MATTER MODELS IN GENERAL RELATIVITY

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Abstract

The *Cauchy problem* (or, *initial value problem*) provides a setting for the analysis of generic solutions to the Einstein field equations parametrised in terms of the initial conditions. In particular, one is interested in showing that the Einstein equations admit a well-posed initial value formulation. The standard strategy to address this issue is to show that the Einstein equations imply evolution equations that are on a *hyperbolic form.* This has been done for the vacuum, dust and Einstein-Euler equations — each treated separately. In the first part of this thesis, we use an orthonormal frame approach to show that one can avoid the details of a specific Einstein - matter model in the construction of a first order symmetric hyperbolic system by introducing an auxiliary field. The frame is Fermi-Walker propagated and coordinates are chosen such as to satisfy the Lagrange condition. It is shown that the solution of the system established is a solution to the Einstein Equations everywhere on the space time by propagation of constraints. Our analysis covers the special cases of dust and perfect fluid, and we also provide a discussion of self-gravitating elastic matter. In the second part of the thesis, we study the conformal Einstein field equations and show the future stability of N self-gravitating dust bodies in a space time with positive cosmological constant. This is achieved in three parts. First, we show that the choice of density function representing N dust bodies — when ascribed as initial data on the conformal boundary ($\mathscr{I}^{+}$) — is a solution to the conformal field equations. This result is obtained using a theorem by H. Friedrich and the Fredholm alternative. We then show stability for small conformal time for this data, which is equivalent to an infinite physical time. Finally, using a theorem by Choquet-Bruhat, we give sufficient conditions for the Einstein constraint equations to admit a solution representing N bodies of dust, and show that the geodesics are future complete.

Declaration

I, Mikael Normann, confirm that the research included within this thesis is my own work or that where it has been carried out in collaboration with, or supported by others, that this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below. I attest that I have exercised reasonable care to ensure that the work is original, and does not to the best of my knowledge break any UK law, infringe any third partys copyright or other Intellectual Property Right, or contain any confidential material. I accept that the College has the right to use plagiarism detection software to check the electronic version of the thesis. I confirm that this thesis has not been previously submitted for the award of a degree by this or any other university. The copyright of this thesis rests with the author and no quotation from it or information derived from it may be published without the prior written consent of the author.

Details of collaboration and publications. Parts of this work have been completed in collaboration with Dr. Juan A. Valiente Kroon and Dr. Shabnam Beheshti, and are published in the following papers:

- *Evolution Equations For a Wide Range of Einstein-Matter Systems*
  Normann, M., Valiente Kroon, J.A., Gen Relativ Gravit 52, 103 (2020).
- *Future Stability of Self-gravitating Dust Balls in an Expanding Universe*
  Normann, M., Valiente Kroon, J.A., Beheshti, S., In preparation

Signature:
Date:

Acknowledgement

It is with an overwhelming gratitude for a great many blessed people that I now write these last words of my completed PhD work; and I am aware that these inky (or pixelled) letters cannot give proper due to any of them. And some are left out altogether unless I should fill too many pages; but they are certainly not left out of my heart.

I would like to give my sincere and heartfelt gratitude to my supervisor *Dr. Juan A. Valiente Kroon* who, not only *accepted* a student lacking proper academic experience and mathematical knowledge, but also faithfully administered supervision such as to lift the student from stupor to PhD level expertise. The student is, of course, me. Whatever I may now posses of mathematical sophistication is wholly a result of the graceful and excellent supervision Dr. Valiente Kroon exercised. I will always look fondly on the many hours spent at St. Georges or other lovely coffee shops in London discussing mathematics (and all the other nonsense). Thank you for some wonderful years, and for enduring my endless questions!

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I have done my research at Queen Mary University of London. Thank you for your friendship, spiritual mentoring and delightful challenges these past years. The Lord willing, we will not depart our separate ways yet!

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At last, but not least, I must acknowledge someone very special: my wife, my companion and dearest friend, *Sarah*. How should a man express his gratitude and love for a woman who has literally sustained him, served him, loved him, fed him and endured him even as he has not adequately given what is duly hers? The formal and boring prose of academic writing would not do; and the informal attitude of the letter seems to make the matter too causal and familiar. Only the highest forms of prose can be used in this matter – the poem. But, mind you, I am no literary giant, so precisely *how* close it meets the attempt, I cannot say. But here it goes:

They say the world has seven wonders, I say there are ten:
when her trembling heart says 'yes', again and again and again,
when her aching fingers, with precision writes the golden pen,
when her soft and tender voice sings the trill of the wren.

No man has seen a proper wonder, until he has beheld my wife,
when need knock on her door, she drives it away, risking her life
she never let go of her duty even through life's bitter strife,
she is industrious, she can balance on the edge of the knife.

She multiplies her sorrows for the sake of another,
when the pain is unbearable, she still serves her brother,
her children will praise her, for she did take the 'bother,'
and they will say "Lord, make me like my mother."

May you, my dear, be commended on earth and heaven above,
may the tales tell stories long and proper of your love,
and in any case, for what it is worth, I want you to know,
that I am so grateful, and forever in love with you my dove.

“Then I saw all the work of God, that man cannot find out the work that is done under the sun. However much man may toil in seeking, he will not find it out. Even though a wise man claims to know, he cannot find it out.

King Salomon

Contents

Chapter 1
Introduction

1.0.1 The Manifold model of spacetime

Up until the time of Newton and Leibnitz, there were two main types of *scientia*. On the one hand, there was the analytic scientia whereunder Mathematics fell. Thus, the study of Mathematics was more about developing one's character rather than its applicability in the World. Then there was the observational scientia which typically was labeled *natural philosophy*. That these two forms of study could be mixed to form what we call today *mathematical physics*, was far from obvious; and it should perhaps not be obvious for us either. It was not until Newton that Mathematics and natural philosophy combined in such a way that the laws of nature could be written down in terms of mathematical equations, typically in the form of differential equations. This thesis follow in that school of thought and assumes that such an undertaking is fruitful and warranted. The idea, then, is to model our experiences living in what we call a *gravitational field* in a mathematical way such that we can analyse the mathematical structure and finally translate the results back into the realm of experience. I say experience and not reality, because to suggest that Mathematics is somehow less real based on the observation that it cannot be touched, seen or heard, seems to me to beg the popular question of the unreasonable effectiveness of Mathematics [1]. Reality — in my opinion — has more faces than Brahma. And Mathematics is one of them. The question of how to know, or shall we say, how to gain access to reality as a whole, has been a long standing question going back to ancient Greek Philosophy. In this thesis — as far as I am aware — we shall be concerned with two such faces of reality and their interplay. The one I shall hereafter refer to as *Gnosisynesis*[a] and concerns the understanding of reality from experience. This is the basis for all the empirical sciences, including Physics. The second face of reality I will call *Logosophia*[b]. This is the wisdom one obtains by studying the world of Logic and reason. Naturally, this is the foundation for subjects such as Logic and Mathematics. In such a view of the world, the interplay and interaction of Gnosisynesis and Logosophia — or, in particular, natural philosophy and Mathematics — becomes rather a natural field of study as part of the study of Reality.

Returning to the subject of this thesis, which is Gravitation, we now make the following observation from Gnosisynesis: the notion of here and there is intimately linked with the notion of now and then. When a friend ask you to meet him at the lamp post behind the bank at 11pm, there is no ambiguity (given you know the place and how to read a clock.) If, on the other hand your friend only tells you to meet him at the lamp post, without specifying the 'when', you would complain. Every thing we see, touch and feel has an imaginary flag associated with it which contains the information we call "when". The information given by the flag we typically associate with an invisible part of reality we call *time*. We will call the 'thing' together with its flag an *event*. For example, the flight of a bumble bee is a sum of uncountable many events. This is all Gnosisynesis — i.e. arguments derived from experiences. We now move to the branch of Logosophia we call Mathematics to analyse the situation further. In Mathematics, there is the notion of *points*, and the collection of points are called *sets*. We make an assumption:

> Assumption 1.1. Each event may be represented by a point.

As a consequence, one can construct a set with each element representing a material particle — i.e. the smallest thing you would be able to see — and its associated "flag". This set, we shall call *spacetime* and is henceforth labeled $\mathscr{M}$. We have thus created a correspondence between objects living

[a]This word is made up of two Greek words: *Gnosis* — which means knowledge gained by experience — and *Synesis* — understanding

[b]From the greek words *Logos* — reason — and *Sophis* — wisdom

in the mathematical realm with things of the sensible world. Furthermore, by this one assumption, we claim to represent the whole of reality as known by Gnosisynesis through Logosophia. It may not be warranted, but insofar as the experience of Gravitation is concerned, it is remarkably successful.

So far, $\mathscr{M}$ is a set. A set, has no structure, it is more like a jar of powder that may be shaken, poured out, rearranged, and still remain the same powder. Clearly the reality of Gnosisynesis is not very well represented by a bag of flour! What we observe has order, it has structure. Thus, we introduce the additional structure of a *topology* $\mathscr{O}$ on $\mathscr{M}$.

Definition 1 (Topology). A topology $\mathscr{O}_{\mathscr{M}}$ on a set $\mathscr{M}$ is a subset of $\mathscr{M}$ such that

1) $\emptyset \in \mathscr{O}_{\mathscr{M}}$ and $\mathscr{M} \in \mathscr{O}_{\mathscr{M}}$,
2) Given any two sets $V, U \in \mathscr{O}_{\mathscr{M}}$, then $V \cap V \in \mathscr{O}_{\mathscr{M}}$,
3) Given any index set α, and let $U_\alpha \in \mathscr{O}_{\mathscr{M}}$, then $\bigcup\limits_{\alpha} U_\alpha \in \mathscr{O}_{\mathscr{M}}$.

We shall label a spacetime $\mathscr{M}$ with the additional structure of topology by $(\mathscr{M}, \mathscr{O}_{\mathscr{M}})$. Thus spacetime is a topological space. We will hereafter refer to a set $\mathscr{U} \in \mathscr{O}_{\mathscr{M}}$ as an *open set*. An important set which will be used frequently throughout is the set labelled as $\mathbb{R}^d$, defined as

$$\mathbb{R}^d \equiv \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{\text{d times}}.$$

Thus a point $p \in \mathbb{R}^d$ may be labelled by numbers in $\mathbb{R}$ — i.e. $p = (p_1, p_2, \ldots p_d)$ with $p_i \in \mathbb{R}$. A very important topology which is often assumed when dealing with the set $\mathbb{R}^d$, is the *standard topology*.

Definition 2 (Standard topology). Let $\mathscr{M} = \mathbb{R}^d$ and $\mathscr{O}_{std} \subseteq \mathscr{P}(\mathbb{R}^d)$. We define the standard topology of M as

$$\mathscr{O}_{st} \equiv \{\mathscr{U} \in \mathbb{R}^d | \forall p \in \mathscr{U}, \exists r | B_r(p) \subseteq \mathscr{U}\}.$$

In the above $\mathscr{P}$ is the *power set* of M — i.e. the set of all subsets of M — and $B_r(p)$ is the *soft ball* defined as

Definition 3 (soft ball). For any $p \in \mathbb{R}^d$ and $r \in \mathbb{R}^+$

$$B_r(p) \equiv \{(q_1, \ldots, q_d) | \sum_{i=1}^{d} (q_i - p_i)^2 < r^2\}.$$

Thus, we learn that topology is something which needs to be chosen. It is not a priori provided. We further introduce the notion of *a map*. This is an object that associates with every point in a set M some point U in the set N. The set U can either be equal to N or a subset. We write this as $U \subseteq N$. We say that M is the domain and N the target. Of course, the domain may itself be a subset of another larger set. A map ϕ is represented schematically in the following way

$$\phi : M \to N.$$

A map is called *surjective* if all the points in the target are mapped onto. It is said to be *injective* if all the points in the target are mapped unto only once. It is *bijective* if it is surjective and injective. Let us for a moment move back into the reality of Gnosisynesis and consider a mirror. When I move my right hand to comb my hair, the mirror produces an image in which the right hand is also moved. But my nose, which is pointing towards the mirror, is — in the mirror picture — pointing towards me! The mirror, in other words, produces an image of me which is inverted inside out. This is a everyday example of a map. And this map is said to be *continuous* — i.e. the image produced by the map has no holes and jumps. It does not produce an image of my face with a big black cavity in the middle. More formally, we have,

Definition 4 (Continuity). Let $(\mathscr{M}, \mathscr{O}_{\mathscr{M}})$ and $(\mathscr{N}, \mathscr{O}_{\mathscr{N}})$ be two topological spaces. Then a map $\phi : \mathscr{M} \to \mathscr{N}$ is said to be continuous with respect to $\mathscr{O}_{\mathscr{M}}$ and $\mathscr{O}_{\mathscr{N}}$ if, $\forall\ V \in \mathscr{O}_{\mathscr{N}}$

$$Preim_{\phi}(V) \in \mathscr{O}_{\mathscr{M}},$$

where,

$$Preim_{\phi} \equiv \{p \in \mathscr{M} | \phi(p) \in V\}.$$

A map is thus continuous if and only if the preimages of all open sets are open. **The notion of a continuous curve in spacetime thus depends on the chosen topology**.

Theorem 1 (Composition of continuous maps). If $\phi : \mathscr{M} \to \mathscr{N}$ and $\gamma : \mathscr{N} \to \mathscr{Q}$ are continuous maps, then the composition map $\gamma \circ \phi : \mathscr{M} \to \mathscr{Q}$ is also continuous.

Important for physics on spacetime is the property that a subset $\mathscr{S} \subset \mathscr{M}$ can "inherit" the topology of $\mathscr{M}$.

Definition 5 (Subset topology). Let $(\mathscr{M}, \mathscr{O}_{\mathscr{M}})$ be a topological space and $\mathscr{S} \subset \mathscr{M}$. Then

$$\mathscr{O}|_{\mathscr{S}} \equiv \{\mathcal{U} \cap \mathscr{S} | \mathcal{U} \in \mathscr{O}_{\mathscr{M}}\}$$

is also a topology, and is called the subset topology.

From Gnosisynesis we again establish a part of experienced reality: the vertex of two walls and the floor produces a frame which the table, the chair and my writing desk is measured in relation to. I can take a stick, and measure the floor-distance between the walls and the objects. For instance my chair may be 5 stick-lenghts away from the one wall and 3 from the other as measured in "straight" lines along the floor. This completely determines the location of my chair in relation to my room. I could extend this to my entire neighbourhood and even further. We thus establish the following Gnosisynesis fact: given any specific location, one can always establish a reference frame in relation to which to measure distances for any objects in the neighbourhood of that location. This experience can be described in the logosophic description of reality by making use of the topological space $(\mathbb{R}^4, \mathscr{O}_{std})$. The idea is that one wants to be able to prescribe to every neighbourhood of every point $p \in \mathscr{M}$ the numbers $(q_1, q_2, q_3, q_4) = q^\mu \in \mathbb{R}$. That is, one want to make sure there exists in every neighbourhood $\mathcal{U} \in \mathscr{M}$ a continuous map $x : \mathscr{M} \to \mathbb{R}^4$ with a continous inverse.

Definition 6 (Topological manifold). A topological space $(\mathscr{M}, \mathscr{O}_{\mathscr{M}})$ is said to be a *topological manifold* if $\forall p \in \mathscr{M}$ $\exists$ an open subset $\mathscr{U} \in \mathscr{O}_{\mathscr{M}}$ and a map $x : \mathscr{U} \to \mathscr{V} \in \mathbb{R}^d$ such that

1) x is invertible — i.e. $x^{-1}(\mathscr{V}) = \mathscr{U}$,
2) x and x^{-1} are continous w.r.t. $\mathscr{O}_{\mathscr{M}}$ and $\mathscr{O}_{std}$.

We call the pair $(\mathscr{U}, x)$ a *chart*. It is, of course, desirable due to our lived experience, that we shall postulate the condition that

$$\mathscr{M} = \bigcup_\alpha \mathscr{U}_\alpha.$$

This assumption defines an *atlas* over $\mathscr{M}$

$$\mathscr{A} \equiv \{(\mathscr{U}_\alpha, x) | \bigcup_\alpha \mathscr{U}_\alpha = \mathscr{M}\}.$$

Observe from the above discussion that the choice of charts on $\mathscr{M}$ is completely arbitrary. In other words, there is nothing intrinsic or fundamental about the charts; they are merely a way to label points $p \in \mathscr{M}$ with numbers. What these numbers are — i.e. what maps we use — are up to choice and fancy. Consider two overlapping charts $(\mathscr{U}, x)$ and $(\mathscr{V}, y)$ on $\mathscr{M}$. We then have two maps from the region $\mathscr{U} \cup \mathscr{V}$ into $\mathbb{R}^d$, and we can define the *transition map* — see Figure 1.1

$$y \circ x^{-1} : \mathbb{R}^d \to \mathbb{R}^d.$$

We may now introduce the notion of *coordinates* on $\mathscr{M}$ in the following way. The map $x : \mathscr{M} \to \mathbb{R}^d$ maps a point $p \in \mathscr{M}$ to $x(p) \in \mathbb{R}^d$ — i.e. one

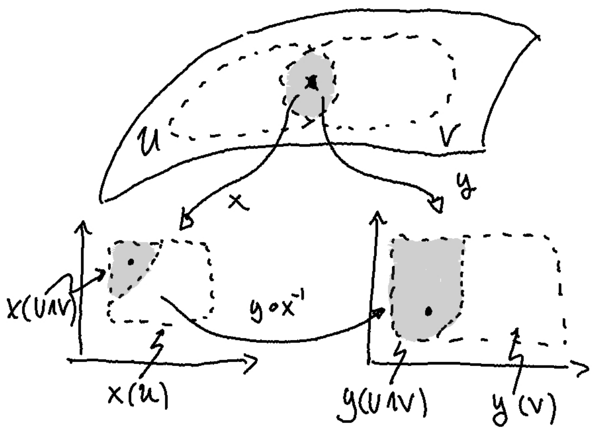

Figure 1.1: The transition map between two regions of $\mathbb{R}^d$.

has

$$x(p) = (x^1(p), x^2(p), \ldots, x^d(p))$$

such that,

$$\begin{aligned} x^1(p) &= q_1, \\ x^2(p) &= q_2, \\ x^3(p) &= q_3, \\ x^4(p) &= q_4. \end{aligned}$$

It is thus equivalent to consider the collection of $i = 1, 2, 3.., d$ maps $x^i(p) : \mathscr{U} \to \mathbb{R}$ instead of the one map $x(p) : \mathscr{U} \to \mathbb{R}^d$. The x^i are called the coordinate functions/maps of the point $p \in \mathscr{U}$. The advantage of introducing charts and thus coordinate maps on $\mathscr{M}$ is that one can define notions on $\mathscr{M}$ by using the already well established calculus in $\mathbb{R}^d$. This comes with a warning: **any geometric notion must be independent of the chosen chart**. One such notion which is important in order for $(\mathscr{M}, \mathscr{O}_{\mathscr{M}})$ to be compatible with the Gnosisynesis view of spacetime, is the *differentiabillity* of a map or curve.

Definition 7. Two charts $(\mathscr{U}, x)$ and $(\mathscr{V}, y)$ are said to be differentiable if either, $\mathscr{U} \cap \mathscr{V} = \emptyset$, or $\mathscr{U} \cap \mathscr{V} \neq \emptyset$ and

1) $y \circ x^{-1} : \mathbb{R}^d \to \mathbb{R}^d$,
2) $x \circ y^{-1} : \mathbb{R}^d \to \mathbb{R}^d$,

are differentiable.

We then restrict the atlas on $\mathscr{M}$ to consist of only differentiable charts — that is:

Definition 8. An atlas $\mathscr{A}_d$ is said to be a differentiable atlas if any two charts in $\mathscr{A}_d$ are differentiable.

Furthermore, we define

Definition 9. A differentiable manifold is a triple $(\mathscr{M}, \mathscr{O}_{\mathscr{M}}, \mathscr{A}_d)$.

Hence, we have argued, so far, that the logosophic description of spacetime is a differentiable manifold $(\mathscr{M}, \mathscr{O}_{\mathscr{M}}, \mathscr{A}_d)$. But nothing has been said about *how* differentiable the manifold is. It turns out — see [2] for more details and proof — that if a manifold is once differentiable and continuous (C^1), it is also smooth (C^∞).

Theorem 2 (Adapted from Whitney). Any $C^{k\leq 1}$ - atlas of a topological manifold contains a C^∞ - atlas.

Thus, if one can argue that a manifold is C^1 one can be sure that it is in fact a C^∞ - manifold. The next structure necessary to equip spacetime with, is the notion of a *straight curve*. To this end, one introduce a map $\boldsymbol{\nabla} : \mathscr{T}\mathscr{M} \to \mathscr{T}\mathscr{M}$, where $\mathscr{T}\mathscr{M}$ denotes the tensor bundle on $\mathscr{M}$ — i.e. the collection of all tensors defined at every point on $\mathscr{M}$. Furthermore, we introduce the notation $T\mathscr{M}$ to denote the collection of *tangent bundles* on $\mathscr{M}$ — i.e. the space of all tangent vectors on $\mathscr{M}$. The space of all smooth functions on $\mathscr{M}$ is denoted by $C^\infty(\mathscr{M})$.

Definition 10. A linear connection $\boldsymbol{\nabla}$ maps a (p, q)-tensor in $\mathscr{T}\mathscr{M}$ to a (p, q)-tensor in $\mathscr{T}\mathscr{M}$ such that $\forall \boldsymbol{u}, \boldsymbol{v} \in T\mathscr{M}$, $\boldsymbol{T}, \boldsymbol{W} \in \mathscr{T}\mathscr{M}$, $t, f \in C^{\infty}(\mathscr{M})$ and $a, b \in \mathbb{R}$ the following is satisfied,

1) Linearity in $\boldsymbol{T}$ and $\boldsymbol{W}$:

$$\nabla_{\boldsymbol{u}}(a\boldsymbol{T} + b\boldsymbol{W}) = a\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{T} + b\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{W} \tag{1.1}$$

2) Linearity in $\boldsymbol{u}, \boldsymbol{v}$:

$$\boldsymbol{\nabla}_{f\boldsymbol{u}+\boldsymbol{v}}\boldsymbol{T} = f\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{T} + \nabla_{\boldsymbol{v}}\boldsymbol{T} \tag{1.2}$$

3) Leibnitz rule:

$$\boldsymbol{\nabla}_{\boldsymbol{u}}(\boldsymbol{T} \otimes \boldsymbol{W}) = \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{T} \otimes \boldsymbol{W} + \boldsymbol{T} \otimes \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{W} \tag{1.3}$$

4) Consistency with the notion of tangent vectors as directional derivatives of scalar fields:

$$\boldsymbol{\nabla}_{\boldsymbol{u}} f = \boldsymbol{u}(f) \tag{1.4}$$

Finally, I should go back to the example of my chair and table in relation to the walls and floor. The act of measuring distances is an experience we take for granted. Given any rigid object with a definite length, one can use it to measure the distances between objects and in relation to a chosen reference frame. As such it is necessary that we equip the manifold model of spacetime with the property of measuring distances. This is achieved by introducing a metric $\boldsymbol{g}$ on $\mathscr{M}$ such that the smooth manifold model of spacetime is now a metric space. But one has many options as to what kind of metric. The specific nature of each metric space is determined uniquely by the signature of the metric. A positive definite metric is said to be *Riemannian*, wheareas a metric with signature $(-, ..., +, .., +, 0, ...,)$ is *Pseudo-Riemannian.* More specifically, it is called *Lorentzian* when its signature is $(-, +, ...)$ or $(+, -, ...)$. Again, appealing to Gnosisynesis, we establish that our experience of causality and the constancy of the speed of light, that the metric must be of a Lorentzian type.

Definition 11. Given $(\mathscr{M}, \mathscr{O}_{\mathscr{M}}, \mathscr{A}_d)$ with $T\mathscr{M}$ the space of all tangent vectors, the metric tensor on $\mathscr{M}$ is a (0,2) - tensor field satisfying $\forall \boldsymbol{X}, \boldsymbol{Y} \in T\mathscr{M}$,

1) Symmetry: $\boldsymbol{g}(\boldsymbol{X}, \boldsymbol{Y}) = \boldsymbol{g}(\boldsymbol{Y}, \boldsymbol{X})$,
2) $\boldsymbol{g}$ is Lorentzian.

In addition to the metric structure, one requires all time like curves to be *oriented*. That is, no closed time like[c] curves are permitted. To allow for such curves would mean in practice that one can travel back in time by going into the future.

Definition 12 (Time orientation). Let $(\mathscr{M}, \boldsymbol{g})$ be a smooth, Lorentzian manifold. Then a time orientation is given by a smooth vector field $\boldsymbol{T}$ that,

1) is non-vanishing everywhere,
2) is time like everywhere.

We thus make the assumption that our description of spacetime in terms of Logosophia, is described by a oriented, smooth Lorentzian manifold. In the rest of this thesis we shall write $(\mathscr{M}, \boldsymbol{g})$ for the manifold model of spacetime, and thus suppress the explicit choices we have made regarding the topology of $\mathscr{M}$. It is worth noting, that what topology is "correct" for the description of our universe is still an open question — e.g. see [3] . We end this section with a summary of the assumptions made in order to arrive at the manifold model of spacetime:

1) spacetime is four dimensional.
2) spacetime is endowed with a topology.
3) spacetime is locally Euclidean and covered by an atlas.

[c]See Section 1.0.2

4) There exists an atlas such that spacetime is everywhere smooth.
5) spacetime has the causal structure of a Lorentzian metric.
6) spacetime is time oriented.

1.0.2 Gravity as the geometry of the spacetime manifold

The previous section briefly outlined the idea behind the attempt at describing Gnosisynesis in terms of Logosophia — or, in more familiar terms — Cosmology in terms of Mathematics. But we said nothing about the form of $\boldsymbol{g}$ — apart from its signature — and the various fields living on spacetime, and by which rules they are governed. This is what we will shortly discuss in this section.

One postulates that the objects of physical interest can be described in terms of tensor fields on the spacetime manifold. But this postulate itself, suggest that the metric tensor field $\boldsymbol{g}$ should have a physical significance. One understands from the structure of the metric that it involves the following concepts:

1) *Causal structure.* One postulates that light travels on curves such that the tangent vectors to these curves are *null* — i.e. $\boldsymbol{g}(\boldsymbol{X}, \boldsymbol{Y}) = 0$. The set of all such curves at a point $p \in \mathscr{M}$ make up what is called the *null cone* — see Figure 1.2. All other objects follow *timelike* curves — i.e. $\boldsymbol{g}(\boldsymbol{X}, \boldsymbol{Y}) = -1$ and lies within the null cone. *Spacelike* curves are generally viewed as not observable due to the postulated/experienced speed limit of light.
2) *Geometry of spacetime.* The invariant quantity ds called *the line element* and defined in local coordinates x^μ as,

$$ds^2 \equiv g_{\mu\nu} dx^\mu \otimes dx^\nu,$$

has units of length. Consequently, it is understood that the metric tensor is a geometric quantity. In terms of Gnosisynesis, one may associate the metric with what is loosely called "shape".

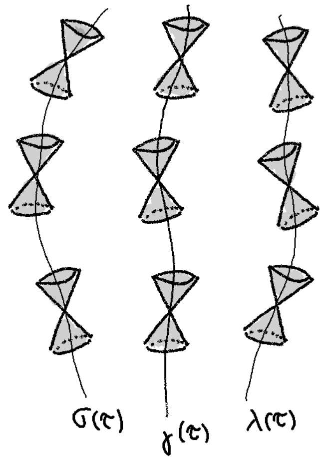

Figure 1.2: Null cones in a spacetime $(\mathscr{M}, g)$ and time like curves $\sigma(\tau)$, $\gamma(\tau)$ and $\lambda(\tau)$.

Furthermore, the metric give rise to a connection $\boldsymbol{\nabla}$ such that

$$\boldsymbol{\nabla g} = 0.$$

Such a connection is said to be *metric compatible*. The connection $\boldsymbol{\nabla}$ is said to be *torsionfree* if it satisfies, for a scalar field ϕ, the additional property,

$$(\boldsymbol{\nabla_u \nabla_v} - \boldsymbol{\nabla_v \nabla_u})\,\phi = 0.$$

Definition 13 (Geodesic). Given a curve $\gamma(\tau)$ on $(\mathscr{M}, \boldsymbol{g})$, and let $\boldsymbol{u}$ be the tangent vector along $\gamma(\tau)$ and $\boldsymbol{\nabla}$ metric compatible and torsion free. The curve γ is said to be a *geodesic* if one has

$$\boldsymbol{\nabla_u u} = a\boldsymbol{u}, \qquad a \in \mathbb{R}.$$

One may choose τ such that $a = 0$. Such curves are said to be *affine parametrised*.

Remark 1.2. Observe that if is not torsion free, then the curve $\gamma(\tau)$ is *auto-parallel* if it is extremised.

The action of gravity on a particle is **the deviation of geodesics from Minkowski geodesics.** There is thus a direct relationship between the metric tensor $\boldsymbol{g}$ and the geometry on $(\mathscr{M}, \boldsymbol{g})$. The significance of Einstein's theory of gravity is that it relates this curvature of the spacetime manifold with the other fields present on the manifold. More precisely, given a tensor field $\boldsymbol{T}$ which depends on fields describing the matter content in the Universe, one has that

$$\boldsymbol{G} = \kappa \boldsymbol{T}, \tag{1.5}$$

where $\boldsymbol{G}$ is the *Einstein tensor* and depends on the metric and its second derivatives, and $\boldsymbol{T}$ is the *energy momentum tensor* representing the matter distribution. In other words, the Einstein field equations relate the matter fields on the spacetime manifold with its curvature as described by the metric. Hence, given a certain matter distribution, one solves these equations for the metric. In other words, one is interested in what geometry the system has under a certain matter distribution.

1.1 The problems studied herein

A closer look at equation (1.5) shows that in local coordinates x^μ it can be written as a second order differential equation in the metric. In general, the tensor $\boldsymbol{T}$ is also a function of the metric tensor. It is therefore a nonlinear second order differential equation one must solve. One is thus faced with two serious challenges. What form should the energy momentum tensor $\boldsymbol{T}$ take for particular matter models and how would these different choices affect the stability and solubility of the equation? In this thesis *we provide a tensor $\boldsymbol{T}$ representative for a relativistic, elastic material.* The result is obtained by introducing a frame field to spacetime and to construct a mapping between spacetime and an additional body manifold. The energy momentum tensor is then found by varying the action with respect to the frame field components — see section 3.4. The rest of the thesis is focused around the second question. The natural framework to address this question is to study the *Cauchy problem* — see section 2.4 for details — of equation (1.5). *We provide sufficient conditions to guarantee the existence of solutions and their local stability for a general class of tensors* $\boldsymbol{T}$. We refer to section 4.1 for more details on the motivation and background of this study. Finally, we employ conformal methods — see Section 2.5 for an introduction on the topic — to prove future stability for the case when $\boldsymbol{T}$ represents *dust* — see Section 3.2 — and with the existence of a positive cosmological constant. We refer to Section 5.1 for a more detailed introduction to the problem and a precise statement of the results.

In Chapter 2 we introduce the mathematical background material to follow the argumentations found in this thesis. In Section 2.1 we provide the notation and conventions, in Section 2.2 we introduce the frame formalism employed throughout and in Section 2.3 we introduce the theory of symmetric hyperbolic differential equations. We introduce the Cauchy problem in general relativity in Section 2.4 and provide a brief introduction to conformal methods in Section 2.5. Finally, in Section 2.6 we establish the necessary definitions and terminology used in the theory of hypersurfaces in Section 2.6. In Chapter 3 we give a brief discussion on the most important matter models in general relativity. In Chapter 4 we use an orthonormal frame approach to provide a general framework for the first order hyperbolic reduction of the Einstein equations coupled to a fairly generic class of matter models. Our analysis covers the special cases of dust and perfect fluid. We also provide a discussion of self-gravitating elastic matter. We also show the propagation of the constraints of the Einstein-matter system. Finally, in Chapter 5 we consider a system representing self-gravitating balls of dust in an expanding Universe. It is demonstrated that one can prescribe data for such a system at infinity and evolve it backward in time without the development of shocks or singularities. The resulting solution to the Einstein-λ-dust equations exists for an infinite amount of time in the asymptotic region of the spacetime. Furthermore, we find that if the density is small compared to the Cosmological constant, then it is possible to construct Cosmological solutions to the Einstein constraint equations on a standard Cauchy hypersurface representing self-gravitating balls of dust. If, in addition, the density is assumed to be sufficiently small, then this initial data gives rise to a future geodesically complete solution to the Einstein-λ-dust equations admitting a smooth conformal extension at infinity which can be regarded as a perturbation of de Sitter spacetime.

Chapter 2

Theoretical foundations and conventions

In this chapter we provide the mathematical background material used throughout the thesis. It is attempted to keep the discussion as introductory and self contained as possible without making it too long. We provide suggested reading for further study and more details where natural.

2.1 Notation and conventions

2.1.1 Abstract index notation

The abstract index notation was invented by *Roger Penrose* — see [4] for details — and is a notation system which allows for tensor manipulations and representation used when dealing with tensor coordinate components, but without introducing coordinates. Thus, one get the best from both the worlds of coordinate representation and coordinate-free notation. The construction is as follows.

Let $\mathscr{F}$ be an element in the space of C^∞ functions $\mathcal{X}$ on a manifold $\mathscr{M}$, and let $\mathscr{V}$ be the vector space with elements $\boldsymbol{V}$ operating on the elements $\mathscr{F}$ — i.e $\boldsymbol{V}(\mathscr{F}) \in \mathbb{R}$. With a vector space is also associated a natural dual vector space $\mathscr{V}^*$ with elements $\boldsymbol{\omega}$. The idea is now to introduce a space consisting of an infinite set of labels

$$\mathscr{L} \equiv \{a, b, c, ..., a_0, b_0, c_0, ..., a_1, b_2, c_2, ...\},$$

such that for any $x \in \mathscr{L}$ one may construct the vector spaces $\mathscr{V}^x$ and $\mathscr{V}_x^*$, respectively given by

$$\mathscr{V}^x \equiv \mathscr{V} \times \{x\}, \qquad \mathscr{V}^*_x \equiv \mathscr{V}^* \times \{x\}.$$

Thus for any element $\boldsymbol{V} \in \mathscr{V}$ and $a \in \mathscr{L}$, one is given a unique vector $V^a \in \mathscr{V}^a$; and similarly for the dual. As such, each element V^x satisfies the axioms of vectors. It is important to note, that the vectors V^a and V^b are elements of different vector spaces. Hence, it is not allowed to write $V^a + V^b$, which is in agreement with the coordinate tensor notation. Since the elements x of $\mathscr{L}$ can only belong to one unique vector, it is not allowed to write V^aU^a. One could, however, write for two elements $V^a \in \mathscr{V}^a$ and $\boldsymbol{U}^b \in \mathscr{V}^b$,

$$V^aU^b \equiv U^a \otimes V^b.$$

However, the tensor product is not in general commutative, and we want a notation in which $V^aU^b = U^bV^a$. Thus one defines a new product between the vector spaces which is essentially a commutative version of the tensor product. It can then be shown that one can construct a space $\mathscr{T}^{abc...}{}_{def...}$ which is spanned by elements on the form

$$V^aU^bZ^c...\omega_d\alpha_e\beta_f.... \tag{2.1}$$

Thus any element $T^{abc...}{}_{def...} \in \mathscr{T}^{abc...}{}_{def...}$ is a linear combination of (2.1). Note that the labels of $\mathscr{T}^{abc...}{}_{def...}$ may be freely permuted while not those of the elements $T^{abc...}{}_{def...}$. In this manner, one can construct the entire system of tensors $\{\mathscr{T}\}$

$$\{\mathscr{T}\} = \{\mathscr{F}, \mathscr{V}^a, \mathscr{T}^a{}_b, ..., \mathscr{T}^{cde...}{}_{fgh...}\}.$$

The operations on $\{\mathscr{T}\}$ are addition, multiplication, index substitution and contraction, respectively:

$$\mathscr{T}^{cde...}{}_{fgh...} + \mathscr{T}^{cde...}{}_{fgh...} \to \mathscr{T}^{cde...}{}_{fgh...}, \tag{2.2}$$

$$\mathscr{T}^{cde...}{}_{fgh...} \times \mathscr{T}^{xyz...}{}_{wmn...} \to \mathscr{T}^{cdexyz...}{}_{fghwmn...}, \tag{2.3}$$

$$\mathscr{T}^{cde...}{}_{fgh...} \to \mathscr{T}^{ade...}{}_{fgh...}, \tag{2.4}$$

$$\mathscr{T}^{cde...}{}_{fgh...} \to \mathscr{T}^{de...}{}_{gh...}. \tag{2.5}$$

Finally, for any $v^a \in \mathscr{V}^a$ and $\omega_b \in \mathscr{V}_b^*$ one can define the *inner product* by,

$$v^a \omega_a \equiv \langle \boldsymbol{\omega}, \boldsymbol{V} \rangle.$$

As with any vector space, one can introduce a basis and write V^a and ω_a in terms of coordinates in relation to these basis vectors.

2.1.2 Coordinate indices

The first part of the Latin alphabet $\{a, b, c, d, ...\}$ will be used as abstract labels. For coordinates, we will use greek letters $\{\mu, \nu, \gamma, ...\}$ for spacetime coordinates x^μ with $\mu = \{0, 1, 2, 3\}$. We will use the middle part of the Latin alphabet $\{i, j, k, ...\}$ for spacetime coordinates x^i with $i = \{1, 2, 3\}$. Occasionally, the first part of the greek alphabet — i.e. $\{\alpha, \beta, ..\}$ will be used for spacetime coordinates taking the values $1, 2, 3$. We will use bold Latin letters $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, ..., \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}, ...\}$ for frame indices and where $\boldsymbol{a}, \boldsymbol{b}, .. = 0, 1, 2, 3$ and $\boldsymbol{i}, \boldsymbol{j}, ... = 1, 2, 3$. Finally, capital latin letters refer to a summation index, and not a coordinate — i.e. $I, J, K, ...$ run from 1 to n.

Remark 2.1. *At times it will be convenient to use bold symbols for vectors and tensors — e.g.* $\boldsymbol{u}, \boldsymbol{T}$ *and* $\boldsymbol{g}$ *for* u^a, T^{ab} *and* g_{ab}*, respectively. This is in particular true when the focus of the discussion is on structural properties as opposed to the details of the equations. For this reason you may find discussions in this thesis which depart from the abstract index notation.*

2.2 Orthonormal frames

So far we have set up a theory which claims to give a precise mathematical account of every event in the Universe. This means that me observing the moon and little Eliana mesmerised by the passing train are included in the formalism of the spacetime manifold. In other words, we need a rigorous definition of what events constitute what we typically call *observers*.

