$, and $0 \in T$, Lemma A.1 implies that there exists a $z \in \ker M(0)$ such that $z(v) = b(v)$ for all $v \in U = \hat{N}_G(v_0)$. The fact that $M_{\overline{D}}(0) = C^*(G,\boldsymbol{p}_0)$ and the defininition of $\boldsymbol{p}_0$ now imply that $z|_{V_0}$ is a bad motion of $(G_0+e_1+e_2,\boldsymbol{p}|_{V_0})$ so (34) holds. □

B Calculation in the proof of Lemma 5.7

We use the same notation for the polynomials $D_{i,j}$ and $\Delta_{i,j,k}$, and the polynomial map b as in the proof of Lemma 5.7. We need to show that:

the graph on $N_G(v_0)$ with edge set $\{v_iv_j : D_{v_i,v_j}\cdot(b(v_i)-b(v_j)) = 0\}$ is a star.

Since $\boldsymbol{p}$ is generic and the C_2^1-motion $\boldsymbol{q}$ of (G,p) defined in the conclusion of Lemma 5.1 satisfies $\boldsymbol{q}(v_i) = b(v_i)$ for all $v_i \in N_G(v_0)$, it will suffice to show that

the graph on $N_G(v_0)$ with edge set $\{v_iv_j : D_{v_i,v_j}\cdot(\boldsymbol{q}(v_i)-\boldsymbol{q}(v_j)) = 0\}$ is a star,

where we are abusing notation by continuing to use D_{ij} for the real number obtained from the polynomial $D(i,j)$ by the substitution $(X_i,Y_i) \to (x_i,y_i) = \boldsymbol{p}(v_i)$ for all $v_i \in \hat{N}_G(v_0)$. The hypotheses that $\boldsymbol{q}$ is a C_2^1-motion of $(G,\boldsymbol{p})$ and v_0 is a type $(\star)$ vertex in G imply that $D_{v_i,v_5}\cdot(\boldsymbol{q}(v_i)-\boldsymbol{q}(v_5)) = 0$ for all $1 \le i \le 4$. Hence it only remains to check that $\delta_{ij} := D_{v_i,v_j}\cdot(\boldsymbol{q}(v_i)-\boldsymbol{q}(v_j)) \neq 0$ for all $1 \le i < j \le 4$. We can do this for each $\delta_{i,j}$ as follows, where we use c_{ij} and c'_{ij} to denote non-zero constants.

$$\delta_{1,2} = D_{1,2}\cdot(\boldsymbol{q}(v_1)-\boldsymbol{q}(v_2)) = c_{1,2}D_{1,2}\cdot(D_{2,4}\times D_{2,5}) = c_{1,2}\begin{vmatrix} D_{1,2} \\ D_{2,4} \\ D_{2,5}\end{vmatrix} \neq 0.$$

$$\delta_{1,3} \neq 0 \quad \text{(by a symmetric argument).}$$

$$\begin{aligned}\delta_{1,4} &= D_{1,4}\cdot(\boldsymbol{q}(v_1)-\boldsymbol{q}(v_4)) = -\alpha\Delta_{1,2,4}D_{1,4}\cdot(D_{3,4}\times D_{4,5}) - \beta\Delta_{1,3,4}D_{1,4}\cdot(D_{2,4}\times D_{4,5})\\ &= -\alpha\Delta_{1,2,4}\begin{vmatrix} D_{1,4} \\ D_{3,4} \\ D_{4,5}\end{vmatrix} - \beta\Delta_{1,3,4}\begin{vmatrix} D_{1,4} \\ D_{2,4} \\ D_{4,5}\end{vmatrix}\\ &= -\Delta_{2,5,0}\Delta_{0,1,5}\Delta_{0,5,3}\Delta_{4,5,1}\Delta_{2,4,5}\Delta_{1,2,4}\Delta_{4,3,5}\Delta_{4,1,3}(\Delta_{2,0,4}\Delta_{0,3,1}-\Delta_{3,0,4}\Delta_{0,2,1})\\ &\neq 0,\end{aligned}$$

where the third equation follows from the Vandermonde identity and the last relation holds since $\Delta_{2,0,4}\Delta_{0,3,1} \neq \Delta_{3,0,4}\Delta_{0,2,1}$. (To see this, consider the special position where $\boldsymbol{p}(v_3)$ is on the line through $\boldsymbol{p}(v_0)$ and $\boldsymbol{p}(v_1)$.

