

## THE COMPLEXITY OF COMPUTING THE SIGN OF THE TUTTE POLYNOMIAL\*

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**Abstract.** We study the complexity of computing the sign of the Tutte polynomial of a graph. As there are only three possible outcomes (positive, negative, and zero), this seems at first sight more like a decision problem than a counting problem. Surprisingly, however, there are large regions of the parameter space for which computing the sign of the Tutte polynomial is actually  $\#P$ -hard. As a trivial consequence, approximating the polynomial is also  $\#P$ -hard in this case. Thus, *approximately* evaluating the Tutte polynomial in these regions is as hard as *exactly* counting the satisfying assignments to a CNF Boolean formula. For most other points in the parameter space, we show that computing the sign of the polynomial is in FP, whereas approximating the polynomial can be done in polynomial time with an NP oracle. As a special case, we completely resolve the complexity of computing the sign of the chromatic polynomial—this is easily computable at  $q = 2$  and when  $q \leq 32/27$ , and is NP-hard to compute for all other values of the parameter  $q$ .

**Key words.** computational complexity, Tutte polynomial,  $\#P$ -completeness

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**1. Introduction.** The Tutte polynomial of an undirected<sup>1</sup> graph is a two-variable polynomial that captures many interesting properties of the graph, such as (by making appropriate choices of the two variables) the number of  $q$ -colorings, the number of nowhere-zero  $q$ -flows, the number of acyclic orientations, and the probability that the graph remains connected when edges are deleted at random.

Much work [2, 4, 3, 5, 11, 20] has studied the difficulty of evaluating the polynomial (exactly or approximately) when the values of the variables are fixed, and a graph is given as input.

Our early paper [3] identified a large region of points where the approximate evaluation of the polynomial is NP-hard and a short hyperbola segment along which approximate evaluation is even  $\#P$ -hard. Thus, an approximation of the polynomial at a point on this short hyperbola segment would enable one to *exactly* solve a problem in  $\#P$ . Kuperberg [12, Theorem 1.3] uses quantum results to show similar (classical)  $\#P$ -hardness for all points  $(x, y)$  in the negative quadrant satisfying  $(x - 1)(y - 1) > 4$ . In this paper, we show that, in fact, for most of the NP-hard points identified in [3], approximation is  $\#P$ -hard. Moreover, it is  $\#P$ -hard for a very simple reason: determining the sign of the polynomial—i.e., whether the evaluation of the polynomial

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<sup>1</sup>All graphs in this paper are undirected, so we shall drop the qualifier in what follows.

is positive, negative, or zero—is #P-hard. This seems surprising since determining the sign of the polynomial is nearly a decision problem (there are only three possible outcomes) but it is #P-hard nearly everywhere (at all of the red points in the plane in Figure 1).

Past work [9] has studied the sign of the Tutte polynomial—in particular, Jackson and Sokal sought to determine for which choices of the two variables the sign is “trivial” in the sense that it does not depend on the input graph (or it depends only very weakly on the input graph, for example when it depends only on the number of vertices in the graph).

To illustrate how our work fits in with the work of Jackson and Sokal, we start with an important univariate case. The *chromatic polynomial*  $P(G; q)$  of an  $n$ -vertex graph  $G$  is the unique degree- $n$  polynomial in the variable  $q$  such that  $P(G; q)$  is the number of proper  $q$ -colorings of  $G$ . Jackson [8, Theorem 5] showed that for  $q \in (1, 32/27]$  the sign of  $P(G; q)$  depends upon  $G$  in an essentially trivial way. In particular, for every connected simple graph with  $n \geq 2$  vertices and  $b$  blocks,  $P(G; q)$  is nonzero with sign  $(-1)^{n+b-1}$ . The sign of  $P(G; q)$  is also known to be a trivial function of  $G$  for  $q \leq 1$ . (See, for example, [9, Theorem 1.1].) Jackson [8, Theorem 12] demonstrated the significance of the value  $32/27$  by constructing an infinite family of graphs such that  $P(G; q) = 0$  at a value of  $q$  which is arbitrarily close to  $32/27$ . In fact, Jackson and Sokal conjectured [9, Conjecture 10.3(e)] that the value  $32/27$  is a phase transition in the sense that, for every  $q$  above this critical value, the sign of  $P(G; q)$  is a nontrivial function of  $G$ . In particular, they conjectured that for any fixed  $q > 32/27$ , and all sufficiently large  $n$  and  $m$ , there are 2-connected graphs  $G$  with  $n$  vertices and  $m$  edges that make  $P(G; q)$  nonzero with either sign.

It turns out that this intuition is correct (see Corollary 56) and that  $q = 32/27$  is, in some sense, a phase transition for the complexity of computing the sign of  $P(G; q)$ :

- As was known, for  $q \leq 32/27$ , the sign of  $P(G; q)$  is a trivial function of  $G$ , which is easily computed.
- At  $q = 2$ , the evaluation  $P(G; q)$  is the number of 2-colorings of  $G$ . The sign of  $P(G; q)$  is positive if  $G$  is bipartite and is 0 otherwise. Thus, the sign of  $P(G; q)$  is not a trivial function of  $G$ , but  $P(G; q)$  is still easily computed in polynomial time.
- For every  $q > 32/27$  except  $q = 2$ , computing the sign of  $P(G; q)$  is NP-hard.

However, the full version of Jackson and Sokal’s conjecture turns out to be incorrect. See Observations 39 and 41 for counterexamples.

While computing the sign of  $P(G; q)$  is NP-hard for every  $q \neq 2$  which is greater than  $32/27$ , the precise complexity of computing the sign does actually depend upon  $q$ . We show (see Corollary 56) that for each fixed *noninteger*  $q > 32/27$ , the complexity of computing the sign of  $P(G; q)$  is #P-hard. This means that a polynomial-time algorithm for computing the sign of  $P(G; q)$ , given  $G$ , would give a polynomial-time algorithm for exactly solving every problem in #P. On the other hand, for integers  $q > 2$ , the problem of computing the sign of  $P(G; q)$  is merely NP-complete.<sup>2</sup>

As one would expect, both of these results have ramifications for the complexity of approximating  $P(G; q)$ . A fully polynomial randomized approximation scheme (FPRAS) for evaluating  $P(G; q)$ , given  $G$ , can be used as a polynomial-time random-

<sup>2</sup>As there are three potential outcomes, determining the sign cannot be NP-complete in a strict sense. However, in this case, one of the outcomes (negative) is impossible, so we can view the determination of the sign as an NP-problem by identifying positive with “accept” and zero with “reject.” This view will be taken throughout the paper.

ized algorithm for computing the sign of  $P(G; q)$ . Thus, we can immediately deduce that if  $q$  is a noninteger greater than  $32/27$ , then there is no FPRAS for  $P(G; q)$  unless there is a randomized polynomial-time algorithm for exactly solving every problem in #P. See section 2.4 for a more thorough discussion of this claim.

On the other hand, for integer values  $q > 32/27$ , we show that the problem of evaluating  $P(G; q)$  is in the complexity class #P $_{\mathbb{Q}}$ , which is defined as follows.

*Definition.* FP is the class of functions computable by polynomial-time algorithms. We say that a function  $f : \Sigma^* \rightarrow \mathbb{Q}$  is in the class #P $_{\mathbb{Q}}$  if  $f(x) = a(x)/b(x)$ , where  $a, b : \Sigma^* \rightarrow \mathbb{N}$ , and  $a \in \#P$  and  $b \in \text{FP}$ .

If  $f$  is in #P $_{\mathbb{Q}}$ , then there is an approximation scheme for  $f$  that runs in polynomial time, using an oracle for an NP predicate (for a more detailed discussion, see [3, section 2.2]). Thus, it is presumably much easier to approximate  $P(G; q)$  when  $q$  is an integer greater than  $32/27$ , as compared to a noninteger.

All of these considerations generalize smoothly to the Tutte polynomial, which we now define. Since we will later need the multivariate generalization [16] of the polynomial, we use the “random cluster” formulation of the Tutte polynomial, which for a graph  $G = (V, E)$  is defined as a polynomial in indeterminates  $q$  and  $\gamma$  as follows:

$$(1) \quad Z(G; q, \gamma) = \sum_{A \subseteq E} q^{\kappa(V, A)} \gamma^{|A|},$$

where  $\kappa(V, A)$  denotes the number of connected components in the graph  $(V, A)$ . The chromatic polynomial discussed earlier is related to the Tutte polynomial via the identity [9, equation (2.15)]  $P(G; q) = Z(G; q, -1)$ .

In fact, Tutte defined the Tutte polynomial using a different, two-variable parameterization, in terms of variables  $x$  and  $y$ . This polynomial is defined for a graph  $G = (V, E)$  by

$$(2) \quad T(G; x, y) = \sum_{A \subseteq E} (x - 1)^{\kappa(V, A) - \kappa(V, E)} (y - 1)^{|A| - |V| + \kappa(V, A)}.$$

It is well known (see, for example, [16, equation (2.26)]) that when  $q = (x - 1)(y - 1)$  and  $\gamma = y - 1$  we have

$$(3) \quad T(G; x, y) = (y - 1)^{-|V|} (x - 1)^{-\kappa(V, E)} Z(G; q, \gamma).$$

This paper studies the complexity of computing the sign of the (random cluster) Tutte polynomial. The definitive statement of our results requires a number of formal definitions and is presented as Theorem 1 in section 5. However, an informal description of Theorem 1 appears in Figure 1, which illustrates the the  $(x, y)$  plane divided into a number of regions A–M according to their complexity.<sup>3</sup> The colors depict the complexity of computing the sign of the polynomial for a fixed point  $(x, y)$ . If the point  $(x, y)$  is colored red, then the problem of computing the sign is #P-hard. If the point  $(x, y)$  is colored green, then the problem of computing the sign is in FP.

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<sup>3</sup>For convenience, our proofs use the random cluster formulation of the Tutte polynomial (1). However, in order to make our results easily comparable to other results in the literature, such as [3, 11], we classify points using the  $(x, y)$ -coordinatization of (2). This is without loss of generality since it is easy to go from one coordinate system to the other using (3). However, the reader should note that if  $y = 1$ , then  $\gamma = 0$  and  $q = (x - 1)(y - 1) = 0$ , so computing  $Z(G; q, \gamma)$  is trivial, whereas the complexity of computing  $T(G; x, y)$  is unclear. In general, any two-parameter version of the Tutte polynomial will omit some points. This issue is discussed further in [6, section 1].

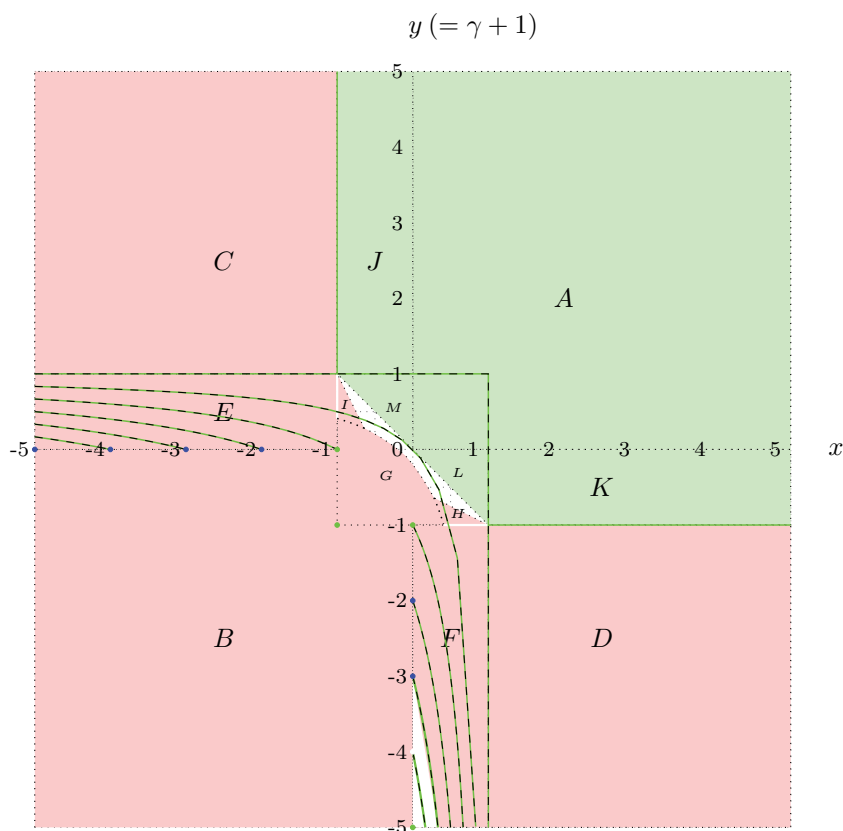


FIG. 1. An illustration of Theorem 1. Computing the sign of the Tutte polynomial is #P-hard at red points, is NP-complete at blue points, and is in FP at green points. We have not resolved the complexity at white points. At red points, approximating the Tutte polynomial is also #P-hard. At blue and green points, it can be done in polynomial time with an NP oracle. Guide for the greyscale (print) version: The red points appear as a darker grey in regions B, C, D, E, F, G, H, and I. The green points appear as a lighter grey in regions A, J, K, L, and M and also as dashed hyperbola segments and at the points  $(-1, 0)$ ,  $(-1, -1)$ ,  $(0, -1)$ , and  $(0, -5)$ . The blue points are  $(-2, 0)$ ,  $(-3, 0)$ ,  $(-4, 0)$ ,  $(-5, 0)$ ,  $(0, -2)$ , and  $(0, -3)$ .

Finally, if the point  $(x, y)$  is colored blue, then the problem of computing the sign is NP-complete. (There are still some points for which we have not resolved the complexity—these are colored white.)

To resolve any ambiguities in Figure 1, a formal description of the regions appearing there is provided in Figure 2. For each region of interest, the condition for a point  $(x, y)$  to belong to that region is given. Note that  $q$  is used to denote  $(x - 1)(y - 1)$ .

Once again, there are ramifications for the complexity of approximating the Tutte polynomial. Since an FPRAS for  $Z(G; q, \gamma)$  gives a randomized algorithm for computing its sign, we can again deduce that there is no FPRAS for points that are colored red (unless there is a randomized polynomial-time algorithm for exactly solving every problem in #P). By contrast, for all of the points that are colored green or blue, we also show that the problem of computing  $Z(G; q, \gamma)$  is in the complexity class #P $_{\mathbb{Q}}$ . Thus, the polynomial can be approximated in polynomial time using an NP oracle.

In order to reach into some of the regions, for example, F, it has been necessary to

- Region A:  $x \geq 0$  and  $y \geq 0$ .
- Region B:  $\min(x, y) \leq -1$  and  $\max(x, y) < 0$ .
- Region C:  $x < -1$  and  $y > 1$ .
- Region D:  $x > 1$  and  $y < -1$ .
- Region E:  $x \leq -1$  and  $0 < y \leq 1$ .
- Region F:  $0 < x \leq 1$  and  $y \leq -1$ .
- The boundary between regions B and E:  $x \leq -1$  and  $y = 0$ .
- The boundary between regions B and F:  $x = 0$  and  $y \leq -1$ .
- Region G:  $\max(|x|, |y|) < 1$  and  $q > 32/27$ .
- Region H:  $\max(|x|, |y|) < 1$  and  $q \leq 32/27$  and  $x < -2y - 1$ .
- Region I:  $\max(|x|, |y|) < 1$  and  $q \leq 32/27$  and  $y < -2x - 1$ .
- Region J:  $-1 \leq x < 0$  and  $y \geq 1$ .
- Region K:  $x \geq 1$  and  $-1 \leq y < 0$ .
- Region L:  $0 < x < 1$  and  $-x < y < 0$ .
- Region M:  $0 < y < 1$  and  $-y < x < 0$ .
- The rest: There are some remaining unresolved points. These points (simultaneously) satisfy all of the following inequalities:  $\max(|x|, |y|) < 1$ ,  $y < -x$ ,  $q \leq 32/27$ ,  $y \geq -2x - 1$ ,  $x \geq -2y - 1$ , and  $q \neq 1$ .

FIG. 2. A formal description of the regions appearing in Theorem 1 and Figure 1. For each region of interest, we give the condition for a point  $(x, y)$  to belong to that region. Throughout we use  $q$  to denote  $(x - 1)(y - 1)$ .

use gadgets that go beyond the series-parallel graphs that have so far proved adequate in this area. For example, exploring region F has necessitated the use of a gadget based on the Petersen graph.

Our classification is not complete and leaves some areas unresolved (colored white in Figure 1). Although the methods could no doubt be pushed a little further, at the expense of adding further complexity to the proofs, it seems likely that a complete classification is some way off. For example, showing that the sign of the Tutte polynomial is hard to compute at the point  $(0, -4)$  would necessarily provide a counterexample to Tutte’s long-standing 5-flow conjecture. In the other direction, it is difficult to conceive of an efficient algorithm for deciding the sign that would not at the same time resolve the conjecture.