Definition 14 (Observer). An observer is a time like worldline $\gamma(s)$ together with a choice of an orthonormal frame $\{e_{\boldsymbol{a}}(s)\} \in T_{\gamma(s)}(\mathscr{M})$ such that,

$$\boldsymbol{g}(e_{\boldsymbol{a}}, e_{\boldsymbol{b}}) = \eta_{\boldsymbol{ab}}, \qquad e_{\boldsymbol{0}} \equiv \boldsymbol{U}, \tag{2.6}$$

where,

$$\eta_{\boldsymbol{ab}} \equiv \mathrm{diag}(-1, 1, 1, 1).$$

The choice of $\{e_{\boldsymbol{a}}\}$ uniquely specifies a dual basis $\{\omega^{\boldsymbol{b}}\} \in T^*(\mathscr{M})$ satisfying,

$$e_{\boldsymbol{a}}{}^{a}\omega^{\boldsymbol{b}}{}_{a} = \delta_{\boldsymbol{a}}{}^{\boldsymbol{b}}.$$

Thus, any tensor field $\boldsymbol{T} \in \mathscr{M}$ written in an orthonormal frame basis, represents the reading of a particular measurement $\boldsymbol{T}(e_{\boldsymbol{a}}, e_{\boldsymbol{b}})$ made by the observer at a point $p \in \mathscr{M}$. For instance, let $\boldsymbol{V}$ be the tangent vector to a massive particle world line δ meeting an observer γ at a point p — i.e. $\delta(\tau) = \gamma(s) = p$ — , then the three numbers $v^{\boldsymbol{i}} \equiv \omega^{\boldsymbol{i}}(\boldsymbol{V})$ represents the velocity of the particle as measured by the observer at the point p. In terms of local coordinates $\overline{x} = \{x^\mu\}$ the frame and dual frame fields can be expanded as

$$e_{\boldsymbol{a}}{}^{a} = e_{\boldsymbol{a}}{}^{\mu}(\partial_\mu)^a, \qquad \omega^{\boldsymbol{b}}{}_{b} = \omega^{\boldsymbol{b}}{}_{\nu}(\mathrm{d}x^\nu)_b.$$

In the remainder of this section we will introduce important tensor fields and relations in the frame formalism which will be used later in the thesis. Throughout we follow the conventions in [5] —see Chapters 2 and 12.

2.2.1 Frame covariant derivatives

Given ∇_a, the Levi-Civita connection of the metric g_{ab}, we denote by $\nabla_{\boldsymbol{a}}$ the associated directional derivative along the vector frame $e_{\boldsymbol{a}}$. The connection coefficients of ∇_a with respect to the frame $\{e_{\boldsymbol{a}}\}$ are defined by the relation

$$\nabla_{\boldsymbol{a}} e_{\boldsymbol{b}}{}^{c} = \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{d}}{}_{\boldsymbol{b}} e_{\boldsymbol{d}}{}^{c} \tag{2.7}$$

so that

$$\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = \omega^{\boldsymbol{c}}{}_{b} \nabla_{\boldsymbol{a}} e_{\boldsymbol{b}}{}^{b}.$$

The metric compatibility of the connection ∇_a is expressed by the condition

$$\eta_{\boldsymbol{db}} \Gamma_{\boldsymbol{c}}{}^{\boldsymbol{d}}{}_{\boldsymbol{a}} + \eta_{\boldsymbol{ad}} \Gamma_{\boldsymbol{c}}{}^{\boldsymbol{d}}{}_{\boldsymbol{b}} = 0. \tag{2.8}$$

Given a vector v^a with components $v^{\boldsymbol{a}} \equiv \omega^{\boldsymbol{a}}{}_{a} v^a$ we define

$$\nabla_{\boldsymbol{a}} v^{\boldsymbol{b}} \equiv e_{\boldsymbol{a}}{}^{a} \omega^{\boldsymbol{b}}{}_{b} (\nabla_a v^b).$$

A direct computation then shows that

$$\nabla_{\boldsymbol{a}} v^{\boldsymbol{b}} = \partial_{\boldsymbol{a}} v^{\boldsymbol{b}} + \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} v^{\boldsymbol{c}}$$

where $\partial_{\boldsymbol{a}} \equiv e_{\boldsymbol{a}}{}^{\mu} \partial_\mu$ is the directional partial derivative along $e_{\boldsymbol{a}}{}^{a}$.

Remark 2.2. Note that it is common in the literature to write $e_{\boldsymbol{a}}[v^{\boldsymbol{b}}]$ instead of $\partial_{\boldsymbol{a}} v^{\boldsymbol{b}}$. We have chosen the latter to make the exposition more accessable to readers with less familiarity with the frame formulation. But be warned that $\partial_{\boldsymbol{a}}$ is not the same as the partial derivative!

Similarly, for a covector α_a with components $\alpha_{\boldsymbol{a}} \equiv e_{\boldsymbol{a}}{}^{a} \alpha_a$ one defines

$$\nabla_{\boldsymbol{a}} \alpha_{\boldsymbol{b}} \equiv e_{\boldsymbol{a}}{}^{a} e_{\boldsymbol{b}}{}^{b} (\nabla_a \alpha_b),$$

so that

$$\nabla_{\boldsymbol{a}} \alpha_{\boldsymbol{b}} \equiv \partial_{\boldsymbol{a}} \alpha_{\boldsymbol{b}} - \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} \alpha_{\boldsymbol{c}}.$$

The above calculus can be extended in the obvious way to tensors of arbitrary rank.

2.2.2 Curvature

Given the connection ∇_a, the *torsion tensor* $\Sigma_a{}^c{}_b$ and *Riemann curvature tensor* $R^c{}_{dab}$ are defined in the usual way through the relations

$$\begin{aligned}
\nabla_a\nabla_b\phi - \nabla_b\nabla_a\phi &= \Sigma_a{}^c{}_b\nabla_c\phi,\\
\nabla_a\nabla_b v^c - \nabla_b\nabla_a v^c &= R^c{}_{dab}v^d + \Sigma_a{}^d{}_b\nabla_d v^c.
\end{aligned}$$

When applied to a covector α_a, the commutator of covariant derivatives is given by

$$\nabla_a\nabla_b\alpha_c - \nabla_b\nabla_a\alpha_c = -R^d{}_{cab}\alpha_d + \Sigma_a{}^d{}_b\nabla_d v^c. \tag{2.9}$$

For a *torsion-free* connection, one has that

$$\Sigma_a{}^c{}_b = 0. \tag{2.10}$$

The connection ∇ is called the *Levi-Civita connection* of $\boldsymbol{g}$ if it satisfies (2.8) and (2.10). In what follows we will assume the connection to be Levi-Civita. Consequently, one naturally has that the *Riemann tensor with all indices down* R_{cdab} has all the usual symmetries. A calculation —see [5] for details— shows that the components of the above tensors with respect to the frame can be expressed as

$$\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}e_{\boldsymbol{c}}{}^c = [e_{\boldsymbol{a}}, e_{\boldsymbol{b}}]^c - (\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} - \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{a}})e_{\boldsymbol{c}}{}^c, \tag{2.11}$$

$$\begin{aligned}
R^{\boldsymbol{c}}{}_{\boldsymbol{dab}} = \partial_{\boldsymbol{a}}\Gamma_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}} - \partial_{\boldsymbol{b}}\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}} + \Gamma_{\boldsymbol{f}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}(\Gamma_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{a}} - \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{b}})\\
+ \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}}\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{f}} - \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}}\Gamma_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{f}} - \Sigma_{\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{b}}\Gamma_{\boldsymbol{f}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}.
\end{aligned} \tag{2.12}$$

Bianchi identities

For reference we list the general first and second Bianchi identities for a general covariant derivative ∇_a:

$$R^d{}_{[cab]} + \nabla_{[a}\Sigma_b{}^d{}_{c]} + \Sigma_{[a}{}^e{}_b\Sigma_{c]}{}^d{}_e = 0, \tag{2.13}$$

$$\nabla_{[a}R^{d}{}_{|e|bc]} + \Sigma_{[a}{}^{f}{}_{b}R^{d}{}_{|e|c]f} = 0. \tag{2.14}$$

The frame version of the above expressions can be readily obtained by simply replacing the abstract indices by frame indices and interpreting the resulting expression in the light of the frame calculus introduced in the previous subsection. Further details on the derivation of the above expressions can be found in Chapter 2 of [5]. Furthermore, we recall that the Riemann tensor admits the *irreducible decomposition*

$$R^{\boldsymbol{c}}{}_{\boldsymbol{dab}} = C^{\boldsymbol{c}}{}_{\boldsymbol{dab}} + 2(\delta^{\boldsymbol{c}}{}_{[\boldsymbol{a}}L_{\boldsymbol{b}]\boldsymbol{d}} - \eta_{\boldsymbol{d}[\boldsymbol{a}}L_{\boldsymbol{b}]}{}^{\boldsymbol{c}}), \tag{2.15}$$

with $C^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ the components of the *Weyl tensor* and

$$L_{\boldsymbol{ab}} \equiv R_{\boldsymbol{ab}} - \frac{1}{6}R\eta_{\boldsymbol{ab}} \tag{2.16}$$

denotes the components of the *Schouten tensor*.

2.3 Symmetric hyperbolic differential equations

This discussion follows closely that given in [5] and the interested readers are referred there and the references therein for more details.

2.3.1 Basic notions

In what follows, let $\overline{x} = \{x^{\mu}\}$ be coordinates in a neighbourhood $\mathscr{U} \subset \mathbb{R}^4$. In these coordinates, consider the quasi-linear evolution equation of the form,

$$\boldsymbol{A}^{\mu}\left(\overline{x}, \boldsymbol{u}\right)\partial_{\mu}\boldsymbol{u} = \boldsymbol{B}\left(\overline{x}, \boldsymbol{u}\right). \tag{2.17}$$

In what follows we let $\boldsymbol{A}^{\mu}$ be $N \times N$ matrices. Furhtermore, we assume that the components of $\boldsymbol{u}$ are scalar functions, and we let $\boldsymbol{u} : \mathbb{R}^4 \to \mathbb{R}$.

The *principal part*[d] of (2.17) is

$$\boldsymbol{A}^{\mu}\left(\overline{x},\boldsymbol{u}\right)\partial_{\mu}\boldsymbol{u},$$

and for a covector $\boldsymbol{\xi} \in T_p^*(\mathscr{U})$ at a point p, with coordinates $\overline{x}\,(p)$, one defines the *symbol* of (2.17) as

$$\boldsymbol{\sigma}\left(\overline{x},\boldsymbol{u},\boldsymbol{\xi}\right) \equiv \boldsymbol{A}^{\mu}\left(\overline{x},\boldsymbol{u}\right)\xi_{\mu}.$$

A straight forward calculation shows that the symbol is an invariant of the equation — i.e. $\boldsymbol{\sigma}$ is invariant under a general coordinate transformation $\overline{x} \to \overline{x}'$.

Definition 15 (Symmetric hyperbolic system.). The system (2.17) is said to be *symmetric hyperbolic* at $(\overline{x},\boldsymbol{u})$ if:

1) The matricies $\boldsymbol{A}^{\mu}$ are Hermitian — i.e. $\boldsymbol{A}^{\mu} = (\boldsymbol{A}^{\mu})^*$
2) There exists a covector $\boldsymbol{\xi}$ such that the symbol $\boldsymbol{\sigma}$ is positive definite.

Remark 2.3. *Observe that for $\boldsymbol{u}(p) \in \mathbb{R}$, it follows that $\boldsymbol{A}^{\mu}$ symmetric and $\boldsymbol{\sigma}$ positive definite is sufficient for symmetric hyperbolicity.*

Let ϕ be a smooth scalar on $\mathscr{M}$. We may then construct a hypersurface $\mathscr{S}$ on $\mathscr{U} \subset \mathscr{M}$ as follows

$$\mathscr{S} \equiv \{p \in \mathscr{U}\,|\,\phi\,(p) = 0\}.$$

We assume that $\boldsymbol{d}\phi \neq 0$ such that there is a well defined normal on $\mathscr{S}$. We say that $\mathscr{S}$ is — with respect to a solution $\boldsymbol{u}$ — *spacelike* if $\boldsymbol{\sigma}\left(\overline{x},\boldsymbol{u},\boldsymbol{d}\phi\right) > 0$, *timelike* if $\boldsymbol{\sigma}\left(\overline{x},\boldsymbol{u},\boldsymbol{d}\phi\right) < 0$ and $det\,(\boldsymbol{\sigma}) \neq 0$, and finally *characteristic* if $det\,(\boldsymbol{\sigma}) = 0$.

[d] The principal part is the term involving the highest order of derivatives; it determines the properties of the equation.

2.3.2 Initial data

Let $\underline{x} = \{x^i\}$ and the coordinates $\overline{x} = \{x^0, \underline{x}\}$ on $\mathscr{U}$ be such that $\mathscr{S}$ is represented by $x^0 = 0$. An *initial data set* for (2.17) on $\mathscr{S}$ which is spacelike — i.e. the symbol is positive definite — consists of a set of functions $\boldsymbol{u}_* = \boldsymbol{u}|_{\mathscr{S}}$. In these *adapted coordinates* one can write (2.17) in the form,

$$\boldsymbol{A}^0\left(0, \underline{x}, \boldsymbol{u}_*\right)\left(\partial_0 \boldsymbol{u}\right)|_{\mathscr{S}} + \boldsymbol{A}^i\left(0, \underline{x}, \boldsymbol{u}_*\right)\left(\partial_i \boldsymbol{u}\right)|_{\mathscr{S}} = \boldsymbol{B}\left(0, \underline{x}, \boldsymbol{u}_*\right).$$

But since the ∂_i are restricted to $\mathscr{S}$ and assuming $\boldsymbol{u}_* = \boldsymbol{u}|_{\mathscr{S}}$, then $\left(\partial_i \boldsymbol{u}\right)|_{\mathscr{S}} = \partial_i \boldsymbol{u}_*$. Thus, one has that,

$$\boldsymbol{A}^0\left(0, \underline{x}, \boldsymbol{u}_*\right)\left(\partial_0 \boldsymbol{u}\right)|_{\mathscr{S}} + \boldsymbol{A}^i\left(0, \underline{x}, \boldsymbol{u}_*\right)\partial_i \boldsymbol{u}_* = \boldsymbol{B}\left(0, \underline{x}, \boldsymbol{u}_*\right),$$

may be interpreted as an algebraic system for $\left(\partial_0 \boldsymbol{u}\right)|_{\mathscr{S}}$ if $\boldsymbol{A}^0\left(0, \underline{x}, \boldsymbol{u}_*\right)$ can be inverted — i.e. if

$$det\left(\boldsymbol{A}^0\right) \neq 0.$$

On the other hand, if $det\left(\boldsymbol{A}^0\right) = 0$, it implies there are a set of constraint equations to be satisfied by $\boldsymbol{u}_*$.

An initial value problem for (2.17) with data prescribed on $\mathscr{S}$ which is nowhere characteristic or time like with respect to $\boldsymbol{u}_*$ will be called a *Cauchy initial value problem*. One can say it is well posed if:

1) There exist solutions to all initial data.
2) The solutions depends continually on the initial data.
3) The solutions are uniquely determined by the initial data.

In what follows it will be established through a series of theorems (uniqueness, existence and stability) that the Cauchy initial value problem for symmetric hyperbolic systems are well posed.

2.3.3 Uniqueness and domain of dependence

Theorem 3. [Uniqueness of solutions to symmetric hyperbolic systems.]
Let $\mathscr{G}$ be a lens shaped domain — see Figure 2.1. If $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two solutions to the symmetric hyperbolic system,

$$\boldsymbol{A}^{\mu}\left(\overline{x}, \boldsymbol{u}\right) \partial_{\mu} \boldsymbol{u} = \boldsymbol{B}\left(\boldsymbol{x}, \boldsymbol{u}\right), \qquad \boldsymbol{u}|_{\mathscr{S}_0} = \boldsymbol{u}_*$$

then $\boldsymbol{u}_1 = \boldsymbol{u}_2$ on $\mathscr{G}$.

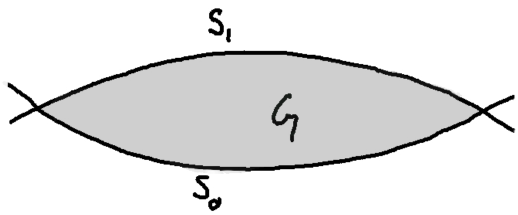

Figure 2.1: A lens shaped domain. $\mathscr{G} \subset \mathbb{R}^4$ with compact closure and $\partial\mathscr{G} = \mathscr{S}_0 \bigcup \mathscr{S}_1$, where $\mathscr{S}_0$ and $\mathscr{S}_1$ are space like with respect to a solution $\boldsymbol{u}$.

Since, any point sufficiently close to $\mathscr{S}$ is contained in a lens shaped domain, Theorem 3 shows that a solution $\boldsymbol{u}$ to a symmetric hyperbolic system is uniquely determined by the initial data on $\mathscr{S}$, as long as $\boldsymbol{u}$ is in the neighbourhood of $\mathscr{S}$.

Definition 16. [Domain of dependence]
Let $\mathscr{R} \subset \mathscr{S}$. The domain of dependence $D(\mathscr{R})$ — Figure 2.2 is all the points $p \in \mathscr{U} \subset \mathbb{R}^4$ such that the value of a solution $\boldsymbol{u}$ to a system of the form (2.17) is uniquely determined by the initial data restricted to $\mathscr{R}$ — i.e. $\mathbf{u}_*|_{\mathscr{R}}$.

Definition 16 means that the Cauchy problem for a symmetric hyperbolic system can be localised in space — i.e. a solution $\boldsymbol{u}$ to a symmetric hyperbolic system can be uniquely determined by initial data on a proper subset of $\mathscr{S}$. This is called *the localisability property* of symmetric hyperbolic systems, and it is this **property of symmetric hyperbolic systems**

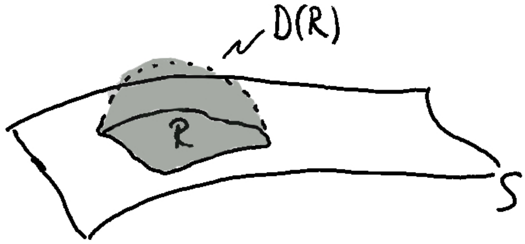

Figure 2.2: Domain of dependence of a region $\mathscr{R} \subset \mathscr{S}$.

which distinguishes them from other PDEs. A global knowledge of $\mathscr{S}$ is thus not necessary to solve the Cauchy problem for such systems, as a solution $\boldsymbol{u}$ in $D(\mathscr{R})$ is independent of initial data from outside of the region $\mathscr{R}$.

2.3.4 Local Existence for Symmetric Hyperbolic Systems

In what follows let $\underline{x} \equiv \{x^i\}$ denote some particular Cartesian coordinate system in $\mathbb{R}^3$, and let d^3x be the volume element. Furthermore, let the components of $\boldsymbol{w}$ be smooth, real functions — i.e. we let $\boldsymbol{w} : \mathbb{R}^3 \to \mathbb{R}^N$ — and let the space of such functions be denoted $C^\infty(\mathbb{R}^3, \mathbb{R}^N)$. On $C^\infty(\mathbb{R}^3, \mathbb{R}^N)$ one defines the *Sobolev norm*:

$$\|\boldsymbol{w}\|^2_{\mathbb{R}^3,m} \equiv \sum_{k=0}^{m} \left(\sum_{\alpha=1}^{3} \int_{\mathbb{R}^3} |\partial^k_\alpha \boldsymbol{w}|^2 d^3x \right), \tag{2.18}$$

with $m \in \mathbb{N}$, $\boldsymbol{w} = (w_1, ..., w_N)$ and $|\boldsymbol{w}|^2$ denoting the usual norm in $\mathbb{R}^N$. By restricting the set of functions to be $\{\boldsymbol{w} \in C^\infty(\mathbb{R}^3, \mathbb{R}^N) \mid \|\boldsymbol{w}\| < \infty\}$ and including the limit points of its Cauchy sequence, one has the *Sobolev space* $H^m(\mathbb{R}^3, \mathbb{R}^N)$. In what follows, we will restrict the discussion to functions $\boldsymbol{w} \in H^m(\mathbb{R}^3, \mathbb{R}^N)$.

We define the *open ball* $B_\varepsilon(\boldsymbol{w}_\bullet)$ of radius ε and centred at $\boldsymbol{w}_\bullet \in H^m(\mathbb{R}^3, \mathbb{R}^N)$

as,

$$B_\varepsilon(\boldsymbol{w}_\bullet) \equiv \left\{\boldsymbol{w} \in H^m(\mathbb{R}^3, \mathbb{R}^N) \mid \|\boldsymbol{w} - \boldsymbol{w}_\bullet\|_{\mathbb{R}^3,m} < \varepsilon\right\}.$$

It is convenient to consider the solutions $\boldsymbol{u}$ of (2.17) as $\boldsymbol{u}(t,\cdot) : [0,T] \to H^m(\mathbb{R}^3, \mathbb{R}^N)$.

Proposition 2.4. [extension of functions on compact space] Let $\mathscr{R} \subset \mathbb{R}^3$ be bounded with smooth boundary $\partial\mathscr{R}$. Then there exists a linear operator $\mathcal{E} : H^m(\mathscr{R}, \mathbb{R}^N) \to H^m(\mathbb{R}^3, \mathbb{R}^N)$ such that for each $\boldsymbol{u} \in \mathscr{R}$

1) $\mathcal{E}\boldsymbol{u} = \boldsymbol{u}$ almost everywhere
2) $\mathcal{E}\boldsymbol{u}$ has support in a open bounded set $\mathscr{R}' \subset \mathscr{R}$.
3) There exists a constant C depending only on $\boldsymbol{u}$ and $\mathscr{R}$ such that $\|\mathcal{E}\boldsymbol{u}\|_{\mathbb{R}^3,m} \leq C\|\boldsymbol{u}\|_{\mathscr{R},m}$.

$\mathcal{E}\boldsymbol{u}$ is called an *extension* of $\boldsymbol{u}$ to $\mathbb{R}^3$.

We can now state the existence theorem for symmetric hyperbolic systems.

Theorem 4. [Local existence of solutions to symmetric hyperbolic systems]
Consider the Cauchy problem

$$\begin{aligned} &\boldsymbol{A}^0(t,\underline{x},\boldsymbol{u})\partial_0\boldsymbol{u} + \boldsymbol{A}^i(t,\underline{x},\boldsymbol{u})\partial_i\boldsymbol{u} = \boldsymbol{B}(t,\underline{x},\boldsymbol{u}),\\ &\boldsymbol{u}(0,\underline{x}) = \boldsymbol{u}_*(\underline{x}) \in H^m(\mathbb{R}^3,\mathbb{R}^N), \qquad m \geq 4, \end{aligned}$$

for a quasilinear symmetric hyperbolic system. If there exists a $\delta > 0$ such that $\boldsymbol{A}^0(t,\underline{x},\boldsymbol{u}_*)$ is positive definite with lower bound δ for all $p \in \mathbb{R}^3$, then there exists a $T > 0$ and a unique solution $\boldsymbol{u}$ to the Cauchy problem defined on $[0,T] \times \mathbb{R}^3$ such that $\boldsymbol{u}$ has regularity $m-2$ and $\boldsymbol{A}^0(t,\underline{x},\boldsymbol{u})$ is positive definite with lower bound δ for $[t,\underline{x}] \in [0,T] \times \mathbb{R}^3$.

Remark 2.5. *Observe that the matrix $\boldsymbol{A}^0(t, \underline{x}, \boldsymbol{u})$ may fail to be positive definite except at the point $(t, \underline{x})$; but as one is interested in uniqueness and existence beyond one point, it is necessary to ensure that if it is positive definite at a point p it remains so in the neighbourhood $\mathscr{U}$ of p up to a "distance" $\delta > 0$. This is sometimes formulated as "bounded away from zero by δ" and other times "...with lower bound δ." But the meaning is the same.*

2.3.5 Cauchy stability

Cauchy stability is the idea that initial data which are "close" should lead to solutions with similar "closeness." It is crucial for a differential equation to admit Cauchy stability in order to be useful as a model for a physical system, since the initial data as measured by an observer has a certain uncertainty. Mathematically the "closeness" mentioned above, is formulated in terms of Sobolev norms. In what follows let $\mathscr{D}$ denote a bounded open subset of $H^m(\mathbb{R}^3, \mathbb{R}^N)$ such that for all $\boldsymbol{w} \in \mathscr{D}$, $\boldsymbol{A}^0(t, \underline{x}, \boldsymbol{w})$ is positive definite bounded away from zero by δ for all $p \in \mathbb{R}^4$. From [6] we adopt the following theorem.

Theorem 5. [Cauchy stability for symmetric hyperbolic systems]
Let $\boldsymbol{u}_* \in \mathscr{D}$ be initial data for the symmetric hyperbolic system. Then:

1) There exist an $\varepsilon > 0$ such that T can be chosen as the common existence time for all initial data in $B_\varepsilon(\boldsymbol{u}_*) \in \mathscr{D}$.

2) If the solution $\boldsymbol{u}$ with initial data $\boldsymbol{u}_*$ exists on $[0, T] \times \mathbb{R}^3$, then a solution exists on $[0, T] \times \mathbb{R}^3$ for all initial data in $B_\varepsilon(\boldsymbol{u}_*) \in \mathscr{D}$ given ε is sufficiently small.

3) If ε and T are chosen as in 1) and given a sequence ${\boldsymbol{u}_*}^n \in B_\varepsilon(\boldsymbol{u}_*) \in \mathscr{D}$ such that

$$\|{\boldsymbol{u}_*}^n - \boldsymbol{u}_*\|_{\mathbb{R}^3, m} \to 0, \qquad as \quad n \to \infty.$$

Then for the solutions $\boldsymbol{u}^n(t,\cdot)$ with $\boldsymbol{u}^n(0,\cdot) = \boldsymbol{u_*}^n$, one has that

$$\|\boldsymbol{u}^n(t,\cdot) - \boldsymbol{u}(t,\cdot)\|_{\mathbb{R}^3,m} \to 0, \qquad as \quad n \to \infty,$$

uniformly for $t \in [0,T] \times \mathbb{R}^3$.

Point 1) in Theorem 5 states that one can always find a common $T > 0$ such that in the region $[0,T] \times \mathbb{R}^3$ a solution $\boldsymbol{u}$ for all the initial conditions in $B_\varepsilon(\boldsymbol{u}_*) \in \mathscr{D}$ is guaranteed by applying Theorem 4. Observe that T is the existence time of a known background data $\boldsymbol{u}_*$.

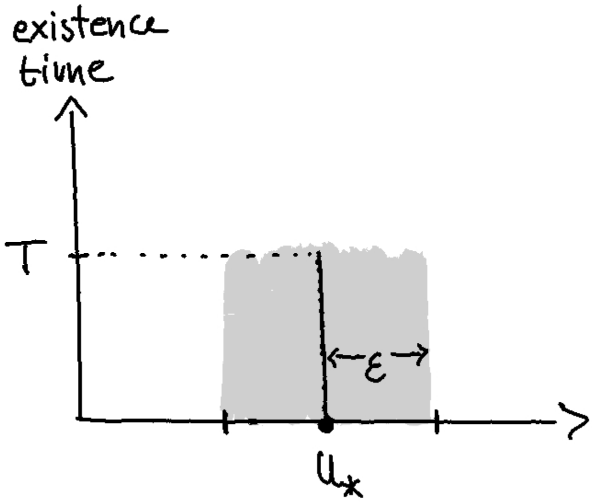

Figure 2.3: Common existence time T.

Point 2) gives the assurance that if a solution $\boldsymbol{u}$ with initial data $\boldsymbol{u}_*$ is already known, then the existence of solutions to all the initial data sufficiently close to $\boldsymbol{u}_*$ is guaranteed for some time interval $[0,T]$. In other words, one can choose the existence time for the known solution as the common existence time given that $\varepsilon > 0$ is sufficiently small.

Finally, point 3) is the statement of *Cauchy stability*. Given a common existence time T and a sufficiently small $\varepsilon > 0$, then data close to the reference initial data give rise to solutions close to the reference solution.

2.3.6 The Cauchy problem on manifolds

In the previous subsections we explored the Cauchy problem for a symmetric hyperbolic system in $\mathbb{R}^4$ and established it as well posed through a series of theorems. In General Relativity, however, the tensor quantities in study are defined on a manifold rather than in $\mathbb{R}^4$. In what follows, we shall present a way to relate these tensors to sections of $\mathbb{R}^4$ such that the theorems established in the previous subsection may be applied, and thus obtain a theorem which ensures the well posedness of the Cauchy problem even for systems defined on a manifold.

In what follows let $\mathscr{S}$ be a 3-dimensional, compact and oriented manifold. Due to its compactness there exists a *finite cover*[e] — i.e. given $\mathscr{R} \subset \mathscr{S}$ then $\bigcup_{I=1}^{n} \mathscr{R}_I = \mathscr{S}$. By introducing coordinates $\underline{x}_I = \{x_I{}^\alpha\}$ with $\alpha = \{1, 2, 3\}$ on each of the patches $\mathscr{R}_I$, we get a map which relates $\mathscr{R}_I$ with a corresponding subset $\mathscr{B}_I$ in $\mathbb{R}^3$ — see Figure 2.4. We assume $\mathscr{S}$ is smooth and therefore the change from $\underline{x}_I$ to $\underline{x}_J$ in overlapping regions $\mathscr{R}_I \cap \mathscr{R}_J$ is smooth.

We further assume that a smooth set of functions $\boldsymbol{u}_* : \mathscr{S} \to \mathbb{R}^N$ has been prescribed on $\mathscr{S}$ and let $\boldsymbol{u}_{*I}$ denote the restriction of $\boldsymbol{u}_*$ unto a particular patch $\mathscr{R}_I$. By the coordinate maps $\underline{x}_I$, the functions $(\boldsymbol{u}_{*I})^\alpha : \mathbb{R}^3 \to \mathbb{R}^N$. Clearly, $(\boldsymbol{u}_{*I})^\alpha$ are the $\boldsymbol{u}_{*I}$ in their local coordinates. Since $(\boldsymbol{u}_{*I})^\alpha \in H^m(\mathbb{R}^3, \mathbb{R}^N)$ we can now apply the results from the previous subsections. We apply first proposition 2.4 to extend the initial data $(\boldsymbol{u}_{*I})^\alpha$ to the whole of $\mathbb{R}^3$ in a controlled manner. This is necessary as the exsistence and stabillity theorems are only applicable for systems where

[e] A manifold is said to be compact if every open cover of the manifold has a finite sub cover. A finite sub cover is essentially the idea that there exists a subset which covers the open set completely but with finite number of subsets. See [7] for a rigorous exposition.

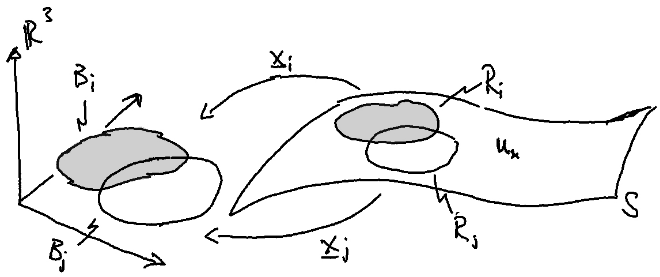

Figure 2.4: A compact manifold $\mathscr{S}$ with patches, and their coordinate maps into $\mathbb{R}^3$.

initial data is prescribed throughout $\mathbb{R}^3$. We also define

$$\|\boldsymbol{u}\|_{\mathscr{S},m} \equiv \sum_{I=1}^{n} \|(\boldsymbol{u}_{*I})^{\alpha}\|_{\mathbb{R}^3,m}.$$

Assuming that $\boldsymbol{A}^0(0,\underline{x},\mathcal{E}(\boldsymbol{u}_{*I})^{\alpha})$ is positive definite with lower bound $\delta > 0$, one gets a unique solution $(\boldsymbol{u}_I)^{\alpha}$ with initial data $(\boldsymbol{u}_I)^{\alpha}(0,\underline{x}) = \mathcal{E}(\boldsymbol{u}_{*I})^{\alpha}(\underline{x})$ and with exsistence interval $[0,T]$. See Figure 2.5 for an illustration.

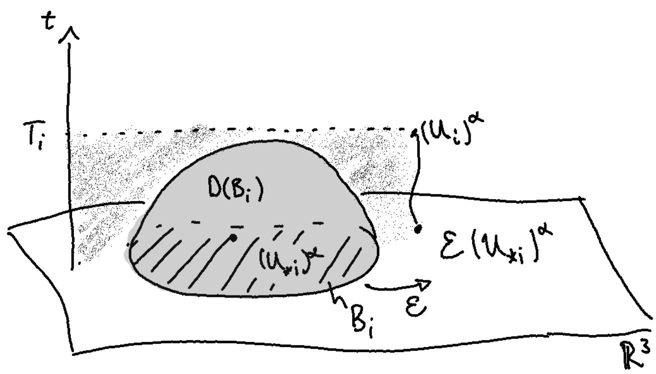

Figure 2.5: The extension and existence of solutions to the initial data $(\boldsymbol{u}_{*I})^{\alpha}$ on $\mathbb{R}^3$.

It follows that $D(\mathscr{B}_I) \subset [0,T] \times \mathscr{B}_I$, and in the intersection between two patches $\mathscr{B}_I \cap \mathscr{B}_J$ that $(\boldsymbol{u}_{*I})^{\alpha} = (\boldsymbol{u}_{*J})^{\alpha}$ — see Figure 2.6. Consequently, the respective solutions $(\boldsymbol{u}_I)^\alpha$ and $(\boldsymbol{u}_J)^\alpha$ also coincides in $D(\mathscr{B}_I \cap \mathscr{B}_J)$ — i.e. one has that $D(\mathscr{B}_I \cap \mathscr{B}_J) = D(\mathscr{B}_I) \cap D(\mathscr{B}_J)$. We may now patch together all the $\mathscr{B}_I$'s and obtain the existence of a solution in the region $[0,T] \times \Sigma$, where we define $\Sigma \equiv \bigcup_I^n \mathscr{B}_I$ and $T \equiv min(T_I)$. By the inverse of $\underline{x}_I$ we find equivalent solutions in the region $[0,T] \times \mathscr{S}$.

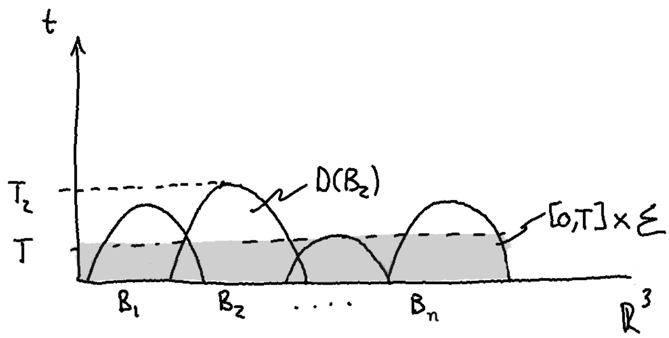

Figure 2.6: The domain in $\mathbb{R}^3$ where existence of solutions of the initial data $(\boldsymbol{u}_{*I})^\alpha$ is guaranteed.

From the above discussion one can formulate a fairly general theorem of existence and stabillity:

Theorem 6. (Existence and stability for symmetric hyperbolic systems on compact spatial sections)
Consider the following Cauchy problem on a 3-dimensional compact, spatial manifold $\mathscr{S}$:

$$\boldsymbol{A}^0(t,\underline{x},\boldsymbol{u})\partial_0\boldsymbol{u} + \boldsymbol{A}^i(t,\underline{x},\boldsymbol{u})\partial_i\boldsymbol{u} = \boldsymbol{B}(t,\underline{x},\boldsymbol{u}),$$
$$\boldsymbol{u}(0,\underline{x}) = \boldsymbol{u}_*(\underline{x}) \in H^m(\mathbb{R}^3,\mathbb{R}^N), \qquad m \geq 4,$$

If there exists a $\delta > 0$ such that $\boldsymbol{A}^0(t,\underline{x},\boldsymbol{u}_*)$ is positive definite and bounded away from zero by δ, then for all $p \in \mathscr{S}$

1) There exists a $T > 0$ and a unique solution $\boldsymbol{u}$ on $[0,T] \times \mathscr{S}$ such

that $\boldsymbol{u}$ is C^{m-2} and $\boldsymbol{A}^0(t, \underline{x}, \boldsymbol{u})$ is positive definite bounded away from zero.

2) There exist an $\varepsilon > 0$ such that for any $\boldsymbol{u}_*$ in $B_\varepsilon(\boldsymbol{u}_*) \in \mathcal{D}$ there exist a common existence time T.

3) If a solution $\boldsymbol{u}$ exist on $[0, T] \times \mathscr{S}$ for $T > 0$ and with initial data $\boldsymbol{u}_*$, then the solutions to all of the initial data in $B_\varepsilon(\boldsymbol{u}_*) \in \mathcal{D}$ exists on $[0, T] \times \mathscr{S}$ if ε is sufficiently small.

4) If $\varepsilon > 0$ and T are chosen as in 1) and given a sequence ${\boldsymbol{u}_*}^n \in B_\varepsilon(\boldsymbol{u}_*) \in \mathscr{D}$ such that

$$\|{\boldsymbol{u}_*}^n - \boldsymbol{u}_*\|_{\mathscr{S},m} \to 0, \qquad as \quad n \to \infty.$$

Then for the solutions $\boldsymbol{u}^n(t, \cdot)$ with $\boldsymbol{u}^n(0, \cdot) = {\boldsymbol{u}_*}^n$, one has that

$$\|\boldsymbol{u}^n(t, \cdot) - \boldsymbol{u}(t, \cdot)\|_{\mathscr{S},m} \to 0, \qquad as \quad n \to \infty,$$

uniformly for $t \in [0, T] \times \mathscr{S}$.

2.4 The Cauchy problem in general relativity

In the previous section the Cauchy problem was given for systems given on a background manifold $\mathbb{R}^4$. By introducing local coordinates, the Einstein field equation is a set of second order partial differential equations in the metric. The formulation of the Cauchy problem for these equations are far from straight forward. There are essentially two problems one faces:

1) The equations are non-linear and self interacting: one solves for the gravitational field and the spacetime upon which it propagates, simultaneously.

2) The diffeomorphism invariance of the theory allows for uniqueness only up to a diffeomorphism — i.e. a solution $\boldsymbol{g}$ of the equations are physically equivalent to any other solution $\tilde{\boldsymbol{g}} = \phi_* \boldsymbol{g}$, where $\phi : \mathscr{M} \to \hat{\mathscr{M}}$ is an isomorphism and ϕ_* denotes the *push forward*.

A consequence of 1) is that one has no information of the domain of dependence of the surface where initial data is prescribed and where the solution is to be determined. In other words the spacetime where the solution is to be propagated, is itself part of the solution. It is for these reasons, that the Cauchy problem takes on a different form than that mentioned in the previous section. In what follows I will only briefly sketch out the idea. The interested reader is referred to [8], [9] and [10] for more details.

