Valuated matroid polytopes and linking system composition

by

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A thesis submitted to the University of London for the degree of
Doctor of Philosophy

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August 2021

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Abstract

Valuated matroids are a generalisation of matroids; matroids themselves being an abstraction of the notion of independence. Valuated matroids have many equivalent definitions including via independent sets and circuits, and in this thesis we show that a valuated matroid has an equivalent definition in terms of a rank function which we construct by analogy with the matroid rank function by looking at matroid and valuated matroid polytopes. We separately construct a hyperoperation which is an extension of a previously studied operation of composing valuated matroids, this being the composition of valuated linking systems. The composition of valuated linking systems can be seen as a generalisation of matrix multiplication to tropical linear spaces. In particular, the hyperoperation we introduce has been influenced by viewing matrices as representing linear spaces, which we can do by looking at their row space, and consequently by how these relate to Plücker coordinates. Working tropically, since tropical linear spaces are equivalent to valuated matroids, which are also known as tropical Plücker vectors, we create the hyperoperation by using the parallels with matrices representing linear spaces over a field. We describe the hyperproduct completely for small rank, where this operation forms a hypergroup. In higher rank we investigate what known matroid subdivisions it contains, as well as also showing that it does not form a fan, and nor is it convex in general. We also conjecture this hyperoperation forms a hypergroup for higher rank, and present some investigation towards this.

Acknowledgments

This is the one place where I get to speak in my own tone. This has been a difficult process as can be imagined, made all the more difficult by the appearance of SARS-CoV-2. I'm grateful for having gone through this period of my life, not only have I grown mathematically and learnt a great deal, but I've gained new meta-skills as well as a new perspective on the world. These new skills will help me as I move on from the mathematical world, but no doubt will heavily influence the landscape and my outlook on it wherever I end up.

There are many people I could acknowledge since the person writing this now is a function of all the people he has interacted with, but I'll keep these short and sweet. I'd like to begin with thanking my supervisor Alex Fink. Without your guidance and encouragement, as well as your endless enthusiasm, this thesis would not have been possible. I don't think a simple thanks can do justice to just how far above and beyond I believe you have gone throughout these 4 years. Thanks to all those I've had conversations of a mathematical nature with. Thanks to all those I have socialised with over the past few years. Thanks to both my mum, and to Elizabeth Painter for proof reeding this thesis, you've definitely made numerous positive contributions. Thanks to both my parents for being there when needed, and providing me with board and lodge at well below market rates throughout.

Table of Contents

Chapter 1

Introduction

Matroids are a way of characterising the general notion of independence. There are numerous axiom systems that are used to define a matroid, including being defined using independent sets, flats, bases, and rank functions. However, the equivalence between many pairs of these axiom systems isn't immediately obvious. We refer to these equivalent definitions as *cryptomorphic*, this informally meaning that two axiom systems are equivalent but their equivalence isn't immediately apparent. These cryptomorphic definitions allow us to have multiple perspectives on matroids, and this lets us work with them in various ways. One concrete way of obtaining a matroid is through the notion of linear independence over a real vector space; this is a way of defining a matroid through the use of independent sets.

Matroids were first formally introduced by Whitney in 1935. However, in the decades prior the notion of abstract independence had been studied in varying areas, such as the study of semimodular lattices by Dedekind and Birkhoff, and the exchange properties of bases which were studied by those including Grassmann and Steinitz. Whitney himself first introduced matroids using independent sets and showed their equivalence with both bases and circuits.

Numerous generalisations have arisen since matroids were first introduced including oriented matroids, Coxeter matroids and valuated matroids. This thesis utilises valuated matroids, these being a generalisation of matroids which extends the notion of independence to working over a field with non-Archimedean valuation. Valuated matroids are matroids which also have an associated valuation to each basis of the matroid where the valuations satisfy exchange conditions. Similarly to matroids there are cryptomorphic definitions, including valuated matroids on independent sets, circuits, as well as a lattice theoretic description. Valuated matroids were first introduced by Dress and Wenzel in 1992 [1], and have use in multiple different contexts outside of pure mathematics, these include being used within economics, and in particular looking at gross substitutes [2],[3]. Dress and Wenzel introduced valuated matroids in an analogous way to how oriented matroids were generalised from matroids. However, instead of working over an ordered field as for oriented matroids, they worked over fields with non-Archimedean valuation.

The original work in this thesis belongs to two themes. The first, Chapter 3, is devoted to giving an alternative characterisation of valuated matroids, this time giving a way to view them by using a rank function. The rank function we define is built up through analogy with the non-valuated matroid rank function and how they relate to associated polyhedral structures.

Associated with both matroids and valuated matroids are the matroid basis polytope and the valuated matroid basis polytope, respectively. These are both cryptomorphic ways of defining matroids and valuated matroids, and enable a completely polyhedral view of matroids and valuated matroids. In particular, this gives us a relation between the matroid polytope and the matroid rank function.

We are able to recover the rank function of a matroid from its matroid basis polytope. For a matroid, M, its basis polytope, $P(M)$, is the convex hull of the indicator vectors of its bases. The rank function r is recovered in the following way:

$$r(A) = \max_{B \in \mathcal{B}} \{|A \cap B|\} = \max_{v \in P(M)} \langle v, e_A \rangle$$

where v is a vertex of $P(M)$, and $\mathcal{B}$ is the set of bases of M. This is possible since the maximum of a linear function of a polytope is attained at at least one of the vertices of the polytope, therefore we need only consider the vertices of this polytope in order to be maximised, and both equalities are by definition.

By analogy with how the matroid polytope and the matroid rank function are related we produce a candidate rank function for a valuated matroid, namely:

$$r(A)(c) = \max_{v \in P(M)} \langle (e_A, c), (v, h) \rangle = \max_{B \in \mathcal{B}} \{|A \cap B| + c\rho(B)\}.$$

Our objective throughout Chapter 3 is to motivate and prove the following result. The six conditions are supposed to be reminiscent of the three conditions of a matroid rank function which can be seen in Theorem 2.2.2. The inclusion of the rescaling is to account for some ρ which aren't valuated matroids but which do in fact satisfy the six conditions, and this is a way of excluding them. These rescalings act on the valuated matroid polytope by a linear transformation which only changes its last coordinate, that is, the associated valuation; the formal definition of this can be seen in Definition 44. We let $\rho^{\mathbf{x}}$ denote a rescaling of a valuated matroid ρ by $\mathbf{x}$, and $PLF(\mathbb{R}_{\leq 0}, \mathbb{R})$ denotes the set of piecewise linear functions mapping from $\mathbb{R}_{\leq 0}$ to $\mathbb{R}$.

Theorem 1.0.1. *Let $r^\rho : 2^E \to PLF(\mathbb{R}_{\leq 0}, \mathbb{R})$ be defined by $r^\rho(A)(c) = \max_{B \in \mathcal{B}} \{|A \cap B| + c\rho(B)\}$. Then ρ is a valuated matroid of rank s on the ground set E if and only if $\rho \in (\mathbb{R} \cup \{\infty\})^{\binom{E}{s}}$ is such that for all $\boldsymbol{x} = (x_a : a \in E)$, $r^{\rho^{\boldsymbol{x}}}$ satisfies the following six conditions:*

1. *for all $A \subseteq E$, $r^{\rho^{\boldsymbol{x}}}(A)$ is convex, and continuous.*

2. *for all $A \subseteq E$, each linear piece of $r^{\rho^{\boldsymbol{x}}}(A)$ takes an integer value at 0.*

3. *$r^{\rho^{\boldsymbol{x}}}(\emptyset)$ is linear and $r^{\rho^{\boldsymbol{x}}}(\emptyset)(0) = 0$.*

4. *$r^{\rho^{\boldsymbol{x}}}(E)$ is linear.*

5. $r^{\rho^x}(A)$ *is increasing in* A, *and* $r^{\rho^x}(A \cup b) \leq r^{\rho^x}(A) + 1$.

6. $r^{\rho^x}(A)$ *is submodular in* A.

Chapter 4 takes on an alternative focus. Here we introduce an operation which is reminiscent of the general linear group, but instead of working over a field we work over the tropical semiring. Working over a field the general linear group of degree n is given by the set of $n \times n$ matrices along with the operation of matrix multiplication. Over the tropical semiring where we use tropical multiplication over square matrices we can see that it is not a group, but a monoid, and hence not akin to the general linear group.

The tropical semiring is given by $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ with operations defined by $x \oplus y = \min(x, y)$ and $x \odot y = x + y$. The geometry over this semiring is well studied, and is known helpfully as tropical geometry. Geometry can be done similarly to working over a field, where we have analogous objects such as tropical polynomials and tropical hypersurfaces. Note that there is a notion of tropicalising varieties which provides a way to view varieties over the tropical semiring.

Tropical varieties are well studied with numerous results analogous to geometry over a field. For example, there is a notion of stable intersection [4], there is a scheme theoretic version of tropical geometry [5] [6], and there is a relation between spaces being tropically convex and being a tropical linear space [7].

In particular, we are able to define the notion of tropical linear spaces, akin to linear spaces. Moreover, all linear spaces tropicalise to form a tropical linear space, but the converse isn't always true. Tropical linear spaces themselves turn out to be a cryptomophism for valuated matroids. Since valuated matroids are a generalisation of matroids, this allows us to tie matroids back to the study of geometry over the tropical semiring.

The *Dressian*, $\mathrm{Dr}(k, n)$, the set of all valuated matroids of size n and rank k, can be viewed as the polyhedral fan of the regular subdivisions of the hypersimplex $\Delta(k, n)$ such that each cell is a matroid polytope. The polyhedral structure of the Dressian, $\mathrm{Dr}(k, n)$, has been fully described for all k when $n = 1, 2$ as well as for $k \leq 8$ when $n = 3$. In particular, $\mathrm{Dr}(2, n)$ is described by a set of phylogenetic trees [8],[9],[10].

Working over a field we have the *Grassmannian*, $\mathrm{Gr}(k, n)$; this being the set of k-dimensional subspaces of an n-dimensional vector space. The Grassmannian is able to be embedded in a projective variety by using the Plücker embedding. This embedding satisfies relations known as the Plücker relations, and it has associated Plücker coordinates. There is tropicalisation of the Grassmannian called the tropical Grassmannian. This can be extended to the Dressian. The difference between the two is that the former parametrises tropicalised linear spaces, whereas the Dressian also parametrises tropical linear spaces. However, when $k = 2$ the Dressian and the tropical Grassmannian do in fact coincide.

Now coming back to matrices, we are able to view a matrix as representing a linear transformation. When describing a linear transformation given by a matrix A, instead of using the matrix we can view it via its graph, this being the row space of the matrix $[I \mid A]$. Working over a field $\mathbb{K}$ we are able to describe any vector space V of $\mathbb{K}^n$ where $\dim V = r$ as the row space of some $r \times n$ matrix A of rank r. These give alternative interpretations of a matrix, and these are both geometric descriptions. These geometric descriptions are used to make an analogy between tropical geometry and plain old geometry over a field.

Alternatively, consider the maximal minors of $[I \mid A]$. The vector obtained, consisting of $\binom{n}{r}$ entries, is called a *Plücker vector*, and satisfies relations known as *Plücker relations.* From this Plücker vector we are able to recover V, by constructing the matrix A. We get the matrix A by considering entries of the Plücker vector labelled by $r - 1$ entries from $1, \ldots, n$ and 1 entry from $n + 1, \ldots, 2n$, and thus V may be viewed as either a row space of a matrix or as a Plücker vector.

However, working over the tropical semifield we are not necessarily able to describe a vector space using the same descriptions as those when we consider a vector space over a field. Similarly to working over a field we have that each tropical linear space V, where $\dim V = r$, has a vector of length $\binom{n}{r}$ consisting of tropical numbers, satisfying *tropical Plücker relations*, and any vector of this form defines a valuated matroid and is also known as a *tropical Plücker vector*. However, such a matrix from which these tropical numbers arise as tropical matrix minors need not exist. This is due to the realisability of tropical linear spaces, which itself is related to representable matroids of which can be seen in Example 9.

Since tropically we don't always have a matrix representing a tropical linear space we use tropical Plücker vectors in order to create an operation which is reminiscent of matrix multiplication. We will note in Chapter 4 that we can get some relations between valuated matroids and matrices, so we can utilise this. We call $T\bar{G}L_n$ the set of all valuated matroids we are interested in.

We define our hyperoperation in Chapter 4 as:

Definition 1. Let $\mathcal{M}_1, \mathcal{M}_2 \in T\bar{G}L_n$ and let z be the set of elements of $\mathrm{Dr}(n, 3n)$ such that the projection of any element of z to the coordinates $E_1 \cup E_2$ is $\mathcal{M}_1$ and the projection of any element of z to the coordinates $E_2 \cup E_3$ is $\mathcal{M}_2$. We define $\mathcal{M}_1 \boxdot \mathcal{M}_2$ to be the projection of z to coordinates $E_1 \cup E_3$. We also define $\mathcal{M}_1 \hat{\boxdot} \mathcal{M}_2$ to be the set of z on the coordinates $E_1 \cup E_2 \cup E_3$.

This hyperoperation is our proposed analogue to the general linear group. Within Chapter 4 the central question we investigate is the following conjecture. The rest of the study stems from investigations into this, and throughout we investigate the structure of this hyperoperation.

Conjecture 1.0.2. *Let $\mathcal{M}_1, \mathcal{M}_2 \in T\bar{G}L_n$. We claim that $\mathcal{M}_1 \boxdot \mathcal{M}_2$ forms a hypergroup.*

Throughout Section 4.3.2 we show that this hyperoperation has an identity element in addition to having an inverse. In particular, we note that it also associative for $n = 1, 2$, and consequently both $(T\bar{G}L_1, \boxdot)$ and $(T\bar{G}L_2, \boxdot)$ are hypergroups. Later within Section 4.5 we show that certain strategies for proving associativity for larger n do not hold in general.

Further throughout this chapter we note that this defined hyperoperation has an extant single valued operation as an element; that given by the composition of valuated linking systems which we first introduce in Section 4.1. Linking systems are an alternative way of viewing matroids. We give some context surrounding the study of these, and give some results in Section 4.4 regarding the flats of this as well as the flats of an extension of the composition of valuated linking systems which we introduce in Section 4.3.1.

The rest of Chapter 4 is devoted to giving further results regarding the structure of our hyperproduct. We begin in Section 4.3.3 by outlining the the $n = 2$ case, and in particular, we give a full description of $\mathcal{M}_1 \boxdot \mathcal{M}_2$ for $\mathcal{M}_1, \mathcal{M}_2 \in T\bar{G}L_2$. We will see that for any given $\mathcal{M}_1 \boxdot \mathcal{M}_2$, that it is a subset of the Dressian, so always has some underlying structure, however in Section 4.3.7 and Section 4.3.8 we will see that it is not always convex, nor is it always a fan. We also show in Section 4.3.4 that $(T\bar{G}L_n, \boxdot)$ is in general noncommutative.

Chapter 2

Preliminaries

2.1 Polytopes

Here we introduce some material that we need regarding polytopes and other polyhedral structures. For a general overview see Ziegler [11].

Definition 2. [11] A point set $K \subseteq \mathbb{R}^d$ is *convex* if for any two points $\mathbf{x}, \mathbf{y} \in K$ then K also contains the straight line segment $[\mathbf{x}, \mathbf{y}] = \{\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \mid 0 \leq \lambda \leq 1\}$.

Remark 1. This is defined similarly to the convexity of a function. Let $f : X \to Y$ be a function. Then f is *convex* if for all $0 \leq t \leq 1$ and for all $x_1, x_2 \in X$ then $f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$.

Definition 3. [11] Clearly the intersection of convex sets is convex, and $\mathbb{R}^d$ itself is convex. Thus for any $K \subseteq \mathbb{R}^d$, the "smallest" convex set containing K, called the *convex hull* of K, can be constructed as the intersection of all convex sets that contain K:

$$\operatorname{conv}(K) := \bigcap \{K' \subseteq \mathbb{R}^d \mid K \subseteq K', K' \text{ is convex}\}.$$

Definition 4. [11] A *(convex) polytope* is the convex hull of a finite set of points in $\mathbb{R}^d$.

Remark 2. Just as for matroids there is more than one way that a polytope can be introduced. All the polytopes we consider throughout this thesis are convex and so we drop the word convex.

Theorem 2.1.1. *[11] A polytope $P \subseteq \mathbb{R}^d$ may also be presented as a bounded intersection of finitely many closed halfspaces in $\mathbb{R}^d$.*

Proposition 2.1.2. *[11] Let $P \subseteq \mathbb{R}^d$ be a polytope.*

1. *Every polytope is the convex hull of its vertices: $P = \mathrm{conv}(\mathrm{vert}(P))$.*

2. *If a polytope can be written as the convex hull of a finite point set then the set contains all the vertices of the polytope: $P = \mathrm{conv}(V)$ implies that $\mathrm{vert}(P) \subseteq V$.*

We now introduce a couple of archetypal examples of polytopes, but before that we introduce affine subspaces.

Definition 5. [11] A is an *affine subspace* of $\mathbb{R}^d$ if it is a translate of some linear subspace of $\mathbb{R}^d$.

Definition 6. [11] We define a *d-simplex* as the convex hull of any $d+1$ affinely independent points in some $\mathbb{R}^n$ $(n \geq d)$ and thus a d-simplex is a polytope of dimension d with $d+1$ vertices.

For the d-simplex we use the *standard d-simplex*, denoted Δ_d, with $d+1$ vertices in $\mathbb{R}^{d+1}$

$$\Delta_d := \{\mathbf{x} \in \mathbb{R}^{d+1} \mid \sum_{i=1}^{d+1} x_i = 1, x_i \geq 0\} = \mathrm{conv}\{e_1, \ldots, e_{d+1}\}.$$

Definition 7. [11] The *hypersimplex* $\Delta(d,k)$ in $\mathbb{R}^d$ is defined by

$$\Delta(d,k) = \mathrm{conv}\{\mathbf{x} \in \{0,1\}^d \mid \sum_{i=1}^{d} x_i = k\} = \{\mathbf{x} \in \mathbb{R}^d \mid 0 \leq x_i \leq 1 \text{ for } 1 \leq i \leq d, \sum_{i=1}^{d} x_i = k\}.$$

Definition 8. [11] Let $P \subseteq \mathbb{R}^p, Q \subseteq \mathbb{R}^q$ be polytopes. We define the *product* of P and

Q, $P \times Q$, to be the set

$$\{\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} \mid \mathbf{x} \in P, \mathbf{y} \in Q\}.$$

Definition 9. [11] The *Minkowski sum* of two sets $P, Q \subseteq \mathbb{R}^d$ is defined by

$$P + Q := \{\mathbf{x} + \mathbf{y} \mid \mathbf{x} \in P, \mathbf{y} \in Q\}.$$

Remark 3. Both the Minkowski sum and the product of polytopes allow us to produce new polytopes from existing polytopes.

Definition 10. [11] Let $P \subseteq \mathbb{R}^d$ be a polytope. A linear inequality $\mathbf{cx} \leq c_0$ is *valid* for P if it is satisfied for all points $\mathbf{x} \in P$. A *face* of P is any set of the form

$$F = P \cap \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{cx} = c_0\}$$

where $\mathbf{cx} \leq c_0$ is a valid inequality for P. The *dimension* of a face is the dimension of its affine hull: $\dim(F) := \dim(\text{aff}(F))$. We note that P and $\emptyset$ are faces of P, and the faces of dimensions $0, 1, \dim(P) - 2$ and $\dim(P) - 1$ are called *vertices*, *edges*, *ridges* and *facets* respectively.

Definition 11. [11] A *lattice polytope* is a polytope whose vertices all have integer Cartesian coordinates

Definition 12. [11] We define the *cone* over a set Y by

$$\text{cone}(Y) = \{\lambda_1 \mathbf{y}_1 + \cdots + \lambda_k \mathbf{y}_k \mid \{\mathbf{y}_1, \ldots, \mathbf{y}_k \subseteq Y, \lambda_i \geq 0\}.$$

Definition 13. [11] A *polyhedron*, $P \subseteq \mathbb{R}^d$, is the minkowski sum of a convex hull of a finite set of points plus a conical combination of vectors. This can be expressed as

$$P = \text{conv}(V) + \text{cone}(Y) \text{ for some } V \in \mathbb{R}^{d \times n}, Y \in \mathbb{R}^{d \times n'}.$$

Remark 4. Just as for polytopes, polyhedra can also be expressed as an intersection of closed halfspaces. We also note that all polytopes are examples of polyhedra, but polyhedra do not need to be bounded.

Definition 14. [11] A *polyhedral complex* C is a finite collection of polyhedra in $\mathbb{R}^d$ such that:

1. the empty polyhedron is in C
2. if $P \in C$, then all the faces of P are also in C
3. the intersection $P \cap Q$ of two polyhedra $P, Q \in C$ is a face of both P and of Q.

C is a *polytopal complex* if all the polyhedra in C are bounded.

Definition 15. [11] A *fan* in $\mathbb{R}^d$ is a family $F = \{C_1, C_2, \ldots, C_N\}$ of nonempty polyhedral cones, with the following two properties:

1. Every nonempty face of a cone in F is also a cone in F.
2. The intersection of any two cones in F is a face of both.

Definition 16. [11] Let P be a polytope. A *subdivision* of P is a polytopal complex C with the underlying space $|C| = P$. The subdivision is a *triangulation* if all the polytopes are simplices.

Definition 17. [12] Given two subdivisions A_1, A_2 of P, we say that A_1 *refines* A_2 if every element of A_1 is contained in some element of A_2. A subdivision if *coarsest* if it does not refine any proper subdivision.

Definition 18. [11] A subdivision C of a polytope $Q \subseteq \mathbb{R}^d$ is *regular* if and only if it arises from a polytope in the following way.

1. The polytope Q is the image $\pi(P) = Q$ of the polytope P, via the canonical

projection map

$$\pi : \mathbb{R}^{d+1} \to \mathbb{R}^{d} \quad \begin{pmatrix} \mathbf{x} \\ x_{d+1} \end{pmatrix} \mapsto \mathbf{x}$$

which "deletes the last coordinate".

2. C is the set of all *lower faces* of P, projected down to Q, that is,

$$C = \{\pi(F) \mid F \text{ is a lower face of } P\}$$

where the lower faces of P are the faces F that satisfy $\mathbf{x} - \lambda \mathbf{e}_{d+1} \notin P$ for each $\mathbf{x} \in F$ and $\lambda > 0$.

Alternatively C is the family of all faces of P that can be "seen" from $-T\mathbf{e}_{d+1}$ for $T \to \infty$ large enough.

Definition 19. Let $P \subseteq \mathbb{R}^d$ be a polytope. We say that a vector $\mathbf{x}$, indexed by each vertex of P, *induces* a regular subdivision on P by creating a new polytope Q with vertices defined by (v_i, x_i) and considering C as the set of lower faces of Q. Then C is the regular subdivision *induced* by $\mathbf{x}$.

2.2 Matroids and valuated matroids

2.2.1 Matroids

Having informally introduced matroids in the introduction to the thesis we now formally state various cryptomorphic definitions of matroids, as well as standard results which will be of use to us in Chapter 3 and Chapter 4. For a general overview of matroids see Oxley [13] or Welsh [14].

Definition 20. [13] A *matroid* M is an ordered pair $(E, \mathcal{I})$ consisting of a finite set E and a collection $\mathcal{I}$ of subsets of E satisfying the following three conditions:

1. $\emptyset \in \mathcal{I}$.

2. If $I \in \mathcal{I}$ and $I' \subseteq I$ then $I' \in \mathcal{I}$.

3. If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$ then there is an element $e \in I_2 \backslash I_1$ such that $I_1 \cup e \in \mathcal{I}$.

We say that M is a matroid *on* the set E.

Definition 21. [13] We call a maximal independent set in a matroid M a *basis* or *base* of M, the collection of which we denote $\mathcal{B}$ or $\mathcal{B}(M)$.

We now give some of the numerous equivalent ways to define a matroid, some of the ways we give include via bases and in terms of a rank function.

Theorem 2.2.1. *([13], Theorem 1.2.3) Let E be a finite set and $\mathcal{B}$ be a set of subsets of E. Then $\mathcal{B}$ is the collection of bases of a matroid on E if and only if it has the following properties:*

1. *$\mathcal{B}$ is non-empty (non-emptiness property).*

2. *If B_1 and B_2 are in $\mathcal{B}$ and $x \in B_1 \backslash B_2$, then there is an element $y \in B_2 \backslash B_1$ such that $(B_1 \backslash x) \cup y \in \mathcal{B}$ (exchange axiom).*

Remark 5. All bases of a matroid $(E, \mathcal{B})$ have the same cardinality. Any element $e \in E$ which does not belong to any basis is called a *loop* of a matroid. Let n be the size of bases of M, then a non-basis is any set of size n which isn't a basis. We denote the set of non-bases of M as $\mathcal{NB}$ or $\mathcal{NB}(M)$.

Theorem 2.2.2. *([13], Corollary 1.3.4) Let E be a finite set. A function $r : 2^E \to \mathbb{Z}_{\geq 0}$ is the rank function of a matroid on E if and only if r has the following properties:*

1. *If $X \subseteq E$ then $0 \leq r(X) \leq |X|$.*

2. *If $X \subseteq Y \subseteq E$ then $r(X) \leq r(Y)$.*

3. *If $X, Y \subseteq E$ then $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$ (submodular inequality).*

Then $\mathcal{I}$ is the collection of subsets $X \subseteq E$ for which $r(X) = |X|$. Then $(E, \mathcal{I})$ is a *matroid with rank function* r.

We can also alternatively define the rank function of a matroid using the following theorem.

Theorem 2.2.3. *[14], Theorem 2] Let* E *be a finite set. A function* $r : 2^E \to \mathbb{Z}_{\geq 0}$ *is the rank function of a matroid on* E *if and only if* r *has the following properties:*

1. $r(\emptyset) = 0$
2. *If* $X \subseteq E, y \in E$, *then* $r(X) \leq r(X \cup y) \leq r(X) + 1$
3. *If* $X, Y \subseteq E$ *then* $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$ *(submodular inequality).*

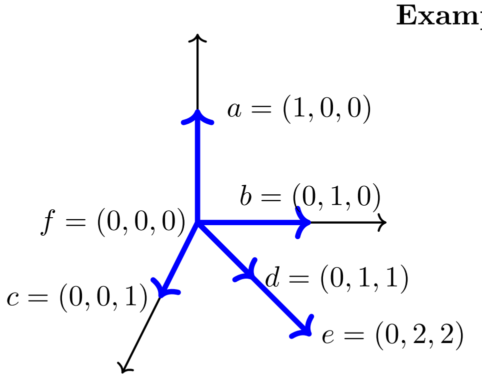
Remark 6. We denote the rank function of a matroid M as either r_M or simply just r if it is clear which matroid it is associated with.

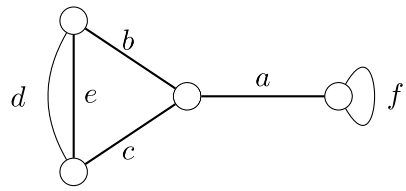
Definition 22. [13] Let M be a matroid. Any minimal dependent set of M is called a *circuit* of M the collection of which we denote $\mathcal{C}$ or $\mathcal{C}(M)$.

Remark 7. The collection of circuits again can give us yet another way to define a matroid. Explicitly this can be seen in [13] Corollary 1.1.5.

We now give an example of a matroid which we see arises from different mathematical settings.

Example 1.





Goal: Choose linear independent vectors

Goal: Choose sets of edges with no cycles

Given either of these configurations we can construct matroids. In the linear algebra setting consider all linearly independent sets of vectors, and in the graph theory setting consider sets of edges with no cycles. In either case we have the following independent sets, denoted by $\mathcal{I}$:

$$\begin{aligned}
\text{Size } 0: &\quad \emptyset \\
\text{Size } 1: &\quad a, b, c, d, e \\
\text{Size } 2: &\quad ab, ac, ad, ae, bc, bd, be, cd, ce \\
\text{Size } 3: &\quad abc, abd, abe, acd, ace
\end{aligned}$$

We are able to see that these satisfy the independence axioms of being a matroid. Similarly we have the set of bases are $\mathcal{B} = \{abc, abd, abe, acd, ace\}$, and these again satisfy the basis axioms of being a matroid. We also note that f is a loop of this matroid, and is the only such loop in the matroid.

Example 2. [13] Let $m, n \in \mathbb{Z}_{\geq 0}$ with $m \leq n$. Let E be an n-element set, and $\mathcal{B}$ be the collection of m-element subsets of E. It can be easily verified that $\mathcal{B}$ is the collection of bases of a matroid on E. We denote this matroid $U(m, n)$ and it is called the *uniform matroid* of rank m on an n-element set.

Definition 23. [15] Let f be a set function on E, that is, a function defined on the collection of all subsets of E. The function f is called *submodular* if for all $T, U \subseteq E$ we have

$$f(T) + f(U) \geq f(T \cap U) + f(T \cup U).$$

We also have a more local characterisation of a submodular function. For all $A \subseteq E$, $a, b \in E$ then f is *submodular* if

$$f(A \cup a) + f(A \cup b) \geq f(A) + f(A \cup a \cup b).$$

Remark 8. We see in Section 2.2.1.1 that we are able to associate a polymatroid with any submodular function. In particular, a matroid is an example of a polymatroid, and we also note this submodular property is the same as those in the matroid rank function axioms.

Definition 24. [13] A matroid which is obtained from the matrix A by considering linear independence of columns over some field $\mathbb{F}$ is called the *vector matroid* of A. If M is isomorphic to the vector matroid of a matrix A over a field $\mathbb{F}$, then M is *representable over* $\mathbb{F}$. A matroid that is representable over some field is called *representable*.

Example 3. We define the Vámos matroid V_8 to be a rank four matroid on $E = \{a, a', b, b', c, c', d, d'\}$ such that all subsets of E of size ≤ 4 are independent, except for the sets $\{a, a', b, b'\}, \{a, a', c, c'\}, \{a, a', d, d'\}, \{b, b', c, c'\}, \{b, b', d, d'\}$. Then by Proposition 2.2.26 of [13] we have that the Vámos matroid is not representable, and thus cannot be represented as a matrix over any field.

Definition 25. [13] The *nullity* of a set X in a matroid $M = (E, r_M)$ is $\text{nullity}(X) = |X| - r_M(X)$.

Definition 26. [13] The *corank* of a set X in a matroid $M = (E, r_M)$ is $\text{corank}(X) = r_M(E) - r_M(X)$.

Remark 9. We note that some authors use corank to be the function as defined in Proposition 2.2.5.

Theorem 2.2.4 ([13], Theorem 2.1.1)**.** *Let $M = (E, \mathcal{B})$ be a matroid and $\mathcal{B}^*(M) := \{E(M) \backslash B \mid B \in \mathcal{B}(M)\}$. Then $\mathcal{B}^*(M)$ is the set of bases of a matroid on $E(M)$.*

Definition 27. [13] The matroid whose ground set is $E(M)$ and whose set of bases is $\mathcal{B}^*(M)$, is called the *dual* of M and is denoted by M^*.

Proposition 2.2.5. *([13], Proposition 2.1.9) For all subsets X of the ground set E of* *a matroid M^*, the dual of a matroid M, we have*

$$r^*(X) = r(E \backslash X) + |X| - r(M),$$

where r^* is the rank function of M^*.

Definition 28. [13] The *closure* function on a matroid $M = (E, r)$ from 2^E into 2^E is defined for all $X \subseteq E$ by

$$\bar{X} = \{x \in E \mid r(X \cup x) = r(X)\}.$$

Definition 29. [13] Let $M = (E, \mathcal{I})$ be a matroid and suppose $X \subseteq E$. Let $\mathcal{I}|X = \{I \subseteq X \mid I \in \mathcal{I}\}$. Then $(X, \mathcal{I}|X)$ is a matroid, and we call this matroid the *restriction of M to X* or the *deletion of $E \backslash X$ from M*. It is denoted $M|X$ or $M \backslash (E \backslash X)$.

Definition 30. [13] If a matroid M is obtained from a matroid N by deleting a non-empty subset of the ground set of N, then N is called an *extension* of M.

Definition 31. [13] Let M be a matroid on E, and let $T \subseteq E$. We define the *contraction of T from M*, denoted M/T, by

$$M/T = (M^* \backslash T)^*.$$

Proposition 2.2.6. *[13] Let M be a matroid on E with rank function r_M, and let $T \subseteq E$. We are able to write the rank functions for both the deletion of T and the contraction of T from M. Let $X \subseteq E \backslash T$*

$$r_{M \backslash T}(X) = r_M(X) \quad \textit{and} \quad r_{M/T} = r_M(X \cup T) - r_M(T).$$

Definition 32. [13] In a matroid M on E, a subset $X \subseteq E$ for which $\bar{X} = X$ is called a *flat* of M. We denote the set of flats of M by $\mathcal{F}(M)$.

Remark 10. Again, we are able to define a matroid in terms of flats. A definition of which can be seen in Exercise 11 of Section 1.4 of [13].

Definition 33. [13] For a matroid M we denote by $\mathcal{L}(M)$ the set of flats of M ordered by inclusion. We call $\mathcal{L}(M)$ the *lattice of flats* in light of the following lemma.

Lemma 2.2.7. *[13] For any matroid M, the $\mathcal{L}(M)$ is a lattice, and for all flats X, Y of M we have*

$$X \wedge Y = X \cap Y \quad \text{and} \quad X \vee Y = X \cup Y.$$

We now introduce a polyhedral description of a matroid.

Definition 34 ([16])**.** Let $M = (E, \mathcal{B})$ be a matroid. The *matroid basis polytope* associated with M is $P(M) = \text{conv}\{e_B \mid B \in \mathcal{B}(M)\}$.

Remark 11. Matroid basis polytopes give us yet another cryptomorphic way of viewing a matroid. There is a similar construction, the matroid independence polytope, and this is built in a similar way but from e_I for $I \in \mathcal{I}(M)$. In particular, it can be seen that the matroid basis polytope of M is a face of the matroid independence polytope of M. Throughout this thesis we also refer to the matroid basis polytope as the matroid polytope.

Proposition 2.2.8. *([13], Proposition 4.2.14) Let M_1 and M_2 be matroids on disjoint sets E_1 and E_2. Let $E = E_1 \cup E_2$ and $\mathcal{B} = \{B_1 \cup B_2 \mid B_1 \in \mathcal{B}(M_1), B_2 \in \mathcal{B}(M_2)\}$. Then $(E, \mathcal{B})$ is a matroid.*

Remark 12. This operation given in Proposition 2.2.8 is called the *direct sum* of M_1 and M_2 and is denoted by $M_1 \oplus M_2$.

Definition 35. [13] We say that a bijection ϕ is a *weak map* from matroids M_1 to M_2 if $\phi^{-1}(I)$ is independent in M_1 for every independent set I of M_2.

Definition 36. [13] The collection of matroids on a fixed set E can be partially ordered, the *weak order* on the set is the partial order on the set by taking $M_2 \leq M_1$ if the identity map on E is a weak map from M_1 to M_2.

Definition 37. [15] Let $M = (E, \mathcal{I})$ be a matroid and let k be a natural number. Define $\mathcal{I}' = \{I \in \mathcal{I} \mid |I| \leq k\}$. Then $(S, \mathcal{I}')$ is a matroid which is called the *k-truncation* of M. We call the k-truncation of the dual matroid of M the *k-cotruncation* of M.

Theorem 2.2.9 (Matroid union theorem)**.** *([15], Corollary 42.1a) Let $M_1 = (S_1, r_1), \ldots, M_k = (S_k, r_k)$ be matroids. Let $S = S_1 \cup \cdots \cup S_k$. The collection of subsets of S that are of the form $X_1 \cup \cdots \cup X_k$, where X_i is independent in M_i, form the independent sets of a matroid on S. The rank function of this matroid is*

$$r(X) = \min_{Y \subseteq X}(|X \backslash Y| + r_1(S_1 \cap Y) + \cdots + r_k(S_k \cap Y)).$$

2.2.1.1 Polymatroids

In this section we look at submodular functions in more generality and introduce some results we will later need.

Definition 38. [15] Let f be a submodular function on E. Define the following polyhedra associated with f

$$P_f := \{x \in \mathbb{R}^S \mid x \geq \mathbf{0}, x(U) \leq f(U) \text{ for each } U \subseteq S\}$$
$$EP_f := \{x \in \mathbb{R}^S \mid x(U) \leq f(U) \text{ for each } U \subseteq S\}$$

We call P_f the *polymatroid associated with* f, and EP_f the *extended polymatroid associated with* f.

Definition 39. A vector x in EP_f (or P_f) is called a *base vector* of EP_f (or of P_f) if $x(S) = f(S)$. A *base vector* of f is a base vector of EP_f. The set of all base vectors of f is called the *base polytope* of EP_f or of f. It is a face of EP_f, and is denoted by B_f. So

$$B_f = \{x \in \mathbb{R}^S \mid x(U) \leq f(U) \text{ for all } U \subseteq S, x(S) = f(S)\}$$

2.2.2 Valuated matroids

We now provide background material on valuated matroids, a generalisation of matroids first introduced by Dress and Wenzel. In this subsection we introduce valuated matroids as a purely combinatorial structure in an analogous way to how we introduced matroids. In Section 2.3 we look at different ways to view valuated matroids.

Definition 40. [6] Let E be a finite set, $s \in \mathbb{N}$. Denote by $\binom{E}{s}$ the collection of subsets of E of size s. A *valuated matroid* $\mathcal{M}$ of rank s on the ground set E is the function $\rho_{\mathcal{M}} : \binom{E}{s} \to \mathbb{R} \cup \{\infty\}$ satisfying the following:

1. There exists $B \in \binom{E}{s}$ such that $\rho(B) \neq \infty$.

2. For every $B, B' \in \binom{E}{s}$ and every $\mathbf{u} \in B \backslash B'$ there exists $\mathbf{v} \in B' \backslash B$ such that

$$\rho_{\mathcal{M}}(B) + \rho_{\mathcal{M}}(B') \geq \rho_{\mathcal{M}}(B \backslash \mathbf{u} \cup \mathbf{v}) + \rho_{\mathcal{M}}(B' \backslash \mathbf{v} \cup \mathbf{u}).$$

Remark 13. Valuated matroids are a generalisation of matroids. We are able to view a matroid as a valuated matroid where $\rho_{\mathcal{M}} : \binom{E}{s} \to \{0, \infty\}$. By letting bases of a matroid have valuation 0 and non-bases valuation ∞ we are able to directly compare the above definition with Theorem 2.2.1. The first condition in the above corresponds to the non-emptiness property, and the second condition in the above corresponds to the basis exchange property from Theorem 2.2.1. Hence, valuated matroids can be viewed as an extension of matroids.

Throughout this thesis we refer to valuated matroids as either $\mathcal{M}$ or just by their function $\rho_{\mathcal{M}}$ depending on exactly what information we are interested in at that particular moment in time.

Given a valuated matroid $\mathcal{M}$ of rank s on $\{1, \ldots, n\}$, then throughout this thesis when we give an explicit representation of $\rho_{\mathcal{M}}$ we label the sets in lexicographic order. That is, $\rho_{\mathcal{M}} = (\rho_{\mathcal{M}}(12345 \cdots s), \rho_{\mathcal{M}}(123 \cdots s+1), \ldots, \rho_{\mathcal{M}}(n-s+1 \cdots n))$.

Definition 41. [6], [17] Let $\mathcal{M}$ be a valuated matroid. The *underlying matroid* of $\mathcal{M}$, which we denote by M, is the matroid that has $\{B \subseteq E \mid \rho(B) \neq \infty\}$ as its collection of bases. The *initial matroid* of $\mathcal{M}$ is the matroid which has $\{B \in \mathcal{B}(M) \mid \rho(B)$ attains the minimum$\}$ as its set of bases.

Example 4. We give our first example of a valuated matroid which isn't itself a matroid. We define the valuated matroid $\mathcal{M}$ of rank 2 on $E = \{1, 2, 3, 4\}$ by $\rho_{\mathcal{M}} = (0, 3, 6, 2, 5, \infty)$. Then $\mathcal{M}$ clearly satisfies the first axiom of being a valuated matroid. We verify the second when we consider $B = 13$ and $B' = 24$. Firstly let $u = 1$. Then we have the following if $v = 2$

$$8 = 3 + 5 = \rho_{\mathcal{M}}(13) + \rho_{\mathcal{M}}(24) \geq \rho_{\mathcal{M}}(23) + \rho_{\mathcal{M}}(14) = 2 + 6 = 8.$$

Now if $u = 3$ then if $v = 4$ we have

$$8 = 3 + 5 = \rho_{\mathcal{M}}(13) + \rho_{\mathcal{M}}(24) \geq \rho_{\mathcal{M}}(14) + \rho_{\mathcal{M}}(23) = 6 + 2 = 8.$$

The other choices of B, B' follow similarly. We note that the underlying matroid is given by $M = (E, \mathcal{B}) = (\{1, 2, 3, 4\}, \{12, 13, 14, 23, 24\})$, and the initial matroid is given by $M = (E, \mathcal{B}) = (\{1, 2, 3, 4\}, \{12\})$.

Definition 42. [4] For a valuated matroid $\mathcal{M}$ define the associated *valuated matroid basis polytope* as $P(\mathcal{M}) = \text{conv}\{(e_B, \rho_B) : B \in \mathcal{B}(M)\} + (\mathbf{0}, \mathbb{R}_{\geq 0})$.

Remark 14. This polytope can be related back to the valuated matroid in a few ways. One is by directly looking at the final coordinate of the points in the polytope, and retrieving the valuation function. More interestingly we can follow Speyer's constructions of subdivisions of $\Delta(k, n)$. If we consider the final coordinate to be a height function to make a regular subdivision we get from Proposition 2.2 of [4] that every face of this regular subdivision is matroidal, this is the content of Lemma 2.3.5.

We have some basic operations on valuated matroids just like on usual matroids.

Definition 43. [18] Let $\mathcal{M}$ be a valuated matroid of rank s on E, and let $X \subseteq E$. The *restriction* of $\mathcal{M}$ to X, is the function $\rho'_{M|X}$ which assigns to each $J \subseteq X$ the minimum of $\rho_{\mathcal{M}}(Z)$, where $J \subseteq Z \subseteq E$, where we just consider the sets $J \subseteq X$ such that $|J|$ is maximal where there is at least one J of this size such that $\rho'_{M|X}(J) \neq \infty$.

Definition 44. Given a vector ρ with entries in $\mathbb{R} \cup \{\infty\}$, and letting $\mathbf{x} = (x_a : a \in E)$, we define a rescaling of ρ by

$$\rho^{\mathbf{x}}(A) = \rho(A) + \sum_{a \in A} x_a.$$

Remark 15. For any ρ which defines a valuated matroid on the ground set E, and for any $\mathbf{x} = (x_a : a \in E)$, we have for any rescaling that $\rho^{\mathbf{x}}$ is also a valuated matroid.

Definition 45. We define the initial matroid operation $\text{in}_{\mathbf{x}}$ of $\mathcal{M}$ to be the initial matroid associated with $\rho^{\mathbf{x}}$. When $x = 0$ this is simply just the initial matroid of $\mathcal{M}$.

2.3 Tropical geometry

Tropical geometry in the simplest sense is the study of polynomials and their geometry over the tropical semiring. We introduce the tropical semiring in Section 2.3.2. However, there are deeper interpretations than just the simple description, in particular, we can ask how tropical geometry relates to algebraic geometry. Specifically, one aim of the study of tropical geometry is about transforming questions about algebraic varieties into questions about polyhedral complexes. Given polynomial equations which define an algebraic variety there is a process, called tropicalisation, to turn these polynomial equations into tropical equations, these tropical equations being convex piecewise linear functions. Similarly there is a process of tropicalisation to turn these algebraic varieties into polyhedral complexes. It turns out by the Fundamental Theorem of Tropical Geometry that both these ways coincide, that is, it doesn't matter if we tropicalise the variety or the polynomial equations which define a variety. This allows a concrete connection between algebraic and tropical geometry.

