

# EXISTENCE AND NON-EXISTENCE OF GLOBAL SOLUTIONS FOR SEMILINEAR HEAT EQUATIONS AND INEQUALITIES ON SUB-RIEMANNIAN MANIFOLDS, AND FUJITA EXPONENT ON UNIMODULAR LIE GROUPS

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*Dedicated to Tokio Matsuyama on the occasion of his 60<sup>th</sup> birthday*

ABSTRACT. In this paper we study the global well-posedness of the following Cauchy problem on a sub-Riemannian manifold  $M$ :

$$\begin{cases} u_t - \mathfrak{L}_M u = f(u), & x \in M, t > 0, \\ u(0, x) = u_0(x), & x \in M, \end{cases}$$

for  $u_0 \geq 0$ , where  $\mathfrak{L}_M$  is a sub-Laplacian of  $M$ . In the case when  $M$  is a connected unimodular Lie group  $\mathbb{G}$ , which has polynomial volume growth, we obtain a critical Fujita exponent, namely, we prove that all solutions of the Cauchy problem with  $u_0 \not\equiv 0$ , blow up in finite time if and only if  $1 < p \leq p_F := 1 + 2/D$  when  $f(u) \simeq u^p$ , where  $D$  is the global dimension of  $\mathbb{G}$ . In the case  $1 < p < p_F$  and when  $f : [0, \infty) \rightarrow [0, \infty)$  is a locally integrable function such that  $f(u) \geq K_2 u^p$  for some  $K_2 > 0$ , we also show that the differential inequality

$$u_t - \mathfrak{L}_M u \geq f(u)$$

does not admit any nontrivial distributional (a function  $u \in L^p_{loc}(Q)$  which satisfies the differential inequality in  $\mathcal{D}'(Q)$ ) solution  $u \geq 0$  in  $Q := (0, \infty) \times \mathbb{G}$ . Furthermore, in the case when  $\mathbb{G}$  has exponential volume growth and  $f : [0, \infty) \rightarrow [0, \infty)$  is a continuous increasing function such that  $f(u) \leq K_1 u^p$  for some  $K_1 > 0$ , we prove that the Cauchy problem has a global, classical solution for  $1 < p < \infty$  and some positive  $u_0 \in L^q(\mathbb{G})$  with  $1 \leq q < \infty$ . Moreover, we also discuss all these results in more general settings of sub-Riemannian manifolds  $M$ .

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## 1. INTRODUCTION AND RESULTS

Let  $M$  be a sub-Riemannian manifold with a smooth volume  $\mu$ . Recall that if  $X_1, \dots, X_k$  is a local orthonormal frame, then the horizontal gradient on  $M$  is defined as

$$\nabla\varphi = \sum_{i=1}^k X_i(\varphi)X_i,$$

where  $X_i(\varphi)$  is the Lie derivative of  $\varphi$  in the direction of  $X_i$ . Denote by  $\operatorname{div}_\mu X$  the divergence of a vector field  $X$  with respect to a volume  $\mu$ , and it is defined by the identity  $L_X\mu = (\operatorname{div}_\mu X)\mu$ , where  $L_X$  is the Lie derivative with respect to  $X$ . The sub-Laplacian associated with the sub-Riemannian structure is defined as the divergence of the gradient, that is,  $\mathfrak{L}_M\varphi = \operatorname{div}_\mu(\nabla\varphi)$ , and it can be written in a local orthonormal frame  $X_1, \dots, X_k$  as

$$\mathfrak{L}_M = \sum_{i=1}^k (X_i^2 + (\operatorname{div}_\mu X_i)X_i). \quad (1.1)$$

Therefore, the sub-Laplacian  $\mathfrak{L}_M$  is the natural generalisation of the Laplace-Beltrami operator defined on a Riemannian manifold. Note that the sub-Laplacian  $\mathfrak{L}_M$  always can be expressed as the sum of squares of the elements of the orthonormal frame plus a first order term that depends on the choice of the volume  $\mu$ .

Let  $h_t(x, y)$  denote the heat kernel for  $u_t - \mathfrak{L}_M u = 0$  for  $x, y \in M$  and  $t > 0$ , that is, for every  $y \in M$  the function  $u(t, x) := p_t(x, y)$ ,  $x \in M$ ,  $t > 0$  is a classical solution to the heat equation  $u_t - \mathfrak{L}_M u = 0$  in  $(0, \infty) \times M$ . Its existence, smoothness, symmetry, and positivity are guaranteed by classical results, see for instance [Str86] or [BBN12, Section 2.2] and [BBN16, Section 2.1] for more references.