The Cauchy problem in general relativity thus takes on the following form. One is given an abstract 3-dim manifold $\mathscr{S}$ with prescribed initial data $\boldsymbol{\omega}$ — see section 2.6 for the explicit form of the data — and ask whether there exists a map θ such that the spacetime $(\mathscr{M}, \boldsymbol{g}, \theta)$ is a development of $(\boldsymbol{\omega}, \mathscr{S})$ — i.e.

1) $D(\theta(\mathscr{S})) = \mathscr{M}$,

2) $\boldsymbol{g}$ satisfies the Einstein field equations and agrees with $\theta(\boldsymbol{\omega})$.

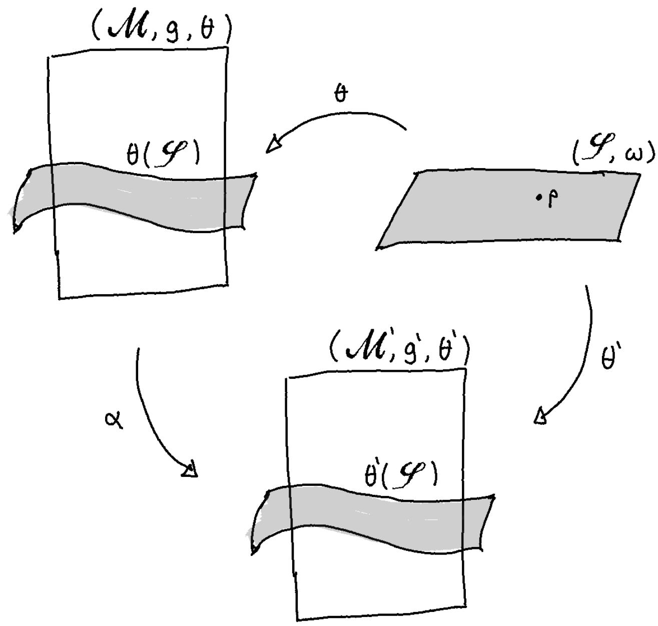

Figure 2.7: Two developments of $(\boldsymbol{\omega}, \mathscr{S})$ of which $(\mathscr{M}', \boldsymbol{g}', \theta')$ is an extension of $(\mathscr{M}, \boldsymbol{g}, \theta)$.

This is illustrated in Figure 2.7. A spacetime which satisfy the above is called *globally hyperbolic* and $\mathscr{S}$ is called a *Cauchy surface* of $\mathscr{M}$. If there are two developments $(\mathscr{M}, \boldsymbol{g}, \theta)$ and $(\mathscr{M}', \boldsymbol{g}', \theta')$ of $(\boldsymbol{\omega}, \mathscr{S})$, then $(\mathscr{M}', \boldsymbol{g}', \theta')$ will be called an *extension* of $(\mathscr{M}, \boldsymbol{g}, \theta)$ if there exists a diffeomorphism α such that ,

1) $\theta^{-1} \circ \alpha^{-1} \circ \theta'(p) = \mathbf{id}(p)$ for any point $p \in \mathscr{S}$,
2) $\boldsymbol{g} = \alpha^* \boldsymbol{g}'$,

where α^* denotes the pull back under α. This ensures that one has uniqueness up to a diffeomorphism.

In order to show that the Einstein equations are well posed, one can advance in various ways. The standard strategy to address this issue is to choose some gauge in which the Einstein equations imply evolution equations that are of a *hyperbolic form*. Physical considerations associated to causality lead to the expectation of the Einstein equations admitting a hyperbolic formulation despite the fact that the immediate form of the equations is not manifestly hyperbolic due to general covariance. Thus, it is necessary to find a subset of the Einstein equations which indeed admits hyperbolicity. This procedure is called *hyperbolic reduction* —see [5] and [11] for details; for an overview of the different reduction methods, see [12].

The well-posedness of the vacuum Einstein equations was first shown in [13] —and later in the case for dust and the Einstein-Euler by the same author [14]. These results were obtained using a *harmonic gauge* to reduce the field equations to a form which is mixed first-second order hyperbolic (Leray hyperbolicity). In [15] this method is extended to show existence of solutions locally for a self-gravitating, relativistic elastic body with compact support. Furthermore, in [16] well-posedness of a viscous fluid coupled to the Einstein equations is presented and in [17] a viable first order system is constructed. In [18] the concept of *first order symmetric hyperbolic* (FOSH) equations was developed. The same author showed later [19] that the Einstein-Euler system could be put on a FOSH form. In [20] a different approach, which makes use of a formulation in terms of frame fields, is employed to construct evolution equations for the Einstein-Euler system which also are on the form of a FOSH system. This method has the advantage that the reduced equations are symmetric hyperbolic while still maintaining a Lagrangian[f] form —which is important in order to keep track of a boundary in the case of matter distributions with compact support.

[f]We mean by this that $e_{\mathbf{0}}$ coincides with the four velocity of the particle trajectories. See 4.3 for details

2.5 The conformal Einstein field equations

The knowledge of the light cone structure of a spacetime is sufficient in determining the metric up to a positive factor. This can be seen by the following observations. Given $\boldsymbol{V}$, $\boldsymbol{U}$ in $T_p\mathscr{M}$, then using the properties of the metric, we have

$$\boldsymbol{g}(\lambda\boldsymbol{U}+\boldsymbol{V},\lambda\boldsymbol{U}+\boldsymbol{V}) = \lambda^2\boldsymbol{g}(\boldsymbol{U},\boldsymbol{U}) + 2\lambda\boldsymbol{g}(\boldsymbol{U},\boldsymbol{V}) + \boldsymbol{g}(\boldsymbol{V},\boldsymbol{V}).$$

If we now equate the above equation to zero, we obtain a second order polynomial for the parameter λ:

$$\lambda^2\boldsymbol{g}(\boldsymbol{U},\boldsymbol{U}) + 2\lambda\boldsymbol{g}(\boldsymbol{U},\boldsymbol{V}) + \boldsymbol{g}(\boldsymbol{V},\boldsymbol{V}) = 0. \tag{2.19}$$

If the null cone structure of $\boldsymbol{g}$ is known — i.e. the structure of the vectors $\boldsymbol{X}$ obeying the equation $\boldsymbol{g}(\boldsymbol{X},\boldsymbol{X}) = 0$ — and one let $\boldsymbol{V}$ and $\boldsymbol{U}$ be timelike and spacelike vectors, respectively, a simple calculation gives an expression for the relative lengths of $\boldsymbol{V}$ and $\boldsymbol{U}$

$$\lambda^2 = -\frac{\boldsymbol{g}(\boldsymbol{V},\boldsymbol{V})}{\boldsymbol{g}(\boldsymbol{U},\boldsymbol{U})}.$$

Let $\boldsymbol{W}$ and $\boldsymbol{Y}$ be any two non-null vectors in $T_p\mathscr{M}$, then using the above relation, one can determine the metric by the equation

$$-\boldsymbol{g}(\boldsymbol{W},\boldsymbol{Y}) = \frac{1}{2}\left(\boldsymbol{g}(\boldsymbol{W},\boldsymbol{W}) + \boldsymbol{g}(\boldsymbol{Y},\boldsymbol{Y}) - \boldsymbol{g}(\boldsymbol{W}+\boldsymbol{Y},\boldsymbol{W}+\boldsymbol{Y})\right).$$

In other words, the null cone structure at each point in spacetime gives essentially the same information as the metric up to a conformal factor.

Remark 2.6. If $\boldsymbol{W}+\boldsymbol{Y}$ turns out to be null, one can arbitrarily re-scale one of the vectors e.g $\boldsymbol{W}+2\boldsymbol{Y}$ to ensure that $\boldsymbol{g}(\boldsymbol{W}+\boldsymbol{Y},\boldsymbol{W}+\boldsymbol{Y})$ is time like or space like.

The light cone structure itself is preserved under a conformal rescaling of the metric. Furthermore, by studying the conformal structure, one can relate the light cone structure to the global aspects of solutions to the Einstein field equations. This is the main motivation for the study of the conformal Einstein field equations: one hope to uncover the large scale structure of a solution.

Two spacetimes $(\mathscr{M}, \boldsymbol{g})$ and $(\tilde{\mathscr{M}}, \tilde{\boldsymbol{g}})$ are conformally related if their respective metrics are related by a conformal factor $\Omega = \Omega(\boldsymbol{x}) > 0$,

$$\boldsymbol{g} = \Omega^2 \tilde{\boldsymbol{g}}, \qquad \boldsymbol{g}^\sharp = \Omega^{-2} \tilde{\boldsymbol{g}}^\sharp. \tag{2.20}$$

In the above, and throughut, a $\sharp$ represent the contravariant form of the metric. Thus, two conformally related spacetimes have the same causal structure — i.e. a trajectory which is time like, space like or null with respect to $\tilde{\boldsymbol{g}}$ is so also with respect to $\boldsymbol{g}$. In what follows we shall call $(\tilde{\mathscr{M}}, \tilde{\boldsymbol{g}})$ the *interior* spacetime. Any tensor fields defined on the interior spacetime will be so indicated by placing a tilde on top of it — e.g. $\tilde{\boldsymbol{u}}$, $\tilde{\boldsymbol{T}}$ etc. The spacetime without a tilde will be called the *conformal spacetime*. Note that in the literature it is common to refer to the two spacetimes as the *physical* and *unphysical* spacetimes, respectively. I will avoid such terminology as it is ambiguous what constitute a 'physical' and 'non-physical' manifold; and I will not enter into that debate herein apart from what is already mentioned earlier.

We assume Ω to be a smooth function on $\mathscr{M}$ and playing the role of a *boundary defining function*. More precicely, we define the *conformal boundary* $\mathscr{I}^+$ as all the points where Ω vanish — i.e. $\mathscr{I} \equiv \{p \in \mathscr{M} \mid \Omega(p) = 0\}$. One can then define the conformal spacetime as the union of the interior spacetime with the boundary — i.e. one have

$$\mathscr{M} \equiv \tilde{\mathscr{M}} \cup \mathscr{I}.$$

It is customary to let $\mathscr{I} = \mathscr{I}^+ \cup \mathscr{I}^-$ where $\mathscr{I}^+$ and $\mathscr{I}^-$ are called *future* and *past null infinity*, respectively. This represent the "end points" and "origin", respectvely, of all null geodesics. In what follows, we shall be concerned about future null infinity only, and hence simply use $\mathscr{I}^+$ instead of $\mathscr{I}$. Since Ω is a smooth function, the differential form $\boldsymbol{d}\Omega$ defines the normal on $\mathscr{I}^+$. The geometry of $\mathscr{M}$ determine whether the hyper surface $\mathscr{I}^+$ is timelike, null or spacelike. In the second part of this thesis — i.e. Chapter 5 — we shall only consider geometries of the form $\mathbb{R} \times \mathbb{S}^3$.[g]. In such a geometry, one finds that

$$\boldsymbol{g}(\boldsymbol{d}\Omega, \boldsymbol{d}\Omega) < 0,$$

which implies that $\mathscr{I}^+$ is a spacelike hypersurface.

The two metrics $\boldsymbol{g}$ and $\tilde{\boldsymbol{g}}$ define, respectively, two derivative operators $\boldsymbol{\nabla}$ and $\tilde{\boldsymbol{\nabla}}$; and these are related to one another by

$$\left(\boldsymbol{\nabla} - \tilde{\boldsymbol{\nabla}}\right)\boldsymbol{\omega} = \boldsymbol{Q} \cdot \boldsymbol{\omega}, \tag{2.21}$$

where $\boldsymbol{\omega}$ is any one form and $\boldsymbol{Q}$ is a symmetric 3-rank tensor defined in local coordinates (x^μ) by

$$\boldsymbol{Q} \equiv \Omega^{-1}\left(\nabla_\delta \Omega g_{\mu\nu} g^{\delta\gamma} - \nabla_\mu \Omega \delta^\gamma{}_\nu - \nabla_\nu \Omega \delta^\gamma{}_\mu\right) \mathbf{d}x^\mu \otimes \boldsymbol{\partial}_\gamma \otimes \mathbf{d}x^\nu.$$

From equations (2.20) and (2.21) one can derive how all metric tensors transform under a conformal mapping. It suffices, at the moment, to point out that the divergencefree condition for the energy momentum tensor $\tilde{\boldsymbol{T}}$, is not invariant under a conformal rescaling. More precisely, one has that,

$$\boldsymbol{\nabla} \cdot \boldsymbol{T} = -T_g\ \Omega^{-1}\boldsymbol{d}\Omega, \tag{2.22}$$

where T_g denotes the trace of $\boldsymbol{T}$ with respect to the metric $\boldsymbol{g}$. It is for this reason that until recently it was thought that one could only treat trace-free matter models on conformal spacetime. Friedrich has, however, showed that regular Einstein field equations can be formulated also in the case for dust [21]. This will be discussed more in detail in Chapter 5.

[g]This is because we want to use the deSitter solution as background data

2.6 The geometry of hypersurfaces

If one is to have any hopes of studying the Cauchy problem for the Einstein field equations, it is necessary to have a notion of *initial hypersurfaces* on which the initial data may be prescribed. Here we will introduce what a hyper-surface is as well as the theoretical framework which will be employed later in the thesis. We start with a definition of a *submanifold*

Definition 17 (Submanifold). Let $(\mathscr{M}, \mathscr{O}_{\mathscr{M}}, \mathscr{A}_{\mathscr{M}})$ be a smooth manifold and $\mathscr{N}$ a set. Furthermore, let $\phi : \mathscr{N} \to \mathscr{M}$ be a surjective map such that $\phi(\mathscr{N}) \subseteq \mathscr{M}$, and $(\mathscr{U}, x)$ be any chart in $\mathscr{A}_{\mathscr{M}}$. We say

$$\mathscr{S} \equiv (\mathscr{N}, \mathscr{O}|_{\mathscr{N}}, \mathscr{A}_{\mathscr{N}})$$

is a submanifold of $(\mathscr{M}, \mathscr{O}_{\mathscr{M}}, \mathscr{A})$ if $\exists$ a y such that $\forall\ q \in \mathcal{U} \cap \mathscr{S} \in \mathscr{O}|_{\mathscr{N}}$,

$$x \circ y^{-1}(q) = \mathbf{id}(q).$$

Put simply: a submanifold $\mathscr{S}$ of an already existing manifold $\mathscr{M}$ with a certain structure $\mathscr{O}_{\mathscr{M}}, \mathscr{A}_{\mathscr{M},\dots}$, is a manifold of equal or less dimension than that of $\mathscr{M}$, and which has inherited the structure from $\mathscr{M}$ via the map ϕ.

It is clear from the above definition, that the tangent space $T(\mathscr{S})$ of the submanifold $\mathscr{S}$ is itself a sub-space of the tangent space $T(\mathscr{M})$ of $\mathscr{M}$. In other words, it is a distribution $\mathscr{P}$ over $\mathscr{M}$:

Definition 18 (Distribution). A distribution $\mathscr{P}$ over a smooth manifold $\mathscr{M}$ is a vector space such that at each point $p \in \mathscr{M}$ one has that $\mathscr{P}|_p \subseteq T_p(\mathscr{M})$.

The space time we construct in Chapter 4 gives a congruence and thus a distribution, rather than a hypersurface. A manifold $\mathscr{S}$ is said to be an *integrable manifold* of $\mathscr{P}$ if $\forall p \in \mathscr{S}$ one has $\mathscr{P}|_p = T_p(\mathscr{S})$. A useful theorem due to *Frobenius* give the condition for a distribution to be integrable:

> Definition 19 (Integrable distribution). A distribution $\mathscr{P}$ is integrable iff $\forall$ $\boldsymbol{V}, \boldsymbol{U} \in \mathscr{P}$, one also has that $[\boldsymbol{V}, \boldsymbol{U}] \in \mathscr{P}$.

In the above, we mean by $[\boldsymbol{V}, \boldsymbol{U}]$ the commutator between $\boldsymbol{V}$ and $\boldsymbol{U}$ — i.e.

$$[\boldsymbol{V}, \boldsymbol{U}] \equiv \boldsymbol{V}\boldsymbol{U} - \boldsymbol{U}\boldsymbol{V}.$$

A sub-manifold of dimension m is called a *hyper-surface* if $m = n - 1$, with n the dimension of the manifold it is embedded in. We also call $\mathscr{S}$ *space like* or *time like* if its norm $\boldsymbol{\nu}$ is time like or space like, respectively, with respect to the metric $\boldsymbol{g}$ by the operation of the *pull back* — see [5] for more details. The sub-manifold $\mathscr{S}$ will naturally inherit a metric from $(\mathscr{M}, \boldsymbol{g})$,

$$\boldsymbol{h} \equiv \phi^* \boldsymbol{g}.$$

For $\mathscr{S}$ space like, we have that $\forall$ $\boldsymbol{V} \neq 0 \in T(\mathscr{S})$ and $\boldsymbol{U} \neq 0 \in T(\mathscr{M})$ such that $\boldsymbol{g}(\boldsymbol{U}, \boldsymbol{U}) = -1$, that $\boldsymbol{h}(\boldsymbol{U}, \boldsymbol{V}) = 0$. From now on — unless otherwise stated — we will assume $(\mathscr{S}, \boldsymbol{h})$ to be a 3-dimensional, space like hyper-surface of spacetime $(\mathscr{M}, \boldsymbol{g})$. If the connection $\boldsymbol{\nabla}$ is metric compatible and torsion free, then the connection $\boldsymbol{D}$ associated with $\boldsymbol{h}$ on $\mathscr{S}$ is given by

$$\boldsymbol{D} = \phi^* \boldsymbol{\nabla}.$$

And obviously one has that

$$\boldsymbol{D}\boldsymbol{h} = 0.$$

The metric and connection on $\mathscr{S}$ respectively define the *extrinsic* and *intrinsic curvature* of $\mathscr{S}$ via

$$2D_{[a}D_{b]}v^c \equiv R^c{}_{dab}[\boldsymbol{h}]v^d, \tag{2.23}$$

$$K_{ab} \equiv h_a{}^c \nabla_c \nu_b, \tag{2.24}$$

where $\nu_b = g_{ab}\nu^a$ is the normal on $\mathscr{S}$. It is readily showed that the intrinsic curvature tensors $R^c{}_{dab}[\boldsymbol{g}]$ and $R^c{}_{dab}[\boldsymbol{h}]$ are related by

$$R_{abcd}[\boldsymbol{g}] = R_{abcd}[\boldsymbol{h}] + K_{ac}K_{bd} - K_{ad}K_{bc}, \tag{2.25}$$

$$u^b R_{abcd}[\boldsymbol{g}] = D_c K_{da} - D_d K_{ca}. \tag{2.26}$$

The first is the *Gauss-Codazzi equation* and the last is called the *Codazzi - Mainardi equation*. Given that the Einstein equations (5.60) hold everywhere on $(\mathscr{M}, \boldsymbol{g})$, equations (2.25) and (2.26) imply a set of constraint equations which must be satisfied on the hypersurface $(\mathscr{S}, \boldsymbol{h})$

$$R^a{}_a[\boldsymbol{h}] + K^a{}_a K^b{}_b - K_{ab}K^{ab} = 2(\rho - \lambda), \tag{2.27}$$

$$D^a K_{ab} - D_b K^a{}_a = T_{ab}\nu^a \equiv j_b. \tag{2.28}$$

One can say that a hypersurface $(\mathscr{S}, \boldsymbol{h})$ with data (ρ, j_b) satisfying equations (5.60) and (5.61) constitutes an initial data set for the Einstein field equations (1.5). Finally, we should mention the seminal work of Yvonne Choquet-Bruhat [13] wherin she proves the existence of a Cauchy development of the Einstein constraint data. More precisely,

Theorem 7 (Existence of a development of initial data). Given the initial data set $\boldsymbol{\omega} = (\mathscr{S}, \boldsymbol{h}, \boldsymbol{K}, \rho, \boldsymbol{j})$ satisfying the Einstein constraint equations (2.27) and (2.28), then there exists a corresponding Cauchy development $(\mathscr{M}, \boldsymbol{g})$ satisfying the Einstein field equation (4.1).

Chapter 3

Matter models

In the introductory chapters we have mostly discussed the structure and assumptions behind Einsteins theory of gravitation as well as some mathematical techniques to be used later. We have not, however, elaborated on the meaning and implications of the energy momentum tensor $\boldsymbol{T}$ appearing at the right hand side of equation (1.5). In this chapter we will give a brief overview of the most important matter models in General relativity as well as a more extensive treatment of relativistic elastic material.

3.1 Vacuum

In the case of vacuum one has that $\boldsymbol{T} = 0$ everywhere and equation (1.5) will then be the *vacuum Einstein field equations*

$$R_{ab} = 0.$$

From the irreducible decomposition of the Riemann tensor (2.15), one see that the vacuum equation implies that

$$R^c{}_{dab} = C^c{}_{dab}. \tag{3.1}$$

In other words, the components of the Riemann tensor which are non-vanishing when $R_{ab} = 0$ are the components of the Weyl tensor. It is therefore generally agreed that the Weyl tensor contains the information of gravitational waves away from any matter sources. Hence the vacuum equation is very important in the study of the propagation of the gravitational field.

3.2 Dust

The next simplest form of the energy momentum tensor is that of *dust*. Given a matter distribution $\mathscr{B}$ in a space time $(\mathscr{M}, \boldsymbol{g})$ such that for each point $p \in \mathscr{B}$ there is a vector $u^a \in T_p(\mathscr{B})$ which is tangent to the world line γ of the dust particles passing through p, then one defines the energy momentum tensor for dust to be

$$T^{ab} \equiv \rho u^a u^b,$$

with ρ a smooth positive function. The positivity condition of ρ is due to the following reasons. Firstly, it is obvious that for $\rho = 0$ the energy momentum tensor would reduce to that of vacuum. It is customary to interpret ρ as the density of energy and matter. It is then assumed that one cannot have negative energy or matter density, hence the case of $\rho < 0$ is ruled out. Since, galaxies in the Universe can be viewed as noninteracting "particles", it is common to model the large scale Universe with an energy momentum tensor of dust.

3.3 Perfect fluid

In the case of dust, the material particles do not interact with one another — that is, there are no pressure terms. But in the interior of a star, one expects the molecules to be interacting with each other, hence it is necessary to add a term representing this interaction. This form of the energy momentum tensor is called a *perfect fluid* and is defined as

$$T^{ab} \equiv \rho u^a u^b + p h^{ab},$$

with h_{ab} the projector metric, and p a function representing the pressure between matter particles. For $p = 0$ the above reduces to the case of dust; and hence dust is a special case of a perfect fluid. In order to specify what sort of fluid one is dealing with, it is necessary to give an *equation of state*

$p = p(\rho)$ — i.e. how the density depends on the pressure.

3.4 Elastic fluid

In cases such as Neutron stars, where the gravitational pressure is immense, one expect the crust to be of a solid-like material. This can best be modelled by an *elastic* energy momentum tensor. In what follows we will derive the frame components of such a tensor. The discussion is based on the approach found in [22]; but essential to the derivation is the use of orthonormal frames, which is new. Let $\mathcal{B}$ — called the *body manifold*— be

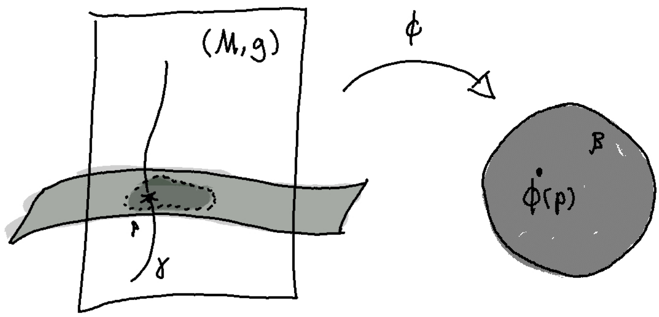

Figure 3.1: The map ϕ between spacetime and the body manifold is an essential part in constructing the relativistic theory of elasticity.

a 3-dimensional manifold representing the ensemble of particles making up the elastic body. The key object in relativistic elasticity is a map

$$\phi : \mathcal{M} \to \mathcal{B},$$

such that if $\overline{x} = (x^\mu)$ and $\overline{X} = (X^M)$ are, respectively, coordinates on the spacetime and body manifold we then have

$$\phi^M(x^\mu) = X^M.$$

As the manifolds $\mathcal{M}$ and $\mathcal{B}$ have, respectively, dimension 4 and 3, the map ϕ is non-injective (not one-to-one). In the following it will be assumed that the inverse image $\phi^{-1}(\overline{X})$ of a point on $\mathcal{B}$ with coordinates $\overline{X} = (X^M)$ is a timelike curve on $\mathcal{M}$. We denote the tangent vector to the curve $\gamma : \mathbb{R} \to \mathcal{M}$ with $\gamma \equiv \phi^{-1}(\overline{X})$ by $\boldsymbol{u}$. If we assume γ to be parametrised by its proper time, then we have that

$$\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{u}) = -1.$$

The curve γ describes the worldline of the particle of the point on $\mathcal{B}$ with coordinates $\overline{X}$.

The map ϕ represents the *configuration* of the elastic body. This means that ϕ associates to each spacetime event a material particle. In other words, ϕ relates the physical state of a material body with the notion of an event in spacetime. The deformation of the elastic body is represented by the *deformation gradient*, defined by in terms of the coordinates at $\mathcal{M}$ and $\mathcal{B}$ by

$$\phi^M{}_\mu = \partial_\mu \phi^M.$$

For a fixed value of the coordinate indices on the body manifold, the components $\phi^M{}_\mu$ give rise to a covector field $\phi^M{}_a$ on $\mathcal{M}$ satisfying the condition

$$\phi^M{}_a u^a = 0.$$

We introduce the *strain* of the material by applying the *push-forward* to the inverse metric tensor $g^{\mu\nu}$ on $\mathcal{M}$. Its components are given by

$$h^{MN} \equiv \phi^M{}_\mu \phi^N{}_\nu g^{\mu\nu}.$$

The body manifold is assumed to be equipped with a volume form V_{ABC} which allows us to construct a scalar function n interpreted as the *particle density number* of the material via the relation

$$n^2 = \frac{1}{3!} \det\left(h^{MN}\right).$$

This interpretation of n is justified by the observation that the equation for particle conservation hold —that is, one has that

$$\nabla_\mu (n u^\mu) = 0.$$

In order to formulate a frame version of the energy momentum tensor of a relativistic elastic material, we begin by consider a frame $\{\boldsymbol{E_A}\}$ on $\mathcal{B}$ with associated coframe $\{\boldsymbol{\Omega^B}\}$. As we have not introduced a metric on $\mathcal{B}$, we do not assume any orthonormality condition on the frame and coframe.

> Remark 3.1. Note that although the frame $\{\boldsymbol{E_A}\}$ and coframe $\{\boldsymbol{\Omega^B}\}$ on $\mathcal{B}$ may not be orthonormal, the frame $\{e_{\boldsymbol{a}}\}$ and coframe $\{\omega^{\boldsymbol{b}}\}$ on $\mathcal{M}$ are $\boldsymbol{g} - orthonormal$.

The map ϕ, defines the pullback ϕ^* which can be used to pull-back the coframe to $\mathcal{M}$. More precisely, one has that

$$\boldsymbol{\Lambda^B} \equiv \phi^* \boldsymbol{\Omega^B}, \qquad \Lambda^{\boldsymbol{B}}{}_{\boldsymbol{a}} = \langle \boldsymbol{\Lambda^B}, e_{\boldsymbol{a}} \rangle.$$

As the map ϕ is surjective and has maximal rank, the set of covectors $\{\boldsymbol{\Lambda^B}\}$ is linearly independent. The fields $\{\Lambda^{\boldsymbol{B}}{}_{\boldsymbol{a}}\}$ will be used, in the sequel, to describe the configuration of the material body. The coefficients $\Lambda^{\boldsymbol{A}}{}_{\boldsymbol{a}}$ are *orthogonal* with respect to u^a —that is

$$\Lambda^{\mathbf{A}}{}_{\boldsymbol{a}} u^{\boldsymbol{a}} = 0.$$

We denote the determinant of the frame field as e. It is related to the determinant of the metric tensor by $e = \sqrt{-g}$. Furthermore, we have

$$\delta e = \omega_{\boldsymbol{a}}{}^{\mu} \delta e^{\mu}_{\boldsymbol{a}}, \tag{3.2a}$$

$$\delta \Lambda^{\boldsymbol{A}}{}_{\boldsymbol{a}} = \Lambda^{\boldsymbol{A}}{}_{\mu} \delta e_{\boldsymbol{a}}{}^{\mu}. \tag{3.2b}$$

In the above δ is understood as an infinitesimal variation. More precisely, for a function f we have,

$$\delta f \equiv \frac{\partial f}{\partial x^\mu} \delta x^\mu.$$

Equation (3.2a) can be obtained by using *Jacobi's formula* for a square matrix given by,

$$\frac{\partial \det(\boldsymbol{A})}{\partial A_{\mu\nu}} = det(\boldsymbol{A})(\boldsymbol{A}^{-1})_{\mu\nu},$$

and recalling that $\omega^{\boldsymbol{a}}{}_{\mu} = (e^{-1})^{\boldsymbol{a}}{}_{\mu}$. Equation (3.2b) follows from observing that

$$\Lambda^{\boldsymbol{A}}{}_{\boldsymbol{a}} = \Lambda^{\boldsymbol{A}}{}_{\mu} e_{\boldsymbol{a}}{}^{\mu}, \qquad \frac{\partial e_{\boldsymbol{a}}{}^{\nu}}{\partial e_{\boldsymbol{c}}{}^{\mu}} = \delta^{\boldsymbol{c}}{}_{\boldsymbol{a}} \delta^{\nu}{}_{\mu}.$$

In terms of the above fields we construct a Lagrangian of the form $L = L\left(\Lambda^{\boldsymbol{A}}{}_{\boldsymbol{b}}, e_{\boldsymbol{a}}{}^{\mu}\right)$. The action thus reads

$$S = \int \mathcal{L}\left(\Lambda^{\boldsymbol{A}}{}_{\boldsymbol{b}}, e_{\boldsymbol{a}}{}^{\mu}\right) d^4x,$$

where we have defined the *Lagrangian density*

$$\mathcal{L}\left(\Lambda^{\boldsymbol{A}}{}_{\boldsymbol{b}}, e_{\boldsymbol{a}}{}^{\mu}\right) \equiv eL\left(\Lambda^{\boldsymbol{A}}{}_{\boldsymbol{b}}, e_{\boldsymbol{a}}{}^{\mu}\right).$$

The variation of the action yields

$$\begin{aligned}
\delta S &= \int \left(\frac{\partial e}{\partial e_{\boldsymbol{a}}{}^{\mu}} \delta e_{\boldsymbol{a}}{}^{\mu} L + e \frac{\partial L}{\partial e_{\boldsymbol{a}}{}^{\mu}} \delta e_{\boldsymbol{a}}{}^{\mu} + e \frac{\partial L}{\partial \Lambda^{\boldsymbol{A}}{}_{\boldsymbol{b}}} \delta \Lambda^{\boldsymbol{A}}{}_{\boldsymbol{b}} \right) d^4x \\
&= \int \left(\omega^{\boldsymbol{a}}{}_{\mu} L + \frac{\partial L}{\partial e_{\boldsymbol{a}}{}^{\mu}} + \frac{\partial L}{\partial \Lambda^{\boldsymbol{A}}{}_{\boldsymbol{a}}} \Lambda^{\boldsymbol{A}}{}_{\mu} \right) e \delta e_{\boldsymbol{a}}{}^{\mu} d^4x \\
&= \int \mathcal{T}^{\boldsymbol{a}}{}_{\mu} e \delta e_{\boldsymbol{a}}{}^{\mu} d^4x,
\end{aligned}$$

where we have made use of equations (3.2a) and (3.2b) and defined

$$\mathcal{T}^{\boldsymbol{a}}{}_{\mu} \equiv \omega^{\boldsymbol{a}}{}_{\mu} L + \frac{\partial L}{\partial e_{\boldsymbol{a}}{}^{\mu}} + \frac{\partial L}{\partial \Lambda^{\boldsymbol{A}}{}_{\boldsymbol{a}}} \Lambda^{\boldsymbol{A}}{}_{\mu}.$$

By multiplying with $e_{\boldsymbol{c}}{}^{\mu} \eta^{\boldsymbol{ac}}$, and applying the chain rule to the second term, we obtain

$$\mathcal{T}^{\boldsymbol{ab}} = \eta^{\boldsymbol{ab}} L + 2 \frac{\partial L}{\partial \Lambda^{\boldsymbol{A}}{}_{\boldsymbol{a}}} \Lambda^{\boldsymbol{A}}{}_{\boldsymbol{c}} \eta^{\boldsymbol{bc}}. \tag{3.3}$$

Assuming that the Lagrangian may be written on the form (see [23] for details)

$$L = \rho = n\epsilon,$$

we find that

$$\frac{\partial L}{\partial \Lambda^{\boldsymbol{A}}{}_{\boldsymbol{b}}} = n\frac{\partial \epsilon}{\partial h^{\boldsymbol{AB}}}\frac{\partial h^{\boldsymbol{AB}}}{\partial \Lambda^{\boldsymbol{A}}{}_{\boldsymbol{b}}} + \epsilon\frac{\partial n}{\partial \Lambda^{\boldsymbol{A}}{}_{\boldsymbol{b}}}$$

with

$$\frac{\partial h^{\boldsymbol{AB}}}{\partial \Lambda^{\boldsymbol{D}}{}_{\boldsymbol{a}}} = 2\eta^{\boldsymbol{ac}}\Lambda^{(\boldsymbol{A}}{}_{\boldsymbol{c}}\delta^{\boldsymbol{B})}{}_{\boldsymbol{D}}, \qquad \frac{\partial n}{\partial \Lambda^{\boldsymbol{D}}{}_{\boldsymbol{a}}} = nh_{\mathbf{AB}}\eta^{\boldsymbol{ac}}\Lambda^{(\boldsymbol{A}}{}_{\boldsymbol{c}}\delta^{\boldsymbol{B})}{}_{\boldsymbol{d}}.$$

Substituting the above expressions back into equation (3.3) we find an expression for the components of the energy-momentum tensor of the form

$$T^{\boldsymbol{ab}} = n\epsilon\eta^{\boldsymbol{ab}} + \Pi^{\boldsymbol{ab}} \tag{3.4}$$

where, in the following, $\Pi^{\boldsymbol{ab}}$ will be known as the components of the *Cauchy stress tensor* and is given by

$$\Pi_{\boldsymbol{ab}} \equiv 2n\tau_{\mathbf{AB}}\Lambda^{\boldsymbol{A}}{}_{\boldsymbol{a}}\Lambda^{\boldsymbol{B}}{}_{\boldsymbol{b}} + \epsilon nh_{\mathbf{AB}}\Lambda^{\boldsymbol{A}}{}_{\boldsymbol{a}}\Lambda^{\boldsymbol{B}}{}_{\boldsymbol{b}}, \tag{3.5}$$

where $\tau_{\boldsymbol{AB}}$ is the *second Piola-Kirchoff stress tensor* defined by.

$$\tau_{\boldsymbol{AB}} \equiv \frac{\partial \epsilon}{\partial h^{\boldsymbol{AB}}}.$$

We further make the reasonable assumption that

$$h_{\boldsymbol{ab}} = h_{\boldsymbol{AB}}\Lambda^{\boldsymbol{A}}{}_{\boldsymbol{a}}\Lambda^{\boldsymbol{B}}{}_{\boldsymbol{b}},$$

where $h_{\boldsymbol{ab}}$ as usual denotes the frame components of the projector metric. To show that this is reasonable, we note the following: the equation holds identically both under multiplication of $u^{\boldsymbol{a}}$ and $\eta^{\boldsymbol{ca}}\Lambda^{\boldsymbol{C}}{}_{\boldsymbol{a}}$ — in the latter case, one has to invoke the definition of $h_{\boldsymbol{AB}}$ on the right hand side of the equation to show this. Secondly, on a spatial hypersurface $\mathcal{S} \in \mathcal{M}$ the map ϕ is a diffeomorphism which implies that the object $h_{\boldsymbol{ab}}$ defined on $\mathcal{S}$ is physically equivalent to the corresponding object defined on $\mathcal{B}$ via ϕ. Using this assumption in (3.5) we obtain the desired form of the energy momentum tensor. Namely, one has that

$$T_{\boldsymbol{ab}} = \rho u_{\boldsymbol{a}} u_{\boldsymbol{b}} + \varPi_{\boldsymbol{ab}}, \tag{3.6}$$

with

$$\varPi_{\boldsymbol{ab}} \equiv 2\rho\eta_{\boldsymbol{ab}} + 2n\tau_{\mathbf{AB}}\varLambda^{\boldsymbol{A}}{}_{\boldsymbol{a}}\varLambda^{\boldsymbol{B}}{}_{\boldsymbol{b}}. \tag{3.7}$$

The form of (3.7) is the same as the form of the tensor S_{ab} defined in [22]; but whereas the latter is defined in terms of the coordinate-dependent fields $F^{A}{}_{\mu}$, in our treatment we have the frame dependent fields $\varLambda^{\boldsymbol{A}}{}_{\boldsymbol{a}}$. In view of definition 14 the fields $\varLambda^{\boldsymbol{A}}{}_{\boldsymbol{a}}$ has a direct physical interpretation as the deformation gradient of the body as measured by a co-moving observer.

Remark 3.2. Observe that the energy momentum tensor obtained in the above treatment is the canonical energy momentum tensor. But it has been shown in [23] that for the elastic case, the metric and canonical energy momentum tensors are the same. Thus, the equations of motion can either be obtained by variation of the action or the divergence free condition.

Chapter 4

Evolution equations for Einstein-matter systems

4.1 Introduction

Einstein's theory of General Relativity provides us with the most appropriate tool for studying the dynamics of self gravitating objects. It is therefore of clear interest to study the structural properties of the Einstein field equations and to provide a framework for studying their solutions. The *Cauchy problem* provides a setting for the analysis of generic solutions to the field equations parametrised in terms of the initial conditions —for details, see [8, 10, 24]. In particular, one is interested in showing that the Einstein equations admit a well-posed initial value formulation — see the discussion in Section 2.4 for more details. See also [25] for a lucid discussion on the Cauchy problem.