Part of the reason for studying tropical geometry is that the study of polyhedral geometry can be seen as a sort of linear optimisation question, which is in some sense easier than algebraic geometry. However, working out all of the combinatorial data of the face structure of a tropical variety is harder than just a simple optimisation problem. Throughout this thesis we are broadly interested in tropical linear spaces, this being a superset of the tropicalisation of all linear spaces in the algebraic sense. The geometry of linear spaces is already relatively easy, but by considering tropical linear spaces we get a link to matroids and the combinatorics they bring with them, and this allows us to view the Bergman fan associated with a matroid as being related to something from a field.

2.3.1 Fields

In this section we introduce valuations on a field, in addition to some canonical examples of fields and their natural choice of valuation.

Definition 46. [17] Let $\mathbb{K}$ be a field. A *valuation* on $\mathbb{K}$ is a function $\mathrm{val} : \mathbb{K} \to \mathbb{R} \cup \{\infty\}$ satisfying the following:

1. $\mathrm{val}(a) = \infty$ if and only if $a = 0$
2. $\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b)$
3. $\mathrm{val}(a + b) \geq \min\{\mathrm{val}(a), \mathrm{val}(b)\}$ for all $a, b \in \mathbb{K}$

A valuation val is *trivial* if $\mathrm{val}(a) = 0$ for all $a \in \mathbb{K}\backslash\{0\}$, otherwise it is *non-trivial*.

Remark 16. Tropical geometry is concerned with non-trivial valuations in order for the Fundamental Theorem of Tropical Geometry, which we give in Theorem 2.3.2, to be informative, since we see that the tropicalisation of any algebraic variety only gives points whose coordinates are in the image of a given valuation.

Definition 47. [17] Take some valuation val on $\mathbb{K}$. Consider the set of all field elements with non-negative valuation $R = \{c \in \mathbb{K} \mid \mathrm{val}(c) \geq 0\}$. This is a local ring, and hence has a unique maximal ideal $m_{\mathbb{K}} = \{c \in \mathbb{K} \mid \mathrm{val}(c) > 0\}$. Now the quotient ring $K = R/m_{\mathbb{K}}$ is a field, called the *residue field* of $(\mathbb{K}, \mathrm{val})$.

Definition 48. [17] A *Laurent polynomial* in one variable over a field $\mathbb{F}$ is any expression of the form

$$p = \sum_k p_k x^k \quad p_k \in \mathbb{F}$$

where $k \in \mathbb{Z}$ and only finitely many p_k are nonzero.

Definition 49. [17] The *ring of Laurent polynomials* over $\mathbb{K}$, denoted $\mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, is defined by taking the set of all Laurent polynomials over $\mathbb{K}$ in variables $x_1^{\pm 1}, \ldots, x_n^{\pm 1}$, and then using the operations of polynomial addition and multiplication finitely many times to get new elements. We call elements of this ring *Laurent Polynomials* in n variables.

Definition 50. [19] Let $\mathbb{F}$ be a field. We define the *field of formal Laurent series*, $\mathbb{F}((t))$ as the set of all objects $t^e h(t)$ where $e \in \mathbb{Z}$ and $h(x) \in \mathbb{F}[[t]]$, where $\mathbb{F}[[t]]$ is the field of formal power series.

Example 5. [17] We introduce the field of Puiseux series with coefficients in $\mathbb{C}$. The elements of this field are

$$c(t) = c_1 t^{a_1} + c_2 t^{a_2} + c_3 t^{a_3} + \cdots$$

where c_i are nonzero complex numbers and $a_1 < a_2 < \cdots$ are rational numbers which have a common denominator. We denote by $\mathbb{C}\{\{t\}\}$ the field of Puiseux series over $\mathbb{C}$ which we are able to write as

$$\mathbb{C}\{\{t\}\} = \bigcup_{n \geq 1} \mathbb{C}((t^{1/n}))$$

where $\mathbb{C}((t^{1/n}))$ is the field of Laurent series in the formal variable $t^{1/n}$.

The valuation which we associate with this field is the following. Let $\mathrm{val} : \mathbb{C}\{\{t\}\} \to \mathbb{R} \cup \{\infty\}$ be given by mapping a nonzero scalar $c(t) \in \mathbb{C}\{\{t\}\} \backslash \{0\}$ to the lowest exponent

a_1 that appears in the series expansion of $c(t)$, and the 0 function maps to ∞.

Remark 17. This example also works for any other field in the place of $\mathbb{C}$ and follows in a similar fashion. In particular, we note that the Puiseux series is a natural way to extend $\mathbb{C}$ to an algebraically closed non-trivially valued field.

2.3.2 Tropicalisation

Throughout this section of the preliminaries we concern ourselves with study over the *tropical semiring* $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ with operations defined by $x \oplus y = \min(x, y)$ and $x \odot y = x + y$. In this section we present how to turn algebraic varieties into tropical varieties.

Definition 51. [17] Let $x_1, \ldots, x_n$ be variables which represent elements in $\mathbb{T}$. A *tropical monomial* is any tropical product of these variables where repetition of variables is permissible. A *tropical polynomial* is a finite linear combination of tropical monomials.

Lemma 2.3.1. *[17] The tropical polynomial functions in n variables $x_1, \ldots, x_n$ are precisely the piecewise-linear concave functions on $\mathbb{R}^n$ with integer coefficients.*

Definition 52. [17] Let $\mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ denote the ring of Laurent polynomials over $\mathbb{K}$, and let $f = \sum_{\mathbf{u} \in \mathbb{Z}^n} c_{\mathbf{u}} x^{\mathbf{u}}$ be a Laurent polynomial. The *tropicalisation* of f, denoted $\operatorname{trop}(f)$, is defined by

$$\operatorname{trop}(f)(\mathbf{w}) = \min_{\mathbf{u} \in \mathbb{Z}^n} (\operatorname{val}(c_{\mathbf{u}}) + \mathbf{u} \cdot \mathbf{w}).$$

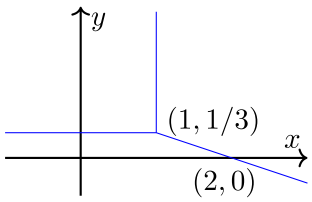
Example 6. Let $f = t + t^2 x^{-1} + y^3 \in \mathbb{C}\{\{t\}\}[x, y]$. We compute the tropicalisation of f under the natural valuation given in Example 5.

$$\operatorname{trop}(f)(x, y) = \min(\operatorname{val}(t), \operatorname{val}(t^2) + (-1, 0) \cdot (x, y), \operatorname{val}(1) + (0, 3) \cdot (x, y)) = \min(1, 2 - x, 3y)$$

Definition 53. [17] The *tropical hypersurface* trop($V(f)$) is the set

$$\{\mathbf{w} \in \mathbb{R}^n \mid \text{ the minimum in } \operatorname{trop}(f) \text{ is achieved at least twice}\}.$$

Example 7. We again consider $f = t+t^2x^{-1}+y^3 \in \mathbb{C}\{\{t\}\}[x,y]$ and we wish to compute the tropical hypersurface. We have from Example 6 that $\operatorname{trop}(f)(x,y) = (1, 2-x, 3y)$. Now it can be seen that the minimum of these is attained so that we have the following hypersurface



Definition 54. [17] Let I be an ideal in the Laurent polynomial ring $\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, and let $X = V(I)$ be its variety in the algebraic torus T^n. The *tropicalisation* of the variety X, denoted trop(X), is

$$\operatorname{trop}(X) = \bigcap_{f \in I} \operatorname{trop}(V(f)).$$

By *tropical variety* we mean any subset of the form trop(X) where X is a subvariety of T^n.

Remark 18. We could use an alternative definition of tropical variety where we call any pure weighted balanced polyhedral complex a tropical variety [7]. This is a generalisation of the definition of a tropical variety we give above. In particular, tropical linear spaces, which we introduce in Section 2.3.5, don't always fit our more narrow definition but do fit the more general definition.

Definition 55. [17] Finite intersections of tropical hypersurfaces are known as *tropical prevarieties*.

Remark 19. This is different to algebraic geometry where the intersection of algebraic hypersurfaces is a variety. Tropically we do not have the same, instead only have that certain intersections of tropical hypersurfaces are varieties, rather than every single one. This is since not all of these intersections will satisfy necessary "balancing conditions" which are required of tropical varieties. We see in Definition 56 that tropically we do have a way of intersecting two tropical varieties.

Theorem 2.3.2. *[17](Fundamental Theorem of Tropical Geometry) Let I be an ideal in $\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and $X = V(I)$ its variety in the algebraic torus T^n. Then the following are equivalent:*

1. *The tropical variety* $\mathrm{trop}(X)$
2. *the closure in $\mathbb{R}^n$ of the set of coordinatewise valuations of points in X:*

$$\mathrm{val}(X) = \{(\mathrm{val}(u_1, \dots, \mathrm{val}(u_n)) \mid (u_1, \dots, u_n) \in X\}$$

Definition 56. [17] Let Σ_1 and Σ_2 be pure weighted balanced polyhedral complexes in $\mathbb{R}^n$. The *stable intersection*, $\Sigma_1 \cap_{st} \Sigma_2$, is the polyhedral complex

$$\Sigma_1 \cap_{st} \Sigma_2 = \bigcup_{\sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2, \dim(\sigma_1 + \sigma_2) = n} \sigma_1 \cap \sigma_2.$$

Remark 20. Given that throughout this thesis we consider tropical linear spaces and Bergman fans, and both of these are balanced, we don't concern ourselves too much with the precise details of how stable intersection works, but it is left here as an example that such an operation exists. Essentially a pure balanced polyhedral complex is a polyhedral complex which has associated weights satisfying a condition that it balances, as well as every polyhedron which is not the face of any other in Σ is such that these polyhedrons have the same dimension.

Definition 57. [20] The *tropical projective space*, $\mathbb{TP}^n$, is defined by

$$\mathbb{TP}^n = \mathbb{T}^{n+1} \backslash \{\infty\} / \sim$$

where the equivalence relation $\sim$ is given by $(x_0, x_1, \ldots, x_n) \sim (x_0+\lambda, x_1+\lambda, \ldots, x_n+\lambda)$ for all $\lambda \in \mathbb{R}$.

Remark 21. We comment that $\mathbb{TP}^n$ is topologically a simplex [8]. Both Definition 53 and Definition 54 can be generalised to the tropical projective setting. The only difference we need to concern ourselves with is ensuring that the functions f which we tropicalise are homogeneous in order to ensure that they can be evaluated at a point of $\mathbb{TP}^n$. Stable intersection can also be defined on the tropical projective space but we don't concern ourselves with the details since we don't need to use this throughout this thesis.

2.3.3 Grassmannian

Definition 58 ([21], Def 14.4)**.** The *Grassmannian*, $\mathrm{Gr}(k, n)$, is the set of $\mathbb{P}^{\binom{n}{k}} - 1$ consisting of the set of k-dimensional subspaces of an n-dimensional vector space.

We let $\mathrm{sgn}(j, I, J)$ denote $(-1)^l$ where l is the number of elements $j' \in J$ with $j < j'$ plus the number of elements of $i \in I$ with $i < j$.

Definition 59. [17] We define the Plücker relation generated by σ and τ by

$$\mathcal{P}_{\sigma,\tau} = \sum_j \mathrm{sgn}(j : \sigma, \tau) \cdot p_{\sigma \cup j} \cdot p_{\tau \backslash j} = 0$$

where $|\sigma| = d - 1$ and $|\tau| = d + 1$ and $\sigma, \tau \subset [m]$, $\sigma \not\subset \tau$, and j runs over elements of τ.

Proposition 2.3.3. *([17], Proposition 2.2.10) The Plücker ideal is generated by the Plücker relations:*

$$I_{k,n} = \langle \mathcal{P}_{I,J} \mid I, J \subseteq [n], |I| = k - 1, |J| = k + 1 \rangle.$$

The Grassmannian, $\mathrm{Gr}(k,n)$*, is the subvariety of* $\mathbb{P}^{\binom{n}{k}-1}$ *defined by this ideal.*

2.3.4 The Dressian and the tropical Grassmannian

We begin this section by introducing the Dressian, which can be viewed tropically as analogous to the Grassmannian in that it parametrises tropical linear spaces. The Dressian is the tropical prevariety obtained by intersecting hypersurfaces given by the tropicalisation of Plücker relations, and this in general does not coincide with the tropical Grassmannian, that being the tropicalisation of the Grassmannian which we also introduce in this section.

Definition 60. The *Dressian* $\mathrm{Dr}(d,n)$ is the set of all valuated matroids of rank d on a set of size n.

Definition 61. Fix $\sigma, \tau \subseteq [n]$, with $|\sigma| = d-1, |\tau| = d+1$. Then the *tropical Plücker relations* of $\mathrm{Dr}(d,n)$ are given by

$$\bigoplus_j \rho(\sigma \cup j) \odot \rho(\tau \backslash j)$$

where j runs over indices in τ.

Remark 22. In the tropical Plücker relations given above we allow for $\rho(\omega)$ to take infinite values as well as finite values. By considering the intersections of these tropical hypersurfaces we get a subset of tropical projective space, and this is exactly $\mathrm{Dr}(n,d)$ as defined in Definition 60.

Proposition 2.3.4. *The set of valuated matroids* $\mathrm{Dr}(d,n)$ *is a tropical projective prevariety.*

We now introduce the tropical Grassmannian, this being the tropicalisation of the Grasssmanian. Recall from Definition 59 that the Plücker relations are given by

$$\mathcal{P}_{\sigma,\tau} = \sum_j \mathrm{sgn}(j : \sigma, \tau) \cdot p_{\sigma \cup j} \cdot p_{\tau \backslash j} = 0$$

where $|\sigma| = d-1$ and $|\tau| = d+1$ and $\sigma, \tau \subset [n]$, $\sigma \not\subset \tau$, and j runs over elements of τ, and that the Plücker ideal is generated by the Plücker relations is given by

$$I_{d,n} = \langle \mathcal{P}_{I,J} \mid I, J \subseteq [n], |I| = d-1, |J| = d+1 \rangle.$$

Definition 62. The tropicalisation of $V(I_{d,n})$ is the *tropical Grassmannian*, which we write $\operatorname{trop}\operatorname{Gr}(d,n) := \operatorname{trop}(I_{d,n})$.

2.3.4.1 Stratification of the Dressian and the tropical Grassmannian

We now follow Maclagan-Sturmfels [17] who introduce both the tropical Grassmannian as well as the Dressian through the Dressian of a matroid and the tropical Grassmannian of a matroid. These are both stratifications of our definitions of the Dressian and tropical Grassmannian on which we comment more on in Remark 23. We now proceed to introduce the tropical Grassmannian of a matroid, as well as the Dressian of a matroid. Before we introduce both of these we relate ourselves back to matroids which we introduced in Section 2.2.1. Let $M = (E, \mathcal{B})$ be a matroid of rank d on $\{1, \ldots, n\}$. For any $B \in \mathcal{B}(M)$ we introduce a variable $\rho(B)$. The resulting polynomial ring over a field K is $K[\boldsymbol{\rho}] := K[\rho(B) \mid B \text{ is a basis of } M]$. The ideal in $K[\boldsymbol{\rho}]$ which is obtained from the Plücker ideal of Proposition 2.3.3 by setting all variables not indexing a basis to zero is

$$I_M := (I_{d,n} + \langle \rho(B) \mid B \text{ is not a basis of } M \rangle) \cap K[\boldsymbol{\rho}].$$

The quadratic Plücker relations which generate I_M are

$$\sum_j \operatorname{sgn}(j : \sigma, \tau) \cdot \rho(\sigma \cup j) \cdot \rho(\tau \backslash j) = 0 \tag{2.1}$$

where $\sigma, \tau \subseteq [n], |\sigma| = d-1$, σ is independent in M, $|\tau| = d+1$ and τ contains a basis of M, and the sum is over j such that both $\sigma \cup j$ and $\tau \backslash j$ are bases of M.

We call $V(I_M)$ the *realisation space* of the matroid M.

Definition 63. [17] The tropicalisation of the realisation space is called the *tropical Grassmannian* of M, which we write $\operatorname{trop}\operatorname{Gr}_M := \operatorname{trop}(I_M)$.

Definition 64. [17] Fix $\sigma, \tau \subset [n]$, with $|\sigma| = d-1$, σ independent in M, $|\tau| = d+1$, $\sigma \not\subset \tau$, and τ contains a basis of M. The tropicalisation of the Plücker relations given in Equation (2.1) is given by

$$\bigoplus_j \rho(\sigma \cup j) \odot \rho(\tau \backslash j) \tag{2.2}$$

where j runs over the elements in τ such that $\sigma \cup j$ and $\tau \backslash j$ are bases of M.

Each of the relations given by Equation (2.2) defines a tropical hypersurface in $\mathbb{R}^{|\mathcal{B}|}/\mathbb{R}1$, this being an instance of a tropical projective space. The intersection of these hypersurfaces is a tropical prevariety denoted by Dr_M and called the *Dressian* of the matroid M.

Remark 23. We note that $\operatorname{trop}\operatorname{Gr}_M \subseteq \operatorname{Dr}_M$ for all matroids M, and that the former depends on the underlying field K whereas the Dressian doesn't. We note that the points of Dr_M are precisely valuated matroids, and in particular, $\operatorname{Dr}(d,n)$ can be partitioned into different spaces Dr_M depending on which sets of coordinates are zero in the field, and hence tropically infinite. Recalling that $\mathbb{TP}^n$ is topologically a simplex, we note that $\operatorname{Dr}(d,n)$ is stratified into Dr_M by faces of the simplex, with the faces being determined by which coordinates are infinity. The fact that the Dressian arises from tropical Plücker vectors, leads to the use of the name *tropical Plücker vector* for a valuated matroid. We use both of these names throughout this thesis.

2.3.5 Tropical linear spaces and combinatorial connection to matroids

Definition 65. [17] For each point ω in the Dressian $\operatorname{Dr}(n,d)$ (that is, each valuated matroid ω) we construct a tropical linear space L_w as follows. Take $\tau \subseteq [n]$ where $|\tau| = d+1$ and $\operatorname{rank}(\tau) = d$, and letting $L_\tau(\omega)$ denote the tropical hyperplane in $\mathbb{TP}^n$

defined by

$$\bigoplus_{j\in\tau} \omega_{\tau\setminus j} \odot u_j = \min_{j\in\tau}(\omega_{\tau\setminus j} + u_j).$$

Our linear space L_ω is defined as the intersection of these tropical hyperplanes. That is, $L_\omega := \bigcap_\tau L_\tau(\omega)$.

A *tropical linear space* in $\mathbb{TP}^n$ is any prevariety of the form L_w where w is in $\mathrm{Dr}(d,n)$.

Example 8. We concretely construct the tropical linear space defined by a point of the Dressian. Let $\mathcal{M} \in \mathrm{Dr}(2,4)$ be where $\rho_\mathcal{M} = (0,3,6,2,5,\infty)$ as in Example 4. Now we use Definition 65 in order to determine $L_\mathcal{M}$. So we have that $L_\mathcal{M}$ is the intersection of the following four tropical hyperplanes

$$\min(u_1 + \rho_\mathcal{M}(23), u_2 + \rho_\mathcal{M}(13), u_3 + \rho_\mathcal{M}(12)) = \min(u_1 + 2, u_2 + 3, u_3)$$
$$\min(u_1 + \rho_\mathcal{M}(24), u_2 + \rho_\mathcal{M}(14), u_4 + \rho_\mathcal{M}(12)) = \min(u_1 + 5, u_2 + 6, u_4)$$
$$\min(u_1 + \rho_\mathcal{M}(34), u_3 + \rho_\mathcal{M}(14), u_4 + \rho_\mathcal{M}(13)) = \min(u_1 + \infty, u_3 + 6, u_4 + 3)$$
$$\min(u_2 + \rho_\mathcal{M}(34), u_3 + \rho_\mathcal{M}(24), u_4 + \rho_\mathcal{M}(23)) = \min(u_2 + \infty, u_3 + 5, u_4 + 2)$$

By careful examination of the intersection of these tropical hyperplanes the tropical linear space is given by

$$(0,-1,2,5) + e_{u_1} \cup (0,-1,2,5) + e_{u_2} \cup (0,-1,2,5) + e_{u_{34}}$$

We recall that this is in the tropical projective space whereby elements are partitioned into equivalence classes by the equivalence relation we introduced.

We now give some combinatorial ways to recognise tropical linear spaces as well as valuated matroids.

Definition 66. [17] A subdivision of a matroid polytope $P(M)$ is a *matroid subdivision* if every cell of the subdivision is a matroid polytope.

Lemma 2.3.5. *[17] Let $M = ([n], \mathcal{B})$ be a matroid and $w \in \mathbb{R}^{|\mathcal{B}|}/\mathbb{R}1$. Then w lies in* *the Dressian if and only if w induces a matroid subdivision on the matroid basis polytope $P(M)$ associated with the matroid M.*

Lemma 2.3.6. *[17] The tropical linear space defined by a point $w \in \mathrm{Dr}(d,n)$ is given by*

$$L_w = \{u \in \mathbb{TP}^n \mid \mathrm{in}_u(w) \textit{ has no loops}\}.$$

Remark 24. If we consider the tropical linear space of a non-valuated matroid M then Lemma 2.3.6 gives us a way to view a tropical linear space as dual to the matroid polytope $P(M)$, and in particular, it consists of certain well-chosen cones in the normal fan of $P(M)$ [22].

Lemma 2.3.7. *([18], Lemma 4.1.11) Let $\mathcal{M}$ be a valuated matroid of rank s on E. The tropical linear space associated with the restriction of $\mathcal{M}$ to X is the image of $L_{\mathcal{M}}$ under the natural projection $\mathbb{TP}^E \to \mathbb{TP}^X$.*

Definition 67. [7] Let $\mathcal{F}$ be the set of flats of a matroid M on a set E. For any chain of flats $A = (F_1, \ldots, F_d)$ where $\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_d = E$ and $F_i \in \mathcal{F}$ for all i. We define a polyhedral cone

$$\mathrm{cone}(A) := \{-\sum_{i=1}^{d-1} \lambda_i e_{F_i} \mid \lambda_1, \ldots, \lambda_{d-1} \geq 0\}.$$

Working through all chains of flats of M the corresponding cones form a fan, and the support of this fan is the *Bergman fan of M*, and is denoted $B(M)$.

Proposition 2.3.8 ([17],Proposition 4.4.2)**.** *If M is a matroid then $B(M)$ is a tropical linear space.*

Remark 25. As a consequence of $B(M)$ being a tropical linear space, and under the duality relating tropical linear spaces and matroid polytopes then by Lemma 2.3.5 we have given a valuated matroid $\mathcal{M}$ then if $\mathrm{in}_x(\mathcal{M}) = M$ then some translate of $B(M)$ agrees with the tropical linear space of $\mathcal{M}$ near x.

Proposition 2.3.9. *([8],Remark 2.4) If* $(\rho(B))_{B\in\binom{E}{s}}$ *is not a* tropical Plücker vector, *then either:*

1. *it fails a three-term tropical Plücker relation, or*

2. $\{B \mid \rho(B) \neq \infty\}$ *is not the set of bases of a matroid.*

Remark 26. Proposition 2.3.9 enables us to check if we have a tropical Plücker vector by checking a smaller list of equations than every single Plücker relation. In particular, the only relations we need to check are the smallest meaningful ones. However, a consequence of this is that we are required to ensure that ρ', where ρ' is the valauted matroid representation of part 2 of the statement, lies in Dr_M for some matroid M.

2.3.6 Tropicalised linear spaces and realisability

Definition 68. [17] A *tropicalised linear space* over K is tropical variety of the form $\mathrm{trop}(X)$, where X is a linear space in $T^n_K \cong (K^*)^{n+1}/K^*$. By this we mean that X is cut out by homogenous linear forms in $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

Proposition 2.3.10 ([17],Proposition 4.4.2)**.** *Every tropicalised linear space over* K *is a tropical linear space.*

Proposition 2.3.11. *The tropical Grassmannian parametrises the set of all tropicalised linear spaces.*

Definition 69. [23] A tropical linear space is *realisable* with respect to a given valuation if it is a tropicalisation of some linear space under that valuation. Else it is *non-realisable.*

Remark 27. [23] For matroids, a realisable tropical linear space via $K \to \{0, \infty\}$ is exactly a representable matroid over K.

Example 9. Consider the Vámos matroid which was introduced in Example 3. Using the remark above and that the Vámos matroid isn't representable gives us an example of a non-realisable tropical linear space.

Chapter 3

Valuated matroid rank function

3.1 Introduction

In this chapter we introduce a new cryptomorphic description of a valuated matroid. This cryptomorphism comes in the form of a rank function, which we construct using an analogy with non-valuated matroids, and how their rank function relates to their basis polytope.

We begin by giving the motivation and definition of the rank function for a valuated matroid. Then we progress to show in Theorem 3.2.1 that this rank function has six key properties which have parallels to the three conditions necessary to be a matroid rank function, on which we comment more in Remark 29. In Section 3.3 we show that the converse is true, that is, given a function satisfying the six conditions, and for every rescaling, then we have a valuated matroid. We require the rescaling condition since there are examples of vectors which aren't valuated matroids but which in fact satisfy the six conditions given by Theorem 3.2.1. We give an explicit example of this in Example 10. Together these give us the requisite result.

We are able to draw a relation between linear optimisation and the rank function we introduce in this chapter. Let E be a set, s be an integer, and ρ be a "cost function" where we associate a value $\rho(A)$ to each set $A \in \binom{E}{s}$. If we assume that the cost function ρ is a valuated matroid, and we consider s to be the rank of this valuated matroid, then we are able to view the process which we are proposing for the rank function to be something akin to linear optimisation. The sum we want to maximise is the following sum for A; $\sum_{i \in A} v_i + c \cdot \rho(A)$. The fact that we consider only non-positive c gives credence to using the word loss. We note that if $v_i = 1$ for all i then it exactly coincides with our proposed candidate rank function, else we are considering a rescaling of that. This is an easy optimisation problem when ρ is a valuated matroid due to a variant of the greedy algorithm [24].

3.2 Alternative characterisation

3.2.1 Introducing our candidate

We recall the relevant parts of the introduction to the thesis. The matroid rank function of a matroid M can be recovered from its basis polytope $P(M)$ by noting

$$r(A) = \max_{B \in \mathcal{B}} \{|A \cap B|\} = \max_{v \in P(M)} \langle v, e_A \rangle. \tag{3.1}$$

This is since if $P \in \mathbb{R}^n$ is a polytope, then P is determined by $(\mathbb{R}^n)^* \to \mathbb{R}$ where $f \mapsto \max_{x \in P} f(X)$. This function is called the *support function.* In particular, if P is a matroid polytope we need not consider $(\mathbb{R}^n)^* \to \mathbb{R}$ since it suffices to consider $(\{0,1\}^n)^* \to \mathbb{R}$ in order to determine the polytope. Further to this, we recall from the introduction that the maximum on the right hand side of (3.1) is attained at a vertex v, so we need only consider the vertices of the polytope.

Given a valuated matroid ρ and its associated valuated matroid basis polytope $P(\rho)$ then by how the matroid basis polytope and the matroid rank function are related we can produce a candidate rank function for a valuated matroid. Recall from the preliminaries the main difference between a matroid basis polytope and a valuated matroid basis polytope is in the valuated case that for each basis we have an extra coordinate representing its valuation. Define $PLF(A, B)$ to be the set of piecewise linear functions mapping from A to B, and let $r^\rho : 2^E \to PLF(\mathbb{R}_{\leq 0}, \mathbb{R})$ be defined by:

$$r^\rho(A)(c) = \max_{(v,h)\in P(\rho)} \langle (e_A, c), (v, h)\rangle = \max_{B\in\mathcal{B}} \{|A \cap B| + c\rho(B)\}.$$

We consider only non-positive values of c since the maximum of the terms which make up $r^\rho(A)(c)$ would not always be a finite real number if $c > 0$, since h can take arbitrarily large values.

As we shall see Theorem 3.2.1 gives us six properties for our newly defined candidate rank function for a valuated matroid. We are able to draw parallels between this new construction for valuated matroids and the rank function for non-valuated matroids in Remark 29.

Remark 28. It can be checked that $(PLF(\mathbb{R}_{\leq 0}, \mathbb{R} \cup \{-\infty\}), +, \cdot)$ with $+$ defined by pointwise maximum, and the $\cdot$ operation being defined by pointwise addition forms a semiring, where the respective identity elements are the functions $-\infty$ and 0.

This is noteworthy since if we only consider functions of $PLF(\mathbb{R}_{\leq 0}, \mathbb{R})$ which are piecewise linear convex whose piece have non-negative integer y-intercepts, similar to those we find in Theorem 3.2.1, then it is suggestive that they might be related to a semiring of tropical polynomials in one variable. For a tropical polynomial all the pieces have non-negative integer slope but can have an y-intercept, so there can be expected to be some sort of isomorphism somehow exchanging the slope and the y-intercept between these semirings, or at least related structures to these.

3.2.2 Properties of our candidate

We now introduce a key theorem, whereby we see six properties which our newly defined candidate rank function r^ρ satisfies when it has a valuated matroid as its input. Later we see that given these properties as well as an extra seventh condition, then it necessarily comes from a valuated matroid.

Theorem 3.2.1. *Let $r^\rho : 2^E \to PLF(\mathbb{R}_{\leq 0}, \mathbb{R})$ be defined by $r^\rho(A)(c) = \max_{B \in \mathcal{B}}\{|A \cap B| + c\rho(B)\}$. If ρ is a valuated matroid then the following six conditions hold:*

1. *for all $A \subseteq E$, $r^\rho(A)$ is convex, and continuous.*

2. *for all $A \subseteq E$, each linear piece of $r^\rho(A)$ takes an integer value at 0.*

3. *$r^\rho(\emptyset)$ is linear and $r^\rho(\emptyset)(0) = 0$.*

4. *$r^\rho(E)$ is linear.*

5. *$r^\rho(A)(c)$ is increasing in A, and $r(A \cup b)(c) \leq r(A)(c) + 1$ for all $b \in E$.*

6. *$r^\rho(A)(c)$ is submodular in A.*

Remark 29. These six conditions allow us to explicitly draw parallels with the matroid rank function. Matroid rank functions are submodular, nondecreasing, and don't increase too quickly. These six conditions describe submodular set functions which are nondecreasing with values in $PLF(\mathbb{R}_{\leq 0}, \mathbb{R})$, and if we put a suitable partial order on that set, it also has values at singletons bounded by the constant function 1. The partial order we impose on $PLF(\mathbb{R}_{\leq 0}, \mathbb{R})$ is the pointwise ordering of functions. Furthermore, these six conditions give us two matroids which we see in Theorem 3.3.2, and from this we see that we are able to recover the usual matroid rank function axioms from these six conditions.

In the valuated case we have that $r(\emptyset)$ is linear and $r(\emptyset)(0) = 0$, and this captures the same idea as $r(\emptyset) = 0$ in the non-valuated case. Another property of our candidate is that it is trying to capture the idea that the codomain is the integers as opposed to the reals, say. We will refer throughout the rest of the chapter to these conditions as the six conditions of Theorem 3.2.1 or simply just the six conditions.

Proof of Theorem 3.2.1. We split this proof into six parts based on which of the conditions of Theorem 3.2.1 we are proving and we work in sequential order.

1. Let $A \subseteq E$ be fixed, and $\theta \in [0,1]$.

$$\begin{aligned}
r^\rho(A)(\theta x + (1-\theta)y) &= \max_{B\in\mathcal{B}}\{|A\cap B| + (\theta x + (1-\theta)y)\rho(B)\} \\
&= \max_{B\in\mathcal{B}}\{|A\cap B| + \theta x\rho(B) + (1-\theta)y\rho(B)\} \\
&= \max_{B\in\mathcal{B}}\{\theta|A\cap B| + \theta x\rho(B) + (1-\theta)|A\cap B| + (1-\theta)y\rho(B)\} \\
&\leq \max_{B\in\mathcal{B}}\{\theta|A\cap B| + \theta x\rho(B)\} + \max_{B\in\mathcal{B}}\{(1-\theta)|A\cap B| + (1-\theta)y\rho(B)\} \\
&= \theta\max_{B\in\mathcal{B}}\{|A\cap B| + x\rho(B)\} + (1-\theta)\max_{B\in\mathcal{B}}\{|A\cap B| + y\rho(B)\} \\
&= \theta r^\rho(A)(x) + (1-\theta)r^\rho(B)(y).
\end{aligned}$$

So for a given $A \subseteq E$ $r^\rho(A)$ is convex. Continuity of $r^\rho(A)$ follows from continuity of the max function.

2. Let $A \subseteq E$ be fixed. Each linear piece can be expressed in the following form for a given $B \in \mathcal{B}$

$$|A\cap B| + c\rho(B).$$

Firstly, this is linear. At $c = 0$ we have

$$|A\cap B| + 0\cdot\rho(B) = |A\cap B| \in \mathbb{Z} \quad \forall B \in \mathcal{B}.$$

So we have that each linear piece of $r^\rho(A)$ takes an integer value at 0.

3. We have

$$\begin{aligned}
r^\rho(\emptyset)(c) &= \max_{B\in\mathcal{B}}\{|\emptyset\cap B| + c\cdot\rho(B)\} \\
&= \max_{B\in\mathcal{B}}\{c\cdot\rho(B)\} \\
&= c\cdot\rho(B_0) \quad \text{for some } B_0 \in \mathcal{B}
\end{aligned}$$

and so $r^\rho(\emptyset)$ is linear, and in particular $r^\rho(\emptyset)(0) = 0$.

4. Similarly, we have

$$\begin{aligned} r^\rho(E)(c) &= \max_{B\in\mathcal{B}}\{|E\cap B| + c\cdot\rho(B)\} \\ &= \max_{B\in\mathcal{B}}\{|B| + c\cdot\rho(B)\} \\ &= |B_0| + c\cdot\rho(B_0) \quad \text{for some } B_0\in\mathcal{B} \end{aligned}$$

and so $r^\rho(E)$ is linear.

5. Let $A\subseteq A'\subseteq E$, then for each $B\in\mathcal{B}$ we have

$$|A\cap B| + c\cdot\rho(B) \leq |A'\cap B| + c\cdot\rho(B).$$

Thus $r^\rho(A)$ is increasing in A. Similarly for any given $B\in\mathcal{B}$ we have

$$|(A\cup b)\cap B| + c\cdot\rho(B) \leq |A\cap B| + c\cdot\rho(B) + 1.$$

Thus we have that $r^\rho(A\cup b)\leq r^\rho(A)+1$.

6. We use the local version of submodularity given in Definition 23. We need to show for any $A\subseteq E, a_1, a_2\in E$ that

$$r^\rho(A\cup a_1) + r^\rho(A\cup a_2) \geq r^\rho(A) + r^\rho(A\cup\{a_1,a_2\}).$$

So let $A\subseteq E$ and $a_1,a_2\in E$ be fixed. Firstly, for a given c, suppose that B maximises $|A\cap B| + c\cdot\rho(B) = r^\rho(A)(c)$, and B' maximises $|(A\cup\{a_1,a_2\})\cap B'| + c\cdot\rho(B') = r^\rho(A\cup\{a_1,a_2\})(c)$, noting that if there is more than one maximiser for either we just choose a single one. We now split into cases based on whether a_1,a_2 are contained in B and B'.

Firstly, suppose that $a_1,a_2\notin B'$. Then we have

$$|(A\cup\{a_1,a_2\})\cap B'| + c\cdot\rho(B') = |A\cap B'| + c\cdot\rho(B') \leq |A\cap B| + c\cdot\rho(B).$$

Since r^ρ satisfies condition 5, that is, it is an increasing function we have $r^\rho((A \cup \{a_1, a_2\})(c) = r^\rho(A)(c)$. We also have that $r^\rho(A \cup a_1)(c), r^\rho(A \cup a_2)(c) = r^\rho(A)(c)$ since r^ρ is an increasing function, and thus we have

$$r^\rho(A \cup a_1)(c) + r^\rho(A \cup a_2)(c) \geq r^\rho(A)(c) + r^\rho(A \cup \{a_1, a_2\})(c).$$

Now suppose that $a_1 \in B', a_2 \notin B'$. Then we have

$$|(A \cup \{a_1, a_2\}) \cap B'| + c \cdot \rho(B') = |(A \cup a_1) \cap B'| + c \cdot \rho(B').$$

So $r^\rho(A \cup \{a_1, a_2\})(c) = r^\rho(A \cup a_1)(c)$, and since r is an increasing function we also have that $r^\rho(A)(c) \leq r^\rho(A \cup \{a_2\})(c)$ and hence

$$r^\rho(A \cup a_1)(c) + r^\rho(A \cup a_2)(c) \geq r^\rho(A)(c) + r^\rho(A \cup \{a_1, a_2\})(c).$$

The same argument as above works for $a_1 \notin B', a_2 \in B'$.

Now we consider the case where $a_1, a_2 \in B'$. We split this into four cases

1. $a_1 \in B, a_2 \in B$

2. $a_1 \in B, a_2 \notin B$

3. $a_1 \notin B, a_2 \in B$

4. $a_1 \notin B, a_2 \notin B$

In Case (1) we have $1 + r^\rho(A)(c) = r^\rho(A \cup a_1)(c)$ and $1 + r^\rho(A)(c) = r^\rho(A \cup a_2)(c)$. We also have $r^\rho(A \cup \{a_1, a_2\})(c) \leq r^\rho(A \cup a_1)(c) + 1, r^\rho(A \cup a_2)(c) + 1$, and so

$$r^\rho(A \cup a_1)(c) + r^\rho(A \cup a_2)(c) \geq r^\rho(A)(c) + r^\rho(A \cup \{a_1, a_2\})(c).$$

In case (2) we have $1 + r^\rho(A)(c) = r^\rho(A \cup a_1)(c)$, and $r^\rho(A \cup \{a_1, a_2\})(c) \leq r^\rho(A \cup a_2)(c) + 1$. So we have

$$r^\rho(A \cup a_1)(c) + r^\rho(A \cup a_2)(c) + 1 \geq 1 + r^\rho(A)(c) + r^\rho(A \cup \{a_1, a_2\})(c)$$

and so

$$r^\rho(A \cup a_1)(c) + r^\rho(A \cup a_2)(c) \geq r^\rho(A)(c) + r^\rho(A \cup \{a_1, a_2\})(c).$$

Case (3) uses exactly the same argument as case (2).

Now consider case (4). We use the basis valuation exchange axiom in order to show the required submodular relation. Take $a_1 \in B' \backslash B$, then there exists $a_3 \in B \backslash B'$ such that

$$\rho(B) + \rho(B') \geq \rho((B \cup a_1) \backslash a_3) + \rho((B' \cup a_3) \backslash a_1).$$

Since $c \leq 0$ we have $c \cdot \rho(B) + c \cdot \rho(B') \leq c \cdot \rho((B \cup a_1) \backslash a_3) + c \cdot \rho((B' \cup a_3) \backslash a_1)$. We now proceed to show

$$|A \cap B| + |(A \cup \{a_1, a_2\}) \cap B'| \leq |(A \cup a_1) \cap ((B \cup a_1) \backslash a_3)| + |(A \cup a_2) \cap ((B' \cup a_3) \backslash a_1)|.$$

Firstly, we have

$$|(A \cup a_1) \cap ((B \cup a_1) \backslash a_3)| = |A \cap ((B \cup a_1) \backslash a_3)| + 1 = |A \cap (B \backslash a_3)| + 1.$$

Now consider if $a_3 \in A$ or $a_3 \notin A$. If $a_3 \in A$ then

$$|(A \cup a_1) \cap ((B \cup a_1) \backslash a_3)| = |A \cap B| - 1 + 1 = |A \cap B|.$$

If $a_3 \notin A$.

$$|(A \cup a_1) \cap ((B \cup a_1) \backslash a_3)| = |A \cap B| + 1.$$

Now consider

$$|(A \cup a_2) \cap ((B' \cup a_3) \backslash a_1)| = |(A \cup a_2) \cap (B' \cup a_3)|.$$

If $a_3 \in A$

$$|(A \cup a_2) \cap ((B' \cup a_3) \backslash a_1)| = |(A \cup a_2) \cap B'| + 1.$$

If $a_3 \notin A$

$$|(A \cup a_2) \cap ((B' \cup a_3) \backslash a_1)| = |(A \cup a_2) \cap B'|.$$

So if $a_3 \in A$ we have

$$\begin{aligned} |(A \cup a_1) \cap ((B \cup a_1) \backslash a_3)| + |(A \cup a_2) \cap ((B' \cup a_3) \backslash a_1)| &= |A \cap B| + |(A \cup a_2) \cap B'| + 1 \\ &\geq |A \cap B| + |(A \cup \{a_1, a_2\}) \cap B'| \end{aligned}$$

and if $a_3 \notin A$ we have

$$\begin{aligned} |(A \cup a_1) \cap ((B \cup a_1) \backslash a_3)| + |(A \cup a_2) \cap ((B' \cup a_3) \backslash a_1)| &= |A \cap B| + 1 + |(A \cup a_2) \cap B'| \\ &\geq |A \cap B| + |(A \cup \{a_1, a_2\}) \cap B'|. \end{aligned}$$

Thus we have

$$\begin{aligned} |(A \cup a_1) \cap ((B \cup a_1) \backslash a_3)| + c \cdot \rho((B \cup a_1) \backslash a_3) + |(A \cup a_2) \cap ((B' \cup a_3) \backslash a_1)| \\ + c \cdot \rho((B' \cup a_3) \backslash a_1) \\ \geq |A \cap B| + c \cdot \rho(B) + |(A \cup \{a_1, a_2\}) \cap B'| + c \cdot \rho(B'). \end{aligned}$$

Thus we have

$$r^\rho(A \cup a_1)(c) + r^\rho(A \cup a_2)(c) \geq r^\rho(A)(c) + r^\rho(A \cup \{a_1, a_2\})(c).$$

So in all cases we have shown submodularity. □

Therefore by combining Theorem 3.2.1 and Remark 15 about the fact that rescalings of valuated matroids are still valuated matroids we have the following result.

Theorem 3.2.2. *Let $r^\rho : 2^E \to PLF(\mathbb{R}_{\leq 0}, \mathbb{R})$ be defined by $r^\rho(A)(c) = \max_{B \in \mathcal{B}}\{|A \cap B| + c \cdot \rho(B)\}$. If $\rho \subseteq (\mathbb{R} \cup \{\infty\})^{\binom{E}{s}}$ is a valuated matroid then for all $\boldsymbol{x} = (x_a : a \in E)$, we have that $r^{\rho^{\boldsymbol{x}}}(A)(c) := \max_{B \in \mathcal{B}}\{|A \cap B| + c \cdot \rho^{\boldsymbol{x}}(B)\}$ satisfies the six conditions of Theorem 3.2.1.*

3.3 Converse

We show that the converse of Theorem 3.2.2 is true. This gives us exactly the result we need, that is, we have a cryptomorphism for valuated matroids in terms of a rank function. Firstly we prove a useful result.

Theorem 3.3.1. *Given r which satisfies the six conditions of Theorem 3.2.1 and letting $s = r(E)(0)$, then there is a vector $\rho \in (\mathbb{R} \cup \{\infty\})^{\binom{E}{s}}$ such that*

$$r(A)(c) = \max_i \{i + c\rho(B) \mid B \in \binom{E}{s}, |A \cap B| \geq i\}.$$

Before we prove this result we introduce some results which are useful for the proof of Theorem 3.3.1.