Let us consider the following Cauchy problem:

$$\begin{cases} u_t - \mathfrak{L}_M u = f(u), & x \in M, t > 0, \\ u(0, x) = u_0(x), & x \in M, \end{cases} \quad (1.2)$$

for  $u_0 \geq 0$ . Recall that

$$e^{t\mathfrak{L}_M} u_0(x) = \int_M h_t(x, y) u_0(y) d\mu_y, \quad x \in M, \quad t > 0.$$

In this paper, we show a sufficient condition on the initial data which guarantees the existence of global solutions of (1.2) on  $M$ :

**Theorem 1.1.** *Let  $M$  be a sub-Riemannian manifold. Let  $f : [0, \infty) \rightarrow [0, \infty)$  be a continuous increasing function such that  $f(u) \leq K_1 u^p$  for some positive constant  $K_1 > 0$  with  $1 < p < \infty$ . Let  $0 \leq u_0 \in L^q(M)$  with  $1 \leq q < \infty$  and assume that*

$$\int_0^\infty \|e^{s\mathfrak{L}_M} u_0\|_{L^\infty(M)}^{p-1} ds < \frac{1}{K_1(p-1)}. \quad (1.3)$$

*Then there exists a non-negative continuous curve  $u : [0, \infty) \rightarrow L^q(M)$  which is a global solution to (1.2) with initial value  $u_0$ . Moreover, we have*

$$(e^{t\mathfrak{L}_M} u_0)(x) \leq u(t, x) \leq C(e^{t\mathfrak{L}_M} u_0)(x), \quad \forall x \in M, \quad \forall t \geq 0, \quad (1.4)$$

for some  $C > 1$  (depending on  $u_0$ ). For example, (1.4) holds with

$$C = \left( 1 - K_1(p-1) \int_0^\infty \|e^{s\mathfrak{L}_M} u_0\|_{L^\infty(M)}^{p-1} ds \right)^{-\frac{1}{p-1}}.$$

**Remark 1.2.** We refer to [Zha99] and the references therein for the case when  $\mathfrak{L}_M$  is the Laplace-Beltrami operator,  $M$  is a noncompact, complete Riemannian manifold with polynomial volume growth, which include those with nonnegative Ricci curvatures. For the case of Riemannian manifolds with negative sectional curvature, we refer to [Pun12] as well as references therein.

When we know the behavior of the heat kernel, then one can see that to satisfy the condition (1.3) there appears a condition for the parameter  $p$ , which is usually called a Fujita exponent. For example, when  $M$  is a connected unimodular Lie group with polynomial volume growth of order  $D$ , since we have by (2.4) the estimate

$$\|e^{t\mathfrak{L}} u_0\|_{L^\infty(\mathbb{G})} \geq ct^{-D/2}, \quad t \rightarrow \infty,$$

for any nontrivial  $u_0 \geq 0$ , then we see that the condition (1.3) cannot be satisfied for  $p \leq p_F := 1 + 2/D$ . Here and in the sequel, when  $M$  is a unimodular Lie group, we will simplify the notation by writing  $\mathfrak{L}$  instead of  $\mathfrak{L}_M$ .

Usually, since the heat kernel is tightly connected to the volume growth, we will demonstrate below results on unimodular Lie groups, where only two situations may occur for the volume growth: polynomial and exponential. We will also discuss the obtained results in more general settings in Section 3.

Let  $\mathbb{G}$  be a connected unimodular Lie group, endowed with the Haar measure, and let  $X = \{X_1, \dots, X_k\}$  be a Hörmander system of left invariant vector fields. Let  $\rho(x, y)$  be the Carnot-Carathéodory distance  $\mathbb{G} \times \mathbb{G} \ni (x, y) \mapsto \rho(x, y)$  associated with  $X$ . We denote by  $\rho(x)$  the distance from the unit element of the group to  $x \in \mathbb{G}$ . Let  $V(t)$  be the volume of the ball  $B(x, t)$  centred at  $x \in \mathbb{G}$  and of radius  $t > 0$  for this distance. In this case, since the left invariant vector fields on  $\mathbb{G}$  are divergence free with respect to the (right) Haar measure, as a consequence of (1.1) the sub-Laplacian associated to the Haar measure has the form of “sum of squares”, that is,

$$\mathfrak{L} := \sum_{i=1}^k X_i^2.$$

Then, on  $\mathbb{G}$ , the Cauchy problem (1.2) becomes

$$\begin{cases} u_t - \mathfrak{L}u = f(u), & x \in \mathbb{G}, \quad t > 0, \\ u(0, x) = u_0(x), & x \in \mathbb{G}, \end{cases} \quad (1.5)$$

for  $u_0 \geq 0$ .

