The Tutte polynomial via lattice point counting

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May 11, 2021

Abstract

We recover the Tutte polynomial of a matroid, up to change of co-ordinates, from an Ehrhart-style polynomial counting lattice points in the Minkowski sum of its base polytope and scalings of simplices. Our polynomial has coefficients of alternating sign with a combinatorial interpretation closely tied to the Dawson partition. Our definition extends in a straightforward way to polymatroids, and in this setting our polynomial has Kálmán's internal and external activity polynomials as its univariate specialisations.

1 Introduction

The Tutte polynomial $T_M(x, y)$ of a matroid M, formulated by Tutte for graphs and generalised to matroids by Crapo, is perhaps the invariant most studied by researchers in either field, on account of its diverse applications. Most straightforwardly, any numerical function of matroids or graphs (or function valued in another ring) which can be computed by a deletion-contraction recurrence can also be obtained as an evaluation of the Tutte polynomial. Examples include the number of bases and the number of independent sets in a given matroid, and the number of acyclic orientations of a graph.

A matroid is expediently encoded by its *matroid (base) polytope*, the convex hull of the indicator vectors of the bases. One useful feature of this polytope is that no other *lattice points*, i.e. points with integer coordinates, are caught in the convex hull. Therefore, counting the lattice points in the base polytope of M tells us the number of bases of M, which equals $T_M(1, 1)$.

The Tutte polynomial can be written as a generating function for bases of a matroid according to their *internal* and *external activity*, two statistics that count elements which permit no basis exchanges of certain shapes. Kálmán [12] observed that the active elements of a basis can be readily discerned from the matroid polytope: given a lattice point of this polytope, one inspects in which ways one can increment one coordinate, decrement another, and remain in the polytope. The lattice points one reaches by incrementing a coordinate are those

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contained in the Minkowski sum with the standard simplex $\mathrm{Conv}\{e_1, \ldots, e_n\}$, and decrementing a coordinate is similarly encoded by the upside-down simplex $\mathrm{Conv}\{-e_1, \ldots, -e_n\}$.

Our main result, Theorem 3.1, proves that the bivariate matroid polynomial that counts the lattice points in the Minkowski sum of the matroid polytope and scalings of the above simplices contains the same information as the Tutte polynomial. To be precise, either polynomial is an evaluation of the other.

Kálmán's interest in this view of activity arose from the question of enumerating spanning trees of bipartite graphs according to their vector of degrees at the vertices of one colour. The set of such vectors is rarely a matroid polytope, but it is always a *polymatroid* polytope. Polymatroids were introduced in Edmonds' work in optimisation circa 1970, as an extension of matroids formed by relaxing one matroid rank axiom so as to allow singletons to have arbitrarily large rank. Postnikov's *generalised permutohedra* [19] are virtually the same object. (Integer generalised permutohedra are exactly translates of polymatroid polytopes, and they are polymatroid polytopes just when they lie in the closed positive orthant.)

One would like an analogue of the Tutte polynomial for polymatroids. Unlike the case for matroids, the simultaneous generating function for internal and external activity does not answer to this desire, as it is not even an invariant of a polymatroid: Example 6.4 shows that it varies with different choices of ordering of the ground set. Only the generating functions for either activity statistic singly are in fact invariants, as Kálmán proved, naming them $I_r(\xi)$ and $X_r(\eta)$. Our lattice point counting polynomial can be applied straightforwardly to a polymatroid, and we prove in Theorem 6.3 that it specialises to $I_r(\xi)$ and $X_r(\eta)$. That is, the invariant we construct is a bivariate analogue of Kálmán's activity polynomials, which answers a question of [12].

In seeking to generalise the Tutte polynomial to polymatroids one might be interested in other properties than activity, especially its universal property with respect to deletion-contraction invariants. Like matroids, polymatroids have a well-behaved theory of *minors*, analogous to graph minors: for each ground set element e one can define a deletion and contraction, and knowing these two determines the polymatroid. The deletion-contraction recurrence for the Tutte polynomial reflects this structure. The recurrence has three cases, depending on whether e is a loop, a coloop, or neither of these. The number of these cases grows quadratically with the maximum possible rank of a singleton, making it correspondingly hairier to write down a universal invariant. In 1993 Oxley and Whittle [18] addressed the case of *2-polymatroids*, where singletons have rank at most 2, and prove that the corank-nullity polynomial is still universal for a form of deletion-contraction recurrence. The general case, with a natural assumption on the coefficients of the recurrence but one slightly stronger than in [18], is addressed using coalgebraic tools in [5], fitting into the tradition of applications of coalgebras and richer structures in combinatorics: see [1, 14] for other work of particular relevance here.

For our polymatroid invariant we give a recurrence that involves not just the deletion and contraction, but a whole array of "slices" of which the deletion

and the contraction are the extremal members (Theorem 5.6). We do not know a recurrence relation where only the deletion and the contraction appear.

We would be remiss not to mention the work of Kálmán and Postnikov [13] proving the central conjecture of [12], that swapping the two colours in a bipartite graph leaves $I_r(\xi)$ unchanged. Their proof also exploits Ehrhart-theoretic techniques, but the key polytope is the *root polytope* of the bipartite graph. We expect that it should be possible to relate this to our machinery via the Cayley trick. Oh [16] has also investigated a similar polyhedral construction, as a way of proving Stanley's pure O-sequence conjecture for cotransversal matroids.

This paper is organised as follows. Section 2 introduces the definitions of our main objects. In Section 3 we begin by explaining the construction of our polynomial for matroids, followed by how this is related to the Tutte polynomial (our main theorem, Theorem 3.1). In Section 4, we give a geometric interpretation of the coefficients of our polynomial, by way of a particular subdivision of the relevant polytope, which has a simple interpretation in terms of Dawson partitions. In Section 5, we discuss the extension to polymatroids, including properties our invariant satisfies in this generality. Section 6 is dedicated to the relationship to Kálmán's univariate activity invariants.

Acknowledgments

2   Preliminaries

We assume the reader has familiarity with basic matroid terminology, and recommend [17] as a reference for this material. Given a set E, let $\mathcal{P}(E)$ be its power set.

Definition 2.1. A *polymatroid* $M = (E, r)$ on a finite *ground set* E consists of the data of a rank function $r : \mathcal{P}(E) \to \mathbb{Z}^+ \cup \{0\}$ such that, for $X, Y \in \mathcal{P}(E)$, the following conditions hold:

P1. $r(\emptyset) = 0$

P2. If $Y \subseteq X$, then $r(Y) \leq r(X)$

P3. $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$

A *matroid* is a polymatroid such that $r(i) \leq 1$ for all $i \in E$. Like matroids, polymatroids can be defined *cryptomorphically* in other equivalent ways, the way of most interest to us being as polytopes (Definition 2.8). We will not be pedantic about which axiom system we mean when we say "polymatroid".

The following three definitions of activity for polymatroids are from [12]. They anticipate what will be our main perspective on polymatroids in this paper, namely viewing them as polytopes (Definition 2.8). The ambient vector space for these polytopes is $\mathbb{R}^E$, where the finite set E is the ground set. For a set $U \subseteq E$, let $\mathbf{e}_U \in \mathbb{R}^E$ be the indicator vector of U, and abbreviate $\mathbf{e}_{\{i\}}$ by $\mathbf{e}_i$.

Definition 2.2. A vector $\mathbf{x} \in \mathbb{Z}^E$ is called a *base* of a polymatroid $M = (E, r)$ if $\mathbf{x} \cdot \mathbf{e}_E = r(E)$ and $\mathbf{x} \cdot \mathbf{e}_S \leq r(S)$ for all subsets $S \subseteq E$.

Let $\mathcal{B}_M$ be the set of all bases of a polymatroid $M = (E, r)$.

Definition 2.3. A *transfer* is possible from $u_1 \in E$ to $u_2 \in E$ in the base $\mathbf{x} \in \mathcal{B}_M$ if by decreasing the u_1-component of $\mathbf{x}$ by 1 and increasing its u_2-component by 1 we get another base.

Like matroids, polymatroids have a base exchange property [10, Theorem 4.1]. If $\mathbf{x}$ and $\mathbf{y}$ are in $\mathcal{B}_M$ and $\mathbf{x}_i > \mathbf{y}_i$ for some $i \in E$, then there exists l such that $\mathbf{x}_l < \mathbf{y}_l$ and $\mathbf{x} - \mathbf{e}_i + \mathbf{e}_l$ is again in $\mathcal{B}_M$, or in other words, such that a transfer is possible from i to l in $\mathbf{x}$.

Fix a total ordering of the elements of E.

Definition 2.4.   i. We say that $u \in E$ is *internally active* with respect to the base x if no transfer is possible in x *from* u to a smaller element of E.

 ii. We say that $u \in E$ is *externally active* with respect to x if no transfer is possible in x *to* u from a smaller element of E.

For $x \in \mathcal{B}_M$, let the set of internally active elements with respect to x be denoted with $\text{Int}(x)$, and let $\iota(x) = |\text{Int}(x)|$; likewise, let the set of externally active elements be denoted with $\text{Ext}(x)$ and $\varepsilon(x) = |\text{Ext}(x)|$. Let $\bar{\iota}(x), \bar{\varepsilon}(x)$ denote the respective numbers of inactive elements.

When M is a matroid, the following definitions of activity are more commonly used, analogous to Tutte's original formulation using spanning trees of graphs.

Definition 2.5. Take a matroid $M = (E, r)$. Let B be a basis of M.

 i. We say that $e \in E - B$ is *externally active* with respect to B if e is the smallest element in the unique circuit contained in $B \cup e$, with respect to the ordering on E.

 ii. We say that $e \in B$ is *internally active* with respect to B if e is the smallest element in the unique cocircuit in $(E \setminus B) \cup e$.

Remark 2.6. In the cases where it is set forth, namely $e \in E - B$ for external activity and $e \in B$ for internal activity, Definition 2.5 agrees with Definition 2.4. But it will be crucial that we follow Definition 2.4 where Definition 2.5 doesn't apply: when M is a matroid and B a basis thereof, we consider all elements $e \in B$ externally active, and all elements $e \in E - B$ internally active, with respect to B.

Definition 2.7. Let $M = (E, r)$ be a matroid with ground set E and rank function $r : \mathcal{P}(E) \to \mathbb{Z}^+ \cup \{0\}$. The *Tutte polynomial* of M is

$$T_M(x, y) = \sum_{S \subseteq E} (x - 1)^{r(M) - r(S)} (y - 1)^{|S| - r(S)}. \qquad (2.7.1)$$

The presentation of the Tutte polynomial in Definition 2.7 is given in terms of the *corank-nullity polynomial*: up to a change of variables, it is the generating function for subsets S of the ground set by their corank $r(M) - r(S)$ and nullity $|S| - r(S)$. When M is a matroid, the Tutte polynomial is equal to a generating function for activities:

$$T_M(x, y) = \frac{1}{x^{|E| - r(E)} y^{r(E)}} \sum_{B \in \mathcal{B}_M} x^{\iota(B)} y^{\varepsilon(B)}. \qquad (2.7.2)$$

The unfamiliar denominator in this formula appears on account of Remark 2.6. Although on its face the right hand side of the formula depends on the ordering imposed on E, its equality with the right hand side of equation 2.7.1 shows that there is no such dependence. For polymatroids, activity invariants can be defined as well: see Definition 6.1 and following discussion.

Let $r : \mathcal{P}(E) \to \mathbb{Z}^+ \cup \{0\}$ be a rank function, and $M = (E, r)$ the associated polymatroid. The *extended polymatroid* of M is defined to be the polytope

$$EP(M) = \{\mathbf{x} \in \mathbb{R}^E \mid \mathbf{x} \geq 0 \text{ and } \mathbf{x} \cdot \mathbf{e}_U \leq r(U) \text{ for all } U \subseteq E\}.$$

Definition 2.8. The *polymatroid (base) polytope* of M is a face of the extended polymatroid:

$$P(M) = EP(M) \cap \{\mathbf{x} \in \mathbb{R}^E \mid \mathbf{x} \cdot \mathbf{e}_E = r(E)\} = \operatorname{conv} \mathcal{B}_M.$$

Either one of these polytopes contains all the information in the rank function. In fact, they can be used as cryptomorphic axiomatisations of polymatroids: a polytope whose vertices have nonnegative integer coordinates is a polymatroid polytope if and only if all its edges are parallel to a vector of the form $\mathbf{e}_i - \mathbf{e}_j$ for some $i, j \in E$. Extended polymatroids permit a similar characterisation; moreover, they can be characterised as those polytopes over which a greedy algorithm correctly optimises every linear functional with nonnegative coefficients, which was the perspective of their inventor Edmonds [7].

3   Our invariant

This section describes the construction of our matroid polynomial, to be denoted Q'_M, which counts the lattice points of a particular Minkowski sum of polyhedra, and explains its relation to the Tutte polynomial. In anticipation of Section 5 we set out the definition in the generality of polymatroids.