3.3.1 Preliminaries for proof of Theorem 3.3.1

Theorem 3.3.2. *Given r which satisfies the six conditions of Theorem 3.2.1 then we have a matroid M_0 with rank function r_0 defined for all $A \subseteq E$ by $r_0(A) = r(A)(0)$, and also a matroid M_∞ with rank function r_∞ defined for all $A \subseteq E$ by $r_\infty(A) := \lim_{c \to -\infty} r(A)(c) - r(\emptyset)(c)$.*

Proof. We first show we have a matroid at $c = 0$ which we do by showing the conditions of Theorem 2.2.2 are satisfied. Note that $r(A)(0)$ is non-negative integer valued, since by condition 3 $r(\emptyset)(0) = 0$, by condition 5 r is increasing, and by condition 2 we get that it is integer valued. For any $A \subseteq E$ we have $0 \leq r(A)(0) \leq |A|$, this is since $r(\emptyset)(0) = 0$ and $r(A)(c) \leq r(A \cup a)(c) \leq r(A)(c) + 1$. We also have if $A \subseteq D \subseteq E$ then

$r(A)(0) \leq r(D)(0)$, and submodularity comes since condition 6 of the six conditions gives pointwise submodularity , and in particular, holds at $c = 0$.

For c sufficiently negative and for any given $A \subseteq E$, we have that $r(A)$ is linear and parallel to both $r(E)$ and $r(\emptyset)$. The reason for this is because by observing that for each successive linear segment of $r(A)(c)$ as we take more negative values of c that the value of the segments at $c = 0$ are decreasing for each of these successive linear segments, but the value of any linear segment at $c = 0$ is always non-negative so can only change finitely often and therefore terminates. We can see that it is always non-negative since we have that the linear segment for $r(\emptyset)$ passes through $c = 0$ at 0, and if for any A the value of any linear segment of $r(A)$ takes a negative value at $c = 0$ then by condition 2 and the second part of condition 5 we would have a violation. So define T for all $A \subseteq E$ by $T(A) := \lim_{c \to -\infty} r(A)(c) - r(\emptyset)(c)$. We claim that T is a matroid rank function.

Firstly, $T(\emptyset) = 0$, and so by using $r(A) \leq r(A \cup a) \leq r(A) + 1$, we also have for any $A \subseteq E$ that $0 \leq T(A) \leq |A|$. Note also T is integer valued since for all sufficiently small c, $r(A)(c)$ and $r(\emptyset)(c)$ are parallel and that by condition 2 of the six conditions that each linear piece takes an integer value. As before we can simply read off the conditions that if $A \subseteq D$ then $T(A) \leq T(D)$ and that T is submodular in A. □

Remark 30. The matroids M_0 and M_∞ are the underlying and the initial matroid respectively associated with r^ρ when ρ is a valuated matroid. We note that both M_0 and M_∞ have the same rank, which we call s.

We now introduce a description of the gradients of the linear segments of a given $r(A)$. Conditions (1) and (2) of the six conditions of Theorem 3.2.1 give us that $r(A)$ is the maximum of some given linear functions, say, $f_0(A)$ with constant term s, $f_1(A)$ with constant term $s-1$, $f_2(A)$ with constant term $s-2$ and so on. The other conditions give that we only require a finite number of these linear functions in order to determine $r(A)$. This process can cease at $f_k(A)$ where k is the corank of A in the matroid M_∞, as defined in Theorem 3.3.2, which we denote as $\mathrm{corank}_\infty(A) = k$. Just as in the proof of Theorem 3.3.2 this process terminates or else we reach a contradiction. Similarly, we will denote the nullity of a set A in the matroid M_∞ as $\text{nullity}_\infty(A)$. Furthermore, each $f_i(A)$ is uniquely determined if we demand that $r(A)$ agree with each $f_i(A)$ at at least one point. We then define $S_i(A)$ to be the gradient of $f_i(A)$ for $i \leq k$, and when $i > k$ we say $S_i(A) = S_k(A)$.

Given r which satisfies the six conditions of Theorem 3.2.1 we are able to interpret some of the six conditions in terms of gradients of linear segments.

Lemma 3.3.3. *Given r which satisfies the six conditions of Theorem 3.2.1 then for any $A \subseteq E, a, b \in E$ we are able to express conditions* 5 *and* 6 *using the gradients of linear segments $S_i(A)$.*

Conditions 5 (increasing properties) give us

$$r(A) \leq r(A \cup a) \iff S_k(A \cup a) \leq S_k(A)\ \forall k$$
$$r(A \cup a) \leq r(A) + 1 \iff S_k(A) \leq S_{k-1}(A \cup a)\ \forall k$$

Condition 6 (submodularity) gives us

$$r(A \cup a) + r(A \cup b) \geq r(A) + r(A \cup a \cup b) \iff$$
$$\forall j : \min_{i_1+i_2=j} S_{i_1}(A \cup a) + S_{i_2}(A \cup b) \leq \min_{i_1+i_2=j} S_{i_1}(A) + S_{i_2}(A \cup a \cup b)$$

Proof. We firstly show we have $r(A) \leq r(A \cup a) \iff S_k(A \cup a) \leq S_k(A)\ \forall k$. Let $r(A) \leq r(A \cup a)$ and assume for contradiction that we have $S_k(A \cup a) > S_k(A)$. This means that we have $f_k(A \cup a)(c) < f_k(A)(c)\ \forall c < 0$. Since for some c_0 $f_k(A \cup a)(c_0) = r(A \cup a)(c_0)$ we then have

$$r(A \cup a)(c_0) = f_k(A \cup a)(c_0) < f_k(A)(c_0) \leq r(A)(c_0)$$

and so we have a contradiction, and so $S_k(A \cup a) \leq S_k(A)$.

Now we show the other implication. Let $S_k(A \cup a) \leq S_k(A)$ and assume for contradiction that $r(A) > r(A \cup a)$. Then there exists c_0 such that $f_k(A)(c_0) = r(A)(c_0)$, and thus we have

$$f_k(A)(c_0) = r(A)(c_0) > r(A \cup a)(c_0) \geq f_k(A \cup a)(c_0).$$

Thus $f_k(A)(c_0) > f_k(A \cup a)(c_0)$, and so $S_k(A) < S_k(A \cup a)$ and hence a contradiction.

Now we wish to show $r(A \cup a) \leq r(A) + 1 \iff S_k(A) \leq S_{k-1}(A \cup a)\ \forall k$. Let $r(A \cup a) \leq r(A) + 1$ and assume for contradiction that $S_k(A) > S_{k-1}(A \cup a)$, and thus for all $c < 0$ we have $f_k(A)(c) + 1 < f_{k-1}(A \cup a)(c)$. At some c_0 we have the property that $f_{k-1}(A \cup a)(c_0) = f_k(A \cup a)(c_0)$, and thus

$$r(A \cup a)(c_0) = f_k(A \cup a)(c_0) = f_{k-1}(A \cup a)(c_0) > f_k(A)(c_0) + 1 \geq r(A)(c_0) + 1.$$

Hence we have $r(A \cup a)(c_0) > r(A)(c_0) + 1$ which is a contradiction so we have $S_k(A) \leq S_{k-1}(A \cup a)$.

Now let $S_k(A) \leq S_{k-1}(A \cup a)$ and assume for contradiction that $r(A \cup a) > r(A) + 1$. Then for some c_0 we have $r(A \cup a)(c_0) = f_k(A \cup a)(c_0) = f_{k-1}(A \cup a)(c_0)$, and thus

$$f_k(A \cup a)(c_0) = f_{k-1}(A \cup a)(c_0) = r(A \cup a)(c_0) > r(A)(c_0) + 1 \geq f_k(A)(c_0) + 1.$$

So $f_{k-1}(A \cup a)(c_0) > f_k(A)(c_0) + 1$, and thus $S_{k-1}(A \cup a) < S_k(A)$, and therefore we have a contradiction.

We now wish to show the new submodularity description. We have

$$\begin{aligned}
r(A)(c) + r(A')(c) &= \max_{i \in \mathbb{N}, i \leq \text{corank}_\infty(A)} (S_i(A)c + i) + \max_{i \in \mathbb{N}, i \leq \text{corank}_\infty(A')} (S_i(A')c + i) \\
&= \max_{i_1, i_2 \in \mathbb{N}, i_1 \leq \text{corank}_\infty(A), i_2 \leq \text{corank}_\infty(A')} (S_{i_1}(A)c + i_1 + S_{i_2}(A')c + i_2) \\
&= \max_{i_1, i_2 \in \mathbb{N}, i_1 \leq \text{corank}_\infty(A), i_2 \leq \text{corank}_\infty(A')} ((S_{i_1}(A) + S_{i_2}(A'))c + i_1 + i_2) \\
&= \max_{j \in \mathbb{N}, j \leq \text{corank}_\infty(A) + \text{corank}_\infty(A')} (\min_{i_1 + i_2 = j} (S_{i_1}(A) + S_{i_2}(A'))c + j)
\end{aligned}$$

noting that we obtain a minimum in the final part since $c \leq 0$, and so by using this and $r(A) \leq r(A \cup a) \iff S_k(A \cup a) \leq S_k(A)\ \forall k$ we have

$$r(A \cup a) + r(A \cup b) \geq r(A) + r(A \cup a \cup b) \iff$$

$$\forall j : \min_{i_1 + i_2 = j} S_{i_1}(A \cup a) + S_{i_2}(A \cup b) \leq \min_{i_1 + i_2 = j} S_{i_1}(A) + S_{i_2}(A \cup a \cup b). \qquad \square$$

Lemma 3.3.4. *Given r which satisfies the six conditions of Theorem 3.2.1. If A is a flat of M_0 then there exists $i \in E \backslash A$ such that $r(A) + 1 = r(A \cup i)$.*

Proof. Let A be a flat of M_0. There are at least $s - r_\infty(A)$ different elements i such that $r_\infty(A \cup i) = r_\infty(A) + 1$. This comes from the fact that r_∞ is a matroid rank function of rank s. Choose Q of these, $i_1, \ldots, i_Q$, such that $r_\infty(A \cup i_1 \cup \cdots \cup i_Q) = s$ and where $s - Q = r_\infty(A)$. These are the set of *candidates* which we consider to be the i in the statement of the lemma. We recall that since A is a flat of M_0 that for every candidate i_j that $r_0(A \cup i_j) = r_0(A) + 1$.

We imagine increasing c from $-\infty$ to 0 and see what happens to our set of candidates $i_1, \ldots, i_Q$. If two consecutive segments of $r(A)$ agree with $f_j(A)$ and $f_k(A)$, we say that $r(A)$ *bends* $j - k$ times between those two segments, we also add that there are $\text{corank}_0(A)$ many bends at $c = 0$. In particular, $r(A)$ has $\text{corank}_\infty(A)$ bends in total. Whilst c increases over a segment of $r(A)$ with no bends, we maintain that all candidates remain as candidates as by convexity and condition 5 we have that $r(A \cup i_j)$ must be equal to $r(A) + 1$ for all remaining candidates.

Let's look at what happens at a bend of $r(A)$. In this case candidates are allowed to go straight, by which we mean that after the bend $r(A)+1 \neq r(A \cup i_j)$, but only at a rate of one per bend. Before we consider the general case assume that we only have a single bend and that two go straight, say $r(A \cup i_1)$ and $r(A \cup i_2)$. However, this can't happen since otherwise we have by submodularity of $r(A)$ that $r(A \cup i_1 \cup i_2)$ will be forced to be non-convex. This works in all cases due to the way that we've chosen $i_1, \ldots, i_Q$. Since we only have a maximum of $s - 1 - \text{corank}_\infty(A)$ bends before the bends at $c = 0$ in total, and $s - Q = \text{corank}_\infty(A)$ candidates in total, we never reach a situation whereby we run out of candidates.

Now assume more generally that $r(A)$ bends $j - k$ times between consecutive segments. We claim that at most $j - k$ candidates cease to be so. Assume we lose $j - k + 1$ candidates, say $i_1, \ldots, i_{j-k+1}$.

In a similar way to the case where there is one bend we get the following from the submodularity axiom:

$$(j - k)r(A) + r(A \cup i_1 \cup i_{j-k+1}) \leq r(A \cup i_1) + \cdots + r(A \cup i_{j-k+1}). \tag{3.2}$$

Before the bends assume $r(A)$ is on a linear segment passing through $s - j$ at $c = 0$, and afterwards is on a linear segment passing through $s - k$ at $c = 0$. Similarly, all of $r(A \cup i_1), \ldots, r(A \cup i_{j-k+1})$ lie on a linear segment passing through $s - j + 1$ before the bend, and also $r(A \cup i_1 \cup \cdots \cup i_{j-k+1})$ lies on a linear segment passing through $s - k + 1$ at $c = 0$.

In order to maximise the chances of the inequality given by Equation (3.2) being satisfied, we minimise the left hand side and maximise the right hand side, so we assume after the bend that $r(A \cup i_1), \ldots, r(A \cup i_{j-k+1})$ all lie on a linear segment passing through $s - k$ at $c = 0$. This is the maximal value, since if they lie on a linear segment passing through $s-k+1$ then they remain as candidates. In addition we assume that $r(A \cup i_1 \cup \cdots \cup i_{j-k+1})$ is on a linear segment passing through $c = 0$ at $s-k+1$. We cannot have that $r(A \cup i_1 \cup \cdots \cup i_{j-k+1})$ is on a linear segment passing through any value lower than $s-k+1$ at $c = 0$ since this we require that A is a flat of the matroid at $c = 0$.

Let $c = c_0$ be the point at which we have the bends we are considering. At this point we have $1 + r(A)(c_0) = r(A \cup i_1)(c_0), \ldots, r(A \cup i_{j-k+1})(c_0)$ and $j-k+1+r(A)(c_0) = r(A \cup i_1 \cup \cdots \cup i_{j-k+1})(c_0)$. Similarly, if we look at the linear segment that we are on immediately after the bend and where it passes through $c = 0$ we have $r(A)(0) = r(A \cup i_1)(0), \ldots, r(A \cup i_{j-k+1})(0)$ and $1 + r(A)(0) = r(A \cup i_1 \cup \cdots \cup i_{j-k+1})(0)$.

We now consider whether the submodular relation holds for these linear segments at $c = c_0$ and $c = 0$. In particular, we will use these to show that r is not submodular immediately after the bend. At $c = c_0$ we have that both sides of Equation (3.2) are equal, whereas at $c = 0$ Equation (3.2) isn't satisfied. Since these are all linear functions we have immediately after the bend that the submodular inequality isn't satisfied, and since at this point the linear segment coincides with r we have the required result since we can only lose at most one candidate per bend. □

Corollary 3.3.5. *Let r be such that it satisfies the six conditions of Theorem 3.2.1. If A is any set such that $r_0(A) < s$ then there exists $i \in E \backslash A$ such that $r(A)+1 = r(A \cup i)$.*

Proof. Let $A' := \bar{A}$, so that A' is a proper flat of the matroid at $c = 0$. So by Lemma 3.3.4 take $f \notin A'$ so that $r(A' \cup f) = 1 + r(A')$. Then by submodularity we have $r(A \cup f) \geq 1 + r(A)$ since

$$r(A') + r(A \cup f) \geq r(A' \cup f) + r(A)$$
$$r(A') + r(A \cup f) \geq r(A') + 1 + r(A)$$
$$r(A \cup f) \geq 1 + r(A),$$

and by the increasing by less than 1 property we have $r(A \cup f) \leq 1 + r(A)$, and hence

$r(A \cup f) = 1 + r(A)$. $\square$

Definition 70. Let r satisfy the six conditions of Theorem 3.2.1, and let M_∞ be the associated matroid given by Theorem 3.3.2. Then for each $1 \leq l \leq s$ we define the following

$$\mathcal{B}(M_\infty, l) := \{B \in \binom{E}{s} \mid |B \cap B_0| \geq s - l \text{ for some } B_0 \in \mathcal{B}(M_\infty)\}.$$

Remark 31. $B(M_\infty, l)$ are the bases of a matroid since this is an l-truncation of an l-cotruncation of M_∞, and these operations, dual and truncation, commute with one and other.

Lemma 3.3.6. *Let r satisfy the six conditions of Theorem 3.2.1, and let $A \subseteq E$, where $A = (B_0 \cup C) \backslash D$, for $B_0 \in \mathcal{B}(M_\infty)$, $C \subseteq E \backslash B_0, |C| = \text{nullity}_\infty A = t$, $D \subseteq B_0, |D| = \text{corank}_\infty A = k$, and $k \geq t$. Then there exists $a_1, \ldots, a_{k-t} \notin A$ such that $r(A \cup a_1 \cup \cdots \cup a_{k-t}) = r(A) + k - t$ and $A \cup a_1 \cup \cdots \cup a_{k-t} \in \mathcal{B}(M_\infty, t)$.*

Proof. We separate into cases depending on whether $k = t$ or $k > t$. If $k = t$, then $|A| = s$. By the definition of nullity we have

$$\begin{aligned} |A| - r_\infty(A) &= t \\ \implies s - r_\infty(A) &= t \\ \implies s - t &= r_\infty(A) \end{aligned}$$

and so $A \in \mathcal{B}(M_\infty, t)$.

Now consider $k > t$. Then $|A| = s + t - k$. By using Corollary 3.3.5 we have

$$\exists\, a_1 \notin A \text{ s.t } r(A \cup a_1) = r(A) + 1$$
$$\exists\, a_2 \notin A \cup a_1 \text{ s.t } r(A \cup a_1 \cup a_2) = r(A \cup a_1) + 1$$
$$\vdots$$
$$\exists\, a_{k-t} \notin A \cup a_1 \cup \cdots \cup a_{k-t-1} \text{ s.t } r(A \cup a_1 \cup \cdots \cup a_{k-t-1}) = r(A \cup a_1 \cup \cdots \cup a_{k-t-2}) + 1.$$

So we have $r(A \cup a_1 \cup \cdots \cup a_{k-t-1}) = r(A) + k - t$.

Since $\text{nullity}_\infty(A) = t$ we have

$$|A| - r_\infty(A) = t$$
$$\implies s + t - k - r_\infty(A) = t$$
$$\implies s - k = r_\infty(A)$$

So we have $r_\infty(A \cup a_1 \cup \cdots \cup a_{k-t-1}) = r_\infty(A) + k - t = s - k + k - t = s - t$, and hence $A \cup a_1 \cup \cdots \cup a_{k-t} \in \mathcal{B}(M_\infty, t)$. □

Lemma 3.3.7. *Let r be a system of functions satisfying the six conditions of Theorem 3.2.1. We are able to obtain a new system of functions r' which satisfy the six conditions of Theorem 3.2.1 in such a way to get $B_k \in \mathcal{B}(M_\infty, k)$ to be dependent at $c = 0$, for any given $B_k \in \mathcal{B}(M_\infty, k)$, and there exists c_0 such that the functions r' and r agree for all $c \leq c_0$ where c_0 is some point to the left of where $f_0(B_k)$ intersects $f_1(B_k)$.*

Remark 32. We think of this process as an augmentation of r in that we are attempting to apply a process to r in order to disrupt it in a way that is somewhat minimal, and where possible we want r and r' to resemble each other. We also note that we have some freedom as to how we chose c_0 in most cases, but for our purposes the choice of c_0 makes little difference.

Proof. Let r be a system of functions satisfying the six conditions. Let M_∞ and M_0 be the two extremal matroids as defined in Theorem 3.3.2. We define M_0^U to be the uniform matroid of the same rank, s, as M_0 on the same ground set.

Let M be any matroid such that we have a weak ordering $M_\infty \leq M \leq M_0^U$. Let $c_0 \leq 0$ be such that for all sets $A \subseteq E$ the constant term of the linear function which agrees with $r(A)$ at $c = c_0$ is at most $\text{rank}_M(A)$.

Define a new system of functions r' by

$$r'(A)(c) = \begin{cases} r(A)(c) \text{ when } c \leq c_0 \\ \text{rank}_M(A) \text{ when } c = 0 \\ \text{use linear interpolation to fill in between } c_0 \text{ and } 0. \end{cases}$$

This new system r' satisfies the six conditions. These properties easily follow. For example, convexity follows from our choice of c_0, since the function is convex for $c \leq c_0$ and $\text{rank}_M(A) \geq r(A)(c_0)$. Axioms (5) and (6) are true if $c \leq c_0$ since r' agrees with r. If $c = 0$ then they are true since we have a matroid at 0. Linearly interpolating between the two still makes them true.

Now we want to choose M such that we get the behaviour that we want with respect to the basis B_k of M_0, that is we wish to choose M such that $\text{rank}_M(B_k) = s - 1$.

Pick c_0 such that $r(B_k)(c_0)$ is on the segment of $r(B_k)$ with constant term $s - 1$ and define a new function f by

$$f(A) = \text{constant term of the segment of } r(A) \text{ near } c_0.$$

If we have a tie of two or more segments we do the following. If $A = B_k$ then set $f(B_k) = s - 1$, else set $f(A)$ to be the minimal constant term we can obtain by this process. We will alter f to obtain a matroid rank function and this rank function will be the one we give the M which we choose. Note that this is a well defined process and that $f(A)$ is bounded between $r_\infty(A)$ and $r_0(A)$ (and also below $\text{rank}_{M_0^U}(A)$).

We implement the following changes in order to fix any of the matroid rank axioms which may have been broken by this construction.

1. If $f(A) > f(A \cup a)$ then change $f(A \cup a)$ to have the same value as $f(A)$.

2. If $f(A) < f(A \cup a) - 1$ then change $f(A)$ to have the same value as $f(A \cup a) - 1$.

3. If $f(A \cup a) + f(A \cup b) < f(A) + f(A \cup \{a, b\})$ then increase either $f(A \cup a)$ or $f(A \cup b)$ so that the left hand side has the same value as the right hand side.

Let g be the resultant function after making all of these changes. Then g is a matroid rank function. Moreover, we have $r_\infty \leq f \leq g$ and we also have that $g \leq \text{rank}_{M_0^U}$ since at no point are we able to get $f(A) > s$. Note that the second inequality shows that this process terminates since we are only able to make finitely many changes before exceeding $\text{rank}_{M_0^U}$.

Now we show why we need not increase $f(B_k)$ from $s-1$ to s. Firstly consider (1). If $f(B_k) < f(B_k \backslash b)$ then $f(B_k \backslash b) > s-1$. However this can't be the case since f is bounded above by $\text{rank}_{M_0^U}$. Now consider (2). If $f(B_k) < f(B_k \cup a) - 1$ then, since $f(B_k) = s-1$, we have $f(B_k \cup a) > s$. Since f is bounded above by $\text{rank}_{M_0^U}$ this situation cannot occur. Finally consider condition (3). If $f(B_k) + f(B_k \backslash b \cup a) < f(B_k \backslash b) + f(B_k \cup a)$ then by the fact that $f(B_k) = s - 1$, and we've already applied conditions (1) and (2), then the only way we can have this violation is if: $f(B_k) = s - 1, f(B_k \backslash b \cup a) = s - 1, f(B_k \backslash b) = s - 1, f(B_k \cup a) = s$. In this case we are able to increase $f(B_k \backslash b \cup a)$ from $s - 1$ to s to remove this violation, without needing to resort to increasing $f(B_k)$. Hence we have shown that we need not increase $f(B_k)$.

So by letting M be the matroid associated with the matroid rank function g we are able to construct a new system r' of functions satisfying the six conditions where B_k is dependent at 0. $\square$

Proposition 3.3.8. *Given* $r : 2^E \to PLF(\mathbb{R}_{\leq 0}, \mathbb{R})$ *which satisfy the six conditions given by Theorem 3.2.1, then the dual defined by* $r^*(A) := r(E \backslash A) + |A| - r(E)$ *satisfies the six conditions of Theorem 3.2.1.*

Proof. We first show condition 1, namely convexity and continuity. Firstly, r^* is continuous since it is the addition of continuous functions. Now for convexity.

$$\begin{aligned} r^*(A)(\theta x + (1-\theta)y) &= r(E\backslash A)(\theta x + (1-\theta)y) + |A| - r(E)(\theta x + (1-\theta)y) \\ &\leq \theta r(E\backslash A)(x) + (1-\theta)r(E\backslash A)(y) + |A| - \theta r(E)(x) - (1-\theta)r(E)(y) \\ &= \theta r(E\backslash A)(x) + \theta|A| - \theta r(E)(x) + (1-\theta)r(E\backslash A)(y) \\ &\quad + (1-\theta)|A| - (1-\theta)r(E)(y) \\ &= \theta r^*(E\backslash A)(x) + (1-\theta)r^*(E\backslash A)(y). \end{aligned}$$

When computing the first inequality we recall that $r(E)$ is a linear function, so the negation of this doesn't impact the inequality.

Now to show that each linear piece takes integer value at 0. Consider $r^*(A) = r(E\backslash A) + |A| - r(E)$. Since $|A|, r(E\backslash A)(0), r(E)(0) \in \mathbb{Z}$, then $r^*(A)$ must take integer value at 0.

Now we show condition 3, that is to say $r^*(\emptyset)$ is linear and $r^*(\emptyset)(0) = 0$.

$$r^*(\emptyset) = r(E\backslash\emptyset) + |\emptyset| - r(E) = r(E) - r(E) = 0.$$

Next we show condition 4, namely that $r^*(E)$ is linear. We have

$$r^*(E) = r(E\backslash E) + |E| - r(E) = r(\emptyset) + |E| - r(E) = |E| - s$$

which is linear.

Now to show condition 5, namely that $r^*(A) \leq r^*(A \cup a)$ and $r^*(A \cup a) \leq r^*(A) + 1$.

Firstly note that

$$\begin{aligned} r^*(A \cup a) - r^*(A) &= r(E \backslash (A \cup a)) + |A \cup a| - r(E) - r(E \backslash A) - |A| + r(E) \\ &= r(E \backslash (A \cup a)) - r(E \backslash A) + 1 \end{aligned}$$

By axiom 5 we have $r(E\backslash(A \cup a)) - r(E\backslash A) \geq -1$ and so

$$r^*(A \cup a) - r^*(A) \geq 0 \implies r^*(A \cup a) \geq r^*(A)$$

Now for the other part:

$$\begin{aligned} r^*(A \cup a) - r^*(A) &= r(E \backslash (A \cup a)) + |A \cup a| - r(E) - r(E \backslash A) - |A| + r(E) \\ &= r(E \backslash (A \cup a)) - r(E \backslash A) + 1. \end{aligned}$$

Since by the other part of axiom 5 $r(E\backslash(A\cup a)) - r(E\backslash A) \leq 0$ we thus have $r^*(A\cup a) \leq r^*(A) + 1$.

Now to show condition 6, namely submodularity.

$$\begin{aligned} r^*(A \cap B) + r^*(A \cup B) &= r(E\backslash(A \cap B)) + |A \cap B| - r(E) + r(E\backslash(A \cup B)) + |A \cup B| - r(E) \\ &\leq r(E\backslash A) + |A \cap B| - r(E) + r(E\backslash B) + |A \cup B| - r(E) \\ &= r(E\backslash A) + |A| - r(E) + r(E\backslash B) + |B| - r(E) \\ &= r^*(A) + r^*(B). \end{aligned}$$

□

3.3.2 Proof of Theorem 3.3.1

Now that we've set up the apparatus we will put this all to use and prove Theorem 3.3.1.

Proof of Theorem 3.3.1. First we define the vector $\rho \in (\mathbb{R} \cup \{\infty\})^{\binom{E}{s}}$. For any set $B \in \binom{E}{s}$ where $\epsilon > 0$ is sufficiently small define $\rho(B)$ by

$$\rho(B) = \begin{cases} \text{slope of } r(B)(c) \text{ for } -\epsilon < c < 0 \text{ if it is finite and } r(B)(0) = r(E)(0) \\ \infty \text{ if } r(B)(0) < r(E)(0). \end{cases}$$

Since the six conditions of Theorem 3.2.1 are satisfied, we have by Theorem 3.3.2 that there is a matroid M_∞. This matroid has the set of bases $\{B \subseteq E \mid \rho(B) \text{ is minimal}\}$. This is since $r(E)$ is linear, and is such that $r(E) \geq r(A)$ for any set, and so for any basis $B_0 \in M_\infty$ we have that $r_0(E) = r_0(B_0)$ and $r_\infty(E) = r_\infty(B_0)$. So if $\rho(B_0)$ is not minimal then this would mean that a gradient of a later segment of $r(B_0)$ would be less than $\rho(B_0)$, and this would not be a linear function, so at some point would either violate convexity or be greater than $r(E)$. Hence for any $B_0 \in \mathcal{B}(M_\infty)$ we have $r(B_0)$ is linear, and in particular $r(B_0) = s + c\rho(B_0)$.

So for any $B_0 \in \mathcal{B}(M_\infty)$ we can write $r(B_0)(c) = s + c\rho(B_0) = |B_0 \cap B_0| + c \cdot \rho(B_0)$, and so $r(B_0)(c) = \max_i\{i + c \cdot \rho(B) \mid B \in \binom{E}{s}, |B_0 \cap B| \geq i\}$ for all $B_0 \in \mathcal{B}(M_\infty)$. This maximum is coming from the fact that s is the maximum value that occurs for an intersection between B_0 and any $B \in \binom{E}{s}$, and the value of $\rho(B_0)$ is minimal up to ties.

Given $A \subseteq E$ we can write $A = (B_0 \cup C) \backslash D$, for some $B_0 \in \mathcal{B}(M_\infty)$, $C \subseteq E \backslash B_0, |C| = \text{nullity}_\infty A$, $D \subseteq B_0, |D| = \text{corank}_\infty A$.

Given $r(A)$, we consider some upper and lower bounds on this function. By noting that $B_0 \backslash D \subseteq A \subseteq B_0 \cup C$ we have

$$r(B_0 \backslash D) = s - |D| + c\rho(B_0)$$
$$r(B_0 \cup C) = s + c\rho(B_0)$$

and hence we have the following bounds, where both of the lower bounds come from using $r(B_0 \backslash D) = s - |D| + c\rho(B_0)$. The upper bound for the first comes from applying the second part of the fifth condition of Theorem 3.2.1, and the upper bound for the second comes from using $r(B_0 \cup C) = s + c\rho(B_0)$.

$$\begin{aligned} s - |D| + c\rho(B_0) \leq r(A) \leq s - |D| + |C| + c\rho(B_0) \\ s - |D| + c\rho(B_0) \leq r(A) \leq s + c\rho(B_0). \end{aligned} \tag{3.3}$$

The width of these bounds is $\min\{|C|, |D|\}$. We firstly deal with the cases when $|C| \leq$

$|D|$. We will deal later with the cases when $|C| > |D|$ by utilising duality. It is useful to note that $|A| \leq s$ when $|C| \leq |D|$. We make an inductive argument whereby we increase on the size of $|C|$, and begin with $|C| = 0$ as our base case as well as also explicitly carrying out the $|C| = 1$ case to help illustrate the inductive step we use.

If $|C| = 0$ then by the first inequality of Equation (3.3) we have $r(A) = s - |D| + c\rho(B_0)$, and thus $r(A) = |A \cap B_0| + c\rho(B_0)$, and thus we can write $r(A)$ as $r(A)(c) = \max_i\{i + c\rho(B) \mid B \in \binom{E}{s}, |A \cap B| \geq i\}$. In particular, when $|D| = 0$ this corresponds to the case we considered earlier, that is, when $A \in \mathcal{B}(\mathcal{M}_\infty)$.

If $|C| = 1$ then $r(A)$ has at most one bend. If $r(A)$ has no bends we can use Corollary 3.3.5 to get $A \cup a_1 \cup \cdots \cup a_l$ which contains a basis of M_∞.

Now what happens if we have one bend, and hence two linear pieces. Firstly assume that $|C| = |D| = 1$. Note that $A \in \mathcal{B}(M_\infty, 1)$, so let $B_1 := A$. The linear piece going through s at $c = 0$ can be written as $s + c \cdot \rho(B_1) = |B_1 \cap B_1| + c \cdot \rho(B_1)$, where $\rho(B_1)$ has been defined as the initial gradient of $r(A)$.

The linear piece going through $s - 1$ can be written as $s - 1 + c \cdot \rho(B_0) = |B_1 \cap B_0| + c \cdot \rho(B_0)$ for some $B_0 \in \mathcal{B}(M_\infty)$. So by combining the two we are able to write $r(A)(c) = \max\{i + c\rho(B) \mid B \in \binom{E}{s}, |A \cap B| \geq i\}$.

Now consider when $|C| = 1, |D| = k > 1$. By Lemma 3.3.6 there exists $a_1, \ldots, a_{k-1}$ such that $r(A) + k - 1 = r(A \cup a_1 \cup \cdots \cup a_{k-1})$, and we let $B_1 := A \cup a_1 \cup \cdots \cup a_{k-1} \in \mathcal{B}(M_\infty, 1)$.

Consider $r(A)$, one linear piece goes through $s - k + 1$, and the other goes through $s - k$. Note that $|A| = s - k + 1$. The linear piece going through $s - k + 1$ can be written as $s - k + 1 + c \cdot \rho(B_1) = |A \cap B_1| + c \cdot \rho(B_1)$. This has the requisite gradient by how $\rho(B_1)$ is defined and the fact that $r(A) + k - 1 = r(B_1)$, and $|A \cap B_1| = s - k + 1$.

The linear piece going through $s - k$ can be written as $s - k + c \cdot \rho(B_0) = |A \cap B_0| + c \cdot \rho(B_0)$ for some $B_0 \in \mathcal{B}(M_\infty)$. So we can write $r(A)(c) = \max\{i + c \cdot \rho(B) \mid B \in$

$\binom{E}{s}, |A \cap B| \geq i\}$. This is because for any other B we cannot have $\rho(B) < \rho(B_1)$ and $|A \cap B| = s - k + 1$.

We proceed inductively, consider $|C| = k$, and assume as inductive hypothesis that each A such that $|C| \leq k - 1$ and $|C| \leq |D|$ can be written in the requisite way. In this case we have $\leq k$ bends. If there are $0, 1, \ldots, k - 1$ bends then we can apply Corollary 3.3.5 to get $A \cup a_1 \cup \cdots \cup a_l$ which contains a basis of $\mathcal{B}(M_\infty, k - 1)$. Else we have k bends.

Firstly assume that $|C| = |D| = k$. Let $B_k := A \in \mathcal{B}(M_\infty, k)$. We show that every gradient of each linear segment of $r(B_k)$ except for the initial gradient has been used before, that is, $S_i(B_k) = S_j(B_l)$ where $l < k$ and $i > 0$.

We use Lemma 3.3.7 to obtain an augmentation of r, say r', which remains identical to r for c sufficiently negative. However, in this new system we make it such that B_k is dependent at $c = 0$ in order to utilise Corollary 3.3.5 to show the needed result.

By translating the dual version of Corollary 3.3.5 into our S notation about the gradients of linear segments we have that there exists h such that $S'_i(B_k) = S'_i(B_k \backslash h) \quad \forall i$ in our augmentation r'. Why is this so? Consider $r'(B_k)$ which we know is not of full rank at $c = 0$, and thus its dual, $r'^*(E \backslash B_k)$, is not of full rank at $c = 0$. Thus by Corollary 3.3.5 there exists h such that $r'^*(E \backslash B_k) + 1 = r'^*(E \backslash B_k \cup h)$. Thus we have

$$r'^*(E \backslash B_k) = r'(B_k) + |E \backslash B_k| - r'(E) = r'(B_k) + n - 2s$$
$$r'^*(E \backslash B_k \cup h) = r'(B_k \backslash h) + |E \backslash B_k \cup h| - r'(E) = r'(B_k \backslash h) + n - 2s + 1$$

and thus $r'(B_k) = r'(B_k \backslash h)$, and consequently we have $S'_i(B_k) = S'_i(B \backslash h)$.

By this construction for any $c \leq c_0$, where c_0 has been defined through our construction of r by using Lemma 3.3.7, we have $r(B_k \backslash h)(c) = r'(B_k \backslash h)(c)$ and $r(B_k)(c) = r'(B_k)(c)$. Since we have $S'_i(B_k) = S'_i(B_k \backslash h)\ \forall i$, we have that $r'(B_k) = r'(B_k \backslash h)$, and so on our restricted domain, $c \leq c_0$ we have $r(B_k) = r(B_k \backslash h)$.

Since $B_k \backslash h \subseteq B_{k-1} \in \mathcal{B}(M_\infty, k-1)$ we have, as an inductive step, that $r(B_k \backslash h)$ can be written as

$$r(B_k \backslash h)(c) = \max_i \{i + c\rho(B) \mid B \in \binom{E}{s}, |B_k \backslash h \cap B| \geq i\}$$

and since $S'_i(B_k) = S'_i(B_k \backslash h)$ we have that for $c \leq c_0$ that

$$r(B_k)(c) = \max_i \{i + c\rho(B) \mid B \in \binom{E}{s}, |B_k \cap B| \geq i\}.$$

Now consider $c > c_0$. By construction the only difference between $r(B_k)$ and $r'(B_k)$ is that there is no linear segment coming from s at $c = 0$ in the $r'(B_k)$ case. So when $c > c_0$ the only segments that are seen are those passing through $s - 1$ and s at 0, so since $\rho(B_k)$ has been defined appropriately we are able to write our function over the entire domain as

$$r(B_k)(c) = \max_i \{i + c\rho(B) \mid B \in \binom{E}{s}, |B_k \cap B| > i\}.$$

Now consider when $|C| = k, |D| = t > k$. Apply Lemma 3.3.6 to obtain $A \cup a_1 \cup \cdots \cup a_{t-k} \in \mathcal{B}(M_\infty, k)$, such that $r(A \cup a_1 \cup \cdots \cup a_{t-k}) = r(A) + t - k$. Let $B_k := A \cup a_1 \cup \cdots \cup a_{t-k}$. Since $r(B_k) = \max_i\{i + c\rho(B) \mid B \in \binom{E}{s}, |B_k \cap B| \geq i\}$ we thus have $r(A) = \max_i\{i + c\rho(B) \mid B \in \binom{E}{s}, |A \cap B| \geq i\}$ else we would end up violating $r(A \cup a) \leq r(A) + 1$ and convexity.

We have now considered all $A \subseteq E$ such that $|A| \leq s$. We complete the proof by utilising the dual matroid. By Proposition 3.3.8 we have that this dual satisfies the six conditions of Theorem 3.2.1.

Let $\rho^*(E \backslash B) = \rho(B)$ and note

$$|A \cap B| \geq i \iff |(E \backslash A) \cap B| \leq s - i \iff |(E \backslash A) \cap (E \backslash B)| \geq |E| - |A| - s + i.$$

Using this fact and also that for $|A| \leq s$ we have

$$r(A) = \max_i \{i + c \cdot \rho(B) | B \in \binom{E}{s}, |A \cap B| \geq i|\}$$

and in particular this would hold for r^*. So we have for any $|A| \geq s$ that

$$\begin{aligned} r(A) &= r^*(E \backslash A) + |A| - r^*(E) \\ &= -|E| + |A| + s + r^*(E \backslash A) \\ &= -|E| + |A| + s + \max_j \{j + c \cdot \rho^*(\hat{B}) \mid \hat{B} \in \binom{E}{|E| - s}, |(E \backslash A) \cap \hat{B}| \geq j\} \\ &= \max_i \{i + c \cdot \rho^*(E \backslash B) \mid E \backslash B \in \binom{E}{|E| - s}, |(E \backslash A) \cap (E \backslash B)| \geq |E| - |A| - s + i\} \\ &= \max_i \{i + c \cdot \rho(B) \mid B \in \binom{E}{s}, |(E \backslash A) \cap (E \backslash B)| \geq |E| - |A| - s + i\} \end{aligned}$$

where $j := |E| - |A| - s + i$, $\hat{B} := E \backslash B$.

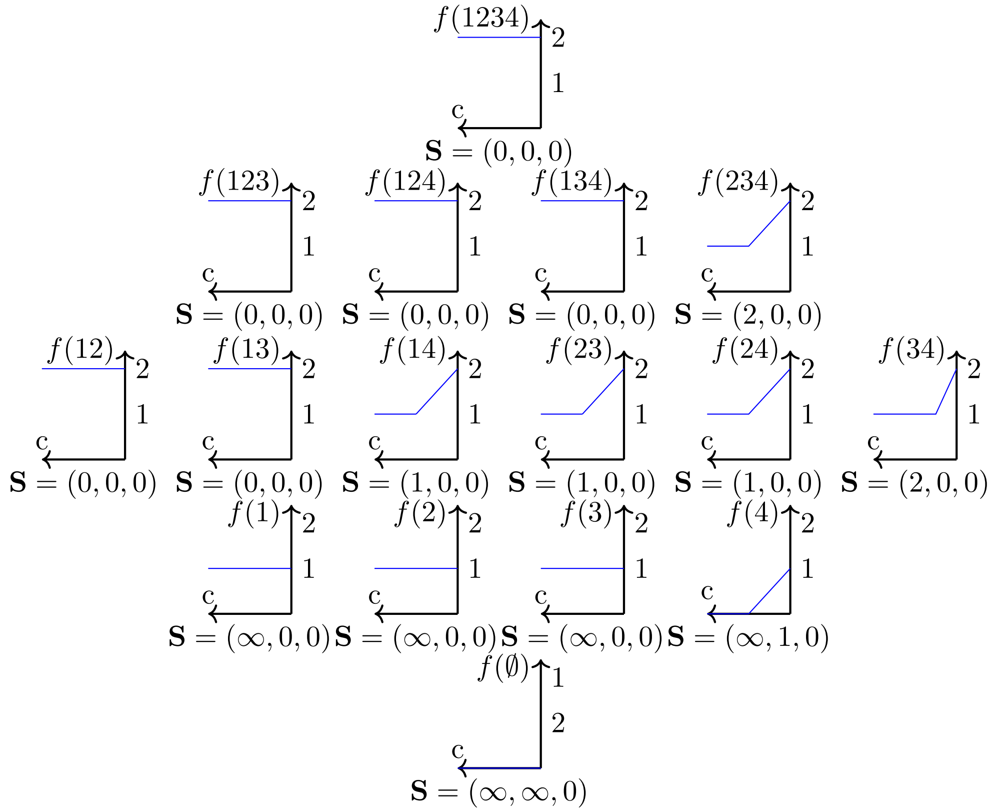
So we have shown we can write $r(A) = \max_i \{i + c \cdot \rho(B) \mid B \in \binom{E}{s}, |A \cap B| \geq i|\}$. □

3.3.3 Consequences of Theorem 3.3.1

We introduce a result in order to show that we have a relationship between valuated matroids and our six conditions from Theorem 3.2.1, when we add a 7th condition to our initial six. Firstly, we show that the six conditions from Theorem 3.2.1 can be satisfied even when ρ is not a valuated matroid.

Example 10. Consider the following ρ when $E = \{1, 2, 3, 4\}$ and $s = 2$. Let $\rho(12) = 0, \rho(13) = 0, \rho(14) = 1, \rho(23) = 1, \rho(24) = 1, \rho(34) = 2$. This is not a valuated matroid since we have that the minimum of $\rho(12) + \rho(34), \rho(13) + \rho(24), \rho(14) + \rho(23)$ is uniquely attained by $\rho(13) + \rho(24)$. Then $r^\rho : 2^E \to PLF(\mathbb{R}_{\leq 0}, \mathbb{R})$ defined by $r^\rho(A)(c) = \max_{B \in \mathcal{B}} \{|A \cap B| + c\rho(B)\}$ satisfies the six conditions of Theorem 3.2.1. The first five conditions follow in the same way as to the proof of Theorem 3.2.1. The sixth condition, submodularity, can be shown by exhaustively testing on a case by case basis. We display below what $r^\rho(A)$ looks like for each $A \subseteq \{1234\}$. The top row consists of A

of size 4, the second row consists of sets of size 3, and so on until the bottom row consists of $A = \emptyset$. From left to right in each row the sets are labelled in lexicographic order. Beneath each graph we give the gradients of the linear segments. We let $f(A) = r^\rho(A)(c)$ on the axes, where $A \subseteq E$.