**2. Preliminaries.**

**2.1. The Tutte polynomial.** It will be helpful to define the multivariate version of the random cluster formulation of the Tutte polynomial. Let  $\gamma$  be a function that assigns a (rational) weight  $\gamma_e$  to every edge  $e \in E$ . We refer to  $\gamma$  as a “weight function.” We define

$$Z(G; q, \gamma) = \sum_{A \subseteq E} q^{\kappa(V, A)} \prod_{e \in A} \gamma_e.$$

Given a graph  $G = (V, E)$  with distinguished nodes  $s$  and  $t$ ,  $Z_{st}(G; q, \gamma)$  denotes the contribution to  $Z(G; q, \gamma)$  arising from edge-sets  $A$  in which  $s$  and  $t$  are in the same component of  $(V, A)$ . That is,

$$Z_{st}(G; q, \gamma) = \sum_{A \subseteq E: s \text{ and } t \text{ in same component}} q^{\kappa(V, A)} \prod_{e \in A} \gamma_e.$$

Similarly,  $Z_{s|t}$  denotes the contribution arising from edge-sets  $A$  in which  $s$  and  $t$  are in different components, so  $Z(G; q, \gamma) = Z_{st}(G; q, \gamma) + Z_{s|t}(G; q, \gamma)$ .

**2.2. Implementing new edge weights, series compositions, and parallel compositions.** Our treatment of implementations, series compositions, and parallel compositions is completely standard and is taken from [5, section 2.1]. The reader who is familiar with this material can skip this section (which is included here for completeness).

Let  $W$  be a set of (rational) edge weights and fix a value  $q$ . Let  $w^*$  be a weight (which may not be in  $W$ ) which we want to “implement.” Suppose that there is a graph  $\Upsilon$ , with distinguished vertices  $s$  and  $t$  and a weight function  $\hat{\gamma} : E(\Upsilon) \rightarrow W$ , such that

$$(4) \quad w^* = qZ_{st}(\Upsilon; q, \hat{\gamma})/Z_{s|t}(\Upsilon; q, \hat{\gamma}).$$

In this case, we say that  $\Upsilon$  and  $\hat{\gamma}$  implement  $w^*$  (or even that  $W$  implements  $w^*$ ).

The purpose of “implementing” edge weights is this. Let  $G$  be a graph with weight function  $\gamma$ . Let  $f$  be some edge of  $G$  with weight  $\gamma_f = w^*$ . Suppose that  $W$  implements  $w^*$ . Let  $\Upsilon$  be a graph with distinguished vertices  $s$  and  $t$  with a weight function  $\hat{\gamma} : E(\Upsilon) \rightarrow W$  satisfying (4). Construct the weighted graph  $G'$  by replacing edge  $f$  with a copy of  $\Upsilon$  (identify  $s$  with either endpoint of  $f$  (it doesn't matter which one), and identify  $t$  with the other endpoint of  $f$  and remove edge  $f$ ). Let the weight function  $\gamma'$  of  $G'$  inherit weights from  $\gamma$  and  $\hat{\gamma}$  (so  $\gamma'_e = \hat{\gamma}_e$  if  $e \in E(\Upsilon)$  and  $\gamma'_e = \gamma_e$  otherwise). Then the definition of the multivariate Tutte polynomial gives

$$(5) \quad Z(G'; q, \gamma') = \frac{Z_{s|t}(\Upsilon; q, \hat{\gamma})}{q^2} Z(G; q, \gamma).$$

So, as long as  $q \neq 0$  and  $Z_{s|t}(\Upsilon; q, \hat{\gamma})$  is easy to evaluate, evaluating the multivariate Tutte polynomial of  $G'$  with weight function  $\gamma'$  is essentially the same as evaluating the multivariate Tutte polynomial of  $G$  with weight function  $\gamma$ .

Two especially useful implementations are series and parallel compositions. These are explained in detail in [9, section 2.3]. So we will be brief here. Parallel composition is the case in which  $\Upsilon$  consists of two parallel edges  $e_1$  and  $e_2$  with endpoints  $s$  and  $t$  and  $\hat{\gamma}_{e_1} = w_1$  and  $\hat{\gamma}_{e_2} = w_2$ . It is easily checked from (4) that  $w^* = (1+w_1)(1+w_2) - 1$ . Also, the extra factor in (5) cancels, so in this case  $Z(G'; q, \gamma') = Z(G; q, \gamma)$ .

Series composition is the case in which  $\Upsilon$  is a length-2 path from  $s$  to  $t$  consisting of edges  $e_1$  and  $e_2$  with  $\hat{\gamma}_{e_1} = w_1$  and  $\hat{\gamma}_{e_2} = w_2$ . It is easily checked from (4) that  $w^* = w_1w_2/(q + w_1 + w_2)$ . Also, the extra factor in (5) is  $q + w_1 + w_2$ , so in this case  $Z(G'; q, \gamma') = (q + w_1 + w_2)Z(G; q, \gamma)$ . It is helpful to note that  $w^*$  satisfies

$$\left(1 + \frac{q}{w^*}\right) = \left(1 + \frac{q}{w_1}\right) \left(1 + \frac{q}{w_2}\right).$$

We say that there is a “shift” from  $(q, \alpha)$  to  $(q, \alpha')$  if there is an implementation of  $\alpha'$  consisting of some  $\Upsilon$  and  $\hat{w} : E(\Upsilon) \rightarrow W$  where  $W$  is the singleton set  $W = \{\alpha\}$ . This is the same notion of “shift” that we used in [3]. Taking  $y = \alpha + 1$  and  $y' = \alpha' + 1$  and defining  $x$  and  $x'$  by  $q = (x - 1)(y - 1) = (x' - 1)(y' - 1)$ , we equivalently refer to this as a shift from  $(x, y)$  to  $(x', y')$ . It is an easy but important observation that shifts may be composed to obtain new shifts. So, if we have shifts from  $(x, y)$  to  $(x', y')$  and from  $(x', y')$  to  $(x'', y'')$ , then we also have a shift from  $(x, y)$  to  $(x'', y'')$ .

The  $k$ -thickening of [11] is the parallel composition of  $k$  edges of weight  $\alpha$ . It implements  $\alpha' = (1 + \alpha)^k - 1$  and is a shift from  $(x, y)$  to  $(x', y')$  where  $y' = y^k$  (and  $x'$  is given by  $(x' - 1)(y' - 1) = q$ ). Similarly, the  $k$ -stretch is the series composition of  $k$  edges of weight  $\alpha$ . It implements an  $\alpha'$  satisfying

$$1 + \frac{q}{\alpha'} = \left(1 + \frac{q}{\alpha}\right)^k.$$

It is a shift from  $(x, y)$  to  $(x', y')$  where  $x' = x^k$ . (In the classical bivariate  $(x, y)$  parameterization, there is effectively one edge weight, so the stretching or thickening is applied uniformly to every edge of the graph.)

Since it is useful to switch freely between  $(q, \alpha)$  coordinates and  $(x, y)$  coordinates, we also refer to the implementation in (4) as an implementation of the point  $(x, y) = (q/w^* + 1, w^* + 1)$  using the points

$$\{(x, y) = (q/w + 1, w + 1) \mid w \in W\}.$$

Thus, if  $q = (x_1 - 1)(y_1 - 1) = (x_2 - 1)(y_2 - 1)$ , then the series composition of  $(x_1, y_1)$  and  $(x_2, y_2)$  implements the point

$$\left(\frac{q}{y_1 y_2 - 1} + 1, y_1 y_2\right),$$

and the parallel composition of these implements the point

$$\left(x_1 x_2, \frac{q}{x_1 x_2 - 1} + 1\right).$$

We make extensive use of series and parallel composition, and the above identities will be employed without comment.

**2.3. Computational problems.** For fixed rational numbers  $q, \gamma$ , and  $\gamma_1, \dots, \gamma_k$ , we consider the following computational problems<sup>4</sup> from [3].

*Name* TUTTE( $q, \gamma$ ).

*Instance* A graph  $G = (V, E)$ .

*Output* The rational number  $Z(G; q, \gamma)$ .

*Name* TUTTE( $q; \gamma_1, \dots, \gamma_k$ ).

*Instance* A graph  $G = (V, E)$  and a weight function  $\gamma : E \rightarrow \{\gamma_1, \dots, \gamma_k\}$ .

*Output* The rational number  $Z(G; q, \gamma)$ .

We also consider variants in which the goal is to compute the sign of the Tutte polynomial.

*Name* SIGNTUTTE( $q, \gamma$ ).

*Instance* A graph  $G = (V, E)$ .

*Output* Determine whether the sign of  $Z(G; q, \gamma)$  is positive, negative, or 0.

*Name* SIGNTUTTE( $q; \gamma_1, \dots, \gamma_k$ ).

*Instance* A graph  $G = (V, E)$  and a weight function  $\gamma : E \rightarrow \{\gamma_1, \dots, \gamma_k\}$ .

*Output* Determine whether the sign of  $Z(G; q, \gamma)$  is positive, negative, or 0.

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<sup>4</sup>In [3] we referred to these as MULTITUTTE( $q, \gamma$ ) and MULTITUTTE( $q; \gamma_1, \dots, \gamma_k$ ), respectively, but we use the shorter names here since there is no confusion.

**2.4. Randomized algorithms and approximate counting.** A *randomized algorithm* for a computational problem takes an instance of the problem and returns a result. We require that for each instance, and each run of the algorithm, the probability that the result is equal to the correct output for the given instance is at least  $\frac{3}{4}$ .

A *randomized approximation scheme* is an algorithm for approximately computing the value of a function  $f : \Sigma^* \rightarrow \mathbb{R}$ . The approximation scheme has a parameter  $\varepsilon > 0$  which specifies the error tolerance. A *randomized approximation scheme* for  $f$  is a randomized algorithm that takes as input an instance  $x \in \Sigma^*$  (e.g., an encoding of a graph  $G$ ) and an error tolerance  $\varepsilon > 0$  and outputs a number  $z \in \mathbb{Q}$  (a random variable of the “coin tosses” made by the algorithm) such that, for every instance  $x$ ,

$$\Pr [e^{-\varepsilon} \leq z/f(x) \leq e^\varepsilon] \geq \frac{3}{4},$$

where, by convention,  $0/0 = 1$ . (The slight modification of the more familiar definition is to ensure that functions  $f$  taking negative values are dealt with correctly.)

The randomized approximation scheme is said to be a *fully polynomial randomized approximation scheme*, or *FPRAS*, if it runs in time bounded by a polynomial in  $|x|$  and  $\varepsilon^{-1}$ .

Completeness of a problem in  $\#P$  is defined with respect to polynomial-time Turing reduction. Suppose  $\text{SIGNTUTTE}(q, \gamma)$  is  $\#P$ -hard for some setting of the parameters  $q, \gamma$ . Then, clearly,  $\text{SIGNTUTTE}(q, \gamma) \in \text{FP}$  would imply  $\#P = \text{FP}$ . In addition, the existence of a polynomial-time randomized algorithm for  $\text{SIGNTUTTE}(q, \gamma)$  would imply the existence of a polynomial-time randomized algorithm for every problem in  $\#P$ . The reasoning is as follows. Suppose the randomized algorithm for  $\text{SIGNTUTTE}(q, \gamma)$  has failure probability at most  $\frac{1}{4}$ . By a standard powering argument, the failure probability can be reduced so that it is exponentially small in the input size. But a polynomial-time Turing reduction makes only polynomially many oracle calls, so the probability that even a single one produces the wrong answer is exponentially small, and certainly less than  $\frac{1}{4}$ . As an immediate consequence, an FPRAS for  $\text{TUTTE}(q, \gamma)$  would again imply the existence of a polynomial-time randomized (but exact in the event of success) algorithm for every problem in  $\#P$ .

**3.  $\#P$ -hardness of computing the sign of the Tutte polynomial—the multivariate case.** We use the fact that the following problem is  $\#P$ -complete. This was shown by Provan and Ball [15].

*Name*  $\#$ MINIMUM CARDINALITY  $(s, t)$ -CUT.

*Instance* A graph  $G = (V, E)$  and distinguished vertices  $s, t \in V$ .

*Output*  $|\{S \subseteq E : S \text{ is a minimum cardinality } (s, t)\text{-cut in } G\}|$ .

**LEMMA 2.** *Suppose  $q > 1$  and that  $\gamma_1 \in (-2, -1)$  and  $\gamma_2 \notin [-2, 0]$ . Then  $\text{SIGNTUTTE}(q; \gamma_1, \gamma_2)$  is  $\#P$ -hard.*

*Proof.* We will give a Turing reduction from  $\#$ MINIMUM CARDINALITY  $(s, t)$ -CUT to  $\text{SIGNTUTTE}(q; \gamma_1, \gamma_2)$ .

Let  $G, s, t$  be an instance of  $\#$ MINIMUM CARDINALITY  $(s, t)$ -CUT. Assume without loss of generality that  $G$  has no edge from  $s$  to  $t$ . Let  $n = |V(G)|$  and  $m = |E(G)|$ . Assume without loss of generality that  $G$  is connected and that  $m \geq n$  is sufficiently large. Let  $k$  be the size of a minimum cardinality  $(s, t)$ -cut in  $G$ , and let  $C$  be the number of size- $k$   $(s, t)$ -cuts.



The following calculations are more general than necessary so that we can reuse them in the proof of Lemma 3 (where  $q < 1$  and  $q$  may even be negative). Let

$$M^* = \max \left( \left( 8 \max \left( |q|, \frac{1}{|q|} \right) \right)^m, \frac{2}{|q-1|} \right).$$

Let  $h$  be the smallest integer such that  $(\gamma_2 + 1)^h - 1 > M^*$ , and let  $M = (\gamma_2 + 1)^h - 1$ . Note that we can implement  $M$  from  $\gamma_2$  via an  $h$ -thickening, and  $h$  is at most a polynomial in  $m$ .

Let  $\delta = (2 \max(|q|, |q|^{-1}))^m / M$ . Let  $\mathbf{M}$  be the constant weight function which gives every edge weight  $M$ . We will use the following facts:

$$(6) \quad M^m q - \delta M^m |q| \leq Z_{st}(G; q, \mathbf{M}) \leq M^m q + \delta M^m |q|$$

and

$$(7) \quad CM^{m-k} q^2 (1 - \delta) \leq Z_{s|t}(G; q, \mathbf{M}) \leq CM^{m-k} q^2 (1 + \delta).$$

Fact (6) follows from the fact that each of the (at most  $2^m$ ) terms in  $Z_{st}(G; q, \mathbf{M})$ , other than the term with all edges in  $A$ , has absolute value at most  $M^{m-1} (\max(|q|, 1))^n$  and  $2^m M^{m-1} (\max(|q|, 1))^n \leq \delta M^m |q|$ . Fact (7) follows from the fact that all terms in  $Z_{s|t}(G; q, \mathbf{M})$  are complements of  $(s, t)$ -cuts. Each term that is not a complement of a size- $k$   $(s, t)$ -cut has absolute value at most  $M^{m-k-1} q^2 (\max(|q|, 1))^n$  and

$$2^m M^{m-k-1} q^2 (\max(|q|, 1))^n \leq \delta CM^{m-k} q^2.$$

For a parameter  $\varepsilon$  in the open interval  $(0, 1)$  which we will tune below, let  $\gamma' = -1 - \varepsilon \in (-2, -1)$ . We will discuss the implementation of  $\gamma'$  below. Let  $G'$  be the graph formed from  $G$  by adding an edge from  $s$  to  $t$ . Let  $\gamma$  be the edge-weight function for  $G'$  that assigns weight  $M$  to every edge of  $G$  and assigns weight  $\gamma'$  to the new edge. Then, using the definition of the Tutte polynomial,

$$(8) \quad \begin{aligned} Z(G'; q, \gamma) &= Z_{st}(G; q, \mathbf{M})(1 + \gamma') + Z_{s|t}(G; q, \mathbf{M}) \left( 1 + \frac{\gamma'}{q} \right) \\ &= -\varepsilon Z_{st}(G; q, \mathbf{M}) + Z_{s|t}(G; q, \mathbf{M}) \left( 1 - \frac{1 + \varepsilon}{q} \right). \end{aligned}$$

Now suppose  $\varepsilon = M^{-2m}$ . Then

$$Z(G'; q, \gamma) = -M^{-2m} Z_{st}(G; q, \mathbf{M}) + Z_{s|t}(G; q, \mathbf{M}) \left( 1 - \frac{1 + M^{-2m}}{q} \right).$$

Now since  $M > 2/(q - 1)$  and  $M \geq 1$ , we have  $1 - (1 + M^{-2m})/q \geq (1 - 1/q)/2$ . (Note that  $M$  is bounded away from 1, so  $M^{-2m}$  can be made as small as we need by taking  $m$  sufficiently large.) So, using (6) and (7),

$$Z(G'; q, \gamma) \geq ((1 - 1/q)/2) CM^{m-k} q^2 (1 - \delta) - M^{-2m} M^m q (1 + \delta),$$

which is positive since  $k \leq m$ . On the other hand, using the definition of  $M$  and facts (6) and (7) above, we can confirm that when  $\varepsilon = 1$ ,  $Z(G'; q, \gamma)$  is negative. Also, when  $\varepsilon = q - 1$  we have  $Z(G'; q, \gamma) = -(q - 1) Z_{st}(G; q, \mathbf{M})$ , which again is negative.