Recall that we have  $V(t) \simeq t^d$  for  $t \in (0, 1)$ , where  $d = d(\mathbb{G}, X) \in \mathbb{N}$  is the local dimension. In the case  $t \geq 1$ , as we mentioned above, only two situations may occur, independently of the choice of  $X$  (see e.g. [CRT01] or [Gui73]): either  $\mathbb{G}$  has polynomial volume growth of order  $D$ , which means that there exists the global dimension  $D = D(\mathbb{G}) \in \mathbb{N}_0$  (i.e.  $\mathbb{N} \cup \{0\}$ ) such that  $V(t) \simeq t^D$ ,  $t \geq 1$ , or  $\mathbb{G}$  has exponential volume growth, that is, there exist positive constants  $c_1, C_1, c_2$  and  $C_2$  such that  $c_1 e^{c_2 t} \leq V(t) \leq C_1 e^{C_2 t}$  for  $t \geq 1$ . Note that (see e.g. [CRT01, Page 285]) the dimension  $D$  at infinity depends only on the group  $\mathbb{G}$  but not on the system

X. Let us also recall that the closed subgroups of nilpotent Lie groups, connected Type  $R$  Lie groups, motion groups, the Mautner group and compact groups are all examples of polynomial growth groups (see e.g. [Sch93, Section 1.5]). For examples of the unimodular Lie groups with exponential volume growth we can refer e.g. [CM96, Section 2] and references therein.

Let us now state the main results on  $\mathbb{G}$ :

**Theorem 1.3.** *Let  $\mathbb{G}$  be a connected unimodular Lie group with polynomial volume growth of order  $D$  and let  $1 < p < \infty$ .*

- (i) *Let  $p < p_F = 1 + 2/D$ . Let  $f : [0, \infty) \rightarrow [0, \infty)$  be a locally integrable function such that  $f(u) \geq K_2 u^p$  for some positive constant  $K_2 > 0$ . Then the differential inequality*

$$u_t - \mathfrak{L}u \geq f(u) \tag{1.6}$$

*does not admit any nontrivial distributional solution  $u \geq 0$  in  $(0, \infty) \times \mathbb{G}$ .*

- (ii) *Let  $p = p_F = 1 + 2/D$ . Let  $f : [0, \infty) \rightarrow [0, \infty)$  be a locally integrable function such that  $f(u) \geq K_2 u^p$  for some positive constant  $K_2 > 0$ . Then the equation*

$$u_t - \mathfrak{L}u = f(u) \tag{1.7}$$

*does not admit any nontrivial distributional solution  $u \geq 0$  in  $(0, \infty) \times \mathbb{G}$ .*

- (iii) *Let  $p > p_F = 1 + 2/D$ . Let  $f : [0, \infty) \rightarrow [0, \infty)$  be a continuous increasing function such that  $f(u) \leq K_1 u^p$  for some positive constant  $K_1 > 0$ . Then, for any  $1 \leq q < \infty$  the Cauchy problem (1.5) has a global, classical solution for some positive  $u_0 \in L^q(\mathbb{G})$ .*

**Remark 1.4.** By a distributional solution, we mean in Parts (i) and (ii) of Theorem 1.3 a function  $u \in L^p_{loc}(Q)$  which satisfies (1.6) and (1.7) in  $\mathcal{D}'(Q)$ , respectively, where  $Q := (0, \infty) \times \mathbb{G}$ .

**Theorem 1.5.** *Let  $\mathbb{G}$  be a connected unimodular Lie group with exponential volume growth and let  $1 < p < \infty$ . Let  $f : [0, \infty) \rightarrow [0, \infty)$  be a continuous increasing function such that  $f(u) \leq K_1 u^p$  for some positive constant  $K_1 > 0$ . Then, for any  $1 \leq q < \infty$  the Cauchy problem (1.5) has a global, classical solution for some positive  $u_0 \in L^q(\mathbb{G})$ .*

**Remark 1.6.** Consider problem (1.5) with  $f(u) \simeq u^p$ . Then, combining these Theorems 1.3 and 1.5, one comes to the following interesting conclusion: In the case  $D = 0$  (e.g. when  $\mathbb{G}$  is a compact group), that is, the case when the volume growth at infinity is constant, we see from Theorem 1.3 that the Cauchy problem (1.5) does not admit any nontrivial distributional solution  $u \geq 0$  in  $(0, \infty) \times \mathbb{G}$  for  $1 < p < \infty$ . In the case of polynomial volume growth, there exists a global, classical solution of (1.5) for  $p > p_F = 1 + 2/D$  and some positive  $u_0 \in L^q(\mathbb{G})$  with  $1 \leq q < \infty$ . When  $\mathbb{G}$  has exponential volume growth, then the Cauchy problem (1.5) has a global, classical solution for  $p > 1$  and some positive  $u_0 \in L^q(\mathbb{G})$  with  $1 \leq q < \infty$  by Theorem 1.5. For compact Lie groups the same kind of phenomenon (blow-up in finite time for all  $p > 1$  under suitable sign assumptions for the Cauchy data) has been recently proved also for the semilinear wave and damped wave equations, see [Pal21a, Pal21b].