[3] Since $(G_0+F+e_1+e_2,\boldsymbol{p}|_{V_0})$ is minimally C_2^1-rigid, Lemma 2.2 implies that $\tilde{C}(G_0+F+e_1+e_2,\boldsymbol{p}|_{V_0})$ is row independent. We may now use elementary matrix manipulations and the fact that $\boldsymbol{p}_0(u_0) = \boldsymbol{p}(u_1)$ to deduce that $C^*(G+F,\boldsymbol{p}_0)$ is row independent and hence non-singular, see the proof of [21, Theorem 10.2.7] for more details.

Then the left term is zero while the right term is non-zero when the other points are in a generic position.)

$$\begin{aligned}
\delta_{2,3} &= D_{2,3}\cdot(\boldsymbol{q}(v_2)-\boldsymbol{q}(v_3)) = D_{2,3}\cdot(\beta\Delta_{1,3,2}D_{2,4}\times D_{2,5}-\alpha\Delta_{1,2,3}D_{3,4}\times D_{3,5})\\
&= -\Delta_{1,2,3}\left(\beta\begin{vmatrix} D_{2,3}\\ D_{2,4}\\ D_{2,5}\end{vmatrix}+\alpha\begin{vmatrix} D_{3,2}\\ D_{3,4}\\ D_{3,5}\end{vmatrix}\right)\\
&= \Delta_{1,2,3}\Delta_{2,3,4}\Delta_{2,4,5}\Delta_{2,5,3}\Delta_{3,4,5}\Delta_{2,5,0}\Delta_{0,1,5}\Delta_{0,5,3}(\Delta_{3,0,4}\Delta_{0,2,1}-\Delta_{2,0,4}\Delta_{0,3,1})\neq 0.\\
\delta_{2,4} &= D_{2,4}\cdot(\boldsymbol{q}(v_2)-\boldsymbol{q}(v_4))\\
&= D_{2,4}\cdot(c_{24}D_{2,4}\times D_{2,5}-\alpha\Delta_{1,2,4}D_{3,4}\times D_{4,5}-c'_{24}D_{2,4}\times D_{4,5})\\
&= -\alpha\Delta_{1,2,4}\begin{vmatrix} D_{2,4}\\ D_{3,4}\\ D_{4,5}\end{vmatrix}\neq 0.\\
\delta_{3,4} &\neq 0 \quad \text{(by a symmetric argument).}
\end{aligned}$$

C Projective Transformations

We give a new analysis of the projective invariance of C_2^1-rigidity from the viewpoint of C_2^1-motions, and then use this to prove Lemma 5.9.

Let $\mathbb{S}^{3\times 3}$ be the set of symmetric 3×3 matrices. We will use the fact that $\mathrm{Trace}(AB)=\sum_{1\le i,j\le 3} a_{ij}b_{ij}$ holds for any $A=(a_{ij}), B=(b_{ij})\in\mathbb{S}^{3\times 3}$.

C.1 C_2^1-motions in the projective setting

Recall that each point $p_i=(x_i,y_i)^\top\in\mathbb{R}^2$ is associated with the point $p_i^\uparrow=[x_i,y_i,1]^\top$ in 2-dimensional real projective space $\mathbb{P}^2$. This allows us to associate a framework $(G,\boldsymbol{p}^\uparrow)$ in $\mathbb{P}^2$ with any framework $(G,\boldsymbol{p})$ in $\mathbb{R}^2$. We will consider $(G,\boldsymbol{p}^\uparrow)$ to be a framework in $\mathbb{R}^3$ in which the 'z-component' of every vertex is non-zero and define a projective C_2^1-motion as a new kind of motion of such a framework.