Thus we have a range from  $\varepsilon = M^{-2m}$  to  $\varepsilon = \min(1, q - 1)$  of length at most 1 in which  $Z(G'; q, \gamma)$  changes sign. The idea is to perform binary search on this range to find an  $\varepsilon$  where  $Z(G'; q, \gamma) = 0$ . For this value of  $\varepsilon$ , we have  $\varepsilon Z_{st}(G; q, \mathbf{M}) = Z_{s|t}(G; q, \mathbf{M})(1 - \frac{1+\varepsilon}{q})$ . It turns out that, given this identity, estimates (6) and (7) above will give us enough information to calculate  $C$ .

As one would expect, there are small technical complications. Since we are somewhat constrained in what values  $\varepsilon$  we can implement, we won't be able to discover the exact value of  $\varepsilon$  that we need, but we will be able to approximate it sufficiently closely to compute  $C$  exactly from (6) and (7). Suppose for a moment that we are able, for a given  $\varepsilon \in (M^{-2m}, \min(1, q - 1))$ , to compute the sign of  $Z(G'; q, \gamma)$ . Our basic strategy will be binary search, subdividing the initial interval  $\lceil m^2 \lg M \rceil$  times, so eventually we'll get an interval of width at most  $M^{-m^2}$  which contains an  $\varepsilon$  where  $Z(G'; q, \gamma) = 0$ .

To do this, we need to address the issue of computing the sign of  $Z(G'; q, \gamma)$  using an oracle for  $\text{SIGNTUTTE}(q; \gamma_1, \gamma_2)$ . We have already seen above that it is easy to implement the weight  $M$  using  $\gamma_2$  (and that the implementation has polynomial size)—we now need to consider the implementation of  $\gamma' = -1 - \varepsilon$  (where  $\varepsilon \in (M^{-2m}, \min(1, q - 1))$  is the particular value that is being queried).

Let  $y' = -\varepsilon$  be the point that we desire to implement. Let  $y_1 = \gamma_1 + 1$ . Note that  $y_1 \in (-1, 0)$ . Let  $j$  be the smallest odd integer so that  $|y_1|^j < \varepsilon$ . Let  $T^- = |y_1|^{-2}$  and  $T^+ = |y_1|^{-3}$ . Let  $T = -\varepsilon/y_1^{j+2}$ . Note that  $1 < T^- \leq T \leq T^+$ .

Let  $(x_2, y_2) = (q/\gamma_2 + 1, \gamma_2 + 1)$ . Note that  $y_2 \notin [-1, 1]$ . We will define a small quantity  $\pi$  below. Looking ahead to Lemma 5, we see that, from the point  $(x_2, y_2)$ , we can implement a point  $(x'', y'')$  with  $T - \pi \leq y'' \leq T$ . The size of the graph used to implement  $(x'', y'')$  is at most a polynomial in  $\log(\pi^{-1})$ . It does not depend upon  $T$ , though it does depend on the fixed bounds  $T^-$  and  $T^+$ . Now implement  $y'$  by a parallel composition of  $y''$  and  $j + 2$  copies of  $y_1$ . (We can do this parallel composition because  $j$  is only polynomially big in  $m$ .) Note that  $-\varepsilon \leq y' \leq -\varepsilon + \pi|y_1|^{j+2}$ , so of course  $-\varepsilon \leq y' \leq -\varepsilon + \pi$ .

Thus, in the binary search, we may not be able to query the exact value of  $\varepsilon$  that we want to, but we can query a value that is between  $\varepsilon - \pi$  and  $\varepsilon$ .

Recall that our goal is to end up with a subinterval of the initial interval  $(M^{-2m}, \min(1, q - 1))$  such that the subinterval has width at most  $M^{-m^2}$  and contains an  $\varepsilon$  where  $Z(G'; q, \gamma) = 0$ . We do this by setting  $\pi = M^{-m^2}/3$  so that  $\pi$  is only a third as large as the smallest subinterval width (where we stop the binary search). We also adjust the binary search, subdividing the original interval up to  $\lceil m^2 \log_{3/2} M \rceil$  times rather than  $\lceil m^2 \log_2 M \rceil$  times, to make up for the fact that we might end up with (crudely) at most two-thirds of the interval after one iteration, rather than half. The result, then, is that we can find a subinterval of width at most  $M^{-m^2}$  which contains an  $\varepsilon$  where  $Z(G'; q, \gamma) = 0$ .

Now let  $\varepsilon$  be an endpoint of this subinterval. Let

$$\rho = 2^m \max(|q|, 1)^m M^m M^{-m^2}.$$

Since  $\varepsilon \geq M^{-2m}$  and  $m$  is sufficiently large, we have  $\rho \leq \varepsilon M^m |q| 4^{-m}$ . From the definition of Tutte polynomial,  $Z(G'; q, \gamma)$  is linear as a function of  $\gamma'$  (and hence of  $\varepsilon$ ), and the coefficient of  $\gamma'$  is a sum of  $2^m$  terms, each bounded in absolute value by  $\max(|q|, 1)^n M^m \leq \max(|q|, 1)^m M^m$ . Since  $\gamma'$  is within distance  $M^{-m^2}$  of the zero of  $Z(G'; q, \gamma)$ , we see that  $|Z(G'; q, \gamma)| \leq \rho$ .

Now using (8), (6), and (7), we have

$$\frac{-\rho + \varepsilon M^m q(1 - \delta)}{\left(1 - \frac{1+\varepsilon}{q}\right) M^{m-k} q^2 (1 + \delta)} \leq C \leq \frac{\rho + \varepsilon M^m q(1 + \delta)}{\left(1 - \frac{1+\varepsilon}{q}\right) M^{m-k} q^2 (1 - \delta)},$$

so, since  $\delta \leq 4^{-m}$ ,

$$(9) \quad \frac{(1 - 2 \cdot 4^{-m}) \varepsilon M^m q}{\left(1 - \frac{1+\varepsilon}{q}\right) M^{m-k} q^2 (1 + 4^{-m})} \leq C \leq \frac{\varepsilon M^m q (1 + 2 \cdot 4^{-m})}{\left(1 - \frac{1+\varepsilon}{q}\right) M^{m-k} q^2 (1 - 4^{-m})}.$$

Now the point is that  $C$  is an integer between 1 and  $2^m$ . Even though the value of  $k$  is not known, the fact that  $M > 4^m$  means that there can only be one integer  $k$  such that the above interval contains an integer between 1 and  $2^m$  (so  $k$  can easily be deduced). All of the other quantities in the lower and upper bounds in (9) are known. Now let  $R = \frac{\varepsilon M^k}{q - (1 + \varepsilon)}$ , so (9) becomes

$$(10) \quad \left(\frac{1 - 2 \cdot 4^{-m}}{1 + 4^{-m}}\right) R \leq C \leq R \left(\frac{1 + 2 \cdot 4^{-m}}{1 - 4^{-m}}\right).$$

Now,  $R < 2^{m+1}$  since otherwise the left-hand side of (10) is greater than  $2^m$ . Also, multiplying through by  $(1 + 4^{-m})(1 - 4^{-m})$ , the width of the interval is at most  $6 \cdot 4^{-m} R < 1$ , so the width of the interval in (10) is less than 1, and so the (integral) value of  $C$  can be calculated exactly.  $\square$

We have a similar lemma for  $q < 1$ .

LEMMA 3. *Suppose  $q < 1$  and  $q \neq 0$  and that  $\gamma_1 \in (-1, 0)$  and  $\gamma_2 \notin [-2, 0]$ . Then  $\text{SIGNTUTTE}(q; \gamma_1, \gamma_2)$  is #P-hard.*

*Proof.* The situation is very similar to that of Lemma 2.

We start with the situation  $0 < q < 1$ . In this case, we follow the proof of Lemma 2. Then facts (6) and (7) hold, as before. For the tunable parameter  $\varepsilon \in (0, 1)$ , we let  $\gamma' = -1 + \varepsilon \in (-1, 0)$ . Implementing  $G'$  as in the proof of Lemma 2, we have

$$(11) \quad Z(G'; q, \gamma) = \varepsilon Z_{st}(G; q, \mathbf{M}) + Z_{s|t}(G; q, \mathbf{M}) \left(1 - \frac{1 - \varepsilon}{q}\right).$$

Now, suppose  $\varepsilon = M^{-2m}$ . Then since  $M > 2/(1 - q)$  and  $M \geq 1$ , we have

$$1 - (1 - M^{-2m})/q \leq \frac{1}{2}(1 - 1/q) < 0.$$

Using facts (6) and (7), we find that  $Z(G'; q, \gamma)$  is negative. On the other hand, at  $\varepsilon = 1 - q$ ,  $Z(G'; q, \gamma)$  is positive.

To implement  $\gamma'$ , let  $y' = \varepsilon$  be the point that we desire to implement. Let  $y_1 = \gamma_1 + 1$ . Note that  $y_1 \in (0, 1)$ . Now proceed as in the proof of Lemma 2, with  $T = \varepsilon/y_1^{j+2}$ , and  $T^-$  and  $T^+$  as before. Once again we find a subinterval of  $(M^{-2m}, 1 - q)$  of width at most  $M^{-m^2}$  which contains an  $\varepsilon$  where  $Z(G'; q, \gamma) = 0$ , so we let  $\varepsilon$  be an endpoint of this subinterval and we conclude that  $|Z(G'; q, \gamma)| \leq \rho$ . Now we finish as in the proof of Lemma 2.

The argument for  $q < 0$  also follows the proof of Lemma 2. Here,  $Z_{st}(G; q, \mathbf{M})$  is negative and  $Z_{s|t}(G; q, \mathbf{M})$  is positive. Taking  $\gamma' = -1 + \varepsilon$ , as above, we still have (11). Now, suppose  $\varepsilon = M^{-2m}$ . Then by (11),  $Z(G'; q, \gamma) \geq M^{-2m} Z_{st}(G; q, \mathbf{M}) +$

$Z_{sit}(G'q, \mathbf{M})$ , which is positive. On the other hand, at  $\varepsilon = 1$ ,  $Z(G'; q, \gamma)$  is negative. Now the implementation of  $\gamma'$  proceeds as above, except that we use Lemma 7 (working from points  $(x_1, y_1)$  and  $(x_2, y_2)$ ) instead of Lemma 5.

So we find a subinterval of  $(M^{-2m}, 1)$  of width at most  $M^{-m^2}$  which contains an  $\varepsilon$  where  $Z(G'; q, \gamma) = 0$ . Letting  $\varepsilon$  be an endpoint of this subinterval, we conclude that  $|Z(G'; q, \gamma)| \leq \rho$ . Now we finish as in the proof of Lemma 2.  $\square$

**4. Implementing new edge weights.** In this section, we collect the information that we need about implementing edge weights within various regions of the Tutte plane. The following straightforward lemmas are useful.

LEMMA 4. *Suppose  $q > 0$  and that  $(x, y)$  is a point with  $x < -1$ . Then  $(x, y)$  can be used to implement a point  $(x', y')$  with  $y' > 1$ .*

*Proof.* A 2-stretch from  $(x, y)$  suffices since it implements the point  $(x', y') = (x^2, (x+y)/(1+x))$ . If  $x < -1$  and  $q = (x-1)(y-1)$  is positive, then  $y < 1$ , so  $x+y$  and  $1+x$  are both negative. Since  $y < 1$ , we conclude that  $-y > -1$ , so  $-x-y > -1-x$  and  $y' > 1$ .  $\square$

We will use the following lemma, which is [5, Lemma 3.26]. The lemma in [5] was stated for  $q > 5$  (which was all that was needed in that paper), but the proof uses only  $q > 0$ . The statement in [5] was in terms of the coordinates  $q$  and  $\gamma$ , but we have translated it to  $(x, y)$  coordinates here since that is how it will be used here. Finally, the statement of the lemma in [5] allowed the implementation to use two additional points  $(x'_2, y'_2)$  and  $(x'_3, y'_3)$  (this was to make the statement of the lemma match other lemmas in that paper). However, these additional points were not used in the proof, so we don't include them here.

LEMMA 5 (see [5, Lemma 3.26]). *Suppose that  $(x_1, y_1)$  is a point with  $y_1 \notin [-1, 1]$  and that  $q = (x_1 - 1)(y_1 - 1) > 0$ . Suppose that  $T^-$  and  $T^+$  satisfy  $1 < T^- \leq T^+$ . Given a target edge-weight  $T \in [T^-, T^+]$  and a positive value  $\pi$  which is sufficiently small with respect to  $x_1, y_1, T^-$ , and  $T^+$ , a point  $(x, y)$  with  $T - \pi \leq y \leq T$  can be implemented using the point  $(x_1, y_1)$ . The size of the graph  $\Upsilon$  used to implement  $(x, y)$  is at most a polynomial in  $\log(\pi^{-1})$ . (This upper bound on the size of  $\Upsilon$  does not depend on  $T$ , though it does depend on the fixed bounds  $T^-$  and  $T^+$ .)*

By duality of  $x$  and  $y$ , we have the following corollary.

COROLLARY 6. *Suppose that  $(x_1, y_1)$  is a point with  $x_1 \notin [-1, 1]$  and that  $q = (x_1 - 1)(y_1 - 1) > 0$ . Suppose that  $T^-$  and  $T^+$  satisfy  $1 < T^- \leq T^+$ . Given a target edge-weight  $T \in [T^-, T^+]$  and a positive value  $\pi$  which is sufficiently small with respect to  $x'_1, y'_1, T^-$ , and  $T^+$ , a point  $(x, y)$  with  $T - \pi \leq x \leq T$  can be implemented using the point  $(x_1, y_1)$ . The size of the graph  $\Upsilon$  used to implement  $(x, y)$  is at most a polynomial in  $\log(\pi^{-1})$ . (This upper bound on the size of  $\Upsilon$  does not depend on  $T$ , though it does depend on the fixed bounds  $T^-$  and  $T^+$ .)*

We will also use the following related lemma, which is [5, Lemma 3.27]. Once again, we translated to  $(x, y)$  coordinates and eliminated unused points.

LEMMA 7 (see [5, Lemma 3.27]). *Suppose that  $(x_1, y_1)$  is a point with  $y_1 \notin [-1, 1]$  and  $(x_2, y_2)$  is a point with  $y_2 \in (-1, 1)$ . Suppose that  $q = (x_1 - 1)(y_1 - 1) = (x_2 - 1)(y_2 - 1) < 0$ . Suppose that  $T^-$  and  $T^+$  satisfy  $1 < T^- \leq T^+$ . Given a target edge-weight  $T \in [T^-, T^+]$  and a positive value  $\pi$  which is sufficiently small with respect to  $x_1, y_1, x_2, y_2, T^-$ , and  $T^+$ , a point  $(x, y)$  with  $T - \pi \leq y \leq T$  can be implemented using the points  $(x_1, y_1)$  and  $(x_2, y_2)$ . The size of the graph  $\Upsilon$  used to implement  $(x, y)$  is at most a polynomial in  $\log(\pi^{-1})$ . (This upper bound on the size of  $\Upsilon$  does not depend on  $T$ , though it does depend on the fixed bounds  $T^-$  and  $T^+$ .)*

The reader may find it useful to consult Figure 1, and the formal definitions listed early in section 6, to see the relevant regions of the  $(x, y)$  plane that we consider.

**4.1. Region B.** The following four lemmas prepare the conditions for applying Lemma 2 to points in region B. Note that the value  $q = (x - 1)(y - 1)$  exceeds 1 in this region.