Concerning the existence in Part (iii) of Theorem 1.3, we actually have the following much stronger property:

**Theorem 1.7.** *Let  $\mathbb{G}$  be a connected unimodular Lie group with polynomial volume growth of order  $D$ . Consider problem (1.5) with  $p_F < p < \infty$ ,  $u_0 \in L^q(\mathbb{G})$  with  $1 \leq q < \infty$ , and let  $\gamma > 0$ . Let  $f : [0, \infty) \rightarrow [0, \infty)$  be a continuous increasing function such that  $f(u) \leq K_1 u^p$  for some positive constant  $K_1 > 0$ . There exists  $\varepsilon = \varepsilon(\gamma) > 0$  such that, if*

$$0 \leq u_0(x) \leq \varepsilon h_\gamma(x), \quad x \in \mathbb{G}, \quad (1.8)$$

*then there exists a non-negative continuous curve  $u : [0, \infty) \rightarrow L^q(\mathbb{G})$  which is a global solution to (1.5) with initial value  $u_0$ . Moreover, we have*

$$u(t, x) \leq C h_{t+\gamma}(x), \quad x \in \mathbb{G}, \quad t \in (0, \infty), \quad (1.9)$$

*for some  $C = C(\gamma) > 0$ .*

Similarly, concerning Theorem 1.5, we have the following stronger property:

**Theorem 1.8.** *Let  $\mathbb{G}$  be a connected unimodular Lie group with exponential volume growth. Consider problem (1.5) with  $1 < p < \infty$ ,  $u_0 \in L^q(\mathbb{G})$  with  $1 \leq q < \infty$ , and let  $\gamma > 0$ . Let  $f : [0, \infty) \rightarrow [0, \infty)$  be a continuous increasing function such that  $f(u) \leq K_1 u^p$  for some positive constant  $K_1 > 0$ . There exists  $\varepsilon = \varepsilon(\gamma) > 0$  such that, if*

$$0 \leq u_0(x) \leq \varepsilon h_\gamma(x), \quad x \in \mathbb{G}, \quad (1.10)$$

*then there exists a non-negative continuous curve  $u : [0, \infty) \rightarrow L^q(\mathbb{G})$  which is a global solution to (1.5) with initial value  $u_0$ . Moreover, we have*

$$u(t, x) \leq C h_{t+\gamma}(x), \quad x \in \mathbb{G}, \quad t \in (0, \infty), \quad (1.11)$$

*for some  $C = C(\gamma) > 0$ .*

In the abelian case  $\mathbb{G} = (\mathbb{R}^n, +)$ , the phenomenon of finite time blow up was first considered by H. Fujita in [Fuj66], where in the proof Gaussian test functions depending only on  $x$  (given by the heat kernel with  $t$  as a parameter) were involved, hence requiring more regularity of the solutions in time. Since then, many people devoted themselves to this problem. For example, we refer to [MP01] and [QP07], where the proof is based on rescalings of a simple, compactly supported test-function, depending on  $x$  and  $t$ . A related proof can be found in [BP85], where the test-functions were obtained by solving an adjoint problem. We also refer to [HM04, LP76, Qui91, SW97, Wei80, Wei81] and the references therein for the case of  $\mathbb{G} = (\mathbb{R}^n, +)$ , as well as [JKS16], [GP21] on the Heisenberg group and [Pas98] on stratified Lie groups. In the case of stratified Lie groups the blow-up result has been studied also in [GP19]. There is huge literature on such Euclidean problems that we do not even attempt to review here.

For a comparison principle for weak solutions of  $p$ -Laplacian heat equation in a bounded domain, we refer to recent works [LZZ18] when  $\mathbb{G} = (\mathbb{R}^n, +)$ , and to [RS18] and [RY20] when  $\mathbb{G}$  is a graded Lie group, the latter two also allowing more general hypoelliptic differential operators (Rockland operators).

The paper is structured as follows. In Section 2 we give the proof of the main results. Finally, these results are discussed on more general settings in Section 3.

