Management of a Capital Stock by Strotz's Naive Planner

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Abstract

The capital management problem posed by R. H. Strotz is analyzed for the case of the "naive" planner who fails to anticipate changes in his own preferences. By imposing progressively stronger restrictions on the primitives of the problem — namely, the discounting function, the utility index function, and the investment technology — the planner's behavior is characterized first as the solution to an ordinary differential equation and then via explicit formulae. Inasmuch as these characterizations leave the discounting function essentially unrestricted, the theory can accommodate, in particular, decision makers who discount time according to the hyperbolic and "quasi-hyperbolic" curves used in applied work and said to be supported by psychological studies. Comparative statics of the model are discussed, as are extensions of the analysis to allow for credit constraints, limited foresight, and partial commitment.

JEL classification: D91, E21

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1. Introduction

When Robert H. Strotz published his investigation (1955–1956) of "Myopia and Inconsistency in Dynamic Utility Maximization," he credited Allais, Hayek, and Samuelson with having already raised, or having at least (p. 165) "alluded to" the questions that he went on to address. But despite this generosity, it is Strotz's paper that has come to be considered the cornerstone of the branch of economic theory that studies agents whose

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preferences change over time, and whose voluntary choices at one moment they might themselves disagree with at another.

It may be that Strotz's contribution is still remembered so long after its appearance mainly because the framework in which he formulated the issue of time inconsistency remains, for the most part, that in which it is studied today. His paper considered a choice at variable time τ among feasible consumption plans C with the goal of maximizing a utility function of the form (p. 167)

$$\int_0^T \lambda(t-\tau)u(C(t),t)\mathrm{d}t, \tag{1}$$

and in this setting he showed that time consistency of the decision maker's behavior is equivalent to log-linearity of the discounting function λ (a specification that he dubbed, p. 172, "the harmony case"). In doing so, Strotz of course also demonstrated what is needed to model a time-*inconsistent* decision maker. And in addition he introduced the important distinction between the cognitive assumptions of "sophistication" (i.e., awareness of any inconsistency) and "naivete" (the absence of such awareness), discussing each of these two possibilities in turn.[2]

If the first few papers to follow up on Strotz's essay (namely, those of Pollak, 1968; Peleg and Yaari, 1973; and Goldman, 1980) dealt primarily with the sophisticated agent, it is only because his analysis of this case (pp. 173–175) was soon found to be defective (see Pollak, 1968, pp. 207–208) and led to an extended and productive debate about the nature of strategic equilibrium in dynamic "intrapersonal" games.[3] Less easy to understand is why, when applied work incorporating time inconsistency finally began to appear — with notable contributions by Laibson (1997, 1998), Barro (1999), Harris and Laibson (2001, 2004), and Krusell and Smith (2003) — sophistication continued to be imposed with at most a cursory acknowledgement that there might be an alternative. This custom seems to have originated in the first of the applied papers just mentioned, where Laibson declared (p. 451) that it had become "standard practice to formally model a consumer as a sequence of temporal selves making choices in a dynamic game (e.g., Pollak [1968], Peleg and Yaari [1973], and Goldman [1980])." But the papers cited here are, of course, precisely the three mentioned above in connection with the *purely theoretical* project of correcting the defective Strotzian analysis; these authors do not even take up the question of whether the sophisticated agent is an appropriate modelling device for any particular applied problem; and thus it appears that assuming sophistication has *become* standard practice simply by being *described* as such, and with less than adequate consideration of its relative merits vis-a-vis naivete in the economic contexts of interest.

In the context of savings behavior, Laibson (1997, p. 444) does offer the argument that observed large holdings of illiquid assets constitute evidence that investors sometimes "prefer to constrain their own future choices"; and as examples of such "golden egg" investments he mentions real estate, business equity, durable goods, pensions, and other retirement (e.g., IRA, Keogh, and 401k) plans. There is undoubtedly some truth to this claim, though Laibson himself concedes there to be other reasons to purchase a house or a Honda than merely (p. 443) to remove money from one's bank account before one has a chance to spend it. And rather than as intrapersonal commitment devices

[2] This terminology is apparently due to Pollak (1968).
[3] This question has been addressed more recently by Asheim (1997).

designed by sophisticates to achieve self-control, it may be more realistic to interpret matched savings and tax-advantaged retirement plans as incentive schemes designed by paternalistic employers and governments to manipulate the behavior of their employees and constituents.

Two excellent points of departure for thinking about these issues are the Richard T. Ely lectures delivered, respectively, by Thomas Schelling (1984) and George Akerlof (1991). The first deals exclusively with sophistication, the second with naivete, and the two together suffice to dismiss any claim that one or the other assumption is universally valid.[4] These lectures, which merit being read in their entirety, will not be summarized here. But it is worth pointing out that while the problem of intertemporal resource allocation is mentioned only in passing (p. 6) in the course of Schelling's engaging tour of sophisticated behavior, it features prominently in Akerlof's (pp. 6–7) discussion of naive decision making.

Yet another source of insightful commentary on Strotz's cognitive dichotomy is the more recent work of O'Donoghue and Rabin (1999a,b, 2001) on the opposing phenomena of procrastination and preproperation. In the first of the cited papers, these authors take on the task of (p. 104) "explicitly comparing [the two] competing assumptions," and having done so warn (p. 119) that while "[p]eople clearly have some degree of sophistication, ... economists should be cautious when [working] solely with [this] assumption." At the same time, they observe that presuming awareness of changing preferences does seem entirely natural from the perspective of rational choice orthodoxy; indeed, another eminent economic theorist has suggested that conscious or unconscious imperatives of professional self-preservation have had as much to do with the traditional bias towards sophistication as any reasoned argument in its favor.

Our intention in this paper, however, is not to adjudicate territorial disputes between the two assumptions in question — this being a task for empirical or experimental work informed by cognitive psychology. Rather, our aim is simply to provide an alternative to the established methodology based on sophistication by investigating the pure theory of the naive agent; understanding, of course, that any satisfactory picture of the role of time inconsistency in human decision making is likely to combine elements of both approaches.[5] Happily, filling the vacuum left by the neglect of the naivete assumption will require simply that we carry out the analysis of this case hinted at decades ago by Strotz himself, and for this reason the reader may wish to glance again at the relevant section (pp. 170–171) of his paper.

More concretely, our goal is to characterize the behavior of a naive planner faced with a generalized version of the Strotzian capital management problem mentioned above. At each moment, taking into account the level of the capital stock inherited from earlier moments, such an agent will identify that feasible consumption plan which is optimal

[4] Schelling writes (p. 1, emphasis added) that "a person in evident possession of her faculties and knowing what she is talking about will rationally seek to prevent, to compel, or to alter her own later behavior — to restrict her own options in violation of *what she knows will be her preference* at the time the behavior is to take place." Akerlof, with other situations in mind, declares (p. 17, emphasis added) that a "modern view of behavior, based on twentieth-century anthropology, psychology, and sociology is that individuals have utilities that do change and, in addition, *they fail fully to foresee those changes* or even recognize that they have occurred."

[5] O'Donoghue and Rabin (2001) have already taken a step in this direction by allowing for what they call "partial naivete."

with respect to his current preferences — a plan that we too can identify by solving the ordinary differential equation (ODE) that expresses the decision maker's first-order condition for optimality. Failing to anticipate any impending change in his own outlook, the planner will use this currently-optimal plan to choose his instantaneous rate of consumption; or, equivalently, the rate of increase of his stock of capital. The resulting relationship between the level of the capital stock (appearing in the current resource constraint) and its time derivative (the current choice variable) then sets up a second ODE, and it is by solving *this* equation that we can obtain the actual, "historical" capital and consumption paths.

Starting with a completely nonparametric problem, defined and discussed in Section 2, we shall carry out the analysis sketched above by adopting at each stage the weakest set of functional-form assumptions that will allow us to proceed analytically.[6] This method will lead naturally to a sequence of characterizations of the planner's behavior: first, in Section 3.1, as the output of a schematic algorithm; second, in Section 3.2, as the solution to an initial value problem; third, in Section 3.3, via explicit formulae for the historical capital path; and fourth, in Section 3.4, via simplified formulae that may or may not be expressible in closed form. In Section 4 we examine some comparative statics of the model, testing whether changes in various parameters (such as the interest rate) can be shown to shift the consumption profile either forward or backward in time. Section 5.1 illustrates how savings behavior is affected when the planner becomes subject to a simple credit constraint. And finally, Section 5.2 shows both how the analysis can be generalized to allow for limited foresight, and how this generalization can be used to model a decision maker with the capacity to partially (but not fully) commit to the ex-ante-optimal plan.

2. Model

2.1. *The planner's problem*

The *calendar date* t will advance as a continuous variable from the initial date 0 through a fixed planning horizon $T > 0$. At calendar date t, the *scheduling date* s will range over the interval $[t, T]$, allowing the planner to consider both his present and future behavior. Each possible pattern of anticipated future consumption will be encoded in a *consumption schedule* $x : [t, T] \to \Re$, and the planner's date-t preferences over candidate schedules will be assumed to admit a utility representation of the form

$$U^t := \int_t^T \delta(s, t) u(x(s), s) \mathrm{d}s \tag{2}$$

(cf. Equation 1) for some strictly positive-valued *discounting function* δ and some *index function* u that is both strictly increasing and strictly concave in its first (consumption) argument.[7]

[6] Cf. Wolfram (2003, pp. 952–953): "In many calculations, it is . . . worthwhile to go as far as you can symbolically, and then resort to numerical methods only at the very end [in order to avoid] the problems that can arise in purely numerical computations."

