Index relations and fusion rules: Explorations of Supersymmetric, Conformal, and Topological Field Theories

Linfeng.Li

Supervisor: Matthew Buican

CRST SPA, Queen Mary

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Abstract

This thesis explores the world of quantum field theories through an analytic approach. It focuses on three special types of quantum field theories: supersymmetric, conformal and topological ones. The necessary background knowledge is introduced in chapter one, then two types of problems are studied in the next three chapters: index relations and fusion rules.[1]

For index relations we study certain exactly marginal gaugings involving arbitrary numbers of Argyres-Douglas (AD) theories and show that the resulting Schur indices are related to those of certain Lagrangian theories of class $\mathcal{S}$ via simple transformations. By writing these quantities in the language of 2D topological quantum field theory (TQFT), we easily read off the S-duality action on the flavor symmetries of the AD quivers and also find expressions for the Schur indices of various classes of exotic AD theories appearing in different decoupling limits. The TQFT expressions for these latter theories are related by simple transformations to the corresponding quantities for certain well-known isolated theories with regular punctures (e.g., the Minahan-Nemeschansky E_6 theory and various generalizations). We then reinterpret the TQFT expressions for the indices of our AD theories in terms of the topology of the corresponding 3D mirror quivers, and we show that our isolated AD theories generically admit renormalization group (RG) flows to interacting superconformal field theories (SCFTs) with thirty-two (Poincaré plus special) supercharges. Motivated by these examples, we argue that, in a sense we make precise, the existence of RG flows to interacting SCFTs with thirty-two supercharges is generic in a far larger class of 4D $\mathcal{N} = 2$ SCFTs arising from compactifications of the 6D $(2,0)$ theory on surfaces with irregular singularities.

Then we study fusion rules in modular tensor categories. We first relate fusion rules to the mathematical conjecture of Arad and Herzog (AH) in group theory: in finite simple groups, the product of two conjugacy classes of length greater than one is never a single conjugacy class. We discuss implications of this conjecture for non-abelian anyons in $2+1$-dimensional discrete gauge theories. Thinking in this way suggests closely related statements about finite simple groups and their associated discrete gauge theories. We prove these statements and give physical intuition for their validity. Finally, we explain that the lack of certain dualities in theories with non-abelian finite simple gauge groups provides a non-trivial check of the AH conjecture.

We also study the implications of the anyon fusion equation $a \times b = c$ on global properties of $2+1$D topological quantum field theories (TQFTs). Here a and b are anyons that fuse together to give a unique anyon, c. As is well known, when at least one of a and b is abelian, such equations describe aspects of the one-form symmetry of the theory. When a and b are non-abelian, the most obvious way such fusions arise is when a TQFT can be resolved into a product of TQFTs with trivial mutual braiding, and a and b lie in separate factors. More generally, we argue that the appearance of such fusions for non-abelian a and b can also be an indication of zero-form symmetries in a TQFT, of what we term "quasi-zero-form symmetries" (as in the case of discrete gauge

[1]Chapter two, three , four are based on the papers [34],[35],[36] respectively.

theories based on the largest Mathieu group, M_{24}), or of the existence of non-modular fusion subcategories. We study these ideas in a variety of TQFT settings from (twisted and untwisted) discrete gauge theories to Chern-Simons theories based on continuous gauge groups and related cosets. Along the way, we prove various useful theorems.

Contents

1 Introduction

Exact solutions in physics are rare, and generic ones are always difficult. Even in the classical world, for example,the three-body problem in Newtonian mechanics can not be solved exactly, one has to rely on approximations, perturbative expansions, numerical simulations, and global analysis to extract information and make predictions, not to mention extremely difficult problems such as Navier-Stokes equations in fluid mechanics.

In order to make progress, one usually studies solvable toy models first to obtain some general feature, and then one uses such models as a starting point for more realistic treatments or approximations.These solvable toy models differ from generic ones mainly because they have certain kinds of symmetry, which give rise to integrals of motion, i.e. conserved quantities such as energy and parity. Because of these constrains , the problems are simplified, sometimes even completely solved.

Classical celestial mechanics is the first and one of the most prominent examples of this strategy, here one first studies the completely solvable Kepler problem, then adds corrections and perturbations to account for more realistic situations, and finally one performs numerical calculations to make actual predictions. As a result, in practice the motion of planets in the solar system can be predicted in a very precise way.

In quantum field theory, the solutions are even more difficult to find; and sometimes the problems of finding solutions themselves are not well defined. So in early applications of quantum field theory, one was mainly concerned with perturbative calculations, or practically, diagrammatics of scattering amplitudes and Green functions. The toy models here are effective field theories with nice symmetry properties, especially gauge theories with appropriate gauge groups. Sometimes the toy models are obvious, as are free-field representations of underlying symmetry groups such as quantum electrodynamics, and sometimes clever physical intuition is needed, for example in the BCS theory of superconductivity.

Based on such calculations, QFT is very successful. In particle physics it gives rise to the standard model, and in condensed matter physics it describes phase transitions of various systems.

One may tend to consider larger symmetry groups to construct more complicated models to deal with more complicated problems, especially as a way to unify the fundamental interactions in the standard model, or even to include gravity using higher dimensional spacetime models. It turns out, however, that the global symmetry group of a QFT cannot be chosen arbitrarily. Under reasonable assumptions, the Coleman-Mandula theorem states that in a QFT with a well-defined S-matrix and massive particles the symmetry group has to be a direct product of the Poincaré group and the internal global symmetry group, hence no mixing. One can also have local symmetry groups, or gauge groups, the corresponding QFTs are usually called gauge theories.These theories are of central importance in modern physics and mathematics, and it turns out that different types of gauge groups can lead to drastically different theories, so in order to be consistent with experiments the gauge group is not arbitrary either.

Especially, on the global symmetry side, the Coleman-Mandula theorem is a serving restriction, but like many no-go theorems, there are ways out. First there is a Zen-like way to bypass the Coleman-Mandula theorem; that is, we simply drop our assumptions, such that we can consider theories without an S-matrix or massive particles. After all, Green functions,scattering amplitudes or mass spectrum are not the most fundamental objects in QFT, they are defined by correlators under some assumptions, although usually they represent the most common and important observables in a theory, it is not necessary the case. We can have non-trivial QFTs without an S-matrix or massive particles, among other theories, there are two important types: topological quantum field theory(TQFT) and conformal field theory(CFT)

In TQFT, the actions and observables are of a topological type, hence the correlators are topological invariants of the underlying manifold. Unlike generic QFT, TQFT is rigorously defined and the exact calculation of physical quantities is possible.

In CFT, the spacetime symmetry group is enlarged to conformal groups, which leads to new kinds of constraints and the so-called bootstrap method, which is essentially non-perturbative in nature. In dimension two, conformal symmetry algebra becomes infinite dimensional, hence a very powerful tool to study and even define the theory, and in many of the important cases complete solutions are obtained.

Secondly, there is also a loophole in the original formulation of Coleman–Mandula theorem, where conserved charges are assumed to be scalars, by allowing spinor charges, supersymmetry(SUSY) is possible. With SUSY field theory, many new phenomena have been discovered and accurate results are now possible. The results obtained by SUSY are especially relevant to understanding RG flow, so it is beneficial to study the SUSY model and try to obtain some general lessons.

All three types of QFT have important applications both in physics and mathematics, and they relate to each other. In this thesis, we are going to discuss some of those relations and applications. For reference, in this chapter we will introduce some basic facts of those theories, along with some concrete examples which are needed later, while the relevant background knowledge in mathematics and physics is also reviewed briefly in the appendix.

1.1 Topological quantum field theory

Topological quantum field theory, or TQFT, is a special kind of quantum field theory in which the correlators are topological invariants. It originated in the interplay between lower dimensional topology and gauge theory in the 1980s, and has since become an increasingly important subject both in physics and mathematics.

- TQFTs arise naturally in effective field theoretical descriptions of topologically non-trivial physical systems, usually associated with highly degenerate ground states, e.g. Chern-Simons theory and quantum Hall effects, in particular, some of these systems have important practical applications, such as quantum computation and information, as well.

- TQFT is mathematically well defined and rich in content and also suitable for rigorous analytic study. This offers important insights and clues for axiomatic quantum field theory in general. When compared to a generic QFT, TQFT is easy to solve but is non-trivial, hence it can be used as a toy model to test new ideas and techniques in QFT.

- Topology is one of the pillars of modern mathematics, where the finding and calculation of topological invariants is the core problem. Since TQFT provides a novel but useful way to deal with this, it has numerous applications in mathematics.

In this section, we first give an elementary introduction to anyons through toy models of Chern-Simons theory[132, 133, 60] , then we introduce general TQFTs with the path integral formalism[116, 4, 45], finally we discuss Chern-Simons theory in a more general setting and show our previous models are all special examples of it[63]. The relevant data, definitions, notations and classification can be found in [112, 156, 139]

1.1.1 Anyons

Before formal constructions, let us discuss a simple and concrete example of topological quantum field theory. The action for the usual electrodynamics in $d = 4$ is

$$I_{\mathrm{EM}}^{4\ \mathrm{d}}[A] = \frac{1}{2}\int d^4x \left(\mathbf{E}^2 - \mathbf{B}^2 + \mathbf{A}\cdot\mathbf{J} + A_0\rho\right) \tag{1.1}$$

where the equation of motion is the usual Maxwell equation

$$\begin{array}{c} \nabla\cdot\mathbf{E} = \rho, \nabla\times\mathbf{B} - \frac{\partial\mathbf{E}}{\partial t} = \mathbf{J} \\ \nabla\cdot\mathbf{B} = 0, \nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0 \end{array} \tag{1.2}$$

Now suppose we are interested in electrodynamics in $d = 3$ spacetime, where the most general Lorentz and gauge invariant, homogeneous and isotropic action is

$$I_{\mathrm{EM}}^{3\ \mathrm{d}}[A] = \frac{1}{2}\int d^3x \left(\mathbf{E}^2 - B^2 + \mathbf{A}\cdot\mathbf{J} + A_0\rho + m\varepsilon^{\mu\upsilon\rho}A_\mu\partial_\upsilon A_\rho\right) \tag{1.3}$$

the additional new term, the so called Chern-Simons term, is

$$I_{\mathrm{CS}} = \frac{m}{2}\int d^3x\varepsilon^{\mu\upsilon\rho}A_\mu\partial_\upsilon A_\rho \tag{1.4}$$

now the equation of motion is

$$\begin{aligned} \boldsymbol{\nabla}\times\mathbf{E} + \frac{\partial B}{\partial t} &= 0, \\ \boldsymbol{\nabla}\cdot\mathbf{E} + mB &= \rho \\ \boldsymbol{\nabla}\times B - \frac{\partial\mathbf{E}}{\partial t} + m\begin{pmatrix} E^2 \\ -E^1 \end{pmatrix} &= \mathbf{J}. \end{aligned} \tag{1.5}$$

This set of equations has some surprising topological features differing from the usual Maxwell-equations, which can be illustrated by the Aharonov-Bohm effect.

To show this, suppose we have a constant magnetic field B along z direction, confined inside an impenetrable solenoid passing through the $x-y$ plane. Let us also assume that $E_z = 0$, then effectively we are dealing with $d = 3$ electrodynamics. If we have a charge Q on the $x-y$ plane traveling around the solenoid through a counter C, the wave function of the charged particle will acquire the Aharonov-Bohm phase

$$\varphi = \frac{Q}{c\hbar} \oint_C \mathbf{A} \cdot d\mathbf{r} \tag{1.6}$$

If we ignore the z direction completely, imagine a three dimensional cylinder spacetime where particles traveling out their world lines, then the section of the magnetic solenoid on the $x-y$ plane will behave as a point particle if the solenoid is infinitely thin , and unlike usual electrodynamics, now the elementary particle is a charge-flux composited object because if $m \neq 0$ in (1.5) ,as a consequence we would have $\int d^2\mathbf{x}\nabla \cdot \mathbf{E} + m \int d^2\mathbf{x}B = \int d^2\mathbf{x}\rho$, then associated with each charge Q, we have a magnetic flux

$$\Phi = \frac{Q}{m} \tag{1.7}$$

Then our particle is a charge-flux composite, suppose we have two such particles, topologically exchanging them twice is equivalent to a circulation plus a translation, but the latter is irrelevant here, due to the Aharonov-Bohm effect, such a particle will effectively have a statistical angle φ_s where

$$2\varphi_s = Q\Phi = \frac{Q^2}{m} \tag{1.8}$$

In particular, if a particle's charge circles around its flux, we effectively have a spin

$$s = \frac{Q^2}{4\pi m} \tag{1.9}$$

So in general, $\varphi_s \neq 0, \pi$ and s is not a half integer,so that such a particle is neither a boson nor a fermion, it is instead an example of (abelian) anyon.

Suppose as illustrated in figure 1 we have r of those anyons with world lines C_i, and the product of their Wilson loops is a physical observable.

$$W(L) = \prod_{i=1}^{r} \exp\left(iQ_i \oint_{C_i} A\right) \tag{1.10}$$

Its vacuum expectation is a correlator

$$\langle W(L)\rangle = \frac{\langle\psi_0|W(L)|\psi_0\rangle}{\langle\psi_0 \mid \psi_0\rangle} = \frac{\int \mathcal{D}AW(L)e^{iI_{\mathrm{CS}}[A]}}{\int \mathcal{D}Ae^{iI_{\mathrm{CS}}[A]}} \tag{1.11}$$

We can indeed calculate the path integral, and it turns out, quite surprisingly

$$\langle W(L)\rangle = \exp\left(\frac{i}{2m}\sum_{i,j} Q_iQ_j\Phi\left(C_i, C_j\right)\right) \tag{1.12}$$

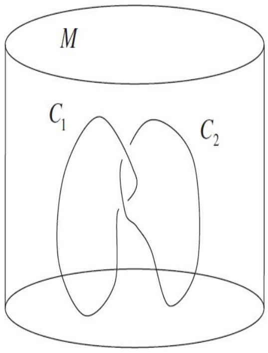


Figure 1: Worldlines of anyons in three dimensional spacetime[132]

with

$$\frac{1}{4\pi}\oint_{C_i} dx^{\mu}\oint_{C_j} dy^{v}\varepsilon^{\mu\nu\rho}\frac{(x-y)^{\rho}}{|x-y|^{3}} = \Phi\left(C_i, C_j\right) \in \mathbb{Z} \tag{1.13}$$

is the Gauss (but not Gaussian) integral , named after Gauss, who introduced this long ago as one of the earliest topological invariants in the history of mathematics. It calculates the linking number $\Phi\left(C_i, C_j\right)$, which counts how many times two curves wind with each other counting orientation. Notice that it is not obvious at all that the integral will produce an integer, but it certainly does so, and it is satisfying to see that this simple TQFT at least reproduced this classical result and actually explained it in a physical way.

The above toy model provides a simple example of anyons, which are more general realizations of the principle of indistinguishable particles than bosons and fermions. As we have mentioned, two exchanges of free identical particles are essentially the same as one particle winding around another through a loop C. While in spatial dimension three or above C is always null homotopic, hence $\exp(2\varphi_s) = 1$, this gives two solutions $\varphi_s = 0, \pi$ only, corresponding to boson and fermion. In spatial dimension one, C is not defined, and particles must pass through each other to change their positions, hence any phase factor associated with such a process can be explained either by identical free particle symmetry or as a result of interactions with the same particles, thus the notion of free identical particles is in a sense ambiguous. Only in spatial dimension two, C can be a non-trivial homotopy class, and anyon appears.

Given a set of anyons $\mathcal{A} = \{a, b, \cdots\}$, there are two basic kinds of operations we can apply to them, namely, fusion and braiding, those are abstraction of the processes such as the ones illustrated in figure 2

- Fusion

  If we bring two anyons a, b together and identify the properties of this composite object as a blackbox, then it behaves as a collection of anyons. To be concrete, suppose we

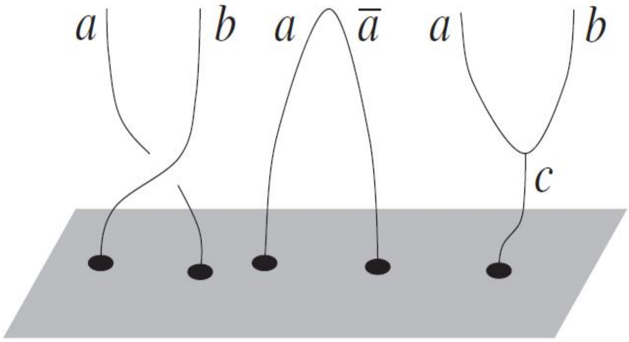


Figure 2: Two dimensional operations on anyons with corresponding worldlines in three dimensional spacetime(assuming time flows downwards): (a) exchange of a, b (b)creation of a particle-antiparticle pair $a, \bar{a}$ from vacuum (c) fuse a, b into c [132]

have some quantum fields Φ_a, Φ_b represent two anyons, if we take their OPE we will find a bunch of anyons with appropriate properties such as scaling dimensions and conserved internal quantum numbers. This process is called fusion, and symbolically we have

$$a \times b = \sum_{c \in \mathcal{A}} N_{ab}^c c, \quad N_{ab}^c \in \mathbb{N} \tag{1.14}$$

where at least one N_{ab}^c is nonzero. In principle there might be more than one c in the RHS, if given a this happens for at least one b, we call a as non-abelian anyon, otherwise it is an abelian anyon, i.e. it gives a unique out come c when fused with any b. If $\mathcal{A}$ contains abelian anyons only,we say $\mathcal{A}$ is abelian as well. This nomenclature is due to the fact that the abelian anyons form a finite abelian group with respect to fusion.

- Braiding

  In three-dimensional spacetime, exchanging identical particles is equivalent to braiding their world lines. In particular, if we braid a, b we will have phase factors R_{ab}^c depending on c, it is convenient to view them as the diagonal entries of a diagonal matrix R_{ab}, and there should be a representation B_{ab} of the braiding group related to those braiding processes, see fig 4.

To be consistent, the set $\mathcal{A} = \{a, b, \cdots\}$ should not be selected arbitrarily, and we should expect fusion and braiding to have some reasonable properties

- Fusion is commutative

  Since we just view the composite $a \times b$ as a blackbox, there is no preferred way to define left or right fusion, we demand that

$$a \times b = b \times a \tag{1.15}$$

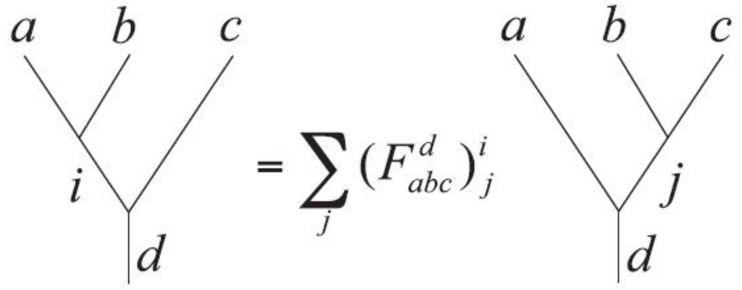


Figure 3: Illustration of F-move, here we have two ways to fuse a, b, c into d [132]

hence

$$N^c_{ab} = N^c_{ba} \tag{1.16}$$

In the special case of abelian $\mathcal{A}$, we have finite abelian groups, and here fusion is just group multiplication, so in a sense we can view the theory of anyon as some sort of quantum generalization of finite abelian groups.

- Fusion is associative

  Again if we fuse a, b, c together, we just identify the whole system as a blackbox so we expect that the order of fusion does not matter, and

$$(a \times b) \times c = a \times (b \times c) \tag{1.17}$$

  hence

$$\sum_{x\in\mathcal{A}} N^x_{ab} N^d_{xc} = \sum_{x\in\mathcal{A}} N^x_{bc} N^d_{ax} \tag{1.18}$$

  Suppose we are fusing a, b, c to get d, then there are several ways to do this, and just like crossing symmetry in usual quantum field theory, we can introduce the so called F-matrix F^d_{abc} to account for this, see fig 3.

  Also, if fusion is associative , it should be compatible with braiding as well, essentially this means that F-move and R-move are compatible with each other, see fig 5,6.

  As a consequence, there are two types of constrains, namely the pentagon equation

$$\left(F^5_{12c}\right)^d_a \left(F^5_{a34}\right)^c_b = \sum_e \left(F^d_{234}\right)^c_e \left(F^5_{1e4}\right)^d_b \left(F^b_{123}\right)^e_a \tag{1.19}$$

  and the hexagon equation

$$\sum_b \left(F^4_{231}\right)^c_b R^4_{1b} \left(F^4_{123}\right)^b_a = R^c_{13} \left(F^4_{213}\right)^c_a R^a_{12} \tag{1.20}$$

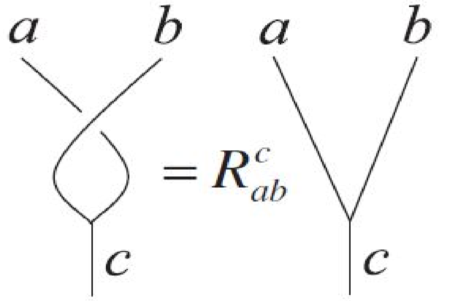


Figure 4: Illustration of R-move [132]

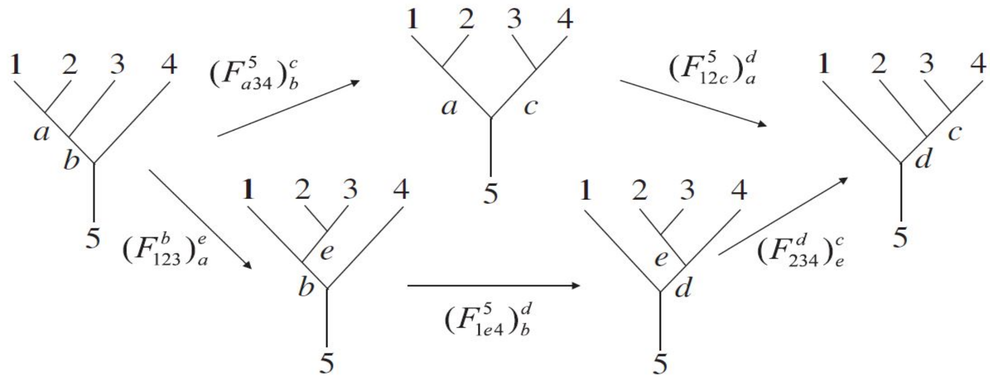


Figure 5: Pentagon equation [132]

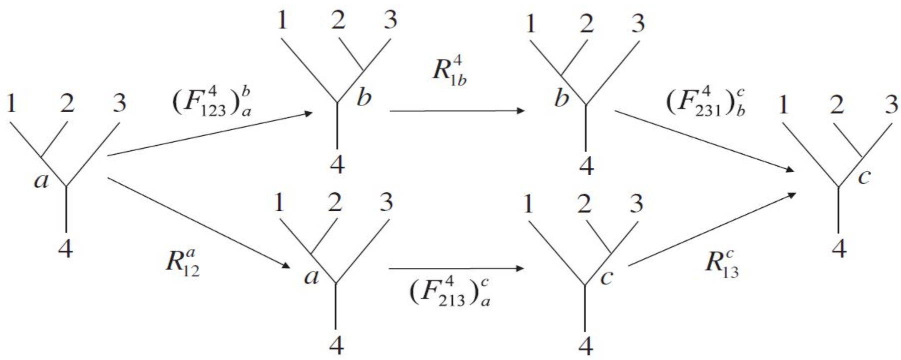


Figure 6: Hexagon equation [132]

We also obtain the following explicit expression for the braiding matrix B as

$$B_{ab} = {F^{d}_{acb}}^{-1} R_{ab} F^{d}_{acb} \tag{1.21}$$

- Vaccum and antiparticle exist

  As in ordinary quantum field theory, we demand that a vacuum exists and is unique, we usually adopt a multiplicative notation to label it as 1, so

$$a \times 1 = 1 \times a = a \tag{1.22}$$

  hence

$$N^{c}_{a1} = N^{c}_{1a} = \delta_{ac} \tag{1.23}$$

  Then we interpret the antiparticle $\bar{a}$ of anyon a as the unique anyon fused with it to give vacuum

$$a \times \bar{a} = 1 + \cdots \tag{1.24}$$

  hence

$$N^{1}_{a\bar{a}} = 1 \tag{1.25}$$

  An anyon can be its own antiparticle, in particular $1 = \bar{1}$, and of course $\bar{\bar{a}} = a$ as particle-antiparticle pairing is unique.

  We can also assign an orientation to the world lines of anyons such that antiparticles travel backward in time as in usual quantum field theory, and then using crossing symmetry we can argue that[2]

$$\theta(a) = R^{1}_{a\bar{a}} \tag{1.26}$$

  effectively defines a spin like quantity $\theta : \mathcal{A} \to U(1)$, we will call it as topological spin, and usually it is understood as $\theta(a) = \exp 2\pi i h_a$ with $h_a : \mathcal{A} \to \mathbb{Q}/\mathbb{Z}$ the scaling dimensions (defined $\bmod 1$ only)

- Quantum dimension and Hilbert space

  We want to construct some Hilbert spaces to describe fusion processes such as the ones in figure 7, just like in particle physics, nuclear physics or chemistry. Suppose we have a, b as the reagents of fusion, then the various c's appear in $a \times b$ as the products of this reaction, and it is quite reasonable to define a Hilbert space to describe this process such that

[2] this form is gauge dependent as indeed $\theta_a R^{1}_{aa} = \nu_a = \pm 1$ and $R^{1}_{aa} = (R^{1}_{a\bar{a}})^{-1}$, more generally, one should define $\theta(a) = d_a^{-1} \widetilde{\mathrm{Tr}} R_{aa}$ as in eq(211) of [112]

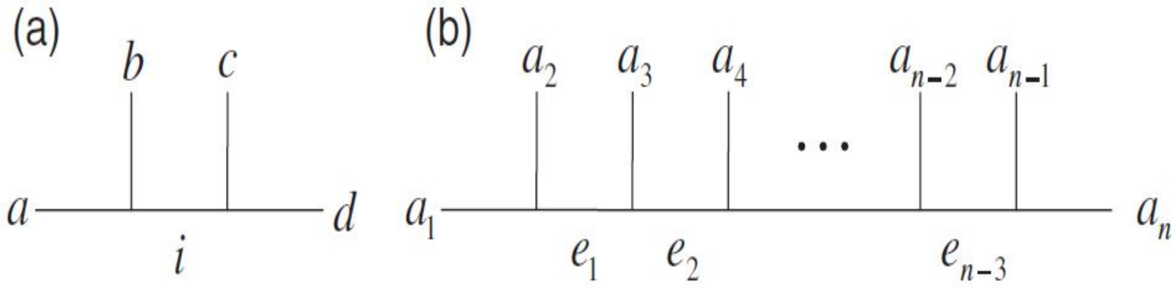


Figure 7: Fusion processes(assume time flows rightwards): (a) a, b fuse into the middle state i and split into c, d (b)A generic fusion process [132]

$$\dim\left(\mathcal{M}_{(3)}\right) = N_{ab}^{c} \tag{1.27}$$

For consistency, obviously, we need to define $\dim\left(\mathcal{M}_{(1)}\right) = 0$ and $\dim\left(\mathcal{M}_{(2)}\right) = 1$, then we can define the Hilbert space of n-anyon fusion $\mathcal{M}_{(n)}$ as the set of all possible outcomes

$$\dim\left(\mathcal{M}_{(n)}\right) = \sum_{e_1 \ldots e_{n-3}} N_{a_1 a_2}^{e_1} \ldots N_{e_{n-3} a_{n-1}}^{a_n} \tag{1.28}$$

In particular, if we have a specific anyon a, we want to define an intrinsic dimension to measure the complexity of fusion outcomes associated with this anyon, we can take n copy of a and construct $\mathcal{M}_{(n)}$, We expect the following asymptotic for $n \to \infty$ exists

$$\dim\left(\mathcal{M}_{(n)}\right) \propto d_a^n \tag{1.29}$$

This $d_a \geq 1$ is called the quantum dimension of a, in general it is just a real number without being integral or even rational, but by definition an anyon is abelian iff $d_a = 1$.

Quantum dimensions have a very nice property: they form a representation of the fusion algebra

$$d_a d_b = \sum_c N_{ab}^{c} d_c \tag{1.30}$$

- Modular data

  Finally we will introduce some concepts which will be very important later, they will be constructed from the previous concepts we have introduced.

First we can assign an global quantum dimension for $\mathcal{A}$ as

$$\mathcal{D} = \sqrt{\sum_{a\in\mathcal{A}} d_a^2} \tag{1.31}$$

And some sort of averaged topological spin as

$$\Theta = \mathcal{D}^{-1}\sum_a d_a^2\theta_a = \exp\frac{i\pi c}{4} \tag{1.32}$$

Where the number c is called chiral central charge, just like h_a it is not defined absolutely, but mod8 only[3]

then we can define the (topological) T-matrix as

$$T_{ab} = \delta_{ab}\theta_a \tag{1.33}$$

and the (topological) S-matrix as

$$S_{ab} = \frac{1}{\mathcal{D}}\sum_c N_{a\bar{b}}^c \frac{\theta_c}{\theta_a\theta_b} d_c \tag{1.34}$$

and the charge conjugation matrix as

$$C_{ab} = \delta_{\bar{a}b} \tag{1.35}$$

Then we get the following modular representation

$$(ST)^3 = \Theta C, \quad S^2 = C, \quad C^2 = I \tag{1.36}$$

Those matrices have many nice properties, for example S is both symmetric and unitary, its elements are algebraic numbers, it also gives some useful identities

$$d_a = \frac{S_{a1}}{S_{11}}, \quad \mathcal{D} = \frac{1}{S_{11}} \tag{1.37}$$

and we can define the monodromy scalar as the 1-1 entry of $B^2 = F^{-1}R^2F$

$$M_{ab} = \frac{S_{ab}S_{11}}{S_{1a}S_{1b}} \tag{1.38}$$

When $|M_{ab}| = 1$ we can interpret it as the overall phase factor we get by braiding a and b, see figure8

[3] In case of MTC from $d = 2$ RCFT, we usually absorb Θ into the definition of T matrix as $T_{ab} = \delta_{ab}\exp 2\pi i(h_a - \frac{c}{24})$, then c is defined mod24

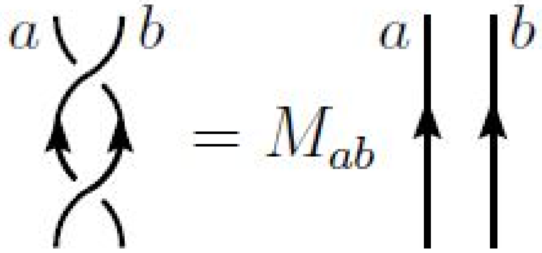


Figure 8: Illustration of the monodromy phase factor [25]

In particular for a theory $\mathcal{A}$ of abelian anyons only, we have

$$M_{ab} = \frac{S_{ab}}{S_{1b}} = \frac{\theta(a \times b)}{\theta(a)\theta(b)} = \sqrt{|\mathcal{A}|} S_{ab} \tag{1.39}$$

So the twists θ alone provide full information of this theory $\mathcal{A}$

Finally,we have the following Verlinde formula that relates the S matrix to the fusion rules

$$N_{ab}^{c} = \sum_{x} \frac{S_{ax} S_{bx} S_{\bar{c}x}}{S_{1x}} \tag{1.40}$$

This formula is highly non-trivial as it equates a complicated combination of algebraic numbers to a natural number.

With these new weapons in our hands we can reexamine our 3d electrodynamics toy model more carefully, for our purpose it is convenient to define the level as $k = 4\pi m$ and rewrite (1.4) as

$$I_{CS} = \int_M \frac{k}{4\pi} A \wedge \mathrm{d}A \tag{1.41}$$

with A as an $U(1)$ gauge field on compact three manifold M, for this theory to be well defined we must have gauge invariance under $A_\mu \to A_\mu^U = U^\dagger A_\mu U - iU^\dagger \partial_\mu U$, which leads to

$$I_{\mathrm{CS}}\left[A^U\right] = I_{\mathrm{CS}}[A] + 2\pi k n \tag{1.42}$$

where the integer n appears as the winding number expressed as the integral

$$n = \omega[U] = \frac{1}{24\pi^2} \int_M d^3x \varepsilon^{\mu\nu\rho} \operatorname{tr}\left(U^\dagger \partial_\mu U U^\dagger \partial_\nu U U^\dagger \partial_\rho U\right) \tag{1.43}$$

so we must have

$$k \in \mathbb{Z} \tag{1.44}$$

The primaries are given by Wilson lines with different representations of $U(1)$

$$W_\alpha(\gamma) := \exp\left[i\alpha \int_\gamma A\right], \quad \alpha \in \mathbb{Z} \tag{1.45}$$

with dimensions

$$\theta(\alpha) = e^{2\pi i h_\alpha}, \quad h_\alpha = \frac{\alpha^2}{2k} \tag{1.46}$$

For simplicity we will restrict to positive k as we have identified it as the mass m in our toy model, and call this theory as $U(1)_k$ Chern-Simons theory, depending on $k(\bmod 2)$ there are two possibilities:

- k is odd[4]

  $$\mathcal{A} \cong \mathbb{Z}_k = \{0, 1, \ldots, k-1\} \tag{1.47}$$

  since this theory is abelian we just label the vacuum as 0 instead of 1, and also use an additive notation for fusion rules

  $$\alpha \times \beta = \alpha + \beta \mod k \tag{1.48}$$

- k is even [5]

  $$\mathcal{A} \cong \mathbb{Z}_{2k} = \{0, 1, \ldots, 2k-1\} \tag{1.49}$$

  with

  $$\alpha \times \beta = \alpha + \beta \mod 2k \tag{1.50}$$

We see that the $U(1)_k$ theory is fixed by the level k, it has a very simple set of fusion rules, which is essentially addition in a finite abelian group, so this theory must be abelian. In order to get nonabelian anyons, we can start with a nonabelian group and build up some fusion rules, but since fusion is commutative we can not use the group multiplication directly, instead we can rely on the physical picture of flux charge composites in our toy model and try to make some appropriate modifications.

For simplicity we restrict ourselves to finite nonabelian groups to avoid difficulties involved with analysis. For example, we pick the simplest of all nonabelian groups $G = S_3$, and now without explicit Lagrangian and action we just imagine that we have some gauge field A taking values in G, and just as before we can set up an Aharonov–Bohm experiment to obtain some flux charge composites. Let us adopt the usual terminology of electrodynamics, it is reasonable to imagine that there are three kinds of objects in this theory, all realized as worldlines of anyons

[4]these theories are spin TQFTs, so in order for them to be well defined, the corresponding spin structures must exist

[5]these theories by themselves are non-spin, but we can always tensor them with a trivial spin TQFT $\{1, \psi\}$ to form corresponding spin TQFTs such that $\mathcal{A} \cong \mathbb{Z}_k \times \mathbb{Z}_2$ and $(\alpha, \beta) \times (\alpha', \beta') = (\alpha + \alpha' \bmod k, \beta + \beta' \bmod 2)$

- electric charge

  These are just the Wilson lines as before, and by electric charge we mean an irreducible representation π of G because we work with representations rather than abstract groups, and this is how we measure charge in reality. These operators should also have trivial magnetic flux in order to behave as pure charges; alternatively they should all have fluxes with the identity element $\mathbb{I}$ of G. A reasonable choice for the fusion rules is the decomposition of tensor representations of G.

- magnetic flux

  Such operators should correspond to the abstract group element g of G, as we have defined charges as representations π and we measure charges by observing their behavior $\pi(g)$ under the influence of magnetic fluxes. But it should be noted that in order to carry out measurements, one has to set up some reference or standard, this again should be realized as a particular abstract group element h of G, and different observables can have different standards h, h', so they agree only on the conjugacy class $[g]$. Hence we define magnetic flux lines as conjugacy classes $[g]$. Again such operators should all have trivial charges as the trivial representation $1 : G \to \{\mathbb{I}\}$of G. And as multiplication between conjugacy classes is commutative, we can just use it to implement the fusion rules.

- dyon

  Of course more generally, we have flux charge composites, but we should be very careful when defining them in a consistent manner. Since such an operator carries both charge and flux, we can just treat it as a blackbox, put it somewhere and then set up some Aharonov–Bohm experiments with pure charge or pure flux to detect the flux and charge of this blackbox through interference patterns. But there is a crucial difference with the abelian $U(1)_k$ theory, suppose we use some flux $[b]$ to detect the charge of this dyon, since the dyon itself has flux $[a]$, exchange them will change the system as $R : |a, b\rangle \to |aba^{-1}, a\rangle$. So if we perform an Aharonov–Bohm experiment by sending $[b]$ through the dyon to measure its charge , the two paths followed by $[b]$ are inequivalent as they differ by one loop around $[a]$ as $P_1 - P_2 = C$, we can tell which path $[b]$ travels by measuring the flux of the resulting system, hence there is no interface. In order to avoid this we need $aba^{-1} = b$, so a dyonic line is given by a conjugacy class of abstract group element $[g]$, as well as an irreducible representation π of N_g the centralizer of g, due to phase ambiguity, this representation is projective only in general. For dyonic lines, fusion rules are more complicated, both decomposition of tensor representation and group multiplication are involved.

For $G = S_3$, we have the following table of anyons, where we have three conjugacy classes with representatives $e, (12), (123)$,and the corresponding centralizers N_g are S_3, Z_2, Z_3, for S_3 we have the one dimensional trivial and sign representations $+, -$ as well as the unique two dimensional representation 2, while representations of Z_2, Z_3 are all one dimensional and labeled by their eigenvalues in $\mathbb{C}$(where $\omega_3 = \exp(\frac{2\pi i}{3})$)[133].

Type	Flux	Charge	$d_{([g],\pi_g^\omega)}$
A	e	$[+]$	1
B	e	$[-]$	1
C	e	$[2]$	2
D	(12)	$[+]$	3
E	(12)	$[-]$	3
F	(123)	$[1]$	2
G	(123)	$[\omega_3]$	2
H	(123)	$[\bar{\omega}_3]$	2

We can perform the above constructions for any finite nonabelian group G, the resulting theory is called the discrete gauge theory of G. Just like the $k \in \mathbb{Z}$ requirement in $U(1)_k$ Chern-Simons theory, there are additional consistency constraints in discrete gauge theory as well: here we need to specify the Dijkgraaf-Witten twist as a cohomology class of third order group cohomology

$$\omega \in H^3(G, U(1)) \tag{1.51}$$

as well as the related cohomology class of second order group cohomology

$$\eta_g(h,k) := \frac{\omega(g,h,k)\omega(h,k,g)}{\omega(h,g,k)} \in H^2\left(N_g, U(1)\right), \quad h,k \in N_g \tag{1.52}$$

which appears in $\pi_g^\omega(h)\pi_g^\omega(k) = \eta_g(h,k)\pi_g^\omega(hk)$ as phase factors.

If $\omega = 0$ then $\eta_g = 0$ automatically for every g, hence all the representations associated with dyonic lines are true representations instead of projective ones. Such a theory is referred to as untwisted, otherwise it is twisted. When $\omega \neq 0$ but $\eta_g = 0$, we still have linear instead of projective π_g^ω, and for magnetic flux lines to be well defined, we must have $\eta_g = 0$ as well because even if we only need the flux lines to have trivial representation, we need to have a true representation instead of a projective one. Finally, for charges since $[g] = 1$, the representations are always linear, even in twisted theories.

More generally, even if $g \neq 1$ and ω is non-trivial, we may still have linear representations.[6] As an example, we can consider $G = PSL(2,4)$ and the $\mathbb{Z}_3$ centralizer of the length twenty conjugacy class. In this case, we have $H^2(\mathbb{Z}_3, U(1)) = \mathbb{Z}_1$, so the resulting η_g (with g in the length twenty conjugacy class) is cohomologically trivial no matter the choice of $\omega \in H^3(PSL(2,4), U(1)) = \mathbb{Z}_6 \times \mathbb{Z}_{10}$.

The most important things for us to focus on in what follows are the fusion coefficients

[6]More precisely, if η_g is a non-trivial 2-coboundary, we will obtain projective representations that are in one-to-one correspondence with linear representations. We can remove these projective factors via a symmetry gauge transformation of the type described in [14]. Note that while linear representations can be one-dimensional (e.g., if the centralizer is an abelian group), projective representations resulting from η_g cohomologically non-trivial are necessarily higher dimensional.

appearing in

$$([g],\pi_g^\omega)\times([h],\pi_h^\omega)=\sum_{k,\pi_k^\omega} N_{([g],\pi_g^\omega),([h],\pi_h^\omega)}^{([k],\pi_k^\omega)}([k],\pi_k^\omega) \tag{1.53}$$

To arrive a description of such a process we must combine conjugacy classes and representations. In particular, we need to multiply elements in $[g]$ and $[h]$ and determine the corresponding conjugacy classes. At the same time, we must decompose the product of irreducible representations of the corresponding centralizers into irreducible representations of centralizers of G. A simple prescription for doing this is given in [14]

$$\begin{aligned} N_{([g],\pi_g^\omega),([h],\pi_h^\omega)}^{([k],\pi_k^\omega)} \quad = \quad & \sum_{(t,s)\in N_g\backslash G/N_h} m(\pi_k^\omega|_{N_{^tg}\cap N_{^sh}\cap N_k},\, {}^t\pi_g^\omega|_{N_{^tg}\cap N_{^sh}\cap N_k}\otimes {}^s\pi_h^\omega|_{N_{^tg}\cap N_{^sh}\cap N_k} \\ & \otimes \pi^\omega_{(^tg,^sh,k)}) \end{aligned} \tag{1.54}$$

where the sum is over the double coset, we define ${}^tg := t^{-1}gt$, and ${}^t\pi_g^\omega|_{N_{^tg}\cap N_{^sh}\cap N_k}\otimes {}^s\pi_h^\omega|_{N_{^tg}\cap N_{^sh}\cap N_k}\otimes \pi^\omega_{(^tg,^sh,k)}$ and $\pi_k^\omega|_{N_{^tg}\cap N_{^sh}\cap N_k}$ are restrictions of irreducible representations of $N_{^tg}$, $N_{^sh}$, and N_k to the triple intersections of these normalizers. These restrictions are generally (though crucially for us below not always) reducible representations of $N_{^tg}\cap N_{^sh}\cap N_k$. The $m(a,b)$ function computes inner products of the representations a and b (we will fill in further details of this function as needed later in this section). Crucially, a and b must be the same type of representation (i.e., they should both be linear or else transform with the same set of projective weights) in order to be meaningfully compared.

We can determine the projectivity of the ${}^t\pi_g^\omega$, ${}^s\pi_h^\omega$, and π_k^ω representations by a computation in the relevant cohomology as in (1.52). The representation $\pi^\omega_{(^tg,^sh,k)}$ is one dimensional (it is a representation of the action of symmetries on the one-dimensional $V^k_{^tg^sh}$ fusion space in the G-SPT) and ensures that the arguments entering $m(a,b)$ involve the same type of representations. Therefore, $\pi^\omega_{(^tg,^sh,k)}$ satisfies

$$\pi^\omega_{(^tg,^sh,k)}(\ell)\pi^\omega_{(^tg,^sh,k)}(m) \quad = \quad \frac{\eta_k(\ell,m)}{\eta_{^tg}(\ell,m)\eta_{^sh}(\ell,m)}\cdot\ \pi^\omega_{(^tg,^sh,k)}(\ell m) \tag{1.55}$$

A more basic quantity of interest to us in what follows is the modular data of the discrete gauge theory. It is given by [101]

$$\begin{aligned} S_{([g],\pi_g^\omega),([h],\pi_h^\omega)} &= \frac{1}{|G|}\sum_{\substack{k\in[g],\ \ell\in[h],\\ k\ell=\ell k}} \chi^k_{\pi_g^\omega}(\ell)^*\chi^\ell_{\pi_h^\omega}(k)^* \\ \theta_{([g],\pi_g^\omega)} &= \frac{\chi_{\pi_g^\omega}(g)}{\chi_{\pi_g^\omega}(e)} \end{aligned} \tag{1.56}$$

where we define $\chi^h_{\pi_g^\omega}(\ell)$ as follows

$$\chi^{xgx^{-1}}_{\pi_g^\omega}(xhx^{-1}) := \frac{\eta_g(x^{-1},xhx^{-1})}{\eta_g(h,x^{-1})}\chi_{\pi_g^\omega}(h) \tag{1.57}$$

Here, θ is the topological spin, and S is the modular S matrix. It follows from these definitions that quantum dimensions are given by

$$d_{([g],\pi_g^\omega)} = \frac{S_{([g],\pi_g^\omega)([1],1)}}{S_{([1],1)([1],1)}} = |[g]| \cdot |\pi_g^\omega| \tag{1.58}$$

where $|[g]|$ is the size of the conjugacy class, and $|\pi_g^\omega|$ is the dimension of the representation. Non-abelian anyons have $d_{([g],\pi^\omega)} > 1$. As a consequence, they must satisfy

$$([g], \pi_g^\omega) \times ([g^{-1}], (\pi_g^\omega)^*) = ([1], 1) + \cdots \tag{1.59}$$

where the ellipses necessarily contain additional terms (otherwise we would have $d_{([g],\pi^\omega)} = 1$), 1 is the trivial representation of G, and $(([g^{-1}], (\pi_g^\omega)^*)$ is the anyon conjugate to $([g], \pi_g^\omega)$.

We may write a dictionary between the non-abelian Wilson lines, flux lines, and dyons discussed in the previous sections and the objects discussed in this section as follows

$$\begin{aligned} \mathcal{W}_{\pi_1} \quad &\leftrightarrow \quad ([1], \pi_1)\ , \quad |\pi_1| > 1\ , \\ \mu_{[g]} \quad &\leftrightarrow \quad ([g], 1_g^\epsilon)\ , \quad |[g]| > 1\ , \\ \mathcal{L}_{([h],\pi_h^\omega)} \quad &\leftrightarrow \quad ([h], \pi_h^\omega)\ , \quad |[h]| \cdot |\pi_h^\omega| > 1\ . \end{aligned} \tag{1.60}$$

We have dropped the ω superscript from π_1 in order to emphasize, as discussed above, that Wilson lines always transform under linear representations of G. We include an ϵ superscript on the trivial representation of the flux line because these objects only exist when the relevant η_g in (1.52) is trivial in cohomology. This triviality means that $\eta_g(h, k)$ can be expressed in terms of a one co-chain as follows: $\eta_g(h, k) = \frac{\epsilon_g(h)\epsilon_g(k)}{\epsilon_g(h\cdot k)}$.[7]

Finally, let us mention two canonical examples of nonabelian anyons as well, more examples can be found in[139]

- Fibonacci anyons

  this theory is named by the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$ which appears naturally in it, it is the simplest system capable for universal quantum computation through braiding alone.

  Anyon types: $\{1, \tau\}$

  Fusion rules: $\tau^2 = 1 + \tau$

  Quantum dimensions: $\{1, \varphi\}$

  Twists: $\theta_1 = 1, \theta_\tau = e^{\frac{4\pi i}{5}}$

  Total quantum order: $D = 2\cos\left(\frac{\pi}{10}\right) = \frac{\sqrt{5}}{2\sin\left(\frac{\pi}{5}\right)}$

  Topological central charge: $c = \frac{14}{5}$ Braidings: $R_1^{\tau\tau} = e^{-\frac{4\pi i}{5}}, R_\tau^{\tau\tau} = e^{\frac{3\pi i}{5}}$

[7]Note that 1_g^ϵ is the irreducible projective representation of N_g whose character is proportional to the trivial representation of N_g.

S-matrix: $S = \frac{1}{\sqrt{2+\varphi}} \begin{pmatrix} 1 & \varphi \\ \varphi & -1 \end{pmatrix}$,

F-matrices: $F_{\tau}^{\tau,\tau,\tau} = \begin{pmatrix} \varphi^{-1} & \varphi^{-1/2} \\ \varphi^{-1/2} & -\varphi^{-1} \end{pmatrix}$

- Ising anyons

  this is the anyon theory behind two dimensional Ising model

  Anyon types: $\{1, \sigma, \psi\}$

  Fusion rules: $\sigma^2 = 1 + \psi, \sigma\psi = \psi\sigma = \sigma, \psi^2 = 1$

  Quantum dimensions: $\{1, \sqrt{2}, 1\}$ Twists: $\theta_1 = 1, \theta_\sigma = e^{\frac{\pi i}{8}}, \theta_\psi = -1$

  Total quantum order: $D = 2$

  Topological central charge: $c = \frac{1}{2}$

  Braidings: $R_1^{\sigma\sigma} = e^{-\frac{\pi i}{8}}, R_1^{\psi\psi} = -1, R_\sigma^{\psi\sigma} = R_\sigma^{\sigma\psi} = -i, R_\psi^{\sigma\sigma} = e^{\frac{3\pi i}{8}}$

  S-matrix: $S = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$,

  F-matrices: $F_\sigma^{\sigma,\sigma,\sigma} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, F_\sigma^{\psi,\sigma,\psi} = (-1), F_\psi^{\sigma,\psi,\sigma} = (-1)$

1.1.2 TQFT as a path integral

The $d = 3$ electrodynamics model and the discrete gauge theory in the last section are all examples of Chern-Simons theory, which is one of the most important classes of TQFT. Just like these examples, for TQFT in general the correlators are topological invariants, concretely, suppose we have a theory defined by some action $S(\phi_i)$ of fields ϕ_i on a Riemannian manifold with metric $g_{\mu\nu}$, then for the correlators between operators $\mathcal{O}_\alpha(\phi_i)$

$$\left\langle \mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \cdots \mathcal{O}_{\alpha_p} \right\rangle = \int [D\phi_i] \, \mathcal{O}_{\alpha_1}(\phi_i) \, \mathcal{O}_{\alpha_2}(\phi_i) \cdots \mathcal{O}_{\alpha_p}(\phi_i) \exp(-S(\phi_i)) \tag{1.61}$$

We should have[8]

$$\frac{\delta}{\delta g^{\mu\nu}} \left\langle \mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \cdots \mathcal{O}_{\alpha_p} \right\rangle = 0 \tag{1.62}$$

[8]If one begins with a metric independent classical action, due to possible anomaly the resulting quantum theory may or may not be topological, so this naive approach may sometimes fail. For example given the action in (1.69), if G is compact, the quantum theory is topological, but with the non-compact $G = SL(n, \mathbb{R})$,it is not known whether this theory is topological or not, see for example the online lecture Quantization, Gauge Theory, and the Analytic Approach to Geometric Langlands 1 by Edward Witten at QUANTUM FIELDS, GEOMETRY AND REPRESENTATION THEORY 2021 (ONLINE).

There are two ways to satisfy this, and the resulting TQFTs are called Schwarz type or Witten type respectively.[115]

For Scgwarz type theories, one simply begin with a metric independent action, i.e. the energy momentum tensor vanishes

$$\frac{\delta S}{\delta g_{\mu\nu}} \equiv T_{\mu\nu} = 0 \tag{1.63}$$

Then there is no local interaction in this theory, and for observables $\mathcal{O}_\alpha(\phi_i)$ we can just pick topological ones such as Wilson lines and surface defects, then

$$\frac{\delta}{\delta g_{\mu\nu}} \mathcal{O}_\alpha(\phi_i) = 0 \tag{1.64}$$

So trivially (1.62) is satisfied.

For Witten type theories, suppose we have a symmetry generated by δ such that

$$\delta S = 0, \quad \frac{\delta}{\delta g_{\mu\nu}} \mathcal{O}_\alpha(\phi_i) = \delta O_\alpha^{\mu\nu}(\phi_i), \quad T_{\mu\nu}(\phi_i) = \delta G_{\mu\nu}(\phi_i) \tag{1.65}$$

then we have

$$\frac{\delta}{\delta g^{\mu\nu}} \left\langle \mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \cdots \mathcal{O}_{\alpha_p} \right\rangle = -\int [D\phi_i]\, \delta\left(\mathcal{O}_{\alpha_1}(\phi_i)\, \mathcal{O}_{\alpha_2}(\phi_i) \cdots \mathcal{O}_{\alpha_p}(\phi_i)\, G_{\mu\nu} \exp\left(-S(\phi_i)\right)\right) = 0 \tag{1.66}$$

This is formally true when we regard every term with δ as infinitesimal, one can also impose

$$\delta^2 = 0 \tag{1.67}$$

such that the action is δ exact

$$S(\phi_i) = \delta\Lambda(\phi_i) \tag{1.68}$$

Then one construct suitable BRST cohomology classes to define $\mathcal{O}_\alpha(\phi_i)$, for this reason, Witten type theories are also called cohomological type theories. Witten type TQFTs appear naturally in extended supersymmetric field theories where by 'twisting' a modified version of SUSY transformation is constructed and it plays the role of δ

In this thesis we will focus on Chern-Simons theory , which is of Schwarz type. Here we will show that both the discrete gauge theory and three dimensional electrodynamics we have discussed the in last section can be viewed as special cases of Chern-Simons theory.

We will assume our spacetime is a three-dimensional oriented manifold M ,the gauge symmetry is given by a compact group G, which is either a finite group or a Lie group, the gauge field A is realized as (the pull back on M of) a connection on a G principal bundle E.

When G is a Lie group, locally A is a $\mathfrak{g}$ vector valued one form, if in addition ,just like in our three dimensional electrodynamics toy model, E is trivial, then A is well defined on M as a $\mathfrak{g}$ vector valued one form. We can define the Chern-Simons functional as

$$S(A) = \frac{k}{8\pi^2} \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) \tag{1.69}$$

Then just as before we have

$$Z(M) = \int \mathscr{D}Ae^{2\pi iS(A)} \tag{1.70}$$

which enforces $k \in \mathbb{Z}$

In particular when G is simply connected, E must be trivial, this includes the most common examples of simply connected G with a semisimple $\mathfrak{g}$ given by an ADE label.

But for our purposes we want to include nontrivial E as well, so we can generalize (1.69) as follow: if we first pick a four dimensional manifold B such that $M = \partial B$ and assume that A can be extended to B, then the following functional is well defined, and reduced to (1.69) for trivial E

$$S(A) = \frac{k}{8\pi^2} \int_B \mathrm{Tr}(F \wedge F) \tag{1.71}$$

But in general such B may not exist , so we prefer to find an intrinsic way to define a functional on M, using a cobordism argument, given M we can always construct $B = M \times I$ with $\partial B = M \cup (-M)$, then suppose we have A'' on B that interpolates between A on $M \times \{0\}$ and A' on $M \times \{1\}$, then

$$S(A) - S(A') = \frac{k}{8\pi^2} \int_B \mathrm{Tr}(F'' \wedge F'') \tag{1.72}$$

But (1.70) tells us that the RHS depends only on $S(A) - S(A') \mod$ but not on the particular choice of A'', so if we can fix it as an integration constant depends intrinsically on M, E, we would be able to define a functional as the topological action.

Indeed there is a natural way to characterize E, given by the map $\lambda : \pi_1(M) \to G$, then we can define the topological action S as a functional assign each pair (M, λ) a number $S(\lambda) \in \mathbb{R}/\mathbb{Z}$, which satisfies the following two consistency conditions

- We define $S \sim S'$ up to a functional depends only on $\lambda|_{\partial M}$, physically this means transition amplitudes $e^{2\pi iS}$ and $e^{2\pi iS'}$ are the same up to redefinition of external states
- If $\partial M = 0$ and $\exists B : \partial B = M$, with $\lambda_B : \pi_1(B) \to G$ and $\lambda_B|_{\partial M} = \lambda$ then $S(\lambda) = 0$, physically this is the factorization property $e^{2\pi iS(M_1 \# M_2)} = e^{2\pi iS(M_1)} \cdot e^{2\pi iS(M_2)}$ for connected sum of manifolds

It turns out $S(\lambda)$'s are in one to one correspondence with cohomology classes of classifying space BG

$$H^3(BG, \mathbb{R}/\mathbb{Z}) \tag{1.73}$$

While for finite groups we have

$$H^3(BG, \mathbb{R}/\mathbb{Z}) \cong H^4(BG, \mathbb{Z}) \tag{1.74}$$

So by a topological Lagrangian we merely mean an element of the abelian group $H^4(BG, \mathbb{Z})$

When G is a simply connected Lie group, we have

$$H^4(BG, \mathbb{Z}) \cong \mathbb{Z} \tag{1.75}$$

And $\frac{1}{8\pi^2} \operatorname{Tr}(F \wedge F)$ is the generator [1], so the action is classified by $[k]$ directly as an element of $H^4(BG, \mathbb{Z}) \cong \mathbb{Z}$ as well

More general type of compact G with finite $\pi_1(G)$ can be built up by these two extreme cases by the following two exact sequences

$$1 \rightarrow G_0 \rightarrow G \rightarrow \Gamma \rightarrow 1 \tag{1.76}$$

Where Γ must be finite for compact G, and

$$1 \rightarrow \pi_1(G) \rightarrow \widetilde{G} \rightarrow G \rightarrow 1 \tag{1.77}$$

And indeed for those G, $S(\lambda)$ is determined by an element of

$$H^4(BG, \mathbb{Z}) \tag{1.78}$$

To be more precise, recall that we have the map $\gamma : M \rightarrow BG$, whose homotopic classes $[\gamma]$ are in one to one correspondence with the equivalent classes of G-principal bundle $[E]$, hence given λ there is an associated γ, and it is known that each $\omega \in H^4(BG, \mathbb{Z})$ will determine an real three cochain $\beta \in H^3(BG, \mathbb{R})$ with

$$\delta\beta = \Omega\left(F_u\right) - \omega \tag{1.79}$$

Where $\Omega(F) = \frac{k}{8\pi^2} \operatorname{Tr} F \wedge F$ and F_u is the curvature of the universal connection A_u on the universal bundle

Then we have a three cochain $\alpha = \beta(\mathrm{mod} 1)$ in $C^3(BG, \mathbb{R}/\mathbb{Z})$ such that its pullback $\alpha_A = \gamma^*\alpha$ on M is a well defined cohomology class in $H^3(M, \mathbb{R}/\mathbb{Z})$, the topological action S is realized as the natural pairing between cohomolgy and homology classes

$$S = \langle \alpha_A, [M] \rangle \tag{1.80}$$

For a Lie group this is an integral, while for a finite group it is just a finite sum.

Now we have defined Chern-Simons theory in a general but abstract way, the main point is that we can treat compact Lie groups and finite groups in an equal footing. This is very important for us as we are going to deal with these two cases in later applications.

1.2 Conformal field theory

A conformal field theory is a quantum field theory with conformal symmetry. It originates in several research areas such as the critical phenomena in statistical physics , the high energy behavior of quantum chromodynamics, and the world-sheet description in string theory, where the underlying physical systems are scale invariant. Since the 1980s it has become a highly developed branch in theoretical and mathematical physics, many new techniques are employed and new applications are explored, it appears in a vast range of topics both in physics and mathematics. For example, the AdS/CFT correspondence is used to analyze blackhole as well as hydrodynamics, CFT inspired ideas are used in number theory as well as probability theory. Here we will only review some of the basic ingredients, and more systematic developments can be found in standard textbooks such as [71, 24].

The importance of CFT arises from several different but related aspects:

- CFT indeed describes some special but important physical systems, either directly ,for example in phase transition and in string world-sheet,or via indirect means such as AdS/CFT correspondence. In particular, some nontrivial models can be solved exactly or numerically by CFT, which provides non-perturbative ways to deal with these problems, e.g. in bootstrap analysis of the three-dimensional Ising model.CFT is very rich in a range of applications, and it provides a bridge between different areas of physics.
- Viewed as a special kind of QFT, CFT is finite, that is, the beta function vanishes exactly. Therefor, CFTs provide natural candidates for renormalization group fixed points of various kinds, hence knowledge about CFT provides constraints and information on generic QFTs and RG processes, for example Zamolodchikov's c-theorem.
- It involves several important branches of mathematics combined in a specific way, and this makes CFT useful and inspiring in many important mathematical problems. For example, the 2d CFT on the torus provides a physical interpretation for some known mathematical identities of theta functions, and thus also inspires modern generalizations of the classical results.

In this section, we first discuss CFT with generic ($d > 2$) spacetime dimension[145, 141, 148], the important special case $d = 2$ is then analyzed separately[145, 61, 24, 23], finally we analyze the field content of CFT from a representation theory viewpoint and introduce the concept of modularity[145, 61, 24, 23].

1.2.1 Conformal symmetry and conformal field theory: $d > 2$

But what is conformal symmetry? It is the symmetry associated with conformal transformations.By definition, a conformal transformation is an invertible mapping of spacetime([9]) such that it leaves the metric invariant up to a scale:

$$x \to x', \quad g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x) \tag{1.81}$$

[9]viewed as an pseudo-Riemann manifold with signature (p, q), and for simplicity assumed to be flat, in most cases we just need Lorentzian/Euclidean $\mathbb{R}^{p,q}$ or torus

So locally it is just a combination of (pseudo) rotation and dilation, i.e.

$$J=\frac{\partial x'^{\mu}}{\partial x^{\nu}}=b(x)M_{\nu}^{\mu}(x), b(x)=\sqrt{\Lambda(x)} \tag{1.82}$$

Obviously, all Poincaré transformations are conformal, and dilation is conformal. In addition we also have the so called special conformal transformation(SCT), taking together they form the conformal group, in $\mathbb{R}^{p,q}, \quad p+q>2$ spacetime it is

$$\operatorname{Conf}\left(\mathbb{R}^{p,q}\right) \cong SO(p+1,q+1) \tag{1.83}$$

In particular for Euclidean spacetime $\mathbb{R}^d$, $p=d, q=0, d>2$, the conformal group is $\operatorname{Conf}\left(\mathbb{R}^{d,0}\right)$ and we have the following isomorphism

$$\operatorname{Conf}\left(\mathbb{R}^{d,0}\right) \cong SO(d+1,1) \tag{1.84}$$

As a consequence we find the dimension of $\operatorname{Conf}\left(\mathbb{R}^{d,0}\right)$ is

$$\frac{(d+2)(d+1)}{2} \tag{1.85}$$

It is generated by:

$$\begin{aligned} x'^{\mu} &= x^{\mu}+a^{\mu} \\ x'^{\mu} &= M^{\mu}{}_{\nu} x^{\nu} \\ x'^{\mu} &= \alpha x^{\mu} \\ x'^{\mu} &= \tfrac{x^{\mu}-b^{\mu}x^2}{1-2b\cdot x+b^2x^2} \end{aligned} \tag{1.86}$$

with the following Lie algebras

$$\begin{aligned} & P_{\mu}=-i\partial_{\mu} \\ & D=-ix^{\mu}\partial_{\mu} \\ & L_{\mu\nu}=i\left(x_{\mu}\partial_{\nu}-x_{\nu}\partial_{\mu}\right) \\ & K_{\mu}=-i\left(2x_{\mu}x^{\nu}\partial_{v}-x^2\partial_{\mu}\right) \end{aligned} \tag{1.87}$$

and commutators

$$\begin{aligned} {\left[D, P_{\mu}\right] } & =iP_{\mu} \\ {\left[D, K_{\mu}\right] } & =-iK_{\mu} \\ {\left[K_{\mu}, P_{\nu}\right] } & =2i\left(\eta_{\mu\nu}D-L_{\mu\nu}\right) \\ {\left[K_{\rho}, L_{\mu v}\right] } & =i\left(\eta_{\rho\mu}K_{v}-\eta_{\rho\nu}K_{\mu}\right) \\ {\left[P_{\rho'} L_{\mu\nu}\right] } & =i\left(\eta_{\rho\mu}P_{\nu}-\eta_{\rho\nu}P_{\mu}\right) \\ {\left[L_{\mu\nu}, L_{\rho\sigma}\right] } & =i\left(\eta_{\nu\rho}L_{\mu\sigma}+\eta_{\mu\sigma}L_{\nu\rho}-\eta_{\mu\rho}L_{\nu\sigma}-\eta_{\nu\sigma}L_{\mu\rho}\right) \end{aligned} \tag{1.88}$$

In the spirit of the last section, if we extend the covariance condition of an ordinary QFT to a conformal group, we will get a CFT(conformal field theory).

Since the conformal group contains the Poincaré group, in any local CFT the energy momentum tensor $T_{\mu\nu}$ always exists and is symmetric, conformal symmetry further enforces it to be traceless[10]

$$T^{\mu}_{\mu} = 0 \tag{1.89}$$

And indeed when $d > 2$, under some mild assumptions, given a QFT, Poincaré and scale invariance plus the energy momentum tensor being traceless will enforce full conformal invariance, but counter examples do exist, there are scale invariant but not conformal theories.[11] Since RG fixed points are scale invariant theories, it is quite natural for RG fixed points to be CFTs.

Due to scale invariance, the operator D plays a primary role in CFT through radial quantization, where the spacetime foliation is given by the family of concentric spheres S^{d-1}, and the radial direction plays the role of time, hence D is the Hamiltonian, to be more precise we have propagator

$$U = e^{iD\Delta\tau} \tag{1.90}$$

where $\tau = \log r$, and associated with each sphere a Hilbert space, particularly for the Hilbert space at the origin such states are classified by representations of D, i.e. the scaling dimension, or simply dimension Δ

$$D|\Delta\rangle = i\Delta|\Delta\rangle \tag{1.91}$$

and of $L_{\mu\nu}$, i.e. the spin l

$$L_{\mu\nu}|\Delta, l\rangle_{\{s\}} = (\Sigma_{\mu\nu})^{\{t\}}_{\{s\}}\,|\Delta, l\rangle_{\{t\}} \tag{1.92}$$

From the conformal algebra we know that P_μ, K_μ, D together forms an oscillator like system where P_μ, K_μ are ladder operators, hence we have

$$|\Delta\rangle \xrightarrow{P_\mu} |\Delta+1\rangle \xrightarrow{P_\nu} |\Delta+2\rangle\cdots \tag{1.93}$$

and

$$0 \xleftarrow{K_\mu} |\Delta\rangle \xleftarrow{K_\nu} |\Delta+1\rangle\cdots \tag{1.94}$$

We will call states with the property $K_\mu|\Delta\rangle = 0$ as primary states, and states obtained from primary states by P_μ as descendant states.

On the other hand, we can imagine that a state at spacetime point x are created by a field operator $\mathcal{O}(x)$, then again they are classified by representations

$$\begin{aligned}
&[P_\mu, \mathcal{O}(x)] = -i\partial_\mu\mathcal{O}(x)\\
&[D, \mathcal{O}(x)] = -i\left(\Delta + x^\mu\partial_\mu\right)\mathcal{O}(x)\\
&[L_{\mu\nu}, \mathcal{O}(x)] = -i\left(\Sigma_{\mu\nu} + x_\mu\partial_\nu - x_\nu\partial_\mu\right)\mathcal{O}(x)\\
&[K_\mu, \mathcal{O}(x)] = -i\left(2x_\mu\Delta + 2x^\lambda\Sigma_{\lambda\mu} + 2x_\mu\left(x^\rho\partial_\rho\right) - x^2\partial_\mu\right)\mathcal{O}(x)
\end{aligned} \tag{1.95}$$

[10]see appendix A.2 for the definition of local field, and there are indeed well defined CFTs with non-local fields, in these theories T may not exist, for example in long range Ising model

[11]See [65] for more details

In particular at the origin we have

$$[K_\mu, \mathcal{O}(0)] = 0 \tag{1.96}$$

and

$$\begin{aligned} &[P_\mu, \mathcal{O}(x)] = -i\partial_\mu \mathcal{O}(x) \\ &[D, \mathcal{O}(0)] = -i\Delta \mathcal{O}(0) \\ &[M_{\mu\nu}, \mathcal{O}(0)] = -i\Sigma_{\mu\nu}\mathcal{O}(0) \end{aligned} \tag{1.97}$$

we will call such $\mathcal{O}(0)$ operators as primary operators with dimension Δ and spin l. Equivalently, this says that the corresponding field $\phi(x)$ transfers properly under conformal transformations

$$\phi(x) \to \widetilde{\phi}(x') = \frac{1}{b(x)^\Delta} R\left[M_\nu^\mu(x)\right]\phi(x) \tag{1.98}$$

In summary we have the so called state-operator correspondence as

$$|\Delta\rangle = \mathcal{O}_\Delta(0)|0\rangle \tag{1.99}$$

and general states are superposition of primaries and their descendants

$$|\Psi\rangle = \mathcal{O}_\Delta(x)|0\rangle = e^{iPx}\mathcal{O}_\Delta(0)e^{-iPx}|0\rangle = e^{iPx}|\Delta\rangle = \sum_n \frac{1}{n!}(iPx)^n|\Delta\rangle \tag{1.100}$$

It turns out conformal symmetry also gives extra constraints on Δ, l for unitary or reflection positive theory, i.e. there is a lower bound for scaling dimension Δ, in the most common case of a gauge invariant primary in a spin-ℓ traceless symmetric tensor representation, it is[12]

$$\begin{aligned} &\Delta = 0 \text{ (unit operator), or} \\ &\Delta \geq \begin{cases} \frac{d-2}{2} & \ell = 0 \\ \ell + d - 2 & \ell > 0 \end{cases} \end{aligned} \tag{1.101}$$

this is called unitary bound. In particular for scalar the bound is saturated by free fields

$$\Box\mathcal{O}(x) = 0 \tag{1.102}$$

for fields with spin the bound is saturated by conserved currents $\mathcal{J}^\mu$, and it also works backwards so

$$\Delta = \ell + d - 2 \quad \text{iff} \quad \partial_\mu \mathcal{J}^\mu = 0 \tag{1.103}$$

[12] For an operator $\mathcal{O}$ in representation $R_\mathcal{O}$, the lower bound is the minimum of $\frac{1}{2}\left(-\operatorname{Cas}\left(V \otimes R_\mathcal{O}\right) + \operatorname{Cas}(V) + \operatorname{Cas}\left(R_\mathcal{O}\right)\right)$, where Cas denotes the Casimir invariant of the corresponding representation and $V = V_1$ is the vector representation, if $R_\mathcal{O} = V_\ell$ then due to the fact $V \otimes V_\ell = V_{\ell-1} \oplus \ldots \quad (\ell > 0)$ and $\operatorname{Cas}\left(V_{\ell-1}\right) = \ell(\ell+d-2)$,we have $\Delta \geq \frac{1}{2}\left(-\operatorname{Cas}\left(V_{\ell-1}\right) + \operatorname{Cas}(V) + \operatorname{Cas}\left(V_\ell\right)\right) = \ell+d-2$, see [148, 141]. While for a generic $d = 4$ primary in representation (j_1, j_2) the bound is $j_1 + j_2 + 2 - \delta_{j_1 j_2, 0}$[92]. Here we should emphasize that this holds for gauge invariant operators only, for example the gauge field A_μ in $d = 4$ Maxwell theory is in vector representation,but it has $\Delta = 1$ instead of 3. Further details, possible subtleties and improvements for specific values of d and types of representations are discussed in [120]

Now since we have defined states and operators in CFT, the natural objects to study are the correlators, again the conformal symmetry gives strong constraints. For simplicity we will consider an Euclidean spacetime without boundary, and for simplicity we just consider scalars, then the one point function vanishes

$$\langle \phi(x) \rangle = 0 \tag{1.104}$$

And the two point function is fixed up to an operator dependent constant

$$\langle \phi_1(x_1)\, \phi_2(x_2) \rangle = \begin{cases} \frac{C_{12}}{|x_1 - x_2|^{2\Delta_1}} & \text{if },\ \Delta_1 = \Delta_2 \\ 0 & \text{if} \quad \Delta_1 \neq \Delta_2 \end{cases} \tag{1.105}$$

The three point function is also fixed up to an operator dependent constant

$$\langle \phi_1(x_1)\, \phi_2(x_2)\, \phi_3(x_3) \rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{23}^{\Delta_2+\Delta_3-\Delta_1} x_{13}^{\Delta_3+\Delta_1-\Delta_2}} \tag{1.106}$$

The four point function is not fixed, but it is of a particular type:

$$\langle \phi_1(x_1) \ldots \phi_4(x_4) \rangle = f\left(\frac{x_{12}x_{34}}{x_{13}x_{24}}, \frac{x_{12}x_{34}}{x_{23}x_{14}} \right) \prod_{i<j}^{4} x_{ij}^{\Delta/3-\Delta_i-\Delta_i} \tag{1.107}$$

where $\Delta = \sum_{i=1}^{4} \Delta_i$

To better understand this, we can use the technique of operator product expansion,or OPE for short, suppose we are given a family of operators $\mathcal{O}_i(x)$, if we apply $\mathcal{O}_i(x)\mathcal{O}_j(0)$ to vacuum, we will get a superposition of primaries and descendants

$$\mathcal{O}_i(x)\mathcal{O}_j(0)|0\rangle = \sum C_{ijk}(x, P)\mathcal{O}_k(0)|0\rangle \tag{1.108}$$

This can be viewed as an equation for operators valid only when $\mathcal{O}_1$ and $\mathcal{O}_2$ are close enough and no other operators are inserted nearby

$$\mathcal{O}_i(x_1)\, \mathcal{O}_j(x_2) = \sum_k C_{ijk}(x_{12}, \partial_2)\, \mathcal{O}_k(x_2) \tag{1.109}$$

Then using OPE, we can calculate two point functions from three point functions, and the four point functions can be reduced to the sum of three point functions as well. But we can either expand as $1 \longleftrightarrow 2, 3 \longleftrightarrow 4$ or $1 \longleftrightarrow 4, 2 \longleftrightarrow 3$, just like the crossing symmetry in scattering amplitude, these two expansions must agree with each other. This put constraints on these coefficients C_{ijk}, which is the bootstrap equation,see figure 9. The problem of finding solutions to and extract information from such equations is called conformal bootstrap, it is of great importance and has a lot of applications, but we will not discuss it here as we do not use it in further development.[13]

[13]See [148] for more details on bootstrap

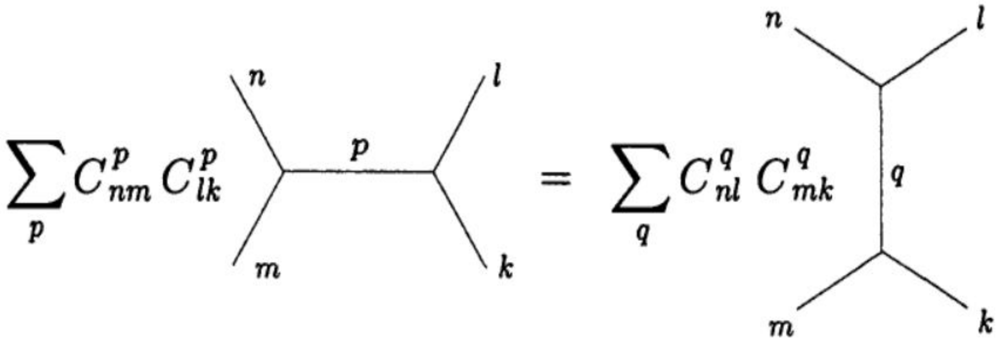


Figure 9: Crossing symmetry and bootstrap equation [71]

1.2.2 Conformal symmetry and conformal field theory: $d = 2$

Two dimensional systems are very special and important in mathematics, and many of them are related to each other in surprising ways; for example, beginning with classical complex analysis, we have two dimensional manifolds, algebraic curves and modular forms, all of which are related to the concept of Riemann surfaces.

And in physics, they provide a lot of nontrivial but solvable models such as the two-dimensional Ising model, which are cornerstones for physics in general. For our purposes, there are three important features associated with two dimensional conformal field theory:

- At dimension two the conformal Lie algebra gets enhanced to an infinite dimensional algebra, and after complexation and central extension it becomes the Virasoro algebra, which is the cornerstone of 2d CFT

- We can rewrite 2d CFT using a holomorphic formalism, so that many rich and powerful techniques in complex analysis become available. In a sense we just get a quantum version of classical complex analysis,this idea eventually will lead to a formal axiomatization of 2d CFT as vertex operator algebra (or VOA for short).

- Due to the special nature of 2d topology, we can define CFT for some general spacetime manifolds. Especially where the torus plays an essential role, and CFT on torus leads to modularity with associated modular data, this will have many surprising consequences such as the Verlinde formula.

Due to their importance, in this section we will focus on Virasoro algebra and introduce the basic concepts first, leaving the more formal discussions of modularity to the next section, VOA is also reviewed in the appendix A.2.

According to our general results in the last section, the two dimensional conformal group for Euclidean spacetime is

$$\text{Conf}\left(\mathbb{R}^{2,0}\right) \cong \text{SO}(3,1) \tag{1.110}$$

which is just the Lorentz group.

Since we are in dimension two, it is quite natural to use complex coordinates $z = x^0 + ix^1, \bar{z} = x^0 - ix^1$, and complex analysis tells us that

$$\text{Conf}\left(\mathbb{R}^{2,0}\right) \cong \text{SO}(3,1) \cong \text{PSL}(2,\mathbb{C}) \tag{1.111}$$

So for two dimensional Euclidean theory, the conformal transformations are just Möbius transformations $\varphi(z) = \frac{az+b}{cz+d}$, those transformations are injective and holomorphic, defined on the whole $\mathbb{C}$ with at most one exceptional point(hence holomorphic on the Riemann sphere), they are called global conformal transformations.

Also from complex analysis, we know that locally every holomorphic function is conformal in the sense of an angle and orientation keeping tranformation. It is obvious now, at least locally, that we have $ds^2 = dzd\bar{z} \mapsto \frac{\partial f}{\partial z}\frac{\partial \bar{f}}{\partial \bar{z}} dzd\bar{z}$ for infinite small transformations, but unlike Möbius transformations we can not define a general $f(z)$ globally as it may have more than one singular points. While in quantum theory symmetry is defined by the Lie algebra rather than the Lie group, so here it is quite natural to extend our definition of conformal transformation to include them. In this sense, we say that the conformal algebra at dimension two is enhanced to an infinite dimensional one. Precisely, we can first use power series expansions to write

$$\begin{aligned} z' &= z + \epsilon(z) = z + \textstyle\sum_{n\in\mathbb{Z}} \epsilon_n \left(-z^{n+1}\right) \\ \bar{z}' &= \bar{z} + \bar{\epsilon}(\bar{z}) = \bar{z} + \textstyle\sum_{n\in\mathbb{Z}} \bar{\epsilon}_n \left(-\bar{z}^{n+1}\right) \end{aligned} \tag{1.112}$$

and define the generators

$$l_n = -z^{n+1}\partial_z \quad \text{and} \quad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \tag{1.113}$$

then they form the so called Witt algebra

$$\begin{aligned} [l_m, l_n] &= (m-n)l_{m+n} \\ \left[\bar{l}_m, \bar{l}_n\right] &= (m-n)\bar{l}_{m+n} \\ \left[l_m, \bar{l}_n\right] &= 0 \end{aligned} \tag{1.114}$$

We see here that a key feature of Witt algebra is that it contains two decoupled identical copies $\{l_n\}$ and $\{\bar{l}_n\}$, it is a custom in complex analysis and geometry to view $z, \bar{z}$ as independent with $z = \bar{z}$ interpreted as reality condition, and call them as holomorphic and antiholomorphic parts(or chiral and anti-chiral). For simplicity we usually only write formulas for the holomorphic part only with the understanding that the corresponding antiholomorphic version is obvious.[14]

[14]this is not true in boundary conformal field theory, but in this thesis we consider bulk theory only

Also we find that the global conformal transformations are generated by

$$\{l_{-1}, l_0, l_1\} \tag{1.115}$$

with $l_{-1}, \bar{l}_{-1}$ for translation, $l_1, \bar{l}_1$ for SCT, $l_0 + \bar{l}_0$ for dilation, and $i\left(l_0 - \bar{l}_0\right)$ for rotation. it should be noted that in two dimensional Minkowski spacetime, we have

$$\operatorname{Conf}\left(\mathbb{R}^{1,1}\right) \cong \operatorname{Diff}_+(\mathbb{S}) \times \operatorname{Diff}_+(\mathbb{S}) \tag{1.116}$$

This group is indeed infinite dimensional, and its Lie algebra, after complexification, contains the Witt algebra as a dense subset.[15] And the finite subgroup

$$\mathrm{SO}(2,2)/\{\pm 1\} \cong \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R}) \subset \operatorname{Conf}\left(\mathbb{S}^{1,1}\right) \tag{1.117}$$

is the Minkowski spacetime analogue of $\mathrm{PSL}(2, \mathbb{C})$. So in summary, for dimension two, we should consider the infinite dimensional Witt algebra instead of the finite one $\{l_{-1}, l_0, l_{+1}\}$.

The above considerations are pure classical, and after quantization it is modified, and the Witt algebra W becomes Virasoro algebra Vir. Mathematically, this is characterized by central extensions of the Witt algebra W

$$0 \longrightarrow \mathbb{C} \longrightarrow Vir \longrightarrow W \longrightarrow 0 \tag{1.118}$$

with

$$H^2(\mathrm{~W}, \mathbb{C}) \cong \mathbb{C} \tag{1.119}$$

and

$$\text{Vir } = W \oplus \mathbb{C} Z \tag{1.120}$$

then we have

$$\begin{gathered}
{\left[L_n, L_m\right]=(n-m) L_{n+m}+\delta_{n+m, 0} \frac{n}{12}\left(n^2-1\right) Z} \\
{\left[L_n, Z\right]=0 \quad \text { for } \quad n, m \in \mathbb{Z}}
\end{gathered} \tag{1.121}$$

It is important to note that after central extension, the generators of global transformations $\{L_{-1}, L_0, L_1\}$ still form a subalgebra.

We can view the central charge Z as an ordinary c-number c ,i.e. identify it with its eigenvalue, so we just write

$$\left[L_n, L_m\right]=(n-m) L_{n+m}+\frac{c}{12}\left(n^3-n\right) \delta_{n+m, 0} \tag{1.122}$$

This number c is a definition datum for a CFT, it is also known as conformal anomaly, this is because if we put the underlying CFT in a curved spacetime with genus g and curvature R then we would have

$$\left\langle T_\mu^\mu(x)\right\rangle_g=\frac{c}{24 \pi} R(x) \tag{1.123}$$

[15]But as Lie group $\operatorname{Diff}_+(\mathbb{S})$ has no complexification, hence the 'infinite dimensional conformal group' does not exist, for more information see [145]

physically, it measures the effective number of degrees of freedom of the underlying theory, as if we put the CFT on an infinite cylinder with circumference $\ell = 1/T$ in the (imaginary)time direction, then the specific heat of this system is given by

$$\lim_{\ell \to \infty} \frac{C(T)}{\ell} = \frac{\pi}{3} cT \tag{1.124}$$

Central charge is important also because of the following famous theorem for generic $d = 2$ QFT with coupling constants g_i at the energy scale μ

Theorem 1.1 (C-theorem[164, 70])**.** *There exists a function $C(g_i, \mu)$ of the coupling constants which is decreasing along the RG flow and it is stationary only at the fixed points. Moreover, at the fixed points the $C(g_i^*, \mu) = C_*$ function is equal to the central charge of the CFT of the fixed point*

More generally for $d > 2$ CFT on a generic manifold M with possible boundary ∂M, we can have different types of anomalies, along with some theorems on RG behaviors which generalize the C-theorem here[97, 72, 113]. For example with $d = 4$ we have the bulk a term as $a \sim \int_{S^4} \langle T^\mu_\mu \rangle$ and the A-theorem roughly says that $a_{IR} < a_{UV}$ [114].

Since we now have an enlarged set of conformal transformations, the concept of a primary should modified correspondingly.For a general field $\phi(z, \bar{z})$ under scaling transformation $z \mapsto \lambda z$, if we have

$$\phi(z, \bar{z}) \mapsto \phi'(z, \bar{z}) = \lambda^h \bar{\lambda}^{\bar{h}} \phi(\lambda z, \bar{\lambda} \bar{z}) \tag{1.125}$$

then $(h, \bar{h})$ are defined as the conformal (or scaling) dimensions of $\phi(z, \bar{z})$, they are related to Δ, l as $h = \frac{1}{2}(\Delta + l) \quad \bar{h} = \frac{1}{2}(\Delta - l)$

Given any conformal transformation $z \mapsto f(z)$ if we always have

$$\phi(z, \bar{z}) \mapsto \phi'(z, \bar{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi(f(z), \bar{f}(\bar{z})) \tag{1.126}$$

or in infinitesimal form

$$\delta_{\epsilon, \bar{\epsilon}} \phi(z, \bar{z}) = \left(h \partial_z \epsilon + \epsilon \partial_z + \bar{h} \partial_{\bar{z}} \bar{\epsilon} + \bar{\epsilon} \partial_{\bar{z}}\right) \phi(z, \bar{z}) \tag{1.127}$$

then we will define $\phi(z, \bar{z})$ as a primary(or Virasoro primary) field, if this holds only for global conformal transformations, we will define $\phi(z, \bar{z})$ as a quasi-primary field. It is obvious that quasi-primary is primary with respect to our earlier definition for conformal fields with $d > 2$, but Virasoro primary is unique for $d = 2$ CFT, it is indeed an infinite sum of quasi primaries.

As we have mentioned, the Witt algebra, hence the Virasoro algebra, has holomorphic and anti-holomorphic parts, so we will call a field $\phi(z)$ depending on z only as holomorphic(chiral) field and similarly $\phi(\bar{z})$ as anti-holomorphic(anti-chiral), then $\phi(z),$ have conformal dimensions $h, \bar{h}$ respectively, so we can just treat them separately.

We can employ radical quantization as in general conformal field theory, and now because spacetime is two dimensional, the foliation is given by concentric circles on complex plane,

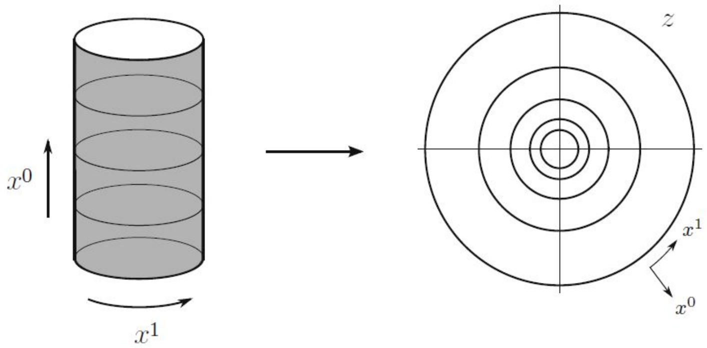


Figure 10: The conformal map from cylinder to complex plane [24]

see figure10. In analogue with the Fourier expansion on the cylinder we can obtain the Laurent expansion on the complex plane

$$\phi(z,\bar{z}) = \sum_{n,\bar{m}\in\mathbb{Z}} z^{-n-h}\bar{z}^{-\bar{m}-\bar{h}}\phi_{n,\bar{m}} \tag{1.128}$$

where the Laurent modes $\phi_{n,\bar{m}}$ become operators after quantization, then using operator-state correspondence we can define an in state as

$$|\phi\rangle = \lim_{z,\bar{z}\to 0}\phi(z,\bar{z})|0\rangle = \phi_{-h,-\bar{h}}|0\rangle \tag{1.129}$$

with Hermitian conjugation

$$(\phi_{n,\bar{m}})^{\dagger} = \phi_{-n,-\bar{m}} \tag{1.130}$$

we also have an out state

$$\langle\phi| = \lim_{\bar{w},w\to\infty} w^{2h}\bar{w}^{2\bar{h}}\langle 0|\phi(w,\bar{w}) = \langle 0|\phi_{+h,+\bar{h}} \tag{1.131}$$

For a holomorphic field, this is simply $\phi(z) = \sum_{n\in\mathbb{Z}} z^{-n-h}\phi_n$ and $|\phi\rangle = |h\rangle = \phi_{-h}|0\rangle$

In order to calculate correlators we also need to define commutators and OPEs. To do this, first notice that now the time ordering is represented as radial ordering.[16]

$$R(A(z)B(w)) := \begin{cases} A(z)B(w) \text{ for } |z| > |w| \\ B(w)A(z) \text{ for } |w| > |z| \end{cases} \tag{1.132}$$

and using the contour sum in figure 11, the equal time commutator is

$$\begin{aligned} \oint dz[A(z),B(w)] &= \oint_{|z|>|w|} dz A(z)B(w) - \oint_{|z|<|w|} dz B(w)A(z) \\ &= \oint_{\mathcal{C}(w)} dz R(A(z)B(w)) \end{aligned} \tag{1.133}$$

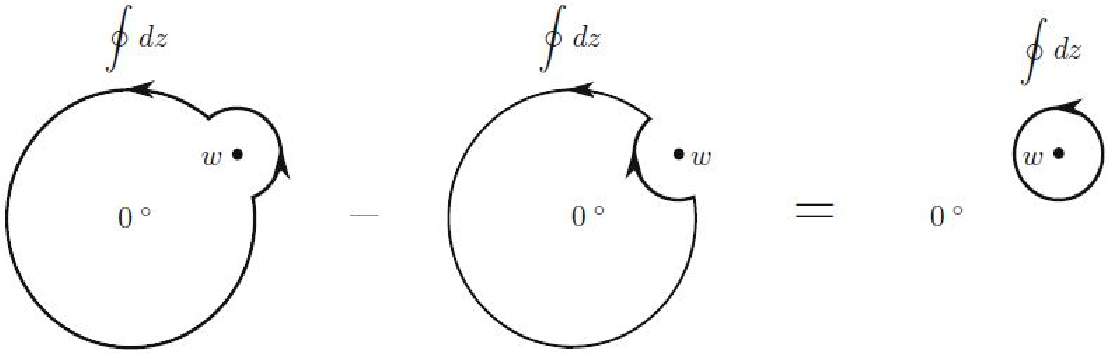


Figure 11: The contour sum at LHS is equivalent to the one on RHS [24]

And normal ordering is defined by $N(\chi\phi)(w) = \oint_{\mathcal{C}(w)} \frac{dz}{2\pi i} \frac{\phi(z)\chi(w)}{z-w}$,or in terms of modes

$$N(\chi\phi)_n = \sum_{k>-h^\phi} \chi_{n-k}\phi_k + \sum_{k\le -h^\phi} \phi_k \chi_{n-k} \tag{1.134}$$

such that ϕ_n's with $n > -h$ are annihilation operators and ϕ_n's with $n \le -h$ are creation operators. In this way we can define OPEs and calculate the correlators between primaries just as before, again correlators$\langle X\rangle = \langle \phi_1(w_1, \bar{w}_1)\dots\phi_N(w_N, \bar{w}_N)\rangle$ are constrained by conformal symmetry, hence $n < 4$ points functions are fixed in the same form, while $n \ge 4$ points functions are determined by bootstrap equation.