In light of the Example 10 we devise a way of giving an extra condition so that we are able to have a bijection between valuated matroids and functions r^ρ of the form we have been considering. The extra condition that we introduce is one in terms of rescaling of our vector ρ.

Proposition 3.3.9. *Given a vector* $\rho \in (\mathbb{R} \cup \{\infty\})^{\binom{E}{s}}$ *which isn't a valuated matroid, but where* $r^\rho : 2^E \to PLF(\mathbb{R}_{\leq 0}, \mathbb{R})$ *defined by* $r^\rho(A)(c) = \max_{B \in \mathcal{B}}\{|A \cap B| + c\rho(B)\}$ *satisfies the six conditions given by Theorem 3.2.1, then we are able to find* $\boldsymbol{x} = (x_a : a \in E)$ *such that* $r^{\rho^{\boldsymbol{x}}}$ *does not satisfy the six conditions of Theorem 3.2.1.*

Proof. We use the fact from Proposition 2.3.9 that ρ is a valuated matroid if it satisfies all three-term tropical Plücker relations and that $\{B \mid \rho(B) \neq \infty\}$ are the set of bases of some matroid. In the case where ρ does not satisfy the condition that $\{B \mid \rho(B) \neq \infty\}$ is the set of bases of a matroid then it clearly follows that r^ρ does not satisfy the six conditions since if the six conditions hold then this implies that we have a matroid at $c = 0$, and thus this fails in this case so this can't happen.

Instead consider ρ such that it does not satisfy all three-term tropical Plücker relations. Firstly, we show that we can find a rescaling of ρ such that $r^{\rho^{\mathbf{x}}}$ does not satisfy the six conditions when $s = n$ and $|E| = n + 2$. Note that we need not consider when $|E| = n$ or $|E| = n + 1$ when $s = n$ since no three-term tropical Plücker relations can be formed, and similarly no three-term tropical Plücker relation can be formed when $|E| < 4$, so assume that $n \geq 2$.

So there is some three-term tropical Plücker relation such that the minimum is uniquely attained, say the one given by $\sigma = \{1, \dots, n-1\}$ and $\tau = \{1, \dots, n-2, n, n+1, n+2\}$. Explicitly the minimum of the following terms is attained exactly once

$$\rho(1, 2, \dots, n-1, n) + \rho(1, 2, \dots, n-2, n+1, n+2),$$

$$\rho(1, 2, \dots, n-1, n+1) + \rho(1, 2, \dots, n-2, n, n+2),$$

$$\rho(1, 2, \dots, n-1, n+2) + \rho(1, 2, \dots, n-2, n, n+1).$$

Letting $A = \{1, \dots, n-2\}$ we have

$$\rho(A, n-1, n) + \rho(A, n+1, n+2), \rho(A, n-1, n+1) + \rho(A, n, n+2),$$

$$\rho(A, n-1, n+2) + \rho(A, n, n+1)$$

and we assume that the unique minimum is attained by $\rho(A, n-1, n+2) + \rho(A, n, n+1)$.

We will show that we can violate the submodular inequality by rescaling by a suitably chosen $\mathbf{x}$. Consider sets $C = \{A, n-1\}, a = \{n\}, b = \{n+1\}$. We consider the submodular relation in terms of the gradients of linear segments as given in Lemma 3.3.3, and in particular, we consider when $j = 1$.

$$\min_{i_1+i_2=1} S_{i_1}(C \cup a) + S_{i_2}(C \cup b) \leq \min_{i_1+i_2=1} S_{i_1}(C) + S_{i_2}(C \cup a \cup b)$$

The potential values of the $S_i s$ appearing on the right hand side of the submodular inequality are

$$\begin{aligned} S_0(C) &: \infty \\ S_1(C) &: \rho(C, n), \rho(C, n+1), \rho(C, n+2) \\ S_0(C \cup n \cup n+1) &: \rho(D), \text{ where } D \in \binom{C \cup n \cup n+1}{n} \\ S_1(C \cup n \cup n+1) &: \rho(D), \text{ where } D \in \binom{E}{n} \end{aligned}$$

and on the other side of the inequality

$$\begin{aligned} S_0(C \cup n) &: \rho(C \cup n) \\ S_1(C \cup n) &: \rho(C, n), \rho(D, n+1), \rho(D, n+2) \text{ where } D \in \binom{C \cup n}{n-1} \\ S_0(C \cup n+1) &: \rho(C \cup n+1) \\ S_1(C \cup n+1) &: \rho(C, n+1), \rho(D, n), \rho(D, n+2) \text{ where } D \in \binom{C \cup n+1}{n-1} \end{aligned}$$

Using these we are able to note that if $\rho(C, n+2) < \rho(C, n), \rho(C, n+1)$ and $\rho(A, n, n+1) < \rho(D)$ where $D \in \binom{C \cup n \cup n+1}{n} \setminus \{A, n, n+1\}$, that is, $S_1(C) = \rho(C, n+2)$ and $S_0(C \cup n \cup n+1) = \rho(A, n, n+1)$, and also if we are able to have the following

$$S_1(C \cup n) = \rho(A, n, n+2) \text{ and } S_1(C \cup n+1) = \rho(A, n+1, n+2).$$

Then for $j = 1$ in the submodular inequality we have

$$\begin{aligned} \rho(A, n-1, n+2) + \rho(A, n, n+1) \geq \min\{&\rho(A, n-1, n) + \rho(A, n+1, n+2), \\ &\rho(A, n-1, n+1) + \rho(A, n, n+2)\} \end{aligned}$$

and this would be a contradiction of our assumption that $\rho(A, n-1, n+2)+\rho(A, n, n+1)$ is the unique minimum of the tropical Plücker relation.

So we rescale ρ in order to ensure those four conditions are satisfied. We importantly note that after any rescaling of ρ that each of the three the terms of a tropical Plücker relation are rescaled by the same amount, so in particular we maintain the same unique minimum of this tropical Plücker relation for both ρ and any such $\rho^{\mathbf{x}}$.

In order to get this contradiction we firstly rescale on $n+2$ to get $\rho^{\mathbf{x}_1}(A, n-1, n+2) < \rho^{\mathbf{x}_2}(A, n-1, n), \rho^{\mathbf{x}_1}(A, n-1, n+1)$. Next we rescale on $n-1$ to get $\rho^{\mathbf{x}_2}(A, n, n+1) < \rho^{\mathbf{x}_2}(D)$ where $D \in \binom{C \cup n \cup n+1}{n} \backslash \{A, n, n+1\}$. Note that this rescaling on $n-1$ doesn't change the previous inequality since all terms are rescaled by the same amount. Lastly, we rescale in order to get the final conditions for $S_1(C \cup n)$ and $S_1(C \cup n+1)$. Let's look more in depth at how we obtain the condition for $S_1(C \cup n)$. The minimum cannot be given by $\rho^{\mathbf{x}_2}(A, n-1, n)$ as $\rho^{\mathbf{x}_2}(A, n-1, n+2)$ is an element of $S_1(C \cup n+1)$. So that means that the minimum is given by one of the following

$$\rho(A, n, n+1), \rho(A, n-1, n+2), \rho(A, n, n+2), \rho(\binom{A}{n-3}, n-1, n, n+2)$$

So again, if we rescale on $n+2$ and $n-1$ we can obtain the minimum as $\rho^{\mathbf{x}_3}(A, n, n+2)$. A similar process can be done for $S_1(C \cup n+1)$. We note that all of these rescalings are compatible. The rescaling that we choose on ρ is $\mathbf{x} = (x_a \mid a \in E)$ to be all 0s except for the entries for x_{n+2} and x_{n-1} where these are chosen so that we get the above four inequalities. This argument works even when we have ∞s in the tropical Plücker relation we are considering. This is since the entries in ρ we wish to show are smaller than something else have to be finite. This is since we assume $\rho(A, n-1, n+2)+\rho(A, n, n+1)$ is the unique minimum and hence it is finite.

Now we see for a given $s = n$ that we are able to take larger sets E. By taking $|E| > n+2$, say $|E| = n+l$ with the same without loss of generality conditions on the tropical Plücker relations we end up with the same potential sets for $S_0(C), \ldots, S_1(C \cup n+1)$

except that there are additional terms which involve $n+3,\dots,l$ for which we can rescale on without impacting on our necessary initial conditions. □

Example 11. Now we consider r^ρ from Example 10 and show that we can find a rescaling for which r^{ρ^x} does not satisfy the six conditions. Given $\rho = (0,1,0,1,1,2)$. Let $\mathbf{x} = (2,0,0,-2)$ then $\rho^{\mathbf{x}} = (2,3,0,1,1,0)$. Now we show that the submodular inequality is no longer satisfied. Take the submodular inequality from Lemma 3.3.3 for when $j=1$. Letting $A=1, a=2, b=3$, we have

$$\min_{i_1+i_2=1} S_{i_1}(A\cup a) + S_{i_2}(A\cup b) \le \min_{i_1+i_2=j} S_{i_1}(A) + S_{i_2}(A\cup a\cup b)$$

$$\min\{S_0(12)+S_1(13), S_1(12)+S_0(13)\} \le \min\{S_0(1)+S_1(123), S_1(1)+S_0(123)\}$$

$$\min\{2+0, 0+3\} \le 0+0$$

$$\min\{2,3\} \le 0$$

So the submodular inequality is not satisfied.

Theorem 3.3.10. *Let $\rho \in (\mathbb{R}\cup\{\infty\})^{\binom{E}{s}}$ be such that for all $\boldsymbol{x} = (x_a : a \in E)$, r^{ρ^x} satisfies the six conditions of Theorem 3.2.1. Then ρ is a valuated matroid of rank s on the ground set E.*

Proof. This is a direct consequence of Proposition 3.3.9 and Theorem 3.3.1. □

Theorem 3.3.11. *Let $r^\rho : 2^E \to PLF(\mathbb{R}_{\le 0}, \mathbb{R})$ be defined by $r^\rho(A)(c) = \max_{B\in\mathcal{B}}\{|A\cap B| + c\rho(B)\}$. Then ρ is a valuated matroid of rank s on the ground set E if and only if $\rho \in (\mathbb{R}\cup\{\infty\})^{\binom{E}{s}}$ is such that for all $\boldsymbol{x} = (x_a : a \in E)$, r^{ρ^x} satisfies the six conditions of Theorem 3.2.1.*

Proof. One direction is just by using Theorem 3.2.2, and the other is by Theorem 3.3.10.

□

Chapter 4

Hyperproduct

The set of all $n \times n$ matrices with tropical entries, denoted $M_n(\mathbb{T})$, is known to be a semiring, and hence a multiplicative monoid. In fact, $M_n(\mathbb{T})$ is only a semigroup when we only consider finite entries, and there is a isomorphism between every maximal subgroup of these finite matrices with the full linear automorphism group of a related tropical polytope [25]. We would like to know if there is any further multiplicative structure on $M_n(\mathbb{T})$ or any other related objects, for example some group structure. Our objective throughout this chapter is to extend the multiplicative monoid $M_n(\mathbb{T})$ to a hyperoperation, whose structure we investigate. We introduce an extension to our hyperoperation in two steps.

Throughout this chapter we study the hyperoperation which we introduce in Section 4.2. This hyperoperation is introduced in two stages, the first, replacing tropical matrices with tropical Plücker vectors is done in Section 4.1, where the product is given by the previously studied composition of valuated linking systems. We replace this single-valued product by a multivalued product which we define in Section 4.2. In particular, within Section 4.3.1 we note the single-valued composition given by composition of valuated linking systems is always an element of our newly defined hyperproduct. The rest of the chapter is devoted to the investigation of the hyperproduct, as well as some results about the single-valued product.

In Section 4.3 we show that the hyperoperation which we define satisfies all the necessary properties of being a hypergroup except for being associative, which has so far only been shown to be true in the case whereby our input valuated matroids have rank one or two. However, in Section 4.5 we do show that some naive proof strategies for showing that we have associativity when considering valuated matroids of larger rank cannot be used.

Also in Section 4.3 we give some results as to the structure of this hyperoperation and show what some of the solutions set look like. In addition, we show that it isn't convex and nor is it a fan in general, which are shown in Section 4.3.7 and Section 4.3.8 respectively. We also outline a case study when considering input valuated matroids of rank 2 in Section 4.3.3. Lastly, in Section 4.4 we look at how our monoid product associated with our hyperoperation which we introduce in Section 4.1 relates to previous studied material. We give an alternative description of this monoid product and an extension of in terms of a matroid rank function when we consider non-valuated versions. This enables us to consider the flats of both of these constructions, and this gives us cryptomorphic definitions of both of these.

4.1 $T\bar{G}L_n$ and monoid structure

Let V be a vector space. Working over a field $\mathbb{K}$, with $V \subseteq \mathbb{K}^{2n}$, $\dim V = n$, we have that V is a graph of a function if we have a matrix $A \subseteq \mathbb{K}^{n\times 2n}$ whose row space is V such that $A_{1,\dots,n\times 1,\dots,n}$ is invertible. If we further have that $A_{1,\dots,n\times n+1,\dots,2n}$ is invertible this gives us that V is a graph of an invertible function.. This implies in the latter case that the Plücker coordinates $P_{1,\dots,n}$ and $P_{n+1,\dots,2n}$ are nonzero.

Classically, given linear functions f, g represented by matrices B, C respectively, the composition fg can be written in terms of matrices as BC. However, when viewing the same problem tropically this does not necessarily work, since as noted in the introduction tropical linear spaces don't have to have their tropical Plücker vectors arising from a matrix.

In addition, for linear maps over a field f, g represented by matrices B, C respectively, then we don't necessarily have that $\text{trop}(B)\,\text{trop}(C) = \text{trop}(BC)$. We illustrate this with an example. Let $f(\mathbf{x}) = B\mathbf{x}$ and $g(\mathbf{x}) = C\mathbf{x}$ where $B = \begin{pmatrix} t & t^5 \\ t^2 & t^4 \end{pmatrix}$ and $C = \begin{pmatrix} t^1 & t^7 \\ t^3 & -t^5 \end{pmatrix}$, where these matrices have entries in the Puiseux series. When we tropicalise these matrices we obtain $\text{trop}(B) = \begin{pmatrix} 1 & 5 \\ 2 & 4 \end{pmatrix}$ and $\text{trop}(C) = \begin{pmatrix} 1 & 7 \\ 3 & 5 \end{pmatrix}$, and so $\text{trop}(B)\,\text{trop}(C) = \begin{pmatrix} 2 & 8 \\ 3 & 9 \end{pmatrix}$. Whereas $BC = \begin{pmatrix} t^2 + t^8 & t^8 - t^{10} \\ t^3 + t^7 & 0 = t^9 - t^9 \end{pmatrix}$, and so $\text{trop}(BC) = \begin{pmatrix} 2 & 8 \\ 3 & \infty \end{pmatrix}$, and thus we can see that $\text{trop}(BC) \neq \text{trop}(B)\,\text{trop}(C)$.

These potential issues mean that we consider the problem in terms of Plücker vectors, and consider the same problem but where $P_{1,\dots,n} \neq \infty$ and $P_{n+1\dots 2n} \neq \infty$. The ∞s arise since they are the multiplicative identity of the tropical semiring.

By following this approach we are able to arrive at a previously studied operation, the composition of valuated linking systems, which have been studied in literature before [26] [18]. Valuated linking systems can be identified as a valuated matroid with a fixed basis, and are a valuated counterpart to linking systems introduced by Schrijver in 1976 [27].

Before defining valuated linking systems we introduce a structure related to $M_n(\mathbb{T})$. Take the set of points in $\text{Dr}(n, 2n)$ such that the Plücker coordinates of $\{1, \dots, n\}$ and $\{n+1, \dots, 2n\}$ aren't ∞. We denote this $T\bar{G}L_n$. This naming has been influenced by the definition given in Frenk's thesis [18] of the *tropical linear monoid*, which he denotes $TGL_n(\mathbb{T})$. This has been defined as the submonoid of $Hom(\mathbb{T}^n, \mathbb{T}^n)$ consisting of the morphisms (valuated linking systems) that map $\mathbb{T}^n$ to $\mathbb{T}^n$. It can be seen that $TGL_n(\mathbb{T})$ and $T\bar{G}L_n$ are defined equivalently except that in the definition of $TGL_n(\mathbb{T})$ we need not have $\{n+1,\ldots,2n\}$ not being ∞ and that $TGL_n(\mathbb{T})$ is defined in terms of linking systems whereas $T\bar{G}L_n$ is defined in terms of valuated matroids.

Throughout this chapter in the context of elements of $T\bar{G}L_n$ we let $E_i = \{(i-1)\times n+1,\ldots,(i-1)\times n+n\}$, and we say that $\mathcal{M}_i \in T\bar{G}L_n$ is on basis set $E_i \cup E_{i+1}$ when required. We allow ourselves to abbreviate $E_i \cup E_j$ as E_{ij}.

We now draw an explicit relation between $M_n(\mathbb{T})$ and $T\bar{G}L_n$ in the case where we normalise our elements of $T\bar{G}L_n$ so that the $P_{1,\ldots,n} = 0$.

$$\Gamma : M_n(\mathbb{T}) \hookrightarrow T\bar{G}L_n \quad \text{by} \quad \mathcal{A} \mapsto \text{tropical Plücker vector given by maximal minors of } [I|\mathcal{A}]$$

and

$$e : T\bar{G}L_n \twoheadrightarrow M_n(\mathbb{T}) \quad \text{by} \quad (P) \mapsto [P_{1,\ldots,\hat{i},\ldots,n,n+j}]_{i,j},$$

where $\hat{i}$ means we omit the element indexed by i.

Proposition 4.1.1. *We note some properties of* Γ *and* e.

1. Γ *is injective but not surjective.*

2. e *is not injective but is surjective.*

We also note that $e \circ \Gamma = id$.

Proof. Firstly we show that Γ is injective. Take distinct $\mathcal{A},\mathcal{B} \in M_n(\mathbb{T})$ such that $\Gamma(\mathcal{A}) = \Gamma(\mathcal{B}) = \mathcal{M} \in T\bar{G}L_n$. Since $\mathcal{M}$ is equal to the Plücker vector of both $[I|\mathcal{A}]$ and $[I|\mathcal{B}]$ we can say that all maximal minors of both matrices are equal. So consider maximal minors using the columns $I\backslash i \cup j$. These describe the (i,j) entry of $\mathcal{A}$ and $\mathcal{B}$ and these agree. So we have an injection.

This is not surjective since it only outputs elements of $T\bar{G}L_n$ such that the entries of the Plücker vector corresponding to E_1 and E_2 are the minimal values that can be associated. For example there is no matrix with maps to $\rho_{\mathcal{M}} = (0, 0, 0, 0, 0, 10) \in T\bar{G}L_2$ in the $n = 2$ case.

Now why is e not injective? Take $\mathcal{M}, \mathcal{N} \in T\bar{G}L_n$ where

$$\rho_{\mathcal{M}}(s) = 0 \; \forall s \neq \{n+1, \ldots, 2n\}, \quad \rho_{\mathcal{M}}(n+1, \ldots, 2n) = 1$$

and $\rho_{\mathcal{N}}(s) = 0 \; \forall s$. Then we have $e(\mathcal{M}) = \mathbf{0}_{n\times n}$ and $e(\mathcal{N}) = \mathbf{0}_{n\times n}$.

We now show how surjectivity works in this situation. For any matrix $\mathcal{A} \in M_n(\mathbb{T})$ if we choose $\mathcal{M} \in T\bar{G}L_n$ such that it is formed by taking the maximal tropical minors of $[I \mid \mathcal{A}]$ then we have the correct conclusion. □

We extend this relationship between matrices with tropical entries and $T\bar{G}L_n$ by introducing the following maps. Again allowing for normalising such that $P_{1,\ldots,n} = 0$.

$$\pi_k : T\bar{G}L_n \to M_{\binom{n}{k}} \quad \text{defined by} \quad \pi_k(P) \mapsto [P_{\{1,\ldots,n\}\setminus I \cup (n+J)}]_{I,J}. \tag{4.1}$$

noting that $\pi_1 = e$.

Remark 33. We can see that the maps defined in Equation (4.1) relate to exterior powers of a vector space. Firstly let's think about this when having a field-version equivalent of the maps π_k. In this case $\pi_1(P)$ is the matrix of f, the linear map associated with this, and $\pi_k(P)$ is the matrix of $\bigwedge^k f$. Tropically we may be able to take this analogy further since a tropical analogue of exterior product has been defined [28].

We now introduce a product on $T\bar{G}L_n$ which is related to the multiplication of tropical matrices in $M_n(\mathbb{T})$. Given $\mathcal{M}, \mathcal{N} \in T\bar{G}L_n$ we are able to apply π_k to $\mathcal{M}, \mathcal{N}$ and to obtain matrices $\mathcal{A}, \mathcal{B}$ such that $\pi_k(\mathcal{M}) = \mathcal{A}, \pi_k(\mathcal{N}) = \mathcal{B}$. Since $\mathcal{A}, \mathcal{B}$ are tropical matrices, and hence elements of a multiplicative monoid, we can construct $\mathcal{AB}$, and we'd like the product of $\mathcal{M}$ and $\mathcal{N}$ to be of the form $\pi_k(\mathcal{M} \circ \mathcal{N}) = AB$, for some operation $\circ$. This is the influence for the monoid we wish to associate with our valuated matroids $\mathcal{M}, \mathcal{N}$.

By utilising a previously studied product, the composition of valuated linking systems [18] [29], we are able to show that there does exist such an operation. We show in Proposition 4.1.4 that composition of valuated linking systems corresponds to our notion of monoid product.

Definition 71. [18] Let R and S be disjoint finite sets. A function $\lambda : P(R) \times P(S) \to \mathbb{T}$ is called a *valuated linking system* on (R, S) over $\mathbb{T}$ when the map $\mu_\lambda : P(R \cup S) \to \mathbb{T}$ defined by

$$\mu_\lambda(X) = \lambda(R \backslash X, S \cap X)$$

is a valuated matroid on $R \cup S$ over $\mathbb{T}$ satisfying $\mu_\lambda(R) = 0$. The map μ_λ is referred to as the *graph or representation matroid* of λ.

Remark 34. Valuated linking systems are sometimes referred to as valuated bimatroids.

We now formally introduce composition of valuated linking systems which can be shown to be our monoid product.

Theorem 4.1.2. *[18][29] Let κ and λ be valuated linking systems on (R, S) and (S, T) respectively. Define the map $\lambda \circ \kappa : P(R) \times P(S) \to \mathbb{T}$ by*

$$(X, Z) \mapsto \min_{Y \subseteq S} \kappa(X, Y) + \lambda(Y, Z).$$

Then this composition gives a valuated linking system.

We give a restatement of Theorem 4.1.2 in the language of valuated matroids which we will make use of throughout this chapter.

Proposition 4.1.3. *Given valuated linking systems κ and λ on (R, S) and (S, T), then* *we are able to write the composition* $\lambda \circ \kappa$ *in terms of valuated matroids as*

$$\rho_{\mathcal{M}}(B) = \min_{Y \subseteq S} \rho_{\mathcal{M}_1}((B \cap R) \cup Y) + \rho_{\mathcal{M}_2}((B \cap T) \cup S \backslash Y)$$

where $\mathcal{M}$ *is the representation matroid of* $\lambda \circ \kappa$*,* $\mathcal{M}_1$ *is the representation matroid of* κ*, and* $\mathcal{M}_2$ *is the representation matroid of* λ*.*

Proof. Let the valuations of $\mathcal{M}, \mathcal{M}_1$ and $\mathcal{M}_2$ be denoted by $\rho_{\mathcal{M}}, \rho_{\mathcal{M}_1}$ and $\rho_{\mathcal{M}_2}$ respectively. Then

$$\begin{aligned} \rho_{\mathcal{M}}(B) = (\lambda \circ \kappa)(R \backslash B, T \cap B) &= \min_{Y \subseteq S} \kappa(R \backslash B, Y) + \lambda(Y, T \cap B) \\ &= \min_{Y \subseteq S} \rho_{\mathcal{M}_1}((B \cap R) \cup Y) + \rho_{\mathcal{M}_2}((B \cap T) \cup S \backslash Y). \end{aligned}$$

□

Proposition 4.1.4. *Each* π_k *defined in Equation* (4.1) *is a monoid morphism from the monoid of valuated linking systems under valuated linking system composition to the monoid of tropical matrices under tropical matrix multiplication.*

Proof. Take $\mathcal{M}_1, \mathcal{M}_2 \in T\bar{G}L_n$, with associated vectors $\rho_{\mathcal{M}_1}, \rho_{\mathcal{M}_2}$. After rescaling by a multiple of $\mathbb{1}$ to make the first coordinate of the tropical Plücker vectors of $\rho_{\mathcal{M}_1}$ and $\rho_{\mathcal{M}_2}$ to be 0 we have associated valuated linking systems κ on (E_1, E_2) and λ on (E_2, E_3) respectively, namely we have,

$$\kappa(I, J) = \rho_{\mathcal{M}_1}(E_1 \backslash I \cup J) \quad \lambda(I, J) = \rho_{\mathcal{M}_2}(E_2 \backslash I \cup J).$$

We show the following diagram commutes.

$$\begin{array}{ccc} (TGL_n)^2 & \xrightarrow{\ \circ\ } & TGL_n \\ {\scriptstyle \pi_k}\downarrow & & \downarrow{\scriptstyle \pi_k} \\ (M_{\binom{n}{k}})^2 & \xrightarrow{\ \times\ } & M_{\binom{n}{k}} \end{array}$$

Let $\lambda \circ \kappa = \gamma$. Using the composition of valuated linking systems we have,

$$\mu_\gamma(E_1 \backslash I \cup J) = \gamma(I,J) = \min_{Y \subseteq E_2} (\kappa(I,Y) + \lambda(Y,J)).$$

So after each application of π_k each entry of the new matrix looks like,

$$(\gamma(I,J))_{I,J} = \min_{Y \subset E_2} (\kappa(I,Y) + \lambda(Y,J)) = \min_{Y \subset E_2, |Y|=k} (\kappa(I,Y) + \lambda(Y,J)).$$

Now we apply π_k first and then compose by tropical matrix multiplication. Firstly we have matrices $A = \lambda(I,J)_{I,J}$ and $B = \kappa(I,J)_{I,J}$. Now what is $BA = C$? It is $C_{I,J} = \min_{|Y'|=k}(\kappa(I,Y') + \lambda(Y',J))$. □

Remark 35. Proposition 4.1.4 is the motivation for calling the composition of valuated linking systems, given by $\mathcal{M} \circ \mathcal{N}$, the monoid product of $\mathcal{M}, \mathcal{N} \in T\bar{G}L_n$. This corresponds in some respect to matrix multiplication of tropical matrices, which themselves form a multiplicative monoid. Throughout the rest of this chapter we refer to both Theorem 4.1.2 and Proposition 4.1.3 as either composition of valuated linking systems, or valuated linking system composition. The distinction as to whether we use Theorem 4.1.2 and Proposition 4.1.3 comes from whether we are using valuated linking systems or valuated matroids.

We look further into composition of valuated linking systems, and the extension of valuated linking systems which we introduce in Section 4.3.1. In Section 4.4 we consider both of these compositions in terms of their rank functions when we consider non-valuated input, and in addition to that we investigate the flats of both.

4.2 Introducing the hyperproduct

As we have formally introduced a relationship between $M_n(\mathbb{T})$ and $T\bar{G}L_n$ in Section 4.1, we are now ready to introduce a hyperoperation on $T\bar{G}L_n$, which is our candidate hypergroup. We later see that this hyperoperation does contain our monoid product given by

Proposition 4.1.3 as an element, so it is truly an extension of the single-valued product.

Classically, suppose that we have two subspaces of $\mathbb{K}^{2n}$, V, W, say, which are graphs of functions. We write elements of $\mathbb{K}^{2n}$ as pairs (u, v), such that u, v are elements of $\mathbb{K}^n$. We are able to write the graph of the function which is the composition of V and W as $\{(u, w)|(u, v) \in V, (v, w) \in W\}$.

If we take this definition and try to tropicalise it, the set described isn't a tropical linear space. In order to try to avoid this issue we introduce the "double graph" $\{(u, v, w)|(u, v) \in V, (v, w) \in W\}$, which is an n-dimensional subspace of $\mathbb{K}^{3n}$. The projection of this double graph to the coordinates E_{12} is V, the projection to the coordinates E_{23} is W, and the projection to the coordinates E_{13} is the graph of the composition. Since projection onto a subset S of coordinates corresponds to discarding Plücker coordinates whose index set is not a subset of S, and the latter is a tropically well-behaved, we are able to think about tropicalising this "double graph" idea. Similarly to tropicalising the graph of a function if we tropicalise the double graph then we don't necessarily get a tropical linear space. However, we maintain the idea of projecting coordinates away as motivation moving forward.

As mentioned, if we try to tropicalise the definition of the graph of a function or the "double graph" of a function, the set described isn't a tropical linear space. This is because tropical linear spaces don't pass the vertical line test, that is, if $u \in \mathbb{T}^{E_1}$ is fixed, then there need not be at most one $v \in \mathbb{T}^{E_2}$ such that $(u, v) \in V$.

Working over a field we are able to look at the same problem in terms of matrices, which we are only able to do some of the time tropically. Since $P_{1,\ldots,n} \neq 0$ we have that V is the row space of a matrix of the form $[I|A]$ and W is the row space of a matrix of the form $[I|B]$. By using invertible row operations we are able to write $[I|B]$ as $[A|AB]$. Hence we have that the "double graph" has the matrix $[I|A|AB]$, and we can obtain each of the matrices for the three significant projections by erasing one of the three blocks.

Now we attempt to circumvent this issue by taking an approach inspired by correspondences from algebraic geometry. Given correspondences V in $E_1 \times E_2$ and W in $E_2 \times E_3$ we can compose to form a new correspondence on $E_1 \times E_3$. Then we can use the pullback for this. We might need to use the moving lemma. Now tropically we basically have a moving lemma. This is the fan displacement rule used to compute stable intersections.

Akin to the composition of correspondences take tropical linear spaces V and W on E_{12} and E_{23} respectively such that both project to the same space on E_2. Then we can compute the pullback tropical linear spaces/sets on E_{123} such that the projections to E_{12} and E_{23} are V, W respectively.

Before we define our candidate hypergroup operation we give the definition of a hypergroup that we will be considering throughout. In literature hypergroups are also found under the names *multigroup* [30]. Commonly the definition includes the notion that a hypergroup is commutative, whereas the definition we use does not require that condition.

Definition 72. [31] A *hypergroup* is a tuple $(G, \boxdot, 1)$ such that:

1. $1 \boxdot x = x \boxdot 1 = \{x\}$ for all $x \in G$ (Identity).
2. For every $x \in G$ there exists a unique element $x^{-1} \in G$ such that $(x^{-1})^{-1} = x$ and where $1 \in x \boxdot x^{-1}$ and $1 \in x^{-1} \boxdot x$ (Inverse).
3. For $x, y, z \in G$ we have $x \boxdot (y \boxdot z) = (x \boxdot y) \boxdot z$ (Associativity).

A natural way of defining a potential group operation inspired by all of this is as follows. Let $\mathcal{M}_1, \mathcal{M}_2 \in T\bar{G}L_n$. Let z be an element of $\mathrm{Dr}(3, n)$ such that the projection of z to the coordinates E_{12} is $\mathcal{M}_1$ and the projection of z to the coordinates E_{23} is $\mathcal{M}_2$. Then we define $\mathcal{M}_1 \boxdot \mathcal{M}_2$ to be the projection of z to coordinates E_{13}. However the issue with this definition is that there may be multiple such z, so instead we consider the whole set of z which satisfy the conditions, thus defining a hyperoperation.

Philosophically a hyperproduct is a reasonable notion to use. When working with equations over the tropical semiring then straightforward equality, that is whether the left hand side is equal to the right hand side, is quite often not as well behaved as the notion of having the minimum being attained at least twice. Tropically we use the notion that the minimum is attained at least twice when defining hypersurfaces, and even when defining tropical linear spaces by our use of intersecting tropical hyperplanes. Baker and Bowler [32] realised that by giving $\mathbb{T}$ a hyperfield structure then the notion of the minimum being attained twice can be obtained from the hyperring axioms.

We now formally define our hyperoperation inspired by this which gives us our candidate hypergroup on $T\bar{G}L_n$.

Definition 73. Let $\mathcal{M}_1, \mathcal{M}_2 \in T\bar{G}L_n$ and let z be the set of elements of $\mathrm{Dr}(n, 3n)$ such that the projection of z to the coordinates E_{12} is $\mathcal{M}_1$ and the projection of z to the coordinates E_{23} is $\mathcal{M}_2$. We define $\mathcal{M}_1 \boxdot \mathcal{M}_2$ to be the projection of z to coordinates E_{13}. We also define $\mathcal{M}_1 \hat{\boxdot} \mathcal{M}_2$ to be the set of z on the coordinates E_{123}.

Conjecture 4.2.1. *Let $\mathcal{M}_1, \mathcal{M}_2 \in T\bar{G}L_n$. We claim that $(T\bar{G}L_n, \boxdot)$ forms a hypergroup.*

Remark 36. Conjecture 4.2.1 is true for $n = 1$ and $n = 2$. In the $n = 1$ case we let $\mathcal{M}_1, \mathcal{M}_2 \in T\bar{G}L_1$, and after rescaling to make the first coordinate of both be 0, we are able to write $\rho_{\mathcal{M}_1} = (0, a)$ and $\rho_{\mathcal{M}_2} = (0, b)$ for some $a, b \in \mathbb{R}$. Thus we have that $\mathcal{M}_1 \boxdot \mathcal{M}_2 = (0, a + b)$. Consequently we have that $(T\bar{G}L_1, \boxdot)$ is a hypergroup as defined by Definition 72, and further to this it is an actual group since this operation is simply an analogue of tropical multiplication.

4.3 Basic properties

We begin now to investigate the hyperproduct we defined in Definition 73. We begin in Section 4.3.1 by showing that the composition of valuated linking systems is an element of this hyperoperation. We do this by defining the extension of valuated linking system composition which is an element on $\mathrm{Dr}(n, 3n)$, and then show that this restricts to the composition of valuated linking systems. This firstly demonstrates that our hyperproduct is an extension of the previously studied single-valued product, and in particular that for any $\mathcal{M}_1, \mathcal{M}_2 \in T\bar{G}L_n$ we have that $\mathcal{M}_1 \boxdot \mathcal{M}_2$ is non-empty. Secondly, we will use this in order to prove some of the axioms of the hypergroup.

Section 4.3.2 is concerned with showing which axioms of being a hypergroup the hyperproduct satisfies. We show that for any n that $(T\bar{G}L_n, \boxdot)$ has an identity element and also that it has an inverse element. Lastly, we comment on the associativity of the hyperproduct.

We then proceed in Section 4.3.3 to investigate the hyperproduct when $n = 2$. We will look into the structure of the hyperproduct. We also give a proof for associativity of $(T\bar{G}L_2, \boxdot)$.

Within Section 4.3.4 we show that the hyperproduct is noncommutative in general for $n \geq 2$. This is why we are interested in noncommutative hypergroups. We then in Section 4.3.5 and Section 4.3.6 look to see in larger cases how we are able to see new solutions to our hyperproduct. Two ways we investigate are by using hyperplane splits and Stiefel subdivisions. We end this section by showing that our hyperproduct is not always a fan but before that we show that it is not convex in general.

4.3.1 Non-emptiness of the hyperproduct

In order to show that Conjecture 4.2.1 is a hypergroup we need to show that $(T\bar{G}L, \boxdot)$ satisfies the definition of a hypergroup as given in Definition 72. Throughout this subsection we show that for any $\mathcal{M}_1, \mathcal{M}_2 \in T\bar{G}L_n$ that there is at least one element z such that $z \in \mathcal{M}_1 \boxdot \mathcal{M}_2$. By doing this we show that our putative hypergroup is non-empty, and thus satisfies one of the necessary conditions of being a hypergroup. In particular, we will show that for any $\mathcal{M}_1, \mathcal{M}_2 \in T\bar{G}L_n$ that $\mathcal{M}_1 \circ \mathcal{M}_2 \in \mathcal{M}_1 \boxdot \mathcal{M}_2$, and hence our hyperproduct defined by Definition 73 has an already well studied element sitting within it for any choice of $\mathcal{M}_1, \mathcal{M}_2 \in T\bar{G}L_n$, and thus our defined hyperproduct is truly an extension of the composition of valuated linking systems.

Theorem 4.3.1. *Given $\mathcal{M}_1, \mathcal{M}_2 \in T\bar{G}L_n$, where these have valuations $\rho_{\mathcal{M}_1}$ and $\rho_{\mathcal{M}_2}$ respectively. We claim that $\mathcal{M} \in \mathrm{Dr}(n, 3n)$ is a valuated matroid with valuation $\rho_{\mathcal{M}}$ defined by*

$$\rho_{\mathcal{M}}(B) = \min_{L_1, L_2 | L_1 \cup L_2 = E_2, L_1 \cap L_2 = B \cap E_2} \rho_{\mathcal{M}_1}((B \cap E_1) \cup L_1) + \rho_{\mathcal{M}_2}((B \cap E_3) \cup L_2),$$

where $\mathcal{M}_1$ is defined on E_{12} and $\mathcal{M}_2$ is defined on E_{23}. We call this operation the extension of valuated linking system composition *and denote this $\mathcal{M} = \mathcal{M}_1 \star \mathcal{M}_2$.*

Remark 37. This gives us an element of the hyperproduct since it correctly restricts to $\mathcal{M}_1$ and $\mathcal{M}_2$ on E_{12} and E_{23} respectively. In particular, this gives $\mathcal{M}_1 \circ \mathcal{M}_2 \in \mathcal{M}_1 \boxdot \mathcal{M}_2$ since restricting $\mathcal{M}_1 \star \mathcal{M}_2$ to E_{13} precisely gives $\mathcal{M}_1 \circ \mathcal{M}_2$.

We now explicitly show the equivalence of the valuated linking system composition, $\mathcal{M}_1 \circ \mathcal{M}_2$, with above construction of Theorem 4.3.1 of $\mathcal{M}_1 \star \mathcal{M}_2$ when we take as input $B \subseteq E_1 \cup E_3$, that is such that $B \cap E_2 = \emptyset$. Consider $\rho_{\mathcal{M}_1 \star \mathcal{M}_2}$ from Theorem 4.3.1 when $B \subseteq E_{13}$, we have

$$\rho_{\mathcal{M}_1 \star \mathcal{M}_2}(B) = \min_{L_1, L_2 | L_1 \cup L_2 = E_2, L_1 \cap L_2 = B \cap E_2 = \emptyset} \rho_{\mathcal{M}_1}((B \cap E_1) \cup L_1) + \rho_{\mathcal{M}_2}((B \cap E_3) \cup L_2).$$

For any given $L' \subseteq E_2$ we can simplify the minimality condition so that

$$\rho_{\mathcal{M}_1 \star \mathcal{M}_2}(B) = \min_{L' \subseteq E_2} \rho_{\mathcal{M}_1}((B \cap E_1) \cup L') + \rho_{\mathcal{M}_2}((B \cap E_3) \cup E_2 \backslash L')$$

and this expression is equivalent to valuated linking system composition as given in Proposition 4.1.3.

Before continuing to the proof we show that Theorem 4.3.1 gives us a way to view the composition of valuated linking systems as lying in the pullback of sets when we view these valuated matroids as tropical linear spaces.

Definition 74. We define the *pullback of sets* of functions $f : X \to Z, g : Y \to Z$ as the set $X \times_Z Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$.

Let $\mathcal{M}_1, \mathcal{M}_2 \in T\bar{G}L_n$, and thus we have associated tropical linear spaces V and W respectively. Let E_2 denote the tropical linear space associated with the matroid $U(n, n) \in \mathrm{Dr}(n, n)$. We claim that the extension of valuated linking system composition gives us an element of the pullback of $f : V \to E_2$ and $g : W \to E_2$, where both f and g are given by projection of coordinates to those E_2 is defined on. Then we have the following commutative diagram

$$\begin{array}{ccc} V \times_{E_2} W & \xrightarrow{\text{proj}} & V \\ {\scriptstyle \text{proj}}\downarrow & & \downarrow{\scriptstyle f} \\ W & \xrightarrow{\quad g \quad} & E_2 \end{array}$$

Now we have that $V \times_{E_2} W$ is given by

$$V \times_{E_2} W = \{(v, w) \in V \times W \mid f(v) = g(w)\}.$$

Let $V \star W$ be the tropical linear space of $\mathcal{M}_1 \star \mathcal{M}_2$. Then by definition we have that $V \star W \in V \times_{E_2} W$, and this nicely follows because projecting away coordinates from a tropical linear space is akin to restricting the valuated matroid rank function, which we can see correctly restricts to V, W and E_2.

Proof of Theorem 4.3.1. Let $\mathcal{M}_1, \mathcal{M}_2$ be our input valuated matroids and $\mathcal{M}$ be the output matroid of our proposed monoid product. In order to show that $\mathcal{M}$ is a valuated matroid we need each tropical Plücker relation, where each $|I| = n - 1$ and $|J| = n + 1$, to satisfy the condition that

$$\min\{\rho_{\mathcal{M}}(I \cup j) + \rho_{\mathcal{M}}(J \backslash j) \mid j \in J \backslash I\} \text{ occurs at least twice.} \tag{4.2}$$

By the definition of $\mathcal{M} = \mathcal{M}_1 \star \mathcal{M}_2$ via the extension of the composition of valuated linking systems given in the statement of Theorem 4.3.1, a typical term in the minimum in Equation (4.2) is

$$\rho_{\mathcal{M}_1}(A) + \rho_{\mathcal{M}_2}(B) + \rho_{\mathcal{M}_1}(C) + \rho_{\mathcal{M}_2}(D) \tag{4.3}$$

where $|A| = |B| = |C| = |D| = n, A, C \subseteq E_{12}, B, D \subseteq E_{23}$ and $e_A + e_B = e_{I \cup j} + e_{E_2}$ and $e_C + e_D = e_{J \setminus j} + e_{E_2}$. For brevity we write $\rho_{\mathcal{M}_1}(A) + \rho_{\mathcal{M}_2}(B) + \rho_{\mathcal{M}_1}(C) + \rho_{\mathcal{M}_2}(D)$ as $t(A, B, C, D)$.

Since $\mathcal{M}_1$ and $\mathcal{M}_2$ are valuated matroids, and hence satisfy all their tropical Plücker relations, we have the following consequences: as a consequence of $\mathcal{M}_1$ we have that

$$\min\{\rho_{\mathcal{M}_1}(K \cup i) + \rho_{\mathcal{M}_2}(B) + \rho_{\mathcal{M}_1}(L \setminus i) + \rho_{\mathcal{M}_2}(D) \mid i \in L \setminus K\} \text{ occurs at least twice,} \tag{4.4}$$

where $K, L \subseteq E_{12}, |K| = n - 1, |L| = n + 1, \ B, D \subseteq E_{23}, \ e_K + e_B = e_I + e_{E_2}$ and $e_L + e_D = e_J + e_{E_2}$.