[7] Similar representations were proposed by Samuelson (1937) and axiomatized by Koopmans (1960) and Fishburn and Rubinstein (1982). Frederick et al. (2002) provide an extensive review of theoretical, experimental, and observational studies relating to this functional form.

Note that the representation in Equation 2 imposes additive separability of the total utility into increments contributed by the various scheduling dates, as well as multiplicative separability of the increment contributed by date s into a *discount factor* $\delta(s,t)$ independent of the rate of consumption and a *utility index* $u(x(s),s)$ independent of the calendar date.[8] The *discount rate* applied at calendar date t to scheduling date s can be calculated via the relation

$$\rho(s,t) := \frac{-\delta_1(s,t)}{\delta(s,t)} \tag{3}$$

(where, as throughout this paper, the subscript denotes differentiation with respect to the indicated argument). And the convenient normalization $\delta(t,t) = 1$ then leads to the inverse relation

$$\delta(s,t) = \exp \int_t^s [-\rho(v,t)] \, \mathrm{d}v. \tag{4}$$

Given a consumption schedule x, the associated *capital schedule* k will evolve according to the law of motion

$$k_1(s) = f(k(s),s) - x(s), \tag{5}$$

where the *technology* f supplying the anticipated return on accumulated capital will be assumed to be both weakly increasing and weakly concave in its first (capital stock) argument.[9] Using Equation 5 to change variables, we can express the planner's date-t objective function as

$$U^t = \int_t^T \delta(s,t) u(f(k(s),s) - k_1(s), s) \mathrm{d}s, \tag{6}$$

to be maximized now by choice of the capital path k.

Our goal is to determine the *history* h of the capital stock over the domain $[0,T]$. At date t, this history will constrain the planner's choice of k to among those satisfying

$$k(t) = \begin{cases} h(0) =: K & \text{for } t = 0, \\ h(t) & \text{for } t > 0; \end{cases} \tag{7}$$

and we shall impose also the terminal condition

$$k(T) = 0. \tag{8}$$

(See Figure 1.) Hence, in summary, the planner's problem at calendar date t is to select a capital path k that maximizes the objective function in Equation 6 subject to the constraints in Equations 7–8.

2.2. *The question of consistency*

Strotz inquires (p. 171): "Under what circumstances will an individual who continuously re-evaluates his planned course of consumption confirm his earlier choices and

[8] Thus, as Strotz puts it (p. 168), the planner can express his enthusiasm for consuming from his stock of champagne on the date of his birth by assigning a high value to the utility index u(two glasses of champagne per diem, planner's birth date), but this enthusiasm can neither increase nor decrease as the date approaches.

[9] Note that, as implied by Equation 5, capital is consumable ("like rabbits," according to Phelps and Pollak, 1968, p. 187).

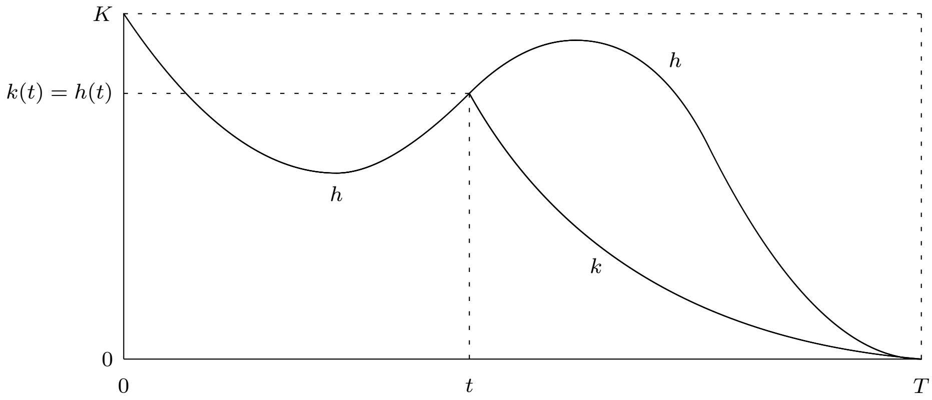

Fig. 1. Equations 7–8. A candidate capital path k considered at calendar date t must connect the points $\langle t, h(t)\rangle$ and $\langle T, 0\rangle$.

follow out the consumption plan originally selected?" Or, in our terminology: Under what conditions will the solutions to the planner's problem at different calendar dates coincide?

The answer to this question (noted already by Burness, 1976) is that coincidental plans will arise if and only if there exists a function σ satisfying

$$\delta(s,t) = \frac{\sigma(s)}{\sigma(t)}; \tag{9}$$

that is, if and only if the discounting function is (multiplicatively) *separable.*[10] To see that this property is necessary (its sufficiency is immediate), observe that any capital schedule k^t rendering U^t maximal must satisfy the Euler equation

$$\delta(s,t)u_1(f(k^t(s),s) - k_1^t(s),s)f_1(k^t(s),s) + \frac{\mathrm{d}\left[\delta(s,t)u_1(f(k^t(s),s) - k_1^t(s),s)\right]}{\mathrm{d}s} = 0 \tag{10}$$

at each $s \in [t,T]$.[11] Equation 10 has first integral

$$\delta(s,t)u_1(f(k^t(s),s) - k_1^t(s),s)\exp\int_s^T \left[-f_1(k^t(v),v)\right]\mathrm{d}v = \tau(t), \tag{11}$$

where the constant $\tau(t)$ can be interpreted as the reciprocal of the shadow price of utility in terms of date-T consumption. And if the schedule k^0 remains optimal at each calendar date $t \in (0,T)$ then we can let

$$\sigma(s) = \left[u_1(f(k^0(s),s) - k_1^0(s),s)\exp\int_s^T \left[-f_1(k^0(v),v)\right]\mathrm{d}v\right]^{-1}, \tag{12}$$

whereupon Equation 9 holds since $\tau(t) = \delta(t,t)/\sigma(t) = 1/\sigma(t)$.

[10] Separability is equivalent to the *calendar invariance* condition that, for each $0 \le t < \bar{t} \le s \le T$ and $\epsilon > 0$, we have $\delta(s+\epsilon,t)/\delta(s,t) = \delta(s+\epsilon,\bar{t})/\delta(s,\bar{t})$.

[11] See, e.g., Chiang (1992, pp. 28–36), Gel'fand and Fomin (2000, pp. 14–18), or Luenberger (1969, pp. 179–183). Equation 10 is in fact both necessary and sufficient for the optimality of k^t since $G(k(s),k_1(s),s) := \delta(s,t)u(f(k(s),s) - k_1(s),s)$ is concave in $\langle k(s),k_1(s)\rangle$. And this concavity follows, in turn, from the inequalities $G_{11} = \delta\left[u_{11}f_1^2 + u_1 f_{11}\right] \le 0$, $G_{22} = \delta u_{11} \le 0$, and $G_{11}G_{22} - G_{12}G_{21} = \delta^2 u_1 u_{11} f_{11} \ge 0$. (See, e.g., Chiang, 1992, pp. 81–91.)

With regard to the representation in Equation 2, it is common to assume (as does Strotz) that the discounting function is *stationary*; i.e., that there exists a function σ such that

$$\delta(s,t) = \sigma(s-t).\text{[12]} \tag{13}$$

Under this assumption, the planner's point of view undergoes a rigid translation with the advance of the calendar date, and in a sense the future never actually arrives at the present.[13]

A third possible assumption is that of *log-linearity*, a property that δ is said to exhibit whenever it admits a function σ such that

$$\delta(s,t) = e^{-\sigma(t)[s-t]}.\text{[14]} \tag{14}$$

Such a function, when it exists, supplies the common discount rate applied by the planner at a given calendar date to all future scheduling dates.

The latter two assumptions on the discounting function are together sufficient for intertemporal consistency; indeed, any two of separability, stationarity, and log-linearity jointly imply the third. Thus standard, "exponential" discounting (which satisfies all three assumptions) is the unique stationary specification of δ that leads to time-consistent planning.

3. Analysis

3.1. *The general case*

When the discounting function is *in*separable and hence intertemporal consistency fails, the observed behavior will generally have the "unpleasant feature," pointed out by Blackorby et al. (1973, p. 239), that "*ex post* [it] makes no sense from any point of view." The reason for this unpleasantness is that while at each calendar date t our naive planner will be following a capital schedule k^t that appears for the moment to be optimal, he will adhere to this plan only for an instant and will soon find that he prefers a new "optimal" schedule $k^{t+\mathrm{d}t}$. Accordingly, although the choice of k^t is made with each future scheduling date taken into consideration, this choice affects the history h only by determining the instantaneous rate

$$h_1(t) = k_1^t(t) \tag{15}$$

of capital accumulation at calendar date t. (See Figure 2.)