3.1 Construction

Given a polytope $P \subseteq \mathbb{R}^E$, we will write $L(P) := P \cap \mathbb{Z}^E$ for its set of lattice points. Let Δ be the standard simplex in $\mathbb{R}^E$ of dimension $|E| - 1$, that is

$$\Delta = \mathrm{conv}\{\mathbf{e}_i : i \in E\},$$

and ∇ be its reflection through the origin, $\nabla = \{-x : x \in \Delta\}$. The faces of Δ are the polyhedra

$$\Delta_S = \mathrm{conv}\{\mathbf{e}_i : i \in S\}$$

for all nonempty subsets S of E; similarly, the faces of ∇ are the polyhedra ∇_S given as the reflections of the Δ_S. (We exclude the empty set as a face of a polyhedron.)

We consider $P(M) + u\Delta + t\nabla$ where $M = (E, r)$ is any polymatroid and $u, t \in \mathbb{Z}^+ \cup \{0\}$. We are interested in the lattice points in this sum. These can be interpreted as the vectors that can be turned into bases of M by incrementing a coordinate t times and decrementing one u times. By Theorem 7 of [15], the number

$$Q_M(t, u) := \#L(P(M) + u\Delta + t\nabla) \qquad (3.0.1)$$

of lattice points in the sum is a polynomial in t and u, of degree $\dim(P(M) + u\Delta + t\nabla) = |E| - 1$. We will mostly work with this polynomial after a change of variables: letting the coefficients c_{ij} be defined by

$$Q_M(t, u) = \sum_{i,j} c_{ij} \binom{u}{j} \binom{t}{i},$$

we use these to define the polynomial

$$Q'_M(x, y) = \sum_{ij} c_{ij} (x - 1)^i (y - 1)^j.$$

The change of variables is chosen so that applying it to any lattice point count $\#L(u\Delta_X + t\nabla_Y)$ yields $x^i y^j$, where Δ_X and ∇_Y are faces of Δ and ∇ of respective dimensions i and j. This is behind the combinatorial interpretation of the coefficients of Q' in Theorem 4.1, as well as the relationship with Kálmán's activity polynomials in Theorem 6.3. The latter result is the one which first suggested the above change of basis to the authors.

3.2 Relation to the Tutte polynomial

For the remainder of Section 3, we assume that M is a matroid. The main theorem of this section is that $Q'_M(x, y)$ is an evaluation of the Tutte polynomial, and in fact one that contains precisely the same information. As such, the Tutte polynomial can be evaluated by lattice point counting methods. Our first formula is a power series expression for $T_M(x, y)$ with the $Q_M(t, u)$ as coefficients.

Theorem 3.1. *Let $M = (E, r)$ be a matroid. Then*

$$T_M(x, y) = (xy - x - y)^{|E|}(-x)^{r(M)}(-y)^{|E|-r(M)} \cdot$$

$$\sum_{u,t \geq 0} Q_M(t, u) \cdot \left(\frac{y - xy}{xy - x - y} \right)^t \left(\frac{x - xy}{xy - x - y} \right)^u$$

Our proof of Theorem 3.1 arrives first at the relationship between $Q'_M(x, y)$ and the Tutte polynomial.

Theorem 3.2. *Let $M = (E, r)$ be a matroid. Then we have that*

$$Q'_M(x, y) = \frac{x^{|E|-r(M)}y^{r(M)}}{x + y - 1} \cdot T_M \left(\frac{x + y - 1}{y}, \frac{x + y - 1}{x} \right) \qquad (3.2.1)$$

An observation is in order before we embark on the proof. Since lattice points and their enumeration are our foremost concerns in this work, we prefer not to have to think of the points of our polyhedra with non-integral coordinates. It is the following lemma that lets us get away with this.

Lemma 3.3 ([20, Corollary 46.2c])**.** *Let P and Q be generalised permutohedra whose vertices are lattice points. Then if $x \in P + Q$ is a lattice point, there exist lattice points $p \in P$ and $q \in Q$ such that $x = p + q$.*

By repeated use of the lemma, if $q \in P(M) + u\Delta + t\nabla$ is a lattice point, then q has an expression of the form

$$q = \mathbf{e}_B + \mathbf{e}_{i_1} + \cdots + \mathbf{e}_{i_u} - \mathbf{e}_{j_1} - \cdots - \mathbf{e}_{j_t}$$

for some $\mathbf{e}_B \in P(M)$. Since all of the summands in $P(M) + u\Delta + t\nabla$ are translates of matroid polytopes, where the scalings are treated as repeated Minkowski sums, this can also be proved using the matroid partition theorem, as laid out by Edmonds [6].

Proof of Theorem 3.2. Lemma 3.3 guarantees that any lattice point q in $P(M) + i\Delta + j\nabla$ can be written

$$q = \mathbf{e}_B + \mathbf{e}_{x_1} + \cdots + \mathbf{e}_{x_i} - \mathbf{e}_{y_1} - \cdots - \mathbf{e}_{y_j} \qquad (3.3.1)$$

for some $\mathbf{e}_B \in P(M)$. Given a lattice point q, consider which i and j permit such an expression. First, the sum of the coordinates of the right hand side of (3.3.1) is $r(M) + i - j$, which implies

$$i - j = q_1 + \cdots + q_n - r(M).$$

Next, note that if $q \in P(M) + i\Delta + j\nabla$ then also $q \in P(M) + (i+1)\Delta + (j+1)\nabla$, because $\mathbf{e}_a - \mathbf{e}_a$ can be added to the right hand side of (3.3.1) for any $a \in E$. So there exist naturals i_0, j_0 such that $q \in P(M) + i\Delta + j\nabla$ if and only if $i = i_0 + c$ and $j = j_0 + c$ for some $c \geq 0$. Call the expression (3.3.1) *minimal* if $(i, j) = (i_0, j_0)$.

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In the case that $q = \mathbf{e}_S$ has coordinates 0 or 1, the summands in a minimal expression (3.3.1), besides $\mathbf{e}_B$, are all of the form $\mathbf{e}_x$ for $x \notin S$ or $-\mathbf{e}_y$ for $y \in S$, without repetition. So determining (i_0, j_0) in this case reduces to a question about sets: what's the fewest number of elements of E we can add to, and remove from, a basis of M to obtain S? The answer is

$$(i_0, j_0) = (|S| - r(S), r(M) - r(S)) = (\mathrm{null}(S), \mathrm{cork}(S)).$$

Removing an element can decrease rank by at most 1, and adding one cannot decrease rank, so $j_0 \geq r(M) - r(S)$. This bound is attainable by choosing B to be a basis containing a largest independent subset of S.

We return to the case of $q \in \mathbb{Z}^E$ general. The coordinates of $\mathbf{e}_B$ are 0 or 1, so if $q_a > 1$ for some $a \in E$, then in any expression (3.3.1) at least $q_a - 1$ of the indices x_k equal a, and if $q_a < 0$ then at least $-q_a$ of the indices y_k equal a. Subtracting $q_a - 1$ instances of $\mathbf{e}_a$ for each a with $q_a > 1$, and $-q_a$ instances of $-\mathbf{e}_a$ for each a with $q_a < 0$, leaves an expression of form (3.3.1) where the left hand side is $\mathbf{e}_S$ with $S = S(q) := \{1 \leq i \leq n \mid q_i > 0\}$. If the expression for q was minimal, so is this new expression for $\mathbf{e}_S$. Therefore, we have

$$(i_0, j_0) = \Big(\mathrm{null}(S(q)) + \sum_{a \in S(q)} (q_a - 1), \ \mathrm{cork}(S(q)) - \sum_{a \in E - S(q)} q_a \Big). \quad (3.3.2)$$

This gives us the enumeration

$$\#L(P(M) + i\Delta + j\nabla) = \sum_{c \geq 0} \sum_{S \subseteq E} \binom{i - c + |S| - \mathrm{null}(S) - 1}{|S| - 1}$$
$$\times \binom{j - c + |E - S| - \mathrm{cork}(S) - 1}{|E - S| - 1}. \quad (3.3.3)$$

The outer sum is over the quantity c above, so that $(i_0, j_0) = (i - c, j - c)$. The inner sum partitions the lattice points $q \in P(M) + i\Delta + j\nabla$ according to $S = S(q)$, i.e. according to which coordinates of q are positive. Subject to the sign conditions the vector q may be any vector satisfying the equalities in (3.3.2). That is, $(q_a - 1 : a \in S)$ is a tuple of $|S|$ non-negative integers that sum to $i - c - \mathrm{null}(S)$, and $(-q_a : a \in E - S)$ is a tuple of $|E - S|$ non-negative integers that sum to $j - c - \mathrm{cork}(S)$. The two binomial coefficients in (3.3.3) are the number of ways to choose these tuples of non-negative integers.

Using Equation (3.3.3), the generating function

$$\sum_i \sum_j \#L(P(M) + i\Delta + j\nabla) v^i w^j$$

equals

$$\sum_i \sum_j \sum_{c \geq 0} \sum_S \binom{i - c + |S| - \mathrm{null}(S) - 1}{|S| - 1} v^{i-c}$$
$$\times \binom{j - c + |E - S| - \mathrm{cork}(S) - 1}{|E - S| - 1} w^{j-c} (vw)^c.$$

Using the identity $\sum_i \binom{i+a}{b} x^i = \dfrac{x^{b+a}}{(1-x)^{b+1}}$ simplifies this to

$$\sum_c \sum_S \frac{v^{\mathrm{null}(S)}}{(1-v)^{|S|}} \cdot \frac{w^{\mathrm{cork}(S)}}{(1-w)^{|E-S|}} \cdot (vw)^c$$

which we can write as

$$\sum_S \frac{v^{\mathrm{null}(S)}}{(1-v)^{\mathrm{null}(S)-\mathrm{cork}(S)+r(M)}} \cdot \frac{w^{\mathrm{cork}(S)}}{(1-w)^{\mathrm{cork}(S)-\mathrm{null}(S)+|E|-r(M)}} \cdot \sum_c (vw)^c.$$

Collecting like exponents together, we end up with

$$
\begin{aligned}
\sum_{i,j} \#L(P(M)+i\Delta+j\nabla)v^i w^j &= \frac{1}{1-vw} \cdot \frac{1}{(1-v)^{r(M)}(1-w)^{|E|-r(M)}} \\
&\quad \times \sum_S \left(\frac{v(1-w)}{1-v}\right)^{\mathrm{null}(S)} \left(\frac{w(1-v)}{1-w}\right)^{\mathrm{cork}(S)} \\
&= \frac{1}{1-vw} \cdot \frac{1}{(1-v)^{r(M)}(1-w)^{|E|-r(M)}} \\
&\quad \times T_M\left(\frac{w(1-v)}{1-w}+1, \frac{v(1-w)}{1-v}+1\right) \\
&= \frac{1}{1-vw} \cdot \frac{1}{(1-v)^{r(M)}(1-w)^{|E|-r(M)}} \\
&\quad \times T_M\left(\frac{1-vw}{1-w}, \frac{1-vw}{1-v}\right) \quad (3.3.4)
\end{aligned}
$$

where T_M is the Tutte polynomial of M. Now it remains to be shown that the left-hand side contains an evaluation of our polynomial Q'_M. Using our original definition of Q_M, Equation (3.0.1), we have that

$$
\begin{aligned}
\sum_{i,j} \#L(P(M)+i\Delta+j\nabla)v^i w^j &= \sum_{i,j,k,l} c_{kl}\binom{i}{l}\binom{j}{k}v^i w^j \\
&= \sum_{k,l} c_{kl} \cdot \frac{v^l}{(1-v)^{l+1}} \cdot \frac{w^k}{(1-w)^{k+1}}.
\end{aligned}
$$

If we let $\dfrac{w}{1-w} = x-1$ and $\dfrac{v}{1-v} = y-1$, then

$$
\begin{aligned}
\sum_{i,j} \#L(P(M)+i\Delta+j\nabla)v^i w^j &= \sum_{k,l} c_{kl} \cdot \frac{v^l}{(1-v)^{l+1}} \cdot \frac{w^k}{(1-w)^{k+1}} \\
&= \frac{1}{(1-v)(1-w)} \sum_{k,l} c_{kl}(x-1)^k(y-1)^l \\
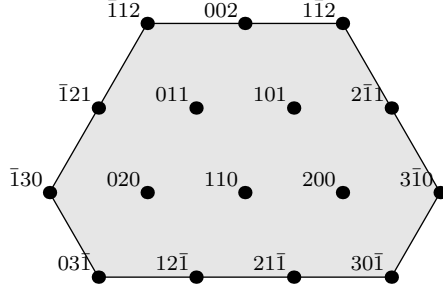&= \frac{Q'_M(x,y)}{(1-v)(1-w)}.
\end{aligned}
$$

Figure 1: The polytope $P(M) + u\Delta + t\nabla$ of Example 3.5. The coordinates are written without parentheses or commas, and $\bar{1}$ means -1.