The energy momentum tensor is of central importance in $d = 2$ CFT, and has many nice properties. First we should notice that in the Euclidean plane the traceless condition for the energy momentum tensor becomes $T_{00} + T_{11} = 0$, as a consequence, in complex coordinates T also separates into holomorphic and anti-holomorphic part

$$T_{zz}(z, \bar{z}) =: T(z), \quad T_{\overline{zz}}(z, \bar{z}) =: \bar{T}(\bar{z}) \tag{1.135}$$

with

$$T_{zz} = \frac{1}{2}\left(T_{00} - iT_{10}\right), \quad T_{\overline{zz}} = \frac{1}{2}\left(T_{00} + iT_{10}\right) \tag{1.136}$$

In this notation, for a general correlator $\langle X\rangle = \langle \phi_1(w_1, \bar{w}_1)\dots\phi_N(w_N, \bar{w}_N)\rangle$ we have the following conformal Ward identity

$$\delta_{\epsilon,\bar{\epsilon}}\langle X\rangle = -\frac{1}{2\pi i}\oint_C dz\epsilon(z)\langle T(z)X\rangle + \frac{1}{2\pi i}\oint_C d\bar{z}\widetilde{\epsilon}(\bar{z})\langle \bar{T}(\bar{z})X\rangle \tag{1.137}$$

where $\oint_C$ enriches all the field positions $(w_i, \bar{w}_i)$ appear in $\langle X\rangle$. In particular,for a single primary $\langle X\rangle = \langle \phi(w, \bar{w})\rangle$ we should recover (1.127),using the residue theorem, we find the following OPEs

[16] we usually omit the explicit radial ordering symbol R if it is clear from the context.

$$
\begin{aligned}
T(z)\phi(w,\bar{w}) &= \tfrac{h}{(z-w)^2}\phi(w,\bar{w}) + \tfrac{1}{z-w}\partial_w\phi(w,\bar{w}) + \ldots \\
\bar{T}(\bar{z})\phi(w,\bar{w}) &= \tfrac{\bar{h}}{(\bar{z}-\bar{w})^2}\phi(w,\bar{w}) + \tfrac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\phi(w,\bar{w}) + \ldots
\end{aligned}
\tag{1.138}
$$

which can be used as an alternative definition for primary, and using those OPEs, we can write the conformal Ward identity in differential notation

$$
\begin{aligned}
&\langle T(z)\phi_1(w_1,\bar{w}_1)\ldots\phi_N(w_N,\bar{w}_N)\rangle \\
&= \sum_{i=1}^{N}\left(\frac{h_i}{(z-w_i)^2} + \frac{1}{z-w_i}\partial_{w_i}\right)\langle\phi_1(w_1,\bar{w}_1)\ldots\phi_N(w_N,\bar{w}_N)\rangle
\end{aligned}
\tag{1.139}
$$

It is important to point out that $T(z)$ itself is not primary, instead it is only a quasi primary and we have

$$
T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \ldots \tag{1.140}
$$

To see this, using the Laurent expansion

$$
T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n \quad \text{where} \quad L_n = \frac{1}{2\pi i}\oint dz z^{n+1}T(z) \tag{1.141}
$$

because

$$
Q_\epsilon = \frac{1}{2\pi i}\oint dz\epsilon(z)T(z), \quad \delta_\epsilon\Phi(w) = -\left[Q_\epsilon,\Phi(w)\right] \tag{1.142}
$$

hence for $\epsilon(z) = \sum_{n\in\mathbf{Z}} z^{n+1}\epsilon_n$ we have

$$
Q_\epsilon = \sum_{n\in\mathbf{Z}}\epsilon_n L_n \tag{1.143}
$$

so we should identify L_n as the Virasoro generator in (1.122), then the OPE (1.140) is equivalent to (1.122). Indeed, under a finite transformation $w = f(z)$, by explicit calculation we have

$$
T'(z) = \left(\frac{\partial f}{\partial z}\right)^2 T(f(z)) + \frac{c}{12}S(f(z),z) \tag{1.144}
$$

where the part deviates from being a primary is called Schwarzian derivative

$$
S(w,z) = \frac{1}{(\partial_z w)^2}\left(\left(\partial_z w\right)\left(\partial_z^3 w\right) - \frac{3}{2}\left(\partial_z^2 w\right)^2\right) \tag{1.145}
$$

Using L_n we can construct the Hilbert space associated with a primary. Begin with $|\phi\rangle = |h\rangle = \phi_{-h}|0\rangle$, and notice that

$$
[L_m,\phi_n] = ((h-1)m-n)\phi_{m+n} \tag{1.146}
$$

so $L_0|h\rangle = [L_0, \phi_{-h}]\,|0\rangle = h|h\rangle$ and $L_n|\phi\rangle = [L_n, \phi_{-h}]\,|0\rangle = (h(n+1)-n)\phi_{-h+n}|0\rangle = 0$ for $n > 0$, while applying L_{-n} repeatedly generates descendants of the form

$$L_{-k_1}L_{-k_2}\cdots L_{-k_n}|h\rangle \quad (1 \le k_1 \le \cdots \le k_n) \tag{1.147}$$

For $|\psi\rangle = L_{-k_1}L_{-k_2}\cdots L_{-k_n}|h\rangle$ we have $L_0\,|\psi\rangle = (k+h)\,|\psi\rangle$ where $k = \sum_i k_i$ is defined as the level of $|\psi\rangle$, for example the first few descendants are

Field	State	Level
$\phi(z)$	$\phi_{-h}\|0\rangle = \|h\rangle$	0
$\partial\phi$	$L_{-1}\phi_{-h}\|0\rangle$	1
$\partial^2\phi$	$L_{-1}L_{-1}\phi_{-h}\|0\rangle$	2
$N(T\phi)$	$L_{-2}\phi_{-h}\|0\rangle$	2
$\partial^3\phi$	$L_{-1}L_{-1}L_{-1}\phi_{-h}\|0\rangle$	3
$N(T\partial\phi)$	$L_{-2}L_{-1}\phi_{-h}\|0\rangle$	3
$N(\partial T\phi)$	$L_{-3}\phi_{-h}\|0\rangle$	3

...

The collection of $|h\rangle$ along with its descendants is a collection of representations of Virasoro algebra, i.e. a Verma module with the primary as the irreducible highest weight representation, we will denote it as $V(c,h)$. Using operator-state correspondence, we can also represent L_{-n}'s as differential operators act on correlators

$$\left\langle \widehat{L}_{-n}\phi(w)\phi_1(w_1)\ldots\phi_N(w_N)\right\rangle = \mathcal{L}_{-n}\left\langle \phi(w)\phi_1(w_1)\ldots\phi_N(w_N)\right\rangle \tag{1.148}$$

with

$$\mathcal{L}_{-n} = \sum_{i=1}^{N}\left(\frac{(n-1)h_i}{(w_i-w)^n} - \frac{1}{(w_i-w)^{n-1}}\partial_{w_i}\right) \tag{1.149}$$

Finally, here we give two basic examples of 2d CFT, first we have the free boson defined by the action

$$\mathcal{S} = \frac{1}{4\pi}\int dzd\bar{z}\partial X\cdot\bar{\partial}X \tag{1.150}$$

There are two types of primary, one type is $j(z) = i\partial X(z,\bar{z}), \bar{j}(\bar{z}) = i\bar{\partial}X(z)$ with $(h,\bar{h}) = (1,1)$ and the following OPE

$$\langle j(z)j(w)\rangle = \frac{1}{(z-w)^2} \tag{1.151}$$

another type is so the called vertex operator, it is an operator of the form $V(z,\bar{z}) =: e^{i\alpha X(z,\bar{z})}:$ with dimension $(h,\bar{h}) = \left(\frac{\alpha^2}{2}, \frac{\alpha^2}{2}\right)$ with OPE

$$\langle V_{-\alpha}(z,\bar{z})V_{\alpha}(w,\bar{w})\rangle = \frac{1}{(z-w)^{\alpha^2}(\bar{z}-\bar{w})^{\alpha^2}} \tag{1.152}$$

In this theory $c = 1$ and the energy momentum tensor is $T(z) = \frac{1}{2}N(jj)(z)$, with a similar antiholomorphic part. Correspondingly, the free fermion is defined by the action

$$\mathcal{S} = \frac{1}{4\pi}\int dzd\bar{z}(\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi}) \tag{1.153}$$

where we must specify the (anti)periodic condition

$$\begin{aligned}\psi\left(e^{2\pi i}z\right) &= +\psi(z) \quad \text{Neveu-Schwarz sector (NS),}\\ \psi\left(e^{2\pi i}z\right) &= -\psi(z) \quad \text{Ramond sector (R).}\end{aligned} \tag{1.154}$$

with $r \in \mathbb{Z}+\frac{1}{2}$ Neveu-Schwarz sector (NS), $r \in \mathbb{Z}$ Ramond sector (R) for the mode index. ψ and $\bar{\psi}$ are primaries with conformal dimensions $(h,\bar{h}) = \left(\frac{1}{2},0\right)$, and $(h,\bar{h}) = \left(0,\frac{1}{2}\right)$ respectively, with the following OPE

$$\psi(z)\psi(w) = \frac{1}{z-w} \tag{1.155}$$

In this theory $c = \frac{1}{2}$ and the energy momentum tensor is $T(z) = \frac{1}{2}N(\psi\partial\psi)$, with a similar antiholomorphic part.

1.2.3 Representations, symmetries and modularity

We have defined the basic vocabulary of $d = 2$ CFT in the last section, and we see that the key data are the central charge c and the primaries $(h,\bar{h})$. In ordinary QFT,we have some particles interacting with each other that we want to study, so we first construct fields with appropriate transformation properties of the symmetry group of the interaction, and then identify particles as quanta of corresponding fields, hence also labeled by representations of the symmetry algebras, so in essence a QFT is a collection of representations. Although due to scale invariance, there is no cluster decomposition so we can not identify primaries $(h,\bar{h})$ as massless particles in the usual sense, but it is still reasonable to say that a CFT is just a collection of primaries as irreducible representations of conformal algebras. Especially for 2d CFT, it is a collection of Verma modules of Virasoro algebra, and due to chirality it has a tensor structure.

This allows us to identify a 2d CFT as an inner product space $\mathfrak{H}$, which is a direct sum of tensor products of Verma modules as representations of $Vir_c \times \overline{Vir}_{\bar{c}}$ [18, 122]

$$\mathfrak{H} = \bigoplus_{h,\bar{h}} V(h,c) \otimes \bar{V}(\bar{h},\bar{c}) \tag{1.156}$$

such that

- Vacuum

  $|0\rangle$ with $(h,\bar{h}) = (0,0)$ exists and is unique, it is invariant under global conformal transformations

- Operator-state correspondence

  To each vector $\alpha \in \mathfrak{H}$, there is an associated operator $\Phi_\alpha(z)$, in particular $|0\rangle$ corresponds to unit operator. These field operators have OPEs, and for a given $\Phi_\alpha(z)$ there is a conjugate $\Phi_{\alpha'}$ such that $\Phi_\alpha\Phi_{\alpha'}$ contains a descendant of the unit operator.

- Primary

  Chrial/anti-chrial primaries $|h\rangle$ and $|\bar{h}\rangle$ are highest weight vectors of $V(h,c)$ and $\bar{V}(\bar{h},\bar{c})$ respectively, with Virasoro algebra generators L_n and $\bar{L}_n$ act on them.

- Analyticity

  The correlators of chiral fields are analytic functions of z

- Modularity

  The correlators and the one loop partition functions are modular invariant, i.e. $V(h,c)$ and $\bar{V}(\bar{h},\bar{c})$ should be paired in a restricted but not arbitrary way.

The first four properties are obvious, while the last one will be discussed later, now it is enough to say that if $\bar{V}=V$, i.e. $c=\bar{c}, h=\bar{h}$, then modularity is always satisfied, such theories are called diagonal, and for simplicity we will treat the holomorphic part $V(c,h)$ only.

The core concept underlying the above formalism is the chiral operator $T(z)$, which is physically the energy momentum tensor. We know that in QFTs it is common to have conserved currents other than the energy momentum tensor, in $d=2$ CFTs they are represented by chiral operators $W(z)$'s, for example the current operator $j(z)$ in the free boson theory (1.150) . Given $W(z)$, there is usually an associated larger symmetry algebra $\mathcal{W}$ containing the Virasoro algebra as a subalgebra, where we can take the Laurent modes W_n of $W(z)$ as the generators of $\mathcal{W}$ with a consistent set of commutators $[W_n, L_m], [W_n, W_m]$, for example $[L_0, W_n] = -nW_n$, this is called W-algebra. Using W-algebra, W-primaries are defined and realized as special combinations of Virasoro primaries, with a similar construction for the antiholomorphic part $\bar{\mathcal{W}}$, we can have the diagonal theory $\mathcal{W}\times\overline{\mathcal{W}}$ and direct sums of corresponding modules

$$\mathfrak{H}_W = \bigoplus_{i,\bar{i}} M_i \otimes \bar{M}_{\bar{i}} \tag{1.157}$$

In principle the possible number of primaries in a CFT can be infinite, and this is indeed the case in generic settings, but for some special theories there are finite many primaries only, or sometimes we can reorganizing infinite many Virasoro algebra modules into finite W-algebra ones, then we have rational conformal field theories, the term rational originates in the fact that c,h are rational numbers in those theories, we will denote it as

$$\mathcal{H} = \bigoplus_{h\in I, h\in \bar{I}} M(c,h)\otimes M(c,\bar{h}), \quad |I|, |\bar{I}| < \infty \tag{1.158}$$

and using $M(c,h)$ to denote its (reduced) Verma module or more general W-algebra module. In this section we will introduce two different kinds of RCFT, the (Virasoro) minimal model and the WZW model, where the former involves Virasoro algebra only and the later is a special type of W-algebra induced by current operators $J(z)$.

Before we get to concrete constructions, it is useful to first introduce the notation of the fusion rule for $d = 2$ CFT. We will only cover the basic notations here, a more abstract formalism using category is also summarized briefly in the appendix A.3.1, for a more systematic development see [73]. Indeed $d = 2$ CFTs have some properties very similar with $d = 3$ TQFTs, i.e. we can define brading and fusion processes through OPEs. In CFT, OPEs or correlators calculation is a basic problem, and once we have calculated all four point functions, the theory is solved, practically the OPEs contain almost all the information we want. However, OPEs are basis dependent, and we have to specify the field points x_i as well as some normalization constants to obtain C_{ij}^k. This problem is similar to the Clebsch–Gordan coefficients calculation, but in that case we know there is a basis independent way to encode the information, that is, using irreducible decomposition of tensor products of representations. In this sense fusion rules are just the basis independent reformulation of OPEs based on the following two facts:

- As we have seen above, fields are organized into direct sums of modules of corresponding algebras, in particular primaries ϕ_i are just highest weight modules and the descendants are determined by them to form families $[\phi_i]$, so we only need to consider OPEs between primaries
- Like Ward identities, the OPEs

$$\phi_i(z,\bar{z})\phi_j(w,\bar{w}) = \sum_k C_{ij}^k(z,w,\bar{z},\bar{w})\phi_k(w,\bar{w}) \tag{1.159}$$

  are operator equations valid inside correlators and they are closed, hence constitute a closed associative operator algebra

$$[\phi_i] \times [\phi_j] = \sum_k \mathcal{N}_{ij}^k[\phi_k] \tag{1.160}$$

Technically, we define fusion as follow:

- If we have

$$\phi_i(z,\bar{z})\phi_j(w,\bar{w}) = \sum_{k\in I} C_{ij}^k(z-w)^{-\Delta_i-\Delta_j+\Delta_k}(\bar{z}-\bar{w})^{-\bar{\Delta}_i-\bar{\Delta}_j+\bar{\Delta}_k}\left[\phi_k(w,\bar{w})+\ldots\right] \tag{1.161}$$

  or equivalently $C_{ij}^k = \lim_{z,\bar{z}\to\infty} z^{-2\Delta_k}\bar{z}^{-2\bar{\Delta}_k}\langle\phi_i(0,0)\phi_j(1,1)\phi_k(z,\bar{z})\rangle$ is nonvanishing, we say $\mathcal{N}_{ij}^k > 0$, hence it counts the number of distinct coupling constants between primaries appear on the RHS of the OPE.

  To obtain the exact values of $\mathcal{N}_{ij}^k$ we need to specific overall normalization constants. While in simple theories such as Virasoro minimal models , this is unnecessary as $\mathcal{N}_{ij}^k = 0, 1$ only, however, in more complicated examples, it is possible to have $\mathcal{N}_{ij}^k > 1$ and we need extra labels such as $C_{ij}^{k;(1)}\eta_{ij}^{k;(1)} + C_{ij}^{k;(2)}\eta_{ij}^{k;(2)} + \ldots$ to do proper counting.

In practice, such calculations are rather cumbersome, and one usually begins with a set of well defined fusion rules based on representation theoretical considerations and using it to calculate the OPEs, but in principle one can obtain $\mathcal{N}_{ij}^k$ from C_{ij}^k and vice versa.

- Then it is obvious that $\mathbf{1}\phi_i(z,\bar{z}) = \phi_i(z,\bar{z})$ so the vacuum serves as the identity $\mathbf{1} \times [\phi_i] = [\phi_i]$ of this algebra, and the antiparticle or involution ϕ_i^+ of ϕ_i is defined by

$$\left\langle \phi_i(z,\bar{z})\phi_i^+(w,\bar{w})\right\rangle = (z-w)^{-2\Delta_i}(\bar{z}-\bar{w})^{-2\bar{\Delta}_i} \tag{1.162}$$

such that $[\phi_i] \times [\phi_i^+] = \mathbf{1} + \cdots$

Fusion rules are indeed closely related to conformal blocks:

- Follow the bootstrap rule, the four-point function

$$\mathcal{F}(z,\bar{z}) \equiv \mathcal{F}_{ijkl}(z,\bar{z}) = \langle \phi_i(z,\bar{z})\phi_j(0,0)\phi_k(1,1)\phi_l(\infty,\infty)\rangle \tag{1.163}$$

is given by a sum over products of chiral and anti-chiral blocks

$$\mathcal{F}(z,\bar{z}) = \sum_{m=1}^{M}\sum_{\bar{m}=1}^{\bar{M}} a_{m\bar{m}}\mathcal{F}_m(z)\overline{\mathcal{F}}_{\bar{m}}(\bar{z}) \tag{1.164}$$

where the number $M \equiv M_{ijkl}$ of blocks is

$$M = \sum_{n\in I}\mathcal{N}_{ij}^n\mathcal{N}_{nkl} \tag{1.165}$$

with an appropriate normalization we also have $a_{m\bar{m}} = a_m\delta_{\bar{m},\sigma(m)}$ with some permutation σ and $a_m = C_{ij}^m C_{mkl}$.

- We can also define F-move and braiding matrices F, B by

$$\begin{aligned}
\mathcal{F}_{ilkj,p}(z) &= \sum_m \mathrm{F}_{pm}\begin{bmatrix} jk \\ il \end{bmatrix}\mathcal{F}_{ijkl,m}\left(z^{-1}\right)\\
\mathcal{F}_{ikjl,n}(z) &= \sum_m \mathrm{B}_{nm}\begin{bmatrix} jk \\ il \end{bmatrix}\mathcal{F}_{ijkl,m}(1-z),
\end{aligned} \tag{1.166}$$

Then we will have hexagon and pentagon equations, this eventually leading to the categories we discussed in appendix A.3.1. And we can verify that just like the fusion rules of anyons, the CFT fusion rules are well defined and consistent. However it should be pointed out that the fusion rules we have constructed are not identical to irreducible decomposition of tensor products of representations of Virasoro algebra or W-algebra as in that case quantities such as central charges c_i, c_k should sum up rather than stay the same. The point is, when view fusion rules as tensor products, the collection of modules in a CFT are the objects of a rigid braided monoidal category, while for RCFT it is indeed a modular tensor category, hence we have a pure algebraic way to characterize them.

As the simplest example of RCFT, given $\mathfrak{H}$, using the commutation relations of Virasoro algebra, the norm $\langle\chi|\chi\rangle$ of a generic state $|\chi\rangle$, either primary or descendant, can be calculated, and it is possible that $\langle\chi|\chi\rangle \leq 0$

- If $\langle\chi|\chi\rangle = 0$, then $|\chi\rangle$ is a null state, we may want to quotient out all of such $|\chi\rangle$ by identifying $|\psi\rangle \sim |\psi\rangle + \alpha |\chi\rangle$ as physical states to obtain a new theory $\mathfrak{H}'$, this new theory may again contain null states $|\chi'\rangle$ so we repeat quotient out them, for some special $\mathfrak{H}$ if we do this in a very careful and consistent eventually we can arrive at a RCFT

- If $\langle\chi|\chi\rangle < 0$ the theory is non-unitary, [17] if we want unitary theories we have to adjust the value of c, h, it terms out that either

$$c \geq 1, \quad h \geq 0 \tag{1.167}$$

or they take some special discrete rational values only, labeled by a natural number m

$$\begin{aligned} c &= 1 - \tfrac{6}{m(m+1)} \\ h_{rs}(m) &= \tfrac{[(m+1)r - ms]^2 - 1}{4m(m+1)} \quad (1 \leq r < m, 1 \leq s < r) \end{aligned} \tag{1.168}$$

Combine those two facts we can construct a series of RCFT, namely the minimal models:

$$\mathcal{H}_{min} = \bigoplus_{\substack{1 \leq < p' \\ 1 \leq s < p}} M\left(c, h_{r,s}\right) \otimes \bar{M}\left(c, h_{r,s}\right) \tag{1.169}$$

where $p > p' \geq 2$ are two coprime integers and

$$\begin{aligned} c &= 1 - 6\frac{(p-p')^2}{pp'} \\ h_{r,s} &= \frac{(pr - p's)^2 - (p-p')^2}{4pp'} \end{aligned} \tag{1.170}$$

the theory is unitary iff $p = p' + 1 = m + 1$.

In minimal models we can label a generic primary as $\phi_{(r,s)}$ where the following identification rule is understood

$$\phi_{(r,s)} = \phi_{(p'-r,p-s)} \tag{1.171}$$

Minimal models are completely solvable and the fusion rules are known explicitly as

$$\boldsymbol{\phi}_{(r,s)} \times \boldsymbol{\phi}_{(m,n)} = \sum_{\substack{k = 1 + |r-m| \\ k + r + m = 1 \pmod 2}}^{\min(r+m-1, 2p'-1-r-m)} \sum_{\substack{l = 1 + |s-n| \\ l + s + n = 1 \pmod 2}}^{\min(s+n-1, 2p-1-s-n)} \boldsymbol{\phi}_{(k,l)} \tag{1.172}$$

[17] it should be noted that this does not mean nonphysical as many physical systems such as Lee-Yang model are described by non-unitary theories

the $mod2$ restrictions are equivalent to increment by 2 instead of 1, for simplicity, sometimes we will simply use $\sum'$ to denote a sum with this restriction, and use (r,s) for $\boldsymbol{\phi}_{(r,s)}$, k_{max}, l_{max} and k_{min}, l_{min} for the upper and lower bounds of k, l, and the model it self is denoted as $\mathcal{M}(p, p')$. Minimal models also have a lot of applications, in particular the first few ones are well known models for two dimensional phase transitions, $M(5,2)$ the simplest but non-unitary one is the Lee-Yang model, while $M(4,3)$ the simplest unitary one is the Ising model, then we have $M(5,4)$ as the tricritical Ising model and $M(6,5)$ as the three-state Potts model.

Up to now we have assumed that our spacetime is the complex plane $\mathbb{C}$, or its conformal compactification, the Riemann sphere S^2. We may also want to define CFTs on general two dimensional manifolds, especially on oriented compacted ones, as we can always perform conformal compactification. We know from topology that two dimensional oriented compacted manifolds are just compact Riemann surfaces and they are labeled by their genus g, with $g = 0$ for Riemann sphere S^2, $g = 1$ for torus $\mathcal{T}_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, and $g = n$ for the generic case as a sphere with n handles, or identically, as a connected sum of n tori. Physically, CFTs on $g > 0$ surfaces correspond to perturbative expansions of string amplitudes, where the $g = 1$ case of one loop processes is especially important, these CFTs on $\mathcal{T}_\tau$ can also be interpreted as periodic systems on both space and time directions, which are common in the context of statistical mechanics. The set of consistency constrains for a CFT to be well defined on $\mathcal{T}_\tau$ is called modularity, and more generally we will have sewing conditions as consistency constraints for CFT on higher genus surfaces. Basically, we want the partition function on $\mathcal{T}_\tau$

$$Z(\tau) = \mathrm{Tr}\left(q^{L_0 - c/24}\bar{q}^{\bar{L}_0 - c/24}\right) \tag{1.173}$$

with $q = \exp 2\pi i\tau, \bar{q} = \exp -2\pi i\bar{\tau}$ to be modular invariant.

To analyze the action of modular group $\mathrm{PSL}(2, \mathbf{Z})$ on $Z(\tau)$,we first introduce the formal graded-dimension of $V(c,h)$ as

$$\chi_h(\tau) = \mathrm{Tr}_{V(c,h)}\left(q^{L_0 - c/24}\right) \tag{1.174}$$

usually, this is divergent and represents a formal sum only, but for RCFT it reduces to a well defined function with appropriate modular properties, and since the number of primaries is finite in RCFT we can rewrite the partition function as a finite sum

$$Z(\tau) = \sum_{h,\bar{h}} \mathcal{M}_{h,\bar{h}}\chi_h(\tau)\bar{\chi}_{\bar{h}}(\bar{\tau}) \tag{1.175}$$

with $\mathcal{M}_{h,\bar{h}}$ counts the possible multiplicity of $M(c,h) \otimes M(c,\bar{h})$ in $\mathcal{H}$, for simplicity we can always order the index set I of primaries and assume χ_0 corresponds to vacuum, i.e. we denote the RCFT and its partition function as

$$\mathcal{H} = \bigoplus_{i\in\mathcal{I},\bar{i}\in\bar{\mathcal{I}}} \mathcal{M}_{i,\bar{i}} M_i \otimes \overline{M_{\bar{i}}}, \quad Z = \sum_{i\in\mathcal{I},\bar{i}\in\bar{\mathcal{I}}} \mathcal{M}_{i,\bar{i}}\chi_i(\tau)\overline{\chi_{\bar{i}}(\tau)} \tag{1.176}$$

then under modular transformation $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z})$ we demand that

$$\chi_i(\gamma\tau) = \sum_j \rho(\gamma)_{ij}\chi_j(\tau) \tag{1.177}$$

with some modular representation$\rho : PSL(2,\mathbb{Z}) \to GL(V)$ on the space of characters, in particular for the generator $\mathcal{S} : \tau \to -1/\tau, \quad \mathcal{T} : \tau \to \tau + 1$ we just omit ρ and write

$$\chi_i\left(-\frac{1}{\tau}\right) = \sum_{j=0}^{N-1} \mathcal{S}_{ij}\chi_j(\tau), \quad \chi_i(\tau+1) = \sum_{j=0}^{N-1} \mathcal{T}_{ij}\chi_j(\tau) \tag{1.178}$$

then modularity just means

$$Z(\tau) = Z(\mathcal{S}\tau) = Z(\mathcal{T}\tau) \tag{1.179}$$

And this reduces to construct the natural number valued matrix $\mathcal{M}_{i,\bar{i}}$ compatible with the actions of $\mathcal{S}, \mathcal{T}$.

$$\begin{aligned} &\mathcal{M}_{0,0} = 1 \\ &\mathcal{M}\mathcal{T} = \mathcal{T}\mathcal{M} \\ &\mathcal{M}S = \mathcal{S}\mathcal{M} \end{aligned} \tag{1.180}$$

Obviously, the identity matrix is a trivial solution and it corresponds to diagonal theories, but other solutions do exist. For example for minimal models, the complete solutions are known, and quite surprisingly it has an ADE pattern classification.

We can choose a basis such that $\mathcal{T}$ is diagonal

$$\mathcal{T}_{ij} = \delta_{ij}e^{2\pi i\left(h_i - \frac{c}{24}\right)} \tag{1.181}$$

usually the representation of $\mathcal{S}$ in this basis is refereed as the S-matrix. For example, for minimal models we have

$$S_{rs;\rho\sigma} = 2\sqrt{\frac{2}{pp'}}(-1)^{1+s\rho+r\sigma}\sin\left(\pi\frac{p}{p'}r\rho\right)\sin\left(\pi\frac{p'}{p}s\sigma\right) \tag{1.182}$$

S-matrix has the very important property such that it is related to the fusion rules as

$$\mathcal{N}_{ij}{}^k = \sum_m \frac{\mathcal{S}_{im}\mathcal{S}_{jm}\overline{\mathcal{S}}_{mk}}{\mathcal{S}_{0m}} \tag{1.183}$$

This is the CFT version of the Verlinde formula, we see for example from (1.182), that the elements of S matrix are usually complicated, in general they are not rational numbers, but the special combinations on the right hand side of Verlinde formula indeed reproduce the natural numbers on the left hand side. This formula has many generalizations and the appearance of those natural numbers can be interpreted as dimension counting of certain spaces of representations.[145]

The (Virasoro) minimal models we have introduced above are 'minimal' in the sense that they are the only RCFT realization of a CFT with conformal symmetry, i.e. Virasoro algebra, only. But a CFT may have extra symmetries hence a larger algebra with Virasoro algebra as a subalgebra, it is possible to combine infinite many Virasoro algebra irreducible representations to form a single irreducible representation of this larger algebra, and an RCFT can be constructed with respect to such representations.The WZW model we have mentioned in the discussion of Chern-Simons theory is a typical example of this kind, where the larger algebra is an affine Lie algebra $\widehat{g}_k$ at level k, another common example is super conformal field theory where we have super conformal algebra, there are other options beyond those two as well, such as W-algebra, but we are mainly interested in the WZW model and we here will introduce more details about it.

In the free boson example, we have dimension $h=1, \bar{h}=1$ primary $j(z), \bar{j}(\bar{z})$, which is very special as it is related to a conserved current. We will call such a chiral field with $h=1$ as (chiral) current, and similar for anti-chiral case. The most general OPE between two currents are

$$J^a(z)J^b(w) \sim \frac{k\delta_{ab}}{(z-w)^2} + \sum_c if_{abc}\frac{J^c(w)}{(z-w)} \tag{1.184}$$

or in terms of Laurent modes

$$\left[J_n^a, J_m^b\right] = \sum_c if_{abc}J^c_{n+m} + kn\delta_{ab}\delta_{n+m,0} \tag{1.185}$$

but those are just the commutators of an affine Lie algebra $\widehat{\mathfrak{g}}_k$ with level k, and we indeed have the following concrete realization of such currents through WZW model

$$S^{\mathrm{WZW}} = \frac{k}{16\pi}\int d^2x\,\mathrm{Tr}'\left(\partial^\mu g^{-1}\partial_\mu g\right) + k\Gamma \tag{1.186}$$

where the first part is a nonlinear sigma model with $g(x)$ a bonsonic field takes values in the (semisimple) Lie group G of $\mathfrak{g}$ as the finite part of $\widehat{\mathfrak{g}}_k$ with some representation, and the trace is normalized by the Dynkin index as $\mathrm{Tr}' = \frac{1}{x_{\mathrm{rep}}}\mathrm{Tr}$ and $x_\lambda = \frac{\dim|\lambda|(\lambda,\lambda+2\rho)}{2\dim\mathfrak{g}}$ for convenience. And the WZW term is given by

$$\Gamma = \frac{-i}{24\pi}\int_B d^3y\epsilon_{\alpha\beta_\gamma}\,\mathrm{Tr}'\left(\widetilde{g}^{-1}\partial^\alpha\widetilde{g}\widetilde{g}^{-1}\partial^\beta\widetilde{g}\widetilde{g}^{-1}\partial^\gamma\widetilde{g}\right) \tag{1.187}$$

with B some tree manifold such that ∂B is the compactification of our two dimensional spacetime, and $\widetilde{g}(x)$ extends $g(x)$ to B. Then the equation of motion in complex coordinate is

$$\partial_z\left(g^{-1}\partial_{\bar{z}}g\right) = 0 \tag{1.188}$$

So we can define the conserved currents as

$$\begin{aligned} J(z) &\equiv -kJ_z(z) = -k\partial_z g g^{-1} \\ \bar{J}(\bar{z}) &\equiv kJ_{\bar{z}}(\bar{z}) = kg^{-1}\partial_z g \end{aligned} \tag{1.189}$$

Then in terms of the Lie algebra generator t^a we have $J = \sum_a J^a t^a$ and similar expansion for anti-chiral part. And we have the energy momentum tensor as the sum of the normal ordered products of current

$$T(z) = \frac{1}{2\left(k + C_{\mathfrak{g}}\right)} \sum_{a=1}^{\dim \mathfrak{g}} : \left(J^a J^a\right)(z) : \tag{1.190}$$

where $C_{\mathfrak{g}}$ is the dual Coxeter number of $\mathfrak{g}$, this is known as Sugawara construction, and it gives the central charge

$$c = \frac{k \dim \mathfrak{g}}{k + C_{\mathfrak{g}}} \tag{1.191}$$

Using Sugawara construction, we can also represent the Virasoro algebra as

$$L_n = \frac{1}{2(k + C_{\mathfrak{g}})} \sum_a \sum_m : J^a_m J^a_{n-m} : \tag{1.192}$$

and the full algebra is

$$\begin{aligned} \left[L_n, L_m\right] &= (n-m)L_{n+m} + \tfrac{c}{12}\left(n^3 - n\right)\delta_{n+m,0} \\ \left[L_n, J^a_m\right] &= -mJ^a_{n+m} \\ \left[J^a_n, J^b_m\right] &= \textstyle\sum_c i f_{abc} J^c_{n+m} + kn\delta_{ab}\delta_{n+m,0} \end{aligned} \tag{1.193}$$

We identify the highest weight representation $\widehat{\lambda}$ of $\widehat{\mathfrak{g}}_k$ with finite part λ as the highest weight representation of $\widehat{\mathfrak{g}}$. For simplicity we just call it as λ, then a WZW primary is a field such that

$$\begin{aligned} J^a_0 \left|\phi_\lambda\right\rangle &= -t^a_\lambda \left|\phi_\lambda\right\rangle \\ J^a_n \left|\phi_\lambda\right\rangle &= 0 \quad \text{for} \quad n > 0 \end{aligned} \tag{1.194}$$

It has dimension

$$h_\lambda = \frac{\sum_a t^a_\lambda t^a_\lambda}{2(k + C_{\mathfrak{g}})} = \frac{(\lambda, \lambda + 2\rho)}{2(k + C_{\mathfrak{g}})} \tag{1.195}$$

where $\rho = \frac{1}{2}\sum_{\alpha \in \Delta_+} \alpha$ is the Weyl vector as the half sum of positive roots of $\mathfrak{g}$.

In WZW models the general correlators between primaries are solutions of the Knizhnik–Zamolodchikov Equation

$$\left[\partial_{z_i} + \frac{1}{k + C_{\mathfrak{g}}} \sum_{i \neq i} \frac{\sum_a t^a_i \otimes t^a_i}{z_i - z_i}\right] \left\langle \phi_1\left(z_1\right) \cdots \phi_n\left(z_n\right)\right\rangle = 0 \tag{1.196}$$

An appropriate character $\chi_{\widehat{\lambda}}$ can be defined for $\widehat{\lambda}$ in terms of affine Lie algebra data as

$$\chi_{\widehat{\lambda}} = e^{-m_{\lambda}\delta} \operatorname{ch}_{\widehat{\lambda}} = \frac{\sum_{w \in W} \epsilon(w) \Theta_{w(\widehat{\lambda} + \widehat{\rho})}}{\sum_{w \in W} \epsilon(w) \Theta_{w\widehat{\rho}}} \tag{1.197}$$

along with the corresponding S-matrix[18]

[18]see standard textbook such as[71] for more details

$$\mathcal{S}_{\widehat{\lambda}\widehat{\mu}} = i^{|\Delta_+|} \left|P/Q^\vee\right|^{-\frac{1}{2}} (k + C_{\mathfrak{g}})^{-r/2} \sum_{w \in W} \epsilon(w) e^{-2\pi i (w(\lambda+\rho), \mu+\rho)(k+C_{\mathfrak{g}})} \tag{1.198}$$

while in our later calculations we only need the following special case of $\widehat{\mathfrak{g}} = \widehat{\mathfrak{su}}(2)_k$ theory S matrix

$$\mathcal{S}_{\widehat{\lambda}\widehat{\mu}} = \left[\frac{2}{k+2}\right]^{\frac{1}{2}} \sin\left[\frac{\pi\,(\lambda_1+1)\,(\mu_1+1)}{(k+2)}\right] \quad 0 \le \lambda_1, \mu_1 \le k \tag{1.199}$$

The fusion rules of generic WZW models are quite complicated, so in the following we will just mention some general features of them, and as typical examples we will focus on $\widehat{\mathfrak{g}} = \widehat{\mathfrak{su}}(N)_k$ A-series. The fusion rules are closely related to the corresponding tensor representation decomposition rules in Lie algebras, in some sense they are truncated version of tensor representation decomposition rules, hence the latter can be viewed as the $k \to \infty$ classical limits.

$$\mathcal{N}_{\lambda\mu}^{\nu} = \lim_{k\to\infty} \mathcal{N}_{\widehat{\lambda}\widehat{\mu}}^{(k)\widehat{v}} \tag{1.200}$$

and

$$\mathcal{N}_{\widehat{\lambda}\widehat{\mu}}^{(k)\widehat{v}} \le \mathcal{N}_{\widehat{\lambda}\widehat{\mu}}^{(k+1)\widehat{v}} \tag{1.201}$$

indeed we have a finer result as

$$\mathcal{N}_{\widehat{\lambda}\widehat{\mu}}^{(k)\widehat{v}} = \begin{cases} \max(i) \text{ such that } k \ge k_0^{(i)} \text{ and } \mathcal{N}_{\lambda\mu\nu} \neq 0 \\ 0 \text{ if } k < k_0^{(1)} \text{ or } \mathcal{N}_{\lambda\mu\nu} = 0 \end{cases} \tag{1.202}$$

and this defines the so called threshold level $k_0^{(i)}$, after which the fusion of $\widehat{\lambda} \times \widehat{\mu}$ will produce $\widehat{\nu}$ exactly as tensor representation decomposition. More explicitly, the fusion coefficient $\mathcal{N}_{\widehat{\lambda}\widehat{\mu}}^{(k)\widehat{v}}$ is given by the following Kac-Walton formula as a special kind of alternative sums of $\mathcal{N}_{\lambda\mu}^{\nu}$

$$\mathcal{N}_{\widehat{\lambda}\widehat{\mu}}^{(k)\widehat{v}} = \sum_{\substack{w \in \widehat{W} \\ w\cdot v \in P_+}} \mathcal{N}_{\lambda\mu}{}^{w\cdot v} \epsilon(w) \tag{1.203}$$

the fusion coefficient also has a symmetry property associated with the action $A \in \mathcal{O}(\widehat{\mathfrak{g}})$ of the outer automorphism group

$$\mathcal{N}_{A(\widehat{\lambda})A'(\widehat{\mu})}^{AA'(\widehat{v})} = \mathcal{N}_{\widehat{\lambda}\widehat{\mu}}^{\widehat{v}} \tag{1.204}$$

except for $\widehat{E}_8$ with $k = 2$, in all WZW models $A \in \mathcal{O}(\widehat{\mathfrak{g}})$ can be realized as fusion with a simple current i.e. an abelian anyon a which acts in fusion rules as permutations:

$$\begin{aligned} \widehat{\lambda} \times \widehat{\mu} &= \textstyle\sum_i m_i \widehat{\nu}_i \\ a^m \widehat{\lambda} \times a^n \widehat{\mu} &= a^{m+n} \left(\textstyle\sum_i m_i \widehat{\nu}_i\right) = \textstyle\sum_i m_i a^{m+n} \widehat{\nu}_i \end{aligned} \tag{1.205}$$

and indeed we can identify a with $A(\mathbb{I})$, for example in $\widehat{\mathfrak{g}} = \widehat{\mathfrak{su}}(N)_k$, by table 14.1 of [71] we know $\mathcal{O}(\widehat{\mathfrak{g}}) = \mathbb{Z}_N$ where a is of the form $[0, \cdots, k, \cdots, 0]$, and we can pick $a = [0, k, 0, \cdots, 0]$ as the single generator of $\mathcal{O}(\widehat{\mathfrak{g}}) = \mathbb{Z}_N$.

For $\widehat{\mathfrak{g}} = \widehat{\mathfrak{su}}(2)_k$ the fusion rules are very simple, and the result is usually expressed in spin basis $j = \frac{\lambda}{2}$ where $\widehat{\lambda} = [k-\lambda, \lambda]$, we have the textbook result on tensor representation decomposition

$$j_1 \otimes j_2 = \sum_{j=|j_1-j_2|}^{j_1+j_2} j \tag{1.206}$$

and the truncated version

$$j_1 \otimes j_2 = \sum_{j=|j_1-j_2|}^{\min(j_1+j_2, k-j_1-j_2)} j \tag{1.207}$$

with $a = [0, k]$ and $a \times \widehat{\lambda}$ as group action $a[k-\lambda, \lambda] = [\lambda, k-\lambda]$

WZW models can be used to construct other RCFTs through the so called coset construction, historically this method is tailor-made to perform explicit realizations of unitary minimal models, but it is used wildly beyond this purpose and provides very fruitful applications. To start, we begin with an affine Lie algebra $\widehat{\mathfrak{g}}$ and one of its subalgebra $\widehat{\mathfrak{p}}$ with embedding index x_e, and we observe that the difference of their energy momentum tensors $T_{\widehat{\mathfrak{g}}} - T_{\widehat{\mathfrak{p}}}$ also behaves as a energy momentum tensor and defines an representation of the Virasoro algebra with generators $L_m^{(g/p)} \equiv L_m^g - L_m^p$

$$\left[L_m^{(g/p)}, L_n^{(g/p)}\right] = (m-n)L_{m+n}^{g/p} + \left(c\left(\widehat{\mathfrak{g}}_k\right) - c\left(\widehat{\mathfrak{p}}_{x_e k}\right)\right)\frac{(m^3-m)}{12}\delta_{m+n.0} \tag{1.208}$$

and central charge

$$c\left(\widehat{\mathfrak{g}}_k/\widehat{\mathfrak{p}}_{x_e k}\right) = \frac{k\dim\mathfrak{g}}{k+C_{\mathfrak{g}}} - \frac{x_e k\dim\mathfrak{p}}{x_e k + C_{\mathfrak{p}}} \tag{1.209}$$

This is known as Goddard-Kent-Olive (GKO) construction, and we obtain a quotient RCFT with energy momentum tensor $T_{\widehat{\mathfrak{g}}} - T_{\widehat{\mathfrak{p}}}$, it is referred as the coset $\widehat{\mathfrak{g}}_k/\widehat{\mathfrak{p}}_{x_e k}$

As the weights in $\widehat{\mathfrak{g}}$ split into direct sums of weight in $\widehat{\mathfrak{p}}$ through the following branch rule

$$\widehat{\lambda} \mapsto \bigoplus_{\widehat{\mu}} b_{\widehat{\lambda}\widehat{\mu}}\widehat{\mu} \tag{1.210}$$

after taking trace $b_{\widehat{\lambda}\widehat{\mu}}$'s behave as characters,to be more precise we have

$$\chi_{\{\widehat{\lambda},\widehat{\mu}\}}(\tau) = e^{2\pi i\tau(m_{\widehat{\lambda}} - m_{\widehat{\mu}})} b_{\widehat{\lambda}\widehat{\mu}}(\tau) \tag{1.211}$$

where $m_{\widehat{\lambda}} = \frac{|\lambda+\rho|^2}{2(k+C_{\mathfrak{g}})} - \frac{|\rho|^2}{2C_{\mathfrak{g}}}$ is the modular anomaly. So the fields in a coset theory are given by pairs of the form $\{\widehat{\lambda}, \widehat{\mu}\}$, where $\widehat{\lambda}, \widehat{\mu}$ are representations in $\widehat{\mathfrak{g}}_k, \widehat{\mathfrak{p}}_{x_e k}$ respectively, but we can not combine them arbitrarily, instead there are two kinds of constraints:

- Field selection rule

  The finite parts of $\widehat{\lambda}, \widehat{\mu}$'s have to satisfy the following projection condition in root lattice

$$\mathcal{P}\lambda - \mu \in \mathcal{P}\mathcal{Q} \tag{1.212}$$

- Field identification rule

  Fields related to each other by outer automorphisms $A \mapsto \widetilde{A}$ for $A \in \mathcal{O}(\widehat{\mathfrak{g}})$, $\widetilde{A} \in \mathcal{O}(\widehat{\mathfrak{p}})$ are identified

$$\{\widehat{\lambda}; \widehat{\mu}\} \sim \{A\widehat{\lambda}; \widetilde{A}\widehat{\mu}\} \tag{1.213}$$

In a coset theory the S, T matrices are just products up to a complex conjugation

$$\begin{aligned} \mathcal{S}_{\{\widehat{\lambda};\widehat{\mu}\},(\widehat{\lambda}';\widehat{\mu}')\}} &= \mathcal{S}^{(k)}_{\widehat{\lambda}\widehat{\lambda}'} \overline{\mathcal{S}}^{(kx_\ell)}_{\widehat{\mu}\widehat{\mu}'} \\ \mathcal{T}_{\{\widehat{\lambda};\widehat{\mu}\}\cdot(\widehat{\lambda}';\widehat{\mu}')\}} &= T^{(k)}_{\widehat{\lambda}\widehat{\lambda}'} \overline{T}^{(kx_e)}_{\widehat{\mu}\widehat{\mu}'} \end{aligned} \tag{1.214}$$

as a consequence the modular matrix are

$$\mathcal{M} = \mathcal{M}^{(k)} \mathcal{M}^{(kx_e)} \tag{1.215}$$

and by the Verlinde formula the fusion matrix are

$$\mathcal{N}^{\{\widehat{\lambda}'',\widehat{\mu}''\}}_{\{\widehat{\lambda},\widehat{\mu}\},\{\widehat{\lambda}',\widehat{\mu}'\}} = \mathcal{N}^{(k)\widehat{\lambda}''}_{\widehat{\lambda}\widehat{\lambda}'} \mathcal{N}^{(kx_e)\widehat{\mu}''}_{\widehat{\mu}\widehat{\mu}'} \tag{1.216}$$

so the fusion matrices are also products.

For our purpose, we will consider only the special case of diagonal embedding $\mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$, with coset $\frac{\widehat{\mathfrak{g}}_{k_1} \oplus \widehat{\mathfrak{g}}_{k_2}}{\widehat{\mathfrak{g}}_{k_1+k_2}}$ and central charge

$$c = \dim \mathfrak{g} \left(\frac{k_1}{k_1 + C_\mathfrak{g}} + \frac{k_2}{k_2 + C_\mathfrak{g}} - \frac{k_1 + k_2}{k_1 + k_2 + C_\mathfrak{g}} \right) \tag{1.217}$$

where the selection rule is

$$\lambda + \mu - v \in Q \tag{1.218}$$

and the identification rule is

$$\{\widehat{\lambda}, \widehat{\mu}; \widehat{\nu}\} \sim \{A\widehat{\lambda}, A\widehat{\mu}; A\widehat{\nu}\} \quad \forall A \in \mathcal{O}(\widehat{\mathfrak{g}}) \tag{1.219}$$

For example, with $k + 2 = p \geq 3$ the unitary minimal model $\mathcal{M}(p+1, p)$ is realized as

$$\frac{\widehat{su}(2)_k \oplus \widehat{\mathrm{su}}(2)_1}{\widehat{su}(2)_{k+1}} \tag{1.220}$$

More generally we can construct

$$\frac{\widehat{su}(N)_k \oplus \widehat{\mathrm{su}}(N)_l}{\widehat{su}(N)_{k+l}} \tag{1.221}$$

This works as long as N is relative prime to k, l, otherwise our field identification rule will have fixed points or points with shorter orbitals under the action of $\mathcal{O}(\widehat{\mathfrak{g}})$, then we have to treat such fields separately, the simplest nontrivial example of this kind is given by

$$\frac{\widehat{su}(2)_2 \oplus \widehat{\mathrm{su}}(2)_4}{\widehat{su}(2)_6} \tag{1.222}$$

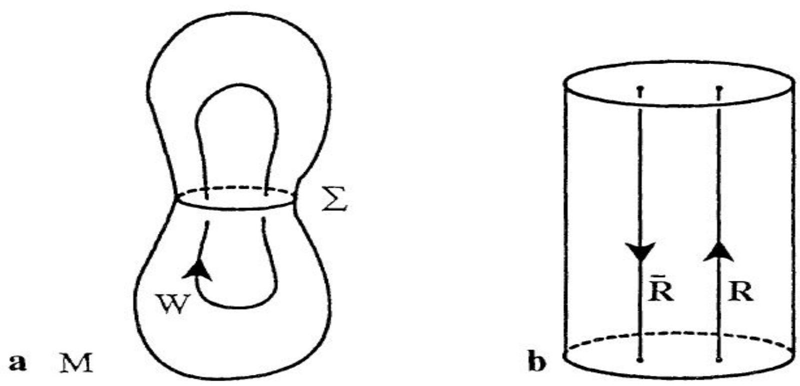


Figure 12: Illustration of the 3d to 2d correspondence between G_k^{CS} and G_k^{WZW} .[157]

where the field $\{\lambda, \mu, \nu\} = \{1, 2, 3\}$ is invariant under the action of $\mathcal{O}(\widehat{\mathfrak{g}})$, we have to resolve it as

$$f \to f_1 + f_2 : \quad \{1, 2, 3\} \to \{1, 2, 3\}^1 + \{1, 2, 3\}^2 \tag{1.223}$$

and the corresponding S matrix should be modified appropriately to obtain a well defined theory, indeed we have a new $\widetilde{S}$, if we denote generic fields other than the fixed point f as $a, b, \cdots$ then

$$\widetilde{S}_{ab} = 2S_{ab}, \quad \widetilde{S}_{f_1 a} = \widetilde{S}_{f_2 a} = S_{fa} \tag{1.224}$$

and

$$\widetilde{S}_{f_i f_j} = \frac{1}{2} \begin{pmatrix} S_{ff} + 1 & S_{ff} - 1 \\ S_{ff} - 1 & S_{ff} + 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{1.225}$$

This example will be studied further in chapter four.

Finally, let us briefly mention the correspondence between WZW model and Chern-Simons theory. The basic idea is very simple, given a three dimensional manifold M and a Chern-Simons theory with a simple ADE gauge group G of level k, at least locally we can cut M through a (compact) Riemann surface Σ so at least locally it behaves as $\mathbb{R} \times \Sigma$, where the $\mathbb{R}$ can be viewed as the direction of time so Σ represents a spatial slice. Now suppose we have a Wilson line W in representation R, it may pierce this spatial slice Σ and leave marked points P_i on M,and depending on the orientation we can view these points as particles carrying representations R or $\bar{R}$, then we simply label them as R_i or $\bar{R}_i$. They are indeed well defined anyons, and can be identified as primaries of a G_k WZW model on Σ insert at P_i's, see fig 12. For convenience we will call these two theories as G_k^{CS} and G_k^{WZW} respectively.

Indeed, on the G_k^{WZW} side, the Lagrangian of (1.187) is labeled by an element of

$$H^3(G, \mathbb{Z}) \tag{1.226}$$

Similarly it induces a differential character[19] $\alpha \in \widehat{H}^2(G, \mathbb{R}/\mathbb{Z})$ such that we have the

[19] those are differential forms such that $\langle \alpha, \partial B \rangle = \int_B \Omega(\text{mod} 1)$ for some $k+1$ form Ω instead of vanish

natural pairing

$$S(g) = \langle g^*\alpha, [\Sigma] \rangle \tag{1.227}$$

But using the bundle map

$$G \to EG \to BG \tag{1.228}$$

We have

$$\tau : H^k(BG, F) \to H^{k-1}(G, F) \tag{1.229}$$

And in particular

$$\tau : H^4(BG, \mathbb{Z}) \to H^3(G, \mathbb{Z}) \tag{1.230}$$

While we know from (1.75) $H^4(BG, \mathbb{Z})$ classify G_k^{CS}'s classical actions, so at least at classical level, this correspondence is valid , and physically this means we take the endpoints of the world lines of anyons as objects living in a two dimensional boundary of spacetime and form an interacting theory through the projections of braiding and fusion in three dimensions.[20]

Indeed this correspondence holds even after quantization as G_k^{CS} and G_k^{WZW} will get the same Hilbert space $\mathscr{H}_\Sigma$:

- On the G_k^{CS} side, we first construct the moduli space $\mathscr{M}$ of classical solutions with gauge fixing constrains, it turns out it is finite dimensional and compact. By introducing a complex structure J, M_J becomes Kähler, essentially it is the moduli space of some special line bundles, then $\mathscr{H}_\Sigma$ is the corresponding space of sections and it does not depend on J.The above construction is for the partition function only, more generally, we need to consider Riemann surface with marked points (Σ, P_i, R_i) to deal with correlators.[157]

- On the G_k^{WZW} side, the same object M_J and $\mathscr{H}_\Sigma$ appear again, but as the space of conformal blocks on Σ. Now the correlators are linear combinations of products of holomorphic and antiholomorphic blocks, and when genus is nonzero suitable gluing conditions have to be implemented, which eventually lead to the abstract form (1.157) of $\mathscr{H}_\Sigma$ as a direct sum of representations of affine Lie algebra $\widehat{\mathfrak{g}}_k$[146].

For example when $G = SU(N)$, $\mathcal{M}$ is the moduli space of all stable rank N holomorphic vector bundles L's of vanishing first Chern class $c_1(L) = 0$, and $\mathscr{H}_\Sigma$ is the space of global holomorphic sections of $L^{\otimes k}$. And as a consequence the Verlinde formula can be viewed as a dimension counting formula of these spaces, see[145] chapter 11.

While the above discussion holds at general level, it is very abstract, concretely there are several ways to realize this correspondence, an elementary one is as follow(for simplicity, assume $\Sigma_n = M_2$ as the $d = 2$ Minkowski space and $G = SU(2)$)[43]:

[20] this map τ is not onto, so only a subset of WZW models are obtained, see [93, 63]

- Imagine a stack of Σ_n with a discrete transverse dimension of a N sites periodic chain, where we put on each layer a G_k^{WZW} theory with field g_n. Then introduce a set of gauge fields $A_{\mu,n}$ coupling between two layers such that g_n is left-coupled to $A_{\mu,n}$ and right-coupled to $A_{\mu,n+1}$, so we have

$$S_n = kW\left[g_n\right] + I\left[g_n, A_{\pm,n}, A_{\pm,n+1}\right] \tag{1.231}$$

  where kW denotes the original WZW action, and I is a suitable interaction term. Although the individual term S_n is not gauge invariant, the overall action $S = \sum_n S_n$ is.

- Then by identifying

$$g_n = \exp\left(-\int_{x^3}^{x^3+a} A_3\left(x^+, x^-, x^3\right) dx^3\right) \simeq -aA_3 \tag{1.232}$$

  and taking the limit of infinite many sites $N \to \infty$ and infinite small lattice spacing $a \to 0$ with constant Na, we have

$$\lim_{a\to 0, N\to\infty} S = \frac{k}{2\pi}\int_{M_2\times S^1} d^3x\epsilon^{ijk}\,\mathrm{Tr}\left[A_i\partial_j A_k - \frac{2}{3}A_iA_jA_k\right] \equiv S_{CS} \tag{1.233}$$

- By a careful analysis, a discrete version of Wilson loop can be defined as

$$R_j\left(x^+, x^-\right) = Tr_j\prod_{n=1}^{N} e^{-aA_3\left(x^+,x^-,na+a/2\right)} \tag{1.234}$$

  where j label the spin, then we have the following continuum limit

$$R_j\left(x^+, x^-\right) \longrightarrow_{a\to 0} \mathrm{Tr}_j\, P\left(e^{-\int_C dx^\mu A_\mu}\right) \tag{1.235}$$

  The key point is that these R_j operators are living in WZW theories, so by calculating their correlators and taking limits we can calculate Chern-Simons correlators.

Because of this CS-WZW correspondence, we will mainly use the language of Chern-Simons theory in later chapters, especially in section 4.3 where we analyze G_k^{CS} theories and the corresponding cosets.

1.3 Supersymetry

Supersymmetry, or SUSY in short, is an extension of spacetime symmetry, it bypasses the Coleman–Mandula theorem by using spinor charges instead of the usual scalar ones, and as a result bosons and fermions are related to each other. Since its discovery in the 1970s, suspersymmetry has been used widely in physics and mathematics.

- Supersymmetric extensions of physical theories are studied intensively, especially in gauge theory and string theory, this generates a lot of surprising results and deep insights, even for non-supersymmetric systems. In many important situations, supersymmetry will bring in highly nontrivial constraints into the theoretical models such that full or partial analytic solutions are possible, and those solutions in turn will reveal some generic features of those theories, in particular one can study the process of supersymmetry breaking and see what remains. And unlike TQFT and CFT, some supersymmetric QFTs have cluster decomposition property hence scattering amplitudes exist, which makes them useful as more realistic toy models of generic quantum field theories with scattering interactions.
- Supersymmetry has found a wide range of applications. For example, in particle physics, supersymmetry will lead to amplitude cancellations, hence modify the RG behavior, and through this it provides a natural way to resolve the Higgs hierarchy problem of the standard model, it also makes the coupling constants converge in the grand unification scale, and more generally it is used to construct extensions of the standard model. As another example, supersymmetry is mandatory for consistent tachyon free string theory, the resulting superstrings have central importance in the development of string theory along with its applications. Finally supersymmetry is also used widely in other areas of physics such as quantum mechanics, condensed matter physics and statistical mechanics.
- Supersymmetry is also important in mathematics, it extends some mathematical objects found in quantum field theories to supersymmetric ones such as super versions of groups, Lie algebras, manifolds and bundles, which are rich in structures with surprising new features. For example, using supersymmetric quantum mechanics, a simple proof of the Atiyah–Singer index theorem is found, this also gives it a nice physical interpretation. And the very fact of the existence of supersymmetry in a QFT is indeed closely related to the algebraic and geometric properties of the underlying spacetime manifold, as a very basic example, the amount of supersymmetries allowed in a QFT depends on the spacetime dimensions and spin structures, as another example, complex manifolds, algebraic curves or surfaces appear naturally as vacuum moduli spaces of supersymmetric QFTs and superstrings.

In this section we first introduce the basic notations regarding supersymmetry[147, 5], then analyze the structure of moduli space of supersymetric gauge theories[152, 103, 8], finally we introduce the superconformal index[153, 74].

1.3.1 Quantum field theory with supersymmetry

To illustrate the general idea, we will begin with a toy model, i.e. the supersymmetric version of the harmonic oscillator in quantum mechanics.

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2 + i\bar{\psi}\dot{\psi} - \bar{\psi}\psi \tag{1.236}$$

where we simply add the Lagrangians of the usual bosonic harmonic oscillator and its Grassmann number version together, then after quantization we have

$$H = \frac{1}{2}p^2 + \frac{1}{2}x^2 + \bar{\psi}\psi = \left(a_B^\dagger a_B + a_F^\dagger a_F\right) \tag{1.237}$$

where $\left[a_B, a_B^\dagger\right] = 1$ and $\left\{a_F, a_F^\dagger\right\} = 1$ with

$$a_B^\dagger = \frac{1}{\sqrt{2}}(-ip + x), \quad a_B = \frac{1}{\sqrt{2}}(ip + x), \quad a_F^\dagger = \bar{\psi}, \quad a_F = \psi \tag{1.238}$$

We can solve this model exactly to obtain the partition function by simply multiplying the partition functions of bosonic and fermionic oscillators

$$Z = Z_B Z_F = \frac{1 + e^{-\beta}}{1 - e^{-\beta}} \tag{1.239}$$

If we define $Q = a_B^\dagger a_F$ and $\bar{Q} = a_F^\dagger a_B$, which are in some sense just like the square roots of the Hamiltonian H, then the following super algebra is obtained

$$[Q, H] = [\bar{Q}, H] = 0, \quad \{Q, \bar{Q}\} = H \tag{1.240}$$

In this system, excited sates are $|\chi_B, n\rangle \equiv \left(a_B^\dagger\right)^n |0\rangle, \quad |\chi_F, m\rangle \equiv \left(a_B^\dagger\right)^m a_F^\dagger|0\rangle$, we also find that the fermionic number $F = \sum a_F^\dagger a_F$ is conserved mod2, hence we have a discrete $\mathbb{Z}_2$ symmetry given by $(-1)^F$, this induces a natural $2-$ grading for excited sates such that there is a bijection between the bosonic and the fermionic parts, notice that here the vacuum is bosonic and unique.

Now if we turn on some generic interaction by the introducing the following superpotential $W(x)$

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}W'(x)^2 + i\bar{\psi}\dot{\psi} - W''(x)\bar{\psi}\psi \tag{1.241}$$

instead of the harmonic superpotential $W(x) = x^2/2$, we can not solve this system exactly, but since $W(x)$ preserves supersymmetry, it should keep the boson fermion correspondence, except for possible zero modes, which is captured by the following Witten index

$$I = \mathrm{Tr}(-1)^F e^{-\beta H} \tag{1.242}$$

Usually, there are different vacua , and some of them break SUSY, if SUSY is not broken, Witten index must vanish. More generally this index depends only on the structure of vacua, and in a sense it classifies the vacua since its value is invariant under deformations as long as the associated Hilbert space remains the same. So to calculate it we can either take $\beta \to 0$ or $\beta \to \infty$, and by knowing this index and identifying different expressions under different limits, a lot of information can be gained.

In the same spirit, we would like to study supersymmetric quantum field theories, just like our treatment of CFT, in this section we will analyze superalgebra and its representations in general first, and introduce physical realizations of such representations as supersymmetric quantum fields. Then in the next section we will study the moduli space of these theories and the related dualities. Finally we will combine supersymmetry with conformal symmetry to obtain some SCFTs and introduce the related superconformal indexes.[21]

As the first step we can generalize (1.240) to the super Poincaré algebra

$$\begin{aligned} &[Q_\alpha, J^{\mu\nu}] = (\sigma^{\mu v})_\alpha{}^\beta Q_\beta, \quad \left[\bar{Q}_{\dot{\alpha}}, J^{\mu\nu}\right] = \epsilon_{\dot{\alpha}\dot{\beta}} (\bar{\sigma}^{\mu v})^{\dot{\beta}}{}_{\dot{\gamma}} \bar{Q}^{\dot{\gamma}} \\ &[Q_\alpha, P^\mu] = 0, \quad \left[\bar{Q}_{\dot{\alpha}}, P^\mu\right] = 0 \\ &\left\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\right\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu, \quad \{Q_\alpha, Q_\beta\} = \left\{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\right\} = 0 \end{aligned} \tag{1.243}$$

In addition, we can assign an $U(1)$ charge to the superalgebra generators $Q, \bar{Q}$ by the following automorphism

$$Q_\alpha \mapsto Q'_\alpha = e^{i\alpha} Q_\alpha, \quad \bar{Q}_{\dot{\alpha}} \mapsto \bar{Q}'_{\dot{\alpha}} = e^{-i\alpha} \bar{Q}_{\dot{\alpha}} \tag{1.244}$$

This is called the $U(1)$ R symmetry

$$[Q_\alpha, R] = Q_\alpha, \quad \left[\bar{Q}_{\dot{\alpha}}, R\right] = -\bar{Q}_{\dot{\alpha}} \tag{1.245}$$

The above super Poincaré algebra can be viewed as a trivial central extension of the usual Poincaré algebra, more generally one can have $\mathcal{N}$ pairs of super generators with non-vanishing central charges as

$$\left\{Q^a_\alpha, \bar{Q}_{b\dot{\beta}}\right\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^a_b, \quad \left\{Q^a_\alpha, Q^b_\beta\right\} = \varepsilon_{\alpha\beta} Z^{ab}, \quad \left\{\bar{Q}_{a\dot{\alpha}}, \bar{Q}_{b\dot{\beta}}\right\} = \varepsilon_{\dot{\alpha}\dot{\beta}} \bar{Z}_{ab} \tag{1.246}$$

with more general R symmetry groups as subgroups of $U(\mathcal{N})$

$$Q^a_\alpha \mapsto Q^{a'}_\alpha = R^a_b Q^b_\alpha, \quad \bar{Q}_{a\dot{\alpha}} \mapsto \bar{Q}'_{a\dot{\alpha}} = \bar{Q}_{b\dot{\alpha}} \left(R^\dagger\right)^b_a \tag{1.247}$$

We can construct particle states as finite dimensional representations of super Poincaré algebra, they are called multiplets, but since the super Poincaré algebra is larger, such multipets are given by combinations of usual particle states, and it is obvious that by symmetry there should be a Boson-Fermion pairing, just as in ordinary quantum field theory we distinguish the massless and massive cases

[21] For more information about spinor and superalgebra, see standard textbooks such as [5, 147].

- Massless representation

  We go to a lightlike frame $p^\mu = (E, 0, 0, E)$ and define helicity as in Poincaré algebra, now the super algebra generators satisfy $\left\{Q_1, \bar{Q}_{\mathrm{i}}\right\} = 4E$ with $Q_2 = \bar{Q}_2 = 0$, then we take the state $|\Omega\rangle$ of minimal helicity λ, we find that it is annihilated by Q_1, so $\bar{Q}_{\mathrm{i}}$ is the creation operator, along with the CPT conjugations we have

$$|p^\mu, \pm\lambda\rangle\,, \quad \left|p^\mu, \pm\left(\lambda + \frac{1}{2}\right)\right\rangle \tag{1.248}$$

  The $\lambda = 0$ case is called chiral multiplet

  The $\lambda = \frac{1}{2}$ case is called vector multiplet

  More generally if $\mathcal{N} > 1$,there is still a unique $|\Omega\rangle$ and we just act $\bar{Q}_{a\mathrm{i}}$ with $a = 1, \ldots, \mathcal{N}$ on it to obtain $2^{\mathcal{N}}$ different states

- Massive representation

  Similarly, we go to the rest frame $p^\mu = (m, 0, 0, 0)$ of the particle and define mass and spin, if the central charges are trivial, we have $Z = 0$ and $\left\{Q^a_\alpha, \bar{Q}_{b\dot{\beta}}\right\} = 2m\delta^a_b\,(\sigma_0)_{\alpha\dot{\beta}} = 2m\delta^a_b \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{\alpha\dot{\beta}}$ so now we have twice the number of creation operators

$$a^b_\alpha = \frac{Q^b_\alpha}{\sqrt{2m}}, \quad \left(a^\dagger\right)^a_{\dot{\alpha}} = \frac{\bar{Q}^a_{\dot{\alpha}}}{\sqrt{2m}} \tag{1.249}$$

  Again we can find the state $|\Omega\rangle$ with lowest spin so in total we have $2^{2\mathcal{N}}$ states. If $Z \neq 0$ we may pick a special basis and write it in the Jordan canonical form such that

$$Z^{ab} = \begin{pmatrix} 0 & q_1 & 0 & 0 & 0 & \cdots & & & \\ -q_1 & 0 & 0 & 0 & 0 & \cdots & & & \\ 0 & 0 & 0 & q_2 & 0 & \cdots & & & \\ 0 & 0 & -q_2 & 0 & 0 & \cdots & & & \\ 0 & 0 & 0 & 0 & \ddots & & & & \\ \vdots & \vdots & \vdots & \vdots & & \ddots & & & \\ & & & & & & & 0 & q_{\frac{N}{2}} \\ & & & & & & & -q_{\frac{N}{2}} & 0 \end{pmatrix} \tag{1.250}$$

  Then redefine $\widetilde{Q}^j_{\alpha\pm} \equiv \left(Q^{2j-1}_\alpha \pm (Q^{2j}_\alpha)^\dagger\right)$ to have

$$\begin{aligned} \left\{\widetilde{Q}^i_{\alpha+}, \left(\widetilde{Q}^j_{\beta+}\right)^\dagger\right\} &= \delta^i_j \delta^\beta_\alpha\,(2m + q_j) \\ \left\{\widetilde{Q}^i_{\alpha-}, \left(\widetilde{Q}^j_{\beta-}\right)^\dagger\right\} &= \delta^i_j \delta^\beta_\alpha\,(2m - q_j) \end{aligned} \tag{1.251}$$

By construction we have $|q_j| \le 2m$ for $j = 1, \ldots, \mathcal{N}/2$, this is known as Bogomolnyi-Prasad-Sommerfield bound or BPS bound for short, if it happens that for k of j we have $|q_j| = 2m$ we will have $2^{2(\mathcal{N}-k)}$ states only, we call such multiplets as $1/2^k$ BPS multiplets, in summary we have:

$k = 0$ $\quad 2^{2\mathcal{N}}$ states $\quad$ long multiplet

$0 < k < \frac{\mathcal{N}}{2}$ $\quad 2^{2(\mathcal{N}-k)}$ states short multiplet

$k = \frac{\mathcal{N}}{2}^2$ $\quad 2^{\mathcal{N}}$ states ultrashort multiplet

To have supersymmetric quantum fields, we can construct infinite dimensional field representations of super Poincaré algebra, using operator valued distributions to realize them and make sure they obey the basic axioms, in parallel of the formal constructions in ordinary quantum field theory introduced in appendix B.2.1.

In practice, one usually begins with simple Lagrangian theories, which are supersymmetric generalizations of well known quantum field theories, especially the $d = 4$ ones in the standard model. In the following we will assume $d = 4$, if it is not the case we will give d explicitly. As a simple example, consider a massless free complex scalar and a massless free left-handed Weyl spinor, add their Lagrangians together

$$\mathcal{L} = -\partial_\mu \phi^* \partial^\mu \phi - i\bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi \tag{1.252}$$

we find this Lagrangian is invariant under

$$\delta_\epsilon \phi = \sqrt{2}\epsilon\psi, \quad \delta_\epsilon \psi_\alpha = \sqrt{2}i \left(\sigma^\mu \bar{\epsilon}\right)_\alpha \partial_\mu \phi \tag{1.253}$$

which generates some spinor charges as $i\delta_\epsilon \phi = [\epsilon\mathcal{Q} + \bar{\epsilon}\overline{\mathcal{Q}}, \phi]$, $\quad i\delta_\epsilon \psi = [\epsilon\mathcal{Q} + \bar{\epsilon}\overline{\mathcal{Q}}, \psi]$, but those transformations are not closed off-shell, we have $[\delta_\epsilon, \delta_\eta]\, \phi = 2i \left(\eta\sigma^\mu\bar{\epsilon} - \epsilon\sigma^\mu\bar{\eta}\right) \partial_\mu \phi$ and $[\delta_\epsilon, \delta_\eta]\, \psi = 2i \left(\eta\sigma^\mu\bar{\epsilon} - \epsilon\sigma^\mu\bar{\eta}\right) \partial_\mu \psi - 2i \left(\bar{\epsilon}\bar{\sigma}^\mu \partial_\mu \psi\right) \eta + 2i \left(\bar{\eta}\bar{\sigma}^\mu \partial_\mu \psi\right) \epsilon$ so the equation of motion must be used to obtain $[\delta_\epsilon, \delta_\eta] = 2i \left(\eta\sigma^\mu\bar{\epsilon} - \epsilon\sigma^\mu\bar{\eta}\right) \partial_\mu$ as well. To fix this we can introduce an auxiliary complex scalar F

$$\mathcal{L}_{\text{kin}} = -\partial_\mu \phi^* \partial^\mu \phi - i\bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi + F^* F \tag{1.254}$$

and define

$$\delta_\epsilon \phi = \sqrt{2}\epsilon\psi, \quad \delta_\epsilon \psi_\alpha = +\sqrt{2}\epsilon_\alpha F + \sqrt{2}i \left(\sigma^\mu \bar{\epsilon}_\alpha\right) \partial_\mu \phi, \quad \delta_\epsilon F = \sqrt{2}i\bar{\epsilon}\bar{\sigma}^\mu \partial_\mu \psi. \tag{1.255}$$

The above Lagrangian is the kinematic part of the Wess-Zumino model, where supersymmetry appears in the first time, in the full model, a mass term

$$\mathcal{L}_{\text{mass}} = m \left(-\frac{1}{2}\psi\psi + \bar{\psi}\bar{\psi} + F\phi + F^*\phi^* \right) \tag{1.256}$$

and a interaction term

$$\mathcal{L}_{\text{int}} = g \left(\phi^2 F + \phi^{*2} F^* - \psi\psi\phi - \bar{\psi}\bar{\psi}\phi\right) \tag{1.257}$$

are added.

Similarly one can construct supersymmetric versions of QED and Yang-Mills theories by such ad hoc trials, since we have only a handful set of renormalizable Lagrangians, and supersymmetry gives us more stringent constraints, this method works well in many situations.

But for the convenience of a theoretical treatment, the so called superspace formalism is used widely, to be concrete assuming we have a $\mathcal{N}=1$ supersymmetric QFT in $(3+1)$ -dimensional flat spacetime, then we add Weyl spinor coordinates as some sort of quantum spacetime dimensions to the usual coordinates of Minkowski spacetime, this defines a superspace labeled by

$$z^A = \left(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}\right) \tag{1.258}$$

More generally this can be viewed as a trivialization on a coordinate chart, similar constructions can be done for generic spaces compatible with SUSY and define supermanifolds, superbundles and so on. Then as a central extension we obtain the super Poincaré group

$$G(x,\theta,\bar{\theta}) = e^{-ix_\mu P^\mu + i\theta Q + i\bar{\theta}\bar{Q}} \tag{1.259}$$

where

$$G(0,\xi,\bar{\xi})G(x,\theta,\bar{\theta}) = G\left(x^\mu + i\theta\sigma^\mu\bar{\xi} - i\xi\sigma^\mu\bar{\theta}, \theta+\xi, \bar{\theta}+\bar{\xi}\right) \tag{1.260}$$

and

$$\begin{aligned} \mathcal{Q}_\alpha &= \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu \\ \overline{\mathcal{Q}}^{\dot{\alpha}} &= \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\beta}}\epsilon^{\dot{\beta}\dot{\alpha}}\partial_\mu \end{aligned} \tag{1.261}$$

It is also convenient to introduce

$$\begin{aligned} \mathcal{D}_\alpha &= \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu \\ \overline{\mathcal{D}}_{\dot{\alpha}} &= -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \end{aligned} \tag{1.262}$$

Then a generic super field is defined as a formal Taylor expansion in terms of spinor coordinates

$$\begin{aligned} \mathbf{F}(x,\theta,\bar{\theta}) =& f^{(1)}(x) + \theta f^{(2)}(x) + \bar{\theta}\bar{f}^{(3)}(x) + \theta^2 f^{(4)}(x) + \bar{\theta}^2 f^{(5)}(x) \\ &+ \theta\sigma^\mu\bar{\theta}f^{(6)}_\mu + \theta^2\bar{\theta}\bar{f}^{(7)} + \bar{\theta}^2\theta f^{(8)} + \theta^2\bar{\theta}^2 f^{(9)}(x) \end{aligned} \tag{1.263}$$

where $f^{(i)}$ are usual fields with appropriate symmetries:

- $f^{(1)}(x), f^{(4)}(x), f^{(5)}(x), f^{(9)}(x)$ are all scalars
- $f^{(2)}(x), f^{(8)}(x)$ and $f^{(3)}(x), f^{(7)}(x)$ are left- and right-handed Weyl spinors
- $f^{(6)}(x)$ is a vector field

- $f^{(9)}$ is auxiliary and usually refereed as the $D-$ term, so does $f^{(4)}, f^{(5)}$, they are refereed as the $F-$ terms.

and under a supersymmetry transformation we have

$$\delta_\epsilon \mathbf{F}(x,\theta,\bar\theta) = (\epsilon\mathcal{Q} + \bar\epsilon\overline{\mathcal{Q}})\mathbf{F}(x,\theta,\bar\theta) \tag{1.264}$$

There are two important special cases of super field

- Chiral superfield

  It is the SUSY version of quark field, defined by the chirality constraint

$$\overline{\mathcal{D}}_{\dot\alpha}\Phi(x,\theta,\bar\theta) = 0 \tag{1.265}$$

  And similarly anti-chiral superfield is defined by

$$\mathcal{D}_\alpha \Phi^\dagger = 0 \tag{1.266}$$

  Then we have the following generic Lagrangian

$$\mathcal{L} = K\left(\Phi^k, \Phi^{k\dagger}\right)_{|\theta^2\bar\theta^2} + \Big(W\left(\Phi^k\right)_{|\theta^2} + W^\dagger\left(\Phi^{k\dagger}\right)_{|\bar\theta^2}\Big) \tag{1.267}$$

  where the Kähler potential K is a real function, and the superpotential $W(W^\dagger)$ a (anti)holomorphic function, those terms are named by the corresponding complex geometric structures associated with this theory. Usually by renormalizability in $d=4$ spacetime one has

$$K = \Phi^\dagger\Phi, \quad W = \frac{m}{2}\Phi^2 + \frac{g}{3}\Phi^3 \tag{1.268}$$

- Vector superfield

  It is the SUSY version of gauge field, defined by the covariant reality constraint

$$V(x,\theta,\bar\theta) = V^\dagger(x,\theta,\bar\theta) \tag{1.269}$$

  with gauge transformation

$$V \mapsto V + \Phi + \Phi^\dagger \tag{1.270}$$

  Then we obtain the following SUSY version of (chiral/anti-chiral) field strengths

$$W_\alpha = -\frac{1}{4}\overline{\mathcal{D}\mathcal{D}}\left(e^{-V}\mathcal{D}_\alpha e^V\right), \quad \bar W_{\dot\alpha} = \frac{1}{4}\mathcal{D}\mathcal{D}\left(e^{V}\overline{\mathcal{D}}_{\dot\alpha} e^{-V}\right) \tag{1.271}$$

  with gauge transformations

$$W_\alpha \mapsto e^{-i\Lambda}W_\alpha e^{i\Lambda}, \quad \bar W_{\dot\alpha} \mapsto e^{-i\bar\Lambda}\bar W_{\dot\alpha} e^{i\Lambda} \tag{1.272}$$

and the following super Yang-Mills Lagrangian

$$\mathcal{S} = \frac{1}{8\pi^2}\int \mathrm{d}^4x\,\mathrm{Im}\,\mathrm{Tr}\left(\tau\int \mathrm{d}^2\theta\,\mathrm{Tr}\left(W^\alpha W_\alpha\right)\right) \tag{1.273}$$

where the super coupling constant is a combination of Yang-Mills coupling and theta term

$$\tau = \frac{\vartheta}{2\pi} + i\frac{4\pi}{g^2_{\mathrm{YM}}} \tag{1.274}$$

More generally, one can define $\tau = (\tau_{IJ})$ as a matrix of couplings between different gauge fields if more than one species are present.

The above superspace formalism is defined for $\mathcal{N}=1$ theories only, for $\mathcal{N}>1$ theories we need to introduce $2\mathcal{N}$ Weyl spinor coordinates $\theta_a, \bar{\theta}_a$ and define corresponding superspace and superfields, this is rather cumbersome as there are a lot of auxiliary fields. In practice, one usually takes a middle way approach by using compatible $\mathcal{N}=1$ superfields and combining them together to form $\mathcal{N}>1$ multiplets. For later applications we are mainly interested in the $\mathcal{N}=2$ case, and here we give two basic examples of $\mathcal{N}=2$ multiplets, they are all made up by two $\mathcal{N}=1$ multiplets, and the overall R-symmetry is $SU(2)_R \times U(1)_R$, but the two multiplets are charged only under different parts of it.

- $\mathcal{N}=2$ Vector multiplet

$$\begin{array}{ccccl} \lambda_\alpha & \longleftrightarrow & A_\mu & & \mathcal{N}=1 \text{ vector multiplet,} \\ \updownarrow & & \updownarrow & & \\ \Phi & \longleftrightarrow & \widetilde{\lambda}_\alpha & & \mathcal{N}=1 \text{ chiral multiplet.} \end{array}$$

where the arrows are super transformations, and the horizontal ones are explicit in $\mathcal{N}=1$ formalism, while the vertical ones are implicit. Here all the $\mathcal{N}=1$ multiplets are in adjoint representations of the gauge group G, and we have a $\mathrm{SU}(2)_R$ symmetry acting on λ_α and $\widetilde{\lambda}_\alpha$, a $U(1)_R$ symmetry for the scalar Φ. The Lagrangian is

$$\frac{\mathrm{Im}\,\tau}{4\pi}\int d^4\theta\,\mathrm{tr}\,\Phi^\dagger e^{[V,\cdot]}\Phi + \int d^2\theta\frac{-\mathrm{i}}{8\pi}\tau\,\mathrm{tr}\,W_\alpha W^\alpha + cc. \tag{1.275}$$

- $\mathcal{N}=2$ Hyper multiplet

$$\begin{array}{ccccl} Q & \longleftrightarrow & \psi & & \mathcal{N}=1 \text{ chiral multiplet,} \\ \updownarrow & & \updownarrow & & \\ \widetilde{\psi}^\dagger & \longleftrightarrow & \widetilde{Q}^\dagger & & \mathcal{N}=1 \text{ anti-chiral multiplet.} \end{array}$$

Here the $N=1$ multiplets are in same representation R of the overall symmetry group $G \times G_f$, and we we have a $\mathrm{SU}(2)_R$ symmetry acting on Q and $\widetilde{Q}^\dagger$, as a consequence of this, a complex mass matrix μ in the adjoint representation of G_f with $[\mu, \mu^\dagger] = 0$, we have the following Lagrangian

$$\int d^4\theta \left(Q^{\dagger i} e^V Q_i + \widetilde{Q}^i e^{-V} \widetilde{Q}^{\dagger}{}_i\right) + \left(\int d^2\theta \widetilde{Q}^i \Phi Q_i + cc.\right) + (\sum_i \int d^2\theta \mu_i^j \widetilde{Q}^i Q_j + cc.) \tag{1.276}$$

As more symmetries are added , more constraints are put in, the $\mathcal{N} = 2$ Lagrangian is indeed fixed, with τ and μ as the only two (matrices of) parameters.

More generally, without using explicit Lagrangians, one can work with representations and BPS conditions directly, just like in ordinary QFTs. In particular, with $\mathcal{N} > 1$ SUSY, there are a lot of short and semi-short multiplets defined by algebraic constraints, these multiplets transform into each other under the action of $Q, \bar{Q}$, for more details, see for example [64].

Finally let us mention the method of compactification or dimensional reduction[22] , one can begin with a higher dimensional $\mathcal{N} = 1$ theory, and use dimension reduction and SUSY breaking to obtain $\mathcal{N} > 1$ theories at lower dimensions, for example we can begin with the following $\mathcal{N} = 1, d = 10$ super Yang-Mills theory

$$\mathcal{S}_{10D} = \int \mathrm{d}^{10}x \,\mathrm{Tr}\left(-\frac{1}{2}F_{mn}F^{mn} + \frac{i}{2}\bar{\Psi}\Gamma^m D_m \Psi\right) \tag{1.277}$$

with a T^6 compactification where m, n are ten dimensional coordinates and μ, ν are four dimensional ones.

$$A_m = \left(A_\mu\left(x^\nu\right), \phi_i\left(x^\nu\right)\right) \tag{1.278}$$

From a four dimensional viewpoint, we will produce six extra scalars , and a similar reduction applies to Ψ as well. By reorganizing the those fields and their super partners appropriately, we will arrive at the $\mathcal{N} = 4, d = 4$ super Yang-Mills theory

$$\begin{aligned}\mathcal{S}_{\mathcal{N}=4} = \int \mathrm{d}^4x \,\mathrm{Tr}\Bigg[&\int \mathrm{d}^4\theta \Phi^{i\dagger} e^V \Phi^i e^{-V} + \frac{1}{8\pi}\,\mathrm{Im}\left(\tau \int \mathrm{d}^2\theta W_\alpha W^\alpha\right) \\ &+ \left(i g_{\mathrm{YM}} \frac{\sqrt{2}}{3!} \int \mathrm{d}^2\theta \epsilon_{ijk} \Phi^i \left[\Phi^j, \Phi^k\right] + cc.\right)\Bigg]\end{aligned} \tag{1.279}$$

By adding extra terms break part of the $\mathcal{N} = 4$ supersymmetry, $\mathcal{N} = 2, 1$ theories can be generated. Similarly,we can begin with a $\mathcal{N} = 1, d = 6$ super Yang-Mills theory, and compact it on a torus T to obtain a $\mathcal{N} = 2, d = 4$ theory, if we compact it further on S^1, we will get a $\mathcal{N} = 4, d = 3$ theory

[22]By compactification, mathematicians usually mean the process of making a topological space compact, while physicists, especially in string theorists, use it to mean the reduction of extra dimensions(as a compact manifold) as well, we use this term mainly in the first sense in previous sections on TQFT and CFT, but in this section we mainly use it in the second sense

1.3.2 Moduli space and duality

With the basic notations we have introduced we can now analyze supersymmetric quantum fields more carefully, first recall that at one-loop the beta function is

$$\beta(g)=E\frac{d}{dE}g=-\frac{g^{3}}{(4\pi)^{2}}b=-\frac{g^{3}}{(4\pi)^{2}}\left[\frac{11}{3}C(\text{ adj })-\frac{2}{3}C\left(R_{f}\right)-\frac{1}{3}C\left(R_{s}\right)\right] \tag{1.280}$$

A key consequence of SUSY is that the above expression is one loop exact, so using the complex coupling τ, we can introduce the complexified dynamical scale Λ by the following equality valid at all orders

$$\Lambda^{b}=E^{b}e^{2\pi i\tau(E)} \tag{1.281}$$

As SUSY reorganizes the scalars, spinors and vectors into multiplets, the β function is constrained. On the reverse, if for some physical reason such as asymptotic freedom or conformal invariance we require β to be negative or zero, the range of possible SUSYs is also restricted. For example with $(N)=2$, given the representation R of the chiral multiplet in the hypermultiplet we would have

$$b=2C(\text{ adj })-C(R) \tag{1.282}$$

By adjusting R we can make b vanish, for example taking the $\mathcal{N}=4,d=4$ super Yang-Mills theory, or assuming $\mathcal{N}=2,d=4$ with gauge group $SU(N)$ and matter $N_f=2N$.In those cases the coupling is exact marginal and we have conformal fixed points where supersymmetry is compatible with conformal symmetry, the resulting theories are super conformal field theories(SCFTs). To be precise, we have to extend the notion of superalgebra to superconformal algebra, to do this we just introduce the special conformal supercharges S_α^a and $S_{a\dot{\alpha}}$ as the super partners of K_μ, then we have, for example[23]

$$\begin{aligned}
\left\{Q_{\alpha}^{a},Q_{\beta}^{b}\right\}&=\left\{S_{\alpha a},S_{\beta b}\right\}=\left\{Q_{\alpha}^{a},\bar{S}_{\dot{\beta}}^{b}\right\}=0\\
\left\{Q_{\alpha}^{a},\bar{Q}_{\dot{\beta}b}\right\}&=2\left(\sigma^{\mu}\right)_{\alpha\dot{\beta}}P_{\mu}\delta_{b}^{a}\\
\left\{S_{\alpha}^{a},\bar{S}_{\dot{\beta}b}\right\}&=2\left(\sigma^{\mu}\right)_{\alpha\dot{\beta}}K_{\mu}\delta_{b}^{a}\\
\left\{Q_{\alpha}^{a},S_{\beta b}\right\}&=\varepsilon_{\alpha\beta}\left(\delta_{b}^{a}D+R_{b}^{a}\right)+\frac{1}{2}\delta_{b}^{a}J_{\mu v}\left(\sigma^{\mu v}\right)_{\alpha\beta}
\end{aligned} \tag{1.283}$$

Then we can define a superconformal primary as[24]

$$\left[S_{\alpha}^{a},\mathcal{O}\right\}=0,\quad\left[\bar{S}_{a\dot{\alpha}},\mathcal{O}\right\}=0 \tag{1.284}$$

In particular it has a special kind of descendant called super descendant

$$\mathcal{O}'=[Q,\mathcal{O}\},\quad\Delta_{\mathcal{O}'}=\Delta_{\mathcal{O}}+\frac{1}{2} \tag{1.285}$$

[23] for the full algebra, see for example appendix B3 of [5]

[24] here by [, } we mean a commutator or an anticommutator depends on the situation, when both operators are fermionic it is anticommute, otherwise it is commute

it is a conformal primary(in the usual sense) as now

$$[K_\mu, \mathcal{O}'] = [K_\mu, [Q, \mathcal{O}\}] = 0 \tag{1.286}$$

One important type of superconformal primaries is chiral primary, where in addition satisfies

$$\exists a, \alpha : [Q^a_\alpha, \mathcal{O}\} = 0 \tag{1.287}$$

These are all BPS operators, and their scaling dimensions are related to their spins and R charges, hence stable under RG, as

$$0 = [\{S, Q\}, \mathcal{O}(0)] = [L + D + R, \mathcal{O}(0)] \sim (\Delta + R + \mathcal{J})\mathcal{O}(I) \tag{1.288}$$

Now let us analyze the RG behaviors of super gauge theory more carefully. For concreteness, assume that at UV we have a non-abelian gauge theory $\mathcal{T}$ with Lie group G and Lie algebra $\mathfrak{g}$ of rank r, a set of parameters including couplings and masses $\{g_k\}$, then at IR it flows to one of the following:

- A gapped theory where all local interactions are frozen and all fields are massive, but it can still be a nontrivial TQFT.
- A gapless theory, usually it is a CFT, and it can be free or interacting. Some of these theories allow for exact marginal deformations, hence forming a family, or a conformal manifold if it is parameterized by some suitable coupling constants as coordinates, otherwise they are isolated and have no exact marginal deformations.

For our present purposes, we are primarily interested in the latter case. And by the Coleman-Gross theorem[52], for any $U(1)$ gauge theory when the couplings are small enough the IR theory is free. Since we can break G down completely to $U(1)^r$, a free CFT at IR should be common for generic G as well. While interacting CFT at IR is limited to specific combinations of masses, couplings and field representations, it is highly non-trivial. Also the existence of conformal manifolds and isolated points are highly non-trivial, they encode key information of underlying theories.

If in addition we have supersymmetry and it is unbroken at IR, we may either have free abelian super gauge fields or isolated interacting SCFTs. In the former case we have a collection of real scalars ϕ^i , Weyl spinors ψ^a_α and $U(1)$ vectors A^I_μ , where we have put spacetime labels on bottom and internal labels on top. In the latter case we have some (super)conformal primary operators $\mathcal{O}$ along with their (super) descendants, and for this to occur, the parameters $\{g_k\}$ must take special values, and the field content, i.e. the representations R_x for multiplets x must be chosen in special ways.

In particular, we are interested in the vacua of such IR theories as to a large extent they characterize the underlying theories, the space of all possible vacuums is called the moduli space of vacuum, or simply moduli space, it captures the IR behavior of the UV theory in a geometric way. Usually, vacuum is determined by the minimal value of the scalar potential $V(\phi)$ in $\mathcal{L} = -V(\phi) + \frac{1}{2}g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j$, it is given by $\phi^i = C(g_k)$ as a set of real constant numbers formed by g_k, usually the minimal value is set to be zero, hence we get a structure of Riemann manifold with metric g_{ij} for the moduli space $\mathcal{M}_0 = \{\phi^i : V(\phi) = 0\}$, if we consider gauge symmetry as well, we should have $\mathcal{M} = \mathcal{M}_0/G'$ where G' is the gauge group at IR, say $G' = U(1)^r$ with maximally broken symmetry.

Assuming $d = 4$, if supersymmetry is present, $\mathcal{M}$ is enhanced to a Kähler manifold, this is due to the fact that now the kinematic part $\frac{1}{2}g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j$ is replaced by the Kähler potential $K(\Phi^k, \Phi^{k\dagger})$ And for $\mathcal{N} = 2$ theory, $\mathcal{M}$ is enhanced further to a hyperKähler manifold as now $\mathcal{K}$ splits into two pieces depending on the hyper and the vector multiplets vevs separately

$$\mathcal{K} = \mathcal{K}_H\left(\phi_n^i, \bar{\phi}_n^i\right) + \mathcal{K}_V\left(\Phi^I, \bar{\Phi}^I\right) \tag{1.289}$$

As a consequence we now have in classical sense, at least locally in the moduli space a product structure

$$\mathcal{M} = \mathcal{M}_H \times \mathcal{M}_V \tag{1.290}$$

We can classify it further as

- Coulomb branch

  Where $\mathcal{M}_H$ shrinks to a point hence is trivial

  $$\mathcal{M} = \mathcal{M}_V \tag{1.291}$$

  This branch is parameterized by the VEVs of Coulomb branch operators. Where by definition a Coulomb branch operator is always a $SU(2)_R$ singlet, but charged under $U(1)_R$,it is a special type of chiral operator annihilated by all anti-chiral super charges

  $$\left[\bar{Q}^a_{\dot{\alpha}}, \mathcal{O}_I\right] = 0 \tag{1.292}$$

  It has scaling dimension $\Delta(\mathcal{O}_I) = -r(\mathcal{O}_I)$ as the negative of its $U(1)_R$ R-charge As a hyperKähler manifold, it is of the form $\mathcal{M}_V = \mathcal{M}/U(1)^r_{\mathbb{C}}$, as the usual case, under quantum corrections, its metric is modified, Coulomb branch may not exist in some free theories but it usually exists in interacting theories, however this is still an open problem. Physically, say for $G = SU(N)$, then $\operatorname{tr}\Phi^n$ are Coulomb operators, so only scalars which are superpartners of gauge fields are condensed, and we get the usual Coulomb phase of gauge theory.

  $$\Phi = diag(a^1, a^2, \cdots, a^{N-1}, -\sum_{I=1}^{N-1} a^I) \tag{1.293}$$

  We have $r = N - 1$ abelian $U(1)$ gauge field A^I_μ

- Higgs branch

  Where $\mathcal{M}_V$ shrinks to a point hence is trivial

$$\mathcal{M} = \mathcal{M}_H \tag{1.294}$$

  This branch is parameterized by VEVs of Higgs branch operators. Where by definition a Higgs branch operator is always a $U(1)_R$ singlet, but charged under $SU(2)_R$ as the lowest or highest components, it is a special type of BPS operator where[25]

$$Q^{\alpha}_{(A_0}\mathcal{O}_{A_1A_2\cdots A_n)} = 0 \quad \bar{Q}^{\dot{\alpha}}_{(A_0}\mathcal{O}_{A_1A_2\cdots A_n)} = 0 \tag{1.295}$$

  It has scaling dimension $\Delta(\mathcal{O}_{A_1A_2\cdots A_n}) = R(\mathcal{O}_{11\cdots 1}) = n$ as its $SU(2)_R$ R-charge of highest weight component As a hyperKähler manifold, it is of the form $\mathcal{M}_V = \mathcal{M}/G'_{\mathbb{C}}$, a key feature of SUSY is that, using holomorphy arguments, quantum corrections will not changes the Higgs branch. This is a rather strong constraint, and unlike Coulomb branch, in many examples Higgs branch do not exist. Physically, now $\Phi = 0$ and the mass matrix μ is turned on and diagonalized[26]

$$\mu = diag(\mu_1, \mu_2, \cdots) \tag{1.296}$$

  Here only scalars that are not connected to gauge fields by supersymmetries get condensed, and this corresponds to the usual Higgs phase of gauge theory.