Using Equations 7 and 15 to calculate

$$\tau(t) = u_1(f(h(t),t) - h_1(t),t) \exp \int_t^T \left[-f_1(k^t(v),v)\right] \mathrm{d}v \tag{16}$$

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[12] Stationarity is equivalent to the *absolute invariance* condition that, for each $0 \le t < \bar{t} \le T$ and $\epsilon > 0$, we have $\delta(t+\epsilon,t) = \delta(\bar{t}+\epsilon,\bar{t})$.

[13] Samuelson (1937, p. 160) writes that "as the individual moves along in time there is a sort of perspective phenomenon in that his view of the future in relation to his instantaneous time position remains invariant, rather than his evaluation of any particular year." (See also the commentary of Rosenstein-Rodan, 1934, cited by Strotz, p. 170.)

[14] Log-linearity is equivalent to the *scheduling invariance* condition that, for each $0 \le t \le s < \bar{s} \le T$ and $\epsilon > 0$, we have $\delta(s+\epsilon,t)/\delta(s,t) = \delta(\bar{s}+\epsilon,t)/\delta(\bar{s},t)$.

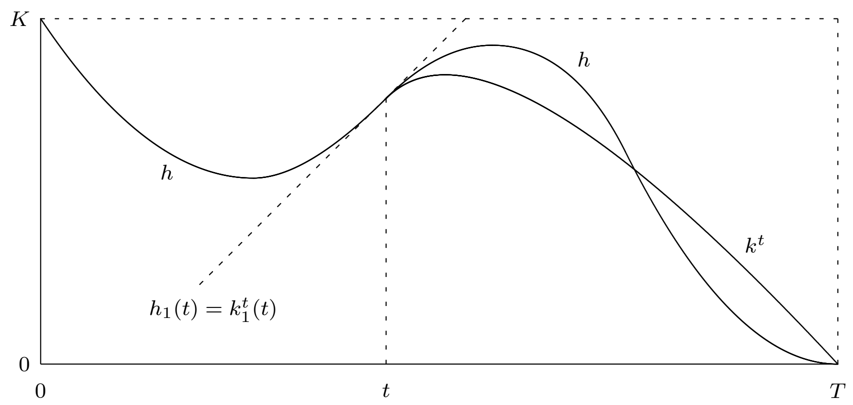

Fig. 2. Equation 15. At calendar date t, the history h must be tangent to the (temporarily) optimal capital schedule k^t.

allows us to rewrite Equation 11 as

$$\underbrace{\delta(s,t)u_1(f(k^t(s),s) - k_1^t(s),s)}_{\text{discounted marginal utility of consumption}} \quad \underbrace{\exp\int_t^s f_1(k^t(v),v)\mathrm{d}v}_{\text{price deflator}} = \cdots$$

$$\cdots = u_1(f(h(t),t) - h_1(t),t), \quad (17)$$

an ODE in k^t that in combination with the terminal condition

$$k^t(T) = 0 \quad (18)$$

can in principle be solved to yield an expression of the form

$$k^t(s) = \Theta(s,t,h(t),h_1(t)), \quad (19)$$

valid for $s \in [t,T]$. Setting $s = t$ in Equation 19 leads to the relation

$$h(t) = \Theta(t,t,h(t),h_1(t)), \quad (20)$$

an ODE in h that in combination with the initial condition

$$h(0) = K \quad (21)$$

can (again, in principle) be solved to yield the historical path of the capital stock. And the associated consumption path m can then be deduced from the law of motion

$$m(t) := f(h(t),t) - h_1(t). \quad (22)$$

3.2. *Linear technologies*

Section 3.1 provides a recipe for computing the capital history h from the primitives of the planner's problem; namely, the discounting function δ, the index function u, and the technology f. As we have seen, this computation requires the solution, in sequence, of two ordinary differential equations, the first (Equation 17) determining k^t as a function of the scheduling date and the second (Equation 20) determining h as a function of the calendar date. The immediate difficulty in carrying out these tasks analytically is that the unknown path k^t in Equation 17 affects both the marginal utility of consumption and the price deflator, and may thus be in a complicated relationship with its derivative k_1^t. This ODE will be linear, however, if the technology is itself linear; that is, under the specification

$$f^1(z,s) := \alpha(s)z + \beta(s) \tag{23}$$

for given functions α and β supplying, respectively, a (time-varying) "interest rate" and a flow of "exogenous income." In this case the price deflator

$$\Pi(s,t) := \exp \int_t^s \alpha(v)\mathrm{d}v \tag{24}$$

is independent of the optimal capital path, and since the strictly decreasing function $u_1(\cdot, s)$ admits an inverse map $u_1(\cdot, s)^{\mathrm{inv}}$, we can put Equation 17 into the standard form

$$k_1^t(s) + [-\alpha(s)]k^t(s) = \beta(s) - u_1(\cdot, s)^{\mathrm{inv}} \left(\frac{u_1(\alpha(t)h(t) + \beta(t) - h_1(t), t)}{\delta(s,t)\Pi(s,t)} \right), \tag{25}$$

with integrating factor

$$\exp \int_0^s [-\alpha(v)]\, \mathrm{d}v = \Pi(s,0)^{-1}. \tag{26}$$

Solving the terminal value problem formed by Equations 18 and 25 then leads to the formula

$$k^t(s) = \int_s^T \left[u_1(\cdot, v)^{\mathrm{inv}} \left(\frac{u_1(\alpha(t)h(t) + \beta(t) - h_1(t), t)}{\delta(v,t)\Pi(v,t)} \right) - \beta(v) \right] \frac{\mathrm{d}v}{\Pi(v,s)} \tag{27}$$

for k^t on the interval $[t, T]$, and hence to the expression

$$h(t) = \int_t^T \left[u_1(\cdot, v)^{\mathrm{inv}} \left(\frac{u_1(\alpha(t)h(t) + \beta(t) - h_1(t), t)}{\delta(v,t)\Pi(v,t)} \right) - \beta(v) \right] \frac{\mathrm{d}v}{\Pi(v,t)} \tag{28}$$

for the relation in Equation 20.

Equation 28, the first-order ODE at the heart of this paper, characterizes the history of the capital stock for the general linear-technology planning problem formulated above. The assumption of linearity is of course a strong one, and rules out important applications such as to (nonlinear versions of) the optimal growth model formulated by Ramsey. But nonlinear capital management problems are for the most part intractable analytically even in the time-consistent case, and it is not our purpose at present to extend the existing theory along this dimension. Rather, we wish to broaden the theory to allow for inseparable discounting functions (see Section 2.2), assuming naivete on the part of the decision maker instead of the usual sophistication (see Section 1). And in order to do so we shall restrict attention for the time being to linear-technology scenarios.

3.3. *Tractable index functions*

We have seen that the assumption of a linear technology reduces the task of determining the history of the capital stock to that of solving the initial value problem formed by Equations 21 and 28. This can be accomplished numerically for a wide range of index functions, but to proceed analytically we shall need to adopt a parametric specification of $u_1(\cdot, t)$ that will allow us to extract its argument $\alpha(t)h(t) + \beta(t) - h_1(t)$ from the integral in Equation 28. And it turns out that two familiar utility specifications have just this property.

The first case we shall consider is that of the *exponential* (or — in the context of choice under uncertainty — constant absolute risk aversion) index function defined, for some mapping $\gamma : [0,T] \to (0,\infty)$, by

$$u^{\mathrm{e}}(z,s) := -\gamma(s)e^{-z/\gamma(s)}, \tag{29}$$

and with associated marginal utility function

$$u_1^{\mathrm{e}}(z,s) = e^{-z/\gamma(s)} \tag{30}$$

(see Figure 3, left panel) and elasticity of intertemporal substitution

$$\eta^{\mathrm{e}}(z,s) := \frac{-u_1^{\mathrm{e}}(z,s)}{zu_{11}^{\mathrm{e}}(z,s)} = \frac{\gamma(s)}{z}. \tag{31}$$

With this specification Equation 28 simplifies to

$$h(t) = [\alpha(t)h(t) + \beta(t) - h_1(t)] \int_t^T \frac{\gamma(v)\mathrm{d}v}{\gamma(t)\Pi(v,t)} + \cdots$$
$$\cdots + \int_t^T \frac{[\gamma(v)\log[\delta(v,t)\Pi(v,t)] - \beta(v)]\,\mathrm{d}v}{\Pi(v,t)}, \tag{32}$$

which with the definition

$$\Phi^{\mathrm{e}}(t) := \left[\int_t^T \frac{\gamma(v)\mathrm{d}v}{\gamma(t)\Pi(v,t)}\right]^{-1} \tag{33}$$

can be put into the standard form

$$h_1(t) + [\Phi^{\mathrm{e}}(t) - \alpha(t)]\,h(t) = \beta(t) + \Phi^{\mathrm{e}}(t)\int_t^T \frac{[\gamma(v)\log[\delta(v,t)\Pi(v,t)] - \beta(v)]\,\mathrm{d}v}{\Pi(v,t)}. \tag{34}$$