So, from Equation (3.3.4), we have that

$$\frac{Q'_M(x,y)}{(1-v)(1-w)} = \frac{1}{1-vw} \cdot \frac{1}{(1-v)^{r(M)}(1-w)^{|E|-r(M)}} \cdot T_M\left(\frac{1-vw}{1-w}, \frac{1-vw}{1-v}\right).$$

Solving for w and v in terms of x and y gives that $w = \dfrac{x-1}{x}, v = \dfrac{y-1}{y}$.
Substitute these into the above equation to get

$$Q'_M(x,y) = \frac{x^{|E|-r(M)}y^{r(M)}}{x+y-1} \cdot T_M\left(\frac{x+y-1}{y}, \frac{x+y-1}{x}\right). \qquad \square$$

We can invert this formula by setting $x' = \dfrac{x+y-1}{y}, y' = \dfrac{x+y-1}{x}$, rearranging, and then relabelling.

Theorem 3.4. *Let $M = (E,r)$ be a matroid. Then*

$$T_M(x,y) = -\frac{(xy-x-y)^{|E|-1}}{(-y)^{r(M)-1}(-x)^{|E|-r(M)-1}} \cdot Q'_M\left(\frac{-x}{xy-x-y}, \frac{-y}{xy-x-y}\right)$$
$$\tag{3.4.1}$$

We conjecture that there is a relationship between our formula for the Tutte polynomial and the algebro-geometric formula for the Tutte polynomial in [9]. The computations on the Grassmannian in that work are done in terms of $P(M)$, the moment polytope of a certain torus orbit closure, and Δ and ∇ are the moment polytopes of the two dual copies of $\mathbb{P}^{n-1}$, the K-theory ring of whose product $\mathbb{Z}[x,y]/(x^n, y^n)$ is identified with the ambient ring of the Tutte polynomial.

Example 3.5. Let M be the matroid on ground set $[3] = \{1,2,3\}$ with $\mathcal{B}_M = \{\{1\},\{2\}\}$. When $u = 2$ and $t = 1$, the sum $P(M) + u\Delta + t\nabla$ is the polytope of Figure 1, with 16 lattice points.

To compute $Q_M(x, y)$, it is enough to count the lattice points in $P(M) + u\Delta + t\nabla$ for a range of u and t, and interpolate. Since Q_M is a polynomial of degree 2, it is sufficient to take t and u nonnegative integers with sum at most 2. These are the black entries in the table below:

$t \setminus u$	0	1	2
0	2	5	9
1	5	10	16
2	9	16	24

The unique degree ≤ 2 polynomial with these evaluations is

$$Q_M(t, u) = \binom{t}{2} + 2tu + \binom{u}{2} + 3t + 3u + 2,$$

so

$$Q'_M(x, y) = (x - 1)^2 + 2(x - 1)(y - 1) + (y - 1)^2 + 3(x - 1) + 3(y - 1) + 2$$
$$= x^2 + 2xy + y^2 - x - y.$$

Finally, by Theorem 3.4,

$$T_M(x, y) = -\frac{(xy - x - y)^2}{(-y)^0(-x)^1} \cdot \left(\frac{y^2 + 2xy + x^2}{(xy - x - y)^2} + \frac{y + x}{xy - x - y} \right)$$
$$= xy + y^2$$

which is indeed the Tutte polynomial of M. $\diamond$

Given Theorem 3.4, we can now prove Theorem 3.1.

Theorem 3.1. *Let $M = (E, r)$ be a matroid. Then*

$$T_M(x, y) = (xy - x - y)^{|E|}(-x)^{r(M)}(-y)^{|E|-r(M)}.$$
$$\sum_{u,t \geq 0} Q_M(t, u) \cdot \left(\frac{y - xy}{xy - x - y} \right)^t \left(\frac{x - xy}{xy - x - y} \right)^u$$

Proof. Consider the power series $\Sigma := \sum\limits_{u,t \geq 0} Q_M(t, u) \, a^t b^u$. Note that

$$\sum_{u,t \geq 0} \binom{t}{i}\binom{u}{j} a^t b^u = \frac{1}{ab} \cdot \left(\frac{a}{1 - a} \right)^{i+1} \left(\frac{b}{1 - b} \right)^{j+1}.$$

We can thus write Σ as

$$\frac{1}{ab} \sum_{i,j} c_{ij} \left(\frac{a}{1 - a} \right)^{i+1} \left(\frac{b}{1 - b} \right)^{j+1}.$$

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Substituting $a = (v-1)/v$ and $b = (w-1)/w$ turns this into

$$\Sigma = \frac{vw}{(v-1)(w-1)} \sum_{i,j} c_{ij}(v-1)^{i+1}(w-1)^{j+1}$$
$$= vw \sum_{i,j} c_{ij}(v-1)^i(w-1)^j$$
$$= vw\, Q'_M(v,w).$$

We can now apply Theorem 3.2:

$$\sum_{u,t\geq 0} Q_M(t,u) \left(\frac{v-1}{v}\right)^t \left(\frac{w-1}{w}\right)^u$$
$$= vw\, Q'_M(v,w)$$
$$= \frac{v^{|E|-r(M)+1}w^{r(M)+1}}{v+w-1} \cdot T_M\left(\frac{v+w-1}{w}, \frac{v+w-1}{v}\right).$$

Substitute $v = -x/(xy - x - y)$ and $w = -y/(xy - x - y)$ to get the stated result. $\square$

A further substitution and simple rearrangement gives the following corollary, included for the sake of completeness.

Corollary 3.6. *Let $M = (E, r)$ be a matroid. Then*

$$\sum_{u,t\geq 0} Q_M(t,u)v^t w^u = \frac{1}{(1-v)^{|E|-r(M)}(1-w)^{r(M)}(1-vw)} \cdot T_M\left(\frac{1-vw}{1-v}, \frac{1-vw}{1-w}\right).$$

Being a Tutte evaluation, Q' must have a deletion-contraction recurrence. We record the form it takes. See Theorem 5.6 for a more general discussion.

Proposition 3.7. *Let $M = (E, r)$ be a matroid with $|E| = n$. Then, for $e \in E$,*

i. $(xy - x - y)Q'_M(x,y) = xQ'_{M\setminus e}(x,y) + yQ'_{M/e}(x,y)$ when e is not a loop or coloop, and

ii. $Q'_M(x,y) = (x+y-1)Q'_{M/e}(x,y) = (x+y-1)Q'_{M\setminus e}(x,y)$ otherwise.

Proof. Part *ii* is a consequence of Proposition 5.1 below (which does not depend on the present section). When e is a (co)loop, $M = M_e \oplus M\setminus e = M_e \oplus M/e$, where M_e is the restriction of M to $\{e\}$ (or the equivalent contraction).

For part *i*, recall that if e is neither a loop nor a coloop, then $E(M\setminus e) = E - e = E(M/e)$, $r(M\setminus e) = r(M)$, and $r(M/e) = r(M) - 1$. Take the equation $T_M(x,y) = T_{M\setminus e}(x,y) + T_{M/e}(x,y)$ and rewrite it in terms of Q', as per

Theorem 3.4:

$$-\frac{(xy-x-y)^{n-1}}{(-y)^{r(M)-1}(-x)^{n-r(M)-1}}\cdot Q'_M(x,y)=$$

$$-\frac{(xy-x-y)^{n-2}}{(-y)^{r(M)-1}(-x)^{n-r(M)-2}}\cdot Q'_{M\setminus e}(x,y)-\frac{(xy-x-y)^{n-2}}{(-y)^{r(M)-2}(-x)^{n-r(M)-1}}\cdot Q'_{M/e}(x,y)$$

Multiplying through by $-\dfrac{(-y)^{r(M)-1}(-x)^{n-r(M)-1}}{(xy-x-y)^{n-2}}$ gives the result.   □

4   Coefficients

Some coefficients of the Tutte polynomial provide structural information about the matroid in question. Let $b_{i,j}$ be the coefficient of $x^i y^j$ in $T_M(x,y)$. The best-known case is that M is connected only if $b_{1,0}$, known as the *beta invariant*, is non-zero; moreover, $b_{1,0}=b_{0,1}$ when $|E|\geq 2$. Not every coefficient yields such an appealing result, though of course they do count the bases with internal and external activity of fixed sizes. In like manner, we are able to provide a enumerative interpretation of the coefficients of $Q'_M(x,y)$, which is the focus of this section.

In order to do this, we will make use of a regular subdivision of $P(M)+u\Delta+t\nabla$. The following definition is standard: see for example [4, Chapter 5]. A polyhedral complex whose vertices are drawn from a point configuration P in $\mathbb{R}^d$ is a *regular* subdivision of P if there are heights $h_i\in\mathbb{R}$ for every point $p_i\in P$ such that the cells of the subdivision are given by the projections (which drop the $d+1$st coordinate) of the lower faces of the polytope $\mathrm{Conv}\{(p_i,h_i)\in\mathbb{R}^{d+1}\mid p_i\in P\}$. A *lower* face is one which is the set of maxima of some linear function $\langle a,x\rangle$, where $a:\mathbb{R}^E\times\mathbb{R}\to\mathbb{R}$ is a linear functional with last coordinate $a(0,1)=-1$. In this paper, when we speak of a regular subdivision of a polytope, we will mean a regular subdivision of the set of all its lattice points.

Fix positive reals $\alpha_1<\cdots<\alpha_n$ and $\beta_1<\cdots<\beta_n$. Our regular subdivision $\mathcal{F}$ of $P(M)+u\Delta+t\nabla$ will be that determined by the heights

$$h_i=\min\{\alpha_{i_1}+\cdots+\alpha_{i_u}+\beta_{j_1}+\cdots+\beta_{j_t}:$$
$$p_i=x+\mathbf{e}_{i_1}+\cdots+\mathbf{e}_{i_u}-\mathbf{e}_{j_1}-\cdots-\mathbf{e}_{j_t}\text{ for some }x\in L(P(M))\}$$

assigned to the lattice points p_i. In other words, the "lifted" polytope whose lower faces are being projected is

$$\mathscr{L}=(P(M)\times\{0\})+\mathrm{Conv}\{(u\mathbf{e}_i,u\alpha_i)\}+\mathrm{Conv}\{(-t\mathbf{e}_i,t\beta_i)\}\subseteq\mathbb{R}^E\times\mathbb{R}.$$

We will write $\mathcal{F}$ as $\mathcal{F}(t,u)$ when we need to make the dependence on the parameters explicit. Note however that the structure of the face poset of $\mathcal{F}$ does not depend on t and u as long as these are positive.

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This Minkowski decomposition of $\mathscr{L}$ implies that $\mathcal{F}$ is also a *mixed* subdivision of $P(M) + u\Delta + t\nabla$: see [4, Section 9.2.2] for the definition of mixed subdivision, and Theorem 1.3.5 of the same work for the implication. In other words, each cell of $\mathcal{F}$ bears a canonical decomposition as a Minkowski sum of a face of $P(M)$, a face of $u\Delta$, and a face of $t\nabla$. When we name a face of $\mathcal{F}$ as a sum of three polytopes $F + G + H$, we mean to invoke this canonical decomposition. Moreover, these decompositions are compatible between faces: if $m_a(P)$ denotes the face of a polytope P on which a linear functional a is maximised, then the canonical decomposition for $m_a(F + G + H)$ is $m_a(F) + m_a(G) + m_a(H)$.

We now state the main result of this section:

Theorem 4.1. *Take the regular mixed subdivision $\mathcal{F}$ of $P(M) + u\Delta + t\nabla$ as described above. The unsigned coefficient $|[x^i y^j]Q'_M|$ counts the cells $F + G + H$ of $\mathcal{F}$ where $i = \dim(G)$, $j = \dim(H)$, and F is a vertex of $P(M)$ and there exists no cell $F' + G + H$ where $F' \supsetneq F$.*

The rest of this section is dedicated to its proof. The key idea will be to cover the lattice points of $P(M) + u\Delta + t\nabla$ by certain special "top degree" faces of $\mathcal{F}$ (Proposition 4.8).

Definition 4.2. A maximal cell $F + G + H$ of the mixed subdivision $\mathcal{F}$ is a *top degree face* when F is a vertex of $P(M)$.

We first describe the top degree faces more carefully. All top degree faces are of dimension $|E| - 1$ and have the form $\{\mathbf{e}_B\} + u\Delta_X + t\nabla_Y$. By Lemma 3.3, if $p \in \{\mathbf{e}_B\} + u\Delta_X + t\nabla_Y$ is a lattice point, then p has an expression of the form

$$p = \mathbf{e}_B + \mathbf{e}_{i_1} + \cdots + \mathbf{e}_{i_u} - \mathbf{e}_{j_1} - \cdots - \mathbf{e}_{j_t}. \tag{4.2.1}$$

The subdivision $\mathcal{F}$ determines a height function $h(x)$ on the lattice points x of $P(M) + u\Delta + t\nabla$, where $h(x)$ is the minimum real number such that $(x, h(x)) \in \mathscr{L}$. This height function is

$$h(x) := \min\{\alpha_{i_1} + \cdots + \alpha_{i_u} + \beta_{j_1} + \cdots + \beta_{j_t} \mid x - \mathbf{e}_{i_1} - \cdots - \mathbf{e}_{i_u} + \mathbf{e}_{j_1} + \cdots + \mathbf{e}_{j_t} \in L(P(M))\}.$$

If x is a lattice point of a top degree face then choosing the i_k and j_l in accord with (4.2.1) achieves the minimum.