And as consequence of $\mathcal{M}_2$ we have that

$$\min\{\rho_{\mathcal{M}_1}(A) + \rho_{\mathcal{M}_2}(K \cup i) + \rho_{\mathcal{M}_1}(C) + \rho_{\mathcal{M}_2}(L \setminus i) \mid i \in L \setminus K\} \text{ occurs at least twice,} \tag{4.5}$$

where $K, L \subseteq E_{23}, |K| = n - 1, |L| = n + 1, \ A, C \subseteq E_{12}, \ e_A + e_K = e_I + e_{E_2}$ and $e_C + e_L = e_J + e_{E_2}$.

Now in each instance of Equation (4.4) some of the terms we see arise as terms of Equation (4.2), and in particular, the terms we see are those when $j \in E_{12}$. There may be some remaining "extra terms" $\rho_{\mathcal{M}_1}(A) + \rho_{\mathcal{M}_2}(B) + \rho_{\mathcal{M}_1}(C) + \rho_{\mathcal{M}_2}(D)$ where some element of E_2 appears in both A and B, but neither C nor D; this can't happen in Equation (4.2) but can arise in Equation (4.4) for some $i \in E_2$.

Likewise, we can do similarly in the case of Equation (4.5). Each term of Equation (4.2) arises in exactly one way from either an instance of Equation (4.4) or an instance of Equation (4.5), whilst each "extra term" comes once from both Equation (4.4) and Equation (4.5), and this accounts for all the terms in Equation (4.4) and Equation (4.5).

We can index these tropical Plücker relations arising from Equation (4.4) and Equation (4.5). If $|A| = n-1, |B| = n, |C| = n+1, |D| = n$, then there is a relation $\mathrm{rel}_1(A, B, C, D)$ among the terms arising from adding $\rho_{\mathcal{M}_2}(B) + \rho_{\mathcal{M}_2}(D)$ to each term of the tropical Plücker relation for sets A and C in $\mathcal{M}_1$. Similarly for $|A| = n, |B| = n-1, |C| = n, |D| = n+1$ there is a relation $\mathrm{rel}_2(A, B, C, D)$ coming from tropical Plücker relations in $\mathcal{M}_2$. Let Rel be the set of all $\mathrm{rel}_k(A, B, C, D)$ $k = 1, 2$ where $e_A + e_B = e_I + e_{E_2}, e_C + e_D = e_J + e_{E_2}$.

We denote by T the set of all $t(A, B, C, D)$ with $|A| = |B| = |C| = |D| = n$ such that either $\mathrm{rel}_1(A\backslash i, B, C \cup i, D)$ or $\mathrm{rel}_2(A, B\backslash i, C, D \cup i)$ is in Rel for some i. In particular, T is the set of terms relevant to this argument.

Terms $t(A, B, C, D)$ in T are "extra terms" if and only if there is some $i \in E_2$ such that $i \in A, B$ but not in C nor in D. This can be rewritten as saying that a term is an "extra term" if and only if it belongs to both a rel_1 and a rel_2; otherwise it belongs to a rel_1 or a rel_2, but not both.

Let $T_{\min}$ be the subset of T consisting of all the terms whose evaluation is minimal. No relation in Rel contains exactly one term of $T_{\min}$, so given any term of $T_{\min}$ and any relation containing it we are able to get another term.

Given $T_{\min}$ we can form a hypergraph where the vertices are all the elements of $T_{\min}$, and the hyperedges are that we have an edge if elements of $T_{\min}$ are all in the same relation of Rel. Here we are using the notion that a *hypergraph* H is an ordered pair (V, E), where V and E are disjoint finite sets such that $V \neq 0$ together with a function $\psi : E \to 2^V$. Elements of V are called *vertices*, and elements of E are called *(hyper)edges* [33]. We call $H' = (V', E')$ a *hypersubgraph* of a hypergraph H if $V' \subseteq V$ and $E' \subseteq E$

[33]. We note that each vertex labelled by an "extra term" is incident to both a rel_1 and a rel_2 edge, and thus has two hyperedges incident to it.

This hypergraph formed from T_{min} must have a hypersubgraph G which is either a cycle, or a path between two non-extra vertices. If the path is between two non-extra vertices which come from different terms of the tropical Plücker relation Equation (4.2) then Equation (4.2) attains its minimum at least twice, and hence we have proved the result.

If not, we claim that G has two vertices $t(A_1, B_1, C_1, D_1)$ and $t(A_2, B_2, C_2, D_2)$ with $e_{A_1} + e_{B_1} = e_{A_2} + e_{B_2}$. If G is a path then choose the two endpoints as these vertices, since by our assumption that they come from the same term of the tropical Plücker relation from Equation (4.2) then this gives us vertices satisfying the above condition.

Otherwise, if G is a cycle, pick a vertex v in the cycle and an edge incident to it. There is an element i corresponding to this incidence in rel_1 and rel_2, and in fact it's the same i for both. So if $v = t(A_1, B_1, C_1, D_1)$, then i is in both A_1 and B_1. One neighbour of v has i in the A coordinate but not in the B, and the other has i not in the A coordinate but i in the B.

In $G \backslash v$ in order to transition between these two states there must be another vertex w such that i is in both the A and B coordinates. Thus v and w are the vertices sought. Now why does this w necessarily exist? The fact that in v we have i in both A_1 and B_1, we necessarily have i in $e_I + e_{E_2}$ since we have $e_{A_1 \backslash i} + e_{B_1} = e_I + e_{E_2}$ and $e_{A_1} + e_{B_1 \backslash i} = e_I + e_{E_2}$. This means that we have to pass through the state whereby i lies in both the A and B parts, and hence we have $e_{A_2} + e_{B_2} = e_{I \cup i} + e_{E_2}$, for some $w = t(A_2, B_2, C_2, D_2)$.

So in either case we have found two vertices $t(A_1, B_1, C_1, D_1)$ and $t(A_2, B_2, C_2, D_2)$ in T_{min} such that $e_{A_1} + e_{B_1} = e_{A_2} + e_{B_2}$. Consequently we also have $t(A_1, B_1, C_2, D_2)$ and $t(A_2, B_2, C_1, D_1)$ in T_{min}. To see this firstly observe that $t(A_1, B_1, C_1, D_1) + t(A_2, B_2, C_2, D_2) = t(A_1, B_1, C_2, D_2) + t(A_2, B_2, C_1, D_1)$. Secondly, realise that we have the following relations: Firstly,

$$e_{A_1} + e_{B_1} = e_{A_2} + e_{B_2}.$$

Then by considering the relations for v in both Rel_1 and Rel_1

$$e_{A_1\backslash i} + e_{B_1} = e_I + e_{E_2}, \quad e_{C_1\cup i} + e_{D_1} = e_J + e_{E_2} \text{ from Rel}_1$$

$$e_{A_1} + e_{B_1\backslash i} = e_I + e_{E_2}, \quad e_{C_1} + e_{D_1\cup i} = e_J + e_{E_2} \text{ from Rel}_2$$

Similarly consider the relations for w in both Rel_2 and Rel_2

$$e_{A_2\backslash j} + e_{B_2} = e_I + e_{E_2} \quad e_{C_2\cup j} + e_{D_2} = e_J + e_{E_2} \text{ from Rel}_1$$

$$e_{A_2} + e_{B_2\backslash j} = e_I + e_{E_2} \quad e_{C_2} + e_{D_2\cup j} = e_J + e_{E_2} \text{ from Rel}_2$$

and thus by these we have

$$e_{A_1\backslash j} + e_{B_1} = e_I + e_{E_2} \quad e_{C_2\cup j} + e_{D_2} = e_{J\cup j} + e_{E_2}$$

and equivalently with the other new vertex. Now both new terms must be in $T_{\min}$ else one of them would have strictly lesser value than elements of $T_{\min}$.

Before we proceed further we show the following properties. These will enable us to use an iterative argument which will allow us to show the needed result:

1. For any two vertices $v_0 = t(A_1, B_1, C_1, D_1)$ and $v_1 = t(A_2, B_2, C_2, D_2)$ in one of these paths or cycles we have $|A_1 \cap C_1| = |A_2 \cap C_2|$.

2. Now for the two vertices which we sought, namely $v_0 = t(A_1, B_1, C_1, D_1)$ and $v_1 = t(A_2, B_2, C_2, D_2)$ where $e_{A_1} + e_{B_1} = e_{A_2} + e_{B_2}$, then $|A_1 \cap C_1| < |A_1 \cap C_2|$.

Let's first show (1). Consider vertices $v_0 = t(A_1, B_1, C_1, D_1)$ and $v_1 = t(A_2, B_2, C_2, D_2)$. Now in any edge in this subgraph G which is in rel_2 then the A and C entries stay constant. If an edge is a relation in rel_1 then the tropical Plücker relation must have $A_1 \cap C_1$ in both τ and σ which generate the tropical Plücker relation for $\mathcal{M}_1$, and no other element which is in both τ and σ. Therefore showing for any vertices $v_0 = t(A_1, B_1, C_1, D_1)$ and $v_1 = t(A_2, B_2, C_2, D_2)$ in either of these paths or cycles that $|A_1 \cap C_1| = |A_2 \cap C_2|$.

We move on to looking at the second statement. Given vertices $v = t(A_1, B_1, C_1, D_1)$ and $w = t(A_2, B_2, C_2, D_2)$ with the property that $e_{A_1} + e_{B_1} = e_{A_2} + e_{B_2}$, we can get from v to w via a series of edges from Rel. If the edge is from rel_2 then this relates to a tropical Plücker relation from M_2, then the A and C coordinates do not change, so we need only concern ourselves with edges from rel_1 which correspond to relations in M_1, of which we always get at least one.

For any tropical Plücker relation in M_1 that we encounter as an edge we must have $A_1 \cap C_1$ in our sets τ and σ, so this bounds us in the right direction. We must also have $e_\tau + e_\sigma = e_{A_1} + e_{C_1}$, else otherwise τ and σ will not be generating terms. So combining these facts gives us the result that we want. Namely that for $v_0 = t(A_1, B_1, C_1, D_1)$ and $v_1 = t(A_2, B_2, C_2, D_2)$ where $e_{A_1} + e_{B_1} = e_{A_2} + e_{B_2}$, that $|A_1 \cap C_1| < |A_1 \cap C_2|$.

We proceed to where we were prior to this useful digression. Given this first subgraph G we either find two minimisers of the Equation (4.2) in $T_{\min}$, as required, else we get $v_0 = t(A_1, B_1, C_1, D_1)$ and $v_1(A_2, B_2, C_2, D_2)$ such that $e_{A_1} + e_{B_1} = e_{A_2} + e_{B_2}$, where $|A_1 \cap C_1| = |A_2 \cap C_2| = k$. Then from this we know that $t(A_1, B_1, C_2, D_2)$ and $t(A_2, B_2, C_1, D_1)$ are vertices of $T_{\min}$ such that $|A_1 \cap C_2| > k$.

Now using one of the new vertices, namely $v_0 = t(A_1, B_1, C_2, D_2)$ we can again construct a suitable cycle or path. To do this, pick an edge from v_0 to, say, v_1, where v_1 is an incident vertex. Pick a different edge from v_1 to v_2 and so on. At some point this process will either reach a vertex v_j which is non-extra, or a vertex which has been encountered before, hence creating a cycle.

Now if v_0 is itself an extra vertex, then there is another edge to a vertex, say v_{-1}, and so on. Now we get a path if both of these processes stop by reaching a non-extra term, else we get a cycle.

Now in either case we get a path or a cycle. If we have the minimum being attained in two separate parts of Equation (4.2) we are done. Else we have that for any vertex in the cycle or path that the intersection between the A and the C parts are constant, and we iterate the process until we get appropriate minimums. Now this process must terminate since the size of the intersection with A and C must increase with each iteration and it is bounded above. □

Corollary 4.3.2. *Let N_1 be a matroid on $E_1 \cup E_2$ such that $E_2 \in \mathcal{B}(N_1)$, and let N_2 be a matroid on the ground set $E_2 \cup E_3$ such that $E_2 \in \mathcal{B}(N_2)$. Let N be a matroid on the ground set $E_1 \cup E_2 \cup E_3$ such that $N|(E_1 \cup E_2) = N_1$ and $N|(E_2 \cup E_3) = N_2$. Then B is a basis of N if there exists L_1, L_2 such that $L_1 \cup L_2 = E_2, L_1 \cap L_2 = B \cap E_2$ where $(B \cap E_i) \cup L_i$ is a basis of N_i for $i = 1, 2$.*

Proof. Direct consequence of Theorem 4.3.1 □

4.3.1.1 Interaction between extension of valuated linking system composition and the initial matroid operation

We present a result that says that the initial matroid operation and the extension of valuated linking system composition commutes. As we have mentioned before we have that the initial matroid of a valuated matroid is able to provide a local description of the tropical linear space. We are able to build on this in Theorem 4.4.19 where we explicitly give the flats of the extension of valuated linking system composition in the case where we consider non-valuated input. This allows us to immediately see the Bergman fan in the non-valuated case.

Theorem 4.3.3. *Given valuated matroids $\mathcal{M}_1, \mathcal{M}_2 \in T\bar{G}L_n$ and some $\boldsymbol{x}$ such that $E_2 \in \mathcal{B}(\mathrm{in}_{\boldsymbol{x}}(\mathcal{M}_1)), \mathcal{B}(\mathrm{in}_{\boldsymbol{x}}(\mathcal{M}_2))$. Then $\mathrm{in}_{\boldsymbol{x}}(\mathcal{M}_1 \star \mathcal{M}_2) = \mathrm{in}_{\boldsymbol{x}}(\mathcal{M}_1) \star \mathrm{in}_{\boldsymbol{x}}(\mathcal{M}_2)$*

Proof. We prove this by showing that the bases of $\mathrm{in}_{\mathbf{x}}(\mathcal{M}_1 \star \mathcal{M}_2)$ and $\mathrm{in}_{\mathbf{x}}(\mathcal{M}_1) \star \mathrm{in}_{\mathbf{x}}(\mathcal{M}_2)$ coincide.

Firstly, given $B \in \mathcal{B}(\text{in}_{\mathbf{x}}(\mathcal{M}_1 \star \mathcal{M}_2))$ we wish to show that $B \in \mathcal{B}(\text{in}_{\mathbf{x}}(\mathcal{M}_1) \star \text{in}_{\mathbf{x}}(\mathcal{M}_2))$. So we have that $\rho_{\mathcal{M}_1 \star \mathcal{M}_2}(B) + \sum_{i \in B} x_i \leq \rho_{\mathcal{M}_1 \star \mathcal{M}_2}(B') + \sum_{i \in B'} x_i$ for all $B' \in \binom{E_{123}}{s}$, and there exists L_1, L_2 such that $L_1 \cup L_2 = E_2$ and $L_1 \cap L_2 = B \cap E_2$ where

$$\begin{aligned}\rho^{\mathbf{x}}_{\mathcal{M}_1 \star \mathcal{M}_2}(B) &= \rho_{\mathcal{M}_1}(B \cap E_1 \cup L_1) + \rho_{\mathcal{M}_2}(B \cap E_3 \cup L_2) + \sum_{i \in B} x_i + \sum_{i \in E_2} x_i \\ &\leq \min_{\substack{L_1, L_2 | \\ L_1 \cup L_2 = E_2, L_1 \cap L_2 = B \cap E_2}} \rho_{\mathcal{M}_1}(B \cap E_1 \cup L_1) + \rho_{\mathcal{M}_2}(B \cap E_3 \cup L_2) + \sum_{i \in B} x_i + \sum_{i \in E_2} x_i\end{aligned}$$

Note by the condition that $E_2 \in \mathcal{B}(\text{in}_{\mathbf{x}}(\mathcal{M}_1)), \mathcal{B}(\text{in}_{\mathbf{x}}(\mathcal{M}_2))$ that we have that $E_2 \in \mathcal{B}(\text{in}_{\mathbf{x}}(\mathcal{M}_1 \star \mathcal{M}_2))$, and thus

$$\rho^{\mathbf{x}}_{\mathcal{M}_1 \star \mathcal{M}_2}(E_2) = \rho_{\mathcal{M}_1}(E_2) + \rho_{\mathcal{M}_2}(E_2) + \sum_{i \in E_2} x_i + \sum_{i \in E_2} x_i$$

So consequentially, by taking the sum of $\rho^{\mathbf{x}}_{\mathcal{M}_1 \star \mathcal{M}_2}(B)$ and $\rho^{\mathbf{x}}_{\mathcal{M}_1 \star \mathcal{M}_2}(E_2)$, we have that $B \cap E_1 \cup L_1$ and $B \cap E_3 \cup L_2$ are bases of $\text{in}_{\mathbf{x}}(\mathcal{M}_1 \star \mathcal{M}_2)$ where

$$\rho^{\mathbf{x}}_{\mathcal{M}_1 \star \mathcal{M}_2}(B \cap E_1 \cup L_1) = \rho_{\mathcal{M}_1}(B \cap E_1 \cup L_1) + \rho_{\mathcal{M}_2}(E_2) + \sum_{i \in B \cap E_1 \cup L_1} x_i + \sum_{i \in E_2} x_i$$

and

$$\rho^{\mathbf{x}}_{\mathcal{M}_1 \star \mathcal{M}_2}(B \cap E_3 \cup L_2) = \rho_{\mathcal{M}_1}(E_2) + \rho_{\mathcal{M}_2}(B \cap E_3 \cup L_2) + \sum_{i \in B \cap E_3 \cup L_2} x_i + \sum_{i \in E_2} x_i.$$

Now since $\rho^{\mathbf{x}}_{\mathcal{M}_1 \star \mathcal{M}_2}(E_2) = \rho^{\mathbf{x}}_{\mathcal{M}_1 \star \mathcal{M}_2}(B \cap E_1 \cup L_1) = \rho^{\mathbf{x}}_{\mathcal{M}_1 \star \mathcal{M}_2}(B \cap E_3 \cup L_2)$ we thus have that $B \in \mathcal{B}(\text{in}_{\mathbf{x}}(\mathcal{M}_1) \star \text{in}_{\mathbf{x}}(\mathcal{M}_2))$.

Now for the other direction, namely that for any $B \in \mathcal{B}(\text{in}_{\mathbf{x}}(\mathcal{M}_1) \star \text{in}_{\mathbf{x}}(\mathcal{M}_2))$ then $B \in \mathcal{B}(\text{in}_{\mathbf{x}}(\mathcal{M}_1 \star \mathcal{M}_2)$. So for $B \in \mathcal{B}(\text{in}_{\mathbf{x}}(\mathcal{M}_1) \star \text{in}_{\mathbf{x}}(\mathcal{M}_2))$ we have that there exists L_1, L_2 where $L_1 \cup L_2 = E_2$ and $L_1 \cap L_2 = B \cap E_2$ such that $B \cap E_1 \cup L_1 \in \mathcal{B}(\text{in}_{\mathbf{x}}(\mathcal{M}_1))$ and

$B \cap E_3 \cup L_2 \in \mathcal{B}(\text{in}_{\mathbf{x}}(\mathcal{M}_2))$. So therefore we have

$$\begin{aligned}\rho^{\mathbf{x}}_{\mathcal{M}_1 \star \mathcal{M}_2}(B) &= \min_{\substack{L_1, L_2 | \\ L_1 \cup L_2 = E_2, L_1 \cap L_2 = B \cap E_2}} \rho_{\mathcal{M}_1}(B \cap E_1 \cup L_1) + \rho_{\mathcal{M}_2}(B \cap E_3 \cup L_2) + \sum_{i \in B} x_i + \sum_{i \in E_2} x_i \\ &= \rho_{\mathcal{M}_1}(B \cap E_1 \cup L_1) + \rho_{\mathcal{M}_2}(B \cap E_3 \cup L_2) + \sum_{i \in B} x_i + \sum_{i \in E_2} x_i\end{aligned}$$

as well as $\rho_{\mathcal{M}_1 \star \mathcal{M}_2}(B) \leq \rho_{\mathcal{M}_1 \star \mathcal{M}_2}(B')$ for all $B \in \binom{E_{123}}{s}$, or else we contradict minimality of $\rho^{\mathbf{x}}_{\mathcal{M}_1}(B \cap E_1 \cup L_1)$ and $\rho^{\mathbf{x}}_{\mathcal{M}_2}(B \cap E_3 \cup L_2)$. This gives us the necessary result. □

4.3.2 Some axioms of the hyperproduct

We will show that some of the axioms of being a hypergroup as given in Definition 72 are satisfied by our hyperproduct as defined in Definition 73. If we were able to show that all the axioms are satisfied then this would give us group like structure on a structure which can be related to $M_n(\mathbb{T})$, and this would give us an extension of the monoid of matrices with tropical entries under tropical multiplication.

We begin by showing that there is an identity element in $(T\bar{G}L_n, \boxdot)$, this being the first condition we need to show in order to show that we have a hypergroup. We introduce $\mathbf{1} = \bigoplus_{i=1,\ldots,n} U_i$ where U_i is the uniform matroid of rank 1 on 2 elements defined on $\{i, n+i\}$ as our proposed identity element.

Theorem 4.3.4. *Let $\mathcal{M} \in T\bar{G}L_n$. Then given $\mathbf{1}$ defined by $\mathbf{1} = \bigoplus_{i=1,\ldots,n} U_i$, where U_i is the rank 1 uniform matroid on $\{i, n+i\}$, we have $\mathbf{1} \boxdot \mathcal{M} = \mathcal{M} \boxdot \mathbf{1} = \{\mathcal{M}\}$, and hence $\mathbf{1}$ is the identity element of $(T\bar{G}L_n, \boxdot)$.*

Proof. We prove that $\mathbf{1} \boxdot \mathcal{M} = \mathcal{M}$, and note that the argument for $\mathcal{M} \boxdot \mathbf{1} = \mathcal{M}$ is similar.

Define $d(B)$, where $B \in \binom{E_{123}}{n}$, to be the distance function defined as $d(B) = |B \cap E_1|$. By combining Proposition 4.2.24 of [18] and Theorem 4.3.1 we have $\mathcal{M} \in \mathbf{1} \boxdot \mathcal{M}$, so it remains to show that is it the only such element of the hyperproduct. It can more directly be seen that $\mathcal{M} \in \mathbf{1} \boxdot \mathcal{M}$ by directly examining the formula for the composition of valuated linking systems of $\mathbf{1}$ and $\mathcal{M}$. Let any $\tau \in \mathbf{1}\hat{\boxdot}\mathcal{M}$, which we know restricts to $\mathbf{1}$ and $\mathcal{M}$, be normalised so $\rho_\tau(E_2) = 0$, and hence also restricts to some $z \in \mathbf{1} \boxdot \mathcal{M}$.

We look at the tropical Plücker relations of τ to show that for each B there is only a single choice for its valuation in order for it to be able to correctly restrict to both $\mathbf{1}$ and $\mathcal{M}$. We work through all such B not in either $\binom{E_{12}}{n}$ or $\binom{E_{23}}{n}$, since these are already uniquely defined based upon their values from $\mathbf{1}$ and $\mathcal{M}$, inductively based on the size of $d(B)$.

If $d(B) = 0$ then for any such B we must have a single valuation since these are precisely the elements of $\mathcal{M}$, and thus there is only a single possible valuation.

Inductively, assume that all B where $d(B) \leq k$ are such that B can only have a single fixed valuation. Let $B = X \cup Y \cup Z$ be such that $X \subseteq E_1, Y \subseteq E_2, Z \subseteq E_3$, $d(B) = k+1$, and $|Z| > 0$, else $B \in E_{12}$, and hence we know it has a fixed single valuation.

Consider the tropical Plücker relation formed by $\sigma = \{X\backslash x \cup Y \cup Z\}, \phi = \{E_2 \cup x\}$. Every term of this relation is infinite except for at most two, namely:

$$\rho_\tau(X \cup Y \cup Z) + \rho_\tau(E_2) \quad \text{and} \quad \rho_\tau(X\backslash x \cup Y \cup Z \cup (x+n)) + \rho_\tau(E_2\backslash(x+n) \cup x).$$

This is since in every other term of the tropical Plücker relation we have the term $\rho_\tau(E_2\backslash y \cup x)$ such that $y \neq x+n$, and thus every $E_2\backslash y \cup x)$ is a set B such that $d(B) \leq k$ is fixed by the inductive hypothesis. Therefore it must be equal to the value coming from the extension of composition of valuated linking systems. We have for any B such that $\{i, n+i\} \in B$ that $\rho_\tau(B) = \infty$, where this comes from the extension of valuation linking system composition.

This shows exactly what is required. The valuation of $\tau(E_2)$ is 0, so the valuation of $B = X \cup Y \cup Z$ is defined by the sum of the valuations of $X\backslash x \cup (x+n) \cup Y \cup Z$ and $E_2\backslash(x+n) \cup x$, recalling that $x \in X$ and $x+n \notin X$.

Now we see that $X\backslash x \cup (x+n) \cup Y \cup Z$ is such that $d(X\backslash x \cup (x+n) \cup Y \cup Z) = k$ so that has already been uniquely determined. We also note that $E_2\backslash(x+n) \cup x$ is in $\mathbf{1}$ so it is also fixed, so we are done.

The other implication follows similarly, that is, $\mathcal{M} \boxdot \mathbf{1} = \mathcal{M}$. □

We move on to showing the existence of inverses in $(T\bar{G}L_n, \boxdot)$, that is, we show that the second condition of Definition 72 is satisfied. We claim that the inverse of $\mathcal{M}_1 \in T\bar{G}L_n$, defined on E_{12}, is simply the valuated matroid on E_{12} such that E_1 and E_2 have been permuted.

Theorem 4.3.5. *For any $\mathcal{M} \in T\bar{G}L_n$ there exists a unique element $\mathcal{M}^{-1} \in T\bar{G}L_n$ such that $(\mathcal{M}^{-1})^{-1} = \mathcal{M}$, and where $\mathbf{1} \in \mathcal{M} \boxdot \mathcal{M}^{-1}$ and $\mathbf{1} \in \mathcal{M}^{-1} \boxdot \mathcal{M}$.*

Proof. We firstly show that $\mathbf{1} \in \mathcal{M} \boxdot \mathcal{M}^{-1}$. Since $\mathbf{1} \boxdot \mathcal{M} = \mathcal{M}$, we have some $\tau \in \mathbf{1} \hat{\boxdot} \mathcal{M}$, where the projection to E_{12} is 1, to E_{23} it is $\mathcal{M}$, and to E_{13} it is $\mathcal{M}$. In particular, we can choose this τ such that $\tau = \mathbf{1} \star \mathcal{M}$.

Take $\mathbf{1} \star \mathcal{M}$ and permute E_2 and E_3. This gives us a new valuated matroid τ' on E_{123}. We note τ' restricts to $\mathbf{1}$ on E_{13}, $\mathcal{M}$ on E_{12}, and a valuated matroid, which we'll call $\mathcal{M}^{-1}$, on E_{23}. We also note that clearly $(\mathcal{M}^{-1})^{-1} = \mathcal{M}$. So this gives us $\mathbf{1} \in \mathcal{M} \boxdot \mathcal{M}^{-1}$.

Now to show $\mathbf{1} \in \mathcal{M}^{-1} \boxdot \mathcal{M}$, which follows similarly. Given that $\mathcal{M} \boxdot \mathbf{1} = \mathcal{M}$. We again have a valuated matroid, namely $\mathcal{M} \star \mathbf{1} \in \mathcal{M} \hat{\boxdot} 1$ on E_{123}. Similarly permute, but this time on E_1 and E_2. This gives us a new valuated matroid on E_{123} which restricts to $\mathcal{M}^{-1}$ on E_{13}, $\mathbf{1}$ on E_{13}, and $\mathcal{M}$ on E_{23}, and hence $\mathbf{1} \in \mathcal{M}^{-1} \boxdot \mathcal{M}$. Again this has $(\mathcal{M}^{-1})^{-1} = \mathcal{M}$.

Now why does this in fact give us a unique element to be our inverse of $\mathcal{M}$. This comes from the fact that $\mathcal{M} = \mathbf{1} \hat{\boxdot} \mathcal{M}$, that is, is it the unique element of the hyperproduct. The argument is the same as in the proof of Theorem 4.3.4. □

The final property which we need to prove in order to show that $(T\bar{G}L_n, \boxdot)$ is a hypergroup is that it is associative. Currently we can show that the hyperproduct is associative in small cases but for larger cases this status is currently unknown. We present in Section 4.5 an argument to show that we are not able to generalise a naive approach for larger n. However, we are able to show that we have associativity for $(T\bar{G}L_1, \boxdot)$ and $(T\bar{G}L_2, \boxdot)$.

As a consequence of these we have the following result, and in particular, some small evidence towards Conjecture 4.2.1.

Theorem 4.3.6. *Let $n = 1, 2$. Then $(T\bar{G}L_n, \boxdot)$ is a hypergroup.*

Proof. These are a consequence of Theorem 4.3.4, Theorem 4.3.5, as well as in the $n = 1$ case Remark 36 and in the $n = 2$ case Corollary 4.3.11. □

4.3.3 n=2 case study

We investigate the structure of the hyperproduct of $\mathcal{M}_1$ and $\mathcal{M}_2$ where $\mathcal{M}_1, \mathcal{M}_2 \in T\bar{G}L_2$. We begin by giving a result regarding the structure of $\mathcal{M}_1 \boxdot \mathcal{M}_2$ for any fixed $\mathcal{M}_1, \mathcal{M}_2 \in T\bar{G}L_2$. We split this result into cases depending on the structure of $\mathcal{M}_1$ and $\mathcal{M}_2$, and in particular which terms of each of their three term tropical Plücker relations are minimal. Then we investigate the structure of $\mathcal{M}_1 \hat{\boxdot} \mathcal{M}_2$ where $\mathcal{M}_1, \mathcal{M}_2$ range over sets of $T\bar{G}L_2$ under given images of e. As we shall see this process is motivated by multiplication of tropical matrices. We end this section by giving a proof that $(T\bar{G}L_2, \boxdot)$ is associative, and hence a hypergroup.

Theorem 4.3.7. *Let $\mathcal{M}_1, \mathcal{M}_2 \in T\bar{G}L_2$. We give in Appendix A a full description of $\mathcal{M}_1 \boxdot \mathcal{M}_2$. This description given has been separated into cases depending on whether or not we are able to increase on the values of $p_{15}, p_{16}, p_{25}, p_{26}$ where these are from $\rho_{\mathcal{M}_1 \circ \mathcal{M}_2} = (p_{12}, p_{15}, p_{16}, p_{25}, p_{26}, p_{56})$.*

Now we begin to investigate the structure of $\mathcal{M}_1 \hat{\boxdot} \mathcal{M}_2$ for $\mathcal{M}_1, \mathcal{M}_2 \in T\bar{G}L_2$ with given images under e. In this case fix $\mathcal{A}, \mathcal{B} \in M_2(\mathbb{T})$ and compute their preimages under e from Section 4.2. Let $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $\mathcal{B} = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$.

Consider which $\mathcal{M}_1, \mathcal{M}_2 \in T\bar{G}L_2$ give $\mathcal{A}$ and $\mathcal{B}$ under e. The potential $\mathcal{M}_1$ and $\mathcal{M}_2$ we get such that $e(\mathcal{M}_1) = \mathcal{A}$ and $e(\mathcal{M}_2) = \mathcal{B}$ are given by

$$\rho_{\mathcal{M}_1} = (x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}) = (0, C, D, A, B, \Delta)$$
$$\rho_{\mathcal{M}_2} = (y_{12}, y_{13}, y_{14}, y_{23}, y_{24}, y_{34}) = (0, G, H, E, F, \Delta')$$

where we fix the 12 coordinate to be 0 just as we do when we define e in Section 4.1. We do this by tropically multiplying each entry of the tropical Plücker vector by a scalar which we are able to do since we still recover the same tropical linear subspace. We also use the Δs since this entry is supposed to be suggestive of the classical determinant.

In order to do something akin to multiplying the tropical matrices $\mathcal{A}$ and $\mathcal{B}$ we first make y_{12} equal to x_{34}, which we are able to do since they are both non infinite, and so by tropically multiplying each term in $\rho_{\mathcal{M}_2}$ by Δ we obtain $\rho_{\mathcal{M}_2} = (\Delta : G + \Delta : H + \Delta : E + \Delta : F + \Delta : \Delta' + \Delta)$. This will enable us to find each element $z \in \mathcal{M}_1 \hat{\boxdot} \mathcal{M}_2$.

Every element z of $\mathcal{M}_1 \hat{\boxdot} \mathcal{M}_2$ must be of the form

$$(z_{12} : z_{13} : z_{14} : z_{15} : z_{16} : z_{23} : z_{24} : z_{25} : z_{26} : z_{34} : z_{35} : z_{36} : z_{45} : z_{46} : z_{56}) =$$
$$(0 : C : D : z_{15} : z_{16} : A : B : z_{25} : z_{26} : \Delta : G + \Delta : H + \Delta : E + \Delta : F + \Delta : \Delta' + \Delta),$$

where $z_{15}, z_{16}, z_{25}, z_{26}$ are unknowns.

We solve for all possible values $z_{15}, z_{16}, z_{25}, z_{26}, \Delta, \Delta'$ so that we have $(z_{12}, z_{15}, z_{16}, z_{25}, z_{26}, z_{56}) \in T\bar{G}L_2$ corresponding to an element of $\mathcal{M}_1 \boxdot \mathcal{M}_2$. In this case we allow for Δ and Δ' to be unknowns to allow this to be more similar to the tropical monoid product of our matrices $\mathcal{A}$ and $\mathcal{B}$, and every $\mathcal{M}_1, \mathcal{M}_2$ of this form is what gives us $\mathcal{A}$ and $\mathcal{B}$.

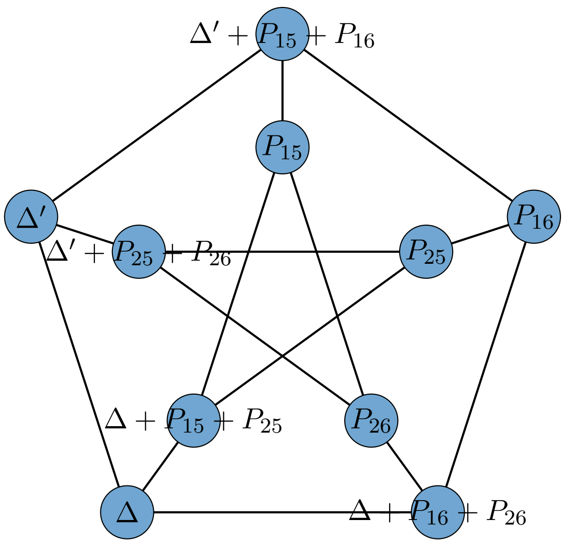
In particular, the monoid product of $\mathcal{M}_1$ and $\mathcal{M}_2$, $\mathcal{M}_1 \circ \mathcal{M}_2$, is given by

$$(P_{12}, P_{15}, P_{16}, P_{25}, P_{26}, P_{56}) = (0, P_{15}, P_{16}, P_{25}, P_{26}, \Delta + \Delta'),$$

where $P_{15}, P_{16}, P_{25}, P_{26}$ are given by valuated linking system composition.

When considering $\mathcal{M}_1 \boxdot \mathcal{M}_2$ we know that we always have an element given by valuated linking system composition. We look to see what other elements there are in the hyperproduct, for example, can we alter the value of P_{15} to make it $P_{15} + 5$, whilst retaining that $(0, P_{15}+5, P_{16}, P_{25}, P_{26}, \Delta+\Delta')$ is an element of the hyperproduct. These new elements of $\mathcal{M}_1 \boxdot \mathcal{M}_2$ will be obtained from the element given by valuated linking system composition by adding positive multiples of vectors which have all entries being either 0 or 1. This can be seen by observing the 15 tropical Plücker relations for z.

We now present a specific case of the structure of $\mathcal{M}_1 \hat{\boxdot} \mathcal{M}_2$. Let $\mathcal{A} = \mathcal{B} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. The following is a description of $\bigcup_{\mathcal{M}_1, \mathcal{M}_2 | e(\mathcal{M}_1) = e(\mathcal{M}_2) = \mathcal{A}} \mathcal{M}_1 \hat{\boxdot} \mathcal{M}_2$, where this shows a fan with apex at $(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)$, and each of these show cones in particular directions.



This graph we obtain uses the following notation to indicate we are increasing by a certain amount. $\Delta + P_{15} + P_{25}$ means increase all of Δ, P_{15} and P_{25} by the same amount, and the rest are similar. If there is a line connecting them then that means we can do both simultaneously. This structure can be seen by examining the 15 tropical Plücker relations given by z.

Theorem 4.3.8. *Let $\mathcal{A}, \mathcal{B} \in M_2(\mathbb{T})$. In Appendix B we have categorised*

$$\bigcup_{\mathcal{M}_1,\mathcal{M}_2 | e(\mathcal{M}_1)=\mathcal{A}, e(\mathcal{M}_2)=\mathcal{B}} \mathcal{M}_1 \hat{\boxdot} \mathcal{M}_2$$

Remark 38. Once we go beyond the 2-dimensional case then the picture becomes significantly more complicated, and we cannot describe it in such detail. This is a consequence of $\mathcal{M}_1 \boxdot \mathcal{M}_2$ being a subset of the Dressian $\mathrm{Dr}(3, 9)$ even in the next simplest case and this hasn't yet been fully described.

4.3.3.1 Associativity of $(T\bar{G}L_2, \boxdot)$

We now show that $(T\bar{G}L_2, \boxdot)$ is associative. Along with the results from Section 4.3.2 which give us an identity element and the existence of inverses this means that $(T\bar{G}L_2, \boxdot)$ is a hypergroup. The argument we use to show associativity does not generalise for $n \geq 3$. We shall see in Section 4.5 that in general we cannot easily use any approach of this type to show associativity of our hyperoperation for larger n.

Before we proceed with the argument we introduce some key concepts, results and definitions which we will require in order to show that $(T\bar{G}L_2, \boxdot)$ is associative.

4.3.3.1.1 Preliminaries for associativity of $(T\bar{G}L_2, \boxdot)$

Definition 75. [17] Given a tree T where associated with each edge is a positive length. Between any two leaves v_i and v_j there is a unique path in our tree T. We define the distance $d(v_i, v_j)$ between the two leaves as the sum of all the edge lengths along this path. This gives a finite metric space on the leaves of T. Specifically, any metric space which arises from a metric tree T is called a *tree metric*

Proposition 4.3.9. *[17] Given a rank 2 valuated matroid $\mathcal{M}$ on E which has no loops then there is a metric tree T with distance function d. This tree, T, is a labelling of its vertices by elements of E, and the distance function d is such that there is a a vector $(x_i \mid i \in E)$ such that*

$$\rho_{\mathcal{M}}(ij) = x_i + x_j - d(v_i, v_j)$$

where $d(v_i, v_j)$ is said to be $-\infty$ if $\rho_{\mathcal{M}}(ij) = \infty$.

Remark 39. We are able to view leaf edges as having negative lengths in T by altering the values of x_i and x_j, and we can also note that altering the values of x_i and x_j is equivalent to accomplishing a rescaling of a tropical Plücker vector.

In the case which we are considering we have no loops since E_i, E_{i+1} are bases of $\mathcal{M}_i$, and thus we obviously don't have any loops.

4.3.3.1.2 Process for creating a matrix representation of elements of $T\bar{G}L_2$

We give a procedure in order to show that for any rank 2 valuated matroid $\mathcal{M}$ with no loops that we are able to give an explicit matrix representation. We need only find a lift of any rescaling of $\rho_{\mathcal{M}}$, and then we are easily able to accomplish a rescaling of this by just multiplying the i-th column by a field element of valuation x_i for each i.

Let $\mathcal{M}$ be a loopless rank 2 valuated matroid with an associated metric tree T given by Proposition 4.3.9. Pick an internal vertex r of T and declare it the root. Extend all the leaf edges so that every distance $d(r, v_i)$ is a constant d_r which is independent of i, and call this new tree T'. Extending the lengths of the leaf edges has the same effect on the tropical Plücker vector as a rescaling. We label each edge e of T' by an element $c_e \in K$, for some field K, where this has a valuation on it such that $\operatorname{val}(c_e) = 0$. In particular, we require that each edge is labelled in such a way that $\operatorname{res}(c_e) \neq \operatorname{res}(c_f)$ where e and f have the same source vertex. In order to do this we choose as our field the generalised Puiseux series, and this allows us to always choose valuations on our edges such that the their values in the residue field aren't the same and in particular, this enables us to label our finite tropical numbers by real numbers [34].

Now we define the matrix M whose ith column, for each $i \in E$, is the column vector $(1, a_i)$, where $a_i = \sum_{j=0}^{k-1} c_{e_j} t^{2d(r,u_j)}$ such that on the unique path from r to v_i the vertices encountered are $u_0 = r, u_1, ..., u_k = v_i$, with e_j being the edge from u_j to u_{j+1}.

We now note what the maximal minors of M, the matrix corresponding to the tropical Plücker vector given by the metric tree T', actually are. The determinant of the minor on columns i, j is $a_j - a_i$, whose valuation is $2d(r, u_k)$ where u_k is the last vertex shared by the paths from r to v_i and from r to v_j. This is since all the terms prior to the kth cancel, but the kth does not by the fact that we demand distinct residues on our edge labels. On the other hand, in the unique path from v_i to v_j in T', u_k is the unique nearest point to r, and so $-d(v_i, v_j) = -2d_r + d(r, v_i) + d(r, v_j) - d(v_i, v_j) = -2d_r + 2d(r, u_k)$ where the second equality is the one using the metric geometry. So the vector of distances $-d(v_i, v_j)$ agrees with the Plücker vector of our matrix up to a global additive constant $-2d_r$ which can be ignored for valuated matroids.

So from this we have a matrix which describes a rescaling of $\rho_{\mathcal{M}}$. In order to get a matrix which represents $\rho_{\mathcal{M}}$ we just multiply the columns which correspond to the correct rescaling.

4.3.3.1.3 After the set-up We use the process outlined in Paragraph 4.3.3.1.2 to construct a matrix representation of a loopless rank 2 valuated matroid in order to show that we have associativity of $(T\bar{G}L_2, \boxdot)$. In order to do so given valuated matroids $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3 \in T\bar{G}L_2$ we show that we can find a valuated matroid $\mathcal{R} \in \mathrm{Dr}(2, 8)$ such that it restricts to $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$. We show in Section 4.5 that this form of argument does not work in general in the case where $n \geq 3$.

Let $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3 \in T\bar{G}L_2$ be valuated matroids on edge sets E_{12}, E_{23}, E_{34} respectively. An element of $(\mathcal{M}_1 \boxdot \mathcal{M}_2) \boxdot \mathcal{M}_3$ is a valuated matroid $\mathcal{Q}$ on E_{14} such that there exists valuated matroids $\mathcal{A}$ on E_{123} and $\mathcal{B}$ on E_{134} with restrictions back to

$\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{Q}$ and that $\mathcal{A}$ and $\mathcal{B}$ agree on their common restriction to E_{13}. Now an element of $\mathcal{M}_1 \boxdot (\mathcal{M}_2 \boxdot \mathcal{M}_3)$ is $\mathcal{Q}'$ so that there exists matroids $\mathcal{A}'$ and $\mathcal{B}'$ on E_{234} and E_{124} with the correct restrictions, akin to $\mathcal{A}$ and $\mathcal{B}$. We will call the pairs $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}', \mathcal{B}')$ *certificates* which show that $\mathcal{Q}$ and $\mathcal{Q}'$ are elements of $(\mathcal{M}_1 \boxdot \mathcal{M}_2) \boxdot \mathcal{M}_3$ and $\mathcal{M}_1 \boxdot (\mathcal{M}_2 \boxdot \mathcal{M}_3)$ respectively.