- Mixed branch

  The generic case, where only the Coulomb factor gets modified under quantum correction

We have seen that the existence of supersymmetry will enhance the moduli space and give them rich geometric structures, but we have not considered the isolated SCFT fixed points yet or possible BPS saturated points. From the viewpoint of moduli space, the vacua of the theories are singular points, and their very existence will change the geometry drastically. In particular singular points will induce monodromies around them, but physically we expect that the underlying physics should be invariant, to resolve this problem we need to use duality, i.e. different descriptions for the same physical content. A particularly nice feature of supersymmetry is that the dual of a theory is still supersymmetric with the same $\mathcal{N}$ but different G and fields, usually it can be regarded as another point in moduli space. Singular points can cause ramifications as well, and so there might be associated cover spaces of moduli space.

[25]In the notation of [64], it is a $\widehat{\mathcal{B}}_R$ type operator such that $Q^1_\alpha|R,r\rangle^{\text{hw}}_{\dot{\alpha}_1\ldots\dot{\alpha}_{2\bar{\jmath}}} = 0$ and $\bar{Q}_{2\dot{\alpha}}|R,r\rangle^{\text{hw}}_{\alpha_1\ldots\alpha_{2j}} = 0$ with $r = j = \bar{\jmath} = 0$

[26]Notice that even in Coulomb branch we can diagonalize the mass matrix as well for our convenience

To illustrate this point, let us begin with the electromagnetic duality in classical electrodynamics, where for a free field without a source we have

$$dF = d \star F = 0 \tag{1.297}$$

Then introduce the dual field as

$$F_D = \frac{4\pi}{e^2} \star F, \quad \frac{4\pi}{e^2}\frac{4\pi}{e_D^2} = 1 \tag{1.298}$$

Under this transformation, electric charge γ_e and magnetic charge γ_m are exchanged as

$$(\gamma_e, \gamma_m) \xrightarrow{S} \quad (-\gamma_m, \gamma_e) \tag{1.299}$$

For two dyons, there is an invariant symplectic pairing(Dirac pairing)

$$< \gamma, \gamma' >= \gamma_e \gamma'_m - \gamma'_e \gamma_m \tag{1.300}$$

More generally, we can introduce a θ term and suppose that e, θ all depends on a neutral scalar ϕ , then for the following Lagrangian

$$\frac{1}{2e(\phi)^2} F_{\mu v} F_{\mu\nu} + \frac{\theta(\phi)}{16\pi^2} F_{\mu\nu} \widetilde{F}_{\mu v} \tag{1.301}$$

W have

$$\begin{gathered} \partial_{[\mu} F_{\nu\rho]} = 0, \\ \partial_\mu \left[\frac{4\pi}{e(\phi)^2} F_{\mu\nu} + \frac{\theta(\phi)}{2\pi} \widetilde{F}_{\mu\nu} \right] = 0. \end{gathered} \tag{1.302}$$

Now we define the dual field as

$$F_D = \frac{4\pi}{e(\phi)^2} \star F - \frac{\theta}{2\pi} F \tag{1.303}$$

And the dual coupling by

$$\tau(\phi) = \frac{4\pi i}{e(\phi)^2} + \frac{\theta(\phi)}{2\pi}, \quad \tau_D(\phi) = \frac{4\pi i}{e_D(\phi)^2} + \frac{\theta_D(\phi)}{2\pi} \tag{1.304}$$

with

$$\tau_D(\phi) = -\frac{1}{\tau(\phi)} \tag{1.305}$$

Then the action is invariant if we replace all the original quantities by their dual correspondences, explicitly the equation of motion is

$$\begin{gathered} \partial_{[\mu} F_{D, v\rho]} = 0 \\ \partial_\mu \left[\frac{4\pi}{e_D(\phi)^2} F_{D,\mu v} + \frac{\theta_D(\phi)}{2\pi} \widetilde{F}_{D,\mu v} \right] = 0 \end{gathered} \tag{1.306}$$

Since θ is an angle, we can shift it to $\theta(\phi)+2\pi$, or in terms of complex coupling

$$\tau(\phi)_{\text{new }}=\tau(\phi)_{\text{old }}+1 \tag{1.307}$$

Then

$$\left(\gamma_e, \gamma_m\right) \xrightarrow{T} \quad\left(\gamma_m+\gamma_e, \gamma_m\right) \tag{1.308}$$

Again $<\gamma, \gamma^{\prime}>$ is invariant under this transformation. These two kinds of transformations are just the S, T transformations of the modular group on the complex coupling τ, more generally we have

$$\left(\begin{array}{ll}a & b \\ c & d\end{array}\right) \in \mathrm{SL}(2, \mathbb{Z}): \tau \rightarrow \frac{d \tau+b}{c \tau+a} \tag{1.309}$$

This is the simplest example of S-duality, where we have a dual transformation between a strongly coupled theory and a weakly coupled theory.

Now consider the Coulomb branch of our $\mathcal{N}=2$super gauge theory $\mathcal{T}$, with $U(1)^r$ at IR we have a symplectic pairing between vectors

$$<\gamma, \gamma^{\prime}>=\gamma_e \cdot \gamma_m^{\prime}-\gamma_e^{\prime} \cdot \gamma_m \tag{1.310}$$

And the above duality transformations generalize to

$$\begin{gathered}\tau_{I J} \rightarrow\left(A_I^L \tau_{L M}+B_{I M}\right)\left(C^{J N} \tau_{N M}+D^J{ }_M\right)^{-1} \\ M \equiv\left(\begin{array}{cc}A_I{ }^K & B_{I L} \\ C^J K & D^J\end{array}\right) \in S p(2 n, \mathbf{Z}) .\end{gathered} \tag{1.311}$$

where the coupling matrix is given by a holomorphic prepotential $\mathcal{F}$

$$\tau_{I J}=\frac{\partial^2 \mathcal{F}}{\partial a^I \partial a^J} \tag{1.312}$$

If we define the dual variable a_I^D for

$$a_I^D=\frac{\partial \mathcal{F}}{\partial a^I} \tag{1.313}$$

Then including the flavor charge, we have the following central charge function

$$Z_\gamma=a \cdot \gamma_e+a^D \cdot \gamma_m+\mu \cdot \gamma_f \tag{1.314}$$

with BPS bound on mass

$$M \geq Z_\gamma \tag{1.315}$$

When the bound is saturated, we have free hypermuliplets with $M=Z_\gamma$ hence extra massless states if $Z_\gamma=0$, as a result there are nontrivial monodromies around such points. Near certain points, both a, a^D are small so the electric and magnetic particles are all very light, and so we have conformal fixed points and the monodromies are even more complicated.

As a concrete example, let us apply these ideas to the simplest non-abelian gauge theory, namely the pure $SU(2)$ gauge theory. In this case $r = 1, N_f = 0$, this theory has no Higgs branch. For Coulomb branch $SU(2)$ is broken down to $U(1)$, and we have

$$\left[\Phi^\dagger, \Phi\right] = 0, \quad \Phi = \mathrm{diag}(a, -a) \tag{1.316}$$

as well as

$$F_{\mu v}^{\mathrm{SU(2)}} = \mathrm{diag}\left(F_{\mu v}^{\mathrm{U(1)}}, -F_{\mu v}^{\mathrm{U(1)}}\right), \quad \tau^{\mathrm{U(1)}} = 2\tau^{\mathrm{SU(2)}} \tag{1.317}$$

We will simply denote this $\tau^{\mathrm{U(1)}}$ as $\tau(a)$,and $\tau^{\mathrm{SU(2)}}$ at UV as τ_{UV}, then the invariant combination of scales is

$$\Lambda^4 = \Lambda_{UV}^4 e^{2\pi i \tau_{UV}} \tag{1.318}$$

with one loop expansion valid at $|a| \gg |\Lambda|$

$$\begin{aligned} \tau(a) &= 2\tau_{UV} - \frac{8}{2\pi i} \log \frac{a}{\Lambda_{UV}} + \cdots \\ &= -\frac{8}{2\pi i} \log \frac{a}{\Lambda} + \cdots \end{aligned} \tag{1.319}$$

and

$$a_D = -\frac{8a}{2\pi i} \log \frac{a}{\Lambda} + \cdots \tag{1.320}$$

If we introduce

$$u = \frac{1}{2}\left\langle \mathrm{tr}\, \Phi^2 \right\rangle = a^2 + \cdots \tag{1.321}$$

and the following phase

$$u = e^{i\theta}|u|, \quad \theta = 0 \sim 2\pi \tag{1.322}$$

then the mass

$$M = |\gamma_e a + \gamma_m a_D| = \left|(a, a_D)\begin{pmatrix} \gamma_e \\ \gamma_m \end{pmatrix}\right| \tag{1.323}$$

transform as

$$\begin{pmatrix} \gamma_e \\ \gamma_m \end{pmatrix} \to \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \gamma_e \\ \gamma_m \end{pmatrix} \tag{1.324}$$

this can be viewed as a monodromy at $u = \infty$ with

$$M_\infty = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix} \tag{1.325}$$

It must be compensated by some singularities at finite points, indeed there are two symmetric singular points at finite $u = \pm u_0$, to see this we assign the R-charges

$$\begin{array}{r|c} R = 0 & A \\ 1 & \lambda \quad \lambda \\ 2 & \Phi \end{array} \tag{1.326}$$

then the T transformation gives us

$$\theta \to \theta + 2\pi, \quad \Phi \to e^{\pi i/2}\Phi \tag{1.327}$$

At IR it is the following symmetry

$$\theta_{IR} \to \theta_{IR} + 4\pi, \quad u \to -u \tag{1.328}$$

so we must have

$$M_\infty = M_+ M_- \tag{1.329}$$

Indeed we can solve this to find

$$M_+ = STS^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad M_- = T^2STS^{-1}T^{-2} = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix} \tag{1.330}$$

These two singular points $\pm u_0$ correspond to a $(0,1)$ monopole point and a $(2,1)$dyon point respectively.

All of the above information is encoded in the following equation

$$\Sigma : \quad \Lambda^2 z + \frac{\Lambda^2}{z} = x^2 - u \tag{1.331}$$

where we have two auxiliary complex variables x, z, by adding $z = \infty$, where z parameterizes the UV moduli space C, in this case, just the Riemann sphere , and the curve Σ defined by this equation is a parametrization of the IR moduli space, called the Seiberg-Witten curve, which gives us the covering in figure 13:

$$\Sigma \xrightarrow{2:1} C \tag{1.332}$$

We can introduce the differential

$$\lambda = x\frac{dz}{z} \tag{1.333}$$

Then a, a_D are periods along different homology classes

$$a = \frac{1}{2\pi i}\oint_A \lambda, \quad a_D = \frac{1}{2\pi i}\oint_B \lambda \tag{1.334}$$

We also have $u_0 = \pm 2\Lambda^2$ for $z = \mp 1$, by studying those integrals near singular points we can calculate all the monodromies. This method extends to general theory with semisimple $\mathfrak{g}$ and flavor N_f. For example , take $SU(2)$ theory with $N_f = 1$, this theory does not have Higgs branch as well , while for Coulomb branch the S-W curve is

$$\Sigma : \quad \frac{2\Lambda(x-\mu)}{z} + \Lambda^2 z = x^2 - u \tag{1.335}$$

with S-W differential

$$\lambda = x\frac{dz}{z} \tag{1.336}$$

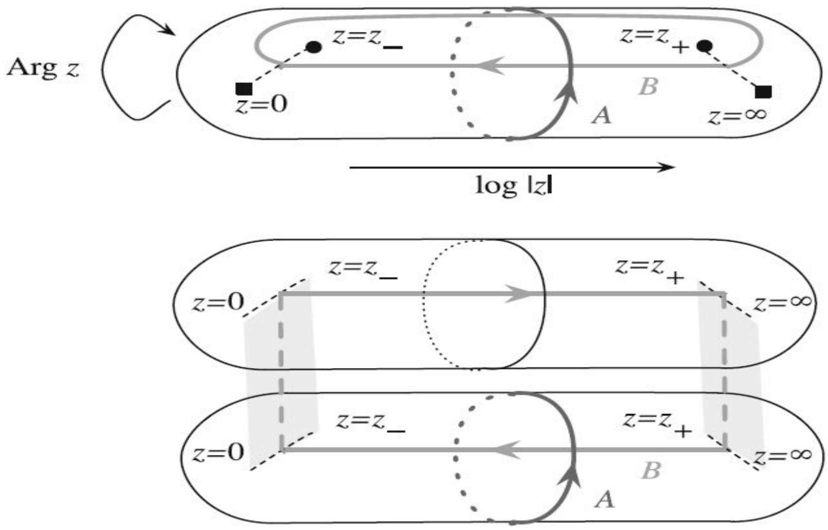


Figure 13: The UV curve and Seiberg-Witten curve for pure $SU(2)$ theory [152]

Then there are three finite singular points

$$M_\infty = M_3 M_2 M_1 \tag{1.337}$$

A special new feature of this theory is that under certain conditions, say when $\mu = -\frac{3}{2}\Lambda, u = 3\Lambda^2$ these singular points will collide to $z = -1$, and also in u plane two singular points will collide as shown in figure 14

In this case the $A - B$ cycles can be made arbitrarily small hence

$$a = a_D = 0 \tag{1.338}$$

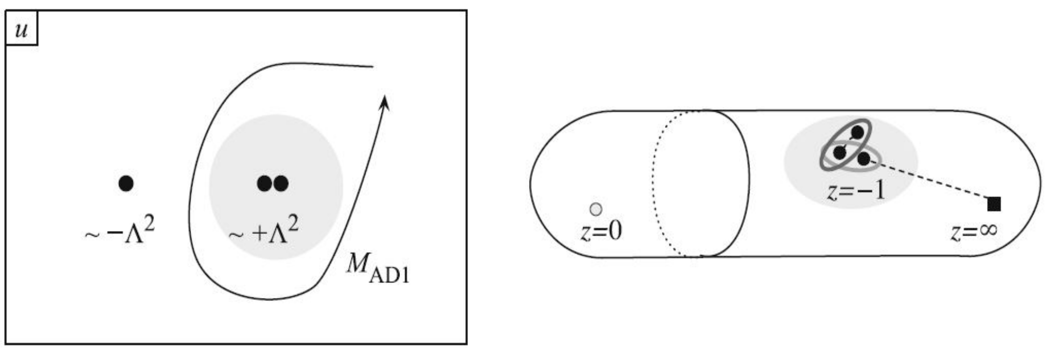


Figure 14: The Argyres–Douglas point of $N_f = 1$ theory, on u plane and z UV curve [152]

Now we have a conformal field theory, this is a simple example of the Argyres-Douglas CFT, using the S-W curve we can find the scaling dimensions by rewriting it as[27]

$$\lambda^2 = \frac{1 + \mu z^2 + uz^3}{z^7} dz^2 \tag{1.339}$$

and assign scaling dimension 1 to λ since it integrates to a, a^D as the masses of BPS particles, then

$$[\mu] = \frac{4}{5}, \quad [u] = \frac{6}{5} \tag{1.340}$$

More generally, one can have the so called class S theories, where by string theoretical constructions a class of six dimensional $(2, 0)$ SCFTs is given, this class has an ADE classification. Begin with one of those theories with simply laced Lie algebra $\mathfrak{g}$, and reduce it to five dimensions on a circle, a $\mathcal{N} = 2$ super Yang Mills theory is obtained, by choosing different boundary conditions, or equivalently a twisted compactification of the original theory on a punctured Riemann surface $C_{g,s}$ with genus g and s punctures [28], one can obtain a $\mathcal{N} = 2$ theory with $\mathfrak{g}$ symmetry, where $C_{g,s}$ is its UV curve. Different theories correspond to different types of punctured Riemann surfaces, and we can assign a system of ODEs to $C_{g,s}$, a system known as the Hitchin system, where different punctures are singular points of it, hence are classified as regular singular, irregular singular and so on. Hitchin system encodes the integrability conditions on the hypercomplex structures of moduli space, as a result the S-W curve can be obtained from it as well. In particular if $\mathfrak{g} = A_{N-1}$, a rank N cover Σ of a punctured sphere $C_{0,s}$ is obtained as the S-W curve, with some minor modifications this works for all ADE types. But the types and numbers of punctures are restricted by consistency, and only some of them correspond to well defined theories at dimension four, in particular the corresponding (generalized)Argyres-Douglas CFTs are limited in number and can be classified.[158]

We can reduce $\mathcal{N} = 2, d = 4$ theories on S^1 further to $\mathcal{N} = 4, d = 3$ ones with a large R symmetry group

$$SU(2)_L \times SU(2)_R \tag{1.341}$$

From the six dimensional viewpoint, $SU(2)_R$ is the original R symmetry group, while $SU(2)_L$ is the rotation group of the three compact dimensions. For these theories we have a special kind of duality, known as three dimensional mirror symmetry. This is a mirror symmetry in the sense that we can begin with a pair of mirror three (complex) dimensional Calabi-Yau manifolds $\mathcal{M}$ and $\mathcal{M}'$ with an appropriate $d = 10$ superstring theory, then the reductions on $\mathcal{M} \times S^1$ and $\mathcal{M}' \times S^1$ are dual three dimensional theories. Under this duality:

- $SU(2)_L$ exchanges with $SU(2)_R$
- Coulomb branch exchanges with Higgs branch

[27] here we use $dz^2 = dz \otimes dz$ and so on
[28] without twisting the $\mathcal{N} = 4$ super Yang Mills is generated

- Mass terms$\sum_i \int d^2\theta \mu_i^j \widetilde{Q}^i Q_j + cc$ exchanges with Fayet-Iliopoulos D terms $S_{FI} = \xi \int d^4\theta V$

In the simplest case of a simply laced $\mathfrak{g}$, the symmetry group of the mirror theory is

$$K_G \equiv \left(\prod_{i=0}^{r} U\left(n_i\right) \right) / U(1) \tag{1.342}$$

where i runs over the nodes of extended Dynkin diagrams, with n_i as the Dynkin index and by definition $n_0 = 1$. As the group K_G is already very complicated, in practice it is convenient to give the product of gauge groups like this kind a graphic representation, such a graph is called a (linear) quiver, constructed as follows:[29]

- Nodes

  Assign a gauge symmetry group $U(n)$ a circle node $\bigcirc$ with number n

  Assign a global symmetry group $U(p)$ a square node $\square$ with number p

- Edges

  Assign a bifundamental hypermultiplet $(\mathbf{n}, \overline{\mathbf{m}}) \oplus (\overline{\mathbf{n}}, \mathbf{m})$ of $U(n) \times U(m)$ fermion an edge — from circle node n to m

  Assign p fundamental hypermultiplets of $U(n)$ an edge — from n to p

Follow similar rules, we have, for example, more complicated quivers in the figures 25,26. Using some group theoretical arguments, given a quiver, some algorithms can be constructed to calculate the R charges hence the scaling dimensions for some special operators. In appendix B.1.3, we will use these algorithms to calculate the lower bound on the scaling dimension of a monopole operator.By definition, a monopole operator is an local operator that creates a classical magnetic monopole solution at a particular point [30], it is a local operator only in dimension three, in generic dimensions it is a defect. In $\mathcal{N} = 4, d = 3$ theories, monopole operators with suitable BPS conditions are K_G Coulomb branch duals of hypermultiplets in G Higgs branch, in particular for a free hypermultiplet we must have scaling dimension $\frac{1}{2}$, and the corresponding monopole operator has scaling dimension $\frac{3}{2}$.[26]

1.3.3 Superconformal index

Most super Yang Mills theories or SCFTs are hard to solve, even the basic field contents are usually hard to identify. So just like in quantum mechanics, it is useful to focus on stable quantities under deformations, and extract information from them. The superconformal index is a direct generalization of the Witten index, which dependents on the structure of vacua, its value is invariant under deformations as long as the associated Hilbert space remains the same. For a d dimensional SCFT, it is defined as

[29] there are several similar but different quivers used in the study of super gauge theory, and while the conventions vary, here we refer to the quiver and convention used in [81]

[30] in this case actually a vortex configuration

$$\mathcal{I}(\mu_i) = \mathrm{Tr}(-1)^F \prod_i \mu_i^{C_i} e^{-\beta\delta}, \quad \delta := \left\{\mathcal{Q}, \mathcal{Q}^\dagger\right\} \tag{1.343}$$

where F is the fermionic number operator, the trace is taken over over the Hilbert space of the radially quantized theory on S^{d-1}. Here $\{C_i\}$ is a maximal set of commuting charges, e.g. quantum numbers of overall flavor symmetries, for consistency they must commute with $\mathcal{Q}, \mathcal{Q}^\dagger$ as well, then we can introduce a set of formal fugacities $\{\mu_i\}$ to keep track of them. And $\mathcal{Q}$ a particular supercharge along with its conjugate $\mathcal{Q}^\dagger$, i.e. the super conformal charge $\mathcal{S} = \mathcal{Q}^\dagger$, where for unitary theories we must have $\delta := \left\{\mathcal{Q}, \mathcal{Q}^\dagger\right\} \geqslant 0$.

There are two interpretations for this index

- As cohomology

  It is obvious that $\delta^2 = 0$ then we can construct cohomology classes as $\ker\delta/\mathrm{im}\delta$, and here the superconformal index receives contributions only from the representatives of these classes of short multiplets, it indeed counts them with signs. It is possible for several short multiplets to cancel each other out as they can be combined into a long multiplet, just like a bosonic zero mode and a fermionic zero mode cancel each other out in the Witten index, so only the special short multiplets which are protected hence cannot be combined into long multiplets contribute.[31]

  Just like the Witten index, we expect this index to be stable, formally it should be invariant under all exact marginal superconformal deformations, however this fails in general, but if we restrict to theories with a discrete spectrum and finite dimensional space of fixed $\{C_i\}$, it still holds.

- As partition function

  Alternatively, we can also view this index as the partition function on $S^{d-1} \times S^1$ with suitable twisted boundary conditions on S^1.

  In this sense there should exist an extension of this index to the non-conformal but supersymmetric case, and still stable under RG flows with some reasonable restrictions.

For our interest, we will specific to $\mathcal{N} = 2, d = 4$ theories. In this case we can write the superconformal index explicitly as

$$\mathcal{I}(p, q, t; \mathbf{x}) := \mathrm{Tr}(-1)^F \left(\frac{t}{pq}\right)^r p^{j_{12}} q^{j_{34}} t^R \prod_i x_i^{f_i} e^{-\beta\delta_{2\dot{-}}} \tag{1.344}$$

and

$$2\delta_{2\dot{-}} := \left\{\widetilde{\mathcal{Q}}_{2\dot{-}}, \widetilde{\mathcal{Q}}^\dagger_{2\dot{-}}\right\} = E - 2j_2 - 2R + r \geq 0 \tag{1.345}$$

with the following conventions:

[31]See appendix B of [15] for more details.

- $\mathbf{x} = (x_1, \cdots, x_i, \cdots)$ are fugacities associated flavor symmetries, when we want to specify different subgroups we will also use $\mathbf{y}, \mathbf{z}$ and so on.

- p, q, t are fugacities associated with linear combinations of rotation and R symmetries

- j_1, j_2 are generators of the S^3 rotation symmetry $SU(2)_1 \times SU(2)_2$, and $j_{12} := j_2 - j_1, \quad j_{34} := j_2 + j_1$

- E ('energy' in radical quantization) denote the scaling dimension and R for R charge.

- the following formal convergence condition is assumed

$$|p| < 1, \quad |q| < 1, \quad |t| < 1, \quad |x_i| = 1, \quad \left|\frac{pq}{t}\right| < 1 \tag{1.346}$$

Just as Witten index, one can calculate the superconformal indexes from free theories and using the invariance property to obtain the indexes of interested non-trivial theories. More generally, begin with a theory $\mathcal{T}$ with known index, one can gauging a subgroup G of its flavor symmetry to obtain a new SCFT $\mathcal{T}_G$, their indexes are related as

$$\mathcal{I}\left[\mathcal{T}_G\right] = \int [d\mathbf{z}]_G \mathcal{I}_V(\mathbf{z}) \mathcal{I}[\mathcal{T}](\mathbf{z}) \tag{1.347}$$

where $\mathcal{I}_V$ is the index of a free vector multiplet in adj of G, and $[d\mathbf{z}]_G$ the invariant Haar measure. Using this method, one can calculate the superconformal indexes for class S theories. In particular, given a theory with symmetry $\mathfrak{g}$ and UV curve $\mathcal{C}_{g,s}$, when all the punctures are regular singular, we can assign a map $\Lambda : \mathfrak{su}(2) \to \mathfrak{g}$ for each puncture, and the overall flavor symmetry group is a direct sum of centralizers $\mathfrak{h} \subset \mathfrak{g}$ as $\oplus_{i=1}^{s} \mathfrak{h}_i$, this system is denoted as $T\left[\mathfrak{g}; \mathcal{C}_{g,s}; \{\Lambda_i\}\right]$, then the index has a third interpretation as a TQFT correlator on $\mathcal{C}_{g,s}$

$$\mathcal{I}^g\left[p, q, t; \mathbf{x}_i\right] = \left\langle \mathcal{O}\left(\mathbf{x}_1\right) \ldots \mathcal{O}\left(\mathbf{x}_s\right)\right\rangle_{\mathcal{C}_{g,s}} \tag{1.348}$$

with $\mathcal{O}\left(\mathbf{x}_i\right)$ a local operator living at puncture i, we can glue C_{g_1,s_1+1} and C_{g_2,s_2+1} to $C_{g,s}, g = g_1 + g + 2, s = s_1 + s_2$, hence we have

$$\mathcal{I}^g\left[\mathbf{x}_1, \ldots \mathbf{x}_s\right] = \int [d\mathbf{y}]_G \mathcal{I}^{g_1}\left[\mathbf{x}_j, \mathbf{y}\right] \mathcal{I}_V(\mathbf{y}) \mathcal{I}^{g_2}\left[\mathbf{y}, \mathbf{x}_k\right], \quad j \in s_1, k \in s_2 \tag{1.349}$$

If we introduce a basis of functions $\{\psi_\alpha\}$

$$\int [d\mathbf{x}]_G \mathcal{I}_V(\mathbf{x}) \psi_\alpha(\mathbf{x}) \psi_\beta(\mathbf{x}) = \delta_{\alpha\beta} \tag{1.350}$$

and the following integral transformation

$$\mathcal{O}_\alpha := \int [d\mathbf{x}]_G \mathcal{I}_V(\mathbf{x}) \psi_\alpha(\mathbf{x}) \mathcal{O}(\mathbf{x}) \tag{1.351}$$

Then using standard gluing arguments, we can define the three punctures sphere indexes

$$\mathcal{I}^{g=0}_{\alpha_1\alpha_2\alpha_3} =: C_{\alpha_1\alpha_2\alpha_3} \tag{1.352}$$

As basic building blocks, they satisfy

$$C_{\alpha_1\alpha_2\beta}C_{\beta\alpha_3\alpha_4} = C_{\alpha_1\alpha_3\gamma}C_{\gamma\alpha_2\alpha_4} \tag{1.353}$$

hence can be viewed as three point function coefficients, or structure constants, now the gluing property is simply

$$\mathcal{I}^{g}_{\alpha_1,\ldots\alpha_s} = \sum_\beta \mathcal{I}^{g_1}_{\{\alpha_j\}\beta}\mathcal{I}^{g_2}_{\beta\{\alpha_k\}} \tag{1.354}$$

And there exists a particular basis, the so-called Frobenius basis $\{\psi_\lambda(\mathbf{x})\}$, such that

$$C_{\lambda_1\lambda_2\lambda_3} = C_\lambda \delta_{\lambda\lambda_1}\delta_{\lambda\lambda_2}\delta_{\lambda\lambda_3} \tag{1.355}$$

and

$$\mathcal{I}^{g}_{\lambda\ldots\lambda} = C_\lambda^{2g-2+s} \tag{1.356}$$

and more generally

$$\mathcal{I}^g\left[\mathbf{x}_1,\ldots\mathbf{x}_s\right] = \sum C_\lambda^{2g-2+s}\psi_\lambda\left(\mathbf{x}_1\right)\ldots\psi_\lambda\left(\mathbf{x}_s\right) \tag{1.357}$$

For more general type of punctures, we have

$$\mathcal{I} = \sum_\lambda C_\lambda^{2g-2}\prod_{j=1}^{s}\phi_\lambda^{\Lambda_i}\left(\mathbf{y}_{\Lambda_i}\right) \tag{1.358}$$

where $\left\{\phi_\lambda^{\Lambda i}\left(\mathbf{y}_{\Lambda_i};p,q,t\right)\right\}$ is a system of 'wave functions' on those punctures. With the above TQFT interpretation, now the topologically equivalent gluing processes on UV curves can be explained as S-dualities for the corresponding four dimensional field theories.

In practice, it would be hard to calculate those wavefunctions and the corresponding indices, one usually taking some limits to extract partial information. If we rewrite the superconformal index as

$$\mathcal{I}(q,p,t) = \mathrm{Tr}(-1)^F p^{\frac{1}{2}\delta^1_-}q^{\frac{1}{2}\delta^1_+}t^{R+r}e^{-\beta'\delta_{2\dot{-}}}\prod_i x_i^{f_i} \tag{1.359}$$

where

$$\begin{aligned} 2\delta^1_+ &:= \left\{\mathcal{Q}^1_+,\left(\mathcal{Q}^1_+\right)^\dagger\right\} = E+2j_1-2R-r \geqslant 0 \\ 2\delta^1_- &:= \left\{\mathcal{Q}^1_-,\left(\mathcal{Q}^1_-\right)^\dagger\right\} = E-2j_1-2R-r \geqslant 0 \end{aligned} \tag{1.360}$$

and let $q=t$, this is refereed as the Schur limit, and the corresponding index ,the so called Schur index is

$$\mathcal{I}_{\mathcal{T}}(q;\mathbf{x}) \equiv \mathrm{Tr}_{\mathcal{H}}(-1)^F q^{E-R}\prod_{i=1}^{\mathrm{rank}\, G_F}\left(\mathbf{x}_i\right)^{f_i}, \tag{1.361}$$

now the cohomology condition is

$$\delta^1_- = \delta_{\dot{2}}^{\dot{-}} = 0 \tag{1.362}$$

or equivalently

$$\widehat{L}_0 := \frac{E - (j_1 + j_2)}{2} - R = 0, \quad \mathcal{Z} := j_1 - j_2 + r = 0 \tag{1.363}$$

in particular, for unitary theories $\widehat{L}_0 \geqslant \frac{|\mathcal{Z}|}{2}$ so we need $\widehat{L}_0 = 0$ only, a operator satisfies this condition is refereed as Schur operator. We will calculate the Schur indexes for special types of Argyres-Douglas CFTs in chapter two, and analyze the corresponding TQFT interpretations and RG flows.

2 Index relations and SUSY enhancement

2.1 Peculiar Index Relations, 2D TQFT,and Universality of SUSY Enhancement

In this chapter, we begin by focusing on a particularly simple—yet surprisingly rich—class of strongly interacting 4D $\mathcal{N} = 2$ SCFTs called the $D_2(SU(N))$ theories, with $N = 2n+1$ an odd integer [46, 47]. These theories are often imagined as arising in type IIB string theory[32] at local Calabi-Yau singularities and are part of a larger class of theories called the $D_p(G)$ theories, where G is the ADE flavor symmetry of the SCFT. However, using the methods of [158], we will primarily think of these theories as coming from twisted compactifications of the 6D $(2, 0)$ theory on Riemann surfaces with an irregular puncture.[33]

While the strongly coupled $D_2(SU(2n + 1))$ SCFTs are of Argyres-Douglas (AD) type[34] and therefore lack $\mathcal{N} = 2$ Lagrangians, they behave in various surprising ways like collections of free hypermultiplets:

- The role of the $D_2(SU(3))$ theory in the S-duality studied in [29, 32, 33] is reminiscent of the role played by some of the hypermultiplets in the S-duality of $\mathcal{N} = 2$ $SU(3)$ Supersymmetric Quantum Chromodynamics (SQCD) with $N_f = 6$ flavors [10].
- The so-called "Schur" limits of the 4D $\mathcal{N} = 2$ superconformal indices of the $D_2(SU(2n+1))$ theories are related to the Schur indices of free hypermultiplets by a simple rescaling of the superconformal fugacity and a specialization of the flavor fugacities [160, 150].
- The (partially refined) Schur indices of the $D_2(SU(2n + 1))$ theories can be computed via theories of free non-unitary hypermultiplets with wrong statistics in 4D [30, 31].

[32] Although note that the simplest example, $D_2(SU(3))$, was originally constructed in [9].

[33] Depending on the realization, the twisted compactification may or may not be accompanied by an extra regular singularity.

[34] In other words, they have $\mathcal{N} = 2$ chiral operators (i.e., operators annihilated by the anti-chiral half of $\mathcal{N} = 2$ superspace sometimes called "Coulomb branch" operators) of non-integer scaling dimension.

AD theory	Class $\mathcal{S}$ fixture analog	Flow to 32 supercharges
$D_2(SU(2n+1))$	$Y_{\rm simple},\ Y_{\rm full},\ Y_{\rm full}$; (free)	no
$R_{0,p}^{2,AD}$	$Y_2^{(1)},\ Y_{\rm full},\ Y_{\rm full}$; (interacting)	yes
$T_{(m_1,m_2,m_3)}^{2,AD}$	$Y_{m_1}^{(1)},\ Y_{m_2}^{(1)},\ Y_{m_3}^{(1)}$; (interacting)	yes

Table 1: Three important classes of isolated SCFTs we study in this paper are in the leftmost column (note that we assume, without loss of generality, that $m_3 \geq m_2 \geq m_1$; these quantities obey further constraints discussed in the main text). The middle column indicates the corresponding regular puncture class $\mathcal{S}$ fixture (specified by a triple of Young diagrams) in the sense described in Sec. 2.3.1, where $Y_k^{(\ell)}$ is the Young diagram shown in Fig. 20. The parenthetical comment in this column indicates whether the class $\mathcal{S}$ fixture is interacting or not. The final column indicates if the theory admits an RG flow, of the type described in the main text, to an interacting SCFT with thirty-two (Poincaré plus special) supercharges. The above AD relatives of interacting class $\mathcal{S}$ fixtures always admit such flows while relatives of free fixtures do not. All the above theories can be realized as type III in the nomenclature of [158]. In Sec. 2.4.2, we vastly generalize these results.

Given these parallels, it is interesting to ask if at least some of these close relations with Lagrangian theories persist upon conformally gauging subgroups of the flavor symmetry of the $D_2(SU(2n+1))$ theories. As we will see below, the answer to this question is a resounding, "yes." In particular, we will show that the Schur indices of an infinite set of theories gotten by gauging various diagonal flavor symmetries of collections of arbitrarily large numbers of $D_2(SU(2n+1))$ theories and hypermultiplets are related to the Schur indices of certain Lagrangian theories of class $\mathcal{S}$ [79] by simple transformations. Rephrasing these relations in the language of 2D TQFT allows us to efficiently study the action of S-duality on the flavor symmetries of the $D_2(SU(2n+1))$ quiver gauge theories (see [162, 163, 159] for recent discussions of other S-duality properties of these theories).

Beyond the action on flavor symmetries, one of the most interesting aspects of $\mathcal{N} = 2$ S-duality is the emergence of exotic isolated theories at cusps in the space of exactly marginal gauge couplings. For example, Argyres and Seiberg found the exotic E_6 Minahan-Nemeschansky theory in $SU(3)$ SQCD with $N_f = 6$ emerging at a dual cusp with a weakly coupled $SU(2) \subset E_6$ gauge group [10]. This construction was then vastly generalized to find new classes of isolated non-Lagrangian $\mathcal{N} = 2$ SCFTs (e.g, see [79, 49]).

As we will see, the TQFT relations we find between the AD quivers and their Lagrangian cousins lead to an interesting new expression for the Schur index of the exotic AD analog of the E_6 theory, the so-called "$\mathcal{T}_X$" SCFT, arising via the S-duality studied in [29, 32, 33]. Moreover, we are able to find the Schur indices for infinitely many generalizations of the $\mathcal{T}_X$ theory arising via various AD generalizations of S-dualities involving only regular punctures. For example, we find indices for AD analogs of the $R_{0,p}$ theories (with $p \in \mathbb{Z}_{\geq 0,\rm odd}$) arising via the S-dualities studied in [49]. We call these theories $R_{0,p}^{2,AD}$ SCFTs. In all cases, the AD index expressions we find are related to those of their regular puncture relatives (e.g., see [77]) by simple transformations on the fugacities. We term these types of AD theories "AD fixtures" in reference to the terminology for the corresponding isolated theories arising from three-punctured spheres in class $\mathcal{S}$ (e.g., see the terminology in [49]). In this context, one may also think of the $D_2(SU(2n+1))$ theories as AD relatives of free regular puncture fixtures. On the other hand, the $R_{0,n}^{2,AD}$ SCFTs (and other theories we construct below) are AD relatives of interacting regular puncture fixtures (see Table. 2.1).

However, the TQFT index expressions we find for these isolated exotic theories are rather illuminating in their own right. For example, unlike the usual expressions for regular puncture theories, the AD indices feature products over TQFT wave functions that are not independent. We then interpret this lack of independence in terms of the topology of the corresponding quivers of the 3D mirrors associated with the AD theories [158]. As we will see, the quiver topology of our AD relatives of interacting fixtures is characterized by a loop of non-abelian gauge nodes in the 3D mirror. This loop has interesting physical consequences: it guarantees that one can take these isolated AD theories, compactify them on S^1, and flow (up to free decoupled matter fields) to interacting theories with thirty-two (Poincaré plus special) superchages (thereby generalizing the examples in [33]).[35] We believe that these latter fixed points uplift to 4D $\mathcal{N}=4$ theories, but we leave a detailed study of this correspondence to future work.[36]

Based on the generic existence of RG flows with enhancement to thirty-two supercharges in the exotic isolated AD theories we study,[37] we ask more generally when such flows can occur. As we will see, the existence of these types of flows is in fact generic in the space of 4D $\mathcal{N}=2$ SCFTs (with known 3D Lagrangian mirrors) obtained by compactifying the 6D $(2,0)$ theory on a Riemann surface with an irregular singularity (we may or may not add an additional regular singularity).[38] Combined with the results of [118, 117, 2, 21, 3, 87, 1, 20, 88], our work here and in [33] suggests that AD theories naturally live along RG flows with accidental SUSY.[39] We discuss further implications of these ideas in the conclusions.

[35]Like their free AD fixture counterparts, the $D_2(SU(2n+1))$ fixtures do not admit RG flows via vevs and relevant deformations to interacting theories with thirty-two supercharges (note that we do not consider turning on additional gauge couplings in these flows).

[36]See also [11] for examples of $\mathcal{N}=2 \to \mathcal{N}=4$ enhancement (in the case of theories with integer dimensional Coulomb branch operators).

[37]In fact, this enhancement can also occur in AD quivers. Indeed, these theories also have indices with non-independent wave functions, and some of the general results we prove below apply to these theories as well. The fact that we gauge some symmetries to build these theories means that the 3D mirror interpretation of their indices is more subtle.

[38]In this sense, the word "exotic" for our isolated AD theories is inappropriate. Indeed, although flows to thirty-two supercharges of the type we describe are not common among the AD theories often studied in the literature, we will see that this is because such theories are actually rather special.

[39]Although note that here and in [33] we imagine that the accidental SUSY enhancement arises along RG flows emanating from the AD theories in the UV. On the other hand, in [118, 117, 2, 21, 3, 87, 1, 20, 88] the accidental SUSY enhancement mainly arises for flows ending on AD theories in the IR.

The outline of the rest of this chapter is as follows. In the next section we give more details regarding the $D_2(SU(N))$ theories, the resulting quiver gauge theories, and the index relations between these quivers and certain Lagrangian theories of class $\mathcal{S}$. We then move on to construct the 2D TQFT expressions for our indices and study S-duality using these expressions. We conclude this section by computing indices for various exotic type III AD fixtures that arise via S-duality and relating them to indices of better-known theories consisting purely of regular punctures. In the following section, we analyze the implications of these expressions for the quivers of the corresponding 3D mirrors. We then move on to a discussion of the resulting RG flows with accidental supersymmetry enhancement to thirty-two supercharges and conclude by proving a theorem on the universality of such flows in the class of theories arising from compactification of the $(2,0)$ theory on surfaces with irregular punctures and known 3D mirrors.

Note that throughout our discussion below, we will use the following shorthand to refer to the $D_2(SU(N))$ theories in order to ease notational burden:

$$AD_N \equiv D_2(SU(N)) \ , \quad N \in \mathbb{Z}_{\geq 0,\text{odd}} \ . \tag{2.1}$$

2.2 Conformal gauging of $AD_N \equiv D_2(SU(N))$ theories

In this section we introduce relevant technical aspects of the $AD_N \equiv D_2(SU(N))$ SCFTs (with N odd) and the quiver theories built by conformally gauging them. In particular, we first construct an intermediate building block, $\mathcal{T}^{(\ell)}_{n_1,n_2}$, and then construct the main quiver theories of interest, $\mathcal{T}^{(\ell)}_{n_1,n,n_2}$. We then move on to construct Schur indices for these quivers and relate them to Schur indices of certain Lagrangian theories.

2.2.1 More details of the AD quiver building blocks

The AD_N theories are a class of isolated strongly coupled 4D $\mathcal{N}=2$ SCFTs. Their Coulomb branch chiral rings are generated by operators of dimensions $\frac{N}{2}-i$ for $0 \leq i \leq \lceil \frac{N}{2} \rceil - 2$ [46]. Since N is odd, these theories have $\mathcal{N}=2$ chiral primaries (i.e., "Coulomb branch" operators) of non-integer dimension and are therefore of AD type.[40] The conformal anomalies of AD_N are given by $a_{AD_N} = \frac{7}{96}(N^2-1)$ and $c_{AD_N} = \frac{1}{12}(N^2-1)$. Most importantly for us in what follows, the flavor symmetry of AD_N is $SU(N)$, and the corresponding flavor central charge is given by

$$k_{SU(N)} = N \ , \tag{2.2}$$

where a fundamental hypermultiplet of $SU(N)$ contributes as $k_{SU(N)} = 2$.

For reference, here we also briefly review the 6d-4d construction of this theory [158, 150]. Using the notation of [150], we begin with a $d=6, \mathcal{N}=(2,0)$ SCFT of ADE type

[40] In particular, AD_3 is identical to the H_2 Argyres-Douglas theory [9] and is sometimes also called the (A_1, D_4) theory [48].

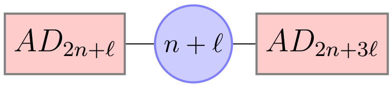


Figure 15: The quiver diagram of two conformally gauged AD_N SCFTs. The left box stands for an $AD_{2n+\ell}$ theory, the right box stands for an $AD_{2n+3\ell}$ theory, and the middle circle stands for an $SU(n+\ell)$ vector multiplet diagonally gauging the two AD theories. Here n is an integer, and ℓ is an odd integer. This is the simplest example of a conformally gauged AD building block for the more complicated class of quivers we will focus on (see Fig. 17).

J, and compactify it on a Riemann surface Σ with irregular punctures of (2.3), then by consistency we must have $g = 0$ so Σ is a Riemann sphere with punctures, there are two further possibilities for conformal $d = 4$ theories:

- Σ is a sphere with an irregular singularity,
- Σ is a sphere with an irregular singularity plus one regular singularity.

Where the singularities are defined by a Hitchin system on Σ[41]:

- We have a holomorphic one form Φ on Σ such that the irregular singularity appears as
$$\Phi = \frac{T_k}{z^{2+\frac{k}{b}}} + \dots \tag{2.3}$$
where T_k is an semi-simple element of J , then we can label this irregular singularity by these two numbers b, k as $J^b[k]$,
- We can also add a regular singular term to the above and in this case there is a flavor symmetry group associated with the regular singular point, we will use the label Y to denote it.[42] In particular, when this regular singular point is absent, we say it is null, and denote it as $Y = \{\emptyset\}$, otherwise we say it is full, and denote it as $Y = F$.

In summary, our $d = 4$ theory is labeled by the pair $(J^b[k], Y)$, then we have
$$AD_N \equiv \left(A^N_{N-1}[2-N], F\right) \ , \quad N \in \mathbb{Z}_{\geq 0,\text{odd}} \ . \tag{2.4}$$

More generally, a $D_p(G)$ with $\mathfrak{g} = J$ theory is realized by setting $b = h^\vee, Y = F$ with some suitable k relating to p, J, where $h^\vee$ is the dual Coxeter number of J.

From the isolated AD_N theories, we can construct an intermediate building block for the theories we are interested in as follows. Consider $AD_{2n+\ell}$ and $AD_{2n+3\ell}$ for a positive integer

[41] Recall that a regular singular point is an order one pole, otherwise it is irregular.

[42] For $J = A_{N-1}$, Y corresponds to a Young diagram $Y = \left[n_1^{h_1}, \dots, n_s^{h_s}\right]$ with $\sum_i h_i n_i = N$, and the flavor symmetry is $G_Y = \left(\prod_{i=1}^s U(h_i)\right)/U(1)$.

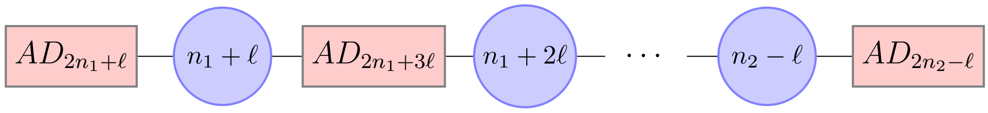


Figure 16: The quiver diagram of the $\mathcal{T}^{(\ell)}_{n_1,n_2}$ building block for the larger quiver we will consider in Fig. 17 and focus on in the next section. Here ℓ is a positive odd integer, and n_1 and n_2 are two integers such that $(n_2-n_1)/\ell$ is a positive integer. The gauge group of the quiver is $SU(n_1+\ell)\times SU(n_1+2\ell)\times\cdots\times SU(n_2-\ell)$. The flavor symmetry is $U(n_1)\times U(1)^{\frac{n_2-n_1}{\ell}-2}\times U(n_2)$.

n and an *odd* positive integer ℓ (so that $2n+\ell$ and $2n+3\ell$ are odd). These theories have $SU(2n+\ell)$ and $SU(2n+3\ell)$ flavor symmetries respectively. We can couple an $SU(n+\ell)$ vector multiplet to these SCFTs by gauging a diagonal $SU(n+\ell)$ flavor symmetry. The flavor central charge (2.2) implies that this gauging is exactly marginal. The resulting theory is an $\mathcal{N}=2$ SCFT described by the quiver diagram in Fig. 15 and has $U(n)\times U(n+2\ell)$ flavor symmetry.

Given this flavor symmetry, we can further gauge an $SU(n+2\ell)\subset U(n+2\ell)$ subgroup. This gauging is exactly marginal when the $SU(n+2\ell)$ vector multiplet is coupled to an additional $AD_{2n+5\ell}$ theory in such a way that the residual flavor symmetry of the $AD_{2n+5\ell}$ sector is $U(n+3\ell)$. The resulting theory now has $U(n)\times U(1)\times U(n+3\ell)$ flavor symmetry.

By continuing this procedure, we obtain a series of conformal linear quiver theories whose matter sector is comprised of various AD_N theories. The quiver diagram for these theories is shown in Fig. 16, where the gauge group is $SU(n_1+\ell)\times SU(n_1+2\ell)\times\cdots\times SU(n_2-\ell)$ for a positive odd integer, ℓ, and two integers, n_1 and n_2, such that $(n_2-n_1)/\ell$ is a positive integer. We denote this theory by $\mathcal{T}^{(\ell)}_{n_1,n_2}$, and it has $U(n_1)\times U(1)^{\frac{n_2-n_1}{\ell}-2}\times U(n_2)$ flavor symmetry.[43] From the quiver diagram, we see that the flavor central charge of the $SU(n_1)$ and $SU(n_2)$ subgroups are $2n_1+\ell$ and $2n_2-\ell$, respectively.

2.2.2 The main quiver theories of interest: the $\mathcal{T}^{(\ell)}_{n_1,n,n_2}$ SCFTs

Now we come to the main quiver theories of interest that are built from the above SCFTs and also from fundamental hypermultiplets. To be more explicit, let us take $\mathcal{T}^{(\ell)}_{n_1,n}$, $\mathcal{T}^{(\ell)}_{n_2,n}$, and ℓ fundamental hypermultiplets of $SU(n)$.[44] By the discussion in the previous subsection, if we gauge a diagonal $SU(n)$ flavor subgroup of these theories, the beta function vanishes:

$$\beta=(2n-\ell)+(2n-\ell)+2\ell-4n=0\ , \tag{2.5}$$

[43]Note that, when we write $\mathcal{T}^{(\ell)}_{n_1,n_2}$, we always have $n_1<n_2$ so that $(n_2-n_1)/\ell$ is a positive integer.

[44]Note that n, n_1, n_2, and ℓ are positive integers such that ℓ is odd, and $(n-n_1)/\ell$ and $(n-n_2)/\ell$ are positive integers.

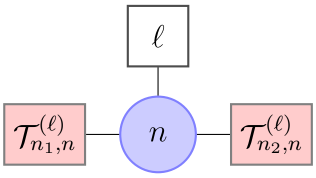


Figure 17: The diagram for the main quiver theory of interest: the $\mathcal{T}^{(\ell)}_{n_1,n,n_2}$ theory. The middle $SU(n)$ diagonally gauges the $SU(n)$ flavor subgroups of $\mathcal{T}^{(\ell)}_{n_1,n}$, $\mathcal{T}^{(\ell)}_{n_2,n}$, and $\ell \geq 1$ fundamental hypermultiplets (recall that $\ell \in \mathbb{Z}_{\text{odd}}$). This theory has $U(n_1) \times U(n_2) \times U(\ell) \times U(1)^{\frac{2n-n_1-n_2}{\ell}-2}$ flavor symmetry.

where $\mathcal{T}^{(\ell)}_{n_1,n}$ and $\mathcal{T}^{(\ell)}_{n_2,n}$ both contribute $2n-\ell$, the ℓ fundamental hypermultiplets contribute 2ℓ, and the $SU(n)$ vector multiplet contributes $-4n$. The resulting theory is an $\mathcal{N}=2$ SCFT described by the quiver diagram in Fig. 17 and has $U(n_1) \times U(n_2) \times U(\ell) \times U(1)^{\frac{2n-n_1-n_2}{\ell}-2}$ flavor symmetry. We denote this theory by $\mathcal{T}^{(\ell)}_{n_1,n,n_2}$, where the middle n in the subscript stands for the largest rank of the simple components of the gauge group.

2.2.3 Schur index

In this subsection, we construct the Schur indices of the $\mathcal{T}^{(\ell)}_{n_1,n,n_2}$ SCFTs from the various building blocks described previously. As we will see, these quantities turn out to be closely related to the indices of certain Lagrangian theories of class $\mathcal{S}$.

To understand these statements, first recall that the Schur index of a general $\mathcal{N}=2$ SCFT, $\mathcal{T}$, is defined as [77, 78][45]

$$\mathcal{I}_{\mathcal{T}}(q;\mathbf{x}) \equiv \mathrm{Tr}_{\mathcal{H}}(-1)^F q^{E-R} \prod_{i=1}^{\mathrm{rank}\, G_F} (\mathbf{x}_i)^{f_i} \ , \tag{2.6}$$

where $\mathcal{H}$ is the Hilbert space of local operators of $\mathcal{T}$, E is the scaling dimension, R is the Cartan generator of $SU(2)_R$ normalized so that the fundamental representation has eigenvalues $\pm\frac{1}{2}$, G_F is the flavor symmetry of the theory, and f_i is the i^{th} Cartan generator of G_F (i.e., the i^{th} flavor charge).

In the case of AD_N, the Schur index was conjectured to be [160] (see also the mathematical results in [107, 54])[46]

$$\mathcal{I}_{AD_N}(q;\mathbf{x}) = P.E.\left[\frac{q}{1-q^2}\chi^{SU(N)}_{\text{adj}}(\mathbf{x})\right] \ , \tag{2.7}$$

where $\mathbf{x} = (x_1, \cdots, x_N)$ subject to $\prod_{i=1}^N x_i = 1$ is the fugacity for the $SU(N)$ flavor symmetry, $\chi^{SU(N)}_{\text{adj}}(\mathbf{x})$ is the character of the adjoint representation, and $P.E.$ is the "plethystic

[45] The Schur index is a particular limit of a more general superconformal index [109, 137].

[46] The $N=3$ case is also discussed in [39, 38, 53, 37], and the formula in (2.7) agrees with the formula found in these references.

exponential." This latter quantity is defined as

$$P.E.[f(x_1, \cdots, x_M)] \equiv \exp\left(\sum_{p=1}^{\infty} f(x_1^p, \cdots, x_M^p)\right) . \tag{2.8}$$

Let us focus on the case $N = 2n_1 + \ell$ for a positive integer n_1 and an odd positive integer ℓ, since these theories enter the quivers we are interested in. In order to make contact with the index of the $\mathcal{T}^{(\ell)}_{n_1,n_2}$ SCFT, it is useful to consider the splitting of the $SU(2n_1+\ell)$ fugacity, $\mathbf{x}$, into those for the $SU(n_1) \times SU(n_1+\ell) \times U(1) \subset SU(2n_1+\ell)$ subgroup. In particular, $\mathbf{x}$ splits into $\mathbf{y} = (y_1, \cdots, y_{n_1})$, $\mathbf{z} = (z_1, \cdots, z_{n_1+\ell})$, and a such that $\prod_{i=1}^{n_1} y_i = \prod_{i=1}^{n_1+\ell} z_i = 1$.[47] In terms of these variables, the Schur index (2.7) for $N = 2n_1 + \ell$ is[48]

$$\mathcal{I}_{AD_{2n_1+\ell}}(q; \mathbf{y}, \mathbf{z}, a) = P.E.\left[\frac{q}{1-q^2}\left(1 + \chi^{SU(n_1)}_{\text{adj}}(\mathbf{y}) + \chi^{SU(n_1+\ell)}_{\text{adj}}(\mathbf{z})\right)\right] \times \mathcal{I}^{n_1\times(n_1+\ell)}_{\text{bfund}}(q^2; \mathbf{y}, \mathbf{z}, a) . \tag{2.11}$$

where $\chi^{SU(N)}_R$ is the character of an $SU(N)$ representation R, "adj" stands for the adjoint representation, and $\mathcal{I}^{N\times M}_{\text{bfund}}(q, \mathbf{y}, \mathbf{z}, a)$ is the Schur index of a bifundamental hypermultiplet of $SU(N) \times SU(M)$

$$\mathcal{I}^{N\times M}_{\text{bfund}}(q; \mathbf{y}, \mathbf{z}, a) \equiv P.E.\left[\frac{q^{\frac{1}{2}}}{1-q}\left(a\,\chi^{SU(N)}_{\text{fund}}(\mathbf{y})\chi^{SU(M)}_{\text{afund}}(\mathbf{z}) + a^{-1}\chi^{SU(N)}_{\text{afund}}(\mathbf{y})\chi^{SU(M)}_{\text{fund}}(\mathbf{z})\right)\right] , \tag{2.12}$$

with "fund" and "afund" being fundamental and anti-fundamental representations, respectively. Note that the last factor of (2.11) is identical to the Schur index of a bifundamental hypermultiplet of $SU(n_1) \times SU(n_1 + \ell)$ with q replaced by q^2. This expression will be important in our discussions below.

The index of the $\mathcal{T}^{(\ell)}_{n_1,n_2}$ building block Let us now evaluate the Schur indices of the $\mathcal{T}^{(\ell)}_{n_1,n_2}$ quiver building blocks we will eventually use to construct the Schur indices of the quivers of ultimate interest. Since the $\mathcal{T}^{(\ell)}_{n_1,n_2}$ SCFTs are obtained by conformally gauging AD_N theories, their indices are evaluated as integrals of products of the indices associated with each sector of the quivers.

---

[47] The precise relation between $\mathbf{x}$ and $(\mathbf{y}, \mathbf{z}, a)$ is given by

$$a = \left(\prod_{i=1}^{n_1} x_i\right)^{\frac{1}{n_1}} , \quad y_i = x_i/a \quad \text{for} \quad i = 1, \cdots, n_1 , \quad z_i = x_i a \quad \text{for} \quad i = n_1+1, \cdots, 2n_1+\ell . \tag{2.9}$$

[48] For $n_1 = 1$, we instead have

$$\mathcal{I}_{AD_{\ell+2}}(q; \mathbf{z}, a) = P.E.\left[\frac{q}{1-q^2}\left(1 + \chi^{SU(\ell+1)}_{\text{adj}}(\mathbf{z}) + a\chi^{SU(\ell+1)}_{\text{afund}}(\mathbf{z}) + a^{-1}\chi^{SU(\ell+1)}_{\text{fund}}(\mathbf{z})\right)\right] . \tag{2.10}$$

To describe this gauging, let $\mathbf{z}_0$, $\mathbf{z}_{\frac{n_2-n_1}{\ell}}$, and $\vec{a} \equiv (a_1, \cdots, a_{\frac{n_2-n_1}{\ell}})$ be fugacities for $SU(n_1)$, $SU(n_2)$, and $U(1)^{\frac{n_2-n_1}{\ell}}$ subgroups of the $\mathcal{T}^{(\ell)}_{n_1,n_2}$ flavor symmetry, respectively. Then the quiver diagram in Fig. 16 implies that

$$\begin{aligned}&\mathcal{I}_{\mathcal{T}^{(\ell)}_{n_1,n_2}}(q;\mathbf{z}_0,\vec{a},\mathbf{z}_{\frac{n_2-n_1}{\ell}})\\&=\int\left(\prod_{i=1}^{\frac{n_2-n_1}{\ell}-1} d\mu_i(\mathbf{z}_i)\,\mathcal{I}^{SU(n_1+i\ell)}_{\text{vec}}(q;\mathbf{z}_i)\right)\left(\prod_{i=0}^{\frac{n_2-n_1}{\ell}-1}\mathcal{I}_{AD_{2n_1+(2i+1)\ell}}(q;\mathbf{z}_i,\mathbf{z}_{i+1},a_{i+1})\right)\ , \quad (2.13)\end{aligned}$$

where the integral is taken over $SU(n_1+\ell)\times SU(n_1+2\ell)\times\cdots\times SU(n_2-\ell)$, $d\mu_i$ is the Haar measure on $SU(n_1+i\ell)$, $\mathbf{z}_i$ for $1\le i\le\frac{n_2-n_1}{\ell}-1$ is the $SU(n_1+i\ell)$ fugacity associated with $d\mu_i$, and

$$\mathcal{I}^{SU(N)}_{\text{vec}}(q;\mathbf{z})\equiv P.E.\left[\frac{-2q}{1-q}\chi^{SU(N)}_{\text{adj}}(\mathbf{z})\right]\ , \quad (2.14)$$

is the index contribution from an $SU(N)$ vector multiplet.

Note that, up to adjoint-valued pre-factors (whose role we will clarify below) and a $q\to q^2$ fugacity rescaling, the Schur indices of the $AD_{2n_1+\ell}$ SCFTs in (2.11) are just the indices of bifundamental hypermultiplets. As a result, the indices of the $\mathcal{T}^{(\ell)}_{n_1,n_2}$ SCFTs will also have a close connection with those of Lagrangian theories. Indeed, using the identities (B.2) and (2.11), one can rewrite (2.13) as

$$\begin{aligned}\mathcal{I}_{\mathcal{T}^{(\ell)}_{n_1,n_2}}(q;\mathbf{z}_0,\vec{a},\mathbf{z}_{\frac{n_2-n_1}{\ell}})=\frac{1}{(q;q^2)^{\frac{n_2-n_1}{\ell}}}\,P.E.\left[\frac{q}{1-q^2}\left(\chi^{SU(n_1)}_{\text{adj}}(\mathbf{z}_0)+\chi^{SU(n_2)}_{\text{adj}}(\mathbf{z}_{\frac{n_2-n_1}{\ell}})\right)\right]&\\ \times\,\mathcal{I}_{\mathcal{L}^{(\ell)}_{n_1,n_2}}(q^2;\mathbf{z}_0,\vec{a},\mathbf{z}_{\frac{n_2-n_1}{\ell}})\ ,& \quad (2.15)\end{aligned}$$

where

$$\begin{aligned}&\mathcal{I}_{\mathcal{L}^{(\ell)}_{n_1,n_2}}(q,\mathbf{z}_0,\vec{a},\mathbf{z}_{\frac{n_2-n_1}{\ell}})\\&\equiv\int\left(\prod_{i=1}^{\frac{n_2-n_1}{\ell}-1} d\mu_i(\mathbf{z}_i)\mathcal{I}^{SU(n_1+i\ell)}_{\text{vec}}(q;\mathbf{z}_i)\right)\prod_{i=0}^{\frac{n_2-n_1}{\ell}-1}\mathcal{I}^{(n_1+i\ell)\times(n_1+(i+1)\ell)}_{\text{bfund}}(q;\mathbf{z}_i,\mathbf{z}_{i+1},a_{i+1})\ , \quad (2.16)\end{aligned}$$

is the Schur index of the Lagrangian theory described by the quiver in Fig. 18. Note that this quiver has the same gauge group as in Fig. 16, but its matter sector is composed purely of fundamental and bifundamental hypermultiplets.[49] The expression (2.15) shows that the Schur index of $\mathcal{T}^{(\ell)}_{n_1,n_2}$ has a close connection with that of $\mathcal{L}^{(\ell)}_{n_1,n_2}$ (we need only multiply by adjoint-valued prefactors and rescale $q\to q^2$).

Let us briefly comment on the plethystic exponential pre-factor in front of $\mathcal{I}_{\mathcal{L}^{(\ell)}_{n_1,n_2}}$ on the RHS of (2.15). This term is inherited from the AD theories at the ends of the quiver and

[49]The $\mathcal{L}^{(\ell)}_{n_1,n_2}$ theory has the same flavor symmetry as $\mathcal{T}^{(\ell)}_{n_1,n_2}$ unless there is an accidental enhancement. Therefore, its Schur index is a function of the same set of fugacities as $\mathcal{I}_{\mathcal{T}^{(\ell)}_{n_1,n_2}}$.

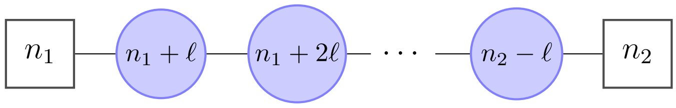


Figure 18: The quiver diagram of the Lagrangian theory we call $\mathcal{L}^{(\ell)}_{n_1,n_2}$. Each edge connecting two nodes stands for a bifundamental hypermultiplet, and each box labeled by "n" stands for n fundamental hypermultiplets. The flavor symmetry of $\mathcal{L}^{(\ell)}_{n_1,n_2}$ is generically the same as that of $\mathcal{T}^{(\ell)}_{n_1,n_2}$.

is independent of the abelian flavor fugacities, $\vec{a}$. On the other hand, this pre-factor does depend on the fugacities, $\mathbf{x}$ and $\mathbf{y}$, for the non-abelian flavor subgroup. The role of this dependence can be understood by noting that $\mathcal{I}_{\mathcal{T}^{(\ell)}_{n_1,n_2}}$ and $\mathcal{I}_{\mathcal{L}^{(\ell)}_{n_1,n_2}}$ satisfy recursive relations. Indeed, $\mathcal{I}_{\mathcal{T}^{(\ell)}_{n_1,n_2}}$ satisfies

$$\mathcal{I}_{\mathcal{T}^{(\ell)}_{n_1,n_2}}(q;\mathbf{x},\vec{a},\mathbf{y}) = \int_{SU(n_2-i\ell)} d\mu(\mathbf{z})\, \mathcal{I}_{\mathcal{T}^{(\ell)}_{n_1,n_2-i\ell}}(q;\mathbf{x},\vec{b},\mathbf{z})\, \mathcal{I}^{SU(n_2-i\ell)}_{\text{vec}}(q;\mathbf{z})\, \mathcal{I}_{\mathcal{T}^{(\ell)}_{n_2-i\ell,n_2}}(q;\mathbf{z},\mathbf{y},\vec{c}) \ , \tag{2.17}$$

where $1 \leq i \leq \frac{n_2-n_1}{\ell}$, and $\vec{a} = (b_1,\cdots,b_{\frac{n_2-n_1}{\ell}-i}, c_1,\cdots,c_i)$. There is a similar recursive relation for $\mathcal{I}_{\mathcal{L}^{(\ell)}_{n_1,n_2}}$, where all $\mathcal{I}_{\mathcal{T}^{(\ell)}_{n,m}}$ are replaced with $\mathcal{I}_{\mathcal{L}^{(\ell)}_{n,m}}$. These two recursive relations are consistent with (2.15) if the $P.E.$ factor is present in the relation (2.15).[50]

The indices of the $\mathcal{T}^{(\ell)}_{n_1,n,n_2}$ quivers Let us now assemble our previous results and compute the Schur indices of the quivers we will ultimately be interested in for our discussion below—the $\mathcal{T}^{(\ell)}_{n_1,n,n_2}$ SCFTs. To begin, we let $(\mathbf{x}_1,a_1)$, $(\mathbf{x}_2,b_1)$, and $(\mathbf{y},c)$ denote the fugacities for the flavor $U(n_1)$, $U(n_2)$, and $U(\ell)$ subgroups, respectively. We also let $(a_2,\cdots,a_{\frac{n-n_1}{\ell}})$ and $(b_2,\cdots,b_{\frac{n-n_2}{\ell}})$ represent the fugacities for the residual $U(1)^{\frac{2n-n_1-n_2}{\ell}-2}$ flavor subgroup. From its quiver description in Fig. 17, we see that the Schur index of $\mathcal{T}^{(\ell)}_{n_1,n,n_2}$ can be evaluated as

$$\begin{aligned}\mathcal{I}_{\mathcal{T}^{(\ell)}_{n_1,n,n_2}}(q;\mathbf{x}_1,\vec{a},(\mathbf{y},c),\vec{b},\mathbf{x}_2) = \int_{SU(n)} & d\mu(\mathbf{z})\, \mathcal{I}^{SU(n)}_{\text{vec}}(q;\mathbf{z})\mathcal{I}^{\ell\times n}_{\text{bifund}}(q;\mathbf{y},\mathbf{z},c) \\ & \times \mathcal{I}_{\mathcal{T}^{(\ell)}_{n_1,n}}(q;\mathbf{x}_1,\vec{a},\mathbf{z})\mathcal{I}_{\mathcal{T}^{(\ell)}_{n_2,n}}(q;\mathbf{x}_2,\vec{b},\mathbf{z}) \ ,\end{aligned} \tag{2.18}$$

where $\vec{a} \equiv (a_1,\cdots,a_{\frac{n-n_1}{\ell}})$ and $\vec{b} \equiv (b_1,\cdots,b_{\frac{n-n_2}{\ell}})$.

As in the case of $\mathcal{T}^{(\ell)}_{n_1,n_2}$, this Schur index is also related to the index of a quiver gauge theory with a Lagrangian description. Indeed, using (B.2), (B.4) and (2.15), one can rewrite

[50]The flavor-independent part of the pre-factor multiplying $\mathcal{I}_{\mathcal{L}^{(\ell)}_{n_1,n_2}}$ in (2.15), $(q;q^2)^{-\left(\frac{n_2-n_1}{\ell}\right)}$, is present in order to make up the difference in $a-c$ between the Lagrangian and non-Lagrangian theories in the Cardy limit of the index.

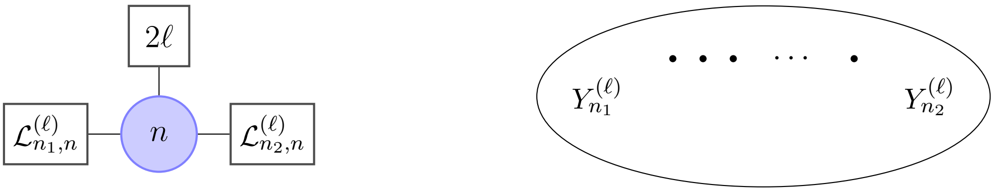


Figure 19: The left quiver is a weak coupling description of the Lagrangian theory $\mathcal{L}^{(\ell)}_{n_1,n,n_2}$. The gauge group is the same as that in Fig. 17, but the matter sector is composed purely of fundamental and bifundamental hypermultiplets. The rank of the flavor symmetry group of $\mathcal{L}^{(\ell)}_{n_1,n,n_2}$ is larger than that of $\mathcal{T}^{(\ell)}_{n_1,n,n_2}$ by ℓ. The $\mathcal{L}^{(\ell)}_{n_1,n,n_2}$ theory is obtained by compactifying the 6D $(2,0)$ A_{n-1} theory on the punctured sphere shown in the right picture. The sphere has $\frac{2n-n_1-n_2}{\ell}$ simple punctures (represented by black points) and two additional regular punctures associated with $Y^{(\ell)}_{n_1}$ and $Y^{(\ell)}_{n_2}$. The complex structure moduli space of this punctured sphere is identified as the conformal manifold of $\mathcal{L}^{(\ell)}_{n_1,n,n_2}$.

(2.18) as[51]

$$\mathcal{I}_{\mathcal{T}^{(\ell)}_{n_1,n,n_2}}(q;\mathbf{x}_1,\vec{a},(\mathbf{y},c),\vec{b},\mathbf{x}_2)=\frac{1}{(q;q^2)^{\frac{2n-n_1-n_2}{\ell}}}P.E.\left[\frac{q}{1-q^2}\left(\chi^{SU(n_1)}_{\text{adj}}(\mathbf{x}_1)+\chi^{SU(n_2)}_{\text{adj}}(\mathbf{x}_2)\right)\right]$$
$$\times\,\mathcal{I}_{\mathcal{L}^{(\ell)}_{n_1,n,n_2}}(q^2;\mathbf{x}_1,\vec{a},(\mathbf{y},cq^{\frac{1}{2}}),(\mathbf{y},cq^{-\frac{1}{2}}),\vec{b},\mathbf{x}_2)\ , \qquad (2.19)$$

where

$$\mathcal{I}_{\mathcal{L}^{(\ell)}_{n_1,n,n_2}}(q;\mathbf{x}_1,\vec{a},(\mathbf{y}_1,c_1),(\mathbf{y}_2,c_2),\vec{b},\mathbf{x}_2)$$
$$\equiv\int d\mu(\mathbf{z})\,\mathcal{I}^{SU(n)}_{\text{vec}}(q;\mathbf{z})\,\mathcal{I}_{\mathcal{L}^{(\ell)}_{n_1,n}}(q;\mathbf{x}_1,\vec{a},\mathbf{z})\,\mathcal{I}_{\mathcal{L}^{(\ell)}_{n_2,n}}(q;\mathbf{x}_2,\vec{b},\mathbf{z})\prod_{i=1}^{2}\mathcal{I}^{\ell\times n}_{\text{bifund}}(q;\mathbf{y}_i,\mathbf{z},c_i)\ , \qquad (2.20)$$

is the Schur index of a Lagrangian theory described by the quiver diagram in Fig. 19. We call this quiver gauge theory $\mathcal{L}^{(\ell)}_{n_1,n,n_2}$.

Note that the flavor symmetry of $\mathcal{L}^{(\ell)}_{n_1,n,n_2}$ is $U(n_1)\times U(n_2)\times U(2\ell)\times U(1)^{\frac{2n-n_1-n_2}{\ell}-2}$. In (2.20), $(\mathbf{x}_1,a_1)$ and $(\mathbf{x}_2,b_1)$ are fugacities for the $U(n_1)$ and $U(n_2)$ flavor subgroups respectively, while $(\mathbf{y}_1,\mathbf{y}_2,c_1,c_2)$ are fugacities for the $U(2\ell)$ flavor subgroup. Note that the flavor symmetry of $\mathcal{L}^{(\ell)}_{n_1,n,n_2}$ is *not* the same as the flavor symmetry of $\mathcal{T}^{(\ell)}_{n_1,n,n_2}$. Indeed, the rank of the flavor symmetry of $\mathcal{L}^{(\ell)}_{n_1,n,n_2}$ is larger than that of $\mathcal{T}^{(\ell)}_{n_1,n,n_2}$ by ℓ. Therefore, in the relation (2.19), 2ℓ fugacities for the $U(2\ell)$ flavor subgroup of $\mathcal{L}^{(\ell)}_{n_1,n,n_2}$ are restricted to ℓ fugacities $(\mathbf{y},c)$. Finally, note that the $P.E.$ factor depending on $\mathbf{x}_1$ and $\mathbf{x}_2$ plays the same role as in the case of $\mathcal{T}^{(\ell)}_{n_1,n_2}$.

[51] In the case of $n_i=1$, the factor $\chi^{SU(n_i)}_{\text{adj}}$ is replaced with 0. In the case of $n_i=0$, it is replaced by -1.

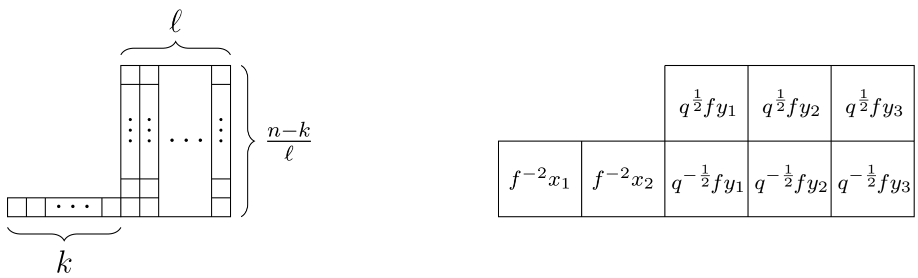


Figure 20: The left picture shows the Young diagram $Y_k^{(\ell)}$ with n boxes. Here k and ℓ are non-negative integers such that $\frac{n-k}{\ell}$ is a positive integer. There are k columns of height one and ℓ columns of height $\frac{n-k}{\ell}$. We also use the shorthand notation $Y_{\text{simple}} \equiv Y_1^{(1)}$ and $Y_{\text{full}} \equiv Y_0^{(n)} = Y_{n-1}^{(1)}$ in the main text. The right picture shows how the $SU(n)$ fugacity $\mathbf{w}$ in (2.26) is related to the $SU(k) \times SU(\ell) \times U(1)$ fugacities $(\mathbf{x}, \mathbf{y}, f)$ (in the particular case of $n = 8, k = 2$, and $\ell = 3$), where $w_1, \cdots, w_n$ are assigned to the boxes.

2.3 TQFT expressions and S-duality

2.3.1 TQFT expressions for the Schur indices and S-duality

In this section we begin by focusing on the $\mathcal{T}^{(\ell)}_{n_1,n,n_2}$ SCFTs and studying the resulting S-dualities via the connection with $\mathcal{L}^{(\ell)}_{n_1,n,n_2}$ discussed in the previous section. In particular, this connection leads us to simple TQFT expressions for the Schur indices of the $\mathcal{T}^{(\ell)}_{n_1,n,n_2}$ SCFTs and makes it straightfoward to read off the action of S-duality on the corresponding abelian flavor symmetries.[52] Moreover, as we will see, the TQFT approach gives rise to interesting new expressions for indices of certain exotic AD building blocks that appear at certain cusps in the conformal manifolds of the $\mathcal{T}^{(\ell)}_{n_1,n,n_2}$ theories.

One useful aspect of the Lagrangian quiver theory, $\mathcal{L}^{(\ell)}_{n_1,n,n_2}$, is that it can be obtained by compactifying the 6D (2,0) A_{n-1} theory on a sphere with $\frac{2n-n_1-n_2}{\ell}$ simple punctures and two additional regular punctures associated with $Y^{(\ell)}_{n_1}$ and $Y^{(\ell)}_{n_2}$ (see Fig. 19) [79]. This fact implies that the superconformal index of $\mathcal{L}^{(\ell)}_{n_1,n,n_2}$ can be computed via a TQFT on the sphere [75]. In this context, its Schur index, $\mathcal{I}_{\mathcal{L}^{(\ell)}_{n_1,n,n_2}}$, is written as a correlation function of q-deformed Yang-Mills (q-YM) theory [77]. Moreover, since the compactificaiton of the 6D $(2,0)$ theory involves only regular punctures, the TQFT expression for $\mathcal{I}_{\mathcal{L}^{(\ell)}_{n_1,n,n_2}}$ is particularly simple.

On the other hand, AD theories arise from the compactifications of the $(2,0)$ theory with one irregular puncture and, depending on the case, at most one additional regular puncture. The resulting TQFT index expressions tend to be considerably more elaborate [39, 40].

However, the simple TQFT expression for $\mathcal{I}_{\mathcal{L}^{(\ell)}_{n_1,n,n_2}}$ and the relation (2.19) imply that the Schur index of the non-Lagrangian quiver theory $\mathcal{T}^{(\ell)}_{n_1,n,n_2}$ also has a simple TQFT expression.

[52] The generalization of this discussion to $\mathcal{T}^{(\ell)}_{n_1,n_2}$ is straightforward but involves extra decoupled hypermultiplets.

Indeed, applying the transformation in (2.19) to the q-YM expression for $\mathcal{I}_{\mathcal{L}^{(\ell)}_{n_1,n,n_2}}$, we obtain

$$
\begin{aligned}
&\mathcal{I}_{\mathcal{T}^{(\ell)}_{n_1,n,n_2}}(q;\mathbf{x}_1,\vec{a},(\mathbf{y},c),\vec{b},\mathbf{x}_2)\\
&=\frac{1}{(q;q^2)^{\frac{2n-n_1-n_2}{\ell}}}P.E.\left[\frac{q}{1-q^2}\left(\chi^{SU(n_1)}_{\text{adj}}(\mathbf{x}_1)+\chi^{SU(n_2)}_{\text{adj}}(\mathbf{x}_2)\right)\right]\\
&\times\sum_{R:\ \text{irreps of }\mathfrak{su}(n)}\frac{f_R^{Y^{(\ell)}_{n_1}}(q^2;\mathbf{x}_1,\mathbf{y},e_0)\Big(\prod_{i=1}^{\frac{n-n_1}{\ell}}f_R^{Y_{\text{simple}}}(q^2;e_i)\Big)\Big(\prod_{j=1}^{\frac{n-n_2}{\ell}}f_R^{Y_{\text{simple}}}(q^2;f_j)\Big)f_R^{Y^{(\ell)}_{n_2}}(q^2;\mathbf{x}_2,\mathbf{y}^*,f_0)}{\big(C_R(q^2)\big)^{\frac{2n-n_1-n_2}{\ell}}}\ ,
\end{aligned}
\tag{2.21}
$$

where $\mathbf{y}^*\equiv(y_1^{-1},\cdots,y_\ell^{-1})$ and

$$
C_R(q)\equiv\frac{\prod_{\ell=1}^{n-1}(1-q^\ell)^{n-\ell}}{(q;q)_\infty^{n-1}}\chi_R^{SU(n)}(q^{\frac{n-1}{2}},q^{\frac{n-3}{2}},\cdots,q^{-\frac{n-1}{2}})\ . \tag{2.22}
$$

The parameters e_i and f_i are functions of $\vec{a}$, $\vec{b}$, c, and q satisfying

$$
(e_0)^n\equiv q^{\frac{n_1}{2}}c^{n_1}\prod_{i=1}^{\frac{n-n_1}{\ell}}(a_i)^{-n_1}\ ,\qquad (f_0)^n=q^{\frac{n_2}{2}}c^{-n_2}\prod_{j=1}^{\frac{n-n_2}{\ell}}(b_j)^{n_2}\ , \tag{2.23}
$$

$$
(e_i)^n=q^{\frac{\ell}{2}}c^\ell(a_i)^{n_1+(i-1)\ell}\prod_{k=i+1}^{\frac{n-n_1}{\ell}}(a_k)^{-\ell}\ ,\qquad (f_j)^n=q^{\frac{\ell}{2}}c^{-\ell}(b_j)^{-n_2-(j-1)\ell}\prod_{k=j+1}^{\frac{n-n_2}{\ell}}(b_k)^{\ell}\ , \tag{2.24}
$$

for $1\le i\le\frac{n-n_1}{\ell}$ and $1\le j\le\frac{n-n_2}{\ell}$.[53] The "wave function" f_R^Y depends on the Young diagram Y, and $Y_k^{(\ell)}$ is the n-box Young diagram with k columns of height one and ℓ columns of height $\frac{n-k}{\ell}$ (see Fig. 20). We use the short-hand notation $Y_{\text{simple}}\equiv Y_1^{(1)}$ and $Y_{\text{full}}\equiv Y_{n-1}^{(1)}=Y_0^{(n)}$. The wave function f_R^Y for $Y=Y_k^{(\ell)}$ is given by [77, 16]

$$
f_R^{Y_k^{(\ell)}}(q;\mathbf{x},\mathbf{y},f)\equiv K^{Y_k^{(\ell)}}(q;\mathbf{x},\mathbf{y},f)\chi_R^{SU(n)}(\mathbf{w})\ , \tag{2.26}
$$

$$
\begin{aligned}
K^{Y_k^{(\ell)}}(q;\mathbf{x},\mathbf{y},f)\equiv P.E.\Bigg[&\frac{q}{1-q}\chi^{SU(k)}_{\text{adj}}(\mathbf{x})+\frac{q^{\frac{1}{2}\left(\frac{n-k}{\ell}+1\right)}}{1-q}f^{-\frac{n}{k}}\chi^{SU(k)}_{\text{fund}}(\mathbf{x})\chi^{SU(\ell)}_{\text{afund}}(\mathbf{y})\\
&+\frac{q^{\frac{1}{2}\left(\frac{n-k}{\ell}+1\right)}}{1-q}f^{\frac{n}{k}}\chi^{SU(k)}_{\text{afund}}(\mathbf{x})\chi^{SU(\ell)}_{\text{fund}}(\mathbf{y})+\frac{q(1-q^{\frac{n-k}{\ell}})}{(1-q)^2}\chi^{U(\ell)}_{\text{adj}}(\mathbf{y})\Bigg]\ ,
\end{aligned}
\tag{2.27}
$$

[53] Note that not all e_i and f_j are independent. Indeed, we see that there is one constraint on them:

$$
\left(\prod_{i=0}^{\frac{n-n_1}{\ell}}e_i\right)\left(\prod_{j=0}^{\frac{n-n_2}{\ell}}f_j\right)=q\ . \tag{2.25}
$$

where $\mathbf{w}$ is an $SU(n)$ fugacity such that $w_i \equiv f^{-\frac{n-k}{k}} x_i$ for $1 \le i \le k$ and $w_{k+\frac{n-k}{\ell}(i-1)+j} \equiv q^{\frac{1}{2}\left(\frac{n-k}{\ell}+1\right)-j} f y_i$ for $1 \le i \le \ell$ and $1 \le j \le \frac{n-k}{\ell}$ (see Fig. 20), and $\chi_{\text{adj}}^{U(\ell)}(\mathbf{y}) = \chi_{\text{adj}}^{SU(\ell)}(\mathbf{y}) + 1$.[54] Note here that the wave function factors in (2.21) are *not* directly given by (2.26) but involve the rescaling $q \to q^2$.

Note also that the expression (2.21) for the Schur index of $\mathcal{T}^{(\ell)}_{n_1,n,n_2}$ is invariant under the permutations of $(e_1, \cdots, e_{\frac{n-n_1}{\ell}}, f_1, \cdots, f_{\frac{n-n_2}{\ell}})$. It turns out that such permutations are realized by reparameterizing a_i, b_j and c. Indeed, $e_i \longleftrightarrow e_{i+1}$ is realized by

$$a_i \to (a_i)^{\frac{\ell}{n_2+i\ell}} (a_{i+1})^{\frac{n_2+(i+1)\ell}{n_2+i\ell}} , \qquad a_{i+1} \to (a_i)^{\frac{n_2+(i-1)\ell}{n_2+i\ell}} (a_{i+1})^{-\frac{\ell}{n_2+i\ell}} , \tag{2.30}$$

with the other fugacities kept fixed. Similarly, $f_i \longleftrightarrow f_{i+1}$ is realized by a transformation of b_i and b_{i+1}. Finally, $e_{\frac{n-n_1}{\ell}} \longleftrightarrow f_{\frac{n-n_2}{\ell}}$ is realized by

$$\begin{aligned} a_{\frac{n-n_1}{\ell}} &\to (a_{\frac{n-n_1}{\ell}})^{\frac{\ell}{n}} (b_{\frac{n-n_2}{\ell}})^{\frac{\ell}{n}-1} c^{-\frac{2\ell}{n}} , \qquad b_{\frac{n-n_2}{\ell}} \to (a_{\frac{n-n_1}{\ell}})^{\frac{\ell}{n}-1} (b_{\frac{n-n_2}{\ell}})^{\frac{\ell}{n}} c^{-\frac{2\ell}{n}} , \\ c &\to (a_{\frac{n-n_1}{\ell}})^{\frac{\ell}{n}-1} (b_{\frac{n-n_2}{\ell}})^{\frac{\ell}{n}-1} c^{1-\frac{2\ell}{n}} , \end{aligned} \tag{2.31}$$

with the other fugacities kept fixed. Note that all these transformations keep e_0 and f_0 invariant, and therefore preserve the wave functions $f_R^{Y^{(\ell)}_{n_1}}(q^2; \mathbf{x}_1, \mathbf{y}, e_0)$ and $f_R^{Y^{(\ell)}_{n_2}}(q^2; \mathbf{x}_2, \mathbf{y}^*, f_0)$. This discussion shows that the Schur index of $\mathcal{T}^{(\ell)}_{n_1,n,n_2}$ is invariant under the action of $S_{\frac{2n-n_1-n_2}{\ell}}$. As discussed below, this invariance can be regarded as a natural generalization of an S_{2n} symmetry of the index of $\mathcal{T}^{(1)}_{1,n,1} = (A_{2n-1}, A_{2n-1})$, which was identified in [40] as the action of the S-duality group (see also [55]). It is therefore natural to interpret the above $S_{\frac{2n-n_1-n_2}{\ell}}$ invariance as a consequence of the S-duality invariance of $\mathcal{T}^{(\ell)}_{n_1,n,n_2}$.

In the next section, we carefully study two special cases, $\mathcal{T}^{(n)}_{0,n,0}$ and $\mathcal{T}^{(1)}_{1,n,1}$, and show that, from various S-dual descriptions of these theories, one can read off the Schur indices of various infinite series of exotic type III AD theories that decouple at cusps in the space of gauge couplings.

2.3.2 S-duality of the $\mathcal{T}^{(n)}_{0,n,0}$ SCFTs and AD analogs for $R_{0,n}$ theories

In this and next section we perform a more thorough analysis of two sets of examples of the S-dualities discussed in the previous section. In particular, we construct indices for exotic AD fixtures that arise in certain decoupling limits of the $\mathcal{T}^{(n)}_{0,n,0}$ and $\mathcal{T}^{(1)}_{1,n,1}$ SCFTs. The first set of examples gives rise to theories that generalize the $\mathcal{T}_X$ theory discussed in [33] and are

[54] For Y_{simple} and Y_{full}, this expression reduces to

$$f_R^{Y_{\text{simple}}}(q; f) = P.E.\left[\frac{q^{\frac{n}{2}}}{1-q}(f^n + f^{-n})\right] \frac{\prod_{\ell=1}^{n-2}(1-q^\ell)^{n-\ell-1}}{(q;q)_\infty^{n-1}} \chi_R^{SU(n)}(f q^{\frac{n-2}{2}}, \cdots, f q^{-\frac{n-2}{2}}, f^{1-n}) , \tag{2.28}$$

$$f_R^{Y_{\text{full}}}(q; \mathbf{x}) = P.E.\left[\frac{q}{1-q}\chi_{\text{adj}}^{SU(n)}(\mathbf{x})\right] \chi_R^{SU(n)}(\mathbf{x}) . \tag{2.29}$$

Figure 21: S-dual descriptions for $\mathcal{L}^{(n)}_{0,n,0}$ (left) and $\mathcal{T}^{(n)}_{0,n,0}$ (right). In the left quiver, an $SU(2)$ gauge group is coupled to a fundamental hypermultiplet and an isolated SCFT called $R_{0,n}$. In the right quiver, an $SU(2)$ gauge group is coupled to AD_3 (playing the role of the hypermultiplet) and an exotic fixture we call $R^{2,\mathrm{AD}}_{0,n}$ (this latter theory is a type III theory in the nomenclature of [158]).

AD analogs of the $R_{0,n}$ theories studied in [49]. Some of the theories in the second set of examples are AD analogs of other regular puncture fixtures (although, we will see there are some interesting subtleties in this analysis).

Let us first focus on the $\mathcal{T}^{(n)}_{0,n,0}$ theory, where $n \geq 3$ is an odd positive integer. The quiver diagrams of $\mathcal{T}^{(n)}_{0,n,0}$ and $\mathcal{L}^{(n)}_{0,n,0}$ are shown in Fig. 17 and Fig. 19, respectively. For $\ell = n$ and $n_1 = n_2 = 0$, the TQFT expression (2.21) reduces to

$$\mathcal{I}_{\mathcal{T}^{(n)}_{0,n,0}}(q;\mathbf{y},c) = \sum_{R\text{: irreps of }\mathfrak{su}(n)} \frac{f^{Y_{\text{full}}}_R(q^2;\mathbf{y})f^{Y_{\text{simple}}}_R(q^2;q^{\frac{1}{2}}c)f^{Y_{\text{simple}}}_R(q^2;q^{\frac{1}{2}}c^{-1})f^{Y_{\text{full}}}_R(q^2;\mathbf{y}^*)}{\left(C_R(q^2)\right)^2}\,, \tag{2.32}$$

where $\mathbf{y}$ and c are fugacities for $SU(n) \subset U(n)$ and $U(1) \subset U(n)$ subgroups of the flavor $U(n)$ symmetry, respectively.[55] Note that, unlike in the case of regular puncture theories, the two full puncture wave functions are not independent of each other since they have conjugate fugacities (the same statement applies for the simple puncture wave functions). We will discuss some implications of this fact in the context of the isolated theories that emerge from cusps in the $\mathcal{T}^{(n)}_{0,n,0}$ gauge coupling space.[56]

Let us now discuss the different S-duality frames of the $\mathcal{T}^{(n)}_{0,n,0}$ SCFTs and the exotic fixtures that appear at certain cusps in the gauge coupling constant space. In order to proceed, it is useful to first review the corresponding story for the $\mathcal{L}^{(n)}_{0,n,0}$ theories. To that end, recall that the $\mathcal{L}^{(n)}_{0,n,0}$ theory has another S-dual description in terms of the quiver diagram on the left of Fig. 21, where the $SU(2)$ gauge group is coupled to a fundamental hypermultiplet and an isolated SCFT / fixture called $R_{0,n}$ [49]. The flavor symmetry of $R_{0,n}$ is generically $SU(2) \times SU(2n)$, which is enhanced to E_6 in the case $n = 3$.[57] The gauge coupling, τ', of the dual description is related to the coupling, τ, of the original description by $\tau' = \frac{1}{1-\tau}$. In terms of the punctured sphere on the right of Fig. 19, this description

[55] Note that, for $n_1 = n_2 = 0$, the first line of the RHS in (2.21) reduces to 1. Moreover, the Young diagrams $Y^{(\ell)}_{n_1}$ and $Y^{(\ell)}_{n_2}$ reduce to $Y_{\text{full}} \equiv Y^{(n)}_0 = Y^{(1)}_{n-1}$. We also note that $e_0 = f_0 = 1$ in this case.