The motivation for our study is provided by the observation that the energy momentum tensor for a perfect fluid, elastic matter —see [22, 26, 27] for details— and bulk viscosity —e.g. see [28–32] and references therein— may be put on a form consisting of a part involving the 4-velocity $\boldsymbol{u}$ and energy density ρ and a part involving a spatial symmetric tensor $\boldsymbol{\Pi}$. Thus, by "hiding" the specific matter variables in the tensor $\boldsymbol{\Pi}$ one cannot differentiate between elastic matter, perfect or viscous fluid by considering the energy momentum tensor alone. By employing a hyperbolic reduction of the Einstein field equations coupled to an energy-momentum tensor of such a general form, we provide the necessary conditions for such a matter model to form First order symmetric hyperbolic (FOSH) evolution equations. *We show that one can avoid the details of the specific matter models in the construction of a FOSH system by introducing an auxilliary* *field.*

The procedure we employ to obtain these evolution equations is similar to that of [20] and may be described as follows: we introduce a frame field to replace the metric tensor as a variable and fix the gauge by choosing Lagrangian coordinates —i.e. one of the vectors of the frame field is chosen as to coincide with the 4-velocity of the particle trajectories; we also let the rest of the frame be Fermi propagated. By virtue of the Bianchi identity and assuming the connection to be Levi-Civita we show that the solution to a set of new field equations constructed with so called *zero-quantities* implies the existence of a metric solution to the Einstein field equations. A subset of these equations provides the symmetric hyperbolic evolution equations. As part of this construction, it turns out to be necessary to introduce an auxilliary field to remove derivatives of the energy-momentum tensor from the principal part of the evolution equation of some of the geometric fields. The evolution equation of $\boldsymbol{\Pi}$ —which encodes the matter fields— is given in terms of the electric decomposition of the auxilliary field. Finally, we make use of the evolution equations, Cartan's identity and the Bianchi identities to show the propagation of constraints. It is important to stress that due to the generality of the procedure, we do not provide an equation defining ρ. It is therefore necessary to provide an equation of state (or the equivalent) when using our equations for a specific matter model. We treat dust and perfect fluid as examples at the end and briefly discuss elastic matter.

A limitation of our procedure is the requirement of $\boldsymbol{\Pi}$ being a purely spatial tensor — indeed, without this requirement the energy momentum tensor would take its most general decomposed form. The difficulty of allowing $\boldsymbol{\Pi}$ to have timelike components resides in the procedure of keeping the hyperbolicity of the theory. We have used the spatial property of $\boldsymbol{\Pi}$ extensively in the process of eliminating problematic derivative terms from the principal part of the equations. We also assume that the equations of motion for a matter system may be entirely determined by the divergence-free condition of the energy-momentum tensor. Thus, any matter models which require additional equations to close the evolution of the matter variables, are not considered herein.

Lastly, we should mention that a very good discussion of the Einstein-Euler-entropy system is found in [33] where a complete discussion of the arguments of the framework put forward in [20] is given.

4.2 The Einstein equations

In this work we consider the Einstein equations

$$R_{ab} - \frac{1}{2}g_{ab}R = \kappa T_{ab} \tag{4.1}$$

with energy-momentum tensor on the form

$$T_{ab} = \rho u_a u_b + \Pi_{ab}. \tag{4.2}$$

where ρ is a positive function of the matter fields. We require Π_{ab} to be a symmetric and purely spatial tensor —i.e.

$$\Pi_{ab}u^a = 0, \tag{4.3a}$$

$$\Pi_{ab} = \Pi_{(ab)}. \tag{4.3b}$$

We do not put any further restrictions on $\boldsymbol{\Pi_{ab}}$ other than that it satisfies the *divergence-free condition* of (4.2)

$$\nabla^a T_{ab} = 0. \tag{4.4}$$

Remark 4.1. *An energy momentum tensor of the form given in* (4.2) *is of a very general form and the conditions* (4.3a), (4.3b) *are not stringent restrictions. Thus, the power of the formalism developed herein lies in its* *generality: given an equation for ρ in terms of the matter fields, one can ignore the matter specific equations of motion and instead solve equations for Π_{ab}. The equations obtained will then be symmetric hyperbolic. This assumes that one can extract the complete set of equations of motion for the matter fields from* (4.4)*.*

A projection formalism

At each point in the spacetime manifold $\mathcal{M}$ the flow lines give rise to a tangent space which can be split into parts in the direction of $\boldsymbol{u}$ and those orthogonal. This means that without implying a foliation, we may decompose every tensor defined at each point $p \in \mathcal{M}$ into its orthogonal and timelike part. This may be done by contracting with u^a and the *projector* defined as

$$h_{\boldsymbol{a}}{}^{\boldsymbol{b}} \equiv \eta_{\boldsymbol{a}}{}^{\boldsymbol{b}} + u_{\boldsymbol{a}} u^{\boldsymbol{b}}, \qquad u^a = u^{\boldsymbol{a}} e_{\boldsymbol{a}}{}^a.$$

Remark 4.2. *In order to prevent confusion around notation and unnecessarily messy calculations, we will sometimes henceforth write $\boldsymbol{e_a}$ and $\boldsymbol{\omega^a}$ instead of $e_{\boldsymbol{a}}{}^a$ and $\omega^{\boldsymbol{a}}{}_a$. That is, wherever convenient we resort to the indexfree notation.*

Thus, a tensor $T_{\boldsymbol{ab}}$ may be split into its time-like, mixed and space-like parts given, respectively, by

$$T_{\boldsymbol{00}} = u^{\boldsymbol{a}} u^{\boldsymbol{b}} T_{\boldsymbol{ab}}, \qquad T'_{\boldsymbol{0c}} = u^{\boldsymbol{a}} h^{\boldsymbol{b}}{}_{\boldsymbol{c}} T_{\boldsymbol{ab}}, \qquad T'_{\boldsymbol{cd}} = h^{\boldsymbol{a}}{}_{\boldsymbol{c}} h^{\boldsymbol{b}}{}_{\boldsymbol{d}} T_{\boldsymbol{ab}},$$

where $'$ denotes that the remaining indices have been projected so that the object is spatial —e.g. $T'_{\boldsymbol{a0}} u^{\boldsymbol{a}} = 0$. Decomposing $\nabla_a u^b$ we obtain

$$\nabla_{\boldsymbol{a}} u^{\boldsymbol{b}} = \chi_{\boldsymbol{a}}{}^{\boldsymbol{b}} + u_{\boldsymbol{a}} a^{\boldsymbol{b}}, \tag{4.5}$$

where $\chi_{\boldsymbol{a}}{}^{\boldsymbol{b}}$ and $a^{\boldsymbol{b}}$ are the components of the *Weingarten tensor* and 4-acceleration, respectively, defined by

$$\chi_{\boldsymbol{a}}{}^{\boldsymbol{b}} \equiv h_{\boldsymbol{a}}{}^{\boldsymbol{c}} \nabla_{\boldsymbol{c}} u^{\boldsymbol{b}}, \qquad a^{\boldsymbol{b}} \equiv u^{\boldsymbol{c}} \nabla_{\boldsymbol{c}} u^{\boldsymbol{b}}.$$

Alternatively, the spatial frame components of the Weingarten tensor and 4-acceleration can be respectively expressed by,

$$\chi_{\boldsymbol{i}}{}^{\boldsymbol{j}} = \langle \boldsymbol{\omega}^{\boldsymbol{j}}, \nabla_{\boldsymbol{i}} \boldsymbol{e_0} \rangle = \Gamma_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{0}}, \qquad a^{\boldsymbol{i}} = \langle \boldsymbol{\omega}^{\boldsymbol{i}}, \boldsymbol{a} \rangle. \tag{4.6}$$

In the literature (e.g. see [25] p.217) the trace, trace-free and antisymmetric part of (4.5) is called, respectively, the expansion, shear and the twist of the fluid. By decomposing (4.4) we obtain an equivalent system of equations in terms of the above quantities

$$\nabla^{\boldsymbol{a}} \Pi_{\boldsymbol{ab}} = -a_{\boldsymbol{b}}\rho + u_{\boldsymbol{b}} \Pi_{\boldsymbol{ac}} \chi^{\boldsymbol{ac}}, \tag{4.7a}$$

$$u^{\boldsymbol{a}} \nabla_{\boldsymbol{a}} \rho = -\rho\chi - \Pi_{\boldsymbol{ab}} \chi^{\boldsymbol{ab}}. \tag{4.7b}$$

The decomposition of the *4-volume* is

$$\epsilon_{\boldsymbol{abcd}} = -2\left(u_{[\boldsymbol{a}} \epsilon_{\boldsymbol{b}]\boldsymbol{cd}} - \epsilon_{\boldsymbol{ab}[\boldsymbol{c}} u_{\boldsymbol{d}]} \right), \qquad \epsilon_{\boldsymbol{bcd}} = \epsilon_{\boldsymbol{abcd}} u^{\boldsymbol{a}}. \tag{4.8}$$

Given a tensor T_{abc} which is antisymmetric in its two last indices, we may construct the *electric* and *magnetic* parts with respect to u^a. In frame indices this is, respectively, defined by

$$E_{\boldsymbol{cd}} \equiv T_{\boldsymbol{abe}} h_{\boldsymbol{c}}{}^{\boldsymbol{a}} h_{\boldsymbol{d}}{}^{\boldsymbol{b}} u^{\boldsymbol{e}}, \qquad B_{\boldsymbol{cd}} \equiv T^{*}{}_{\boldsymbol{abe}} h_{\boldsymbol{c}}{}^{\boldsymbol{a}} h_{\boldsymbol{d}}{}^{\boldsymbol{b}} u^{\boldsymbol{e}},$$

where the *Hodge dual operator*, denoted by *, is defined by

$$T^{*}{}_{\boldsymbol{abe}} \equiv -\frac{1}{2} \epsilon^{\boldsymbol{mn}}{}_{\boldsymbol{be}} T_{\boldsymbol{amn}},$$

and has the property that

$$T^{**}{}_{\boldsymbol{abc}} = -T_{\boldsymbol{abc}}.$$

Depending on the symmetries and rank of the tensor, the above definition for electric and magnetic decomposition may vary slightly. Central for our discussion is that $E_{\boldsymbol{ab}}$ and $B_{\boldsymbol{ab}}$ are spatial and symmetric.

4.3 Gauge considerations

The gauge to be considered in our hyperbolic reduction procedure for the Einstein field equations follows the same considerations as in [20]. In particular, we make the following choices:

i. ***Orientation of the frame.*** We align the time-leg of the frame with the flow vector u^a tangent to the worldlines of the particle —that is, we set

$$u^a = e_{\mathbf{0}}{}^a.$$

ii. ***Basis in a coordinate system.*** Given a coordinate system $x = (x^\mu)$ we expand the basis vectors as

$$e_{\boldsymbol{a}}{}^a = e_{\boldsymbol{a}}{}^\mu \boldsymbol{\partial}_\mu{}^a. \tag{4.9}$$

Given an initial hypersurface, $\mathcal{S}_\star$, then the coordinates (x^j) defined on $\mathcal{S}_\star$ remain constant along the flow and, thus, specify the frame.

iii. ***Lagrangian condition.*** The implementation of a Lagrangian gauge is equivalent to requiring that $e_{\mathbf{0}}{}^a = \partial_t{}^a$ where t is a suitable parameter along the world-lines of the material —e.g. the *proper time*. In terms of the components of the frame, this condition is equivalent to requiring that

$$e_{\mathbf{0}}{}^\mu = \delta_{\mathbf{0}}{}^\mu. \tag{4.10}$$

iv. ***Fermi Propagation of the frame.*** We require the vector fields $e_{\boldsymbol{a}}{}^a$ to be *Fermi propagated* along the direction of $e_{\mathbf{0}}{}^a$ —i.e.

$$\nabla_{\mathbf{0}} e_{\boldsymbol{a}}{}^a + \boldsymbol{g}\left(\boldsymbol{e}_{\boldsymbol{a}}, \nabla_{\mathbf{0}} e_{\mathbf{0}}\right) \boldsymbol{e}_{\mathbf{0}} - \boldsymbol{g}\left(\boldsymbol{e}_{\boldsymbol{a}}, \boldsymbol{e}_{\mathbf{0}}\right) \nabla_{\mathbf{0}} \boldsymbol{e}_{\mathbf{0}} = 0.$$

By using equations (2.7) and (2.8) the Fermi propagation of the frame implies the following conditions on the connection coefficients:

$$\Gamma_{\mathbf{0}}{}^{\boldsymbol{i}}{}_{\boldsymbol{j}} = 0, \tag{4.11a}$$

$$\Gamma_{\mathbf{0}}{}^{\mathbf{0}}{}_{\mathbf{0}} = 0, \tag{4.11b}$$

for $\boldsymbol{i}, \boldsymbol{j} = 1, 2, 3$. A frame satisfying the above equation is a frame where $e_{\boldsymbol{0}}{}^a = u^a$ and $\{e_{\boldsymbol{i}}{}^a\}$ is orthonormal at every point along the trajectory for which u^a is the tangent vector.

4.4 Zero-quantities

In the subsequent discussion it will prove convenient to introduce, as a book-keeping device, the *zero-quantities*

$$\Delta^{\boldsymbol{d}}{}_{\boldsymbol{abc}} \equiv \hat{R}^{\boldsymbol{d}}{}_{\boldsymbol{abc}} - \rho^{\boldsymbol{d}}{}_{\boldsymbol{abc}}, \tag{4.12a}$$

$$F_{\boldsymbol{bcd}} \equiv \nabla_{\boldsymbol{a}} F^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}, \tag{4.12b}$$

$$N_{\boldsymbol{cab}} \equiv Z_{\boldsymbol{cab}} - 2\nabla_{[\boldsymbol{a}} \Pi_{\boldsymbol{b}]\boldsymbol{c}}, , \tag{4.12c}$$

where $L_{\boldsymbol{ce}}$ denotes the components of the Schouten tensor as defined by equation (2.16). Moreover, by $\hat{R}^{\boldsymbol{d}}{}_{\boldsymbol{abc}}$ it is understood the expression for the Riemann tensor in terms of the connection coefficients $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$ and its frame derivatives. We have also defined

$$\rho^{\boldsymbol{d}}{}_{\boldsymbol{abc}} \equiv \hat{C}^{\boldsymbol{d}}{}_{\boldsymbol{abc}} + 2\eta^{\boldsymbol{d}}{}_{[\boldsymbol{b}} \hat{L}_{\boldsymbol{c}]\boldsymbol{a}} - 2\eta_{\boldsymbol{a}[\boldsymbol{b}} \hat{L}_{\boldsymbol{c}]}{}^{\boldsymbol{d}}, \tag{4.13a}$$

$$F^{\boldsymbol{c}}{}_{\boldsymbol{abd}} \equiv \hat{C}^{\boldsymbol{c}}{}_{\boldsymbol{abd}} - 2\eta^{\boldsymbol{c}}{}_{[\boldsymbol{b}} \hat{L}_{\boldsymbol{d}]\boldsymbol{a}}, \tag{4.13b}$$

$$Z_{\boldsymbol{cab}} \equiv 2\nabla_{[\boldsymbol{a}} \Pi_{\boldsymbol{b}]\boldsymbol{c}}, \tag{4.13c}$$

$$\hat{L}_{\boldsymbol{ab}} \equiv T_{\boldsymbol{ab}} - \frac{1}{3}\eta_{\boldsymbol{ab}} T, , \tag{4.13d}$$

where $\hat{C}^{\boldsymbol{d}}{}_{\boldsymbol{abc}}$ is defined as having the same symmetries as the components of the Weyl tensor $C^{\boldsymbol{d}}{}_{\boldsymbol{abc}}$.

> Remark 4.3. *The components $\rho^{\boldsymbol{d}}{}_{\boldsymbol{abc}}$ are known as the* algebraic curvature *and encode the decomposition of the Riemann curvature tensor in terms of the Weyl and Schouten tensors while $F^{\boldsymbol{c}}{}_{\boldsymbol{abc}}$ are the components of the* Friedrich tensor. *The latter provides a convenient way to encode the second Bianchi identity for the curvature.*

Remark 4.4. *The tensor $Z_{\boldsymbol{cab}}$, hereafter to be referred to as the Z-tensor, is introduced so that the evolution equations of the electric and magnetic part of the Weyl tensor can be expressed in terms of lower order terms —i.e. preventing any derivatives of $\Pi_{\boldsymbol{ab}}$ to appear in the equations and hence keeping their hyperbolicity.*

In terms of the objects introduced in the previous paragraphs, the Einstein field equations (4.1) can be encoded in the conditions

$$\nabla^{\boldsymbol{a}} T_{\boldsymbol{ab}} = 0, \tag{4.14a}$$

$$\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{b}} = 0, \tag{4.14b}$$

$$\Delta^{\boldsymbol{d}}{}_{\boldsymbol{abc}} = 0, \tag{4.14c}$$

$$F_{\boldsymbol{bcd}} = 0. \tag{4.14d}$$

More precisely, one has the following result:

Lemma 1. For a given ρ, let $(\hat{L}_{\boldsymbol{ab}}, e^{\mu}{}_{\boldsymbol{a}}, \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}, \hat{C}^{\boldsymbol{d}}{}_{\boldsymbol{abc}})$ be a solution to equations (4.14a)-(4.14d) for which the metric compatibility condition (2.8) holds. Then $(\hat{L}_{\boldsymbol{ab}}, e^{\mu}{}_{\boldsymbol{a}}, \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}, \hat{C}^{\boldsymbol{d}}{}_{\boldsymbol{abc}})$ implies the existence of a metric $\boldsymbol{g}$ solution to the Einstein field equations (4.1) with energy-momentum tensor defined by the components $T_{\boldsymbol{ab}}$. Moreover, the fields $\hat{C}^{\boldsymbol{d}}{}_{\boldsymbol{abc}}$ are, in fact, the components of the Weyl tensor of $\boldsymbol{g}$.

Remark 4.5. *Note that equations* (4.14a)-(4.14d) *do not provide a closed system of evolution equations for the unknowns of our system. They are only the necessary equations for giving Lemma 1.*

Proof. The frame $\{\boldsymbol{e_a}\}$ obtained from the solution to equation (4.14b) implies, in turn, by the condition $\langle \boldsymbol{\omega^b}, \boldsymbol{e_a} \rangle = \delta_{\boldsymbol{a}}{}^{\boldsymbol{b}}$ the existence of a coframe $\{\boldsymbol{\omega^b}\}$ from which one can construct a metric tensor $\boldsymbol{g}$ via the relation

$$\boldsymbol{g} = \eta_{\boldsymbol{ab}} \boldsymbol{\omega^a} \otimes \boldsymbol{\omega^a}.$$

Since the coefficients $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$ satisfy the no-torsion and metric compatibility conditions (4.14b) and (2.8), then they must coincide with the connection coefficients of the metric $\boldsymbol{g}$ with respect to the frame $\{\boldsymbol{e_a}\}$. Moreover, by equation (2.12) we have that

$$\hat{R}^{\boldsymbol{d}}{}_{\boldsymbol{abc}} = R^{\boldsymbol{d}}{}_{\boldsymbol{abc}},$$

where $R^{\boldsymbol{d}}{}_{\boldsymbol{abc}}$ denotes the frame components of the Riemann curvature tensor. Using the Riemann decomposition as defined by equation (2.15) together with equation (4.14c) we obtain

$$C^{\boldsymbol{d}}{}_{\boldsymbol{abc}} + 2\eta^{\boldsymbol{a}}{}_{[\boldsymbol{b}}L_{\boldsymbol{c}]\boldsymbol{a}} - 2\eta_{\boldsymbol{a}[\boldsymbol{b}}L_{\boldsymbol{c}]}{}^{\boldsymbol{a}} = \hat{C}^{\boldsymbol{d}}{}_{\boldsymbol{abc}} + 2\eta^{\boldsymbol{a}}{}_{[\boldsymbol{b}}\hat{L}_{\boldsymbol{c}]\boldsymbol{a}} - 2\eta_{\boldsymbol{a}[\boldsymbol{b}}\hat{L}_{\boldsymbol{c}]}{}^{\boldsymbol{a}}. \quad (4.15)$$

Taking the trace of equation (4.15) with respect to the indices $\boldsymbol{b}$ and $\boldsymbol{d}$ and using the trace-free property of the Weyl tensor and $\hat{C}^{\boldsymbol{d}}{}_{\boldsymbol{abc}}$ we obtain

$$L_{\boldsymbol{ca}} + \frac{1}{2}\eta_{\boldsymbol{ca}}L^{\boldsymbol{d}}{}_{\boldsymbol{d}} = \hat{L}_{\boldsymbol{ca}} + \frac{1}{2}\eta_{\boldsymbol{ca}}\hat{L}^{\boldsymbol{d}}{}_{\boldsymbol{d}}. \quad (4.16)$$

Finally, taking the trace of equation (4.16) and using equations (4.4) and (2.14), we get the identity

$$L^{\boldsymbol{d}}{}_{\boldsymbol{d}} = \hat{L}^{\boldsymbol{d}}{}_{\boldsymbol{d}}.$$

The latter shows that $\hat{L}_{\boldsymbol{ab}}$ are, in fact, the components of the Schouten tensor of the metric $\boldsymbol{g}$. Using the definition of the Schouten tensor in terms of the Ricci tensor, equation (2.16), it follows readily that the metric $\boldsymbol{g}$ satisfies the Einstein field equations with an energy-momentum tensor defined by the components $T_{\boldsymbol{ab}}$. Returning to equation (4.16) we conclude by the uniqueness of the decomposition of the Riemann tensor that the fields $\hat{C}^{\boldsymbol{d}}{}_{\boldsymbol{abc}}$ are, in fact, the components of the Weyl tensor of $\boldsymbol{g}$.

Remark 4.6. *In the following to ease the notation, and in a slight abuse of notation we simply write $C^{\boldsymbol{d}}{}_{\boldsymbol{abc}}$ instead of $\hat{C}^{\boldsymbol{d}}{}_{\boldsymbol{abc}}$.*

4.5 Evolution equations

Given the gauge conditions introduced in Section 4.3, the next step in our analysis involves the extraction of a suitable (symmetric hyperbolic) evolution system from equations (4.14a)-(4.14d). We do this in a number of steps.

4.5.1 Equations for the components of the frame

The evolution equations for the components of the frame $e_{\boldsymbol{a}}{}^{\mu}$ are obtained from the *no-torsion condition* (4.14b). In order to do so we exploit the freedom available in the choice of the frame and require it to be adapted to the world-lines of the material particles and the gauge conditions outlined above.

Making use of the expansion (4.9) in equation (4.14b) one readily finds that

$$e_{\boldsymbol{a}}{}^{\mu}\partial_{\mu}e_{\boldsymbol{b}}{}^{\nu} - e_{\boldsymbol{b}}{}^{\mu}\partial_{\mu}e_{\boldsymbol{a}}{}^{\nu} = \left(\Gamma_{\boldsymbol{a}}{}^{\mathbf{c}}{}_{\boldsymbol{b}} - \Gamma_{\boldsymbol{b}}{}^{\mathbf{c}}{}_{\boldsymbol{a}}\right) e_{\mathbf{c}}{}^{\nu}.$$

Setting $\boldsymbol{a} = 0$ in the above expression and making use of the Lagrangian gauge condition *(iii)* we obtain

$$\partial_{\mathbf{0}}e_{\boldsymbol{b}}{}^{\nu} - \left(\Gamma_{\mathbf{0}}{}^{\mathbf{c}}{}_{\boldsymbol{b}} - \Gamma_{\boldsymbol{b}}{}^{\mathbf{c}}{}_{\mathbf{0}}\right) e_{\mathbf{c}}{}^{\nu} = 0. \tag{4.17}$$

This last equation will be read as an evolution equation for the frame coefficients $e_{\boldsymbol{b}}{}^{\nu}$ with $\boldsymbol{b} = 1, 2, 3$. As it only contains derivatives along the flow lines of the matter, it is, in fact, a transport equation along the world-lines. Observe that for $\boldsymbol{b} = 0$ the equation is satisfied automatically —recall that as a consequence of the Lagrangian condition (4.10) the coefficients $e_{\mathbf{0}}{}^{\mu}$ are already fixed.

Remark 4.7. *Assuming that the gauge conditions* (i), (ii) *and* (iii) *above hold, equation* (4.17) *can be succinctly written as*

$$\Sigma_{\mathbf{0}}{}^{\mathbf{c}}{}_{\boldsymbol{b}} = 0.$$

This observation will be of use in the discussion of the propagation of the constraints.

4.5.2 Evolution equations for the connection coefficients

The evolution equations for the frame components are given in terms of the frame connection coefficients. Due to the Fermi propagation and the metric compatibility, equation (2.8), the independent, non-zero components of the connection coefficients are $\Gamma_i{}^k{}_j$, $\Gamma_0{}^0{}_j$ and $\Gamma_i{}^0{}_j$. Evolution equations for $\Gamma_i{}^k{}_j$ and $\Gamma_i{}^0{}_j$ may be extracted from the equation for the algebraic curvature (4.14c). More precisely, we consider the condition

$$\Delta^{\boldsymbol{a}}{}_{\boldsymbol{bc0}} = 0. \tag{4.18}$$

But $\Delta^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}$ inherits the symmetries of the Riemann tensor — in particular

$$\Delta_{\boldsymbol{abcd}} = \Delta_{[\boldsymbol{ab}][\boldsymbol{cd}]}.$$

As a consequence one has that

$$\Delta^{\boldsymbol{a}}{}_{\boldsymbol{b00}} = 0, \qquad \Delta^{\boldsymbol{0}}{}_{\boldsymbol{0cd}} = 0$$

are satisfied identically. Thus, the non trivial components of $\Delta^{\boldsymbol{a}}{}_{\boldsymbol{bc0}}$ are

$$\Delta^{\boldsymbol{i}}{}_{\boldsymbol{jk0}}, \qquad \Delta^{\boldsymbol{0}}{}_{\boldsymbol{ij0}}, \qquad \Delta^{\boldsymbol{k}}{}_{\boldsymbol{0j0}},$$

where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}, \ldots = 1, 2, 3$. It is readily verified that from the metric compatibility condition — i.e. equation (2.8) — one can further reduce the independent components of the connection. More precisely, one has that

$$\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{b}} = 0, \qquad \Gamma_{\boldsymbol{i}}{}^{\boldsymbol{0}}{}_{\boldsymbol{j}} = \Gamma_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{0}}, \qquad \Gamma_{\boldsymbol{0}}{}^{\boldsymbol{0}}{}_{\boldsymbol{i}} = \Gamma_{\boldsymbol{0}}{}^{\boldsymbol{k}}{}_{\boldsymbol{0}}\eta_{\boldsymbol{ik}}.$$

Consequently, it follows that

$$\Delta^{\boldsymbol{0}}{}_{\boldsymbol{ij0}} = \eta_{\boldsymbol{ik}}\Delta^{\boldsymbol{k}}{}_{\boldsymbol{0j0}}.$$

Hence, the condition (4.18) is equivalent to imposing the conditions

$$\Delta^{i}{}_{\boldsymbol{jk0}} = 0, \tag{4.19}$$

$$\Delta^{\mathbf{0}}{}_{\boldsymbol{ij0}} = 0. \tag{4.20}$$

From equation (4.19) and (4.20) we may extract the evolution equation for $\Gamma_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}}$ and $\Gamma_{\boldsymbol{i}}{}^{\mathbf{0}}{}_{\boldsymbol{j}}$, respectively. Using equations (4.12a) and (4.13a), the above equations take the form

$$\hat{R}^{\boldsymbol{i}}{}_{\boldsymbol{jk0}} = C^{\boldsymbol{i}}{}_{\boldsymbol{jk0}} + 2\eta^{\boldsymbol{i}}{}_{[\boldsymbol{j}}L_{\mathbf{0}]\boldsymbol{k}} - 2\eta_{\boldsymbol{k}[\boldsymbol{j}}L_{\mathbf{0}]}{}^{\boldsymbol{i}}, \tag{4.21}$$

$$\hat{R}^{\mathbf{0}}{}_{\boldsymbol{jk0}} = C^{\mathbf{0}}{}_{\boldsymbol{jk0}} + 2\eta^{\mathbf{0}}{}_{[\boldsymbol{j}}L_{\mathbf{0}]\boldsymbol{k}} - 2\eta_{\boldsymbol{k}[\boldsymbol{j}}L_{\mathbf{0}]}{}^{\mathbf{0}}. \tag{4.22}$$

The Lagrangian gauge — i.e. gauge condition *(iii)* — and the condition that the frame remains orthonormal along $e_{\mathbf{0}}$ — i.e. equation (2.6) — imply that

$$L_{\mathbf{0}\boldsymbol{i}} = 0, \qquad \eta_{\mathbf{0}\boldsymbol{i}} = 0.$$

Substituting these conditions back into equation (4.21) and (4.22) gives,

$$\hat{R}^{\boldsymbol{i}}{}_{\boldsymbol{jk0}} = C^{\boldsymbol{i}}{}_{\boldsymbol{jk0}}, \tag{4.23}$$

$$\hat{R}^{\boldsymbol{i}}{}_{\boldsymbol{jk0}} = C^{\mathbf{0}}{}_{\boldsymbol{ij0}} + L_{\boldsymbol{ij}} - \eta_{\boldsymbol{ij}}L_{\mathbf{00}}. \tag{4.24}$$

Finally, the Riemann tensor can be expressed in terms of the connection coefficients via equation (2.12). Making use of gauge condition *(iv)*, we obtain

$$\partial_{\mathbf{0}}\Gamma_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}} = C^{\boldsymbol{j}}{}_{\boldsymbol{ki0}} + \Gamma_{\boldsymbol{l}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}}\Gamma_{\boldsymbol{i}}{}^{\boldsymbol{l}}{}_{\mathbf{0}} + \Gamma_{\mathbf{0}}{}^{\boldsymbol{j}}{}_{\mathbf{0}}\Gamma_{\boldsymbol{i}}{}^{\mathbf{0}}{}_{\boldsymbol{k}} - \Gamma_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\mathbf{0}}\Gamma_{\mathbf{0}}{}^{\mathbf{0}}{}_{\boldsymbol{k}}, \tag{4.25}$$

$$\begin{aligned}\partial_{\mathbf{0}}\Gamma_{\boldsymbol{j}}{}^{\mathbf{0}}{}_{\boldsymbol{i}} - \partial_{\boldsymbol{j}}\Gamma_{\mathbf{0}}{}^{\mathbf{0}}{}_{\boldsymbol{i}} = {} & \Gamma_{\mathbf{0}}{}^{\mathbf{0}}{}_{\boldsymbol{i}}\Gamma_{\mathbf{0}}{}^{\mathbf{0}}{}_{\boldsymbol{j}} - \Gamma_{\boldsymbol{k}}{}^{\mathbf{0}}{}_{\boldsymbol{i}}\Gamma_{\boldsymbol{j}}{}^{\boldsymbol{j}}{}_{\mathbf{0}} - \Gamma_{\boldsymbol{k}}{}^{\mathbf{0}}{}_{\boldsymbol{i}}\Gamma_{\boldsymbol{j}}{}^{\boldsymbol{k}}{}_{\mathbf{0}} \\ & + C^{\mathbf{0}}{}_{\boldsymbol{ij0}} + L_{\boldsymbol{ij}} - \eta_{\boldsymbol{ij}}L_{\mathbf{00}}. \end{aligned} \tag{4.26}$$

Remark 4.8. *Assuming that the gauge condition* (iii) *and* (iv) *holds and that* $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$ *is Levi-Civita, equations* (4.25) *and* (4.26) *are equivalent to*

$$\Delta^{\boldsymbol{c}}{}_{\boldsymbol{ab0}} = 0.$$

Observe again, that the resulting equations are, in fact, transport equations along the world-line of the material particles.

The evolution equation for the remaining connection coefficients — i.e. $\Gamma_{\boldsymbol{0}}{}^{\boldsymbol{0}}{}_{\boldsymbol{i}} = \Gamma_{\boldsymbol{0}}{}^{\boldsymbol{i}}{}_{\boldsymbol{0}}$ — will be obtained by splitting $\Pi_{\boldsymbol{ab}}$ into its trace and trace-free part

$$\Pi_{\boldsymbol{ab}} = \Pi_{\{\boldsymbol{ab}\}} + \frac{1}{4}\Pi\eta_{\boldsymbol{ab}},$$

where $\Pi_{\{\boldsymbol{ab}\}}$ denotes the trace-free part of $\Pi_{\boldsymbol{ab}}$. Plugging this into (4.7a) and (4.7b), we obtain

$$\nabla^{\boldsymbol{a}}\Pi_{\{\boldsymbol{ab}\}} = -\frac{1}{4}\nabla_{\boldsymbol{b}}\Pi - \rho a_{\boldsymbol{b}} + u_{\boldsymbol{b}}\Pi_{\{\boldsymbol{ac}\}}\chi^{\boldsymbol{ac}} + \frac{1}{4}\Pi\chi u_{\boldsymbol{b}}, \tag{4.27a}$$

$$u^{\boldsymbol{a}}\nabla_{\boldsymbol{a}}\rho = -\rho\chi - \Pi_{\{\boldsymbol{ab}\}}\chi^{\boldsymbol{ab}} - \frac{1}{4}\Pi\chi. \tag{4.27b}$$

In the above and throughout we have put $\Pi \equiv \Pi^{\boldsymbol{a}}{}_{\boldsymbol{a}}$. Since $\nabla_{[\boldsymbol{d}}\nabla_{\boldsymbol{b}]}\Pi = 0$, we obtain from equation (4.27a) that

$$J_{\boldsymbol{db}} = 0,$$

with

$$\begin{aligned} J_{\boldsymbol{db}} \equiv& -2\rho\nabla_{[\boldsymbol{b}}a_{\boldsymbol{d}]} + 2a_{[\boldsymbol{d}}\nabla_{\boldsymbol{b}]}\rho + 2\Pi_{\{\boldsymbol{ac}\}}\chi^{\boldsymbol{ac}}\nabla_{[\boldsymbol{d}}u_{\boldsymbol{b}]} + 2u_{[\boldsymbol{b}}\nabla_{\boldsymbol{d}]}\left(\Pi_{\{\boldsymbol{ac}\}}\chi^{\boldsymbol{ac}}\right) \\ &+ \frac{1}{2}\Pi\chi\nabla_{[\boldsymbol{d}}u_{\boldsymbol{b}]} + \frac{1}{2}u_{[\boldsymbol{b}}\nabla_{\boldsymbol{d}]}\left(\Pi\chi\right) - 2\nabla_{[\boldsymbol{d}}\nabla^{\boldsymbol{a}}\Pi_{\{\boldsymbol{b}]\boldsymbol{a}\}}. \end{aligned}$$

The last term may be written

$$\begin{aligned} \nabla_{[\boldsymbol{d}}\nabla^{\boldsymbol{a}}\Pi_{\{\boldsymbol{b}]\boldsymbol{a}\}} &= -R^{\boldsymbol{m}}{}_{\boldsymbol{bd}}{}^{\boldsymbol{a}}\Pi_{\{\boldsymbol{ma}\}} + R^{\boldsymbol{m}}{}_{\boldsymbol{db}}{}^{\boldsymbol{a}}\Pi_{\{\boldsymbol{ma}\}} + 2R^{\boldsymbol{m}}{}_{[\boldsymbol{d}}\Pi_{\{\boldsymbol{a}]\boldsymbol{m}\}} + \nabla^{\boldsymbol{a}}Z_{\boldsymbol{adb}}, \\ &= 2R^{\boldsymbol{m}}{}_{[\boldsymbol{d}}\Pi_{\{\boldsymbol{a}]\boldsymbol{m}\}} + \nabla^{\boldsymbol{a}}Z_{\boldsymbol{adb}}, \end{aligned} \tag{4.28}$$

where we have used the symmetry of the Riemann tensor in the last step. A straight forward calculation using the definition of the Z-tensor and equations (4.14d), (4.12b) and (4.13b) shows that

$$Z_{adb} = \nabla_m C^m{}_{adb} + P_{adb},$$

where

$$P_{adb} \equiv \frac{2}{3}\nabla_{[d}(\eta_{b]a}T) - 2\nabla_{[d}(\rho u_{b]} u_a).$$

Using that

$$\nabla_a \nabla_b C^{ab}{}_{cd} = 0, \qquad \nabla_{[a}\nabla_{b]}T = 0,$$

we have

$$\nabla^a Z_{adb} = -2\nabla^a \nabla_{[d}(\rho u_{b]} u_a). \tag{4.29}$$

Substituting this back into (4.28), we may now write the $\{0, \boldsymbol{i}\}$ components of J_{db} as

$$\begin{aligned} J_{0i} = &-2\rho\nabla_0 a_i - R_0{}^j \Pi_{ij} - 2a_i\rho\chi - 2a_i \Pi_{ij}\chi^{ij} \\ &+ a^j\rho\chi_{ij} - \rho\nabla_j \chi_i{}^j + \rho\nabla_i\chi. \end{aligned} \tag{4.30}$$

In the above expression we have used the Lagrangian gauge condition to set $u_i = 0$. Finally, using the definition for $\chi^i{}_j$ and a_i in terms of the connection coefficients — see (4.6) — we readily obtain

$$\begin{aligned} 3\partial_0 \Gamma_0{}^0{}_i - \eta^{jk}\partial_i \Gamma_j{}^0{}_k = &-2a_i\chi + a^j\chi_{ij} - \partial^j \Gamma_j{}^0{}_i + \Gamma_j{}^k{}_i\chi_k{}^j - \Gamma_j{}^j{}_k\chi_i{}^k \\ &- \frac{1}{\rho}\left(R_0{}^j\Pi_{ij} + 2a_i\Pi_{ij}\chi^{ij}\right) - \Gamma_j{}^0{}_0\chi^j{}_i - \Gamma_0{}^0{}_i\chi \\ &+ \Gamma_0{}^j{}_i\Gamma_0{}^0{}_j + \eta^{jk}\Gamma_i{}^l{}_j\Gamma_l{}^0{}_k + \eta^{jk}\Gamma_i{}^l{}_k\Gamma_j{}^0{}_l. \end{aligned} \tag{4.31}$$

Equations (4.25), (4.26) and (4.31) are of a form which is known to be symmetric hyperbolic — we refer again to [34] for details.

4.5.3 Evolution equations for the decomposed Z-tensor

It is well known that in vacuum the Bianchi equation leads to a symmetric hyperbolic equation for the independent components of the Weyl tensor. By contrast, an inspection of the definition of the Friedrich tensor F_{abcd}, equation (4.12b), reveals that the condition $F_{abc} = 0$ involves both derivatives of C_{abcd} and the matter variables. This potentially destroys the symmetric hyperbolicity of the equation for the components of the Weyl tensor. In the following we will show that it is possible to deal with this difficulty by providing two auxiliary fields —the Z-tensor and σ-tensor as defined by equations (4.13c) and (4.38), respectively.