The content of Theorem 4.3.10 is the following. Given $\mathcal{Q} \in (\mathcal{M}_1 \boxdot \mathcal{M}_2) \boxdot \mathcal{M}_3$ and a certificate $(\mathcal{A}, \mathcal{B})$ we show that we can find a certificate $(\mathcal{A}', \mathcal{B}')$ showing that $\mathcal{Q} \in \mathcal{M}_1 \boxdot (\mathcal{M}_2 \boxdot \mathcal{M}_3)$. By symmetry this shows that $(\mathcal{M}_1 \boxdot \mathcal{M}_2) \boxdot \mathcal{M}_3 = \mathcal{M}_1 \boxdot (\mathcal{M}_2 \boxdot \mathcal{M}_3)$. This is achieved by finding a valuated matroid on E_{1234} which satisfies the correct restrictions.

Theorem 4.3.10. *Let $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3 \in T\bar{G}L_2$ be valuated matroids on edge sets E_{12}, E_{23}, E_{34} respectively. Given $\mathcal{Q} \in (\mathcal{M}_1 \boxdot \mathcal{M}_2) \boxdot \mathcal{M}_3$ along with a certificate $(\mathcal{A}, \mathcal{B})$ then there exists a certificate $(\mathcal{A}', \mathcal{B}')$ showing that $\mathcal{Q} \in \mathcal{M}_1 \boxdot (\mathcal{M}_2 \boxdot \mathcal{M}_3)$.*

Proof. We show that we have a 2×8 matrix whose maximal minors give a rank 2 valuated matroid on 8 elements which correctly restricts to $\mathcal{A}$ and $\mathcal{B}$ on ground sets E_{123} and E_{134} respectively. We are able to use this valuated matroid which we can restrict to appropriate ground sets to form $\mathcal{A}'$ and $\mathcal{B}'$ and this gives us our required certificates.

We split this into two cases as to whether we have to deal with ∞s or not. Firstly, consider the case where we have no ∞s arising in the valuations of either $\mathcal{A}$ or $\mathcal{B}$.

Now $\mathcal{A}$ and $\mathcal{B}$ are loopless since we have that E_1, E_2, E_3 are bases of $\mathcal{A}$ and E_1, E_3, E_4 are bases of $\mathcal{B}$, and these cover the respective ground sets. Since $\mathcal{A}, \mathcal{B}$ are rank 2 and loopless then we can write them both as metric trees by Proposition 4.3.9. Call these $T_{\mathcal{A}}$ and $T_{\mathcal{B}}$ respectively. Let r be an internal vertex of the subtree of $T_{\mathcal{A}}$ on E_{13}. Then by the process outlined in Paragraph 4.3.3.1.2 make all leaf edges of $T_{\mathcal{A}}$ a constant distance d_r from this r. This gives us a tree associated with some rescaling of $\mathcal{A}$. Do this also for $T_{\mathcal{B}}$.

By using the process described in Paragraph 4.3.3.1.2 we have a matrix associated with each tree, and importantly we are able to choose the internal vertices r such that the matrices agree on the columns for E_{13}. They agree if we choose the same r for both which lives in the subtree for E_{13}, and the rescaling to get back our original tropical Plücker vectors is the same for leaf edges in both $\mathcal{A}$ and $\mathcal{B}$, plus the paths are the same in both.

We are able to rescale on each of these matrices to obtain matrices which correspond to our valuated matroids. Both matrices still agree on their overlaps to the columns for E_{13} since we don't end up rescaling on these vertices. So we can merge the matrices to get our required 2×8 matrix. This shows associativity since we can get $(\mathcal{A}', \mathcal{B}')$ by restricting this 2×8 matrix to the appropriate columns.

Now for the case where we have some ∞s in our tropical Plücker vectors. We use the notation that if two nodes have the same endpoint then the distance between them is ∞. So as before we can choose an internal vertex r in E_{13}. Then apply the same process as before. Then this all works as before. □

Corollary 4.3.11. $(T\bar{G}L_2, \boxdot)$ *is associative.*

Proof. Simple consequence of Theorem 4.3.10. □

4.3.4 Noncommutativity of hyperproduct for all $n \geq 2$

Theorem 4.3.12. *For all $n \geq 2$ we have that there exist $\mathcal{M}, \mathcal{N} \in T\bar{G}L_n$ such that $\mathcal{M} \boxdot \mathcal{N} \neq \mathcal{N} \boxdot \mathcal{M}$.*

Proof. We first prove this when $n = 2$ and then extend the argument for all $n \geq 3$.

Let $n = 2$. Given $\rho_{\mathcal{M}} = (0, 0, \infty, \infty, 0, 0), \rho_{\mathcal{N}} = (0, 0, \infty, 0, 0, 0) \in T\bar{G}L_2$, we show that $\mathcal{M} \boxdot \mathcal{N} \neq \mathcal{N} \boxdot \mathcal{M}$.

By valuated linking system composition we have

$$\rho_{\mathcal{M}\circ\mathcal{N}} = (0,0,0,0,\infty,0) \quad \text{and} \quad \rho_{\mathcal{N}\circ\mathcal{M}} = (0,\infty,0,0,0,0).$$

Then we are able to note by using arguments from Section 4.3.3 that $\mathcal{M}\circ\mathcal{N} \notin \mathcal{N}\boxdot\mathcal{M}$ and $\mathcal{N}\circ\mathcal{M} \notin \mathcal{M}\boxdot\mathcal{N}$, and hence $\mathcal{M}\boxdot\mathcal{N} \neq \mathcal{N}\boxdot\mathcal{M}$.

Now we move on to considering when $n \geq 3$. We define $\mathcal{M} \in T\bar{G}L_n$ in the following way. Let $\rho_{\mathcal{M}'} = (0,0,\infty,\infty,0,0)$ on $\{1,2,n+1,n+2\}$. We define $\mathcal{M}$ by

$$\mathcal{M} = \mathcal{M}' \oplus U(1,2) \text{ on } \{3,n+3\} \oplus \cdots \oplus U(1,2) \text{ on } \{n,2n\}.$$

Similarly we define $\mathcal{N} \in T\bar{G}L_n$ in an analogous way. Let $\rho'_{\mathcal{N}} = (0,0,\infty,0,0,0)$ on $\{1,2,n+1,n+2\}$. Then define $\mathcal{N}$ by

$$\mathcal{N} = \mathcal{N}' \oplus U(1,2) \text{ on } \{3,n+3\} \oplus \cdots \oplus U(1,2) \text{ on } \{n,2n\}.$$

In order to show the correct conclusion we consider the valuation given to $2, 2n+2,\ldots,3n$ for any $\tau \in \mathcal{M}\hat{\boxdot}\mathcal{N}$ and for any $\tau \in \mathcal{N}\hat{\boxdot}\mathcal{M}$ and we show that they can never coincide, and hence we obtain that $\mathcal{M}\boxdot\mathcal{N} \neq \mathcal{N}\boxdot\mathcal{M}$.

Look at $\rho_\tau(2,2n+2,\ldots,3n)$ for any $\tau \in \mathcal{M}\hat{\boxdot}\mathcal{N}$. Since τ is a valuated matroid which restricts correctly to $\mathcal{M}$ and $\mathcal{N}$ we have that it satisfies the tropical Plücker relations. In particular, it satisfies the one with generating functions $\theta = \{2n+2,\ldots,3n\}$ and $\sigma = \{2,n+1,\ldots,2n\}$. Therefore we have that the minimum of the following terms is attained at least twice

$$\rho_\mathcal{T}(2, 2n+2, \dots, 3n) + \rho_\mathcal{T}(n+1, \dots, 2n),$$
$$\rho_\mathcal{T}(n+1, 2n+2, \dots, 3n) + \rho_\mathcal{T}(2, n+2, \dots, 2n),$$
$$\rho_\mathcal{T}(n+2, 2n+2, \dots, 3n) + \rho_\mathcal{T}(2, n+1, n+3, \dots, 2n),$$
$$\rho_\mathcal{T}(n+3, 2n+2, \dots, 3n) + \rho_\mathcal{T}(2, n+1, n+2, n+4, \dots, 2n), \dots$$

All the terms are infinite except for those given by $\tau(2, 2n+2, , \dots, 3n) + \tau(n+1, \dots, 2n)$, and since $\tau(n+1, \dots, 2n)$ is finite, we therefore have that $\tau(2, 2n+2, , \dots, 3n)$ is infinite else we have a unique minimum.

Now we show that the valuation of $\{2, 2n+2, \dots, 3n\}$ is 0 in $\mathcal{N} \circ \mathcal{M}$, and thus we cannot have $\mathcal{M} \boxdot \mathcal{N} = \mathcal{N} \boxdot \mathcal{M}$. So by the definition of $\mathcal{N} \circ \mathcal{M}$ we can see that

$$\begin{aligned}\rho_{\mathcal{N}\circ\mathcal{M}}(2, 2n+2, \dots, 3n) = \min\{&\rho_\mathcal{N}(2, n+1, \dots, 2n-1) + \rho_\mathcal{M}(2n, 2n+2, \dots, 3n),\\ &\rho_\mathcal{N}(2, n+1, \dots, 2n-2, 2n) + \rho_\mathcal{M}(2n-1, 2n+2, \dots, 3n),\\ &\rho_\mathcal{N}(2, n+1, \dots, 2n-3, 2n-1, 2n) + \rho_\mathcal{M}(2n-2, 2n+2, \dots, 3n), \dots,\\ &\rho_\mathcal{N}(2, n+2, \dots, 2n) + \rho_\mathcal{M}(n+1, 2n+2, \dots, 3n)\}\end{aligned}$$

and we note that $\rho_\mathcal{N}(2, n+1, n+3, \dots, 2n) = 0$ and $\rho_\mathcal{M}(n+2, 2n+2, \dots, 3n) = 0$ and thus we have that $\rho_{\mathcal{N}\circ\mathcal{M}}(2, 2n+2, \dots, 3n)$ is finite, and hence we have $\mathcal{M} \boxdot \mathcal{N} \neq \mathcal{N} \boxdot \mathcal{M}$. □

4.3.5 Hyperplane splits

We introduce the notion of a hyperplane split in order to show how we can utilise them to obtain more elements of a hyperproduct from the element given by the composition of valuated linking systems.

4.3.5.1 Preliminaries

Definition 76. [35] A *split* of a polytope is a subdivision with precisely two maximal cells.

Remark 40. The splits of $\Delta(d, n)$ are necessarily regular, and the cells are matroid polytopes ([36], Lemma 7.4). Since the splits are regular subdivisions they enable us to look at valuated matroids.

Proposition 4.3.13. *([35], Proposition 4) For any proper non-empty subset $S \subset [n]$ and any positive integer $\mu < d$ with $d - |S| < \mu < n - |S|$ the (S, μ)-hyperplane equation*

$$\mu \sum_{i \in S} x_i = (d - \mu) \sum_{j \notin S} x_j$$

defines a split of $\Delta(d, n)$. Conversely, each split of $\Delta(d, n)$ arises this way. We are able to write this inhomogeneously as $\sum_{i \in S} x_i = d - \mu$, by taking $\sum_i x_i = d$.

Definition 77. [35] Two splits of a polytope P are *compatible* if their hyperplanes do not meet in a relatively interior point of P.

Remark 41. This allows us to say that two splits of $\Delta(d, n)$ are compatible if and only if there is a matroid subdivision refining both of them.

Proposition 4.3.14. *([36], Proposition 5.4) Two splits (S, μ) and (S, μ') of $\Delta(d, n)$ are compatible if and only if one of the following holds:*

$$|S \cap S'| \leq d - \mu - \mu' \qquad |S \backslash S'| \leq \mu' - \mu$$
$$|S' \backslash S| \leq \mu - \mu' \qquad |[n] \backslash S \backslash S'| \leq \mu + \mu' - d$$

So how do these hyperplane splits link to tropical Plücker vectors? Since hyperplane splits of $\Delta(d, n)$ give rise to regular matroid subdivisions, and the height functions of regular matroid subdivisions are tropical Plücker vectors, we'd like to know what the height functions are for these hyperplane splits.

From [35] we have the following. Let M be a rank d matroid on n elements. The

k-corank vector of M is the map

$$\rho_k(M) : \binom{[n]}{k} \to \mathbb{N}, \quad S \mapsto d - rk_M(S).$$

The regular subdivision of $\Delta(k, n)$ with lifting function $\rho_k(M)$ is the k-corank subdivision induced by the matroid M.

Lemma 4.3.15. *([35],Lemma 27) Let M be a rank d matroid on n elements. The k-corank vector of M is a tropical Plücker vector of rank k on n elements.*

When we consider $k = d$ then this allows us to look at regular matroid subdivisions of $\Delta(d, n)$ induced by M, and hence the d-corank vector we obtain is a tropical Plücker vector.

For any hyperplane split of $\Delta(d, n)$ we have a matroid polytope lying to either side. Choose one of these, say $P(M')$. To this we are able to associate the corank function of M', and hence we get an associated tropical Plücker vector to any hyperplane split of $\Delta(d, n)$. We also get another associated tropical Plücker vector by considering the matroid polytope lying to the other side of the split.

4.3.5.2 Investigations into hyperplane splits

We now investigate hyperplane splits of $0 \boxdot 0$. This is done in order to be able to utilise these splits to give new elements of our hyperproduct of $0 \boxdot 0$ for general n. In order to utilise these hyperplane splits we need to ensure that the corank vectors, that is, the tropical Plücker vectors, belong to $0 \boxdot 0$. We need to ensure they restrict to zero vectors on $\{1, \ldots, 2n\}$ and on $\{n+1, \ldots, 3n\}$. For any hyperplane split of $\Delta(n, 3n)$ we have matroid polytopes lying to each side of a (S, μ)-hyperplane. We choose the matroid polytope lying on the side which contains vertices of $\{1, \ldots, 2n\}$ and $\{n+1, \ldots, 3n\}$. Then we get an associated corank vector, which is our tropical Plücker vector. We are able to add this termwise to our zero vector to get a new solution. The criteria we need to ensure that this happens is given by

$$\begin{aligned} |\{1, \ldots, 2n\} \cap S| \leq d - \mu \\ |\{n+1, \ldots, 3n\} \cap S| \leq d - \mu \end{aligned} \tag{4.6}$$

where we are considering a split of $\Delta(n, 3n)$ coming from the (S, μ)-hyperplane defining a split of $\Delta(n, 3n)$. These conditions follows as a result of the fact that there needs to be no vertices from $\binom{\{1,\ldots,2n\}}{n}$ or $\binom{\{n+1,\ldots,3n\}}{n}$ appearing on one side of the (S, μ)-hyperplane in order to get the correct corank vector.

Remark 42. Beyond finding out more about the structure of $0 \boxdot 0$ in and of itself, it might also be hoped that this greater understanding would better help us understand $\mathcal{M} \boxdot \mathcal{N}$ for $\mathcal{M}, \mathcal{N} \in T\bar{G}L_3$. Based on a guess that potentially "many" rays of $\mathrm{Dr}(3, 9)$ might lie in $0 \boxdot 0$, then call this set of rays R_0. Then for some other $\mathcal{M} \boxdot \mathcal{N}$ we might be able to understand it by the finding a set R_1 of potentially fewer rays "near" $\mathcal{M} \circ \mathcal{N}$, and constructing $\mathcal{M} \boxdot \mathcal{N}$ by building a subcone complex of $\mathrm{Dr}(n, 3n)$ on rays $R_0 \cup R_1$ and intersecting it with the linear space that forces the Plücker coordinates to be correct. However, part of this relies on the intuition that $\mathcal{M} \circ \mathcal{N}$ is supposed to be the minimal element of $\mathcal{M} \boxdot \mathcal{N}$ so that $\mathcal{M} \boxdot \mathcal{N}$ is contractible and thus has a retraction onto the composition of valuated linking systems.

We now look at the compatibility condition in the case of $0 \boxdot 0$. So we convert Proposition 4.3.14 into language which is more useful in our case. This compatibility condition determines whether or not we are able to rescale using multiple such hyperplane splits simultaneously.

Lemma 4.3.16. *Let ρ_1 be the corank function for the split of $\Delta(n, 3n)$ on the (S, μ) hyperplane, and ρ_2 being a corank function for the split of $\Delta(n, 3n)$ on the (S', μ') hyperplane, such that the criteria given by Equation* (4.6) *is satisfied. Then these splits are* *compatible if and only if exactly one of the following conditions is satisfied*

$$|S \cap S'| + d \leq 2d - \mu - \mu'$$

$$|S \backslash S| \leq \mu' - \mu$$

$$|S' \backslash S'| \leq \mu - \mu'$$

Proof. This proof utilises Proposition 4.3.14. We show that the fourth condition given cannot ever be satisfied in our case.

$$\begin{aligned} |E_{123} \backslash S \backslash S'| = 3d - |S \cup S'| &= 3d - (|S| + |S'| - |S \cap S'|) \\ &= 3d - |S| - |S'| + |S \cap S'| \\ &> 3d - (2d - \mu) - (2d - \mu') + |S \cap S'| \\ &= \mu + \mu' - d + |S \cap S'| \\ &\geq \mu + \mu - d \end{aligned}$$

□

Example 12. Each of the three cases of Lemma 4.3.16 is able to arise. We firstly show that (2) and (3) can occur. Consider the $n = 4$ case, with hyperplane splits given by $(S, \mu) = (\{1, 2, 9, 10\}, 1)$ and $(S', \mu') = \{1, 2, 9, 10\}, 2)$. Firstly note that they are both hyperplane splits since

$$4 - |\{1, 2, 9, 10\}| = 0 < 1, 2 < 8 = 12 - |\{1, 2, 9, 10\}|$$

whilst also observing that the corank vectors restrict correctly to 0 vectors on $\{1, \ldots, 2n\}$ and $\{n + 1, \ldots, 3n\}$

$$|\{1, 2, 9, 10\} \cap E_{12}| = 2 \leq 4 - 1 = 3 \quad \text{and} \quad |\{1, 2, 9, 10\} \cap E_{23}| = 2 \leq 4 - 1 = 3$$

$$|\{1, 2, 9, 10\} \cap E_{12}| = 2 \leq 4 - 2 = 2 \quad \text{and} \quad |\{1, 2, 9, 10\} \cap E_{23}| = 2 \leq 4 - 2 = 2$$

Now by Corollary 5.6 of [36] we have that these hyperplane splits are compatible, and so we can see that the following will show that this is a case where only (2) arises, and similarly if we interchange the roles of S and S' we get similarly for (3).

$$|S' \backslash S| = |\{1,2,9,10\} \backslash \{1,2,9,10\}| = |\emptyset| = 0 \leq 2-1$$

Now we show an example of (1) arising. Let $n = 3$ and the hyperplane splits be given by $(S, \mu) = (\{1,2,7\}, 1)$ and $(S', \mu') = (\{3,8,9\}, 1)$. Then they themselves are hyperplane splits since

$$3 - |\{1,2,7\}| = 3 - 3 = 0 < 1 < 6 = 9 - |\{1,2,7\}|$$
$$3 - |\{3,8,9\}| = 3 - 3 = 0 < 1 < 6 = 9 - |\{3,8,9\}|$$

Also note that these enable us to have corank vectors which restrict correctly.

$$|\{1,2,7\} \cap E_{12}| = 2 \leq 3-1 = 2 \quad \text{and} \quad |\{1,2,7\} \cap E_{23}| = 1 \leq 3-1 = 2$$
$$|\{3,8,9\} \cap E_{12}| = 1 \leq 3-1 = 2 \quad \text{and} \quad |\{3,8,9\} \cap E_{23}| = 2 \leq 3-1 = 2$$

Now this does satisfy condition (1): $|\{1,2,7\} \cap \{3,8,9\}| = 0 \leq 3-1-1 = 1$ whereas it doesn't satisfy (2): $|\{1,2,7\} \backslash \{3,8,9\}| = 3 \leq 1-1 = 0$ nor (3): $|\{3,8,9\} \backslash \{1,2,7\}| = 3 \leq 1-1 = 0$

Lemma 4.3.17. *Let $(S_1, \mu_1), \ldots, (S_i, \mu_i)$ be hyperplane splits of $\Delta(n, 3n)$, with respective corank functions $\rho_1, \ldots, \rho_i$, such that each of these hyperplane splits satisfies the conditions given in Equation* (4.6) *and that all of the hyperplane splits are pairwise compatible. Then all of the splits are compatible.*

Proof. Since $\rho_1, \ldots, \rho_i$ are tropical Plücker vectors without infinities, we have that the underlying matroid is the uniform matroids, and hence by Proposition 2.3.9 all that leaves to show is that the three-term relations are satisfied. Any of these corank vectors restricted to a rank 2 minor on 4 elements is, up to rescaling, either all zero entries, else a single positive coordinate. In order to make coordinates P_I and P_J positive and still have a tropical Plücker vector then either $I = J$ or $I = \{1,2,3,4\} \backslash J$, and thus additions to these coordinates is perfectly fine, since they are in the same part of any three-term relation. □

Example 13. Consider $\bigcup_{\mathcal{M}_1,\mathcal{M}_2|e(\mathcal{M}_1)=e(\mathcal{M}_2)=0} \mathcal{M}_1 \hat{\boxdot} \mathcal{M}_2$ which we saw in Section 4.3.3, along with the Petersen graph we obtain. It can be seen that all of these rays come from hyperplane splits. Recall that any such $\rho_{\mathcal{M}_1}$ is of the form $(0,0,0,0,0,\Delta)$ and $\rho_{\mathcal{M}_2}$ is of the form $(0,0,0,0,0,\Delta')$, where Δ,Δ' is some number greater than or equal to 0. We apply a rescaling on every $\rho_{\mathcal{M}_1}$ by $(\Delta/2,\Delta/2,-\Delta/2,-\Delta/2)$, so that the 12 coordinate becomes Δ and the 34 coordinate becomes 0. Call these new functions $\rho^{\mathbf{x}}_{\mathcal{M}_1}$. So if we consider the following hyperplane splits we get the rays in the directions required.

$$(\{12\},1),(\{15\},1),(\{16\},1),(\{25\},1),(\{26\},1),(\{56\},1)$$

This gives us the rays in the directions we want with just the single elements. Now the following

$$(\{125\},1),(\{126\},1),(\{156\},1),(\{256\},1)$$

Now these give the points including 3 increasing simultaneously. We have the compatibility conditions are satisfied, for example, $(\{25\},1)$ and $(\{25,26,56\},1)$ are compatible due to Corollary 5.6 of [36]. Similarly, we can see that those like $(\{15\},1)$ and $(\{26\},1)$ are compatible by checking the conditions of Lemma 4.3.16. These allow us to do what we want.

We can rescale back by $(-\Delta+x_{12}/2,-\Delta+x_{12}/2,\Delta+x_{12}/2,\Delta+x_{12}/2,0,0)$ where x_{12} is whatever we end up increasing the Δ coordinate by in this construction.

4.3.6 Stiefel subdivisions

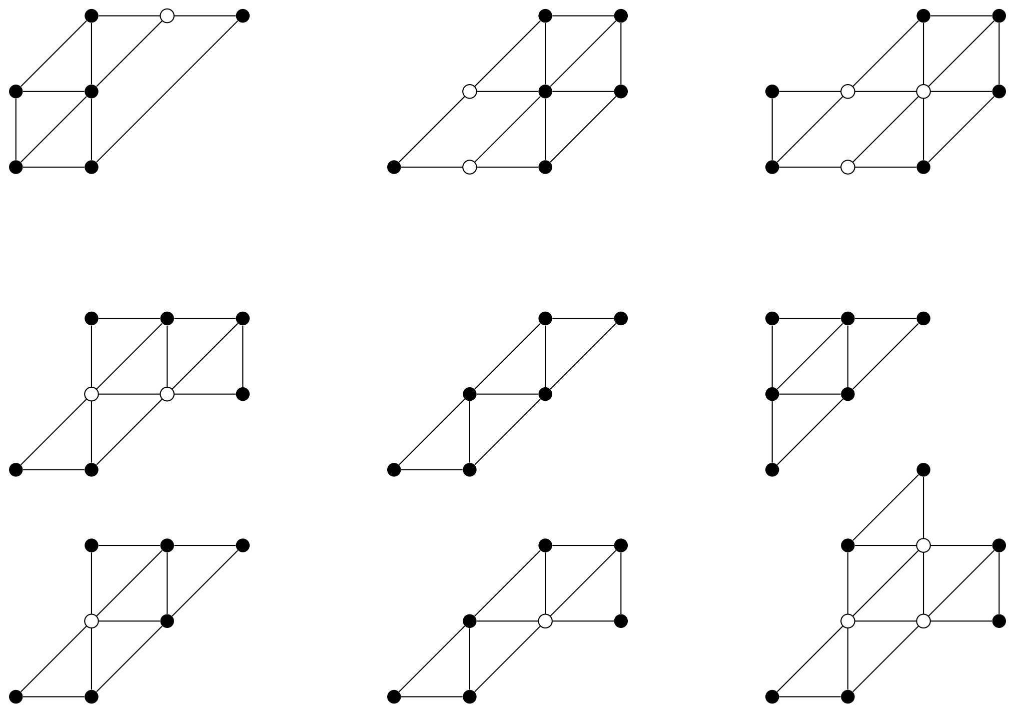
Throughout this section we attempt to use Stiefel subdivisions in order to attempt to find further elements of the hyperproduct which don't come from hyperplane splits. We will show in Theorem 4.3.19 that when $n=3$ in the case $0 \boxdot 0$ that we aren't able to utilise Stiefel subdivisions in order to see further elements of the hyperproduct.

We note that $\mathrm{Dr}(3,9)$, of which this is a subset, is the smallest such Dressian which hasn't been completed described as a list of its faces as a polyhedral complex, however we are able to describe some of its rays through the use of hyperplane splits and Stiefel subdivisions. Finding all rays of $\mathrm{Dr}(3,9)$ would be equivalent to finding all the coarsest subdivisions of the uniform matroid polytope, and thus by investigating Stiefel subdivisions we can look for more potential coarsest subdivisions which satisfy the criteria we require of them. We build on a result of Schröter [37] where they attempt to describe a cell complex structure on a subset of $\mathrm{Dr}(3,9)$, where they categorise all Stiefel subdivisions which do not come from hyperplane splits.

Definition 78. ([38], Definition 3.1) Let $M_{\hat{\mathbb{T}}}$ be the set of tropical matrices whose support contains a matching. The *tropical Stiefel map* is the map $\pi : M_{\hat{\mathbb{T}}} \to \operatorname{trop}\mathrm{Gr}(d,n)$ such that $\pi(A)_J$ is the $([d], J)$ tropical minor of A, that is, if $A = (a_{ij})$, then

$$\pi(A)_J = \min\{\sum_{(i,j)\in\lambda} a_{ij} \mid \lambda \text{ is a matching from } [d] \text{ to } J\}.$$

Example 14. ([37], Example 4.4 + Proposition 4.6 + Figure 5) The following nine rigid tropical point configurations are all coarsest subdivisions of $\Delta_2 \times \Delta_5$. The tropical Stiefel map induces coarsest matroid subdivisions of $\Delta(3,9)$.



Proposition 4.3.18. *([37], Proposition 4.6) The nine liftings illustrated as tropical point configurations as given in Example 14 (also Figure 5 of [37]) all lead to coarsest regular subdivisions of* $\Delta(3,9)$*. These are, up to symmetry, all the coarsest regular subdivisions which are induced by the Stiefel map and not by a hyperplane split.*

Theorem 4.3.19. *Let* $\mathcal{M},\mathcal{N} \in T\bar{G}L_3$ *be such that* $\rho_{\mathcal{M}} = \rho_{\mathcal{N}} = 0$*. All coarsest subdivisions in* $\mathcal{M} \boxdot \mathcal{N}$ *which are Stiefel subdivisions are also given by hyperplane splits.*

Proof. Take the nine rigid tropical point configurations given in Proposition 4.3.18, and consider any rescaling or dilation of these. Using the convention given in [37] we get the Stiefel subdivisions given by 3×9 matrices with tropical entries, where three of the columns are those of the tropical identity matrix, and the other six are the points of these nine point configurations after rescaling and dilation. Dilating a matrix A, which we can do by multiplying the whole matrix by a positive constant in the usual sense, dilates the valuated matroid. Tropically scaling a row of A, that is adding to each entry of a row in the usual sense, adds a constant to each entry of the tropical Plücker vector, and so does nothing to the valuated matroid. Scaling a column of A, similarly adding to each entry of column in the usual sense, accomplishes a rescaling in the way we outlined in Definition 44. Let A' be any of these matrices we obtain after rescaling and dilation.

We are able to assume that each entry in A' is non-negative if we rescale the tropical Plücker vector so that its minimum term is sufficiently large. One way to accomplish this is by rescaling on all rows by a sufficiently large number, say a. Apply this process and call the matrix we obtain A. This means that we have all three columns of the matrix $\begin{pmatrix} a & \infty & \infty \\ \infty & a & \infty \\ \infty & \infty & a \end{pmatrix}$, where a is some non-negative real number, appearing as columns of A.

Now take A and restrict it to columns indexed by $\{1, \ldots, 6\}$ and $\{4, \ldots, 9\}$ and call these matrices L and R respectively.

We split this into cases depending on how many columns of the tropical identity matrix appear as columns of L. We begin by looking at the case whereby all 3 columns of the tropical identity matrix are in L. Up to permutations of columns we have that L is of the form $\begin{pmatrix} a & \infty & \infty & b & c & d \\ \infty & a & \infty & e & f & g \\ \infty & \infty & a & h & i & j \end{pmatrix}$, where $b, c, d, e, f, g, h, i, j$ are unknowns.

We show that none of these matrices L can restrict correctly to $\rho_{U(3,6)}$ up to renormalisation. In order to yield a valuated matroid of this form we require that all of the unknowns are equal to a, and thus this means that L cannot come from any A of the form we are considering. Concretely, calling the valuated matroid given by the maximal minors of L ρ_L then $\rho_L(123) = 3a$, and thus for any other maximal minor we require it to be equal to $3a$. For example, $\rho_L(124) = \min\{a+a+h, a+e+\infty, \infty+\infty+h, \infty+e+\infty, b+\infty+\infty, b+a+\infty\} = 3a$ and thus $h = a$. The same reasoning can be applied for any other minor. Clearly this gives that any L is of the form such that multiple columns would have to be equal, and this cannot arise from any tropical point configuration given in Proposition 4.3.18 as all the points are distinct.

We now consider having two columns of the tropical identity matrix as columns of L. As a consequence of having the maximal minors of L yield a renormalisation of $\rho_{U(3,6)}$ we need to have up to permutation of rows and columns that L is a matrix of the form $\begin{pmatrix} a & \infty & & & & \\ \infty & a & & & & \\ \infty & \infty & b & b & b & b \end{pmatrix}$ where b is some non-negative real number. Thus we require a tropical Plücker vector with all entries being $2a+b$. So we have a matrix of the form, again up to permutation of rows and columns, $\begin{pmatrix} a & \infty & > & a & a & a \\ \infty & a & & & & \\ \infty & \infty & b & b & b & b \end{pmatrix}$ where $>$ is a number that is larger than or equal to a. We can only have at most one entry of the top row, in this case, to be larger than a, else the maximal minors do not yield a renormalisation of $(0, \dots, 0)$. Thus to get the correct tropical Plücker vector from this require the matrix to be of the form $\begin{pmatrix} a & \infty & > & a & a & a \\ \infty & a & > & a & a & a \\ \infty & \infty & b & b & b & b \end{pmatrix}$ or $\begin{pmatrix} a & \infty & > & a & a & a \\ \infty & a & a & > & a & a \\ \infty & \infty & b & b & b & b \end{pmatrix}$ up to permutation of row and columns.

Both of these cases are not possible as restrictions of any A since we have two identical columns and this again cannot arise from any tropical point configuration given in Proposition 4.3.18 as all the points are distinct, and thus we are done here.

If a single column of L is that of the tropical identity matrix then we are able to say that R contains at least two columns of the tropical identity matrix. By similar arguments to earlier we get that R cannot be a restriction of any A.

Now if zero columns of L are from the tropical identity matrix, then we are able to say that R contains all three columns of the tropical identity matrix, and hence by similar arguments to before we get that this case doesn't arise as a restriction of any A. $\square$

4.3.7 Non-convexity of the hyperproduct

We show that we don't have convexity of the hyperproduct in general. We consider $0 \boxdot 0$ and show that in this case we don't have convexity when $n \geq 3$.

Definition 79. A subset S of $\mathbb{R}^n$ is *tropically convex* if the set S contains the point $\min(a + \mathbf{x}, b + \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in S$ and all $a, b \in \mathbb{R}^n$.

Theorem 4.3.20. *Let $n \geq 3$. Given $U(n, 2n) \in T\bar{G}L_n$ then $U(n, 2n) \boxdot U(n, 2n)$ is not convex.*

Proof. We aim to find a combination of hyperplane splits that we can use to get new elements from the composition of valuated linking systems such that we can use them to show that we don't have tropical convexity. Consider the hyperplane $x_1 + x_{2n+2} = 1$. This is a hyperplane split since $|S| = 2$, so we have $n - 2 < \mu < 3n - 2$. So our choice of $\mu = n - 1$ works. So we increase all coordinates such that $x_1 + x_{2n+2} > 1$ by 5.

Also consider the hyperplane $x_1 + x_2 + x_{2n+1} + x_{2n+2} = 2$. Again this gives a hyperplane split since $|S| = 4$ and $n - 4 < \mu < 3n - 4$ and so $\mu = n - 2$ works for $n \geq 3$. So we increase all coordinates such that $x_1 + x_2 + x_{2n+1} + x_{2n+2} > 2$ by 2.

We note that both of these hyperplane splits satisfy the conditions from Equation (4.6), so we have the vertices we wish to stay fixed are all to one side of the hyperplane.

Call the tropical Plücker vector related to the first hyperplane split ρ_1 and the second hyperplane split ρ_2. Then we look at tropical convexity between these points when $a = 0$ and $b = 2$, that is we are consider the point given by $\rho_3(A) = \min\{\rho_1(A), \rho_2(A) + 2\}$ where $A \in \binom{2n}{n}$, and see whether it is an element of the hyperproduct.

Look at the tropical Plücker relation of ρ_3 given by $\sigma = \{1, \ldots, n-1\}$, $\tau = \{1, 3, 4, 5, \ldots, n-1, 2n+1, 2n+2, 2n+3\}$. Let $C = \{1, 3, 4, 5, \ldots, n-1\}$, then the tropical Plücker relation is explicitly given by

$$\min(\rho_3(C,2,2n+1)+\rho_3(C,2n+2,2n+3), \rho_3(C,2,2n+2)+\rho_3(C,2n+1,2n+3),$$
$$\rho_3(C,2,2n+3)+\rho_3(C,2n+1,2n+2)) = \min(0+2,4+0,0+4)$$

So the tropical Plücker relation has a unique minimum, and so ρ_3 does not define a valuated matroid, and hence is not an element of $U(n,2n) \boxdot U(n,2n)$. ☐

4.3.8 Not a fan

Proposition 4.3.21. *In general we do not have that the hyperproduct of two elements of $T\bar{G}L_2$ is a fan.*

Remark 43. As noted when $n = 2$ we do sometimes see a fan structure. We also still do have some polyhedral complex structure on our hyperproduct which comes from our solution set being a subset of the Dressian, and this structure is inherited from the Dressian $\mathrm{Dr}(n, 3n)$ by intersecting it with a coordinate subspace.

Proof. Let $\mathcal{M}_1, \mathcal{M}_2 \in T\bar{G}L_2$ be defined by $\rho_{\mathcal{M}_1} = (0,0,0,0,0,0)$ and $\rho_{\mathcal{M}_2} = (0,0,0,0,0,10)$. We have that $\mathcal{M}_1 \circ \mathcal{M}_2 \in \mathcal{M}_1 \boxdot \mathcal{M}_2$, where $\rho_{\mathcal{M}_1 \circ \mathcal{M}_2} = (0,0,0,0,0,10)$. Then we can increase in the direction $e_{15} + e_{16}$ up to 10 and this still satisfies the Plucker relations. However, if we try to increase by 11, say, we violate some of the tropical Plücker relations. For instance the one given by $\tau = \{1\}, \sigma = \{3,5,6\} : p_{13} + p_{56}, p_{15} + p_{36}, p_{16} + p_{35} = 10, 11, 11$. ☐

4.4 Relation to (valuated) linking system composition

4.4.1 Introduction

We recall from Section 4.3.1 that valuated linking system composition gives us an element of our hyperproduct $\mathcal{M}_1 \boxdot \mathcal{M}_2$ for $\mathcal{M}_1, \mathcal{M}_2 \in T\bar{G}L_2$. We begin with Section 4.4.2 by introducing linking systems, the non-valuated counterparts of valuated linking systems, along with introducing a way of composing linking systems. In Section 4.4.3 we review valuated linking systems which we introduced earlier in this chapter.

In Section 4.4.4 we give a description of the flats of the composition of linking systems, and in Section 4.4.5.3 we give a description of the flats of the extension of linking systems. The main intention for investigating the flats of both the composition of linking systems and also the extension thereof was in order to be able to think about the linear spaces of both of these new valuated matroids in terms of Bergman fans. In order to inspect tropical linear spaces locally we can use the initial matroid, which gives us a non-valuated counterpart to a valuated matroid. As previously mentioned in Remark 25 Bergman fans give us a local description of tropical linear spaces and we initially investigated the flats in order to see if we could use these in order to build up a global picture of any tropical linear space.

In Section 4.4.5.1 we give the rank function of the extension of the composition of valuated linking systems when we consider it to be $\{0, \infty\}$-valued. This is done in order to give a description of the flats of the extension of linking system composition which we give in Section 4.4.5.3. In order to obtain the rank function we give two separate proofs, one polyhedral, and the other more matroidal, the latter of which we give in Section 4.4.5.2.

4.4.2 Linking systems

We introduce linking systems, the non-valuated counterparts to valuated linking systems. Linking systems were first introduced in 1976 by Schrijver [27] as a way to generalise the relation between matroids with bipartite graphs and direct graphs. Linking systems are also known as bimatroids and this naming comes from the independent discovery in 1978 by Kung [39] who named them as such. Kung's investigations are about viewing a relation between matroids and results in invariant theory. Just like matroids there are various cryptomorphic ways to define a linking system, and in fact, a linking system is essentially equivalent to a matroid.

Following Schrijver [27] we note that linking systems generalise theorems relating matroids with bipartite graphs and directed graphs. In particular, Schrijver introduced linking systems in order to generalise theorems such as the following to over a linking system as opposed to just a bipartite graph.

Theorem 4.4.1. *[27] Let (X, Y, E) be a bipartite graph and let $(X, \mathcal{I})$ be a matroid. Define $\mathcal{I}'$ as the set of all subsets $Y' \subseteq Y$ such that there is a matching in the bipartite graph between some independent subset of X and Y'. Then $(Y, \mathcal{I}')$ is a matroid.*

4.4.2.1 Preliminaries

There are numerous ways to induce a linking system, such as by bipartite graphs, direct graphs and by matroids, and we will give a concrete way of defining one via a bipartite graph in Theorem 4.4.3. Before we do that we formally introduce them.

Definition 80. [27] A *linking system* is a triple (X, Y, Λ), where X and Y are finite sets and $\emptyset \neq \Lambda \subseteq \mathcal{P}(X) \times \mathcal{P}(Y)$ such that:

1. if $(X', Y') \in \Lambda$, then $|X'| = |Y'|$
2. if $(X', Y') \in \Lambda$ and $X'' \subseteq X'$, then $(X'', Y'') \in \Lambda$ for some $Y'' \subseteq Y'$
3. if $(X', Y') \in \Lambda$ and $Y'' \subseteq Y'$, then $(X'', Y'') \in \Lambda$ for some $X'' \subseteq X'$
4. if $(X_1, Y_1) \in \Lambda$ and $(X_2, Y_2) \in \Lambda$ then there exists an $(X', Y') \in \Lambda$ such that $X_1 \subseteq X' \subseteq X_1 \cup X_2$ and $Y_2 \subseteq Y' \subseteq Y_1 \cup Y_2$

Remark 44. We can see that $(\emptyset, \emptyset) \in \Lambda$.

Before we introduce an alternative definition we say that for a linking system (X, Y, Λ) that its *linking function* λ is defined by

$$\lambda(X', Y') = \max\{|X''| \mid (X'', Y'') \in \Lambda \text{ for some } X'' \subseteq X' \text{ and } Y'' \subseteq Y'\}$$

for $X' \subseteq X$ and $Y' \subseteq Y$. A linking system is determined by its linking function since

$(X', Y') \in \Lambda$ if and only if $\lambda(X', Y') = |X'| = |Y'|$.

Akin to how we can define matroids in terms of the rank function we can define linking systems in terms of the linking function.

Theorem 4.4.2. *[27] A* linking system *is a triple (X, Y, λ), where X and Y are finite sets and λ is an integer valued function defined on $\mathcal{P}(X) \times \mathcal{P}(Y)$ such that:*

1. $0 \leq \lambda(X', Y') \leq \min\{|X'|, |Y'|\}$ *for* $X' \subseteq X, Y' \subseteq Y$
2. *if* $X'' \subseteq X'$ *and* $Y'' \subseteq Y'$, *then* $\lambda(X'', Y'') \leq \lambda(X', Y')$ *for* $X' \subseteq X, Y' \subseteq Y$.
3. $\lambda(X' \cap X'', Y' \cup Y'') + \lambda(X' \cup X'', Y' \cap Y'') \leq \lambda(X', Y') + \lambda(X'', Y'')$ *for* $X', X'' \subseteq X$ *and* $Y', Y'' \subseteq Y$.

Theorem 4.4.3. *[27] Let (X, Y, E) be a bipartite graph and define Λ by $(X', Y') \in \Lambda$ if and only if there is a matching in E between $X' \subseteq X$ and $Y' \subseteq Y$. Then (X, Y, Λ) is a linking system.*

Now we state a theorem relating linking systems with matroids which have a fixed base.

Theorem 4.4.4. *[27] Let X and Y be disjoint finite sets. Then there is a one to one correspondence between linking systems (X, Y, Λ) and matroids $(X \cup Y, \mathcal{B})$ such that $X \in \mathcal{B}$. This relation is given by $(X', Y') \in \Lambda$ if and only if $(X \backslash X') \cup Y' \in \mathcal{B}$ for $X' \subseteq X, Y' \subseteq Y$. Similarly, the corresponding linking function λ and the matroid rank function r are related by*

$$r(X' \cup Y') = \lambda(X \backslash X', Y') + |X'| \text{ for } X' \subseteq X, Y' \subseteq Y.$$

Theorem 4.4.5. *[27] Let $(X, \mathcal{I})$ be a matroid, (X, Y, Λ) be a linking system, and let*

$$\mathcal{I} \star \Lambda = \{Y' \subseteq Y \mid (X', Y') \in \Lambda \text{ for some } X' \in \mathcal{I}\}.$$

Then $(Y, \mathcal{I} \star \Lambda)$ *is a matroid.*

Remark 45. This theorem is a generalisation of Theorem 4.4.1.

Now we introduce a way of defining a product of linking systems which again was first introduced by Schrijver.