Selecting a suitable integrating factor

$$\Psi^{\mathrm{e}}(t) := \exp\int_0^t [\Phi^{\mathrm{e}}(v) - \alpha(v)]\,\mathrm{d}v, \tag{35}$$

the initial value problem in question is then solved by the capital history

$$h(t) = \frac{K}{\Psi^{\mathrm{e}}(t)} + \cdots$$
$$\cdots + \int_0^t \frac{\Psi^{\mathrm{e}}(v)}{\Psi^{\mathrm{e}}(t)}\left[\beta(v) + \Phi^{\mathrm{e}}(v)\int_v^T \frac{[\gamma(w)\log[\delta(w,v)\Pi(w,v)] - \beta(w)]\,\mathrm{d}w}{\Pi(w,v)}\right]\mathrm{d}v. \tag{36}$$

The second tractable case is that of the *power* (or constant relative risk aversion) index function defined, for some "shift" map $\kappa : [0,T] \to \Re$ and parameter $p > 0$, by

$$u^{\mathrm{p}}(z,s) := \begin{cases} [z-\kappa(s)]^{1-1/p} / [1-1/p] & \text{for } p \neq 1, \\ \log[z-\kappa(s)] & \text{for } p = 1; \end{cases} \tag{37}$$

with associated marginal utility

$$u_1^{\mathrm{p}}(z,s) = [z-\kappa(s)]^{-1/p} \tag{38}$$

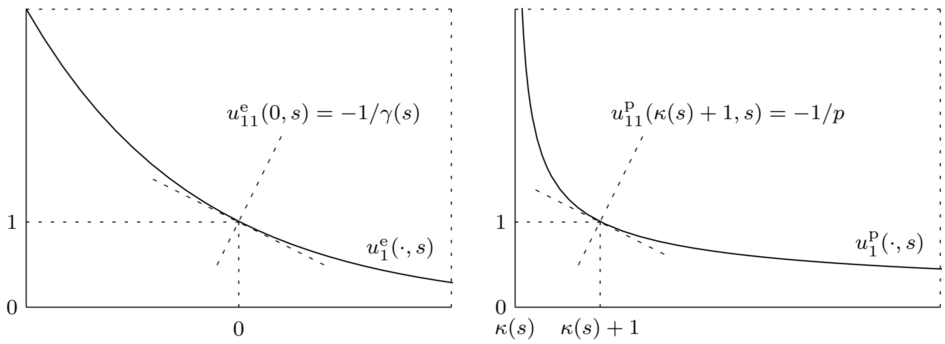

Fig. 3. Marginal utility curves for the exponential (left panel) and power (right panel) index specifications. Given the scheduling date s, each curve is strictly decreasing and approaches zero at high rates of consumption. The parameters $\gamma(s)$ and p equal the slopes of the lines normal to the marginal utility curve at, respectively, 0 for the exponential case and $\kappa(s)+1$ for the power case.

(see Figure 3, right panel) and elasticity of intertemporal substitution

$$\eta^{\mathrm{p}}(z,s) = p\left[1 - \kappa(s)/z\right]. \tag{39}$$

Defining the shifted income function

$$\overrightarrow{\beta}(s) := \beta(s) - \kappa(s), \tag{40}$$

Equation 28 simplifies under this specification to

$$h(t) = \left[\alpha(t)h(t) + \overrightarrow{\beta}(t) - h_1(t)\right]\int_t^T \frac{\delta(v,t)^p \mathrm{d}v}{\Pi(v,t)^{1-p}} - \int_t^T \frac{\overrightarrow{\beta}(v)\mathrm{d}v}{\Pi(v,t)}, \tag{41}$$

which with the definition

$$\Phi^{\mathrm{p}}(t) := \left[\int_t^T \frac{\delta(v,t)^p \mathrm{d}v}{\Pi(v,t)^{1-p}}\right]^{-1} \tag{42}$$

can be put into the standard form

$$h_1(t) + \left[\Phi^{\mathrm{p}}(t) - \alpha(t)\right]h(t) = \overrightarrow{\beta}(t) - \Phi^{\mathrm{p}}(t)\int_t^T \frac{\overrightarrow{\beta}(v)\mathrm{d}v}{\Pi(v,t)}. \tag{43}$$

Selecting the integrating factor

$$\Psi^{\mathrm{p}}(t) := \exp\int_0^t \left[\Phi^{\mathrm{p}}(v) - \alpha(v)\right]\mathrm{d}v, \tag{44}$$

our initial value problem is then solved by the capital history

$$h(t) = \frac{K}{\Psi^{\mathrm{p}}(t)} + \int_0^t \frac{\Psi^{\mathrm{p}}(v)}{\Psi^{\mathrm{p}}(t)}\left[\overrightarrow{\beta}(v) - \Phi^{\mathrm{p}}(v)\int_v^T \frac{\overrightarrow{\beta}(w)\mathrm{d}w}{\Pi(w,v)}\right]\mathrm{d}v. \tag{45}$$

3.4. *Hyperbolic cake consumption*

Equations 36 and 45 (together with the definitions in Equations 24, 33/42, 35/44, and 40) provide explicit formulae for the path of a capital stock managed by a naive planner

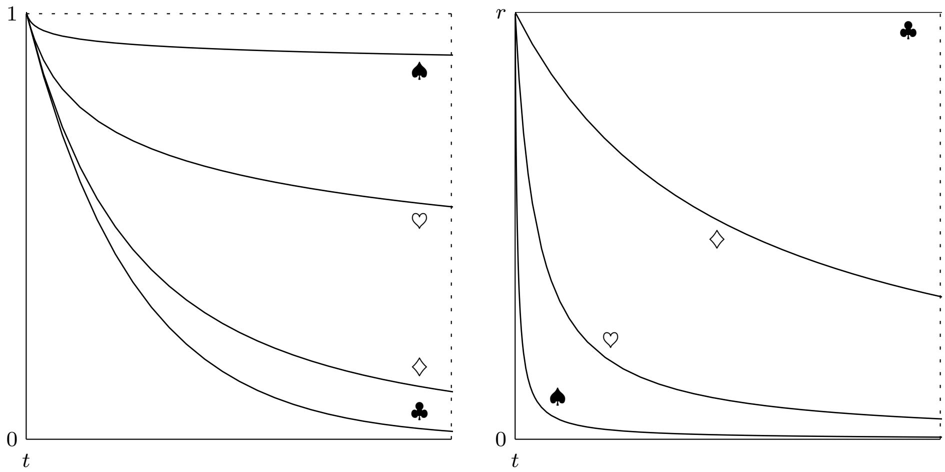

Fig. 4. Hyperbolic discount factor (left panel) and rate (right panel) curves; computed for $q \to 0$ (♣), $q = 1$ (◇), $q = 10$ (♡), and $q = 100$ (♠).

with access to a linear technology and possessing, respectively, exponential and power utility. It should be noted that, in contrast to much existing work on the sophisticated case, *these formulae impose no conditions whatsoever on the discounting function*, which need not even be stationary. In this sense the naive planning model provides a convenient laboratory for experimenting with alternative assumptions about time preference.

One notable nonstandard specification of δ, advocated by Loewenstein and Prelec (1992), among others, is the (stationary) hyperbolic form

$$\delta^{\mathrm{h}}(s,t) := \left[\frac{1}{1+q\,[s-t]}\right]^{r/q} \tag{46}$$

with parameters $r \geq 0$ and $q > 0$. The associated discount rate function

$$\rho^{\mathrm{h}}(s,t) = \frac{r}{1+q\,[s-t]} \tag{47}$$

is decreasing in the scheduling date — a phenomenon often generated in discrete-time models using the "quasi-hyperbolic" specification introduced by Phelps and Pollak (1968) and popularized by Laibson (1997). And since

$$\delta^{\mathrm{e}}(s,t) := e^{-r[s-t]} = \lim_{q \to 0} \delta^{\mathrm{h}}(s,t), \tag{48}$$

the parameter q controls the degree of distortion relative to exponential discounting with rate r. (See Figure 4; cf. Loewenstein and Prelec, 1992, p. 581.)