We first define some important quantities and identities used in the following discussion. The Z-tensor has the symmetries

$$Z_{[abc]} = 0, \qquad Z_{abc} = Z_{a[bc]}.$$

The symmetry of the Z-tensor thus allows for a decomposition in terms of its electric and magnetic parts defined respectively as

$$\Psi_{ac} \equiv Z_{ebd} u^d h_a{}^e h_c{}^b, \qquad \Phi_{ac} \equiv Z^*_{ebd} u^d h_a{}^e h_c{}^b,$$

where, Z^*_{ebd}, is the dual Z-tensor defined in the customary way. The electric and magnetic part of the Z-tensor are symmetric tensors defined on the orthogonal space of $\boldsymbol{u}$ —i.e. one has that

$$\Psi_{ac} = \Psi_{(ac)}, \qquad \Psi_{ac} u^a = 0, \qquad \Phi_{ac} = \Phi_{(ac)}, \qquad \Phi_{ac} u^a = 0.$$

As such, the Z-tensor and its dual may be expressed in terms of the spatial fields Ψ_{ab} and Φ_{ab}

$$Z_{cab} = \Psi_{cb} u_a - \Psi_{ca} u_b - \epsilon_{ab}{}^e \Phi_{ce} + u_c \Pi_{db} \chi_a{}^d - u_c \Pi_{da} \chi_b{}^d, \tag{4.32a}$$

$$Z^*{}_{amn} = \tfrac{1}{2} \Psi_{ac} \epsilon_{mn}{}^c + u_{[m} \Phi_{n]a} + \tfrac{1}{2} \epsilon_{mnd} \Pi_c{}^d \chi_a{}^c - \tfrac{1}{2} \epsilon_{mnc} \Pi_{ad} \chi^{cd}. \tag{4.32b}$$

By plugging the definition for the Z-tensor into the definitions of Ψ_{ab} and Φ_{ab}, respectively, we obtain an evolution equation for the matter tensor Π_{ab} in terms of Ψ_{ab} together with a constraint equation. Namely, one has that

$$u^a \nabla_a \Pi_{fm} = a^a u_m \Pi_{fa} + a^a u_f \Pi_{ma} - \Pi_{ma} \chi_f{}^a - \Psi_{fm}, \tag{4.33a}$$

$$\epsilon_f{}^b{}_a \mathcal{D}_b \Pi_m{}^a = \epsilon_{fba} u_m \Pi_c{}^a \chi^{bc} + \Phi_{fm}. \tag{4.33b}$$

where $\mathcal{D}_b$ denotes the *Sen connection* defined as,

$$\mathcal{D}_b \Pi_{cd} \equiv h_b{}^a \nabla_a \Pi_{cd}.$$

It is worth noting that due to the $1+3$ split of space time, we do not have a spatial metric on the 3-dimensional surfaces. Hence, we cannot define a spatial derivative — i.e. a spatial metric satisfying the metric compatibility condition does not exist on the three surfaces. Equation (4.33b) is regarded as a constraint equation.

> Remark 4.9. *Note that equation* (4.33b) *will always hold as long as the definition of the Z-tensor (i.e. equation* (4.13c)*) propagates. This will be shown in Section 4.6.*

In order to close the system and to ensure hyperbolicity a set of evolution equations for the fields Ψ_{ab} and Φ_{ab} are needed. In the rest of this section we shall develop these equations and show they form a first order symmetric hyperbolic system.

The evolution equations for Ψ_{ab} is obtained by taking the divergence of the Z-tensor —i.e we have the equation

$$\nabla^b Z_{cab} = 2\nabla^b \nabla_{[a} \Pi_{b]c}.$$

Expanding the above equation and using the decomposition of the Z-tensor —i.e. equation (4.32a)— we obtain an equation of the form,

$$\begin{aligned} u^b \nabla_b \Psi_{ac} - \epsilon_a{}^{bd} \mathcal{D}_d \Phi_{cb} = {} & u^a \nabla_b \Psi_c{}^b - \Pi_{ad} \chi^{bd} \nabla_b u_c - u_c \chi^{bd} \nabla_b \Pi_{ad} \\ & - u_c \Pi_{ad} \nabla_b \chi^{bd} - \nabla_b \nabla_a \Pi_c{}^b + \nabla_b \nabla^b \Pi_{ac} \\ & + \Psi_{cb} \nabla^b u_a + \Phi_c{}^b \nabla_d \epsilon_{ab}{}^d + u_c \chi_a{}^b \nabla_d \Pi_b{}^d \\ & + \Pi_{bd} \chi_a{}^b \nabla^d u_c + u_c \Pi_{bd} \nabla^d \chi_a{}^b. \end{aligned} \tag{4.34}$$

It is necessary to write the terms which involve derivatives of Ψ_{ab} and Π_{ab} in terms of lower order terms. The divergence of Ψ_{ab} can be obtained from equation (4.33a),

$$
\begin{aligned}
\nabla_a \Psi_m{}^a = & -a^a \Psi_{ma} - a_a a^a u_m \rho + a^a a^f u_m \Pi_{af} \\
& + a^a \Pi_{ma} \chi^f{}_f + a^a \Pi_{af} \chi^f{}_m - a^a \Pi_{af} \chi_m{}^f \\
& + u^a \Pi_{mf} \nabla_a a^f - a^a \Pi_{mf} \chi_a{}^f - \chi^{af} \nabla_a \Pi_{mf} \\
& - \Pi_{mf} \nabla_a \chi^{af} - \chi^{af} \nabla_f \Pi_{ma} - u^a \nabla_f \nabla_a \Pi_m{}^f \\
& + u_m \Pi_{af} \nabla^f a^a
\end{aligned} \tag{4.35}
$$

In the above calculation we have made use of equations (4.33a), (4.5) and (4.7a). A straight forward calculation using the definition for the Riemann tensor — i.e. equation (2.9) — and equations (4.7a), (4.7b) as well as (4.33a), gives the relation

$$
\begin{aligned}
u^a \nabla_b \nabla_a \Pi_c{}^b = & -u^a R_{cbam} \Pi^{bm} - u^a R_a{}^b \Pi_{cb} + a_c \rho \chi \\
& - \Psi_{ab} u_c \chi^{ab} - u_c \Pi_{bm} \chi_a{}^m \chi^{ab} - u^a \rho \nabla_a a_c \\
& + u^a u_c \Pi_{bm} \nabla_a \chi^{bm}.
\end{aligned} \tag{4.36}
$$

Using equations (4.35), (4.36), (2.9), (4.33a), (4.5), (4.7a) and (4.7b) in equation (4.34), and by contracting with the projector — i.e $h^a{}_m h^c{}_n$ — we obtain after some algebraic manipulations the desired form of the evolution equation for Ψ_{ab},

$$
u^b \nabla_b \Psi_{mn} - \epsilon_m{}^{bd} \mathcal{D}_d \Phi_{nb} = \mathcal{W}_{mn}, \tag{4.37}
$$

where $\mathcal{W}_{ab}$ denotes the lower order terms and is explicitly given by

$$
\begin{aligned}
\mathcal{W}_{mn} = & -a^a \Psi_{na} u_m - a^a \Psi_{ma} u_n + a^a \epsilon_{nab} \Phi_m{}^b \\
& + u^a u^b u_m u_n \sigma_{ab} + u^a u_n \sigma_{ma} + \sigma_{mn} + u^a u_m \sigma_{na} \\
& + R_{manb} \Omega^{ab} + u^a u_n R_{mbac} \Omega^{bc} + u^a u_m R_{nbac} \Omega^{bc} \\
& + u^a u^b u_m u_n R_{acbd} \Omega^{cd} + R_n{}^a \Omega_{ma} + u^a u_n R_a{}^b \Omega_{mb} \\
& - \Psi_{mn} \chi - a_m u_n \rho \chi + \epsilon_{nac} u_m \Phi_b{}^c \chi^{ab} \\
& - a_m u_n \Omega_{ab} \chi^{ab} - \Omega_{nb} \chi_a{}^b \chi^a{}_m + \Psi_{ma} \chi^a{}_n \\
& - a^a u_m \rho \chi_{na} + \Omega_{ab} \chi^b{}_m \chi_n{}^a - \Omega_{ab} \chi^{ab} \chi_{nm} \\
& + u^a u_n \rho \nabla_a a_m + \rho \nabla_n a_m + a_m \nabla_n \rho.
\end{aligned}
$$

In the above, we have defined

$$\sigma_{ab} \equiv \nabla^c \nabla_c \Pi_{ab}. \tag{4.38}$$

Observe that the derivatives in equation (4.38) are covariant differentiation in the direction of the frame e_c. More precisely one has

$$\begin{aligned} \eta^{ab} \nabla_{e_a} \nabla_{e_b} \boldsymbol{\Pi} &= \omega^b (e_b[\Pi_{ac}])\, \omega^a \otimes \omega^c + L.O.T \\ &= 4\Pi_{ac}\, \omega^a \otimes \omega^c + L.O.T, \end{aligned} \tag{4.39}$$

where we have observed that $omega^a$ belong to the dual space of e_a and may thus be defined such that,

$$\omega^a[e_b] = \delta^a{}_b.$$

The lower order terms consist of connection coefficients. **Hence, the tensor σ_{ab} appearing in expression (4.37) consists of only lower order terms**. Note also that the derivatives of χ_{ab} and a_a may be expressed in terms of the connection coefficients and thus dealt with by (4.31) and (4.26) —i.e. we have

$$\chi_{ab} = h_a{}^m h_b{}^n \Gamma_m{}^{\mathbf{0}}{}_n, \qquad a_a = h_{ac} \Gamma_{\mathbf{0}}{}^c{}_{\mathbf{0}}.$$

In obtaining equation (4.37) we have used standard tensor manipulations involving the commutation of derivatives using the Riemann tensor and frequently making use of the spatial property of Π_{ab} to get rid of derivatives.

To obtain the evolution equation for the field Φ_{ab}, we first provide two preliminary identities obtained by again using the commutator of the covariant derivative — i.e. equation (2.9). We have

$$\begin{aligned} \epsilon_{amn} u^b \nabla^n \nabla_b \Pi_c{}^m &= \epsilon_{amn} u^b R_d{}^{mn}{}_b \Pi_c{}^d - \epsilon_{amn} u^b R^d{}_c{}^n{}_b \Pi_d{}^m \\ &\quad + \epsilon_{amn} u^b \nabla_b \nabla^n \Pi_c{}^m, \end{aligned} \tag{4.40}$$

$$\epsilon_{amn} u^b \nabla^n \nabla^m \Pi_{cb} = \epsilon_{anm}\, u^b R^d{}_b{}^{nm} \Pi_{cd}. \tag{4.41}$$

We then proceed in a similar way as with the field Ψ_{ab} and considering the dual equation

$$\nabla^b Z^*{}_{cab} = -\epsilon_{ab}{}^{de}\nabla^b\nabla_{[d}\Pi_{e]c}.$$

By applying the decomposition (4.32b) and (4.8), the above identities — i.e. (4.40) and (4.41) — and contracting with $h^a{}_p h^c{}_l$ we obtain after a few manipulations the evolution equation on the desired form,

$$u^b\nabla_b\Phi_{lp} + \epsilon_{pbm}\mathcal{D}^m\Psi_l{}^b = \mathcal{U}_{lp}, \tag{4.42}$$

with $\mathcal{U}_{ac}$ denoting the lower order terms —explicitly given by

$$\begin{aligned}\mathcal{U}_{lp} = &-a^a\Psi_l{}^b\epsilon_{pab} - a^a u_p\Phi_{la} - a^a u_l\Phi_{pa} + \epsilon_{pb}{}^m u^a R_{lcam}\Pi^{bc}\\ &+ \epsilon_{pc}{}^d u^a u^b u_l R_{ambd}\Pi^{cm} - \epsilon_p{}^{cm}u^a R_{abcm}\Pi_l{}^b - \epsilon_p{}^{cm}u^a R_{acbm}\Pi_l{}^b\\ &- \Phi_{lp}\chi^a{}_a - \Psi_b{}^c\epsilon_{pac}u_l\chi^{ab} - a_l\epsilon_{pac}\Pi_b{}^c\chi^{ab} + \Phi_{la}\chi^a{}_p.\end{aligned}$$

Equations (4.37) and (4.42) are of a form known to be symmetric hyperbolic — see [34] for a more detailed discussion. We note that we have made ample use of the suite `xAct`[h] to obtain $\mathcal{W}$ and $\mathcal{U}$.

Remark 4.10. *The presence of a spatial derivative of ρ in equations* (4.37) *means that an equation for ρ is necessary to ensure the hyperbolicity of the equations. Furthermore, the choice of ρ must at least be C^1. In other words our treatment does not allow for discontinuities in the matter source.*

Remark 4.11. *In the expressions for $\mathcal{W}_{ac}$ and $\mathcal{U}_{ac}$ it is understood that wherever $R^d{}_{abc}$ appears, it is to be evaluated using the decomposition in terms of the $C^d{}_{abc}$ and $\hat{L}_{ab}$.*

4.5.4 Evolution equations for the decomposed Weyl tensor

The construction of suitable evolution equations for the components of the Weyl tensor follows a similar approach as in the previous discussion.

[h] See http://www.xact.es for more information.

Again, the strategy is to decompose the Weyl tensor into parts orthogonal to the 4-velocity —i.e. one needs to understand the form the Weyl tensor takes on the orthogonal space to the 4-velocity.

Due to the symmetries of the Weyl tensor, the essential components are encoded in what are called the *electric* and *magnetic parts of the Weyl tensor* defined, respectively, as

$$E_{\boldsymbol{ac}} \equiv C_{\boldsymbol{ebfd}} u^{\boldsymbol{b}} u^{\boldsymbol{d}} h_{\boldsymbol{a}}{}^{\boldsymbol{e}} h_{\boldsymbol{c}}{}^{\boldsymbol{f}}, \qquad B_{\boldsymbol{ac}} \equiv C^{*}_{\boldsymbol{ebfd}} u^{\boldsymbol{b}} u^{\boldsymbol{d}} h_{\boldsymbol{a}}{}^{\boldsymbol{e}} h_{\boldsymbol{c}}{}^{\boldsymbol{f}},$$

where $C^{*}_{\boldsymbol{abcd}}$ denotes components the *Hodge dual* of the Weyl tensor. In terms of these spatial tensors, the frame components of the Weyl tensor and its dual admit the decomposition

$$C_{\boldsymbol{abcd}} = -2\left(l_{\boldsymbol{b}[\boldsymbol{c}} E_{\boldsymbol{d}]\boldsymbol{a}} - l_{\boldsymbol{a}[\boldsymbol{c}} E_{\boldsymbol{d}]\boldsymbol{b}}\right) - 2\left(u_{[\boldsymbol{c}} B_{\boldsymbol{d}]\boldsymbol{p}} \epsilon^{\boldsymbol{p}}{}_{\boldsymbol{ab}} + u_{[\boldsymbol{a}} B_{\boldsymbol{b}]\boldsymbol{p}} \epsilon^{\boldsymbol{p}}{}_{\boldsymbol{cd}}\right), \quad (4.43)$$

$$C^{*}_{\boldsymbol{abcd}} = 2u_{[\boldsymbol{a}} E_{\boldsymbol{b}]\boldsymbol{p}} \epsilon^{\boldsymbol{p}}{}_{\boldsymbol{cd}} - 4E^{\boldsymbol{p}}{}_{[\boldsymbol{a}} \epsilon_{\boldsymbol{b}]\boldsymbol{p}[\mathbf{c}} u_{\boldsymbol{d}]} - 4u_{[\boldsymbol{a}} B_{\boldsymbol{b}][\mathbf{c}} u_{\boldsymbol{d}]} - B_{\boldsymbol{pq}} \epsilon^{\boldsymbol{p}}{}_{\boldsymbol{ab}} \epsilon^{\boldsymbol{q}}_{\boldsymbol{cd}}, \quad (4.44)$$

—see e.g. [5] for details. For convenience we have written

$$l_{\boldsymbol{ab}} \equiv h_{\boldsymbol{ab}} - u_{\boldsymbol{a}} u_{\boldsymbol{b}}.$$

Similarly, we decompose the Friedrich tensor — defined by (4.12b) — and its dual in terms of their spatial, mixed and temporal components,

$$\begin{aligned}
F_{\boldsymbol{abcd}} = {} & u_{\boldsymbol{a}} u_{\boldsymbol{b}} u_{\boldsymbol{d}} F'_{\boldsymbol{000c}} - u_{\boldsymbol{b}} u_{\boldsymbol{d}} F'_{\boldsymbol{a00c}} - u_{\boldsymbol{a}} u_{\boldsymbol{d}} F'_{\boldsymbol{0b0c}} + u_{\boldsymbol{d}} F'_{\boldsymbol{ab0c}} \\
& - u_{\boldsymbol{a}} u_{\boldsymbol{b}} u_{\boldsymbol{c}} F'_{\boldsymbol{000d}} + u_{\boldsymbol{b}} u_{\boldsymbol{c}} F'_{\boldsymbol{a00d}} + u_{\boldsymbol{a}} u_{\boldsymbol{c}} F'_{\boldsymbol{0b0d}} - u_{\boldsymbol{c}} F'_{\boldsymbol{ab0d}} \qquad (4.45) \\
& + u_{\boldsymbol{a}} u_{\boldsymbol{b}} F'_{\boldsymbol{00cd}} - u_{\boldsymbol{b}} F'_{\boldsymbol{a0cd}} - u_{\boldsymbol{a}} F'_{\boldsymbol{0bcd}} + F'_{\boldsymbol{abcd}}
\end{aligned}$$

$$\begin{aligned}
F^{*}_{\boldsymbol{abcd}} = {} & u_{\boldsymbol{a}} u_{\boldsymbol{b}} u_{\boldsymbol{d}} F^{*\prime}{}_{\boldsymbol{000c}} - u_{\boldsymbol{b}} u_{\boldsymbol{d}} F^{*\prime}{}_{\boldsymbol{a00c}} - u_{\boldsymbol{a}} u_{\boldsymbol{d}} F^{*\prime}{}_{\boldsymbol{0b0c}} + u_{\boldsymbol{d}} F^{*\prime}{}_{\boldsymbol{ab0c}} \\
& - u_{\boldsymbol{a}} u_{\boldsymbol{b}} u_{\boldsymbol{c}} F^{*\prime}{}_{\boldsymbol{000d}} + u_{\boldsymbol{b}} u_{\boldsymbol{c}} F^{*\prime}{}_{\boldsymbol{a00d}} + u_{\boldsymbol{a}} u_{\boldsymbol{c}} F^{*\prime}{}_{\boldsymbol{0b0d}} - u_{\boldsymbol{c}} F^{*\prime}{}_{\boldsymbol{ab0d}} \qquad (4.46) \\
& + u_{\boldsymbol{a}} u_{\boldsymbol{b}} F^{*\prime}{}_{\boldsymbol{00cd}} - u_{\boldsymbol{b}} F^{*\prime}{}_{\boldsymbol{a0cd}} - u_{\boldsymbol{a}} F^{*\prime}{}_{\boldsymbol{0bcd}} + F^{*\prime}{}_{\boldsymbol{abcd}},
\end{aligned}$$

where $F^{*}_{\boldsymbol{abcd}}$ is defined in the usual way. In order to obtain evolution equations for $E_{\boldsymbol{ab}}$ and $B_{\boldsymbol{ab}}$, we make use of the following decomposition of the Bianchi identity (4.14d) and its dual:

$$F_{bcd} = u_b\left(F'_{0c0}u_d - F'_{0d0}u_c\right) + 2F'_{b0[c}u_{d]} - u_b F'_{0cd} + F'_{bcd}, \tag{4.47a}$$

$$F^*_{bcd} = u_b\left(F'^*_{0c0}u_d - F'^*_{0d0}u_c\right) + 2F'^*_{b0[c}u_{d]} - u_b F'^*_{0cd} + F'^*_{bcd}. \tag{4.47b}$$

The term $F_{a0b} = -F_{ab0}$ in equation (4.47a) gives the evolution equations for E_{ab}. Using the definition for F_{abc} and F_{abcd} — i.e. equation (4.12b) and (4.13b) — and using the decomposition (4.45) we get an expression (rather long) which involves terms of the Weyl tensor and its derivative. Consequently we make use of the decomposition (4.43). Furthermore, using the definition for the Schouten tensor — i.e. (2.16) — and observing that F_{abc} is a zero quantity, we finally obtain after a few manipulations,

$$\begin{aligned}
u^c\nabla_c E_{ab} - \epsilon_{aef}D^f B_b{}^e = {} & \tfrac{1}{2}\kappa\Psi_{ab} - a^c E_{bc}u_a - a^c E_{ac}u_b \\
& - \tfrac{1}{6}\kappa\Psi^c{}_c h_{ab} + a^c\epsilon_{cef}B_a{}^e h_b{}^f - \kappa a^c u_b \Pi_{ac} \\
& - \kappa a^c u_a \Pi_{bc} + \tfrac{1}{2}\kappa\Pi_{bc}\chi_a{}^c - \tfrac{1}{2}\kappa\rho\chi_{ba} - E_{ac}\chi_b{}^c \\
& - \tfrac{1}{2}\kappa\Pi_{ac}\chi_b{}^c + 2E_{bc}\chi^c{}_a - 2E_{ab}\chi^c{}_c + \tfrac{1}{6}\kappa\rho h_{ab}\chi^c{}_c \\
& - E_{cd}h_{ab}\chi^{cd} + \epsilon_{cdef}B_b{}^e h_a{}^f\chi^{cd} + \epsilon_{dfa}u_b B_e{}^f\chi^{de}.
\end{aligned} \tag{4.48}$$

Similarly, the symmetric part of the term F^*_{ab0} in equation (4.47b) gives the evolution equations for B_{ab}. The steps to obtain the evolution equation are almost identical to that for the field E_{ab} except that one substitute the dual equations (4.47b), (4.46) and (4.44) in place of (4.47a), (4.45) and (4.43). Another difference is that it is necessary to symmetrize about the free indices to obtain the final equation —namely,

$$\begin{aligned}
u^d\nabla_d B_{ab} + D^f E_{(b}{}^d\epsilon_{a)df} = {} & -\tfrac{1}{2}a^d E_b{}^f\epsilon_{adf} - \tfrac{1}{2}a^d E_a{}^f\epsilon_{bdf} \\
& - a^d u_b B_{ad} - a^d u_a B_{bd} - \tfrac{1}{2}\kappa\Phi_{ab} - \tfrac{1}{4}\kappa a^d\epsilon_{bdf}\Pi_a{}^f \\
& - \tfrac{1}{4}\kappa a^d\epsilon_{adf}\Pi_b{}^f + \tfrac{1}{2}B_{bd}\chi_a{}^d + \tfrac{1}{2}B_{ad}\chi_b{}^d + B_{bd}\chi^d{}_a \\
& + B_{ad}\chi^d{}_b - 2B_{ab}\chi^d{}_d - \tfrac{1}{2}E_f{}^c\epsilon_{bdc}u_a\chi^{df} \\
& - \tfrac{1}{2}E_f{}^c\epsilon_{adc}u_b\chi^{df} - B_{df}h_{ab}\chi^{df}.
\end{aligned} \tag{4.49}$$

In the above calculations we have also made use of the equations (4.33a), (4.33b) and (4.7b). We note that equations (4.48) and (4.49) are on the same form as the one given in [20] and constitutes a symmetric hyperbolic system of equations. We refer once again to [34] for an explicit discussion.

Remark 4.12. *The standard approach to show that equations* (4.48) *and* (4.49) *constitute a symmetric hyperbolic system ignores the tracefreeness of the fields $E_{\boldsymbol{ab}}$ and $B_{\boldsymbol{ab}}$ and list 12 components in a vector. Thus,* a posteriori *it is necessary to show that the fields are tracefree if they were so initially. This is discussed in Section 4.7.*

4.5.5 Summary

We summarise the results of the long computations of this section by the following proposition:

Proposition 4.13. The evolution equations for the matter fields as expressed by ρ and $\Pi_{\boldsymbol{ab}}$ are respectively given by (4.7b) and (4.33a), and $\Pi_{\boldsymbol{ab}}$ satisfy the constraints (4.7a) and (4.33b). Furthermore, the evolution equations for the geometric fields $e_{\boldsymbol{a}}{}^{\mu}$, $\Gamma_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}}$, $\Gamma_{\boldsymbol{i}}{}^{\boldsymbol{0}}{}_{\boldsymbol{j}}$, $\Gamma_{\boldsymbol{0}}{}^{\boldsymbol{0}}{}_{\boldsymbol{i}}$, $E_{\boldsymbol{ab}}$, $B_{\boldsymbol{ab}}$ and the auxiliary fields $\Psi_{\boldsymbol{ab}}$, $\Phi_{\boldsymbol{ab}}$ are given, respectively, by

$$\Sigma_{\boldsymbol{0}}{}^{\boldsymbol{e}}{}_{\boldsymbol{b}} = 0, \tag{4.50a}$$

$$\Delta^{\boldsymbol{d}}{}_{\boldsymbol{ca0}} = 0, \tag{4.50b}$$

$$J_{\boldsymbol{i0}} = 0, \tag{4.50c}$$

$$F_{\boldsymbol{ab0}} = 0, \tag{4.50d}$$

$$F^{*}_{(\boldsymbol{ab})\boldsymbol{0}} = 0, \tag{4.50e}$$

$$\nabla^{\boldsymbol{b}} N'_{\boldsymbol{cab}} = 0, \tag{4.50f}$$

$$\nabla^{\boldsymbol{b}} N'^{*}_{\boldsymbol{cab}} = 0. \tag{4.50g}$$

The above evolution equations constitutes a symmetric hyperbolic system. The remaining equations from (4.14a)-(4.14d) are considered constraint equations.

4.6 Propagation equations

In order to complete our analysis of the evolution system, we need to show that the equations that have been discarded in the process of hyperbolic reduction (i.e. the *constraints*) propagate. In this section we will, therefore, construct a subsidiary system for the zero-quantities $Q_{\boldsymbol{a}}$, $N_{\boldsymbol{abc}}$, $\Sigma^{\boldsymbol{e}}{}_{\boldsymbol{ab}}$, $\Delta^{\boldsymbol{d}}{}_{\mathbf{cab}}$ and $F_{\mathbf{abc}}$. The task is then to show that either the Lie derivative of the constraints vanish, or that it may be written in terms of zero-quantities. A key observation in this strategy is the fact that several of the zero-quantities can be regarded as differential forms with respect to a certain subset of their indices —thus, *Cartan's identity* can be readily be used to compute the Lie derivative in a very convenient way.

Remark 4.14. *In what follows one should be careful when evaluating the covariant derivative of a tensor fields in frame coordinates. The following order should be employed: first, evaluate the tensorial expression for the derivative of the tensor, then write the expression in a frame basis, and lastly do any contractions if necessary — e.g contracting with the four velocity.*

4.6.1 Propagation of Divergence-free condition

The divergence free condition gives rise to two equations: on the one hand, equation (4.85b) is an evolution equation for ρ and the other hand equation (4.85a) a constraint for $\boldsymbol{\Pi}$. As such, the latter needs to be shown to hold on the whole space time if satisfied on an initial hypersurface.

We first define the zero quantity

$$Q_{\boldsymbol{b}} \equiv \nabla^{\boldsymbol{a}} \Pi_{\boldsymbol{ab}} + a_{\boldsymbol{b}}\rho - u_{\boldsymbol{b}} \Pi_{\boldsymbol{ac}} \chi^{\boldsymbol{ac}}.$$

As is obvious from the above, $Q_{\boldsymbol{b}} = 0$ must hold for the Einstein equations to be satisfied. By contracting with u^a it is readily shown that $Q_{\boldsymbol{0}} = Q_{\boldsymbol{b}} u^{\boldsymbol{b}} = 0$. Thus it is sufficient to only consider $Q_{\boldsymbol{i}}$ in what follows. A simple calculation shows that

$$\begin{aligned} 2\nabla_{[\boldsymbol{d}} Q_{\boldsymbol{b}]} = {} & 2\nabla^{\boldsymbol{a}} Z_{\boldsymbol{adb}} + 2\nabla_{[\boldsymbol{d}} \left(a_{\boldsymbol{b}]} \rho \right) + 2R_{\boldsymbol{e}}{}^{\boldsymbol{a}}{}_{[\boldsymbol{d}|\boldsymbol{a}|} \Pi_{\boldsymbol{b}]}{}^{\boldsymbol{e}} \\ & - 2\nabla_{[\boldsymbol{d}} \left(u_{\boldsymbol{b}]} \Pi_{\boldsymbol{ac}} \chi^{\boldsymbol{ac}} \right) + 4\Sigma_{[\boldsymbol{d}}{}^{\boldsymbol{c}}{}_{|\boldsymbol{a}|} \nabla_{\boldsymbol{c}} \Pi_{\boldsymbol{b}]}{}^{\boldsymbol{a}}, \end{aligned}$$

where we have used the commutation property of the connection followed by the definition of the Z-tensor, as well as

$$2R^{\boldsymbol{e}}{}_{[\boldsymbol{ad}]\boldsymbol{a}} \Pi_{\boldsymbol{e}}{}^{\boldsymbol{a}} = 0.$$

By using equation (4.29) and multiplying through with $u^{\boldsymbol{d}}$, followed by applying equations (4.12a) and (4.30), we obtain the propagation equation

$$u^{\boldsymbol{a}} \nabla_{\boldsymbol{a}} Q_{\boldsymbol{i}} = -Q^{\boldsymbol{j}} \chi_{\boldsymbol{ij}} + \Delta_{\boldsymbol{0}}{}^{\boldsymbol{kj}}{}_{\boldsymbol{k}} \Pi_{\boldsymbol{ij}} + 2\Sigma_{\boldsymbol{i}}{}^{\boldsymbol{l}}{}_{\boldsymbol{j}} \Pi_{\boldsymbol{k}}{}^{\boldsymbol{j}} \chi_{\boldsymbol{l}}{}^{\boldsymbol{k}} + 2\Sigma_{\boldsymbol{i}}{}^{\boldsymbol{0}}{}_{\boldsymbol{j}} \Pi_{\boldsymbol{k}}{}^{\boldsymbol{j}} a^{\boldsymbol{k}}. \tag{4.51}$$

Note that $\rho_{\boldsymbol{j}}{}^{\boldsymbol{k}}{}_{\boldsymbol{0k}} = 0$ due to the divergence free property of Weyl and $T_{\boldsymbol{0i}} = 0$ as a consequence of the gauge. We have also made use of the evolution equation $\Sigma_{\boldsymbol{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{a}} = 0$.

4.6.2 Propagation equations for the torsion

For fixed value of the index $\boldsymbol{e}$, the torsion $\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{b}}$ can be regarded as the components of a 2-form —namely, one has that

$$\boldsymbol{\Sigma}^{\boldsymbol{e}} \equiv \Sigma_{[\boldsymbol{b}}{}^{\boldsymbol{e}}{}_{\boldsymbol{c}]} \boldsymbol{\omega}^{\boldsymbol{b}} \otimes \boldsymbol{\omega}^{\boldsymbol{c}}.$$

Using *Cartan's identity* to compute its Lie derivative along the vector u^a one finds that

$$\mathcal{L}_{\boldsymbol{u}} \boldsymbol{\Sigma}^{\boldsymbol{e}} = i_{\boldsymbol{u}} \mathrm{d} \boldsymbol{\Sigma}^{\boldsymbol{e}} + \mathrm{d}(i_{\boldsymbol{u}} \boldsymbol{\Sigma}^{\boldsymbol{e}}). \tag{4.52}$$

The second term in the right-hand side of the above equation can be seen to vanish as a consequence of the evolution equations (cf. Remark 4.7) while the first one involves the exterior derivative of the torsion which can be manipulated using the general form of the Bianchi identity. For clarity, these computations are done explicitly using frame index notation.

Following the general discussion given above consider the expression $\nabla_{[\mathbf{0}}\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}]}$ which roughly corresponds to the first term in the right-hand side of equation (4.52). Expanding the expression one readily finds that

$$
\begin{aligned}
3\nabla_{[\mathbf{0}}\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}]} &= \nabla_{\mathbf{0}}\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} + \nabla_{\boldsymbol{a}}\Sigma_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\mathbf{0}} + \nabla_{\boldsymbol{b}}\Sigma_{\mathbf{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{a}} \\
&= \nabla_{\mathbf{0}}\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} - \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\mathbf{0}}\Sigma_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{e}} - \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{e}}{}_{\mathbf{0}}\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{e}}.
\end{aligned}
$$

Now, we compute $\nabla_{[\mathbf{0}}\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}]}$ in a different way using the general expression for the first Bianchi identity (i.e. the form this identity takes in the presence of torsion):

$$
R^{\boldsymbol{d}}{}_{[\boldsymbol{cab}]} = -\nabla_{[\boldsymbol{a}}\Sigma_{\boldsymbol{b}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}]} - \Sigma_{[\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{b}}\Sigma_{\boldsymbol{c}]}{}^{\boldsymbol{d}}{}_{\boldsymbol{e}}.
$$

Setting $\boldsymbol{a} = 0$ and making use of the zero-quantity defined in (4.12a) to eliminate the components of the Riemann curvature tensor one finds that

$$
\begin{aligned}
3\nabla_{[\mathbf{0}}\Sigma_{\boldsymbol{b}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}]} &= -\Delta^{\boldsymbol{d}}{}_{[\boldsymbol{c}\mathbf{0}\boldsymbol{b}]} - \Sigma_{[\mathbf{0}}{}^{\boldsymbol{e}}{}_{\boldsymbol{b}}\Sigma_{\boldsymbol{c}]}{}^{\boldsymbol{d}}{}_{\boldsymbol{e}} \\
&= -\Delta^{\boldsymbol{d}}{}_{\mathbf{0}\boldsymbol{bc}}
\end{aligned}
$$

where we have used the fact that

$$
\rho^{\boldsymbol{d}}{}_{[\boldsymbol{cab}]} = 0,
$$

and the evolution equations (4.50a) and (4.50b) in the last step. From the above discussion it follows that the *propagation equation* is

$$
\nabla_{\mathbf{0}}\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\mathbf{0}}\Sigma_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{e}} + \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{e}}{}_{\mathbf{0}}\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{e}} - \Delta^{\boldsymbol{d}}{}_{\mathbf{0}\boldsymbol{bc}}. \tag{4.53}
$$

> Remark 4.15. *The main structural feature of equation* (4.53) *is the fact that it is homogeneous in the zero-quantities* $\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$ *and* $\Delta^{\boldsymbol{d}}{}_{\boldsymbol{abc}}$.

4.6.3 Propagation equations for the geometric curvature

Next we turn to equation (4.14c). For this we observe that the zero-quantity $\Delta^{\boldsymbol{d}}{}_{\boldsymbol{cab}}$ for fixed values of $\boldsymbol{d}$ and $\boldsymbol{c}$ can be regarded as the components of a 2-form on the indices $\boldsymbol{a}$ and $\boldsymbol{b}$. Using again Cartan's identity one finds that

$$\mathcal{L}_{\mathbf{u}}\boldsymbol{\Delta}^{\boldsymbol{d}}{}_{\boldsymbol{c}} = i_{\boldsymbol{u}}\mathrm{d}\boldsymbol{\Delta}^{\boldsymbol{d}}{}_{\boldsymbol{c}} + \mathrm{d}\left(i_{\boldsymbol{u}}\boldsymbol{\Delta}^{\boldsymbol{d}}{}_{\boldsymbol{c}}\right).$$

Now, the last term in the right-hand side vanishes due to the evolution equation for the connection coefficients (see Remark 4.8), while the first term takes the form

$$i_{\mathbf{u}}\mathrm{d}\boldsymbol{\Delta}^{\boldsymbol{d}}{}_{\boldsymbol{c}} = \nabla_{[\boldsymbol{0}}\Delta^{\boldsymbol{d}}{}_{|\boldsymbol{c}|\boldsymbol{ab}]}\boldsymbol{\omega}^{\boldsymbol{a}} \otimes \boldsymbol{\omega}^{\boldsymbol{b}}.$$

As in the case of the torsion, the strategy is to rewrite this expression in terms of zero-quantities only. For convenience in the following calculations we set

$$S_{\boldsymbol{ab}}{}^{\boldsymbol{cd}} \equiv \delta_{\boldsymbol{a}}{}^{\boldsymbol{c}}\delta_{\boldsymbol{b}}{}^{\boldsymbol{d}} + \delta_{\boldsymbol{a}}{}^{\boldsymbol{d}}\delta_{\boldsymbol{b}}{}^{\boldsymbol{c}} - \eta_{\boldsymbol{ab}}\eta^{\boldsymbol{cd}}. \tag{4.54}$$

From equations (2.14) and (4.13a) it readily follows that

$$\nabla_{[\boldsymbol{a}}\Delta^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]} = \nabla_{[\boldsymbol{a}}C^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]} - S^{\boldsymbol{df}}{}_{\boldsymbol{e}[\boldsymbol{b}}\nabla_{\boldsymbol{a}}\hat{L}_{\boldsymbol{c}]\boldsymbol{f}} - \Sigma^{\boldsymbol{f}}{}_{[\boldsymbol{ab}}R^{\boldsymbol{d}}_{|\boldsymbol{e}|\boldsymbol{c}]\boldsymbol{f}}.$$

To simplify the calculations, we multiply by $\epsilon_{\boldsymbol{l}}{}^{\boldsymbol{abc}}$. The first term yields

$$\begin{aligned}
\epsilon_{\boldsymbol{l}}{}^{\boldsymbol{abc}}\nabla_{[\boldsymbol{a}}C^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]} &= \epsilon_{\boldsymbol{l}}^{\boldsymbol{abc}}\nabla_{\boldsymbol{a}}C^{\boldsymbol{d}}{}_{\boldsymbol{ebc}} \\
&= \nabla_{\boldsymbol{a}}\left(\epsilon_{\boldsymbol{l}}{}^{\boldsymbol{abc}}C^{\boldsymbol{d}}{}_{\boldsymbol{ebc}}\right) \\
&= 2\nabla_{\boldsymbol{a}}C^{*\boldsymbol{d}}{}_{\boldsymbol{el}}{}^{\boldsymbol{a}} \\
&= -2\nabla_{\boldsymbol{a}}C^{*\boldsymbol{a}}{}_{\boldsymbol{l}}{}^{\boldsymbol{d}}{}_{\boldsymbol{e}}.
\end{aligned} \tag{4.55}$$

In the above, we have used that

$$\nabla_{\boldsymbol{a}}\epsilon_{\boldsymbol{l}}{}^{\boldsymbol{abc}} = 0.$$

The second term require a few more steps. Using the definition (4.54) we have

$$\begin{aligned}
\epsilon_l{}^{abc} S_{e[b}{}^{df}\nabla_a \hat{L}_{c]f} &= \epsilon_l{}^{abc}\delta_e{}^d\delta_b{}^f\nabla_a\hat{L}_{cf} + \epsilon_l{}^{abc}\delta_e{}^f\delta_b{}^d\nabla_a\hat{L}_{cf} \\
&\quad - \epsilon_l{}^{abc}\eta^{\mathbf{df}}\eta_{eb}\nabla_a\hat{L}_{cf} \\
&= \delta_e{}^d\nabla_a(\epsilon_l{}^{abc}\hat{L}_{cf}) \\
&\quad + \epsilon_l{}^{afc}\delta_e{}^f\nabla_a\hat{L}_{cf} - \epsilon_l{}^a{}_e{}^c\eta^{df}\nabla_a\hat{L}_{cf} \\
&= -\epsilon_l{}^{dac}\nabla_a\hat{L}_{ce} + \epsilon_{le}{}^{ac}\eta^{df}\nabla_a\hat{L}_{cf}
\end{aligned} \tag{4.56}$$

where we have made use of the symmetry of the Schouten tensor in the last step. Now, from equations (4.12b), (4.13b) and (4.14d) we readily obtain that

$$\epsilon_l{}^{dbc}\nabla_{[b}\hat{L}_{c]a} = -\epsilon_l{}^{dbc}F_{abc} + \epsilon_l{}^{dbc}\nabla_m C^m{}_{abc}.$$

Plugging this result back into (4.56), we obtain after making use of the definition for the dual that

$$\begin{aligned}
\epsilon_l{}^{abc} S_{e[b}{}^{df}\nabla_a \hat{L}_{c]f} &= \epsilon_l{}^{dac}F_{eac} - \eta^{df}\epsilon_{le}{}^{ac}F_{fac} \\
&\quad + \eta^{df}2\nabla_a C^{*a}{}_{fle} - 2\nabla_a C^{*bma}{}_{el}{}^d.
\end{aligned} \tag{4.57}$$

Putting the result for calculation (4.57) and (4.55) together, we obtain

$$\begin{aligned}
\epsilon_l{}^{abc}\nabla_a\Delta^d{}_{ebc} &= \epsilon_l{}^{dac}F_{eac} - \eta^{df}\epsilon_{le}{}^{ac}F_{fac} \\
&\quad + \eta^{df}\left(-2\nabla_a C^{*a}{}_{lfe} + 2\nabla_a C^{*a}{}_{fle} - 2\nabla_a C^{*a}{}_{elf}\right) \\
&= \epsilon_l{}^{dac}F_{eac} - \eta^{df}\epsilon_{le}{}^{ac}F_{fac} \\
&\quad - 2\eta^{df}\left(\nabla_a C^{*a}{}_{lfe} + \nabla_a C^{*a}{}_{fel} + \nabla_a C^{*a}{}_{elf}\right) \\
&= \epsilon_l{}^{dac}F_{eac} - \eta^{df}\epsilon_{le}{}^{ac}F_{fac},
\end{aligned}$$

where we made use of equation (2.13) in the last step. Multiplying by $\epsilon^l{}_{mnp}$, we recover the equation in its original form. Thus, the right-hand side of the propagation equation for $\Delta^d{}_{c[ab]}$ is given by

$$\nabla_{[\mathbf{0}}\Delta^d{}_{|e|bc]} = -\eta_{e[\mathbf{0}}F^d{}_{bc]} + \eta^d{}_{[\mathbf{0}}F_{|e|bc]}. \tag{4.58}$$

But we also have that,

$$\nabla_{[\mathbf{0}}\Delta^d{}_{|e|bc]} = \nabla_{\mathbf{0}}\Delta^d{}_{ebc} - \Gamma_c{}^f{}_{\mathbf{0}}\Delta^d{}_{efb} - \Gamma_b{}^f{}_{\mathbf{0}}\Delta^d{}_{ecf}.$$

Plugging the above result back into equation (4.58), we obtain the final propagation equation

$$\nabla_{\mathbf{0}}\Delta^{\boldsymbol{d}}{}_{\boldsymbol{ebc}} = -\eta_{\boldsymbol{e}[\mathbf{0}}F^{\boldsymbol{d}}{}_{\boldsymbol{bc}]} + \eta^{\boldsymbol{d}}{}_{[\mathbf{0}}F_{|\boldsymbol{e}|\boldsymbol{bc}]} + \Gamma_{\boldsymbol{c}}{}^{\boldsymbol{f}}{}_{\mathbf{0}}\Delta^{\boldsymbol{d}}{}_{\boldsymbol{efb}} + \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\mathbf{0}}\Delta^{\boldsymbol{d}}{}_{\boldsymbol{ecf}}. \tag{4.59}$$

> Remark 4.16. *As in the case of the propagation equation for the torsion the main conclusion of the previous discussion is that the propagation equation for the zero-quantitity $\Delta^{\boldsymbol{d}}{}_{\boldsymbol{abc}}$ is homogeneous on zero-quantities.*

4.6.4 Propagation of the N-tensor

It is also necessary to show that $N_{\boldsymbol{abc}}$ —see equation (4.12c)— propagates. The strategy will be different than what has been employed in the above discussions; rather, we will follow the strategy employed for the propagation of the Friedrich tensor in [11].