The polytope $\mathscr{L}$ is the Minkowski sum of $P(M) \times \{0\}$ and

$$\mathscr{T} = \mathrm{Conv}\{(u\mathbf{e}_i, u\alpha_i)\} + \mathrm{Conv}\{(-t\mathbf{e}_i, t\beta_i)\},$$

and the faces of $\mathscr{L}$ that project to top degree faces of $\mathcal{F}$ are exactly those having as a mixed summand a lower face of $\mathscr{T}$ of the maximal dimension, $|E| - 1$. So we analyse these faces of $\mathscr{T}$.

The projection of an $(|E| - 1)$-dimensional lower face of $\mathscr{T}$ has the form $\Pi = u\Delta_X + t\nabla_Y$. If i and j were distinct elements of $X \cap Y$, then Π would have edges of the form $\mathrm{Conv}\{x, x + k(\mathbf{e}_i - \mathbf{e}_j)\}$ whose preimages in the lower face of $\mathscr{T}$ would not be edges, since they would contain the sum of the nonparallel

segments $\mathrm{Conv}\{(u\mathbf{e}_i, \alpha_i), (u\mathbf{e}_j, \alpha_j)\}$ and $\mathrm{Conv}\{(-t\mathbf{e}_i, \beta_i), (-t\mathbf{e}_j, \beta_j)\}$. Therefore we must have $|X \cap Y| \le 1$. Together with the fact that the dimensions of Δ_X and ∇_Y sum to $|E| - 1$, this implies that $X \cup Y = E$ and $|X \cap Y| = 1$. In fact the conditions on the α and β imply that $X \cap Y = \{1\}$, because replacing equal subscripts $i_k = j_l > 1$ by 1 decreases the quantity

$$\{\alpha_{i_1} + \cdots + \alpha_{i_u} + \beta_{j_1} + \cdots + \beta_{j_t} \mid x = \mathbf{e}_{i_1} + \cdots + \mathbf{e}_{i_u} - \mathbf{e}_{j_1} - \cdots - \mathbf{e}_{j_t}\},$$

which is the height function on lattice points of $\mathscr{T}$. These conditions $X \cup Y = E$ and $|X \cap Y| = 1$ are sufficient for Π to be a projection of of an $(|E| - 1)$-dimensional lower face of $\mathscr{T}$. Indeed, $m_a(\mathscr{T})$ projects to Π for $a : \mathbb{R}^E \times \mathbb{R} \to \mathbb{R}$ where $a(0, 1) = -1$ and

$$a(\mathbf{e}_i, 0) = \begin{cases} 0 & i = 1 \\ \alpha_i - \alpha_1 & i \in X \setminus \{1\} \\ \beta_1 - \beta_i & i \in Y \setminus \{1\}. \end{cases} \qquad (4.2.2)$$

The choice of a is unique up to adding a multiple of the functional $\mathbf{1} : \mathbb{R}^E \times \mathbb{R} \to \mathbb{R}$ that sums the coordinates of the $\mathbb{R}^E$ factor.

We thus have exactly $2^{|E|-1}$ top degree faces, one for each choice of X and Y as above – each element except 1 is either in X but not Y, or it is in Y but not X. The top degree faces are $\{\mathbf{e}_B\} + u\Delta_X + t\nabla_Y$ where $(\mathbf{e}_B, 0)$ is the vertex of $P(M) \times \{0\}$ maximised by a. The choice of the α and β imply that a has distinct coordinates, so that $m_a(P(M) \times \{0\})$ is indeed a vertex. The next lemma describes B combinatorially.

Lemma 4.3. *Take subsets X and Y of E with $X \cup Y = E$ and $X \cap Y = \{1\}$. The basis B such that $\{\mathbf{e}_B\} + u\Delta_X + t\nabla_Y$ is a top degree face is the unique basis B such that no elements of X are externally inactive and no elements of Y are internally inactive with respect to B, with reversed order on E.*

Proof. The basis B can be found using the simplex algorithm for linear programming on $P(M)$, applied to the linear functional a. The edges of $P(M)$ correspond to basis exchanges: they are exactly $\mathrm{Conv}\{\mathbf{e}_B, \mathbf{e}_{B'}\}$ where $B' = B \setminus \{b\} \cup \{x\}$ for some $b \in B$, $x \in E \setminus B$. So the simplex algorithm dictates that we should make basis exchanges to increase $a(\mathbf{e}_B, 0)$ until this is no longer possible. Using this we may rewrite the simplex algorithm turning the polyhedral geometry into matroid combinatorics. The result is Algorithm 4.4 below, in which we have also made the choices made about which basis exchange to use when many are possible.

Algorithm 4.4.

Input: M, X, Y.

Output: the basis B of M called for in Lemma 4.3.

(1) Let $i = 0$. Let B_0 be any basis of M.

(2) Define new bases $B_1, B_2, \ldots$ iteratively, as follows. Let the elements of B_i be $b_1, \ldots, b_r$, where $b_1 < \cdots < b_r$. For each $j = 1, \ldots, r$:

    (a) If there exists $x \in X$ greater than b_j such that $B_i \setminus \{b_j\} \cup \{x\}$ is a basis of M, then choose the maximal such x, let $B_{i+1} = B_i \setminus \{b_j\} \cup \{x\}$, increment i, and repeat step (2). Call this a move of type (a).

    (b) If not, and $b_j \in Y$, and there exists z less than b_j such that $B_i \setminus \{b_j\} \cup \{z\}$ is a basis of M, then choose the minimal such z, let $B_{i+1} = B_i \setminus \{b_j\} \cup \{z\}$, increment i, and repeat step (2). Call this a move of type (b).

(3) Terminate and return $B = B_i$.

The statement of the lemma is proved by noting that, by the definition of activity, an element x permits a move of type (a) if and only if it is externally inactive with respect to B, and an element z permits a move of type (b) if and only if it is internally inactive with respect to B, both activity statements being interpreted with reversed order on E.

For self-containedness, the remainder of this proof is a purely combinatorial analysis of Algorithm 4.4, avoiding reliance on the simplex algorithm. Claim 4.6 proves that Algorithm 4.4 terminates and can only return a single basis B of M. Then claim 4.7 proves that B agrees with the basis selected by the functional a of (4.2.2).

We will write the symmetric difference of two sets B and C as $B \triangle C$. Given $B, C \subseteq E$, we say that $B > C$ in *lexicographic order* if and only if the largest element of $B \triangle C$ belongs to B. Equivalently, if $B = \{b_1, b_2, \ldots\}$ and $C = \{c_1, c_2, \ldots\}$ with $b_1 > b_2 > \cdots$ and $c_1 > c_2 > \cdots$, then $B > C$ in lexicographic order if and only if $(b_1, b_2, \ldots) > (c_1, c_2, \ldots)$ according to the usual lexicographic order on sequences.

Claim 4.5. *In Algorithm 4.4, $B_i \triangle Y$ is strictly lexicographically increasing with i.*

Proof of Claim 4.5. If B_{i+1} is produced by a move of type (a), the maximum element of
$$(B_{i+1} \triangle Y) \triangle (B_i \triangle Y) = B_{i+1} \triangle B_i = \{x, b_j\}$$
is x. We have $x \in B_{i+1}$ and $x \notin Y$, so $x \in (B_{i+1} \triangle Y)$. Likewise if B_{i+1} is produced by a move of type (b), the maximum of
$$(B_{i+1} \triangle Y) \triangle (B_i \triangle Y) = B_{i+1} \triangle B_i = \{z, b_j\}$$
is b_j, which is in Y but not B_{i+1}, so again $b_j \in (B_{i+1} \triangle Y)$.   ■

Claim 4.6. *Algorithm 4.4 terminates and gives an output independent of B_0.*

Note that the choice of B_0 is the only nondeterministic part of the algorithm.

Proof of Claim 4.6. We have shown in the previous corollary that the sequence $(B_i \triangle Y)$ is strictly increasing in i. As M has a finite number of bases, this sequence must be finite, and thus the algorithm terminates. We now need to show that there is a unique basis at which the algorithm can terminate: we call these bases "termination bases" in the rest of the proof.

The structure $(E, \{B_i \triangle Y \mid B_i \in \mathcal{B}_M\})$ is what is known as a *delta-matroid*, a generalisation of a matroid allowing bases to have different sizes. This delta-matroid is a *twist* of M by the set Y [2]. The subsets $B_i \triangle Y$ are called the feasible sets. It is a result of Bouchet ([2]) that feasible sets of largest size form the bases of a matroid.

In order to show uniqueness of termination bases, we will first show that if B is a termination basis and $B \triangle Y$ is not of largest size, then any basis B' with $|B' \triangle Y| > |B \triangle Y|$ is terminal. Suppose this is not the case. As $|B \triangle Y| = |B| + |Y| - 2|B \cap Y|$, this requires that $|B' \cap Y| < |B \cap Y|$.

As B' is not a termination basis, there is either an element $b \in B'$ such that $(B' - b) \cup c \in \mathcal{B}$ for some element $c \in X$, where $c > b$, or there is an element $b \in Y \cap B$ such that $(B' - b) \cup a \in \mathcal{B}$ for some $a < b$. If we have $b, c \in X$, then $|((B' - b) \cup c) \cap Y| = |B' \cap Y|$. If $b \in Y$, then $|((B' - b) \cup c) \cap Y| < |B' \cap Y|$. If $a, b \in Y$, then $|((B' - b) \cup a) \cap Y| = |B' \cap Y|$. Finally, if $b \in Y$ and $a \in X$, then $|((B' - b) \cup a) \cap Y| < |B' \cap Y|$. In every case we have a contradiction.

As the algorithm terminates, we know that after a finite number of such exchanges, we produce B from B'. Let the bases constructed in each step form a chain

$$B', B_1, B_2, \ldots, B_n, B.$$

From above, we have that $|B' \cap Y| \geq |B_1 \cap Y| \geq \cdots \geq |B_n \cap Y| \geq |B \cap Y|$. This contradicts the initial assumption that $|B' \cap Y| < |B \cap Y|$.

Now assume the algorithm can terminate with two bases B_1, B_2. Take $B_1 \triangle Y$ and $B_2 \Delta Y$, and choose the earliest element $b \in B_1 \triangle Y - B_2 \triangle Y$ (assuming this comes lexicographically first in $B_1 \triangle Y$). If $b \in X$, then $b \in B_1 - B_2$. If $b \in Y$, then $b \in B_2 - B_1$. Similarly, if $c \in B_2 \triangle Y - B_1 \triangle Y$, if $c \in X$ then $c \in B_2 - B_1$, or if $c \in Y$ then $c \in B_1 - B_2$.

Apply the delta-matroid exchange algorithm to $B_1 \triangle Y$ and $B_2 \triangle Y$ to get that $(B_1 \triangle Y) \triangle \{b, c\}$ is a feasible set, for some element $c \in (B_1 \triangle Y) \triangle (B_2 \triangle Y)$. Given we have a twist of a matroid, we must have that $(B_1 \triangle Y) \triangle \{b, c\} = B_3 \triangle Y$ for some basis B_3, and so $|(B_1 \triangle \{b, c\}| = |B_3| = |B_1|$ as $\triangle$ is associative. This means we must have that exactly one of $\{b, c\}$ is in B_1. If $b \in X$, $(B_1 \triangle Y - b) \cup c = ((B_1 - b) \cup c) \triangle Y$, so $(B_1 - b) \cup c \in \mathcal{B}$ and we must have $c \in X$ by the above paragraph. As b was the earliest element different in either basis, we must have $c > b$, and so B_1 was not a termination basis of the original algorithm. If $b \in Y$, $(B_1 \triangle Y - b) \cup c = B_1 \triangle ((Y - b) \cup c)$. But we cannot change Y, so must have $[(B_1 - c) \cup b] \triangle Y$ and $c \in Y$. This means that again B_1 was not a termination basis, as we are replacing an element of Y with a smaller one. This completes the proof of Claim 4.6. ∎

Claim 4.7. *Let B_i and B_{i+1} be two bases of M found consecutively by the algorithm. Then $\sum\limits_{b \in B_i} a_b < \sum\limits_{b \in B_{i+1}} a_b$ for all i. That is, the sum $\sum\limits_{b \in B_i} a_b$ is increasing with i.*

Proof of Claim 4.7. Moves of type (a) replace an element b of a basis with a larger element c in X, so regardless of whether b was in X or Y, this must increase the sum as $a_b > 0$. Moves of type (b) replace an element $y \in Y$ in the basis with a smaller element d. If $d \in Y$, we are replacing a_y with a smaller negative, as $|a_d| < |a_y|$. If $d \in X$, we are replacing a negative a_y with a positive a_d. So $\sum\limits_{b \in B} a_b$ is increasing in every case. ∎

To complete the proof of Lemma 4.3, if Algorithm 4.4 picks B_0 to be the basis B^* maximising $\sum_{b \in B^*} a_b$, then Claim 4.7 implies that the algorithm must return $B = B^*$. So by the uniqueness in Claim 4.6, the algorithm must return $B = B^*$ in every case, and this is the geometric content of the statement of the lemma. □

Top degree faces are useful for counting lattice points in $P(M) + u\Delta + t\nabla$ because they form a cover.