LEMMA 8. *Suppose  $(x, y)$  is a point with  $x < -1$  and  $y < -1$ . Then we can use  $(x, y)$  to implement a point  $(x_1, y_1)$  with  $y_1 \in (-1, 0)$  and a point  $(x_2, y_2)$  with  $y_2 \notin [-1, 1]$ .*

*Proof.* Let  $q = (x - 1)(y - 1)$ . Let  $j$  be an odd positive integer which is sufficiently large such that  $|x|^j + 1 > q$ . Implement  $(x', y') = (x^j, q/(x^j - 1) + 1)$  from  $(x, y)$  with a  $j$ -stretch. Note that  $y' \in (0, 1)$ . Now, for a sufficiently large positive integer  $k$ , implement  $(x_1, y_1)$  using the parallel composition of  $(x, y)$  with  $k$  copies of  $(x', y')$  so that  $y_1 = y'^k y \in (-1, 0)$ . Finally, let  $(x_2, y_2) = (x, y)$ .  $\square$

LEMMA 9. *Suppose  $(x, y)$  is a point with  $x < -1$  and  $y = -1$ . Then we can use  $(x, y)$  to implement a point  $(x_1, y_1)$  with  $y_1 \in (-1, 0)$  and a point  $(x_2, y_2)$  with  $y_2 \notin [-1, 1]$ .*

*Proof.* Let  $j$  be a sufficiently large odd integer such that  $q/(|x|^j + 1) < 1$ . Implement  $(x', y')$  using a  $j$ -stretch from  $(x, y)$  so that  $y' = q/(x^j - 1) + 1 \in (0, 1)$ . Implement  $(x_1, y_1)$  by taking a parallel composition of  $(x', y')$  and  $(x, y)$  so that  $y_1 = -y'$ . Finally, implement  $(x_2, y_2)$  from  $(x, y)$  using Lemma 4.  $\square$

LEMMA 10. *Suppose  $(x, y)$  is a point with  $x < -1$  and  $-1 < y < 0$ . Then we can use  $(x, y)$  to implement a point  $(x_1, y_1)$  with  $y_1 \in (-1, 0)$  and a point  $(x_2, y_2)$  with  $y_2 \notin [-1, 1]$ .*

*Proof.* We let  $(x_1, y_1) = (x, y)$ . We implement  $(x_2, y_2)$  from  $(x, y)$  using Lemma 4.  $\square$

LEMMA 11. *Suppose  $(x, y)$  is a point with  $-1 \leq x < 0$  and  $y < -1$ . Then we can use  $(x, y)$  to implement a point  $(x_1, y_1)$  with  $y_1 \in (-1, 0)$  and a point  $(x_2, y_2)$  with  $y_2 \notin [-1, 1]$ .*

*Proof.* Implement  $(x_a, y_a)$  by a 2-thickening of  $(x, y)$ . Note that  $y_a = y^2 > 1$ , and therefore, since  $q > 0$ ,  $x_a > 1$  as well. Let  $j$  be an integer that is sufficiently large such that  $|x| \cdot x_a^j + 1 > q$ . Implement  $(x_b, y_b)$  by a series composition of  $(x, y)$  with  $j$  copies of  $(x_a, y_a)$  so that

$$y_b = q/(xx_a^j - 1) + 1 \in (0, 1).$$

Let  $k$  be a sufficiently large integer such that  $|y|y_b^k \in (0, 1)$ . Implement  $(x_1, y_1)$  by a parallel composition of  $(x, y)$  and  $k$  copies of  $(x_b, y_b)$  so that  $y_1 = yy_b^k$ . Finally, let  $(x_2, y_2) = (x, y)$ .  $\square$

**4.2. Regions G, H, and I.** We next consider the problem of implementing edge-weights starting from a point in the “vicinity of the origin,” which corresponds to points with  $|x| < 1$  and  $|y| < 1$ . In the vicinity of the origin, we have  $0 < q < 4$ . As noted in the introduction, there is a “phase transition” at  $q = 32/27$ , so we start by considering  $q > 32/27$ .

LEMMA 12. *Suppose  $(x, y)$  is a point with  $|x| < 1$  and  $|y| < 1$  and  $q = (x - 1)(y - 1) > 32/27$ . Then  $(x, y)$  can be used to implement a point  $(x', y')$  with  $y' > 1$ .*

*Proof.* We will use the “diamond operation” of Jackson and Sokal [9, section 8]. This corresponds to choosing the graph  $\Upsilon$  with vertex-set  $\{s, t, u, v\}$  and edge-set  $\{(s, u), (u, t), (s, v), (v, t)\}$ . ( $\Upsilon$  is a parallel composition of two paths from  $s$  to  $t$ , each of which is formed from the series composition of two edges.) If we start with the

weight function  $\widehat{\gamma}$  that assigns weight  $\gamma$  to every edge of  $\Upsilon$ , then it is easy to check (see [9, equation (8.1)]) that the implemented weight  $w^*$  from (4) is  $\frac{\gamma^2(\gamma^2+4\gamma+2q)}{(q+2\gamma)^2}$ . Equivalently, the point implemented from  $(x, y)$  (which we denote as  $(\diamond_{q,1}(x, y), \diamond_{q,2}(x, y))$ ) is given by

$$(\diamond_{q,1}(x, y), \diamond_{q,2}(x, y)) = \left( \frac{x + x^2 + x^3 + y}{1 + 2x + y}, \frac{(x + y)^2}{(1 + x)^2} \right).$$

The diamond operation is well defined as long as  $x \neq -1$  and  $y \neq -1 - 2x$ . Jackson and Sokal [9, Lemma 8.5(c)] prove that if you start from a point  $(x_1, y_1)$  with  $y_1 < 1$  and  $q > 32/27$  and apply a sequence of diamond operations for  $j = 1, 2, \dots$  with  $(x_{j+1}, y_{j+1}) = (\diamond_{q,1}(x_j, y_j), \diamond_{q,2}(x_j, y_j))$ , then for each  $j$ , we have  $y_{j+1} > y_j$  and there is a  $k$  such that  $y_k \geq 1$ . Their analysis allows the situation  $x_{k-1} = -1$ , so the terminating point has  $y_k = \infty$  (which would not give an implementation of a *finite*  $y_k > 1$ , which we require), and it also allows  $y_{k-1} = -1 - 2x_{k-1}$ , which gives  $y_k = 1$  (whereas we require  $y_k > 1$ ).

We start with  $(x_1, y_1) = (x, y)$  and apply the sequence of diamond operations until we reach a point  $(x_j, y_j)$  with  $y_j > 1$ . However, there are two exceptions.

First, suppose, for some  $j$ , that  $y_j = -1 - 2x_j$ . Then instead of taking  $(x_{j+1}, y_{j+1}) = (\diamond_{q,1}(x_j, y_j), \diamond_{q,2}(x_j, y_j))$  we define  $(x_{j+1}, y_{j+1})$  as follows: We let  $(x'_1, y'_1) = (x_j^2, -1)$  be the point implemented by a series composition of two copies of  $(x_j, y_j)$ . We then let  $(x'_2, y'_2) = (x_j^4, (x_j^2 - 1)/(x_j^2 + 1))$  be the point implemented by a series composition of two copies of  $(x'_1, y'_1)$ . Finally, we let  $(x_{j+1}, y_{j+1}) = (1 - x_j^{-2} + x_j^2, (1 - x_j^2)/(1 + x_j^2))$  be the point implemented by a parallel composition of  $(x'_1, y'_1)$  and  $(x'_2, y'_2)$ . Note that  $y_{j+1} - y_j = 2(x_j^3 + x_j + 1)/(x_j^2 + 1)$ . Now note that  $q = 2 - 2x_j^2$ , so, since  $q \geq 32/27$ , we have  $x_j > -0.64$ . Thus,  $y_{j+1} - y_j$  is positive, as required (the denominator is always positive, and the numerator is positive for  $x_j \geq -0.68$ ). Note that exceptional points  $(x_j, y_j)$  where  $y_j = -1 - 2x_j$  arise at most twice during the sequence of points  $(x_1, y_1), (x_2, y_2), \dots$  since the hyperbola  $(x - 1)(y - 1) = q$  intersects the line  $y = -1 - 2x$  only in at most two places. Also,  $y_{j+1} \neq 1$ , so the sequence does not terminate incorrectly at  $(x_{j+1}, y_{j+1})$ .

For the second exception, suppose that we get to a point  $(x_j, y_j)$  with  $x_j = -1$ . Then  $(x_j, y_j) = (-1, -q/2 + 1)$ . Now,  $j \neq 1$  since we start in the vicinity of the origin (so we don't have  $x_1 = -1$ ). If  $(x_j, y_j)$  was obtained as a result of the exceptional case above, then  $q < 2$  (since then  $q = 2 - 2x_{j-1}^2$  and  $x_{j-1} \neq 0$  since that would imply  $y_{j-1} = -1$ , contrary to the fact that the  $y$ 's are all strictly above  $-1$ ). Otherwise,  $(x_j, y_j)$  was obtained as the result of a diamond operation. It is not possible that  $x_{j-1} = -y_{j-1}$  since then  $q = (x_{j-1} - 1)(y_{j-1} - 1) = -x_{j-1}^2 + 1 \leq 1$ . Thus, from the definition of the diamond operation,  $y_j > 0$ . Thus, since  $y_j = -q/2 + 1$ , we also have  $q < 2$ . Let  $(x^*, y^*)$  be obtained as a parallel composition of two copies of  $(x_j, y_j)$  and then  $(x'', y'')$  as a series composition of  $(x_j, y_j)$  and  $(x^*, y^*)$ . By direct calculation from the series/parallel formulas,

$$(x^*, y^*) = \left( \frac{-q}{4 - q}, \frac{(q - 2)^2}{4} \right) \quad \text{and} \quad (x'', y'') = \left( \frac{q}{4 - q}, \frac{q^2 - 6q + 4}{2(2 - q)} \right).$$

It can be verified that  $y'' < -1$  in the range  $32/27 \leq q < 2$ . ( $y''$  is monotonically decreasing in  $q$ , and less than  $-1$  at  $q = 32/27$ .) So letting  $(x', y')$  be a parallel composition of two copies of  $(x'', y'')$  we are done since  $y' > 1$ .  $\square$

LEMMA 13. *Consider a point  $(x, y)$  such that  $y < -1 - 2x$  and  $x > -1$ . Then  $(x, y)$  can be used to implement a point  $(x', y')$  with  $y' > 1$ .*

*Proof.* Let  $(x'', y'') = (x^2, \frac{x+y}{1+x})$  be the point implemented by a 2-stretch from  $(x, y)$ . Note that  $y'' < -1$ . Now implement  $(x', y')$  by a 2-thickening of  $(x'', y'')$ .  $\square$

LEMMA 14. *Consider a point  $(x, y)$  such that  $x < -1 - 2y$  and  $y > -1$  and  $q = (x - 1)(y - 1) > 0$ . Then  $(x, y)$  can be used to implement a point  $(x', y')$  with  $y' > 1$ .*

*Proof.* Let  $(x', y') = (\frac{x+y}{1+y}, y^2)$  be the point implemented by a 2-thickening. Note that  $x' < -1$ . Then use Lemma 4.  $\square$

LEMMA 15. *Suppose that  $(x, y)$  is a point satisfying  $\max(|x|, |y|) < 1$  and  $q = (x - 1)(y - 1) > 1$ . Suppose that  $(x, y)$  also satisfies at least one of the following conditions.*

- $q > 32/27$ , or
- $y < -1 - 2x$ , or
- $x < -1 - 2y$ .

*Then  $(x, y)$  can be used to implement a point  $(x_1, y_1)$  with  $-1 < y_1 < 0$ .*

*Proof.* If  $-1 < y < 0$ , then we simply take  $(x_1, y_1) = (x, y)$ . Thus, we can assume  $0 \leq y < 1$ . This implies  $-1 < x < 0$ , and  $q > 32/27$  or  $y < -1 - 2x$ .

By Lemmas 12 and 13, we can implement a point  $(x'_1, y'_1)$  with  $y'_1 > 1$ . Since  $(x'_1 - 1)(y'_1 - 1) = q$ , we also have  $x'_1 > 1$ .

Note that the restrictions on  $x$  and  $y$  imply  $1 < q < 2$ . Choose an even integer  $j$  so that  $x^j < 1 - q/4$ . By Corollary 6 (taking  $T = (1 - q/4)/x^j$  and  $\pi = q/(8x^j)$ , say) the point  $(x'_1, y'_1)$  can be used to implement a point  $(x'', y'')$  with

$$\frac{1 - q/2}{x^j} < x'' < \frac{1}{x^j}.$$

Implement  $(x^*, y^*)$  by taking the series composition of  $(x'', y'')$  with  $j$  copies of  $(x, y)$ . Note that  $y^* = \frac{q}{x''x^{j-1}} + 1 < -1$ .

Now implement  $(x_1, y_1)$  by choosing a sufficiently large integer  $\ell$  and taking the parallel composition of  $(x^*, y^*)$  with  $\ell$  copies of  $(x, y)$  so that  $y_1 = y^*y^\ell$ .  $\square$

**4.3. Regions C and D.**

LEMMA 16. *Suppose  $(x, y)$  is a point satisfying one of the following.*

- $y > 1$  and  $x < -1$ , or
- $x > 1$  and  $y < -1$ .

*Then  $(x, y)$  can be used to implement a point  $(x_1, y_1)$  with  $y_1 \in (0, 1)$ .*

*Proof.* Note that  $q < 0$ . Choose an even number  $j$  such that  $x^j - 1 > |q|$ . Implement  $(x_1, y_1)$  by taking a  $j$ -stretch of  $(x, y)$  so that  $y_1 = q/(x^j - 1) + 1$ .  $\square$

**4.4. Region E.**

LEMMA 17. *Suppose  $(x, y)$  is a point satisfying  $x < -1$  and  $0 < y < 1$  and  $1 < (x - 1)(y - 1) < 2$ . Then  $(x, y)$  can be used to implement a point  $(x_1, y_1)$  with  $-1 < y_1 < 0$ .*

*Proof.* Let  $q = (x - 1)(y - 1)$ . Note that  $1 - q/2 > 0$  since  $q < 2$ . Let  $j$  be a sufficiently large integer such that  $0 < y^j < 1 - q/2$ . Note that  $1 - q < 0$  so that  $1 - q < y^j < 1 - q/2$ . Implement  $(x', y')$  by  $j$ -thickening from the point  $(x, y)$  so that  $x' = q/(y^j - 1) + 1$ . Note that  $-1 < x' < 0$ . Now let  $k$  be an odd integer which is sufficiently large such that  $0 < x(x')^k < 1 - q/2$  so  $1 - q < x(x')^k < 1 - q/2$ . Implement  $(x_1, y_1)$  by taking a series composition of  $(x, y)$  with  $k$  copies of  $(x', y')$  so that  $x_1 = x(x')^k$ . Then  $y_1 = q/(x(x')^k - 1) + 1$  so  $-1 < y_1 < 0$ , as required.  $\square$

LEMMA 18. *Suppose  $(x, y)$  is a point satisfying  $x < -1$  and  $0 < y < 1$ . Suppose that  $q = (x - 1)(y - 1) > 2$  is not an integer. Then  $(x, y)$  can be used to implement a point  $(x', y')$  with  $y' < 0$ .*

*Proof.* Let  $q = (x - 1)(y - 1)$ . Let us first examine what points we can implement from the point  $(x_1, y_1) = (1 - q, 0)$  and from points nearby. We will later show how to implement points near  $(x_1, y_1)$  from the given point  $(x, y)$ . Let  $n = \lfloor q \rfloor + 2$ . Note that  $n \geq 4$  and that  $n - 2 < q < n - 1$ . Let  $\Gamma_n$  be the graph obtained from the complete graph  $K_n$  on  $n$  vertices by deleting some edge  $(s, t)$ . Let  $\gamma$  be the weight function that gives every edge of  $\Gamma_n$  weight  $y_1 - 1 = -1$ . From section 2.2, the graph  $\Gamma_n$  and the weight function  $\gamma$  implement the weight

$$(12) \quad w(q, n) = \frac{qZ_{st}(\Gamma_n; q, -1)}{Z_{s|t}(\Gamma_n; q, -1)}.$$

We wish to calculate some properties of  $w(q, n)$ . Recall from the introduction that  $Z(G; q, -1)$  is equal to the chromatic polynomial  $P(G; q)$ . We will next calculate  $Z_{st}(\Gamma_n; q, -1)$  and  $Z_{s|t}(\Gamma_n; q, -1)$  as polynomials in  $q$  using known facts about the chromatic polynomial. In particular, when  $q$  is a positive integer,  $Z(G; q, -1)$  gives the number of proper  $q$ -colorings of  $G$ .

Now, let  $V$  denote the vertex set of  $K_n$ . We can expand the definition of  $Z(K_n; q, -1)$  as

$$Z(K_n; q, -1) = \sum_{A \subseteq E - (s, t)} \left( q^{\kappa(V, A \cup \{(s, t)\})} (-1)^{|A|+1} + q^{\kappa(V, A)} (-1)^{|A|} \right).$$

If a subset  $A$  connects  $s$  and  $t$ , then  $\kappa(V, A \cup \{(s, t)\}) = \kappa(V, A)$  so that the contribution from this  $A$  is zero. On the other hand, if a subset  $A$  does not connect  $s$  and  $t$ , then  $\kappa(V, A \cup \{(s, t)\}) = \kappa(V, A) - 1$ . Thus,

$$(13) \quad \begin{aligned} Z(K_n; q, -1) &= Z_{st}(\Gamma_n; q, -1)(1 - 1) + Z_{s|t}(\Gamma_n; q, -1)\left(1 - \frac{1}{q}\right) \\ &= Z_{s|t}(\Gamma_n; q, -1)\left(1 - \frac{1}{q}\right). \end{aligned}$$

Note that the factor  $(1 - \frac{1}{q})$  is positive.

Similarly,

$$Z_{st}(\Gamma_n; q, -1) = Z(\Gamma_n; q, -1) - Z_{s|t}(\Gamma_n; q, -1),$$

so we have

$$(14) \quad Z_{st}(\Gamma_n; q, -1) = Z(\Gamma_n; q, -1) - \frac{Z(K_n; q, -1)}{1 - \frac{1}{q}},$$

and

$$(15) \quad Z_{s|t}(\Gamma_n; q, -1) = \frac{Z(K_n; q, -1)}{1 - \frac{1}{q}}.$$

The properties of  $w(q, n)$  that we require will follow from (12), (14), and (15). First note that  $Z(K_n; q, -1) = \prod_{i=0}^{n-1} (q - i)$ . This is clear at positive integer  $q$  since both sides can be interpreted as the number of  $q$ -colorings of an  $n$ -clique. But we know that  $Z(K_n; q, -1)$  is a polynomial in  $q$ , so the two sides must be equal for all  $q$ . Let  $N_{q,n} = \prod_{i=0}^{n-2} (q - i)$ , so  $Z(K_n; q, -1) = N_{q,n}(q - n + 1)$ . Then  $Z(\Gamma_n; q, -1) = N_{q,n}(q - n + 2)$  since, again, both sides may be interpreted as the number of  $q$ -colorings of a certain graph, in this case  $\Gamma_n$ . (If you color the vertices of  $\Gamma_n$  in order, coloring  $s$



last, there are  $q - (n - 2)$  choices for  $s$ , rather than  $q - (n - 1)$  in  $K_n$ .) Then, from (12), (14), and (15),

$$\begin{aligned} w(q, n) &= \frac{qZ_{st}(\Gamma_n; q, -1)}{Z_{s|t}(\Gamma_n; q, -1)} = \frac{qZ(\Gamma_n; q, -1) - q\frac{Z(K_n; q, -1)}{1-1/q}}{\frac{Z(K_n; q, -1)}{1-1/q}} \\ &= \frac{(q-1)Z(\Gamma_n; q, -1) - qZ(K_n; q, -1)}{Z(K_n; q, -1)} \\ &= \frac{(q-1)(q-n+2) - q(q-n+1)}{q-n+1} \\ &= \frac{n-2}{q-n+1}, \end{aligned}$$

where we use the fact that  $q$  is not integral, so  $Z(K_n; q, -1) \neq 0$ .