To try out hyperbolic discounting, let us apply our results from Section 3 to the benchmark "cake-eating" problem in which the interest rate and exogenous income functions (α and β, respectively) are both identically zero. Adopting exponential utility with constant intertemporal substitutability parameter $\gamma(s) = c > 0$, we have

$$h(t) = \frac{K[T-t]}{T} - \frac{cr[T-t]}{q} \int_0^t \left[\int_v^T \frac{\log\left[1+q\,[w-v]\right] \mathrm{d}w}{[T-v]^2}\right] \mathrm{d}v \tag{49}$$

from Equation 36. Evaluating the inner integral and making the substitution $y = q[T-v]$, we obtain

$$h(t) = \frac{K[T-t]}{T} - \frac{cr[T-t]}{q} \int_{q[T-t]}^{qT} \frac{[[1+y]\log[1+y]-y]\,\mathrm{d}y}{y^2}. \tag{50}$$

And evaluating the remaining integral yields the history

$$h(t) = \frac{K[T-t]}{T} + \frac{cr[T-t]}{q} \left[\frac{[1+y]\log[1+y]}{y} + \operatorname{dilog}[-y] \right]_{y=q[T-t]}^{qT}, \tag{51}$$

where $\operatorname{dilog} z$ denotes the dilogarithm of z (defined on the unit sphere as $\sum_{i=1}^{\infty} z^i i^{-2}$ and elsewhere via analytic continuation). Finally, note that as the parameter $q \to 0$ and δ^{h} tends to its exponential limit, the capital history in Equation 51 converges to the textbook cake-eating formula

$$h(t) = \frac{K[T-t]}{T} - \frac{crt[T-t]}{2} \tag{52}$$

for a time-consistent planner (cf. Luenberger, 1969, p. 182).[15]

Alternatively, adopting power utility with shift parameter $\kappa(s) = 0$, we have

$$h(t) = K \exp \int_0^t \left[- \int_v^T [1+q[w-v]]^{-pr/q} \mathrm{d}w \right]^{-1} \mathrm{d}v \tag{53}$$

from Equation 45. Evaluating the inner integral (for $q \neq pr$) yields

$$h(t) = K \exp \int_0^t \frac{[q-pr]\mathrm{d}v}{1-[1+q[T-v]]^{1-pr/q}}. \tag{54}$$

And as $q \to 0$ this path converges to the time-consistent history

$$h(t) = \frac{K[e^{pr[T-t]}-1]}{e^{prT}-1}. \tag{55}$$

The preceding examples demonstrate that even in the simplest of capital management problems, evaluating the integrals in Equations 36 and 45 to obtain closed-form histories may require the use of non-elementary functions or may turn out to be infeasible. And a determined investigation of this issue shows (see Tyson, 2006, pp. 21–23) that our experience here is typical: While closed-form paths can be computed for a broad class of problems under exponential utility, difficult integrals will almost always arise under power utility.[16]

4. Comparative statics

4.1. *Hyperbolic discounting*

Once we have developed an explicit formula for the capital path under a particular set of functional form assumptions, we are in a position to investigate how our planner's

[15] To see this, apply l'Hôpital's Rule three times in succession, making use of the identity $\mathrm{d}[\operatorname{dilog} z]/\mathrm{d}z = -[1/z]\log[1-z]$.

[16] Specifically, given any capital management problem with linear f, constant α, and continuous β; with exponential u and constant γ; and with stationary, continuous δ; the history in Equation 36 can be approximated with arbitrary precision by a path expressible in closed form.

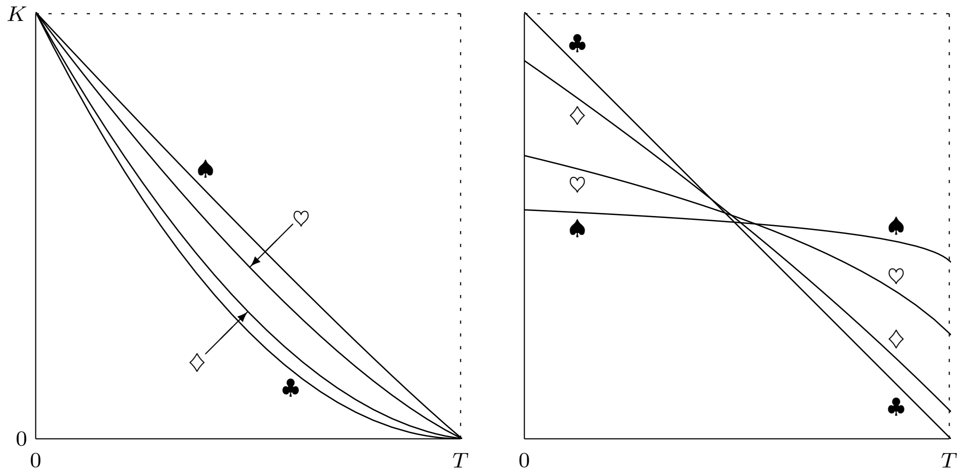

Fig. 5. Capital (left panel) and consumption (right panel) histories for a problem with exponential utility and hyperbolic discounting; computed for $q \to 0$ (♣), $q = 1$ (◇), $q = 10$ (♡), and $q = 100$ (♠).

savings and consumption behavior varies with the parameters of the model. For concreteness we shall focus on the specific question of whether increasing a given parameter shifts the consumption profile forward or backward in time (if either), and for simplicity we shall consider only the case of exponential utility.

Let us say that a parameter z *postpones* (resp., *prepones*) *consumption* if the function m (defined in Equation 22) has increasing (resp., decreasing) differences in $\langle t, z\rangle$; that is, if $\partial m_1(t)/\partial z \geq 0$ (resp., ≤ 0). Accordingly as the monotonicity and the corresponding inequality are strict, let us say that the postponement or preponement is likewise strict. It is then clear from Equation 52 that in the standard cake-eating problem, both the discount rate r and the intertemporal substitutability parameter c strictly prepone consumption. (Given at least some substitutability, stronger myopia leads the planner to consume more cake earlier; while given at least some myopia, increased substitutability has the same effect.) Do these propositions hold more generally in the case of hyperbolic discounting? And what is the comparative static effect of the new, "hyperbolicity" parameter q?

From Equation 51 it is a simple matter to compute the slope

$$m_1(t) = -h_{11}(t) = \frac{-cr \log[1 + q[T - t]]}{q[T - t]} < 0 \tag{56}$$

of the consumption path m and its (negative) partial derivatives with respect to r and c. The first question above can then be answered in the affirmative: These two parameters continue to strictly prepone consumption no matter how much we distort the discounting function relative to the exponential benchmark. Moreover, since

$$\frac{\partial m_1(t)}{\partial q} = \frac{cr}{q}\left[\frac{\log[1 + q[T - t]]}{q[T - t]} - \frac{1}{1 + q[T - t]}\right] > 0 \tag{57}$$

(the sign following from the identity $z \log z > z - 1$, valid for $0 < z \neq 1$), we can settle the second question as well: The hyperbolicity parameter q strictly *postpones* consumption by the planner. (See Figure 5.)

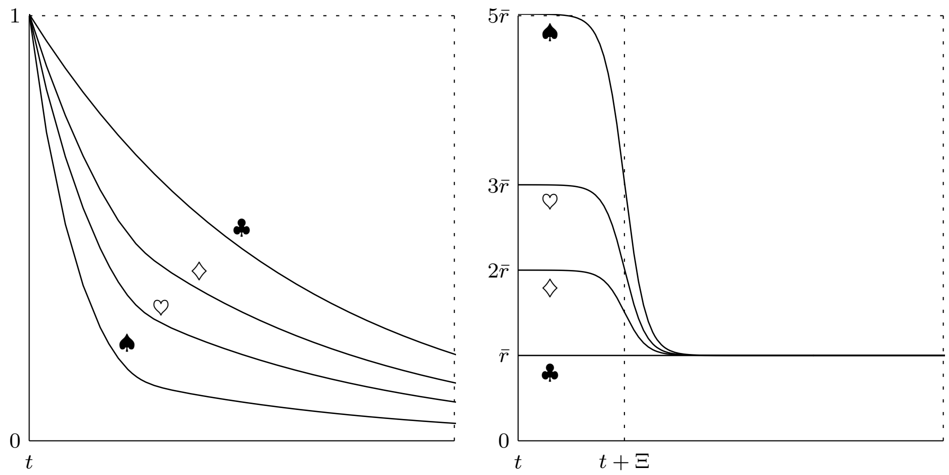

Fig. 6. Generalized quasi-hyperbolic discount factor (left panel) and rate (right panel) curves; computed for $\bar{q} = 0$ (♣), $\bar{q} = \bar{r}$ (♢), $\bar{q} = 2\bar{r}$ (♡), and $\bar{q} = 4\bar{r}$ (♠).

4.2. *Generalized quasi-hyperbolic discounting*

As the conclusion just reached flatly contradicts the conventional wisdom that hyperbolic discounting leads to "undersaving," it is worth taking a moment to examine the quasi-hyperbolic model from which that intuition is derived. Although "beta-delta" discounting was designed by Phelps and Pollak for use in discrete time, it can be translated into the present continuous-time setting via the (stationary) "generalized quasi-hyperbolic" specification

$$\delta^{\mathrm{q}}(s,t) := e^{-[\bar{r}+\bar{q}[\exp[-\xi\Xi]+1]][s-t]} \left[\frac{e^{\xi\Xi}+1}{e^{\xi\Xi}+e^{\xi[s-t]}}\right]^{-[\bar{q}/\xi][\exp[-\xi\Xi]+1]} \tag{58}$$

with parameters $\bar{r}, \bar{q}, \Xi, \xi \geq 0$. The associated function

$$\rho^{\mathrm{q}}(s,t) = \bar{r} + \bar{q}\left[\frac{e^{\xi\Xi}+1}{e^{\xi\Xi}+e^{\xi[s-t]}}\right] \tag{59}$$

makes clear that $\bar{r}+\bar{q}$ and $\bar{r}$ can be interpreted as the rates of discount in, respectively, the short and the long term; that Ξ measures the time interval over which the short-term rate applies; and that ξ controls the speed of the decline in the rate from $\bar{r}+\bar{q}$ to $\bar{r}$ in the vicinity of the scheduling date $t+\Xi$. (See Figure 6.) Ordinary quasi-hyperbolic discounting then appears as the special case in which $\xi \to \infty$,[17] while exponential discounting emerges when either $\bar{q} = 0$ or $\Xi \to \infty$.