Proposition 4.8. *In the subdivision $\mathcal{F}$, each of the lattice points of $P(M) + u\Delta + t\nabla$ lies in a top degree face.*

Proof. Let π be the projection map $\mathbb{R}^E \times \mathbb{R} \to \mathbb{R}^E$. Let x be a lattice point of $P(M) + u\Delta + t\nabla$, and let $x^* = (x, h) \in \mathcal{L}$ be the element of $\pi^{-1}(x)$ with minimal last coordinate. We will show that

$$x^* \in \{(\mathbf{e}_B, 0)\} + \mathrm{Conv}\{(u\mathbf{e}_i, u\alpha_i) : i \in X\} + \mathrm{Conv}\{(-t\mathbf{e}_j, t\beta_j) : j \in Y\} \quad (4.8.1)$$

for some sets X, Y such that

$X \cup Y = E$, $X \cap Y = \{1\}$ and Algorithm 4.4 yields B when given X and Y as input. $\quad (*)$

By Theorem 4.3, the set on the right side of (4.8.1) projects under π to a top degree face, which will prove the proposition.

Lemma 3.3 shows that there exists an expression for x of the form

$$x = \mathbf{e}_B - \mathbf{e}_{i_1} - \cdots - \mathbf{e}_{i_t} + \mathbf{e}_{j_1} + \cdots + \mathbf{e}_{j_u}. \quad (4.8.2)$$

We thus have

$$\tilde{x} = (\mathbf{e}_B, 0) + (-\mathbf{e}_{i_1}, \beta_1) + \cdots + (-\mathbf{e}_{i_t}, \beta_t) + (\mathbf{e}_{j_1}, \alpha_1) + \cdots + (\mathbf{e}_{j_u}, \alpha_u) \quad (4.8.3)$$

for some $\tilde{x}$ with $\pi(\tilde{x}) = x$. To prove the proposition, it is sufficient to show that if there do not exist $X \ni i_1, \ldots, i_t$ and $Y \ni j_1, \ldots, j_u$ such that X, Y, B satisfy conditions (*) above, then $\tilde{x} \neq x^*$. We do this by rewriting the expression (4.8.2) into a different expression for x which decreases the sum of the last coordinates in (4.8.3).

The postconditions of Algorithm 4.4 require that

1. if there exists an element $a \notin B$ such that $a < b \in B$ and $(B - b) \cup a \in \mathcal{B}$ then $b \in X$, and

2. if there exists an element $a \in B$ such that $a < b \notin B$ and $(B - a) \cup b \in \mathcal{B}$ then $b \in Y$.

Choose any subsets X and Y of E, both containing 1. Given these, we will construct X' and Y' such that $X' \cup Y' = E$ and $X \cap Y = \{1\}$.

- Suppose $b \in B$ and $b = i_k$, and there exists $a < b$ with $a \in B$ such that $(B - b) \cup a \in \mathcal{B}_M$. In the expression for x, replace $-\mathbf{e}_b + \mathbf{e}_B$ with $-\mathbf{e}_a + \mathbf{e}_{(B-b)\cup a}$. Add a to Y'.

- Suppose $b \in B$ and $b = j_l$, and there exists $a < b$ with $a \in B$ such that $(B - a) \cup b \in \mathcal{B}_M$. In the expression for x, replace $\mathbf{e}_B + \mathbf{e}_b$ with $\mathbf{e}_{(B-a)\cup b} + \mathbf{e}_a$. Add a to X'.

- Suppose we have an element $i \in X \cap Y$ other than 1. In the expression for x replace $-\mathbf{e}_i + \mathbf{e}_i$ with $-\mathbf{e}_1 + \mathbf{e}_1$. Remove i from both X' and Y'.

The above three operations always replace a term $\pm\mathbf{e}_b$ in the expression for x with another term $\pm\mathbf{e}_a$ with $a < b$. Therefore we replace either α_b with α_a or β_b with β_a in the last coordinates of (4.8.3), decreasing the sum of these coordinates, as claimed. $\qquad\square$

Before we can get to the proof of Theorem 4.1, we first need two results on how these top degree faces interact. Note that in the service of readability we write 1 instead of $\{1\}$ in subscripts. When we say that a polytope contains a basis B, we mean that it contains the indicator vector $\mathbf{e}_B$.

Lemma 4.9. *Take two distinct partitions (X_1, Y_1), (X_2, Y_2) of $[n] \setminus \{1\}$. Let B_1, B_2 be the bases found by Algorithm 4.4 such that we have top degree faces $T_i = \{\mathbf{e}_{B_i}\} + \Delta_{1 \cup X_i} + \nabla_{1 \cup Y_i}$, $i \in \{1, 2\}$. Suppose that $T_1 \cap T_2 \neq \emptyset$. Then $B_1 = B_2$.*

Proof. As we have noted, the combinatorial type of the subdivision $\mathcal{F}(t, u)$ is independent of the values of t and u, as long as these are positive. Also, if $t \leq t'$ and $u \leq u'$, then each cell of $\mathcal{F}(t, u)$ is a subset of the corresponding cell of $\mathcal{F}(t', u')$, up to translation of the latter by $(t' - t - u' + u)\mathbf{e}_1$. Thus if the top degree faces indexed by (X_1, Y_1) and (X_2, Y_2) intersect, they will continue to intersect if t or u are increased. So we may assume that none of t, u, $t - u$ lie in $\{-1, 0, 1\}$, by increasing t and u as necessary.

Because $\mathcal{F}$ is a cell complex, $T_1 \cap T_2$ is a face of $\mathcal{F}$, and it therefore contains a vertex p of $\mathcal{F}$. For each $i = 1, 2$, the point p is the sum of $\mathbf{e}_{B_i}$, a vertex of $u\Delta$, and a vertex of $t\nabla$. Because every subset of the list $1, u, -t$ has a different sum, p can be decomposed as a zero-one vector plus a vertex of $u\Delta$ plus a vertex of $t\nabla$ in only one way, and it follows that $\mathbf{e}_{B_1} = \mathbf{e}_{B_2}$. $\qquad\square$

Lemma 4.10. *Take two distinct partitions $P_1 = (X_1, Y_1)$, $P_2 = (X_2, Y_2)$ of $[n] \setminus \{1\}$ such that their corresponding top degree faces T_1 and T_2 contain a common point p. Now let $P_3 = (X_3, Y_3)$ be a partition of $[n] \setminus \{1\}$ such that $X_1 \cap X_2 \subseteq X_3$ and $Y_1 \cap Y_2 \subseteq Y_3$. Then p is in the top degree face T_3 indexed by P_3, and Algorithm 4.4 finds the same basis B^* for each of P_1, P_2 and P_3.*

Proof. By Lemma 4.9, we have $T_1 = \{\mathbf{e}_{B^*}\} + \Delta_{1 \cup X_1} + \nabla_{1 \cup Y_1}$ and $T_2 = \{\mathbf{e}_{B^*}\} + \Delta_{1 \cup X_2} + \nabla_{1 \cup Y_2}$ where the basis B^* found by Algorithm 4.4 is common to both expressions. The lexicographically greatest set of form $B \triangle Y_1$ is that with $B = B^*$, likewise for $B \triangle Y_2$.

Let $B' \triangle Y_3$ be the lexicographically greatest set of form $B \triangle Y_3$. Our objective is to show that $B' = B^*$. Assume otherwise for a contradiction, and let e be the largest element in $B' \triangle B^* = (B' \triangle Y_3) \triangle (B^* \triangle Y_3)$. By choice of B', we have $e \in B' \triangle Y_3$ and $e \notin B^* \triangle Y_3$. The latter implies that $e \notin B^* \triangle Y_i$ for at least one of $i = 1, 2$; without loss of generality, say $e \notin B^* \triangle Y_1$. Then $B' \triangle Y_1$ is lexicographically earlier than $B^* \triangle Y_1$, because the former but not the latter contains e and they agree in which elements greater than e they contain. This is the desired contradiction.

We conclude that $T_k = \{\mathbf{e}_{B^*}\} + \Delta_{1 \cup X_k} + \nabla_{1 \cup Y_k}$ for each $k = 1, 2, 3$. Expanding $p - \mathbf{e}_{B^*}$ in the basis $\mathbf{e}_2 - \mathbf{e}_1, \ldots, \mathbf{e}_n - \mathbf{e}_1$ of the affine span of the T_k, we see that the coefficient of $\mathbf{e}_i - \mathbf{e}_1$ is nonnegative if $i \in 1 \cup X_k$ and nonpositive if $i \in 1 \cup Y_k$, for $k = 1, 2$. Therefore this coefficient is zero unless $i \in X_1 \cap X_2$ or $i \in Y_1 \cap Y_2$, and this implies $p \in T_3$. $\qquad \square$

The following result is an immediate corollary of Lemma 4.10:

Corollary 4.11. *Define $T_Y = \{\mathbf{e}_B\} + u\Delta_X + t\nabla_Y$. For every face F of the mixed subdivision $\mathcal{F}$, if F is contained in any top degree face, then the set of Y such that F is contained in T_Y is an interval in the boolean lattice.*

We now have all the ingredients we need to prove the main result of this section, restated here:

Theorem 4.1. *Take the regular mixed subdivision $\mathcal{F}$ of $u\Delta + P(M) + t\nabla$ as described above. The unsigned coefficient $|[x^i y^j]Q'_M|$ counts the cells $F + G + H$ of $\mathcal{F}$ where $i = \dim(G)$, $j = \dim(H)$, and F is a vertex of $P(M)$ and there exists no cell $F' + G + H$ where $F' \supsetneq F$.*

Proof. We form a poset P where the elements are the top degree faces and all nonempty intersections of sets of these, ordered by containment. This poset is a subposet of the face lattice of the $(|E| - 1)$-dimensional cube whose vertices correspond to the top degree faces. Proposition 4.8 shows that every lattice point of $P(M) + u\Delta + t\nabla$ lies in at least one face in P. The total number of lattice points is given by inclusion-exclusion on the function on P assigning to each element of P the number of lattice points in that face. For an element $p \in P$ indexing a face $F \in \mathcal{F}$, let $[p] := L(F)$. So we have that

$$Q_M(t, u) = \sum_{i,j} c_{ij} \binom{u}{j} \binom{t}{i} = \sum_{k \geq 1} (-1)^k \sum_{\substack{S \subseteq \mathrm{atoms}(P) \\ |S| = k}} \left[\bigwedge S \right] \qquad (4.11.1)$$

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$$= \sum_{x \in P} \mu(0, x)[x]$$

where μ is the Möbius function. Now, as the face poset of the cubical complex $\mathcal{C}$ is Eulerian, we have that $\mu(x, y) = (-1)^{r(y) - r(x)}$. This means that

$$\sum_{k \geq 1} (-1)^k \sum_{\substack{S \subseteq \mathrm{atoms}(P) \\ |S| = k}} \left[\bigwedge S \right] = \sum_{E \text{ face of } \mathcal{C}} (-1)^{\mathrm{codim} E} \binom{t+i}{i} \binom{u+j}{j} \quad (4.11.2)$$

where E is the product of an i-dimensional face of Δ with a j-dimensional face of ∇, that is, E corresponds to a face of type $F + tG + uH$, where F is a vertex of $P(M)$.

Q'_M expands as

$$Q'_M(x, y) = \sum_{i,j} c_{ij} (x - 1)^i (y - 1)^j = \sum_{i,j,k,l} c_{ij} \binom{i}{k} x^k (-1)^{i-k} \binom{j}{l} y^l (-1)^{j-l}$$

in which the coefficient of $x^k y^l$ is $\sum_{i,j} c_{ij} \binom{i}{k} (-1)^{i-k} \binom{j}{l} (-1)^{j-l}$. To compare this to the count in the lattice, we need to expand $\binom{t}{i}$ (and $\binom{u}{j}$) in the basis of $\binom{t+i}{i}$ (and $\binom{u+j}{j}$). This gives that

$$\binom{t}{i} = \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} \binom{t+k}{k},$$

as proven below.

Claim 4.12. *For any positive integers i, t,*

$$\binom{t}{i} = \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} \binom{t+k}{k}.$$

Proof of Claim 4.12. The Vandermonde identity gives that

$$\binom{t+i}{i} = \sum_{k=0}^{i} \binom{t}{k} \binom{i}{i-k} = \sum_{k=0}^{i} \binom{t}{k} \binom{i}{k}. \quad (4.12.1)$$

We now apply an identity that has been called the "binomial inversion theorem", i.e. Möbius inversion on the lattice L of finite subsets of $\mathbb{N}$. For $K \in L$ let $g_K = \binom{t}{|K|}$, and for $I \in L$ let

$$f_I = \sum_{K \subseteq I} g_K.$$

Then $f_I = \binom{t+|I|}{|I|}$ by (4.12.1). Möbius inversion now gives

$$g_I = \sum_{K \subseteq I} (-1)^{|I \setminus K|} f_K$$

which when $i = |I|$ and $k = |K|$ gives the claim. ∎

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Substitute this into Equation 4.11.1 to get

$$\sum_{k\geq 1}(-1)^k \sum_{\substack{S\subseteq \mathrm{atoms}(P)\\|S|=k}}\left[\bigwedge S\right] = \sum_{i,j,k,l} c_{ij}(-1)^{i-k}\binom{i}{k}\binom{t+k}{k}(-1)^{j-l}\binom{j}{l}\binom{u+l}{l}$$

$$= \sum_{k,l}[x^k y^l]Q'_M(x,y)\binom{t+k}{k}\binom{u+l}{l}.$$

Comparing this to Equation 4.11.2 proves Theorem 4.1. □

The above proof immediately yields as a corollary that the signs of the coefficients of $Q'_M(x,y)$ are alternating.