[56] In fact, some of these implications apply to the gauged theories as well. However, the 3D mirror analysis is more complicated in this case.

[57] Since its Coulomb branch operators are all of integral dimension, the $R_{0,n}$ theory is *not* an AD theory.

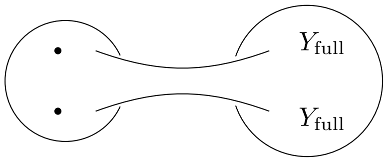


Figure 22: The pants decomposition for the punctured sphere corresponding to the S-dual description of $\mathcal{L}^{(n)}_{0,n,0}$ shown on the left of Fig. 21. The left and right spheres correspond to a fundamental hypermultiplet and $R_{0,n}$ respectively, while the middle cylinder corresponds to the $SU(2)$ vector multiplet.

corresponds to the pants decomposition shown in Fig. 22. This dual description implies that the Schur index of $\mathcal{L}^{(n)}_{0,n,0}$ can also be expressed as

$$\mathcal{I}_{\mathcal{L}^{(n)}_{0,n,0}}(q;(\mathbf{w}_1,c_1),(\mathbf{w}_2,c_2)) = \int_{SU(2)} d\mu(\mathbf{z})\, \mathcal{I}^{SU(2)}_{\text{fund}}(q;\mathbf{z},s)\mathcal{I}^{SU(2)}_{\text{vec}}(q;\mathbf{z})\mathcal{I}_{R_{0,n}}(q;\mathbf{z},r,\mathbf{w}_1,\mathbf{w}_2^*)\ , \tag{2.33}$$

where $(\mathbf{w}_i, c_i)$ are $U(n)$ fugacities as in (2.20), $\mathbf{z} = (z, z^{-1})$ is an $SU(2)$ fugacity, and $s \equiv (c_1c_2)^{\frac{n}{2}}$ and $r \equiv c_1/c_2$ are $U(1)$ fugacities. The last factor in (2.33) is the Schur index of $R_{0,n}$ given by [77]

$$\mathcal{I}_{R_{0,n}}(q;\mathbf{z},r,\mathbf{w}_1,\mathbf{w}_2^*) = \sum_{R:\ \text{irrep of}\ \mathfrak{su}(n)} \frac{f^{Y_2^{(1)}}_R(q;\mathbf{z},r)\, f^{Y_{\text{full}}}_R(q;\mathbf{w}_1)\, f^{Y_{\text{full}}}_R(q;\mathbf{w}_2^*)}{C_R(q)}\ , \tag{2.34}$$

where only an $SU(2)\times U(1)\times SU(n)^2$ subgroup of the flavor symmetry is manifest.

As we discuss in appendix B.1.2, the $\mathcal{T}^{(n)}_{0,n,0}$ theory has a similar S-dual description, which is described by the quiver shown on the right of Fig. 21. The gauge group is again $SU(2)$, which is now coupled to an AD_3 theory (acting as an AD generalization of hypermultiplets) and a type III AD theory in the language of [158]. This type III AD theory is labeled by three Young diagrams $Y_1 = Y_2 = [n-1, n-1, 2]$ and $Y_3 = [2,\cdots,2,1,1]$ with $2n$ boxes and generically has $SU(2)\times SU(n)$ flavor symmetry (the $n=3$ case has $SU(3)\times SU(2)\times SU(2)$ flavor symmetry). We denote this type III theory by $R^{2,\text{AD}}_{0,n}$ since it can be regarded as an AD counterpart of the $R_{0,n}$ fixture. This quiver description implies that the Schur index of $\mathcal{T}^{(n)}_{0,n,0}$ can also be expressed as

$$\mathcal{I}_{\mathcal{T}^{(n)}_{0,n,0}}(q;\mathbf{y},c) = \int_{SU(2)} d\mu(\mathbf{z})\, \mathcal{I}_{AD_3}(q;\mathbf{z},c^n)\mathcal{I}^{SU(2)}_{\text{vec}}(q;\mathbf{z})\mathcal{I}_{R^{2,\text{AD}}_{0,n}}(q;\mathbf{z},\mathbf{y})\ , \tag{2.35}$$

where $\mathcal{I}_{R^{2,\text{AD}}_{0,n}}$ is the Schur index of the $R^{2,\text{AD}}_{0,n}$ theory. Note that previously this index was obtained only for the special case $n=3$ [32], while here we describe it for all odd $n \geq 3$.[58]

[58]The identification of the flavor $U(1)$ fugacity in $\mathcal{I}_{AD_3}(q;\mathbf{z},c^n)$ can be understood as follows. From the

By substituting (2.33) and (2.35) into (2.19) and using the identities (2.11) and (B.2), we obtain

$$\begin{aligned} 0 = \int_{SU(2)} & d\mu(\mathbf{z})\, \mathcal{I}_{\text{fund}}(q^2; \mathbf{z}, c^n)\, \mathcal{I}_{\text{vec}}^{SU(2)}(q^2; \mathbf{z}) \\ & \times \left\{ \mathcal{I}_{R_{0,n}}(q^2; \mathbf{z}, q, \mathbf{y}, \mathbf{y}^*) - P.E.\left[\frac{q}{1-q^2} \left(1 - \chi_{\text{adj}}^{SU(2)}(\mathbf{z}) \right) \right] \mathcal{I}_{R_{0,n}^{2,\text{AD}}}(q; \mathbf{z}, \mathbf{y}) \right\} \ . \end{aligned} \tag{2.36}$$

This equation is solved by

$$\mathcal{I}_{R_{0,n}^{2,\text{AD}}}(q; \mathbf{z}, \mathbf{y}) = P.E.\left[\frac{q}{1-q^2} \left(-1 + \chi_{\text{adj}}^{SU(2)}(\mathbf{z}) \right) \right] \mathcal{I}_{R_{0,n}}(q^2; \mathbf{z}, q, \mathbf{y}, \mathbf{y}^*) \ . \tag{2.37}$$

Indeed, there exists an inversion formula [76] that extracts the integrand of (2.36), which implies that (2.37) is the unique solution to (2.36). Combining (2.37) and (2.34), we obtain the following TQFT expression for the Schur index of $R_{0,n}^{2,\text{AD}}$

$$\mathcal{I}_{R_{0,n}^{2,\text{AD}}}(q; \mathbf{z}, \mathbf{y}) = \frac{1}{(zq; q^2)(z^{-1}q; q^2)} \sum_{R:\ \text{irrep of}\ \mathfrak{su}(n)} \frac{f_R^{Y_2^{(1)}}(q^2; \mathbf{z}, q) f_R^{Y_{\text{full}}}(q^2; \mathbf{y}) f_R^{Y_{\text{full}}}(q^2; \mathbf{y}^*)}{C_R(q^2)} \ . \tag{2.38}$$

Note that, even though the flavor $U(1) \subset U(2)$ fugacity r of $f_R^{Y_2^{(1)}}(q^2; \mathbf{z}, r)$ is set to q, one can show that (2.38) only has integer and half-integer powers of q as it should. Moreover, one can check that for $n > 3$ the index does not have an $\mathcal{O}(q^{\frac{1}{2}})$ term and so the theory does not have free hypermultiplets. In appendix B.1.3 we find another proof of this fact by bounding monopole operator dimensions in the 3D mirror.[59]

For $n = 3$, one can perform a stronger consistency check of the above result. Indeed, the $R_{0,3}^{2,\text{AD}}$ theory was carefully studied in [32], where it was shown that $R_{0,3}^{2,\text{AD}}$ splits into an exotic AD theory called $\mathcal{T}_X$ and a decoupled half-hypermultiplet in the fundamental representation of the flavor $SU(2)$.[60] The Schur index of the $\mathcal{T}_X$ SCFT is then

$$\mathcal{I}_{\mathcal{T}_X}(q; \mathbf{z}, \mathbf{y}) = (zq^{\frac{1}{2}}; q)(z^{-1}q^{\frac{1}{2}}; q)\mathcal{I}_{R_{0,3}^{2,\text{AD}}}(q; \mathbf{z}, \mathbf{y}) \ , \tag{2.39}$$

quiver description in Fig. 19, we see that $\mathcal{T}_{0,n,0}^{(n)}$ has two baryonic Higgs branch operators of dimension n. These operators are charge conjugate to each other and contribute $q^{\frac{n}{2}} c^{\pm n}$ to the Schur index. In the dual description shown in Fig. 21, these operators are realized as the product of a flavor $SU(3)$ moment map in AD_3 and a Higgs branch operator in $R_{0,n}^{2,\text{AD}}$. Indeed, from the 3D mirror of $R_{0,n}^{2,\text{AD}}$ discussed in appendix B.1.3, we see that $R_{0,n}^{(2,\text{AD})}$ has a Higgs branch operator of dimension $(n-2)$ in the $\mathbf{2} \otimes \mathbf{1}$ representation of the flavor $SU(2) \times SU(n)$ symmetry (this operator corresponds to a mirror monopole of scaling dimension $(n-2)/2$). Let us denote it by $\mathcal{O}^a$ with $a = 1, 2$ being the $SU(2)$ index. Let us also denote by $\mathcal{O}_\pm^a$ two flavor $SU(3)$ moment maps in the doublet of $SU(2) \subset SU(3)$, where the subscript stands for the charge under $U(1) \subset SU(3)$. Then we see that $\epsilon_{ab}\mathcal{O}_\pm^a \mathcal{O}^b$ can be identified as the baryonic Higgs branch operators mentioned above. This discussion implies that the flavor $U(1)$ fugacity in $\mathcal{I}_{AD_3}$ is c^n.

[59] Due to the non-trivial quiver topology of the 3D mirror that will be discussed further in the next section, this computation is non-trivial and does not follow directly from the results in [81].

[60] In [32], the $R_{0,3}^{2,\text{AD}}$ theory is denoted as $\mathcal{T}_{3,\frac{3}{2}}$.

where the first two factors comprise the Schur index of the free matter fields. One can check, order by order in q, that (2.39) with (2.38) substituted in is identical to the following expression for the index of $\mathcal{T}_X$ obtained in [32]:

$$\mathcal{I}_{\mathcal{T}_X}(q;\mathbf{z},\mathbf{y}) = \sum_{\lambda=0}^{\infty} q^{\frac{3}{2}\lambda}\, P.E.\left[\frac{2q^2}{1-q} + 2q - 2q^{\lambda+1}\right] \mathrm{ch}^{SU(2)}_{R_\lambda}(q;\mathbf{z})\mathrm{ch}^{SU(3)}_{R_{\lambda,\lambda}}(q;\mathbf{y}) \ , \tag{2.40}$$

where $\mathrm{ch}^{SU(N)}_{R}(q;\mathbf{x})$ is the character of a representation R of $\widehat{\mathfrak{su}(N)}_{-N}$, and R_λ and $R_{\lambda,\lambda}$ are the highest weight representations of $\widehat{\mathfrak{su}(2)}_{-2}$ and $\widehat{\mathfrak{su}(3)}_{-3}$ corresponding to the Dynkin labels $(-2-\lambda,\lambda)$ and $(-3-2\lambda,\lambda,\lambda)$, respectively.

Let us further analyze the two equivalent expressions in (2.39), with (2.38) substituted in, and (2.40). Note that these two expressions have very different origins. Indeed, the expression in (2.40) is written in terms of affine Kac-Moody representations[61] while (2.38) is closely related to the correlator of a TQFT on a sphere with three regular punctures. Moreover, (2.40) takes the form of a sum over a full set of $SU(2)$ representations (with the $SU(3)$ representations restricted in terms of the $SU(2)$ data), while (2.38) takes the form of a sum over a full set of $SU(3)$ representations (here the $SU(2)$ data is fixed in terms of the larger $SU(3)$ data). In spite of these differences, the two formulas both take the form of a product of group theoretical factors of $SU(2)\times SU(3)$. Indeed, the second $SU(3)$ wave function in (2.38) is dependent on the first $SU(3)$ wave function since their fugacities are complex conjugates of each other (therefore, in some sense, both expressions involve restrictions on $SU(3)$ data). In Sec. 2.4.1, we will reinterpret this dependence of the wave functions in terms of the topology of the corresponding 3D mirrors of the $R^{2;AD}_{0,n}$ SCFTs.

2.3.3 S-duality of $\mathcal{T}^{(1)}_{1,n,1} = (A_{2n-1}, A_{2n-1})$ theory

Next let us consider the $\mathcal{T}^{(1)}_{1,n,1}$ theories for positive integer $n \geq 2$. Taking $\ell = n_1 = n_2 = 1$, the TQFT expression (2.21) reduces to

$$\mathcal{I}_{\mathcal{T}^{(1)}_{1,n,1}}(q;\vec{a},c,\vec{b}) = \frac{1}{(q;q^2)^{2n-2}} \sum_{R:\text{ irreps of }\mathfrak{su}(\mathfrak{n})} \frac{\left(\prod_{i=0}^{n-1} f_R^{Y_{\text{simple}}}(q^2;e_i)\right)\left(\prod_{j=0}^{n-1} f_R^{Y_{\text{simple}}}(q^2;f_j)\right)}{\left(C_R(q^2)\right)^{2n-2}} \ , \tag{2.41}$$

where e_i and f_i are determined by (2.23) and (2.24). Note that this index is invariant under the S_{2n} that permutes $e_0,\cdots,e_{n-1}$ and $f_0,\cdots,f_{n-1}$. These permutations are realized by transforming the flavor fugacities as in (2.30) and (2.31), but now for $i=0,\cdots,n-1$. In particular, the permutation symmetry is "accidentally" enhanced in this case from $S_{2(n-1)}$ to S_{2n}.

[61] This expansion is natural considering that the Schur index is related to the vacuum character of the corresponding 2D chiral algebra under the 4D/2D map of [15].

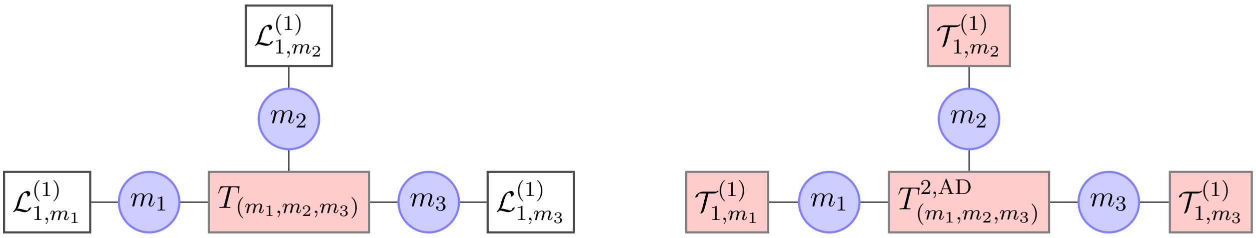


Figure 23: The S-dual descriptions of $\mathcal{L}^{(1)}_{1,n,1}$ and $\mathcal{T}^{(1)}_{1,n,1}$ corresponding to (m_1, m_2, m_3) such that $m_1 + m_2 + m_3 = 2n$ and $1 < m_i < n$. Here a circle with m_i inside stands for an $SU(m_i)$ gauge group.

This S_{2n} invariance can be interpreted as reflecting the S-duality invariance of the theories. Indeed, it has been argued in [29, 161] that the $\mathcal{T}^{(1)}_{1,n,1}$ theories are identical to the so-called (A_{2n-1}, A_{2n-1}) SCFTs [48], whose S-duality group acts on the flavor fugacities through S_{2n} [40]. Our formula (2.41) clarifies how this S_{2n} acts on the $(2n-1)$ flavor fugacities, $(\vec{a}, c, \vec{b})$, of $\mathcal{T}^{(1)}_{1,n,1}$.

As in the case of $\mathcal{T}^{(n)}_{0,n,0}$, other S-dual descriptions of our theories lead us to expressions for the Schur indices of a series of exotic type III AD fixtures. Indeed, by applying the technique developed in [161], we see that the $\mathcal{T}^{(1)}_{1,n,1}$ SCFTs have an S-dual description for each set (m_1, m_2, m_3) of integers such that $2 \leq m_i \leq 2n-4$ and $m_1 + m_2 + m_3 = 2n$. We focus on the case in which $2 \leq m_i < n$ for all $i = 1, 2, 3$ (we will discuss relaxing the condition that $m_i < n$ below). Then this dual description is characterized by the quiver diagram shown on the right of Fig. 23. The quiver has three tails corresponding to three $\mathcal{T}^{(1)}_{1,m_i}$ SCFTs, which are connected to the central node by an $SU(m_i)$ gauge group. The central node corresponds to an isolated type III AD theory labeled by three Young diagrams with n boxes $Y_1 = Y_2 = [m_1, m_2, m_3]$ and $Y_3 = [1, \cdots, 1]$, which we denote by $T^{2,\mathrm{AD}}_{(m_1,m_2,m_3)}$. The flavor symmetry of $T^{2,\mathrm{AD}}_{(m_1,m_2,m_3)}$ is generically $U(1)^2 \times \prod_{i=1}^{3} SU(m_i)$. From this S-dual description of $\mathcal{T}^{(1)}_{1,n,1}$, we see that its Schur index can also be written as

$$\mathcal{I}_{\mathcal{T}^{(1)}_{1,n,1}}(q; \vec{a}, c, \vec{b}) = \int \left(\prod_{i=1}^{3} d\mu(\mathbf{z}_i)\, \mathcal{I}^{SU(m_i)}_{\mathrm{vec}}(q; \mathbf{z}_i)\, \mathcal{I}_{\mathcal{T}^{(1)}_{1,m_i}}(q; \vec{s}_i, \mathbf{z}_i) \right) \mathcal{I}_{T^{2,\mathrm{AD}}_{(m_1,m_2,m_3)}}(q; \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, t_1, t_2) \,, \tag{2.42}$$

where $\vec{s}_i \equiv (s_{i,1}, \cdots, s_{i,m_i-1})$, t_j are some functions of $\vec{a}, c$ and $\vec{b}$, and the last factor is the Schur index of $T^{2,\mathrm{AD}}_{(m_1,m_2,m_3)}$. This latter index has not been worked out in the literature before.

The Lagrangian counterpart, $\mathcal{L}^{(1)}_{1,n,1}$, has a similar S-dual frame described by the quiver diagram on the left of Fig. 23 [79], where the gauge group is the same but each tail now corresponds to $\mathcal{L}^{(1)}_{1,m_i}$. The central node now stands for the theory obtained by compactifying the 6d (2,0) A_{n-1} theory on a sphere with three regular punctures associated with Young diagrams $Y^{(1)}_{m_1}$ [79, 49], which we call the $T_{(m_1,m_2,m_3)}$ theory. The flavor symmetry of $T_{(m_1,m_2,m_3)}$

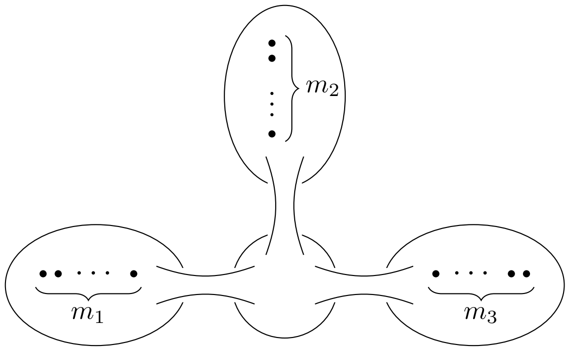


Figure 24: The decomposition of the punctured sphere corresponding to the S-dual description of $\mathcal{L}^{(1)}_{1,n,1}$ shown on the left of Fig. 23. The i-th tail contains m_i (simple) punctures and corresponds to $\mathcal{L}^{(1)}_{1,m_i}$. The three cylinders correspond to $SU(m_i)$ vector multiplets. The middle sphere corresponds to $T_{(m_1,m_2,m_3)}$.

contains $\prod_{i=1}^3 U(m_i)$, and the diagonal $U(1)$ enhances to $SU(2)$.[62] This description of $\mathcal{L}^{(1)}_{1,n,1}$ corresponds to a decomposition of the punctured sphere as in Fig. 24 (note that the punctures are all simple punctures when $\ell = n_1 = n_2 = 1$). This S-dual description implies that the Schur indices of $\mathcal{L}^{(1)}_{1,n,1}$ have integral expressions similar to (2.42) but with the indices of $\mathcal{L}^{(1)}_{1,m_i}$ and $T_{(m_1,m_2,m_3)}$ replacing those of $\mathcal{T}^{(1)}_{1,m_i}$ and $T^{2,\text{AD}}_{(m_1,m_2,m_3)}$. Note that the Schur indices of $T_{(m_1,m_2,m_3)}$ have already been written down in [77] as $\sum_R (C_R(q))^{-1} \prod_{i=1}^3 f_R^{Y^{(1)}_{m_i}}(q;\mathbf{z}_i,v_i)$, where the sum runs over irreducible representations of $\mathfrak{su}(n)$, and $\mathbf{z}_i$ and v_i are fugacities for $SU(m_i) \subset U(m_i)$ and $U(1) \subset U(m_i)$, respectively. Using (2.19), (2.15), and (B.2), one can translate this integral expression for $\mathcal{I}_{\mathcal{L}^{(1)}_{1,n,1}}$ into the following formula for $\mathcal{I}_{\mathcal{T}^{(1)}_{1,n,1}}$:

$$\mathcal{I}_{\mathcal{T}^{(1)}_{1,n,1}}(q;\vec{a},c,\vec{b}) = \frac{1}{(q;q^2)} \int \left(\prod_{i=1}^3 d\mu(\mathbf{z}_i)\, \mathcal{I}^{SU(m_i)}_{\text{vec}}(q;\mathbf{z}_i)\, \mathcal{I}_{\mathcal{T}^{(1)}_{1,m_i}}(q;\vec{u}_i,\mathbf{z}_i) \right) \\ \times P.E.\left[\frac{q}{1-q^2} \sum_{i=1}^3 \chi^{SU(m_i)}_{\text{adj}}(\mathbf{z}_i) \right] \sum_{R\text{: irrep of } \mathfrak{su}(n)} \frac{\prod_{i=1}^3 f_R^{Y^{(1)}_{m_i}}(q^2;\mathbf{z}_i, q^{\frac{m_i}{2n}} v_i)}{C_R(q^2)} \ , \tag{2.43}$$

where $\mathbf{z}_i$ is an $SU(m_i)$ fugacity, and $\vec{u}_i \equiv (u_{i,1},\cdots,u_{i,m_i-1})$ and v_i are $U(1)$ fugacities related to e_k and f_k by

$$(u_{i,k})^{k(k+1)} = \frac{(g_{i,k+1})^{kn}}{(g_{i,1}\cdots g_{i,k})^n} \ , \qquad v_i = q^{-\frac{m_i}{2n}} \prod_{k=1}^{m_i} g_{i,k} \ , \tag{2.44}$$

[62]There can be additional enhancements when $n = m_i + 1$ for at least one i. If this statement holds for all i, then we get the usual E_6 SCFT (i.e., $T_{(2,2,2)} = T_3$).

with $(g_{1,1},\cdots,g_{1,m_1},g_{2,1},\cdots,g_{2,m_2},g_{3,1},\cdots,g_{3,m_3})=(e_0,\cdots,e_{n-1},f_0,\cdots,f_{n-1})$. From (2.23) and (2.24), we see that $u_{i,k}$ and v_i are functions only of flavor fugacities $\vec{a},\vec{b}$ and c and are therefore independent of q. Note also that, since $v_1v_2v_3=1$, only two of the v_i are independent.

We now see that the two expressions (2.42) and (2.43) are consistent if $\vec{s}_i=\vec{u}_i$, $t_i=v_i$ and the Schur index of $T^{2,\mathrm{AD}}_{(m_1,m_2,m_3)}$ is given by

$$\begin{aligned}&\mathcal{I}_{T^{2,\mathrm{AD}}_{(m_1,m_2,m_3)}}(q;\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3,t_1,t_2)\\ &\quad=P.E.\left[\frac{q}{1-q^2}\left(1+\sum_{i=1}^{3}\chi^{SU(m_i)}_{\mathrm{adj}}(\mathbf{z}_i)\right)\right]\sum_{R:\text{ irrep of }\mathfrak{su}(n)}\frac{\prod_{i=1}^{3}f_R^{Y^{(1)}_{m_i}}(q^2;\mathbf{z}_i,q^{\frac{m_i}{2n}}t_i)}{C_R(q^2)}\;,\end{aligned}\tag{2.45}$$

with $t_3\equiv\frac{1}{t_1t_2}$ (as in the case of the $R^{2,AD}_{0,n}$ SCFTs, this fugacity dependence will have consequences for the corresponding 3D mirrors to be discussed in the next section). While we don't have a full proof that this is the only expression consistent with (2.42) and (2.43), we see that it gives a physically meaningful result, since there are only integer and half-integer powers of q (which is necessary for the quantity to be a Schur index of an $\mathcal{N}=2$ SCFT), and it has the expected S_3 symmetry acting on the $\mathbf{z}_i$ and t_i.

Finally, let us note that the expression in (2.43) assumes that $m_i<n$ (at least for the corresponding 4D regular puncture theory to make sense). Indeed, for $m_i\geq n$, we would end up with a Young diagram with m_i columns of height one and one column of non-positive height, $n-m_i\leq 0$.[63]

On the other hand, the expression in (2.42) may in principle make sense for $m_i\geq n$. It would be interesting to understand if we can analytically continue the expression in (2.43) to the regime of $m_i\geq n$ and understand the corresponding regular puncture theory, $T_{(m_1,m_2,m_3)}$, as a non-unitary 4D theory (perhaps generalizing the discussion in [6, 95, 96, 155, 62]).

2.4 RG flows to thirty-two supercharges

2.4.1 Wave function relations and topology of 3D mirrors

In this section, we interpret the TQFT formulas (2.38) and (2.45) for the Schur indices of the $R^{2,\mathrm{AD}}_{0,n}$ and $T^{2,\mathrm{AD}}_{(m_1,m_2,m_3)}$ SCFTs in terms of the corresponding 3D mirrors given in Fig. 26 and Fig. 28 respectively. In the following subsection, we argue that this discussion implies the existence of RG flows with accidental SUSY enhancement to thirty-two (Poincaré plus special) supercharges.

We begin by discussing the TQFT formula for the $R^{2,\mathrm{AD}}_{0,n}$ index, which we reproduce below

[63] If $m_i>n$ for some $i\in\{1,2,3\}$, the decomposition of the punctured sphere shown in Fig. 24 leads to a different S-dual description of $\mathcal{L}^{(1)}_{1,n,1}$ from the one described by the left quiver of Fig. 23. In particular, the central three-punctured sphere corresponds to a different fixture from $T_{(m_1,m_2,m_3)}$. It would be interesting to find an AD analog of this class $\mathcal{S}$ fixture.

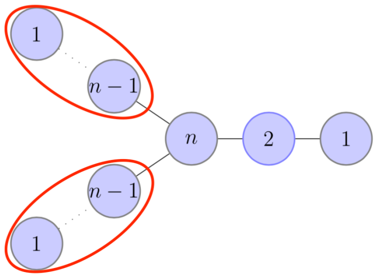


Figure 25: The 3D mirror of the S^1 reduction of the $R_{0,n}$ SCFT. Nodes labeled by "N" represent $U(N)$ gauge nodes and lines between nodes denote bifundamental hypermultiplets (an overall decoupled $U(1)$ is removed). The $R_{0,n}$ index has a TQFT expression with three independent wave functions corresponding to the three quiver tails in the above diagram of the 3D mirror. The two quiver tails circled in red generate monopoles which are responsible for the $SU(n)^2 \subset SU(2n) \times SU(2)$ flavor symmetry of the theory (the $SU(2) \subset SU(2n) \times SU(2)$ factor comes from the third tail, and the balanced central node is responsible for the $U(1) \times SU(n)^2 \to SU(2n) \subset SU(2n) \times SU(2)$ enhancement). When we perform the transformation that takes us from the Schur index of $R_{0,n}$ to that of $R_{0,n}^{2,\mathrm{AD}}$, the two $SU(n)$ tails fuse to form a single $SU(n)$ line of nodes as in Fig. 26.

for ease of reference

$$\mathcal{I}_{R_{0,n}^{2,\mathrm{AD}}}(q;\mathbf{z},\mathbf{y}) = P.E.\left[\frac{q}{1-q^2}\left(-1+\chi_{\mathrm{adj}}^{SU(2)}(\mathbf{z})\right)\right]\mathcal{I}_{R_{0,n}}(q^2;\mathbf{z},q,\mathbf{y},\mathbf{y}^*)\ . \tag{2.46}$$

Using the expression for $\mathcal{I}_{R_{0,n}}$ (2.34) we then have

$$\mathcal{I}_{R_{0,n}^{2,\mathrm{AD}}}(q;\mathbf{z},\mathbf{y}) = \frac{1}{(zq;q^2)(z^{-1}q;q^2)}\sum_{R:\text{ irrep of }\mathfrak{su}(n)}\frac{f_R^{Y_2^{(1)}}(q^2;\mathbf{z},q)f_R^{Y_{\mathrm{full}}}(q^2;\mathbf{y})f_R^{Y_{\mathrm{full}}}(q^2;\mathbf{y}^*)}{C_R(q^2)}\ . \tag{2.47}$$

Let us pay special attention to the transformation on the flavor fugacities when we go from the TQFT expression for $R_{0,n}$ to that for $R_{0,n}^{2,AD}$. At the level of flavor symmetries, recall that $R_{0,n}$ has a $G_{R_{0,n}} = SU(2) \times SU(2n)$ flavor symmetry (which is enhanced to E_6 for $n = 3$) [49]. On the other hand, $R_{0,n}^{2,AD}$ has flavor symmetry $G_{R_{0,n}^{2,AD}} = SU(2) \times SU(n)$ (which is enhanced to $SU(2)^2 \times SU(3)$ for $n = 3$).

The TQFT expression in (2.34) makes manifest a $U(2) \times SU(n)^2 \subset G_{R_{0,n}}$ flavor subgroup via the wave function with $U(2)$ symmetry, $f_R^{Y_2^{(1)}}(q,\mathbf{z},r)$, and the two wave functions with $SU(n)$ symmetry, $f_R^{Y_{\mathrm{full}}}(q,\mathbf{w_1})$ and $f_R^{Y_{\mathrm{full}}}(q,\mathbf{w_2^*})$. These wave functions, and the flavor symmetries they describe, are related to punctures in the A_{n-1} $(2,0)$ theory. The punctures appear in the 3D mirrors of the S^1 compactifications of our 4D theories via the presence of certain quiver tails radiating off a central $SU(n)$ node as in Fig. 25 [49, 19]. In particular,

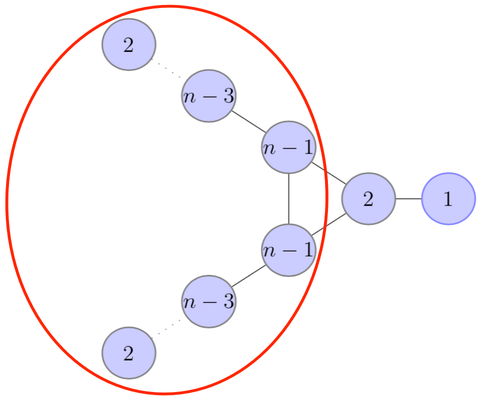


Figure 26: The 3D mirror of the S^1 reduction of the $R_{0,n}^{2,AD}$ theory. Nodes and lines are defined as in Fig. 25. The two quiver tails corresponding to the TQFT wave functions with conjugate fugacities are fused together to give one linear set of nodes generating an $SU(n)$ symmetry. The corresponding nodes are inside the red oval. The remaining quiver tail gives an $SU(2)$ symmetry and is symmetrically fused with the $SU(n)$ nodes via an unbalanced $SU(2)$ node. As a result, the quiver contains a closed non-abelian loop, and the theory can flow to an interacting $\mathcal{N}=8$ SCFT via the procedure described in the text.

the two tails with gauge groups $U(n-1)\times\cdots\times U(1)$ correspond to punctures described by the $f_R^{Y_{\text{full}}}$ wave functions, while the tail with gauge group $U(2)\times U(1)$ corresponds to the puncture described by $f_R^{Y_2^{(1)}}$. Indeed, by the linear quiver rules given in [81], the dimension one monopole operators with fluxes supported on, say, one of the $U(n-1)\times\cdots\times U(1)$ tails give rise to multiplets containing the additional symmetry currents that enhance the corresponding $U(1)^{n-1}$ topological symmetry to $SU(n)$.[64] This statement follows from the fact that the corresponding line of nodes is "balanced," i.e., each $U(n_c)$ node has $n_f=2n_c$ flavors. A similar phenomenon occurs in the other $U(n-1)\times\cdots\times U(1)$ tail and the $U(2)\times U(1)$ tail, thereby giving rise to the $U(2)\times SU(n)^2\subset G_{R_{0,n}}$ non-abelian symmetry (the $U(1)\times SU(n)^2\to SU(2n)$ enhancement occurs because of monopole operators with flux through the central $U(n)$ node).

Given this discussion and the relations between (2.34) and (2.47), let us give an explanation for the form of the quiver tails for the 3D mirror of $R_{0,n}^{2,AD}$ shown in Fig. 26. First, note that the two independent $SU(n)$ TQFT $R_{0,n}$ wave functions in (2.34) are no longer independent in (2.47). Indeed, we must set $w_1=w_2=y$ (in addition to taking $q\to q^2$) and so there is just one independent set of $SU(n)$ fugacities. Since the two $SU(n)$ wave

[64]Recall that any 3D $U(n_c)$ gauge group has a corresponding topological symmetry current, $j_\mu=\epsilon_{\mu\nu\rho}F^{\nu\rho}$, where $F^{\nu\rho}$ is the field strength corresponding to the trace part of $U(n_c)$. Note that this is a global flavor symmetry acting on the Coulomb branch. In the direct reduction (i.e., the mirror of the mirror quivers we are discussing), the topological symmetry (along with any additional enhanced symmetry via monopole operators) acts on the Higgs branch and descends from the usual 4D flavor symmetry.

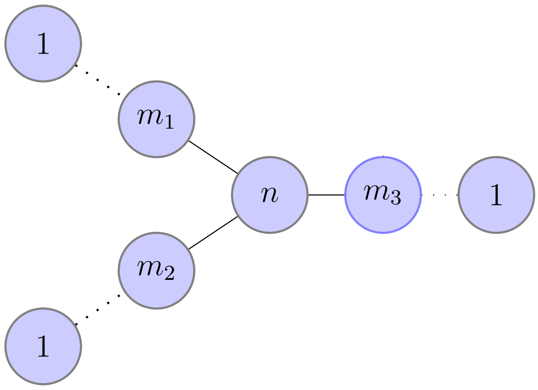


Figure 27: The 3D mirror of the S^1 reduction of the $T_{(m_1,m_2,m_3)}$ theory. Nodes and lines are defined as in Fig. 25. The three quiver tails correspond to TQFT wave functions carrying $U(m_i)$ global symmetry.

functions are no longer independent, it is natural that in going from Fig. 25 to Fig. 26 we should fuse the two previously independent quiver tails into a single tail giving rise to a single $SU(n)$ symmetry.[65] Indeed, note that the line of nodes in the red oval in Fig. 26 have a bifundamental connecting the two previously independent tails and consist of $n-1$ total balanced nodes. By the rules of [81], this line of nodes gives rise to the $SU(n) \subset G_{R^{2,AD}_{0,n}}$ symmetry.

Since the two previously independent $SU(n)$ wave functions are now related by complex conjugation of fugacities, non-chirality demands that that their corresponding line of nodes connects to the quiver tail corresponding to the $SU(2)$ wave function in a symmetric fashion. Indeed, the loop of nodes appearing in Fig. 26 is precisely such a symmetric connection. The shortening of the remaining tail reduces the $U(2)$ global symmetry factor to $SU(2)$ and also ensures that the line of nodes generating the $SU(n)$ symmetry are indeed balanced. Note that this loop topology of the $R^{2,AD}_{0,n}$ quiver will be important in arguing for flows to theories with thirty-two (Poincaré plus special) supercharges in the next section.

Next let us discuss the case of $T^{2,AD}_{(m_1,m_2,m_3)}$. For ease of reference, we again write the TQFT

[65] In fact, since the wave functions have conjugate fugacities, it is tempting to write $f_R^{Y_{\rm full}}(q^2,y^*) = f_{\bar{R}}^{Y_{\rm full}}(q^2,y)$, where $\bar{R}$ is the $SU(n)$ representation conjugate to R. We may then write the product of $SU(n)$ wave functions in (2.47) as

$$f_R^{Y_{\rm full}}(q^2,\mathbf{y})f_R^{Y_{\rm full}}(q^2,\mathbf{y}^*) = f_R^{Y_{\rm full}}(q^2,\mathbf{y})f_{\bar{R}}^{Y_{\rm full}}(q^2,\mathbf{y}) = \mathcal{I}_V^{-\frac{1}{2}}(q^2,\mathbf{y}) \sum_{R'\in R\otimes\bar{R}} f_{R'}^{Y_{\rm full}}(q^2,\mathbf{y}) \ , \tag{2.48}$$

where $\mathcal{I}_V$ is the Schur index of the $SU(n)$ vector multiplet. The appearance of a single wave function suggests that the $SU(n)$ symmetry should be associated with a single line of nodes in the 3D mirror. Moreover, the additional inverse factor of $\mathcal{I}_V^{-\frac{1}{2}}$ reminds us that this symmetry was associated with two punctures in the original regular puncture theory. At the level of the mirror quiver, this factor reflects the fact that the ranks of the gauge groups in the red oval increase by two between successive nodes in the tails.

expression for the Schur indices of these theories originally appearing in (2.45)

$$\begin{aligned}
&\mathcal{I}_{T^{2,\mathrm{AD}}_{(m_1,m_2,m_3)}}(q;\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3,t_1,t_2) \\
&\quad = P.E.\left[\frac{q}{1-q^2}\left(1+\sum_{i=1}^{3}\chi^{SU(m_i)}_{\mathrm{adj}}(\mathbf{z}_i)\right)\right]\sum_{R\text{: irrep of }\mathfrak{su}(n)}\frac{\prod_{i=1}^{3} f_R^{Y^{(1)}_{m_i}}(q^2;\mathbf{z}_i,q^{\frac{m_i}{2n}}t_i)}{C_R(q^2)}\,, \qquad (2.49)
\end{aligned}$$

where the $U(1)$ fugacities are constrained to satisfy $t_3 = \frac{1}{t_1 t_2}$ (i.e., there is only a $U(1)^2$ abelian symmetry), and $\mathbf{z_i}$ are $SU(m_i)$ fugacities. In the case of the $T_{(m_1,m_2,m_3)}$ theory, we have a generic global symmetry group $G_{T_{(m_1,m_2,m_3)}} \supset U(m_1)\times U(m_2)\times U(m_3)$ (the diagonal $U(1)$ enhances to $SU(2)$), while for the $T^{2,AD}_{(m_1,m_2,m_3)}$ theory, we have $G_{T^{2,AD}_{(m_1,m_2,m_3)}} = SU(m_1)\times SU(m_2)\times SU(m_3)\times U(1)^2$.

The correspondence between TQFT wave functions and quiver tails is clear in the case of the 3D mirror of the reduction of $T_{(m_1,m_2,m_3)}$ in Fig. 27: each $f_R^{Y^{(1)}_{m_i}}$ wave function corresponds to an independent $U(m_i)\times\cdots\times U(1)$ quiver tail of balanced nodes which, by the rules of [81] gives rise to monopole operators leading to the $U(1)^{m_i}\to U(m_i)$ flavor enhancement (again, this statement holds assuming generic m_i such that $m_1+m_2+m_3=2n$ and $m_i<n$).

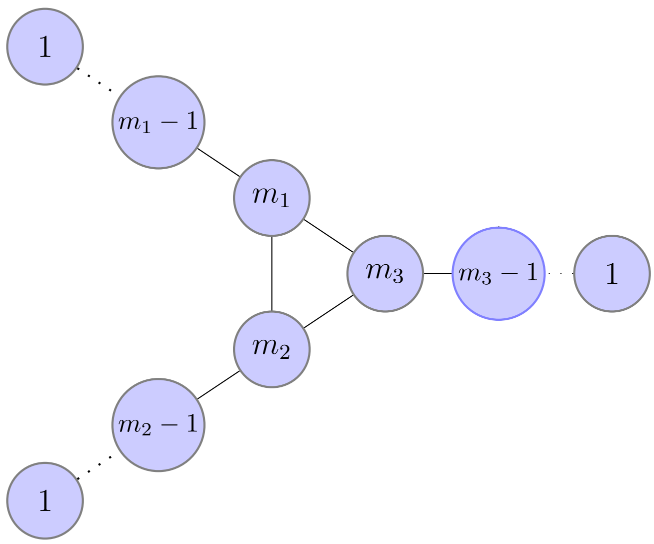


Figure 28: The 3D mirror of the S^1 reduction of the $T^{2,AD}_{(m_1,m_2,m_3)}$ SCFT. Nodes and lines are defined as in Fig. 25. The three $U(m_i)\times\cdots\times U(1)$ quiver tails generate independent $SU(m_i)$ fugacities and correspond to the independent $SU(m_i)$ parts of the three TQFT wave functions. On the other hand, the $U(1)$ parts of the three TQFT wave functions are symmetrically dependent. This dependence is reflected in the quiver by the loop of three $U(m_i)$ nodes.

On the other hand, the $T^{2,AD}_{(m_1,m_2,m_3)}$ theory no longer has independent wave functions carrying $U(m_i)$ flavor symmetry since the t_i fugacities in (2.49) are constrained so that

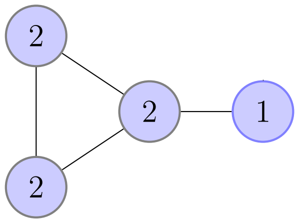


Figure 29: The mirror quiver obtained after performing the Coulomb branch flow from Fig. 26 described in the main text. Nodes and lines are defined as in Fig. 25.

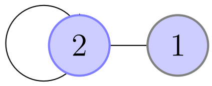


Figure 30: The mirror quiver obtained after performing the Higgs branch flow from Fig. 29 described in the main text (we drop decoupled hypermultiplets). The circular line is an adjoint hypermultiplet, and the remaining lines and nodes are defined as in Fig. 25. This theory flows to 3D $\mathcal{N} = 8$ in the IR.

$t_3 = \frac{1}{t_1 t_2}$. Indeed, only the $SU(m_i)$ parts of the wave functions are still independent. We can then give an argument in favor of the 3D mirror of the S^1 reduction of $T_n^{2,AD}$ shown in Fig. 28. The point is that the three balanced $U(m_i - 1) \times \cdots U(1)$ tails correspond to the independent $SU(m_i)$ parts of the TQFT wave functions, while the loop of three $U(m_i)$ nodes appears because of the constraint on the $U(1)$ parts of the TQFTs wave functions. Again, this difference in the topology of the AD mirror relative to the $T_{(m_1,m_2,m_3)}$ mirror gives rise to the RG flows to theories with thirty-two supercharges that will be discussed further in the next section.

2.4.2 Flows to thirty-two supercharges

As alluded to in the previous section and also in the introduction, one important characteristic of the isolated AD fixtures we are discussing is that, unlike the regular puncture theories they are related to, the AD theories have RG flows (triggered by vevs and genuinely relevant deformations) with accidental SUSY enhancement to interacting theories with thirty-two (Poincaré plus special) supercharges (thereby generalizing the examples in [33]). In the next section, we will argue that such flows are in fact generic in the landscape of AD theories with known 3D mirrors. Note that these flows will proceed via reduction to 3D and via flowing onto the moduli spaces of the resulting theories. We briefly discuss the possibility of uplifting this discussion to 4D at the end of this subsection while postponing a more detailed analysis for future work.

To first understand why the RG flows to interacting theories with thirty-two supercharges occur in the $R_{0,n}^{2,AD}$ and $T_{(m_1,m_2,m_3)}^{2,AD}$ theories discussed above, it is sufficient to compactify these theories on S^1 and consider the corresponding 3D mirrors. Let us start with the mirror in Fig. 26. Flowing to generic points on the Coulomb branch of the two lines of nodes with gauge groups $U(2) \times U(4) \times \cdots \times U(n-3)$ and also onto points of the Coulomb branch of $U(n-1) \times U(n-1)$ with symmetry breaking pattern $U(n-1) \times U(n-1) \to U(2) \times U(2) \times U(1)^{2(n-3)}$, we obtain the quiver in Fig. 29, where we have dropped decoupled $U(1)$ factors. This is the mirror of the lowest rank theory studied in [33], which we know from that reference flows to $\mathcal{N}=8$ via mass terms in the direct reduction. However, it will be useful for our more general discussion below to analyze a purely moduli space flow to $\mathcal{N}=8$ in the mirror theory itself.[66] To that end, consider turning on Higgs branch vevs

$$\langle Q_1 \widetilde{Q}_1 \rangle = \langle Q_2 \widetilde{Q}_2 \rangle = \langle Q_3 \widetilde{Q}_3 \rangle \neq 0 \ , \tag{2.50}$$

where the $Q_i, \widetilde{Q}_i$ pairs correspond to the three edges in the loop of Fig. 29 so that we break $U(2)^3 \to U(2)_{\text{diag}}$ leaving the quiver in Fig. 30 after dropping decoupled fields.[67] To see this, notice that the bifundamentals $Q_i, \widetilde{Q}_i$'s all carry non-trivial representations for two out of the three $U(2)$ factors, we may label them as $(\mathbf{1}, \mathbf{2}, \bar{\mathbf{2}}), (\mathbf{2}, \mathbf{1}, \bar{\mathbf{2}}), (\mathbf{2}, \bar{\mathbf{2}}, \mathbf{1})$ and similarly for their conjugates with $\mathbf{1}, \mathbf{2}, \bar{\mathbf{2}}$ as the trivial/fundamental/anti-fundamental representations respectively. Under the action of a generic group element (g_1, g_2, g_3) of $U(2)^3$, the common VEV u of $\langle Q_1 \widetilde{Q}_1 \rangle, \langle Q_2 \widetilde{Q}_2 \rangle, \langle Q_3 \widetilde{Q}_3 \rangle$ will transform in three ways as $g_2 g_3^{-1} u, g_1 g_3^{-1} u, g_1 g_2^{-1} u$, for consistency we must have $g_1 = g_2 = g_3$. Then we are left with the representation $\mathbf{2} \otimes \bar{\mathbf{2}}$ in $U(2)_{diag}$, but that is precisely the adjoint representation, including the conjugate we have the adjoint hypermultiplet, which is Fig. 30.[68] Similarly this argument generalizes to Fig. 31 and Fig. 32, where we have $U(m_1)$ instead of $U(2)$, but $\mathbf{adj} = \mathbf{n} \otimes \bar{\mathbf{n}}$ is true in all $U(n)$.

In terms of the squark fields, we may imagine turning on vevs

$$\langle Q_i \rangle = \langle \widetilde{Q}_i \rangle = v \mathbb{1}_{2\times 2} \neq 0 \ , \tag{2.51}$$

for $i = 1, 2, 3$. This latter theory flows directly to $\mathcal{N}=8$ in the IR. Therefore, we see that through a combination of Coulomb and Higgs branch flows in the mirror theory, we flow to an interacting 3D $\mathcal{N}=8$ SCFT.

Now consider the $T_{(m_1,m_2,m_3)}^{2,AD}$ SCFTs. Since the $T_{(m_1,m_2,m_3)}^{2,AD}$ mirror clearly flows to the $R_{0,n}^{2,AD}$ mirror via excursions on the Coulomb branch, we can appeal to the above discussion and conclude that there are flows from $T_{(m_1,m_2,m_3)}^{2,AD}$ to theories with thirty-two supercharges. On the other hand, the $T_{(m_1,m_2,m_3)}^{2,AD}$ theories also admit other flows to a richer set of $\mathcal{N}=8$ theories, which we now describe.

[66] The mirror analog of the flow in [33] proceeds by turning on Fayet-Iliopoulos terms.

[67] In the direct reduction, this maneuver corresponds to turning on vevs for the overall $U(1) \subset U(2)$ vector multiplet primary.

[68] Notice that we have an adjoint hypermultiplet of $U(2)$, but at the level of Lie algebra we have $\mathfrak{u}(2) = \mathfrak{u}(1) \oplus \mathfrak{su}(2)$ so we can drop further the decoupled $\mathfrak{u}(1)$ factor and keep the $\mathfrak{su}(2)$ adjoint hypermultiplet only.

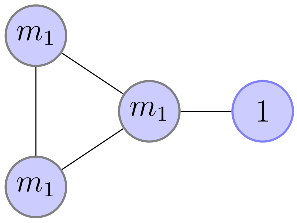


Figure 31: The mirror quiver obtained after performing the Coulomb branch flow from Fig. 28 described in the main text. Nodes and lines are defined as in Fig. 25.

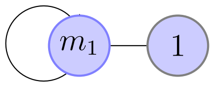


Figure 32: The mirror quiver obtained after performing the Higgs branch flow from Fig. 31 described in the main text (we drop decoupled hypermultiplets). The circular line is an adjoint hypermultiplet, and the remaining lines and nodes are defined as in Fig. 25. This theory flows to 3D $\mathcal{N} = 8$ in the IR.

Without loss of generailty, we will assume that $m_3 \geq m_2 \geq m_1$. Now, let us flow to points on the Coulomb branches of the two quiver tails of length m_i with $i = 3, 2$ such that the corresponding gauge groups break as $U(m_i) \times U(m_i - 1) \times \cdots \times U(1) \to U(m_1) \times U(1)^{m_i - m_1} \times U(1)^{m_i - 1} \times \cdots \times U(1)$ (where the ellipses on the RHS of the breaking contain only abelian gauge groups). Simultaneously, we flow to a generic point on a Coulomb sub-branch in the third tail specified by $SU(m_1 - 1) \times U(m_1 - 2) \times \cdots \times U(1)$ and obtain the theory in Fig. 31. We can then turn on vevs as in (2.50) where the $Q_i, \widetilde{Q}_i$ pairs are now bifundamentals of $U(m_1) \times U(m_1)$.[69] This procedure produces the interacting $\mathcal{N} = 8$ theory described in Fig. 32.

Note that in all RG flows described in this subsection, we flow on both the Coulomb and Higgs branches of the 3D mirror. Therefore, by mirror symmetry, in order to reach the $\mathcal{N} = 8$ fixed points, we flow on both the Higgs and Coulomb branches of the direct S^1 reductions of our 4D SCFTs. It would be interesting to understand if these flows uplift to 4D flows along the Higgs and Coulomb branches of our 4D theories (i.e., if the corresponding 4D RG flows commute with the S^1 reduction as in [33]).

If the flows do uplift, then it would also be interesting to understand if the 3D $\mathcal{N} = 8$ fixed points map to $\mathcal{N} = 4$ theories in 4D. In principle, if the flows are well behaved enough, then the detailed properties of these possible $\mathcal{N} = 4$ fixed points—e.g., if they are of Super-Yang Mills (SYM) type or not—can be studied.

[69] In analogy with the previous case, we turn on vevs $\langle Q_i \rangle = \langle \widetilde{Q}_i \rangle = v \mathbb{1}_{m_1 \times m_1} \neq 0$ for all $i = 1, 2, 3$.

Universality of flows to interacting SCFTs with thirty-two supercharges In this section, we briefly state and prove a theorem governing how universally we may expect the existence of RG flows to interacting theories with thirty-two (Poincaré plus special) supercharges. This discussion is motivated by our TQFT formulae for the Schur indices of the $R_{0,n}^{2,AD}$ and $T_{(m_1,m_2,m_3)}^{2,AD}$ theories and our reinterpretation of these formulae as leading to closed loops of non-abelian nodes in the corresponding 3D mirrors. Indeed, we saw that the existence of such closed loops generically led to RG flows ending on interacting SCFTs with thirty-two supercharges.

Combined with the infinite class of examples in [33], it is then tempting to wonder whether such flows are generic in the class of (untwisted) type III theories (and therefore, perhaps, in the space of $\mathcal{N}=2$ theories coming from compactifications of the $(2,0)$ theory on surfaces with untwisted punctures). In fact, it is straightforward to show this is the case, if we assume the classification of such theories given in [162, 163]. In this classification, the space of type III theories is specified by $N\geq 2$ Young diagrams (the theories discussed above have $N=3$). The $N=2$ theories cannot flow to theories with thirty-two supercharges (we do not consider turning on additional gauge couplings in the UV), and so we focus on the more generic theories with $N\geq 3$.[70] The Young diagrams in question take the form [162, 163]

$$Y_1=Y_2=\cdots=Y_{N-1}=[h_1,h_2,\cdots,h_p]\ ,\quad Y_N=[a_{1,1},\cdots,a_{1,n_1},a_{2,1}\cdots,a_{2,n_2}\cdots a_{p,n_p}]\ ,\tag{2.52}$$

where the column heights h_i and $a_{i,b}$ are non-decreasing (from left to right) positive integers satisfying

$$\sum_{b=1}^{n_b}a_{i,b}=h_i\ .\tag{2.53}$$

The above Young diagrams correspond to the degeneracy of the eigenvalues of the singular terms in the Higgs field one obtains in the Hitchin system describing the type III compactification [158] (although note that in our conventions Y_1 corresponds to the *most* singular piece). At the level of the 3D mirror, the quiver consists of a core with gauge group

$$G=U(h_1)\times U(h_2)\times\cdots\times U(h_p)\ ,\tag{2.54}$$

and $N-2$ bifundamentals between each node.[71] The final Young diagram, Y_N, describes the quiver tails. For example, if the column of height h_b is broken up into $[a_{b,1},\cdots,a_{b,n_b}]$, we attach a tail to $U(h_b)$ with gauge group

$$G_b^{\rm tail}=U(h_b-a_{b,1})\times U(h_b-a_{b,1}-a_{b,2})\times\cdots\times U(h_b-a_{b,1}-\cdots-a_{b,n_{b-1}})\ ,\tag{2.55}$$

[70]Interestingly if one adds a regular singularity one finds, among the $N=2$ theories, 3D mirrors equivalent to the star-shaped quivers found in the case of some theories with regular punctures (and no irregular punctures).

[71]In the case of the $R_{0,n}^{2,AD}$ and $T_{(m_1,m_2,m_3)}^{2,AD}$ theories, the cores are the triangular loops in Fig. 26 and Fig. 28 respectively.

and bifundamentals between each corresponding node (and also a single bifundamental between the $U(h_b - a_{b,1})$ and $U(h_b)$ node). One repeats this procedure for all $b \in \{1, \cdots, p\}$. Given this setup and assumptions, we can prove the following theorem on the universality of non-perturbative flows from sixteen to thirty-two supercharges:

Theorem 2.1. *If the quantities h_3 and n_1 in (2.52) satisfy $h_3, n_1 > 1$, the corresponding type III SCFT flows, up to free decoupled factors, to an interacting theory with thirty-two (Poincaré plus special) supercharges upon compactification to 3D, flowing to certain points on the moduli space of the theory, and, for $N > 3$, turning on mass terms in the 3D mirror.*[72]

Proof: We would like to reduce the 3D mirror to the diagram in Fig. 31 with $m_1 = h_3 > 1$. To accomplish this task, we can first move along the Coulomb branch to reduce our theory to a diagram similar to the one in Fig. 31, but containing $N-2$ bifundamentals between each node. To get to this diagram, first go to generic points on the Coulomb branches of the subset of the core nodes (see (2.54)) characterized by $U(h_4) \times \cdots \times U(h_p) \subset G$ and to generic points on the Coulomb branches of all their tails (if any exist). Next, we go to generic points on the Coulomb branches of the tails of the $U(h_2) \times U(h_3)$ nodes to remove them as well. Then, we go to generic points on the Coulomb branch of the $U(h_1 - a_{1,1} - 1) \times \cdots \times U(h_1 - a_{1,1} - \cdots - a_{1,n_1})$ part of the $U(h_1)$ quiver tail. This procedure leaves us (up to decoupled $U(1)$ factors, which we drop) with a $U(h_1) \times U(h_2) \times U(h_3)$ group of core nodes connected by $N-2$ bifundamentals between each node and a $U(1)$ node connected to $U(h_1)$ via a fundamental. To proceed, we now go to a point on the $U(h_1) \times U(h_2)$ Coulomb branch that breaks the gauge symmetry as $U(h_1) \times U(h_2) \to U(h_3)^2 \times U(1)^{h_1 - h_3} \times U(1)^{h_2 - h_3}$. Up to decoupled $U(1)$'s, we have a diagram equivalent to that in Fig. 31 with $m_1 = h_3$ except for the fact that there are $N-2$ bifundamentals between each non-abelian node. We may add mass terms to remove $N-3$ of the bifundamentals between each node to end up with a diagram identical to the one in Fig. 31. Combined with the Higgs branch flow described below Fig. 31, we flow to an interacting $\mathcal{N} = 8$ theory. Therefore, if we are willing to go on the Coulomb and Higgs branches of the 3D mirror and, at the same time, add mass terms for some of the bifundamentals between the remaining non-abelian nodes, we flow to a theory with thirty-two supercharges.[73] **q.e.d.**

[72] The same caveats described at the end of the previous section apply in lifting these flows to 4D.

[73] Note that adding a regular singularity to the above set of theories does not change the above proof: we can decouple the additional nodes associated with this singularity via flowing to generic points on the corresponding Coulomb branches.

3 Fusion of non-abelian anyons I: Arad-Herzog conjecture

3.1 Arad-Herzog conjecture and its discrete gauge theory cousins

Non-abelian anyons are interesting for a variety of reasons. For example, they naturally appear in quantum field theory descriptions of knot theory [157], they are believed to play an important role in the fractional quantum Hall effect [121], and they underly a topological form of quantum computation [156]. More recently, they have attracted attention as providing possible lessons for quantum gravity [140].

We will be exclusively concerned with non-abelian anyons in a particular type of $2+1$-dimensional topological quantum field theory (TQFT): discrete gauge theories [63, 136]. As we have introduced in 1.1.1, these are gauge theories based on some discrete gauge group, G, along with a Dijkgraaf-Witten 3-cocycle, $\omega \in H^3(G, U(1))$ (when ω is cohomologically non-trivial, the theory is said to be "twisted"). The basic degrees of freedom are anyonic line operators (i.e., operators supported on one-dimensional loci of spacetime that have non-trivial braiding with each other) of three general types:

1. *Wilson lines*, which carry electric charge labeled by a linear irreducible representation of G, π. These operators have trivial magnetic charge.

2. *Magnetic flux lines* carrying magnetic charge labeled by a conjugacy class, $[g]$, of an element $g \in G$ with $g \neq 1$. These operators have trivial electric charge. Depending on the choice of ω, such operators may or may not exist.

3. *Dyonic lines* (or simply dyons) carrying a magnetic charge labeled by a conjugacy class, $[g]$, of an element $g \in G$ with $g \neq 1$ and an electric charge labeled by an, in general, projective representation of the centralizer of g, N_g. In the case of an untwisted gauge theory (i.e., $\omega = 0 \in H^3(G, U(1))$), the representation is linear. Dyons are the most generic type of anyons in discrete gauge theories.

As a physical toy model, one can think of dyons as Aharonov-Bohm systems with charges bound to magnetic flux lines [133].

Our first observation is that the above line operators naturally relate close cousins in group theory: representations to centralizers. Therefore, discrete gauge theory is a natural way to organize and unify ideas in the theory of finite groups.

We will focus exclusively on the case of finite simple groups. Via group extensions, these are the basic building blocks of all finite groups. The celebrated classification of finite simple groups guarantees that any such group fits into the following categories:

1. Abelian groups of prime order

2. Alternating groups

3. Lie groups over finite fields

4. Twenty-six sporadic groups

In spite of this classification, various open problems remain. Of particular interest to us is the following:

Conjecture (Arad-Herzog): Consider a non-abelian finite simple group, G, and non-trivial elements $g, h \in G$. Then,

$$[g] \cdot [h] \neq [gh] \tag{3.1}$$

where $[g]$, $[h]$, and $[gh]$ are conjugacy classes of g, h, and gh respectively [7].
More pithily, Arad and Herzog (AH) conjectured that in non-abelian finite simple groups, the product of non-trivial conjugacy classes cannot be a single conjugacy class.

As we will argue in section 3.2.1, this conjecture implies:

Theorem 3.1. *In a (twisted or untwisted) $2+1$-dimensional discrete gauge theory with a non-abelian finite simple gauge group, the fusion of any two lines carrying non-trivial magnetic flux cannot have a unique fusion outcome.*

In other words, theorem 3.1 asserts we cannot have

$$\mathcal{L}_{([g],\pi_g^\omega)} \times \mathcal{L}_{([h],\pi_h^\omega)} = \mathcal{L}_{([k],\pi_k^\omega)} \ , \quad g, h \neq 1 \tag{3.2}$$

where, generically, all lines (denoted by $\mathcal{L}$) are non-abelian dyons [74]. We will think of this theorem as a first cousin of the AH conjecture.

So far, we have avoided discussing the fusion of Wilson lines. However, in light of (3.2), it is interesting to ask if we can fuse non-abelian Wilson lines to obtain a unique outcome

$$\mathcal{W}_\pi \times \mathcal{W}_{\pi'} = \mathcal{W}_{\pi''} \tag{3.3}$$

As follows from our general discussion in section 1.1.1, (3.3) is equivalent to

$$\chi_\pi \cdot \chi_{\pi'} = \chi_{\pi''} \tag{3.4}$$

where $\chi_\pi, \chi_{\pi'}$, and $\chi_{\pi''}$ are the characters of irreducible linear representations, π, π', and π'', of G with dimension greater than 1. Although it might seem strange that (3.4) is possible (especially if one thinks of $SU(N)$), it turns out that products of irreducible representations of finite simple groups can be irreducible [165].

The corresponding (twisted or untwisted) discrete gauge theory then has a product of Wilson lines as in (3.3). One simple example of this phenomenon involves the fusion of a Wilson line carrying charge in the 8-dimensional representation of A_9 with a Wilson line carrying charge in either of the 21-dimensional representations. Intriguingly, discrete gauge theories based on finite simple groups are prime [127], so they do not consist of separate TQFTs with

[74] We may allow for pure fluxes as well.

trivial mutual braiding. Therefore, (3.3) corresponds to other structural properties of the A_9 discrete gauge theory. We discuss such properties further in chapter four.

Therefore, we learn that a version of the AH conjecture for characters alone cannot hold. However, our physical discussion above suggests studying one more type of fusion

$$\mathcal{W}_\pi \times \mathcal{L}_{([g],\pi_g^\omega)} = \mathcal{L}_{([h],\pi_h^\omega)} , \quad g \neq 1 \tag{3.5}$$

where $\mathcal{W}_\pi$ is a non-abelian Wilson line, and the remaining anyons are non-abelian dyons. As a simpler fusion, we may study

$$\mathcal{W}_\pi \times \mu_{[g]} = \mathcal{L}_{([h],\pi_h^\omega)} , \quad g \neq 1 \tag{3.6}$$

where we have replaced the dyon on the left-hand side of (3.5) with a non-abelian flux line. Here we have implicitly assumed that the flux line also exists in the theory (depending on the twist, this assumption may or may not hold).

This observation brings us to our second cousin of the AH conjecture:

Theorem 3.2. *In any (twisted or untwisted) discrete gauge theory based on a non-abelian finite simple group, G, fusion of the types in* (3.5) *and* (3.6) *is forbidden.*

Intuition: One heuristic intuition behind this theorem is the following. As a consequence of theorem 3.1, theorem 3.2 implies that in discrete gauge theories based on non-abelian simple groups, the only allowed fusions with unique outcomes involving non-abelian anyons are those in (3.3). Wilson lines have trivial braiding amongst themselves [75]. Therefore, even though the fusion in (3.3) does not arise from a factorization of the TQFT into separate theories with trivial mutual braiding, the Wilson lines themselves have trivial mutual braiding.

Just as theorem 1 follows from the AH conjecture, so too theorem 2 follows from a more basic theorem on finite simple groups which we refer to as the third cousin of the AH conjecture:

Theorem 3.3. *Consider any non-abelian finite simple group, G, any irreducible linear representation, π, of G having dimension greater than one, and the centralizer, N_g, of any $g \neq 1$. The restricted representation, $\pi|_{N_g}$, is reducible.*

We refer to theorems 3.1, 3.2, 3.3 as "cousins" of the AH conjecture since they are all related by TQFT.

Note that the above discussion is not relevant for abelian simple groups since they do not have conjugacy classes of length larger than one or representations of dimension larger than one. In other words, the corresponding TQFT fusion rules are those of a discrete finite group. As a result, we focus on non-abelian groups.

Duality: It is also interesting to understand how our above picture is compatible with a type of electric/magnetic duality that often features in discrete gauge theories. For example,

[75] Physically, this last statement is clear from the fact that Wilson lines do not carry magnetic flux.

the S_3 discrete gauge theory has a duality that exchanges the Wilson line charged under the 2-dimensional representation with the line having flux in the 3-cycle conjugacy class [17, 130]. More general examples appear in [130, 126, 99]. Clearly, theorems 3.1, 3.2, and 3.3 can only be compatible with such dualities if the Wilson lines participating in (3.3) are not exchanged with lines carying non-abelian flux. In fact, no such dualities exist in theories based on non-abelian finite simple gauge groups (**Proof:** apply theorem 5.8 of [126] noting that non-abelian simple groups have no non-trivial abelian normal subgroups). This fact is a non-trivial check of the above picture and of the AH conjecture (this latter claim holds since, if theorem 3.1 were not true, then the AH conjecture would be false).

In the next section, we first set up an useful lemma, then derive theorem 3.1 from the AH conjecture and explain the equivalence of theorems 3.2 and 3.3. Finally, we prove our theorems and provide some additional technical details 3.2.2.

3.2 Proofs

To begin, notice that non-abelian anyons have $d_{([g],\pi^\omega)} > 1$ and necessarily satisfy

$$([g],\pi_g^\omega)\times([g^{-1}],(\pi_g^\omega)^*) = ([1],1)+\cdots \tag{3.7}$$

where the ellipses must contain additional terms, 1 is the trivial representation of G, and $(([g^{-1}],(\pi_g^\omega)^*)$ is conjugate to $([g],\pi_g^\omega)$.

As we will see in more detail when we prove theorems 3.1 and 3.2, anyons $([g],\pi_g^\omega)$ and $([h],\pi_h^\omega)$ that fuse to give a unique outcome satisfy

$$|S_{([g]\pi_g^\omega),([h],\pi_h^\omega)}| = \frac{1}{|G|}d_{([g],\pi_g^\omega)}d_{([h],\pi_h^\omega)} \tag{3.8}$$

Here we will use (3.8) to prove the following lemma that will star in our proofs of theorems 3.1 and 3.2:

Lemma 3.4. *The condition* (3.8) *is satisfied iff the conjugacy classes* $[g]$ *and* $[h]$ *commute element-wise and the projective characters satisfy*

$$|\chi_{\pi_g^\omega}(l)| = deg\ \pi_g^\omega \ \ and\ \ |\chi_{\pi_h^\omega}(k)| = deg\ \pi_h^\omega \tag{3.9}$$

$\forall\ l\in[h], k\in[g]$.

Proof: To proceed it is useful to first recall the formula for the modular S matrix

$$S_{([g],\pi_g^\omega),([h],\pi_h^\omega)} \ \ = \ \ \frac{1}{|G|}\sum_{\substack{k\in[g],\ \ell\in[h],\\ k\ell=\ell k}} \chi^k_{\pi_g^\omega}(\ell)^*\chi^\ell_{\pi_h^\omega}(k)^* \ \ , \tag{3.10}$$

where $\chi^h_{\pi_g^\omega}(\ell)$ is defined through the relation

$$\chi^{xgx^{-1}}_{\pi_g^\omega}(xhx^{-1}) := \frac{\eta_g(x^{-1},xhx^{-1})}{\eta_g(h,x^{-1})}\chi_{\pi_g^\omega}(h)\ . \tag{3.11}$$

More importantly, from these definitions, one can check that the quantum dimensions of the anyons are

$$d_{([g],\pi_g^\omega)} = \frac{S_{([g],\pi_g^\omega)([1],1)}}{S_{([1],1)([1],1)}} = |[g]| \cdot \deg\, \pi_g^\omega \ , \tag{3.12}$$

where $|[g]|$ is the size of $[g]$, and $|\pi_g^\omega|$ is the dimension of π_g^ω.

From (3.12), we have $d_{([g],\pi_g^\omega)} d_{([h],\pi_h^\omega)} = |[g]||[h]| \cdot \deg\, \pi_g^\omega \cdot \deg\, \pi_h^\omega$. Substituting in (3.8) and using (3.10), we have

$$\begin{aligned}
\tfrac{1}{|G|} & \ |[g]||[h]| \cdot \deg\, \pi_g^\omega \cdot \deg\, \pi_h^\omega \\
= & \left| \frac{1}{|G|} \sum_{\substack{k\in[g],\ \ell\in[h],\\ k\ell=\ell k}} \chi^k_{\pi_g^\omega}(\ell)^* \chi^\ell_{\pi_h^\omega}(k)^* \right| \\
\leq & \ \frac{1}{|G|} \sum_{\substack{k\in[g],\ \ell\in[h],\\ k\ell=\ell k}} |\chi^k_{\pi_g^\omega}(\ell)||\chi^\ell_{\pi_h^\omega}(k)| \\
\leq & \ \frac{|[g]||[h]|}{|G|} \cdot \deg\, \pi_g^\omega \cdot \deg\, \pi_h^\omega
\end{aligned} \tag{3.13}$$

In the last inequality above, we have used (3.11) as well as the fact that projective characters satisfy $|\chi_{\pi_g^\omega}| \leq \deg\, \pi_g^\omega$ [76]. It is clear that (3.8) is satisfied if and only if the conjugacy classes $[g]$ and $[h]$ commute element-wise and the projective characters satisfy (3.9). □

3.2.1 From fusion to theorem 3.1 and a relation between theorems 3.2 and 3.3

Here we first explain why the AH conjecture implies that, in (twisted and untwisted) discrete gauge theories based on simple groups, the fusion of any two lines carrying magnetic flux must have more than one fusion outcome (i.e., theorem 3.1). We then explain the relation between theorems 3.2 and 3.3.

To that end, we would like to rule out fusion rules of the type

$$([g],\pi_g^\omega) \times ([h],\pi_h^\omega) = ([k],\pi_k^\omega) \ , \quad g,h \neq 1 \ , \tag{3.14}$$

where all magnetic fluxes on the LHS are non-trivial. In fact, equation (1.54) implies the existence of such a fusion is equivalent to:

1. $[g] \cdot [h] = [k] = [h] \cdot [g]$
2. $\exists!\ \pi_k^\omega$ such that $m(\pi_k^\omega|_{N_g\cap N_h\cap N_k}, \pi_g^\omega|_{N_g\cap N_h\cap N_k} \otimes \pi_h^\omega|_{N_g\cap N_h\cap N_k} \otimes \pi^\omega_{(g,h,k)}) = 1$

[76]This statement is guaranteed as long as the projection factors defining the representations are roots of unity, which is satisfied in our case. Indeed, the 3-cocycle $\omega \in H^3(G, U(1))$ can be chosen to be valued in roots of unity without loss of generality.

To see this, recall the fusion formula (1.54). Since the arguments of the $m(\cdot,\cdot)$ function are representations of $N_{^tg} \cap N_{^sh} \cap N_k$, we can decompose them in terms of irreducible representations, $\pi^{\omega(i)}$, of this group

$$
\begin{aligned}
&{}^t\pi^\omega_g|_{N_{^tg}\cap N_{^sh}\cap N_k} \otimes {}^s\pi^\omega_h|_{N_{^tg}\cap N_{^sh}\cap N_k} \otimes \pi^\omega_{(^tg,^sh,k)} \\
&= \sum_i \alpha_i \pi^{\omega(i)} \ , \\
&\pi^\omega_k|_{N_{^tg}\cap N_{^sh}\cap N_k} = \sum_i \alpha_i^{'} \pi^{\omega(i)} \ ,
\end{aligned}
\tag{3.15}
$$

for some non-negative integers $\alpha_i, \alpha_i^{'}$. Then the definition of $m(\cdot,\cdot)$ in [14] implies

$$
\begin{aligned}
&m(\pi^\omega_k|_{N_{^tg}\cap N_{^sh}\cap N_k}, {}^t\pi^\omega_g|_{N_{^tg}\cap N_{^sh}\cap N_k} \otimes {}^s\pi^\omega_h|_{N_{^tg}\cap N_{^sh}\cap N_k} \\
&\otimes \pi^\omega_{(^tg,^sh,k)}) = \sum_i \alpha_i \alpha_i^{'} \ .
\end{aligned}
\tag{3.16}
$$

We know that π^ω_k is an irreducible representation of N_k. Also, $N_{^tg}\cap N_{^sh}\cap N_k$ is a subgroup of N_k. According to the Frobenius reciprocity theorem for projective representations of finite groups [108] [77], we know that, given any irreducible representation, $\pi^{\omega(i)}$, of $N_{^tg}\cap N_{^sh}\cap N_k$, there is always an irreducible representation, π^ω_k, of N_k such that the decomposition of $\pi^\omega_k|_{N_{^tg}\cap N_{^sh}\cap N_k}$ into irreducible representations of $N_{^tg}\cap N_{^sh}\cap N_k$ contains $\pi^{\omega(i)}$. This reasoning shows that, given ${}^t\pi^\omega_g|_{N_{^tg}\cap N_{^sh}\cap N_k}\otimes {}^s\pi^\omega_h|_{N_{^tg}\cap N_{^sh}\cap N_k}\otimes \pi^\omega_{(^tg,^sh,k)}$, there is always some irreducible representation, π^ω_k, such that $m(\pi^\omega_k|_{N_{^tg}\cap N_{^sh}\cap N_k}, {}^t\pi^\omega_g|_{N_{^tg}\cap N_{^sh}\cap N_k}\otimes {}^s\pi^\omega_h|_{N_{^tg}\cap N_{^sh}\cap N_k}\otimes \pi^\omega_{(^tg,^sh,k)})$ is non-zero. It follows that once we choose some conjugacy class, $[k]$, such that $[k] \in [g]\cdot[h]$, there is always some π^ω_k such that $N^{([k],\pi^\omega_k)}_{([g],\pi^\omega_g)([h],\pi^\omega_h)} \neq 0$. Here, $[g]\cdot[h]$ are the conjugacy classes obtained from taking a product of anyons with magnetic charges in $[g]$ and $[h]$.

Hence, in order to have a fusion rule of the type

$$
([g],\pi^\omega_g) \times ([h],\pi^\omega_h) = ([k],\pi^\omega_k) \ , \quad g,h \neq 1 \ ,
\tag{3.17}
$$

where all magnetic fluxes on the LHS are non-trivial, we need the fusion of the orbits $[g]\cdot[h]$ to contain only a single orbit $[k]$ (note that $|[k]|$ need not be equal to $|[g]||[h]|$ [78]). Moreover, commutativity of the fusion rules requires $[k] = [h]\cdot[g]$. Hence, the double coset $N_g\backslash G/N_h$ should have only a single element. (Since the double coset is trivial, we will remove the t,s dependence in the expressions below). We also require that the decomposition of representations $\pi^\omega_k|_{N_g\cap N_h\cap N_k}$ and $\pi^\omega_g|_{N_g\cap N_h\cap N_k}\otimes\pi^\omega_h|_{N_g\cap N_h\cap N_k}\otimes\pi^\omega_{(g,h,k)}$ into irreps of $N_g\cap N_h\cap N_k$ to have only a single irrep (of multiplicity one) in common. That is, if

$$
\begin{aligned}
\pi^\omega_g|_{N_g\cap N_h\cap N_k}\otimes\pi^\omega_h|_{N_g\cap N_h\cap N_k}\otimes\pi^\omega_{(g,h,k)} &= \sum_i \alpha_i\pi^{\omega(i)} \\
\pi^\omega_k|_{N_g\cap N_h\cap N_k} &= \sum_i \alpha_i^{'}\pi^{\omega(i)} \ ,
\end{aligned}
\tag{3.18}
$$

[77] We use this theorem in the twisted case; in the untwisted case we use the usual theorem for linear representations.

[78] In the case of the fusion of pure fluxes, we do require $|[k]| = |[g]||[h]|$.

then there should be only one $i = i_0$ for which $\alpha_{i_0} = \alpha'_{i_0} \neq 0$. Furthermore, we require that $\alpha_{i_0} = 1$.

So, in order to have a fusion of the type (3.2), we arrive at the two desired constraints:

1. $[g] \cdot [h] = [k] = [h] \cdot [g]$

2. $\exists!\ \pi_k^\omega$ such that $m(\pi_k^\omega|_{N_g\cap N_h\cap N_k}, \pi_g^\omega|_{N_g\cap N_h\cap N_k} \otimes \pi_h^\omega|_{N_g\cap N_h\cap N_k} \otimes \pi^\omega_{(g,h,k)}) = 1$

The first constraint is on the conjugacy classes involved, and the second one is on the representations. The AH conjecture implies that (1) is impossible. Therefore, we see that

$$\text{AH conjecture} \Rightarrow \text{no fusions as in (3.2) for simple } G\ .$$

In particular, we see that

$$\mathcal{L}_{([g],\pi_g^\omega)} \times \mathcal{L}_{([h],\pi_h^\omega)} \neq \mathcal{L}_{([k],\pi_k^\omega)}\ , \tag{3.19}$$

where $\mathcal{L}_{([g],\pi_g^\omega)} = ([g], \pi_g^\omega)$, $\mathcal{L}_{([h],\pi_h^\omega)} = ([h], \pi_h^\omega)$, and $\mathcal{L}_{([k],\pi_k^\omega)} = ([k], \pi_k^\omega)$. Therefore:

$$\text{AH conjecture} \Rightarrow \text{Theorem 3.1}\ .$$

This result does not prove theorem 1 since the AH conjecture is still unproven. However, it is a non-trivial consistency check of this conjecture. We will return to theorem 3.1 in section 3.2.2.

Next, let us show how theorem 3.3 implies theorem 3.2. To that end, let us specialize the general fusion in (1.54) to the product of a non-abelian Wilson line, $\mathcal{W}_{\pi_1} = ([1], \pi_1)$, with a non-abelian flux line, $\mu_{[h]} = ([h], 1_h^\epsilon)$. In order to have such a flux line in our theory we should either consider an untwisted discrete gauge theory or a theory in which ω is such that $\eta_h \in H^2(N_h, U(1))$ is cohomologically trivial.

Now, it is not hard to argue that

$$N^{([h],\pi_h^\omega)}_{([1],\pi_1),([h],1_h^\epsilon)} = m(\pi_h^\omega, \pi_1|_{N_h} \otimes 1_h^\epsilon)\ . \tag{3.20}$$

To understand this point, let us specialize the general fusion in (1.54) to the product of a non-abelian Wilson line, $\mathcal{W}_{\pi_1} = ([1], \pi_1)$, with a non-abelian flux line, $\mu_{[h]} = ([h], 1_h^\epsilon)$.Then we find

$$\begin{aligned} N^{([h],\pi_h^\omega)}_{([1],\pi_1),([h],1_h^\epsilon)} &= \sum_{(t,s)\in G\backslash G/N_h} m(\pi_h^\omega, {}^t\pi_1|_{N_h} \otimes {}^s1_h^\epsilon \\ &\quad \otimes \pi^\omega_{(1,h,h)}|_{N_h})\ . \end{aligned} \tag{3.21}$$

In this case, the double coset $G\backslash G/N_h$ is trivial. Hence, we have

$$N^{([h],\pi_h^\omega)}_{([1],\pi_1),([h],1_h^\epsilon)} = m(\pi_h, \pi_1|_{N_h} \otimes 1_h^\epsilon \otimes \pi^\omega_{(1,h,h)}|_{N_h})\ . \tag{3.22}$$

In fact, the representation $\pi^\omega_{(1,h,h)}$ is trivial (this follows from the fixed nature of the V^h_{1h} fusion space in the G-SPT [14]). So the product of representations $\pi_1|_{N_h} \otimes 1_h^\epsilon \otimes \pi^\omega_{(1,h,h)}|_{N_h}$ is isomorphic to $\pi_1|_{N_h} \otimes 1_h^\epsilon$. Therefore, the expression above simplifies to

$$N^{([h],\pi_h^\omega)}_{([1],\pi_1),([h],1_h^\epsilon)} = m(\pi_h^\omega, \pi_1|_{N_h} \otimes 1_h^\epsilon)\ , \tag{3.23}$$

as desired.

Note that π_1 is an irreducible representation of G. Its restriction to N_h is in general reducible. So $m(\pi_h, \pi_1|_{N_h} \otimes 1^{\epsilon}_h)$ gives the multiplicity of the irreducible representation, π_h, in the decomposition of the representation, $\pi_1|_{N_h} \otimes 1^{\epsilon}_h$, into irreducible representations of N_h. If $\pi_1|_{N_h}$ is irreducible, $m(\pi_h, \pi_1|_{N_h} \otimes 1^{\epsilon}_h) = \delta_{\pi_h, \pi_1|_{N_h} \otimes 1^{\epsilon}_h}$. Hence, we have

$$([1], \pi_1) \otimes ([h], 1_h) = ([h], \pi_1|_{N_h} \otimes 1^{\epsilon}_h) \ , \tag{3.24}$$

if and only if $\pi_1|_{N_h}$ is an irreducible representation of N_h.

As a result, theorem 3 implies that we have more than one fusion channel

$$\mathcal{W}_{\pi_1} \times \mu_{[h]} = \mathcal{L}_{([h], \pi^{\omega}_h)} + \cdots \ . \tag{3.25}$$

In fact, we may take the flux, $([h], 1^{\epsilon}_h)$, and replace it with a dyon, $([h], \pi^{\omega}_h)$. Note that, in some theories, such a dyon may exist while the flux line does not. We then find that the right-hand side of (3.23) becomes $m(\widetilde{\pi}^{\omega}_h, \pi_1|_{N_h} \otimes \pi^{\omega}_h)$. Clearly, if the fusion in (3.25) requires more terms on the right-hand side, so too will the fusion with the dyon replacing the flux. This is the content of theorem 3.2.

Similarly, by the logic of this section, if we satisfy theorem 3.2 for the untwisted discrete G gauge theory, we then have that, for any irreducible linear representation, π_1, of G having dimension greater than one, $\pi_1|_{N_h}$ is reducible. This is the content of theorem 3.3, and so:

$$\text{Theorem 3.3} \Leftrightarrow \text{Theorem 3.2} \ .$$

What remains is to prove at least one of these theorems, this will be done in next section. Where we choose to prove theorem 3.2 first since it has a more physical flavor, then a direct proof of theorem 3.3 is also given.

3.2.2 Proofs of the cousin theorems

proof of theorem 3.2 Let us first prove theorem 3.2. To that end, suppose we have a fusion as in (3.6). In section 3.2.1, we argued that, if such a fusion exists, the electric charge of the dyon on the right-hand side is given by a reduction of an irreducible representation of the gauge group G (i.e., $\pi^{\omega}_h = \pi|_{N_g} \otimes 1^{\epsilon}_h$) and $h = g$. Next, note that the S-matrix satisfies [112]

$$S_{\mathcal{W}_{\pi} \bar{\mu}_{[g^{-1}]}} = \frac{1}{|G|} \frac{\theta_{\mathcal{L}_{([g], \pi^{\omega}_g)}}}{\theta_{\mathcal{W}_{\pi}} \theta_{\mu_{[g]}}} d_{\mathcal{L}_{([g], \pi_g)}} = \frac{1}{|G|} \frac{\theta_{\mathcal{L}_{([g], \pi^{\omega}_g)}}}{\theta_{\mathcal{W}_{\pi}} \theta_{\mu_{[g]}}} d_{\mathcal{W}_{\pi}} d_{\mu_{[g]}} \ , \tag{3.26}$$

where $\bar{\mu}_{[g^{-1}]}$ is the conjugate of $\mu_{[g]}$. Therefore,

$$|S_{\mathcal{W}_{\pi} \mu_{[g]}}| = \frac{1}{|G|} d_{\mathcal{W}_{\pi}} d_{\mu_{[g]}} \ . \tag{3.27}$$

Lemma 4 then implies $|\chi_{\pi}(g)| = \deg\ \chi_{\pi}$, where χ_{π} is the character corresponding to the Wilson line's charge, and $\deg\ \chi_{\pi} = |\pi| > 1$ is the dimension of π.

A standard argument in representation theory then says that[79]

$\pi(g) = c \cdot \mathbb{1}_{|\pi|}$, where $\mathbb{1}_{|\pi|}$ is the $|\pi| \times |\pi|$ unit matrix, and c is an n^{th} root of unity (the twist of the dyon). Next, choose some $k \in [G, g] := \langle \ell g \ell^{-1} g^{-1} | \ell \in G \rangle$. Clearly,

$$\begin{aligned} \pi(k) &= \pi(\ell g \ell^{-1} g^{-1}) = \pi(\ell) \cdot c \cdot \mathbb{1}_{|\pi|} \cdot \pi(\ell)^{-1} \cdot c^{-1} \cdot \mathbb{1}_{|\pi|} \\ &= \mathbb{1}_{|\pi|} \ . \end{aligned} \tag{3.28}$$

Since G is a simple group, we can choose $k \neq 1$. As a result, π is an unfaithful representation of G. Therefore, the kernel, $\ker(\pi)$, is a non trivial normal subgroup. Since G is simple, we must have $\ker(\pi) = G$. But then, π cannot be irreducible. Note that we may repeat this proof verbatim by taking $\mathcal{L}_{([g],\pi^\omega_g)}$ instead of the flux line. Therefore fusion of the form in (3.5) is also forbidden. $\square$

By the discussion in section 3.2.1, we have also proved theorem 3.3. Although this proof is mathematical, it has a distinctly TQFT-flavor: notice the prominent role of the modular S matrix (and, to a lesser extent, the twists). We give a direct group theoretical proof of theorem 3.3 in the end of this section.

proof of theorem 3.1 The proof of theorem 3.1 proceeds similarly to that of theorem 3.2.

We would like to show the following fusion is impossible

$$\mathcal{L}_{([g],\pi^\omega_g)} \times \mathcal{L}_{([h],\pi^\omega_h)} = \mathcal{L}_{([k],\pi^\omega_k)} \ , \quad g, h \neq 1 \ , \tag{3.29}$$

where, according to the discussion in section 3.2.2, $[k] = [gh]$. Similarly to the case of theorem 2, we have that

$$\begin{aligned} S_{\mathcal{L}_{([g],\pi^\omega_g)} \mathcal{L}_{([h^{-1}],(\pi^\omega_h)^*)}} &= \frac{1}{|G|} \frac{\theta_{\mathcal{L}_{([gh],\pi^\omega_{gh})}}}{\theta_{\mathcal{L}_{([g],\pi^\omega_g)}} \theta_{\mathcal{L}_{([h],\pi^\omega_h)}}} d_{\mathcal{L}_{([gh],\pi^\omega_{gh})}} \\ &= \frac{1}{|G|} \frac{\theta_{\mathcal{L}_{([gh],\pi^\omega_{gh})}}}{\theta_{\mathcal{L}_{([g],\pi^\omega_g)}} \theta_{\mathcal{L}_{([h],\pi^\omega_h)}}} \cdot \\ &\quad \cdot d_{\mathcal{L}_{([g],\pi^\omega_g)}} d_{\mathcal{L}_{([h],\pi^\omega_h)}} \ , \end{aligned} \tag{3.30}$$

where $\mathcal{L}_{([h^{-1}],(\pi^\omega_h)^*)}$ is the conjugate of $\mathcal{L}_{([h],\pi^\omega_h)}$. Therefore,

$$|S_{\mathcal{L}_{([g],\pi^\omega_g)} \mathcal{L}_{([h],\pi^\omega_h)}}| = \frac{1}{|G|} d_{\mathcal{L}_{([g],\pi^\omega_g)}} d_{\mathcal{L}_{([h],\pi^\omega_h)}} \ . \tag{3.31}$$

This last result allows us, as in the case of theorem 3.2, to use lemma 3.4. We then conclude that for any $\ell \in [g]$ and $m \in [h]$, $\ell m = m \ell$ (i.e., that the two conjugacy classes $[h]$ and $[g]$ commute element-by-element).

[79] Since G is finite, let g be an element of G of order k, then $\pi(g^k) = \pi(g)^k = \mathbb{1}_{|\pi|}$, so the eigenvalues of $\pi(g)$ are all k-th roots of unity and there are deg χ_π of them, while they sum up to $\chi_\pi(g) = \operatorname{Tr} \pi(g)$, if $|\chi_\pi(g)| =$ deg χ_π, all the eigenvalues must be the same, then $\pi(g) = c \cdot \mathbb{1}_{|\pi|}$ in eigenbasis, but this result is basis independent

Now, consider the product of conjugacy classes

$$[g] \cdot [g] = \sum_{[a]} N_{[g][g]}^{[a]} [a] \ , \quad N_{[g][g]}^{[a]} \in \mathbb{Z}_{\geq 0} \ . \tag{3.32}$$

Clearly, we have that all elements on the left hand side commute with all elements of $[h]$. Therefore, the same is true of all elements in the conjugacy classes $[a]$. Now, consider taking pairwise products of all the $[a]$'s with themselves and with $[g]$ and so on. Eventually, we will come to a set of conjugacy classes closed under multiplication. This defines a normal subgroup $K \trianglelefteq G$ in which each element commutes with $[h]$. Since G is simple, we must have that $K = G$. However, this means that $[h]$ commutes with all elements of the group and so we have a non-trivial center. This is a contradiction. $\square$

proof of theorem 3.3 As theorem 3.2 implies theorem 3.3 and vice versa. In this sense, we have already proven theorem 3.3. However, here we would like to give a direct (albeit mathematical) proof:

Since G is a non-abelian simple group, its irreducible representations of dimension larger than one must be faithful (otherwise their kernels would be non-trivial normal subgroups). Now, consider some faithful non-abelian representation, π. Furthermore, take some $g \in G$ such that $g \neq 1$ and consider the centralizer, N_g.

Suppose the restriction $\pi|_{N_g}$ is irreducible. Clearly g is central in N_g. As a result, by Schur's lemma

$$\pi|_{N_g}(g) = c \cdot \mathbb{1}_{|\pi|} \ , \tag{3.33}$$

where c is an n^{th} root of unity. Since this is a restriction of a representation of G, we must also have in the parent group that

$$\pi(g) = c \cdot \mathbb{1}_{|\pi|} \ , \tag{3.34}$$

and so it follows that

$$\pi(hgh^{-1}g^{-1}) = \mathbb{1}_{|\pi|} \ . \tag{3.35}$$

Since the group is simple, $g \neq 1$ cannot be in the (trivial) center of G. As a result, there exists h such that $hgh^{-1}g^{-1} \neq 1$. The result in (3.35) contradicts the fact that π is faithful. $\square$

3.2.3 Special cases

Although we have given full proofs of theorems 3.3 and 3.2, it is amusing to give direct proofs that apply to certain classes of finite simple groups.