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## 2. PROOFS

*Proof of Theorem 1.1.* Here, the argument required for the extension to the sub-Riemannian manifold  $M$  is virtually the same as employed by Weissler in [Wei81, Theorem 3], see also [QP07, Section 20] for more details and other arguments. So, as in [Wei81] we study (1.2) via the following integral equation:

$$u(t) = e^{t\mathfrak{L}_M} u_0 + \int_0^t e^{(t-s)\mathfrak{L}_M} f(u(s)) ds.$$

Denote

$$\omega(t) := \left( 1 - K_1(p-1) \int_0^t \|e^{s\mathfrak{L}_M} u_0\|_{L^\infty(M)}^{p-1} ds \right)^{-\frac{1}{p-1}}.$$

Note that  $\omega(0) = 1$  and  $\omega'(t) = K_1 \|e^{t\mathfrak{L}_M} u_0\|_{L^\infty(M)}^{p-1} (\omega(t))^p$ . Then we have

$$\omega(t) = 1 + K_1 \int_0^t \|e^{s\mathfrak{L}_M} u_0\|_{L^\infty(M)}^{p-1} (\omega(s))^p ds. \quad (2.1)$$

Let  $u : [0, \infty) \rightarrow L^q(M)$  be a continuous curve with  $1 \leq q < \infty$  and  $e^{t\mathfrak{L}_M} u_0 \leq u(t) \leq \omega(t)e^{t\mathfrak{L}_M} u_0$  for all  $t \geq 0$ . Noting this, if we denote

$$\mathfrak{F}u(t) := e^{t\mathfrak{L}_M} u_0 + \int_0^t e^{(t-s)\mathfrak{L}_M} f(u(s)) ds,$$

then using  $f(u) \leq Ku^p$  and the positivity of the heat kernel (see Introduction for the references), we get

$$\begin{aligned} \mathfrak{F}u(t) &\leq e^{t\mathfrak{L}_M} u_0 + K_1 \int_0^t e^{(t-s)\mathfrak{L}_M} (u(s))^p ds \\ &\leq e^{t\mathfrak{L}_M} u_0 + K_1 \int_0^t e^{(t-s)\mathfrak{L}_M} (e^{s\mathfrak{L}_M} u_0)^p (\omega(s))^p ds \\ &\leq e^{t\mathfrak{L}_M} u_0 + K_1 \int_0^t e^{(t-s)\mathfrak{L}_M} (e^{s\mathfrak{L}_M} u_0) \|e^{s\mathfrak{L}_M} u_0\|_{L^\infty(M)}^{p-1} (\omega(s))^p ds \\ &= e^{t\mathfrak{L}_M} u_0 \left( 1 + K_1 \int_0^t \|e^{s\mathfrak{L}_M} u_0\|_{L^\infty(M)}^{p-1} (\omega(s))^p ds \right), \end{aligned} \quad (2.2)$$

which implies with (2.1) that  $e^{t\mathfrak{L}_M} u_0 \leq \mathfrak{F}u(t) \leq \omega(t)e^{t\mathfrak{L}_M} u_0$  for all  $t \geq 0$ .

Now we take the sequence of functions  $\{v_k(t)\}_{k=0}^\infty$  such that  $v_0(t) = e^{t\mathfrak{L}_M} u_0$  and  $v_{k+1}(t) = \mathfrak{F}v_k(t)$ , and show that this sequence converges to the desired solution. Note that since  $e^{t\mathfrak{L}_M} u_0 \leq v_0(t) \leq \omega(t)e^{t\mathfrak{L}_M} u_0$  then by induction and discussing as in (2.2), we get  $e^{t\mathfrak{L}_M} u_0 \leq v_k(t) \leq \omega(t)e^{t\mathfrak{L}_M} u_0$  for each  $k$ . We also note that since  $v_0(t) \leq v_1(t)$  for all  $t \geq 0$ , and  $v_k(t) \leq v_{k+1}(t) \Rightarrow \mathfrak{F}v_k(t) \leq \mathfrak{F}v_{k+1}(t)$  for all  $t \geq 0$  by monotonicity of function  $f$ , then by induction one obtains  $v_k(t) \leq v_{k+1}(t)$  for all  $t \geq 0$ . Then, the dominated convergence theorem implies that  $v_k(t)$  converge in  $L^q(M)$  to a function which we call  $u(t)$ . This with the fact that  $e^{t\mathfrak{L}_M} u_0 \leq v_k(t) \leq \omega(t)e^{t\mathfrak{L}_M} u_0$  for each  $k$  gives  $e^{t\mathfrak{L}_M} u_0 \leq u(t) \leq \omega(t)e^{t\mathfrak{L}_M} u_0$  for all  $t \geq 0$ .