Let $G=(V,E)$ be a graph and $\tilde{\boldsymbol{p}}:V\to\mathbb{R}^3$ such that $\tilde{\boldsymbol{p}}(v_i)=(x_i,y_i,z_i)^\top$ and $z_i\neq 0$ for all $v_i\in V$. We say that $\boldsymbol{Q}:V\to\mathbb{S}^{3\times 3}$ is a *projective C_2^1-motion* of $(G,\tilde{\boldsymbol{p}})$ if

$$\mathrm{Trace}\Big((\tilde{\boldsymbol{p}}(v_i)\times\tilde{\boldsymbol{p}}(v_j))(\tilde{\boldsymbol{p}}(v_i)\times\tilde{\boldsymbol{p}}(v_j))^\top(\boldsymbol{Q}(v_i)-\boldsymbol{Q}(v_j))\Big)=0 \tag{64}$$

for all $v_iv_j\in E$, where $(\tilde{\boldsymbol{p}}(v_i)\times\tilde{\boldsymbol{p}}(v_j))$ denotes the cross product of $\tilde{\boldsymbol{p}}(v_i)$ and $\tilde{\boldsymbol{p}}(v_j)$. A projective C_2^1-motion $\boldsymbol{Q}$ is *trivial* if (64) holds for all $v_i,v_j\in V$. These definitions are inspired by the following observation (and Lemma C.2 below).

Lemma C.1. *For a framework $(G,\boldsymbol{p})$ in $\mathbb{R}^2$, define $\boldsymbol{p}^\uparrow:V\to\mathbb{R}^3$ by $\boldsymbol{p}^\uparrow(v_i)=(\boldsymbol{p}(v_i),1)^\top$. For $\boldsymbol{q}:V\to\mathbb{R}^3$, define $\boldsymbol{Q}:V\to\mathbb{S}^{3\times 3}$ by*

$$\boldsymbol{q}(v_i)=\begin{pmatrix} q_1(v_i)\\ q_2(v_i)\\ q_3(v_i)\end{pmatrix}\mapsto\boldsymbol{Q}(v_i)=\begin{pmatrix} 2q_3(v_i) & -q_2(v_i) & 0\\ -q_2(v_i) & 2q_1(v_i) & 0\\ 0 & 0 & 0\end{pmatrix}\text{ for all } v_i\in V. \tag{65}$$

Then $\boldsymbol{q}$ is a C_2^1-motion of $(G,\boldsymbol{p})$ if and only if $\boldsymbol{Q}$ is a projective C_2^1-motion of $(G,\boldsymbol{p}^\uparrow)$.

Proof. Denote $\boldsymbol{p}(v_i)=(x_i,y_i)^\top$ for all $v_i\in V$. Then

$$(\boldsymbol{p}^\uparrow(v_i)\times\boldsymbol{p}^\uparrow(v_j))(\boldsymbol{p}^\uparrow(v_i)\times\boldsymbol{p}^\uparrow(v_j))^\top=\begin{pmatrix} (y_i-y_j)^2 & -(x_i-x_j)(y_i-y_j) & *\\ -(x_i-x_j)(y_i-y_j) & (x_i-x_j)^2 & *\\ * & * & *\end{pmatrix}.$$

It is now straightforward to check that

$$\mathrm{Trace}\Big((\boldsymbol{p}^{\uparrow}(v_i)\times\boldsymbol{p}^{\uparrow}(v_i))(\boldsymbol{p}^{\uparrow}(v_i)\times\boldsymbol{p}^{\uparrow}(v_i))^{\top}(\boldsymbol{Q}(v_i)-\boldsymbol{Q}(v_j))\Big) = 2D(\boldsymbol{p}(v_i),\boldsymbol{p}(v_j))\cdot(\boldsymbol{q}(v_i)-\boldsymbol{q}(v_j))$$

for all $v_i, v_j \in V$. □

Lemma C.1 gives a linear isomorphism between the C_2^1-motions $\boldsymbol{q}$ of $(G,\boldsymbol{p})$ and the projective C_2^1-motions $\boldsymbol{Q}$ of $(G,\boldsymbol{p}^{\uparrow})$ with the property that the right column and bottom row of $\boldsymbol{Q}$ are both zero.