Theorem 4.4.6. *[27] Let* (X, Y, Λ_1) *and* (Y, Z, Λ_2) *be two linking systems, with linking functions* λ_1 *and* λ_2 *respectively. Define* $\Lambda_1 * \Lambda_2$ *by*

$$\Lambda_1 * \Lambda_2 = \{(X', Z') \mid (X', Y') \in \Lambda_1, (Y', Z') \in \Lambda_2 \textit{ for some } Y' \in Y\}.$$

Then $(X, Z, \Lambda_1 * \Lambda_2)$ *is a linking system, which we call the* composition of linking systems*, with linking function,* $\lambda_1 * \lambda_2$*, given by*

$$(\lambda_1 * \lambda_2)(X', Z') = \min_{Y' \subseteq Y} (\lambda_1(X', Y') + \lambda_2(Y \backslash Y', Z')).$$

Remark 46. It can be seen that the composition of linking systems is in some sense related to Theorem 4.4.5. Assume $f : M \times L \to M$ is the function defined in Theorem 4.4.5 which takes a matroid and a linking system and outputs a new matroid, where M is the set of all matroids on ground set X, and L is the set of all linking systems of the form (X, X, Λ). Since the set of all linking systems forms a monoid under the operation of composition of linking systems we are able to say that f defines a monoid action. This is since we can take the linking system $(X, X, \Lambda) = e$, where every pair of the same size is in Λ as the identity so that we have $f(M, e) = M$. For the compatibility condition we have that $f(M, l_1 * l_2) = f(f(M, l_1), l_2)$ for every matroid M and every pair of linking systems l_1, l_2.

Now we rewrite Theorem 4.4.6 in terms of matroid rank functions.

Theorem 4.4.7. *Let* M *and* N *be matroids on* $E_1 \cup E_2$ *and* $E_2 \cup E_3$*, with rank functions* r_M *and* r_N *respectively, where* $E_2 \in \mathcal{B}(M), \mathcal{B}(N)$*. We define a rank function on* $E_1 \cup E_3$

for any $A \subseteq E_1 \cup E_3$ *by*

$$r_{M\cdot N}(A) = \min_{S\subseteq E_2}(r_M((A\cap E_1)\cup S) + r_N((A\cap E_3)\cup S) - |S|).$$

We claim that $r_{M\cdot N}$ *is a matroid rank function, and we denote this matroid by* $M \cdot N$*.*

Proof. We utilise both Theorem 4.4.4 and Theorem 4.4.6

$$\begin{aligned}
&(\lambda_M * \lambda_N)(E_1\backslash(A\cap E_1), A\cap E_3) + |A\cap E_1| \\
&= \min_{S\subseteq E_2}(\lambda_M(E_1\backslash(A\cap E_1), S) + \lambda_N(E_2\backslash S, A\cap E_3)) + |A\cap E_1| \\
&= \min_{S\subseteq E_2}(r_M((A\cap E_1)\cup S) - |A\cap E_1| + r_N(S\cup(A\cap E_3)) - |S|) + |A\cap E_1| \\
&= \min_{S\subseteq E_2}(r_M((A\cap E_1)\cup S) + r_N((A\cap E_3)\cup S) - |S|) = r_{M\cdot N}(A)
\end{aligned}$$

□

Remark 47. We note that this composition of linking systems coincides with the composition of valuated linking systems when we take non-valuated matroids as input. Similarly to the valuated case we call both the composition given by Theorem 4.4.6 and Theorem 4.4.7 the composition of linking systems and the distinction is left to the reader to distinguish.

4.4.3 Valuated linking systems

Valuated linking systems, also known as valuated bimatroids, were first introduced by Murota as a variant of valuated matroids [26]. Murota introduced valuated linking systems as a way to look into the combinatorics of the degree of the subdeterminants of a rational function matrix. Valuated linking systems also allow, similarly to non-valuated linking systems, that a valuated matroid can be induced by a valuated bipartite graph, this being simply a variant of induction of a valuated matroid by a bipartite graph [29]. This variant of induction by a bipartite graph is simply a generalisation of Theorem 4.4.1 which we saw in Section 4.4.2.

4.4.3.1 Review of preliminaries

Definition. [18] Let R and S be disjoint finite sets. A function $\lambda : P(R) \times P(S) \to \mathbb{T}$ is called a *valuated linking system* on (R, S) over $\mathbb{T}$ when the map $\mu_\lambda : P(R \cup S) \to \mathbb{T}$ defined by

$$\mu_\lambda(X) = \lambda(R \backslash X, S \cap X)$$

is a valuated matroid on $R \cup S$ over $\mathbb{T}$ satisfying $\mu_\lambda(R) = 0$. The map μ_λ is referred to as the *graph or representation matroid* of λ.

As we outlined in Section 4.1 there is a previously studied way of composing valuated linking systems, and we now recall Proposition 4.1.3 which gives us this extant operation in terms of elements of $T\bar{G}L_n$ as input.

Proposition. *Let $\mathcal{M}_1, \mathcal{M}_2 \in T\bar{G}L_n$. Then the composition of valuated linking systems $\mathcal{M}_1 \circ \mathcal{M}_2$ is given by*

$$\rho_{\mathcal{M}}(B) = \min_{L' \subseteq E_2} \rho_{\mathcal{M}_1}((B \cap E_1) \cup L') + \rho_{\mathcal{M}_2}((B \cap E_3) \cup L \backslash L').$$

Similarly to how matroids and bipartite graphs are related we have the following for valuated linking systems.

Proposition 4.4.8. *[18] The weight function of a $\mathbb{T}$-weighted bipartite graph is a valuated linking system.*

4.4.4 The flats of linking system composition

We now look at flats of the matroid given by the composition of linking systems. Before we state the description of the flats we first introduce a piece of notation. Let $L(A)$ be the set of all $S \subseteq E_2$ such that $r_{M \cdot N}(A) = r_M((A \cap E_1) \cup S) + r_N((A \cup E_3) \cup S) - |S|)$, that is, $L(A)$ is the set of all minimisers of $r_{M \cdot N}(A)$.

Theorem 4.4.9. *Let M and N be matroids on $E_1 \cup E_2$ and $E_2 \cup E_3$ with rank functions*

r_M and r_N respectively, where $E_2 \in \mathcal{B}(M), \mathcal{B}(N)$. Let $\mathcal{F}(M)$ and $\mathcal{F}(N)$ denote the flats of M and N respectively. Then the flats of $M \cdot N$ are given by

$$\mathcal{F}(M \cdot N) = \{(F \cup G)\backslash E_2 \mid F \in \mathcal{F}(M), G \in \mathcal{F}(N), F \cap E_2 = G \cap E_2 = \bigcup_{S \in L((F \cup G)\backslash E_2)} S\}$$

Proof. Firstly we show for any given flat A of $M \cdot N$ that $A \in \mathcal{F}(M \cdot N)$. Let A be a proper flat of $M \cdot N$, that is, $1 + r_{M \cdot N}(A) = r_{M \cdot N}(A \cup a)$ for $a \notin A$ and $A \neq E_1 \cup E_3$. We have

$$r_{M \cdot N}(A) = \min_{S \subseteq E_2} (r_M((A \cap E_1) \cup S) + r_N((A \cap E_3 \cup S) - |S|).$$

Let $a \notin A$, and without loss of generality assume that $a \in E_1$. Then for any $S \in L(A)$ we have $r_M((A \cap E_1) \cup S) + 1 = r_M(((A \cup a) \cap E_1) \cup S)$.

We wish to show there is some minimiser $S \in L(A)$ such that for any $q \in E_2\backslash S$ that $r_M((A \cap E_1) \cup S) + 1 = r_M((A \cap E_1) \cup S \cup q)$ and $r_N((A \cap E_3) \cup S) + 1 = r_N((A \cap E_3) \cup S \cup q)$. Assume that $T, T' \in L(A)$. We show $T \cup T'$ is a minimiser. Firstly note

$$r_M((A \cap E_1) \cup T) + r_N((A \cap E_3) \cup T) - |T| = r_M((A \cap E_1) \cup T') + r_N((A \cap E_3) \cup T') - |T'|$$

as well as r_M, r_N being submodular and that $|T| + |T'| = |T \cap T'| + |T \cup T'|$. So we have

$$\begin{aligned} &r_M((A \cap E_1) \cup T) + r_N((A \cap E_3) \cup T) + r_M((A \cap E_1) \cup T') + r_N((A \cap E_3) \cup T') \geq \\ &r_M((A \cap E_1) \cup T \cup T') + r_N((A \cap E_3) \cup T \cup T') + r_M((A \cap E_1) \cup (T \cap T')) \\ &+ r_N((A \cap E_3) \cup (T \cap T')) \end{aligned}$$

The only way the above can happen is if both $T \cup T'$ and $T \cap T'$ are minimisers, else we end up in a situation where $T, T' \notin L(A)$.

So by using the above we can choose an $S \in L(A)$ such that we have a flat in both M and N and a way of doing this is by choosing S to be the union of all elements of $L(A)$.

Now for the other direction. Given some $A = (F \cup G)\backslash E_2$, where $F \in \mathcal{F}(M)$, $G \in \mathcal{F}(N)$ and $F \cap E_2 = G \cap E_2 = \bigcup_{S \in L((F \cup G)\backslash E_2)} S$, we want to show that A is a flat of $M \cdot N$. Assume $a \in E_1$. Then for any $S \in L(A)$ we have

$$r_{M \cdot N}(A) = r_M((A \cap E_1) \cup S') + r_N((A \cap E_3) \cup S') - |S'|.$$

Now by adding the element $a \in E_1$ we have the following evaluation

$$r_M(((A \cup a) \cap E_1) \cup S) + r_N((A \cap E_3) \cup S) - |S|$$

So what is $r_M(((A \cup a) \cap E_1) \cup S)$ compared with $r_M((A \cap E_1) \cup S)$? Since $(A \cap E_1) \cup S \subseteq F$ and $F \in \mathcal{F}(M)$ we have that $r_M(((A \cup a) \cap E_1) \cup S) = r_M((A \cap E_1) \cup S) + 1$. The case where $a \in E_3$ follows similarly. This shows that A is a flat of $M \cdot N$. □

4.4.5 The extension of linking system composition and its flats

4.4.5.1 The rank function of the extension of the composition of linking systems

We give a description of the rank function of the extension of the composition of valuated linking systems as given in Theorem 4.3.1 in the case where we only consider $\{0, \infty\}$-valuated matroids. We give two alternative proofs of this result. Firstly, we give a proof directly using matroid theory, and in particular, we utilise the matroid union theorem. The second proof which we outline in Section 4.4.5.2 takes a polyhedral approach.

Theorem 4.4.10. *Given $M_1, M_2 \in T\bar{G}L_n$, and the extension of the composition of valuated linking systems $M_1 \star M_2$ as given by Theorem 4.3.1. If M_1 and M_2 are both $\{0, \infty\}$-valued then we are able to write the rank function of $M_1 \star M_2$ as*

$$r_{M_1 \star M_2}(A) = \min_{A \cap E_2 \subseteq S \subseteq E_2} r_{M_1}(A \cap E_1 \cup S) + r_{M_2}(A \cap E_3 \cup S) - |S|.$$

We begin with the first proof, which we split up into subsidiary results. Firstly, we introduce some notation. Let $\hat{L}(A)$ be the set of all $S \subseteq E_2$ where S is such that $r_{M_1 \star M_2}(A) = r_{M_1}((A \cap E_1) \cup S) + r_{M_2}((A \cup E_3) \cup S) - |S|)$, that is, the set of all minimisers of $r_{M_1 \star M_2}$.

Proposition 4.4.11. *Let $A \subseteq E_{123}$, where $|E_1| = |E_2| = |E_3| = n$. Let r_1 and r_2 be matroids on E_{12} and E_{23} respectively, such that E_2 is a basis of both. Then r defined by*

$$r(A) = \min_{A \cap E_2 \subseteq S \subseteq E_2} r_1(A \cap E_1 \cup S) + r_2(A \cap E_3 \cup S) - |S|$$

defines a matroid of rank n on E_{123}.

Proof. We show the first condition from Theorem 2.2.3:

$$r(\emptyset) = \min_{\emptyset \subset S \subset E_2} r_1(S) + r_2(S) - |S| = 0.$$

Next we show $r(A) \leq r(A \cup a)$. Split into cases based on where a lives. If $a \in E_{13}$ then assume without loss of generality that $a \in E_1$. For any minimiser $S \in \hat{L}(A \cup a)$ we have

$$\begin{aligned} r(A \cup a) &= r_1((A \cup a) \cap E_1 \cup S) + r_2(A \cap E_3 \cup S) - |S| \\ &\geq r_1(A \cap E_1 \cup S) + r_2(A \cap E_3 \cup S) - |S| \geq r(A). \end{aligned}$$

Now consider $a \in E_2$. If $r(A \cup a) < r(A)$ then for any $S \in \hat{L}(A \cup a)$ we are able to use that as the minimiser of $r(A)$, and then clearly we obtain a contradiction.

Now we show that $r(A \cup a) \leq r(A) + 1$. Again we split this into cases based on where a lives. If $a \in E_{13}$, we again assume without loss of generality that $a \in E_1$. For any

$S \in \hat{L}(A)$ we have

$$\begin{aligned} r(A) + 1 &= r_1(A \cap E_1 \cup S) + r_2(A \cap E_3 \cup S) - |S| + 1 \\ &\geq r_1((A \cup a) \cap E_1 \cup S) + r_2(A \cap E_3 \cup S) - |S| \\ &\geq r(A \cup a). \end{aligned}$$

Now consider $a \in E_2$. For any $S \in \hat{L}(A)$ we have either $a \notin S$ or $(A \cup a) \cap E_2 \subseteq S$. In the former case we have for S that

$$\begin{aligned} r(A) + 1 &= r_1(A \cap E_1 \cup S) + r_2(A \cap E_3 \cup S) - |S| + 1 \\ &\geq r_1(A \cap E_1 \cup S \cup a) + r_2(A \cap E_3 \cup S \cup a) - |S \cup a| \\ &\geq r(A \cup a). \end{aligned}$$

In the case that $(A \cup a) \cap E_2 \subseteq S$ when considering any minimiser of $r(A \cup a)$ we can choose S to be the minimiser. Hence we have $r(A) = r(A \cup a) \leq r(A) + 1$.

We now show submodularity. Given specific minimisers $S \in \hat{L}(X)$ and $T \in \hat{L}(Y)$ we have the following:

$$\begin{aligned} r(X) + r(Y) &= r_1((X \cap E_1) \cup S) + r_2((X \cap E_3) \cup S) - |S| \\ &\quad + r_1((Y \cap E_1) \cup T) + r_2((Y \cap E_3) \cup T) - |T| \\ &\geq r_1(((X \cap Y) \cap E_1) \cup (S \cap T)) + r_2(((X \cap Y) \cap E_3) \cup (S \cap T)) - |S| \\ &\quad + r_1(((X \cup Y) \cap E_1) \cup (S \cup T)) + r_2(((X \cup Y) \cap E_3) \cup (S \cup T)) - |T| \\ &= r_1(((X \cap Y) \cap E_1) \cup (S \cap T)) + r_2(((X \cap Y) \cap E_3) \cup (S \cap T)) - |S \cap T| \\ &\quad + r_1(((X \cup Y) \cap E_1) \cup (S \cup T)) + r_2(((X \cup Y) \cap E_3) \cup (S \cup T)) - |S \cup T| \\ &\geq r(X \cup Y) + r(X \cap Y). \end{aligned}$$

So r defines a matroid rank function where the rank of this matroid is $r(E_1 \cup E_2 \cup E_3) = n$. □

We proceed to show that this rank function is the one which is actually given by the

the extension of linking system composition, that is, we prove Theorem 4.4.10

Proof of Theorem 4.4.10. Denote the basis sets for M_1 and M_2 as E_{12} and E_{23} respectively. By considering the extension of valuated linking system composition in the case where our input valuated matroids are $\{0, \infty\}$-valued then the output is a matroid of rank n on E_{123} where the bases are defined by sets B where there exists $L, L' \subseteq E_2$ such that $L \cup L' = E_2, L \cap L' = E_2 \cap B$ and $B \cap E_1 \cup L \in \mathcal{B}(M_1), B \cap E_3 \cup L' \in \mathcal{B}(M_2)$.

Firstly, we show for a basis B of the form above that it is also a basis of the matroid given in Proposition 4.4.11. Given $B \cap E_1 \cup L \in \mathcal{B}(M_1)$ and $B \cap E_3 \cup L' \in \mathcal{B}(M_2)$ then $|B| = n$, so it is the correct size in order to be a basis of the matroid with rank function r as given by Proposition 4.4.11. Now we need to show that $r(B) = n$. We have

$$r(B) = \min_{B \cap E_2 \subseteq S \subseteq E_2} r_{M_1}(B \cap E_1 \cup S) + r_{M_2}(B \cap E_3 \cup S) - |S|.$$

Note that we have $r_{M_1}(B \cap E_1 \cup B \cap E_2) = |B \cap (E_1 \cup E_2)|$ and $r_{M_2}(B \cap E_2 \cup B \cap E_3) = |B \cap (E_2 \cup E_3)|$. So when $S = B \cap E_2$ we have $r(B) = n$. What happens when we consider a minimiser which is a superset of S? Since any $a \in E_2 \backslash B$ is either in L or L' then we have $r(B) = n$.

Now we show that given a basis B of the matroid given by r from Proposition 4.4.11 that B satisfies the conditions of having appropriate L and L'. In order to do this we utilise matroid union theorem given by Theorem 2.2.9.

Firstly, restrict the potential choices for L to bases of $M_1/(B \cap E_1)\backslash(E_1 \backslash B)$ and similarly restrict the choices for L' to bases of $M_2/(B \cap E_3)\backslash(E_3 \backslash B)$. We do this since we require $B \cap E_1 \cup L \in \mathcal{B}(M_1)$ and $B \cap E_3 \cup L' \in \mathcal{B}(M_2)$. We take the dual of both of these matroid constructions and call them N and N' respectively. Then for any L and L' we have bases L^* and L'^* of N and N' such that $L^* = E_2 \backslash L$ and $L'^* = E_2 \backslash L'$, and the equivalent condition we need them to satisfy becomes $L^* \cup L'^* = E_2 \backslash B$.

We show that the dual of the matroid $(M_1 \star M_2)/((E_1 \cup E_3) \cap B)\backslash((E_1 \cup E_3)\backslash B))$ agrees with the matroid union of N and N'. Denote the matroid $(M_1 \star M_2)/((E_1 \cup E_3) \cap B)\backslash((E_1 \cup E_3)\backslash B))$ by τ'. So what is the rank function of τ'^*?

$$
\begin{aligned}
r_{\tau'^*}(A) &= |A| - r_{\tau'}(E_2) + r_{\tau'}(E_2\backslash A) \\
&= |A| - |B \cap E_2| + r_{M_1 \star M_2}(E_2\backslash A \cup ((E_1 \cup E_3) \cap B)) - r_{M_1 \star M_2}((E_1 \cup E_3) \cap B) \\
&= |A| - n + r_{M_1 \star M_2}(E_2\backslash A \cup ((E_1 \cup E_3) \cap B)) \\
&= |A| - n + \min_{E_2\backslash A \subseteq S \subseteq E_2} (r_{M_1}(E_1 \cap B \cup S) + r_{M_2}(E_3 \cap B \cup S) - |S|).
\end{aligned}
$$

Now we look at the matroid union of N and N'. Let's calculate the matroid rank function for this.

$$
\begin{aligned}
r(A) &= \min_{Y \subseteq A} |A\backslash Y| + r_N(Y) + r_{N'}(Y) \\
&= \min_{Y \subseteq A} |A\backslash Y| + |Y| - (n - |E_1 \cap B|) + r_{M_1}(E_2\backslash Y \cup (E_1 \cap B)) - |E_1 \cap B| \\
&\quad + |Y| - (n - |E_3 \cap B|) + r_{M_2}(E_2\backslash Y \cup (E_3 \cap B)) - |E_3 \cap B| \\
&= \min_{Y \subseteq A} |A| + |Y| - 2n + r_{M_1}(E_2\backslash Y \cup (E_1 \cap B)) + r_{M_2}(E_2\backslash Y \cup (E_3 \cap B)) \\
&= |A| - n + \min_{Y \subseteq A} |Y| - n + r_{M_1}(E_2\backslash Y \cup (E_1 \cap B)) + r_{M_2}(E_2\backslash Y \cup (E_3 \cap B)).
\end{aligned}
$$

Then after changing some notation we can see that the expressions for the rank function of $r_{\tau'^*}$ and that of the matroid union of N and N' are equal.

We show that $E_2\backslash B$ is a basis of the matroid with rank function $r_{\tau'^*}$, and hence it follows that it is a basis of the matroid union of N and N'. Given $E_2\backslash B$ as a basis of the matroid union we clearly have L and L' which satisfy the conditions which we require which comes directly from Theorem 2.2.9.

Let's proceed. We are given that B is a basis of $M_1 \star M_2$. After contracting B by $(E_1 \cup E_3) \cap B$ and then the deletion of $(E_1 \cup E_3)\backslash B$ we have that $B \cap E_2$ is a basis of τ', and hence $E_2\backslash B$ is a basis of the dual. Hence the result follows. □

4.4.5.2 Alternative Approach to proof of Theorem 4.4.10

Now we look at an alternative approach to proving Theorem 4.4.10. This time polyhedrally as opposed to directly using matroid results.

Proposition 4.4.12. *Write $P(M)$ for the polytope of a matroid M. Given $M_1, M_2 \in T\bar{G}L_n$ such that they are $\{0,\infty\}$-valued. Then we get $P(M_1 \star M_2)$ as a subset of $\mathbb{R}^{3n}$ in the following way*

$$P(M_1 \star M_2) = (P(M_1) + P(M_2) - 1_{E_2}) \cap \mathbb{R}^{3n}_{\geq 0} \tag{4.7}$$

Here we regard $P(M_1)$ as confined to the subspace where all of the E_3 coordinates are 0, and similarly with $P(M_2)$ with all of the E_1 coordinates being 0, and where 1_{E_2} is the indicator function of E_2.

We introduce a result which we use in the proof of Proposition 4.4.12.

Corollary 4.4.13. *([15],Theorem 46.2c) Let $P_1, \dots, P_k$ be lattice polymatroids. Then each integer vector in $P_1 + \cdots + P_k$ is a sum $x_1 + \cdots + x_k$ of integer vectors $x_1 \in P_1, \dots, x_k \in P_k$.*

Proof of Proposition 4.4.12. The right hand side of Equation (4.7) describes a lattice polytope and its lattice points are exactly bases of $M_1 \star M_2$. It is a lattice polytope since every vertex of $P(M_1), P(M_2)$ and 1_{E_2} is integral, and by definition of Minkowski sum we have that $P(M_1) + P(M_2) - 1_{E_2}$ describes a lattice polytope. Intersecting with the positive orthant doesn't cause any issues since all the vertex points we see after doing so are integral.

Now why are its lattice points exactly bases of $M_1 \star M_2$? Each $B \in \mathcal{B}(M_1 \star M_2)$ is such that there exist L, L' with $B \cap E_1 \cup L \in \mathcal{B}(M_1)$ and $B \cap E_3 \cup L' \in \mathcal{B}(M_2)$ and $B \cap E_2 = L \cap L'$ and $L \cup L' = E_2$. Given bases of M_1 and M_2 which can be written as $B \cap E_1 \cup L \in \mathcal{B}(M_1)$ and $B \cap E_3 \cup L' \in \mathcal{B}(M_2)$ where L and L' satisfy the conditions above then they are not impacted by the intersection with the positive orthant since all of E_2 is positive before the negation of 1_{E_2}.

Now given any bases of M_1 and M_2 such that they cannot be written as $B \cap E_1 \cup L$ and $B \cap E_3 \cup L'$ for appropriate L, L' we can note that any B of this form is not a lattice point. This is since firstly if $B \cap E_2 \neq L \cap L'$ then this point doesn't appear by the Minkowski sum, and second if $L \cup L' \neq E_2$ then we have negative values when we minus 1_{E_2} and thus these don't appear as lattice points. We have used the result which we stated before the proof, namely Corollary 4.4.13. □

By Proposition 4.4.12 we have the right hand side of Equation (4.7) is a matroid polytope since the left hand side is by definition, and hence the rank function of its matroid is the restriction of its support function to $\{0,1\}$-vectors. We are able to work out the rank function by considering how Minkowski sums and intersections act on support functions. Let's work this through.

Let A be a subset of E_{123}. The restricted support function of $P(M_1) + P(M_2) - 1_{E_2}$ is $A \mapsto r_{M_1}(A \cap E_{12}) + r_{M_2}(A \cap E_{23}) - |A \cap E_2|$. This is true due to the following result.

Theorem 4.4.14. *([15],Theorem 44.4) Let f_1 and f_2 be nondecreasing submodular set functions on S, with $f_1(\emptyset) = 0, f_2(\emptyset) = 0$, and associated polymatroids P_1 and P_2 respectively. Let P by the polymatroid associated with $f := f_1 + f_2$ then we have*

$$P_{f_1+f_2} = P_{f_1} + P_{f_2}.$$

Next we consider what impact intersecting with the positive orthant has on this support function. We firstly introduce a previously studied submodular function [15]. Given a submodular set function f on S, and a vector $a \in \mathbb{R}^S$, we define the set function $(f|_a)$ as

$$(f|_a)(U) = \min_{T \subseteq U}(f(T) + a(U \backslash T)).$$

Proposition 4.4.15. *[15] Let f be a submodular function on S and let $a \in \mathbb{R}^S$, then $(f|_a)$ is a submodular function.*

Proposition 4.4.16. *[15] If f is a submodular set function on S and $f(\emptyset) = 0$, then $EP_{f|_a} = \{x \in EP_f \mid x \leq a\}$.*

Definition 81. Given a submodular function f on E_{123} we define f^* by

$$f^*(A) = f(E_{123} \backslash A) - f(E_{123}).$$

Remark 48. It can be easily checked that f^* is a submodular function. We only use this piece of notation in this section. Outside of this section we use dual in the sense of matroid duality. It can also be seen that if f is the rank function of a matroid M then the polytope of f^* in this regard is the matroid polytope of M^* translated by $(-1, \dots, -1)$.

Now we present a short corollary to Proposition 4.4.16.

Corollary 4.4.17. *Let f be a submodular function on E_{123}. Then $B_{f^*} = -B_f$.*

Proof. Recall the definitions of B_f and B_{f^*}.

$$B_f = \{x \mid x(U) \leq f(U), x(E_{123}) = f(E_{123})\}$$

$$B_{f^*} = \{x \mid x(U) \leq f^*(U), x(E_{123}) = f^*(E_{123})\}$$

So do we have $x \in B_f \iff -x \in B_{f^*}$? Take $x \in B_f$, and thus by noting

$x(E_{123}) = f(E_{123}) = n$, we have

$$
\begin{aligned}
& x(U) \le f(U) \\
\iff & x(U) - n \le f(U) - n \\
\iff & x(U) - x(E_{123}) \le f(U) - n \\
\iff & -x(E_{123} \backslash U) \le f(U) - n \\
\iff & -x(E_{123} \backslash U) \le f^*(E_{123} \backslash U)
\end{aligned}
$$

and clearly $x(E_{123}) = f(E_{123}) \iff -x(E_{123}) = f^*(E_{123})$. □

Now for the real corollary of Proposition 4.4.16. Before we begin we say $f \ge 0$ if $f(A) \ge 0$ for all sets A.

Corollary 4.4.18. *Let f be a submodular function on E_{123} such that $f(\emptyset) = 0$ and $f \ge 0$. Then $B_{(f^*|_0)^*} = \{x \in B_f \mid x \ge 0\}$, and $(f^*|_0)^*$ is given by $(f^*|_0)^*(U) = \min_{U \subseteq C} f(C)$.*

Proof. From Corollary 4.4.17 we have that $-B_f = B_{f^*}$. When applying the $|_0$ operation to f^* we get by Proposition 4.4.16 that $B_{f^*|_0} = \{x \in -B_f \mid x \le 0\}$, and thus by dualising $(f^*|_0)$ we have $B_{(f^*|_0)^*} = -\{x \in -B_f \mid x \le 0\} = \{x \in B_f \mid x \ge 0\}$.

So given a submodular function f on E_{123} such that $f(\emptyset) = 0$ and $f \ge 0$ we want to see what the effect of this three stage process has on f. Firstly by dualising f we get

$$f^*(A) = f(E_{123} \backslash A) - f(E_{123}) = f(E_{123} \backslash A) - n,$$

where we let $f(E_{123}) = n$. Now by applying $|_0$ operation to f^* we have

$$f^*|_0(A) = \min_{C \subseteq A} f^*(C) = \min_{C \subseteq A} f(E_{123} \backslash C) - n.$$

So finally by dualising $f^*|_0$ we obtain

$$\begin{aligned}
(f^*|_0)^*(A) &= f^*|_0(E_{123}\backslash A) - f^*|_0(E_{123}) \\
&= \min_{C\subseteq E_{123}\backslash A} f^*(C) - \min_{C\subseteq E_{123}} f^*(C) \\
&= \min_{C\subseteq E_{123}\backslash A} (f(E_{123}\backslash C) - n) - \min_{C\subseteq E_{123}} (f(E_{123}\backslash C) - n) \\
&= \min_{C\subseteq E_{123}\backslash A} (f(E_{123}\backslash C) - n) - (-n) \\
&= \min_{C\subseteq E_{123}\backslash A} f(E_{123}\backslash C) \\
&= \min_{A\subseteq C} f(C)
\end{aligned}$$

□

From earlier we have that the restricted support function of $P(M) + P(N) - 1_{E_2}$ is $A \mapsto r_{M_1}(A \cap E_{12}) + r_{M_2}(A \cap E_{23}) - |A \cap E_2|$, and so by Corollary 4.4.18 we have that the support of $(P(M_1) + P(M_2) - 1_{E_2}) \cap \mathbb{R}^{3n}_{\geq 0}$ is $A \mapsto \min_{A\subseteq S} r_{M_1}(S \cap E_{12}) + r_{M_2}(S \cap E_{23}) - |S \cap E_2| = \min_{A\cap E_2\subseteq S\subseteq E_2} r_{M_1}(A \cap E_1 \cup S) + r_{M_2}(A \cap E_3 \cup S) - |S|$. We are able to use Corollary 4.4.18 since $f = r_{M_1}(A \cap E_{12}) + r_{M_2}(A \cap E_{23}) - |A \cap E_2|$ is such that $f(\emptyset) = 0$ and $f \geq 0$.

4.4.5.3 The flats of the extension of linking system composition

We utilise our explicit description of the matroid rank function for the extension of valuated linking system composition when we have inputs which are $\{0, \infty\}$-valued in order to give a description of the flats of this function. We will show that the flats of $M_1 \star M_2$ are given by $\{F \cup G \mid F \in \mathcal{F}(M_1), G \in \mathcal{F}(M_2)$ where $F \cap E_2 = G \cap E_2$ is the sole minimiser of $r(F \cup G)\}$. This gives us an alternative way of describing the matroid $M_1 \star M_2$ and in particular, this more easily enables us to look at the Bergman fan of the matroid given by the composition of valuated linking systems.

Theorem 4.4.19. *Let $M_1, M_2 \in T\bar{G}L_n$ be such that they are both $\{0, \infty\}$-valued. The* *set of flats of the extension of valuated linking system composition $M_1 \star M_2$ are given by*

$$\{F \cup G \mid F \in \mathcal{F}(M_1), G \in \mathcal{F}(M_2) \text{ where } \hat{L}(F \cup G) = F \cap E_2 = G \cap E_2\}. \tag{4.8}$$

Proof. Firstly, look at the $\implies$ direction. Given a flat of $M_1 \star M_2$ we can write it as $F \cup G$ where $F \subseteq E_{12}, G \subseteq E_{23}$ and $F \cap E_2 = G \cap E_2$. We need to show that $F \in \mathcal{F}(M_1)$ and $G \in \mathcal{F}(M_2)$, and that $F \cap E_2 = G \cap E_2$ is the sole minimiser of $r(F \cup G)$.

Since $F \cup G$ is a flat we have for any $a \in E_{123} \backslash (F \cup G)$ that $r_{M_1 \star M_2}(F \cup G \cup a) = r_{M_1 \star M_2}(F \cup G) + 1$. In particular, for any $S \in \hat{L}(F \cup G)$ we have

$$r_{M_1 \star M_2}(F \cup G) = r_{M_1}(F \cup S) + r_{M_2}(G \cup S) - |S|.$$

We split into cases as to whether $a \in E_1, E_2$ or E_3. When $a \in E_1$ we have

$$r_{M_1 \star M_2}(F \cup G \cup a) = r_{M_1 \star M_2}(F \cup G) + 1 = r_{M_1}(F \cup S \cup a) + r_{M_2}(G \cup S) - |S|,$$

and so $r_{M_1}(F \cup S \cup a) = r_{M_1}(F \cup S) + 1$. Similarly, if $a \in E_3$ then $r_{M_2}(G \cup S \cup a) = r_{M_2}(G \cup S) + 1$.

Next consider the case whereby $a \in E_2$. Firstly we show that $F \cap E_2 = G \cap E_2$ is the sole minimiser of $r_{M_1 \star M_2}(F \cup G)$. Assume for contradiction that there is another minimiser, say $S \in \hat{L}(F \cup G)$. Take $a \in S \backslash (F \cup G)$. We have

$$r_{M_1 \star M_2}(F \cup G) + 1 = r_{M_1 \star M_2}(F \cup G \cup a) = r_{M_1}(F \cup S) + r_{M_2}(G \cup S) - |S| = r_{M_1 \star M_2}(F \cup G).$$

This is a contradiction. Since this is the case for every $a \in E_2 \backslash (F \cup G)$ we have a single minimiser, namely $F \cap E_2 = G \cap E_2$.

Now we show that $F \in \mathcal{F}(M_1)$ and $G \in \mathcal{F}(M_2)$. Recall from earlier that for any $a \in E_1 \backslash F$ that $r_{M_1}(F \cup a) = r_{M_1}(F) + 1$, and for any $a \in E_3 \backslash G$ that $r_{M_2}(G \cup a) =$

$r_{M_2}(G)+1$. Now consider if $a \in E_2 \backslash (F \cup G)$.

$$r_{M_1 \star M_2}(F \cup G \cup a) = r_{M_1 \star M_2}(F \cup G) + 1 = r_{M_1}(F \cup a) + r_{M_2}(G \cup a) - |S \cup a|$$

and so $r_{M_1}(F \cup a) = r_{M_1}(F) + 1$ and $r_{M_2}(G \cup a) = r_{M_2}(G) + 1$. So this shows that $F \in \mathcal{F}(M_1)$ and $G \in \mathcal{F}(M_2)$.

Now for the opposite implication, namely whether $F \cup G$ satisfying the conditions given in Equation (4.8) is a flat of $M_1 \star M_2$. So $r_{M_1 \star M_2}(F \cup G) = r_{M_1}(F) + r_{M_2}(G) - |F \cap E_2|$. Let $a \in E_{123} \backslash (F \cup G)$. So if $a \in E_1$ then

$$\begin{aligned} r_{M_1 \star M_2}(F \cup G \cup a) &= r_{M_1}(F \cup a) + r_{M_2}(G) - |F \cap E_2| \\ &= r_{M_1}(F) + 1 - r_{M_2}(G) - |F \cap E_2| \\ &= r_{M_1 \star M_2}(F \cup G) + 1 \end{aligned}$$

The argument is similar for $a \in E_3$. Now if $a \in E_2$, we have $r_{M_1 \star M_2}(F \cup G \cup a) = r_{M_1 \star M_2}(F \cup G) + 1$, since else $F \cap E_2 = G \cap E_2$ is not the sole minimiser of $r_{M_1 \star M_2}(F \cup G)$. $\square$

4.5 Associativity

We now show that we cannot use a similar argument to that used in the $n = 2$ section where we find a valuated matroid of rank n on a set of size $4n$ which restricts to the correct matroids on the needed sets. We demonstrate that this cannot be true in the case of particular matroids when $n > 3$ and thus the argument cannot be fully generalised. Let's firstly recall a potential line of argumentation we wish to have used, and why in particular this method is unable to work.

Let $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3 \in T\bar{G}L_n$. An element $\mathcal{Q}$ of $(\mathcal{M}_1 \boxdot \mathcal{M}_2) \boxdot \mathcal{M}_3$ is a valuated matroid on E_{14} such that there are valuated matroids $\mathcal{A}, \mathcal{B}$ on E_{123} and E_{134} respectively such that we have the following restrictions. We have that $\mathcal{A}$ restricted to E_{12} is $\mathcal{M}_1$ and restricted to E_{23} is $\mathcal{M}_2$, and that $\mathcal{B}$ restricted to E_{14} is $\mathcal{Q}$ and restricted to E_{34} is $\mathcal{M}_3$. In addition, we also require that both $\mathcal{A}$ and $\mathcal{B}$ agree on their common restriction to E_{13}.

Similarly we have that $\mathcal{Q}'$ is an element of $\mathcal{M}_1 \boxdot (\mathcal{M}_2 \boxdot \mathcal{M}_3)$ if it is a valuated matroid on E_{14} such that there are valuated matroids $\mathcal{A}', \mathcal{B}'$ on E_{124} and E_{234} respectively such that we have similar restrictions.

So given an element $\mathcal{Q}$ of $(\mathcal{M}_1 \boxdot \mathcal{M}_2) \boxdot \mathcal{M}_3$ we wish to show that $\mathcal{Q}$ is also in $\mathcal{M}_1 \boxdot (\mathcal{M}_2 \boxdot \mathcal{M}_3)$, and by symmetry the other direction follows. One potential avenue to do this is to find a valuated matroid on E_{1234} which restricts to $\mathcal{A}, \mathcal{B}, \mathcal{A}', \mathcal{B}'$. However, we are able to show this isn't possible in general for n large enough.

Before we show that we cannot construct a matroid on E_{1234} to work in general for us, we firstly give a result about incompatible extensions. We show that there exists an example of two rank 3 matroids which have no common extension to a larger matroid.

Proposition 4.5.1. *Let M and N be matroids on ground sets $\{1,2,3,4,5,6,7,8\}$ and $\{1,2,3,4,5,6,7,9\}$ respectively such that the non-bases of each are as follows: $\mathcal{NB}(M) = \{128, 348, 568\}$ and $\mathcal{NB}(N) = \{129, 349, 579\}$. Then there is no common extension to a matroid P on $\{1,2,3,4,5,6,7,8,9\}$.*

Proof. Firstly note that M and N can be built up from $U(3,7)$ in the following way. Consider $U(3,7)$ in terms of projective point arrangements. We are able to view $U(3,7)$ as seven points in a plane such that no three are collinear. Then we create the one point extension by adding the element 8 such that the triples 128,348,568 are collinear, and this is M. We do similarly by adding an element 9 to $U(3,7)$ so that 129,349,579 are collinear, and this is our N.

Note that we can't add both 8 and 9 in these ways simultaneously. Let's say we attempt to add both, then they must lie in the same place since they are on the intersection of lines 12 and 34. By this and that 568 and 578 are collinear, then either 567 is collinear, or 5 is placed in the same place as 8, and thus 125 and 345 are collinear, and neither of these restrict correctly to M. □

Theorem 4.5.2. *Let $n > 3$. Then there exist some matroids $M_1, M_2, M_3 \in T\bar{G}L_n$ and a certificate (A, B) showing that $\mathcal{Q} \in (M_1 \boxdot M_2) \boxdot M_3$ such that there is no common extension of A and B to a matroid of rank n on a set of size $4n$.*

Proof. Now if $n > 3$ then we define M_1, M_2, M_3 in the following way. Denote $E_1 = \{1, 2, \ldots, n\}, E_2 = \{a, \alpha_1, \alpha_2, \alpha_{n-1}\}, E_3 = \{n{+}1, n{+}2, \ldots, 2n{-}1, x\}, E_4 = \{b, \alpha'_1, \ldots, \alpha'_{n-1}\}$. We begin by forming matroid A and B. Define A on $E_1 \cup E_2 \cup E_3$ by the following procedure.

1. Define $U(3, 7)$ on $\{1, 2, 3, 4, n+1, n+2, n+3\}$.

2. Form a new matroid on $\{1, 2, 3, 4, n + 1, n + 2, n + 3, a\}$ by the matroid where $\mathcal{NB} = \{12a, 34a, n + 1n + 2a\}$.

3. Add $x, n+3, \ldots, 2n-1$ in the following way. Add x by taking the old matroid and adding x to the ground set, and taking as the bases $B \cup x$ where B is a basis of the old matroid, as well as any set S of size $|B \cup x|$ where S is a spanning set of the old matroid of size one greater than the old matroid. Do this also for $n+3, \ldots, 2n-1$. Each time we do this we increase the rank by 1

4. Add the elements $\alpha_1, \ldots, \alpha_{n-1}$ one at a time such that $B \backslash b \cup \alpha_i$ is a new basis, for every B which is a basis of the old matroid and every element $b \in B$. This defines our matroid A of rank n on $E_1 \cup E_2 \cup E_3$. Each application of this does not change the rank of the matroid

We then define M_1 to be A restricted to $E_1 \cup E_2$ and M_2 to be A restricted to $E_2 \cup E_3$. Observe that $E_1, E_2 \in \mathcal{B}(M_1)$ and $E_2, E_3 \in \mathcal{B}(M_2)$.

We construct B in a similar way on $E_1 \cup E_3 \cup E_4$ but where b plays the role of a, and each α'_i plays the roles of α_i. The only difference is that in the first step we have that the non-bases are $\{12b, 34b, n+1n+3b\}$. We define M_3 to be B restricted to $E_3 \cup E_4$. We can also see that both A and B restrict to the same matroid on $E_1 \cup E_3$.

So we now wish to look at $(M_1 \boxdot M_2) \boxdot M_3$ to see how to construct a matroid on $E_1 \cup E_2 \cup E_3 \cup E_4$ with correct restrictions.

Let $\mathcal{Q}$ be the restriction of B to $E_1 \cup E_4$. Now we have $\mathcal{Q} \in (M_1 \boxdot M_2) \boxdot M_3$ along with a certificate (A, B). We now show that we can't find a matroid on $E_1 \cup E_2 \cup E_3 \cup E_4$ which correctly restricts to both A and B.

Take A. Restrict this to $\{1, \ldots, 2n-1, x, a\}$ and then contract to $\{1, 2, 3, 4, n+1, n+2, n+3, a\}$. Similarly with B we can restrict to $\{1, \ldots, 2n-1, x, b\}$ and then contract to $\{1, 2, 3, 4, n+1, n+2, n+3, b\}$.

Then by Proposition 4.5.1 there is no common extension of these matroids, and thus there is no common extension of A and B. $\square$

Remark 49. This means that in order to utilise this global certificate argument then we need to be able to be slightly more clever in our choices of certificate.