Now we need to prove that the function  $u(t)$  is a global, classical solution of (1.2). Note that since we have  $v_k(t) \leq \omega(t)e^{t\mathfrak{L}_M}u_0$  and  $f(u) \leq K_1u^p$ , then the functions  $s \mapsto e^{(t-s)\mathfrak{L}_M}f(v_k(s))$  are dominated by

$$\begin{aligned} e^{(t-s)\mathfrak{L}_M}f(v_k(s)) &\leq K_1e^{(t-s)\mathfrak{L}_M}(v_k(s))^p \\ &\leq K_1e^{(t-s)\mathfrak{L}_M}(\omega(s)e^{s\mathfrak{L}_M}u_0)^p \\ &\leq K_1e^{t\mathfrak{L}_M}u_0\|e^{s\mathfrak{L}_M}u_0\|_{L^\infty(M)}^{p-1}(\omega(s))^p \end{aligned}$$

in  $L^1(0, t; L^q(M))$ . These functions converge for every  $0 < s < t$  to  $e^{(t-s)\mathfrak{L}_M}f(u(s))$  monotonically in  $L^q(M)$  since the dominating function is in  $L^q(M)$  for every  $0 < s < t$  and the fact that  $f$  is a continuous function. Then, the dominated convergence theorem for  $L^q$ -valued functions implies that

$$\lim_{k \rightarrow \infty} \int_0^t e^{(t-s)\mathfrak{L}_M}f(v_k(s))ds = \int_0^t e^{(t-s)\mathfrak{L}_M}f(u(s))ds,$$

which gives

$$u(t) = \lim_{k \rightarrow \infty} v_{k+1}(t) = \lim_{k \rightarrow \infty} \mathfrak{F}v_k(t) = \mathfrak{F}u(t),$$

which means that  $u(t)$  is a global solution of (1.2). Continuity of  $u(t)$  in  $L^q(M)$  easily follows by standard arguments.

The proof is complete.  $\square$

Before giving the proof of the main results on unimodular groups, let us briefly recall some necessary notations and some facts from [VCS92].

Recall that the heat kernel  $(t, x) \mapsto h_t(x)$  is a positive fundamental solution of  $u_t - \mathfrak{L}u = 0$ . This heat kernel satisfies the following (see e.g. [VCS92, Section I.3, Page 5] or [CRT01, Section 2.1.1, Page 295]) property:

$$\|h_t\|_{L^1(\mathbb{G})} = 1, \quad \forall t > 0. \quad (2.3)$$

Let  $\rho(x, y)$  be the Carnot-Carathéodory distance  $\mathbb{G} \times \mathbb{G} \ni (x, y) \mapsto \rho(x, y)$  associated with the Hörmander system of left invariant vector fields  $X$ . We also recall that  $\rho$  is symmetric and satisfies the triangle inequality (see e.g. [VCS92, Section III.4, Page 39]).

We will also use the following fundamental properties:

**Theorem 2.1.** [VCS92, VIII.2.9 Theorem] *Let  $\mathbb{G}$  be a connected unimodular Lie group with polynomial volume growth. Then there exist positive constants  $C_1 > 0$  and  $C_2 > 0$  such that*

$$C_1V(\sqrt{t})^{-1} \exp\left(-C_2\frac{(\rho(x))^2}{t}\right) \leq h_t(x) \leq C_2V(\sqrt{t})^{-1} \exp\left(-C_1\frac{(\rho(x))^2}{t}\right), \quad (2.4)$$

for all  $t > 0$  and  $x \in \mathbb{G}$ .

**Theorem 2.2.** [VCS92, VIII.4.3 Theorem] *Let  $\mathbb{G}$  be a connected unimodular Lie group with exponential volume growth. Then for every  $n \geq d$  and  $\varepsilon > 0$ , there exist  $C_{n,\varepsilon}$  such that*

$$|h_t(x)| \leq C_{n,\varepsilon}t^{-\frac{n}{2}} \exp\left(-\frac{(\rho(x))^2}{(4+\varepsilon)t}\right), \quad (2.5)$$

for all  $t > 0$  and  $x \in \mathbb{G}$ , where  $d$  is the local dimension of  $\mathbb{G}$ .

In order to prove Parts (i) and (ii) of Theorem 1.3, it is enough to prove the following result:

**Theorem 2.3.** *Let  $\mathbb{G}$  be a connected unimodular Lie group with polynomial volume growth of order  $D$  and  $1 < p < \infty$ . Let  $f : [0, \infty) \rightarrow [0, \infty)$  be a locally integrable function such that  $f(u) \geq K_2 u^p$  for some positive constant  $K_2 > 0$ . Let  $u_0 : \mathbb{G} \rightarrow [0, \infty]$  be measurable, such that  $u_0 > 0$  in a set of positive measure.*

(i) *If  $p < p_F = 1 + 2/D$ , then there is no nonnegative measurable global solution  $u : [0, \infty) \times \mathbb{G} \rightarrow [0, \infty]$  to the integral inequality*

$$u(t, x) \geq \int_{\mathbb{G}} h_t(y^{-1}x) u_0(y) dy + \int_0^t \int_{\mathbb{G}} h_{t-s}(y^{-1}x) f(u(s, y)) dy ds \quad (2.6)$$