C.2 Projective C_2^1-rigidity

We show that every framework $(G,\tilde{\boldsymbol{p}})$ in $\mathbb{R}^3$ has at least $3|V|+6$ linearly independent projective C_2^1-motions. We use this to define projective C_2^1-rigidity and then show that a given framework $(G,\boldsymbol{p})$ in $\mathbb{R}^2$ is C_2^1-rigid if and only if the corresponding framework $(G,\boldsymbol{p}^{\uparrow})$ in $\mathbb{R}^3$ is projective C_2^1-rigid.

We defined six linearly independent (trivial) C_2^1-motions $\boldsymbol{q}_i^*$ for an arbitrary 2-dimensional framework in (2) and (3). We may apply Lemma C.1 to each of these to obtain the following six linearly independent projective C_2^1-motions for any framework $(G,\tilde{\boldsymbol{p}})$ in $\mathbb{R}^3$.

$$\boldsymbol{Q}_1^*(v_i)=\begin{pmatrix}0&0&0\\0&2&0\\0&0&0\end{pmatrix},\boldsymbol{Q}_2^*(v_i)=\begin{pmatrix}0&-1&0\\-1&0&0\\0&0&0\end{pmatrix},\boldsymbol{Q}_3^*(v_i)=\begin{pmatrix}2&0&0\\0&0&0\\0&0&0\end{pmatrix},$$
$$\boldsymbol{Q}_4^*(v_i)=\begin{pmatrix}0&x_i/z_i&0\\x_i/z_i&2y_i/z_i&0\\0&0&0\end{pmatrix},\boldsymbol{Q}_5^*(v_i)=\begin{pmatrix}2x_i/z_i&y_i/z_i&0\\y_i/z_i&0&0\\0&0&0\end{pmatrix}, \tag{66}$$
$$\boldsymbol{Q}_6^*(v_i)=\begin{pmatrix}2x_i^2/z_i^2&2x_iy_i/z_i^2&0\\2x_iy_i/z_i^2&2y_i^2/z_i^2&0\\0&0&0\end{pmatrix}.$$

It is straightforward to check that each $\boldsymbol{Q}_k^*$, $1\le k\le 6$, satisfies (64) for every pair of vertices $v_i,v_j\in V$, and hence is a trivial projective C_2^1-motion of $(G,\tilde{\boldsymbol{p}})$. We may define a further $3|V|$ linearly independent projective C_2^1-motions for any $(G,\tilde{\boldsymbol{p}})$, three for each vertex, as follows. For each $v_i\in V$, let $\boldsymbol{Q}_{i,1}^*,\boldsymbol{Q}_{i,2}^*,\boldsymbol{Q}_{i,3}^*: V\to\mathbb{S}^{3\times 3}$ by:

$$\boldsymbol{Q}_{i,1}^*(v_i)=\begin{pmatrix}0&x_i&0\\x_i&2y_i&z_i\\0&z_i&0\end{pmatrix},\text{ and }\boldsymbol{Q}_{i,1}^*(v_j)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\text{ for all }v_j\neq v_i;$$
$$\boldsymbol{Q}_{i,2}^*(v_i)=\begin{pmatrix}2x_i&y_i&z_i\\y_i&0&0\\z_i&0&0\end{pmatrix},\text{ and }\boldsymbol{Q}_{i,2}^*(v_j)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\text{ for all }v_j\neq v_i; \tag{67}$$
$$\boldsymbol{Q}_{i,3}^*(v_i)=\begin{pmatrix}-x_i^2&-x_iy_i&0\\-x_iy_i&-y_i^2&0\\0&0&z_i^2\end{pmatrix},\text{ and }\boldsymbol{Q}_{i,3}^*(v_j)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\text{ for all }v_j\neq v_i.$$

It is straightforward to check that each $\boldsymbol{Q}_{i,j}^*$ satisfies (64) for all pairs of vertices $v_i,v_j\in V$, and hence is a trivial motion of $(G,\tilde{\boldsymbol{p}})$. We will refer to the space of projective C_2^1-motions generated by these $3|V|+6$ motions as the *space of trivial motions*, and say that a projective C_2^1-motion of $(G,\tilde{\boldsymbol{p}})$ is *non-trivial* if it does not belong to this space. We define $(G,\tilde{\boldsymbol{p}})$ to be *projectively C_2^1-rigid* if every projective C_2^1-motion of $(G,\tilde{\boldsymbol{p}})$ is a trivial motion, or equivalently, if its space of projective C_2^1-motions has dimension $3|V|+6$.