Returning now to the cake-eating problem and imposing generalized quasi-hyperbolic discounting, we can once again (though with rather more difficulty) calculate the slope

$$m_1(t) = -h_{11}(t) = -c\left[\bar{r} + \bar{q}\left[1+e^{-\xi\Xi}\right]\left[1 + \frac{1}{\xi[T-t]}\log\frac{e^{\xi\Xi}+1}{e^{\xi\Xi}+e^{\xi[T-t]}}\right]\right] < 0 \tag{60}$$

of the consumption path associated with the capital history in Equation 36. As might be expected, both the intertemporal substitutability parameter c and the long-term discount

[17] Here $e^{-\bar{q}\Xi}$ and $e^{-\bar{r}\Xi}$ play the roles of "beta" and "delta," respectively.

rate $\bar{r}$ generate negative partial derivatives and therefore strictly prepone consumption. But here the quasi-hyperbolicity parameter $\bar{q}$ clearly prepones consumption as well — in contrast to the effect of its analog q in the genuinely hyperbolic case but *in agreement with the conventional wisdom* mentioned above. Moreover, this qualitative difference between the hyperbolic and generalized quasi-hyperbolic scenarios is a straightforward consequence of how these two discounting functions are parameterized: In the hyperbolic case, there is less overall discounting of future utility and hence more consumption at later dates for higher values of q (see again Figure 4); while in the quasi-hyperbolic case there is *more* discounting and hence *less* late consumption for higher values of $\bar{q}$ (Figure 6). Thus we see that there is nothing intrinsic to hyperbolic discounting — or, more generally, to declining discount rates — that leads inevitably to preponement of consumption and any consequent undersaving. Rather, it is simply that more discounting of the future *of any kind* tends to prepone consumption, and under different specifications of the function δ parameters such as q and $\bar{q}$ may or may not be associated with such an increase.

4.3. *Positive interest rate*

As a final, more challenging comparative statics exercise, let us investigate the impact on our planner's behavior of the possibility of investment at a constant interest rate $\alpha^{\mathrm{c}}(s) := a > 0$. Relying as usual on Equation 36, we obtain the capital history

$$h(t) = \left[1 - e^{-a[T-t]}\right] \left[\frac{K + cT}{1 - e^{-aT}} + \int_0^t \left[\int_v^T \frac{ace^{-a[w-v]} \log \delta(w,v) \mathrm{d}w}{\left[1 - e^{-a[T-v]}\right]^2}\right] \mathrm{d}v\right] + \cdots \cdots - c\,[T-t]\,. \quad (61)$$

From this can be derived the remarkably simple expression

$$m_1(t) = ah_1(t) - h_{11}(t) = ac\left[1 - \int_t^T \frac{\zeta(v,t)\mathrm{d}v}{e^{a[v-t]} - e^{-a[T-v]}}\right] \quad (62)$$

for the consumption slope, where the function

$$\zeta(s,t) := \frac{\delta_2(s,t)}{\delta(s,t)} \quad (63)$$

returns a kind of alternative discount rate (equal to the ordinary rate $\rho(s,t)$ whenever δ is stationary). And routine differentiation then yields the expressions

$$\frac{\partial m_1(t)}{\partial c} = a\left[1 - \int_t^T \frac{\zeta(v,t)\mathrm{d}v}{e^{a[v-t]} - e^{-a[T-v]}}\right] \quad (64)$$

and

$$\frac{\partial m_1(t)}{\partial a} = c\left[1 - \int_t^T \frac{\zeta(v,t)}{e^{a[v-t]} - e^{-a[T-v]}}\left[\underbrace{1 - \frac{a[T-t]}{e^{a[T-t]} - 1}}_{\oplus}\underbrace{-a[v-t]}_{\ominus}\right]\mathrm{d}v\right] \quad (65)$$

for the responsiveness of the consumption slope to the intertemporal substitutability and interest rate parameters.

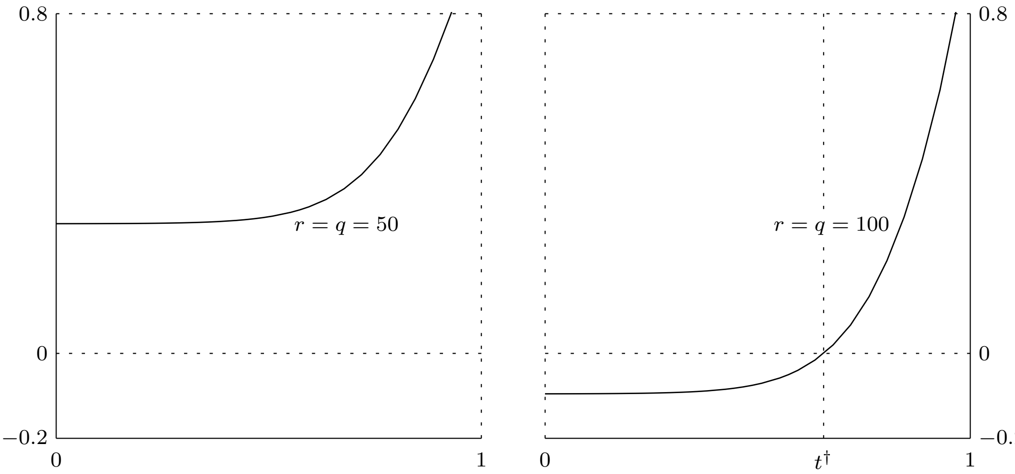

Fig. 7. The partial derivative $\partial m_1(\cdot)/\partial a$ as a function of the calendar date in a problem with exponential utility, hyperbolic discounting, and parameter settings $T = 1$, $K = 1$, $a = 12$, and $c = 1$. When $r = q = 50$ (left panel), this quantity is comfortably above zero and so the interest rate a strictly postpones consumption. But when $r = q = 100$ (right panel), the derivative is negative on the interval $[0, t^\dagger]$, over which a prepones consumption.

Since $m_1(t)$ and $\partial m_1(t)/\partial c$ have the same sign, c postpones or prepones consumption accordingly as $m(t)$ is increasing or decreasing; i.e., accordingly as the postponing effect of the investment opportunity outweighs or is outweighed by the preponing effect of the planner's discounting of future utility. This is most apparent in the exponential case (viz., $\zeta^{\mathrm{e}}(s,t) = r$), where $\partial m_1(t)/\partial c = a[1 - r/a]$ and its sign is determined by a straight comparison between the discount and interest rates (a classical result). But more generally postponement of consumption is consistent with $\zeta(t,t) > a$, as long as $\zeta(\cdot, t)$ falls rapidly enough over the interval $[t, T]$.

The comparative statics of interest rate variation are somewhat more complex. Under exponential discounting the integral in Equation 65 vanishes, yielding $\partial m_1(t)/\partial a = c > 0$ and strict postponement of consumption.[18] This is of course just what one would expect: A higher rate of interest is a reason to invest and to consume relatively more later. When the discount rate $\zeta(\cdot, t)$ is declining, however, the integral in question can be positive, since the contribution of the negative and variable (in v) term $\ominus$ will be underweighted against that of the positive and constant term $\oplus$. And when the discount rate is declining *sharply*, this effect can be strong enough to drag $\partial m_1(t)/\partial a$ below zero and hence to create preponement of consumption. (See Figure 7, right panel, for an example of such a pathological scenario.)

The intuition for the possibility of the interest rate acting to prepone consumption bears a loose analogy to that for the possibility of a Giffen good. The planner consumes both *early* and *late*, and purchases more of the second commodity by choosing higher values of $m_1(t)$. Intertemporal substitutability is his "income"; thus late consumption is "inferior" when $\partial m_1(t)/\partial c < 0$. The "price" of this commodity is $-a$, so that it is "Giffen" when $-\partial m_1(t)/\partial a = \partial m_1(t)/\partial[-a] > 0$. Now it is clear from Equations 64

[18] In fact this assertion is true for any discounting function that is *separable* (i.e., time-consistent; see Section 2.2).

and 65 that $m_1(t)$ is "Giffen" only if it is "inferior," as Marshallian theory demands. And any preponement of consumption resulting from an interest rate increase can then be attributed to the "income effect" associated with this "price change."

Lastly, for the sake of completeness, it is worth recording the specialized forms of Equation 62 for hyperbolic

$$m_1(t) = ac\left[1 - r\int_t^T \frac{\mathrm{d}v}{[1+q\,[v-t]][e^{a[v-t]} - e^{-a[T-v]}]}\right] \tag{66}$$

and for generalized quasi-hyperbolic

$$m_1(t) = c\left[a - \bar{r} - \bar{q}\int_t^T \frac{a[e^{\xi\Xi}+1]\mathrm{d}v}{[e^{\xi\Xi}+e^{\xi[v-t]}][e^{a[v-t]} - e^{-a[T-v]}]}\right] \tag{67}$$

discounting. Using these formulas, the reader can confirm that our earlier conclusions about the impact of the parameters r, q, $\bar{r}$, and $\bar{q}$ on the consumption profile remain valid when the interest rate is nonzero.