Now since  $n > 2$  and  $1 < q < n - 1$ , we can see that the numerator  $n - 2$  is positive, the denominator  $q - n + 1$  is negative, and  $n - 2 > n - q - 1$ , and hence  $w(q, n) < -1$ .

We now have

$$(16) \quad \frac{qZ_{st}(\Gamma_n; q, -1)}{Z_{s|t}(\Gamma_n; q, -1)} < -1.$$

Unfortunately, we are not finished because we cannot necessarily implement the weight  $-1$  exactly from the given point  $(x, y)$ . However, by continuity, (16) implies that there is a small positive  $\varepsilon$  (depending on  $q$  and  $n$ ) such that if  $|z - Z_{st}(\Gamma_n; q, -1)| \leq \varepsilon$  and  $|z' - Z_{s|t}(\Gamma_n; q, -1)| \leq \varepsilon$ , then we have  $\frac{qz}{z'} < -1$ .

To finish, we will show that we can implement an edge-weight  $-1 + \delta$  from  $(x, y)$  so that  $|Z_{st}(\Gamma_n; q, -1 + \delta) - Z_{st}(\Gamma_n; q, -1)| \leq \varepsilon$  and  $|Z_{s|t}(\Gamma_n; q, -1 + \delta) - Z_{s|t}(\Gamma_n; q, -1)| \leq \varepsilon$ . Thus, we can implement an edge-weight less than  $-1$  by using  $\Gamma_n$  with all edge-weights equal to  $-1 + \delta$ .

We finish with the relevant technical details. First, let  $V$  be the vertex set of  $\Gamma_n$ . For any  $\delta \in (0, \varepsilon / (2^m q^n m))$ , note that

$$\begin{aligned} Z_{st}(\Gamma_n; q, -1 + \delta) - Z_{st}(\Gamma_n; q, -1) &= \sum_A q^{\kappa(V,A)} (-1)^{|A|+1} \left(1 - (1 - \delta)^{|A|}\right) \\ &\leq \sum_A q^{\kappa(V,A)} \left(1 - (1 - \delta)^{|A|}\right) \\ &\leq 2^m q^n m \delta \\ &< \varepsilon, \end{aligned}$$

where the sum is over edge-subsets  $A$  with  $s$  and  $t$  in the same component. Similarly,  $Z_{st}(\Gamma_n; q, -1) - Z_{st}(\Gamma_n; q, -1 + \delta) < \varepsilon$  and  $|Z_{s|t}(\Gamma_n; q, -1 + \delta) - Z_{s|t}(\Gamma_n; q, -1)| \leq \varepsilon$ .

It remains to show that we can implement weight  $-1 + \delta$  from the given  $(x, y)$ . Using  $(x, y)$  coordinates, the point we wish to implement is  $(x'', y'') = (1 + q/(\delta - 1), \delta)$ . This can be done using a  $k$ -thickening from  $(x, y)$ , choosing  $k$  to be sufficiently large such that  $y^k \leq \varepsilon / (2^m q^n m)$ .  $\square$

As we shall see shortly, region B consists of those points  $(x, y)$  for which  $\min(x, y) \leq -1$  and  $\max(x, y) < 0$ . Also, region G consists of points  $(x, y)$  with  $\max(|x|, |y|) < 1$  and  $q = (x - 1)(y - 1) > 32/27$ . We use these definitions in the following lemma.

LEMMA 19. *Suppose  $(x, y)$  is a point satisfying  $x < -1$  and  $0 < y < 1$ . Suppose that  $q = (x - 1)(y - 1) > 2$  is not an integer. Then  $(x, y)$  can be used to implement*

a point  $(x', y')$  apart from the special point  $(-1, -1)$  which is either in region  $B$  or in region  $G$ .

*Proof.* By Lemma 18, the point  $(x, y)$  can be used to implement a point  $(x', y')$  with  $y' < 0$ . We know that  $(x', y')$  is not the special point  $(-1, -1)$  since  $q$  is not an integer. If  $(x', y')$  is in region  $B$  or region  $G$ , then we are finished. Otherwise, the point  $(x', y')$  satisfies  $0 \leq x' < 1$  and  $y' \leq -1$ . Let  $j$  be a sufficiently large integer such that  $|y'|y^j < 1$ . Then implement the point  $(x'', y'')$  by taking the parallel composition of  $(x', y')$  with  $j$  copies of  $(x, y)$  so that  $y'' = y'y^j$ . Note that  $-1 < y'' < 0$  so that the point  $(x'', y'')$  is in region  $B$  or  $G$ , as required.  $\square$

**4.5. The flow polynomial.** In order to implement new edge-weights from region  $F$  (and also to show tractability results and NP-completeness results for region  $F$  in section 7.5), we must introduce a specialization of the Tutte polynomial called the flow polynomial.

A  $q$ -flow of an undirected graph  $G = (V, E)$  is defined as follows [16, section 2.4]. Choose an arbitrary direction for each edge. Let  $H$  be any Abelian group of order  $q$ . A  $q$ -flow is a mapping  $\psi : E \rightarrow H$  such that the flow into each vertex is equal to the flow out (doing arithmetic in  $H$ ).

Consider the following polynomial, where the sum is over  $q$ -flows of  $G$  (see [16, equation (2.21)]):

$$F(G; q, u) = \sum_{\psi} \prod_{e \in E} (1 + u\delta(\psi(e), 0)),$$

where  $\delta$  is the Kronecker delta function defined by  $\delta(a, b) = 1$  if  $a = b$  and  $\delta(a, b) = 0$  otherwise. It is a nontrivial fact that  $F(G; q, u)$  depends only on  $q$ , the size of  $H$ , and not on  $H$  itself. This polynomial is related to the Tutte polynomial via the following identity [16, equation (2.22)].

**FACT 20.** *If  $q$  is a positive integer, then  $F(G; q, q/\gamma) = q^{-|V|}(\frac{q}{\gamma})^{|E|} Z(G; q, \gamma)$ .*

The flow polynomial of  $G$ , which we write as  $F(G; q)$ , is given by  $F(G; q, -1)$ . A  $q$ -flow  $\psi$  of a graph  $G = (V, E)$  is said to be nowhere-zero if, for every  $e \in E$ ,  $\psi(e) \neq 0$ . From Fact 20 it is easy to see that if  $q$  is a positive integer, then  $F(G; q) = q^{-|V|}(-1)^{|E|} Z(G; q, -q)$  is the number of nowhere-zero  $q$ -flows of  $G$ .

**4.6. Region F.**

**LEMMA 21.** *Suppose  $(x, y)$  is a point satisfying  $0 < x < 1$  and  $y < -1$  and  $0 < (x - 1)(y - 1) < 1$ . Then  $(x, y)$  can be used to implement a point  $(x_1, y_1)$  with  $0 < y_1 < 1$ .*

*Proof.* Let  $j$  be a sufficiently large positive integer such that  $x^j < 1 - q$ . Implement  $(x_1, y_1)$  by a  $j$ -stretch of  $(x, y)$  so that  $y_1 = q/(x^j - 1) + 1$ .  $\square$

**LEMMA 22.** *Suppose  $(x, y)$  is a point satisfying  $0 < x < 1$  and  $y < -1$  and  $1 < (x - 1)(y - 1) < 2$ . Then  $(x, y)$  can be used to implement a point  $(x_1, y_1)$  with  $-1 < y_1 < 0$ .*

*Proof.* Let  $q = (x - 1)(y - 1)$ . Note that  $1 - q/2 > 0$  since  $q < 2$ . Let  $j$  be a sufficiently large integer such that  $0 < x^j < 1 - q/2$ . Note that  $1 - q < 0$  so that  $1 - q < x^j < 1 - q/2$ . Implement  $(x_1, y_1)$  by a  $j$ -stretch from the point  $(x, y)$  so that  $y_1 = q/(x^j - 1) + 1$ . Note that  $-1 < y_1 < 0$ .  $\square$

**LEMMA 23.** *Suppose  $(x, y)$  is a point satisfying  $0 < x < 1$  and  $y < -1$  for which  $q = (x - 1)(y - 1)$  is not an integer. Suppose  $2 < q < 4$ . Then  $(x, y)$  can be used to implement a point  $(x', y')$  with  $x' < 0$ .*

*Proof.* Let  $q = (x - 1)(y - 1)$ . As in the proof of Lemma 18, we start by examining what points we can implement from the point  $(x', y') = (0, 1 - q)$  and from points nearby.

Suppose that  $G$  is a graph which contains the edge  $(s, t)$ . Let  $\Gamma = G - (s, t)$ . Following the approach of Lemma 18, let

$$(17) \quad w(q) = \frac{qZ_{st}(\Gamma; q, -q)}{Z_{s|t}(\Gamma; q, -q)},$$

which is the weight implemented by  $\Gamma$  with edge-weight  $-q$ .

Then, using reasoning similar to the derivation of (13),

$$(18) \quad \begin{aligned} Z(G; q, -q) &= Z_{st}(\Gamma; q, -q)(1 - q) + \frac{1}{q}Z_{s|t}(\Gamma; q, -q)(q - q) \\ &= Z_{st}(\Gamma; q, -q)(1 - q). \end{aligned}$$

Also,

$$\begin{aligned} Z_{s|t}(\Gamma; q, -q) &= Z(\Gamma; q, -q) - Z_{st}(\Gamma; q, -q) \\ &= Z(\Gamma; q, -q) + Z(G; q, -q)/(q - 1). \end{aligned}$$

Thus, we can use Fact 20 to see that

$$\begin{aligned} w(q) &= \frac{qZ_{st}(\Gamma; q, -q)}{Z_{s|t}(\Gamma; q, -q)} \\ &= -q \left( \frac{Z(G; q, -q)/(q - 1)}{Z(\Gamma; q, -q) + Z(G; q, -q)/(q - 1)} \right) \\ &= -q \left( \frac{Z(G; q, -q)}{(q - 1)Z(\Gamma; q, -q) + Z(G; q, -q)} \right) \\ &= -q \left( \frac{F(G; q)}{F(G; q) - (q - 1)F(\Gamma; q)} \right). \end{aligned}$$

First, suppose  $2 < q < 3$ . Following the reasoning in Lemma 18, we will show below that, for a suitable  $G$ ,  $F(G; q) > 0$  and  $F(\Gamma; q) < 0$ . Together, these imply that the denominator  $F(G; q) - (q - 1)F(\Gamma; q)$  is positive and also that it is larger than the numerator  $F(G; q)$ . Thus,  $w(q) < 0$  and  $w(q) > -q$ . (It is our goal to implement a  $\gamma'$  in the range  $-q < \gamma' < 0$  since, for this  $\gamma'$ ,  $q/\gamma' + 1 < 0$ , so the corresponding  $x$ -coordinate is less than 0.)

By continuity, there is a positive  $\varepsilon$  (which depends upon  $q$  and  $G$ ) such that if  $|z - Z_{st}(\Gamma; q, -q)| \leq \varepsilon$  and  $|z' - Z_{s|t}(\Gamma; q, -q)| \leq \varepsilon$ , then  $-q < \frac{qz}{z'} < 0$ . As in the proof of Lemma 18, we can show that, for a sufficiently small  $\delta \in (0, 1)$ ,  $|Z_{st}(\Gamma; q, -q - \delta) - Z_{st}(\Gamma; q, -q)| \leq \varepsilon$  and  $|Z_{s|t}(\Gamma; q, -q - \delta) - Z_{s|t}(\Gamma; q, -q)| \leq \varepsilon$ . Then we finish by implementing the weight  $-q - \delta$  from the given  $(x, y)$  using a large stretch so that

$$(x'', y'') = (x^k, q/(x^k - 1) + 1) = (\delta/(q + \delta), 1 - q - \delta).$$

For  $3 < q < 4$  the proof will be similar except that we will establish  $F(G; q) < 0$  and  $F(\Gamma; q) > 0$  so that the denominator of the final expression for  $w(q)$  is negative and is larger in absolute value than the numerator.

To complete the proof, we must establish that  $F(G; q)$  and  $F(\Gamma; q)$  have different signs. Let  $G$  be the Petersen graph. Since  $G$  is edge-transitive, the edge  $(s, t)$  may be chosen arbitrarily. It can be verified, e.g., using Maple, that

$$F(G, q) = q^6 - 15q^5 + 95q^4 - 325q^3 + 624q^2 - 620q + 240$$

and

$$F(\Gamma, q) = q^5 - 12q^4 + 58q^3 - 138q^2 + 157q - 66.$$

Now we note that  $F(G; q)$  has four real zeros at  $q = 1, 2, 3, 4$  and two complex zeros, and  $F(G; 2.5) > 0$ . Also,  $F(\Gamma; q)$  has three real zeros at  $q = 1, 2, 3$  and two complex zeros, and  $F(\Gamma; 2.5) < 0$ .  $\square$

*Remark 24.* The construction used in the proof of Lemma 23 breaks down for  $q > 4$  because  $F(G; q)$  and  $F(\Gamma; q)$  have the same signs. It is conceivable that the lemma could be proved for noninteger  $q$  in the range  $4 < q < 6$  by using a generalized Petersen graph rather than a Petersen graph in the construction. Indeed, Jacobsen and Salas have shown [10] that there are generalized Petersen graphs whose flow polynomials have roots between 5 and 6. Given the current state of knowledge, we are pessimistic about the prospects of proving the lemma for all  $q > 4$ . Currently, it is an open question [10] whether there is a uniform upper bound  $Q$  for real zeros of arbitrary bridgeless graphs (so that every bridgeless graph  $G$  would have  $F(G; q) > 0$  for all  $q > Q$ ). If so, then computing the sign of the flow polynomial will be trivial for  $q > Q$ , so computing the sign of the Tutte polynomial will also be trivial for  $y < -Q + 1$  along the  $y$ -axis. If not, then it seems likely that the hardness construction can be extended. (Thus, it doesn't seem to be possible to resolve all of the unresolved points in region F without solving the open problem about flow polynomials.)

**5. The main theorem.** This section is devoted to a formal statement of our results concerning the complexity of  $\text{SIGNTUTTE}(q, \gamma)$  and  $\text{TUTTE}(q, \gamma)$ . In what follows,  $\#\text{P}$ -hardness is defined with respect to polynomial-time Turing reductions. NP-hardness is defined by a many-one reduction from an NP-complete decision problem, whose instance is a “yes instance” if the corresponding instance of  $\text{SIGNTUTTE}(q, \gamma)$  has a positive sign and a “no instance” otherwise. In Figure 1, which is a pictorial representation of our theorem,  $\#\text{P}$ -hard points are depicted in red, NP-complete points are depicted in blue, and FP points are depicted in green. Points depicted in white are unresolved.

Theorem 1 gives a complete description of what we know about the complexity of  $\text{SIGNTUTTE}(q, \gamma)$  and  $\text{TUTTE}(q, \gamma)$ . For consistency with existing work by a variety of authors, we classify the complexity in terms of the  $(x, y)$  parameterization. Throughout, we maintain the connection between the parameterizations  $(q, \gamma)$  and  $(x, y)$  so that always  $\gamma = y - 1$  and  $q = (x - 1)(y - 1)$ .