The following lemma follows from Lemma C.1 (and the formulae for $\boldsymbol{Q}_{i,j}^*$ given above).

Lemma C.2. *Let $(G,\boldsymbol{p})$ be a two-dimensional framework, and let $(G,\boldsymbol{p}^{\uparrow})$ be as defined in Lemma C.1. Then $(G,\boldsymbol{p})$ is C_2^1-rigid if and only if $(G,\boldsymbol{p}^{\uparrow})$ is projectively C_2^1-rigid.*

C.3 Projective invariance

In this subsection we show that the projective C^1_2-rigidity of a framework $(G,\tilde{\boldsymbol{p}})$ is invariant under a pointwise scaling and a non-singular linear transformation of $\mathbb{R}^3$. These two facts combined with Lemma C.2 will imply the projective invariance of C^1_2-rigidity for 2-dimensional frameworks.

We first show that the projective C^1_2-rigidity of $(G,\tilde{\boldsymbol{p}})$ is invariant by the scaling of each point. Choose a non-zero scalar $\lambda_i \in \mathbb{R}$ for each $v_i \in V$, and let $\lambda\tilde{\boldsymbol{p}}$ be the point configuration in $\mathbb{R}^3$ defined by $(\lambda\tilde{\boldsymbol{p}})(v_i) = \lambda_i\tilde{\boldsymbol{p}}(v_i)$ for each $v_i \in V$. Then $\boldsymbol{Q}: V \to \mathbb{S}^{3\times 3}$ is a (non-trivial) motion of $(G,\tilde{\boldsymbol{p}})$ if and only if $\boldsymbol{Q}$ is a (non-trivial) motion of $(G,\lambda\tilde{\boldsymbol{p}})$ since

$$\begin{aligned}&\mathrm{Trace}\left(((\lambda\tilde{\boldsymbol{p}})(v_i)\times(\lambda\tilde{\boldsymbol{p}})(v_j))((\lambda\tilde{\boldsymbol{p}})(v_i)\times(\lambda\tilde{\boldsymbol{p}})(v_j))^\top(\boldsymbol{Q}(v_i)-\boldsymbol{Q}(v_j))\right)\\&=\lambda_i^2\lambda_j^2\mathrm{Trace}\left((\tilde{\boldsymbol{p}}(v_i)\times\tilde{\boldsymbol{p}}(v_j))(\tilde{\boldsymbol{p}}(v_i)\times\tilde{\boldsymbol{p}}(v_j))^\top(\boldsymbol{Q}(v_i)-\boldsymbol{Q}(v_j))\right).\end{aligned}$$

Thus projective C^1_2-rigidity is invariant by the scaling of each point.

To see the invariance under non-singular linear transformations, choose any non-singular matrix $A \in \mathbb{R}^{3\times 3}$ and let C_A be its cofactor matrix. It is well-known that $(Ax)\times(Ay) = C_A(x\times y)$ for any $x,y\in\mathbb{R}^3$. Moreover, C_A is non-singular if and only if A is non-singular. Given $\boldsymbol{Q}: V\to\mathbb{S}^{3\times 3}$ we let $\boldsymbol{Q}_A: V\to\mathbb{S}^{3\times 3}$ by $\boldsymbol{Q}_A(v_i) = C_A^{-\top}\boldsymbol{Q}(v_i)C_A^{-1}$, where $C_A^{-\top} = (C_A^{-1})^\top$. Then, for any $v_i, v_j \in V$,