5. Extensions and applications

5.1. *Credit constraints*

Apart from the boundary conditions listed in Equations 7–8, the planner's problem as defined in Section 2.1 does not restrict his choice of capital path k at calendar date t. In particular, this formulation endows the agent with an infinite capacity to borrow that we may at times wish to rein in; for example via the requirement that

$$k(s) \geq 0 \tag{68}$$

for each $s \in [t, T]$. (More general credit constraints are discussed in Tyson, 2006, pp. 16–20.)

Let us impose the above inequality in the context of retirement savings, assuming for simplicity both that $K = 0$ and that once the agent begins saving (i.e., once the credit constraint becomes slack) he plans to hold strictly positive capital up until the horizon T. Defining

$$t^* := \max\{0 \leq t < T : h(t) = 0\}, \tag{69}$$

we have (when f is linear) that the ODE in Equation 28 holds on the interval $[t^*, T]$ with initial condition $h(t^*) = 0$. And the capital histories

$$h(t) = \begin{cases} 0 & \text{for } t < t^*, \\ \int_{t^*}^t \frac{\Psi^{\mathrm{e}}(v)}{\Psi^{\mathrm{e}}(t)}\left[\beta(v) + \Phi^{\mathrm{e}}(v)\int_v^T \frac{[\gamma(w)\log[\delta(w,v)\Pi(w,v)] - \beta(w)]\mathrm{d}w}{\Pi(w,v)}\right]\mathrm{d}v & \text{for } t \geq t^*; \end{cases} \tag{70}$$

for exponential utility and

$$h(t) = \begin{cases} 0 & \text{for } t < t^*, \\ \int_{t^*}^t \frac{\Psi^{\mathrm{p}}(v)}{\Psi^{\mathrm{p}}(t)}\left[\overrightarrow{\beta}(v) - \Phi^{\mathrm{p}}(v)\int_v^T \frac{\overrightarrow{\beta}(w)\mathrm{d}w}{\Pi(w,v)}\right]\mathrm{d}v & \text{for } t \geq t^*; \end{cases} \tag{71}$$

for power utility can then be obtained as special cases of Equations 36 and 45, respectively.

It remains only to determine the calendar date t^* at which the planner commences saving. By definition this date must satisfy $h(t^*) = 0$, and we have also that

$$h_1(t^*) = k_1^{t^*}(t^*) = \lim_{t \nearrow t^*} k_1^t(t) = \lim_{t \nearrow t^*} 0 = 0 \tag{72}$$

as a consequence of our many tacit smoothness assumptions. Equation 28 then yields the relation

$$0 = \int_{t^*}^{T} \left[u_1(\cdot, v)^{\mathrm{inv}} \left(\frac{u_1(\beta(t^*), t^*)}{\delta(v, t^*)\Pi\,(v, t^*)} \right) - \beta(v) \right] \frac{\mathrm{d}v}{\Pi\,(v, t^*)}, \tag{73}$$

now to be considered an algebraic equation in t^* rather than an ODE. And by specializing this implicit definition to

$$\frac{\beta(t^*)}{\Phi^{\mathrm{e}}(t^*)} = \int_{t^*}^{T} \frac{[\beta(v) - \gamma(v) \log [\delta(v, t^*)\Pi\,(v, t^*)]]\,\mathrm{d}v}{\Pi\,(v, t^*)} \tag{74}$$

in the case of exponential and

$$\frac{\overrightarrow{\beta}(t^*)}{\Phi^{\mathrm{p}}(t^*)} = \int_{t^*}^{T} \frac{\overrightarrow{\beta}(v)\mathrm{d}v}{\Pi\,(v, t^*)} \tag{75}$$

in the case of power utility, we can complete the construction of the credit-constrained capital histories in Equations 70 and 71, respectively.

As an example, consider an agent active between the ages of 20 and 70 (i.e., $T = 50$) and who enters this period with no assets (i.e., $K = 0$). Assume power utility with $p = 1$ and $\kappa(s) = 0$ everywhere. Let the discounting function be generalized quasi-hyperbolic with parameters $\bar{r} = 0.03$, $\bar{q} = 0.05$, $\Xi = 3$, and $\xi = 1$. And finally, impose the linear technology combining the constant interest rate $a = 0.04$ with the exogenous income having "impulse" specification

$$\beta^{\mathrm{i}}(s) := b \exp[-\chi[B - s]^2] \tag{76}$$

and parameters $b = 50$, $B = 20$, and $\chi = 0.0025$.

In the unconstrained case, the decision maker first borrows capital to finance his consumption and then at $t \approx 8$ begins to save part of his rapidly-growing income, building up a "retirement fund" that peaks near $h(36) \approx 190$. (See Figure 8.) In the constrained case, on the other hand, consumption tracks income until $t^* \approx 10$, at which point the planner embarks on a savings program that peaks near $h(34) \approx 232$. The constrained agent thus begins to save later but accumulates more capital than his unconstrained alter ego, consuming more in "old age" than does the latter as a result.

5.2. *Limited foresight and partial commitment*

The current framework can be extended without great difficulty to model decision makers who do not always take the *entire* future into account, but rather consider only the *immediate* future up to a given "planning horizon." Denoting the horizon applicable to calendar date $t < T$ by $\Omega(t) \in (t, T]$, we can express our agent's objective function at this point in time as

$$\hat{U}^t := \int_{t}^{\Omega(t)} \delta(s, t) u(x(s), s) \mathrm{d}s \tag{77}$$

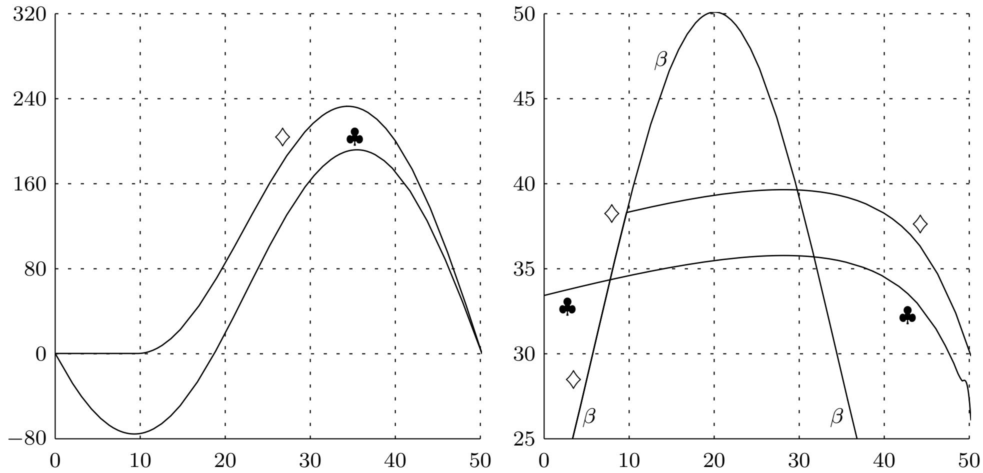

Fig. 8. Capital (left panel) and consumption and income histories (right panel) for an unconstrained problem (♣) and its credit-constrained counterpart (♢).

(cf. Equation 2). Writing $g(s)$ for the target level of capital at scheduling date s (should this date be realized as a planning horizon) and imposing $g(T) = 0$, we can then replace Equation 28 with its limited-foresight generalization

$$h(t) = \frac{g(\Omega(t))}{\Pi\left(\Omega(t), t\right)} + \cdots$$
$$\cdots + \int_t^{\Omega(t)} \left[u_1(\cdot, v)^{\mathrm{inv}} \left(\frac{u_1(\alpha(t)h(t) + \beta(t) - h_1(t), t)}{\delta(v,t)\Pi\left(v,t\right)}\right) - \beta(v)\right] \frac{\mathrm{d}v}{\Pi\left(v,t\right)}. \quad (78)$$

Specializing to exponential utility (see Equation 29) now requires the modified definitions

$$\hat{\Phi}^{\mathrm{e}}(t) := \left[\int_t^{\Omega(t)} \frac{\gamma(v)\mathrm{d}v}{\gamma(t)\Pi\left(v,t\right)}\right]^{-1} \quad (79)$$

and

$$\hat{\Psi}^{\mathrm{e}}(t) := \exp \int_0^t \left[\hat{\Phi}^{\mathrm{e}}(v) - \alpha(v)\right] \mathrm{d}v; \quad (80)$$

and leads to the capital history

$$h(t) = \frac{K}{\hat{\Psi}^{\mathrm{e}}(t)} + \int_0^t \frac{\hat{\Psi}^{\mathrm{e}}(v)}{\hat{\Psi}^{\mathrm{e}}(t)} \left[\frac{\hat{\Phi}^{\mathrm{e}}(v) g(\Omega(v))}{\Pi\left(\Omega(v), v\right)} + \cdots\right.$$
$$\left.\cdots + \beta(v) + \hat{\Phi}^{\mathrm{e}}(v) \int_v^{\Omega(v)} \frac{\left[\gamma(w) \log\left[\delta(w,v)\Pi\left(w,v\right)\right] - \beta(w)\right] \mathrm{d}w}{\Pi\left(w,v\right)}\right] \mathrm{d}v \quad (81)$$

(cf. Equation 36). Similarly, specializing to power utility (see Equation 37) requires the definitions

$$\hat{\Phi}^{\mathrm{p}}(t) := \left[\int_t^{\Omega(t)} \frac{\delta(v,t)^p \mathrm{d}v}{\Pi\left(v,t\right)^{1-p}}\right]^{-1} \quad (82)$$

and

$$\hat{\Psi}^{\mathrm{p}}(t) := \exp \int_0^t \left[\hat{\Phi}^{\mathrm{p}}(v) - \alpha(v)\right] \mathrm{d}v; \tag{83}$$

and leads to the history

$$h(t) = \frac{K}{\hat{\Psi}^{\mathrm{p}}(t)} + \int_0^t \frac{\hat{\Psi}^{\mathrm{p}}(v)}{\hat{\Psi}^{\mathrm{p}}(t)} \left[\frac{\hat{\Phi}^{\mathrm{p}}(v) g(\Omega(v))}{\Pi\left(\Omega(v), v\right)} + \overrightarrow{\beta}(v) - \hat{\Phi}^{\mathrm{p}}(v) \int_v^{\Omega(v)} \frac{\overrightarrow{\beta}(w)\mathrm{d}w}{\Pi\left(w, v\right)}\right] \mathrm{d}v \tag{84}$$

(cf. Equation 45).