**THEOREM 1.**

*The points in the  $(x, y)$  plane are classified as follows:*

- *Region A: Points  $(x, y)$  with  $x \geq 0$  and  $y \geq 0$ . In this region,  $\text{SIGNTUTTE}(q, \gamma) \in \text{FP}$  and  $\text{TUTTE}(q, \gamma) \in \#\text{P}_{\mathbb{Q}}$ . When  $q = 0$ , we have the stronger  $\text{TUTTE}(q, \gamma) \in \text{FP}$ .*
- *Region B: Points  $(x, y)$  with  $\min(x, y) \leq -1$  and  $\max(x, y) < 0$ . In this region,  $\text{SIGNTUTTE}(q, \gamma)$  is  $\#\text{P}$ -hard, except at the point  $(x, y) = (-1, -1)$ , where  $\text{TUTTE}(q, \gamma) \in \text{FP}$ .*
- *Region C: Points  $(x, y)$  with  $x < -1$  and  $y > 1$ . In this region,  $\text{SIGNTUTTE}(q, \gamma)$  is  $\#\text{P}$ -hard.*

- *Region D: Points  $(x, y)$  with  $x > 1$  and  $y < -1$ . In this region,  $\text{SIGNTUTTE}(q, \gamma)$  is  $\#P$ -hard.*
- *Region E: Points  $(x, y)$  with  $x \leq -1$  and  $0 < y \leq 1$ . Note that these points have  $q \geq 0$ . When  $q = 0$ , we have  $\text{TUTTE}(q, \gamma) \in \text{FP}$ . When  $q \neq 0$  is an integer,  $\text{SIGNTUTTE}(q, \gamma) \in \text{FP}$  and  $\text{TUTTE}(q, \gamma) \in \#P_{\mathbb{Q}}$ . When  $q$  is a noninteger,  $\text{SIGNTUTTE}(q, \gamma)$  is  $\#P$ -hard, apart from the line segment with  $x = -1$  and  $11/27 \leq y < 1$ , which is unresolved.*
- *Region F: Points  $(x, y)$  with  $0 < x \leq 1$  and  $y \leq -1$ . Once again, these points have  $q \geq 0$ . When  $q = 0$ , we have  $\text{TUTTE}(q, \gamma) \in \text{FP}$ . When  $q \neq 0$  is an integer,  $\text{SIGNTUTTE}(q, \gamma) \in \text{FP}$  and  $\text{TUTTE}(q, \gamma) \in \#P_{\mathbb{Q}}$ . When  $q$  is a noninteger satisfying  $0 < q < 4$ ,  $\text{SIGNTUTTE}(q, \gamma)$  is  $\#P$ -hard, apart from the line segment with  $y = -1$  and  $11/27 \leq x < 1$ , which is unresolved. Points with noninteger  $q > 4$  are also unresolved.*
- *The boundary between regions B and E: Points  $(x, y)$  with  $x \leq -1$  and  $y = 0$ . Note that  $q \geq 2$ . When  $q$  is not an integer, i.e.,  $x$  is not an integer,  $\text{SIGNTUTTE}(q, \gamma)$  is  $\#P$ -hard. At  $(x, y) = (-1, 0)$  we have  $\text{TUTTE}(q, \gamma) \in \text{FP}$ , while at the rest of the points  $(x, 0)$ , where  $x$  is a negative integer,  $\text{SIGNTUTTE}(q, \gamma)$  is NP-complete and  $\text{TUTTE}(q, \gamma) \in \#P_{\mathbb{Q}}$ .*
- *The boundary between regions B and F: Points  $(x, y)$  with  $x = 0$  and  $y \leq -1$ . Note that  $q \geq 2$ . When  $2 < q < 4$  is not an integer, i.e.,  $-3 < y < -1$  is not an integer,  $\text{SIGNTUTTE}(q, \gamma)$  is  $\#P$ -hard. When  $q > 4$  is not an integer, i.e.,  $y < -3$  is not an integer, the complexity of  $\text{SIGNTUTTE}(q, \gamma)$  and  $\text{TUTTE}(q, \gamma)$  is unresolved. At the points  $(0, -2)$  and  $(0, -3)$ ,  $\text{SIGNTUTTE}(q, \gamma)$  is NP-complete and  $\text{TUTTE}(q, \gamma) \in \#P_{\mathbb{Q}}$ . The complexity at the point  $(0, -4)$  is unresolved. At the rest of the points  $(0, y)$ , where  $y \leq -5$  is a negative integer,  $\text{SIGNTUTTE}(q, \gamma) \in \text{FP}$  and  $\text{TUTTE}(q, \gamma) \in \#P_{\mathbb{Q}}$ .*
- *Region G: Points  $(x, y)$  with  $\max(|x|, |y|) < 1$  and  $q > 32/27$ . In this region,  $\text{SIGNTUTTE}(q, \gamma)$  is  $\#P$ -hard.*
- *Region H: Points  $(x, y)$  with  $\max(|x|, |y|) < 1$ ,  $q \leq 32/27$ , and  $x < -2y - 1$ . In this region,  $\text{SIGNTUTTE}(q, \gamma)$  is  $\#P$ -hard, apart from points with  $q = 1$ , where  $\text{TUTTE}(q, \gamma) \in \text{FP}$ .*
- *Region I: Points  $(x, y)$  with  $\max(|x|, |y|) < 1$ ,  $q \leq 32/27$ , and  $y < -2x - 1$ . In this region,  $\text{SIGNTUTTE}(q, \gamma)$  is  $\#P$ -hard, apart from points with  $q = 1$ , where  $\text{TUTTE}(q, \gamma) \in \text{FP}$ .*
- *Region J: Points  $(x, y)$  with  $-1 \leq x < 0$  and  $y \geq 1$ . In this region,  $\text{SIGNTUTTE}(q, \gamma) \in \text{FP}$  and  $\text{TUTTE}(q, \gamma) \in \#P_{\mathbb{Q}}$ .*
- *Region K: Points  $(x, y)$  with  $x \geq 1$  and  $-1 \leq y < 0$ . In this region,  $\text{SIGNTUTTE}(q, \gamma) \in \text{FP}$  and  $\text{TUTTE}(q, \gamma) \in \#P_{\mathbb{Q}}$ .*
- *Region L: Points  $(x, y)$  with  $0 < x < 1$  and  $-x < y < 0$ . In this region,  $\text{SIGNTUTTE}(q, \gamma) \in \text{FP}$  and  $\text{TUTTE}(q, \gamma) \in \#P_{\mathbb{Q}}$ .*
- *Region M: Points  $(x, y)$  with  $0 < y < 1$  and  $-y < x < 0$ . In this region,  $\text{SIGNTUTTE}(q, \gamma) \in \text{FP}$  and  $\text{TUTTE}(q, \gamma) \in \#P_{\mathbb{Q}}$ .*
- *The rest: There are some remaining unresolved points. These points (simultaneously) satisfy all of the following inequalities:  $\max(|x|, |y|) < 1$ ,  $y < -x$ ,  $q \leq 32/27$ ,  $y \geq -2x - 1$ ,  $x \geq -2y - 1$ , and  $q \neq 1$ .*

*Proof.* The proof follows from the following lemmas, which appear in the rest of the paper.

- Region A: Lemma 37.
- Region B: Corollaries 25–27 and section 7.2.

- Region C: Corollary 30.
- Region D: Corollary 30.
- Region E: Corollaries 31 and 33, and Observation 39.
- Region F: Corollaries 34 and 36, and Observation 41.
- The boundary between regions B and E: Corollary 32 and Observation 40.
- The boundary between regions B and F: Corollary 35 and Observation 42.
- Region G: Corollary 28.
- Region H: Corollaries 28 and 29, and section 7.7.
- Region I: Corollaries 28 and 29, and section 7.7.
- Region J: Corollary 50.
- Region K: Corollary 48.
- Region L: Corollary 53.
- Region M: Corollary 55.  $\square$

All  $\#P$ -hardness results are proved in section 6. Tractability results and NP-completeness results are proved in section 7, where we also show that  $\text{TUTTE}(q, \gamma)$  is in  $\#P_{\mathbb{Q}}$  for these points.

**6.  $\#P$ -hardness.** In this section and the next we use the following shorthand. We say that a point  $(x, y)$  is  $\#P$ -hard, NP-complete, or in FP if, for  $\gamma = y - 1$  and  $q = (x - 1)(y - 1)$ , the corresponding problem  $\text{SIGNTUTTE}(q, \gamma)$  is  $\#P$ -hard, NP-complete, or in FP, respectively.

### 6.1. Points in region B.

**COROLLARY 25.** *Suppose that  $(x, y)$  is a point such that  $\min(x, y) < -1$  and  $\max(x, y) < 0$ . Then  $(x, y)$  is  $\#P$ -hard.*

*Proof.* Note that  $q = (x - 1)(y - 1) > 1$ . The corollary follows from Lemmas 2, 8, 9, 10, and 11.  $\square$

**COROLLARY 26.** *Suppose that  $(x, y)$  is a point satisfying  $x = -1$  and  $-1 < y < 0$ . Then  $(x, y)$  is  $\#P$ -hard.*

*Proof.* A 3-thickening from  $(x, y)$  implements the point

$$(x', y') = \left( \frac{-1 + y + y^2}{1 + y + y^2}, y^3 \right).$$

Now  $x' < -1$  and  $-1 < y' < 0$ , so  $(x', y')$  was already shown to be  $\#P$ -hard by Corollary 25.  $\square$

Similarly, we have the following.

**COROLLARY 27.** *Suppose that  $(x, y)$  is a point satisfying  $y = -1$  and  $-1 < x < 0$ . Then  $(x, y)$  is  $\#P$ -hard.*

### 6.2. Points in regions G, H, and I.

**COROLLARY 28.** *Suppose that  $(x, y)$  is a point satisfying  $\max(|x|, |y|) < 1$  and  $q = (x - 1)(y - 1) > 1$ . Suppose that  $(x, y)$  also satisfies at least one of the following conditions:*

- $q > 32/27$ , or
- $y < -1 - 2x$ , or
- $x < -1 - 2y$ .

*Then  $(x, y)$  is  $\#P$ -hard.*

*Proof.* The corollary follows from Lemmas 2, 12, 13, 14, and 15.  $\square$

**COROLLARY 29.** *Suppose that  $(x, y)$  is a point satisfying  $\max(|x|, |y|) < 1$  and  $q = (x - 1)(y - 1) < 1$ . Suppose that  $(x, y)$  also satisfies at least one of the following conditions:*

- $y < -1 - 2x$ , or
- $x < -1 - 2y$ .

Then  $(x, y)$  is #P-hard.

*Proof.* Note that  $q > 0$ . The corollary follows from Lemmas 3, 13, and 14. We implement the point  $(x_1, y_1)$  required by Lemma 3 by taking a 2-thickening of  $(x, y)$  so that  $y_1 = y^2 \in (0, 1)$ .  $\square$

**6.3. Points in regions C and D.**

COROLLARY 30. *Suppose  $(x, y)$  is a point satisfying one of the following:*

- $y > 1$  and  $x < -1$ , or
- $x > 1$  and  $y < -1$ .

Then  $(x, y)$  is #P-hard.

*Proof.* Note that  $q < 0$ . The corollary follows from Lemmas 3 and 16. The point  $(x_2, y_2)$  required by Lemma 3 is just  $(x, y)$  itself.  $\square$

**6.4. Points with noninteger  $q$  in region E and on the boundary between regions B and E.** Note that  $q$  is an integer when  $(x, y) = (-1, 0)$  and when  $y = 1$ . We will discuss these points in section 7.

COROLLARY 31. *Suppose  $(x, y)$  is a point satisfying  $x < -1$  and  $0 < y < 1$ . Suppose that  $q = (x - 1)(y - 1) > 0$  is not an integer. Then  $(x, y)$  is #P-hard.*

*Proof.* If  $0 < q < 1$ , then the result follows from Lemmas 3 and 4. If  $1 < q < 2$ , then the result follows from Lemmas 2, 17, and 4. So suppose  $q > 2$ . By Lemma 19, the point  $(x, y)$  can be used to implement a point other than the special point  $(-1, -1)$  that is in region B or G. All of these points are known to be #P-hard by Corollaries 25, 26, 27, and 28.  $\square$

COROLLARY 32. *Consider a point  $(x, y)$  satisfying  $x < -1$  and  $y = 0$ . Suppose that  $q = (x - 1)(y - 1)$  is not an integer. Then  $(x, y)$  is #P-hard.*

*Proof.* Note that  $q = (x - 1)(0 - 1) = 1 - x > 0$ . Let

$$(x', y') = \left( x^3, \frac{x + x^2}{1 + x + x^2} \right)$$

be the point implemented by a 3-stretch from  $(x, y)$ . Note that  $x + x^2 > 0$  so that  $0 < y' < 1$ . Also,  $x' < -1$ . Thus,  $(x', y')$  is #P-hard by Corollary 31.  $\square$

COROLLARY 33. *Suppose that  $(x, y)$  is a point satisfying  $x = -1$  and  $0 < y < 11/27$ . Then  $(x, y)$  is #P-hard.*

*Proof.* Note that  $q = (x - 1)(y - 1) > 32/27$ . Implement  $(x', y')$  by a 2-thickening from  $(x, y)$  so that  $(x', y') = (\frac{-1+y}{1+y}, y^2)$ . Note that  $-1 < x' < 0$  and  $0 < y' < 1$  so that  $(x', y')$  is in region G and is #P-hard by Corollary 28.  $\square$

**6.5. Points with noninteger  $q$  in region F and on the boundary between regions B and F.** Note that  $q$  is an integer when  $(x, y) = (0, -1)$  and when  $x = 1$ . We will discuss these points in section 7.

COROLLARY 34. *Suppose  $(x, y)$  is a point satisfying  $0 < x < 1$  and  $y < -1$ . Suppose that  $q = (x - 1)(y - 1)$  is not an integer. Suppose  $0 < q < 4$ . Then  $(x, y)$  is #P-hard.*

*Proof.* If  $0 < q < 1$ , then the result follows from Lemmas 3 and 21. If  $1 < q < 2$ , then the result follows from Lemmas 2 and 22. So suppose  $2 < q < 4$ . By Lemma 23,  $(x, y)$  can be used to implement a point  $(x', y')$  with  $x' < 0$ . The point  $(x', y')$  is in one of the regions E, B, or G. It is not the special point  $(-1, -1)$  from region B since  $q$  is not an integer. It is not the unresolved line segment from region E since  $q > 2$ . Thus,  $(x', y')$  is #P-hard by Corollaries 25, 26, 27, 28, 31, 32, and 33.  $\square$

As we explained in Remark 24, it seems possible that Corollary 34 could be extended, say up to  $q = 6$ , by doing more complicated calculations in the proof of Lemma 23, analyzing the flow polynomial of generalized Petersen graphs, rather than just the flow polynomial of the Petersen graph. However, our lack of knowledge about the zeros of the flow polynomial seems to be a barrier to extending the lemma to cover all  $q$ .

**COROLLARY 35.** *Consider a point  $(x, y)$  satisfying  $x = 0$  and  $y < -1$ . Suppose that  $q = (x - 1)(y - 1)$  is not an integer and that  $q < 4$ . Then  $(x, y)$  is #P-hard.*

*Proof.* Note that  $2 < q < 4$ . Let

$$(x', y') = \left( \frac{y + y^2}{1 + y + y^2}, y^3 \right)$$

be the point implemented by a 3-thickening from  $(x, y)$ . Note that  $y + y^2 > 0$  so that  $0 < x' < 1$ . Also,  $y' < -1$ . Thus,  $(x', y')$  is #P-hard by Corollary 34.  $\square$

**COROLLARY 36.** *Suppose that  $(x, y)$  is a point satisfying  $0 < x < 11/27$  and  $y = -1$ . Then  $(x, y)$  is #P-hard.*

*Proof.* Note that  $q = (x - 1)(y - 1) > 32/27$ . Implement  $(x', y')$  by a 2-stretch from  $(x, y)$  so that  $(x', y') = (x^2, \frac{-1+x}{1+x})$ . Note that  $0 < x' < 1$  and  $-1 < y' < 0$  so that  $(x', y')$  is in region G and is #P-hard by Corollary 28.  $\square$

**7. Tractability results and NP-completeness results.** As we mentioned earlier, we say that a point  $(x, y)$  is in FP if  $\text{SIGN-TUTTE}(q, \gamma)$  can be solved in polynomial time, where  $q = (x - 1)(y - 1)$  and  $\gamma = y - 1$ . These points are depicted in green in Figure 1. For each point in FP, and also for the points that are NP-complete (depicted in blue), we show that  $\text{TUTTE}(q, \gamma)$  is in #P $_{\mathbb{Q}}$ . Thus,  $\text{TUTTE}(q, \gamma)$  can be efficiently approximated using an NP oracle.

**7.1. Points in region A.** The following lemma is implicit in the work of Tutte [18, 19]. The connection is explained explicitly in [3, section 2.3].

**LEMMA 37.** *Suppose  $(x, y)$  is a point satisfying  $\min(x, y) \geq 0$ . Let  $q = (x - 1)(y - 1)$  and  $\gamma = y - 1$ . Then for every graph  $G$ ,  $Z(G; q, \gamma) > 0$  so that  $\text{SIGN-TUTTE}(q, \gamma)$  is in FP. Furthermore,  $\text{TUTTE}(q, \gamma)$  is in #P $_{\mathbb{Q}}$ . In the case  $q = 0$ , we have  $Z(G; q, \gamma) = 0$  and  $\text{TUTTE}(q, \gamma)$  is trivially in FP.*

**7.2. Points in region B.** It is known [11] that  $\text{TUTTE}(4, -2)$  is in FP (so it is certainly in #P $_{\mathbb{Q}}$ ). Thus, the point  $(x, y) = (-1, -1)$  is in FP.

**7.3. Points with integer  $q$  in region E.** The points in region E have  $x \leq -1$  and  $0 < y \leq 1$ . Thus, they have  $q = (x - 1)(y - 1) \geq 0$  and  $\gamma = y - 1$ .