In the subsequent discussion we shall make use of the observation that,

$$\nabla^{\boldsymbol{b}}N'^{*}{}_{\boldsymbol{cab}} = 0, \qquad N'{}_{\boldsymbol{c0b}} = 0, \qquad N'^{*}{}_{\boldsymbol{c0b}} = 0.$$

respectively are equivalent to the evolution equations for $\Psi_{\boldsymbol{ab}}$, $\Phi_{\boldsymbol{ab}}$ and $\Pi_{\boldsymbol{ab}}$ and the constraint equation as given in (4.33b). Furthermore, we define the fields,

$$\xi_{\boldsymbol{ab}} \equiv N'^{*}{}_{\boldsymbol{ab0}}, \qquad \lambda_{\boldsymbol{ab}} \equiv N'{}_{\boldsymbol{0ab}}.$$

By decomposing $N^{*}{}_{\boldsymbol{abc}}$ in terms of the fields $\lambda_{\boldsymbol{ab}}$ and $\xi_{\boldsymbol{ab}}$, we have

$$N^{*}{}_{\boldsymbol{cab}} = \xi_{\boldsymbol{ca}}u_{\boldsymbol{b}} - \xi_{\boldsymbol{cb}}u_{\boldsymbol{a}} + 3\lambda_{\boldsymbol{de}}u_{[\boldsymbol{b}}\epsilon_{\boldsymbol{a}]}{}^{\boldsymbol{de}}u_{\boldsymbol{c}} \tag{4.60}$$

Using the symmetry relation

$$N_{[\boldsymbol{abc}]} = 0,$$

we obtain the expression

$$\lambda_{\boldsymbol{ab}} = N'_{\boldsymbol{ba0}} - N'_{\boldsymbol{ab0}}.$$

But from the evolution equation for $\Pi_{\boldsymbol{ab}}$, we have that $N'_{\boldsymbol{ba0}} = 0$, thus

$$\lambda_{\boldsymbol{ab}} = 0.$$

Applying the above result, and the divergence in equation (4.60) we obtain,

$$u_{\boldsymbol{b}}\nabla^{\boldsymbol{b}}\xi_{\boldsymbol{cd}} = \xi_{\boldsymbol{ca}}a^{\boldsymbol{a}}u_{\boldsymbol{d}} - \xi_{\boldsymbol{cd}}\chi + \xi_{\boldsymbol{cb}}\chi^{\boldsymbol{b}}{}_{\boldsymbol{d}}.$$

To obtain the above we have used the evolution equation for $\Phi_{\boldsymbol{ab}}$ and multiplied through with the projector $h_{\boldsymbol{d}}{}^{\boldsymbol{a}}$ to get rid of a divergence. This is permitted as the field $\xi_{\boldsymbol{ab}}$ is spatial. Thus, we have established the following lemma:

Lemma 2. If the constraint $\xi_{\boldsymbol{ab}} = 0$ — equivalently equation (4.33b) — holds initially, and under the assumption that the evolution equations (4.33a) and (4.50g) holds everywhere on $\mathcal{M}$, then the relation

$$Z_{\boldsymbol{cab}} = 2\nabla_{[\boldsymbol{a}}\Pi_{\boldsymbol{b}]\boldsymbol{c}},$$

also holds everywhere on $\mathcal{M}$.

4.6.5 Propagation equations for the Bianchi identity

Lastly, we need to show propagation of the Bianchi identity, equation (4.14d). Again the strategy is to use the decomposition of the Friedrich tensor and its dual and use the divergence to obtain propagation equations for the constraints.

First, we shall express the divergence of $F_{\boldsymbol{abc}}$ in terms of known zero quantities. Making use of the antisymmetry property of the Weyl tensor about the indices $\boldsymbol{a}$ and $\boldsymbol{b}$, We have,

$$2\nabla^b F_{bcd} = 2\nabla^{[b}\nabla^{a]}C_{abcd} - \nabla^b\nabla_c\hat{L}_{db} + \nabla^b\nabla_d\hat{L}_{cb}.$$

By virtue of the commutator as defined in (2.9), we obtain,

$$\begin{aligned}2\nabla^b F_{bcd} = &-2R^l{}_a{}^{ba}C_{lbcd} - 2R^l{}_b{}^{ba}C_{alcd} - 2R^l{}_c{}^{ba}C_{abld} \\ &- 2R^l{}_d{}^{ba}C_{abcl} + 2R^l{}_d{}^b{}_c\hat{L}_{lb} + 2R^l{}_b{}^b{}_c\hat{L}_{dl} \\ &- R^l{}_c{}^b{}_d\hat{L}_{lb} - R^l{}_b{}^b{}_d\hat{L}_{cl} + 4\Sigma_a{}^l{}_b\nabla_l C^{ab}{}_{cd} \\ &+ \nabla^b\nabla_d\hat{L}_{cb} - \nabla^b\nabla_c\hat{L}_{db}.\end{aligned} \tag{4.61}$$

Similarly, we may rewrite the terms involving the Schouten tensor in the following way:

$$\begin{aligned}\nabla^b\nabla_c\hat{L}_{db} &= \eta^{eb}\left(\nabla_{[e}\nabla_{c]}\hat{L}_{db} + \nabla_c\nabla_e\hat{L}_{db}\right) \\ &= \eta^{eb}\left(-R^l{}_{dec}\hat{L}_{lb} - R^l{}_{bec}\hat{L}_{dl} + \nabla_c\nabla_e\hat{L}_{db}\right) \\ &= -R^l{}_d{}^b{}_c\hat{L}_{lb} - R^l{}_b{}^b{}_c\hat{L}_{dl} + \nabla_c\nabla^b\hat{L}_{db}.\end{aligned} \tag{4.62}$$

Substituting the above result into equation (4.61), we obtain

$$\begin{aligned}2\nabla^b F_{bcd} = &-2R^l{}_a{}^{ba}C_{lbcd} - 2R^l{}_b{}^{ba}C_{alcd} - 2R^l{}_c{}^{ba}C_{abld} \\ &- 2R^l{}_d{}^{ba}C_{abcl} + 2R^l{}_d{}^b{}_c\hat{L}_{lb} + 2R^l{}_b{}^b{}_c\hat{L}_{dl} \\ &- R^l{}_c{}^b{}_d\hat{L}_{lb} - R^l{}_b{}^b{}_d\hat{L}_{cl} + \nabla_d\nabla^b\hat{L}_{cb} - \nabla_c\nabla^b\hat{L}_{db}.\end{aligned} \tag{4.63}$$

Solving equation (4.12a) for the Riemann tensor and substituting into equation (4.63), results in

$$\begin{aligned}2\nabla^b F_{bcd} = &-2\Delta^l{}_a{}^{ba}C_{lbcd} - 2\Delta^l{}_b{}^{ba}C_{alcd} - 2\Delta^l{}_c{}^{ba}C_{abld} \\ &- 2\Delta^l{}_d{}^{ba}C_{abcl} + -2\rho^l{}_a{}^{ba}C_{lbcd} - 2\rho^l{}_b{}^{ba}C_{alcd} \\ &- 2\rho^l{}_c{}^{ba}C_{abld} - 2\rho^l{}_d{}^{ba}C_{abcl} + \Delta^l{}_d{}^b{}_c\hat{L}_{lb} + \Delta^l{}_b{}^b{}_c\hat{L}_{dl} \\ &- \Delta^l{}_c{}^b{}_d\hat{L}_{lb} - \Delta^l{}_b{}^b{}_d\hat{L}_{cl} + \rho^l{}_d{}^b{}_c\hat{L}_{lb} + \rho^l{}_b{}^b{}_c\hat{L}_{dl} \\ &- \rho^l{}_c{}^b{}_d\hat{L}_{lb} - \rho^l{}_b{}^b{}_d\hat{L}_{cl} + \nabla_d\nabla^b\hat{L}_{cb} - \nabla_c\nabla^b\hat{L}_{db}.\end{aligned} \tag{4.64}$$

Next we evaluate each term involving $\rho^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}$. First, we have

$$
\begin{aligned}
\rho^{\boldsymbol{l}}{}_{\boldsymbol{a}}{}^{\boldsymbol{ba}} C_{\boldsymbol{lbcd}} &= C^{\boldsymbol{l}}{}_{\boldsymbol{a}}{}^{\boldsymbol{ba}} C_{\boldsymbol{lbcd}} + S^{\boldsymbol{le}}{}_{\boldsymbol{a}[\boldsymbol{m}} \hat{L}_{\boldsymbol{n}]\boldsymbol{e}} \eta^{\boldsymbol{mb}} \eta^{\boldsymbol{na}} C_{\boldsymbol{lbcd}} \\
&= \left(\delta_{\boldsymbol{a}}{}^{\boldsymbol{l}} \delta_{\boldsymbol{m}}{}^{\boldsymbol{e}} + \delta_{\boldsymbol{a}}{}^{\boldsymbol{e}} \delta_{\boldsymbol{m}}{}^{\boldsymbol{l}} - \eta^{\boldsymbol{le}} \eta_{\boldsymbol{am}}\right) \hat{L}_{\boldsymbol{ne}} \eta^{\boldsymbol{mb}} \eta^{\boldsymbol{na}} C_{\boldsymbol{lbcd}} \\
&\quad - \left(\delta_{\boldsymbol{a}}{}^{\boldsymbol{l}} \delta_{\boldsymbol{n}}{}^{\boldsymbol{e}} + \delta_{\boldsymbol{a}}{}^{\boldsymbol{e}} \delta_{\boldsymbol{n}}{}^{\boldsymbol{l}} - \eta^{\boldsymbol{le}} eta_{\boldsymbol{an}}\right) \hat{L}_{\boldsymbol{me}} \eta^{\boldsymbol{mb}} \eta^{\boldsymbol{na}} C_{\boldsymbol{lbcd}} \\
&= \hat{L}^{\boldsymbol{a}}{}_{\boldsymbol{e}} C_{\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{cd}} + \hat{L}^{\boldsymbol{a}}{}_{\boldsymbol{e}} C_{\boldsymbol{m}}{}^{\boldsymbol{m}}{}_{\boldsymbol{cd}} \delta^{\boldsymbol{e}}{}_{\boldsymbol{a}} - \hat{L}^{\boldsymbol{a}}{}_{\boldsymbol{e}} C^{\boldsymbol{e}}{}_{\boldsymbol{acd}} \\
&\quad - \hat{L}^{\boldsymbol{b}}{}_{\boldsymbol{e}} C^{\boldsymbol{e}}{}_{\boldsymbol{bcd}} - \hat{L}^{\boldsymbol{b}}{}_{\boldsymbol{e}} C^{\boldsymbol{e}}{}_{\boldsymbol{bcd}} + 4\hat{L}^{\boldsymbol{b}}{}_{\boldsymbol{e}} C^{\boldsymbol{e}}{}_{\boldsymbol{bcd}} \\
&= 0,
\end{aligned}
\tag{4.65}
$$

where we have used that $S_{\mathbf{ab}} = S_{(\mathbf{ab})}$ and $C_{\mathbf{abcd}} = C_{[\mathbf{ab}][\mathbf{cd}]}$, as well as the trace-free property of the Weyl tensor. Using the same approach as above, we find that

$$
\rho^{\boldsymbol{l}}{}_{\boldsymbol{c}}{}^{\boldsymbol{ba}} C_{\boldsymbol{abld}} = C^{\boldsymbol{l}}{}_{\boldsymbol{c}}{}^{\boldsymbol{ba}} C_{\boldsymbol{abld}} + \hat{L}^{\boldsymbol{bl}} C_{\boldsymbol{cbld}} - \hat{L}^{\boldsymbol{al}} C_{\boldsymbol{acld}}, \tag{4.66}
$$

$$
\rho^{\boldsymbol{l}}{}_{\boldsymbol{d}}{}^{\boldsymbol{ba}} C_{\boldsymbol{ablc}} = -C^{\boldsymbol{l}}{}_{\boldsymbol{d}}{}^{\boldsymbol{ba}} C_{\boldsymbol{ablc}} + \hat{L}^{\boldsymbol{bl}} C_{\boldsymbol{dblc}} - \hat{L}^{\boldsymbol{al}} C_{\boldsymbol{adlc}}. \tag{4.67}
$$

Interchanging $\boldsymbol{c}$ and $\boldsymbol{l}$ in equation (4.67), will change the overall sign, and thus when we subtract with equation (4.66) we obtain

$$
\begin{aligned}
\rho^{\boldsymbol{l}}{}_{\boldsymbol{d}}{}^{\boldsymbol{ba}} C_{\boldsymbol{ablc}} - \rho^{\boldsymbol{l}}{}_{\boldsymbol{c}}{}^{\boldsymbol{ba}} C_{\boldsymbol{abld}} &= C^{\boldsymbol{l}}{}_{\boldsymbol{c}}{}^{\boldsymbol{ba}} C_{\boldsymbol{abld}} - C^{\boldsymbol{l}}{}_{\boldsymbol{d}}{}^{\boldsymbol{ba}} C_{\boldsymbol{ablc}} \\
&= C_{\boldsymbol{lcba}} C^{\boldsymbol{abl}}{}_{\boldsymbol{d}} - C^{\boldsymbol{l}}{}_{\boldsymbol{d}}{}^{\boldsymbol{ba}} C_{\boldsymbol{ablc}} \\
&= C_{\boldsymbol{balc}} C^{\boldsymbol{l}}{}_{\boldsymbol{d}}{}^{\boldsymbol{ab}} - C^{\boldsymbol{l}}{}_{\boldsymbol{d}}{}^{\boldsymbol{ba}} C_{\boldsymbol{ablc}} \\
&= C^{\boldsymbol{l}}{}_{\boldsymbol{d}}{}^{\boldsymbol{ba}} C_{\boldsymbol{ablc}} - C^{\boldsymbol{l}}{}_{\boldsymbol{d}}{}^{\boldsymbol{ba}} C_{\boldsymbol{ablc}} \\
&= 0.
\end{aligned}
\tag{4.68}
$$

Furthermore, we find that

$$
\begin{aligned}
\rho^{\boldsymbol{l}}{}_{\boldsymbol{b}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} \hat{L}_{\boldsymbol{dl}} &= -2\hat{L}_{\boldsymbol{c}}{}^{\boldsymbol{l}} \hat{L}_{\boldsymbol{dl}} - \hat{L}_{\boldsymbol{b}}{}^{\boldsymbol{b}} \hat{L}_{\boldsymbol{dc}} \\
\rho^{\boldsymbol{l}}{}_{\boldsymbol{b}}{}^{\boldsymbol{b}}{}_{\boldsymbol{d}} \hat{L}_{\boldsymbol{cl}} &= -2\hat{L}_{\boldsymbol{d}}{}^{\boldsymbol{l}} \hat{L}_{\boldsymbol{cl}} - \hat{L}_{\boldsymbol{b}}{}^{\boldsymbol{b}} \hat{L}_{\boldsymbol{cd}}.
\end{aligned}
$$

Consequently, subtracting one from the other leads to

$$\begin{aligned}
\rho^{l}{}_{b}{}^{b}{}_{c}\hat{L}_{dl} - \rho^{l}{}_{b}{}^{b}{}_{d}\hat{L}_{cl} &= 2\hat{L}_{d}{}^{l}\hat{L}_{cl} - 2\hat{L}_{c}{}^{l}\hat{L}_{dl} \\
&= 2\hat{L}_{dl}\hat{L}_{c}{}^{l} - 2\hat{L}_{c}{}^{l}\hat{L}_{d_l} \\
&= 0.
\end{aligned} \tag{4.69}$$

Finally, we have that

$$\begin{aligned}
\rho^{l}{}_{d}{}^{b}{}_{c}\hat{L}_{lb} &= \hat{L}_{m}{}^{l}\hat{L}_{l}{}^{m}\eta_{cd} - \hat{L}_{cd}\hat{L}^{l}{}_{l}, \\
\rho^{l}{}_{c}{}^{b}{}_{d}\hat{L}_{lb} &= \hat{L}_{m}{}^{l}\hat{L}_{l}{}^{m}\eta_{dc} - \hat{L}_{dc}\hat{L}^{l}{}_{l}.
\end{aligned}$$

A straight forward calculation shows that

$$\rho^{l}{}_{d}{}^{b}{}_{c}\hat{L}_{lb} - \rho^{l}{}_{c}{}^{b}{}_{d}\hat{L}_{lb} = 0, \tag{4.70}$$

due to the symmetry of the Schouten and the metric tensor. Substituting the above results back into equation (4.64), we obtain

$$\begin{aligned}
\nabla^{b}F_{bcd} = {} & -\Delta^{l}{}_{a}{}^{ba}C_{lbcd} - \Delta^{l}{}_{b}{}^{ba}C_{alcd} - \Delta^{l}{}_{c}{}^{ba}C_{abld} - \Delta^{l}{}_{d}{}^{ba}C_{abcl} \\
& + \Delta^{l}{}_{d}{}^{b}{}_{c}\hat{L}_{lb} + \Delta^{l}{}_{b}{}^{b}{}_{c}\hat{L}_{dl} - \Delta^{l}{}_{c}{}^{b}{}_{d}\hat{L}_{lb} - \Delta^{l}{}_{b}{}^{b}{}_{d}\hat{L}_{cl} + \nabla_{[d}Q_{c]} \\
& + 4\Sigma_{a}{}^{l}{}_{b}\nabla_{l}C^{ab}{}_{cd} - 2\Sigma_{b}{}^{l}{}_{c}\nabla_{l}S_{d}{}^{b} + 2\Sigma_{b}{}^{l}{}_{c}\nabla_{l}S_{c}{}^{b} + \frac{1}{3}\Sigma^{e}{}_{dc}\nabla_{e}T,
\end{aligned} \tag{4.71}$$

where Q_a is the zero quantity defined in section 4.6.1. Following the strategy outlined in [11], we define the fields

$$p_a \equiv F'_{0a0}, \qquad q_a \equiv F^{*\prime}_{0a0},$$

which encodes the information of the constraint equations of F_{abc} and F^{*}_{abc}, respectively. Thus, the aim is to find evolution equations for p_a and q_a. In terms of the above fields the decomposition (4.47a) takes the form

$$F_{bcd} = 2u_b p_{[c}u_{d]} + h_{b[d}p_{c]} - \frac{1}{2}u_b q_e \epsilon_{cd}{}^{e}, \tag{4.72}$$

where we have used

$$F'_{\mathbf{0}bd} = q_e \epsilon_{bd}{}^{e}, \qquad F'_{bcd} = h_{b[d}p_{c]} + \frac{1}{2}u_b q_e \epsilon_{cd}{}^{e}.$$

To obtain the above, we used the evolution equations —i.e. that $F_{bc\mathbf{0}} = 0$ and $F^{*}{}_{(bc)\mathbf{0}} = 0$ as well as the identity

$$F_{[ab]c} = -\frac{1}{2}F_{cab},$$

which is a direct result of the symmetry properties of F_{abc}. By taking the divergence of the first index of (4.72) and equating with $u^d h_a{}^c$ times (4.71), we obtain a propagation equation for the p_a field, namely

$$\begin{aligned}
2u^b\nabla_b p_a = {} & 2u_a a^c p_c - \epsilon_a{}^{cd} a_c q_d + \chi_a{}^c p_c + 3\chi p_a + 2\Delta^l{}_a{}^{bm} C_{lbc\mathbf{0}} h_a{}^c \\
& + 2\Delta^l{}_b{}^{bm} C_{mlc\mathbf{0}} h_a{}^c + 2\Delta^l{}_c{}^{bm} C_{mbl\mathbf{0}} h_a{}^c + 2\Delta^l{}_{\mathbf{0}}{}^{bm} C_{mbcl} h_a{}^c \\
& + 2\Delta^l{}_c{}^b{}_{\mathbf{0}} \hat{L}_{lb} h_a{}^c + 2\Delta^l{}_b{}^b{}_{\mathbf{0}} \hat{L}_{al} - 2\nabla_{[\mathbf{0}} Q_{c]} h_a{}^d \\
& + 4u^d h_a{}^c \Sigma_a{}^l{}_b \nabla_l C^{ab}{}_{cd} - 2u^d h_a{}^c \Sigma_b{}^l{}_c \nabla_l S_d{}^b + 2u^d h_a{}^c \Sigma_b{}^l{}_c \nabla_l S_c{}^b.
\end{aligned} \tag{4.73}$$

In the above, we have used that $\Sigma_{\mathbf{0}}{}^d{}_b = 0$ everywhere on $\mathcal{M}$. Applying the same procedure to (4.47b) we obtain the propagation equations for the q_a field,

$$\begin{aligned}
u^b\nabla_b q_a - \epsilon_a{}^{cd}\mathcal{D}_c p_d = {} & u_a a^c q_c - \chi q_a + 2p_c \epsilon_{ab} a^b \\
& + \Delta^l{}_a{}^{bm} C^{*}_{lbc\mathbf{0}} h_a{}^c + \Delta^l{}_b{}^{bm} C^{*}_{mlc\mathbf{0}} h_a{}^c + \Delta^l{}_p{}^{bm} C_{mbln} \epsilon_a{}^{pn} \\
& + \Delta^l{}_n{}^{bm} C_{mbpl} \epsilon_a{}^{pn} + \Delta^l{}_p{}^b{}_n \hat{L}_{lb} \epsilon_a{}^{pn} + \Delta^l{}_b{}^b{}_n \hat{L}_{pl} \epsilon_a{}^{pn} \\
& + \nabla_n Q_p \epsilon_a{}^{np} + \quad 4u^m \Sigma_a{}^l{}_b \nabla_l C^{*ab}{}_{ma} + 2\Sigma_b{}^l{}_n \nabla_l S_p{}^b \epsilon_a{}^{np}.
\end{aligned} \tag{4.74}$$

Remark 4.17. *Again, the main observation to be extracted from the previous analysis is that equations* (4.73) *and* (4.74) *are homogeneous in the various zero-quantities. Moreover, their form is analogous to that of the evolution equations* (4.48) *and* (4.49)*. Thus, it can be verified they* *imply a symmetric hyperbolic system. Note also that equation* (4.73) *is different in the principle part compared to equation* (4.74)*. This is due to the fact that the evolution equation* $F_{\boldsymbol{ab0}} = 0$ *is not symmetrized. It is also understood in equation* (4.73) *that one can apply equation* (4.51) *to eliminate the time derivative of* $Q_{\boldsymbol{a}}$.

4.6.6 Main theorem

The homogeneity of the propagation equations for the various zero-quantities implies, from the uniqueness of symmetric hyperbolic systems that if the zero-quantities vanish on some initial hypersurface $\mathcal{S}_\star$ then they will also vanish at later times. We summarise the analysis of the previous subsections in the following statement:

Theorem 8. A solution

$$(e_{\boldsymbol{a}}{}^{\mu}, \Gamma_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}}, \Gamma_{\boldsymbol{0}}{}^{\boldsymbol{0}}{}_{\boldsymbol{k}}, \Gamma_{\boldsymbol{i}}{}^{\boldsymbol{0}}{}_{\boldsymbol{j}}, E_{\boldsymbol{ab}}, B_{\boldsymbol{ab}}, \Psi_{\boldsymbol{ab}}, \Phi_{\boldsymbol{ab}}, \Pi_{\boldsymbol{ab}}, \rho)$$

to the system of evolution equations given, respectively, by equations (4.17), (4.25), (4.31), (4.26), (4.48), (4.49), (4.33a), (4.37), (4.42) and (4.7b) with initial data satisfying the conditions

$$\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} = 0, \qquad \Delta^{\boldsymbol{d}}{}_{\boldsymbol{abc}} = 0, \qquad F_{\boldsymbol{abc}} = 0,$$

on an initial hypersurface $\mathcal{S}_\star$ implies a solution to the Einstein-matter frame equations (4.14a)-(4.14d).

Remark 4.18. *As a consequence of Lemma 1 it follows that a solution of the Einstein matter frame equations implies, in turn, a solution to the standard Einstein-matter field equations* (4.1)*.*

4.7 Examples of matter models

We will in the following exemplify the previous discussion with a number of particular matter models. We shall also note that although the equations given in the following resembles those found in [20], the treatment of the propagation of constraints for dust or perfect fluid was not treated therein. In this thesis we fill this gap.

4.7.1 Dust

The simplest case is of course that of *dust*. In this case $\Pi_{\boldsymbol{ab}} = 0$ and the expression for the energy-momentum tensor, equation (4.2), reduces to

$$T_{\boldsymbol{ab}} = \rho u_{\boldsymbol{a}} u_{\boldsymbol{b}}.$$

Furthermore, as there are no internal interactions, each dust particle follows a geodesic —i.e the following hold

$$\Gamma_{\boldsymbol{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = 0.$$

Consequently, equation (4.7b) reduces to

$$u^{\boldsymbol{a}} \nabla_{\boldsymbol{a}} \rho = -\rho \chi \tag{4.75}$$

and equations (4.25) and (4.26) take the form,

$$\partial_{\boldsymbol{0}} \Gamma_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}} = -\Gamma_{\boldsymbol{l}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}} \Gamma_{\boldsymbol{i}}{}^{\boldsymbol{l}}{}_{\boldsymbol{0}} + C^{\boldsymbol{j}}{}_{\boldsymbol{ki0}}, \tag{4.76}$$

$$\partial_{\boldsymbol{0}} \Gamma_{\boldsymbol{j}}{}^{\boldsymbol{0}}{}_{\boldsymbol{i}} = -2\Gamma_{\boldsymbol{k}}{}^{\boldsymbol{0}}{}_{\boldsymbol{i}} \Gamma_{\boldsymbol{j}}{}^{\boldsymbol{k}}{}_{\boldsymbol{0}} + C^{\boldsymbol{0}}{}_{\boldsymbol{ij0}} - \frac{1}{3}\rho \eta_{\boldsymbol{ij}}. \tag{4.77}$$

Remark 4.19. *Note that the condition* $\Gamma_{\boldsymbol{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = 0$ *implies that* $a_{\boldsymbol{a}} = 0$. *Consequently, the evolution equation* (4.31) *is not necessary.*

Also, we have that $Z_{\boldsymbol{abc}} = 0$. Thus, the discussion of the Z-tensor and its evolution equations are irrelevant —i.e. there is no need for the construction of an auxiliary field. The evolution equations for the Weyl tensor reduce to

$$\begin{aligned} u^{c}\nabla_{\boldsymbol{c}}E_{\boldsymbol{ab}} - \epsilon_{\boldsymbol{aef}}D^{\boldsymbol{f}}B_{\boldsymbol{b}}{}^{\boldsymbol{e}} = & -\tfrac{1}{2}\kappa\rho\chi_{\boldsymbol{ba}} - E_{\boldsymbol{ac}}\chi_{\boldsymbol{b}}{}^{\boldsymbol{c}} + 2E_{\boldsymbol{bc}}\chi^{\boldsymbol{c}}{}_{\boldsymbol{a}} - 2E_{\boldsymbol{ab}}\chi^{\boldsymbol{c}}{}_{\boldsymbol{c}} + \tfrac{1}{6}\kappa\rho h_{\boldsymbol{ab}}\chi^{\boldsymbol{c}}{}_{\boldsymbol{c}} \\ & - E_{\boldsymbol{cd}}h_{\boldsymbol{ab}}\chi^{\boldsymbol{cd}} + \epsilon_{\boldsymbol{cdef}}B_{\boldsymbol{b}}{}^{\boldsymbol{e}}h_{\boldsymbol{a}}{}^{\boldsymbol{f}}\chi^{\boldsymbol{cd}} + \epsilon_{\boldsymbol{dfa}}u_{\boldsymbol{b}}B_{\boldsymbol{e}}{}^{\boldsymbol{f}}\chi^{\boldsymbol{de}}, \end{aligned} \tag{4.78}$$

and

$$\begin{aligned} u^{\boldsymbol{d}}\nabla_{\boldsymbol{d}}B_{\boldsymbol{ab}} + D^{\boldsymbol{f}}E_{(\boldsymbol{b}}{}^{\boldsymbol{d}}\epsilon_{\boldsymbol{a})\boldsymbol{df}} = & \tfrac{1}{2}B_{\boldsymbol{bd}}\chi_{\boldsymbol{a}}{}^{\boldsymbol{d}} + \tfrac{1}{2}B_{\boldsymbol{ad}}\chi_{\boldsymbol{b}}{}^{\boldsymbol{d}} + B_{\boldsymbol{bd}}\chi^{\boldsymbol{d}}{}_{\boldsymbol{a}} + B_{\boldsymbol{ad}}\chi^{\boldsymbol{d}}{}_{\boldsymbol{b}} - 2B_{\boldsymbol{ab}}\chi^{\boldsymbol{d}}{}_{\boldsymbol{d}} \\ & - \tfrac{1}{2}E_{\boldsymbol{f}}{}^{\boldsymbol{c}}\epsilon_{\boldsymbol{bdc}}u_{\boldsymbol{a}}\chi^{\boldsymbol{df}} - \tfrac{1}{2}E_{\boldsymbol{f}}{}^{\boldsymbol{c}}\epsilon_{\boldsymbol{adc}}u_{\boldsymbol{b}}\chi^{\boldsymbol{df}} - B_{\boldsymbol{df}}h_{\boldsymbol{ab}}\chi^{\boldsymbol{df}}. \end{aligned} \tag{4.79}$$

Thus, equations (4.17), (4.76), (4.77), (4.78), (4.79) and (4.75) provide the symmetric hyperbolic evolution equations for the fields $e^{\mu}_{\boldsymbol{a}}$, $\Gamma_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}}$,$\Gamma_{\boldsymbol{i}}{}^{\boldsymbol{0}}{}_{\boldsymbol{j}}$, $E_{\boldsymbol{ab}}$, $B_{\boldsymbol{ab}}$ and ρ, respectively.

4.7.2 Perfect fluid

Before we discuss the details of a perfect fluid, we shall briefly review some important quantities in relativistic thermodynamics.