*such that  $u(t, x) < \infty$  for a.e.  $(t, x) \in (0, \infty) \times \mathbb{G}$ .*

(ii) *If  $p = p_F = 1 + 2/D$ , then there is no nonnegative measurable global solution  $u : [0, \infty) \times \mathbb{G} \rightarrow [0, \infty]$  to the integral equation*

$$u(t, x) = \int_{\mathbb{G}} h_t(y^{-1}x) u_0(y) dy + \int_0^t \int_{\mathbb{G}} h_{t-s}(y^{-1}x) f(u(s, y)) dy ds \quad (2.7)$$

*such that  $u(t, x) < \infty$  for a.e.  $(t, x) \in (0, \infty) \times \mathbb{G}$ .*

Now let us show the following lemma on sub-Riemannian manifold  $M$ , which we will use in the proof of Theorem 2.3 when  $M$  is a connected unimodular Lie group:

**Lemma 2.4.** *Let  $M$  be a sub-Riemannian manifold with  $\|h_t\|_{L^1(M)} \leq 1$ . Let  $1 < p < \infty$  and  $T > 0$ . Let  $f : [0, \infty) \rightarrow [0, \infty)$  be a locally integrable function such that  $f(u) \geq K_2 u^p$  for some positive constant  $K_2 > 0$ . Let  $\vartheta_0 : M \rightarrow [0, \infty]$  and  $\vartheta : [0, T] \times M \rightarrow [0, \infty]$  be measurable and satisfy*

$$\vartheta(t) \geq e^{t\mathfrak{L}_M} \vartheta_0 + \int_0^t e^{(t-s)\mathfrak{L}_M} f(\vartheta(s)) ds \quad (2.8)$$

*a.e. in  $Q_T^M := [0, T] \times M$ . Assume that  $\vartheta(t, x) < \infty$  for a.e.  $(t, x) \in Q_T^M$ . Then we have*

$$t^{\frac{1}{p-1}} \|e^{t\mathfrak{L}_M} \vartheta_0\|_{L^\infty(M)} \leq A_p := (K_2(p-1))^{-\frac{1}{p-1}} \quad (2.9)$$

*for all  $t \in (0, T]$ .*

*Proof of Lemma 2.4.* Note that by virtue of Fubini's theorem for nonnegative measurable functions we will use operations such as interchange of integrals and moving of  $e^{-t\mathfrak{L}_M}$  inside integrals in the proof of this lemma. We notice that

$$e^{t\mathfrak{L}_M} F = e^{(t-s)\mathfrak{L}_M} e^{s\mathfrak{L}_M} F \quad (2.10)$$

for all  $0 < s < t$  and any measurable  $F : M \rightarrow [0, \infty]$ . Also, we obtain from Jensen's inequality and  $\|h_t\|_{L^1(M)} \leq 1$  that

$$e^{t\mathfrak{L}_M} F^p \geq (e^{t\mathfrak{L}_M} F)^p \quad (2.11)$$

for all measurable  $F : M \rightarrow [0, \infty]$ . Now, by redefining  $u$  on a null set, one may assume that (2.8) actually holds everywhere in  $(0, T) \times M$ . By assumption, we have  $\vartheta(\tau, \cdot) < \infty$  a.e. in  $M$  for a.e.  $\tau \in (0, T)$ . Let us fix such  $\tau$  and denote



$M_\tau := \{x \in M : \vartheta(\tau, x) < \infty\}$ . Then, (2.8) with  $f(u) \geq K_2 u^p$ , (2.10) and (2.11) imply for  $t \in [0, \tau]$  that

$$\begin{aligned} e^{(\tau-t)\mathfrak{L}_M} \vartheta(t) &\geq e^{\tau\mathfrak{L}_M} \vartheta_0 + K_2 \int_0^t e^{(\tau-s)\mathfrak{L}_M} (\vartheta(s))^p ds \\ &\geq e^{\tau\mathfrak{L}_M} \vartheta_0 + K_2 \int_0^t (e^{(\tau-s)\mathfrak{L}_M} \vartheta(s))^p ds =: g(t, \cdot), \end{aligned} \quad (2.12)$$

where we have also used that the heat kernel is positive (see Introduction for the references), so that the integration with it preserves inequalities between non-negative functions. Here, from the second inequality in (2.12) noting (2.8) and  $f(u) \geq K_2 u^p$ , we get