Appendix A

Theorem 4.3.7

We look into the structure of $\mathcal{M}_1 \boxdot \mathcal{M}_2$ for given $\mathcal{M}_1, \mathcal{M}_2 \in T\bar{G}L_2$ which have associated valuations $\rho_{\mathcal{M}_1} = (0 = x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34} = \Delta), \rho_{\mathcal{M}_2} = (0 = y_{34}, y_{35}, y_{36}, y_{45}.y_{46}, y_{56} = \Delta')$. Let $\mathcal{M} \in \mathcal{M}_1 \boxdot \mathcal{M}_2$ with associated valuation $\rho_{\mathcal{M}} = (p_{12}, p_{15}, p_{16}, p_{25}, p_{26}, p_{56})$. Any such $\mathcal{M}'$, with valuation $\rho_{\mathcal{M}'} = (0, x_{13}, x_{14}, p_{15}, p_{16}, x_{23}, x_{24}, p_{25}, p_{26}, \Delta, y_{35} + \Delta, y_{36} + \Delta, y_{45} + \Delta, y_{46} + \Delta, \Delta + \Delta')$, which projects to $\mathcal{M}, \mathcal{M}_1$ and $\mathcal{M}_2$, has 15 tropical Plücker relations which we list now.

$$\sigma = \{1\}, \tau = \{234\} : \min(p_{12}+p_{34}, p_{13}+p_{24}, p_{14}+p_{23}) = \min(\Delta, x_{13}+x_{24}, x_{14}+x_{23})$$

$$\sigma = \{1\}, \tau = \{235\} : \min(p_{12}+p_{35}, p_{13}+p_{25}, p_{15}+p_{23}) = \min(y_{35}+\Delta, x_{13}+p_{25}, p_{15}+x_{23})$$

$$\sigma = \{1\}, \tau = \{236\} : \min(p_{12}+p_{36}, p_{13}+p_{26}, p_{16}+p_{23}) = \min(y_{36}+\Delta, x_{13}+p_{26}, p_{16}+x_{23})$$

$$\sigma = \{1\}, \tau = \{245\} : \min(p_{12}+p_{45}, p_{14}+p_{25}, p_{15}+p_{24}) = \min(y_{45}+\Delta, x_{14}+p_{25}, p_{15}+x_{24})$$

$$\sigma = \{1\}, \tau = \{246\} : \min(p_{12}+p_{46}, p_{14}+p_{26}, p_{16}+p_{24}) = \min(y_{46}+\Delta, x_{14}+p_{26}, p_{16}+x_{24})$$

$$\sigma = \{1\}, \tau = \{256\} : \min(p_{12}+p_{56}, p_{15}+p_{26}, p_{16}+p_{25}) = \min(\Delta+\Delta', p_{15}+p_{26}, p_{16}+p_{25})$$

$$\sigma = \{1\}, \tau = \{345\} : \min(p_{13}+p_{45}, p_{14}+p_{35}, p_{15}+p_{34}) = \min(x_{13}+y_{45}, x_{14}+y_{35}, p_{15})$$

$$\sigma = \{1\}, \tau = \{346\} : \min(p_{13}+p_{46}, p_{14}+p_{36}, p_{16}+p_{34}) = \min(x_{13}+y_{46}, x_{14}+y_{36}, p_{16})$$

$$\sigma = \{1\}, \tau = \{356\} : \min(p_{13}+p_{56}, p_{15}+p_{36}, p_{16}+p_{35}) = \min(x_{13}+\Delta', p_{15}+y_{36}, p_{16}+y_{35})$$

$$\sigma = \{1\}, \tau = \{456\} : \min(p_{14}+p_{56}, p_{15}+p_{46}, p_{16}+p_{45}) = \min(x_{14}+\Delta', p_{15}+y_{46}, p_{16}+y_{45})$$

$$\sigma = \{2\}, \tau = \{345\} : \min(p_{23}+p_{45}, p_{24}+p_{35}, p_{25}+p_{34}) = \min(x_{23}+y_{45}+, x_{24}+y_{35}, p_{25})$$

$\sigma = \{2\}, \tau = \{346\} : \min(p_{23} + p_{46}, p_{24} + p_{36}, p_{26} + p_{34}) = \min(x_{23} + y_{46}, x_{24} + y_{36}, p_{26})$

$\sigma = \{2\}, \tau = \{356\} : \min(p_{23}+p_{56}, p_{25}+p_{36}, p_{26}+p_{35}) = \min(x_{23}+\Delta', p_{25}+y_{36}, p_{26}+y_{35})$

$\sigma = \{2\}, \tau = \{456\} : \min(p_{24}+p_{56}, p_{25}+p_{46}, p_{26}+p_{45}) = \min(x_{24}+\Delta', p_{25}+y_{46}, p_{26}+y_{45})$

$\sigma = \{3\}, \tau = \{456\} : \min(p_{34} + p_{56}, p_{35} + p_{46}, p_{36} + p_{45}) = \min(\Delta', y_{35} + y_{46}, y_{36} + y_{45})$

We split these into cases depending on whether $p_{15}, p_{16}, p_{25}, p_{26}$ are free to increase. What we mean by that is for $\rho_{\mathcal{M}_1} = (0 = x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34} = \Delta), \rho_{\mathcal{M}_2} = (0 = y_{34}, y_{35}, y_{36}, y_{45}.y_{46}, y_{56} = \Delta')$ that we have

p_{15} is *free* if $x_{13} + y_{45} = x_{14} + y_{35}$

p_{16} is *free* if $x_{13} + y_{46} = x_{14} + y_{36}$

p_{25} is *free* if $x_{23} + y_{45} = x_{24} + y_{35}$

p_{26} is *free* if $x_{23} + y_{46} = x_{24} + y_{36}$

If it is not free then we refer to it as *fixed*.

Now we split into cases depending on these. Firstly, if $p_{15}, p_{16}, p_{25}, p_{26}$ are all fixed then we have $\mathcal{M}_1 \boxdot \mathcal{M}_2 = \mathcal{M}_1 \circ \mathcal{M}_2$. This can be seen by looking at the 15 relations of $\mathcal{M}_1 \star \mathcal{M}_2$ and noting that is the only solution.

If p_{15} is free, but p_{16}, p_{25}, p_{26} are fixed, what do we have? We have that $\mathcal{M}_1 \boxdot \mathcal{M}_2 = \mathcal{M}_1 \circ \mathcal{M}_2 + \{\lambda e_{15} \mid \lambda \geq 0\}$. This can be observed by looking at the tropical Plücker relations. Similarly we have

1. If p_{16} is free, but p_{15}, p_{25}, p_{26} are fixed then $\mathcal{M}_1 \boxdot \mathcal{M}_2 = \mathcal{M}_1 \circ \mathcal{M}_2 + \{\lambda e_{16} \mid \lambda \geq 0\}$.

2. If p_{25} is free, but p_{15}, p_{16}, p_{26} are fixed then $\mathcal{M}_1 \boxdot \mathcal{M}_2 = \mathcal{M}_1 \circ \mathcal{M}_2 + \{\lambda e_{25} \mid \lambda \geq 0\}$.

3. If p_{26} is free, but p_{15}, p_{16}, p_{25} are fixed then $\mathcal{M}_1 \boxdot \mathcal{M}_2 = \mathcal{M}_1 \circ \mathcal{M}_2 + \{\lambda e_{26} \mid \lambda \geq 0\}$.

Now we consider what happens if p_{15}, p_{16} are free, but p_{25}, p_{26} aren't. Firstly, note that $\Delta = \min(x_{13} + x_{24}, x_{14} + x_{23})$. For contradiction assume this isn't true, then we have

$x_{13} + x_{24} = x_{14} + x_{23}$, and $x_{13} + y_{45} = x_{14} + y_{35}$ and so we have

$$x_{13} + y_{45} + x_{14} + x_{23} = x_{14} + y_{35} + x_{13} + x_{24}$$

$$y_{45} + x_{23} = y_{35} + x_{24}$$

and this implies that p_{25} is not fixed. So by looking at the 15 tropical Plücker relations we have

$$\begin{aligned}\mathcal{M}_1 \boxdot \mathcal{M}_2 = \text{conv}\{&\mathcal{M}_1 \circ \mathcal{M}_2, \mathcal{M}_1 \circ \mathcal{M}_2 \ + (\Delta' - \min(y_{35} + y_{46}, y_{36} + y_{45}))(e_{15} + e_{16})\} \\ &\cup \mathcal{M}_1 \circ \mathcal{M}_2 \ + (\Delta' - \min(y_{35} + y_{46}, y_{36} + y_{45}))(e_{15} + e_{16}) + \{\lambda e_{15} \mid \lambda \geq 0\} \\ &\cup \mathcal{M}_1 \circ \mathcal{M}_2 \ + (\Delta' - \min(y_{35} + y_{46}, y_{36} + y_{45}))(e_{15} + e_{16}) + \{\lambda e_{16} \mid \lambda \geq 0\}\end{aligned}$$

Similarly, if p_{25}, p_{26} are free, but p_{15}, p_{16} are fixed then we have

$$\begin{aligned}\mathcal{M}_1 \boxdot \mathcal{M}_2 = \text{conv}\{&\mathcal{M}_1 \circ \mathcal{M}_2, \mathcal{M}_1 \circ \mathcal{M}_2 \ + (\Delta' - \min(y_{35} + y_{46}, y_{36} + y_{45}))(e_{25} + e_{26})\} \\ &\cup \mathcal{M}_1 \circ \mathcal{M}_2 \ + (\Delta' - \min(y_{35} + y_{46}, y_{36} + y_{45}))(e_{25} + e_{26}) + \{\lambda e_{25} \mid \lambda \geq 0\} \\ &\cup \mathcal{M}_1 \circ \mathcal{M}_2 \ + (\Delta' - \min(y_{35} + y_{46}, y_{36} + y_{45}))(e_{25} + e_{26}) + \{\lambda e_{26} \mid \lambda \geq 0\}\end{aligned}$$

Again similarly if p_{15}, p_{25} are free, but p_{16}, p_{26} are fixed then we have

$$\begin{aligned}\mathcal{M}_1 \boxdot \mathcal{M}_2 = \text{conv}\{&\mathcal{M}_1 \circ \mathcal{M}_2, \mathcal{M}_1 \circ \mathcal{M}_2 \ + (\Delta - \min(x_{13} + x_{24}, x_{14} + x_{23}))(e_{15} + e_{25})\} \\ &\cup \mathcal{M}_1 \circ \mathcal{M}_2 \ + (\Delta - \min(x_{13} + x_{24}, x_{14} + x_{23}))(e_{15} + e_{25}) + \{\lambda e_{15} \mid \lambda \geq 0\} \\ &\cup \mathcal{M}_1 \circ \mathcal{M}_2 \ + (\Delta - \min(x_{13} + x_{24}, x_{14} + x_{23}))(e_{15} + e_{25}) + \{\lambda e_{25} \mid \lambda \geq 0\}\end{aligned}$$

And lastly on this if p_{16}, p_{26} are free, but p_{15}, p_{25} are fixed then we have

$$\begin{aligned}\mathcal{M}_1 \boxdot \mathcal{M}_2 = \text{conv}\{&\mathcal{M}_1 \circ \mathcal{M}_2, \mathcal{M}_1 \circ \mathcal{M}_2 \ + (\Delta - \min(x_{13} + x_{24}, x_{14} + x_{23}))(e_{16} + e_{26})\} \\ &\cup \mathcal{M}_1 \circ \mathcal{M}_2 \ + (\Delta - \min(x_{13} + x_{24}, x_{14} + x_{23}))(e_{16} + e_{26}) + \{\lambda e_{16} \mid \lambda \geq 0\} \\ &\cup \mathcal{M}_1 \circ \mathcal{M}_2 \ + (\Delta - \min(x_{13} + x_{24}, x_{14} + x_{23}))(e_{16} + e_{26}) + \{\lambda e_{26} \mid \lambda \geq 0\}\end{aligned}$$

Now assume that p_{15}, p_{26} are free and the other two fixed. We have $\mathcal{M}_1 \boxdot \mathcal{M}_2 = \mathcal{M}_1 \circ \mathcal{M}_2 + \{\lambda e_{15} \mid \lambda \geq 0\} + \{\lambda e_{26} \mid \lambda \geq 0\}$

We have similarly when p_{16}, p_{25} are free and the other two fixed. In this case we have $\mathcal{M}_1 \boxdot \mathcal{M}_2 = \mathcal{M}_1 \circ \mathcal{M}_2 + \{\lambda e_{16} \mid \lambda \geq 0\} + \{\lambda e_{25} \mid \lambda \geq 0\}$.

Now we note that we cannot have exactly three of them being free. We show an example of one of these. Assume that p_{15}, p_{16}, p_{25} are free. Then we have

$$x_{13} + y_{45} = x_{14} + y_{35} \quad x_{13} + y_{46} = x_{14} + y_{36} \quad x_{23} + y_{45} = x_{24} + y_{35}$$

and using these three equations we can show $x_{23} + y_{46} = x_{24} + y_{36}$ and thus p_{26} is free.

Now if all four of them are free then what? Firstly, let $G = \Delta - \min(x_{13} + x_{24}, x_{14} + x_{23})$, $H = \Delta' - \min(y_{35} + y_{46}, y_{36} + y_{45})$. We split this into cases depending on the values of G and H. Firstly when $G = H = 0$ then we have

$$\begin{aligned}\mathcal{M}_1 \boxdot \mathcal{M}_2 = \mathcal{M}_1 \circ \mathcal{M}_2 &+ \{\lambda e_{15} \mid \lambda \geq 0\} + \{\lambda e_{26} \mid \lambda \geq 0\} \\ &\cup \mathcal{M}_1 \circ \mathcal{M}_2 + \{\lambda e_{16} \mid \lambda \geq 0\} + \{\lambda e_{25} \mid \lambda \geq 0\}\end{aligned}$$

Now what if $G = 0$ but $H > 0$. We have

$$\begin{aligned}\mathcal{M}_1 \boxdot \mathcal{M}_2 = \mathrm{conv}\{&\mathcal{M}_1 \circ \mathcal{M}_2, \mathcal{M}_1 \circ \mathcal{M}_2 + G(e_{15} + e_{16})\} \\ &\cup \mathrm{conv}\{\mathcal{M}_1 \circ \mathcal{M}_2, \mathcal{M}_1 \circ \mathcal{M}_2 + G(e_{25} + e_{26})\} \\ &\cup \mathcal{M}_1 \circ \mathcal{M}_2 + G(e_{15} + e_{16}) + \{\lambda e_{15} \mid \lambda \geq 0\} \\ &\cup \mathcal{M}_1 \circ \mathcal{M}_2 + G(e_{15} + e_{16}) + \{\lambda e_{16} \mid \lambda \geq 0\} \\ &\cup \mathcal{M}_1 \circ \mathcal{M}_2 + G(e_{25} + e_{26}) + \{\lambda e_{25} \mid \lambda \geq 0\} \\ &\cup \mathcal{M}_1 \circ \mathcal{M}_2 + G(e_{25} + e_{26}) + \{\lambda e_{26} \mid \lambda \geq 0\}\end{aligned}$$

Similarly if $G > 0$ but $H = 0$. We have

$$\begin{aligned}\mathcal{M}_1 \boxdot \mathcal{M}_2 = \operatorname{conv}\{&\mathcal{M}_1 \circ \mathcal{M}_2, \mathcal{M}_1 \circ \mathcal{M}_2 + H(e_{15} + e_{25})\} \\ &\cup \operatorname{conv}\{\mathcal{M}_1 \circ \mathcal{M}_2, \mathcal{M}_1 \circ \mathcal{M}_2 + H(e_{16} + e_{26})\} \\ &\cup \mathcal{M}_1 \circ \mathcal{M}_2 + H(e_{15} + e_{25}) + \{\lambda e_{15} \mid \lambda \geq 0\} \\ &\cup \mathcal{M}_1 \circ \mathcal{M}_2 + H(e_{15} + e_{25}) + \{\lambda e_{25} \mid \lambda \geq 0\} \\ &\cup \mathcal{M}_1 \circ \mathcal{M}_2 + H(e_{16} + e_{26}) + \{\lambda e_{16} \mid \lambda \geq 0\} \\ &\cup \mathcal{M}_1 \circ \mathcal{M}_2 + H(e_{16} + e_{26}) + \{\lambda e_{26} \mid \lambda \geq 0\}\end{aligned}$$

Now lastly, what if $G > 0$ and $H > 0$. We split into three cases depending on whether $G > H, G = H, G < H$. Firstly if $G > H$ we have

$$\begin{aligned}\mathcal{M}_1 \boxdot \mathcal{M}_2 = \operatorname{conv}\{&\mathcal{M}_1 \circ \mathcal{M}_2, \mathcal{M}_1 \circ \mathcal{M}_2 + H(e_{15} + e_{16} + e_{25} + e_{26})\} \\ &\cup \operatorname{conv}\{\mathcal{M}_1 \circ \mathcal{M}_2 + H(e_{15} + e_{16} + e_{25} + e_{26}), \mathcal{M}_1 \circ \mathcal{M}_2 + \\ &\quad H(e_{15} + e_{16} + e_{25} + e_{26}) + (G - H)(e_{15} + e_{25})\} \\ &\cup \operatorname{conv}\{\mathcal{M}_1 \circ \mathcal{M}_2 + H(e_{15} + e_{16} + e_{25} + e_{26}), \mathcal{M}_1 \circ \mathcal{M}_2 + \\ &\quad H(e_{15} + e_{16} + e_{25} + e_{26}) + (G - H)(e_{16} + e_{26})\} \\ &\cup, \mathcal{M}_1 \circ \mathcal{M}_2 + H(e_{15} + e_{16} + e_{25} + e_{26}) + (G - H)(e_{15} + e_{25}) + \{\lambda e_{15} \mid \lambda \geq 0\} \\ &\cup, \mathcal{M}_1 \circ \mathcal{M}_2 + H(e_{15} + e_{16} + e_{25} + e_{26}) + (G - H)(e_{15} + e_{25}) + \{\lambda e_{25} \mid \lambda \geq 0\} \\ &\cup, \mathcal{M}_1 \circ \mathcal{M}_2 + H(e_{15} + e_{16} + e_{25} + e_{26}) + (G - H)(e_{16} + e_{26}) + \{\lambda e_{16} \mid \lambda \geq 0\} \\ &\cup, \mathcal{M}_1 \circ \mathcal{M}_2 + H(e_{15} + e_{16} + e_{25} + e_{26}) + (G - H)(e_{16} + e_{26}) + \{\lambda e_{26} \mid \lambda \geq 0\}\end{aligned}$$

Similarly if we have $H > G$ then we have

$$\begin{aligned}
\mathcal{M}_1 \boxdot \mathcal{M}_2 = \operatorname{conv}\{&\mathcal{M}_1 \circ \mathcal{M}_2, \mathcal{M}_1 \circ \mathcal{M}_2 + G(e_{15} + e_{16} + e_{25} + e_{26})\} \\
&\cup \operatorname{conv}\{\mathcal{M}_1 \circ \mathcal{M}_2 + G(e_{15} + e_{16} + e_{25} + e_{26}), \mathcal{M}_1 \circ \mathcal{M}_2 + \\
&\qquad G(e_{15} + e_{16} + e_{25} + e_{26}) + (H - G)(e_{15} + e_{16})\} \\
&\cup \operatorname{conv}\{\mathcal{M}_1 \circ \mathcal{M}_2 + G(e_{15} + e_{16} + e_{25} + e_{26}), \mathcal{M}_1 \circ \mathcal{M}_2 + \\
&\qquad G(e_{15} + e_{16} + e_{25} + e_{26}) + (H - G)(e_{25} + e_{26})\} \\
&\cup \mathcal{M}_1 \circ \mathcal{M}_2 + G(e_{15} + e_{16} + e_{25} + e_{26}) + (H - G)(e_{15} + e_{16}) + \{\lambda e_{15} \mid \lambda \geq 0\} \\
&\cup \mathcal{M}_1 \circ \mathcal{M}_2 + G(e_{15} + e_{16} + e_{25} + e_{26}) + (H - G)(e_{15} + e_{16}) + \{\lambda e_{25} \mid \lambda \geq 0\} \\
&\cup \mathcal{M}_1 \circ \mathcal{M}_2 + G(e_{15} + e_{16} + e_{25} + e_{26}) + (H - G)(e_{25} + e_{26}) + \{\lambda e_{16} \mid \lambda \geq 0\} \\
&\cup \mathcal{M}_1 \circ \mathcal{M}_2 + G(e_{15} + e_{16} + e_{25} + e_{26}) + (H - G)(e_{25} + e_{26}) + \{\lambda e_{26} \mid \lambda \geq 0\}
\end{aligned}$$

Now if $G = H > 0$ then

$$\begin{aligned}
\mathcal{M}_1 \boxdot \mathcal{M}_2 = \operatorname{conv}\{&\mathcal{M}_1 \circ \mathcal{M}_2, \mathcal{M}_1 \circ \mathcal{M}_2 + G(e_{15} + e_{16} + e_{25} + e_{26})\} \\
&\cup \mathcal{M}_1 \circ \mathcal{M}_2 + G(e_{15} + e_{16} + e_{25} + e_{26}) + \{\lambda e_{15} \mid \lambda \geq 0\} \\
&\cup \mathcal{M}_1 \circ \mathcal{M}_2 + G(e_{15} + e_{16} + e_{25} + e_{26}) + \{\lambda e_{16} \mid \lambda \geq 0\} \\
&\cup \mathcal{M}_1 \circ \mathcal{M}_2 + G(e_{15} + e_{16} + e_{25} + e_{26}) + \{\lambda e_{25} \mid \lambda \geq 0\} \\
&\cup \mathcal{M}_1 \circ \mathcal{M}_2 + G(e_{15} + e_{16} + e_{25} + e_{26}) + \{\lambda e_{26} \mid \lambda \geq 0\}
\end{aligned}$$

Appendix B

Theorem 4.3.8

We look into the structure of $\bigcup_{\mathcal{M}_1,\mathcal{M}_2|e(\mathcal{M}_1)=\mathcal{A},e(\mathcal{M}_2)=\mathcal{B}} \mathcal{M}_1 \hat{\boxdot} \mathcal{M}_2$ for all such $\mathcal{A},\mathcal{B} \in M_2(\mathbb{T})$. Akin to Appendix A we split these into cases depending on whether $p_{15}, p_{16}, p_{25}, p_{26}$ are free to increase, again using exactly the same notation. What we mean by that is for $\rho_{\mathcal{M}_1} = (0 = x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34} = \Delta), \rho_{\mathcal{M}_2} = (0 = y_{34}, y_{35}, y_{36}, y_{45} \cdot y_{46}, y_{56} = \Delta')$ that we have

$$p_{15} \text{ is } \textit{free} \text{ if } x_{13} + y_{45} = x_{14} + y_{35}$$

$$p_{16} \text{ is } \textit{free} \text{ if } x_{13} + y_{46} = x_{14} + y_{36}$$

$$p_{25} \text{ is } \textit{free} \text{ if } x_{23} + y_{45} = x_{24} + y_{35}$$

$$p_{26} \text{ is } \textit{free} \text{ if } x_{23} + y_{46} = x_{24} + y_{36}$$

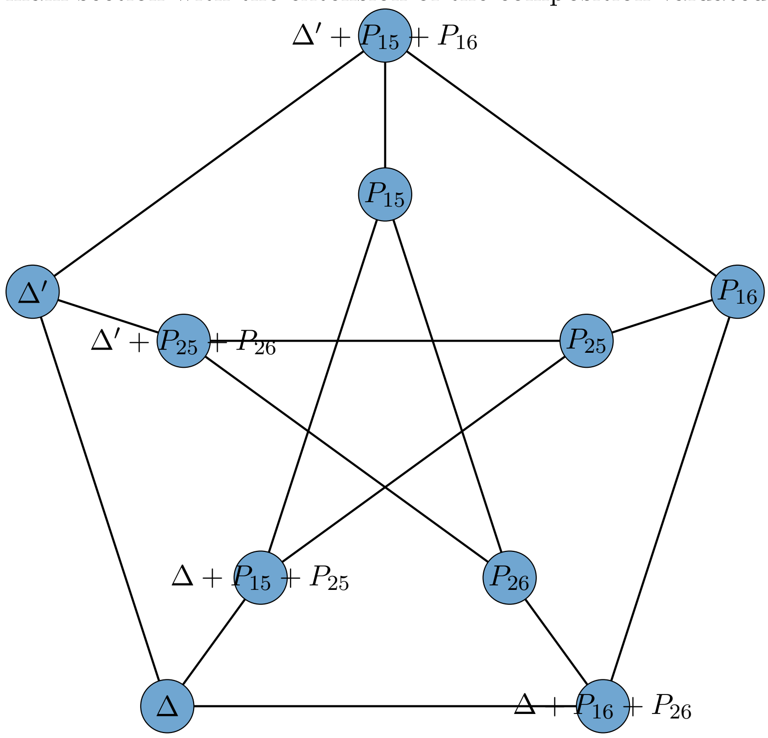
This time we also add the following two

$$\Delta \text{ is } \textit{free} \text{ if } x_{23} + x_{14} = x_{24} + x_{13}$$

$$\Delta' \text{ is } \textit{free} \text{ if } y_{35} + y_{46} = y_{45} + y_{36}$$

If it is not free then we refer to it as *fixed*. We also define $\mathcal{M}_1 \bullet \mathcal{M}_2$ to be the minimal element of $\mathcal{M}_1 \hat{\boxdot} \mathcal{M}_2$ given $\mathcal{A},\mathcal{B} \in M_2(\mathbb{T})$.

We begin by looking at the case where $p_{15}, p_{16}, p_{25}, p_{26}$ are all free. Then necessarily we have that Δ and Δ' are free to increase. This gives the same picture as in the main section with the extension of the composition valuated linking system as the apex.



Now we assume that p_{15}, p_{16}, p_{25} are free. Then Δ' is free since p_{15}, p_{16} are free, and hence $y_{35} + y_{46} = y_{36} + y_{45}$. We can also show that Δ is free. Since p_{15} and p_{25} are free we have

$$x_{13} + y_{45} + x_{24} + y_{35} = x_{14} + y_{35} + x_{23} + y_{45}$$

$$\implies x_{13} + x_{24} = x_{14} + x_{23}$$

$$\implies \Delta \text{ is free}$$

Now from this we can also show that p_{26} is free. By using $p_{25} + p_{16} - p_{15}$ we have

$$x_{23} + y_{45} + x_{13} + y_{46} - (x_{13} + y_{45}) = x_{24} + y_{35} + x_{14} + y_{36} - (x_{14} + y_{35})$$
$$\implies x_{23} + y_{46} = x_{24} + y_{36}$$
$$\implies p_{26} \text{ is free}$$

So it follows that we cannot be in a case whereby exactly three of $p_{15}, p_{16}, p_{25}, p_{26}$ are free.

Now consider the case where p_{15}, p_{16} are free, but p_{25} and p_{26} are fixed. Firstly we have Δ' is free. Now can Δ be free? The answer to this is due to the following. Assume Δ is free. We have the following

$$x_{13} + y_{45} + x_{14} + y_{23} = x_{14} + y_{35} + x_{13} + y_{24}$$
$$\implies y_{45} + y_{23} = y_{35} + x_{24}$$

This implies that p_{25} is free, and this is not true. Hence we know what the solution looks like

So we have in this case

$$\bigcup_{\mathcal{M}_1,\mathcal{M}_2 | e(\mathcal{M}_1)=\mathcal{A}, e(\mathcal{M}_2)=\mathcal{B}} \mathcal{M}_1 \hat{\boxdot} \mathcal{M}_2 = \mathcal{M}_1 \bullet \mathcal{M}_2 + \{\lambda(e_{15} + e_{16} + e_{56}) \mid \lambda \geq 0\} + \{\lambda e_{15} \mid \lambda \geq 0\}$$
$$\cup\, \mathcal{M}_1 \bullet \mathcal{M}_2 + \{\lambda(e_{15} + e_{16} + e_{56}) \mid \lambda \geq 0\} + \{\lambda e_{16} \mid \lambda \geq 0\}$$
$$\cup\, \mathcal{M}_1 \bullet \mathcal{M}_2 + \{\lambda(e_{15} + e_{16} + e_{56}) \mid \lambda \geq 0\} + \{\lambda e_{56} \mid \lambda \geq 0\}$$

If p_{25}, p_{26} are free but p_{15}, p_{16} aren't then

$$\bigcup_{\mathcal{M}_1,\mathcal{M}_2|e(\mathcal{M}_1)=\mathcal{A},e(\mathcal{M}_2)=\mathcal{B}} \mathcal{M}_1 \hat{\boxdot} \mathcal{M}_2 = \mathcal{M}_1 \bullet \mathcal{M}_2 + \{\lambda(e_{25}+e_{26}+e_{56}) \mid \lambda \geq 0\} + \{\lambda e_{25} \mid \lambda \geq 0\}$$

$$\cup\, \mathcal{M}_1 \bullet \mathcal{M}_2 + \{\lambda(e_{25}+e_{26}+e_{56}) \mid \lambda \geq 0\} + \{\lambda e_{26} \mid \lambda \geq 0\}$$

$$\cup\, \mathcal{M}_1 \bullet \mathcal{M}_2 + \{\lambda(e_{25}+e_{26}+e_{56}) \mid \lambda \geq 0\} + \{\lambda e_{56} \mid \lambda \geq 0\}$$

If p_{15}, p_{25} are free but p_{16}, p_{26} aren't then

$$\bigcup_{\mathcal{M}_1,\mathcal{M}_2|e(\mathcal{M}_1)=\mathcal{A},e(\mathcal{M}_2)=\mathcal{B}} \mathcal{M}_1 \hat{\boxdot} \mathcal{M}_2 = \mathcal{M}_1 \bullet \mathcal{M}_2 + \{\lambda(e_{15}+e_{25}+e_{34}) \mid \lambda \geq 0\} + \{\lambda e_{15} \mid \lambda \geq 0\}$$

$$\cup\, \mathcal{M}_1 \bullet \mathcal{M}_2 + \{\lambda(e_{15}+e_{25}+e_{34}) \mid \lambda \geq 0\} + \{\lambda e_{25} \mid \lambda \geq 0\}$$

$$\cup\, \mathcal{M}_1 \bullet \mathcal{M}_2 + \{\lambda(e_{15}+e_{25}+e_{34}) \mid \lambda \geq 0\} + \{\lambda e_{34} \mid \lambda \geq 0\}$$

If p_{16}, p_{26} are free but p_{15}, p_{25} aren't then

$$\bigcup_{\mathcal{M}_1,\mathcal{M}_2|e(\mathcal{M}_1)=\mathcal{A},e(\mathcal{M}_2)=\mathcal{B}} \mathcal{M}_1 \hat{\boxdot} \mathcal{M}_2 = \mathcal{M}_1 \bullet \mathcal{M}_2 + \{\lambda(e_{16}+e_{26}+e_{34}) \mid \lambda \geq 0\} + \{\lambda e_{16} \mid \lambda \geq 0\}$$

$$\cup\, \mathcal{M}_1 \bullet \mathcal{M}_2 + \{\lambda(e_{16}+e_{26}+e_{34}) \mid \lambda \geq 0\} + \{\lambda e_{26} \mid \lambda \geq 0\}$$

$$\cup\, \mathcal{M}_1 \bullet \mathcal{M}_2 + \{\lambda(e_{16}+e_{26}+e_{34}) \mid \lambda \geq 0\} + \{\lambda e_{34} \mid \lambda \geq 0\}$$

Now if p_{15}, p_{26} are free, but p_{16}, p_{25} are fixed then what? Firstly we can note the following:

$$x_{13} + x_{24} + y_{36} + y_{45} = x_{14} + x_{23} + y_{35} + y_{46}$$

So from this we can see that Δ is free if and only if Δ' is free. Now do these imply that p_{16} or p_{25} are free? Assume that Δ, Δ' are free. We have similarly that:

$$x_{13} + y_{45} + x_{23} + y_{46} = x_{14} + y_{35} + x_{24} + y_{36}$$

and so we have that p_{16} is free if and only if p_{25} is free.

Now we use the following by considering p_{15} and Δ'. So if Δ' is free then we have the following

$$x_{13} + y_{45} - (y_{36} - y_{45}) = x_{14} + y_{35} - (y_{35} + y_{46})$$
$$\implies x_{13} - y_{36} = x_{14} - y_{46}$$
$$\implies x_{13} + y_{46} = x_{14} + y_{36}$$

and thus p_{16} is free.

Now the following similar process. If p_{16} is free.

$$x_{13} + y_{45} - (x_{13} + y_{46}) = x_{14} + y_{35} - (x_{14} + y_{36})$$
$$\implies y_{45} - y_{46} = y_{35} - y_{36}$$
$$\implies y_{45} + y_{36} = y_{35} + y_{46}$$

and thus we have that Δ' is free. Hence we have

$$\Delta \text{ free } \iff \Delta' \text{ free } \iff p_{16} \text{ free} \iff p_{25} \text{ free} \tag{B.1}$$

Thus we only have p_{15}, p_{26} being free, else we are in a previously considered situation, and thus

$$\bigcup_{\mathcal{M}_1,\mathcal{M}_2 | e(\mathcal{M}_1)=\mathcal{A}, e(\mathcal{M}_1)=\mathcal{B}} \mathcal{M}_1 \hat{\boxdot} \mathcal{M}_2 = \mathcal{M}_1 \bullet \mathcal{M}_2 + \{\lambda e_{15} \mid \lambda \geq 0\} + \{\lambda e_{26} \mid \lambda \geq 0\}.$$

The case where p_{16} and p_{25} are free is similar. So we have

$$\bigcup_{\mathcal{M}_1,\mathcal{M}_2 | e(\mathcal{M}_1)=\mathcal{A}, e(\mathcal{M}_2)=\mathcal{B}} \mathcal{M}_1 \hat{\boxdot} \mathcal{M}_2 = \mathcal{M}_1 \bullet \mathcal{M}_2 + \{\lambda e_{16} \mid \lambda \geq 0\} + \{\lambda e_{25} \mid \lambda \geq 0\}.$$

Now in the case whereby just p_{15} is free. So if Δ is free then we have

$$x_{13} + y_{45} + x_{14} + x_{23} = x_{14} + y_{35} + x_{13} + x_{24}$$
$$\implies x_{23} + y_{45} = x_{24} + y_{35}$$

and thus p_{25} is free. Similarly we get an issue if Δ' is free. Thus we know what the solution set looks like. Namely

$$\bigcup_{\mathcal{M}_1,\mathcal{M}_2 | e(\mathcal{M}_1)=\mathcal{A}, e(\mathcal{M}_2)=\mathcal{B}} \mathcal{M}_1 \hat{\boxdot} \mathcal{M}_2 = \mathcal{M}_1 \bullet \mathcal{M}_2 + \{\lambda e_{15} \mid \lambda \geq 0\}.$$

In the case where just p_{16} is free we have

$$\bigcup_{\mathcal{M}_1,\mathcal{M}_2 | e(\mathcal{M}_1)=\mathcal{A}, e(\mathcal{M}_2)=\mathcal{B}} \mathcal{M}_1 \hat{\boxdot} \mathcal{M}_2 = \mathcal{M}_1 \bullet \mathcal{M}_2 + \{\lambda e_{16} \mid \lambda \geq 0\}.$$

Just p_{25} we have

$$\bigcup_{\mathcal{M}_1,\mathcal{M}_2 | e(\mathcal{M}_1)=\mathcal{A}, e(\mathcal{M}_2)=\mathcal{B}} \mathcal{M}_1 \hat{\boxdot} \mathcal{M}_2 = \mathcal{M}_1 \bullet \mathcal{M}_2 + \{\lambda e_{25} \mid \lambda \geq 0\}.$$

and just p_{26} we have

$$\bigcup_{\mathcal{M}_1,\mathcal{M}_2 | e(\mathcal{M}_1)=\mathcal{A}, e(\mathcal{M}_2)=\mathcal{B}} \mathcal{M}_1 \hat{\boxdot} \mathcal{M}_2 = \mathcal{M}_1 \bullet \mathcal{M}_2 + \{\lambda e_{26} \mid \lambda \geq 0\}.$$

Now what if none of $p_{15}, p_{16}, p_{25}, p_{26}$ are free. Then Δ and Δ' are able to be free, thus if they are we have our solution set here.

$$\bigcup_{\mathcal{M}_1,\mathcal{M}_2 | e(\mathcal{M}_1)=\mathcal{A}, e(\mathcal{M}_2)=\mathcal{B}} \mathcal{M}_1 \hat{\boxdot} \mathcal{M}_2 = \mathcal{M}_1 \bullet \mathcal{M}_2 + \{\lambda e_{34} \mid \lambda \geq 0\} + \{\lambda e_{56} \mid \lambda \geq 0\}.$$

If only Δ is free then we have

$$\bigcup_{\mathcal{M}_1,\mathcal{M}_2 | e(\mathcal{M}_1)=\mathcal{A}, e(\mathcal{M}_2)=\mathcal{B}} \mathcal{M}_1 \hat{\boxdot} \mathcal{M}_2 = \mathcal{M}_1 \bullet \mathcal{M}_2 + \{\lambda e_{34} \mid \lambda \geq 0\}.$$

If only Δ' is free then we have

$$\bigcup_{\mathcal{M}_1,\mathcal{M}_2 | e(\mathcal{M}_1)=\mathcal{A}, e(\mathcal{M}_2)=\mathcal{B}} \mathcal{M}_1 \hat{\boxdot} \mathcal{M}_2 = \mathcal{M}_1 \bullet \mathcal{M}_2 + \{\lambda e_{56} \mid \lambda \geq 0\}.$$

References

[1] A. W. Dress and W. Wenzel, "Valuated matroids," *Advances in Mathematics*, vol. 93, no. 2, pp. 214 – 250, 1992. [Online]. Available: http://www.sciencedirect.com/science/article/pii/000187089290028J

[2] K. Murota, "Multiple exchange property for M$^\natural$-concave functions and valuated matroids," *Math. Oper. Res.*, vol. 43, no. 3, pp. 781–788, 2018. [Online]. Available: https://doi.org/10.1287/moor.2017.0882

[3] E. Balkanski and R. Paes Leme, "On the construction of substitutes," *Math. Oper. Res.*, vol. 45, no. 1, pp. 272–291, 2020. [Online]. Available: https://doi.org/10.1287/moor.2019.0991

[4] D. E. Speyer, "Tropical linear spaces," *SIAM Journal on Discrete Mathematics*, vol. 22, no. 4, pp. 1527–1558, 2008. [Online]. Available: https://doi.org/10.1137/080716219

[5] J. Giansiracusa and N. Giansiracusa, "Equations of tropical varieties," *Duke Math. J.*, vol. 165, no. 18, pp. 3379–3433, 12 2016. [Online]. Available: https://doi.org/10.1215/00127094-3645544

[6] D. Maclagan and F. Rincón, "Tropical schemes, tropical cycles, and valuated matroids," *Journal of the European Mathematical Society*, vol. 22, 01 2014.

[7] S. Hampe, "Tropical linear spaces and tropical convexity," *Electron. J. Combin.*, vol. 22, no. 4, pp. Paper 4.43, 20, 2015.

[8] S. Herrmann, M. Joswig, and B. Sturmfels, "How to draw tropical planes," *The electronic journal of combinatorics*, vol. 16, 08 2008.

[9] D. Speyer and B. Sturmfels, "The tropical grassmannian," *Advances in Geometry*, vol. 4, 05 2003.

[10] S. Herrmann, M. Joswig, and D. Speyer, "Dressians, tropical grassmannians, and their rays," *Forum Mathematicum*, vol. 26, 12 2011.

[11] G. M. Ziegler, *Lectures on polytopes*, ser. Graduate Texts in Mathematics. Springer-Verlag, New York, 1995, vol. 152. [Online]. Available: https://doi.org/10.1007/978-1-4613-8431-1

[12] J. A. De Loera, J. Rambau, and F. Santos, *Triangulations*, ser. Algorithms and Computation in Mathematics. Springer-Verlag, Berlin, 2010, vol. 25, structures for algorithms and applications. [Online]. Available: https://doi.org/10.1007/978-3-642-12971-1

[13] J. Oxley, *Matroid theory*, 2nd ed., ser. Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, 2011, vol. 21. [Online]. Available: https://doi.org/10.1093/acprof:oso/9780198566946.001.0001

[14] D. J. A. Welsh, *Matroid theory / D. J. A. Welsh.* Academic Press London ; New York, 1976.

[15] A. Schrijver, *Combinatorial optimization. Polyhedra and efficiency. Vol. B*, ser. Algorithms and Combinatorics. Springer-Verlag, Berlin, 2003, vol. 24, matroids, trees, stable sets, Chapters 39–69.

[16] F. Ardila, C. Benedetti, and J. Doker, "Matroid polytopes and their volumes," *Discrete and Computational Geometry*, vol. 43, 10 2008.

[17] D. Maclagan and B. Sturmfels, *Introduction to Tropical Geometry*, ser. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2015, vol. 161.

[18] B. Frenk, "Tropical varieties, maps and gossip." [Online]. Available: h

[19] A. Aparicio Monforte and M. Kauers, "Formal laurent series in several variables," *Expositiones Mathematicae*, vol. 31, no. 4, pp. 350–367, 2013. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0723086913000054

[20] G. Mikhalkin and J. Rau, "Tropical geometry," unplublished manuscript.

[21] E. Miller and B. Sturmfels, *Combinatorial commutative algebra*, ser. Graduate Texts in Mathematics. Springer-Verlag, New York, 2005, vol. 227.

[22] E. M. Feichtner and B. Sturmfels, "Matroid polytopes, nested sets and Bergman fans," *Port. Math. (N.S.)*, vol. 62, no. 4, pp. 437–468, 2005.

[23] J. Mundinger, "The image of a tropical linear space," 08 2018. [Online]. Available: arXiv:1808.02150

[24] A. W. Dress and W. Wenzel, "Valuated matroids: a new look at the greedy algorithm," *Applied Mathematics Letters*, vol. 3, no. 2, pp. 33–35, 1990. [Online]. Available: https://www.sciencedirect.com/science/article/pii/089396599090009Z

[25] Z. Izhakian, M. Johnson, and M. Kambites, "Tropical matrix groups," 2012.

[26] K. Murota, "Finding optimal minors of valuated bimatroids," *Applied Mathematics Letters*, vol. 8, no. 4, pp. 37 – 41, 1995. [Online]. Available: http://www.sciencedirect.com/science/article/pii/089396599500043P

[27] A. Schrijver, "Matroids and linking systems," *Journal of Combinatorial Theory, Series B*, vol. 26, no. 3, pp. 349 – 369, 1979. [Online]. Available: http://www.sciencedirect.com/science/article/pii/009589567990011X

[28] J. Giansiracusa and N. Giansiracusa, "A grassmann algebra for matroids," *manuscripta mathematica*, vol. 156, 05 2018.

[29] K. Murota, *Matrices and matroids for systems analysis*, ser. Algorithms and Combinatorics. Springer-Verlag, Berlin, 2010, vol. 20, corrected paperback reprint of the 2000 original [MR1739147]. [Online]. Available: https://doi.org/10.1007/978-3-642-03994-2

[30] M. Dresher and O. Ore, "Theory of multigroups," *American Journal of Mathematics*, vol. 60, no. 3, pp. 705–733, 1938. [Online]. Available: http://www.jstor.org/stable/2371606

[31] K. Stephenson, "Analytic functions sharing level curves and tracts," *Annals of Mathematics*, vol. 123, no. 1, pp. 107–144, 1986. [Online]. Available: http://www.jstor.org/stable/1971353

[32] M. Baker and N. J. Bowler, "Matroids over hyperfields," *arXiv: Combinatorics*,

2016.

[33] A. Bahmanian and M. Šajna, "Hypergraphs: connection and separation," *Theory and Applications of Graphs*, vol. 2, 04 2015.

[34] T. Markwig, "A field of generalised puiseux series for tropical geometry," *Rendiconti del Seminario Matematico*, vol. 68, 10 2007.

[35] M. Joswig and B. Schröter, "Matroids from hypersimplex splits," *Journal of Combinatorial Theory, Series A*, vol. 151, pp. 254–284, 2017. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0097316517300596

[36] S. Herrmann and M. Joswig, "Splitting Polytopes," *Münster J. Math*, pp. 109–141, 2008.

[37] B. Schröter, "Multi-splits and tropical linear spaces from nested matroids," *Discrete and Computational Geometry*, vol. 61, 04 2019.

[38] A. Fink and F. Rincón, "Stiefel tropical linear spaces," *Journal of Combinatorial Theory, Series A*, vol. 135, pp. 291–331, 2015. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0097316515000710

[39] J. P. Kung, "Bimatroids and invariants," *Advances in Mathematics*, vol. 30, no. 3, pp. 238–249, 1978. [Online]. Available: https://www.sciencedirect.com/science/article/pii/0001870878900385