CA groups For example, there is a large class of groups called "AC" groups or, depending on the literature, "CA" groups. These groups are defined to have abelian centralizers for all conjugacy classes of elements $g \neq 1$. In this case, theorem 3.3 is trivially true: $\pi_1|_{N_g}$ is automatically reducible since $|\pi_1| > 1$.

In particular, the $PSL(2, 2^n)$ groups with $n \geq 2$ are simple AC groups. In fact, these are the only such groups [151]. For $n = 2$, we have $PSL(2, 4) \simeq A_5$. More generally, however, the $PSL(2, 2^n)$ groups are a distinct class of groups.

As a result, we conclude that in all (twisted or untwisted) discrete gauge theories based on AC groups our theorems hold. □

A_n **groups** As a more involved example here we will prove our theorems for A_n groups. In order for A_n to be simple, we require $n = 3$ or $n \geq 5$ (proofs of the AH conjecture exist in the cases discussed here as well [69]). The basic idea is to use Saxl's classification of irreducible characters of A_n, χ^λ, that remain irreducible upon reduction to a subgroup $G < A_n$ [142]. We will argue that such subgroups cannot act as centralizers.

To that end, theorems 3.1 and 3.2 of [142] constrain λ and G to be one of the following (note that $\Omega = \{1, 2, \cdots, n\}$ is the set of elements A_n acts on):

1. $\lambda = (n)$ is the trivial representation.
2. $\lambda = (n-1, 1)$ is the $n-1$ dimensional representation, and G acts 2-transitively on Ω.
3. $\lambda = (n-2, 2)$, $n = 9, 11, 12, 23, 24$ and G is $\mathrm{P\Gamma L}(2, 8)$, M_{11}, M_{12}, M_{23}, M_{24} respectively.
4. $\lambda = (n-2, 1, 1) = (n-2, 1^2)$ and either $n = 2^c$ for some integer, c, with $G = \mathrm{AGL}(c, 2)$ or $n = 11, 12, 12, 16, 22, 23, 24$ with $G = M_{11}, M_{11}, M_{12}, V_{16}A_7, M_{22}, M_{23}, M_{24}$ respectively.
5. $\lambda = (21, 2, 1)$ or $\lambda = (21, 1^3)$, $n = 24$, and $G = M_{24}$.
6. $\lambda = (\lambda_1^a)$ with $a \neq \lambda_1$, $n = a\lambda_1$, and $G = A_{n-1}$ stabilizing a point in Ω.
7. $\lambda = (a^a)$, $n = a^2$, $G \geq A_{n-2}$, the stabilizer of two points in Ω.
8. $\lambda = (a^b, b^{a-b})$, $n = (2a-b)b$, and $G = A_{n-1}$, the stabilizer of a point in Ω.
9. $\lambda = (3^3)$, $n = 9$, and $G = \mathrm{P\Gamma L}(2, 8)$ or $G = \mathrm{AGL}(3, 2)$.
10. $\lambda = (3^2, 2)$, $n = 8$, and $G = \mathrm{AGL}(3, 2)$.

Here we have used partitions of n to label representations of A_n.

In case (1) there is nothing to prove as the Wilson line would be the trivial abelian line. For case (2), the fact that G is 2-transitive on Ω rules it out as a centralizer. To understand this statement, consider non-trivial $\sigma \in A_n$ and $g \in G$. Without loss of generality, we may take $\sigma(1) = 2$. Since G is 2-transitive on Ω, it is also transitive, and we can choose g so that $g(1) = 3$. Now, $\sigma(3) = a \neq 2$. By 2-transitivity, we may further choose g such that

$g(2) = 2 \neq a$. As a result $g^{-1}\sigma g(1) = g^{-1}\sigma(3) = g^{-1}(a) \neq 2 = \sigma(1)$ and so G does not centralize σ.

In case (3) we may check that there is no conjugacy class of length $|A_9|/|P\Gamma L(2,8)|$, $|A_{11}|/|M_{11}|$, $|A_{12}|/|M_{12}|$, $|A_{23}|/|M_{23}|$, or $|A_{24}|/|M_{24}|$ respectively.

We may rule out the possibility of $G = \mathrm{AGL}(c,2)$ in (4) by noting, as in [129], that $\mathrm{AGL}(c,2)$ acts 2-transitively on the c-dimensional vector space over $GF(2) \simeq \mathbb{Z}_2$ [80]. Since this vector space is 2^c dimensional, we may associate vectors with points in Ω, and we are done by the logic that solved case (2). Since $G = V_{16}A_7$ acts 2-transitively on Ω [142], we see this will not work either. We may rule out the remaining possibilities in case (4) by similar logic to that employed in case (3). This logic also rules out case (5).

Let us now consider case (6). Here we may use the fact that non-trivial conjugacy classes in S_n have length at least $n(n-1)/2$. As is well known, conjugacy classes in A_n either have the same length as those in S_n or else they have half the length. As a result, we conclude that non-trivial conjugacy classes in A_n have length at least $n(n-1)/4$. This reasoning implies that A_{n-1} is too large to act as a centralizer in A_n (this statement holds since we can use GAP [85] to explicitly check all cases $n \leq 11$; therefore, we need only worry about the cases $n > 11$). This logic also rules out case (8). Cases (9) and (10) may be ruled out by explicit computation in GAP.

This leaves case (7). Here we may use the fact that A_{n-2} fixes two points in Ω and acts $(n-4)$-transitively on the remaining $n-2$ points in $\Omega' \subset \Omega$. In fact, since we can check with GAP that this scenario doesn't arise for $n \leq 11$, we only need to discuss the case $n > 11$ and use the weaker condition that $(n-4)$-transitivity implies 2-transitivity. Without loss of generality, we can again take non-trivial $\sigma \in A_n$ satisfying $\sigma(1) = 2$. Without further loss of generality, there are three sub-cases to consider:

- case (a):

  $\sigma(2) = 1$ and $\sigma(3) = 4$.

- case (b):

  $\sigma(2) = 3$ and $\sigma(3) = 1$.

- case (c):

  $\sigma(2) = 3$ and $\sigma(3) = 4$.

Let us consider (a) first. Suppose that $1, 2 \in \Omega'$. By transitivity, we can choose $g \in A_{n-2}$ such that $g(1) = x \neq 1, 2$ and $x \in \Omega'$. We then have $\sigma(x) = a \neq 2$. By 2-transitivity, we may choose $g(2) = 2 \neq a$, and we have $g^{-1}\sigma g(1) = g^{-1}\sigma(x) = g^{-1}(a) \neq 2 = \sigma(1)$. Next, suppose that $1 \in \Omega'$ but $2 \notin \Omega'$. Here we are forced to choose $g(2) = 2$, but this doesn't matter. Indeed, the same logic we used when both $1, 2 \in \Omega'$ now works in this case as well. Continuing on, suppose instead that $1 \notin \Omega'$ but $2 \in \Omega'$. Here we are forced to take

[80] In fact, $\mathrm{AGL}(c,2)$ acts generously 3-transitively on the c-dimensional vector space over $\mathbb{Z}_2$ [129].

$g(1) = 1$. Since $2 \in \Omega'$, we are free to choose $g(2) = y \neq 2$. As a result, we have that $g^{-1}\sigma g(1) = g^{-1}\sigma(1) = g^{-1}(2) \neq 2 = \sigma(1)$. To finish our discussion of (a), let us suppose that $1, 2 \notin \Omega'$. Then, $g(1) = 1$ and $g(2) = 2$. So $g^{-1}\sigma g(1) = 2 = \sigma(1)$. However, we have that $3, 4 \in \Omega'$. As a result, we can repeat our logic for the case $1, 2 \in \Omega'$ with $1, 2 \to 3, 4$.

Next consider (b). This case can be treated identically to (a) except for the scenario in which $1, 2 \notin \Omega'$. However, the treatment here is similar. We must have $3 \in \Omega'$ and so we can take $g(3) = x \neq 1, 2, 3$. We also have $\sigma(x) = a \neq 1$. Therefore, $g^{-1}\sigma g(3) = g^{-1}\sigma(x) = g^{-1}(a) \neq 1$ (if $a \in \Omega'$, then this is clear since $1 \notin \Omega'$; if $a = 2$, then $g^{-1}(a) = 2$).

Finally, consider (c). This case may be treated analogously to (a). $\square$

4 Fusion of non-abelian anyons II: axb=c

4.1 $a \times b = c$ fusion rule

Topological quantum field theories (TQFTs) in 2+1 dimensions and their anyonic excitations lie at the heart of important physical [121], mathematical [157], and computational [156] systems and constructions. In principle, these TQFTs can be fully characterized by solving a set of polynomial consistency conditions [123, 13, 112], although proceeding in this way is often quite difficult as a practical matter (however, see [139, 28] for examples of some results; see also [51] for a potentially very different approach). More generally, it is interesting to understand aspects of the global structure of a TQFT and its symmetries without the need to fully solve the theory (e.g., see [125]).

Proceeding in this way, we will study anyonic fusions $a \times b$ that have a unique product anyon, c

$$a \times b = c \ , \quad a,\ b,\ c \in \mathcal{T} \ , \tag{4.1}$$

in a general $2+1$ dimensional TQFT, $\mathcal{T}$.[81] Our main questions is: what does (4.1) tell us about the global structure of $\mathcal{T}$ and its symmetries?

For invertible a and b (i.e., a and b are abelian anyons), fusion rules of the form (4.1) describe the abelian 1-form symmetry group of the theory [80] (the closely related modular S matrix characterizes its 't Hooft anomalies [98]). In the case in which, say, a is abelian and b is non-abelian,[82] the equation (4.1) gives the fixed points of the fusion of anyons in the theory with the one-form generator, a. Such equations have important consequences for anyon condensation / one-form symmetry gauging in TQFT [12, 98] as well as for orbifolding and coset constructions in closely related 2D rational conformal field theories (RCFTs) (e.g., see [104, 144]).

[81] Throughout what follows, we only consider non-spin TQFTs. These are theories that do not require a spin structure in order to be well-defined.

[82] In this case, b is non-invertible, and the fusion $b \times \bar{b} = 1 + \cdots$, where $\bar{b}$ is the anyon conjugate to b, necessarily contains at least one more anyon in the ellipses.

Although these cases will play a role below, we will be more interested in the situation in which both a and b are non-abelian

$$a \times b = c \ , \quad d_a \ , \ d_b > 1 \ . \tag{4.2}$$

Here $d_{a,b}$ denote the quantum dimensions of a and b (given they are larger than one, neither a nor b are invertible). Since both a and b are non-abelian, one typically finds that the right-hand side of (4.2) has multiple fusion products. For example, fusions as in (4.2) do not occur in $SU(2)_k$ Chern-Simons (CS) theory for any value of $k \in \mathbb{N}$.[83] As we will see, when fusions of non-abelian a and b do have a unique outcome, there are consequences for the global structure of $\mathcal{T}$.

The most trivial case in which a fusion of the type (4.2) occurs is when $\mathcal{T}$ factorizes (not necessarily uniquely) as

$$\mathcal{T} = \mathcal{T}_1 \boxtimes \mathcal{T}_2 \ , \tag{4.3}$$

with $\mathcal{T}_1$ and $\mathcal{T}_2$ two separate TQFTs that have trivial mutual braiding, $a \in \mathcal{T}_1$, and $b \in \mathcal{T}_2$.[84] Here "$\boxtimes$" denotes a categorical generalization of the direct product called a "Deligne product" that respects some of the additional structure present in TQFT.

As we will discuss in section 4.3, precisely such a situation arises in the modular tensor categories (MTCs) related to unitary A-type Virasoro minimal models with $c > 1/2$.[85] MTCs are mathematical descriptions of TQFTs, and, for the theories in question, they encapsulate the topological properties of the Virasoro primary fields. One may think of the, say, left-movers in these RCFTs as arising at a 1+1 dimensional interface between 2+1 dimensional CS theories with gauge groups $SU(2)_1 \times SU(2)_k$ and $SU(2)_{k+1}$. In the minimal models, we have

$$\varphi_{(r,1)} \times \varphi_{(1,s)} = \varphi_{(r,s)} \ , \tag{4.4}$$

where $2 \leq r < p-2$ and $2 \leq s < p-1$ are Kac labels that give Virasoro primaries with non-abelian fusion rules (here we have $(r,s) \sim (p-1-r, p-s)$, and $p > 4$ is an integer labeling the unitary minimal model).[86] Thinking in terms of cosets, we will see that (4.4) arises because the Virasoro MTC factorizes as in (4.3).[87]

To gain further insight into more general situations in which (4.2) occurs, it is useful to imagine connecting a fusion vertex involving the a, b, c anyons with a fusion vertex involving the $\bar{a}$, $\bar{b}$, and $\bar{c}$ anyons via a c internal line as in the left diagram of figure 33. Using associativity of fusion (via a so-called $F_{\bar{b}}^{\bar{a}ab}$ symbol) we arrive at the right diagram of figure 33. The relation between these two diagrams can be thought of as a change of basis on the

[83] In section 4.3, we will discuss the situation for more general G_k CS theories.

[84] Note that $\mathcal{T}_{1,2}$ may factorize further. Moreover, a may contain an abelian component in $\mathcal{T}_2$, and b may contain an abelian component in $\mathcal{T}_1$.

[85] Note that in the case of the Ising model ($c = 1/2$), at least one of the anyons in the fusion $a \times b = c$ is abelian (and the corresponding MTC does not factorize). We thank I. Runkel for drawing our attention to the $a \times b = c$ fusion rules for non-abelian fields in Virasoro minimal models.

[86] The abelian field $\varphi_{(p-2,1)} \sim \varphi_{(1,p-1)}$ satisfies the fusion rule $\varphi_{(1,p-1)} \times \varphi_{(1,p-1)} = \varphi_{(1,1)} = 1$.

[87] Note that this factorization does not extend to one of the RCFT.

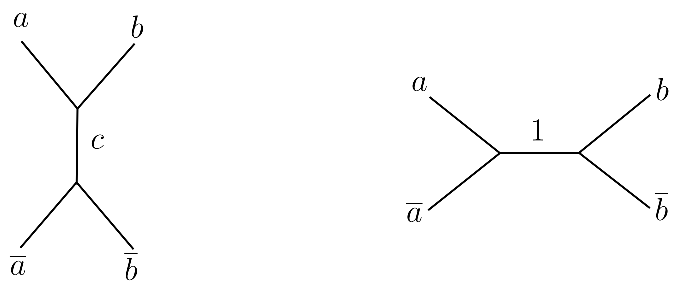


Figure 33: The fusions $a \times b$ and $\bar{a} \times \bar{b}$ have unique outcomes c and $\bar{c}$ respectively. In the left diagram, we connect the corresponding fusion vertices. To get to the diagram on the right, we perform an $F_{\bar{b}}^{\bar{a}ab}$ transformation. Just as the left diagram has a unique internal line, so too does the diagram on the right (in this latter case, the internal line must be the identity).

space of internal states. Since, by construction, the left diagram in figure 33 can only involve a c internal line, the right diagram in figure 33 can also only involve a single internal line. On general grounds, this line must be the identity.[88] This result can also be derived by looking at decomposition of fusion spaces. Consider the fusion space $V^b_{ba\bar{a}}$. It can be decomposed in the following different ways

$$V^b_{ba\bar{a}} \simeq \sum_c V^c_{ba} \otimes V^b_{c\bar{a}} = \sum_x V^x_{a\bar{a}} \otimes V^b_{bx} = \sum_x V^x_{a\bar{a}} \otimes V^{\bar{x}}_{b\bar{b}}, \tag{4.5}$$

where, in the last equality above, we have used the fusion space isomorphism, $V^b_{bx} \simeq V^{\bar{x}}_{b\bar{b}}$. If we have the fusion rule $a \times b = c$, then (4.5) simplifies to

$$V^b_{ba\bar{a}} \simeq V^c_{ba} \otimes V^b_{c\bar{a}} = \sum_x V^x_{a\bar{a}} \otimes V^{\bar{x}}_{b\bar{b}} \tag{4.6}$$

Moreover, we know that V^c_{ba} and $V^b_{c\bar{a}}$ are 1-dimensional. Hence, the dimension of direct sum of fusion spaces $\sum_x V^x_{a\bar{a}} \otimes V^{\bar{x}}_{b\bar{b}}$ should be 1-dimensional. It follows that the sum should be over a single element and that the fusion spaces $V^x_{a\bar{a}}$ and $V^{\bar{x}}_{b\bar{b}}$ should be 1-dimensional. Since the trivial anyon 1 is always an element in the fusions $a \times \bar{a}$ and $b \times \bar{b}$, we have

$$V^b_{ba\bar{a}} \simeq V^c_{ba} \otimes V^b_{c\bar{a}} = V^1_{a\bar{a}} \otimes V^1_{b\bar{b}} \tag{4.7}$$

Therefore, we learn that a fusion rule of the form (4.2) is equivalent to the following

$$\begin{aligned} &a \times \bar{a} = 1 + \sum_{a_i \neq 1} N^{a_i}_{a\bar{a}}\, a_i \ , \quad b \times \bar{b} = 1 + \sum_{b_j \neq 1} N^{b_j}_{b\bar{b}}\, b_j \ , \quad b_j \in b \times \bar{b} \ \Rightarrow \ b_j \notin a \times \bar{a} \ , \\ &a_i \in a \times \bar{a} \ \Rightarrow \ a_i \notin b \times \bar{b} \quad \forall\, i, j \ . \end{aligned} \tag{4.8}$$

In other words, the fusion of $a \times b$ has a unique outcome if and only if the only fusion product that $a \times \bar{a}$ and $b \times \bar{b}$ have in common is the identity.

[88] By rotating the $\bar{a}$, $\bar{b}$, and $\bar{c}$ vertex, we see that $a \times b = c$ is equivalent to requiring $a \times \bar{b} = d$ and $\bar{a} \times b = \bar{d}$ (see figure 34). This logic also explains why, for non-abelian a, it is impossible to have $a \times a = c$ even if $a \neq \bar{a}$.

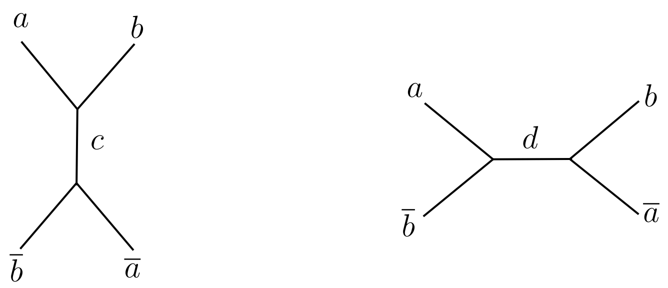


Figure 34: By rotating the bottom vertex in the left diagram of figure 33, we arrive at the above diagram on the left. Again, we have a single internal line labeled by c. We get to the diagram on the right by performing an $F_{\bar{a}}^{\bar{b}ab}$ transformation. Just as the left diagram has a unique internal line, so too does the diagram on the right.

Reformulating the problem as in (4.8) immediately suggests scenarios in which fusions of the form (4.2) occur beyond cases in which $\mathcal{T}$ factorizes into prime TQFTs. For example, if $a \in \mathcal{C}_1 \subset \mathcal{T}$ and $b \in \mathcal{C}_2 \subset \mathcal{T}$ lie in non-modular fusion subcategories of $\mathcal{T}$, $\mathcal{C}_{1,2}$, with trivial intersection (i.e., $\mathcal{C}_1 \cap \mathcal{C}_2 = 1$ only contains the trivial anyon), then we have (4.2) and $\mathcal{T}$ need not factorize.[89] More generally, when $a \in \mathcal{C} \subset \mathcal{T}$ is a member of a non-modular subcategory that does not include b (i.e., $b \notin \mathcal{C}$), we expect it to be more likely to find fusions of the form (4.8) and (4.2) since $a \times \bar{a} \in \mathcal{C}$, but $b \times \bar{b}$ will generally include elements outside $\mathcal{C}$. In fact, we will see that we can often say more when the fusion of a non-abelian Wilson line carrying charge in an unfaithful representation of a discrete gauge group is involved.

Another scenario in which we can imagine (4.8)—and therefore (4.2)—arising is one in which zero-form symmetries act non-trivially on a (i.e., $g(a) \neq a$ for some zero-form generator $g \in G$, where G is the zero-form group) and the $a_i \neq 1$ but not on b.[90] In this case, combinations of a_i that do not form full orbits under G are forbidden from appearing in $b \times \bar{b}$. Given a particular G, this argument may suffice to show that, for all i, $a_i \notin b \times \bar{b}$. More generally, symmetries constrain what can appear as fusion products of $a \times \bar{a}$ and $b \times \bar{b}$. The more powerful these symmetries, the more likely to find fusion rules of the type (4.8).

Interestingly, there is a close connection between the existence of symmetries and the existence of subcategories in TQFT. For example, as we will discuss further in section 4.2.3, for TQFTs corresponding to discrete gauge theories [63, 136], certain "quantum symmetries" or electric-magnetic self-dualities arise when we have particular non-modular subcategories $\mathcal{C}_i \subset \mathcal{T}$ (see [130] for a general theory of such symmetries and [17] for the case of S_3 discrete gauge theory).

[89]In other words, fusion of anyons in $\mathcal{C}_i$ is closed. Moreover, the $\mathcal{C}_i$ inherit associativity and braiding from $\mathcal{T}$, but the Hopf link evaluated on anyons in these subcategories is degenerate (as a matrix). By modularity, the $\mathcal{C}_i$ will have non-trivial braiding with some anyons $x_A \notin \mathcal{C}_{1,2}$ (where A is an index running over such anyons). On the other hand, if the Hopf links for the $\mathcal{C}_i$ are non-degenerate, Müger's theorem [125] guarantees that they will in fact be separate TQFTs and so we are back in the situation of (4.3).

[90]By definition, the symmetry also acts non-trivially on $\bar{a}$ so that $\overline{g(a)} = g(\bar{a}) \neq \bar{a}$. On the other hand, note that one-form symmetry will act trivially on the product $a \times \bar{a}$.

We will also find various other, more subtle, connections between symmetries and fusion rules of the form (4.8) and (4.2). Moreover, we will see that symmetry is ubiquitous: in all the theories with fusion rules of the form (4.8) and (4.2) we analyze, either there is a zero-form symmetry present or else there is, at the very least, a symmetry of the modular data that exchanges anyons (in cases where this action does not lift to the full TQFT, we call these symmetries "quasi zero-form symmetries").

We will study fusions of the above type in two typically very different classes of $2+1$D TQFTs:[91] discrete gauge theories and cosets built out of CS theories with continuous gauge groups (we will refer to these latter theories simply as "cosets"). Discrete gauge theories are always non-chiral, whereas Chern-Simons theories and their associated cosets are typically chiral.[92]

In the context of discrete gauge theories, whenever we have a (full) zero-form symmetry present, we will see that fusion rules of the type (4.8) and (4.2) have simple interpretations in certain parent theories gotten by gauging the zero-form symmetry, G_0. We go from the parent theories back to the original theories by gauging a "dual" one-form symmetry, G_1, that is isomorphic (as a group) to G_0 (see [14] for a more general review of this procedure). In this reverse process, we produce the $a \times b = c$ fusion rules of the corresponding discrete gauge theories via certain fusion fixed points of the one-form symmetry generators in the parent theories.

Similarly, in the context of our coset theories, we will see that fusion rules of the form $a \times b = c$ arise due to certain fixed points in the coset construction (though these fixed points do not generally involve a, b, and c). Cosets corresponding to the Virasoro minimal models lack such fixed points and so, as discussed above, they factorize. On the other hand, more complicated cosets do sometimes have such fixed points, and we will construct an explicit example of such a prime TQFT that has fusion rules of the form (4.8) and (4.2).

To summarize, this discussion leads us to the following questions we will answer in subsequent sections:

1. Does (4.2) imply a factorization of TQFTs

$$\mathcal{T} = \mathcal{T}_1 \boxtimes \mathcal{T}_2 \ , \tag{4.9}$$

   with $a \in \mathcal{T}_1$ and $b \in \mathcal{T}_2$? As has been hinted at above, we will see in sections 4.2 and 4.3 that the answer is generally no.

2. Does (4.2) imply that a belongs to one fusion subcategory and b to another and that the intersection of these subcategories is trivial? In other words, do we have

$$a \in \mathcal{C}_1 \subset \mathcal{T} \ , \quad b \in \mathcal{C}_2 \subset \mathcal{T} \ , \quad \mathcal{C}_1 \cap \mathcal{C}_2 = 1 \ ? \tag{4.10}$$

[91]Note that there are sometimes dualities between theories in these two classes.

[92]By a chiral TQFT, we mean one in which the topological central charge satisfies $c_{\rm top} \neq 0$ (mod 8).

As we will see in section 4.2, the answer is generally no, even if we relax the requirement of trivial intersection. However, we will explicitly construct such examples (with non-modular $\mathcal{C}_{1,2} \subset \mathcal{T}$, where $\mathcal{T}$ is prime) in the case of discrete gauge theories.

3. Does (4.2) imply that a is in some subcategory $\mathcal{C} \subset \mathcal{T}$ that b is not a member of? In other words, do we have
$$a \in \mathcal{C} \subset \mathcal{T} \ , \quad b \notin \mathcal{C} \ ? \tag{4.11}$$
As we will see in section 4.2, the answer is generally no. However, we will argue that such constructions are quite easy to engineer in the context of discrete gauge theories, and we will explain when they arise. We will see that these constructions often have interesting interactions with symmetries.

4. Given a and b as in (4.2), do they have trivial mutual braiding? In other words, do we have
$$\frac{S_{ab}}{S_{0b}} = d_a \ , \tag{4.12}$$
where S is the modular S-matrix? This is true in the context of discrete gauge theories with a simple gauge group [35]. However, non-trivial braiding does arise naturally in the context of the fusion of non-abelian electrically charged lines with non-abelian magnetically charged lines.

5. Given a and b as in (4.2), does $\mathcal{T}$ have a non-trivial zero-form symmetry acting on either a or b? Does the TQFT have a zero-form symmetry that acts more generally? We will see in section 4.2 the answer to both these questions is no. However, in cases in which this is true, it seems to always be related to the existence of a certain fusion fixed point of one-form symmetry generators in a parent TQFT. Of the infinitely many examples of untwisted discrete gauge theories we study, only gauge theories based on the Mathieu groups M_{23} and M_{24} fail to have zero-form symmetries.

6. Given a and b as in (4.2), does $\mathcal{T}$ have a non-trivial symmetry of the modular data? As we will see in sections 4.2 and 4.3, the answer seems to be yes. Clearly, it would be interesting to see if it is possible to define parents of such theories that generalize the relationship in (5). Note that the Mathieu gauge theories discussed in the previous point do have symmetries of their modular data (however, these symmetries do not lift to symmetries of the full TQFTs).

As we will see, many of these questions have simpler answers when studying discrete gauge theories. The reason is that powerful statements in these TQFTs can often be deduced from simple reasoning in the underlying theory of discrete groups. On the other hand, intuition one gains from taking products of representations in various continuous groups, like $SU(N)$, turns out to be somewhat misleading for our questions above.

The plan of this chapter is as follows. In section 4.2, we start with discrete gauge theories and explain how intuition in the theory of finite groups leads us to various answers to the above questions. Along the way, we prove various theorems about discrete gauge theories and fusion rules of the form (4.2) and (4.8) generalizing our work in [35]. Moreover, we discuss the role that subcategories and symmetries of discrete gauge theories play in such fusion rules. In section 4.3, we go to continuous groups and discuss coset theories. We tie the existence of fusion rules of type (4.2) and (4.8) to certain fixed points in the coset construction. We then finish with some conclusions and future directions.

4.2 $a \times b = c$ and discrete gauge theories

In this section we test the ideas presented in the introduction on discrete gauge theories [63, 136]. These TQFTs are characterized by a choice of discrete gauge group, G, and a Dijkgraaf-Witten twist, $\omega \in H^3(G, U(1))$.[93] The basic degrees of freedom are anyonic line operators of the following three types

1. *Wilson lines*, $\mathcal{W}_\pi$, carrying electric charge labeled by a linear representation, π, of G and trivial magnetic charge. This set of operators exists no matter the value of ω.

2. *Magnetic flux lines*, $\mu_{[g]}$, carrying magnetic charge labeled by a conjugacy class, $[g]$, of a representative element, $g \in G$, but having trivial electric charge. In general, their existence depends on the choice of ω.

3. *Dyonic lines*, $\mathcal{L}_{([g],\pi_g^\omega)}$, carrying both magnetic flux and electric charge. In general, they carry a projective representation of G.

These theories have the advantage that we can prove many theorems about them. At the same time, they are very broad and so we can gain some insight into the physical and mathematical questions we are asking.[94]

As we will see in the subsequent subsections, the physics of the various operators listed above is qualitatively different. In order to take the shortest route to answering some of the questions posed in the introduction and in order to establish the existence of fusion rules of the form (4.2) in prime TQFTs, we will start with an analysis of Wilson lines. These objects form a closed fusion subcategory that is particular easy to analyze.[95] As we explain, these are the most "group theoretical" and least anyonic objects in a discrete gauge theory (in addition, as we see from the discussion of the above list of operators, they are the most robust). As a result, we can borrow various useful results from the study of finite groups.

In order to study the physics of other sectors of discrete gauge theories, we will find it convenient to introduce some additional machinery for discussing subcategories (in section 4.2.2) and symmetries (in section 4.2.3). We also discuss quasi zero-form symmetries and

[93] There are redundancies / dualities in this description: see [126].

[94] Discrete gauge theories are, however, necessarily non-chiral. We will consider chiral coset theories in section 4.3.

[95] For other degrees of freedom, the story is more complicated. For example, in section 4.2.2, we will see that in non-abelian discrete gauge theories, full sets of magnetic fluxes do not form fusion subcategories.

their appearance in various discrete gauge theories of interest based on large Mathieu groups. Finally, we move beyond Wilson lines in section 4.2.5 and discuss fusions of the form (4.2) involving non-abelian fluxes, magnetic fluxes, and dyons.

4.2.1 Non-abelian Wilson lines and $a \times b = c$

We would like to recast the problem of constructing discrete gauge theories with fusion rules (4.2) and (4.8) in terms of the closely related problem of finding irreducible products of irreducible finite group representations. To make this connection as direct as possible, it is useful to focus on Wilson lines of the discrete gauge theories we are studying. Indeed, by specializing (1.54) to Wilson lines, we find

$$N_{(1,\pi),(1,\pi')}^{(1,\pi'')} = m(\pi'', \pi\otimes\pi') = \frac{1}{|G|}\sum_{g\in G}\chi_{\pi''}(g)\chi_{\pi}^*(g)\chi_{\pi'}^*(g) = \langle\chi_{\pi''}, \chi_{\pi}\chi_{\pi'}\rangle \ , \quad (4.13)$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product on characters. Therefore, the Wilson lines form a closed fusion subcategory of the discrete gauge theory, $\mathcal{C}_{\mathcal{W}}$. Moreover, the fusion rules of the Wilson lines are those of the representation semiring of the gauge group.[96] Note that $\mathcal{C}_{\mathcal{W}}$ is, in some sense, the "least anyonic" part of the theory: it is easy to check from (1.56) that the Wilson lines are bosonic, so $\theta_{\mathcal{W}_i} = 1$, and that the braiding of Wilson lines amongst themselves is trivial,[97] so $S_{\mathcal{W}_1\mathcal{W}_2} = d_{\mathcal{W}_1}d_{\mathcal{W}_2}/\mathcal{D}$ (here $\mathcal{D} = \sqrt{\sum_{i=1}^N d_i^2}$, and the sum is over all the anyons).[98] To summarize, we see that if we can find representations of some group, G, satisfying

$$\chi_\pi\cdot\chi_{\pi'} = \chi_{\pi''} \ , \quad |\pi|,\ |\pi'|,\ |\pi''| > 1 \ , \quad (4.14)$$

where π, π', and π'' are irreducible, then, in the corresponding G discrete gauge theory, we will have non-abelian Wilson lines satisfying

$$\mathcal{W}_\pi \times \mathcal{W}_{\pi'} = \mathcal{W}_{\pi''} \ . \quad (4.15)$$

Since, by Cayley's theorem, every finite group is isomorphic to a subgroup of the symmetric group, S_N, (for some N) it is natural to start our discussion with S_N. In particular, to check whether π'' is irreducible, we want to perform the group theory analog of the F transformation discussed in the introduction (see figure 33)

$$\langle\chi_\pi\cdot\chi_{\pi'}, \chi_\pi\cdot\chi_{\pi'}\rangle = \langle\chi_\pi^2, \chi_{\pi'}^2\rangle \ , \quad (4.16)$$

where we have used the fact that S_N is ambivalent (g and g^{-1} are in the same conjugacy class for all $g\in S_N$) so that the characters are real. A theorem of Zisser [165] shows that

[96]In fact, we have $\mathcal{C}_{\mathcal{W}} \simeq \mathrm{Rep}(G)$, where $\mathrm{Rep}(G)$ is the category of finite dimensional representations of G over $\mathbb{C}$.

[97]The Wilson lines braid non-trivially with other anyons in the theory (more formally: the Wilson line subcategory is Lagrangian and so the Müger center of $\mathcal{C}_{\mathcal{W}}$ is $\mathcal{C}_{\mathcal{W}}$ itself).

[98]In fact, [59] guarantees that any such subcategory is equivalent to $\mathrm{Rep}(H)$ for some group H.

$\chi_{[N-2,2]} \in \chi_\alpha^2$, where $[N-2,2]$ is a partition of N labeling the corresponding representation of S_N, and α is any irreducible representation of dimension larger than one, $|\alpha| > 1$. Moreover, since S_N is ambivalent, this means that $\chi_{[N]} \in \chi_\alpha^2$, where $\chi_{[N]}$ is the trivial representation of S_N. As a result, we see that the analog of (4.8) yields

$$\chi_\pi \cdot \chi_\pi = \chi_{[N]} + \chi_{[N-2,2]} + \cdots \ , \quad \chi_{\pi'} \cdot \chi_{\pi'} = \chi_{[N]} + \chi_{[N-2,2]} + \cdots \ \Rightarrow \ \langle \chi_\pi \cdot \chi_{\pi'}, \chi_\pi \cdot \chi_{\pi'} \rangle > 1 \ , \quad (4.17)$$

and so products of non-abelian representations of S_N are never irreducible. Therefore, we cannot have (3.3) in S_N discrete gauge theory.

Discrete gauge theories of finite simple groups Since we have $A_N \triangleleft S_N$ (i.e., the alternating group, A_N, is a normal subgroup of S_N), it is natural to consider A_N discrete gauge theories as the next possibility for realizing (4.14) [165] and hence (3.3). Moreover, since A_N is simple, only pure Wilson lines can be involved in fusions of the form (4.2) [35], and the A_N discrete gauge theories are guaranteed to be prime [127] (we will return to the question of primality in greater generality in section 4.2.2). Therefore, finding an example of (3.3) in A_N discrete gauge theories is sufficient to answer question (1) from the introduction in the negative.

To understand if going to A_N is a fruitful direction, we note that there are two types of characters that arise in going from S_N to A_N:

1. Characters that are restrictions of S_N characters satisfying $\chi_\lambda \neq \chi_{[1^N]} \cdot \chi_\lambda$, where $\chi_{[1^N]}$ corresponds to the sign representation of S_N. Let us call these "type A" characters: $\widetilde{\chi}_\lambda := \chi_\lambda|_{A_N}$.
2. Characters that descend from S_N characters satisfying $\chi_\rho = \chi_{[1^N]} \cdot \chi_\rho$. As representations of A_N, they split into two representations of the same dimension, $\rho_\pm$. Let us call these "type B" characters: $\chi_\rho^{(B)} = \chi_{\rho_+} + \chi_{\rho_-} = \chi_\rho|_{A_N}$.

In going from S_N to A_N, we perform a group-theoretical version of gauging the "one-form symmetry" generated by $\chi_{[1^N]}$: we identify characters related by multiplication with $\chi_{[1^N]}$, and we split characters that are invariant under multiplication with $\chi_{[1^N]}$. Clearly, products of type A characters cannot be irreducible since they will always contain $\chi_{[N]}^{(A)}$ and $\chi_{[N-2,2]}^{(A)}$ after performing the F-transformation and computing (4.17).[99]

A little more work in [165] shows that we can obtain (4.14) for A_N if and only if $N = k^2 \geq 9$ by taking the product of the following type A and type B representations

$$\widetilde{\chi}_{[N-1,1]} \cdot \chi_{[k^k]_\pm} = \widetilde{\chi}_{[k^{k-1},k-1,1]} \ . \quad (4.18)$$

Moreover, the $\mathbb{Z}_2$ outer automorphism of A_N acts on the type B characters as

$$g\left(\chi_{[k^k]_\pm}\right) = \chi_{[k^k]_\mp} \ , \quad 1 \neq g \in \mathrm{Out}(A_N) \simeq \mathbb{Z}_2 \ . \quad (4.19)$$

[99] In this discussion, we have implicitly assumed that $N \neq 4$ (although, for $N = 3$, we should take $[N-2,2] \to [2,1]$ to conform to usual conventions). For $N = 4$, we have $\chi_{[N-2,2]}^{(A)} \to \chi_{[N-2,2]}^{(B)} = \chi_{[N-2,2],+} + \chi_{[N-2,2],-}$.

Therefore, at the level of the non-abelian Wilson lines in the corresponding A_N discrete gauge theory, we learn that

$$\mathcal{W}_{[N-1,1]} \times \mathcal{W}_{[k^k]_\pm} = \mathcal{W}_{[k^{k-1},k-1,1]} \ . \tag{4.20}$$

Finally, $\mathrm{Out}(A_N)$ lifts to a full zero-form symmetry of the discrete gauge theory [130], since, according to corollary 7.8 of [130]

$$\mathrm{Aut}^{\mathrm{br}}(\mathcal{Z}(\mathrm{Vec}_{A_N})) \simeq H^2(A_N, U(1)) \rtimes \mathrm{Out}(A_N) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \ , \tag{4.21}$$

where the group on the left hand side is the group of braided tensor auto-equivalences of the MTC underlying the discrete gauge theory, $\mathcal{Z}(\mathrm{Vec}_{A_N})$. As a result, we learn that the symmetries of the discrete gauge theory exchange the $\mathcal{W}_{[k^k]_\pm}$ lines

$$g\left(\mathcal{W}_{[k^k]_\pm}\right) = \mathcal{W}_{[k^k]_\mp} \ , \quad 1 \neq g \in \mathrm{Out}(A_N) \lhd \mathrm{Aut}^{\mathrm{br}}(\mathcal{Z}(\mathrm{Vec}_{A_N}) \ . \tag{4.22}$$

In other words, we have found that, in an infinite number of prime theories, fusion rules of the type (4.2) are generated in pairs related by symmetries of the discrete gauge theory. This discussion shows that TQFTs with fusions of the form (4.2) need not factorize and so the answer to question (1) in the introduction is "no."

Let us now drive home the importance of symmetries in arriving at (4.20) and, at the same time, gain insight that will be useful later. To that end, let us consider gauging the $\mathbb{Z}_2$ outer automorphism symmetry of the A_N discrete gauge theory. Note that this gauging is allowed since the "defectification" obstruction described physically in [14] is trivial here: $H^4(\mathbb{Z}_2, U(1)) = \mathbb{Z}_1$. Moreover, since A_N is simple, the discrete gauge theory has no non-trivial abelian anyons (i.e., $\mathcal{A} = \mathcal{W}_{[N]}$) and so $H^3(\mathbb{Z}_2, \mathcal{A}) = \mathbb{Z}_1$. Therefore, (4.21) is a genuine zero-form symmetry group (as opposed to being a 2-group).

More abstractly, let us consider a generalization of the fusion rules in (1.54) to the case of gauging a zero form group, H, of a more general G-crossed braided theory, $\mathcal{T}_{G^\times}$ (as worked out in [14])

$$N_{([a],\pi_a),([b],\pi_b)}^{([c],\pi_c)} = \sum_{(t,s)\in N_a\backslash H/N_b} m(\pi_c|_{N_{t_a}\cap N_{s_b}\cap N_c}, {}^t\pi_a|_{N_{t_a}\cap N_{s_b}\cap N_c} \otimes {}^s\pi_b|_{N_{t_a}\cap N_{s_b}\cap N_c} \otimes \pi^\omega_{(^ta,^sb,c)}) \ , \tag{4.23}$$

where $a, b, c \in \mathcal{T}_{G^\times}$, $[a] := \{h(a), \ \forall h \in H\}$, $N_a := \{h \in H | h(a) = a\}$, and π_a is a representation of N_a.

In our case at hand, $\mathcal{T}_{G^\times} = \mathcal{Z}(\mathrm{Vec}_{A_N})_{H^\times}$ is the A_N discrete gauge theory extended by surface defects implementing the $H = \mathbb{Z}_2$ global symmetry. Moreover, $a = \mathcal{W}_{[N-1,1]}$, $b = \mathcal{W}_{[k^k]_\pm}$, $N_a = \mathbb{Z}_2$, and $N_b = \mathbb{Z}_1$. As a result, $t = s = 1$, the summation in (4.23) is trivial, the various representations are all restricted to the trivial subgroup, and $\pi^\omega_{a,b,c} = 1$ (this latter statement follows from the fact that the action of H on the V_{ab}^c fusion space via $U_1(a,b,c)$ is trivial). In particular, we have

$$N_{([\mathcal{W}_{[N-1,1]}],\pm),([\mathcal{W}_{[k^k]_\pm}],+)}^{([\mathcal{W}_{[k^{k-1},k-1,1]}],\pm)} = m(\pm|_{\mathbb{Z}_1}, \pm|_{\mathbb{Z}_1} \otimes +|_{\mathbb{Z}_1}) = m(1,1) = 1 \ , \tag{4.24}$$

where $\pm$ denote the two representations of $\mathbb{Z}_2$. Therefore, we learn that when we gauge the outer automorphism group of A_N, we have

$$([\mathcal{W}_{[N-1,1]}], \pm) \times ([\mathcal{W}_{[k^k]_\pm}], +) = ([\mathcal{W}_{[k^{k-1},k-1,1]}], +) + ([\mathcal{W}_{[k^{k-1},k-1,1]}], -) \ , \tag{4.25}$$

which is the TQFT version of the lift of (4.18) to S_N. This is what we expect, since we can always fix our choice of parameters so that gauging $\mathbb{Z}_2$ yields [57]

$$\mathcal{Z}(\mathrm{Vec}_{A_N})_{\mathbb{Z}_2^\times} \overset{\text{gauge}}{\longrightarrow} \mathcal{Z}(\mathrm{Vec}_{A_N \rtimes \mathbb{Z}_2}) = \mathcal{Z}(\mathrm{Vec}_{S_N}) \ , \tag{4.26}$$

where we have used the fact that $S_N \simeq A_N \rtimes \mathbb{Z}_2$.

Finally, from the general rules above, it is not hard to check that the trivial Wilson line in the A_N theory lifts to a $\mathbb{Z}_2$ one-form symmetry in the S_N gauge theory. The resulting non-trivial one-form symmetry generator acts as

$$\begin{aligned} ([\mathcal{W}_{[N]}], -) \times ([\mathcal{W}_{[N-1,1]}], \pm) &= ([\mathcal{W}_{[N-1,1]}], \mp) \ , \\ ([\mathcal{W}_{[N]}], -) \times ([\mathcal{W}_{[k^k]_\pm}], +) &= ([\mathcal{W}_{[k^k]_\pm}], +) \ , \\ ([\mathcal{W}_{[N]}], -) \times ([\mathcal{W}_{[k^{k-1},k-1,1]}], \pm) &= ([\mathcal{W}_{[k^{k-1},k-1,1]}], \mp) \ , \end{aligned} \tag{4.27}$$

where $([\mathcal{W}_{[N]}], -) = \mathcal{W}_{[1^N]}$.

To summarize, we learn that, in order to generate the fusion rule (4.20), we can gauge a $\mathbb{Z}_2$ one-form symmetry in the S_N (with $N = k^2 \geq 9$) discrete gauge theory with fusion rules (4.25) and (4.27). Crucially, we need a fixed point of the one-form symmetry (as in the second line in (4.27)) in order to generate the fusion rule of the form (4.20) in the A_N discrete gauge theory. We will return to the existence of fixed points of various kinds repeatedly throughout this paper.

One may wonder if zero-form gaugings always resolve fusion rules of the form $a \times b = c$ into fusion rules with multiple outcomes. Taking $G = O(5,3)$, one can see the answer is no.[100] Indeed, in this theory, one can check that we have the following analogs of (4.20)

$$\mathcal{W}_{5_i} \times \mathcal{W}_6 = \mathcal{W}_{30_i} \ , \quad i = 1, 2 \ , \tag{4.28}$$

where 5_i are the two five-dimensional representations of $O(5,3)$, 6 is the unique six-dimensional representation, and 30_i are the two complex thirty-dimensional representations (there is also a third, real, thirty-dimensional representation that does not appear in (4.28)). As in the previous case, $\mathrm{Out}(O(5,3)) = \mathbb{Z}_2$ and it acts non-trivially on the Wilson lines involved in the fusion above. In particular, we have

$$\mathcal{W}_{5_1} \leftrightarrow \mathcal{W}_{5_2} \ \text{ and } \ \mathcal{W}_{30_1} \leftrightarrow \mathcal{W}_{30_2} \tag{4.29}$$

under the action of the non-trivial element in $\mathrm{Out}(O(5,3))$. This symmetry lifts to a symmetry of the discrete gauge theory that we can gauge. Doing so, we can choose parameters such that

$$\mathcal{Z}(\mathrm{Vec}_{O(5,3)})_{\mathbb{Z}_2^\times} \overset{\text{gauge}}{\longrightarrow} \mathcal{Z}(\mathrm{Vec}_{O(5,3) \rtimes \mathbb{Z}_2}) \ . \tag{4.30}$$

[100]This is the group $O(5)$ over the field $\mathbb{F}_3$. It has order 25920 and is the smallest simple group whose discrete gauge theory has a fusion of non-abelian Wilson lines with a unique outcome.

We may again apply (4.23) to find

$$N_{([\mathcal{W}_{5_i}],+),(\mathcal{W}_6,\pm)}^{([\mathcal{W}_{30_i}],+)} = m(+|_{\mathbb{Z}_1}, +|_{\mathbb{Z}_1} \otimes \pm|_{\mathbb{Z}_1}) = m(1,1) = 1 \ , \tag{4.31}$$

and conclude

$$([\mathcal{W}_{5_i}],+) \times (\mathcal{W}_6,\pm) = ([\mathcal{W}_{30_i}],+) \ . \tag{4.32}$$

Such a situation arises whenever $N_c = \mathbb{Z}_1 = N_a \cap N_b$. This equality is special since, more generally, we have $N_a \cap N_b \subseteq N_c$.

Non-simple groups and unfaithful higher-dimensional representations Before moving on to discuss other phenomena, let us note that the above discrete gauge theories based on simple groups also provide answers to questions (2) and (3) from the introduction. Indeed, as we will see in greater detail in section 4.2.2, a discrete gauge theory with a simple gauge group has no non-trivial proper fusion subcategories except the subcategory of Wilson lines. Therefore, our above examples are enough to answer questions (2) and (3) generally in the negative (although we will see interesting examples of some of these ideas below).

Let us now consider discrete gauge theories with unfaithful higher-dimensional (i.e., non-abelian) representations. The corresponding gauge groups are necessarily non-simple because the kernel of a non-trivial unfaithful representation is a non-trivial proper normal subgroup. As we will explain at a more pedestrian level below (and in a somewhat more sophisticated way in section 4.2.2), these examples illustrate the appearance of non-trivial fusion subcategories in the Wilson line sector. As a result, they demonstrate some of the ideas—described in the introduction—behind constraints from subcategory structure leading to fusion rules of the type (4.2). In particular, these theories provide examples where ideas in questions (2) and (3) of the introduction are realized.

To that end, let us consider some unfaithful higher-dimensional irreducible representation of the gauge group, $\pi \in \mathrm{Irrep}(G)$. Since π is unfaithful, it has a non-trivial kernel, $\mathrm{Ker}(\pi) \triangleleft G$. Let us also define the set of characters whose kernel includes $\mathrm{Ker}(\pi)$ as follows

$$K_\pi = \left\{ \chi_\rho : \ \chi_\rho|_{\mathrm{Ker}(\pi)} = \deg \chi_\rho \right\} \ , \tag{4.33}$$

where $\deg \chi_\rho = |\rho|$ is the degree of the character. Now, consider $\chi_\lambda, \chi_{\lambda'} \in K_\pi$. We claim $\chi_\lambda \cdot \chi_{\lambda'} \in K_\pi$. To see this, let us study

$$\chi_\lambda|_{\mathrm{Ker}(\pi)} \cdot \chi_{\lambda'}|_{\mathrm{Ker}(\pi)} = \deg \chi_\lambda \cdot \deg \chi_{\lambda'} = \sum_{\lambda''} \chi_{\lambda''}|_{\mathrm{Ker}(\pi)} \leq \sum_{\lambda''} \left| \chi_{\lambda''}|_{\mathrm{Ker}(\pi)} \right| \ . \tag{4.34}$$

Evaluating this expression on the identity element shows that $\deg \chi_\lambda \cdot \deg \chi_{\lambda'} = \sum_{\lambda''} \deg \chi_{\lambda''}$. Therefore, we have $\chi_{\lambda''}|_{\mathrm{Ker}(\pi)} = \deg \chi_{\lambda''}$, and $\lambda'' \in K_\pi$. In particular, we see that

$$\chi_\lambda \cdot \chi_{\lambda'} = \sum_{\lambda'' \in K_\pi} \chi_{\lambda''} \ . \tag{4.35}$$

As a result, the Wilson lines with charges in K_π form a closed fusion subcategory[101]

$$\mathcal{W}_\lambda \times \mathcal{W}_{\lambda'} = \sum_{\lambda'' \in K_\pi} \mathcal{W}_{\lambda''} \in \mathcal{C}_{K_\pi} \simeq \mathrm{Rep}(G/\mathrm{Ker}(\pi)) \ . \tag{4.36}$$

If we now consider the fusion of $\mathcal{W}_\pi \in \mathcal{C}_{K_\pi}$ with a non-abelian Wilson line $\mathcal{W}_\gamma \notin \mathcal{C}_{K_\pi}$, we see that the subcategory structure makes it more likely to find a unique outcome. Indeed, $\mathcal{W}_\pi \times \mathcal{W}_{\bar{\pi}} \in \mathcal{C}_{K_\pi}$ whereas $\mathcal{W}_\gamma \times \mathcal{W}_{\bar{\gamma}}$ will typically include lines not in $\mathcal{C}_{K_\pi}$.

In fact, we can go further if we take $\gamma|_{\mathrm{Ker}(\pi)}$ to be an irreducible representation of $\mathrm{Ker}(\pi)$. Since we are assuming that γ is a higher-dimensional representation, irreducibility of $\gamma|_{\mathrm{Ker}(\pi)}$ implies that $\mathrm{Ker}(\pi)$ is a non-abelian group. Invoking Gallagher's theorem (e.g., see corollary 6.17 of [105]), we see that, for $\gamma, \pi \in \mathrm{Irrep}(G)$, $\gamma \otimes \pi$ is an irreducible representation if the restriction $\gamma|_{\mathrm{Ker}(\pi)}$ is irreducible. Then, we are guaranteed to have the following fusion rule of non-abelian Wilson lines

$$\mathcal{W}_\pi \times \mathcal{W}_\gamma = \mathcal{W}_{\pi\gamma} \ . \tag{4.37}$$

To understand this statement, let us first prove that $\gamma \notin K_\pi$. Suppose this were not the case: then we arrive at a contradiction since $|\gamma| > 1$ would imply that $\gamma|_{\mathrm{Ker}(\pi)}$ is reducible. As a result, $\mathcal{W}_\gamma \notin \mathcal{C}_{K_\pi}$. Let us now consider the product

$$\chi_\gamma \cdot \chi_{\overline{\gamma}} = \chi_1 + \sum_i \chi_{\alpha_i} \ , \tag{4.38}$$

where α_i are irreps of G. Then we have

$$(\chi_\gamma \cdot \chi_{\overline{\gamma}})|_{\mathrm{Ker}(\pi)} = \chi_1|_{\mathrm{Ker}(\pi)} + \sum_i \chi_{\alpha_i}|_{\mathrm{Ker}(\pi)} \ . \tag{4.39}$$

Here, $\chi_1|_{\mathrm{ker}(\pi)}$ corresponds to the trivial irreducible representation of $\mathrm{Ker}(\pi)$, $\chi_{\alpha_i}|_{\mathrm{Ker}(\pi)}$ corresponds to an, in general, reducible representation of $\mathrm{Ker}(\pi)$. Suppose that $\alpha_i|_{\mathrm{Ker}(\pi)}$ contains the trivial irreducible representation of $\mathrm{Ker}(\pi)$ for some i, then we will have at least two copies of the trivial character of $\mathrm{Ker}(\pi)$ on the right hand side of (4.39). However, we know that $(\gamma \otimes \overline{\gamma})|_{\mathrm{Ker}(\pi)} = \gamma|_{\mathrm{Ker}(\pi)} \otimes \overline{\gamma}|_{\mathrm{Ker}(\pi)}$. Therefore, we cannot have more than one copy of the trivial character in the decomposition (4.39). Hence, $\alpha_i|_{\mathrm{Ker}(\pi)}$ cannot contain the trivial representation for any i. It follows that $\alpha_i|_{\mathrm{Ker}(\pi)}(h)$ is non-trivial for at least some $h \in \mathrm{Ker}(\pi)$. Therefore, it is clear that $\mathrm{Ker}(\pi)$ cannot be in the kernel of the representations α_i for any i. This shows that

$$\mathcal{W}_{\alpha_i} \in \mathcal{W}_\gamma \times \mathcal{W}_{\bar{\gamma}} \ \Rightarrow \ \mathcal{W}_{\alpha_i} \notin \mathcal{C}_{K_\pi} \ . \tag{4.40}$$

As a result, the subcategory structure guarantees (4.37).

To better understand the above general discussion (as well as the continuing role of symmetries), let us consider some examples. Note that these results give explicit realizations of

[101]Such Wilson lines recently played an interesting role in [140]. Indeed, when one adds non-topological matter charged under these representations, the corresponding Wilson lines can end on a point. Magnetic flux lines or dyons with flux supported in $\mathrm{Ker}(\pi)$ remain topological while lines carrying other fluxes do not.

the idea in question (3) in the introduction. The simplest discrete gauge theories realizing the above discussion are based on gauge groups of order forty-eight. Interestingly, the existence of subcategory structure in the Wilson line sector, $\mathcal{C}_{\mathcal{W}} \simeq \text{Rep}(G)$, explains the large ratio of orders, Δ_{gap}, between these groups and the smallest simple group, $O(5,3)$, with unique non-abelian fusion outcomes

$$\Delta_{\text{gap}} = \frac{25920}{48} = 540 \gg 1 \ . \tag{4.41}$$

In this section, we will discuss the examples of the binary octahedral group (BOG) and the very closely related general linear group of 2×2 matrices with elements in the finite field $\mathbb{F}_3$, $GL(2,3)$. In appendix B.2.1 we will consider the remaining cases at order forty-eight.

Let us begin with BOG. In this case, we have that 2_1 is an unfaithful (real) two-dimensional representation and that the restrictions of the other (real and faithful) two-dimensional irreducible representations to $\text{Ker}(2_1) = Q_8 \triangleleft BOG$, $2_{2,3}|_{\text{Ker}(2_1)}$, are irreducible. As expected from the general discussion above we have the following Wilson line fusions

$$\mathcal{W}_{2_1} \times \mathcal{W}_{2_2} = \mathcal{W}_{2_1} \times \mathcal{W}_{2_3} = \mathcal{W}_4 \ . \tag{4.42}$$

Similarly to the simple discrete gauge theories discussed in the previous subsection, BOG's $\mathbb{Z}_2$ outer automorphism again lifts to a non-trivial symmetry of the TQFT, and the non-trivial element $g \neq 1$ acts as follows: $g(\mathcal{W}_{2_2}) = \mathcal{W}_{2_3}$.

Let us note that in this case, the role of symmetries is even more pronounced. Indeed, one can check that

$$\begin{aligned} \mathcal{W}_{2_1} \times \mathcal{W}_{2_1} &= \mathcal{W}_1 + \mathcal{W}_{1_2} + \mathcal{W}_2 \in \mathcal{C}_{K_{2_1}} \simeq \text{Rep}(BOG/Q_8) \simeq \text{Rep}(S_3) \ , \\ \mathcal{W}_{2_2} \times \mathcal{W}_{2_2} &= \mathcal{W}_{2_3} \times \mathcal{W}_{2_3} = \mathcal{W}_1 + \mathcal{W}_{3_2} \ , \end{aligned} \tag{4.43}$$

where 1_2 is a non-trivial one-dimensional irreducible representation, and 3_2 is a real three-dimensional irreducible representation.[102] This latter representation satisfies $\chi_{1_2} \cdot \chi_{3_2} = \chi_{3_1}$ (and similarly $\chi_{1_2} \cdot \chi_{3_1} = \chi_{3_2}$). Therefore, we see that $\mathcal{W}_{1_2}$ generates a non-trivial one-form symmetry in the BOG discrete gauge theory and that $\mathcal{W}_{3_{1,2}}$ and $\mathcal{W}_{2_{2,3}}$ form doublets under fusion with this generator while $\mathcal{W}_{2_1}$ is fixed

$$\mathcal{W}_{1_2} \times \mathcal{W}_{3_2} = \mathcal{W}_{3_1} \ , \quad \mathcal{W}_{1_2} \times \mathcal{W}_{2_2} = \mathcal{W}_{2_3} \ , \quad \mathcal{W}_{1_2} \times \mathcal{W}_{2_1} = \mathcal{W}_{2_1} \ . \tag{4.44}$$

This non-trivial orbit structure then implies that $\mathcal{W}_{3_2} \notin \mathcal{W}_{2_1} \times \mathcal{W}_{2_1}$ on symmetry grounds alone. Hence, in this example, both the subcategory structure and the symmetries guarantee the fusion rules (4.42).

Before finishing this example, we should check that $\mathcal{Z}(\text{Vec}_{BOG})$ is indeed prime. After we discuss more formal aspects of subcategory structure in section 4.2.2, we will have more tools to use when answering this type of question. For now, let us prove that the Wilson

[102] Note that since $2_{2,3}$ are faithful representations, a result of Burnside [42] generalized to Wilson lines shows that there exist $n_{1,2} \in \mathbb{N}$ such that $\mathcal{W}_{2_{2,3}}^{\times n_1} \supset \mathcal{W}_{1_2}$ and $\mathcal{W}_{2_{2,3}}^{\times n_2} \supset \mathcal{W}_{2_1}$. Our discussion implies $n_{1,2} > 2$.

lines must all lie in the same TQFT factor.[103] To that end, write down the Wilson lines of the BOG discrete gauge theory

$$\begin{aligned} &\mathcal{W}_1\ ,\ \ \mathcal{W}_{1_2}\ ,\ \ \mathcal{W}_{2_1}\ ,\ \ \mathcal{W}_{2_2}\ ,\ \ \mathcal{W}_{2_3}=\mathcal{W}_{2_2}\times\mathcal{W}_{1_2}\ ,\ \ \mathcal{W}_{3_1}\ ,\ \ \mathcal{W}_{3_2}=\mathcal{W}_{3_1}\times\mathcal{W}_{1_2}\ ,\\ &\mathcal{W}_4=\mathcal{W}_{2_1}\times\mathcal{W}_{2_2}=\mathcal{W}_{2_1}\times\mathcal{W}_{2_3}\ . \end{aligned} \tag{4.45}$$

We can consider two cases: (1) $\mathcal{W}_{3_1}$ is in the same TQFT factor as $\mathcal{W}_{1_2}$ (call this factor $\mathcal{T}_0$) or (2) $\mathcal{W}_{3_1}$ is not in the same TQFT factor as $\mathcal{W}_{1_2}$.

Let us consider case (1) first. From the fusion equation involving $\mathcal{W}_{3_2}$, we immediately see that $\mathcal{W}_{3_2}$ is also in $\mathcal{T}_0$. Note that $\mathcal{W}_{2_1}$ cannot be written as the fusion product of two other Wilson lines. Since there is no Wilson line of quantum dimension six, we also have $\mathcal{W}_{2_1}\in\mathcal{T}_0$. Now, we must clearly have that either $\mathcal{W}_{2_{2,3}}\in\mathcal{T}_0$ or $\mathcal{W}_{2_{2,3}}\notin\mathcal{T}_0$. However, in the latter case we will again have a Wilson line of quantum dimension six. Therefore, we have that $\mathcal{W}_{2_{2,3}}\in\mathcal{T}_0$. Therefore, by the $\mathcal{W}_4$ fusion rule in (4.45), all Wilson lines are in the same TQFT factor.

Let us now consider case (2). Let $\mathcal{W}_{3_1}\in\mathcal{T}_0$ and $\mathcal{W}_{1_2}\in\mathcal{T}_1$ with $\mathcal{Z}(\text{Vec}_{BOG})=\mathcal{T}_0\boxtimes\mathcal{T}_1$. As in case (1), $\mathcal{W}_{2_1}$ cannot be written as the fusion product of two other Wilson lines, and, since there is no Wilson line of quantum dimension six, we have $\mathcal{W}_2\in\mathcal{T}_0$. However, this leads to a contradiction because then $\mathcal{W}_2\times\mathcal{W}_1'\neq\mathcal{W}_2$. As a result, we conclude that all Wilson lines must lie in the same factor of $\mathcal{Z}(\text{Vec}_{BOG})$.

Let us conclude with a brief discussion of the $GL(2,3)$ discrete gauge theory. This gauge group is quite similar to BOG. For the purposes of the above discussion, the only difference is that $2_{2,3}$ become complex conjugate two-dimensional irreducible representations (otherwise, the remaining representations and remaining parts of the character tables are the same). Therefore, (4.42) and (4.44) apply to $\mathcal{Z}(\text{Vec}_{GL(2,3)})$ as well (by identifying these Wilson lines with their relatives in $\mathcal{Z}(\text{Vec}_{GL(2,3))})$. The only change is that in the second line of (4.43), we should take $\mathcal{W}_{2_{2,3}}\times\mathcal{W}_{2_{2,3}}\to\mathcal{W}_{2_2}\times\mathcal{W}_{2_3}$. In particular, the roles of subcategory structure (again $\text{Rep}(S_3)\subset\text{Rep}(GL(2,3))$) as well as outer automorphisms and one-form symmetries is the same in both the BOG and the $GL(2,3)$ discrete gauge theories.

Note that Gallagher's theorem does not exhaust all cases where representations with non-trivial kernel have irreducible products. Another interesting case is given by Gajendragadkar's theorem [82, 128]. If we have a group G which is both π-separable as well as Σ-separable, for two disjoint set of primes π and Σ, then this theorem guarantees that the product of a π-special character with a Σ-special character is irreducible. A character χ is known as π-special if $\chi(1)$ is a product of powers of primes in π (a π number) and if, for every subnormal subgroup N of G, any irreducible constituent θ of $\chi|_N$ is such that $o(\theta)$[104] is a π-number. Hence, the fusion of Wilson lines corresponding to such characters have a unique outcome. Note that, in this case, one of the characters involved in the fusion is not required to be irreducible in the kernel of the other (unlike in Gallagher's theorem).

[103] The same pedestrian arguments used below can be extended to the full set of lines in the theory to prove that $\mathcal{Z}(\text{Vec}_{BOG})$ is prime.

[104] $o(\theta)$ is the order of the determinental character $\det(\chi)$ in the group of linear characters.

Some general lessons and theorems Let us conclude this section with a recapitulation of some of the main points above as well as some general theorems that amplify our discussion:

- In all of the infinitely many examples we studied so far, symmetries played an important role. For example, zero-form symmetries had a non-trivial action on Wilson lines involved in the fusion rules of interest in the A_N (with $N = k^2 \geq 9$) and $O(5,3)$ discrete gauge theories (see (4.22) and (4.29)), and similarly in theories based on BOG, $GL(2,3)$, and the other order forty-eight groups (e.g., see below (4.42) and in appendix B.2.1). We will revisit some of these discussions after introducing further technical tools for symmetries in section 4.2.3.

- We also saw that we can use $\mathbb{Z}_2$ one-form symmetry gauging in the S_N (with $N = k^2 \geq 9$) gauge theory to generate fusion rules involving non-abelian Wilson lines with unique outcomes in the A_N discrete gauge theories. We can constrain when such a situation arises with the following theorem:

  Theorem 4.1 (one-form fixed points). *Consider a TQFT, $\mathcal{T}$, with no fusion rules of the form* (4.2)*. Suppose we can gauge a non-trivial one-form symmetry of this TQFT, H. After performing this gauging, we have fusion rules of the form* (4.2) *only if there are $a \in \mathcal{T}$ such that fusion with at least one of the one-form generators, $\alpha \in \mathrm{Rep}(H)$, yields $\alpha \times a = a$.*

  Proof: Suppose this were not the case. Then, all anyons are organized into full length orbits under fusion with the one-form symmetry generators. When we gauge the one-form symmetry, we identify these orbits as single elements (if the braiding with one-form symmetry generators is trivial, these orbits become genuine lines of the gauged theory; if the braiding is non-trivial, these orbits become lines bounding symmetry-generating surface operators in the gauged theory). Note that all anyons appearing on the right hand side of fusion rules have the same braiding with the one-form symmetry generators. Therefore, the claim follows. □

  As we will see, this theorem will have echoes in the coset theories we describe in the second half of this paper.

- In the case of $O(5,3)$ discrete gauge theories, we saw that we can gauge the outer automorphisms and have fusion rules of form (4.2) in this gauged theory as well. This discussion inspires the following theorem:

  Theorem 4.2 (zero-form fixed points). *Consider a TQFT, $\mathcal{T}$, and suppose we can gauge a non-trivial zero-form symmetry of this TQFT, H. After performing this gauging, we have fusion rules of the form* (4.2) *only if there are non-trivial $a_i \in \mathcal{T}$ such that at least one of the non-trivial elements of the zero-form group fixes a_i.*

Proof: Suppose that all non-trivial elements of the discrete gauge theory leave all the non-trivial anyons unfixed. Now consider anyons $a, b, c \in \mathcal{T}$ such that $c \in a \times b$. From the general discussion around (4.23), we see that $N_{{}^t a} \cap N_{{}^s b} \cap N_c = \mathbb{Z}_1$ and $N_a \backslash H / N_b = H$. Moreover, since the stabilizers are trivial, $\pi_a = \pi_b = \pi_c = 1$ are the trivial representations. We then have

$$N_{([a],1),([b],1)}^{([c],1)} = |H| \cdot m(1,1) = |H| > 1 \ . \tag{4.46}$$

Therefore, we cannot produce fusion rules of the desired type. □

Our discussion of the $O(5,3)$ theory also suggests the following theorem

Corollary 4.2.1. *Consider a TQFT, $\mathcal{T}$, with a fusion rule of the form $a \times b = c$ and a zero-form symmetry, H. If at least one of $\{a, b, c\}$ is unfixed by H, then the only way for $a \times b = c$ to map to a fusion rule with unique outcome in the gauged theory is for c to be unfixed by H.*

Proof: If c is unfixed by H, then $N_c = N_a \cap N_b = \mathbb{Z}_1$. If either a or b are unfixed then $N_a \cap N_b = \mathbb{Z}_1$ as well (although we need not have $N_c = \mathbb{Z}_1$). In any case, (4.23) becomes

$$N_{([a],\pi_a),([b],\pi_b)}^{([c],\pi_c)} = \sum_{(t,s) \in N_a \backslash H / N_b} m(\pi_c|_{\mathbb{Z}_1}, {}^t\pi_a|_{\mathbb{Z}_1} \otimes {}^s\pi_b|_{\mathbb{Z}_1} \otimes \pi^{\omega}_{({}^t a, {}^s b, c)}) \ . \tag{4.47}$$

We have two cases: **(1)** $N_a \backslash H / N_b \neq \mathbb{Z}_1$ or **(2)** $N_a \backslash H / N_b = \mathbb{Z}_1$. Consider case **(1)** first. In this case, all resulting fusion rules will have multiplicity $|N_a \backslash H / N_b| > 1$. Next, consider case **(2)**. If c is fixed by some element of H, then we have at least two possible π_c (one is the trivial representation). This results in a fusion rules with non-unique outcomes. □

- In the case of the BOG and $GL(2,3)$ discrete gauge theories we saw that both one-form symmetries and subcategory structure offered an explanation of the existence of the fusion rules (4.42). The following theorem further explains and generalizes this connection between symmetries and subcategories of the Wilson line sector:

  Theorem 4.3 (subcategories and symmetries)**.** *Consider a finite group, G, with an unfaithful higher-dimensional irreducible representation, π. Moreover, suppose there are one-dimensional representations, π_i, with $\mathrm{Ker}(\pi_i) \trianglerighteq \mathrm{Ker}(\pi)$. Then, in the corresponding (twisted or untwisted) discrete gauge theory, Wilson lines charged under representations, γ, that have $\gamma|_{\mathrm{Ker}(\pi)}$ irreducible transform non-trivially under fusion with the abelian Wilson lines, $\mathcal{W}_{\pi_i}$.*

  Proof: We have that $\mathcal{W}_{\pi_i} \in \mathcal{C}_{K_\pi}$, where $\mathcal{C}_{K_\pi}$ was defined around (4.36) as the subcategory of Wilson lines charged under representations whose kernels contain $\mathrm{Ker}(\pi)$ (see (4.33)). Therefore, we see that the abelian Wilson lines $\mathcal{W}_{\pi_i} \in \mathcal{C}_{K_\pi}$.

By the discussion around (4.40), we also see that all non-identity lines $\mathcal{W}_{\alpha_i} \in \mathcal{W}_\gamma \times \mathcal{W}_{\bar{\gamma}}$ are not elements of $\mathcal{C}_{K_\pi}$. As a result, $\mathcal{W}_{\pi_i} \notin \mathcal{W}_\gamma \times \mathcal{W}_{\bar{\gamma}}$. On the other hand, the trivial line is clearly in $\mathcal{W}_\gamma \times \mathcal{W}_{\bar{\gamma}}$. This logic implies

$$\mathcal{W}_{\pi_i} \times \mathcal{W}_\gamma \times \mathcal{W}_{\bar{\gamma}} \neq \mathcal{W}_\gamma \times \mathcal{W}_{\bar{\gamma}} \ , \tag{4.48}$$

from which the claim in the theorem trivially follows. □

This result tells us that the $\mathcal{W}_\gamma$ must transform under fusion with the one-form symmetry generators while $\mathcal{W}_\pi$ need not. In the case of the BOG and $GL(2,3)$ discrete gauge theories, precisely this mechanism gave a symmetry explanation for the $\mathcal{W}_\pi \times \mathcal{W}_\gamma = \mathcal{W}_{\pi\gamma}$ fusion rule in (4.42). Here we see it is somewhat more general.

- Note that the results of this section answer questions (1)-(3) of the introduction negatively in general. Still, we saw that in the BOG and $GL(2,3)$ discrete gauge theories, the ideas in (3) and (4.11) do apply in some cases. We will return to a proposal for construct a theory satisfying (4.10) in question (2) in section 4.2.2.

4.2.2 Subgroups, subcategories, and primality

In sections 4.2.1, we saw the important role subcategories play in generating fusion rules involving non-abelian Wilson lines with unique outcomes (e.g., they explained the hierarchy in (4.41)). Moreover, understanding the subcategory structure is crucial to resolving the question of whether a particular discrete gauge theory is prime or not. In the case of theories with simple gauge groups (see section 4.2.1), we used results from [127]. In the case of the examples of discrete gauge theories with non-simple groups we studied, we used an argument that does not easily generalize. Therefore, in this section, we review some of the more general results of [127] on subcategories of discrete gauge theories. We then apply these results to generate some useful theorems that will serve us in subsequent sections.

The main power of the results in [127] is that they rephrase questions about subcategories in discrete gauge theories in terms of data of the underlying gauge group. In particular, we have:

Theorem 4.4 ([127])**.** *Fusion subcategories of discrete gauge theories with finite group G are in bijective correspondence with triples, (K, H, B). Here $K, H \trianglelefteq G$ are normal subgroups that centralize each other (i.e., they commute element-by-element), and $B : K \times H \to \mathbb{C}^\times$ is a G-invariant bicharacter. If we have a non-trivial twist, ω, then the same conditions hold except that we demand that B is a G-invariant ω-bicharacter.*

Proof: See proofs of Theorems 1.1 and 1.2 (though they are phrased using different, but equivalent, terminology) of [127]. □

Since B is a bicharacter, it satisfies

$$B(k_1k_2, h) = B(k_1, h) \cdot B(k_2, h) \ , \quad B(k, h_1h_2) = B(k, h_1) \cdot B(k, h_2) \ . \tag{4.49}$$

Here G invariance means that $B(g^{-1}kg, g^{-1}hg) = B(k,h)$ for all $k \in K$, $h \in H$, and $g \in G$. In fact, [127] also give a way to construct the subcategory, $\mathcal{S}(K,H,B)$, in question given the above data:

$$\mathcal{S}(K,H,B) := \text{gen}\left((a,\chi)|\left\{a \in K\cap R\ ,\ \chi \in \text{Irr}(N_a) \text{ s.t. } \chi(h) = B(a,h)\deg\chi\ ,\ \forall h \in H\right\}\right)\ , \tag{4.50}$$

where R is a set of representatives of conjugacy classes, $\text{Irr}(N_a)$ is the set of characters of irreducible representations of the centralizer N_a, and "gen$(\cdots)$" means that the category is generated by the simple objects inside the parenthesis. A normal subgroup is a union of conjugacy classes. Hence, K specifies all the conjugacy classes labelling the anyons in the subcategory $\mathcal{S}(K,H,B)$. Also, all the Wilson lines in $\mathcal{S}(K,H,B)$ are such that the corresponding representations have kernels which contain H.

If we have non-trivial twist, then (4.49) and G-invariance become [127]

$$\begin{aligned} B(k_1k_2,h) &= \eta_h(k_1,k_2)\cdot B(k_1,h)\cdot B(k_2,h)\ ,\ B(k,h_1h_2) = \eta_k^{-1}(h_1,h_2)\cdot B(k,h_1)\cdot B(k,h_2)\ , \\ B(g^{-1}kg,h) &= \frac{\eta_k(g,h)\eta_k(gh,g^{-1})}{\eta_k(g,g^{-1})}B(k,ghg^{-1})\ , \end{aligned} \tag{4.51}$$

where

$$\eta_g(h,k) := \frac{\omega(g,h,k)\cdot\omega(h,k,k^{-1}h^{-1}ghk)}{\omega(h,h^{-1}gh,k)}\ , \tag{4.52}$$

is a generalization of (1.52). For non-trivial twist, we also have that (4.50) becomes

$$\mathcal{S}(K,H,B) := \text{gen}\left((a,\chi)|\left\{a \in K\cap R\ ,\chi \in \text{Irr}_\omega(N_a) \text{ s.t. } \chi(h) = B(a,h)\deg\chi\ ,\forall h \in H\right\}\right)\ , \tag{4.53}$$

where the ω in $\text{Irr}_\omega(N_a)$ is a reminder that we should consider characters with projectivity phase given by (1.52) or (4.52).

We can now immediately see how the subcategories we studied in previous sections arose: $\mathcal{S}(G,\mathbb{Z}_1,1) \simeq \mathcal{Z}(\text{Vec}_G^\omega)$ is the full discrete gauge theory, $\mathcal{S}(\mathbb{Z}_1,G,1)$ is the trivial subcategory, and $\mathcal{S}(\mathbb{Z}_1,\mathbb{Z}_1,1) \simeq \text{Rep}(G) \simeq \mathcal{C}_\mathcal{W}$ is the full subcategory of Wilson lines. In the case of simple discrete gauge theories, we see that, as claimed in section 4.2.1, these are the *only* subcategories. However, in the case of the $\mathcal{Z}(\text{Vec}_{BOG}^\omega)$, $\mathcal{Z}(\text{Vec}_{GL(2,3)}^\omega)$, and other gauge theories based on gauge groups with unfaithful irreducible representations, π, we find additional subcategories: $\mathcal{S}(\mathbb{Z}_1,\text{Ker}(\pi),1) \simeq \text{Rep}(G/\text{Ker}(\pi))$ and $\mathcal{S}(\text{Ker}(\pi),\mathbb{Z}_1,1)$. Using Lemma 3.11 of [127], we have that $\mathcal{S}(\text{Ker}(\pi),\mathbb{Z}_1,1)$ is the Müger center of $\mathcal{S}(\mathbb{Z}_1,\text{Ker}(\pi),1)$.

Since we will study flux lines and dyons below, it is interesting to ask what the above theorems imply for such operators. One immediate consequence is that magnetic flux lines behave very differently from Wilson lines. For example:

Theorem 4.5. *The set of magnetic flux lines, $\mathcal{M}$, of a discrete gauge theory (both untwisted and twisted) with non-abelian gauge group, G, do not form a fusion subcategory. In particular, $\mathcal{M} \not\simeq \text{Rep}(G)$.*

Proof: Suppose the full set of flux lines form a subcategory. Then, we need K to include at least one element of each conjugacy class in order to include all of $\mathcal{M}$ in $\mathcal{S}$. However, since K is a normal subgroup, it must consist of full conjugacy classes. Therefore, $K = G$. Using theorem 4.4, we can label this putative subcategory as $\mathcal{S}(G, H, B)$. Since H has to commute with all elements in G, it has to be a subgroup of the center of the group $Z(G)$. Suppose the group has trivial center. This forces $B = 1$, and $\mathcal{S}(G, \mathbb{Z}_1, 1)$ is the full discrete gauge theory, which means we also include objects with charge. This is a contradiction.

Suppose H is a non-trivial subgroup of $Z(G)$. We know that the function B, being a bicharacter, satisfies $B(e, h) = 1\ \forall h \in H$. So the Wilson line $([e], \pi) \in \mathcal{S}(G, H, B)$ if π has H in its kernel. Recall that the irreducible representations of G/H are in one-to-one correspondence with irreducible representations of G with H in its kernel. Since G is non-abelian, $Z(G) \neq G$. Hence, G/H is a non-trivial group. It follows that there is at least one non-trivial irreducible representation π' of G such that H is in its kernel. Hence, the Wilson line $([e], \pi')$ belongs to the subcategory $\mathcal{S}(G, H, B)$ for any B. A contradiction. $\square$

The fact that $\mathcal{M} \not\simeq \mathrm{Rep}(G)$ has consequences in section 4.2.3. In particular, it explains why electric-magnetic self-dualities are non-trivial to engineer in theories with non-abelian gauge groups and trivial centers.[105] If such a duality exists and involves magnetic flux lines, then they will necessarily be in a $\mathrm{Rep}(G)$-like subcategory with objects carrying electric charge (e.g., see the S_3 discrete gauge theory self-duality [17], where the dimension-two flux line is in a $\mathrm{Rep}(S_3)$ subcategory with both dimension one Wilson lines).

Now, we turn to the question of primality. Here the following theorem of [127] is useful

Theorem 4.6 ([127])**.** *A discrete gauge theory with gauge group, G, is a prime TQFT if and only if there is no triple (K, H, B) with $K, H \triangleleft G$ normal subgroups centralizing each other, $HK = G$, $(G, \mathbb{Z}_1) \neq (K, H) \neq (\mathbb{Z}_1, G)$, and B is a G-invariant bicharacter on $K \times H$ such that $BB^{\mathrm{op}}|_{(K\cap H)\times(K\cap H)}$ is non-degenerate. In the case of non-trivial twisting, ω, the previous conditions still hold, but B is also a G-invariant ω-bicharacter.*

Proof: See proof of theorem 1.3 (though it is phrased using different, but equivalent, terminology) in [127]. $\square$

Note that in the statement of theorem 6, $B^{\mathrm{op}}(h, k) := B(k, h)$ for all $k \in K$ and $h \in H$.

Given this theorem, we may prove the following result that will be useful to us in section 4.2.5:

Theorem 4.7. *If G is a non-direct product group with trivial center, then the corresponding (twisted or untwisted) gauge theory is a prime TQFT.*

Proof: We have a non-direct product group G with trivial center. Let us assume that $\mathrm{Rep}(D(G))$ has a modular subcategory. Then, there exists two normal subgroups, K and H,

[105] In any untwisted abelian gauge theory, this is not an issue as $\mathcal{M} \simeq \mathrm{Rep}(G)$ and there is a canonical electric/magnetic duality.

commuting with each other and satisfying $KH = G$. So, every element of G is a product of an element of K with an element of H. Hence, any element in $K \cap H$ has to commute with all elements of G. Since the center of G is trivial by choice, $K \cap H = \mathbb{Z}_1$. It follows that G has to be a direct product of K and H. A contradiction. Hence, for non-direct product groups G with trivial center, $\text{Rep}(D(G))$ is prime. $\square$

A simple set of examples subject to this theorem include the S_N discrete gauge theories analyzed above and the $\mathbb{Z}_{15} \rtimes \mathbb{Z}_4$ discrete gauge theory we will analyze further in section 4.2.5.

Finally, we conclude with a proposal for engineering an example of a theory of the type envisioned in question (2) in the introduction. In particular, consider a $G \times G$ discrete gauge theory, $\mathcal{Z}(\text{Vec}^{\omega}_{G\times G})$. Clearly, for trivial twisting this is a non-prime theory since $\mathcal{Z}(\text{Vec}_{G\times G}) = \mathcal{Z}(\text{Vec}_G) \boxtimes \mathcal{Z}(\text{Vec}_G)$. Indeed, by theorem 4.6, we can take $K = G \times \mathbb{Z}_1$, $H = \mathbb{Z}_1 \times G$, and $B = 1$. However, if we turn on a twist, $\omega \in H^3(G \times G, U(1))$, we might be able to generate a prime theory. In particular, if we can find G such that ω is non-trivial and does not factorize, then we would have an example of a prime theory with Wilson lines in $\text{Rep}(G \times G) = \text{Rep}(G) \boxtimes \text{Rep}(G)$. Choosing one Wilson line in each $\text{Rep}(G)$ factor and fusing would give a unique fusion outcome.[106] It would be interesting to see if this proposal can be realized. For example, we would like to see if there is an obstruction at the level of the existence of a G-invariant ω-bicharacter (all other requirements of theorem 4.6 can be satisfied). A concrete example of a theory of the type discussed in question (2) is studied in section 4.2.5.

4.2.3 Zero-form symmetries

In sections 4.2.1 and 4.2.1 we saw that zero-form symmetries played an important role in generating fusions rules of the form (4.2). In this section we review some relevant results of [130] and prove a theorem that will be useful to us in section 4.2.5.

In three spacetime dimensions, zero-form symmetries are implemented by dimension two topological defects (recall that one-form symmetries are generated by abelian lines). These defects act on lines that pierce them as in figure 35. We will say the corresponding symmetry group, H, is non-trivial iff it has a generator, $h \in H$, such that there is an anyon $a \in \mathcal{T}$ satisfying $h(a) \neq a$.

Note that the automorphisms of the gauge group G, $\text{Aut}(G)$, are a natural source of symmetries. Indeed, in the context of the G-SPT that we gauge to generate the discrete gauge theory, these automorphisms permute the symmetry defects. Therefore, we expect they will play a role in the discrete gauge theory. To be more precise, recall that we can distinguish between the inner automorphisms $\text{Inn}(G) \trianglelefteq \text{Aut}(G)$, generated by conjugations of the form gxg^{-1} for $x, g \in G$, and outer automorphisms, $\text{Out}(G) := \text{Aut}(G)/\text{Inn}(G)$. Since the discrete gauge theory involves magnetic charges labeled by conjugacy classes and electric charges labeled by representations of centralizers, it is clear that inner automorphisms will

[106] We thank D. Aasen for suggesting the basis for this idea.

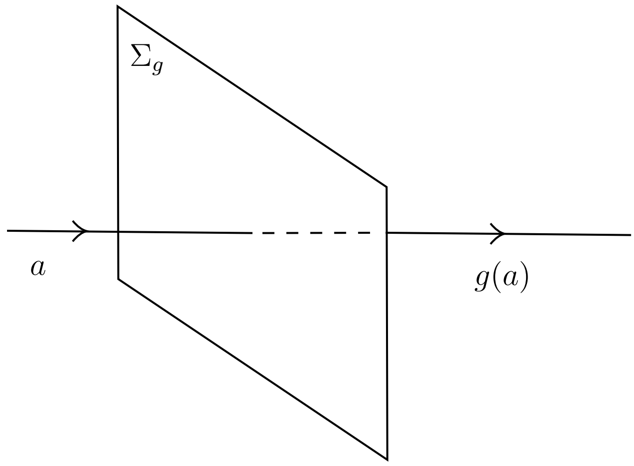


Figure 35: The symmetry defect Σ_g, labelled by a zero-form symmetry group element g, acts on an anyon a.

act trivially on the discrete gauge theory (conjugacy classes are invariant under $\mathrm{Inn}(G)$ and the normalizers of different elements in a conjugacy class are isomorphic). Therefore, we can at best expect $\mathrm{Out}(G)$ to lift to a symmetry of the TQFT. Indeed, this is precisely what happens.

More formally, we have that, in a discrete gauge theory $\mathrm{Out}(G)$ lifts to a part of the group of braided autoequivalences of the discrete gauge theory, $\mathrm{Aut}^{\mathrm{br}}(\mathcal{Z}(\mathrm{Vec}_G))$:

Theorem 4.8 ([130])**.** *The subgroup of braided autoequivalences that fix the Wilson lines* $\mathrm{Stab}(\mathrm{Rep}(G)) \leq \mathrm{Aut}^{\mathrm{br}}(\mathcal{Z}(\mathrm{Vec}_G))$ *takes the form*

$$\mathrm{Stab}(\mathrm{Rep}(G)) \simeq H^2(G, U(1)) \rtimes \mathrm{Out}(G) \ . \tag{4.54}$$

Proof: See the proof of Corollary 6.9 (though it is phrased using different, but equivalent, terminology) in [130]. □

Note that Out(G) generally acts non-trivially on the conjugacy classes. Therefore, it will also generally act non-trivially on the Wilson lines. However, in certain more exotic cases, all of Out(G) preserves conjugacy classes.[107] In such cases, the Wilson lines are fixed. Note that elements $\zeta \in H^2(G, U(1))$ always leave the Wilson lines invariant since they act as follows [130]

$$\zeta(([a], \pi_a)) = ([a], \pi_g \rho_g) \ , \quad \rho_g(x) := \frac{\zeta(x, g)}{\zeta(g, x)} \ , \tag{4.55}$$

where $g \in [a]$ (in particular, $g = 1$ for Wilson lines). Note that $\rho_g(x)$ depends only on the cohomology class of ζ (it is invariant under shifts by a 2-coboundary).

[107] The smallest group that has this feature has order 2^7 [27]. See [56] for an application of groups that have at least some class-preserving outer automorphisms to quantum doubles.

A second set of symmetries involves the exchange of electric and magnetic degrees of freedom. These are electric/manetic self-dualities and are inherently quantum mechanical in nature. These symmetries are closely related to the existence of Lagrangian subcategories. As we briefly mentioned at the beginning of section 4.2.1, a Lagrangian subcategory, $\mathcal{L}$, is a collection of bosons with trivial mutual braiding that is equal to its Müger center (e.g., like the subcategory of Wilson lines, $\mathcal{C}_{\mathcal{W}} \simeq \mathrm{Rep}(G)$) . This latter condition simply means that the only objects that braid non-trivially with *every* element of $\mathcal{L}$ are elements of that subcategory.

To find the set of these symmetries, it turns out to be useful to construct the categorical Lagrangian Grassmannian, $\mathbb{L}(G)$. This is the collection of all Lagrangian subcategories. Each such subcategory, $\mathcal{L}_{(N,\mu)} \simeq \mathrm{Rep}(G_{(N,\mu)})$ with $|G_{(N,\mu)}| = |G|$, is labeled by a normal abelian subgroup, $N \lhd G$, and a G-invariant $\mu \in H^2(N, U(1))$ (the Wilson line subcategory is $\mathcal{L}_{1,1}$). For the purposes of understanding these symmetries, the important subcategory is [130]

$$\mathbb{L} \supseteq \mathbb{L}_0 := \{\mathcal{L} \in \mathbb{L}(G) | \mathcal{L} \simeq \mathrm{Rep}(G)\} \ . \tag{4.56}$$

In particular, we have

Theorem 4.9 ([130])**.** *The action of* $\mathrm{Aut}^{\mathrm{br}}(\mathcal{Z}(\mathrm{Vec}_G))$ *on* $\mathbb{L}_0(G)$ *is transitive. Moreover,*

$$|\mathrm{Aut}^{\mathrm{br}}(\mathcal{Z}(\mathrm{Vec}_G))| = |H^2(G, U(1))| \cdot |\mathrm{Out}(G)| \cdot |\mathbb{L}_0(G)| \ . \tag{4.57}$$

Proof: See proposition 7.6 and corollary 7.7 of [130]. □

Examples of such dualities appear in the S_3 discrete gauge theory [17] and beyond [100].

Let us now apply this theorem to prove a result that will be useful for us below

Theorem 4.10. *If* $G \simeq N \rtimes K$*, where* N *is an abelian group, then the corresponding untwisted discrete gauge theory has an electric-magnetic self-duality.*

Proof: By theorem 4.9, in order to find a self-duality, we need to find a normal abelian subgroup $N \lhd G$ and a G-invariant 2-cocycle, $\mu \in H^2(N, U(1))$. Moreover, we need to find a corresponding $G_{(N,\mu)} \simeq G$. In particular, from remark 7.3 of [130], when μ is trivial, we have that $G_{(N,1)} \simeq \widehat{N} \rtimes G/N$, where $\widehat{N}$ is the character group of N. For an abelian group, $\widehat{N} \simeq N$. Therefore, we have that $G_{(\widetilde{N},1)} \simeq N \rtimes K = G$ as desired. □

This theorem will be useful in our symmetry searches in section 4.2.5. Note that one immediate consequence of the above discussion is that none of the examples discussed above have self-dualities. Indeed, theories with simple gauge groups have no non-trivial normal abelian subgroups. On the other hand, theories like BOG and $GL(2,3)$ have $H^2(BOG, U(1)) \simeq H^2(GL(2,3), U(1)) \simeq \mathbb{Z}_1$ (and similarly for all normal abelian subgroups). Since these groups are not semi-direct products, we conclude they lack self-dualities.

4.2.4 Quasi-zero-form symmetries

In the previous subsections, we have seen that zero-form symmetries play an important role in generating fusion rules for non-abelian anyons with unique outcomes. However, since our interest is simply in the existence of such fusion rules, it is natural that we should generalize our notion of symmetry to include symmetries of the modular data (and hence, by Verlinde's formula, automorphisms of the fusion rules) that don't necessarily lift to symmetries of the TQFT.[108] The basic reason such "quasi zero-form symmetries" as we will call them exist is that the modular data does not define a TQFT (see [119] for a consequence of this fact). In particular, the underlying F and R symbols may not be invariant (up to an allowed gauge transformation) under a quasi zero-form symmetry even if S and T are.

In fact, such "quasi-zero-form symmetries" are common, with charge conjugation being a particular example [58]. Indeed, even in the A_N (with $N = k^2 \geq 9$) theories we discussed in section 4.2.1, such quasi-charge conjugation symmetries exist. These symmetries are in addition to the genuine zero-form symmetries we described when analyzing these examples. In appendix B.2.2, we study the particular case of A_9 discrete gauge theory in more detail and explicitly disentangle the quasi-symmetries from the genuine symmetries.

More generally, there are theories that have no genuine symmetries. One set of examples include discrete gauge theories based on the Mathieu groups. These are simple groups with trivial Out(G) and $H^2(G, U(1))$. Moreover, since these groups have no non-trivial normal abelian subgroups, $\mathbb{L}(G) = \mathbb{L}_0(G) \simeq \text{Rep}(G)$, and so there are no non-trivial self-dualities.