First, if  $y = 1$ , then  $q = 0$ . We will handle this easy case below. So, suppose  $y < 1$  so that  $-1 < \gamma < 0$ . Note that  $q > 0$  so that since we restrict attention to integer  $q$ ,  $q \geq 1$ . Consider the Potts-model partition function for  $G$  (see [16, equation (2.7)]),

$$Z_{\text{Potts}}(G; q, \gamma) = \sum_{\sigma: V \rightarrow [q]} \prod_{e=(u,v) \in E} (1 + \gamma \delta(\sigma(u), \sigma(v))),$$

where  $\delta$  is the Kronecker delta function defined by  $\delta(a, b) = 1$  if  $a = b$  and  $\delta(a, b) = 0$  otherwise. The following well-known fact is due to Fortuin and Kasteleyn (see [16, Theorem 2.3]).

**FACT 38.** *If  $q \geq 1$  is an integer, then  $Z_{\text{Potts}}(G; q, \gamma) = Z(G; q, \gamma)$ .*



The following observation now follows from Fact 38.

**OBSERVATION 39.** *Let  $(x, y)$  be a point with  $x \leq -1$  and  $0 < y \leq 1$ . Let  $q = (x - 1)(y - 1)$  and  $\gamma = y - 1$ . Suppose that  $q$  is an integer.*

- *If  $y = 1$ , then  $Z(G; q, \gamma) = 0$  so that  $\text{SIGNTUTTE}(q, \gamma)$  and  $\text{TUTTE}(q, \gamma)$  are both in FP.<sup>5</sup>*
- *Otherwise,  $Z(G; q, \gamma) > 0$  so that  $\text{SIGNTUTTE}(q, \gamma)$  is in FP. Also,  $\text{TUTTE}(q, \gamma)$  is in  $\#P_{\mathbb{Q}}$ .*

Note that Observation 39 disproves [9, Conjecture 10.3(e)]. Jackson and Sokal conjectured that for every fixed  $x \leq -1$  and  $0 < y < 1$  satisfying  $q = (x - 1)(y - 1) > 32/27$ , for all sufficiently large  $n$  and  $m$ , there are 2-connected graphs with  $n$  vertices and  $m$  edges that make  $Z(G; q, y - 1)$  nonzero with either sign, but this is clearly false when  $q$  is an integer.

**7.4. Points with integer  $q$  on the boundary between regions B and E.**

These points have  $x \leq -1$  and  $y = 0$ . Since  $q = (x - 1)(y - 1) = 1 - x$  is an integer, we conclude that  $x$  is an integer. From Fact 38,  $Z(G; q, -1)$  is the number of proper  $q$ -colorings of  $G$ .

**OBSERVATION 40.** *The point  $(-1, 0)$  is in FP since  $Z(G; 2, -1)$  is equal to the number of 2-colorings of  $G$ , and this can be computed in polynomial time. For integer  $x < -1$ , the point  $(x, 0)$  is NP-complete.  $Z(G; 1 - x, -1)$  is positive if  $G$  has a proper  $(1 - x)$ -coloring and is 0 otherwise.  $\text{TUTTE}(1 - x, -1)$  is in  $\#P_{\mathbb{Q}}$ , so it is in  $\#P_{\mathbb{Q}}$ .*

**7.5. Points with integer  $q$  in region F.** The points in region F have  $0 < x \leq 1$  and  $y \leq -1$ . They have  $q = (x - 1)(y - 1) \geq 0$  and  $\gamma = y - 1$ .

First, if  $x = 1$ , then  $q = 0$ . We will handle this easy case below. So, let us restrict our attention to the range  $0 \leq x < 1$ . This corresponds to  $\gamma \leq -2$  and  $q/\gamma \in (-1, 0)$ . Recall the definition of the flow polynomial from section 4.5. Using Fact 20 we obtain the following observation.

**OBSERVATION 41.** *Let  $(x, y)$  be a point with  $0 < x \leq 1$  and  $y \leq -1$ . Let  $q = (x - 1)(y - 1)$  and  $\gamma = y - 1$ . Suppose that  $q$  is an integer.*

- *If  $x = 1$ , then  $Z(G; q, \gamma) = 0$  so that  $\text{SIGNTUTTE}(q, \gamma)$  and  $\text{TUTTE}(q, \gamma)$  are both in FP.*
- *Otherwise,  $q^{-|V|} \left(\frac{q}{\gamma}\right)^{|E|} Z(G; q, \gamma) > 0$  so that  $\text{SIGNTUTTE}(q, \gamma)$  is in FP. Also,  $\text{TUTTE}(q, \gamma)$  is in  $\#P_{\mathbb{Q}}$ .*

Like Observation 39, Observation 41 provides counterexamples to [9, Conjecture 10.3(3)]. They conjectured that for every fixed  $0 < x \leq 1$  and  $y \leq -1$  satisfying  $q = (x - 1)(y - 1) > 32/27$ , for all sufficiently large  $n$  and  $m$  (including even  $m$ ), there are 2-connected graphs with  $n$  vertices and  $m$  edges that make  $Z(G; q, y - 1)$  nonzero with either sign, but this is clearly false when  $q$  is an integer.

**7.6. Points with integer  $q$  on the boundary between regions B and F.**

These points have  $x = 0$  and  $y \leq -1$ . Since  $q = (x - 1)(y - 1) = 1 - y$  is an integer, we conclude that  $y$  is an integer.

Recall from section 4.5 that if  $q$  is a positive integer, then  $q^{-|V|} (-1)^{|E|} Z(G; q, -q)$  is the number of nowhere-zero  $q$ -flows of  $G$ . A graph has a nowhere-zero 2-flow iff it is Eulerian [1, Theorem 11.21]. Thus, this can be tested in polynomial time. On the other hand, it is NP-complete to test whether a graph has a nowhere-zero 3-flow,

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<sup>5</sup>The case  $y = 1$  is trivial for us because we are using the  $(q, \gamma)$  parameterization, where a single point  $(q, \gamma) = (0, 0)$  corresponds to the line  $(x, 1)$  in the  $(x, y)$  parameterization. This issue is touched on in the introduction.

even if the graph is planar. To see this, note that a planar graph has a nowhere-zero 3-flow iff its dual has a proper 3-coloring, and it is NP-complete to determine whether a planar graph is 3-colorable. It is also NP-complete to test whether a graph has a nowhere-zero 4-flow, even if the graph is cubic. To see this, consider a cubic graph  $G$  and let  $H$  be the Abelian group  $Z_2 \times Z_2$ . A 4-flow maps the edges of  $G$  to  $(0, 1)$ ,  $(1, 0)$ , and  $(1, 1)$ . To be nowhere-zero, it maps one of each to the edges adjacent to each vertex. So the number of nowhere-zero 4-flows is the same as the number of proper 3-edge-colorings of  $G$ . But it is NP-complete to decide whether a graph has such an edge coloring [7]. A “bridge” (or cut-edge) of a graph is an edge whose deletion increases the number of connected components. It is known [1, Corollary 11.26] that no graph with a bridge has a nowhere-zero  $q$ -flow for any integer  $q \geq 2$ . However, Seymour has shown [1, Theorem 11.32] that every bridgeless graph has a nowhere-zero 6-flow. Thus, determining whether a graph has a nowhere-zero  $q$ -flow is in FP for  $q \geq 6$ . We do not know the complexity of determining whether a graph has a nowhere-zero 5-flow. Indeed, it is currently an open question whether there exists a bridgeless graph without a nowhere-zero 5-flow.

**OBSERVATION 42.** *The point  $(0, -1)$  is in FP since  $Z(G; 2, -2)$  is computable from the number of nowhere-zero 2-flows of  $G$ , and this can be computed in polynomial time. The point  $(0, -2)$  is NP-complete since  $Z(G; 3, -3)$  allows one to determine the number of nowhere-zero 3-flows of  $G$ . The point  $(0, -3)$  is NP-complete since  $Z(G; 4, -4)$  allows one to determine the number of nowhere-zero 4-flows of  $G$ . For integer  $y \leq -5$ , the point  $(0, y)$  is in FP since  $Z(G; 1 - y, y - 1)$  is computable from the number of nowhere-zero  $(1 - y)$ -flows of  $G$ . This quantity is positive iff  $G$  has no bridge.  $\text{TUTTE}(1 - x, -1)$  is in  $\#\text{P}$ , so it is in  $\#\text{P}_{\mathbb{Q}}$ .*

**7.7. Points in regions H and I.** It is known [11] that points  $(x, y)$  with  $(x - 1)(y - 1) = 1$  are in FP since  $\text{TUTTE}(1, \gamma)$  is in FP so that  $\text{SIGNTUTTE}(1, \gamma)$  is also in FP.

**7.8. Matroids.** The definitions from section 2 can be generalized from graphs to matroids. To deal with regions J and K (and also with regions L and M in future sections), it is advantageous to work with matroids, rather than with graphs, because we can then exploit a duality between the variables  $x$  and  $y$ . In order to avoid difficulties over how matroids should be presented, we will work with the class of binary matroids. This is a more general class than the class of graphs—every graph corresponds to a binary matroid, but there are binary matroids that do not correspond to graphical matroids.

A matroid  $\mathcal{M}$  is a combinatorial structure defined by a set  $E$  (the “ground set”) together with a “rank function”  $r_{\mathcal{M}} : 2^E \rightarrow \mathbb{N}$  which must satisfy the following conditions (see [14] for details):

1.  $0 \leq r_{\mathcal{M}}(A) \leq |A|$ ,
2.  $A \subseteq B$  implies  $r_{\mathcal{M}}(A) \leq r_{\mathcal{M}}(B)$  (monotonicity), and
3.  $r_{\mathcal{M}}(A \cup B) + r_{\mathcal{M}}(A \cap B) \leq r_{\mathcal{M}}(A) + r_{\mathcal{M}}(B)$  (submodularity).

A subset  $A \subseteq E$  satisfying  $r_{\mathcal{M}}(A) = |A|$  is said to be *independent*. Every other subset  $A \subseteq E$  is said to be *dependent*. A maximal (with respect to inclusion) independent set is a *basis*, and a minimal dependent set is a *circuit*. A circuit with one element is a *loop*.

The multivariate Tutte polynomial of a matroid  $\mathcal{M}$  with ground set  $E$  and rank function  $r_{\mathcal{M}}$  is defined as follows (see [16, equation (1.3)]), where the weight function

$\gamma$  assigns weights to elements of the ground set:

$$(19) \quad \tilde{Z}(\mathcal{M}; q, \gamma) = \sum_{A \subseteq E} q^{-r_{\mathcal{M}}(A)} \prod_{e \in A} \gamma_e.$$

If  $\gamma$  assigns weight  $\gamma$  to every element of  $E$ , then we use  $\tilde{Z}(\mathcal{M}; q, \gamma)$  as shorthand for  $\tilde{Z}(\mathcal{M}; q, \gamma)$ .

Let  $M$  be a matrix over a field  $F$  with row set  $V$  and column set  $E$ .  $M$  is said to “represent” a matroid  $\mathcal{M}$  with ground set  $E$ . The rank  $r_{\mathcal{M}}(A)$  of a set of columns  $A$  in this matroid is defined to be the rank of the submatrix consisting of those columns. A matroid is said to be *representable* over the field  $F$  if it can be represented in this way. It is said to be *binary* if it is representable over the two-element field  $\mathbb{F}_2$ .

The *cycle matroid* of an undirected graph  $G = (V, E)$  is the binary matroid  $\mathcal{M}(G)$  represented by the vertex-edge incidence matrix  $M$  of  $G$  (in which rows are vertices and columns are edges). It can be deduced from the definition above that  $r_{\mathcal{M}(G)}(A) = |V| - \kappa(V, A)$ . The Tutte polynomial of a cycle matroid  $\mathcal{M}(G)$  is very closely connected to the Tutte polynomial of the underlying graph  $G$ . In particular, (see [16, equations (1.2) and (1.3)]),

$$(20) \quad Z(G; q, \gamma) = q^{|V|} \tilde{Z}(\mathcal{M}(G); q, \gamma).$$

Every matroid  $\mathcal{M}$  has a dual matroid  $\mathcal{M}^*$  with the same ground set. Furthermore,  $\mathcal{M}^*$  is binary iff  $\mathcal{M}$  is (see [14]), and a binary matrix representing  $\mathcal{M}^*$  can be efficiently computed from a representation of  $\mathcal{M}$  [17, p. 63]. A *cocircuit* in  $\mathcal{M}$  is a set that is a circuit in  $\mathcal{M}^*$ ; equivalently, a cocircuit is a minimal set that intersects every basis. A cocircuit with one element is a *coloop*. We use the following fact [16, equation (4.14a)].

FACT 43. *Suppose that  $\mathcal{M}$  is a matroid with ground set  $E$  and that  $\gamma$  is a weight function assigning weights to elements in  $E$ . Let  $\mathcal{M}^*$  be the dual of  $\mathcal{M}$ , and let  $\gamma^*$  be the weight function that assigns weight  $q/\gamma_e$  to every ground set element  $e \in E$ . Then*

$$\tilde{Z}(\mathcal{M}^*; q, \gamma) = q^{-r_{\mathcal{M}^*}(E)} \left( \prod_{e \in E} \gamma_e \right) \tilde{Z}(\mathcal{M}; q, \gamma^*).$$

Two important matroid operations are deletion and contraction. Suppose  $e \in E$  is a member of the ground set of matroid  $\mathcal{M}$ . The *contraction*  $\mathcal{M}/e$  of  $e$  from  $\mathcal{M}$  is the matroid on ground set  $E - \{e\}$  with rank function given by  $r_{\mathcal{M}/e}(A) = r_{\mathcal{M}}(A \cup \{e\}) - r_{\mathcal{M}}(\{e\})$  for all  $A \subseteq E - \{e\}$ . The *deletion*  $\mathcal{M} \setminus e$  of  $\{e\}$  from  $\mathcal{M}$  is the matroid on ground set  $E - \{e\}$  with rank function given by  $r_{\mathcal{M} \setminus e}(A) = r_{\mathcal{M}}(A)$  for all  $A \subseteq E - \{e\}$ . Given a matrix representing a matroid  $\mathcal{M}$ , there are efficient algorithms for constructing matrices representing contractions and deletions of  $\mathcal{M}$  [17, Chapter 3]. We use the following fact (see, for example, [16, equation (4.18b)]).

FACT 44. *If  $\mathcal{M}$  is a matroid with a loop  $e$ , then*

$$\tilde{Z}(\mathcal{M}; q, \gamma) = (1 + \gamma_e) \tilde{Z}(\mathcal{M} \setminus e; q, \gamma).$$

We also use a related fact about coloops (see, for example, [13, equation (2.6)]).

FACT 45. *If  $\mathcal{M}$  is a matroid with a coloop  $e$ , then*

$$\tilde{Z}(\mathcal{M}; q, \gamma) = (1 + \gamma_e/q) \tilde{Z}(\mathcal{M}/e; q, \gamma).$$

We introduce two computational problems for binary matroids.

*Name* MATROIDSIGNTUTTE( $q, \gamma$ ).

*Instance* A matrix representing a binary matroid  $\mathcal{M}$  and an edge-weight  $\gamma$ .

*Output* Determine whether the sign of  $\tilde{Z}(\mathcal{M}; q, \gamma)$  is positive, negative, or 0.

*Name* MATROIDTUTTE( $q, \gamma$ ).

*Instance* A matrix representing a binary matroid  $\mathcal{M}$  and an edge-weight  $\gamma$ .

*Output*  $\tilde{Z}(\mathcal{M}; q, \gamma)$ .

**7.9. Points in regions J and K.** The points in regions J and K satisfy  $-1 \leq \min(x, y) < 0$  and  $\max(x, y) \geq 1$ . Let  $q = (x - 1)(y - 1)$  and  $\gamma = y - 1$ . Note that  $q \leq 0$ . It is known (see [9, Theorem 4.1]) that in these regions, the sign of  $Z(G; q, \gamma)$  is essentially a trivial function of  $G$ , apart from some factors arising from loops in the matroid associated with  $G$  and in its dual matroid. We will show that, for all of these points, TUTTE( $q, \gamma$ ) is in  $\#P_{\mathbb{Q}}$ . In fact, we will show that MATROIDTUTTE( $q, \gamma$ ) is in  $\#P_{\mathbb{Q}}$ . Working with matroids, instead of with graphs, will enable us to prove the results for one region (region K) and immediately to deduce the same results for the other region (region J), by duality of the variables  $x$  and  $y$ . (The replacement of  $\gamma_e$  with  $q/\gamma_e$  in Fact 43 is equivalent to swapping  $x$  and  $y$ .)

**7.9.1. Points in region K.** Points in region K have  $x \geq 1$  and  $-1 \leq y < 0$ . Let  $q = (x - 1)(y - 1)$  and  $\gamma = y - 1$ .