Corollary 4.13. $(-1)^{|E|-1}Q'_M(-x,-y)$ *has nonnegative coefficients in* x *and* y.

This is not dissimilar to the Tutte polynomial, whose coefficients are all non-negative. The coefficients of Q'_M, up to sign, have the combinatorial interpretation of counting elements of P of form $\{\mathbf{e}_B\} + u\Delta_X + t\nabla_Y$ by the cardinalities of $X\setminus\{1\}$ and $Y\setminus\{1\}$. In particular the top degree faces are counted by the collection of coefficients of Q'_M of top degree (hence the name), and the degree $|E|-1$ terms of Q'_M are always $(x+y)^{|E|-1}$.

The appearance of basis activities in Lemma 4.3 reveals that P is intimately related to a familiar object in matroid theory, the *Dawson partition* [3]. Give the lexicographic order to the power set $\mathcal{P}(E)$. A partition of $\mathcal{P}(E)$ into intervals $[S_1,T_1],\ldots,[S_p,T_p]$ with indices such that $S_1 < \ldots < S_p$ is a Dawson partition if and only if $T_1 < \ldots < T_p$. Every matroid gives rise to a Dawson partition in which these intervals are $[B\setminus\mathrm{Int}(B), B\cup\mathrm{Ext}(B)]$ for all $B\in\mathcal{B}_M$.

Proposition 4.14. *Let* $[S_1,T_1],\ldots,[S_p,T_p]$ *be the Dawson partition of* M. *The poset* P *is a disjoint union of face posets of cubes* $C_1,\ldots,C_p$ *where the vertices of* C_i *are the top degree faces* $\{\mathbf{e}_B\} + u\Delta_X + t\nabla_Y$ *such that* $X\in[S_i,T_i]$.

The description of the cubes comes from Lemma 4.10. Note that the element 1 is both internally and externally active with respect to every basis, due to it being the smallest element in the ordering. So, even though 1 is in both X and Y, it is in $T_i - S_i$ for all i.

5 Polymatroids

In this section we investigate the invariants Q_M and Q'_M when $M = (E,r)$ is a polymatroid. Many familiar matroid operations have polymatroid counterparts, and we describe the behaviour of Q'_M under these operations. We see that it retains versions of several formulae true of the Tutte polynomial.

For instance, there is a polymatroid analogue of the direct sum of matroids: given two polymatroids $M_1 = (E_1,r_1)$, $M_2 = (E_2,r_2)$ with disjoint ground sets,

their *direct sum* $M = (E, r)$ has ground set $E = E_1 \sqcup E_2$ and rank function $r(S) = r_1(S \cap E_1) + r_2(S \cap E_2)$. This definition extends the usual direct sum of matroids and corresponds to the cartesian product of polytopes.

Proposition 5.1. *Let $M_1 \oplus M_2$ be the direct sum of two polymatroids M_1 and M_2. Then*

$$Q'_{M_1 \oplus M_2}(x, y) = (x + y - 1) \cdot Q'_{M_1}(x, y) \cdot Q'_{M_2}(x, y).$$

Proof. We will need to use distinguished notation for our standard simplices according to the matroid under consideration. So we write $\Delta^{(i)} = \mathrm{Conv}\{\mathbf{e}_j : j \in E_i\}$ for $i = 1, 2$ and reserve the unadorned name Δ for $\mathrm{Conv}\{\mathbf{e}_j : j \in E\}$. Define $\nabla^{(i)}$ and ∇ similarly.

The basic relationship between $M = M_1 \oplus M_2$ and its summands in terms of our lattice point counts is the following equality:

$$\sum_{k=0}^{\min\{t,u\}} Q_M(t - k, u - k) = \sum_{t_1=0}^{t} \sum_{u_1=0}^{u} Q_{M_1}(t_1, u_1) \cdot Q_{M_2}(t - t_1, u - u_1). \quad (5.1.1)$$

The right hand side counts tuples (t_1, u_1, q_1, q_2) where $q_1 \in L(P(M_1) + u_1\Delta^{(1)} + t_1\nabla^{(1)})$ and $q_2 \in L(P(M_2) + (u - u_1)\Delta^{(2)} + (t - t_1)\nabla^{(2)})$. Because the coordinate inclusions of $\Delta^{(1)}$ and $\Delta^{(2)}$ are subsets of Δ, and similarly for ∇, the concatenation $(q_1, q_2) \in \mathbb{Z}^E$ is a lattice point of $P(M) + u\Delta + t\nabla$. The left hand side counts pairs (k, q) where q is a lattice point of $P(M) + (u - k)\Delta + (t - k)\nabla$; this polyhedron is a subset of $P(M) + u\Delta + t\nabla$. To prove equation (5.1.1) we will show that each $q = (q_1, q_2) \in \mathbb{Z}^E$ occurs with the same number of values of k on the left as values of (t_1, u_1) on the right.

If $q \in L(P(M) + u\Delta + t\nabla)$ then there is some maximal integer c such that $q \in L(P(M) + (u - c)\Delta + (t - c)\nabla)$, and then q is counted just $c + 1$ times on the left hand side, namely for $k = 0, 1, \ldots, c$. This c is the same as the c in the proof of Theorem 3.2, and exists for the same reason. Choose an expression

$$q = e_B + \mathbf{e}_{x_1} + \cdots + \mathbf{e}_{x_{u-c}} - \mathbf{e}_{y_1} - \cdots - \mathbf{e}_{y_{t-c}},$$

where B is a basis of M and the x_i and y_i are elements of E. Suppose the x_i and y_i are ordered such that $x_i \in E_1$ if and only if $i \leq t'$ and $y_i \in E_1$ if and only if $i \leq u'$. Then we have

$$q_1 = e_{B \cap E_1} + \mathbf{e}_{x_1} + \cdots + \mathbf{e}_{x_{u'}} - \mathbf{e}_{y_1} - \cdots - \mathbf{e}_{y_{t'}},$$
$$q_2 = e_{B \cap E_2} + \mathbf{e}_{x_{u'+1}} + \cdots + \mathbf{e}_{x_{u-c}} - \mathbf{e}_{y_{t'+1}} - \cdots - \mathbf{e}_{y_{t-c}},$$

and both of these expressions also have the minimal number of $\mathbf{e}_{x_i}$ and $\mathbf{e}_{y_i}$ summands, or else our expression for q would not have been minimal. Thus q_1 is in $P(M_1) + u'\Delta + t'\nabla$ but not $P(M_1) + (u' - 1)\Delta + (t' - 1)\nabla$, and q_2 is in $P(M_1) + (u - u' - c)\Delta + (t - t' - c)\nabla$ but not $P(M_1) + (u - u' - c - 1)\Delta + (t - t' - c - 1)\nabla$. So the possibilities for t_1 and u_1 on the right hand side of (5.1.1) are those that arrange $t_1 \geq t'$ and $t - t_1 \geq t - t' - c$, and the corresponding

equations for the u variables, together with $t_1 - u_1 = t' - u'$. There are exactly $c+1$ solutions here as well, namely $t_1 = t', t'+1, \ldots, t'+c$.

Using equation (5.1.1) within a generating function for t and u, we have that

$$\sum_{t,u} \sum_{k=0}^{\min\{t,u\}} Q_{M_1 \oplus M_2}(t-k, u-k) v^u w^t$$

$$= \sum_{t,u} \sum_{t_1=0}^{t} \sum_{u_1=0}^{u} Q_{M_1}(t_1, u_1) v^{u_1} w^{t_1} \cdot Q_{M_2}(t-t_1, u-u_1) v^{u-u_1} w^{t-t_1}$$

$$= \sum_{t_1,u_1} Q_{M_1}(t_1, u_1) v^{u_1} w^{t_1}$$

$$\cdot \sum_{t-t_1, u-u_1} Q_{M_2}(t-t_1, u-u_1) v^{u-u_1} w^{t-t_1}$$

$$= \left(\sum_{t,u} Q_{M_1}(t, u) v^u w^t \right) \left(\sum_{t,u} Q_{M_2}(t, u) v^u w^t \right).$$

The left-hand side can also be simplified as

$$\sum_{t,u} \sum_{k=0}^{\min\{t,u\}} Q_{M_1 \oplus M_2}(t-k, u-k) v^u w^t$$

$$= \sum_{k \geq 0} \sum_{t,u \geq k} Q_{M_1 \oplus M_2}(t-k, u-k) v^{u-k} w^{t-k} (vw)^k$$

$$= \sum_{k \geq 0} \sum_{t,u} Q_{M_1 \oplus M_2}(t, u) v^u w^t (vw)^k$$

$$= \frac{1}{1-vw} \cdot \sum_{t,u} Q_{M_1 \oplus M_2}(t, u) v^u w^t$$

and thus we have that

$$\frac{1}{1-vw} \cdot \sum_{t,u} Q_{M_1 \oplus M_2}(t, u) v^u w^t = \left(\sum_{t,u} Q_{M_1}(t, u) v^{u_1} w^{t_1} \right) \left(\sum_{t,u} Q_{M_2}(t, u) v^{u_1} w^{t_1} \right).$$

The generating functions below allow us easily to change basis from Q_M to Q'_M in the above:

$$\sum_{t,u} Q_M(t, u) v^u w^t = \sum_{t,u} \sum_{i,j} c_{ij} \binom{u}{j} \binom{t}{i} v^u w^t$$

$$= \sum_{i,j} c_{ij} \frac{v^j}{(1-v)^{j+1}} \cdot \frac{w^i}{(1-w)^{i+1}}$$

$$= \frac{1}{(1-v)(1-w)} \cdot Q'_M \left(\frac{w}{1-w} + 1, \frac{v}{1-v} + 1 \right)$$

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where the last line follows from the definition of Q'_M. Hence we have

$$Q'_{M_1 \oplus M_2}\left(\frac{1}{1-w}, \frac{1}{1-v}\right) = \frac{1-vw}{(1-v)(1-w)} Q'_{M_1}\left(\frac{1}{1-w}, \frac{1}{1-v}\right) Q'_{M_2}\left(\frac{1}{1-w}, \frac{1}{1-v}\right)$$

and substituting $w = \dfrac{x-1}{x}$ and $v = \dfrac{y-1}{y}$ gives the result. $\qquad\square$

Remark 5.2. It follows from Proposition 5.1 that the rescaled matroid invariant $(x + y - 1) \cdot Q'_M(x, y)$ is exactly multiplicative under direct sum. Recasting Theorem 3.2 in terms of this rescaled invariant also eliminates a denominator. And, by Proposition 4.14, its coefficients can be interpreted as counting intervals in the Boolean lattice $\mathcal{P}(E)$ contained in a single part of a Dawson partition according to the ranks of their minimum and maximum, with no need to accord a special role to one element.

Moving on to the next polymatroid operation, it is apparent from the symmetry of Theorem 3.1 under switching x and y that Q'_M, like the Tutte polynomial, exchanges its two variables under matroid duality. The best analogue of duality for polymatroids requires a parameter s greater than or equal to the rank of any singleton; then if $M = (E, r)$ is a polymatroid, its *s-dual* is the polymatroid $M^* = (E, r^*)$ with

$$r^*(S) = r(E) + s|E \setminus S| - r(E \setminus S).$$

The 1-dual of a matroid is its usual dual.

Proposition 5.3. *For any polymatroid $M = (E, r)$ and any s-dual M^* of M, $Q'_{M^*}(x, y) = Q'_M(y, x)$.*

Proof. Let $\phi : \mathbb{R}^E \to \mathbb{R}^E$ be the involution that subtracts every coordinate from s. Definition 2.8 implies that ϕ is a bijection which takes elements of $P(M)$ to elements of $P(M^*)$; it also clearly preserves the property of being a lattice point. This gives that

$$\begin{aligned}
\#L(P(M) + u\Delta + t(-\Delta)) &= \#L(\phi(P(M)) + u\phi(\Delta) + t\phi(-\Delta)) \\
&= \#L(P(M^*) + u(\nabla + 1_E) + t(\Delta + 1_E) \\
&= \#L(P(M^*) + (t - u)1_E + u\nabla + t\Delta) \\
&= \#L(P(M^*) + u\nabla + t\Delta)
\end{aligned}$$

where the last line is true due to the polytope being a translation of the one in the line above. The statement follows. $\qquad\square$

A *valuation* is a function f on a family of polytopes, such that given a subdivision of a polytope P with maximal cells $P_1, \ldots, P_n$ such that P and all the faces of the subdivision lie in our family,

$$f(P) = \sum_{P_i} f(P_i) - \sum_{P_i, P_j} f(P_i \cap P_j) + \ldots + (-1)^{n-1} f(P_1 \cap \cdots \cap P_n).$$

The number of lattice points in a polytope is a valuation: if $f(P) = \#L(P)$, then it is easily checked that each lattice point counted in $f(P)$ also contributes exactly one to the sum on the right hand side. Thus the invariant Q'_M is a valuation as well:

Proposition 5.4. *Let $\mathcal{F}$ be a polyhedral complex whose total space is a polymatroid base polytope $P(M)$, and each of whose faces F is a polymatroid base polytope $P(M(F))$. Then*

$$Q'_M(x,y) = \sum_F (-1)^{\dim(P(M)) - \dim F} Q'_{M(F)}(x,y),$$

where the sum is over faces F of $\mathcal{F}$ not on the boundary of $P(M)$.