The largest Mathieu groups, M_{23} and M_{24} are of particular interest to us since their discrete gauge theories have non-abelian Wilson lines that fuse together to produce a unique outcome.[109] Moreover, of the theories with fusions of type (4.2), these are the only untwisted discrete gauge theories that have no modular symmetries that lift to symmetries of the full TQFTs.

For M_{23} it is not hard to check that

$$\mathcal{W}_{22} \times \mathcal{W}_{45_1} = \mathcal{W}_{990_1} \ , \quad \mathcal{W}_{22} \times \mathcal{W}_{45_2} = \mathcal{W}_{990_2} \ , \tag{4.58}$$

where 22 is the real twenty-two dimensional representation, $45_{1,2}$ are two forty five dimensional complex representations, and $990_{1,2}$ are two nine hundred and ninety dimensional representations. Under charge conjugation

$$\mathcal{W}_{45_1} \leftrightarrow \mathcal{W}_{45_2} \ , \quad \mathcal{W}_{990_1} \leftrightarrow \mathcal{W}_{990_2} \ . \tag{4.59}$$

For M_{24}, we have a particularly rich set of fusions[110]

$$\mathcal{W}_{23} \times \mathcal{W}_{45_1} \ = \ \mathcal{W}_{1035_2} \ , \ \mathcal{W}_{23} \times \mathcal{W}_{45_2} = \mathcal{W}_{1035_3} \ , \ \mathcal{W}_{23} \times \mathcal{W}_{231_1} = \mathcal{W}_{5313}$$

[108] In fact, most generally, we might expect automorphisms of the fusion rules that are not even symmetries of the modular data (e.g., as studied recently in [41]).

[109] By the results of [35], these theories cannot have such fusions involving lines that carry magnetic flux.

[110] It would be interesting to know if our results here have any connection with moonshine phenomena observed involving M_{24} as in [66, 50, 84].

$$\mathcal{W}_{23} \times \mathcal{W}_{231_2} = \mathcal{W}_{5313}\ ,\ \mathcal{W}_{45_1} \times \mathcal{W}_{231_1} = \mathcal{W}_{10395}\ ,\ \mathcal{W}_{45_2} \times \mathcal{W}_{231_1} = \mathcal{W}_{10395}\ ,$$
$$\mathcal{W}_{45_1} \times \mathcal{W}_{231_2} = \mathcal{W}_{10395}\ ,\ \mathcal{W}_{45_2} \times \mathcal{W}_{231_2} = \mathcal{W}_{10395}\ . \tag{4.60}$$

where 23 is a real twenty-three dimensional representation, $45_{1,2}$ are complex forty-five dimensional representations, $231_{1,2}$ are two-hundred and thirty-one dimensional complex representations, and $1035_{2,3}$ are complex one-thousand and thirty-five dimensional representations, 5313 is a real five-thousand three-hundred and thirteen dimensional representation, and 10395 is a real ten-thousand three-hundred and ninety-five dimensional representation. Under charge conjugation, we have

$$\mathcal{W}_{45_1} \leftrightarrow \mathcal{W}_{45_2}\ , \quad \mathcal{W}_{231_1} \leftrightarrow \mathcal{W}_{231_2}\ , \quad \mathcal{W}_{1035_2} \leftrightarrow \mathcal{W}_{1035_3}\ . \tag{4.61}$$

While we have seen similar actions in previous sections, but here the novelty is that charge conjugation is a quasi-symmetry.

More generally, as we will discuss in greater detail below, all other examples of TQFTs that we have found with fusion rules involving non-abelian anyons with unique outcome have at least quasi zero-form symmetries.

Finally, let us conclude this section by discussing how twisting affects the quasi-zero-form symmetries. When the quasi-symmetry is charge conjugation and the group has complex representations, the quasi-symmetry lifts to an action on Wilson lines (see appendix B.2.2 for a discussion in a concrete example). In this case, the quasi-symmetry persists regardless of the twisting.

As a more complicated example, let us consider the case of BOG first discussed in section 4.2.1. This theory only has real conjugacy classes and representations. However, there is still a non-trivial charge conjugation acting on certain dyons since elements in BOG have centralizer groups $\mathbb{Z}_4$, $\mathbb{Z}_6$, and $\mathbb{Z}_8$. These latter groups admit complex representations. However, unlike the spectrum of Wilson lines, the spectrum of dyons generally changes as we change the twist. Therefore, we might imagine that the charge conjugation quasi symmetry can be twisted away.

In fact, this is not the case. The main point is that any twisting $\omega \in H^3(BOG, U(1)) \simeq \mathbb{Z}_{48}$ of the BOG discrete gauge theory is "cohomologically trivial" in the following sense: the $\eta_g(h,k) \in H^2(N_g, U(1))$ phases defined in (1.52) are all trivial. Indeed, this statement follows from the fact that $H^2(N_g, U(1)) = \mathbb{Z}_1$ for all $g \in BOG$. Therefore, none of the anyons are lifted by the twisting, and the characters of BOG change as follows

$$\chi_{\pi_g^\omega}(h) \to \epsilon_g(h) \cdot \chi_{\pi_g^\omega}(h) \tag{4.62}$$

where ϵ_g is a 1-cochain that gives the 2-coboundary, η_g. It is not too hard to check that all choices of the twisting leave us with complex characters. Therefore, the charge conjugation quasi-symmetry persists (here it would be more accurate to term it a "modular symmetry" since it is apriori possible—though we have not checked—that charge conjugation becomes a symmetry of the theory for certain choices of ω).[111]

[111] One may also wonder about the fate of the genuine $\mathrm{Out}(BOG) \simeq \mathbb{Z}_2$ zero-form symmetry under twisting.

4.2.5 Beyond Wilson lines

So far, we have only constructed fusion rules of the form (4.2) using Wilson lines. In the case of gauge theories with simple groups, this is all we can do [35]. However, when we have non-simple gauge groups, the existence of self-dualities discussed in section 4.2.3 as well as the possibility of electric-magnetic dualities between theories with different gauge groups and Dijkgraaf-Witten twists [126, 100] suggests that we should also be able to involve non-abelian anyons carrying flux. Indeed, we will see this is the case.

To that end, let us study a fusion of the form

$$\mathcal{L}_{([g],\pi_g^\omega)} \times \mathcal{L}_{([h],\pi_h^\omega)} = \mathcal{L}_{([k],\pi_k^\omega)} \ , \quad g,h \neq 1 \ , \tag{4.63}$$

Carefully applying the machinery in section 1.1.1 reveals the following contraints[112]

1. $[g]\cdot[h] = [k] = [h]\cdot[g]$
2. $\exists!\ \pi_k^\omega$ such that $m(\pi_k^\omega|_{N_g\cap N_h\cap N_k}, \pi_g^\omega|_{N_g\cap N_h\cap N_k} \otimes \pi_h^\omega|_{N_g\cap N_h\cap N_k} \otimes \pi^\omega_{(g,h,k)}) = 1$

We will apply these constraints in what follows.

For an untwisted discrete gauge theory based on a group G with a non-trivial center $Z(G)$, the constraints above implies that if we have a fusion of Wilson lines giving a unique outcome

$$\mathcal{W}_\pi \times \mathcal{W}_\gamma = \mathcal{W}_{\pi\gamma} \ , \tag{4.64}$$

then we have a fusion of dyons of the form

$$\mathcal{L}_{([g],\pi)} \times \mathcal{L}_{([h],\gamma)} = \mathcal{L}_{([gh],\pi\gamma)} \ , \tag{4.65}$$

where for any $g,h \in Z(G)$. Hence, we can dress the Wilson lines with fluxes from the center of the group to obtain fusion rules involving dyons with unique outcomes. For example, we have already seen that the discrete gauge theories corresponding to BOG and $GL(2,3)$ have Wilson lines fusing to give a unique outcome. Since these two groups have a non-trivial center (isomorphic to $\mathbb{Z}_2$), the above discussion immediately implies the existence of dyonic fusions where the dyons are labelled by the non-trivial element of the centre. In fact, these two types of fusions exhaust all $a\times b = c$ type fusions in both $\mathcal{Z}(\mathrm{Vec}_{BOG})$ and $\mathcal{Z}(\mathrm{Vec}_{GL(2,3)})$.

In the case of the fusion of non-abelian Wilson lines with a unique outcome, we saw that we were not guaranteed to find fusion subcategories beyond the three universal subcategories

First, consider ω corresponding to the order 2 element in $\mathbb{Z}_{48}$. Since $\mathrm{Out}(BOG)$ acts on $H^3(BOG,U(1))$ through $\mathrm{Aut}(H^3(BOG,U(1)))$, ω should be fixed under it. Hence, it seems plausible that the twisted discrete gauge theory corresponding to this choice of ω has $\mathrm{Out}(BOG)$ as a subgroup of its symmetries (while theorem 8 has nothing to say on this point since it assumes untwisted theories, we view the existence of a symmetry in this case as a plausible assumption). In fact, more generally, if the action of $\mathrm{Out}(G)$ leaves $\omega \in H^3(G,U(1))$ invariant up to a 3-coboundary, then it can be shown that this is a symmetry of the modular data of the twisted theory. It would be interesting to understand what happens for other twists as well.

[112] We refer the interested reader to the derivation in section III of [35] for further details.

present in any discrete gauge theory (the theory itself, the trivial TQFT, and the Wilson line sector, $\mathcal{C}_{\mathcal{W}} \simeq \text{Rep}(G)$). On the other hand, when we have fusions of non-abelian anyons carrying flux with a unique outcome, we are guaranteed to have fusion subcategories. When the gauge group has a non-trivial center, $Z(G)$, this statement is trivial.[113] The following theorems guarantee this fact more generally:

Theorem 4.11. *Let G be a non-simple finite non-abelian group. If we have a fusion rule involving two dyons or fluxes giving a unique outcome in the (twisted or untwisted) G gauge theory, then $\mathcal{S}(M_g, \mathbb{Z}_1, 1)$ and $\mathcal{S}(M_h, \mathbb{Z}_1, 1)$ (along with $\mathcal{S}(\mathbb{Z}_1, M_g, 1)$ and $\mathcal{S}(\mathbb{Z}_1, M_h, 1)$) are proper fusion subcategories of the theory. Here, g and h are elements labelling the non-trivial conjugacy classes (of length > 1) involved in the fusion. M_g is the normal subgroup generated by the elements in $[g]$.*

Proof: We have an $a \times b = c$ type fusion rule involving the non-trivial conjugacy classes $[g]$ and $[h]$. Let M_g be the normal subgroup generated by $[g]$. In fact, it has to be a proper normal subgroup. To see this, suppose $M_g = G$. From Lemma 3.4 of [127], we know that $[g]$ and $[h]$ commute element-wise. Hence, $[h]$ commutes with all elements in $M_g = G$. It follows that $[h]$ should be a subset of the elements in $Z(G)$. However, elements of $Z(G)$ form single element conjugacy classes. A contradiction. Hence, M_g has to be a proper normal subgroup of G. Since $g \neq e$, it is clear that M_g is not the trivial subgroup either. We can use the same argument to show that M_h is also a proper non-trivial normal subgroup of G. Therefore, by theorem 4.4, we have fusion subcategories corresponding to the choices $\mathcal{S}(M_g, \mathbb{Z}_1, 1)$ and $\mathcal{S}(M_h, \mathbb{Z}_1, 1)$ (and similarly $\mathcal{S}(\mathbb{Z}_1, M_g, 1)$ and $\mathcal{S}(\mathbb{Z}_1, M_h, 1)$). $\square$

Note that we have, $\mathcal{L}_{([g], \pi_g^\omega)} \in \mathcal{S}(M_g, \mathbb{Z}_1, 1)$ and $\mathcal{L}_{([h], \pi_h^\omega)} \in \mathcal{S}(M_h, \mathbb{Z}_1, 1)$. Generically, we also expect $\mathcal{L}_{([g], \pi_g^\omega)} \notin \mathcal{S}(M_h, \mathbb{Z}_1, 1)$ and $\mathcal{L}_{([h], \pi_h^\omega)} \notin \mathcal{S}(M_g, \mathbb{Z}_1, 1)$. In such situations we have, in the spirit of section 4.2.1, an "explanation" for the fusion rule.

In fact, the reasoning in the proof to theorem 4.11 immediately implies that if $[h]$ has at least one element $h' \in [h]$ such that $[h', h] \neq 1$, then $\mathcal{L}_{([g], \pi_g^\omega)}$ and $\mathcal{L}_{([h], \pi_h^\omega)}$ lie in different subcategories

Corollary 4.11.1. *Given the conditions in theorem 4.11, if there exists $h' \in [h]$ such that $[h', h] \neq 1$, $\mu_{[g]} \in \mathcal{S}(M_g, \mathbb{Z}_1, 1)$, $\mathcal{L}_{([h], \pi_h^\omega)} \notin \mathcal{S}(M_g, \mathbb{Z}_1, 1)$, and similarly for $h \leftrightarrow g$.*

For $a \in M_g$ the fusion subcategory $\mathcal{S}(M_g, \mathbb{Z}_1, 1)$ contains anyons $([a], \pi_a)$ where π_a is any irrep of the centralizer N_a. In an untwisted discrete gauge theory, for a fusion of fluxes labelled by conjugacy classes $[g]$ and $[h]$, we can define fusion subcategories $\mathcal{S}(M_g, M_h, 1)$ and $\mathcal{S}(M_h M_g, 1)$ which have a more restricted set of elements. For $a \in M_g$, the anyon $([a], \pi_a)$ is an element of $\mathcal{S}(M_g, M_h, 1)$ if and only if $M_h \subseteq \text{Ker}(\pi_a)$. Clearly, $([g], 1_g) \in \mathcal{S}(M_g, M_h, 1)$

[113]The discussion in section 4.2.2 guarantees that $\mathcal{S}(Z(G), \mathbb{Z}_1, 1)$ and $\mathcal{S}(\mathbb{Z}_1, Z(G), 1)$ are non-trivial subcategories.

and $([h], 1_h) \in \mathcal{S}(M_h, M_g, 1)$. However, in general, we don't expect $([g], 1_g) \notin \mathcal{S}(M_h, M_g, 1)$ and $([h], 1_h) \notin \mathcal{S}(M_g, M_h, 1)$. We will discuss an example of this below.

If one of the operators involved in the fusion of non-abelian anyons with a unique outcome is a Wilson line, then we also have the following theorem:

Theorem 4.12. *Let G be a non-simple group. If we have a fusion of a Wilson line and a dyon giving a unique outcome, then $\mathcal{S}(Ker(\chi_\pi), \mathbb{Z}_1, 1)$ and $\mathcal{S}(\mathbb{Z}_1, \mathrm{Ker}(\chi_\pi), 1)$ are proper fusion subcategories of the (twisted or untwisted) discrete gauge theory. Here, π is an irrep of G labelling the Wilson line.*

Proof: Suppose $[b]$ is the non-trivial conjugacy labelling the flux line. Let χ_π be the character of an irreducible representation, π, of G labelling the Wilson line. From note 3.5 of [127] we know that χ should be trivial on a subset of elements given by $[G, b]$. Since b is not in the center, $[G, b]$ is guaranteed to have a non-trivial element. Hence, χ_π is not a faithful representation. $\mathrm{Ker}(\chi_\pi)$ is a non-trivial normal subgroup of G. Since χ_π is not the trivial representation, $\mathrm{Ker}(\chi_\pi) \neq G$ is a non-trivial proper normal subgroup. Hence, by theorem 4.4, we have a fusion subcategory given by $\mathcal{S}(\mathrm{Ker}(\chi_\pi), \mathbb{Z}_1, 1)$ and $\mathcal{S}(\mathbb{Z}_1, \mathrm{Ker}(\chi_\pi), 1)$. $\square$

Note that in this case the Wilson line is an element of $\mathcal{S}(\mathbb{Z}_1, \mathrm{Ker}(\chi_\pi), 1)$ while the magnetic flux is not. In this sense, such fusions are "natural." To illustrate the ideas above, let us consider the following examples.

$\mathcal{Z}(\mathbf{Vec}_{\mathbb{Z}_3 \rtimes Q_{16}})$ Let us consider the $\mathbb{Z}_3 \rtimes Q_{16}$ discrete gauge theory. Even though this group has many non-trivial proper normal subgroups, we have $\mathbb{Z}_3 \rtimes Q_{16} \neq HK$ for any proper normal subgroups H, K. Hence, using theorem 4.6, we have that $\mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_3 \rtimes Q_{16}})$ is a prime theory.

This group has a length 2 conjugacy class $[f_3]$ (here we are using the notation of GAP [85], where this group is entry $(48, 18)$ in GAP's small group library) and a 2-dimensional representation 2_3 (the third 2-dimensional representation in the character table of $\mathbb{Z}_3 \rtimes Q_{16}$ on GAP). We have the following fusion of a Wilson line and a flux line giving a unique outcome.

$$\mathcal{W}_{2_3} \times \mu_{[f_3]} = \mathcal{L}_{([f_3], 2_3|_{N_{f_3}})} \ , \tag{4.66}$$

where the restricted representation $2_3|_{N_{f_3}}$ is irreducible.

Since we have a prime theory, the existence of this fusion rule is not due to a Deligne product. However, it can be explained using the subcategory structure of $\mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_3 \rtimes Q_{16}})$. To that end, consider the fusion subcategory $\mathcal{S}(\mathbb{Z}_1, \mathrm{Ker}(2_3)), 1)$. This fusion subcategory contains only Wilson lines. A Wilson line $\mathcal{W}_\pi$ belongs to this subcategory only if $\mathrm{Ker}(2_3)$ is in $\mathrm{Ker}(\pi)$. From the character table of $\mathbb{Z}_3 \rtimes Q_{16}$, we find three representations satisfying this constraint: 1, 1_3 and 2_3. Here 1 is the trivial representation and 1_3 is the third 1-dimensional representation in the character table. Hence, the anyons contained in the fusion subcategory

$\mathcal{S}(\mathbb{Z}_1, \mathrm{Ker}(2_3), 1)$ are the Wilson lines $\mathcal{W}_1$, $\mathcal{W}_{1_3}$ as well as $\mathcal{W}_{2_3}$. Moreover, we can check the following

$$1_3 \times 1_3 = 1; \quad 1_3 \times 2_3 = 2_3; \quad 2_3 \times 2_3 = 1 + 1_3 + 2_3. \tag{4.67}$$

Now let us consider a fusion subcategory corresponding to the triple $\mathcal{S}(M_{f_3}, \mathrm{Ker}(1_2), 1)$ where M_{f_3} is the normal subgroup generated by the elements of the conjugacy class $[f_3]$ and 1_2 is the second 1 dimensional representation in the character table of $\mathbb{Z}_3 \rtimes Q_{16}$. We have $M_{f_3} = \{e, f_3, f_4, f_3 \cdot f_4\}$. A Wilson line $\mathcal{W}_\pi$ belongs to the set of generators of this subcategory only if $\mathrm{Ker}(1_2)$ is in $\mathrm{Ker}(\pi)$. Using the character table we can check that there are only two representations which satisfy this constraint: 1 and 1_2. Moreover, we have $1_2 \times 1_2 = 1$. Hence, the Wilson lines in $\mathcal{S}(M_{f_3}, \mathrm{Ker}(1_2), 1)$ are $\mathcal{W}_1$ and $\mathcal{W}_{1_2}$. Note that the flux line $\mu_{[f_3]}$ belongs to this subcategory.

Hence, we have two fusion subcategories $\mathcal{S}(\mathbb{Z}_1, \mathrm{Ker}(2_3)), 1)$ and $(M_{f_3}, \mathrm{Ker}(1_2), 1)$ with the following structure

$$\begin{aligned} \mathcal{W}_{2_3} \in \mathcal{S}(\mathbb{Z}_1, \mathrm{Ker}(2_3)), 1); \quad \mu_{[f_3]} \in (M_{f_3}, \mathrm{Ker}(1_2), 1); \\ \mathcal{S}(\mathbb{Z}_1, \mathrm{Ker}(2_3)), 1) \cap \mathcal{S}(M_{f_3}, \mathrm{Ker}(1_2), 1) = \{\mathcal{W}_1\} \end{aligned} \tag{4.68}$$

Therefore, the fusions $\mathcal{W}_{2_3} \times \overline{\mathcal{W}}_{2_3}$ and $\mu_{[f_3]} \times \overline{\mu}_{[f_3]}$ have only $\mathcal{W}_1$ in common. This trivial intersection explains the fusion (4.66) and gives an example of the idea behind question (2) in the introduction.

$\mathcal{Z}(\mathbf{Vec}_{\mathbb{Z}_{15} \rtimes \mathbb{Z}_4})$ Let us consider the $\mathbb{Z}_{15} \rtimes \mathbb{Z}_4$ discrete gauge theory. Since the center of the gauge group is trivial and the group involves a semi-direct product, we know from theorem 4.7 that this gauge theory is prime.

This group has a length 5 conjugacy class labelled by the element f_2 and a length 2 conjugacy class labelled by the element f_3 (here we are using the notation of GAP, where this group is entry $(60, 7)$ in GAP's small group library). We also have a length 10 conjugacy class labeled by $f_2 f_3$. It is therefore clear that we have a fusion of flux lines giving a unique outcome corresponding to these conjugacy classes

$$\mu_{[f_2]} \times \mu_{[f_3]} = \mu_{[f_2 f_3]} \ . \tag{4.69}$$

Based on our discussion above, let us consider the groups M_{f_2} and M_{f_3} generated by the elements in the corresponding conjugacy class. It is not too hard to show that

$$M_{f_2} = [e] \cup [f_2] \tag{4.70}$$

$$M_{f_3} = [e] \cup [f_3] \cup [f_4] \tag{4.71}$$

Hence, the fusion subcategories $\mathcal{S}(M_{f_2}, M_{f_3}, 1)$ and $\mathcal{S}(M_{f_3}, M_{f_2}, 1)$ can only have Wilson lines as common elements. The trivial Wilson line $\mathcal{W}_1$ is of course a common element. As we saw in section 4.2.2, a Wilson line, $\mathcal{W}_\pi$, is a member of the fusion subcategory, $\mathcal{S}(M_{f_2}, M_{f_3}, 1)$, only if the condition

$$\chi_\pi(h) := B(e,h) \deg \chi_\pi = \deg \chi_\pi \ , \quad \forall\, h \in M_{f_3} \ , \tag{4.72}$$

is satisfied. Hence, M_{f_3} should be in the kernel of χ_π. Similarly, a Wilson line $\mathcal{W}_{\pi'}$, is a member of $(M_{f_3}, M_{f_2}, 1)$ only if M_{f_2} is in the kernel of $\chi_{\pi'}$. Therefore, the common elements of the two fusion subcategories are given by the Wilson lines $W_{\widetilde{\pi}}$ for which M_{f_2} and M_{f3} are in the kernel of $\chi_{\widetilde{\pi}}$. Using the character table of $\mathbb{Z}_{15} \rtimes \mathbb{Z}_2$, we find that there is only one representation π_{1_2}, which satisfies this constraint.

Consider the fusions

$$\begin{aligned} \mu_{[f_2]} \times \mu_{[f_2^{-1}]} &= \mathcal{W}_1 + \cdots \ , && (4.73)\\ \mu_{[f_3]} \times \mu_{[f_3{}^{-1}]} &= \mathcal{W}_1 + \cdots \ . && (4.74) \end{aligned}$$

We know $\mu_{[f_2]}$ and $\mu_{[f_3]}$ belong to the fusion subcategories $(M_{f_2}, M_{f_3}, 1)$ and $(M_{f_3}, M_{f_2}, 1)$. Therefore, the only anyons common to both fusions above are $\mathcal{W}_1$ and $\mathcal{W}_{1_2}$. We would like to know whether the Wilson line, $\mathcal{W}_{1_2}$, appears on the right hand side of these fusions. To that end, consider the fusion

$$\mathcal{W}_{1_2} \times \mu_{[f_3]} = \mathcal{L}_{([f_3], 1_2|_{N_{f_3}})} \ . \tag{4.75}$$

It turns out that $1_2|_{N_{f_3}}$ is the trivial representation of N_{f_3}. Hence, $\mu_{[f_3]}$ is fixed under fusion with the one-form symmetry generator, $\mathcal{W}_{1_2}$. So it is clear that $\mathcal{W}_{1_2}$ should appear in the fusion $\mu_{[f_3]} \times \mu_{[f_3{}^{-1}]}$. Similarly, consider the fusion

$$\mathcal{W}_{1_2} \times \mu_{[f_2]} = \mathcal{L}_{([f_2], 1_2|_{N_{f_2}})} \ . \tag{4.76}$$

It is easy to check that $1_2|_{N_{f_2}}$ is a non-trivial representation of N_{f_2}. Hence, $\mu_{[f_2]}$ is not fixed under the fusion with $\mathcal{W}_{1_2}$. Since $\mathcal{W}_{1_2}$ is an order two anyon, it cannot appear in the fusion $\mu_{[f_2]} \times \mu_{[f_2{}^{-1}]}$ (because if $\mathcal{W}_{1_2} \subset \mu_{[f_2]} \times \mu_{[f_2{}^{-1}]}$, then multiplying both sides on the left with $\mathcal{W}_{1_2}$ implies that $\mathcal{L}_{([f_2], 1_2)}$ is the inverse of $\mu_{[f_2]}$ which is clearly false).

We have that the fusions $\mu_{[f_2]} \times \mu_{[f_2^{-1}]}$ and $\mu_{[f_3]} \times \mu_{[f_3^{-1}]}$ only have the trivial anyon in common. Hence, the combination of subcategory structure and one-form symmetry explains the fusion rule

$$\mu_{[f_2]} \times \mu_{[f_3]} = \mu_{[f_2 f_3]} \ . \tag{4.77}$$

It is interesting to note that this discussion parallels the one for Wilson lines in section 4.2.1.

This example is additionally illuminating because this theory also has a fusion involving a Wilson and a flux line with unique outcome. Indeed, we have two 2-dimensional representations 2_1 and 2_2 of $\mathbb{Z}_{15} \rtimes \mathbb{Z}_4$ whose restriction to the centralizer $N_{f_2} = \mathbb{Z}_3 \rtimes \mathbb{Z}_4$ are irreducible. Hence, we have the fusion rules

$$\mathcal{W}_{2_1} \times \mu_{[f_2]} = \mathcal{L}_{([f_2], 2_1|_{N_{f_2}})} \ , \quad \mathcal{W}_{2_2} \times \mu_{[f_2]} = \mathcal{L}_{([f_2], 2_2|_{N_{f_2}})} \ . \tag{4.78}$$

Do we have trivial braiding between the anyons involved in this fusion? This question is equivalent to whether the dyons are bosons are not. For $\mathcal{L}_{([f_2],2_i|_{N_{f_2}})}$ to be a boson, we want f_2 to be in the kernel of $2_i|_{f_2}$, which is equivalent to the condition that f_2 be in the kernel of 2_i. Using this condition, we can easily check to see that the anyons $\mathcal{W}_{2_1}$ and $\mu_{[f_2]}$ braid non-trivially with each other, while $\mathcal{W}_{2_2}$ and $\mu_{[f_2]}$ braid trivially with each other.

Moreover, this theory has several fusions involving dyons which give a unique output. For example, consider the dyons $\mathcal{L}_{([f_2],\widetilde{1}_{f_2})}$ and $\mathcal{L}_{([f_3],\widetilde{1}_{f_3})}$, where $\widetilde{1}_{f_2}$ and $\widetilde{1}_{f_3}$ are the unique non-trivial real 1-dimensional representations of $N_{f_2} = \mathbb{Z}_3 \rtimes \mathbb{Z}_4$ and $N_{f_3} = \mathbb{Z}_3 \times D_{10}$, respectively. We have the fusion

$$\mathcal{L}_{([f_2],\widetilde{1}_{f_2})} \times \mathcal{L}_{([f_3],\widetilde{1}_{f_3})} = \mathcal{L}_{([f_2 f_3],\widetilde{1}_{f_2 f_3})} \tag{4.79}$$

where $\widetilde{1}_{f_2 f_3}$ is the unique non-trivial 1-dimensional representation of $N_{f_2 f_3} = \mathbb{Z}_6$.

Let us also explore the zero-form symmetry of this theory. We have $\mathrm{Out}(\mathbb{Z}_{15} \rtimes \mathbb{Z}_4) = \mathbb{Z}_2$ and $H^2(\mathbb{Z}_{15} \rtimes \mathbb{Z}_4) = \mathbb{Z}_1$. From theorem 4.10, we know that this theory features non-trivial self-duality. In fact, the group $\mathbb{Z}_{15} \rtimes \mathbb{Z}_4$ has three non-trivial normal abelian subgroups $\mathbb{Z}_3, \mathbb{Z}_5, \mathbb{Z}_{15}$ all of which have trivial 2nd cohomology group. So we have the Lagrangian subcategories

$$\{\mathcal{L}_{(\mathbb{Z}_1,1)}, \mathcal{L}_{(\mathbb{Z}_3,1)}, \mathcal{L}_{(\mathbb{Z}_5,1)}, \mathcal{L}_{(\mathbb{Z}_{15},1)}\} \tag{4.80}$$

Using remark 7.3 in [130], we have

$$\mathcal{L}_{(N,1)} \simeq \mathrm{Rep}((\mathbb{Z}_{15} \rtimes \mathbb{Z}_4)_{(N,1)}) \simeq \widehat{N} \rtimes (\mathbb{Z}_{15} \rtimes \mathbb{Z}_4)/\widehat{N} \tag{4.81}$$

where $\widehat{N}$ is the group of representations of N and $N = \mathbb{Z}_3, \mathbb{Z}_5, \mathbb{Z}_{15}$. Also, we have the isomorphisms

$$\mathbb{Z}_{15} \rtimes \mathbb{Z}_4 \simeq \mathbb{Z}_3 \rtimes (\mathbb{Z}_5 \rtimes \mathbb{Z}_4) \simeq \mathbb{Z}_5 \rtimes (\mathbb{Z}_3 \rtimes \mathbb{Z}_4) \tag{4.82}$$

Hence, all Lagrangian subcategories above are isomorphic to $\mathrm{Rep}(\mathbb{Z}_{15} \rtimes \mathbb{Z}_4)$. Hence, $|\mathbb{L}_0(\mathbb{Z}_{15} \rtimes \mathbb{Z}_4)| = 4$. From theorem 4.9, we know that $\mathrm{Aut}^{\mathrm{br}}(\mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_{15} \rtimes \mathbb{Z}_4}))$ should act transitively on $\mathbb{L}(\mathbb{Z}_{15} \rtimes \mathbb{Z}_4)$. In fact, we can use proposition 7.11 of [130] to show that $H^2(\mathbb{Z}_{15} \rtimes \mathbb{Z}_4, U(1)) \rtimes \mathrm{Out}(\mathbb{Z}_{15} \rtimes \mathbb{Z}_4) \simeq \mathbb{Z}_2$ acts trivially on $\mathbb{L}_0(\mathbb{Z}_{15} \rtimes \mathbb{Z}_4)$. Using theorem 4.9, we have $|\mathrm{Aut}^{\mathrm{br}}(\mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_{15} \rtimes \mathbb{Z}_4}))| = 8$.

Finally, since $\mathbb{Z}_{15} \rtimes \mathbb{Z}_4$ has complex characters, $\mathcal{Z}(\mathrm{Vec}^{\omega}_{\mathbb{Z}_{15} \rtimes \mathbb{Z}_4})$ has a non-trivial quasi-zero-form symmetry given by charge conjugation.

Symmetry and quasi-symmetry searches We have used the software GAP to search for groups for which the corresponding untwisted discrete gauge theories have fusions rules with unique outcomes. We present our results below. The relevant GAP code is given in Appendix B.2.3.

Fusion of Wilson lines

Irreducible representations of a direct product of groups is the product of representations of the individual groups. Hence, it is natural that the first example with two Wilson lines fusing to give a unique Wilson line is the quantum double of $S_3 \times S_3$ (however, this fusion arises because the discrete gauge theory factorizes; this follows from theorem 4.6). More interesting (non-direct-product) groups with this property only appear at order 48 (see Appendix B.2.1). For groups of order less than or equal to 639 (except orders 384, 512, 576)[114] we have verified that whenever the corresponding untwisted discrete gauge theory has a fusion Wilson lines giving a unique outcome, $\text{Aut}^{\text{br}}\mathcal{Z}(\text{Vec}_G)$ is non-trivial. In this set of groups, there are two which have a trivial automorphism group. They are $S_3 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$ and $(((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$. However, $H^2(S_3 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4), U(1)) = \mathbb{Z}_2$ leading to non-trivial $\text{Aut}^{\text{br}}(\mathcal{Z}(\text{Vec}_{S_3\times(\mathbb{Z}_5\rtimes\mathbb{Z}_4)}))$. The group $(((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ has trivial $H^2(G, U(1))$. So the theory $\mathcal{Z}(\text{Vec}_{(((\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes Q_8)\rtimes\mathbb{Z}_3)\rtimes\mathbb{Z}_2})$ doesn't have classical symmetries. $(((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ has only one abelian normal subgroup $N = \mathbb{Z}_3 \times \mathbb{Z}_3$. Moreover, we have $(((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2 \simeq N \rtimes K$ where $K = GL(2, 3)$. Therefore, using theorem 10, we know that this theory has non-trivial electric-magnetic duality. The groups $S_3 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$ and $(((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ have complex characters, hence the corresponding discrete gauge theories have quasi-zero-form symmetries.

Fusion of flux lines

The simplest example of an untwisted discrete gauge theory with a fusion of two flux lines giving a single outcome is $\mathcal{Z}(\text{Vec}_{S_3\times S_3})$. The conjugacy classes of a direct product is a product of conjugacy classes of the individual groups. Hence, it follows that quantum doubles of direct products naturally have such fusions. As mentioned above, it follows from theorem 4.6 that discrete gauge theories based on direct product groups are non-prime. Therefore, the fusion rules with unique outcome in this case are a consequence of the Deligne product. Since $\text{Out}(S_3 \times S_3) = \mathbb{Z}_2$, $\mathcal{Z}(\text{Vec}_{S_3\times S_3})$ has non-trivial zero-form symmetry.
After $S_3 \times S_3$, we have several groups of order 48 with flux fusions giving unique outcome. The examples discussed in Appendix B.2.1 (except BOG and $GL(2, 3)$) exhaust all such groups of order 48. All of these groups have non-trivial automorphism group, and hence the corresponding discrete gauge theory has non-trivial symmetries. In fact, for groups of order less than or equal to 639 (except orders 384, 512, 576) we have verified that whenever the corresponding untwisted discrete gauge theory has a fusion of flux lines with a unique outcome, $\text{Aut}^{\text{br}}(\mathcal{Z}(\text{Vec}_G))$ is non-trivial. In fact, the only group with a trivial automorphism group in this set is $S_3 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$. We already discussed above that this theory has non-trivial zero-form symmetries as well as non-trivial quasi-zero-form symmetries.

Fusion of a Wilson line with a flux line

The simplest example with a fusion of a Wilson line and a flux line giving a single outcome is $\mathcal{Z}(\text{Vec}_{S_3\times S_3})$. Then we have more examples in order 48. The examples discussed in Appendix B.2.1 (except BOG and $GL(2, 3)$) exhausts all such groups of order 48. For groups of order

[114] We have not checked order 384, 512, 576 due to the huge number of groups (up to isomorphism) with these orders.

less than or equal to 639 (except orders 384, 512, 576) we have verified that whenever the corresponding untwisted discrete gauge theory has a fusion of a Wilson line with a flux line giving a unique outcome, $\mathrm{Aut}^{\mathrm{br}}\mathcal{Z}(\mathrm{Vec}_G)$ is non-trivial. In this set of groups, there are three which have a trivial automorphism group. They are $S_3 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes QD_{16}$ (where QD_{16} is the semi-dihedral group of order 16) and $(((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$. We discussed the groups $S_3 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$ and $(((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ above. The group $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes QD_{16}$ has trivial $H^2(G, U(1))$. So the theory $\mathcal{Z}(\mathrm{Vec}_{(\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes QD_{16}})$ doesn't have classical symmetries. However, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes QD_{16}$ has one abelian normal subgroup $N = \mathbb{Z}_3 \times \mathbb{Z}_3$. Moreover, we have $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes QD_{16} \simeq N \rtimes K$ where $K = QD_{16}$. Therefore, using theorem 4.10, we know that the corresponding untwisted discrete gauge theory has non-trivial electric-magnetic self-duality.

The group $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes QD_{16}$ has complex characters, hence the corresponding discrete gauge theory has quasi-zero-form symmetries.

Fusion of general dyons

Being a Deligne product, $\mathcal{Z}(\mathrm{Vec}_{S_3\times S_3})$ also has fusions involving dyons, and this is the smallest rank theory with such fusions. The next example is in order 48. The examples discussed in Appendix B.2.1 exhausts all such groups of order 48. For groups of order less than or equal to 100 we have verified that whenever the corresponding untwisted discrete gauge theory has a fusion of two dyons giving a unique outcome, $\mathrm{Aut}^{\mathrm{br}}\mathcal{Z}(\mathrm{Vec}_G)$ is non-trivial. In fact, every group in this set has non-trivial automorphism group. Hence, they all have non-trivial classical 0-form symmetries.

4.3 $a \times b = c$ and WZW models

In this section, we turn our attention to a (generally) very different set of theories: TQFTs based on G_k Chern-Simons (CS) theories and cosets thereof (here G is a compact simple Lie group). Unlike the theories discussed in section 4.2, the theories we discuss here are typically chiral (i.e., $c_{\mathrm{top}} \neq 0 \pmod 8$).

In order to gain a sense of what such theories allow us to do in constructing TQFTs with fusion rules of the form (4.2) and (4.8), it is useful to recall the basic representation theory of $SU(2)$. Somewhat surprisingly, this intuition will be quite useful for more general $SU(N)_k$ CS theories. To that end, consider the textbook matter of the fusion of $SU(2)$ spin j_1 and j_2 representations

$$j_1 \otimes j_2 = \sum_{j=|j_1-j_2|}^{j_1+j_2} j \ . \tag{4.83}$$

As in the case of the finite groups in the previous section, we would like to understand if we can have $j_1 \otimes j_2 = j_3$ for $j_1, j_2 > 0$ and fixed j_3 spin. Clearly this is impossible, since we would have $j_1 + j_2 > |j_1 - j_2|$ and the sum (4.83) will have at least two contributions.

While this result is rather trivial, it is useful to recast it using the group theory analog of the F-transformation described in the introduction (as well as in section 4.2 for the case of discrete groups). To that end, we wish to consider

$$j_1 \otimes j_1 = \sum_{j=0}^{2j_1} j \ , \quad j_2 \otimes j_2 = \sum_{k=0}^{2j_2} k \ , \quad |j_{1,2}| > 1 \ , \tag{4.84}$$

where $|j_{1,2}|$ are the dimensions of the representations. In particular, we see that (since $j_{1,2} > 0$) both products in (4.84) must always contain the trivial representation and the adjoint representation. This observation also implies that $j_1 \otimes j_2 \neq j_3$ for fixed j_3 spin.

The discussion around (4.84) easily generalizes to arbitrary compact simple Lie group, G. In particular, let us consider

$$\alpha \otimes \bar{\alpha} = 1 + \sum_{\gamma \neq 1} N_{\alpha\bar{\alpha}}^{\gamma}\ \gamma \ , \quad \beta \otimes \bar{\beta} = 1 + \sum_{\delta} N_{\beta\bar{\beta}}^{\delta}\ \delta \ , \quad |\alpha|,\ |\beta| > 1 \ , \tag{4.85}$$

where α, β and $\bar{\alpha}, \bar{\beta}$ are conjugate higher-dimensional irreducible representations of G, $\mathrm{Irr}(G)$. The number of times the adjoint appears in the product $\alpha \otimes \bar{\alpha}$ is [22]:

$$N_{\alpha\bar{\alpha}}^{\mathrm{adj}} = \left|\left\{\lambda_j^{(\alpha)} \neq 0\right\}\right| \geq 1 \ , \tag{4.86}$$

where $\lambda_j^{(\alpha)}$ are the Dynkin labels of α. Therefore, we learn that for all higher-dimensional representations of G

$$\alpha \otimes \beta \neq \gamma \ , \ \forall\ |\alpha|,\ |\beta| > 1 \ , \ \alpha, \beta, \gamma \in \mathrm{Irr}(G) \ , \tag{4.87}$$

Of course, our interest is in the fusion algebra of G_k. From this perspective, the above discussion is in the limit $k \to \infty$. As we will prove in the next section, taking $G_k = SU(N)_k$ and imposing finite level does not lead to fusions of the form (4.2) or (4.8).

4.3.1 G_k CS theory

Let us now consider the finite-level deformation of the fusion rules discussed in the previous section. These are the fusion rules of Wilson lines in G_k CS theory. We first consider $SU(2)_k$ as it is rather illustrative. We will then generalize to $SU(N)_k$ and comment on more general G_k.

In the case of $SU(2)_k$, (4.83) becomes [86, 61]

$$j_1 \otimes j_2 = \sum_{j=|j_1-j_2|}^{\min(j_1+j_2, k-j_1-j_2)} j \ . \tag{4.88}$$

In addition to truncating the spectrum to the spins $\{0, 1/2, 1, \cdots, k/2\}$, the above deformation abelianizes the spin $k/2$ representation (since $k/2 \otimes k/2 = 0$). However, these changes do not alter the conclusion from the previous section: we cannot write $j_1 \otimes j_2 = j_3$ for j_3 non-abelian irreducible $j_{1,2,3}$. Indeed, consider

$$j_1 \otimes j_1 = \sum_{j=0}^{\min(2j_1, k-2j_1)} j \ , \quad j_2 \otimes j_2 = \sum_{j=0}^{\min(2j_2, k-2j_2)} j \ , \quad j_{1,2} \neq 0, \ \frac{k}{2} \ . \tag{4.89}$$

The conditions $j_{1,2} \neq 0, \ \frac{k}{2}$ are to ensure that the representation is non-abelian. In particular, we again see that the adjoint representation appears in (4.89).

While the fusion rules discussed in [86, 61] apply to more general groups, they are rather difficult to implement. Instead, using proposals suggested in [111, 110] and finally proven in [68], the authors of [154] show that for α an irreducible representation of G_k (with G a compact simple Lie group), we have

$${}^{(k)}N_{\alpha\bar{\alpha}}^{\rm adj} = \left|\left\{\widehat{\lambda}_j^{(\alpha)} \neq 0\right\}\right| - 1 \ , \tag{4.90}$$

where $\widehat{\lambda}_j^{\alpha}$ are the associated affine Dynkin labels.

In particular, for $SU(N)_k$, if $|\alpha| > 1$, then ${}^{(k)}N_{\alpha\bar{\alpha}}^{\rm adj} \geq 1$.[115] Indeed, the abelian representations, γ_i, satisfy a $\mathbb{Z}_N$ fusion algebra and are characterized by $\widehat{\lambda}_j^{(\gamma_i)} = k\delta_{ij}$, where $i \in \{0, 1, ..., N-1\}$. On the other hand, all non-abelian representations have at least two non-zero Dynkin labels. As a result, we learn that

$$\alpha \otimes \beta \neq \gamma \ , \quad \forall \alpha, \beta, \gamma \in {\rm Irr}(SU(N)_k) \ , \quad |\alpha|, |\beta| > 1 \ . \tag{4.91}$$

Therefore, we see that we have the following fusions for non-abelian Wilson lines in $SU(N)_k$ CS TQFT

$$\mathcal{W}_\alpha \times \mathcal{W}_\beta = \mathcal{W}_\gamma + \cdots \ , \quad |\alpha|, |\beta| > 1 \ , \tag{4.92}$$

where the ellipses necessarily include additional Wilson lines. This statement is more generally true in any G_k CS theory (with G a compact and simple Lie group) for which the lines in question correspond to affine representations with at least two non-zero Dynkin labels.

Note that for certain G_k, non-abelian representations can have a single non-vanishing Dynkin label. For example, consider the $(E_7)_2$ CS theory.[116] It has Wilson lines $\mathcal{W}_\tau$ and $\mathcal{W}_\sigma$ with quantum dimensions $\frac{1+\sqrt{5}}{2}$ and $\sqrt{2}$, repectively, and they fuse to give a unique outcome. The existence of this fusion rule follows from the fact that $(E_7)_2$ is not a prime TQFT. In fact, it resolves into the product of prime theories Fib $\boxtimes$ Ising$'$, where Fib is the Fibonacci anyon theory and Ising$'$ is a TQFT with the same fusion rules as the the Ising model.

We can apply the above arguments to learn about global properties of G_k CS theory. For example, we can ask if G_k CS theory is prime or not. The answer is no in general. Indeed, consider the case $G = SU(2)$. For $k \in \mathbb{N}_{\rm even}$, $SU(2)_k$ is prime. However, for $k \in \mathbb{N}_{\rm odd}$, the abelian anyon generating the $\mathbb{Z}_2$ one-form symmetry forms a modular subcategory. By

[115] Here we define $|\alpha|$ to be the quantum dimension.

[116] We thank a referee for pointing out this example.

Müger's theorem [125] (see also [104] for a discussion at the level of RCFT), it then decouples and the theory resolves into a product of two prime theories

$$SU(2)_k \simeq \begin{cases} SU(2)_1 \boxtimes SU(2)_k^{\rm int} \ , & \text{if } k = 1 \ (\text{mod}4) \\ \overline{SU(2)_1} \boxtimes SU(2)_k^{\rm int} \ , & \text{if } k = 3 \ (\text{mod}4) \ . \end{cases} \tag{4.93}$$

where $SU(2)_k^{\rm int}$ is a TQFT built out of the integer spin $SU(2)_k$ representations. Here $\overline{SU(2)_1}$ is the TQFT conjugate to $SU(2)_1$ (these TQFTs are sometimes called the anti-semion and semion theories in the condensed matter literature).

While G_k CS theory is not prime in general, our arguments above readily prove the following:

Claim : Non-abelian Wilson lines in $SU(N)_k$ CS theory must all lie in the same prime TQFT factor. For more general G_k CS theory (with G compact and simple), all Wilson lines corresponding to affine representations with at least two non-zero Dynkin labels must be part of the same prime TQFT factor.

Proof: Suppose this were not the case. Then, we would find fusion rules of the form (4.92) with *no* Wilson lines in the ellipses. □

Clearly, to produce fusion rules of the form (4.2) for non-abelian Wilson lines in the same prime TQFT, we will need to go beyond $SU(N)_k$ CS theory. One way to proceed is to consider coset theories and use some intuition from section 4.2. Indeed, since cosets can have fixed points (which we will describe below), it is natural to think they can lead to fusion rules of the form (4.2).

4.3.2 Virasoro minimal models and some cosets without fixed points

We begin with a discussion of the Virasoro minimal models, as these are simple examples of theories that are related to cosets. While these cosets do not have fixed points, they turn out to produce factorized TQFTs that are nonetheless illustrative. In the next section, we will focus on cosets that have fixed points, and we will see how to engineer fusion rules of the form (4.2).

One way to construct the Virasoro minimal models is to take a three-dimensional spacetime $\mathbb{R} \times \Sigma$ and place $SU(2)_{k-1} \times SU(2)_1$ CS theory on $I \times \Sigma$, where I is an interval in $\mathbb{R}$. We can place $SU(2)_k$ CS theory outside this region. At the two $1+1$ dimensional interfaces between the CS theories (which form two copies of Σ, call them $\Sigma_{1,2}$), we obtain the left and right movers of the RCFT. Here the chiral (anti-chiral) primaries lie where endpoints of Wilson lines from the $SU(2)_k$ and $SU(2)_{k-1} \times SU(2)_1$ theories meet on Σ_1 (Σ_2).

Another way to think about the Wilson lines related to the Virasoro minimal models is to start with $SU(2)_{k-1} \times SU(2)_1$ CS theory and change variables to make an $SU(2)_k$ subsector manifest [106]. Integrating this sector out leaves an effective coset TQFT.

The end result is that the TQFT we are interested in is[117]

$$\mathcal{T}_p = \frac{SU(2)_{p-2} \boxtimes SU(2)_1}{SU(2)_{p-1}} \ , \quad p \geq 3 \ . \tag{4.95}$$

Here, the natural number $p \geq 3$ labels the corresponding Virasoro minimal model (so, for example, $p = 3$ for the Ising model).[118] We may construct the MTC data underlying the RCFT and the coset TQFT by taking products (e.g., see [134])

$$F_{\mathcal{T}_p} = F_{SU(2)_{p-2}} \cdot F_{SU(2)_1} \cdot \bar{F}_{SU(2)_{p-1}} \ , \quad R_{\mathcal{T}_p} = R_{SU(2)_{p-2}} \cdot R_{SU(2)_1} \cdot \bar{R}_{SU(2)_{p-1}} \ . \tag{4.96}$$

In order to make (4.96) precise, we need to explain how the states in $\mathcal{T}_p$ are related to those in the individual $SU(2)_k$ theories that make up the coset. Let us denote the $SU(2)_{p-2}$, $SU(2)_1$, and $SU(2)_{p-1}$ weights as λ, μ, and ν. Then, to build the coset we should identify Wilson lines as follows

$$\mathcal{W}_{\{\lambda,\mu,\nu\}} := \mathcal{W}_\lambda \times \mathcal{W}_\mu \times \mathcal{W}_\nu \simeq (\mathcal{W}_{p-2} \times \mathcal{W}_\lambda) \times (\mathcal{W}_1 \times \mathcal{W}_\mu) \times (\mathcal{W}_{p-1} \times \mathcal{W}_\nu) \ , \tag{4.97}$$

where $\mathcal{W}_{p-2}$, $\mathcal{W}_1$, and $\mathcal{W}_{p-1}$ are abelian Wilson lines transforming in the weight $p-2$ (spin $(p-2)/2$), weight 1 (spin 1/2), and weight $p-1$ (spin $(p-1)/2$) representations of the different TQFT factors.[119] Moreover, in order to be a valid Wilson line in $\mathcal{T}_p$, we should demand that our Wilson lines satisfy

$$\mathcal{W}_{\{\lambda,\mu,\nu\}} \in \mathcal{T}_p \ \Leftrightarrow \ \lambda + \mu - \nu \in Q \ \Leftrightarrow \ \lambda + \mu + \nu = 0 \ (\text{mod } 2) \ , \tag{4.99}$$

where Q is the $SU(2)$ root lattice. This relation guarantees that all lines that remain have trivial braiding with $\mathcal{W}_{\{p-2,1,p-1\}}$ (which is a boson that is in turn identified with the vacuum). It is in terms of these degrees of freedom that (4.96) should be understood.

Before proceeding, let us stop and note that the fusion in (4.97) has no fixed points. Indeed, this statement readily follows from the fact that $SU(2)_1$ is an abelian TQFT, and abelian theories cannot have fixed points since their fusion rules are those of a finite abelian group (in this case $\mathbb{Z}_2$).

[117]This is the TQFT analog of the classic result [90] for the corresponding affine algebras:

$$\text{Vir}_p \simeq \frac{\widehat{\mathfrak{su}}(2)_{p-2} \times \widehat{\mathfrak{su}}(2)_1}{\widehat{\mathfrak{su}}(2)_{p-1}} \ . \tag{4.94}$$

[118]In writing (4.95), we have used the Deligne product to emphasize the fact that the $SU(2)_{p-2} \times SU(2)_1$ CS theory is a product TQFT.

[119]At the level of the corresponding affine algebras, this is the statement that [61]

$$\left\{\widehat{\lambda}, \widehat{\mu}, \widehat{\nu}\right\} \simeq \left\{a\widehat{\lambda}, a\widehat{\mu}, a\widehat{\nu}\right\} \ , \tag{4.98}$$

where the hat denotes affine weights, and a is the generator of the (diagonal) $\mathcal{O}(\widehat{\mathfrak{su}}(2))$ outer automorphism.

Given this groundwork, we claim that $\mathcal{T}_p$ factorizes as follows

$$\mathcal{T}_p \simeq \begin{cases} (SU(2)_{p-2} \boxtimes SU(2)_1)^{\rm int} \boxtimes SU(2)^{\rm int}_{p-1} \ , & \text{if } p = 0 \ (\text{mod } 2) \\ SU(2)^{\rm int}_{p-2} \boxtimes SU(2)^{\rm conj}_{p-1} \ , & \text{if } p = 1 \ (\text{mod } 2) \ . \end{cases} \tag{4.100}$$

The various TQFTs appearing in (4.100) are

$$\begin{aligned} (SU(2)_{p-2} \boxtimes SU(2)_1)^{\rm int} &:= \text{gen}\left(\left\{\mathcal{W}_{\{\lambda,\mu\}} \in SU(2)_{p-2} \boxtimes SU(2)_1 | \ \lambda + \mu = 0 \ (\text{mod } 2)\right\}\right) \ , \\ SU(2)^{\rm int}_{p-1} &:= \text{gen}\left(\{\mathcal{W}_\nu \in SU(2)_{p-1} | \ \nu = 0 \ (\text{mod } 2)\}\right) \ , \\ SU(2)^{\rm conj}_{p-1} &:= \text{gen}\left(\left\{\mathcal{W}_{\{\lambda,\mu,\nu\}} | \ \lambda + \mu + \nu = 0 \ (\text{mod } 2) \ , \ \mathcal{W}_\lambda, \mathcal{W}_\mu \text{ abelian}\right\}\right) \ , \\ SU(2)^{\rm int}_{p-2} &:= \text{gen}\left(\{\mathcal{W}_\lambda \in SU(2)_{p-2} | \ \lambda = 0 \ (\text{mod } 2)\}\right) \ , \end{aligned} \tag{4.101}$$

where "gen$(\cdots)$" means that the TQFT is generated by the Wilson lines enclosed. Notice that in the case that p is even, $p-1$ is odd and $SU(2)^{\rm int}_{p-1}$ is precisely the decoupled TQFT factor required by Müger's theorem in (4.93) containing integer spins (even Dynkin labels). Similar logic applies to $SU(2)^{\rm int}_{p-2}$ in the case that p is odd. The TQFT $SU(2)^{\rm conj}_{p-1}$ has the same fusion rules as $SU(2)_{p-1}$, but it is a different TQFT. Finally, for the case that $p = 3$ (i.e., the Ising model), we see that $\mathcal{T}_3$ does not factorize.[120]

Our strategy to prove the factorization in (4.100) is to construct the various factors and then argue that they are well-defined TQFTs by Müger's theorem [125]. Although we will not pursue it in this paper, this same approach leads to interesting generalizations for cosets built out of groups other than $SU(2)$.

To that end, let us first take the case of $p \geq 3$ odd. Using the result in (4.96), we have that the modular S matrix also takes a product form

$$S_{\{\lambda,\mu,\nu\}\{\lambda',\mu',\nu'\}} = S^{(p-2)}_{\lambda\lambda'} \cdot S^{(1)}_{\mu\mu'} \cdot S^{(p-1)}_{\nu\nu'} \ , \quad \theta_{\{\lambda,\mu,\nu\}\{\lambda',\mu',\nu'\}} = \theta^{(p-2)}_{\lambda\lambda'} \cdot \theta^{(1)}_{\mu\mu'} \cdot \bar{\theta}^{(p-1)}_{\nu\nu'} \ , \tag{4.102}$$

where the superscripts on the righthand sides of the above equations refer to the corresponding factors in the coset (4.95). From the S matrix, Verlinde's formula yields (see also the discussion in [61])

$$N_{\{\lambda,\mu,\nu\}\{\lambda',\mu',\nu'\}}^{\{\lambda'',\mu'',\nu''\}} = N^{(p-2)\lambda''}_{\lambda\lambda'} \cdot N^{(1)\mu''}_{\mu\mu'} \cdot N^{(p-1)\nu''}_{\nu\nu'} \ , \tag{4.103}$$

where, again, the superscripts on the righthand side denote the different coset factors in (4.95). The factor $SU(2)^{\rm int}_{p-2}$ in the second line of (4.100) is clearly closed under fusion. So too is $SU(2)^{\rm conj}_{p-1}$. To have factorization of the TQFT, we need only show that all Wilson lines can be written in this way and, by Müger's theorem, that one of these factors is modular. The second part is trivial: we have already seen that $SU(2)^{\rm int}_{p-2}$ is modular in the discussion surrounding (4.93). We can confirm this statement by looking at the modular S-matrix for $SU(2)_{p-2}$

$$S^{(p-2)}_{\lambda\lambda'} = \sqrt{\frac{2}{p}} \sin\left(\frac{(\lambda+1)(\lambda'+1)\pi}{p}\right) \ . \tag{4.104}$$

[120]Note also that Ising shares the same fusion rules as $SU(2)_2$, though they are not the same TQFTs. For example, the σ fields have different twists.

and taking the submatrix involving the integer spins (even weights).

Therefore, we need only check that all states in the coset (4.95) can be expressed in this way. To that end, we can see that

$$|SU(2)^{\rm int}_{p-2}| = \frac{p-1}{2}\ ,\ |SU(2)^{\rm conj}_{p-1}| = p\ , \tag{4.105}$$

where the norm denotes the number of simple elements within. Therefore, we see that we have $|\mathcal{T}_p| = p(p-1)/2$, which is precisely the number of states in the coset (4.95) (note that in these computations we have used (4.97) and (4.99)) and the corresponding A-type Virasoro minimal model.

To make contact with the fusion rules in (4.4), we need to explain precisely how coset lines map onto the Virasoro primaries. The results above allow us to realize the, say, Virasoro left-movers as states on the boundary of the bulk TQFT, $\mathcal{T}_p \simeq SU(2)^{\rm int}_{p-2} \boxtimes SU(2)^{\rm conj}_{p-1}$ with p odd. Now, we need to see how we can map boundary endpoints of lines in this theory to Virasoro primaries, $\varphi_{(r,s)}$. To that end, by comparing the S-matrix for $\mathcal{T}_p \simeq SU(2)^{\rm int}_{p-2} \boxtimes SU(2)^{\rm conj}_{p-1}$ with the corresponding expressions for those of the Virasoro minimal models, we have that the labels of the Virasoro primary, $\varphi_{(r,s)}$ map as follows (see also [61])

$$r = \lambda + 1\ ,\quad s = \nu + 1\ . \tag{4.106}$$

In particular, we see that the $\varphi_{(r,1)}$ primaries are endpoints of lines in $SU(2)^{\rm int}_{p-2}$ while the $\varphi_{(1,s)}$ are endpoints of lines in $SU(2)^{\rm conj}_{p-1}$. This reasoning explains the fact that non-abelian Virasoro primaries of these types have unique fusion outcomes[121]

$$\varphi_{(r,1)} \times \varphi_{(1,s)} = \varphi_{(r,s)}\ , \tag{4.107}$$

discussed in the introduction (at least for p odd). As an example, we have $\mathcal{T}_3 \simeq$ Ising (i.e., the TQFT is the Ising MTC), which does not factorize. On the other hand, for $p = 5$, we have

$$\mathcal{T}_5 = (G_2)_1 \boxtimes SU(2)^{\rm conj}_4\ , \tag{4.108}$$

where $(G_2)_1$ is the so-called "Fibnonacci" TQFT, and $SU(2)^{\rm conj}_4$ is a TQFT with the same fusion rules and S-matrix as $SU(2)_4$.

Let us now consider $p \geq 4$ even. The modular data and fusion rules still take a product form as in (4.102) and (4.103). Now, however, we should examine the first line in (4.100). Using (4.103), it is again easy to see that both $SU(2)^{\rm int}_{p-1}$ and $(SU(2)_{p-2} \boxtimes SU(2)_1)^{\rm int}$ are separately closed under fusion. Moreover, just as before, we can use the discussion around (4.93) and Müger's theorem to conclude that $SU(2)^{\rm int}_{p-1}$ is indeed a decoupled TQFT as claimed in (4.100).

We should again check that all states in (4.95) can be reproduced. To that end, we have

$$|SU(2)^{\rm int}_{p-1}| = \frac{p}{2}\ ,\ |(SU(2)_{p-2} \boxtimes SU(2)_1)^{\rm int}| = p - 1\ . \tag{4.109}$$

[121]Though, again, we stress that this factorization is not a factorization of RCFT correlators.

As a result, we have $|\mathcal{T}_p| = p(p-1)/2$, which is the correct number of states in the coset (4.95) and the corresponding A-type Virasoro minimal model.

Our mapping is again as in (4.106), but now $\varphi_{(r,1)}$ primaries are endpoints of lines in $(SU(2)_{p-2} \boxtimes SU(2)_1)^{\text{int}}$, and $\varphi_{1,s}$ are endpoints of lines in $SU(2)^{\text{int}}_{p-1}$. This again explains the fusion outcomes in (4.107) for the case of p even as well. As an example, note that

$$\mathcal{T}_4 = \text{Ising}' \boxtimes (F_4)_1 \ , \tag{4.110}$$

where the first factor is a rank three TQFT with the same fusion rules as Ising (and $SU(2)_2$), and the second factor is the time reversal of the Fibonacci theory in (4.108).

As a result, we conclude that, although the TQFTs discussed in this section do have non-abelian anyons fusing to give a unique outcome, this is due to the fact that the corresponding TQFTs factorize.

4.3.3 Beyond Virasoro: cosets with fixed points

In section 4.2 we saw that fixed points of various kinds gave rise to fusion rules of the form (4.8) (in particular, see theorem 1 of section 4.2.1). In the context of cosets, we can also naturally engineer fixed points under the action of fusion with abelian anyons generating identifications of fields. In the case of Virasoro, this didn't happen (see (4.97)). Indeed, this statement followed from the fact that we had an abelian factor in the coset (4.95).

The simplest way to get around this obstacle and generate fixed points is to consider instead

$$\widehat{\mathcal{T}}_p = \frac{SU(2)_{p-2} \boxtimes SU(2)_2}{SU(2)_p} \ , \tag{4.111}$$

where $p \geq 3$ (we should take $p \geq 4$ to avoid the problem of abelian factors). By further identifying some of these coset fields, we get theories related to the $\mathcal{N} = 1$ super-Virasoro minimal models [91, 90]. Note that the case of $p = 3$ corresponds to the $\mathcal{T}_4$ case discussed previously (i.e., to the TQFT related to the tri-critical Ising model).

For the theories in (4.111), we find the following generalization of the identification condition in (4.97)[122]

$$\begin{aligned} \mathcal{W}_{\{\lambda,\mu,\nu\}} &:= \mathcal{W}_\lambda \times \mathcal{W}_\mu \times \mathcal{W}_\nu \simeq (\mathcal{W}_{p-2} \times \mathcal{W}_\lambda) \times (\mathcal{W}_2 \times \mathcal{W}_\mu) \times (\mathcal{W}_p \times \mathcal{W}_\nu) \\ &= \mathcal{W}_{p-2-\lambda} \times \mathcal{W}_{2-\mu} \times \mathcal{W}_{p-\nu} \ , \end{aligned} \tag{4.112}$$

In particular, if $\lambda = (p-2)/2$, $\mu = 1$, and $\nu = p/2$, we can have a fixed point[123]. Of course, if p is odd, we don't have a fixed point. In this case, we can again run logic similar to that used in the Virasoro case to argue that the TQFT factorizes.

[122] We also require that $\lambda + \mu + \nu = 0$ (mod 2) so that the lines in the coset theory have trivial braiding with the bosonic line $\mathcal{W}_{\{p-2,2,p\}}$. This line is in turn identified with the vacuum.

[123] Note that the fixed points discussed in section 4.2 are fixed points under 1-form and 0-form symmetry action. In the coset examples studied here, fixed points refer to field identification fixed points.

However, if p is even, then we need to properly define the coset. In particular, we should resolve the fixed point Wilson line as follows (see [143, 144] for the dual RCFT discussion)

$$\mathcal{W}_{\{(p-2)/2,1,p/2\}} \to \mathcal{W}^{(1)}_{\{(p-2)/2,1,p/2\}} + \mathcal{W}^{(2)}_{\{(p-2)/2,1,p/2\}} \ . \tag{4.113}$$

Let us consider what turns out to be the simplest interesting case, $p = 6$

$$\widehat{\mathcal{T}}_6 = \frac{SU(2)_4 \boxtimes SU(2)_2}{SU(2)_6} \ . \tag{4.114}$$

The fixed point resolution in (4.113) becomes $\mathcal{W}_{\{2,1,3\}} \to \mathcal{W}^{(1)}_{\{2,1,3\}} + \mathcal{W}^{(2)}_{\{2,1,3\}}$. As in the cases of one-form gauging with fixed points discussed in section 4.2, it is natural that there should be a zero-form symmetry exchanging $\mathcal{W}^{(1)}_{\{2,1,3\}} \leftrightarrow \mathcal{W}^{(2)}_{\{2,1,3\}}$.

As a first step to better understand the theory after resolving the fixed point, note that $\widehat{\mathcal{T}}_6$ has the following number of lines

$$|\widehat{\mathcal{T}}_6| = 28 \ . \tag{4.115}$$

Of these fields, twenty-six come from identifying full length-two orbits in (4.112) while two come from resolving the fixed point. In what follows, $\{\lambda, \mu, \nu\}$ will denote fields in full orbits, while labels of the form $\{2, 1, 3\}^{(i)}$ (with $i = 1, 2$) will denote the fixed point lines.

To understand the fusion rules and the question of primality after fixed point resolution, we can compute the S matrix using the algorithm discussed in [143] (let us denote the result by $\widetilde{S}$). It takes the form

$$\begin{aligned} \widetilde{S}_{\{\lambda,\mu,\nu\}\{\lambda',\mu',\nu'\}} &= 2S_{\{\lambda,\mu,\nu\}\{\lambda',\mu',\nu'\}} \ , \quad \widetilde{S}_{\{2,1,3\}^{(i)}\{\lambda',\mu',\nu'\}} = S_{\{2,1,3\}\{\lambda',\mu',\nu'\}} \ , \\ \widetilde{S}_{\{2,1,3\}^{(i)}\{2,1,3\}^{(j)}} &= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \ , \end{aligned} \tag{4.116}$$

where

$$S_{\{\lambda,\mu,\nu\}\{\lambda',\mu',\nu'\}} = S^{(p-2)}_{\lambda\lambda'} \cdot S^{(2)}_{\mu\mu'} \cdot S^{(p)}_{\nu\nu'} \ , \tag{4.117}$$

is the naive generalization of (4.102) to the cosets at hand. Note that the fusion rules we obtain from $\widetilde{S}$ for fields not involving $\{2, 1, 3\}^{(i)}$ are the naive ones we get from S via the restrictions and identifications described above.

The above discussion is sufficient to prove that $\widehat{\mathcal{T}}_6$ is prime. Indeed, we see from (4.116) that the fields that come from identifying length-two orbits have the quantum dimensions they inherit from S. The fixed point resolution fields, on the other hand, have half the quantum dimension of the fixed point field. We therefore have the following four abelian anyons generating a $\mathbb{Z}_2 \times \mathbb{Z}_2$ fusion algebra

$$\mathcal{W}_{\{0,0,0\}} \simeq \mathcal{W}_{\{4,2,6\}} \ , \ \mathcal{W}_{\{4,0,0\}} \simeq \mathcal{W}_{\{0,2,6\}} \ , \ \mathcal{W}_{\{0,2,0\}} \simeq \mathcal{W}_{\{4,0,6\}} \ , \ \mathcal{W}_{\{0,0,6\}} \simeq \mathcal{W}_{\{4,2,0\}} \ . \tag{4.118}$$

By (4.116), we see that the braiding amongst abelian anyons is not affected by taking $S \to \widetilde{S}$. As a result, we see that the four abelian anyons all braid trivially. Therefore, they cannot form a decoupled TQFT.

Wilson lines	Quantum dimensions
$\mathcal{W}_{\{0,0,0\}}, \mathcal{W}_{\{4,0,0\}}, \mathcal{W}_{\{0,2,0\}}, \mathcal{W}_{\{0,0,6\}}$	1
$\mathcal{W}_{\{0,0,2\}}, \mathcal{W}_{\{0,0,4\}}, \mathcal{W}_{\{4,0,2\}}, \mathcal{W}_{\{4,0,4\}}$	$\cot\left(\frac{\pi}{8}\right)$
$\mathcal{W}_{\{1,0,1\}}, \mathcal{W}_{\{1,0,5\}}, \mathcal{W}_{\{3,0,1\}}, \mathcal{W}_{\{3,0,5\}}$	$\sqrt{\frac{3}{2}}\csc\left(\frac{\pi}{8}\right)$
$\mathcal{W}_{\{0,1,3\}}, \mathcal{W}_{\{2,1,3\}^{(1)}}, \mathcal{W}_{\{2,1,3\}^{(2)}}$	$\sqrt{2}\csc\left(\frac{\pi}{8}\right)$
$\mathcal{W}_{\{1,0,3\}}, \mathcal{W}_{\{3,0,3\}}$	$\sqrt{3}\csc\left(\frac{\pi}{8}\right)$
$\mathcal{W}_{\{2,0,0\}}, \mathcal{W}_{\{2,0,6\}}$	2
$\mathcal{W}_{\{0,1,1\}}, \mathcal{W}_{\{0,1,5\}}$	$\csc\left(\frac{\pi}{8}\right)$
$\mathcal{W}_{\{1,1,0\}}, \mathcal{W}_{\{1,1,6\}}$	$\sqrt{6}$
$\mathcal{W}_{\{2,0,2\}}, \mathcal{W}_{\{2,0,4\}}$	$2\cot\left(\frac{\pi}{8}\right)$
$\mathcal{W}_{\{1,1,2\}}, \mathcal{W}_{\{1,1,4\}}$	$\sqrt{6}\cot\left(\frac{\pi}{8}\right)$
$\mathcal{W}_{\{2,1,1\}}$	$2\csc\left(\frac{\pi}{8}\right)$

Table 2: The twenty-eight Wilson lines and associated quantum dimensions in the $\widehat{\mathcal{T}}_6$ TQFT.

Given this discussion, what can a putative factorized theory look like? Since $\widehat{\mathcal{T}}_6$ has order $28 = 7 \cdot 2^2$, we see that the only way to have a non-trivial factorization is to have a factorization of the form $\widetilde{\mathcal{T}}_{14} \boxtimes \widetilde{\mathcal{T}}_2$ into prime TQFTs with rank fourteen and rank two, or $\widetilde{\mathcal{T}}_7 \boxtimes \widetilde{\mathcal{T}}_4$ with prime TQFTs of rank seven and four, or $\widetilde{\mathcal{T}}_7 \boxtimes \widetilde{\mathcal{T}}_2 \boxtimes \widetilde{\mathcal{T}}_2'$ with prime TQFTs of rank seven, two, and two.

Let us consider the first factorization first. Since the abelian anyons (and any subset thereof) cannot form a separate TQFT factor (this factor would be non-modular), the classification in [139] implies that we have either $\widetilde{\mathcal{T}}_2 \simeq (G_2)_1$ or $\widetilde{\mathcal{T}}_2 \simeq (F_4)_1$. In any case, the non-trivial anyon in $\widetilde{\mathcal{T}}_2$ has quantum dimension $d_\tau = (1+\sqrt{5})/2$. It is easy to check that no such quantum dimension can be produced from products of quantum dimensions in the different coset factors (and so restrictions cannot produce them either). Moreover, one can check that the resolved fixed point fields cannot have this quantum dimension either. This same logic applies to the $\widetilde{\mathcal{T}}_7 \boxtimes \widetilde{\mathcal{T}}_2 \boxtimes \widetilde{\mathcal{T}}_2'$ factorization as well.

Therefore, it only remains to consider $\widetilde{\mathcal{T}}_7 \boxtimes \widetilde{\mathcal{T}}_4$. The other factor, $\widetilde{\mathcal{T}}_4$, has four anyons. By [139], this theory is either $(G_2)_2$ or its time reversal. In either case, we cannot produce the requisite $d_\alpha = 2\cos(\pi/9)$ quantum dimension from our coset. Therefore, we conclude that $\widehat{\mathcal{T}}_6$ is indeed a prime TQFT.

Moreover, we find the following fusion rules of non-abelian Wilson lines with unique outcome

$$\begin{aligned}
\mathcal{W}_{\{2,0,0\}} \times \mathcal{W}_{\{0,0,2\}} &= \mathcal{W}_{\{2,0,2\}} \ , \ \mathcal{W}_{\{2,0,0\}} \times \mathcal{W}_{\{0,0,4\}} = \mathcal{W}_{\{2,0,4\}} \ , \\
\mathcal{W}_{\{1,1,0\}} \times \mathcal{W}_{\{0,0,2\}} &= \mathcal{W}_{\{1,1,2\}} \ , \ \mathcal{W}_{\{1,1,0\}} \times \mathcal{W}_{\{0,0,4\}} = \mathcal{W}_{\{1,1,4\}} \ , \\
\mathcal{W}_{\{0,1,1\}} \times \mathcal{W}_{\{2,0,0\}} &= \mathcal{W}_{\{2,1,1\}} \ .
\end{aligned} \tag{4.119}$$

We can obtain additional such fusion rules by taking a product with some of the abelian lines in (4.118).

Just as in the case of discrete gauge theories with fusion rules of the above type, our theory also has a non-trivial symmetry of the modular data. Indeed, from (4.116), it is clear that the $\widetilde{S}$-matrix has a $\mathbb{Z}_2$ symmetry under the interchange

$$g\left(\mathcal{W}_{\{2,1,3\}^{(1)}}\right) = \mathcal{W}_{\{2,1,3\}^{(2)}} \ , \quad 1 \neq g \in \mathbb{Z}_2 \ . \tag{4.120}$$

Note that this symmetry is not charge conjugation since $\widetilde{S}$ is manifestly real. Moreover, since we don't change the twists, this action lifts to a symmetry of the modular data (additionally, it should lift to a symmetry of the full TQFT).

If we wish to make contact with the $\mathcal{N}=1$ minimal model, then we should note that the fermionic $\mathcal{W}_{\{0,2,0\}}$ line corresponds to the supercurrent of the SCFT. We can then organize the Neveu-Schwarz (NS) sector into supermultiplets under fusion with this operator. Doing so (and paying careful attention to the fields in the resolution of the fixed point), we find nine NS sector fields and nine Ramond sector fields as required.

There are many ways to generalize the example we have given here. Indeed, when there are fixed points in the coset construction we expect to often be able to generate fusion rules of the form (4.2). A deeper understanding of these theories and some more general methods to characterize whether the cosets are prime (along the lines of the general criteria we have in the case of discrete gauge theories) would be useful. In any case, we see that, as in the case of discrete gauge theories, symmetry fixed points and zero-form (quasi) symmetries are deeply connected with fusion rules of the form (4.2).

5 Conclusion

In this paper we have introduced three kinds of QFTs, that is TQFT, CFT and SQFT, then discussed three problems associated with those theories

Index relations and SUSY enhancement In chapter two we found various new relations between theories with non-integer scaling dimension $\mathcal{N}=2$ chiral operators (i.e., AD theories) and those with purely integer dimensional $\mathcal{N}=2$ chiral operators (the regular puncture class $\mathcal{S}$ theories). The latter theories have TQFT index expressions that are typically simpler (and more uniformly presented) than those of the former. The additional complication in the TQFT expressions for the case of AD theories (e.g., see [40, 149]) is related to the fact that the corresponding singularities in the compactification from 6D to 4D generally contain more data. However, we saw that we can, in some sense, encode this additional data by taking TQFT data for regular puncture theories (which only have integer dimension $\mathcal{N}=2$ chiral operators) and demanding interdependence of the different TQFT wave functions through intricate fugacity relations. This fugacity interdependence has important physical consequences: a large class of AD theories flow to interacting IR SCFTs with thirty-two (Poincaré plus special) supercharges via flows of the type discussed in Sec.

2.4.2. Using these index relations, we also found expressions for the Schur indices of various classes of exotic type III AD theories.

Clearly, there is a lot more to be said. We conclude with some open problems (and potential solutions):

- It would be interesting to understand if the RG flows we discussed above can be lifted to 4D (for some flows, we know this is the case; e.g., see [33]). If so, then it would be particularly intriguing to try to compute the indices of some of the resulting IR theories and see if they are $\mathcal{N}=4$ theories or not. If they are $\mathcal{N}=4$ theories, then it would be interesting to understand if they are Lagrangian (SYM theories) or not.

- One way to address the above point would be to try to construct better-behaved RG flows in the class described in Sec. 2.4.2. This might involve better understanding the role that monopole operators can play in the corresponding mirror RG flows. Alternatively, this might involve a better understanding of non-abelian mirror symmetry.

- Another approach to the problem in the first bullet point might be as follows. The authors of [135] find $\mathcal{N}=1$ Lagrangians for certain class $\mathcal{S}$ regular puncture theories by considering excursions along $\mathcal{N}=1$ conformal manifolds that include these $\mathcal{N}=2$ SCFTs as special points. In their discussion, the authors find $\mathcal{N}=1$ Lagrangians on certain conformal manifolds containing $\mathcal{N}=2$ SCFTs that have both dimension three Higgs branch and dimension three Coulomb branch operators. Some of the theories discussed in the present article satisfy this condition. Moreover, given the similarity of the Schur indices of our theories to those in the regular puncture class $\mathcal{S}$ case, it would be interesting to see if one can find $\mathcal{N}=1$ Lagrangians for some of the $R_{0,n}^{2,AD}$ and $\mathcal{T}_{(m_1,m_2,m_3)}^{2,AD}$ theories in this manner. Having an $\mathcal{N}=1$ Lagrangian or, at the very least, an $\mathcal{N}=1$ conformal manifold might in turn make it easier to study flows to $\mathcal{N}=4$.

- The ubiquity of RG flows to interacting theories with thirty-two supercharges emanating from compactifications of the 6D $(2,0)$ theory on Riemann surfaces with irregular punctures strongly suggests the existence of another way of understanding these theories via D3 branes probing type IIB / F-theory backgrounds far beyond what has been explored in the literature.

- It would be interesting to understand the most general class of $\mathcal{N}=2$ SCFTs with non-integer dimensional $\mathcal{N}=2$ chiral operators (i.e., Coulomb branch operators) that are involved in RG flows with SUSY enhancement either as UV or IR end points.

- We had to rescale fugacities as $q \to q^2$ in order to find a match between the indices of the AD theories and those of the regular puncture theories. In the process, we had to consider going from the A_{n-1} to the A_{2n-1} 6D $(2,0)$ parent theories. It would be interesting to understand why this is the case and also to see if more general $q \to q^m$ rescalings are meaningful.

- Finally, we saw that there is a close relation between regular puncture class $\mathcal{S}$ fixtures and our AD fixtures. It would be interesting to understand if to each class $\mathcal{S}$ fixture there exists an AD counterpart and, if so, how many such counterparts exist. In addition, we saw that in our class of theories, the AD fixtures with interacting regular puncture relatives admitted RG flows to interacting thirty-two supercharge theories. On the other hand, AD fixtures with free class $\mathcal{S}$ relatives did not admit such flows (even though the corresponding AD theories are strongly interacting). It would be interesting to understand if this story is completely general in the space of theories of class $\mathcal{S}$.

Arad-Herzog conjecture In chapter three we have argued that discrete gauge theory is useful for putting conjectures involving finite simple groups into a broader context. Using this approach, we proved three theorems that TQFT relates to the AH conjecture.

In fact, we may also generalize the discussion in section 3.2.1 and show that the AH conjecture implies that in our theories of interest

$$\mathcal{L}_{([g],\pi_g^\omega)} \times \mathcal{L}_{([h],\pi_h^\omega)} = \sum_{\pi_{gh}^\omega} \mathcal{L}_{([gh],\pi_{gh}^\omega)} \ , \quad g,h \neq 1 \ , \tag{5.1}$$

is not allowed.

Finally, we argued that the lack of electric-magnetic dualities involving discrete gauge theories with non-abelian finite simple groups is a consistency check of our picture above and of the AH conjecture.

One natural question is to better understand to what extent ideas involving non-abelian anyons can be used to prove the AH conjecture (see [124, 94, 44] for recent progress on this conjecture). Since discrete gauge theories feature in various physical systems, perhaps there is a physical proof that awaits.

Another interesting question is to understand to what degree fusion rules of the types we have been discussing constrain global properties of more general TQFTs, which is the topic of chapter four.

$a \times b = c$ **fusion rule** In chapter four, we have seen that the existence of fusions of non-abelian anyons having a unique outcome is intimately connected with the global structure of the corresponding TQFT.

Let us summarize our results for continuous gauge groups (and continuous groups more generally):

- Building on the well-known fact that $SU(2)$ spin addition / fusion of two non-abelian representations (i.e., higher-dimensional / spin non-singlet representations) is reducible (i.e., has multiple outcomes with different total spin), we argued that a similar result holds in all compact simple Lie groups.

- We argued that the result in the previous bullet point on classical groups can be extended to a theorem constraining $SU(N)_k$ CS theory: fusions of non-abelian Wilson lines in these theories do not have unique outcomes. More generally, Wilson lines corresponding to affine representations with at least two non-vanishing Dynkin labels in any G_k CS theory (for G a compact simple Lie group) do not have unique outcomes. These results have implications for the global structure of these theories (the claim in section 4.3.1): the Wilson lines discussed here must all lie in the same prime factor (although G_k CS theories are not prime in general).

- We showed that one way to produce $a \times b = c$ fusions involving non-abelian a and b is to consider cosets. In the case of TQFTs underlying Virasoro minimal models we argued that (as in the $(E_7)_2$ case) such rules arise from factorizations of the TQFTs into multiple prime factors. On the other hand, if we include cosets with fixed points, we can obtain prime theories with such fusion rules.

Next, let us summarize our results for discrete gauge groups (and discrete groups more generally):

- We argued that Zisser's construction of irreducible products of higher-dimensional irreducible A_N representations [165] can be lifted to fusions of non-abelian Wilson lines with unique outcomes in A_N discrete gauge theory. From the perspective of the closely related S_N group and corresponding discrete gauge theory, the A_N result requires certain 1-form symmetry fixed points (where we define "one-form symmetry" in the S_N group to correspond to the $\mathbb{Z}_2 \subset \mathrm{Rep}(\mathrm{S_N})$ generated by the sign representation). We then derived theorem 4.1 that generalizes this relation between the A_N and S_N discrete gauge theories to other TQFTs.

- Going to the S_N discrete gauge theory by gauging the $\mathbb{Z}_2$ 0-form outer automorphism symmetry of the A_N discrete gauge theory resolves the $a \times b = c$ non-abelian fusion rule into fusion rules not of this type. However, we saw that in the case of $O(5,3)$ discrete gauge theory such resolutions do not always occur via automorphism gauging. On the other hand, a symmetry fixed point again plays a role: in the resulting $O(5,3) \rtimes \mathbb{Z}_2$ discrete gauge theory, there is a 0-form symmetry fixed point. We then proved theorem 4.2, which explains why this phenomenon occurs in more general theories. In fact, the $O(5,3) \rtimes \mathbb{Z}_2$ discrete gauge theory relative of the $a \times b = c$ fusion equations in the $O(5,3)$ TQFT described in (4.32) also has a 1-form symmetry fixed point for the anyon appearing on the right hand side. In the original $O(5,3)$ TQFT this latter anyon becomes a set of two anyons related by the 0-form symmetry. Our corollary 4.2.1 generalizes this observation to other TQFTs.

- We showed that one can lift Gallagher's theorem to a statement on the fusion of non-abelian Wilson lines involving unfaithful representations with a unique outcome in TQFT. Moreover, we elucidated the roles that subcategory structure and symmetries play in this result for various specific TQFTs. We then proved theorem 4.3 that generalizes these observations to a broader set of theories. We also argued that this subcategory structure helps explain the large ratio of group orders in (4.41).

- To gain a sense of how magnetic fluxes behave in general discrete gauge theories, we proved theorem 4.5. In particular, we showed that in discrete gauge theories with a non-abelian gauge group, G, the magnetic fluxes do not form a fusion subcategory. This result immediately places constraints on electric-magnetic self-dualities / quantum symmetries that constrain our symmetry searches later in section 2.
- At a more constructive level, we also proved theorem 4.10. This result gives infinitely many generalizations of the well-known electric-magnetic self-duality of the S_3 discrete gauge theory.
- In order to better understand which discrete gauge theories are prime, we proved theorem 4.7. This result allowed us to more easily analyze which prime discrete gauge theories have fusions of non-abelian anyons with unique outcomes.
- In order to get a handle on the structure of discrete gauge theories with fusion rules of our desired type involving anyons carying non-trivial flux, we proved theorem 4.11 and corollary 4.11.1. These results give the subcategory structure that arises when such fusions occur. In turn, this structure gives an explanation of these fusion rules. Theorem 4.12 then partially extends these results to the case in which one of the non-abelian anyons involved is a Wilson line.
- The software GAP was used to analyze the fusion rules of hundreds of untwisted discrete gauge theories. In all the cases we checked, we find that discrete gauge theories with $a \times b = c$ type fusion rules have quasi-zero-form symmetries. This suggests that symmetries of the modular data are a characteristic feature of such fusion rules.

The above discussion leads to various natural questions:

- In the discussion around (4.41) we explained the large hierarchy between the size of simple and non-simple groups whose corresponding discrete gauge theories have non-abelian Wilson lines satisfying (4.2) by using symmetries and subcategory structure. It would be interesting to explore whether other related hierarchies can be explained in a similar way.
- We saw that in almost all the prime untwisted discrete gauge theories we studied, if there was a fusion rule of the form (4.2), then the theory had non-trivial zero-form symmetries. The only exceptions where discrete gauge theories based on the M_{23} and M_{24} Mathieu groups discussed in section 4.2.4. Here we argued that there were zero-form symmetries of the modular data that did not lift to symmetries of the full theory. It would be interesting to understand if gauge theories based on certain finite simple

sporadic groups are the only prime theories with fusion rules of the form (4.2) that exhibit this phenomenon.

- In section 4.3.1, we proved that the non-abelian lines of $SU(N)_k$ CS theory don't have fusion rules of the form (4.2). While $(E_7)_2$ CS theory does have such fusion rules, we do not know of an example of such a fusion in a prime G_k CS theory with G a compact and simple Lie group. It would be interesting to either find an example of such a fusion or prove a more general theorem forbidding one. Given such fusions are common for discrete gauge theories, it would be interesting to understand how these two statements interact with each other.

- As we saw in section 4.3.3, it would be useful to develop new tools to understand primality in theories built on cosets. One promising direction is to study the role of Galois actions in such theories.

A Axiomatic approaches and categorical constructions

Quantum field theory is a rich and deep subject, but up to the present time it is mathematically ill defined and a completely rigorous understanding is lacking.

In practice we usually do not bother with rigorousness , and begin with some classical Lagrangians or Hamiltonians, then do quantization, although the procedure 'quantization' by itself is again mathematically highly nontrivial and sometimes even ill defined, it indeed leads to many testable correct results and deep insights, this is just like how Newton did calculus in his time, where he always has some specific series or functions at hand and only then differentiation or integration are carried out, of course some care must be taken, but such Weierstrass style delta-epsilon issues are usually 'safely' ignored and assumed to be treated systematically somewhere in a textbook.

However, we do not have any such textbooks, although since the very beginning of this subject, a huge amount of effort has been put into rigorous axiomatic constructions. What we can say is that, to some extent, at least in some special cases, this goal has been partially achieved. In this appendix, we will review some of these axiomatic approaches to quantum field theory and discuss some of the theories we have discussed in the text.

We first introduce the notation of operator valued distributions,then use it to develop a reasonable set of basic axioms for generic quantum field theory, this serves as a blue print for further development[145]. Then similarly, we introduce the notation of formal distribution which is tailor made for CFT, then use it to develop the theory of vertex operator algebra as an axiomatic characterization of CFT[145].

Then we turn to categorical constructions, where we introduce the theory of modular tensor categories, which characterize fusion and braiding in an abstract way and hence summaries the common essential algebraic features of $d = 3$ TQFT and $d = 2$ RCFT[156, 13].

We also introduce functorial formalism as an alternative approach, where TQFT is analyzed as the main example[156, 13, 45].

Finally, we summarize some mathematical facts about groups and algebras that are used throughout the context[145, 83, 138].

A.1 Basic axioms of quantum field theory

A.1.1 Operator valued distributions

Here we will first introduce the notation of a distribution, it is a formal generalization of the real smooth function of several variables such that we can take derivatives and Fourier transforms freely without worrying about their very existence first. As an example we use these tools to solve the Klein-Gordon equation, which justifies the validity of the usual 'plane wave expansion' for the free field used in physics textbooks. After this we can get a straight forward generalization to the case of operator valued distribution.

In this section all functions are defined on $\mathbb{R}^n$[124], and assumed to be complex valued and smooth, that is , of the form $f:\mathbb{R}^n\to\mathbb{C}$ with continuous derivatives to any order

Recall that the Scgwarz space is defined as the vector space of rapidly decreasing functions with respect to the following seminorm

$$|f|_{p,k}:=\sup_{|\alpha|\leq p}\sup_{x\in\mathbb{R}^n}|\partial^\alpha f(x)|\left(1+|x|^2\right)^k<\infty \tag{A.1}$$

We will call the elements of $\mathscr{S}=\mathscr{S}\left(\mathbb{R}^n\right)$ as test functions. Then a tempered distribution T is a linear functional $T:\mathscr{S}\to\mathbb{C}$ such that it is continuous with respect to all $|\quad|_{p,k}$, the space of tempered distributions is denoted as $\mathscr{S}'=\mathscr{S}'\left(\mathbb{R}^n\right)$ and equipped with the compact-open topology. We will only consider tempered distributions here so we just call them as distributions.

As examples, first let us note that every measurable and bounded g induces a distribution

$$T_g(f):=\int_{\mathbb{R}^n}g(x)f(x)dx, f\in\mathscr{S} \tag{A.2}$$

Then we also have the delta 'function'

$$\delta_y(f)=f(y)=\int_{\mathbb{R}^n}\delta(x-y)f(x)dx. \tag{A.3}$$

which is indeed a distribution but can not be induced by any g

The main advantage of using distributions is that they are automatically smooth, because we can define the derivatives as

$$\partial^\alpha T(f):=(-1)^{|\alpha|}T\left(\partial^\alpha f\right), f\in\mathscr{S} \tag{A.4}$$

[124] or $\mathbb{R}^{1,n-1}$, all the following discussions hold without any essential difference

so we are actually taking derivatives of f, which are smooth by construction. In particular $\partial^\alpha T_g = T_{\partial^\alpha g}$ when it is defined, and indeed every $T \in \mathscr{S}'$ is a linear combination for some derivatives of continuous$g_\alpha : \mathbb{R}^n \to \mathbb{C}$ of polynomial growth

$$T = \sum_{0 \leq |\alpha| \leq k} \partial^\alpha T_{g_\alpha} \tag{A.5}$$

Similar with derivative, the Fourier transform $\mathscr{F} : \mathscr{S} \to \mathscr{S}$ is defined by its action on the argument of the functional, i.e by its adjoint $\mathscr{F}^\dagger$

$$\mathscr{F}(T)(v) := T(\mathscr{F}(v)) = \mathscr{F}^\dagger(T)(v), v \in \mathscr{S} \tag{A.6}$$

In particular, for T_g it is

$$\mathscr{F}\left(T_g\right)(v) = T_g(\widehat{v}) = \int_{\mathbb{R}^n} \int_{(\mathbb{R}^n)'} g(x) v(p) e^{ix \cdot p} dp dx = T_{\mathscr{F}(g)}(v) \tag{A.7}$$

And for delta function it is

$$\mathscr{F}\left(\delta_0\right) = \int_{\mathbb{R}^D} \delta_0(x) e^{ix \cdot p} dx = 1 \tag{A.8}$$

with inverse

$$\mathscr{F}^{-1}\left(e^{ip \cdot y}\right) = (2\pi)^{-D} \int_{\mathbb{R}^D} e^{ip \cdot (y-x)} dp = \delta(x-y) \tag{A.9}$$

Combine derivative and Fourier transform together, we find

$$\mathscr{F}\left(\partial_k u\right) = -ip_k \mathscr{F}(u) \tag{A.10}$$

Using this formula we can solve linear differential equations formally, suppose we have an inhomogeneous linear PDE induced by a polynomial $P(X) = c_\alpha X^\alpha \in \mathbb{C}\left[X_1, \ldots, X_n\right]$ as

$$P(-i\partial) u = v \tag{A.11}$$

Then we define a fundamental solution for this equation as a distribution G such that

$$P(-i\partial) G = \delta \tag{A.12}$$

and it satisfies

$$P(-i\partial)(G * v) = v \tag{A.13}$$

So the convolution product $G * v$ is a particular solution, but since we can do Fourier transforms, such G is easily found by solving the algebraic equation $PT = 1$ and transforming it back as $G = \mathscr{F}^{-1}(T)$. To be specific, consider the inhomogeneous Klein-Gordon equation

$$\left(\Box + m^2\right) u = v \tag{A.14}$$

Then we have $T = (m^2 - p^2)^{-1}$ which gives us the familiar propagator

$$G(x) = (2\pi)^{-D} \int_{\mathbb{R}^D} \left(m^2 - p^2\right)^{-1} e^{-ix\cdot p} dp \tag{A.15}$$

And for the homogenous Klein Gordon equation

$$\left(\Box + m^2\right) \phi = 0 \tag{A.16}$$

The fundamental solution is generated by

$$D_m(x) := 2\pi i \mathscr{F}^{-1} \left(\left(\operatorname{sgn}(p_0)\, \delta\left(p^2 - m^2\right)\right) (x)\right. \tag{A.17}$$

Which gives the general solution, or the familiar plane wave expansion

$$\phi(t, \mathrm{x}) := (2\pi)^D \int_{\mathbb{R}^{D-1}} \left(a(\mathrm{p}) e^{i(\mathrm{p}\cdot x - \omega(\mathrm{p})t)} + a^*(\mathrm{p}) e^{-i(\mathrm{p}\cdot \mathrm{x} - \omega(\mathrm{p})t)}\right) d\lambda_m(\mathrm{p}) \tag{A.18}$$

where a, a^* are functions in the Scgwarz space of the forward light cone, and λ_m the invariant measure on it.