First, if  $x = 1$ , then  $q = 0$ . We will handle this easy case below. So, let us restrict our attention to the range  $x > 1$ . Then  $q < 0$  and  $-2 \leq \gamma < -1$ . We will use the following lemma, which is similar in spirit to [9, Theorem 4.1].<sup>6</sup>

LEMMA 46. *Suppose that  $q < 0$  and  $\mathcal{M}$  is a loopless matroid. Suppose that  $\gamma$  is a weight function in which every weight  $\gamma_e$  satisfies  $-2 \leq \gamma_e \leq 0$ . Then  $\tilde{Z}(\mathcal{M}; q, \gamma) > 0$  and the problem of computing  $\tilde{Z}(\mathcal{M}; q, \gamma)$  is in  $\#P_{\mathbb{Q}}$ .*

*Proof.* We start with some preprocessing. Before trying to compute  $\tilde{Z}(\mathcal{M}; q, \gamma)$ , we first modify  $\mathcal{M}$ , without changing its Tutte polynomial, to get rid of any size-2 circuits. We do this by parallel composition. So if we have a size-2 circuit containing elements  $e_1$  and  $e_2$ , we replace it with a new element  $e$  which is the parallel composition of the two elements in the circuit. In the matrix representing  $\mathcal{M}$ , the size-2 circuit arises as a pair of identical columns. In the representation of the new matroid, the columns corresponding to elements  $e_1$  and  $e_2$  are deleted and the new element  $e$  corresponds to one of these columns. The new weight  $\gamma_e$  is given by  $\gamma_{e_1} + \gamma_{e_2} + \gamma_{e_1}\gamma_{e_2}$  (see [9, equation (2.34)]). The reason that we want to do this preprocessing is that, in the recursive step, we will want to be able to contract an element of a circuit without creating a loop. The reason that we can do the preprocessing without falsifying the conditions in the statement of the lemma is that the region  $-2 \leq \gamma \leq 0$  maintains itself for parallel composition: If  $-2 \leq \gamma_{e_1} \leq 0$  and  $-2 \leq \gamma_{e_2} \leq 0$ , then  $-2 \leq \gamma_e \leq 0$ .

Now suppose that  $\mathcal{M}$  has no size-2 circuit. Let  $r = r_{\mathcal{M}}$  and  $E = E(\mathcal{M})$ . Then

$$\tilde{Z}(\mathcal{M}; q, \gamma) = \sum_{A \subseteq E} q^{-r(A)} \prod_{e \in A} \gamma_e.$$

*Base case.* If  $r(E) = |E|$ , then, from the axioms of rank functions of matroids, for every  $S \subseteq E$ ,  $r(S) = |S|$ , so

$$\tilde{Z}(\mathcal{M}; q, \gamma) = \sum_{A \subseteq E} q^{-|A|} \prod_{e \in A} \gamma_e = \sum_{A \subseteq E} \prod_{e \in A} \frac{\gamma_e}{q}.$$

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<sup>6</sup>We need to repeat the steps of their proof here because we want to extract computational information in addition to the sign.

The contribution from  $A = \emptyset$  is 1, and the contribution from each other  $A$  is nonnegative. Also,  $\tilde{Z}(\mathcal{M}; q, \gamma)$  can be computed by summing over the sets  $A$ .

*Recursive step.* Pick any  $e$  in a circuit. Then from [16, equation (4.18a)],

$$\tilde{Z}(\mathcal{M}; q, \gamma) = \tilde{Z}(\mathcal{M} \setminus e; q, \gamma) + \frac{\gamma_e}{q} \tilde{Z}(\mathcal{M}/e; q, \gamma).$$

Now the point is that the fraction  $\gamma_e/q$  doesn't change the sign, and it is easy to compute. Also, the two minors  $\mathcal{M} \setminus e$  and  $\mathcal{M}/e$  both satisfy the conditions of the theorem.

Both minors are matroids on ground set  $E \setminus e$ . The rank functions are given by  $r_{\mathcal{M} \setminus e}(A) = r(A)$  and  $r_{\mathcal{M}/e}(A) = r(A \cup e) - 1$ .

To see that  $\mathcal{M}/e$  has no loop, note that  $r_{\mathcal{M}/e}(\{e'\}) = r(\{e, e'\}) - 1$  and, since  $\{e, e'\}$  is not a circuit, by the preprocessing step,  $r(\{e, e'\}) = 2$ .  $\square$

We can now classify the points in region K. See also [9, Theorem 4.1], which shows that the sign is trivial in this region.

LEMMA 47. *Let  $(x, y)$  be a point with  $x \geq 1$  and  $-1 \leq y < 0$ . Let  $q = (x-1)(y-1)$  and  $\gamma = y - 1$ . Then  $\text{MATROIDSIGNTUTTE}(q, \gamma)$  is in FP and  $\text{MATROIDTUTTE}(q, \gamma)$  is in  $\#P_{\mathbb{Q}}$ .*

*Proof.* If  $\mathcal{M}$  has  $k$  loops, then, by Fact 44,  $\tilde{Z}(\mathcal{M}; q, \gamma) = (1 + \gamma)^k \tilde{Z}(\mathcal{M}'; q, \gamma)$ , where  $\mathcal{M}'$  is the matrix formed from  $\mathcal{M}$  by deleting these loops. If  $q = 0$ , then  $\tilde{Z}(\mathcal{M}'; q, \gamma) = 1$ . Otherwise,  $q < 0$ . Now Lemma 46 shows that  $Z(\mathcal{M}'; q, \gamma) > 0$  and can be computed in  $\#P_{\mathbb{Q}}$ .  $\square$

The following corollary follows immediately using (20).

COROLLARY 48. *Let  $(x, y)$  be a point with  $x \geq 1$  and  $-1 \leq y < 0$ . Let  $q = (x-1)(y-1)$  and  $\gamma = y - 1$ . Then  $\text{SIGNTUTTE}(q, \gamma)$  is in FP and  $\text{TUTTE}(q, \gamma)$  is in  $\#P_{\mathbb{Q}}$ .*

**7.9.2. Points in region J.** The following lemma classifies points in region J. See also [9, Theorem 4.4].

LEMMA 49. *Let  $(x, y)$  be a point with  $-1 \leq x \leq 0$  and  $y \geq 1$ . Let  $q = (x-1)(y-1)$  and  $\gamma = y - 1$ . Then  $\text{MATROIDSIGNTUTTE}(q, \gamma)$  is in FP and  $\text{MATROIDTUTTE}(q, \gamma)$  is in  $\#P_{\mathbb{Q}}$ .*

*Proof.* This follows from Fact 43 and from Lemma 47.  $\square$

The following corollary follows immediately using (20).

COROLLARY 50. *Let  $(x, y)$  be a point with  $-1 \leq x \leq 0$  and  $y \geq 1$ . Let  $q = (x-1)(y-1)$  and  $\gamma = y - 1$ . Then  $\text{SIGNTUTTE}(q, \gamma)$  is in FP and  $\text{TUTTE}(q, \gamma)$  is in  $\#P_{\mathbb{Q}}$ .*

**7.10. Points in regions L and M.** We use the following lemma. The statement is a slight generalization of [9, Theorem 5.4]. However, their proof (a straightforward generalization of their proof of [9, Theorem 5.1]) suffices.

LEMMA 51 (Jackson and Sokal). *Let  $\mathcal{M}$  be a matroid with ground set  $E$ , and let  $q \in (0, 1)$ . Suppose that  $\gamma$  is a weight function such that the following hold:*

1.  $\gamma_e > -1$  for every loop  $e$ ;
2.  $\gamma_e < -q$  for every coloop  $e$ ; and
3.  $-1 - \sqrt{1-q} < \gamma_e < -1 + \sqrt{1-q}$  for every normal (i.e., nonloop and non-coloop) element  $e$ .

Then

$$(21) \quad (-1)^{r_{\mathcal{M}}(E)} \tilde{Z}(\mathcal{M}; q, \gamma) > 0$$

and the problem of computing  $\tilde{Z}(\mathcal{M}; q, \gamma)$ , given such a matroid  $\mathcal{M}$ , is in  $\#P_{\mathbb{Q}}$ .

*Proof.* We follow the inductive argument that Jackson and Sokal use to prove (21) for the graphical case. This is the proof of [9, Theorem 5.1]. The induction is on  $m$ , the number of elements in the ground set of  $\mathcal{M}$ . If  $m = 0$ , then  $r_{\mathcal{M}}(E) = 0$  so that  $\tilde{Z}(\mathcal{M}; q, \gamma) = 1$ , so the lemma is true. For  $m > 0$ , there are five cases. We apply these in order, so in each case we assume that the previous cases don't apply.

1. If  $\mathcal{M}$  has a loop  $e$ , then by Fact 44,

$$\tilde{Z}(\mathcal{M}; q, \gamma) = (1 + \gamma_e)\tilde{Z}(\mathcal{M} \setminus e; q, \gamma).$$

Note that  $1 + \gamma_e > 0$  and  $r_{\mathcal{M} \setminus e}(E \setminus e) = r_{\mathcal{M}}(E \setminus e) = r_{\mathcal{M}}(E)$ . Thus, the result follows by induction.

2. If  $\mathcal{M}$  has a coloop  $e$ , then by Fact 45,

$$\tilde{Z}(\mathcal{M}; q, \gamma) = (1 + \gamma_e/q)\tilde{Z}(\mathcal{M}/e; q, \gamma).$$

Note that  $1 + \gamma_e/q < 0$  and  $r_{\mathcal{M}/e}(E \setminus e) = r_{\mathcal{M}}(E) - r_{\mathcal{M}}(e) = r_{\mathcal{M}}(E) - 1$ . Thus, the result follows by induction.

3. Suppose that  $\mathcal{M}$  has a size-2 circuit consisting of edges  $e_1$  and  $e_2$ . Let  $\mathcal{M}'$  be the matroid formed from  $\mathcal{M}$  by deleting  $e_2$ , and let  $\gamma'$  be the weight function that is the same as  $\gamma$  except that  $\gamma'_{e_1}$  is the effective weight from the parallel composition of  $e_1$  and  $e_2$ — $\gamma'_{e_1} = \gamma_{e_1} + \gamma_{e_2} + \gamma_{e_1}\gamma_{e_2}$ . Then, as in the proof of Lemma 46 (see [9, equation (2.34)]),  $\tilde{Z}(\mathcal{M}; q, \gamma) = \tilde{Z}(\mathcal{M}'; q, \gamma')$ . Also,  $r_{\mathcal{M}'}(E \setminus e_2) = r_{\mathcal{M}}(E \setminus e_2) = r_{\mathcal{M}}(E)$ . Finally, Jackson and Sokal show that  $\mathcal{M}'$  and  $\gamma'$  satisfy the conditions of the lemma (so  $\tilde{Z}(\mathcal{M}; q, \gamma)$  can be computed by induction).
4. Suppose that  $\mathcal{M}$  has a size-2 cocircuit consisting of edges  $e_1$  and  $e_2$ . Let  $\mathcal{M}'$  be the matroid formed from  $\mathcal{M}$  by contracting  $e_2$ , and let  $\gamma'$  be the weight function that is the same as  $\gamma$  except that  $\gamma'_{e_1}$  is the effective weight from the series composition of  $e_1$  and  $e_2$ — $\gamma'_{e_1} = \gamma_{e_1}\gamma_{e_2}/(q + \gamma_{e_1} + \gamma_{e_2})$ . Then from [9, equation (2.40)]  $\tilde{Z}(\mathcal{M}; q, \gamma) = \left(\frac{q + \gamma_{e_1} + \gamma_{e_2}}{q}\right)\tilde{Z}(\mathcal{M}'; q, \gamma')$ . Also, Jackson and Sokal show that

$$\left(\frac{q + \gamma_{e_1} + \gamma_{e_2}}{q}\right) < 0.$$

This is what we require since  $r_{\mathcal{M}'}(E \setminus e_2) = r_{\mathcal{M}}(E) - r_{\mathcal{M}}(e_2) = r_{\mathcal{M}}(E) - 1$ . Finally, Jackson and Sokal show that  $\mathcal{M}'$  and  $\gamma'$  satisfy the conditions of the lemma (so  $\tilde{Z}(\mathcal{M}; q, \gamma)$  can be computed by induction).

5. Otherwise, pick any ground set element  $e$  and apply the deletion-contraction identity [9, equation (2.29a)]

$$\tilde{Z}(\mathcal{M}; q, \gamma) = \tilde{Z}(\mathcal{M} \setminus e; q, \gamma) + \frac{\gamma_e}{q}\tilde{Z}(\mathcal{M}/e; q, \gamma).$$

Since  $e$  is not a cocircuit,  $r_{\mathcal{M} \setminus e}(E \setminus e) = r_{\mathcal{M}}(E)$ . As Jackson and Sokal argue,  $\mathcal{M} \setminus e$  and  $\gamma$  satisfy the conditions of the lemma. Also,  $\gamma_e/q < 0$  and  $r_{\mathcal{M}/e}(E \setminus e) = r_{\mathcal{M}}(E) - 1$ . Again, Jackson and Sokal argue that  $\mathcal{M}/e$  and  $\gamma$  satisfy the conditions of the lemma, so the result follows by induction.  $\square$

**7.11. Points in region L.**

LEMMA 52. *Let  $(x, y)$  be a point with  $0 < x < 1$  and  $-x < y < 0$ . Let  $q = (x - 1)(y - 1)$  and  $\gamma = y - 1$ . Then  $\text{MATROIDSIGNTUTTE}(q, \gamma)$  is in FP and  $\text{MATROIDTUTTE}(q, \gamma)$  is in  $\#\text{P}_{\mathbb{Q}}$ .*

*Proof.* Note that  $q = (1 - x)(1 - y) < (1 - x)(1 + x) = 1 - x^2 < 1$ . Also,  $q > (1 - x) > 0$ . Thus,  $q \in (0, 1)$ .

Now since  $y > -x$ , we have  $y(y - 1) < (-x)(y - 1)$  so that  $y^2 - y < x - xy$ , which implies that  $y^2 < x + y - xy = 1 - q$ . This implies that  $y < |y| < \sqrt{1 - q}$  so that  $y > -\sqrt{1 - q}$ . Thus,  $-1 - \sqrt{1 - q} < \gamma < -1 + \sqrt{1 - q}$ .

Finally, since  $0 < x(1 - y)$ , we have  $y < y + x(1 - y) = 1 - q$  so that  $\gamma < -q$ .

Now let  $\mathcal{M}$  be a matroid and let  $\gamma$  be a weight function assigning weight  $\gamma$  to every element the ground set of  $\mathcal{M}$ . If  $\mathcal{M}$  has  $k$  loops, then by Fact 44,  $\tilde{Z}(\mathcal{M}; q, \gamma) = (1 + \gamma)^k \tilde{Z}(\mathcal{M}'; q, \gamma)$ , where  $\mathcal{M}'$  is the matroid formed from  $\mathcal{M}$  by deleting these loops. Note that  $\mathcal{M}'$  and  $\gamma$  satisfy the hypotheses of Lemma 51.  $\square$

The following corollary follows immediately using (20).

COROLLARY 53. *Let  $(x, y)$  be a point with  $0 < x < 1$  and  $-x < y < 0$ . Let  $q = (x - 1)(y - 1)$  and  $\gamma = y - 1$ . Then  $\text{SIGNTUTTE}(q, \gamma)$  is in FP and  $\text{TUTTE}(q, \gamma)$  is in  $\#\text{P}_{\mathbb{Q}}$ .*

**7.12. Points in region M.**

LEMMA 54. *Let  $(x, y)$  be a point with  $0 < y < 1$  and  $-y < x < 0$ . Let  $q = (x - 1)(y - 1)$  and  $\gamma = y - 1$ . Then  $\text{MATROIDSIGNTUTTE}(q, \gamma)$  is in FP and  $\text{MATROIDTUTTE}(q, \gamma)$  is in  $\#\text{P}_{\mathbb{Q}}$ .*

*Proof.* This follows from Fact 43 and Lemma 52.  $\square$

The following corollary follows immediately using (20).

COROLLARY 55. *Let  $(x, y)$  be a point with  $0 < y < 1$  and  $-y < x < 0$ . Let  $q = (x - 1)(y - 1)$  and  $\gamma = y - 1$ . Then  $\text{SIGNTUTTE}(q, \gamma)$  is in FP and  $\text{TUTTE}(q, \gamma)$  is in  $\#\text{P}_{\mathbb{Q}}$ .*

**8. Putting things together for points with  $|y| < 1$ .** Collecting Observations 40 and 39 and Corollaries 25, 26, 28, 31, 32, and 33, we get the following corollary.

COROLLARY 56. *Suppose  $(x, y)$  is a point satisfying  $|y| < 1$  such that  $q = (x - 1)(y - 1) \geq 32/27$ . Let  $\gamma = y - 1$ .*

- *If  $(x, y) = (-1, 0)$ , then  $\text{SIGNTUTTE}(q, \gamma)$  and  $\text{TUTTE}(q, \gamma)$  are in FP.*
- *If  $(x, y) = (x, 0)$  for any integer  $x < -1$ , then  $\text{SIGNTUTTE}(q, \gamma)$  is NP-complete.  $\text{TUTTE}(q, \gamma)$  is in  $\#\text{P}_{\mathbb{Q}}$ .*
- *If  $x \leq -1$  and  $0 < y < 1$  and  $q$  is an integer, then  $Z(G; q, \gamma) > 0$  so that  $\text{SIGNTUTTE}(q, \gamma)$  is in FP. Also,  $\text{TUTTE}(q, \gamma)$  is in  $\#\text{P}_{\mathbb{Q}}$ .*
- *Otherwise,  $\text{SIGNTUTTE}(q, \gamma)$  is  $\#\text{P}$ -hard.*

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