Through Theorem 3.4 – or in fact more directly through Theorem 3.1 – this provides another proof that the Tutte polynomial of a matroid M is a valuation as a function of $P(M)$. This fact was first observed by Speyer in [21]; see [8] for a survey of the theory that has built up around the observation since.

Here is one example of a consequence of valuativity for properties of Q'. Suppose M is a matroid and C is a circuit-hyperplane (that is, both a circuit and a hyperplane) of M. Then as in [11] we can *relax* C, forming a new matroid M' with $\mathcal{B}_{M'} = \mathcal{B}_M \cup \{C\}$. In this case there is a subdivision of $P(M)$ with two maximal faces, $P(M')$ and $P(M'')$ where

$$\mathcal{B}_{M''} = \{B \subseteq E \mid |B| = |C|, |B \setminus C| \leq 1\}.$$

Proposition 5.4 then gives:

Corollary 5.5. *Take a matroid $M = (E,r)$ and let $C \subset E$ be a circuit-hyperplane of M. Let M' be the matroid formed by relaxing C. Then $Q'_M(x,y) = Q'_{M'}(x,y) - x^{n-r(M)-1} y^{r(M)-1}$.*

Now consider deletion and contraction in polymatroids. We have that $P(M\backslash a) = \{p \in P(M) \mid p_a = \underline{k}\}$, where $\underline{k}$ is the minimum value p_a takes (this will be 0 unless a is a coloop), and that $P(M/a) = \{p \in P(M) \mid p_a = \overline{k}\}$, where $\overline{k}$ is the maximum value p_a takes. When M is a matroid, $L(P(M\backslash a))$ and $L(P(M/a))$ partition $L(P(M))$, unless a is a loop or coloop in which case $P(M\backslash a) = P(M/a) = P(M)$. However, when M is a polymatroid, we can have points p in $L(P(M))$ where $\underline{k} < p_a < \overline{k}$. Let $P(N_k) := \{p \in P(M) \mid p_a = k\}$, which is a polymatroid base polytope, and let N_k be its underlying polymatroid. Note that $N_{\underline{k}} = M\backslash a$ and $N_{\overline{k}} = M/a$. We also have that the collection of $L(P(N_k))$ for $k \in \{\underline{k}, \ldots, \overline{k}\}$ partition $L(P(M))$. We will refer to the polytopes $P(N_k)$ as *a-slices* of $P(M)$. When we do not include the deletion and contraction slices $P(M\backslash a)$ and $P(M/a)$, we can talk about *(strictly) interior slices*.

Theorem 5.6. *Let $M = (E,r)$ be a polymatroid and take $a \in E(M)$. Then*

$$Q'_M(x,y) = (x-1)Q'_{M\backslash a}(x,y) + (y-1)Q'_{M/a}(x,y) + \sum_N Q'_N(x,y).$$

where $P(N)$ ranges over a-slices of $P(M)$.

Note that when M is a matroid, the statement simplifies to the formulae given in Lemma 3.7: if a is neither a loop nor coloop, then the a-slices are $P(M\backslash a)$ and $P(M/a)$, so

$$Q'_M(x,y) = (x-1)Q'_{M\backslash a}(x,y) + (y-1)Q'_{M/a}(x,y) + \sum_{N_k} Q'_{N_k}(x,y)$$

$$= (x-1)Q'_{M\backslash a}(x,y) + (y-1)Q'_{M/a}(x,y) + Q'_{M\backslash a}(x,y) + Q'_{M/a}(x,y)$$

$$= xQ'_{M\backslash a}(x,y) + yQ'_{M/a}(x,y).$$

When a is a loop or coloop, $M\backslash a = M/a$, and we have only one a-slice: $P(M\backslash a) = P(M/a)$. So we get that

$$Q'_M(x,y) = (x-1)Q'_{M\backslash a}(x,y) + (y-1)Q'_{M/a}(x,y) + \sum_{N_k} Q'_{N_k}(x,y)$$

$$= (x+y-1)Q'_{M/a}(x,y)$$

as in Lemma 3.7. It follows that Theorem 5.6 can be used to give another proof of Theorem 3.2. The Tutte polynomial of a matroid can be defined by a deletion-contraction recurrence with two cases parallel to the above two formulae for $Q'_M(x,y)$, so it suffices to use Lemma 3.7 to check that the right hand side of (3.2.1) satisfies the defining recurrence.

Proof of Theorem 5.6. In this proof, we make constant use of Lemma 3.3 in order to express lattice points as sums of lattice points. Let M be a polymatroid. If the rank function of M is a matroid rank function summed with a function of the form $S \mapsto \sum_{i \in S} c_i$, then $P(M)$ will be a translate of a matroid polytope, and the same argument as above will hold. Assume now that this is not the case. This means that for some $a \in E(M)$, there will be at least one a-slice of $P(M)$, $P(N_k)$, which is not equal to $P(M/a)$ or $P(M\backslash a)$.

Claim 5.7. *Define R to be the polytope $\{q \in P(M) + u\Delta_E + t\nabla_E \mid q_a = k\}$, and define S to be $P(N_k) + u\Delta_{E-a} + t\nabla_{E-a}$. If R intersects the set of lattice points of $P(M)$, then $R = S$.*

Proof of Claim 5.7. It is clear that the lattice points of S are contained in R. Take a point in R, say $q_1 = p_1 + \mathbf{e}_{i_1} + \cdots + \mathbf{e}_{i_u} - \mathbf{e}_{j_1} - \cdots - \mathbf{e}_{j_t}$. We will show that q_1 is equal to a point of form

$$q_2 = p_2 + \mathbf{e}_{m_1} + \cdots + \mathbf{e}_{m_u} - \mathbf{e}_{n_1} - \cdots - \mathbf{e}_{n_t} \in S,$$

where no m_i or n_j can be equal to a.

If $(p_1)_a = k$, then we simply choose p_2 to be p_1 and choose $m_s = i_s$, $n_s = j_s$ for all $s \in \{1, \ldots, t\}$, with one possible change: if we have $i_r = j_s = a$ in q_1, in q_2 instead set m_r and n_s equal to b, where $b \neq a$ is any other element in E. Note $\mathbf{e}_a$ must always appear paired in this way because $(q_1)_a = k$, and so this change does not affect the fact $q_2 = q_1$.

If $(p_1)_a \neq k$, we will first rewrite the expression for q_1. By the base exchange property for polymatroids ([10, Theorem 4.1]), given p_1 and any point $p_3 \in P(M)$, if $(p_1)_i > (p_3)_i$ there exists l such that $(p_1)_l < (p_3)_l$ and $p_1 - \mathbf{e}_i + \mathbf{e}_l \in P(M)$. Let $(p_1)_a = k + \lambda$, where $\lambda > 0$. Then, by repeatedly applying the exchange property, we get that $p_1 - \lambda \mathbf{e}_a + \mathbf{e}_{l_1} + \cdots + \mathbf{e}_{l_\lambda} \in P(M)$. Then we can find q_2 by setting $p_2 = p_1 - \lambda \mathbf{e}_a + \mathbf{e}_{l_1} + \cdots + \mathbf{e}_{l_\lambda}$, so

$$q_2 = p_2 + \lambda \mathbf{e}_a - \mathbf{e}_{l_1} - \cdots - \mathbf{e}_{l_\lambda} + \mathbf{e}_{i_1} + \ldots + \mathbf{e}_{i_u} - \mathbf{e}_{j_1} - \cdots - \mathbf{e}_{j_t} = q_1.$$

Note that as $(q_1)_a = k$ and $(p_2)_a = k$, there must be λ $-\mathbf{e}_{j_k}$ terms equal to $-\mathbf{e}_a$, so

$$q_2 = p_2 - \mathbf{e}_{l_1} - \cdots - \mathbf{e}_{l_\lambda} + \mathbf{e}_{i_1} + \ldots + \mathbf{e}_{i_u} - \mathbf{e}_{j_1} - \cdots - \mathbf{e}_{j_{t-\lambda}}$$

which is of the correct form, completing the proof of Claim 5.7. ∎

Claim 5.8. *Let N_i be a strictly interior slice of $P(M)$. Then*

$$L(P(M) + u\Delta_E + t\nabla_E) = L(P(M/a) + u\Delta_E + t\nabla_{E-a}) \sqcup$$
$$\bigsqcup_i L(P(N_i) + u\Delta_{E-a} + t\nabla_{E-a}) \sqcup L(P(M \backslash a) + u\Delta_{E-a} + t\nabla_E).$$

Proof of Claim 5.8. Split $L(P(M) + u\Delta_E + t\nabla_E)$ into a collection of sets according to the value of q_a for points $q \in L(P(M) + u\Delta_E + t\nabla_E)$. This is clearly a partition of $L(P(M) + u\Delta_E + t\nabla_E)$. By the previous result, if the convex hull of one of these parts intersects $P(M)$ we can write it as $L(P(N_k) + u\Delta_{E-a} + t\nabla_{E-a})$. Otherwise, we must be able to write the part as $L(P(M/a) + (u - \lambda)\Delta_{E-a} + \lambda \mathbf{e}_a + t\nabla_{E-a})$, where $\lambda > \bar{k}$, or as $L(P(M \backslash a) + t\Delta_{E-a} - \mu \mathbf{e}_a + (t - \mu)\nabla_{E-a})$, where $\mu > \underline{k}$.

We will show that

$$\bigsqcup_\lambda L(P(M/a) + (u - \lambda)\Delta_{E-a} + \lambda \mathbf{e}_a + t\nabla_{E-a}) = L(P(M/a) + u\Delta_E + t\nabla_{E-a}).$$
$$(5.8.1)$$

It is clear that the sets on the left are pairwise disjoint as λ varies, as the a-coordinates in each set must be different. It is also clear that the lattice points contained in the polytope on the left hand side are contained in that of the right hand side. Take a point $q_1 = p_1 + \mathbf{e}_{i_1} + \cdots + \mathbf{e}_{i_u} - \mathbf{e}_{j_1} - \cdots - \mathbf{e}_{j_t}$ contained in $P(M/a) + u\Delta_E + t\nabla_{E-a}$. Let $(q_1)_a = \bar{k} + \mu$, where $\mu > 0$. We need to write q_1 as $p_2 + \mathbf{e}_{m_1} + \cdots + \mathbf{e}_{m_{u-\lambda}} + \lambda \mathbf{e}_a - \mathbf{e}_{n_1} - \cdots - \mathbf{e}_{n_t}$, a lattice point contained in one of the summands on the left hand side. Choose $\mu = \lambda$, $p_2 = p_1$, $\{j_\alpha\} = \{n_\alpha\}$, and $\{i_\beta \mid i_\beta \neq a\} = \{m_\beta\}$ and the equality follows.

The same arguments show that

$$\bigsqcup_\mu L(P(M \backslash a) + u\Delta_{E-a} - \mu \mathbf{e}_a + (t - \mu)\nabla_{E-a}) = L(P(M \backslash a) + u\Delta_{E-a} + t\nabla_E)$$
$$(5.8.2)$$

and the claim follows. ∎

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Claim 5.9. *We have that*

$$\#L(P(M/a) + u\Delta_E + t\nabla_{E-a}) = \sum_{j=0}^{u} \#L(P(M/a) + j\Delta_{E-a} + t\nabla_{E-a})$$

and

$$\#L(P(M\backslash a) + u\Delta_{E-a} + t\nabla_E) = \sum_{i=0}^{u} \#L(P(M\backslash a) + u\Delta_{E-a} + i\nabla_{E-a}).$$

Proof of Claim 5.9. Take the cardinalities of both sides of Equations 5.8.1 and 5.8.2. ∎

Continuing the proof of the theorem, Claim 5.9 implies that

$$Q_M(t,u) = \sum_{N_k} Q_{N_k}(t,u) + \sum_{j=0}^{u} Q_{M/a}(t,j) + \sum_{i=0}^{t} Q_{M\backslash a}(i,u) \qquad (5.9.1)$$

where $k \in \{\underline{k}+1,\dots,\overline{k}-1\}$, that is, N_k is always a strictly interior slice of $P(M)$.