Generalizing our analysis to allow for limited foresight introduces a potential source of time inconsistency quite independent of any possible nonseparability of the discounting function: When the horizon $\Omega(t)$ varies with t, the planner will generally behave in an inconsistent fashion even under the standard specification δ^{e} (see Equation 48), which would otherwise guarantee consistency. This fact can be seen most clearly when there is *no discounting whatsoever*, as in the following example.

Let $T = 100$, $K = 0$, and $\delta(s,t) = 1$ everywhere. Assume power utility with $p = 1/2$ and $\kappa(s) = 0$ everywhere. Impose a linear technology, zero interest, and the impulse-function income (see Equation 76) with parameters $b = 9$, $B = 50$, and $\chi = 1/2500$. Adopt the "stationary" horizon map

$$\Omega^{\mathrm{s}}(t) := \begin{cases} t+\omega & \text{for } t < T-\omega, \\ T & \text{for } t \geq T-\omega; \end{cases} \tag{85}$$

where $\omega > 0$ measures the (constant) scheduling interval considered by the planner. And finally, let $g(s) = 0$ everywhere, so that no provision is made at calendar date t for the period following $\Omega(t)$.

In the problem just posed, setting $\omega = 100$ amounts to assuming unlimited foresight. Under this assumption, since the interest and discount rates are both zero, the agent first borrows and then saves to create a constant consumption path over his "lifetime." (See Figure 9.) As the planning window controlled by ω narrows, perfect consumption smoothing continues to be achieved only on the interval $[T-\omega, T]$, while at earlier dates the decision maker's consumption begins to track his income. Here the permanent income hypothesis fails for the obvious reason that inflows arriving after the horizon $\Omega(t)$ are not taken into account in date t planning, and as a result consumption diverges from the unlimited-foresight benchmark whenever average income over $[t, \Omega(t)]$ differs from average lifetime income.

As well as creating a new source of time inconsistency, the generalization to limited foresight gives us two additional degrees of freedom in specifying the planner's behavior. Firstly, the function Ω need not have the stationary form in Equation 85: It could be used to model foresight that increases or decreases over time; it could consist of a series of checkpoints that must be reached before the horizon changes; or it could be determined endogenously by contemplation costs or other such factors. And secondly, the function g need not be identically zero, as in the above example: It could return a fixed level of savings (e.g., "Keep two years' basic expenses in reserve.") except near T; it could be proportional to the income function β (e.g., "Keep one year's salary in reserve."); or it could be influenced by the planner's income — or lack of income — after the date in question. In short, both Ω and g can be chosen so as to implement a variety of behavioral rules.

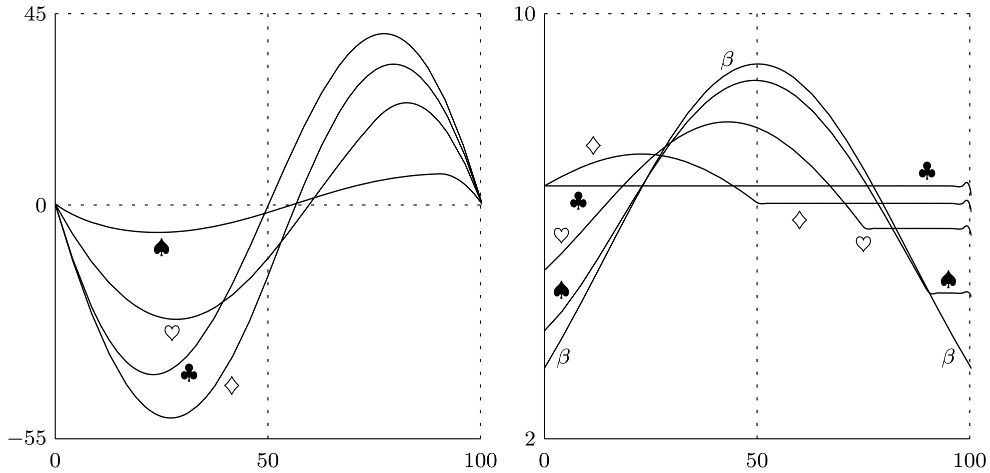

Fig. 9. Capital (left panel) and consumption and income histories (right panel) for a problem with imperfect foresight; computed for $\omega = 100$ (♣), $\omega = 50$ (♢), $\omega = 25$ (♡), and $\omega = 10$ (♠).

Suppose now that we set the target function g equal everywhere to the ex-ante-optimal capital schedule k^0, which solves Equation 10 for $t = 0$ with the boundary conditions $k^0(0) = K$ and $k^0(T) = 0$. (Denote the associated consumption schedule, related to k^0 via Equation 5, by x^0.) Then, at calendar date $t < T$, the agent planning his consumption over the period $[t, \Omega(t)]$ will be subject to the terminal constraint $k(\Omega(t)) = k^0(\Omega(t))$, and it follows that the ratio

$$\wp(t) := \frac{T - \Omega(t)}{T - t} \in [0, 1) \tag{86}$$

can be interpreted as a measure of his commitment to the plan k^0 at this point in time. For instance, when the horizon function is stationary, increasing either the date t or the parameter ω increases the fraction of remaining time over which the planner can exercise discretion, and thus *decreases* the commitment ratio $\wp$.

As an example of partial commitment, implemented in this fashion, consider again the planning problem set out in Section 5.1 above and adopt the stationary horizon map Ω^{s}. In this setting, the ex-ante-optimal consumption plan x^0 is U-shaped due to the discount rate crossing the interest rate from above. (See Figure 10.) Letting $\omega \to 0$ amounts to assuming full commitment to this plan, while setting $\omega = 50$ returns us to the unlimited-foresight case and grants the planner no ability to commit, resulting in significantly less saving. For intermediate values of ω, the capital and consumption paths trace out a homotopy between these two extremes. And it is in such cases that commitment can be described as partial: At each moment our agent feels the pull of the "retirement plan" k^0 formulated at age 20 with the aid of his financial advisor, but at no point is he bound to follow this plan exactly. Indeed, the "salience of the present" causes him to constantly fall short of his ambition to save, with the extent of this failure being determined jointly by the quasi-hyperbolicity and planning window parameters $\bar{q}$ and ω.

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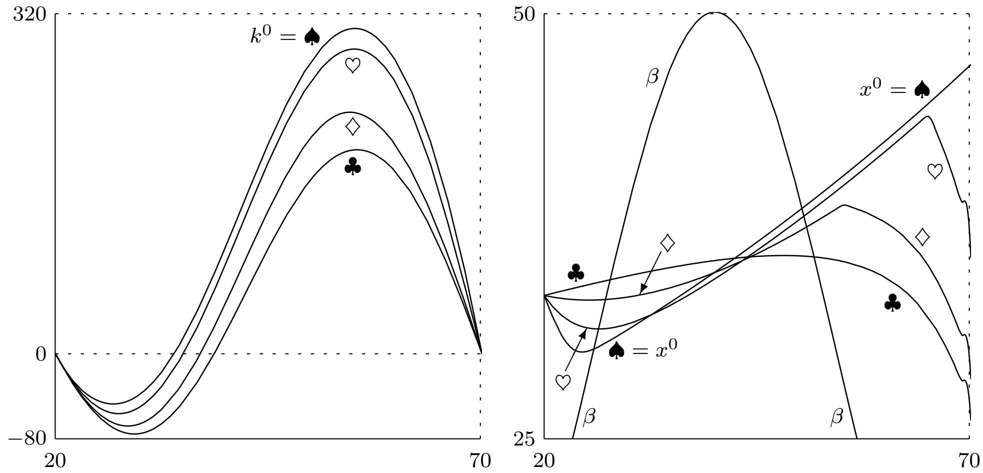

Fig. 10. Capital (left panel) and consumption and income histories (right panel) for a problem with partial commitment; computed for $\omega = 50$ (♣), $\omega = 15$ (◇), $\omega = 5$ (♡), and $\omega \to 0$ (♠).

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