Now we can generalize all the above concepts to the case of operator valued distribution on some Hilbert space $\mathcal{H}$

$$\Phi : \mathscr{S}\left(\mathbb{R}^n\right) \to \mathscr{O} \tag{A.19}$$

This is because we define derivative and Fourier transform only through their actions on the argument $f \in \mathscr{S}\left(\mathbb{R}^n\right)$. But for the field operator in quantum field theory, we should put on some extra constrains, we demand there is a dense subspace $D \subset \mathcal{H}$ such that

1. $\forall f \in \mathscr{S}$ we have $D \subset D_{\Phi(f)}$ so all $\Phi(f)$ are well defined on the common subspace D, this is necessary as usually field operator can not be defined everywhere on whole $\mathcal{H}$

2. The induced map $\mathscr{S} \to \operatorname{End}(D)$, $f \mapsto \Phi(f)|_D$, is linear. This is an abstraction of the idea of linear response in measurement such that the field couples to the classical source linearly.

3. $\forall v \in D, \forall w \in \mathcal{H}$ the assignment $f \mapsto \langle w, \Phi(f)(v)\rangle$ is a tempered distribution, this means that the physical observation processes are characterized by tempered distributions.

It can be shown without too much difficulty that free fields such as the one in A.18 and its generalizations with appropriate symmetry properties are all well defined field operators. And nontrivial examples of interacting fields also exist, but the explicit constructions are much more difficult and technical.[89]

A.1.2 Wightman axioms

Minkowiski formalism and Wightman axioms By a quantum field theory we mean a collection of some quantum fields with certain axioms, naively we tend to consider a quantum field as an operator valued mappings defined on each spacetime point, in physical textbooks it is usually denoted as $\widehat{\Phi}(t,x)$, with the implicit understanding that $\Phi(t,x)$ is the corresponding classical field, also exists, and $\widehat{\Phi}(t,x)$ is its quantization. But careful examination shows that it is better to view quantum fields as operator valued (tempered) distributions, and there are two reasons for this:

- The concept of operator valued field is simply mathematically inconsistent with the axioms we want hence ill defined.
- More importantly, from a physical viewpoint, an object such as $\widehat{\Phi}(t,x)$ is of meta-physical 'Ding an sich' type, in actual experiments one can not measure the quantity $\widehat{\Phi}$ defined precisely at the spacetime point (t,x), rather one only measures things in some finite region Δ of spacetime around (t,x), and what we get is the exception value $<\widehat{\Phi}>_\Delta$, in this way we may view our experiment as a test function f compactly supported on Δ , and the quantum field Φ as a distribution applied to it to generate the number $\Phi(f) =< \widehat{\Phi} >_\Delta$.

For convenience we will still use the notation $\widehat{\Phi}(t,x)$, and usually we omit the hat and denote t,x simply by x as well, so we just write $\Phi(x)$, but with the above understanding in mind.

By axioms, we mean some obvious properties we would like to have for our quantum fields, in the most typical examples of fields in the standard model, we have the following Wightman axioms for fields in Minkowski spacetime:

- Covariance

  We have the Poincaré group P, with an unitary representation U on a Hilbert space H with a vacuum $|\Omega\rangle$, such that the vacuum is invariant and the fields are covariant[125]

$$U(g)\,|\Omega\rangle = |\Omega\rangle \quad U(g)\Phi(x)U(g)^\dagger = \Phi(gx) \tag{A.20}$$

- Locality

  Fields always (anti)commute at spacelike separation[126]

$$\forall (x-y)^2 < 0, \quad [\Phi_1(x), \Phi_2(y)] = 0 \tag{A.21}$$

- Spectrum condition

  The joint spectrum of the momentum P_μ is contained in the forward light cone, i.e. mass is non-negative and causality points toward the future.

[125]Usually, the vacuum is assumed to be unique, so if $\forall g, U(g)\,|v\rangle = |v\rangle$ then $|v\rangle = c\,|\Omega\rangle$ for some constant c

[126]for fermionic fields anti commutators are used instead

Once a quantum field theory is defined, the main objects of study interest are the n-body correlators, or vacuum expectation values(VEVs for short),or in math jargon Wightman distributions

$$W_n(f_1, f_2, \cdots, f_n) = \langle\Omega| \Phi_1(f_1)\Phi_2(f_2)\cdots\Phi_n(f_n) |\Omega\rangle \tag{A.22}$$

Again, we usually stick with physicists' shorthand notation for convenience:

$$W_n(x_1, x_2, \cdots, x_n) =< \Phi_1(x_1)\Phi_2(x_2)\cdots\Phi_n(x_n) > \tag{A.23}$$

or simply just $W_n(x)$ with $x = \{x_1, x_2, \cdots, x_n\}$.

Every correlator W_n satisfies:

- Covariance

  W_n is invariant under $g \in P$

  $$W_n(x) = W_n(gx) \tag{A.24}$$

- Locality

  W_n does not depend on the order of x_i, x_j if they are spacelike separated

  $$\forall (x_i - x_j)^2 < 0, \quad W_n(x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n) = W_n(x_1, \cdots, x_j, \cdots, x_i, \cdots, x_n) \tag{A.25}$$

- Spectrum condition

  By translational invariance, W_n indeed just depends on the $n-1$ variables $x_{j+1} - x_j$ so we can introduce the Fourier transform $M_n(p)$ of $W_n(x)$ where $p = (p_1, \cdots, p_{n-1}), dp = dp_1 \cdots dp_{n-1}$ and

  $$W_n(x) = \int M_n(p) \exp i \sum p_j \cdots (x_{j+1} - x_j) dp \tag{A.26}$$

  Then $M_n(p)$ is supported in the products of forward lightcone

- Positive Definiteness

  For sequence of test functions $f = \{f_1, \cdots, f_n\}, g = \{g_1, \cdots, g_m\}$, define

  $$f \otimes g(x_1, \cdots, x_{m+n}) = f(x_1, \cdots, x_m) g(x_{m+1}, \cdots, x_{m+n}) \tag{A.27}$$

  then for any fixed $k \in \mathbb{N}$

  $$\sum_{m,n=0}^{k} W_{m+n}(\bar{f_m} \otimes g_n) \geq 0 \tag{A.28}$$

  This property is indeed a characterization of unitarity, i.e. in our theory

  $$\langle\psi|\psi\rangle \geq 0 \tag{A.29}$$

operationally speaking, the main purpose of using quantum field theory is to calculate the correlators, and indeed we have the following:

Theorem A.1 (Wightman Reconstruction theorem)**.** *Given a collection of tempered distributions W_n satisfies the above four proprieties, then there exists a Wightman QFT where W_n's are the Wightman distributions of this theory*

Euclidean formalism and reflection positivity The above discussions all focus on Minkowski spacetime, while in many cases, Euclidean spacetime is used instead. To obtain a Euclidean version of a given Wightman QFT, we can do analytic continuation on Wightman distributions by using imaginary time $t \to it$, i.e. by the following identification

$$E := \{(it, x^1, \cdots, x^{d-1}) \in \mathbb{C}^d \sim (\tau, x^1, \cdots, x^{d-1}) \in \mathbb{R}^d\} \tag{A.30}$$

This is always possible unless two points coincide, i.e. $x \in \Delta = \{\{x_1, \cdots, x_n\} : \exists i \neq j, \quad x_i = x_j\}$, so we have the extended version of W_n

$$S_n := W_n|_{E^n \setminus \Delta} \tag{A.31}$$

And these Euclidean correlators are also called the Schwinger functions.

Locality and Covariance generalize to Schwinger functions in an obvious way, but the Spectrum condition and Positive Definiteness are replaced by the following property:

- Reflection Positivity

  Let $\theta : (\tau, x^1, \cdots, x^{d-1}) \to (-\tau, x^1, \cdots, x^{d-1})$ be the Euclidean reflection in time direction, and the operator Θ acts on time ordered test functions $f(x) = f(x_1, \cdots, x_n)$ with $\tau_1 < \tau_2 < \cdots < \tau_n$ as $\Theta f(x) := \bar{f}(\theta x)$, then we have [127]

  $$\sum_{m,n=0}^{k} S_{m+n}(\Theta \bar{f_m} \otimes g_n) \geq 0 \tag{A.32}$$

  From this perspective, given

  $$|\psi\rangle = O(-\tau_1)O(-\tau_2)\cdots O(-\tau_n)|\Omega\rangle \tag{A.33}$$

  we should define

  $$\langle\psi| = \langle\Omega| O(\tau_n)O(\tau_{n-1})\cdots O(\tau_1) \tag{A.34}$$

  and we again need

  $$\langle\psi|\psi\rangle \geq 0 \tag{A.35}$$

On the reverse, we can define a Euclidean QFT at the beginning and derive Schwinger functions as analytic tempered distributions on $E^n \setminus \Delta$, then we have the following theorem:

[127] to avoid confusion, here we use x_i to mean different spacetime points, rather than components of a given point, and τ_i is understood as the zeroth components of x_i

Theorem A.2 (Osterwalder-Schrader theorem)**.** *Schwinger functions of a Euclidean QFT satisfy covariance, locality and reflection positivity are precisely the analytic continuations of Wightman distributions of a Wightman QFT.*

The main lesson here is that unitarity in Euclidean theory needs non-trivial check.

The benefits of the above axiomatic approach are:

- It is weak enough such that it contains only essential features of QFT, all packed together in an economic and universal way, hence these axioms are especially useful as basic criteria.
- It is strong enough such that with those axioms we can prove important generic theorems about QFT directly or by a adding few extra assumptions, for example, CPT theorem, spin-statistics theorem, Coleman–Mandula theorem and so on.

However, the main problem with the above axiomatic approach is that these axioms do not tell us how to construct actual quantum field theories other than trivial ones, i.e free fields such as the one in (A.18) where we can indeed prove that all axioms are satisfied and all fields are well defined. And this construction in general is indeed extremely difficult, especially when the spacetime is four dimensional and gauge symmetries are involved, and this is the famous Yang–Mills existence and mass gap problem. Up to now, only lower dimensional($d < 4$) examples are known.

For special types of QFTs, such as TQFT, CFT and super QFT, similar sets of axioms exist by simply reducing or enlarging the symmetry involved , and adding corresponding consistency conditions. We will not discuss these extensions in detail, instead we will introduce some alternative formalism for these special types in the following sections, where all the alternatives are and indeed must be consistent with their Wightman style axioms.

A.2 Vertex operator algebra

A.2.1 Formal distributions

In a similar spirit to our treatment for operator valued tempered distribution, in this section we will introduce the notation of formal distribution, it mimics complex analysis of one variable in a formal way such that we can talk about series expansions and residues without out worrying about contours, convergence conditions and explicit integration processes. This feature makes it an ideal tool for the mathematical formulation of $d = 2$ CFT, as a result we will define fields in a CFT as a sepcial type of formal distribution.

Let $Z = \{z_1, \ldots, z_n\}$ be a set of formal variables, and R be a $\mathbb{C}$ vector space, then a formal distribution is a formal series expansion with $A_j \in R$

$$A(z_1, \ldots, z_n) = \sum_{j\in\mathbb{Z}^n} A_j z^j = \sum_{j\in\mathbb{Z}^n} A_{j_1,\ldots,j_n} z_1^{j_1} \ldots z_n^{j_n} \tag{A.36}$$

The space of formal distributions is denoted as $R\left[\left[z_1^{\pm},\ldots,z_n^{\pm}\right]\right]=R\left[\left[z_1,\ldots,z_n,z_1^{-1}\ldots,z_n^{-1}\right]\right]$ or $R\left[\left[Z^{\pm}\right]\right]$, it contains the subspace of Laurent polynomials

$$R\left[z_1^{\pm},\ldots,z_n^{\pm}\right]=\left\{A\in R\left[\left[z_1^{\pm},\ldots,z_n^{\pm}\right]\right]\mid\quad \exists k,l:A_j=0\text{ except for }k\leq j\leq l\right\}. \quad (A.37)$$

and the subspace of formal power series

$$R\left[\left[z_1,\ldots,z_n\right]\right]:=\left\{A:A=\sum_{j\in\mathbb{N}^n}A_{j_1,\ldots,j_n}z_1^{j_1}\ldots z_n^{j_n}\right\} \quad (A.38)$$

In particular, for the case of one formal variable only, we also define the subspace of formal Laurent series

$$R((z))=\left\{A\in R\left[\left[z^{\pm}\right]\right]\mid\exists k\in\mathbb{Z}\forall j\in\mathbb{Z}:j<k\Rightarrow A_j=0\right\} \quad (A.39)$$

Formal distributions can be added easily, but multiplication is not always well defined. For $A,B\in R\left[\left[Z^{\pm}\right]\right]$, the usual Cauchy product is well defined whenever A and B are formal Laurent series or when B is a Laurent polynomial. And for $A(z)B(w)\in R\left[\left[Z^{\pm},W^{\pm}\right]\right]$ it is always well defined.

In the following we will mainly focus on the $Z=\{z\}$ case where z can be viewed as the complex coordinate for $d=2$ CFT, here we can formally define the residue of $A\in R\left[\left[z^{\pm}\right]\right],A(z)=\sum_{j\in\mathbb{Z}}A_jz^j$ as

$$\operatorname{Res}_z A(z)=A_{-1}\in R \quad (A.40)$$

and formal derivative

$$\partial\left(\sum_{j\in\mathbb{Z}}A_jz^j\right)=\sum_{j\in\mathbb{Z}}(j+1)A_{j+1}z^j \quad (A.41)$$

Every A will induce the following map

$$\widehat{A}:\mathbb{C}\left[z^{\pm}\right]\to R,\quad \widehat{A}(f(z)):=\operatorname{Res}_z A(z)f(z),\phi\in\mathbb{C}\left[z^{\pm}\right] \quad (A.42)$$

And it turns out that this provides an isomorphism $R\left[\left[z^{\pm}\right]\right]\to\operatorname{Hom}\left(\mathbb{C}\left[z^{\pm}\right],R\right)$ in this sense we can view A as a functional hence a R valued distribution over the ring of formal complex series, which justifies the nomenclature of formal distribution.

Similarly to tempered distributions, true polynomials and series in $\mathbb{C}[z^{\pm}]$ are formal distributions.

Since we also need to define commutators between fields, $\mathbb{C}\left[\left[z^{\pm},w^{\pm}\right]\right]$ appears also naturally, and here the (formal) delta function is defined to be $\delta\in\mathbb{C}\left[\left[z^{\pm},w^{\pm}\right]\right]$ such that

$$\delta(z-w)=\sum_{n\in\mathbb{Z}}z^{n-1}w^{-n}=\sum_{n\in\mathbb{Z}}z^nw^{-n-1}=\sum_{n\in\mathbb{Z}}z^{-n-1}w^n. \quad (A.43)$$

In particular, we are interested in the special type of formal distribution such that $f\in R\left[\left[z^{\pm},w^{\pm}\right]\right]$ with $(z-w)^Nf=0$ for some fixed $N\in\mathbb{N}:N>0$, as we will see later this resemblances locality. These special f's have a very nice property such that they can be written uniquely as linear combinations of derivatives, in a way similar to tempered distributions on $\mathbb{R}^n \to \mathbb{C}$ in (A.5)

$$f(z,w) = \sum_{j=0}^{N-1} c^j(w) D_w^j \delta(z-w), c^j \in R\left[\left[w^{\pm}\right]\right] \tag{A.44}$$

where $D_w^j := \frac{1}{j!}\partial_w^j$ and in addition, we have

$$\forall n : 0 \leq n < N, \quad c^n(w) = \mathrm{Res}_z (z-w)^n f(z,w) \tag{A.45}$$

This formally generalizes the concept of pole and meromorphic function.

Unlike tempered distributions, we do not need to define integration and Fourier transform, because in the world of meromorphic functions all these can be replaced by formal manipulations on poles, especially on Res. The point is that we have made some formal constructions such that actual contour integrals never appear, but the results of integration processes are obtained formally.

We say two formal distributions $A, B \in R[[z^{\pm}]]$ are local with respect to each other if for some $N \in \mathbb{N}$

$$(z-w)^N [A(z), B(w)] = 0 \tag{A.46}$$

By differentiate both sides, it is obvious that if A and B are mutually local, so does ∂A and B and so on, physically this just says if primaries are mutually local so does the descendants.

Now we define normal ordering as(assuming $|z| > |w|$ so the expansion makes sense formally)

$$: A(z)B(w) := \sum_{n\in\mathbb{Z}} \left(\sum_{m<0} A_{(m)} B_{(n)} z^{-m-1} + \sum_{m \geq 0} B_{(n)} A_{(m)} z^{-m-1} \right) w^{-n-1} \tag{A.47}$$

where $A_{(n)} = A_{-n-1} = \mathrm{Res}_z\, A(z) z^n$, then A, B are mutually local iff we have

$$A(z)B(w) = \sum_{j=0}^{N-1} \frac{C^j(w)}{(z-w)^{j+1}} + : A(z)B(w) :\sim \sum_{j=0}^{N-1} \frac{C^j(w)}{(z-w)^{j+1}} \tag{A.48}$$

We also have the following nontrivial result

Lemma A.3 (Dong)**.** *suppose $A(z), B(z), C(z)$ are pair wise mutually local, so does $: A(z)B(z) :$ and $C(z)$*

The point of above discussions is that physically OPEs are well defined for mutually local fields.

Finally, let us define field operators, including both primary and descendant. First we fix $R = \mathrm{End}\, V$ with $b \in R, \quad b : V \to V$ and denote $b(v)$ as $b \cdot v$ or bv, then a field is a formal distribution

$$a \in \mathrm{End}\, V\left[\left[z^{\pm}\right]\right], a = \sum a_{(n)} z^{-n-1} \tag{A.49}$$

such that $\forall v \in V, \quad \exists n_0 \in \mathbb{N} : \quad \forall n \geq n_0$

$$a_{(n)}(v) = a_{(n)} \cdot v = a_{(n)} v = 0 \tag{A.50}$$

If a, b are fields, so does ∂a and $: a(z)b(z) :$ and so on.

Especially,for example in Virasoro algebra, when V has a natural grading ,

$$V = \bigoplus_{n \in \mathbb{Z}} V_n \tag{A.51}$$

with $V_n = \{0\}$ for $n < 0$ and $\dim V_n < \infty$, we say $T \in \mathrm{End}\, V$ is homogenous of degree g if $T(V_n) \subset V_{n+g}$, and a formal distribution is homogenous of conformal weight $h \in \mathbb{Z}$ if all $a_{(k)}$ are homogenous of degree $h - k - 1$, then all homogenous formal distributions are fields. This weight so defined behaves in a reasonable way, for example if a has weight h_a, ∂a has weight $h_a + 1$, and $: a(z)b(z) :$ has weight $h_a + h_b$

A.2.2 $d = 2$ chiral CFT as vertex operator algebra

In parallel with our approach to generic QFT, we now define a CFT as a collection of fields with some axioms, the corresponding mathematical object is called a vertex operator algebra, or simply VOA. This name comes from the vertex operators in free boson theory, which serves as the prototype of VOA.

A vertex operator algebra is a vector space V equipped with an element $\Omega \in V$ as the vacuum, an endomorphism $T \in \mathrm{End}(V)$ as the infinite small translation operator(the energy momentum tensor), a linear map $Y : V \to \mathscr{F}(V)$ to the space of fields[128] as a realization of operator-state correspondence

$$a \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}, a_{(n)} \in \text{ End } V \tag{A.52}$$

It satisfies the following axioms:

- Translation covariance

$$\forall a \in V : \quad [T, Y(a, z)] = \partial Y(a, z) \tag{A.53}$$

- Locality

$$\forall a, b \in V, \exists N \in \mathbb{N} : \quad (z - w)^N [Y(a, z), Y(b, w)] = 0 \tag{A.54}$$

- Vacuum

$$\forall a \in V : T\Omega = 0, Y(\Omega, z) = \mathrm{id}_V, Y(a, z)\Omega|_{z=0} = a \tag{A.55}$$

[128]Here the field is defined in the sense of last section as a special kind of formal distribution

Usually, one assume further that

- Nonnegative Grading

$$V = \oplus_{n=0}^{\infty} V_n \tag{A.56}$$

By direct calculation we have $Ta = a_{(-2)}\Omega$, i.e the standard physical argument 'the energy momentum tensor is a quasi primary as a $h = 2$ descendant of the vacuum', and when the above grading exists, we will assume further that $V_0 = \mathbb{C}\Omega$, and T is homogeneous of degree 1, $Y(a, z)$ is homogeneous of weight m for $a \in V_m$, in accordance with the Virasoro example.

With those axioms, we can view the chiral part of a conformal field as an operator valued formal distribution in the sense of

$$\widehat{Y}(a,) : \mathbb{C}\left[\left[z^{\pm}\right]\right] \to \text{ End } V \tag{A.57}$$

The primary example is, of course the Virasoro algebra, where in section 1.2.2 we have (1.140) and (1.141)

More generally we will call any T satisfies (1.140) and (1.141) as a Virasoro field with central charge c. If we have a vector $v \in V$ such that $Y(v, z) = \sum v_{(n)} z^{-n-1} = \sum L_n^v z^{-n-2}$ is a Virasoro field with central charge c and in addition

- $T = L_{-1}^v$
- L_0^v is diagonalizable.

then we will call this vector as a conformal vector with central charge c, and a VOA with a conformal vector with central charge c as a conformal VOA with central charge c. This merely means that T is indeed an energy momentum tensor as it generates translation , the corresponding scaling dimension and central charge are all well defined.

With those notations we can actually verify that the chiral parts of all those $d = 2$ CFTs we have introduced in section1.2.2 and section1.2.3 are conformal VOAs with corresponding central charges, and all the relevant properties for OPEs, primaries, states and so on are proved as well. Indeed all of these models are special types of representations of VOAs, and correspondingly RCFTs are finite direct sums of irreducible representations. So essentially VOA characterizes the genus zero part of chiral $d = 2$ CFT. With an appropriate construction for the superpartner $G(z)$ of $T(z)$, super VOA can be defined and it again characterize the genus zero part of chiral $d = 2$ SCFT.

We should note that unlike the Wightman axioms, the VOA axioms are very simple, and actual models are easy to construct as all constructions are formal hence problems such as definable domain for operator or convergence issues all disappear. This is possible because we have restricted to $d = 2$ spacetime and given this theory an 'meromorphic' or 'analytic on Riemann sphere' flavor at the very beginning, just like the theory of one complex variable compares with the several real variables case.

A.3 Categories and functors

A.3.1 Modular tensor category

In this section we will introduce some basic facts about categories and functors, with these concepts we can define tensor product, braiding, fusion formally and analyze their relations, which essentially characterize the properties and concepts we have seen in $d = 3$ TQFT and $d = 2$ RCFT in an algebraic way, in particular we can define modulartiy and this characterizes the genus one or higher parts of these theories.

Throughout this section we will work in a characteristic zero field k, and use stylized letters $\mathcal{A}, \mathcal{B}, \cdots$ for categories, capital letters $U, V, \cdots$ for objects, Greek letters $\psi, \varphi, \cdots$ for morphisms.

An additive category $\mathcal{C}$ over k is a category such that

- All $\mathrm{Hom}_{\mathcal{C}}(U, V)$ are k vector space, and all compositions $\mathrm{Hom}_{\mathcal{C}}(V, W) \times \mathrm{Hom}_{\mathcal{C}}(U, V) \to \mathrm{Hom}_{\mathcal{C}}(U, W), \quad (\varphi, \psi) \mapsto \varphi \circ \psi$ are k bilinear
- Zero object 0 exists, and $\mathrm{Hom}_{\mathcal{C}}(0, V) = \mathrm{Hom}_{\mathcal{C}}(V, 0) = 0$ for all V
- Finite sum $\oplus$ exists

An abelian category $\mathcal{A}$ is an additive category such that

- every φ is a composition of an epimorphism followed by a monomorphism, and $\ker \varphi$, $\mathrm{coker}\, \varphi$ always exist. If $\ker \varphi = 0$, then $\varphi = \ker(\mathrm{coker}\, \varphi)$; if $\mathrm{coker}\, \varphi = 0$, then $\varphi = \mathrm{coker}(\ker \varphi)$

As the name suggests, the category of abelian groups *Ab* is an abelian category, and indeed it is the primary one rules over all because we have a metatheorem which says that a proposition that depends only on the above data in a categorical way is true for $\mathcal{A}$ iff the same thing is true for *Ab*. In a sense, this just says that in *Ab* we can formally pretend that we are dealing with abelian groups and perform all the typical constructions without using the actual group structures. Other common examples of abelian categories are the category $Vec(k)$ of k vector spaces, the category $Vec_f(k)$ of finite dimensional k vector spaces, the category $Rep(A)$ of representations of a k algebra A, the category $Rep(G)$ of representations of a group G over k.

Given an abelian category $\mathcal{A}$, an object U of it is simple if every injection $V \hookrightarrow U$ is either 0 or isomorphic, and we call $\mathcal{A}$ as semisimple if every object V is a direct sum of simple objects, that is

$$V \simeq \bigoplus_{i \in I} N_i V_i \tag{A.58}$$

with $N_i \in \mathbb{N} : N_i > 0$ and I runs over all (isomorphism classes of)non-zero simple objects. We will consider semisimple abelian categories only, and assume further that

$$\forall i \in I, \quad \mathrm{End} V_i = k \tag{A.59}$$

And we should notice that by definition $\mathrm{Hom}(V_i, V_j) = 0$ when $i \neq j$.

In abelian categories we have a well defined natural addition as $\oplus$, now we will introduce another kind of category where a natural multiplication, that is , the tensor product $\otimes$ is well defined.

A monoidal category $\mathcal{M}$ is a category equipped with

- a bifunctor

$$\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \tag{A.60}$$

- a functorial isomorphism

$$\alpha_{UVW} : (U \otimes V) \otimes W \xrightarrow{\sim} U \otimes (V \otimes W) \tag{A.61}$$

- a unit object $\mathbf{1}$ with isomorphisms

$$\begin{aligned} \lambda_V &: \mathbf{1} \otimes V \xrightarrow{\sim} V \\ \rho_V &: V \otimes \mathbf{1} \xrightarrow{\sim} V \end{aligned} \tag{A.62}$$

- and satisfies the associativity axiom

  All compositions of α 's, λ 's, ρ 's and their inverses are associative

Just like abelian category is an abstract generalization of abelian group, monoidal category is an abstract generalization of monoid. It also includes $Vec(k)$, $Vec_f(k)$ and $Rep(A)$, in addition, it contains the category of $Rep(\mathfrak{g})$ of representations of a Lie algebra $\mathfrak{g}$ over k(and the finite dimensional $Rep_f(\mathfrak{g})$) . As those examples suggest, a category $\mathcal{C}$ can be both abelian and monoidal, in this case for consistency we will assume further that

- $\mathbf{1}$ is simple and $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) = k$

A notable fact of monoidal category is that at this level we already have the following theorem

Theorem A.4 (MacLane Coherence theorem)**.** *Given* $(\mathcal{C}, \otimes, \alpha, \lambda, \rho)$ *as above,* $\mathcal{C}$ *is monoidal iff it satisfies*

- *Pentagon axiom, see figure36* [129]

- *Triangle axiom see figure37*

A monoidal category $\mathcal{M}$ is strict if instead of isomorphisms we have equalities

$$V \otimes \mathbf{1} = V, \quad \mathbf{1} \otimes V = V, \quad (V_1 \otimes V_2) \otimes V_3 = V_1 \otimes (V_2 \otimes V_3) \tag{A.63}$$

In this case we can omit the brackets altogether, we will always assume this as well, and indeed this is fine because another theorem from MacLane

[129] in particular figure 5 is a special case of this

$$\begin{tikzcd}
& ((V_1 \otimes V_2) \otimes V_3) \otimes V_4 \arrow[dl, "\alpha_{1,2,3} \otimes \mathrm{id}_4"'] \arrow[dr, "\alpha_{12,3,4}"] & \\
(V_1 \otimes (V_2 \otimes V_3)) \otimes V_4 \arrow[d, "\alpha_{1,23,4}"] & & (V_1 \otimes V_2) \otimes (V_3 \otimes V_4) \arrow[d, "\alpha_{1,2,34}"'] \\
V_1 \otimes ((V_2 \otimes V_3) \otimes V_4) \arrow[rr, "\mathrm{id}_1 \otimes \alpha_{2,3,4}"'] & & V_1 \otimes (V_2 \otimes (V_3 \otimes V_4))
\end{tikzcd} .$$

Figure 36: Pentagon axiom [13]

$$\begin{tikzcd}
(V_1 \otimes \mathbf{1}) \otimes V_2 \arrow[rr, "\alpha"] \arrow[dr, "\rho \otimes \mathrm{id}"'] & & V_1 \otimes (\mathbf{1} \otimes V_2) \arrow[dl, "\mathrm{id} \otimes \lambda"] \\
& V_1 \otimes V_2 &
\end{tikzcd}$$

Figure 37: Triangle axiom [13]

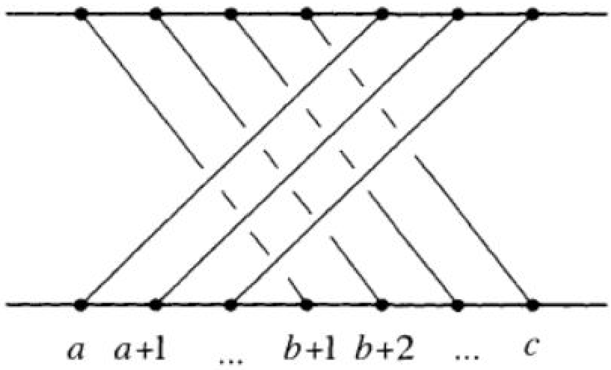


Figure 38: the action of braid b_{AB} on strands [13]

Theorem A.5 (MacLane). *Every monoidal category is equivalent to a strict one*

Now that we have equipped with both $\otimes$ and $\oplus$, it is time to talk about braiding. A braided tensor category $\mathcal{B}$ is a monoidal category equipped with

- a functorial isomorphism called braiding

$$\sigma_{VW} : V \otimes W \xrightarrow{\sim} W \otimes V \tag{A.64}$$

realized as a representation of the generators of the braid groups B_n, through

$$\begin{aligned} \sigma_{AB} : \cdots \otimes (V_{i_a} \otimes \cdots \otimes V_{i_b}) \otimes \left(V_{i_{b+1}} \otimes \cdots \otimes V_{i_c}\right) \otimes \cdots \\ \xrightarrow{\sim} \cdots \otimes \left(V_{i_{b+1}} \otimes \cdots \otimes V_{i_c}\right) \otimes (V_{i_a} \otimes \cdots \otimes V_{i_b}) \otimes \cdots \end{aligned} \tag{A.65}$$

such that all compositions of α 's, λ 's, ρ 's, σ 's and their inverses depend only on its image of the braid groups B_n, see figure 38

Typical examples of braided tensor category are again $Vec(k)$, $Vec_f(k)$ and $Rep(A)$. At this level, we have a similar coherence theorem as follow

Theorem A.6. $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, \lambda, \rho, \sigma)$ *is a braided tensor category iff it satisfies the following*

- *Pentagon axiom*
- *Triangle axiom*
- *Hexagon axiom, see figure39* [130]

$$\begin{tikzcd}
 & V_1 \otimes (V_2 \otimes V_3) \arrow[r, "\sigma_{1,23}"] & (V_2 \otimes V_3) \otimes V_1 \arrow[dr, "\alpha_{2,3,1}"] & \\
(V_1 \otimes V_2) \otimes V_3 \arrow[ur, "\alpha_{1,2,3}"] \arrow[dr, "\sigma_{1,2} \otimes \mathrm{id}_3"'] & & & V_2 \otimes (V_3 \otimes V_1) \; . \\
 & (V_2 \otimes V_1) \otimes V_3 \arrow[r, "\alpha_{2,1,3}"'] & V_2 \otimes (V_1 \otimes V_3) \arrow[ur, "\mathrm{id}_2 \otimes \sigma_{1,3}"'] &
\end{tikzcd}$$

Figure 39: Hexagon axiom [13]

A particularly important case of the braided tensor category is the symmetric braided tensor category, where for all σ we have $\sigma_{WV}\sigma_{VW} = \mathrm{id}_{V\otimes W}$. As examples, $Vec(k)$ and $Vec_f(k)$ are all symmetric.

In $Vec(k)$ we have dual, and dual of dual, all of which are very powerful tools in constructions and calculations, so we generalize those notations as well. Let M be a monoidal category and V an object, the right dual V^* of V is an object with two isomorphims

$$\begin{aligned} e_V &: V^* \otimes V \to \mathbf{1} \\ i_V &: \mathbf{1} \to V \otimes V^*, \end{aligned} \tag{A.66}$$

such that $V \overset{i_V \otimes \mathrm{id}_V}{\longrightarrow} V \otimes V^* \otimes V \overset{\mathrm{id}_V \otimes e_V}{\longrightarrow} V$ is equal to id_V and

$V^* \overset{\mathrm{id}_{V^*} \otimes i_V}{\longrightarrow} V^* \otimes V \otimes V^* \overset{e_V \otimes \mathrm{id}_{V^*}}{\longrightarrow} V^*$ is equal to id_{V^*} These properties are called right rigidity axioms, similarly one can define the left dual ${}^*V$ with

$$\begin{aligned} e'_V &: V \otimes^* V \to \mathbf{1} \\ i'_V &: \mathbf{1} \to {}^*V \otimes V \end{aligned} \tag{A.67}$$

and similar left rigidity axioms. Then M is called rigid if every object has both its left and right duals. The main reason for elaborating on this is that we now have:

$$\begin{aligned} \mathrm{Hom}(U \otimes V, W) &= \mathrm{Hom}\left(U, W \otimes V^*\right) \\ \mathrm{Hom}(U, V \otimes W) &= \mathrm{Hom}\left(V^* \otimes U, W\right) \end{aligned} \tag{A.68}$$

and

$$\mathrm{Hom}(U, V) = \mathrm{Hom}\left(V^*, U^*\right) = \mathrm{Hom}\left(1, V \otimes U^*\right) \tag{A.69}$$

[130]in particular figure 6 is a special case of this

along with

$$
\begin{aligned}
&1^{*}=1={}^{*}1\\
&(V\otimes W)^{*}=W^{*}\otimes V^{*}\\
&\left(\alpha_{V_1V_2V_3}\right)^{*}=\alpha_{V_3^{*}V_2^{*}V_1^{*}}
\end{aligned}
\tag{A.70}
$$

and especially for braided tensor category

$$
\begin{aligned}
&\left(\sigma_{VW}\right)^{*}=\sigma_{V^{*}W^{*}}\\
&e_{V\otimes W}=\left(e_V\otimes e_W\right)\left(\sigma_{W^{*},V^{*}\otimes V}\otimes \mathrm{id}\right)\\
&i_{V\otimes W}=\left(\mathrm{id}\otimes \sigma_{V^{*},W\otimes W^{*}}\right)\left(i_V\otimes i_W\right)
\end{aligned}
\tag{A.71}
$$

All of these suggest that rigidity mimics the notion of antiparticle. But this is not enough, as we expect the antiparticle of an antiparticle is the original particle, that is , we need the notion of dual of dual. A ribbon category $\mathcal{R}$ is a rigid braided tensor category equipped with

- a functorial isomorphism δ

$$
\delta_V : V \xrightarrow{\sim} V^{**}
\tag{A.72}
$$

  such that

$$
\begin{aligned}
\delta_{V\otimes W}&=\delta_V\otimes\delta_W\\
\delta_1&=\mathrm{id},\\
\delta_{V^{*}}&=\left(\delta_V^{*}\right)^{-1}
\end{aligned}
\tag{A.73}
$$

Notice that in any rigid braided tensor category $\mathcal{B}$ we can construct the following

$$
\psi_V : V^{**} \xrightarrow{\sim} V
\tag{A.74}
$$

as

$$
V^{**}\xrightarrow{i\otimes\mathrm{id}} V\otimes V^{*}\otimes V^{**}\xrightarrow{\mathrm{id}\otimes\sigma^{-1}} V\otimes V^{**}\otimes V^{*}\xrightarrow{\mathrm{id}\otimes e} V
\tag{A.75}
$$

In symmetric $\mathcal{B}$, we have $\delta_V=\psi_V^{-1}$ and $\mathcal{B}$ becomes ribbon automatically, but in general this does not happen, and we can define the so called balancing isomorphism to measure how far a ribbon category $\mathcal{R}$ is deviated from this

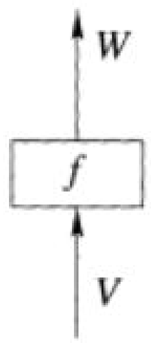
$$
\theta_V=\psi_V\delta_V : V \xrightarrow{\sim} V
\tag{A.76}
$$

this θ satisfies

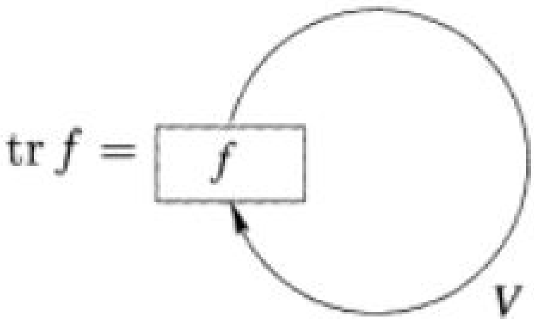
$$
\begin{aligned}
\theta_{V\otimes W}&=\sigma_{WV}\sigma_{VW}\left(\theta_V\otimes\theta_W\right)\\
\theta_1&=\mathrm{id}\\
\theta_{V^{*}}&=\left(\theta_V\right)^{*}
\end{aligned}
\tag{A.77}
$$

Although looks strange, this θ_V is very important, and indeed we will see later that it is closely related to what we called topological twist.

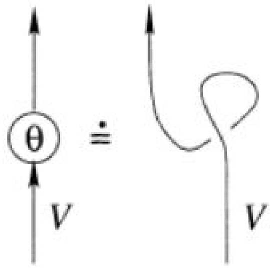
The main point of using ribbon category is that, just like a ribbon we can twist it as we want, so here the usual graphic calculus for braiding is well defined, see figure40. For



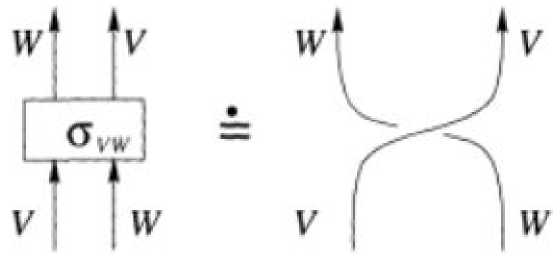
(a) the map f



(b) the trace $\mathrm{tr}\, f$



(c) the map θ_V



(d) the map σ_{VW}

Figure 40: Examples of graphic calculus[13]

example we can represent a map $f : V \to W$ as a line with a box, and by convention id_V will be omitted. Especially for $f \in \mathrm{End}V$ we have

$$\mathbf{1} \xrightarrow{i_V} V \otimes V^* \xrightarrow{f\otimes\mathrm{id}} V \otimes V^* \xrightarrow{\delta_V\otimes\mathrm{id}} V^{**} \otimes V^* \xrightarrow{e_{V^*}} \mathbf{1} \tag{A.78}$$

but since $\mathrm{End}_k(\mathbf{1}) \simeq k$ this defines the trace of f as an element $\mathrm{tr}\, f \in \mathrm{End}_k(\mathbf{1}) \simeq k$, and for id_V it is defined to be $\dim V$. In this graphic language, σ is braiding and θ is twisting in the literal sense.

Now we turn to fusion, a fusion category $\mathcal{F}$ is defined to be a rigid semisimple k linear monoidal category such that

- the index set is finite $|I| < \infty$, i.e. there are finite many (up to isomorphism classes) of simple objects.
- $\mathbf{1}$ is simple

In $\mathcal{F}$, V_i^* is simple when V_i is and $V_i^* \simeq V_{i^*}$ for some $i^* \in I$, so we have a notation of antiparticle here. We can also define fusion rule as

$$V_i \otimes V_j \simeq \bigoplus_k N_{ij}^k V_k \tag{A.79}$$

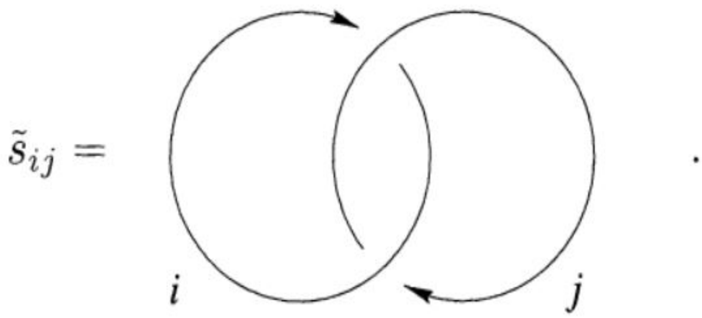


Figure 41: Graphic representation of $\widetilde{s}$ [13]

where the standard properties

$$
\begin{gathered}
N_{ij}^{k} = \dim \operatorname{Hom}\left(V_k, V_i \otimes V_j\right) = \dim \operatorname{Hom}\left(\mathbf{1}, V_i \otimes V_j \otimes V_k^*\right), \\
N_{ij}^{k} = N_{ji}^{k} = N_{ik^*}^{j^*} = N_{i^*j^*}^{k^*}, \quad N_{ij}^{0} = \delta_{ij^*}
\end{gathered}
\tag{A.80}
$$

are satisfied. we also have the twist and quantum dimension

$$
\theta_{V_i} = \theta_i \mathrm{id}_{V_i}, \quad \dim V_i = d_i \tag{A.81}
$$

where

$$
\begin{aligned}
&\theta_0 = 1, \quad \theta_{i^*} = \theta_i \\
&d_0 = 1, \quad d_{i^*} = d_i, \quad d_i d_j = \sum_k N_{ij}^k d_k.
\end{aligned}
\tag{A.82}
$$

Now it is time to bring all of the above together to define a category compatible with braiding and fusion.

To do this, let us begin with a semisimple ribbon category $\mathcal{C}$ with $|I| < \infty$, so it is a ribbon category as well as a fusion category. Then we can define the matrix

$$
\widetilde{s}_{ij} = \theta_i^{-1}\theta_j^{-1} \operatorname{tr} \theta_{V_i^* \otimes V_j} = \theta_i^{-1}\theta_j^{-1} \sum_{k \in I} N_{i^*j}^k \theta_k d_k \tag{A.83}
$$

As a consequence we have

$$
\widetilde{s}_{ij} = \widetilde{s}_{ji} = \widetilde{s}_{i^*j^*} = \widetilde{s}_{j^*i^*}, \quad \widetilde{s}_{i0} = d_i = \dim V_i \tag{A.84}
$$

we say this $\mathcal{C}$ is a modular tensor category, or MTC for short, if

- The matrix $\widetilde{s} = \left(\widetilde{s}_{ij}\right)_{i,j \in I}$ is invertible

In MTC we can define the quantity

$$
p^{\pm} := \sum_{i \in I} \theta_i^{\pm 1} d_i^2 \tag{A.85}
$$

as well as the standard T matrix and charge conjugation matrix C

$$\begin{aligned} T_{ij} &= \delta_{ij}\theta_i \\ C_{ij} &= \delta_{ij^*} \end{aligned} \tag{A.86}$$

where

$$\begin{aligned} (\widetilde{s}T)^3 &= p^+\widetilde{s}^2 \\ \left(\widetilde{s}T^{-1}\right)^3 &= p^-\widetilde{s}^2 C \\ CT = TC, C\widetilde{s} &= \widetilde{s}C, \quad C^2 = 1 \end{aligned} \tag{A.87}$$

and

$$\widetilde{s}^2 = p^+p^-C \tag{A.88}$$

The usual S matrix is obtained from $\widetilde{s}$ as $S := \widetilde{s}/D$ by the following normalization factor, or the total quantum order

$$D = \sqrt{p^+p^-} = \sqrt{\sum \dim^2 V_i} = \frac{1}{S_{00}} \tag{A.89}$$

Then from here we can recover all the modular data and their properties that we have introduced in chapter one. For example if we introduce

$$\zeta := \left(p^+/p^-\right)^{1/6} \tag{A.90}$$

then in RCFT we have

$$\theta_i = e^{2\pi \mathrm{i} h i}, \quad \zeta = e^{2\pi \mathrm{i} c/24} \tag{A.91}$$

as conformal dimensions and central charge.

The main source of MTC is $d = 2$ RCFT, under some reasonable assumptions, given a VOA V for a RCFT, $Rep(V)$ is a MTC, see [102] .Equivalently, it can be obtained from $d = 3$ TQFT, which we will discuss in next section.

A.3.2 Functorial QFT

In this section we will introduce an alternative formalism of quantum field theory: the functorial formalism. Unlike the previous approaches, here we will not attempt to define quantum field operator of any kind, instead we just view quantum field theory as a way to assign collections of quantum states to spacetime submanifolds. Physically, this means we are using the path integral approach instead of the canonical one. But mathematically, we do not know how to define path integrals for fields, and to bypass this difficulty, we again rely on formal constructions. The strategy goes like this, first we merely list some good properties of path integral that we would like to have, then we abstract those properties into compatible axioms in terms of categorical language. To be more specific, we will characterize spacetime submanifolds and collections of quantum states as categories,and define a quantum field theory as a special functor between those categories such that all of our axioms are satisfied. In summary, we have the following map:

$$\text{'geometry/spacetime'} \longrightarrow \text{'algebra/quanta'} \tag{A.92}$$

Typically, we will have a Riemannian manifold M on the geometric side,a Hilbert space H on the algebraic side, and a symmetry group G acting on both sides, a categorical characterization might be quite complicated. However, for certain special cases, the construction is not that hard because some part of the data (M, H, G) becomes trivial or simple enough to deal with, especially, there are two important examples of this kind:

- TQFT, where only the metric independent part of M is needed, H happens to be finite dimensional, and G is trivial. At least for $d = 1, 2, 3$ the constructions are well understood. We will focus on this formalism of TQFT and these lower dimensional cases in the rest of this section.

- $d = 2$ CFT, where M can be taken as a compact Riemann surface hence the metric, conformal and topological structures are all compatible and highly constrained, H can be reorganized into representations of the Virasoro algebra, and G is essentially the modular group. Here everything is under control and formal construction is possible, the resulting theory is equivalent to the one we obtained using VOA approach.[131]

As we have seen, $Vec(k)$ is a symmetric braided tensor category, it can be used on the algebraic or quantum side of our construction, for the geometric or spacetime side we will introduce the category $Cob(n)$ of n dimensional cobordisms, which is also a symmetric braided tensor category. A n dimensional TQFT is defined as a functor $\mathscr{Z}$ compatible with those symmetric braided tensor category structures

$$\mathscr{Z} : Cob(n) \longrightarrow Vec(k) \tag{A.93}$$

Physically, cobordism is an abstract of spacetime slices with common boundary, mathematically it is defined as follow: given two closed manifolds Σ_0, Σ_1 of dimension $n - 1$, a corbordism between them is a n dimensional manifold M such that

$$\partial M = \Sigma_0 \coprod \Sigma_1 \tag{A.94}$$

In order to mimic the concept of in and out states, we will further assume M is oriented and compact, then decorate it with a frame on $\Sigma = \partial M$. To be more precise, suppose $[v_1, \ldots, v_{n-1}]$ is a positive basis for $T_x\Sigma$ at $x \in \Sigma$, then a vector $w \in T_xM$ is defined to be positive normal if $[v_1, \ldots, v_{n-1}, w]$ is positive. A connected component Σ_i of Σ is called in-boundary/out-boundary if a positive normal points inward/outward. We will always assume that we have maps $\Sigma_0 \to M$ and $\Sigma_1 \to M$ such that Σ_0 is diffeomorphically equivalent to

[131] for more details see [146],[101]

$$\begin{tikzcd}
\Sigma_0 \arrow[r] \arrow[dr] & M \arrow[d, dashed, "\varphi"] & \Sigma_1 \arrow[l] \arrow[dl] \\
& M' &
\end{tikzcd}$$

Figure 42: Equivalence of cobordism [4]

$$\begin{tikzcd}
\Sigma_0 \arrow[r] \arrow[dr] & M_0 \arrow[d, dashed, "\varphi_0"] & \Sigma_1 \arrow[l] \arrow[dl] \arrow[r] \arrow[dr] & M_1 \arrow[d, dashed, "\varphi_1"] & \Sigma_2 \arrow[l] \arrow[dl] \\
& M_0' & & M_1' &
\end{tikzcd}$$

(a) the composition process

$$\begin{tikzcd}
\Sigma_0 \arrow[r] \arrow[dr] & M_1 \circ M_0 \arrow[d, dashed, "\varphi"] & \Sigma_2 \arrow[l] \arrow[dl] \\
& M_1' \circ M_0' &
\end{tikzcd}$$

(b) the product $M_1 \circ M_0$

Figure 43: Composition of cobordisms[4]

the in-boundary and Σ_1 is diffeomorphically equivalent to the out-boundary. In summary, an (oriented) cobordism is given by the triple

$$\Sigma_0 \to M \leftarrow \Sigma_1 \tag{A.95}$$

This is also denoted as

$$M : \Sigma_0 \to \Sigma_1 \tag{A.96}$$

Two cobordisms $\Sigma_0 \to M \leftarrow \Sigma_1$ and $\Sigma_0 \to M' \leftarrow \Sigma_1$ between Σ_0, Σ_1 is equivalent iff there exists an orientation-preserving diffeomorphism $\varphi : M \to M'$ makes the diagram in figure 42 commutes

As an example, given an closed Σ of dimension $n-1$, we can construct the cylinder $M = \Sigma \times [0,1]$ such that $\Sigma_0 = \Sigma \times \{0\}, \Sigma_1 = \Sigma \times \{1\}$ with obvious diffeomorphisms, and usually we will denote this as $\Sigma \to \Sigma \times [0,1] \leftarrow \bar{\Sigma}$ This example also provides a way to define composition in $Cob(n)$, suppose we have $\varphi_0 : M_0 \simeq \Sigma_0 \times [0,1], \quad \varphi_1 : M_1 \simeq \Sigma_1 \times [1,2]$, then naturally we have $M_1 \circ M_0 = M_0 \coprod_\Sigma M_1$ as

$$\varphi_1 \coprod_\Sigma \varphi_2 : M_0 \coprod_\Sigma M_1 \to \Sigma \times [0,2] \tag{A.97}$$

More generally one can prove that $M_1 \circ M_0$ is well defined as long as M_0, M_1 have a common boundary component Σ, and it does not depend on particular representatives φ_0, φ_1 chosen for the equivalence classes of cobordisms as in figure 43. $Cob(n)$ consists of Σ's as objects where $\bar{\Sigma}$ plays the role of $*$ dual, and cobordisms M's between them as morphisms, the composition of morphisms is given by cylinder gluing $M_1 \circ M_0 = M_0 \coprod_\Sigma M_1$, it is monoidal by the operation $X \coprod Y$, the braiding is defined by reverse the in/out boundaries

$$\sigma_{X,Y} : X \coprod Y \to Y \coprod X \tag{A.98}$$

and it is symmetric, physically this means micro reversibility.

A symmetric monoidal functor between two symmetric braided tensor categories $(\mathcal{C}, \Box, 1_\mathcal{C})$ and $(\mathcal{D}, \Box, 1_\mathcal{D})$ is a functor $F : \mathcal{C} \to \mathcal{D}$ equipped with natural transformations $\varphi_{X,Y} : FX \Box FY \to F(X \Box Y)$ and a morphism $\varphi : 1_\mathcal{D} \to F1_\mathcal{C}$ such that

- it is compatible with the associativity morphism α's
- it is compatible with the identity morphism 1's
- it is compatible with the braiding morphism σ's

where the explicit diagrams are shown in figure 44

In this sense the functor in (A.93) is defined as a symmetric monoidal functor between $Cob(n)$ and $Vec(k)$, concretely, we have

- A $n-1$ dimensional closed manifold Σ is mapped into a k vector space $\mathcal{Z}(\Sigma)$, and it can be proved that by consistency $\mathcal{Z}(\Sigma)$ is finite dimensional so $\mathcal{Z}$ actually takes values in $Vec_f(k)$ hence physically we have assigned a finite Hilbert space of the topological ground quantum states at every spacetime slice under some suitable foliation.
- Given a corbordism $M : \Sigma_0 \to \Sigma_1$, we can understand this as an evolution process from in-state to out-state
$$\mathcal{Z}(M) : \mathcal{Z}(\Sigma_0) \longrightarrow \mathcal{Z}(\Sigma_1) \tag{A.99}$$
- Consistency with associativity and braiding means
$$\mathcal{Z}(\emptyset) \cong k, \quad \mathcal{Z}(\Sigma_0 \coprod \Sigma_1) \cong \mathcal{Z}(\Sigma_0) \otimes_k \mathcal{Z}(\Sigma_1) \tag{A.100}$$
In particular, if $\Sigma = \partial M$ we can also view it as the cobordism $M : \emptyset \to \Sigma$, then such $\mathcal{Z}(M) : k \to \mathcal{Z}(\Sigma)$ is just a particular vector in $\mathcal{Z}(\Sigma)$, and physically it is a quantum state.
- Consistency with identity means
$$\mathcal{Z}(\Sigma \times [0,1]) = \mathrm{id}_{\mathcal{Z}(\Sigma)} \tag{A.101}$$
physically, this means there is no evolution and the system is left untouched.

$$
\begin{tikzcd}
(FX\Box FY)\Box FZ \arrow[r, "\alpha"] \arrow[d, "\varphi_{X,Y}\Box 1_{FZ}"'] & FX\Box(FY\Box FZ) \arrow[d, "1_{FX}\Box\varphi_{Y,Z}"] \\
F(X\Box Y)\Box FZ \arrow[d, "\varphi_{X\Box Y,Z}"'] & FX\Box F(Y\Box Z) \arrow[d, "\varphi_{X,Y\Box Z}"] \\
F((X\Box Y)\Box Z) \arrow[r, "F\alpha"'] & F(X\Box(Y\Box Z))
\end{tikzcd}
$$

$$
\begin{tikzcd}
FX\Box 1_{\mathcal{D}} \arrow[r, "1_{FX}\Box\varphi"] \arrow[d, "\rho_{FX}"'] & FX\Box F1_{\mathcal{C}} \arrow[d, "\varphi_{X,1_{\mathcal{C}}}"] \\
FX & F(X\Box 1_{\mathcal{C}}) \arrow[l, "F\rho_X"]
\end{tikzcd}
\qquad
\begin{tikzcd}
1_{\mathcal{D}}\Box FY \arrow[r, "\varphi\Box 1_{FY}"] \arrow[d, "\lambda_{FY}"'] & F1_{\mathcal{C}}\Box FY \arrow[d, "\varphi_{1_{\mathcal{C}},Y}"] \\
FY & F(1_{\mathcal{C}}\Box Y) \arrow[l, "F\lambda_Y"]
\end{tikzcd}
$$

Figure 44: The commutative diagrams for symmetric monoidal functor [4]

- Finally, we have the gluing property

$$\mathcal{Z}\left(M_2 \coprod_{\Sigma} M_1\right) = \mathcal{Z}\left(M_2\right) \circ \mathcal{Z}\left(M_1\right) \tag{A.102}$$

which says that a spacetime gluing process corresponds to a combination of state transitions. This property is an essential feature of path integral formalism, i.e.

$$\left(ZM_2 \circ ZM_1\right)(f)\left(\psi_+\right) = \int_{\psi \in \mathcal{F}(\Sigma_2)} \int_{\psi_- \in \mathcal{F}(\Sigma_1)} f\left(\psi_-\right) K\left(\psi_-, \psi\right) K\left(\psi, \psi_+\right) D\psi_- D\psi \tag{A.103}$$

The point is, although the RHS of the above equation is ill defined, we have found a way to characterize the LHS such that it has all the essential properties we want from the RHS.

In lower dimensional cases $d = 1, 2, 3$, the actual constructions of $\mathcal{Z}$ are done, and it is known that

- $d = 1$, $Cob(1)$ is trivially equivalent to $Vec_f(k)$ equipped with $*$, i.e. the category of dual pairs of $Vec_f(k)$
- $d = 2$ $Cob(2)$ are in one to one correspondence with finite dimensional Frobenius algebras A over k. To be more specific, A is commutative and associative, it also equips with an unit and a non-degenerate bilinear trace $\mathrm{tr} : A \to k$
- $d = 3$ in MTC with $p^+/p^- = 1$, a corresponding $d = 3$ TQFT exists and calculate some well define topological invariants.

Finally, as a guide to the more mathematically inclined reader, we summarize certain aspects of discrete gauge theory in a more formal way using the concepts we have developed in this section.

The mathematical framework underlying the physical discussion of [14] used in section1.1.1 is the notion of a G-crossed braided category [67]. The gauging procedure corresponds to the mathematical notion of equivariantization.

In fact, to construct the (twisted or untwisted) discrete gauge theory we can use a simpler notion. Our starting point is the category of G-graded vector spaces, Vec_ω^G, with associator given by the $\mathbb{C}^\times$-valued 3-cocycle, ω. The theory we obtain upon equivariantization is the modular tensor category (MTC) constructed by the process of taking the Drinfeld center [13, 131]. In particular, our gauge theory is just

$$\text{Twisted } G \text{ discrete gauge theory} \leftrightarrow \mathcal{Z}(\mathrm{Vec}_\omega^G)\ .$$

The various operators discussed in the dictionary at the end of section I of the main text correspond to the simple objects of $\mathcal{Z}(\mathrm{Vec}_\omega^G)$ with categorical dimension larger than one. The simple objects corresponding to the trivial conjugacy class in G (what we have called Wilson lines) have trivial topological spin, θ, and are closed under fusion. This means they form a symmetric subcategory. In fact, as is well-known, these simple objects form a Lagrangian subcategory isomorphic to $\mathrm{Rep}(G)$, the category of finite dimensional representations of G over $\mathbb{C}$. In particular, Wilson line fusion rules are those of the corresponding representation semiring. This observation explains the equivalence, mentioned in the introduction of the main text, of the fusion $\mathcal{W}_\pi \times \mathcal{W}_{\pi'} = \mathcal{W}_{\pi''}$ and the character identitiy $\chi_\pi \cdot \chi_{\pi'} = \chi_{\pi''}$.

A.4 Groups and algebras

In this section we will review some concepts and facts about groups, Lie algebras, their representations and cohomologies, which are very important in pure and applied mathematics but usually omitted in textbooks for physicists. For convenience, here we recall some basic notations as follow:

- rings and fields

  R, a ring with unit 1 such that $1 \neq 0$ and $f(1) = 1$ for every ring homomorphism f.

  K, a field, in physics usually it is assumed that it is algebraically closed and char $K = 0$, i.e. $K = \mathbb{C}$,but here we will keep K to be generic.

- groups

  $G, H \cdots$ for groups, and $g, h \cdots$ for group elements, with $[g], [h] \cdots$ the corresponding conjugacy classes. We also define $x^g = gxg^{-1}$ and extend it to subset X of G as $X^g = \{x^g | x \in X\}$, if $\forall g \in G, X^g = X$ we say X is G-invariant.

- Lie algebras

  $\mathfrak{g}, \mathfrak{h} \cdots$ for Lie algebras of simply connected Lie groups G, H, $\alpha, \beta \cdots$ for simple roots, $\lambda, \mu \cdots$ for weights. The corresponding affine version is labeled by an hat, e.g $\widehat{\mathfrak{g}}$, and the level is denoted by k.

  For simple $\mathfrak{g}(A)$ with a $n \times n$ Cartan matrix A we will use the Cartan-Weyl basis e_i, f_i, h_i, for $i = 1, \ldots, n$ such that $[e_i, f_j] = \delta_{ij} h_i, [h_i, e_j] = a_{ij} e_j, [h_i, f_j] = -a_{ij} f_j$, and $[h_i, h_j] = 0$, for all i, j; and $(\mathrm{ad}\, e_i)^{1-a_{ij}} e_j = (\mathrm{ad}\, f_i)^{1-a_{ij}} f_j = 0$ whenever $i \neq j$.

A.4.1 Modules, representations and conjugacy classes

We first introduce some formal generalizations of familiar concepts in linear algebra.

A left R-module ${}_R M$ is an abelian group M on which R acts linearly on the left, i.e. we have a multiplication law $R \times M \to M$, denoted by $(r, m) \mapsto rm$, such that

$$
\begin{aligned}
&r\left(m+m'\right)=rm+rm' \\
&\left(r+r'\right)m=rm+r'm \\
&\left(rr'\right)m=r\left(r'm\right) \\
&1m=m .
\end{aligned}
\tag{A.104}
$$

The corresponding R-homomorphism $f: M \rightarrow N$ is as a map such that for all $m, m' \in M$ and $r \in R$:

$$
\begin{aligned}
&f\left(m+m'\right)=f(m)+f\left(m'\right) \\
&f(rm)=rf(m)
\end{aligned}
\tag{A.105}
$$

If instead R acts on the right, we can similarly define right R-module M_R, for commutative ring ${}_RM$ and M_R are essentially the same thing so we simply call it as R-module M. Sometimes M have left and right actions of different rings R, S, in this case we say it is a R, S-bimodule ${}_RM_S$.

Then we have the category ${}_R\mathbf{Mod}$ of R-modules in an obvious way, and it is easy to verify that ${}_K\mathbf{Mod}$ is $Vec(K)$, and ${}_\mathbb{Z}\mathbf{Mod}$ is $\mathbf{Ab}$ so R-module generalizes both vector space and abelian group, hence ${}_R\mathbf{Mod}$ also has well defined direct sum , direct product, tensor product and so on.

The main purpose of using ${}_R\mathbf{Mod}$ is to define Hom and tensor functors from ${}_R\mathbf{Mod}$ to $\mathbf{Ab}$ [132]:

- covariant Hom functor $T_A=\operatorname{Hom}_R(A, \square)$

  on objects $T_A(B)=\operatorname{Hom}_R(A, B)$ and on morphisms $f: B \rightarrow B'$ we have $T_A(f)$: $\operatorname{Hom}(A, B) \rightarrow \operatorname{Hom}\left(A, B'\right)$ such that $T_A(f): h \mapsto fh$

- contravariant Hom functor $T^B=\operatorname{Hom}_R(\square, B)$ similar with $T_A=\operatorname{Hom}_R(A, \square)$ but reverse the arrows appropriately.

- covariant tensor functors $F_A=A \otimes_R \square$ and $G_B=\square \otimes_R B$

  $F_A(B)=A \otimes_R B \quad$ and $\quad F_A(g)=1_A \otimes g$ for $g: B \rightarrow B'$

  similarly, one also have $G_B(A)=A \otimes_R B \quad$ and $\quad G_B(f)=f \otimes 1_B$ for $f: A \rightarrow A'$

  in R-module there is a canonical isomorphism for $A \otimes B \simeq B \otimes A$, so G, F are essentially the same.

The notation of R-module allows us to do all linear operations formally, but sometimes we also want to be able to perform multiplicative operations, in that case we need the notation of R-algebra. Suppose we have a commutative ring A with a homomorphism $R \rightarrow A$, then

[132] for R, S-bimodules, tensor functors actually take values in $\mathbf{Mod}_S$, and for commutative R hom functors actually take values in ${}_R\mathbf{Mod}$, so in the simplest but quite common case of R, R-bimodules with commutative R, we indeed have functors: ${}_R\mathbf{Mod} \rightarrow {}_R\mathbf{Mod}$. These functors can be defined in general abelian categories as well.

A automatically inherits a natural R-module structure, in addition we can use the ring multiplication law of A, which by construction is consistent with the R-module structure, we say A is a R-algebra or an algebra over R. In particular for a field K, we have K-algebra, it is a formal generalization of matrix algebra. In summary, with R-modules and R-algebras we can formally do linear algebra over generic fields, rings and abelian groups.In the following we will give some applications of these concepts, first for representations of group and Lie algebra, then for conjugacy class multiplication.

As we have said above, we want to do linear algebra formally, in this way we also generalize the concept of linear representation G/g modules to representation. Given a group G, we can formally assign to each g an formal generator e_g, they multiply with each other just as group elements i.e. $e_g e_h = e_{gh}$. Then the group ring $R[G]$ is defined as the R-module $\oplus_{g\in G} R \cdot e_g$ of formal sums of generators over R, where we also assume that $R \cdot e_g = R$ for all $g \in G$, due to the group structure, we indeed have an R-algebra, explicitly:[133]

$$\left(\sum_{h_1\in G} r_{h_1} e_{h_1}\right) \cdot \left(\sum_{h_2\in G} s_{h_2} e_{h_2}\right) := \sum_{g\in G} \left(\sum_{h\in G} r_h s_{h^{-1}g}\right) e_g \tag{A.106}$$

Because $g \mapsto 1 \cdot e_g$ sometimes one just use g to denote both the group element and its formal generator. $R[G]$ is not commutative if G is not, but we can still form $R[G]$-module M. If $R = K$ is a field and $M = V$ is a vector space, then the above structure map naturally induce a map $\widehat{\varphi} : K[G] \longrightarrow \mathrm{End}_K(V)$, which is indeed a $\dim_K(V)$ dimensional representation of G on V, and reversely all linear representations of G arise in this way.

For a discrete group, just like the definition of an R-module, we can simply let elements of G act linearly on M to define G-modules, and use G equivariant group homomorphisms $f : M \to M'$ with $gf(m) = f(gm)$ as morphisms, we have the category of G-module, which is indeed equivalent to the category of $\mathbb{Z}[G]$-module. When $M = V$ these modules again reduce to usual representations. In particular, the special $\mathbb{C}[G]$-module $\mathbb{C}^n$ where $n = |G|$ is associated with the regular representation of G, usually it is referred as group algebra $\mathbb{C}[G]$.

Similarly, for a Lie algebra $\mathfrak{g}$, we can first define the universal enveloping algebra generated by the Cartan-Weyl basis e_i, f_i, h_i over R, explicitly its basis is

$$\left(\prod_{\alpha} f_\alpha^{m_\alpha}\right)\left(\prod_{\alpha} e_\alpha^{n_\alpha}\right)\left(\prod_{i=1}^{r} h_i^{p_i}\right) \tag{A.107}$$

where all the Lie product $xy - yx = [x, y]$ are implemented as relations.

Usually one just takes $R = \mathbb{C}$ and denotes the universal enveloping algebra as $U(\mathfrak{g})$, which is the Lie algebra analogue of $\mathbb{C}[G]$. These $U(\mathfrak{g})$-modules are formal generalizations of Lie algebra representations, where the Verma module in RCFT is a special case . In summary, $U(\mathfrak{g})$ is the algebra where all $[x, y]$ in $\mathfrak{g}$ is realized as $xy - yx$ in $U(\mathfrak{g})$, and $U(\mathfrak{g})$-modules generalize $\mathfrak{g}$ representations.

[133]For finite groups this notation is obvious, while for infinite groups, $x = \sum_{g\in G} r_g e_g$ is defined to finite many non-zero terms only, i.e. $|\{g \in G \mid r_g \neq 0\}| < \infty$.

In chapter 3 and 4 we have discussed the Arad-Herzog conjecture and $a \times b = c$ fusion rules where multiplication of conjugacy classes is involved frequently. First recall that by definition, for a conjugacy class C and an arbitrary group element g, as a set C satisfies $gCg^{-1} = C$, that is $gC = Cg$, hence given two conjugacy classes C_1, C_2, we can multiply them as sets to obtain a commutative multiplication law $C_1 \cdot C_2 = C_2 \cdot C_1$. This multiplication is indeed well defined as $gC_1 \cdot C_2 g^{-1} = gC_1g^{-1} \cdot gC_2g^{-1} = C_1 \cdot C_2$ is again G-invariant, hence is a union of conjugacy classes, so we can simply write it as $[g] \cdot [h]$ by the corresponding representatives $g \in C_1, h \in C_2$ as in chapter 3 and 4.

In discrete gauge theory we have charges as well as fluxes, where Wilson lines are in a sense classical since their existence do not depend on the quantum twist ω and they always form fusion subcategory, hence the decomposition of tensor representations can be viewed as classical limits of quantum fusion rules. On the contrary ,in a nonabelian discrete gauge theory the existence of fluxes depends on ω and their fusion never close(theorem 4.5), although they can be viewed as conjugacy classes of G, their fusion rules are quite different from the multiplication law we have introduced above. For example, looking at table 1.1.1, we have pure fluxes D, F, but they fuse to $D \times F = D + E$, while as conjugacy classes we have $[(12)] \times [(123)] = [(12)]$, because unlike decomposition of tensor representations, this multiplication law does not count multiplicity.

However using the concept of group algebra we can introduce another kind of multiplication law for conjugacy classes in finite group, which indeed counts the multiplicity, but as an operation in G, it knows nothing about the ω, hence if we view this multiplication as classical limit, we will loss some information. To do this, we first label the conjugacy classes as C_μ , then we define a formal element K_μ as the following sum in $\mathbb{C}[G]$

$$K_\mu = \sum_{g \in C_\mu} e_g \tag{A.108}$$

Then we multiply K_μ, K_ν in $\mathbb{C}[G]$, but rewrite the result as sums of K_λ , this is always possible because the result is G-invariant

$$K_\mu \times K_\nu = \sum_\lambda C^\lambda_{\mu\nu} K_\lambda \tag{A.109}$$

For example now we have $K_{[(12)]} \times K_{[(123)]} = 2K_{[(12)]}$, in this way the coefficient $C^\lambda_{\mu\nu}$ is a sum of fusion coefficients with all charges being identified as trivial, compare with $D \times F = D + E$ where D is a flux and E is a dyon, now we forget about their charges completely and end up with two copies of $K_{[(12)]}$.[134]

A.4.2 Cohomologies, central extensions and projective representations

In this section we will introduce the notation of cohomology for abelian category, then study the two special but important cases of group cohomology and Lie algebra cohomology. Using

[134] For an interesting discussion of this multiplication law and its relation with Verlinde formula, see[71] exercise 10.18

these tools we can analyze central extensions and projective representations of groups and Lie algebras.

Recall that a complex in an abelian category $\mathcal{A}$ is a sequence of morphisms (differentials)

$$(\mathbf{C}_\theta, d_\bullet) = \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \tag{A.110}$$

such that $d_n d_{n+1} = 0$ for all $n \in \mathbb{Z}$, usually we will omit d and use $\mathbf{C}$ to denote the complex.

Since $\mathcal{A}$ is abelian, we have well defined ker, im for every d_n, we say $\mathbf{C}$ is exact if $\mathrm{im} d_{n+1} = \ker d_n$ for all n, in particular the following exact complex is called short exact(with redundant 0's omitted)

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0 \tag{A.111}$$

Then we define

$$\begin{aligned} n\text{-chains } &= C_n, \\ n\text{-cycles } &= Z_n(\mathbf{C}) = \ker d_n \\ n\text{-boundaries } &= B_n(\mathbf{C}) = \mathrm{im}\, d_{n+1} \end{aligned} \tag{A.112}$$

and the n-th homology is

$$H_n(\mathbf{C}) = Z_n(\mathbf{C})/B_n(\mathbf{C}) \tag{A.113}$$

All of these are just formal generalizations of the usual singular homology, they work because $\mathcal{A}$ mimics $\mathbf{Ab}$, the key point is that homology measures obstructions for $\mathbf{C}$ as being exact. This is important, because functors such as $\mathrm{Hom}_R(A, \square)$ and $A \otimes_R \square$ in general are not exact, i.e. when applying these functors $T : \mathcal{A} \to \mathcal{A}'$ to an exact complex $\mathbf{C}$, the image complex $T\mathbf{C}$ is not exact, hence there are nontrivial obstructions.

The above formalism is useful for general discussions on formal properties, but in actual calculations, we also need some concrete ways to realize it. Here we will introduce two important examples, group cohomology and Lie algebra cohomology.

Groups in general are not abelian, but we can study G-modules, so we begin with a group G and an abelian group A, where we have a natural action $\sigma : G \to \mathrm{Aut}\, A = \mathrm{Out}\, A$.

We define the collection of maps $\alpha_n : G \times \ldots \times G \to A$ as n-cochain group $C^n(G, A)$ or simply C^n, that is

$$\alpha_n : (g_1, \ldots, g_n) \mapsto \alpha_n (g_1, \ldots, g_n) \in A \tag{A.114}$$

we also define $C^0(G, A) = A$, and now for the coboundary operator $\delta : C^n \to C^{n+1}$, depend on σ acts on left or right, we have two versions

- Coboundary operator for left action

$$\begin{aligned} (\delta\alpha_n)(g_1, \ldots, g_n, g_{n+1}) :=\ & \sigma(g_1)\, \alpha_n(g_2, \ldots, g_n, g_{n+1}) + \\ & + \sum_{i=1}^{n} (-1)^i \alpha_n (g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{n+1}) \\ & + (-1)^{n+1} \alpha_n (g_1, \ldots, g_n) \end{aligned} \tag{A.115}$$

for example the first few ones are

$$
\begin{aligned}
(\delta\alpha_0)(g) &= \sigma(g)\alpha_0 - \alpha_0 \\
(\delta\alpha_1)(g_1, g_2) &= \sigma(g_1)\alpha_1(g_2) - \alpha_1(g_1g_2) + \alpha_1(g_1) \\
(\delta\alpha_2)(g_1, g_2, g_3) &= \sigma(g_1)\alpha_2(g_2, g_3) + \alpha_2(g_1, g_2g_3) - \alpha_2(g_1g_2, g_3) \\
&\quad - \alpha_2(g_1, g_2) \\
(\delta\alpha_3)(g_1, g_2, g_3, g_4) &= \sigma(g_1)\alpha_3(g_2, g_3, g_4) - \alpha_3(g_1g_2, g_3, g_4) + \alpha_3(g_1, g_2g_3, g_4) \\
&\quad - \alpha_3(g_1, g_2, g_3g_4) + \alpha_3(g_1, g_2, g_3)
\end{aligned}
\tag{A.116}
$$

- Coboundary operator for right action

$$
\begin{aligned}
(\delta\alpha_n)(g_1, \ldots, g_n, g_{n+1}) := &(-1)^{n+1}\alpha_n(g_2, \ldots, g_n, g_{n+1}) \\
&+ \sum_{i=1}^{n}(-1)^{i+n+1}\alpha_n(g_1, \ldots, g_{i-1}, g_ig_{i+1}, g_{i+2}, \ldots, g_{n+1}) \\
&+ \alpha_n(g_1, \ldots, g_n)\sigma(g_{n+1})
\end{aligned}
\tag{A.117}
$$

for example the first few ones are

$$
\begin{aligned}
(\delta\alpha_0)(g) &= \alpha_0\sigma(g) - \alpha_0 \\
(\delta\alpha_1)(g_1, g_2) &= \alpha_1(g_2) - \alpha_1(g_1g_2) + \alpha_1(g_1)\sigma(g_2) \\
(\delta\alpha_2)(g_1, g_2, g_3) &= -\alpha_2(g_2, g_3) + \alpha_2(g_1g_2, g_3) - \alpha(g_1, g_2g_3) \\
&\quad + \alpha_2(g_1, g_2)\sigma(g_3) \\
(\delta\alpha_3)(g_1, g_2, g_3, g_4) &= \alpha_3(g_2, g_3, g_4) - \alpha_3(g_1g_2, g_3, g_4) + \alpha_3(g_1, g_2g_3, g_4) \\
&\quad - \alpha_3(g_1, g_2, g_3g_4) + \alpha_3(g_1, g_2, g_3)\sigma(g_4)
\end{aligned}
\tag{A.118}
$$

then it is obvious that $\delta(\alpha + \alpha') = \delta\alpha + \delta\alpha'$ and by explicit calculation one find $\delta \circ \delta = 0$ so δ is indeed well defined, and we have

$$
\begin{aligned}
Z^n_\sigma &:= \ker \delta_n \equiv \{ \text{ cocycles } \} \\
B^n_\sigma &:= \operatorname{im} \delta_{n-1} \equiv \{ \text{ coboundaries } \}
\end{aligned}
\tag{A.119}
$$

and

$$
H^n_\sigma(G, A) := Z^n_\sigma(G, A)/B^n_\sigma(G, A) \tag{A.120}
$$

Note that for a Lie group G, we have the usual de-Rham cohomology $H^n_{dR}(G, A)$ of G as a manifold, which is different from $H^n_\sigma(G, A)$, however they are related in the sense that $H^n_{dR}(BG, A)$ is similar(sometimes isomorphic, e.g. for finite group) to $H^n_\sigma(G, A)$.

For Lie algebra, we similarly begin with $U(\mathfrak{g})$-module, so we have a Lie algebra $\mathfrak{g}$ and a finite dimensional representation ρ on V, then we define the collection of maps $\omega_n : \mathfrak{g} \times \cdots \times \mathfrak{g} \to V$ as n-cochain group $C^n(\mathfrak{g}, V)$ or simply C^n, in particular we define $C^0(\mathfrak{g}, V) = V$. The coboundary operator $s_n : C^n \to C^{n+1}$ again has left and right versions depending on ρ :

- Coboundary operator for left action

$$
\begin{aligned}
(s\omega_n)(X_1,\ldots,X_{n+1}) :=& \sum_{i=1}^{n+1}(-)^{i+1}\rho(X_i)\left(\omega\left(X_1,\ldots,\widehat{X}_i,\ldots,X_{n+1}\right)\right)\\
&+\sum_{\substack{j,k=1\\ j<k}}^{n+1}(-)^{j+k}\omega\left([X_j,X_k],X_1,\ldots,\widehat{X}_j,\ldots,\widehat{X}_k,\ldots,X_{n+1}\right)
\end{aligned}
\tag{A.121}
$$

where $\widehat{X}$ means omit X, and the first few examples are

$$
\begin{aligned}
(s\omega_0)(X_1) &= \rho(X_1)\,\omega_0;\\
(s\omega_1)(X_1,X_2) &= \rho(X_1)\,\omega_1(X_2)-\rho(X_2)\,\omega_1(X_1)-\omega_1([X_1,X_2])\\
(s\omega_2)(X_1,X_2,X_3) =& \rho(X_1)\,\omega_2(X_2,X_3)-\rho(X_2)\,\omega_2(X_1,X_3)\\
&+\rho(X_3)\,\omega_2(X_1,X_2)-\omega_2([X_1,X_2],X_3)\\
&+\omega_2([X_1,X_3],X_2)-\omega_2([X_2,X_3],X_1);\\
(s\omega_3)(X_1,X_2,X_3,X_4) &= \rho(X_1)\,\omega_3(X_2,X_3,X_4)-\rho(X_2)\,\omega_3(X_1,X_3,X_4)\\
&\quad+\rho(X_3)\,\omega_3(X_1,X_2,X_4)-\rho(X_4)\,\omega_3(X_1,X_2,X_3)\\
-\omega_3([X_1,X_2],X_3,X_4)&+\omega_3([X_1,X_3],X_2,X_4)-\omega_3([X_1,X_4],X_2,X_3)\\
-\omega_3([X_2,X_3],X_1,X_4)&+\omega_3([X_2,X_4],X_1,X_3)-\omega_3([X_3,X_4],X_1,X_2)
\end{aligned}
\tag{A.122}
$$

- Coboundary operator for right action

$$
\begin{aligned}
(s\omega)(X_1,\ldots,X_{n+1}) :=& \sum_{i=1}^{n+1}(-)^{i+1}\rho(X_i)\left(\omega\left(X_1,\ldots,\widehat{X}_i,\ldots,X_{n+1}\right)\right)\\
&+\sum_{\substack{j,k=1\\ j<k}}^{n+1}(-)^{j+k+1}\omega\left([X_j,X_k],X_1,\ldots,\widehat{X}_j,\ldots,\widehat{X}_k,\ldots,X_{n+1}\right)
\end{aligned}
\tag{A.123}
$$

where $\widehat{X}$ means omit X, and the first few examples are

$$
\begin{aligned}
(s\omega_0)(X_1) &= \rho(X_1)\,\omega_0;\\
(s\omega_1)(X_1,X_2) &= \rho(X_1)\,\omega_1(X_2)-\rho(X_2)\,\omega_1(X_1)+\omega_1([X_1,X_2]);\\
(s\omega_2)(X_1,X_2,X_3) =& \rho(X_1)\,\omega_2(X_2,X_3)-\rho(X_2)\,\omega_2(X_1,X_3)\\
&+\rho(X_3)\,\omega_2(X_1,X_2)+\omega_2([X_1,X_2],X_3)\\
&-\omega_2([X_1,X_3],X_2)+\omega_2([X_2,X_3],X_1)\\
(s\omega_3)(X_1,X_2,X_3,X_4) &= \rho(X_1)\,\omega_3(X_2,X_3,X_4)-\rho(X_2)\,\omega_3(X_1,X_3,X_4)\\
&\quad+\rho(X_3)\,\omega_3(X_1,X_2,X_4)-\rho(X_4)\,\omega_3(X_1,X_2,X_3)\\
+\omega_3([X_1,X_2],X_3,X_4)&-\omega_3([X_1,X_3],X_2,X_4)+\omega_3([X_1,X_4],X_2,X_3)\\
+\omega_3([X_2,X_3],X_1,X_4)&-\omega_3([X_2,X_4],X_1,X_3)+\omega_3([X_3,X_4],X_1,X_2)
\end{aligned}
\tag{A.124}
$$

and just like the case for group we can define $Z^n_\rho(\mathfrak{g},V)=\ker s_n$ and $B^n_\rho(\mathfrak{g},V)=\mathrm{im} s_n$, then the n-th cohomology group is

$$H^n_\rho(\mathfrak{g},V):=Z^n_\rho(\mathfrak{g},V)/B^n_\rho(\mathfrak{g},V) \tag{A.125}$$

Now we have defined these cohomology groups and we can apply them to study various kinds of problems.

By direct calculation we find the first two group cohomologies have interpretations as follow:

- $H^0_\sigma(G,A)$ This is the subgroup of A contains all elements which are invariant under the action σ, that is , the collection of fixed points.
- $H^1_\sigma(G,A)$ This group characterizes crossed homomorphisms $\alpha_1 : G \to A$ modulo principal homomorphsims. Where a crossed homomorphism is defined by the condition $\alpha_1(g_1g_2)=\sigma(g_1)\alpha_1(g_2)+\alpha_1(g_1)$ and a principal homomorphism is defined by the condition $\alpha_1(g)=(\delta a)(g)=\sigma(g)a-a$.

The corresponding Lie algebra versions are:

- $H^0_\rho(\mathfrak{g},V)$ This is the subspace $V^\mathfrak{g}$ of invariants in V under the action of ρ.
- $H^1_\rho(\mathfrak{g},V)$ This group classifies 1-cochains satisfy $(s\omega)(X_1,X_2)=\rho(X_1)\omega(X_2)-\rho(X_2)\omega(X_1)-\omega([X_1,X_2])=0$ modulo coboundaries $\omega_{cob}(X)=\rho(X)v$. In particular when ρ is trivial, we have $H^1_0(\mathfrak{g},V)=(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^*$ as the group of linear maps vanishing on $[\mathfrak{g},\mathfrak{g}]$.

But what about second order cohomologies? It turns out that $H^2_\sigma(G,A)$ characterizes central extensions, its elements are in one-to-one correspondence with central extensions of G by A. Recall that a central extension of G by abelian group A is a exact sequence

$$1\longrightarrow A\xrightarrow{\imath}E\xrightarrow{\pi}G\longrightarrow 1 \tag{A.126}$$

such that $\imath(A)$ is in the centre of E. For example, trivially the direct product is a central extension:

$$1\longrightarrow A\xrightarrow{i}A\times G\xrightarrow{pr_2}G\longrightarrow 1 \tag{A.127}$$

and a less trivial example is the semidirect product

$$1\longrightarrow H\xrightarrow{\imath}G\ltimes H\xrightarrow{\pi}G\longrightarrow 1 \tag{A.128}$$

Concretely, We have the familiar example of

$$1\longrightarrow\{+1,-1\}\longrightarrow \mathrm{SL}(2,\mathbb{C})\xrightarrow{\pi}\mathrm{SO}(1,3)\longrightarrow 1 \tag{A.129}$$

as well as

$$1\longrightarrow K^\times\xrightarrow{i}\mathrm{GL}(V)\xrightarrow{\pi}\mathrm{PGL}(V)\longrightarrow 1 \tag{A.130}$$

For Lie algebra, $H^n_\rho(\mathfrak{g}, \mathfrak{a})$ also characterize central extensions. Similar with central extension of group, we define a central extension of $\mathfrak{g}$ by an abelian Lie algebra $\mathfrak{a}$ as a exact sequence

$$0 \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0 \tag{A.131}$$

such that $[\mathfrak{a}, \mathfrak{h}] = 0$ where– for notational simplicity– we have identified $\mathfrak{a}$ with its image in $\mathfrak{h}$.

Again the direct sum corresponds to a trivial extension

$$0 \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{a} \oplus \mathfrak{g} \longrightarrow \mathfrak{g} \longrightarrow 0 \tag{A.132}$$

and a central extension of simply connected Lie group will induce a central extension of its Lie algebra.

For concrete examples, we have affine Lie algebras

$$0 \to \mathbb{C} \to \widehat{\mathfrak{g}} \to \mathfrak{g} \to 0 \tag{A.133}$$

as well as the Virasoro algebra

$$0 \longrightarrow \mathbb{C} \longrightarrow Vir \longrightarrow W \longrightarrow 0 \tag{A.134}$$

Finally, let us mention the relation between central extension and projective representation. Recall that a projective representation of G on V is a map $P : G \to \mathrm{GL}(V)$ such that:

- $P(e) = I$
- $\forall g, h, \exists \alpha(g, h) \in \mathbb{C} : \quad P(g)P(h) = \alpha(g, h)P(gh)$

equivalently it is a group homomorphism $\pi : G \to \mathrm{PGL}(V)$.

By simple calculation we find[135]

$$\alpha(h, k)\alpha(g, hk) = \alpha(gh, k)\alpha(g, h) \tag{A.135}$$

so this α is indeed a 2-cocycle.

What is more, two projective representations are equivalent if they differ by a 2-coboundary, that is, when there exist a function $\beta : G \to \mathbb{C}^\times$and an isomorphism $\varphi : V_1 \to V_2$ such that $\forall g \in G, \quad \varphi^{-1}(P_1(\varphi(g))) = \beta(g)P_2(g)$ such that

$$\alpha_2(g, h) = \alpha_1(g, h)\beta(gh)\beta^{-1}(g)\beta^{-1}(h) \tag{A.136}$$

hence we have a well defined class of

$$[\alpha] \in H^2\left(G, \mathbb{C}^\times\right) \tag{A.137}$$

[135]notice that now we have switched to multiplicative notation, which is more convenient here, and we also assume here σ is trivial and omit it in the cohomology group

In summary,$H^2(G,\mathbb{C}^\times)$, also called Schur multiplier, characterize projective representations, it is a finite group when G is, when $[\alpha]=0$ we can lift it to a linear representation on $GL(V)$.

However, for Lie algebra although we can also define projective representation $P:\mathfrak{g}\to \mathrm{End}(V)$ by

$$[P(x),P(y)] = P([x,y]) + c(x,y)I \tag{A.138}$$

with

$$\begin{aligned} c(x,y) &= -c(y,x) \\ c([xy],z) &= c([yz],x) + c([zx],y) = 0, \end{aligned} \tag{A.139}$$

$H^2(\mathfrak{g},\mathbb{C})$ does not characterize these projective representations fully.However, there are several partial results that can be drawn. First, for finite dimensional semisimple $\mathfrak{g}$ all have vanishing $H^2(\mathfrak{g},\mathbb{C})$. Secondly, in the important case of an infinite dimensional projective unitary representation of a simply connected Lie group G , if $H^2(\mathfrak{g},\mathbb{R})=0$ we can always lift it to a linear unitary representation(Bargmann's theorem), and this is the typical case one sees in quantum mechanics for representations of physical observables.

B Supplementary material

B.1 For quivers

B.1.1 Useful identities

In this appendix, we derive useful identities for index contributions from a vector multiplet and a bifundamental hypermultiplet. The index contribution from an $SU(n)$ vector multiplet is given by

$$\mathcal{I}^{SU(N)}_{\text{vec}}(q;\mathbf{z}) = P.E.\left[-\frac{2q}{1-q}\chi^{SU(N)}_{\text{adj}}(\mathbf{z})\right] . \tag{B.1}$$

Using $q/(1-q) = q/(1-q^2) + q^2/(1-q^2)$, we find the following identity

$$\mathcal{I}^{SU(N)}_{\text{vec}}(q;\mathbf{z}) = \mathcal{I}^{SU(N)}_{\text{vec}}(q^2;\mathbf{z}) \times P.E.\left[-\frac{2q}{1-q^2}\chi^{SU(N)}_{\text{adj}}(\mathbf{z})\right] . \tag{B.2}$$

Similarly, for the Schur index of a bifundamental hypermultiplet of $SU(N)\times SU(M)$

$$\mathcal{I}^{N\times M}_{\text{bifund}}(q;\mathbf{y},\mathbf{z},a) = P.E.\left[\frac{q^{\frac{1}{2}}}{1-q}\left(a\chi^{SU(N)}_{\text{fund}}(\mathbf{y})\chi^{SU(M)}_{\text{afund}}(\mathbf{z}) + a^{-1}\chi^{SU(N)}_{\text{afund}}(\mathbf{y})\chi^{SU(M)}_{\text{fund}}(\mathbf{z})\right)\right] , \tag{B.3}$$

we can show the identity

$$\mathcal{I}^{N\times M}_{\text{bifund}}(q;\mathbf{y},\mathbf{z},a) = \mathcal{I}^{N\times M}_{\text{bifund}}(q^2;\mathbf{y},\mathbf{z},aq^{\frac{1}{2}})\mathcal{I}^{N\times M}_{\text{bifund}}(q^2;\mathbf{y},\mathbf{z},aq^{-\frac{1}{2}}) , \tag{B.4}$$

using $q^{\frac{1}{2}}a^{\pm 1}/(1-q) = q(q^{-\frac{1}{2}}+q^{\frac{1}{2}})a^{\pm 1}/(1-q^2)$.