We now work out how the change of basis from Q to Q' transforms the sums in Equation 5.9.1. Write $Q_{M/a}(t,j) = \sum_{i,k} c_{ik} \binom{j}{k}\binom{t}{i}$. We have that

$$\sum_{j=0}^{u} c_{ik} \binom{j}{k}\binom{t}{i} = c_{ik} \binom{u+1}{k+1}\binom{t}{i}$$

$$= c_{ik} \binom{t}{i}\left(\binom{u}{k} + \binom{u}{k+1}\right).$$

Now apply the change of basis to get

$$c_{ik}(x-1)^i((y-1)^k + (y-1)^{k+1}) = c_{ik}(x-1)^i\left((y-1)^k(1+y-1)\right)$$

$$= c_{ik}(x-1)^i(y-1)^k y.$$

Thus

$$\sum_{j=0}^{u} Q_{M/a}(t,j) = yQ'_{M/a}(t,u)$$

and similarly,

$$\sum_{i=0}^{t} Q_{M\backslash a}(i,u) = xQ'_{M\backslash a}(t,u).$$

Substituting these into (5.9.1) gives

$$Q'_M(t,u) = xQ'_{M/a}(t,u) + yQ'_{M/a}(t,u) + \sum_{\text{interior } N_k} Q'_{N_k}(t,u)$$

$$= (x-1)Q'_{M/a}(t,u) + (y-1)Q'_{M/a}(t,u) + \sum_{N_k} Q'_{N_k}(t,u).$$

This completes the proof of Theorem 5.6. □

Unfortunately, when M is a polymatroid, there is no analogue to Corollary 4.13: the coefficients of Q'_M do not have sign independent of M, and thus there can be no straightforward enumerative interpretation of the coefficients. This is a consequence of the failure of Theorem 4.1 for polymatroids. Here is an example to illustrate this.

Example 5.10. The left of Figure 2 displays the subdivision $\mathcal{F}$ for the sum of Example 3.5. We see that the four grey top degree faces contain all the lattice points between them, and the poset P contains two other faces which are pairwise intersections thereof, the horizontal segment on the left with $(X, Y) = (1, 12)$ and the one on the right with $(X, Y) = (12, 1)$. These are indeed enumerated, up to the alternation of sign, by the polynomial $Q'_M(x, y) = x^2 + 2xy + y^2 - x - y$ found earlier.

By contrast, the right of the figure displays $\mathcal{F}$ for the polymatroid M_2 obtained by doubling the rank function of M. The corresponding polynomial is $Q'_{M_2}(x, y) = x^2 + 2xy + y^2 - 1$, in which the signs are not alternating, dashing hopes of a similar enumerative interpretation. In the figure we see that there are lattice points not on any grey face. $\diamond$
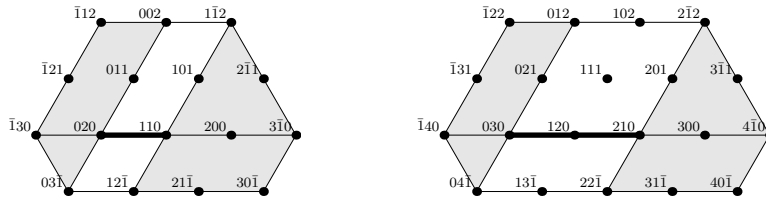

Figure 2: At left, the regular subdivision $\mathcal{F}$ associated to the Minkowski sum of Example 3.5, with $P(M)$ bolded and the top degree faces shaded in grey. At right, the regular subdivision for a related polymatroid, still with $(t, u) = (2, 1)$.

6 Kálmán's activities

One motivation for the particular Minkowski sum we have employed in our definition is that it provides a polyhedral translation of Kálmán's construction of activities in a polymatroid. This section explains the connection. We first recall the definitions of Kálmán's univariate activity invariants of polymatroids. These polynomials do not depend on the order on E that was used to define them [12, Theorem 5.4].

Definition 6.1 ([12]). *Define the* internal polynomial *and* external polynomial *of a polymatroid $M = (E, r)$ by*

$$I_M(\xi) = \sum_{x \in \mathcal{B}_M \cap \mathbb{Z}^E} \xi^{\bar{\iota}(x)} \quad \text{and} \quad X_M(\eta) = \sum_{x \in \mathcal{B}_M \cap \mathbb{Z}^E} \eta^{\bar{\varepsilon}(x)}.$$

Lemma 6.2. *Let P be a polymatroid polytope. At every lattice point $f \in P$, attach the scaled simplex*

$$f + t \operatorname{conv}(\{-e_i \mid i \text{ is internally active in } f\}).$$

This operation partitions $P + t\nabla$ into a collection of translates of faces of $t\nabla$, with the simplex attached at f having codimension $\bar{\iota}(f)$ within P.
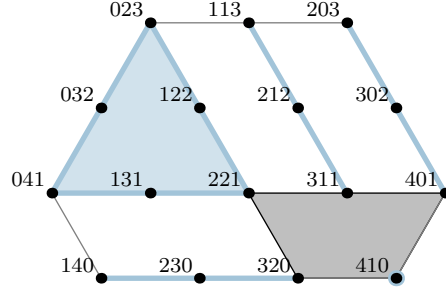

Figure 3: An example of the partition of Lemma 6.2. A translate of P is shown, each of its lattice points lying in the corresponding simplex of the partition.

Figure 3 shows a case of this operation, to illustrate why we speak of "attaching" a simplex. Our polymatroid $P(M)$ is coloured grey, and the polytope drawn is $P(M) + 2\Delta + \nabla$. In the picture we translate $P(M)$ by a multiple of $\mathbf{e}_1$, namely $1\mathbf{e}_1$, in order to allow both polygons to reside in the same plane. Coordinate labels are written without parentheses and commas. The blue areas are faces of the scaled simplices $2\Delta_f$, and can be seen to be a partition of the lattice points. (Don't overlook the blue dot at 410!)

Proof. We will first show that our simplices covers all the lattice points of $P + t\nabla$. Let $g \in P + t\nabla$ be a lattice point. We will find f such that $g \in t\Delta_f$.

Let $g_t = g$. For $i \in \{0, \ldots, t-1\}$, define

$$g_i = \begin{cases} g_{i+1} + \mathbf{e}_1 & \text{if } g_{i+1} + \mathbf{e}_1 \in P + (t-i)\nabla \\ g_{i+1} + \mathbf{e}_2 & \text{if } g_{i+1} + \mathbf{e}_1 \notin P + (t-i)\nabla, g_{i+1} + \mathbf{e}_2 \in P + (t-i)\nabla \\ \quad \vdots \\ g_{i+1} + \mathbf{e}_n & \text{if } g_{i+1} + \mathbf{e}_h \notin P + (t-i)\nabla, \ \forall j \in [n], \ g_{i+1} + \mathbf{e}_n \in P + (t-i)\nabla \end{cases}$$

In other words, at each iteration i, we are adding an element $\mathbf{e}_j$ which is internally active with respect to g_{i+1}. We cannot replace $\mathbf{e}_j$ with $\mathbf{e}_i$ where $i < j$ and remain inside $P + t\nabla$. Let $\mathbf{e}_{j_i}$ be the element added in iteration t. We get that

$$g = g_t = g_0 - \mathbf{e}_{j_1} - \ldots - \mathbf{e}_{j_t} \in P + t\nabla.$$

Note that if we added $\mathbf{e}_i$ at some stage g_s of the iteration, and $\mathbf{e}_j$ at stage g_{s-1}, then $j \geq i$. Thus if we take a tuple $\mathbf{e}_k$ such that $(k_1, \ldots, k_t) < (j_1, \ldots, j_t)$ with

respect to the lexicographic ordering, then $g_0 - \sum_t \mathbf{e}_{j_t} + \sum_t \mathbf{e}_{k_t} \notin P$, so each $\mathbf{e}_j$ is internally active. Thus $g_0 = f$ and the $\mathbf{e}_j$ found define a simplex Δ_f such that $g \in t\Delta_f$.

Now we will show that this operation gives disjoint sets. We have that $\{t\Delta_f\}$ covers $L(P+t\nabla)$, and that $\{(t-1)\Delta_f\}$ partitions $L(P+(t-1)\nabla)$. Thus in order to show that $\{t\Delta_f\}$ is in fact a partition of the lattice points of P, it suffices to prove that if $g_t \in t\Delta_f$, then $g_{t-1} \in (t-1)\Delta_f$. Say that $f = g_t + \mathbf{e}_{i_1} + \cdots + \mathbf{e}_{i_t}$. This means that each element $\mathbf{e}_{i_k}$ is internally active at f for all $k \in \{0, \ldots, t\}$. Now, for a contradiction, let $g_0 = f' \neq f$, so that $g_{t-1} = g_t + \mathbf{e}_i \in (t-1)\Delta_{f'}$. Apply the same iterative process as before to get

$$
\begin{aligned}
f' = g_0 \;\; &= g_{t-1} + \mathbf{e}_{i_1} + \cdots + \mathbf{e}_{i_{t-1}} \\
&= g_t + \mathbf{e}_i + \mathbf{e}_{i_1} + \cdots + \mathbf{e}_{i_{t-1}} \\
&= f - \mathbf{e}_{i_t} + \mathbf{e}_i.
\end{aligned}
$$

Thus $\mathbf{e}_{i_t}$ was internally inactive at f, contradicting our construction of $t\Delta_f$. $\quad\square$

An exterior analogue of Lemma 6.2 also holds, in which ι and ∇ are replaced with ε and Δ. The next theorem is a consequence.

Theorem 6.3. *Let M be a polymatroid. Then $I_M(\xi) = \xi^{|E|-1} \cdot Q'_M(1/\xi, 1)$ and $X_M(\eta) = \eta^{|E|-1} \cdot Q'_M(1, 1/\eta)$.*

Proof. An i-dimensional face of $t\nabla$ – that is, a face with $i+1$ vertices – has

$$
\binom{t+i}{i} = \sum_{k=0}^{i} \binom{i}{k}\binom{t}{i}
$$

lattice points. So if $P = P(M)$, counting lattice points in $P+t\nabla$ by the partition in Lemma 6.2 says

$$
Q_M(t, 0) = \sum_{f \in L(P)} \sum_k \binom{\iota(f)-1}{k}\binom{t}{\iota(f)-1}.
$$

Applying the change of basis,

$$
Q'_M(x, 1) = \sum_{f \in L(P)} \sum_k \binom{\iota(f)-1}{k}(x-1)^{\iota(f)-1} = \sum_{f \in L(P)} x^{\iota(f)-1}
$$

$$
= x^{|E|-1} \sum_{f \in L(P)} (x^{-1})^{\bar{\iota}(f)} = x^{|E|-1} I_M(x^{-1}).
$$

This proves the first equality, with $\xi = x^{-1}$. The second follows in the same way from the exterior analogue of Lemma 6.2. $\quad\square$

As Kálmán observed [12, Remark 5.7], the bivariate enumerator of internal and external activities for polymatroids is not order-independent. Therefore, no formula like

$$
Q'_M(x, y) \overset{?}{=} \sum_{x \in \mathcal{B}_M} \xi^{\iota(x)-1} \eta^{\varepsilon(x)-1}
$$

can hold, since the left hand side is order-independent and the right hand side is not.

Example 6.4. Take the polymatroid with bases $\{(0,2,1),(1,1,1),(1,2,0),(2,1,0),(2,0,1)\}$. Using the natural ordering on [3], we have that $\sum_{x \in \mathcal{B}_M} \xi^{\iota(x)} \eta^{\varepsilon(x)} = \xi^3 \eta + 2\xi^2 \eta^2 + \xi \eta^2 + \xi \eta^3$. If we instead use the ordering $2 < 3 < 1$, the enumerator is $\xi^3 \eta^2 + \xi^2 \eta^2 + \xi^2 \eta + \xi \eta^2 + \xi \eta^3$. $\quad \diamond$
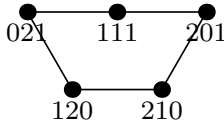

Figure 4: The polymatroid of Example 6.4.

Question 6.5. Section 10 of [12] is dedicated to the behaviour of Kálmán's activity invariants in trinities in the sense of Tutte. One can obtain six hypergraphs from a properly three-coloured triangulation of the sphere by deleting one colour class and regarding the second and third as vertices and hyperedges of a hypergraph. Kálmán considered the relationships between values of his invariants on these six hypergraphs. With the proof of his main conjecture in [13], we know that besides the internal and external invariants I_G and X_G of a hypergraph G with plane embedding, there exists a third invariant Y_G such that the values of the three invariants are permuted by the action of the symmetric group S_3 on the colour classes in the natural way.

Our work has cast I_M and X_M as univariate evaluations of a bivariate polynomial Q'_M, and the content of Proposition 5.3 is that polymatroid duality, i.e. exchanging the deleted and vertex colour classes, exchanges the two variables of Q'_M. Is there a good trivariate polynomial of a three-coloured triangulation of the sphere which similarly encapsulates the above observations on trinities? Permuting the three colours should permute its three variables, and three of its univariate evaluations should be I_G, X_G, and Y_G. We would of course be even happier if Q'_G were among its bivariate evaluations.

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