$$\begin{aligned}&\mathrm{Trace}\left((A\tilde{\boldsymbol{p}}(v_i)\times A\tilde{\boldsymbol{p}}(v_j))(A\tilde{\boldsymbol{p}}(v_i)\times A\tilde{\boldsymbol{p}}(v_j))^\top(\boldsymbol{Q}_A(v_i)-\boldsymbol{Q}_A(v_j))\right)\\&=\mathrm{Trace}\left(C_A(\tilde{\boldsymbol{p}}(v_i)\times\tilde{\boldsymbol{p}}(v_j))(\tilde{\boldsymbol{p}}(v_i)\times\tilde{\boldsymbol{p}}(v_j))^\top C_A^\top C_A^{-\top}(\boldsymbol{Q}(v_i)-\boldsymbol{Q}(v_j))C_A^{-1}\right)\\&=\mathrm{Trace}\left(C_A\left((\tilde{\boldsymbol{p}}(v_i)\times\tilde{\boldsymbol{p}}(v_j))(\tilde{\boldsymbol{p}}(v_i)\times\tilde{\boldsymbol{p}}(v_j))^\top(\boldsymbol{Q}(v_i)-\boldsymbol{Q}(v_j))\right)C_A^{-1}\right)\\&=\mathrm{Trace}\left((\tilde{\boldsymbol{p}}(v_i)\times\tilde{\boldsymbol{p}}(v_j))(\tilde{\boldsymbol{p}}(v_i)\times\tilde{\boldsymbol{p}}(v_j))^\top(\boldsymbol{Q}(v_i)-\boldsymbol{Q}(v_j))\right).\end{aligned}$$

Thus $\boldsymbol{Q}$ is a (non-trivial) projective C^1_2-motion of $(G,\tilde{\boldsymbol{p}})$ if and only if $\boldsymbol{Q}_A$ is a (non-trivial) projective C^1_2-motion of $(G,A\tilde{\boldsymbol{p}})$.

Hence projective C^1_2-rigidity is invariant under both pointwise scaling and non-singular linear transformations. This implies that the C^1_2-rigidity of 2-dimensional frameworks is projectively invariant.

C.4 Proof of Lemma 5.9

Lemma 5.9. *Let $(G,\boldsymbol{p})$ be a generic framework, v_0 be a vertex of degree five with $N_G(v_0) = \{v_1,\dots,v_5\}$, and $(G,\boldsymbol{p}')$ be a projective image of $(G,\boldsymbol{p})$ such that $\boldsymbol{p}'(v_1) = (1,0)$, $\boldsymbol{p}'(v_2) = (0,0)$, $\boldsymbol{p}'(v_3) = (0,1)$, and $\boldsymbol{p}'(v_4) = (1,1)$. If $(G,\boldsymbol{p}')$ has a bad motion at v_0, then $(G,\boldsymbol{p})$ has a bad motion at v_0.*

Proof. Since $\boldsymbol{p}$ is generic, the set of coordinates of the other vertices of $(G,\boldsymbol{p}')$ is algebraically independent over $\mathbb{Q}$. Suppose that $(G,\boldsymbol{p}')$ has a bad motion $\boldsymbol{q}'$ at v_0, i.e., there is a map $b': \hat{N}_G(v_0)\to\mathbb{Q}[X_0,\dots,Y_5]^3$ satisfying (23) and such that $b'_i(\boldsymbol{p}') = \boldsymbol{q}'(v_i)$ for all $v_i \in \hat{N}_G(v_0)$, where $b'_i(\boldsymbol{p}')$ denotes the evaluation of $b'(v_i)$ at $p'(\hat{N}_G(v_0))$. Suppose also that the inverse of the projective map from $(G,\boldsymbol{p})$ to $(G,\boldsymbol{p}')$ is represented by the matrix $A\in\mathbb{R}^{3\times 3}$. Note that each entry of A can be expressed as a rational function of $\boldsymbol{p}(v_1),\dots,\boldsymbol{p}(v_4)$ over $\mathbb{Q}$. We consider the following procedure for converting the motion $\boldsymbol{q}'$ to a motion $\boldsymbol{q}$ of $(G,\boldsymbol{p})$:

$$\begin{aligned}C^1_2\text{-motion } \boldsymbol{q}' \text{ of } (G,\boldsymbol{p}') &\xrightarrow{①} \text{projective } C^1_2\text{-motion } \boldsymbol{Q}' \text{ of } (G,(\boldsymbol{p}')^\uparrow)\\&\xrightarrow{②} \text{projective } C^1_2\text{-motion } \boldsymbol{Q}'_A \text{ of } (G,A(\boldsymbol{p}')^\uparrow)\\&\xrightarrow{③} \text{projective } C^1_2\text{-motion } \boldsymbol{Q}'_A \text{ of } (G,\boldsymbol{p}^\uparrow)\\&\xrightarrow{④} \text{projective } C^1_2\text{-motion } \boldsymbol{Q} \text{ of } (G,\boldsymbol{p}^\uparrow)\\&\xrightarrow{⑤} C^1_2\text{-motion } \boldsymbol{q} \text{ of } (G,\boldsymbol{p}).\end{aligned}$$

An explanation of each step in this procedure is given below.

① We construct the projective C_2^1-motion $\boldsymbol{Q}'$ of $(G,(\boldsymbol{p}')^{\uparrow})$ from $\boldsymbol{q}'$ as in (65).

② $\boldsymbol{Q}'_A$ is a projective C_2^1-motion of $(G,A(\boldsymbol{p}')^{\uparrow})$ as explained in Subsection C.3.

③ Since A represents the projective transformation from $(G,\boldsymbol{p}')$ to $(G,\boldsymbol{p})$, we can scale each point of $(G,A(\boldsymbol{p}')^{\uparrow})$ to obtain $(G,\boldsymbol{p}^{\uparrow})$. As explained in Subsection C.3, $\boldsymbol{Q}'_A$ remains a projective C_2^1-motion after this pointwise scaling.

④ We eliminate non-zero entries in the right column and the bottom row in $\boldsymbol{Q}'_A(v_i)$ by adding (scaled) trivial motions $\boldsymbol{Q}^*_{i,1},\boldsymbol{Q}^*_{i,2},\boldsymbol{Q}^*_{i,3}$ for each $v_i \in V$. This is always possible since $\boldsymbol{Q}^*_{i,1},\boldsymbol{Q}^*_{i,2},\boldsymbol{Q}^*_{i,3}$ are of the form, $\begin{pmatrix} * & * & 0 \\ * & * & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} * & * & 1 \\ * & * & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}$ respectively, by (67). (Note that $z_i = 1$ in $\boldsymbol{p}^{\uparrow}(v_i)$.) Since we only add trivial motions, the resulting $\boldsymbol{Q}: V \to \mathbb{S}^{3\times 3}$ is still a projective C_2^1-motion of $(G,\boldsymbol{p}^{\uparrow})$.

⑤ We construct the C_2^1-motion $\boldsymbol{q}$ of $(G,\boldsymbol{p})$ from $\boldsymbol{Q}$ as in (65).

We can compute each $\boldsymbol{q}(v_i)$ from $\boldsymbol{q}'(v_i)$ by following the above procedure. For each $v_i \in \hat{N}_G(v_0)$, the procedure can be performed at a symbolic level starting from $\boldsymbol{q}'(v_i) = b'_i(\boldsymbol{p}')$, and returning a map $b : \hat{N}_G(v_0) \to \mathbb{Q}(X_0,Y_0,X_1,\ldots,Y_5)^3$ with $\boldsymbol{q}(v_i) = b_i(\boldsymbol{p})$ for each $v_i \in N_G(v_0)$. By scaling appropriately, we may suppose that $b(v_i)$ is a polynomial map for each $v_i \in N_G(v_0)$. Then $\boldsymbol{q}$ is a b-motion of $(G,\boldsymbol{p})$. In addition, for all $v_i,v_j \in V$, $D(\boldsymbol{p}'(v_i),\boldsymbol{p}'(v_j))\cdot(b'_i(\boldsymbol{p}')-b'_j(\boldsymbol{p}')) = 0$ if and only if $D(\boldsymbol{p}(v_i),\boldsymbol{p}(v_j))\cdot(b_i(\boldsymbol{p})-b_j(\boldsymbol{p})) = 0$. This can be verified by checking the corresponding formulae in each step of the above procedure. Since p is generic, this implies that b satisfies (23), and hence $\boldsymbol{q}$ is a bad motion of $(G,\boldsymbol{p})$ at v_0. □

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