Given a material with N different particle species, let n_A denote the number density of a particular species, where $A = \{1, 2, ..., N\}$. Furthermore, we denote by s the entropy density. The energy density of the system is a function of these quantities —i.e. we have

$$\rho = f\left(s, n_1, n_2, ..., n_N\right). \tag{4.80}$$

The function f is called the *equation of state* of the system. Finally, the *first law of Thermodynamics* is given by,

$$d\rho = Tds + \mu^{A}dn_A, \tag{4.81}$$

where,

$$T \equiv \left(\frac{\partial\rho}{\partial s}\right)_{n_A}, \qquad \mu_A \equiv \left(\frac{\partial\rho}{\partial n_A}\right)_{s},$$

denotes the temperature and chemical potential, respectively. In what follows we shall consider a simple perfect fluid — i.e a fluid of only one type of particles ($A = 1$) and with an energy momentum tensor with

$$\Pi_{\boldsymbol{ab}} = p h_{\boldsymbol{ab}}. \tag{4.82}$$

Consequently, we have

$$\Pi_{\{\boldsymbol{ab}\}} = p u_{\boldsymbol{a}} u_{\boldsymbol{b}}, \qquad \Pi = 3p,$$

where p denotes the pressure and is defined by

$$p \equiv n\mu - \rho. \tag{4.83}$$

Throughout we shall assume an equation of state of the form given by (4.80) with $A = 1$ and the law of particle conservation —i.e.

$$u^{\boldsymbol{a}} \nabla_{\boldsymbol{a}} n = -n\chi. \tag{4.84}$$

With these assumptions, equations (4.27a) and (4.27b) reduce to the well known Einstein-Euler equations, given by

$$u_{\boldsymbol{b}} u^{\boldsymbol{a}} \nabla_{\boldsymbol{a}} p + \nabla_{\boldsymbol{b}} p = -\left(\rho + p\right) a_{\boldsymbol{b}}, \tag{4.85a}$$

$$u^{\boldsymbol{a}} \nabla_{\boldsymbol{a}} \rho = -\left(\rho + p\right) \chi. \tag{4.85b}$$

It follows from equations (4.83), (4.85b), (4.84) and (4.81) that the fluid is adiabatic —i.e we have

$$u^{\boldsymbol{a}} \nabla_{\boldsymbol{a}} s = 0. \tag{4.86}$$

From the above discussion it follows that equation (4.31) takes the form

$$\begin{aligned} 3\partial_{\boldsymbol{0}} \Gamma_{\boldsymbol{0}}{}^{\boldsymbol{0}}{}_{\boldsymbol{i}} - \eta^{\boldsymbol{jk}} \partial_{\boldsymbol{i}} \Gamma_{\boldsymbol{j}}{}^{\boldsymbol{0}}{}_{\boldsymbol{k}} = {} & -2a_{\boldsymbol{i}}\chi + a^{\boldsymbol{j}}\chi_{\boldsymbol{ij}} - \partial^{\boldsymbol{j}} \Gamma_{\boldsymbol{j}}{}^{\boldsymbol{0}}{}_{\boldsymbol{i}} + \Gamma_{\boldsymbol{j}}{}^{\boldsymbol{k}}{}_{\boldsymbol{i}} \chi_{\boldsymbol{k}}{}^{\boldsymbol{j}} - \Gamma_{\boldsymbol{j}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}} \chi_{\boldsymbol{i}}{}^{\boldsymbol{k}} \\ & - \frac{1}{\rho}\left(R_{\boldsymbol{0i}} p + 2p a_{\boldsymbol{i}} \chi\right) - \Gamma_{\boldsymbol{j}}{}^{\boldsymbol{0}}{}_{\boldsymbol{0}} \chi^{\boldsymbol{j}}{}_{\boldsymbol{i}} - \Gamma_{\boldsymbol{0}}{}^{\boldsymbol{0}}{}_{\boldsymbol{i}} \chi \\ & + \Gamma_{\boldsymbol{0}}{}^{\boldsymbol{j}}{}_{\boldsymbol{i}} \Gamma_{\boldsymbol{0}}{}^{\boldsymbol{0}}{}_{\boldsymbol{j}} + \eta^{\boldsymbol{jk}} \Gamma_{\boldsymbol{i}}{}^{\boldsymbol{l}}{}_{\boldsymbol{j}} \Gamma_{\boldsymbol{l}}{}^{\boldsymbol{0}}{}_{\boldsymbol{k}} + \eta^{\boldsymbol{jk}} \Gamma_{\boldsymbol{i}}{}^{\boldsymbol{l}}{}_{\boldsymbol{k}} \Gamma_{\boldsymbol{j}}{}^{\boldsymbol{0}}{}_{\boldsymbol{l}}.. \end{aligned} \tag{4.87}$$

Similarly, equation (4.26) takes the form

$$\begin{aligned}\partial_{\boldsymbol{0}}\Gamma_{\boldsymbol{j}}{}^{\boldsymbol{0}}{}_{\boldsymbol{i}} - \partial_{\boldsymbol{j}}\Gamma_{\boldsymbol{0}}{}^{\boldsymbol{0}}{}_{\boldsymbol{i}} &= \Gamma_{\boldsymbol{0}}{}^{\boldsymbol{0}}{}_{\boldsymbol{i}}\Gamma_{\boldsymbol{0}}{}^{\boldsymbol{0}}{}_{\boldsymbol{j}} - \Gamma_{\boldsymbol{k}}{}^{\boldsymbol{0}}{}_{\boldsymbol{i}}\Gamma_{\boldsymbol{j}}{}^{\boldsymbol{j}}{}_{\boldsymbol{0}} - \Gamma_{\boldsymbol{k}}{}^{\boldsymbol{0}}{}_{\boldsymbol{i}}\Gamma_{\boldsymbol{j}}{}^{\boldsymbol{k}}{}_{\boldsymbol{0}} \\ &\quad + C^{\boldsymbol{0}}{}_{\boldsymbol{ij0}} + ph_{\boldsymbol{ij}} - \frac{1}{3}\rho\eta_{\boldsymbol{ij}} + 2p\eta_{\boldsymbol{ij}}. \end{aligned} \tag{4.88}$$

Now, writing equations (4.33a) and (4.33b) in terms of the above definitions we obtain

$$\Psi_{\boldsymbol{ab}} = \chi_{\boldsymbol{ba}} - h_{\boldsymbol{ab}}(\rho + p)\left(1 - \nu^2\right)\chi + h_{\boldsymbol{ab}}\mu n\chi, \tag{4.89a}$$

$$\Phi_{\boldsymbol{ab}} = -\epsilon_{\boldsymbol{a}}{}^{\boldsymbol{d}}{}_{\boldsymbol{b}}(\rho + p)\, a_{\boldsymbol{d}} - \epsilon_{\boldsymbol{acd}}\chi^{\boldsymbol{cd}} p u_{\boldsymbol{b}}, \tag{4.89b}$$

where we have defined the scalar,

$$\nu^2 \equiv \frac{\partial p}{\partial \rho},$$

and used equation (4.83) and the definition of μ to obtain,

$$u^{\boldsymbol{a}}\nabla_{\boldsymbol{a}} p = (\rho + p)\left(1 - \nu^2\right)\chi - \mu n\chi. \tag{4.90}$$

Finally, the evolution equations for $E_{\boldsymbol{ab}}$ and $B_{\boldsymbol{ab}}$ are obtained by substituting equations (4.89a), (4.89b) and (4.82) into the equations (4.48) and (4.49)

$$\begin{aligned} u^{\boldsymbol{c}}\nabla_{\boldsymbol{c}}E_{\boldsymbol{ab}} - \epsilon_{\boldsymbol{acd}}D^{\boldsymbol{d}}B_{\boldsymbol{b}}{}^{\boldsymbol{c}} &= -2a^{\boldsymbol{c}}u_{(\boldsymbol{a}}E_{\boldsymbol{b})\boldsymbol{c}} + a^{\boldsymbol{c}}\epsilon_{\boldsymbol{bcd}}B_{\boldsymbol{a}}{}^{\boldsymbol{d}} - 2\kappa a_{(\boldsymbol{b}}u_{\boldsymbol{a})}p + \kappa p\chi_{[\boldsymbol{ab}]} + \tfrac{1}{2}\kappa\chi_{\boldsymbol{ba}} \\ &\quad - \tfrac{1}{2}\kappa\rho\chi_{\boldsymbol{ba}} + E_{\boldsymbol{ac}}\chi_{\boldsymbol{b}}{}^{\boldsymbol{c}} + 2E_{\boldsymbol{bc}}\chi^{\boldsymbol{c}}{}_{\boldsymbol{a}} - 2E_{\boldsymbol{ab}}\chi^{\boldsymbol{c}}{}_{\boldsymbol{c}} - \tfrac{1}{6}\kappa h_{\boldsymbol{ab}}\chi^{\boldsymbol{c}}{}_{\boldsymbol{c}} \\ &\quad + \tfrac{1}{6}\kappa\rho h_{\boldsymbol{ab}}\chi^{\boldsymbol{c}}{}_{\boldsymbol{c}} + \epsilon_{\boldsymbol{ace}}u_{\boldsymbol{b}}B_{\boldsymbol{d}}{}^{\boldsymbol{e}}\chi^{\boldsymbol{cd}} - E_{\boldsymbol{cd}}h_{\boldsymbol{ab}}\chi^{\boldsymbol{cd}}, \end{aligned} \tag{4.91}$$

$$\begin{aligned} u^{\boldsymbol{d}}\nabla_{\boldsymbol{d}}B_{\boldsymbol{ab}} - D^{\boldsymbol{f}}E_{(\boldsymbol{a}}{}^{\boldsymbol{d}}\epsilon_{\boldsymbol{b})\boldsymbol{df}} &= -\tfrac{1}{2}a^{\boldsymbol{d}}E_{\boldsymbol{b}}{}^{\boldsymbol{f}}\epsilon_{\boldsymbol{adf}} - \tfrac{1}{2}a^{\boldsymbol{d}}E_{\boldsymbol{a}}{}^{\boldsymbol{f}}\epsilon_{\boldsymbol{bdf}} - a^{\boldsymbol{d}}u_{\boldsymbol{b}}B_{\boldsymbol{ad}} \\ &\quad - a^{\boldsymbol{d}}u_{\boldsymbol{a}}B_{\boldsymbol{bd}} + \tfrac{1}{2}B_{\boldsymbol{bd}}\chi_{\boldsymbol{a}}{}^{\boldsymbol{d}} + \tfrac{1}{2}B_{\boldsymbol{ad}}\chi_{\boldsymbol{b}}{}^{\boldsymbol{d}} \\ &\quad + B_{\boldsymbol{bd}}\chi^{\boldsymbol{d}}{}_{\boldsymbol{a}} + B_{\boldsymbol{ad}}\chi^{\boldsymbol{d}}{}_{\boldsymbol{b}} - 2B_{\boldsymbol{ab}}\chi^{\boldsymbol{d}}{}_{\boldsymbol{d}} - \tfrac{1}{2}E_{\boldsymbol{f}}{}^{\boldsymbol{c}}\epsilon_{\boldsymbol{bdc}}u_{\boldsymbol{a}}\chi^{\boldsymbol{df}} \\ &\quad - \tfrac{1}{2}E_{\boldsymbol{f}}{}^{\boldsymbol{c}}\epsilon_{\boldsymbol{adc}}u_{\boldsymbol{b}}\chi^{\boldsymbol{df}} - B_{\boldsymbol{df}}h_{\boldsymbol{ab}}\chi^{\boldsymbol{df}} + \tfrac{1}{4}\kappa\epsilon_{\boldsymbol{bdf}}u_{\boldsymbol{a}}p\chi^{\boldsymbol{df}} \\ &\quad + \tfrac{1}{4}\kappa\epsilon_{\boldsymbol{adf}}u_{\boldsymbol{b}}p\chi^{\boldsymbol{df}}. \end{aligned} \tag{4.92}$$

Equations (4.17), (4.25), (4.87), (4.25),(4.88), (4.91), (4.92), (4.90), (4.85b), (4.86) and (4.84) provide the symmetric hyperbolic system for the fields ($e^{\mu}_{\boldsymbol{a}}$, $\Gamma_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}}$, $\Gamma_{\boldsymbol{0}}{}^{\boldsymbol{0}}{}_{\boldsymbol{k}}$, $\Gamma_{\boldsymbol{i}}{}^{\boldsymbol{0}}{}_{\boldsymbol{j}}$, $E_{\boldsymbol{ab}}$, $B_{\boldsymbol{a},\boldsymbol{b}}$, p, ρ,s,n) respectively.

4.7.3 Elastic matter

The following discussion follows the treatment of relativistic elasticity found in [22].

The energy density of the elastic system is given by

$$\rho = n\epsilon, \tag{4.93}$$

where ϵ is the *stored energy function* of the system. It can be shown[i] that the elastic energy-momentum tensor in frame coordinates can be put on the form of equation (4.2) with an energy density as given by (4.93) and

$$\Pi_{\boldsymbol{ab}} \equiv 2\rho\eta_{\boldsymbol{ab}} + 2n\tau_{\mathbf{AB}}\Lambda^{\boldsymbol{A}}{}_{\boldsymbol{a}}\Lambda^{\boldsymbol{B}}{}_{\boldsymbol{b}}. \tag{4.94}$$

where $\tau_{\boldsymbol{AB}}$ denotes the relativistic Piola-Kirchoff stress tensor and is defined in Chapter 3. Thus, the field $\Lambda^{\boldsymbol{A}}{}_{\boldsymbol{a}}$ is the fundamental material field of the theory. We shall, however, not write explicit equations for these fields, but rather use the formalism described earlier in the chapter. Hence, the information regarding $\Lambda_{\boldsymbol{a}}{}^{\boldsymbol{A}}$ is encoded in the tensor $\Pi_{\boldsymbol{ab}}$ by equation (4.94). Consequently, the symmetric hyperbolic system for the fields ($e^{\mu}_{\boldsymbol{a}}$, $\Gamma_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}}$, $\Gamma_{\boldsymbol{0}}{}^{\boldsymbol{0}}{}_{\boldsymbol{k}}$, $\Gamma_{\boldsymbol{i}}{}^{\boldsymbol{0}}{}_{\boldsymbol{j}}$, $E_{\boldsymbol{ab}}$, $B_{\boldsymbol{a},\boldsymbol{b}}$, $\Psi_{\boldsymbol{ab}}$, $\Phi_{\boldsymbol{ab}}$, $\Pi_{\boldsymbol{ab}}$, ρ) are respectively given by equations (4.17), (4.25), (4.31), (4.26), (4.48),(4.49), (4.33a), (4.37),(4.42) and (4.7b). Equations (4.7a) and (4.33b) are considered constraint equations.

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[i] See Section 3.4.

4.8 Concluding remarks

As stressed previously, we have developed first order symmetric hyperbolic evolution equations for a wide range of matter models which solves the Einstein equations. In fact, any matter model which has zero heat transfer should be covered by our formalism. It should thus be applicable to the development of a theory of neutron stars as a relativistic elastic system. In this case one would proceed as with the perfect fluid case: one needs to write the tensor $\boldsymbol{\Pi}$ in terms of its trace and trace-free parts and provide equations for n and ϵ to close the system. The latter is likely obtained from thermodynamical considerations. Remarkably no other information is needed to solve the system.

The treatment given in this chapter is sufficiently general that showing symmetric hyperbolicity for a given matter model coupled to the Einstein equations, is reduced to the simple task of showing that the system admits an energy momentum tensor on the form (4.2) satisfying (4.3b) and (4.3a). It is understood that an equation of state for ρ is provided.

The analysis of this chapter assumes that suitable initial data for the evolution equations has been provided. The details on how to construct suitable data depend on the particular details of the matter model under consideration. However, there exists a more or less general procedure to construct solutions to the constraint equations of General Relativity coupled to general classes of data —see e.g. [35]. Accordingly, we do not expect the construction of initial data to be a major issue.

The ultimate aim of the formulation of the evolution problem of relativistic self-gravitating systems provided in the present chapter is to make connections with numerical Relativity. There is, however, currently a limited experience in the numerical community regarding the use of frame formulations of General Relativity in simulations —see however [36, 37]. The techniques developed in these references provides an initial stepping stone for the implementation of the equations in the present chapter.

A natural, and in some ways necessary, extension of the analysis in this chapter is the formulation of an initial-boundary value problem for the evolution equations. This analysis requires the identification of proper boundary conditions and evolution equations which ensure the propagation of the constraints. This is a challenging task. However, the seminal work on the initial-boundary value problem for the Einstein vacuum equations given in [38] makes use of a frame formulation for the vacuum Einstein field similar to the one used in the present chapter. Thus, several of the key ideas in that reference may be carried over to the more general setting of matter models. These ideas will be explored elsewhere.

Chapter 5

Future stability of N - bodies in general relativity

5.1 Introduction

Much of Physics is the study of evolution of a system under certain conditions and laws. In Cosmology one is thus interested in the evolution of our Universe from the far past to the distant future. The dominant law governing galaxies and the evolution of the Cosmos is embedded in Einsteins theory of gravitation. In the large scale structure of the Universe, galaxies can be treated as dust —i.e. each galaxy is represented by a "dust" particle— exerting no pressure on the surrounding particles. Given that current observations suggest our Universe is expanding, the setting to investigate the evolution of our Universe is thus the Einstein equations coupled to dust matter with a positive cosmological constant. There is a challenge associated to the study of our Universe at such large scales and its evolution over long time: namely, the global properties of the theory become important. Accordingly, any comprehensive study of a solution to the Einstein field equations should also take into account its global properties.

In the attempt to model astrophysical objects such as stars and galaxies and solar systems, it is necessary to consider solutions of the Einstein field equations which represent an *isolated system*. This is far from a trivial endeavour. In Newtonian gravity and relativistic electrodynamics, one has a flat background metric upon which the fields propagates, and one can meaningfully speak about the fall-off properties of the fields as one moves away from the sources. In these terms an isolated system is a system for which the field strength vanishes at infinity and the source density is zero outside a finite radius. In General Relativity, the metric is part of the unknowns for which one solves the equations. Thus, there is no "background" metric upon which the gravitational field propagate and in terms of which we may define fall-off properties in a meaningful way. Accordingly, attempts at solving the Einstein equations for an isolated system by introducing approximations in terms of a background metric plus perturbations cannot be satisfactory as they disregard the non-linear aspect of the full theory.

A procedure which has proved successful in the study of global properties of spacetimes describing isolated systems was devised by *Roger Penrose* in [39], and involves a conformal compactification of space time —essentially allowing for a treatment of infinity as a three dimensional submanifold; see [5] for details. This allow for a rigorous description of the asymptotic behaviour and global properties of a space time [40–42]. But not all spacetimes allow for a conformal treatment. Accordingly, one is interested in knowing which solutions to the Einstein field equations admit a smooth conformal compactification.

One important aspect of Penrose's conformal method which will be extensively used in the following discussion, is that a small conformal time can represent an infinite amount of physical time. Hence, if the equations describing the conformally rescaled spacetime imply a regular system of evolution equations one could, in turn, apply general results of the theory of partial differential equations to show global existence and stability. The seminal work of H. Friedrich has established that the Einstein vacuum equations [43], including de Sitter-like spacetimes with positive cosmological constant [44, 45], the Einstein-Maxwell-Yang-Mills equations [46] and the Einstein-λ-dust equations [21], all can be described in terms of a set of regular *conformal Einstein field equations* from which, in turn, one can extract a symmetric hyperbolic evolution system for which general theory of hyperbolic differential equations is available —in particular, locally the Cauchy problem is well posed and stability over a small time is guaranteed.

In [47], Y. Choquet-Bruhat & H. Friedrich have established the local existence in time of solutions to the Einstein field equations representing isolated self-gravitating dust bodies. However, the mathematical technology available did not allow to pursue the pressing question of the global existence of solutions. One of the key technical aspects of their analysis is the use of a formulation of the evolution equations which is well behaved independently of whether the density of the dust vanishes or not. This formulation crucially depends on the fact that the flow lines of the dust are geodesics.

A suitable framework for the analysis of global properties of solutions to the Einstein-λ-dust system by means of conformal methods was given in [21]. This setup was used to study the backwards evolution of asymptotic data prescribed on the conformal boundary $\mathscr{I}^+$. The work in [21] is remarkable in that it is one of the few conformal treatments of a matter model with non-vanishing trace of the energy-momentum. The conformal evolution system used in this analysis is well-defined up to and beyond the conformal boundary. Its construction depends crucially on the observation that the flow lines of the dust can be recast as certain conformally invariant curves —the so-called *conformal geodesics*. Moreover, as in the case of the analysis in [47], the evolution system is also regular independently of whether the density vanishes or not. Accordingly, as it will be discussed in this article, it provides an ideal framework to study global properties of the evolution of isolated dust bodies in General Relativity in the presence of a positive Cosmological constant. The analysis of these relativistic self-gravitating matter configurations is a subject of physical relevance as the Cosmological constant is generally believed to be connected with the observed expansion of our Universe, and dust to the solutions to the Einstein field equations are good models for the description of the matter content of the Universe.

In this work we combine the approaches followed in [47] and [21] to provide a toy model of self-gravitating dust balls in an expanding Universe for which it is possible to make assertions regarding global existence and stability. More precisely, we show that in a spacelike conformal boundary (which, for simplicity one can assume as having the topology of $\mathbb{S}^3$) one can prescribe asymptotic data which represents patches of dust on the conformal boundary. Using then the conformal evolution equations one can then show that these configurations would have to exist for some small amount of conformal time —which, when translated into the physical picture corresponds to an infinite amount of physical time. To complement the above *backwards evolution problem*, we provide sufficient conditions for the existence of solutions to the Einstein constraint equations on a standard Cauchy initial hypersurface which represent patches of dust in a de-Sitter-like universe. We further show future stability of these patches, provided the density function satisfies a smallness condition. The resulting spacetime is future geodesically complete. *The above analysis provides a non-trivial example of fairly generic matter configurations which exist arbitrarily into the future.* The physical mechanism ensuring this result is the expansion driven by the Cosmological constant λ.

5.1.1 The Einstein-λ-dust system

In what follows we are concerned with the Einstein-λ-dust system governed by the equations

$$\tilde{R}_{ab} - \left(\frac{1}{2}\tilde{R} - \lambda\right)\tilde{g}_{ab} = \kappa\tilde{T}_{ab}, \tag{5.1}$$

$$\tilde{T}_{ab} = \tilde{\rho}\tilde{U}_a\tilde{U}_b, \tag{5.2}$$

$$\tilde{U}^a\tilde{\nabla}_a\tilde{U}_b = 0, \tag{5.3}$$

$$\tilde{\nabla}_a\tilde{j}^a = 0, \tag{5.4}$$

where $\tilde{R}_{ab}$, $\tilde{T}_{ab}$ and $\tilde{U}_a$ are the Ricci tensor, energy momentum tensor and the four velocity for the metric $\tilde{g}_{ab}$. Furthermore, we have defined the matter current as

$$\tilde{j}^a \equiv \tilde{\rho}\tilde{U}^a,$$

where $\tilde{\rho}$ is a positive function representing the energy-density of the matter. We also let κ and λ be positive constants. In the following we shall set $\kappa = 1$ to simplify the discussion.

> Remark 5.1. *Note that equations* (5.3) *and* (5.4) *are the equations of motion for the matter fields obtained through the divergence-free condition* $\tilde{\nabla}_a \tilde{T}^{ab} = 0$. *In particular, equation* (5.3) *states that the flow lines of the dust matter model are geodesics.*

The objective of this study is to make use of the conformal representation of equations (5.1) - (5.4) to say something about the stability of the system over large time. The first step is to find a system of equations which can be smoothly extended to the conformal boundary. This system has been found in [21]. We will not derive the equations here, but rather sketch out the argument employed in [21] leading to the conformal equations.

As shown in section 2.5, the conformal transformation (2.20) implies the following relationship between the connection ∇_a of g_{ab} and $\hat{\nabla}_a$ of $\hat{g}$ for any one form ω_b,

$$\left(\nabla_a - \tilde{\nabla}_a\right)\omega_b = \Omega^{-1}\left(\nabla_d \Omega g_{ab} g^{dc} - \nabla_a \Omega \delta^c{}_b - \nabla_b \Omega \delta^c{}_a\right)\omega_c. \qquad (5.5)$$

Furthermore we introduce the *conformal 4-velocity* U^a related to the tangent vector of the flow lines $\tilde{U}^a$ in such a way that $g_{ab}U^aU^b = \tilde{g}_{ab}\tilde{U}^a\tilde{U}^b = -1$ — i.e. we have

$$U^a = \Omega^{-1}\tilde{U}^a \qquad U_a = \Omega \tilde{U}_a.$$

Since $\tilde{\rho}$ is a scalar field independent of the metric, its transformation rule can be freely specified. It is convenient to define the *conformal energy density* as

$$\rho = \Omega^{-3}\tilde{\rho}.$$

Using the conformal transformation (2.20) one can thus use (5.5) to obtain the transformation between the Ricci tensor of the conformal metric R_{ab}

and the interior metric $\tilde{R}_{ab}$,

$$\begin{aligned} R_{ab} - \tilde{R}_{ab} = &-2\Omega^{-1}\nabla_a\nabla_b\Omega \\ &- g_{ab}\left(\Omega^{-1}\nabla^c\nabla_c\Omega - 3\Omega^{-2}\nabla^c\Omega\nabla_c\Omega\right), \end{aligned} \tag{5.6}$$

where we have defined,

$$\nabla^a \equiv g^{ab}\nabla_b.$$

Contracting equation (5.6) with g^{ab}, we find the transformation of the Ricci scalar,

$$R - \Omega^{-2}\tilde{R} = -6\Omega^{-1}\nabla^a\nabla_a\Omega + 12\Omega^{-3}\nabla^a\Omega\nabla_a\Omega. \tag{5.7}$$

Remark 5.2. *Observe that equations* (5.6) *and* (5.7) *are singular at the points where* $\Omega = 0$*. Thus, using the form of the Ricci tensor and scalar as above will not directly lead to a set of field equations which extends to the conformal boundary* $\mathcal{I}^+$*. It is, therefore, necessary to find another set of equations which are equivalent to* (5.6) *and* (5.7) *but that extends smoothly to the conformal boundary. Another issue is the freedom in the choice of the conformal factor* Ω*. This gauge freedom means a solution to the equations* (5.6) *and* (5.7) *are, in some sense, not unique. Hence, the new set of equations must be constructed such that one can fix this gauge freedom.*

5.1.2 The conformal regular Einstein-λ-dust system

We refer to [5] for a derivation of the *regular conformal field equations*. In what follows we will only give a short summary of some results from [21] which make up the basis of the next section. We refer the interested reader to the original paper for details.

Again, we introduce a $\boldsymbol{g}$-orthonormal frame field $\{\boldsymbol{e_a}\}$ on $\mathcal{M}$ such that in local coordinates $x = (x^\mu)$ we have that $\boldsymbol{e_a} = e^\mu{}_{\boldsymbol{a}}\partial_\mu$. Furthermore,

$g_{\boldsymbol{ab}} \equiv \boldsymbol{g}(\boldsymbol{e_a}, \boldsymbol{e_b}) = \eta_{\boldsymbol{ab}}$ and we assume the frame connection defined by (2.7) to be such that the metric compatibility condition (2.8) holds. Then, one can recover the interior metric $\tilde{\boldsymbol{g}}$ by the transformation (2.20). More details on the frame formulism is given in Section 2.2. Most equations and tensor fields are henceforth given in terms of frame indices.

Using $\{\boldsymbol{e_a}\}$ as the fundamental geometric unknown, we may write a new set of equations entirely in terms of fields on $\mathcal{M}$ which is equivalent to the system (5.1)-(5.4) in the domain for which $\Omega > 0$ — namely

$$6s\Omega - 3\nabla_{\boldsymbol{a}}\Omega\nabla^{\boldsymbol{a}}\Omega - \lambda = \frac{1}{4}\Omega^3\rho, \tag{5.8}$$

$$\nabla_{\boldsymbol{b}}\nabla_{\boldsymbol{d}}\Omega + \Omega L_{\boldsymbol{bd}} - s g_{\boldsymbol{bd}} = \frac{1}{2}\Omega^2\rho\left(U_{\boldsymbol{b}}U_{\boldsymbol{d}} + \frac{1}{4}g_{\boldsymbol{bd}}\right), \tag{5.9}$$

$$\begin{aligned}\nabla_{\boldsymbol{d}}s + \nabla_{\boldsymbol{a}}\Omega L^{\boldsymbol{a}}{}_{\boldsymbol{d}} = {}& \frac{1}{2}\nabla^{\boldsymbol{a}}\Omega\rho\left(U_{\boldsymbol{a}}U_{\boldsymbol{d}} + \frac{1}{4}g_{\boldsymbol{ad}}\right) + \frac{1}{8}\Omega\rho\nabla_{\boldsymbol{d}}\Omega \\ &+ \frac{1}{24}\Omega^2\nabla_{\boldsymbol{d}}\rho,\end{aligned} \tag{5.10}$$

$$\begin{aligned}2\nabla_{[\boldsymbol{d}}L_{\boldsymbol{c}]\boldsymbol{b}} - \nabla_{\boldsymbol{a}}\Omega W^{\boldsymbol{a}}{}_{\boldsymbol{bdc}} = {}& \Omega\Big(\rho\left(\nabla_{[\boldsymbol{d}}U_{\boldsymbol{c}]}U_{\boldsymbol{b}} + U_{[\boldsymbol{c}}\nabla_{\boldsymbol{d}]}U_{\boldsymbol{b}}\right) \\ &+\nabla_{[\boldsymbol{d}}\rho U_{\boldsymbol{c}]}U_{\boldsymbol{b}} + \frac{1}{3}\nabla_{[\boldsymbol{d}}\rho g_{\boldsymbol{c}]\boldsymbol{b}}\Big) + \rho Z_{\boldsymbol{bdc}},\end{aligned} \tag{5.11}$$

$$\begin{aligned}\nabla_{\boldsymbol{a}}W^{\boldsymbol{a}}{}_{\boldsymbol{bdc}} = {}& \rho\left(\nabla_{[\boldsymbol{d}}U_{\boldsymbol{c}]}U_{\boldsymbol{b}} + U_{[\boldsymbol{c}}\nabla_{\boldsymbol{d}]}U_{\boldsymbol{b}}\right) + \nabla_{[\boldsymbol{d}}\rho U_{\boldsymbol{c}]}U_{\boldsymbol{b}} \\ &+ \frac{1}{3}\nabla_{[\boldsymbol{d}}\rho g_{\boldsymbol{c}]\boldsymbol{b}} + \frac{\rho}{\Omega}Z_{\boldsymbol{bdc}}.\end{aligned} \tag{5.12}$$

The matter equations are given by,

$$U^{\boldsymbol{a}}\nabla_{\boldsymbol{a}}U^{\boldsymbol{d}} = \frac{1}{\Omega}\left(g^{\boldsymbol{da}} + U^{\boldsymbol{d}}U^{\boldsymbol{a}}\right)\nabla_a\Omega, \tag{5.13}$$

$$U^{\boldsymbol{a}}\nabla_{\boldsymbol{a}}\rho = -\rho\chi^{\boldsymbol{a}}{}_{\boldsymbol{a}}. \tag{5.14}$$

In the above, the following fields has been defined:

$$s \equiv \frac{1}{4}\nabla^{\boldsymbol{a}}\nabla_{\boldsymbol{a}}\Omega + \frac{1}{24}\Omega R[\boldsymbol{g}], \tag{5.15}$$

$$W^{\boldsymbol{d}}{}_{\boldsymbol{abc}} \equiv \Omega^{-1}C^{\boldsymbol{d}}{}_{\boldsymbol{abc}}, \tag{5.16}$$

$$Z_{\boldsymbol{bdc}} \equiv \nabla_{[\boldsymbol{d}}\Omega g_{\boldsymbol{c}]\boldsymbol{b}} + 2\nabla_{[\boldsymbol{d}}\Omega U_{\boldsymbol{c}]}U_{\boldsymbol{b}} + U_{[\boldsymbol{d}}g_{\boldsymbol{c}]\boldsymbol{b}}g^{\boldsymbol{ef}}\nabla_{\boldsymbol{e}}\Omega U_{\boldsymbol{f}}. \tag{5.17}$$

Remark 5.3. *The main interest is to find solutions to the system* (5.8)-(5.14) *in the domain* $\Omega > 0$ *which admit a meaningful limit on* $\mathscr{I}^+$. *It was found by Friedrich that a necessary condition to have this type of solutions is that the geodesics generated by* $\tilde{U}$ *approach* $\mathscr{I}^+$ *orthogonally —see [21].*

In the above and in what follows, the frame field is fixed by choosing $e_{\mathbf{0}} = \boldsymbol{U}$ and the *Lagrangian gauge* — i.e. given coordinates x^μ in a neighbourhood $\mathcal{U} \subset \mathcal{M}$ then the frame components of $e_{\mathbf{0}}$ are given by $e_{\mathbf{0}}{}^\mu = \delta_{\mathbf{0}}{}^\mu$. Moreover, *Fermi propagation* of the spatial components of the frame will be employed. More precisely, one has that,

$$\Gamma_{\mathbf{0}}{}^{\boldsymbol{a}}{}_{\boldsymbol{b}} = 0.$$

5.1.3 Regularisation of the equations

In order for the above system to be of use, it is necessary to deal with the singular equations (5.12) and (5.13). To do so, one makes use of the *conformal geodesic equation*,

$$U^a \tilde{\nabla}_a U^b + 2b_a V^a U^b - \tilde{g}_{ac} U^a U^c b^b = 0$$
$$U^a \tilde{\nabla}_a b_b - b_a U^a b_b + \frac{1}{2} \tilde{g}^{ac} b_a b_c U_b - U^a \tilde{L}_{ab} = 0,$$

where $\tilde{L}_{ab}$ is the Shouten tensor for the interior metric and b_a a one form associated with a curve $\gamma(\boldsymbol{\tau})$ for which U^a is the tangent vector. Furthermore, we have defined

$$b^a \equiv g^{ab} b_b, \qquad U_a \equiv g_{ab} U^b.$$

A solution $(b_a(\tau), U^a(\tau))$ to the conformal geodesic equation is called a *conformal geodesic*.

Remark 5.4. *The one form b_a can be thought of as an acceleration associated with U^a. Thus, for $b_a = 0$, a conformal geodesic coincides with a metric geodesic.*

Given local coordinates $x = (x^\mu)$ and a curve parametrised by $\sigma \in \mathbb{R}$, Friedrich defines a new 1-form f_a with coordinate components,

$$f_\nu(\sigma) \equiv b_\nu(\sigma) - \Omega^{-1}\nabla_\nu \Omega|_{x(\sigma)}, \tag{5.18}$$

where b_ν are the components in local coordinates of a one-form satisfying the geodesic equation. If, in addition, the coordinate components of a vector V^b are given by $V^\mu(\sigma) = \dfrac{dx^\mu}{d\sigma}$, then $\left(f_a, V^b\right)$ is a solution to the equations

$$V^a\nabla_a V^b + 2f_a V^a V^b - V^a V_a f^b = 0, \tag{5.19}$$

$$V^a\nabla_a f_b - f_a V^a f_b + \frac{1}{2} f^a f_a V_b - V^a L_{ab} = 0, \tag{5.20}$$

where,

$$f^a \equiv g^{ab} f_a, \qquad V_a \equiv g_{ab} V^b,$$

and L_{ab} is the Schouten tensor with respect to the conformal metric.

Remark 5.5. *Observe that equations* (5.19) *and* (5.20) *involve only conformal fields, as opposed to the conformal geodesic equation.*

By assuming that V^a is related to the tangent vector of a geodesic of the matter particles via

$$V^a = \omega^{-1}\tilde{U}^a, \qquad \omega^{-1} \equiv \frac{dt}{d\sigma},$$

and with the relations

$$V^a V_a = -\theta^{-2}, \qquad \theta = \frac{\omega}{\Omega}, \qquad U^a \nabla_a \theta = \theta U^a f_a,$$

it can then be shown — see [21] for details —, using the definition for f_a, that one obtains a *regularising relation* which in frame indices takes the form

$$\nabla_{\boldsymbol{a}}\Omega = -\left(\nabla_{\boldsymbol{0}}\Omega + \Omega f_{\boldsymbol{0}}U_{\boldsymbol{a}}\right) - \Omega f_{\boldsymbol{a}}. \tag{5.21}$$

Using the above equation in (5.12) and (5.13) one removes the singularities in the system of equations (5.8)-(5.14) which now can be smoothly extended to the conformal boundary. In other words, one has a regular system of field equations. Moreover, it can be shown that these equations imply a symmetric hyperbolic system of equations for the unknowns

$$(e_{\boldsymbol{i}}{}^{\mu}, \Gamma_{\boldsymbol{k}}{}^{\boldsymbol{i}}{}_{\boldsymbol{j}}, f_{\boldsymbol{d}}, \varsigma_{\boldsymbol{ij}}, \xi, \Omega, \Sigma_{\boldsymbol{d}}, s, L_{\boldsymbol{0i}}, L_{\boldsymbol{ij}}, \rho, \omega_{\boldsymbol{ij}}, \omega^{*}{}_{\boldsymbol{ij}}).$$

More precisely, one has the equations

$$\partial_0 e_{\boldsymbol{i}}{}^{\mu} = -f_{\boldsymbol{i}}\delta_{\boldsymbol{0}}{}^{\mu} - \chi_{\boldsymbol{i}}{}^{\boldsymbol{j}} e_{\boldsymbol{j}}{}^{\mu}, \tag{5.22}$$

$$\partial f_{\boldsymbol{0}} = -\frac{1}{2} f_{\boldsymbol{a}} f^{\boldsymbol{a}} + L_{\boldsymbol{00}}, \tag{5.23}$$

$$\partial_{\boldsymbol{0}} f_{\boldsymbol{i}} = L_{\boldsymbol{0i}}, \tag{5.24}$$

$$\partial_{\boldsymbol{0}}\varsigma_{\boldsymbol{ij}} = -\Omega\left(\varsigma_{\boldsymbol{i}}{}^{\boldsymbol{k}}\varsigma_{\boldsymbol{kj}} - \frac{1}{3}\varsigma^{\boldsymbol{kl}}\varsigma_{\boldsymbol{lk}} g_{\boldsymbol{ij}}\right) - \frac{2}{3}\left(\nabla_{\boldsymbol{U}}\Omega\right)^{-1}\left(\Omega\xi - 3s\right)\varsigma_{\boldsymbol{ij}} \tag{5.25}$$

$$- W_{\boldsymbol{0i0j}}, \tag{5.26}$$

$$\partial_{\boldsymbol{0}}\xi = \left(\nabla_{\boldsymbol{U}}\Omega\right)^{-1}\left(\Omega\xi - 3s\right)\left(-\frac{1}{3}\xi + f_{\boldsymbol{i}} f^{\boldsymbol{i}} - L_{\boldsymbol{00}} + \frac{1}{4}\rho\Omega\right)$$

$$- \nabla_{\boldsymbol{U}}\Omega\Omega\varsigma^{\boldsymbol{kl}}\varsigma_{\boldsymbol{lk}} + 3 f^{\boldsymbol{i}} L_{\boldsymbol{i0}} - \frac{3}{4}\rho\nabla_{\boldsymbol{U}}\Omega, \tag{5.27}$$

$$\nabla_{\boldsymbol{0}}\Omega = \Sigma_{\boldsymbol{0}}, \tag{5.28}$$

$$\nabla_{\boldsymbol{0}}\Sigma_{\boldsymbol{d}} = -\Omega L_{\boldsymbol{0d}} + s g_{\boldsymbol{0d}} + \frac{1}{2}\Omega^2\rho\left(U_{\boldsymbol{0}}U_{\boldsymbol{d}} + \frac{1}{4}g_{\boldsymbol{0d}}\right), \tag{5.29}$$

$$\nabla_{\boldsymbol{0}} s = -\nabla^{\boldsymbol{a}}\Omega L_{\boldsymbol{a0}} = \frac{1}{2}\Omega\rho\nabla^{\boldsymbol{a}}\Omega\left(U_{\boldsymbol{0}}U_{\boldsymbol{d}} + \frac{1}{4}g_{\boldsymbol{0d}}\right) + \frac{1}{8}\Omega\rho\nabla_{\boldsymbol{0}}\Omega$$

$$+ \frac{1}{24}\Omega^2\nabla_{\boldsymbol{0}}\rho, \tag{5.30}$$

$$\nabla_{\boldsymbol{0}} L_{\boldsymbol{0i}} = h^{\boldsymbol{ij}}\nabla_{\boldsymbol{j}} L_{\boldsymbol{ik}} + \frac{1}{6}\nabla_{\boldsymbol{i}} R + K^{\boldsymbol{b}}{}_{\boldsymbol{bi}}, \tag{5.31}$$

$$\nabla_{\boldsymbol{0}} L_{\boldsymbol{ii}} = \nabla_{\boldsymbol{i}} L_{\boldsymbol{0i}} + K_{\boldsymbol{i0i}}, \tag{5.32}$$

$$\nabla_{\boldsymbol{0}} L_{\boldsymbol{ij}} = \nabla_{\boldsymbol{i}} L_{\boldsymbol{0j}} + \nabla_{\boldsymbol{j}} L_{\boldsymbol{0i}} + K_{\boldsymbol{i0j}} + K_{\boldsymbol{j0i}}, \tag{5.33}$$

$$\nabla_0 \omega_{\boldsymbol{ij}} + D_{\boldsymbol{k}} {\omega^*}_{\boldsymbol{l}(\boldsymbol{j}} \epsilon_{\boldsymbol{i})}{}^{\boldsymbol{kl}} = L.O.T, \tag{5.34}$$

$$\nabla_{\boldsymbol{0}} {\omega^*}_{\boldsymbol{ij}} - D_{\boldsymbol{k}} \omega_{\boldsymbol{l}(\boldsymbol{j}} \epsilon_{\boldsymbol{i})}{}^{\boldsymbol{kl}} = L.O.T, \tag{5.35}$$

where $L.O.T$ stand for "lower order terms." In the above, the following fields have been defined:

$$\omega_{\boldsymbol{ab}} \equiv W_{\boldsymbol{cdrs}} U^c U^r h^{\boldsymbol{d}}{}_{\boldsymbol{a}} h^{\boldsymbol{s}}{}_{\boldsymbol{b}}, \qquad \omega^*_{\boldsymbol{ab}} \equiv \frac{1}{2} W_{\boldsymbol{cdpq}} \epsilon^{\boldsymbol{mn}}{}_{\boldsymbol{rs}} U^c U^r h^{\boldsymbol{d}}{}_{\boldsymbol{a}} h^{\boldsymbol{s}}{}_{\boldsymbol{b}}, \tag{5.36}$$

$$\varsigma_{\boldsymbol{ij}} \equiv \Omega^{-1} \left(\xi_{\boldsymbol{ij}} - \frac{1}{3} g_{\boldsymbol{ij}} \xi \right), \qquad \xi \equiv \Omega^{-1} \left(\nabla_{\boldsymbol{U}} \Omega \chi + 3s \right), \qquad \Sigma_{\boldsymbol{d}} \equiv \nabla_{\boldsymbol{d}} \Omega. \tag{5.37}$$

We have also made use of the relations

$$\chi_{\boldsymbol{ij}} = \Omega \varsigma_{\boldsymbol{ij}} + \frac{1}{3} \left(\nabla_{\boldsymbol{U}} \Omega \right)^{-1} \left(\Omega \xi - 3s \right) g_{\boldsymbol{ij}}, \tag{5.38}$$

$$\varsigma_{\boldsymbol{ij}} = - \left(\nabla_{\boldsymbol{U}} \Omega \right)^{-1} \left(D_{\boldsymbol{i}} f_{\boldsymbol{j}} - f_{\boldsymbol{i}} f_{\boldsymbol{j}} - L_{\boldsymbol{ij}} - \frac{1}{3} \left(D_{\boldsymbol{k}} f^{\boldsymbol{k}} - f_{\boldsymbol{k}} f^{\boldsymbol{k}} - L_{\boldsymbol{k}}{}^{\boldsymbol{k}} g_{\boldsymbol{ij}} \right) \right), \tag{5.39}$$

$$\xi = - D_{\boldsymbol{i}} f^{\boldsymbol{i}} + f_{\boldsymbol{i}} f^{\boldsymbol{i}} + L_{\boldsymbol{i}}{}^{\boldsymbol{i}} - \frac{3}{8} \Omega \rho, \tag{5.40}$$

$$- L_{\boldsymbol{00}} + g^{\boldsymbol{ij}} L_{\boldsymbol{ij}} = L_{\boldsymbol{b}}{}^{\boldsymbol{b}} = \frac{1}{6} R. \tag{5.41}$$

5.1.4 Relation to the interior field equations

The equations (5.8)-(5.12) and (5.13)-(5.14) are evolution equations to the reduced system. This is, however, only a subset of the full Einstein-frame-equations. The remaining equations are constraints. It is therefore necessary to show that these constraints propagate, which indeed has been done in [21]. We thus have the following theorem adapted from Friedrich:

Theorem 9. A solution

$$\mathbf{u} = \left(e_{\boldsymbol{i}}{}^{\mu}, \Gamma_{\boldsymbol{k}}{}^{\boldsymbol{i}}{}_{\boldsymbol{j}}, f_{\boldsymbol{d}}, \varsigma_{\boldsymbol{ij}}, \xi, \Omega, \Sigma_{\boldsymbol{d}}, s, L_{\boldsymbol{0i}}, L_{\boldsymbol{ij}}, \rho, \omega_{\boldsymbol{ij}}, {\omega^*}_{\boldsymbol{ij}} \right)$$

to the symmetric hyperbolic system (5.22)-(5.35) satisfying the constraint equations associated to the conformal equations (5.8)-(5.12) and (5.13)-(5.14) on an initial hypersurface implies a solution to the Einstein-λ-dust system (5.1)-(5.4) whenever $\Omega \neq 0$.