B.1.2 $\mathrm{III}_{2\times[n-1,n-1,2],[2,\cdots,2,1,1]}$ theory

In this appendix, we argue that theory described by the right quiver in Fig. 21 is equivalent to $\mathcal{T}^{(n)}_{0,n,0}$. To that end, first note that the former theory is equivalent to the type III AD theory associated with three Young diagrams, $Y_1 = Y_2 = [n{-}1, n{-}1, 1, 1]$ and $Y_3 = [2, \cdots, 2, 1, 1]$, in the language of [158]. Indeed, the prescription of [161] suggests that this type III theory has a weak coupling description corresponding to the splitting of $2n$ boxes in $Y_1 = [n{-}1, n{-}1, 1, 1]$ into the two groups, $[1, 1]$ and $[n-1, n-1]$.[136] From the 3d mirror analysis, we see that the sector corresponding to $[1, 1]$ is $D_2(SU(3)) = AD_3$, the one corresponding to $[n-1, n-1]$ is $R^{2,\mathrm{AD}}_{0,n}$, and an $SU(2)$ vector multiplet is coupled to them.[137] Therefore, all we have to show here is that this type III AD theory is equivalent to $\mathcal{T}^{(n)}_{0,n,0}$.

To see the equivalence of the above-mentioned type III theory and $\mathcal{T}^{(n)}_{0,n,0}$, let us consider a weak coupling description of the type III theory corresponding to the splitting of $2n$ boxes in Y_1 into $[n-1, 1]$ and $[n-1, 1]$. From the prescription of [161] and the spectrum of $\mathcal{N} = 2$ chiral operators, we see that the sector corresponding to each $[n-1, 1]$ is the type IV AD theory (in the language of [158]) associated with an irregular puncture labeled by three Young diagrams $Y_1 = Y_2 = [n-1, 1]$ and $Y_3 = [2, \cdots, 2, 1]$, and a full (and therefore regular) puncture.[138] We also see that an $SU(n)$ vector multiplet is coupled to these type IV AD theories as well as an extra fundamental hypermultiplet. Therefore, this weak coupling description corresponds to the quiver diagram in Fig. 45.

Hence, all we need to show is the equivalence of $\mathcal{T}^{(n)}_{0,n,0}$ and the theory described by the quiver in Fig. 45. Note that, for this purpose, it is sufficient to show that the type IV theory involved in the quiver is equivalent to the $\mathcal{T}^{(\ell)}_{0,n} = AD_n$ with $\frac{n-1}{2}$ extra fundamental hypermultiplets of $SU(n)$.[139] In the rest of this appendix, we show that the Seiberg-Witten (SW) curves of these two theories are indeed identical, which strongly suggests the equivalence of these two theories.

Curve of type IV theory Let us first write down the SW curve of the above-mentioned type IV theory. Since the theory is obtained by compactifying the 6d (2,0) A_{n-1} theory on

[136] Here, the idea of [161] is that there exists an S-dual frame for each splitting of boxes in Y_1 into two groups.

[137] Recall that $R^{2,\mathrm{AD}}_{0,n}$ is the type III AD theory associated with $Y_1 = Y_2 = [n-1, n-1, 2]$ and $Y_3 = [2, \cdots, 2, 1, 1]$.

[138] A type IV theory is obtained by compactifying the 6d (2,0) A_{n-1} theory on sphere with an irregular puncture and a regular puncture. These punctures are characterized by the singularity of an $\mathfrak{sl}(n)$-valued meromorphic (1,0)-form, φ, around them. Suppose that a regular puncture is at $z = 0$. Then φ behaves near $z = 0$ as $\varphi \sim (\frac{M}{z} + \cdots)dz$ with $M \in \mathfrak{sl}(n)$, up to conjugation. When the regular puncture is a full puncture, the eigenvalues of M are all different. When an irregular puncture associated with Y_1, Y_2 and Y_3 are at $z = 0$, φ behaves as $\varphi \sim \left(\frac{M_1}{z^3} + \frac{M_2}{z^2} + \frac{M_3}{z} + \cdots\right) dz$ up to conjugation, where $M_1, M_2, M_3 \in \mathfrak{sl}(n)$ and the eigenvalues of M_i are such that the ordered list of the numbers of equal eigenvalues is identical to Y_i.

[139] Recall here that n is odd, and therefore $\frac{n-1}{2}$ is an integer.

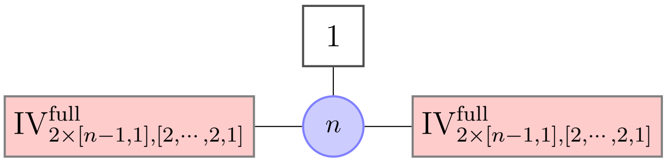


Figure 45: Another weak coupling description of the type III AD theory associated with the Young diagrams $Y_1 = Y_2 = [n-1, n-1, 2]$ and $Y_3 = [2, \cdots, 2, 1, 1]$. The left and right boxes each stand for one copy of the type IV AD theory described in the main text, while the top box stands for a fundamental hypermultiplet. We argue that this quiver theory is identical to $\mathcal{T}^{(n)}_{0,n,0}$.

a sphere with one irregular puncture and a regular puncture, its SW curve is

$$\det(xdz - \varphi) = 0 \ , \tag{B.5}$$

where xdz is the SW 1-form and $\varphi = \varphi_z dz$ is a meromorphic $(1,0)$-form valued in $\mathfrak{sl}(n)$. We take a holomorphic coordinate, z, on the sphere so that the irregular puncture is at $z = \infty$ and the full puncture is at $z = 0$. The Young diagrams characterizing the irregular puncture, $Y_1 = Y_2 = [n-1, 1]$ and $Y_3 = [2, \cdots, 2, 1]$, imply that φ behaves near $z = \infty$ as

$$\varphi \sim dz \left(T_1 z + T_2 + \frac{T_3}{z} + \cdots \ , \right) \tag{B.6}$$

where, up to conjugations, $T_1 = \mathrm{diag}(a, \cdots, a, -(n-1)a)$, $T_2 = \mathrm{diag}(b, \cdots, b, -(n-1)b)$ and $T_3 = \mathrm{diag}(m_1, m_1, m_2, m_2, \cdots, m_{\frac{n-1}{2}}, m_{\frac{n-1}{2}}, -2\sum_{i=1}^{\frac{n-1}{2}} m_i)$. On the other hand, near $z = 0$, φ behaves as

$$\varphi \sim dz \left(\frac{M}{z} + \cdots \right) \ , \tag{B.7}$$

where $M = \mathrm{diag}(M_1, \cdots, M_n)$ such that $\sum_{i=1}^n M_i = 0$. By a change of coordinates that preserves the SW 1-form, the first two matrices can be mapped to $T_1 = \mathrm{diag}(0, \cdots, 0, -1)$ and $T_2 = \mathrm{diag}(0, \cdots, 0, -\widetilde{b})$. Here, m_i and M_i are identified as mass parameters, and $\widetilde{b}$ is identified as a relevant coupling of the type IV theory.

While the masses and couplings of the 4D theory are encoded in the singular terms described above, the vacuum expectation values (vevs) of Coulomb branch operators are encoded in less singular terms. To write down the most general expression for the curve including these vevs, let us consider the first correction, U/z^2, to the terms in the bracket of (B.6), where we parameterize U as $U = \mathrm{diag}(u_1 + v_1, u_1 - v_1, \cdots, u_{n-1} + v_{n-1}, u_{n-1} - v_{n-1}, -2\sum_{i=1}^{\frac{n-1}{2}} u_i)$. The parameters u_i and v_i are not fixed by the boundary conditions, but they are partially restricted so that $\det(x - \varphi_z)$ has only integer powers of x and z. This condition implies that the most general expression for the curve $0 = \det(x - \varphi_z)$ is

$$0 = x^n + x^{n-1}(z + \widetilde{b}) + \sum_{i=2}^{n} x^{n-i}\left((z+\widetilde{b})\frac{t_{i-1}}{z^{i-1}} + \frac{w_{i-1}}{z^{i-1}} + \frac{s_i}{z^i}\right) . \tag{B.8}$$

where s_i, t_i and w_i are combinations of the parameters such that $\prod_{i=1}^{n}\left(x - \frac{M_i}{z}\right) = x^n + \sum_{i=2}^{n} s_i \frac{x^{n-i}}{z^i}$, $\prod_{i=1}^{\frac{n-1}{2}}\left(x - \frac{m_i}{z}\right)^2 = x^{n-1} + \sum_{i=2}^{n} t_{i-1}\frac{x^{n-i}}{z^{i-1}}$ and

$$\frac{1}{z}\sum_{i=1}^{\frac{n-1}{2}} u_i \prod_{j\neq i}\left(x - \frac{m_i}{z}\right)\prod_{k=1}^{\frac{n-1}{2}}\left(x - \frac{m_k}{z}\right) = \sum_{i=2}^{n} w_{i-1}\frac{x^{n-i}}{z^{i-1}} . \tag{B.9}$$

Note that the curve (B.8) can be rewritten as

$$0 = \prod_{i=1}^{n}\left(x - \frac{M_i}{z}\right) + z\prod_{i=1}^{\frac{n-1}{2}}\left(x - \frac{m_i}{z}\right)^2 + \left(\widetilde{b}x^{\frac{n-1}{2}} + \sum_{i=1}^{\frac{n-1}{2}}\widetilde{u}_i\frac{x^{\frac{n-1}{2}-i}}{z^i}\right)\prod_{i=1}^{\frac{n-1}{2}}\left(x - \frac{m_i}{z}\right) , \tag{B.10}$$

where $\widetilde{u}_i$ are defined by

$$\widetilde{b}\prod_{i=1}^{\frac{n-1}{2}}\left(x - \frac{m_i}{z}\right) + \frac{1}{z}\sum_{i=1}^{\frac{n-1}{2}} u_i\prod_{j\neq i}\left(x - \frac{m_i}{z}\right) = \widetilde{b}x^{\frac{n-1}{2}} + \sum_{i=1}^{\frac{n-1}{2}}\widetilde{u}_i\frac{x^{\frac{n-1}{2}-i}}{z^i} . \tag{B.11}$$

Curve of $\mathcal{T}_{0,n}^{(n)}$ with $\frac{n-1}{2}$ fundamental hypers Let us now turn to the SW curve of the AD_n theory with $\frac{n-1}{2}$ extra fundamental hypermultiplets of $SU(n)$. Our strategy is the same as in Appendix B of [29], i.e., we start with the curve of AD_n, weakly gauge its $SU(n)$ flavor symmetry, introduce $\frac{n-1}{2}$ extra fundamental hypermultiplets of $SU(n)$, and then turn off the $SU(n)$ gauge coupling. The SW curve of $AD_n = D_2(SU(n))$ is [46]

$$0 = t^2 + t\sum_{i=0}^{\frac{n-1}{2}} U_i w^i + \prod_{i=1}^{n}(w - M_i) , \tag{B.12}$$

where M_i are the mass parameters associated with the $SU(n)$ flavor symmetry and therefore subject to $\sum_{i=1}^{N} M_i = 0$, U_0 is the relevant coupling of dimension $\frac{1}{2}$, and U_i for $i \geq 1$ are the vevs of Coulomb branch operators. The SW 1-form is given by $\lambda = w\frac{dt}{t}$. When we weakly gauge the $SU(n)$ flavor symmetry, the curve becomes

$$0 = t^2 + t\sum_{i=0}^{\frac{n-1}{2}} U_i w^i + \prod_{i=1}^{n}(w - M_i) + \frac{\Lambda^{\frac{3n}{2}}}{t} , \tag{B.13}$$

where Λ is the corresponding dynamical scale, and M_i is identified with the vevs of the Coulomb branch operators arising from the $SU(n)$ vector multiplet. When we introduce $\frac{n-1}{2}$ extra fundamental hypermultiplets of $SU(n)$, the curve becomes

$$0 = t^2 + t\sum_{i=0}^{\frac{n-1}{2}} U_i w^{\frac{n-1}{2}-i} + \prod_{i=1}^{n}(w - M_i) + \frac{\Lambda^{n+\frac{1}{2}}}{t}\prod_{i=1}^{\frac{n-1}{2}}(w - m_i) . \tag{B.14}$$

In terms of $z \equiv t/\prod_{i=1}^{\frac{n-1}{2}}(w-m_i)$ and $x \equiv w/z$, the curve is

$$0 = z\prod_{i=1}^{\frac{n-1}{2}}\left(x-\frac{m_i}{z}\right)^2 + \left(\sum_{i=0}^{\frac{n-1}{2}} U_i \frac{x^{\frac{n-1}{2}-i}}{z^i}\right)\prod_{i=1}^{\frac{n-1}{2}}\left(x-\frac{m_i}{z}\right) + \prod_{i=1}^{n}\left(x-\frac{M_i}{z}\right) + \frac{\Lambda^{n+\frac{1}{2}}}{z^{n+1}} \ , \quad \text{(B.15)}$$

and the 1-form is $\lambda = xdz$ up to exact terms. We finally turn off the $SU(n)$ gauge coupling by setting $\Lambda = 0$. We then see that the resulting curve is precisely identical to the curve in (B.10), where U_0 is identified as $\widetilde{b}$ and U_i for $i \geq 1$ are identified as $\widetilde{u}_i$. This strongly suggests that the type IV theory discussed in the previous sub-section is identical to the AD_n theory with $\frac{n-1}{2}$ extra decoupled hyper multiplets of $SU(n)$. The last identification then implies the equivalence of $\mathcal{T}^{(n)}_{0,n,0}$ and the theory described by the quiver in Fig. 45.

B.1.3 Monopole dimension bounds

In this appendix, we argue that the dimensions of monopole operators in the 3D mirror SCFTs associated with the $R^{2,AD}_{0,n}$ theories, $\Delta(\mathcal{O}_i)$, satisfy the following bounds

$$\Delta \geq \begin{cases} \frac{1}{2} \ , & n = 3 \\ 1 \ , & n > 3 \ (n \text{ odd}) \ . \end{cases} \quad \text{(B.16)}$$

This result is in agreement with our 4D index analysis in the main text. Indeed, we argued that the $R^{2,AD}_{0,n}$ SCFT only has a decoupled free field sector for $n = 3$. Note that the linear quiver discussion in [81] does not directly apply here since, as discussed around Fig. 26, the mirror quiver contains a closed loop of nodes. Indeed, the fact that the $n = 3$ case has free hypermultiplets even though it is "good" by the naive application of the criteria of [81] motivates us to examine the case for general n more carefully.

While the bound for $n = 3$ follows from the mirror symmetry discussion in [32, 33] (and also the analysis in [29]), we will prove the result in this case and also for all $n > 3$ directly via an analytic monopole analysis in the mirror. To that end, the quantity we wish to bound is

$$\begin{aligned} \Delta \ = \ & -\left(\sum_{A=1}^{\frac{n-1}{2}}\sum_{i_A<j_A}|a^{(A)}_{i_A}-a^{(A)}_{j_A}| + \sum_{B=1}^{\frac{n-1}{2}}\sum_{i_B<j_B}|b^{(B)}_{i_B}-b^{(B)}_{j_B}| + |c_1-c_2|\right) \\ & + \frac{1}{2}\left(\sum_{A=1}^{\frac{n-3}{2}}\sum_{i_A,j_{A+1}}|a^{(A)}_{i_A}-a^{(A+1)}_{j_{A+1}}| + \sum_{B=1}^{\frac{n-3}{2}}\sum_{i_B,j_{B+1}}|b^{(B)}_{i_B}-b^{(B+1)}_{j_{B+1}}|\right) + \frac{1}{2}\left(|c_1|+|c_2|\right) \\ & + \frac{1}{2}\left(\sum_{i,j}|a_i^{\left(\frac{n-1}{2}\right)}-b_j^{\left(\frac{n-1}{2}\right)}| + \sum_{i,j}|c_i-a_j^{\left(\frac{n-1}{2}\right)}| + \sum_{i,j}|c_i-b_j^{\left(\frac{n-1}{2}\right)}|\right) \ , \end{aligned} \quad \text{(B.17)}$$

where $i_A, j_A \in \{1,\cdots,2A\}$, $i_B, j_B \in \{1,\cdots,2B\}$, and $a^{(A)} \in \mathbb{Z}^{2A}, b^{(b)} \in \mathbb{Z}^{2B}, c \in \mathbb{Z}^2$ label the magnetic flux through each gauge node in the quiver (note that we have dropped subscripts

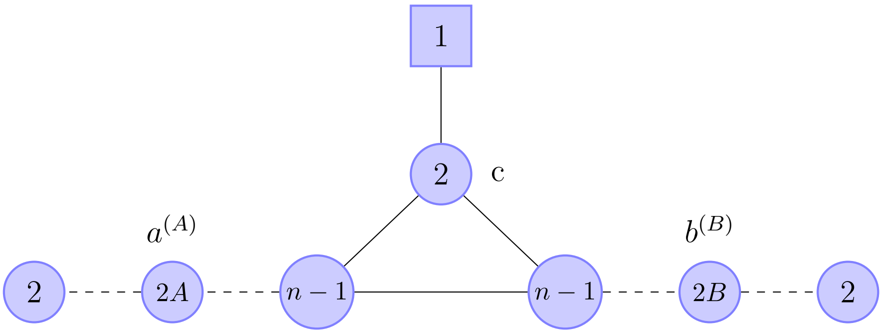


Figure 46: We reproduce the quiver from Fig. 26 but rotated and with labels $a^{(A)} \in \mathbb{Z}^{2A}, b^{(B)} \in \mathbb{Z}^{2B}, c \in \mathbb{Z}^2$ denoting magnetic charges through the corresponding gauge nodes (the nodes to the left of the central $U(2)$ node have fluxes labeled by "a," while those to the right have fluxes labeled by "b").

denoting the particular entry in the flux vector)—see Fig. 46. Note that the negative contributions in (B.17) arise from the gauge nodes while the positive contributions arise from the (bi)fundamentals.

The main strategy in proving (B.16) is repeated use of the triangle inequality to cancel four positive matter contributions to Δ against single gauge contributions (we perform the cancelation between lines and the nodes that they end on). We will start from the leftmost $U(2)$ node in Fig. 46 and then inductively argue that we can cancel all the negative contributions from all the nodes in the left tail up to and including negative contributions from the $U(n-3)$ node that neighbors the left $U(n-1)$ node. By $\mathbb{Z}_2$ symmetry, the corresponding negative contributions from the $U(2)$ to $U(n-3)$ nodes from the right tail will also be cancelled by corresponding matter contributions. We then move on to consider the core of the quiver and prove (B.16).

Before continuing, let us note that we may always use Weyl transformations at each gauge node to arrange that

$$a_1^{(\alpha)} \geq a_2^{(\alpha)} \geq \cdots \geq a_{2\alpha}^{(\alpha)} \ , \quad b_1^{(\beta)} \geq b_2^{(\beta)} \geq \cdots \geq b_{2\beta}^{(\beta)} \ , \quad c_1 \geq c_2 \ , \tag{B.18}$$

for all $\alpha, \beta \in \left\{1, 2, \cdots, \frac{n-1}{2}\right\}$. This maneuver has the effect of removing absolute values from gauge node contributions in (B.17). We may then write the contributions from the $U(2A)$ node as

$$\Delta \supset -\sum_{i=1}^{A}(2(A-i)+1)(a_i^{(A)} - a_{2A+1-i}^{(A)}) \tag{B.19}$$

Note that there are $A^2 = \sum_{i=1}^{A}(2(A-i)+1)$ such contributions in total.

Inductive proof of the canceling of negative contributions from the quiver tails
Let us begin by focusing on the left quiver tail in Fig. 46. We start with the somewhat special

$U(2)$ contributions to Δ and the contributions of the corresponding eight hypermultiplets in the bifundamental of $U(2) \times U(4)$

$$\Delta \supset \Delta_2 = -(a_1^{(1)} - a_2^{(1)}) + \frac{1}{2} \sum_{i,j} |a_i^{(1)} - a_j^{(2)}| \ . \tag{B.20}$$

We can cancel the negative contributions from $U(2)$ against four hypermultiplet contributions by using the triangle inequality twice

$$-(a_1^{(1)} - a_2^{(1)}) + \frac{1}{2} \left(|a_1^{(1)} - a_2^{(2)}| + |a_1^{(1)} - a_3^{(2)}| + |a_2^{(1)} - a_2^{(2)}| + |a_2^{(1)} - a_3^{(2)}| \right) \geq 0 \tag{B.21}$$

This procedure leaves a surplus of four matter contributions we can use to cancel contributions from the adjoining $U(4)$ node. Moreover, since we have not used matter contributions involving $a_{1,4}^{(2)}$, we can use this surplus to cancel one of the most negative terms from $U(4)$ (i.e., one proportional to $a_1^{(2)} - a_4^{(2)}$).

Let us now discuss the $U(4)$ node and adjoining matter contributions more carefully. Since this computation contains contributions from matter fields to the left and right of the gauge node, we can use this discussion to build a base case for an inductive proof of the positivity of contributions to Δ from the left quiver tail. To that end, consider the contributions

$$\begin{aligned} \Delta \ \ \supset \ \ \Delta_4 &= -\sum_{i=1}^{2} (2(2-i)+1)(a_i^{(2)} - a_{5-i}^{(2)}) + \frac{1}{2} \Big(|a_1^{(1)} - a_1^{(2)}| + |a_1^{(1)} - a_4^{(2)}| + \\ &+ \ \ |a_2^{(1)} - a_1^{(2)}| + |a_2^{(1)} - a_4^{(2)}| \Big) + \frac{1}{2} \sum_{k,\ell} |a_k^{(2)} - a_\ell^{(3)}| \ . \end{aligned} \tag{B.22}$$

We may use the surplus contributions in the second term above to cancel one of the contributions from the $U(4)$ gauge node so that

$$\Delta_4 \geq -(2(a_1^{(2)} - a_4^{(2)}) + (a_2^{(2)} - a_3^{(2)})) + \frac{1}{2} \sum_{k,\ell} |a_k^{(2)} - a_\ell^{(3)}| \ . \tag{B.23}$$

Let us now use twelve of the twenty-four $U(4) \times U(6)$ hypermultiplets to cancel the remaining three negative $U(4)$ contributions. To see how this cancelation is done, it is useful to visualize the hypermultiplet contributions via a 4×6 matrix with a "1" indicating an unused matter contribution and a "0" indicating a used matter contribution. We start with

$$\mathbf{L_{4,6}} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \ . \tag{B.24}$$

Our strategy is to leave as surplus the first and last columns while using the remainder of the first and last rows (eight terms in all) to cancel the two $U(4)$ contributions proportional to $a_1^{(2)} - a_4^{(2)}$ (this is done via four applications of the triangle inequality). In other words, we have

$$\mathbf{L_{4,6}} \to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} , \tag{B.25}$$

which leads to the bound

$$\begin{aligned} -2(a_1^{(2)} - a_4^{(2)}) \quad &+ \quad \frac{1}{2}\Big([|a_1^{(2)} - a_2^{(3)}| + |a_1^{(2)} - a_5^{(3)}| + |a_4^{(2)} - a_2^{(3)}| + |a_4^{(2)} - a_5^{(3)}|] \\ &+ \quad [|a_1^{(2)} - a_3^{(3)}| + |a_1^{(2)} - a_4^{(3)}| + |a_4^{(2)} - a_3^{(3)}| + |a_4^{(2)} - a_4^{(3)}|]\Big) \geq 0 \end{aligned} \tag{B.26}$$

We cancel the remaining negative contribution from $U(4)$ by using the middle four entries of $\mathbf{L_{4,6}}$ so that

$$\mathbf{L_{4,6}} \to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{B.27}$$

Indeed, we see that

$$-(a_2^{(2)} - a_3^{(2)}) + \frac{1}{2}[|a_2^{(2)} - a_3^{(3)}| + |a_2^{(2)} - a_4^{(3)}| + |a_3^{(2)} - a_3^{(3)}| + |a_3^{(2)} - a_4^{(3)}|] \geq 0 \ . \tag{B.28}$$

This procedure leaves a surplus of 12 hypermultiplets we can use to cancel negative contributions from $U(6)$.

Now that we have shown how the negative $U(2) \times U(4)$ contributions in the left quiver tail are cancelled, we can move on to the induction hypothesis in our proof. We assume that all the negative contributions in $U(1) \times \cdots \times U(2A)$ have been canceled. In particular, the A^2 negative $U(2A)$ contributions (see the discussion below (B.19)) have been canceled as follows: $\frac{A(A-1)}{2}$ of them from $U(2(A-1)) \times U(2A)$ bifundamentals and $\frac{A(A+1)}{2}$ of them from $U(2A) \times U(2(A+1))$ bifundamentals.

Let us understand these statements in more detail. In particular, we should first focus on the $\mathbf{L_{2(A-1),2A}}$ generalization of (B.27) we get after finishing the cancelation of terms in $U(2(A-1))$. This matrix has its first column filled with 1's. The next column has all 1's except in the first and last row which are 0. For $2 \leq p \leq A$, the p^{th} column consists of zeros in positions i such that $1 \leq i \leq p-1$ and $2A - p \leq i \leq 2(A-1)$ with 1's everywhere else. This discussion specifies half the matrix. The remaining half is set by demanding that $\mathbf{L_{2(A-1),2A}}$ is symmetric under reflections through a line running between columns A and $A+1$, i.e.

$$\mathbf{L_{2(A-1),2A}} \to \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 & 0 & 1 \\ 1 & 1 & 0 & \cdots & \cdots & 0 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 0 & \cdots & \cdots & 0 & 1 & 1 \\ 1 & 0 & 0 & \cdots & \cdots & 0 & 0 & 1 \end{pmatrix} . \tag{B.29}$$

By using the $2A(A-1)$ hypermultiplet contributions corresponding to the 1's in (B.29), we assume we cancel $\frac{A(A-1)}{2}$ of the negative $U(2A)$ contributions via repeated applications of the triangle innequality.

Next we move to $\mathbf{L_{2A,2(A+1)}}$. This is a $2A \times 2(A+1)$ matrix full of 1's. Now, as in the $U(4)$ case, we leave the first column alone. In the p^{th} column, with $2 \leq p \leq A+1$, we set to zero all rows i such that $1 \leq i \leq p-1$ and $2(A+1)-p \leq i \leq 2A$. This procedure again specifies the left half of the matrix. The right half is fixed by requiring the matrix to be symmetric under reflection through a line running between columns $A+1$ and $A+2$, i.e.

$$\mathbf{L_{2A,2(A+1)}} \to \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 & 0 & 1 \\ 1 & 1 & 0 & \cdots & \cdots & 0 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 0 & \cdots & \cdots & 0 & 1 & 1 \\ 1 & 0 & 0 & \cdots & \cdots & 0 & 0 & 1 \end{pmatrix} . \tag{B.30}$$

We have therefore set to zero $2A(A+1)$ entries, and we assume we can use the corresponding hypermultiplet contributions to cancel the remaining $\frac{A(A+1)}{2}$ negative contributions in $U(2A)$ via repeated use of the triangle inequality.

Given these assumptions, we now show that we can cancel the negative contributions in $U(2(A+1))$ and complete our inductive proof. The negative contributions in this case take the form

$$\Delta \supset -\sum_{i=1}^{A+1} (2(A+1-i)+1)(a_i^{(A+1)} - a_{2(A+1)+1-i}^{(A+1)}) \tag{B.31}$$

Let us now focus on the matter contributions from the first and last columns in (B.30). We have

$$\begin{aligned} \frac{1}{2} \quad \Big(\quad & [|a_1^{(A)} - a_1^{(A+1)}| + |a_1^{(A)} - a_{2(A+1)}^{(A+1)}| + |a_{2A}^{(A)} - a_1^{(A+1)}| + |a_{2A}^{(A)} - a_{2(A+1)}^{(A+1)}|] \\ + \quad & [|a_2^{(A)} - a_1^{(A+1)}| + |a_2^{(A)} - a_{2(A+1)}^{(A+1)}| + |a_{2A-1}^{(A)} - a_1^{(A+1)}| + |a_{2A-1}^{(A)} - a_{2(A+1)}^{(A+1)}|] \\ + \quad & \cdots + [|a_A^{(A)} - a_1^{(A+1)}| + |a_A^{(A)} - a_{2(A+1)}^{(A+1)}| + |a_{A+1}^{(A)} - a_1^{(A+1)}| + |a_{A+1}^{(A)} - a_{2(A+1)}^{(A+1)}|]\Big) \\ \geq \quad & A(a_1^{(A+1)} - a_{2(A+1)}^{(A+1)}) \ , \end{aligned} \tag{B.32}$$

where, in the last line, we have repeatedly used the triangle inequality. Working inward, a similar computation shows that the contributions from columns p and $2(A+1)-p+1$ are bounded from below by $(A+1-p)(a_p^{(A+1)} - a_{2(A+1)-p+1}^{(A+1)})$. Therefore, after using the $2A(A+1)$ 1's in (B.30), we have the following remaining negative contributions from $U(2(A+1))$

$$\Delta \supset -\sum_{i=1}^{A+1} (A+2-i)(a_i^{(A+1)} - a_{2(A+1)+1-i}^{(A+1)}) \tag{B.33}$$

To cancel the remaining negative terms, we must use the $U(2(A+1)) \times U(2(A+2))$ bifundamental contributions captured by $\mathbf{L_{2(A+1),2(A+2)}}$. In particular, this latter matrix

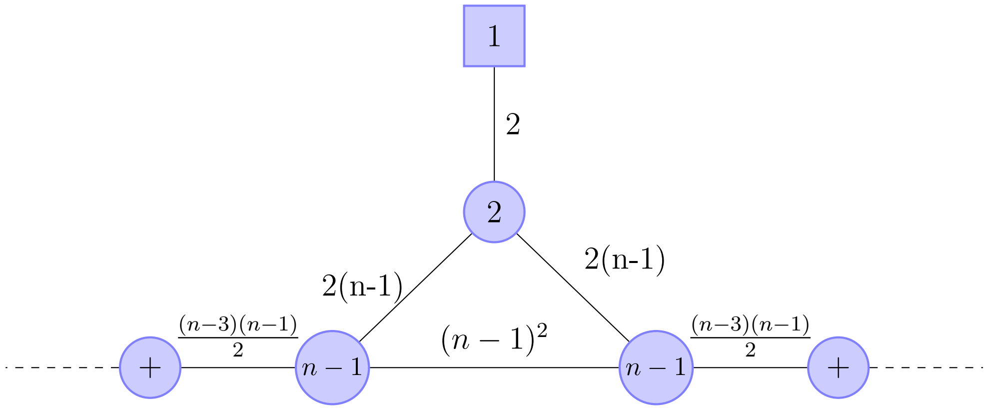


Figure 47: After cancelling the negative contributions from the left and right quiver tails, we put a "+" in each corresponding node. We are left over with $\frac{(n-3)(n-1)}{2}$ surplus contributions to Δ from bifundamentals of $U(n-3)\times U(n-1)$ in each quiver tail, and this has been encoded in the corresponding numbers on the tail links emanating from the $U(n-1)$ nodes. The remaining numbers associated with the core links indicate the total number of unused (bi)fundamental contributions to Δ.

has 1's in all $2(A+1)\times 2(A+2)$ entries. Let us use entries 2 through $2A+3$ of the first and last rows to cancel the $-(A+1)(a_1^{(A+1)}-a_{2(A+1)}^{(A+1)})$ contribution in (B.33). Indeed, we see

$$\begin{aligned}
\frac{1}{2}\quad \Big(\quad & [|a_1^{(A+1)}-a_2^{(A+2)}|+|a_1^{(A+1)}-a_{2A+3}^{(A+2)}|+|a_{2(A+1)}^{(A+1)}-a_2^{(A+2)}|+|a_{2(A+1)}^{(A+1)}-a_{2A+3}^{(A+2)}|] \\
+\quad & [|a_1^{(A+1)}-a_3^{(A+2)}|+|a_1^{(A+1)}-a_{2(A+1)}^{(A+2)}|+|a_{2(A+1)}^{(A+1)}-a_3^{(A+2)}|+|a_{2(A+1)}^{(A+1)}-a_{2(A+1)}^{(A+2)}|] \\
+\quad & \cdots+[|a_1^{(A+1)}-a_{A+2}^{(A+2)}|+|a_1^{(A+1)}-a_{A+3}^{(A+2)}|+|a_{2(A+1)}^{(A+1)}-a_{A+2}^{(A+2)}|+|a_{2(A+1)}^{(A+1)}-a_{A+3}^{(A+2)}|]\Big) \\
\geq\quad & (A+1)(a_1^{(A+1)}-a_{2(A+1)}^{(A+1)})\ ,
\end{aligned} \tag{B.34}$$

where we have repeatedly used the triangle inequality. Proceeding in a similar fashion with rows p and $2(A+1)-p+1$ (but now using entries $p+1$ through $2(A+2)-p$ of each row), we find that each contribution is bounded from below by $(A+2-p)(a_p^{(A+1)}-a_{2(A+1)+1-p}^{(A+1)})$.

Therefore, we have succeeded in cancelling all the negative contributions of $U(2(A+1))$. Note that, after canceling the $U(2(A+1))$ contributions, we have $2(A+1)(A+2)$ contributions from bifundamentals of $U(2(A+1))\times U(2(A+2))$ left over as surplus. By $\mathbb{Z}_2$ symmetry, we have now proven that all the non-core nodes of the quiver have their negative contributions to Δ canceled, and we are left over with $\frac{(n-3)(n-1)}{2}$ bifundamental contributions of $U(n-3)\times U(n-1)$ in both gauge tails of Fig. 46. In particular, we have shown

$$\begin{aligned}
\Delta\ \geq\ & -\left(\sum_{i<j}|a_i^{(\frac{n-1}{2})}-a_j^{(\frac{n-1}{2})}|+\sum_{i<j}|b_i^{(\frac{n-1}{2})}-b_j^{(\frac{n-1}{2})}|+|c_1-c_2|\right) \\
& +\ \frac{1}{2}\left(\sum_{i,j\in\mathcal{S}_a}|a_i^{(\frac{n-3}{2})}-a_j^{(\frac{n-1}{2})}|+\sum_{i,j\in\mathcal{S}_b}|b_i^{(\frac{n-3}{2})}-b_j^{(\frac{n-1}{2})}|\right)+\frac{1}{2}\left(|c_1|+|c_2|\right)
\end{aligned}$$

$$+ \quad \frac{1}{2}\left(\sum_{i,j}|a_i^{\left(\frac{n-1}{2}\right)} - b_j^{\left(\frac{n-1}{2}\right)}| + \sum_{i,j}|c_i - a_j^{\left(\frac{n-1}{2}\right)}| + \sum_{i,j}|c_i - b_j^{\left(\frac{n-1}{2}\right)}|\right) \ , \tag{B.35}$$

where the first line contains the only remaining negative contributions (i.e., those from the core $U(n-1)\times U(n-1)\times U(2)$ nodes of the quiver), the first two sums in the second line are restricted to lie in the sets $\mathcal{S}_{a,b}$ that run over the surplus $U(n-3)\times U(n-1)$ nodes in the left and right tails respectively (the "a" and "b" subscripts distinguish these tails), and the final line contains bifundamentals from the core of the quiver. This discussion is summarized in Fig. 47.

Analyzing the quiver core and proving (B.16) To complete our proof, we now proceed to the quiver core in Fig. 47. In particular, let us begin by canceling some of the negative contributions to Δ from the left $U(n-1)$ node

$$\Delta \supset -\sum_{i=1}^{\frac{n-1}{2}}(n-2i)\left(a_i^{\left(\frac{n-1}{2}\right)} - a_{n-i}^{\left(\frac{n-1}{2}\right)}\right) \ . \tag{B.36}$$

First we use the remaining $\frac{(n-3)(n-1)}{2}$ bifundamental contributions of $U(n-3)\times U(n-1)$ as in the discussion above (B.33) to cancel some of the $U(n-1)$ contributions and obtain

$$\Delta \supset -\sum_{i=1}^{\frac{n-1}{2}}\left(\frac{n+1}{2} - i\right)\left(a_i^{\left(\frac{n-1}{2}\right)} - a_{n-i}^{\left(\frac{n-1}{2}\right)}\right) \ . \tag{B.37}$$

Without loss of generality, we may also choose to use the $2(n-1)$ bifundamentals of $U(2)\times U(n-1)$ to cancel more of these negative contributions.[140] Indeed, repeated use of the triangle inequality shows that

$$\frac{1}{2}\sum_i\left(|c_1 - a_i^{\left(\frac{n-1}{2}\right)}| + |c_2 - a_i^{\left(\frac{n-1}{2}\right)}|\right) \geq \sum_{i=1}^{\frac{n-1}{2}}\left(a_i^{\left(\frac{n-1}{2}\right)} - a_{n-i}^{\left(\frac{n-1}{2}\right)}\right) \ . \tag{B.38}$$

As a result, we have that the remaining negative contributions from $U(n-1)$ are

$$\Delta \supset -\sum_{i=1}^{\frac{n-3}{2}}\left(\frac{n-1}{2} - i\right)\left(a_i^{\left(\frac{n-1}{2}\right)} - a_{n-i}^{\left(\frac{n-1}{2}\right)}\right) \ . \tag{B.39}$$

Let us now use some of the $U(n-1)\times U(n-1)$ bifundamentals to cancel the remaining negative contributions in (B.39). To that end, consider using entries 2 through $n-2$ in the first and last rows of $\mathbf{L_{n-1,n-1}}$. We have

$$\frac{1}{2} \quad \Big(\quad [|a_1^{\left(\frac{n-1}{2}\right)} - b_2^{\left(\frac{n-1}{2}\right)}| + |a_1^{\left(\frac{n-1}{2}\right)} - b_{n-2}^{\left(\frac{n-1}{2}\right)}| + |a_{n-1}^{\left(\frac{n-1}{2}\right)} - b_2^{\left(\frac{n-1}{2}\right)}| + |a_{n-1}^{\left(\frac{n-1}{2}\right)} - b_{n-2}^{\left(\frac{n-1}{2}\right)}|]$$

[140]This choice of cancellation will make some of the later inequalities we derive look less manifestly $\mathbb{Z}_2$ symmetric, but this choice does not affect the final result.

$$
\begin{aligned}
&+ \quad [|a_1^{\left(\frac{n-1}{2}\right)} - b_3^{\left(\frac{n-1}{2}\right)}| + |a_1^{\left(\frac{n-1}{2}\right)} - b_{n-3}^{\left(\frac{n-1}{2}\right)}| + |a_{n-1}^{\left(\frac{n-1}{2}\right)} - b_3^{\left(\frac{n-1}{2}\right)}| + |a_{n-1}^{\left(\frac{n-1}{2}\right)} - b_{n-3}^{\left(\frac{n-1}{2}\right)}|] \\
&+ \quad \cdots + [|a_1^{\left(\frac{n-1}{2}\right)} - b_{\frac{n-1}{2}}^{\left(\frac{n-1}{2}\right)}| + |a_1^{\left(\frac{n-1}{2}\right)} - b_{\frac{n+1}{2}}^{\left(\frac{n-1}{2}\right)}| + |a_{n-1}^{\left(\frac{n-1}{2}\right)} - b_{\frac{n-1}{2}}^{\left(\frac{n-1}{2}\right)}| + |a_{n-1}^{\left(\frac{n-1}{2}\right)} - b_{\frac{n+1}{2}}^{\left(\frac{n-1}{2}\right)}|]\Big) \\
&\geq \quad \left(\frac{n-1}{2} - 1\right)\left(a_1^{\left(\frac{n-1}{2}\right)} - a_{n-1}^{\left(\frac{n-1}{2}\right)}\right) \qquad \text{(B.40)}
\end{aligned}
$$

Similarly, we see that the contributions from rows $p \geq 2$ and $n-p$ are bounded from above by $\left(\frac{n-1}{2} - p\right)\left(a_p^{\left(\frac{n-1}{2}\right)} - a_{n-p}^{\left(\frac{n-1}{2}\right)}\right)$. As a result, we have countered all negative contributions from the left $U(n-1)$ node.

We must still counter the negative contributions from the remaining $U(n-1) \times U(2)$ nodes with contributions from $2(n-1)$ bifundamentals of $U(n-1) \times U(2)$, $\frac{n(n-1)}{2}$ bifundamentals of $U(n-1) \times U(n-1)$, and $\frac{(n-3)(n-1)}{2}$ bifundamentals of $U(n-1) \times U(n-3)$ (from the right quiver tail in Fig. 47). Proceeding in analogy with the discussion for the other $U(n-1)$ node in (B.37), we use the remaining $U(n-1) \times U(n-3)$ bifundamentals to get rid of some of the $U(n-1)$ contributions. We are left with

$$
\Delta \supset -\sum_{i=1}^{\frac{n-1}{2}} \left(\frac{n+1}{2} - i\right)\left(b_i^{\left(\frac{n-1}{2}\right)} - b_{n-i}^{\left(\frac{n-1}{2}\right)}\right) \ . \qquad \text{(B.41)}
$$

Now we may use the remaining contributions from the $U(n-1) \times U(n-1)$ bifundamentals to cancel the negative contribution in (B.41).[141]

We start with the first and last columns of 1's remaining in $\mathbf{L_{n-1,n-1}}$ and find the following bound via repeated uses of the triangle inequality

$$
\begin{aligned}
&\frac{1}{2} \quad \Big(\quad [|a_1^{\left(\frac{n-1}{2}\right)} - b_1^{\left(\frac{n-1}{2}\right)}| + |a_1^{\left(\frac{n-1}{2}\right)} - b_{n-1}^{\left(\frac{n-1}{2}\right)}| + |a_{n-1}^{\left(\frac{n-1}{2}\right)} - b_1^{\left(\frac{n-1}{2}\right)}| + |a_{n-1}^{\left(\frac{n-1}{2}\right)} - b_{n-1}^{\left(\frac{n-1}{2}\right)}|] \\
&+ \quad [|a_2^{\left(\frac{n-1}{2}\right)} - b_1^{\left(\frac{n-1}{2}\right)}| + |a_2^{\left(\frac{n-1}{2}\right)} - b_{n-1}^{\left(\frac{n-1}{2}\right)}| + |a_{n-2}^{\left(\frac{n-1}{2}\right)} - b_1^{\left(\frac{n-1}{2}\right)}| + |a_{n-2}^{\left(\frac{n-1}{2}\right)} - b_{n-1}^{\left(\frac{n-1}{2}\right)}|] \\
&+ \quad \cdots + [|a_{\frac{n-1}{2}}^{\left(\frac{n-1}{2}\right)} - b_1^{\left(\frac{n-1}{2}\right)}| + |a_{\frac{n-1}{2}}^{\left(\frac{n-1}{2}\right)} - b_{n-1}^{\left(\frac{n-1}{2}\right)}| + |a_{\frac{n+1}{2}}^{\left(\frac{n-1}{2}\right)} - b_1^{\left(\frac{n-1}{2}\right)}| + |a_{\frac{n+1}{2}}^{\left(\frac{n-1}{2}\right)} - b_{n-1}^{\left(\frac{n-1}{2}\right)}|]\Big) \\
&\geq \quad \left(\frac{n-1}{2}\right)\left(b_1^{\left(\frac{n-1}{2}\right)} - b_{n-1}^{\left(\frac{n-1}{2}\right)}\right) \qquad \text{(B.42)}
\end{aligned}
$$

Similarly, we find that the remaining contributions from columns $p \geq 2$ and $n-p$ can be bounded from above as $\left(\frac{n+1}{2} - p\right)\left(b_p^{\left(\frac{n-1}{2}\right)} - b_{n-p}^{\left(\frac{n-1}{2}\right)}\right)$. Therefore, we cancel all the remaining negative contributions in (B.41).

We are left with one final source of negative contributions, those from the top $U(2)$ node

$$
\Delta \supset -(c_1 - c_2) \ . \qquad \text{(B.43)}
$$

[141]Note that we have more such bifundamentals left over than we used in the cancelation of the contributions from the left $U(n-1)$ node since we chose to use the left $U(2) \times U(n-1)$ bifundamentals in the cancelation of the contributions from the left $U(n-1)$ node.

However, we still have all $2(n-1)$ bifundamentals of the right $U(2)\times U(n-1)$ left to cancel them. This is more than enough since

$$\frac{1}{2}\left(|b_{\frac{n-1}{2}}^{\left(\frac{n-1}{2}\right)}-c_1|+|b_{\frac{n-1}{2}}^{\left(\frac{n-1}{2}\right)}-c_2|+|b_{\frac{n+1}{2}}^{\left(\frac{n-1}{2}\right)}-c_1|+|b_{\frac{n+1}{2}}^{\left(\frac{n-1}{2}\right)}-c_2|\right)\geq c_1-c_2\ . \tag{B.44}$$

As a result, we have proven that

$$\Delta\geq\frac{1}{2}\left(|c_1|+|c_2|\right)+\frac{1}{2}\sum_{j\neq\frac{n\pm1}{2},i}|c_i-b_j^{\left(\frac{n-1}{2}\right)}|\ . \tag{B.45}$$

While our choice of cancelation below (B.37) has the effect of making this inequality less manifestly $\mathbb{Z}_2$ symmetric (the contributions of the "a" side of the quiver have already been taken into account in the above bound), this choice does not affect our conclusions.

To prove (B.16), we need only consider a few simple cases. For $c_1=c_2=0$, we know that all monopole operators have $\Delta\geq1$ by [81] since the quiver effectively reduces to a linear quiver and all nodes are "good." Moreover, if $|c_i|\geq2$ for either $i=1$ or $i=2$, then clearly $\Delta\geq1$. Similar statements hold if $|c_1|=|c_2|=1$. Therefore, we need only consider the case where, without loss of generality, $|c_1|=1$ and $c_2=0$. We then have

$$\Delta\geq\frac{1}{2}+\frac{1}{2}\sum_{j\neq\frac{n\pm1}{2}}|b_j^{\left(\frac{n-1}{2}\right)}|+\frac{1}{2}\sum_{j\neq\frac{n\pm1}{2}}|c_1-b_j^{\left(\frac{n-1}{2}\right)}| \tag{B.46}$$

For $n=3$, this bound reduces to (B.16) since the second and third terms are trivial. For $n>3$, if we choose any of the $b_j^{\left(\frac{n-1}{2}\right)}\neq0$, then $\Delta\geq1$ due to contributions from the second term in (B.46). However, if we set all $b_j^{\left(\frac{n-1}{2}\right)}=0$, then the third term leads to $\Delta\geq1$. Therefore, we have proven (B.16).

B.2 For discrete gauge theories

B.2.1 Wilson line $a\times b=c$ in gauge theories with order forty-eight discrete gauge group

Let us study groups of order 48 for which the corresponding discrete gauge theories have Wilson line $a\times b=c$ type fusions[142].

$(48,15)$ $((\mathbb{Z}_3\times D_8)\rtimes\mathbb{Z}_2)$;

$$\mathcal{W}_{2_2}\times\mathcal{W}_{2_4}\ =\ \mathcal{W}_4\ ,\ \mathcal{W}_{2_2}\times\mathcal{W}_{2_5}=\mathcal{W}_4\ ,\ \mathcal{W}_{2_3}\times\mathcal{W}_{2_4}=\mathcal{W}_4\ ,\ \mathcal{W}_{2_3}\times\mathcal{W}_{2_5}=\mathcal{W}_4$$

[142] We won't discuss the direct product groups $S_3\times S_3, D_8\times S_3$ and $Q_8\times S_3$ which also have such fusions (the corresponding discrete gauge theories factorize). Since we have already discussed the case of BOG and $GL(2,3)$, we won't be discussing them here

$$\mathcal{W}_{2_4} \times \mathcal{W}_{2_6} \;=\; \mathcal{W}_4 \ , \ \mathcal{W}_{2_4} \times \mathcal{W}_{2_7} = \mathcal{W}_4 \ , \ \mathcal{W}_{2_5} \times \mathcal{W}_{2_6} = \mathcal{W}_4 \ , \ \mathcal{W}_{2_5} \times \mathcal{W}_{2_7} = \mathcal{W}_4 \ . \tag{B.47}$$

We have $\mathrm{Out}((\mathbb{Z}_3 \times D_8) \rtimes \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2$. Let r_1 and r_2 be the generators of this group. They act on the Wilson lines involved in the fusion above as follows

$$r_1 : \mathcal{W}_{2_2} \leftrightarrow \mathcal{W}_{2_2};\ \mathcal{W}_{2_3} \leftrightarrow \mathcal{W}_{2_3};\ \mathcal{W}_{2_4} \leftrightarrow \mathcal{W}_{2_4};\ \mathcal{W}_{2_5} \leftrightarrow \mathcal{W}_{2_5};\ \mathcal{W}_{2_6} \leftrightarrow \mathcal{W}_{2_7}; \tag{B.48}$$
$$r_1 : \mathcal{W}_{2_2} \leftrightarrow \mathcal{W}_{2_2};\ \mathcal{W}_{2_3} \leftrightarrow \mathcal{W}_{2_3};\ \mathcal{W}_{2_4} \leftrightarrow \mathcal{W}_{2_5};\ \mathcal{W}_{2_6} \leftrightarrow \mathcal{W}_{2_6};\ \mathcal{W}_{2_7} \leftrightarrow \mathcal{W}_{2_7}; \tag{B.49}$$

Since this group has complex characters we also have a non-trivial quasi-zero-form symmetry given by complex conjugation. $\mathcal{Z}(\mathrm{Vec}_{(\mathbb{Z}_3 \times D_8)\rtimes\mathbb{Z}_2})$ also has all other $a \times b = c$ type fusions (involving fluxes and dyons) discussed in this paper.

$(48,16)$ $((\mathbb{Z}_3 : Q_8) \rtimes \mathbb{Z}_2)$; This has fusions identical to (B.47). The only difference is that now $\mathcal{W}_{2_4}$ and $\mathcal{W}_{2_5}$ are conjugates. The outer automorphism group and symmetry action is identical to $\mathcal{Z}(\mathrm{Vec}_{(\mathbb{Z}_3 \times D_8)\rtimes\mathbb{Z}_2})$. Since this group has complex characters we also have a non-trivial quasi-zero-form symmetry given by complex conjugation. We additionally have all other $a \times b = c$ type fusions (involving fluxes and dyons) discussed in this paper.

$(48,17)$ $((\mathbb{Z}_3 \times Q_8) \rtimes \mathbb{Z}_2)$; This has identical character table to $(48,16)$, so same fusion rules. The properties are identical to the two cases above.

$(48,18)$ $(\mathbb{Z}_3 \rtimes Q_{16})$; Identical characters to $(48,15)$, so shares (B.47). The discussion is identical to the case above.

$(48,39)$ $((\mathbb{Z}_4 \times S_3) \rtimes \mathbb{Z}_2)$;

$$\begin{aligned}
\mathcal{W}_{2_1} \times \mathcal{W}_{2_5} &= \mathcal{W}_4 \ , \ \mathcal{W}_{2_1} \times \mathcal{W}_{2_6} = \mathcal{W}_4 \ , \ \mathcal{W}_{2_2} \times \mathcal{W}_{2_5} = \mathcal{W}_4 \ , \ \mathcal{W}_{2_2} \times \mathcal{W}_{2_6} = \mathcal{W}_4 \\
\mathcal{W}_{2_3} \times \mathcal{W}_{2_5} &= \mathcal{W}_4 \ , \ \mathcal{W}_{2_3} \times \mathcal{W}_{2_6} = \mathcal{W}_4 \ , \ \mathcal{W}_{2_4} \times \mathcal{W}_{2_5} = \mathcal{W}_4 \ , \ \mathcal{W}_{2_4} \times \mathcal{W}_{2_6} = \mathcal{W}_4 \ .
\end{aligned} \tag{B.50}$$

We have $\mathrm{Out}((\mathbb{Z}_4 \times S_3) \rtimes \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2$. Let r_1 and r_2 be the generators of this group. They act on the Wilson lines involved in the fusion above as follows

$$r_1 : \mathcal{W}_{2_1} \leftrightarrow \mathcal{W}_{2_1};\ \mathcal{W}_{2_2} \leftrightarrow \mathcal{W}_{2_2};\ \mathcal{W}_{2_3} \leftrightarrow \mathcal{W}_{2_3};\ \mathcal{W}_{2_4} \leftrightarrow \mathcal{W}_{2_4};\ \mathcal{W}_{2_5} \leftrightarrow \mathcal{W}_{2_6}; \tag{B.51}$$
$$r_1 : \mathcal{W}_{2_1} \leftrightarrow \mathcal{W}_{2_2};\ \mathcal{W}_{2_3} \leftrightarrow \mathcal{W}_{2_3};\ \mathcal{W}_{2_4} \leftrightarrow \mathcal{W}_{2_4};\ \mathcal{W}_{2_5} \leftrightarrow \mathcal{W}_{2_5};\ \mathcal{W}_{2_6} \leftrightarrow \mathcal{W}_{2_6}; \tag{B.52}$$

Since this group has complex characters we also have a non-trivial quasi-zero-form symmetry given by complex conjugation. $\mathcal{Z}(\mathrm{Vec}_{(\mathbb{Z}_4 \times S_3)\rtimes\mathbb{Z}_2})$ also have all other $a \times b = c$ type fusions (involving fluxes and dyons) discussed in this paper.

$(48,41)$; $((\mathbb{Z}_4 \times S_3) \rtimes \mathbb{Z}_2)$

Fusion of Wilson lines giving unique output is same as (B.50). We have $\mathrm{Out}((\mathbb{Z}_4 \times S_3) \rtimes \mathbb{Z}_2) = D_{12}$.

Since this group has complex characters we also have a non-trivial quasi-zero-form symmetry given by complex conjugation. $\mathcal{Z}(\mathrm{Vec}_{(\mathbb{Z}_4 \times S_3)\rtimes\mathbb{Z}_2})$ also have all other $a \times b = c$ type fusions (involving fluxes and dyons) discussed in this paper.

B.2.2 Genuine zero-form symmetries and quasi-zero-form symmetries in A_9 discrete gauge theory

Recall from section 4.2.1 that A_9 is the simplest example of an A_N (with $N = k^2 \geq 9$) discrete gauge theory with fusion rules involving non-abelian Wilson lines having unique outcome. Here our goal is to disentangle the genuine zero form symmetries

$$\mathrm{Aut}^{\mathrm{br}}(\mathcal{Z}(\mathrm{Vec}_{A_9})) \simeq H^2(A_9, U(1)) \rtimes \mathrm{Out}(A_9) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \ , \tag{B.53}$$

from a charge conjugation quasi zero-form symmetry [58].

Let us first discuss the outer automorphisms. To that end, recall that A_9 has an outer automorphism corresponding to conjugation by odd elements of $S_9 \triangleright A_9$. Acting with the outer automorphism generated by $(89) \in S_9$, we see that the following lines are exchanged

$$\mathcal{L}_{([(123456789)],\pi_p)} \leftrightarrow \mathcal{L}_{([(123456798)],\pi_p)} \ , \quad \mathcal{L}_{([(12345)(678)],\pi_n)} \leftrightarrow \mathcal{L}_{([(12345)(679)],\pi_n)} \ , \tag{B.54}$$

where the relevant conjugacy classes are listed in table 3, and $0 \leq p \leq 8$, $0 \leq n \leq 14$ label representations of the corresponding $\mathbb{Z}_9$ and $\mathbb{Z}_{14}$ centralizers (they are also listed in table 3).

In fact, as described in the main text, the symmetry in (B.54) generates an action on some of the Wilson lines involved in (4.20)

$$\mathcal{W}_{[3^3]_+} \leftrightarrow \mathcal{W}_{[3^3]_-} \ . \tag{B.55}$$

This action can be read off from the character table of A_9 or, equivalently, from the braiding

$$\begin{aligned}
\frac{S_{\mathcal{W}_{[3^3]_+}\mathcal{L}_{([(12345)(678)],\pi_n)}}}{S_{\mathcal{W}_1\mathcal{L}_{([(12345)(678)],\pi_n)}}} &= \chi_{[3^3]_+}([(12345)(678)])^* = -\frac{1}{2}(1 - i\sqrt{15}) \ , \\
\frac{S_{\mathcal{W}_{[3^3]_-}\mathcal{L}_{([(12345)(678)],\pi_n)}}}{S_{\mathcal{W}_1\mathcal{L}_{([(12345)(678)],\pi_n)}}} &= \chi_{[3^3]_-}([(12345)(678)])^* = -\frac{1}{2}(1 + i\sqrt{15}) \ , \\
\frac{S_{\mathcal{W}_{[3^3]_+}\mathcal{L}_{([(12345)(679)],\pi_n)}}}{S_{\mathcal{W}_1\mathcal{L}_{([(12345)(679)],\pi_n)}}} &= \chi_{[3^3]_+}([(12345)(679)])^* = -\frac{1}{2}(1 + i\sqrt{15}) \ , \\
\frac{S_{\mathcal{W}_{[3^3]_-}\mathcal{L}_{([(12345)(679)],\pi_n)}}}{S_{\mathcal{W}_1\mathcal{L}_{([(12345)(679)],\pi_n)}}} &= \chi_{[3^3]_-}([(12345)(679)])^* = -\frac{1}{2}(1 - i\sqrt{15}) \ .
\end{aligned} \tag{B.56}$$

Note that, since the $[(12345)(678)]$ and $[12345)(679)]$ conjugacy classes are complex, we also have a non-trivial $\mathbb{Z}_2$ charge conjugation that acts on the modular data and swaps $\mathcal{W}_{[3^3]_+} \leftrightarrow \mathcal{W}_{[3^3]_-}$ and $\mathcal{L}_{([(123456789)],\pi_p)} \leftrightarrow \mathcal{L}_{([(123456798)],\pi_p)}$. Recall from the discussion in (4.55) that elements of $H^2(A_9, U(1)) \simeq \mathbb{Z}_2$ act trivially on the Wilson lines. Hence, we learn that charge conjugation cannot be a genuine symmetry of the TQFT (this statement is also confirmed by the analysis in [58]).

However, this is not a contradiction with what we have written, because $\mathrm{Out}(A_9)$ also interchanges the real conjugacy classes $[(123456789)]$ and $[(123456798)]$ along with the corresponding lines in (B.54). Since charge conjugation leaves these degrees of freedom untouched, it is a distinct operation.

Conjugacy class	Length	Centralizer
1	1	A_9
[(12)(34)]	378	SmallGroup$(480, 951)$
[(12)(34)(56)(78)]	945	SmallGroup$(192, 1493)$
[(123)]	168	SmallGroup$(1080, 487)$
[(123)(45)(67)]	7560	SmallGroup$(24, 10)$, $(D_8 \times \mathbb{Z}_3)$
[(123)(456)]	3360	SmallGroup$(54, 13)$
[(123)(456)(789)]	2240	SmallGroup$(81, 7)$, $((\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3)$
[(1234)(56)]	7560	SmallGroup$(24, 5)$, $(S_3 \times \mathbb{Z}_4)$
[(1234)(567)(89)]	15120	SmallGroup$(12, 2)$, $(\mathbb{Z}_{12})$
[(1234)(5678)]	11340	SmallGroup$(16, 13)$, (central product D_8, Z_4)
[(12345)]	3024	SmallGroup$(60, 9)$
[(12345)(67)(89)]	9072	SmallGroup$(20, 5)$, $(\mathbb{Z}_{10} \times \mathbb{Z}_2)$
[(12345)(678)]	12096	SmallGroup$(15, 1)$, $(\mathbb{Z}_{15})$
[(12345)(679)]	12096	SmallGroup$(15, 1)$, $(\mathbb{Z}_{15})$
[(123456)(78)]	30240	SmallGroup$(6, 2)$, $(\mathbb{Z}_6)$
[(1234567)]	25920	SmallGroup$(7, 1)$, $(\mathbb{Z}_7)$
[(123456789)]	20160	SmallGroup$(9, 1)$, $(\mathbb{Z}_9)$
[(123456798)]	20160	SmallGroup$(9, 1)$, $(\mathbb{Z}_9)$

Table 3: The eighteen conjugacy classes of A_9, their order, and their centralizers (recall that the centralizers of elements in the same conjugacy class are isomorphic). The centralizer is labeled by its GAP ID (for sufficiently small groups) as "SmallGroup(a, b)" along with a more common name in certain cases.

Note that in the A_9 discrete gauge theory we can also turn on a large variety of twists

$$\omega \in H^3(A_9, U(1)) \simeq \mathbb{Z}_2 \times \mathbb{Z}_3^2 \times \mathbb{Z}_4 \simeq \mathbb{Z}_6 \times \mathbb{Z}_{12} \ . \tag{B.57}$$

Since the charge conjugation quasi-symmetry is a property of the Wilson line fusion rules, it remains regardless of the twist.

B.2.3 GAP code

The following GAP code defines the function checkdyon() which takes in a group as an argument. It checks for $a \times b = c$ type fusions for non-abelian anyons $a, b, c \in \mathcal{Z}(\mathrm{Vec}_G)$ and ouputs all such fusions. Moreover, if such fusions exist, it outputs $\mathrm{Out}(G)$ as well as $H^2(G, U(1))$. Note that it requires the package HAP to function.

In order to define checkdyon() we need to first define the functions comconj() and conjprof().

```
> conjcom:=function(a,b)
> local com,i,j;
> com:=[];
> for i in [1..Size(AsList(a))] do
> for j in [i..Size(AsList(b))] do
> Append(com, [AsList(a)[i]*AsList(b)[j]*Inverse(AsList(b)[j]*AsList(a)[i])]);
> od; od;
> return DuplicateFreeList(com)=[AsList(a)[1]*Inverse(AsList(a)[1])]; end;
```

This function takes two conjugacy classes of a group G as inputs and outputs true if they commute element-wise and false otherwise. Now, let us define the function conjprod()

```
> conjprod:=function(a,b,c)
> local prod,i,j,k;
> prod:=[];
> for i in [1..Size(AsList(a))] do
> for j in [i..Size(AsList(b))] do
> for k in [1..Size(c)] do
> if AsList(a)[i]*AsList(b)[j] in AsList(c[k]) then
> Append(prod, [k]); break; fi; od; od; od;
> if Size(DuplicateFreeList(prod))=1 then
> return DuplicateFreeList(prod)[1]; else return 0; fi; end;
```

This function takes three arguments. The first two arguments a, b are two conjugacy classes of a group G and the third argument c is the set of all conjugacy classes of G. The function outputs an integer $k > 1$ if the product of two input conjugacy is a single conjugacy class (which is at position k in the list of conjugacy classes c). The function outputs 0 otherwise.

Using these two functions, we finally define the checkdyon() function.

```
checkdyon:=function(G)
> local cn,i,j,k,a,l,cen1,cen2,cen3,cenint,irrcenint,irrcen1,irrcen2,irrcen3,
cen1res,cen2res,cen3res,x,y,z,w,a1,a2,A,I,F,R;
> cn:=ConjugacyClasses(G);
> a:=0;
> for i in [1..Size(cn)] do
> for j in [i..Size(cn)] do
> if conjcom(cn[i],cn[j]) then
> k:=conjprod(cn[i],cn[j],cn);
> if k<>0 then
> cen1:=Centralizer(G,AsList(cn[i])[1]);
> cen2:=Centralizer(G,AsList(cn[j])[1]);
> cen3:=Centralizer(G,AsList(cn[k])[1]);
> cenint:=Intersection(cen1,cen2,cen3);
> irrcen1:=Irr(cen1);
> irrcen2:=Irr(cen2);
> irrcen3:=Irr(cen3);
> cen1res:=RestrictedClassFunctions(irrcen1,cenint);
> cen2res:=RestrictedClassFunctions(irrcen2,cenint);
> cen3res:=RestrictedClassFunctions(irrcen3,cenint);
> irrcenint:=Irr(cenint);
> for x in [1..Size(cen1res)] do
> for y in [1..Size(cen2res)] do
> if Size(AsList(cn[i]))*DegreeOfCharacter(cen1res[x])>1 and
 Size(AsList(cn[j]))*DegreeOfCharacter(cen2res[y])>1 then
```

```
> for z in [1..Size(cen3res)] do
> a1:=[ ]; a2:=[ ];
> for w in [1..Size(irrcenint)] do
> Append(a1,[ScalarProduct(irrcenint[w],cen1res[x]*cen2res[y])]);
> Append(a2,[ScalarProduct(irrcenint[w],cen3res[z])]);
> od;
> if a1*a2=1 and
Size(AsList(cn[i]))*DegreeOfCharacter(cen1res[x])*
Size(AsList(cn[j]))*DegreeOfCharacter(cen2res[y])=
Size(AsList(cn[k]))*DegreeOfCharacter(cen3res[z]) then
> a:=1;
> Print(IdSmallGroup(G), " ", StructureDescription(G), "\n");
> Print("Anyon a: ", cn[i], " , ", irrcen1[x], "\n");
> Print("Anyon b: ", cn[j], " , ", irrcen2[y], "\n");
> Print("Anyon c: ", cn[k], " , ", irrcen3[z], "\n","\n");
> fi; od; fi; od;od; fi; fi; od; od;
> if a=1 then
> A:=AutomorphismGroup(G);
> I:=InnerAutomorphismsAutomorphismGroup(A);
> F:=FactorGroup(A,I);
> Print("Out(G): ",StructureDescription(F), "\n");
> R:=ResolutionFiniteGroup(G,3);
> Print("H2(G,U(1)): ",Homology(TensorWithIntegers(R),2),"\n");
> Print("\n","\n"); fi;
> end;
```

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References

[1] Prarit Agarwal. On dimensional reduction of 4d N=1 Lagrangians for Argyres-Douglas theories. 2018.

[2] Prarit Agarwal, Kazunobu Maruyoshi, and Jaewon Song. $\mathcal{N}$ =1 Deformations and RG flows of $\mathcal{N}$ =2 SCFTs, part II: non-principal deformations. *JHEP*, 12:103, 2016. [Addendum: JHEP04,113(2017)].

[3] Prarit Agarwal, Antonio Sciarappa, and Jaewon Song. $\mathcal{N}$ =1 Lagrangians for generalized Argyres-Douglas theories. *JHEP*, 10:211, 2017.

[4] Dylan G.L. Allegretti. Notes on topological quantum field theory.

[5] Martin Ammon and Johanna Erdmenger. *Gauge/gravity duality: Foundations and applications*. Cambridge University Press, Cambridge, 4 2015.

[6] Dionysios Anninos, Thomas Hartman, and Andrew Strominger. Higher Spin Realization of the dS/CFT Correspondence. *Class. Quant. Grav.*, 34(1):015009, 2017.

[7] Z. Arad and M. Herzog. Lecture notes in mathematics. 112, 1985.

[8] PC ARGYRES. Supersymmetric effective actions in four dimensions. 1999.

[9] Philip C. Argyres, M. Ronen Plesser, Nathan Seiberg, and Edward Witten. New N=2 superconformal field theories in four-dimensions. *Nucl. Phys.*, B461:71–84, 1996.

[10] Philip C. Argyres and Nathan Seiberg. S-duality in N=2 supersymmetric gauge theories. *JHEP*, 12:088, 2007.

[11] Philp Argyres, Matteo Lotito, Yongchao Lu, and Mario Martone. Geometric constraints on the space of $\mathcal{N} = 2$ SCFTs. Part III: enhanced Coulomb branches and central charges. *JHEP*, 02:003, 2018.

[12] FA Bais and JK Slingerland. Condensate-induced transitions between topologically ordered phases. *Physical Review B*, 79(4):045316, 2009.

[13] Bojko Bakalov and Alexander A Kirillov. *Lectures on tensor categories and modular functors*, volume 21. American Mathematical Soc., 2001.

[14] Maissam Barkeshli, Parsa Bonderson, Meng Cheng, and Zhenghan Wang. Symmetry Fractionalization, Defects, and Gauging of Topological Phases. *Phys. Rev. B*, 100(11):115147, 2019.

[15] Christopher Beem, Madalena Lemos, Pedro Liendo, Wolfger Peelaers, Leonardo Rastelli, and Balt C. van Rees. Infinite Chiral Symmetry in Four Dimensions. *Commun. Math. Phys.*, 336(3):1359–1433, 2015.

[16] Christopher Beem, Wolfger Peelaers, Leonardo Rastelli, and Balt C. van Rees. Chiral algebras of class S. *JHEP*, 05:020, 2015.

[17] Salman Beigi, Peter W Shor, and Daniel Whalen. The quantum double model with boundary: condensations and symmetries. *Communications in mathematical physics*, 306(3):663–694, 2011.

[18] A. A. Belavin, Alexander M. Polyakov, and A. B. Zamolodchikov. Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory. *Nucl. Phys. B*, 241:333–380, 1984.

[19] Francesco Benini, Yuji Tachikawa, and Dan Xie. Mirrors of 3d Sicilian theories. *JHEP*, 09:063, 2010.

[20] Sergio Benvenuti and Simone Giacomelli. Lagrangians for generalized Argyres-Douglas theories. *JHEP*, 10:106, 2017.

[21] Sergio Benvenuti and Simone Giacomelli. Supersymmetric gauge theories with decoupled operators and chiral ring stability. *Phys. Rev. Lett.*, 119(25):251601, 2017.

[22] A.D. Berenstein and A.V. Zelevinsky. Tensor Product Multiplicities and Convex Polytopes in Partition Space. *J. Geom. Phys.*, 5:453, 1989.

[23] Ralph Blumenhagen, Dieter Lust, and Stefan Theisen. *Basic concepts of string theory*. Theoretical and Mathematical Physics. Springer, Heidelberg, Germany, 2013.

[24] Ralph Blumenhagen and Erik Plauschinn. *Introduction to conformal field theory: with applications to String theory*, volume 779. 2009.

[25] Parsa Bonderson, Kirill Shtengel, and J.K. Slingerland. Interferometry of non-abelian anyons. *Annals of Physics*, 323(11):2709–2755, Nov 2008.

[26] Vadim Borokhov, Anton Kapustin, and Xinkai Wu. Monopole operators and mirror symmetry in three dimensions. *Journal of High Energy Physics*, 2002(12):044–044, Dec 2002.

[27] Peter Brooksbank and Matthew Mizuhara. On groups with a class-preserving outer automorphism.

[28] Paul Bruillard, Julia Plavnik, and Eric Rowell. Modular categories of dimension p^3m with m square-free. *Proceedings of the American Mathematical Society*, 147(1):21–34, 2019.

[29] Matthew Buican, Simone Giacomelli, Takahiro Nishinaka, and Constantinos Papageorgakis. Argyres-Douglas Theories and S-Duality. *JHEP*, 02:185, 2015.

[30] Matthew Buican and Zoltan Laczko. Nonunitary Lagrangians and unitary non-Lagrangian conformal field theories. *Phys. Rev. Lett.*, 120(8):081601, 2018.

[31] Matthew Buican and Zoltan Laczko. Rationalizing CFTs and Anyonic Imprints on Higgs Branches. *JHEP*, 03:025, 2019.

[32] Matthew Buican, Zoltan Laczko, and Takahiro Nishinaka. $\mathcal{N} = 2$ S-duality revisited. *JHEP*, 09:087, 2017.

[33] Matthew Buican, Zoltan Laczko, and Takahiro Nishinaka. Flowing from 16 to 32 Supercharges. *JHEP*, 10:175, 2018.

[34] Matthew Buican, Linfeng Li, and Takahiro Nishinaka. Peculiar Index Relations, 2D TQFT, and Universality of SUSY Enhancement. *JHEP*, 01:187, 2020.

[35] Matthew Buican, Linfeng Li, and Rajath Radhakrishnan. Non-Abelian Anyons and Some Cousins of the Arad-Herzog Conjecture. 12 2020.

[36] Matthew Buican, Linfeng Li, and Rajath Radhakrishnan. $a \times b = c$ in $2+1$D TQFT. *Quantum*, 5:468, 2021.

[37] Matthew Buican and Takahiro Nishinaka. Argyres-Douglas Theories, the Macdonald Index, and an RG Inequality. *JHEP*, 02:159, 2016.

[38] Matthew Buican and Takahiro Nishinaka. Argyres–Douglas theories, S^1 reductions, and topological symmetries. *J. Phys.*, A49(4):045401, 2016.

[39] Matthew Buican and Takahiro Nishinaka. On the superconformal index of Argyres–Douglas theories. *J. Phys.*, A49(1):015401, 2016.

[40] Matthew Buican and Takahiro Nishinaka. On Irregular Singularity Wave Functions and Superconformal Indices. *JHEP*, 09:066, 2017.

[41] Matthew Buican and Rajath Radhakrishnan. Galois conjugation and multiboundary entanglement entropy. *JHEP*, 12:045, 2020.

[42] W. Burnside. *Theory of groups of finite order (2nd Ed.)*. Dover Publications, Inc., New York, 1955.

[43] Daniel C. Cabra and Gerardo L. Rossini. Explicit connection between conformal field theory and 2+1 chern–simons theory. *Modern Physics Letters A*, 12(23):1687–1697, Jul 1997.

[44] C.M. Campbell, C.W. Parker, M.R. Quick, E.F. Robertson, and C.M. Roney-Dougal. *Groups St Andrews 2017 in Birmingham*. London Mathematical Society Lecture Note Series. Cambridge University Press, 2019.

[45] Nils Carqueville and Ingo Runkel. Introductory lectures on topological quantum field theory. *Banach Center Publications*, 114:9–47, 2018.

[46] Sergio Cecotti and Michele Del Zotto. Infinitely many N=2 SCFT with ADE flavor symmetry. *JHEP*, 01:191, 2013.

[47] Sergio Cecotti, Michele Del Zotto, and Simone Giacomelli. More on the $\mathcal{N} = 2$ superconformal systems of type d p (g). *Journal of High Energy Physics*, 2013(4), Apr 2013.

[48] Sergio Cecotti, Andrew Neitzke, and Cumrun Vafa. R-Twisting and 4d/2d Correspondences. 2010.

[49] Oscar Chacaltana and Jacques Distler. Tinkertoys for Gaiotto Duality. *JHEP*, 11:099, 2010.

[50] Miranda C.N. Cheng, John F.R. Duncan, and Jeffrey A. Harvey. Umbral Moonshine. *Commun. Num. Theor. Phys.*, 08:101–242, 2014.

[51] Gil Young Cho, Dongmin Gang, and Hee-Cheol Kim. M-theoretic Genesis of Topological Phases. *JHEP*, 11:115, 2020.

[52] Sidney Coleman and David J. Gross. Price of asymptotic freedom. *Phys. Rev. Lett.*, 31:851–854, Sep 1973.

[53] Clay Cordova and Shu-Heng Shao. Schur Indices, BPS Particles, and Argyres-Douglas Theories. *JHEP*, 01:040, 2016.

[54] Thomas Creutzig. W-algebras for Argyres-Douglas theories. 2017.

[55] Thomas Creutzig. Logarithmic W-algebras and Argyres-Douglas theories at higher rank. *JHEP*, 11:188, 2018.

[56] Shawn X. Cui, Dawei Ding, Xizhi Han, Geoffrey Penington, Daniel Ranard, Brandon C. Rayhaun, and Zhou Shangnan. Kitaev's quantum double model as an error correcting code. 8 2019.

[57] Shawn X Cui, César Galindo, Julia Yael Plavnik, and Zhenghan Wang. On gauging symmetry of modular categories. *Communications in Mathematical Physics*, 348(3):1043–1064, 2016.

[58] Alexei Davydov. Unphysical diagonal modular invariants. *Journal of Algebra*, 446:1–18, 2016.

[59] Pierre Deligne. Catégories tensorielles. *Moscow Mathematical Journal*, 2(2):227–248, 2002.

[60] Diego Delmastro and Jaume Gomis. Symmetries of abelian chern-simons theories and arithmetic, 2021.

[61] P. Di Francesco, P. Mathieu, and D. Senechal. *Conformal Field Theory*. Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 1997.

[62] Robbert Dijkgraaf, Ben Heidenreich, Patrick Jefferson, and Cumrun Vafa. Negative Branes, Supergroups and the Signature of Spacetime. *JHEP*, 02:050, 2018.

[63] Robbert Dijkgraaf and Edward Witten. Topological Gauge Theories and Group Cohomology. *Commun. Math. Phys.*, 129:393, 1990.

[64] F.A. Dolan and H. Osborn. On short and semi-short representations for four-dimensional superconformal symmetry. *Annals of Physics*, 307(1):41–89, Sep 2003.

[65] Anatoly Dymarsky, Kara Farnsworth, Zohar Komargodski, Markus A. Luty, and Valentina Prilepina. Scale invariance, conformality, and generalized free fields, 2014.

[66] Tohru Eguchi, Hirosi Ooguri, and Yuji Tachikawa. Notes on the K3 Surface and the Mathieu group M_{24}. *Exper. Math.*, 20:91–96, 2011.

[67] Pavel Etingof, Dmitri Nikshych, Victor Ostrik, et al. Fusion categories and homotopy theory. *arXiv preprint arXiv:0909.3140*, 2009.

[68] Alex J. Feingold and Stefan Fredenhagen. A New perspective on the Frenkel-Zhu fusion rule theorem. *J. Algebra*, 320:2079–2100, 2008.

[69] Elsa Fisman and Zvi Arad. A proof of szep's conjecture on nonsimplicity of certain finite groups. *Journal of Algebra*, 108(2):340–354, 1987.

[70] E. Fradkin. *Quantum Field Theory: An Integrated Approach*. Princeton University Press, 2021.

[71] P. Francesco, P. Di Francesco, P. Mathieu, and D. Sénéchal. *Conformal Field Theory*. Graduate Texts in Contemporary Physics. Springer, 1997.

[72] Daniel Friedan and Anatoly Konechny. On the boundary entropy of one-dimensional quantum systems at low temperature. *Phys. Rev. Lett.*, 93:030402, 2004.

[73] Jürgen Fuchs. Fusion rules in conformal field theory. *Fortschritte der Physik/Progress of Physics*, 42(1):1–48, 1994.

[74] Abhijit Gadde. Lectures on the superconformal index, 2020.

[75] Abhijit Gadde, Elli Pomoni, Leonardo Rastelli, and Shlomo S. Razamat. S-duality and 2d Topological QFT. *JHEP*, 03:032, 2010.

[76] Abhijit Gadde, Leonardo Rastelli, Shlomo S. Razamat, and Wenbin Yan. The Superconformal Index of the E_6 SCFT. *JHEP*, 08:107, 2010.

[77] Abhijit Gadde, Leonardo Rastelli, Shlomo S. Razamat, and Wenbin Yan. The 4d Superconformal Index from q-deformed 2d Yang-Mills. *Phys. Rev. Lett.*, 106:241602, 2011.

[78] Abhijit Gadde, Leonardo Rastelli, Shlomo S. Razamat, and Wenbin Yan. Gauge Theories and Macdonald Polynomials. *Commun. Math. Phys.*, 319:147–193, 2013.

[79] Davide Gaiotto. N=2 dualities. *JHEP*, 08:034, 2012.

[80] Davide Gaiotto, Anton Kapustin, Nathan Seiberg, and Brian Willett. Generalized Global Symmetries. *JHEP*, 02:172, 2015.

[81] Davide Gaiotto and Edward Witten. S-Duality of Boundary Conditions In N=4 Super Yang-Mills Theory. *Adv. Theor. Math. Phys.*, 13(3):721–896, 2009.

[82] Dilip Gajendragadkar. A characteristic class of characters of finite π-separable groups. *Journal of algebra*, 59(2):237–259, 1979.

[83] T. Gannon. *Moonshine Beyond the Monster: The Bridge Connecting Algebra, Modular Forms and Physics*. Cambridge monographs on mathematical physics. Cambridge University Press, 2006.

[84] Terry Gannon. Much ado about Mathieu. *Adv. Math.*, 301:322–358, 2016.

[85] GAP. GAP group: GAP-groups, algorithms, and programming, Version 4.4 (2004). http://www.gap-system.org.

[86] Doron Gepner and Edward Witten. String Theory on Group Manifolds. *Nucl. Phys.*, B278:493–549, 1986.

[87] Simone Giacomelli. RG flows with supersymmetry enhancement and geometric engineering. 2017.

[88] Simone Giacomelli. Infrared enhancement of supersymmetry in four dimensions. *JHEP*, 10:041, 2018.

[89] J. Glimm and A. Jaffe. *Quantum Physics: A Functional Integral Point of View*. Springer New York, 1987.

[90] P. Goddard, A. Kent, and David I. Olive. Virasoro Algebras and Coset Space Models. *Phys. Lett. B*, 152:88–92, 1985.

[91] P. Goddard, A. Kent, and David I. Olive. Unitary Representations of the Virasoro and Supervirasoro Algebras. *Commun. Math. Phys.*, 103:105–119, 1986.

[92] Benjamin Grinstein, Kenneth Intriligator, and Ira Z. Rothstein. Comments on unparticles. *Physics Letters B*, 662(4):367–374, May 2008.

[93] SERGEI GUKOV, EMIL MARTINEC, GREGORY MOORE, and ANDREW STROMINGER. Chern–simons gauge theory and the ads3/cft2 correspondence. *From Fields to Strings: Circumnavigating Theoretical Physics*, page 1606–1647, Feb 2005.

[94] Robert M. Guralnick and Alexander Moretó. Conjugacy classes, characters and products of elements, 2018.

[95] Thomas Hertog, Gabriele Tartaglino-Mazzucchelli, Thomas Van Riet, and Gerben Venken. Supersymmetric dS/CFT. *JHEP*, 02:024, 2018.

[96] Thomas Hertog, Gabriele Tartaglino-Mazzucchelli, and Gerben Venken. Spinors in Supersymmetric dS/CFT. 2019.

[97] Christopher P. Herzog and Kuo-Wei Huang. Boundary conformal field theory and a boundary central charge. *Journal of High Energy Physics*, 2017(10), Oct 2017.

[98] Po-Shen Hsin, Ho Tat Lam, and Nathan Seiberg. Comments on One-Form Global Symmetries and Their Gauging in 3d and 4d. 2018.

[99] Yuting Hu and Yidun Wan. Electric-magnetic duality in twisted quantum double model of topological orders. *Journal of High Energy Physics*, 2020(11), Nov 2020.

[100] Yuting Hu and Yidun Wan. Electric-magnetic duality in twisted quantum double model of topological orders. *arXiv preprint arXiv:2007.15636*, 2020.

[101] Yuting Hu, Yidun Wan, and Yong-Shi Wu. Twisted quantum double model of topological phases in two dimensions. *Phys. Rev. B*, 87(12):125114, 2013.

[102] Y.-Z. Huang. Vertex operator algebras, the verlinde conjecture, and modular tensor categories. *Proceedings of the National Academy of Sciences*, 102(15):5352–5356, Apr 2005.

[103] K. Intriligator and N. Seiberg. Mirror symmetry in three dimensional gauge theories. *Physics Letters B*, 387(3):513–519, Oct 1996.

[104] Kenneth A. Intriligator. Bonus Symmetry in Conformal Field Theory. *Nucl. Phys. B*, 332:541–565, 1990.

[105] I Martin Isaacs. *Character theory of finite groups*, volume 69. Courier Corporation, 1994.

[106] J.M. Isidro, J.M.F. Labastida, and A.V. Ramallo. Coset constructions in Chern-Simons gauge theory. *Phys. Lett. B*, 282:63–72, 1992.

[107] Victor G Kac and Minoru Wakimoto. A remark on boundary level admissible representations. *Comptes Rendus Mathematique*, 355(2):128–132, 2017.

[108] G. Karpilovsky. *Projective Representations of Finite Groups*. Finite groups. M. Dekker, 1985.

[109] Justin Kinney, Juan Martin Maldacena, Shiraz Minwalla, and Suvrat Raju. An Index for 4 dimensional super conformal theories. *Commun. Math. Phys.*, 275:209–254, 2007.

[110] A.N. Kirillov, P. Mathieu, D. Senechal, and M.A. Walton. Crystallizing the depth rule for WZNW fusion coefficients. In *19th International Colloquium on Group Theoretical Methods in Physics*, 9 1992.

[111] A.N. Kirillov, P. Mathieu, D. Senechal, and M.A. Walton. Can fusion coefficients be calculated from the depth rule? *Nucl. Phys. B*, 391:651–674, 1993.

[112] Alexei Kitaev. Anyons in an exactly solved model and beyond. *Annals of Physics*, 321(1):2–111, 2006.

[113] Igor R. Klebanov, Silviu S. Pufu, and Benjamin R. Safdi. F-theorem without supersymmetry. *Journal of High Energy Physics*, 2011(10), Oct 2011.

[114] Zohar Komargodski and Adam Schwimmer. On renormalization group flows in four dimensions. *Journal of High Energy Physics*, 2011(12), Dec 2011.

[115] J. M. F. Labastida and Carlos Lozano. Lectures in topological quantum field theory. *AIP Conf. Proc.*, 419(1):54–93, 1998.

[116] Jose Labastida and Carlos Lozano. *Lectures on topological quantum field theory*, volume 419, page 54. 01 1998.

[117] Kazunobu Maruyoshi and Jaewon Song. $\mathcal{N} = 1$ deformations and RG flows of $\mathcal{N} = 2$ SCFTs. *JHEP*, 02:075, 2017.

[118] Kazunobu Maruyoshi and Jaewon Song. Enhancement of Supersymmetry via Renormalization Group Flow and the Superconformal Index. *Phys. Rev. Lett.*, 118(15):151602, 2017.

[119] Michaël Mignard and Peter Schauenburg. Modular categories are not determined by their modular data. *arXiv preprint arXiv:1708.02796*, 2017.

[120] Shiraz Minwalla. Restrictions imposed by superconformal invariance on quantum field theories. *Adv. Theor. Math. Phys.*, 2:783–851, 1998.

[121] Gregory W. Moore and N. Read. Nonabelions in the fractional quantum Hall effect. *Nucl. Phys. B*, 360:362–396, 1991.

[122] Gregory W. Moore and Nathan Seiberg. Classical and Quantum Conformal Field Theory. *Commun. Math. Phys.*, 123:177, 1989.

[123] Gregory W. Moore and Nathan Seiberg. LECTURES ON RCFT. In *1989 Banff NATO ASI: Physics, Geometry and Topology Banff, Canada, August 14-25, 1989*, pages 1–129, 1989. [,1(1989)].

[124] Jamshid Moori* and Hung P. Tong-Viet†. Products of conjugacy classes in simple groups. *Quaestiones Mathematicae*, 34(4):433–439, 2011.

[125] Michael Müger. On the structure of modular categories. *Proceedings of the London Mathematical Society*, 87(2):291–308, 2003.

[126] Deepak Naidu. Categorical morita equivalence for group-theoretical categories. *Communications in Algebra*, 35(11):3544–3565, 2007.

[127] Deepak Naidu, Dmitri Nikshych, and Sarah Witherspoon. Fusion subcategories of representation categories of twisted quantum doubles of finite groups. *International Mathematics Research Notices*, 2009(22):4183–4219, 2009.

[128] Gabriel Navarro. New properties of the π-special characters. *Journal of Algebra*, 187(1):203 – 213, 1997.

[129] Peter M. Neumann. Generosity and Characters of Multiply Transitive Permutation Groups. *Proceedings of the London Mathematical Society*, s3-31(4):457–481, 11 1975.

[130] Dmitri Nikshych and Brianna Riepel. Categorical lagrangian grassmannians and brauer–picard groups of pointed fusion categories. *Journal of Algebra*, 411:191–214, 2014.

[131] Viktor Ostrik. Module categories over the Drinfeld double of a finite group. *International Mathematics Research Notices*, 2003(27):1507–1520, 01 2003.

[132] J.K. Pachos. *Introduction to Topological Quantum Computation*. Introduction to Topological Quantum Computation. Cambridge University Press, 2012.

[133] John Preskill. Lecture notes for physics 219: Quantum computation. 01 1999.

[134] P. Ramadevi, T.R. Govindarajan, and R.K. Kaul. Knot invariants from rational conformal field theories. *Nucl. Phys. B*, 422:291–306, 1994.

[135] Shlomo S. Razamat and Gabi Zafrir. N=1 conformal dualities. 2019.

[136] P. Roche, V. Pasquier, and R. Dijkgraaf. QuasiHopf algebras, group cohomology and orbifold models. *Nucl. Phys. B Proc. Suppl.*, 18:60–72, 1990.

[137] Christian Romelsberger. Counting chiral primaries in N = 1, d=4 superconformal field theories. *Nucl. Phys.*, B747:329–353, 2006.

[138] J.J. Rotman. *An Introduction to Homological Algebra.* Universitext. Springer New York, 2008.

[139] Eric Rowell, Richard Stong, and Zhenghan Wang. On classification of modular tensor categories. *Communications in Mathematical Physics*, 292(2):343–389, 2009.

[140] Tom Rudelius and Shu-Heng Shao. Topological Operators and Completeness of Spectrum in Discrete Gauge Theories. 6 2020.

[141] Slava Rychkov. *EPFL Lectures on Conformal Field Theory in D¿= 3 Dimensions.* SpringerBriefs in Physics. 2016.

[142] J. Saxl. The complex characters of the symmetric groups that remain irreducible in subgroups. *Journal of Algebra*, 111(4B):718, 1987.

[143] A.N. Schellekens and S. Yankielowicz. Field Identification Fixed Points in the Coset Construction. *Nucl. Phys. B*, 334:67–102, 1990.

[144] A.N. Schellekens and S. Yankielowicz. Simple Currents, Modular Invariants and Fixed Points. *Int. J. Mod. Phys. A*, 5:2903–2952, 1990.

[145] M. Schottenloher. *A Mathematical Introduction to Conformal Field Theory: Based on a Series of Lectures given at the Mathematisches Institut der Universität Hamburg.* Lecture Notes in Physics Monographs. Springer Berlin Heidelberg, 2008.

[146] G. B. Segal. *The Definition of Conformal Field Theory*, pages 165–171. Springer Netherlands, Dordrecht, 1988.

[147] M. Shifman. *Advanced Topics in Quantum Field Theory: A Lecture Course.* Cambridge University Press, 2023.

[148] David Simmons-Duffin. Tasi lectures on the conformal bootstrap, 2016.

[149] Jaewon Song. Superconformal indices of generalized Argyres-Douglas theories from 2d TQFT. *JHEP*, 02:045, 2016.

[150] Jaewon Song, Dan Xie, and Wenbin Yan. Vertex operator algebras of Argyres-Douglas theories from M5-branes. *JHEP*, 12:123, 2017.

[151] Michio Suzuki. Finite groups with nilpotent centralizers. *Transactions of the American Mathematical Society*, 99(3):425–470, 1961.

[152] Yuji Tachikawa. N=2 supersymmetric dynamics for pedestrians. *Lecture Notes in Physics*, 2015.

[153] J. Teschner. *New Dualities of Supersymmetric Gauge Theories*. Mathematical Physics Studies. Springer International Publishing, 2015.

[154] Andrew Urichuk and Mark A. Walton. Adjoint affine fusion and tadpoles. *J. Math. Phys.*, 57(6):061702, 2016.

[155] Cumrun Vafa. Non-Unitary Holography. 2014.

[156] Zhenghan Wang. *Topological quantum computation*. Number 112. American Mathematical Soc., 2010.

[157] Edward Witten. Quantum Field Theory and the Jones Polynomial. *Commun. Math. Phys.*, 121:351–399, 1989. [,233(1988)].

[158] Dan Xie. General Argyres-Douglas Theory. *JHEP*, 01:100, 2013.

[159] Dan Xie and Wenbin Yan. Schur sector of Argyres-Douglas theory and W-algebra. 2019.

[160] Dan Xie, Wenbin Yan, and Shing-Tung Yau. Chiral algebra of Argyres-Douglas theory from M5 brane. 2016.

[161] Dan Xie and Shing-Tung Yau. New N = 2 dualities. 2016.

[162] Dan Xie and Shing-Tung Yau. Argyres-Douglas matter and N=2 dualities. 2017.

[163] Dan Xie and Ke Ye. Argyres-Douglas matter and S-duality: Part II. *JHEP*, 03:186, 2018.

[164] A. B. Zamolodchikov. Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory. *JETP Lett.*, 43:730–732, 1986. [Pisma Zh. Eksp. Teor. Fiz.43,565(1986)].

[165] Ilan Zisser. Irreducible products of characters in a_n. *Israel Journal of Mathematics*, 84(1-2):147–151, 1993.