In the following we will consider two different types of initial hypersurfaces for the conformal evolution equations (5.22)-(5.35):

(i) the conformal boundary $\mathscr{I}^+$;

(ii) standard Cauchy hypersurface $\mathcal{S}_\star$.

As it will be seen in more detail in what follows, initial data for the conformal evolution equations on a standard hypersurface can be obtained from the solution of the Hamiltonian and momentum constraints implied by the Einstein-λ-dust system (5.1)-(5.4).

5.2 Backward evolution of self-gravitating dust balls

The purpose of this section is to study the (backward) evolution of asymptotic initial data for the conformal Einstein-λ-dust system which describes a collection of self-gravitating dust balls.

The following result is obtained by assuming certain gauge choices on a hyper surface $\mathcal{S}$ which later is interpreted to be the conformal boundary $\mathscr{I}^+$. These are

$$s = 0, \qquad \chi_{\boldsymbol{ij}} = 0, \qquad \nabla_{\boldsymbol{a}}\nabla_{\boldsymbol{b}}\Omega = 0, \qquad L_{\boldsymbol{0i}} = L_{\boldsymbol{i0}} = 0.$$

Note, however, that these choices may not be satisfied if one evolves a solution from $\mathcal{M}$ to $\mathcal{S}$. In that case, the reader is referred to the original article [21] for details.

5.2.1 Asymptotic initial data for self-gravitating dust balls

All throughout it is assumed that the initial hyper surface $\mathscr{I}^+$ representing the conformal boundary is a compact 3-manifold. For (asymptotic) initial data prescribed on a hyper surface corresponding to the conformal boundary the following holds:

> Lemma 3. [Adopted from [21]] Any smooth initial data set for the conformal evolution equations (5.22)-(5.35) is uniquely determined on $\mathcal{I}^+$ by a Riemannian metric h_{ij}, the density $\rho \geq 0$, the acceleration f_i and symmetric, $\boldsymbol{h}$-tracefree tensor field ω_{ij}, which are arbitrary up to the relation
> $$D^i\omega_{ij} = \frac{1}{3}D_j\rho - \rho f_j, \tag{5.42}$$
> on $\mathcal{I}^+$, and where D denotes the Levi-Civita operator defined by h_{ij}.

Observe that ρ is allowed to be zero. This suggest to consider a density profile which represents patches of dust in an otherwise empty space. In the case of a strictly positive density function, the data $\rho, \boldsymbol{\omega}$ and $\boldsymbol{h}$ can be prescribed freely, and equation (5.42) is read as a defining equation for the acceleration $\boldsymbol{f}$, unless one has further conditions on $\boldsymbol{f}$ such as hypersurface orthogonality etc., in which case the equation must be treated as a differential equation. **In the following it will be shown how the above result can be used to construct asymptotic initial data representing a collection of dust balls.**

The starting observation of our analysis is the fact that equation (5.42) in Lemma 3 is an underdetermined condition (3 equations) for the 5 independent components of the tracefree tensor ω_{ij}. Nevertheless, this type of divergence equations are well understood in the context of the analysis of the momentum constraint —see e.g. [35].

In the following it will be convenient to define

$$\varpi_{ij} \equiv D_i s_j + D_j s_i - \frac{2}{3}h_{ij}D^k s_k + \Psi'_{ij}, \tag{5.43}$$

where Ψ'_{ij} is a symmetric $\boldsymbol{h}$-tracefree tensor field which may be freely specified and s_i is an arbitrary covector field. A direct computation shows that ϖ_{ij} is a solution of (5.42) if s_i satisfies

$$\Delta_h s_j + D^i D_j s_i - \frac{2}{3} D_j D_k s^k = k_j - D^i \Psi'_{ij}, \tag{5.44}$$

where we have defined

$$k_i \equiv \frac{1}{3} D_j \rho - \rho f_j.$$

Equation (5.44) is of elliptic type —in particular, it provides 3 equations for the 3 components of s_i. It is convenient to reformulate the above equations by defining the operators

$$\begin{aligned}
&\boldsymbol{\delta}\left(\boldsymbol{\omega}\right)_j \equiv D^i \omega_{ij}, \\
&\boldsymbol{L}\left(\boldsymbol{s}\right)_{ij} \equiv D_i s_j + D_j s_i - \frac{2}{3} h_{ij} D^k s_k, \\
&\mathcal{L}\left(\boldsymbol{s}\right)_j \equiv \Delta_h s_j + D^i D_j s_i - \frac{2}{3} D_j D_k s^k.
\end{aligned}$$

We shall refer to these throughout as the *divergence operator*, the *conformal Killing operator* and the *vector Laplacian operator*, respectively. It is readily seen that the vector Laplacian operator is a result of the composition of the divergence and conformal Killing operator —i.e. we have

$$\mathcal{L}\left(\boldsymbol{s}\right) = \left(\boldsymbol{\delta} \circ \boldsymbol{L}\right)\left(\boldsymbol{s}\right). \tag{5.45}$$

In terms of the above definitions, equations (5.42) and (5.44) take the simple form

$$\boldsymbol{\delta}\left(\boldsymbol{\omega}\right)_j = k_j, \tag{5.46}$$

$$\mathcal{L}\left(\boldsymbol{s}\right)_j = k_j - D^i \Psi'_{ij}. \tag{5.47}$$

A solution $\boldsymbol{s}$ of equation (5.47) solves equation (5.46) if the symmetric tracefree tensor $\boldsymbol{\omega}$ is of the form given by (5.43). To solve the elliptic equation for the covector $\boldsymbol{s}$ we make use of the following [48]:

Fact 5.6 (Fredholm alternative). Given any $\boldsymbol{u}$ and $\boldsymbol{v} \in L^2$, then there exists a solution $\boldsymbol{u}$ of the elliptic equation

$$\mathcal{L}(\boldsymbol{u}) = \boldsymbol{F}$$

if there exists a $\boldsymbol{v}$ which solves $\mathcal{L}^*(\boldsymbol{v}) = 0$ and satisfy the L^2-inner product

$$\langle \boldsymbol{v}, \boldsymbol{F} \rangle = \int_{\mathcal{S}} h^{ij} v_i F_j \mathrm{d}\mu = 0. \tag{5.48}$$

The operators $\boldsymbol{\delta}$ and $\boldsymbol{L}$ can be regarded as formal adjoints of each other under the standard L^2-inner product over a compact 3-manifold $\mathcal{S}$. It then follows that their composition, the operator $\mathcal{L}$, is self-adjoint —that is

$$\langle \boldsymbol{u}, \mathcal{L}(\boldsymbol{s}) \rangle = \langle \mathcal{L}(\boldsymbol{u}), \boldsymbol{s} \rangle.$$

Proof. By the definition of the adjoint and the L^2-inner product we have that

$$\begin{aligned} \langle \boldsymbol{\delta}(\boldsymbol{\omega}), \boldsymbol{s} \rangle &= \langle \boldsymbol{\delta}^*(\boldsymbol{s}), \boldsymbol{\omega} \rangle \\ &= \int_{\mathcal{S}} \boldsymbol{\delta}^*(\boldsymbol{s})^{\boldsymbol{ij}} \omega_{\boldsymbol{ij}} d\mu, \end{aligned} \tag{5.49}$$

where we have used the definition for the L^2-inner product. Hence,

$$\int_{\mathcal{S}} \boldsymbol{\delta}(\boldsymbol{\omega})_{\boldsymbol{j}} s^{\boldsymbol{j}} d\mu = \int_{\mathcal{S}} \boldsymbol{\delta}^*(\boldsymbol{s})^{\boldsymbol{ij}} \omega_{\boldsymbol{ij}} d\mu. \tag{5.50}$$

Evaluating the left hand side of the above equation by using the definition of the divergence operator and equation (5.46), and using integration by parts, we find

$$\begin{aligned} \int_{\mathcal{S}} \boldsymbol{\delta}(\boldsymbol{\omega})_{\boldsymbol{j}} s^{\boldsymbol{j}} d\mu &= -\int_{\mathcal{S}} D^{\boldsymbol{i}} \kappa_{\boldsymbol{i}} d\mu + \int_{\mathcal{S}} \omega_{\boldsymbol{ij}} D^{\boldsymbol{i}} s^{\boldsymbol{j}} d\mu \\ &= -\int_{\partial\mathcal{S}} n^{\boldsymbol{i}} \kappa_{\boldsymbol{i}} dA + \int_{\mathcal{S}} \omega_{\boldsymbol{ij}} D^{\boldsymbol{i}} s^{\boldsymbol{j}} d\mu. \end{aligned} \tag{5.51}$$

In the above, $n^{\boldsymbol{i}}$ is the components of the normal on ∂S, dA the infinitesimal surface area and we have defined

$$\kappa_{\boldsymbol{i}} \equiv \omega_{\boldsymbol{ij}} s^{\boldsymbol{j}}.$$

Remark 5.7. *We are interested in studying the fields* $\boldsymbol{\omega}$ *and* $\boldsymbol{s}$ *on the conformal boundary of space time — i.e. we let* $S = \mathcal{I}^+$ *— as such, the boundary* ∂S *is the empty set; more precisely, we have that* $\partial \mathcal{I}^+ = \{\mathbf{0}\}$*. Thus, the boundary term vanish,*

$$\int_{\partial S} n^{\boldsymbol{i}} \kappa_{\boldsymbol{i}} dA = 0.$$

Hence, we have

$$\int_{\mathcal{S}} \boldsymbol{\delta}^* (\boldsymbol{s})^{\boldsymbol{ij}} \omega_{\boldsymbol{ij}} d\mu = \int_{\mathcal{S}} \omega_{\boldsymbol{ij}} D^{\{\boldsymbol{i}} s^{\boldsymbol{j}\}} d\mu,$$

with $^{\{ij\}}$ denoting the symmetric trace free components. This is a consequence of the summation with the symmetric trace free tensor field ω_{ij}. Direct inspection shows that

$$\boldsymbol{\delta}^* (\boldsymbol{s})^{\boldsymbol{ij}} = D^{\{\boldsymbol{i}} s^{\boldsymbol{j}\}}. \tag{5.52}$$

Expanding the above, we have

$$\begin{aligned} \boldsymbol{\delta}^* (\boldsymbol{s})^{\boldsymbol{ij}} &= \frac{1}{2} \left(D_{\boldsymbol{i}} s_{\boldsymbol{j}} + D_{\boldsymbol{j}} s_{\boldsymbol{i}} - \frac{2}{3} h_{\boldsymbol{ij}} D^{\boldsymbol{k}} s_{\boldsymbol{k}} \right) \\ &= \frac{1}{2} \boldsymbol{L} (\boldsymbol{s})^{\boldsymbol{ij}} . \end{aligned} \tag{5.53}$$

Similarly, we have for the operator $\boldsymbol{L}$

$$\begin{aligned} \langle \boldsymbol{L} (\boldsymbol{s}) , \boldsymbol{\omega} \rangle &= \langle \boldsymbol{L}^* (\boldsymbol{s}) , \boldsymbol{\omega} \rangle \\ &= \int_{\mathcal{S}} \boldsymbol{\delta}^* (\boldsymbol{s})^{\boldsymbol{ij}} \omega_{\boldsymbol{ij}} d\mu \end{aligned} \tag{5.54}$$

The left hand side can be expanded using the definition for the conformal killing operator and the L^2- inner product

$$\langle \boldsymbol{L} (\boldsymbol{s}) , \boldsymbol{\omega} \rangle = \int_{\mathcal{S}} \left(D_{\boldsymbol{i}} s_{\boldsymbol{j}} + D_{\boldsymbol{j}} s_{\boldsymbol{i}} - \frac{2}{3} h_{\boldsymbol{ij}} D^{\boldsymbol{k}} s_{\boldsymbol{k}} \right) \omega^{\boldsymbol{ij}} dV. \tag{5.55}$$

Using the chain rule and observing that $\boldsymbol{\omega}$ is trace free we readily obtain

$$
\begin{aligned}
\langle \boldsymbol{L}(\boldsymbol{s}), \boldsymbol{\omega} \rangle &= -\int_{\mathcal{S}} D_{\boldsymbol{i}} \omega^{\boldsymbol{ij}} dV + \int_{\mathcal{S}} D_{\boldsymbol{i}} \kappa^{\boldsymbol{i}} dV \\
&\quad - 2\int_{\mathcal{S}} D^{\boldsymbol{k}} (\omega \kappa_{\boldsymbol{k}}) dV + \int_{\mathcal{S}} s_{\boldsymbol{k}} D^{\boldsymbol{k}} \omega dV \\
&= -\int_{\mathcal{S}} D_{\boldsymbol{i}} \omega^{\boldsymbol{ij}},
\end{aligned}
\tag{5.56}
$$

where again $\kappa^{\boldsymbol{i}} \equiv \omega^{\boldsymbol{ij} s_{\boldsymbol{j}}}$ and ω denotes the $\boldsymbol{h}$-trace free part of $\boldsymbol{\omega}$. In the last step we have also made use of the divergence theorem and Remark 5.7. In order to make use of the Fredholm alternative to establish the existence of solutions to equation (5.47) it is necessary to identify the Kernel of the operator $\mathcal{L}$. For this, it is observed that

$$
0 = \langle \boldsymbol{v}, \mathcal{L}(\boldsymbol{v}) \rangle = \langle \boldsymbol{v}, (\boldsymbol{\delta} \circ \boldsymbol{L})(\boldsymbol{v}) \rangle \tag{5.57}
$$

$$
= \langle \boldsymbol{\delta}^*(\boldsymbol{v}), \boldsymbol{L}(\boldsymbol{v}) \rangle = \langle \boldsymbol{L}(\boldsymbol{v}), \boldsymbol{L}(\boldsymbol{v}) \rangle. \tag{5.58}
$$

Consequently, any element of the Kernel of $\mathcal{L}$ satisfies the equation $\boldsymbol{L}(\boldsymbol{v}) = 0$ —that is, the Kernel consists of conformal Killing vectors. Thus, if the pair $(\mathcal{S}, \boldsymbol{h})$ does not have conformal Killing vectors (this is the generic situation) then there are no obstructions to the existence of solutions to equation (5.47). On the other hand, if conformal Killing vectors are present then the Kernel orthogonality condition in the Fredholm alternative, equation (5.48), has to be satisfied.

The discussion of the previous paragraph is summarised in the following result where all the relevant fields are assumed to be suitably smooth:

Lemma 4. Let $\mathcal{S}$ denote a compact 3-dimensional manifold. Given a (Riemannian) metric h_{ij}, a $\boldsymbol{h}$-tracefree tensor Ψ'_{ij}, a covector f_i and a scalar ρ over $\mathcal{S}$ then one of the following holds:

(i) if $(\mathcal{S}, h_{ij})$ admits no conformal Killing vectors then the tracefree tensor ϖ_{ij} given by equation (5.43) gives a solution to the asymptotic constraint (5.42);

(ii) if $(\mathcal{S}, h_{ij})$ admits conformal Killing vectors then ϖ_{ij} given by equation (5.43) gives a solution to the asymptotic constraint (5.42) if and only if

$$\int_{\mathcal{S}} v^{i}\left(k_{i} - D^{j}\Psi'_{ij}\right) d\mu = 0$$

for any conformal Killing vector v^i.

Remark 5.8. *In the present context, the simplest example of a pair $(\mathcal{S}, h_{ij})$ with conformal Killing vectors is the 3-sphere $\mathbb{S}^3$ with the round metric. In this case one has, in fact, the maximal number of conformal Killing vectors (10) for a 3-dimensional manifold.*

Remark 5.9. *The freely specifiable data given by the tracefree tensor Ψ'_{ij} can be thought of as a candidate describing some gravitational wave content [49].*

In order to construct initial data representing a collection of balls of dust, let Σ_i, $i = 1, \dots, n$ denote n compact open subsets on $\mathcal{S}$ and consider a smooth non-negative scalar field ρ over $\mathcal{S}$ with support on the union of the sets Σ_i. That is, we require that

$$\begin{cases} \rho > 0, & \rho \in \bigcup\limits_{i=1}^{n} \Sigma_i, \\ \rho = 0, & \rho \in \mathcal{I}^{+} / \bigcup\limits_{i=1}^{n} \Sigma_i. \end{cases} \tag{5.59}$$

Lemma 4 gives the conditions for the existence of solution to the asymptotic constraint (5.42) for this type of density profile ρ and a given choice of metric h_{ij} and fields Ψ'_{ij} and f_i.

Remark 5.10. *Let ρ be a smooth non-negative scalar field given by* (5.59)*. Furthermore, let $\boldsymbol{\Psi}'$ be a symmetric, $\boldsymbol{h}$-tracefree spatial tensor field, $\boldsymbol{h}$ the projector metric and $\boldsymbol{f}$ a one form, then we say that $(\rho, \boldsymbol{h}, \boldsymbol{\Psi}', \boldsymbol{f})$ is* ***n-body dust asymptotic data*** *if either condition (i) or (ii) of Lemma 4 holds on $\mathcal{I}^+$.*

5.2.2 Evolution of the asymptotic data

The asymptotic data constructed in the previous subsection can be readily combined with the conformal evolution equations of Section 5.1.3 to obtain the asymptotic region of a spacetime with positive Cosmological constant containing a collection of n balls of dust. The key observation here is that as we are working in the conformal picture, any interval of time of the conformal boundary represents an infinite time domain from the physical perspective. The existence result can be stated as follows:

Theorem 10. Given a choice of asymptotic data representing a collection of n dust balls, there exists a time $\tau > 0$ such that the conformal Einstein-λ-dust equations have a unique smooth solution on the slab $[0, \tau) \times \mathcal{S}$ associated to this data. This solution implies, in turn, a solution to the (interior) Einstein-λ-dust system on $(0, \tau) \times \mathcal{S}$ for which the hypersurface $\{0\} \times \mathcal{S}$ corresponds to the conformal boundary $\mathcal{I}^+$.

Remark 5.11. *By restricting the existence time further, if necessary, it is possible to ensure that the congruence of conformal geodesics on which our gauge is based remains non-intersecting for the interval $[0, \tau]$. This, in turn, ensures that dust balls in the initial asymptotic configuration do not intersect each other in the past.*

In terms of Physics, Theorem 10, suggest the following: if, in the infinite far future of an expanding universe, one is given a matter distribution representing patches of dust balls, then one can evolve this system backward in time for as long as one wish, and still have that the patches of dust remain non-intersecting.

5.3 Forward evolution of dust balls

In this section we consider the more physically realistic setting of the evolution of dust balls from a standard Cauchy hypersurface in a spacetime with positive Cosmological constant. Our strategy is to consider this setting as a perturbation of the de Sitter spacetime in order to make a statement of the future global existence of the dust balls. As in the case of the backwards evolution we start by constructing suitable initial data.

5.3.1 Standard Cauchy initial data for self-gravitating dust balls

Let $\tau \in \tilde{\mathscr{M}}$ be a positive function such that for $t \in \mathbb{R}$, $\tau(p) = t$ gives the level surfaces $\tilde{\mathcal{S}}_t$. We denote by $\tilde{\mathcal{S}}_\star \subset \tilde{\mathscr{M}}$ the hypersurface which coincides with the level surface $\tau(p) = 0$, and interpret this as an initial hypersurface at some fiduciary time. The *Einstein constraints* on $\tilde{\mathcal{S}}_\star$ are given by

$$r[\tilde{\boldsymbol{h}}] + \tilde{K}^2 - \tilde{K}_{ij}\tilde{K}^{ij} = 2\left(\tilde{\rho} - \lambda\right), \tag{5.60}$$

$$\tilde{D}^i\tilde{K}_{ij} - \tilde{D}_j\tilde{K} = -\tilde{j}_j, \tag{5.61}$$

where $\tilde{\boldsymbol{h}}$ and $\tilde{K}_{ij}$ denote, respectively, the intrinsic metric and extrinsic curvature of $\tilde{\mathcal{S}}_\star$, $\tilde{D}$ is the Levi-Civita connection of metric $\tilde{\boldsymbol{h}}$ and $r[\tilde{\boldsymbol{h}}]$ its Ricci scalar. Moreover, $\tilde{\rho}$ and $\tilde{j}_i$ denote, respectively, the *energy-density* and *flux current* of the matter content.

The constraints (5.60) and (5.61) will be solved using the *conformal method of Licnerowicz-York* —see e.g. [35]. In the following, let Σ be a conformal factor defined on $\mathscr{M}$. Following the discussion in [5], Chapter 11, let $h_{ij} = \Omega^2\tilde{h}_{ij}$. Implementing this rescaling in equations (5.60)-(5.61) leads to

$$\begin{aligned}
&2\Omega D_iD^i\Omega - 3D_i\Omega D^i\Omega + \frac{1}{2}\Omega^2 r + 3\Sigma^2 \\
&+\frac{1}{2}\Omega^2\left(K^2 - K_{ij}K^{ij}\right) - 2\Omega\Sigma K = \Omega^4\rho - \lambda, \qquad (5.62)\\
&\Omega^3D^iK_{ij} - 2K^i{}_jD_i\Omega - \Omega D_kK + 2D_k\Sigma = \Omega^3 j_k, \qquad (5.63)
\end{aligned}$$

where

$$\rho \equiv \Omega^{-4}\tilde{\rho}, \qquad j_k \equiv \Omega^{-3}\tilde{j}_k, \qquad \Sigma \equiv \nu^i D_i\Xi,$$

and $\Omega = \Xi|_{\tilde{\mathcal{S}}_\star}$. Now, by setting $\Omega = \theta^{-2}$, equation (5.62) leads to the manifestly elliptic equation

$$\mathcal{L}_{\boldsymbol{h}}\theta = \frac{1}{8}\theta\left(K^2 - K_{\boldsymbol{ij}}K^{\boldsymbol{ij}}\right) - \frac{1}{2}\theta^2\Sigma K - \frac{1}{4}\theta^5\left(\theta^{-8}\rho - \lambda\right), \tag{5.64}$$

where we have defined the *Yamabe operator*

$$\mathcal{L}_{\boldsymbol{h}}\theta \equiv D_i D^i\theta - \frac{1}{8}r[\boldsymbol{h}]\theta.$$

Now, defining

$$\psi_{ij} \equiv \theta^4 K_{\{ij\}}, \qquad K_{\{ij\}} = K_{ij} - \frac{1}{3}Kh_{ij},$$

it follows that equation (5.63) leads to the equation

$$D^i\psi_{ij} = \frac{2}{3}\theta^6 D_j\tilde{K} - 2\theta^6 D_j\Sigma + j_j. \tag{5.65}$$

Remark 5.12. *We will consider equations* (5.64) *and* (5.65) *in the particular case that*

$$K = \Sigma = 0.$$

It can be readily verified that the above conditions imply that $\tilde{\mathcal{S}}_\star$ is a maximal hypersurface.

In order to put equation (5.65) in an elliptic form, we make use of the *York splitting* —i.e. given an arbitrary covector field X_i, we consider solutions ψ_{ij} of the form

$$\psi_{ij} = (L_{\boldsymbol{h}}X)_{ij} + \psi'_{ij}, \tag{5.66}$$

where ψ'_{ij} is a freely specifiable symmetric and tracefree tensor field, and $L_{\boldsymbol{h}}\boldsymbol{X}$ is the *conformal Killing operator* defined by

$$(L_{\boldsymbol{h}}X)_{ij} \equiv D_i X_j + D_j X_i - \frac{2}{3}h_{ij}D_k X^k.$$

For simplicity, we set $\psi'_{ij} = 0$, so that substituting (5.66) into equation

(5.65), we obtain the elliptic equation

$$D^i \left(L_{\boldsymbol{h}} X\right)_{ij} = j_j. \tag{5.67}$$

We thus seek to show that there exist a solution to the elliptic equations (5.64) and (5.67) which represents initial data for a de Sitter-like spacetime with an energy density function given by (5.59) —so that it can be regarded as describing a collection of dust balls.

With regards to the solution to equation (5.67) we adapt the following result from [35], Chapter VII, Section 6:

Proposition 5.13. Let $h_{ab} \in H^2(\tilde{\mathcal{S}}_\star)$ and ξ^a be, respectively, a Riemannian metric and a conformal Killing vector over $\tilde{\mathcal{S}}_\star$. Then equation (5.67) has a solution $X^a \in H^2(\tilde{\mathcal{S}}_\star)$ if $j^a \in L^2(\tilde{\mathcal{S}}_\star)$ and

$$\int_{\tilde{\mathcal{S}}_\star} h_{ab} j^a \xi^b dV = 0,$$

where dV denotes the volume form of the metric h_{ab}. The solution is determined up to the addition of a conformal Killing vector. Furthermore, the solution is unique if one imposes

$$\int_{\tilde{\mathcal{S}}_\star} X^a \xi_a dV = 0.$$

In that case there exists a positive constant C such that

$$\|X^a\|^2_{L^2} \leq C \|j^a\|^2_{L^2}.$$

Now, setting $K = \Sigma = 0$ and using tracefree tensor ψ_{ij} defined in equation (5.66), the Lichnerowicz equation (5.64) can be written as

$$D_i D^i \theta - a\theta + b\theta^{-7} + c\theta^5 = 0, \tag{5.68}$$

where

$$a \equiv \frac{1}{8}\boldsymbol{r}[\boldsymbol{h}], \qquad b \equiv \frac{1}{8}\psi_{ij}\psi^{ij}, \qquad c \equiv \frac{1}{4}\left(\tilde{\rho} - \lambda\right), \quad \tilde{\rho} = \Omega^4 \rho = \theta^{-8}\rho.$$

Following the theory developed in [35] Chapter VII, Sections 5, 6 and 7 (see also [50]) the above equation has a unique solution $\theta > 0$ if $b \geq 0$ and $c < 0$. Since one readily has that $\psi_{ij}\psi^{ij} > 0$, the only condition to be imposed is

$$\tilde{\rho} < \lambda.$$

Thus, one has the following

Proposition 5.14. For $r[\boldsymbol{h}] > 0$, $\psi_{ij}\psi^{ij} > 0$ and $\lambda > 0$, the condition $\tilde{\rho} < \lambda$ is a sufficient condition for the existence of a unique solution θ to the Lichnerowicz equation (5.68).

Together, Propositions 5.13 and 5.14 ensure the existence of a large class of solutions to the Einstein constraint equations representing an arbitrary configuration of dust balls at some fiduciary time. For this, as in the asymptotic problem, one chooses the density $\tilde{\rho}$ as in eqution (5.59) — the method for the construction of solutions to the Einstein constraints described above works irrespectively from the fact that the density is only non-zero on a finite number of subsets of $\tilde{\mathcal{S}}_\star$. If, in addition, one chooses the metric $\boldsymbol{h}$ as a constant multiple of the round metric on $\mathbb{S}^3$ —as in the case of the de Sitter spacetime— one can then regard the dust balls as matter-sourced perturbation of the de Sitter spacetime. The size of $\tilde{\rho}$ as described in terms of Sobolev norms controls the closeness of θ to the value 1 (the de Sitter value). This observation is of importance in the discussion of the stability of solutions to the evolution problem.

Remark 5.15. *For the purpose of simplicity of presentation of the subsequent discussion it is convenient to consider a setting in which the initial current vector* $\boldsymbol{j}$ *vanishes. This choice of free data is consistent with the 4-velocity* $\boldsymbol{u}$ *being orthogonal to the initial hypersurface* $\tilde{\mathcal{S}}_\star$. *This choice is made throughout the whole hypersurface regardless of whether the density vanishes or not in a given region. For this choice, if the density vanishes all over the initial hypersurface, then one obtains* trivial data *corresponding to the de Sitter spacetime.*

Following the discussion in [5] Chapter 11, from a solution to the Einstein constraint equations it is possible to obtain a solution to the conformal Einstein field equations by algebraic manipulations and differentiation. The deviation of this data from (vacuum) data for the de Sitter spacetime is controlled by the size of the current $\boldsymbol{j}$ and the density $\tilde{\rho}$.

5.3.2 Long time evolution

In this section we discuss the evolution of the initial data given by Propositions 5.13 and 5.14. In particular, we discuss how the ideas used in the stability of the de Sitter spacetime [44] (see also [5], Chapter 15) can be used to obtain a future global existence statement for the dust balls if the initial density is sufficiently small.

In the following let $\mathbf{u}$ denote a solution to the conformal evolution equations discussed in Section 5.1.3. Moreover, let $\mathring{\mathbf{u}}$ denote the solution to these evolution equations with $\rho = 0$ (i.e. vanishing density) and the 4-velocity u^a chosen so that it is tangent to timelike geodesics in the interior spacetime —see Remark 5.15. Denote by $\mathbf{u}_\star$ and $\mathring{\mathbf{u}}_\star$ the associated initial data on some fiduciary initial hypersurface $\mathcal{S}_\star$. The solution $\mathbf{u}$ provides a conformal representation of the de Sitter spacetime which is smooth up to and beyond the conformal boundary $\mathscr{I}^+$. In particular, it has vanishing rescaled Weyl tensor. For concreteness assume that the conformal boundary for this (background) solution is given by the condition $\tau = \tau_\infty$, for τ_∞ some constant. To this background solution one can readily apply the standard theory of stability for symmetric hyperbolic equations — see [6]; also [5]— to ensure the existence of nearby solutions (in the sense of Sobolev spaces) to the evolution equations with a similar existence time. Accordingly, these solutions extend up to and beyond the conformal boundary. This amounts to a future global existence result. More precisely, one has the following:

Theorem 11. Let $\mathbf{u}_\star$ denote smooth initial data for the conformal Einstein-λ-dust evolution equations on a compact manifold $\mathcal{S}_\star$ describing a configuration of dust balls as given by Propositions 5.13 and 5.14. There exists $\varepsilon > 0$ such that for any initial data $\mathbf{u}_\star$ such that

$$\|\mathbf{u}_\star - \mathring{\mathbf{u}}_\star\|_m < \varepsilon, \qquad m \geq 5,$$

there exists a smooth solution $\mathbf{u}$ to the conformal evolution equations over the domain

$$\mathcal{M} \equiv [\tau_\star, \tau_\infty] \times \mathcal{S},$$

$\mathcal{S} \approx \mathcal{S}_\star$. Moreover, given a sequence of initial data $\mathbf{u}_\star^{(n)}$, as above, such that

$$\left\|\mathbf{u}_\star^{(n)} - \mathring{\mathbf{u}}_\star\right\|_m \to 0, \qquad \text{as} \quad n \to \infty,$$

one has that the corresponding solutions satisfy

$$\left\|\mathbf{u}^{(n)}(\tau, \cdot) - \mathring{\mathbf{u}}(\tau, \cdot)\right\|_m \to 0, \qquad \text{as} \quad n \to \infty.$$

The solution $\mathbf{u}$ implies, in turn, a future geodesically complete solution to the (interior) Einstein-λ-dust system for which $\mathcal{I}^+$ corresponds to future (timelike) infinity.

Proof. The proof of this result follows the same structure of that of the stability of the de Sitter spacetime [44, 45] —see also [5], Chapter 15. Here we provide a brief outline of the main ideas. As already mentioned, the evolution equations (5.22)-(5.35) imply a symmetric hyperbolic evolution system for the components of the unknown vector $\mathbf{u}$. Now writing $\mathbf{u} = \mathring{\mathbf{u}} + \breve{\mathbf{u}}$ where $\mathring{\mathbf{u}}$ denotes the *background* de Sitter solution, it follows that the *perturbation* $\breve{\mathbf{u}}$ also satisfies a symmetric hyperbolic evolution system. Existence of solutions for this system follows from the theory developed in [6]. Moreover, as the perturbed initial data $\breve{\mathbf{u}}_\star$ is small (in the sense of Sobolev spaces), it follows then from Cauchy stability that its existence interval includes the time τ_∞ —so that the development includes the conformal factor. Finally, a *propagation of the constraints* argument ensures the solution to the reduced evolution system implies a solution to the interior Einstein-λ-dust system.

Remark 5.16. *From the discussion leading to Propositions 5.13 and 5.14, it follows that the size (in the Sobolev norm) of the initial data* $\mathbf{u}_\star$ *is controlled by the initial value of the density over* $\mathcal{S}_\star$*. In particular, if* $\rho_\star = 0$ *then* $\mathbf{u}_\star = \mathring{\mathbf{u}}_\star$*. Accordingly, Theorem 11 states that the initial configuration of dust balls will exist globally into the future if the density is sufficiently small —that is, if the dust making up the balls is sufficiently diluted.*

Remark 5.17. *The spacetimes arising from Theorem 11 can be readily shown to be geodesically complete. The simplest manner of doing this is to make use of the theory developed in [51]. The required estimates needed to establish geodesic completeness follow from the closeness (in the sense of Sobolev spaces) of the solution provided by Theorem 11 and the background exact de Sitter solution. In the present case it is possible to show even more: as the background 4-velocity* $\mathring{u}^a$ *is chosen to be tangent to a congruence of non-intersecting conformal geodesics, it follows that if the perturbed solutions given by Theorem 11 are the flow lines of* u^a*, then they are also non-intersecting. This observation shows, in addition, that the various members of an arbitrary configuration of dust balls never intersect in the future.*

The purpose of this chapter is the development of a model of self-gravitating bodies in General Relativity for which it is possible to make statements of long-term existence. As mentioned in the introduction, the well-posedness and local existence in time of self-gravitating balls of dust has been given in [47]. These self-gravitating bodies possess a smooth boundary (in the sense that the density is assumed to go to zero smoothly). This observation, combined with an evolution law for the 4-velocity which is well defined even in the regions where the density vanishes allows one to obtain a suitable evolution system for which existence theory is available.

The analysis of the Einstein-λ-dust system in [21] provides a conformal analogue to this system and thus it allows one to implement an argument establishing long-term existence of dust ball configurations. The physical mechanism making it possible to run this argument is the *acceleration* provided by the Cosmological constant λ. It should be mentioned that an extension of this result to the setting where $\lambda = 0$ is made much more challenging by the fact that in this scenario, and following a conformal point of view, timelike geodesics converge at future timelike infinity i^+. Accordingly, any attempt to analyse the long-term existence of matter configurations is tied to the development of a suitable description of this asymptotic point.

Chapter 6

Conclusions and Outlook

We provide here a short summary of the key findings of this thesis and how these findings motivate the study of further interesting problems.

6.0.1 Conclusions

In this thesis we have applied two very different forms of analysis to the Einstein field equations. The first method described in Chapter 4, employ the theory of hyperbolic differential equations in order to show that the Einstein field equations coupled to a generic matter model admits a well posed Cauchy problem. This result is obtained by showing that the Einstein-matter equations reduce to a first order symmetric hyperbolic set of evolution equations, and that a solution to these equations is also a solution to the original Einstein-matter system. It is understood in the above that an equation for the matter density ρ is given and we require that the energy momentum tensor take the form of that of dust plus a spatial tensor field Π_{ab}. We treat the specific matter models of dust and perfect fluid, and provide a short discussion on elasticity. The generality employed in the hyperbolic reduction in this thesis gives a framework to analyse the Cauchy problem for future Einstein-matter systems not mentioned herein. The process of showing well posedness of a wide range of Einstein-matter systems is thus much simplified.

In Chapter 5 we employ conformal methods to show future stability of the Einstein-λ-dust equations with a energy density representing balls of dust. We show that one can evolve the balls of dust backward in time from future infinity as well as from an initial hypersurface into the infinite future without the geodesics ever forming acoustic shocks. That is, the geodesics will be future and past complete for the respective situations, given that the energy density is sufficiently small. It is understood that λ represents the cosmological constant and is assumed to be positive. The Cosmological Universe studied in this chapter is a toy model. That is to say, it may not be directly applied to study the Cosmology of the observable Universe. However, our finding is still of interest to the study of physical Cosmology in that we prove — from a mathematical point of view — that a positive cosmological constant λ may act as an acceleration which can keep self gravitating dust balls from interacting with one another.

6.0.2 Outlook

That the Einstein equations coupled to a wide range of matter systems can now be written in terms of a system of first order hyperbolic differential equations opens the door for many interesting problems. Firstly, it is a first step in solving the initial boundary value problem (IBVP) for Einstein-matter systems. Since the system found herein is written in a frame formulism, it is suggestive to do a similar approach as that found in [38], which discusses the IBVP for the Einstein-vacuum equations.

Another avenue for further study is to investigate the conformal analog of the evolution equations found in this thesis, in a similar spirit as the seminal work of Friedrich in [43], [44], [46] and [21]. One would hope to recover a system of symmetric hyperbolic evolution equations which extends in a regular fashion to the conformal boundary. To date, no such system exists beyond dust. If one is successful in finding such a system, it opens the door for global existence and stability results.

The findings in this thesis may also be applicable to Astrophysics — in particular in the study of neutron stars. It should be rather straight forward to find the explicit evolution equations for the Einstein-elastic system. An interesting project would then be to investigate the details of such a system. In particular, to recover the elastic constants and compare with [52]. Are there any obvious advantages with the frame-elastic equations?

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