$$\begin{aligned} g(\tau, \cdot) &\leq e^{\tau\mathfrak{L}_M} \vartheta_0 + K_2 \int_0^\tau e^{(\tau-s)\mathfrak{L}_M} (\vartheta(s))^p ds \\ &\leq e^{\tau\mathfrak{L}_M} \vartheta_0 + \int_0^\tau e^{(\tau-s)\mathfrak{L}_M} f(\vartheta(s)) ds \\ &\leq \vartheta(\tau, \cdot), \end{aligned} \quad (2.13)$$

and so  $g(t, x) < \infty$  for all  $(t, x) \in M_\tau \times [0, \tau]$ . Fixing  $x \in M_\tau$ , we see that the function  $\phi(t) := g(t, x)$  is absolutely continuous on  $[0, \tau]$ , and that (2.12) implies

$$\phi'(t) = K_2 (e^{(\tau-t)\mathfrak{L}_M} \vartheta(t))^p(x) \geq K_2 (\phi(t))^p \quad (2.14)$$

for a.e.  $t \in [0, \tau]$ . For fixed  $x \in M_\tau$  we have  $\phi(t) = g(t, x) \geq (e^{\tau\mathfrak{L}_M} \vartheta_0)(x) > 0$  by the definition of the function  $g$  in (2.12), therefore, we can rewrite (2.14) as  $[\phi^{1-p}]' \leq -K_2(p-1)$ . By integrating this inequality over  $[0, \tau]$ , we obtain

$$[(e^{\tau\mathfrak{L}_M} \vartheta_0)(x)]^{1-p} = \phi^{1-p}(0) \geq \phi^{1-p}(\tau) + K_2(p-1)\tau \geq K_2(p-1)\tau, \quad (2.15)$$

which implies  $\tau^{1/(p-1)} \|e^{\tau\mathfrak{L}_M} \vartheta_0\|_{L^\infty(M)} \leq (K_2(p-1))^{-1/(p-1)}$ . In particular, this means that  $e^{t\mathfrak{L}_M} \vartheta_0 \in L^\infty(M)$  for a.e.  $t \in (0, T)$ . Since we know that  $t \mapsto \|e^{t\mathfrak{L}_M} v\|_{L^\infty(M)}$  is continuous for  $v \in L^\infty(M)$  and  $t > 0$ , then (2.10) yields that the function

$$t \mapsto t^{\frac{1}{p-1}} \|e^{t\mathfrak{L}_M} \vartheta_0\|_{L^\infty(M)}$$

is continuous in  $(0, T)$ , hence (2.9).  $\square$

**Corollary 2.5.** *Let  $M$  be a sub-Riemannian manifold with  $\|h_t\|_{L^1(M)} \leq 1$ . Let  $1 < p < \infty$  and  $T > 0$ . Let  $f : [0, \infty) \rightarrow [0, \infty)$  be a locally integrable function such that  $f(u) \geq K_2 u^p$  for some positive constant  $K_2 > 0$ . Let  $\vartheta_0 : M \rightarrow [0, \infty]$  and  $\vartheta : [0, T] \times M \rightarrow [0, \infty]$  be measurable and satisfy*

$$\vartheta(t) = e^{t\mathfrak{L}_M} \vartheta_0 + \int_0^t e^{(t-s)\mathfrak{L}_M} f(\vartheta(s)) ds$$

a.e. in  $Q_T^M = [0, T] \times M$ . Then we have

$$\|t^{\frac{1}{p-1}} e^{t\mathfrak{L}_M} \vartheta(\tau)\|_{L^\infty(M)} \leq A_p := (K_2(p-1))^{-\frac{1}{p-1}}$$

for all  $t \in (0, T - \tau]$  and a.e.  $\tau \in (0, T)$ .

*Proof of Corollary 2.5.* Denote  $\tilde{\vartheta}(t) := \vartheta(t+\tau)$ . Then, by (2.10) and Fubini's theorem we have for a.e.  $\tau \in (0, T)$  and a.e.  $t \in (\tau, T)$  that

$$\begin{aligned} \tilde{\vartheta}(t) &= e^{(t+\tau)\mathfrak{L}_M} \vartheta_0 + \int_0^{t+\tau} e^{(t+\tau-s)\mathfrak{L}_M} f(\vartheta(s)) ds \\ &= e^{t\mathfrak{L}_M} e^{\tau\mathfrak{L}_M} \vartheta_0 + \int_0^\tau e^{t\mathfrak{L}_M} e^{(\tau-s)\mathfrak{L}_M} f(\vartheta(s)) ds + \int_\tau^{t+\tau} e^{(t+\tau-s)\mathfrak{L}_M} f(\vartheta(s)) ds \\ &= e^{t\mathfrak{L}_M} \left( e^{\tau\mathfrak{L}_M} \vartheta_0 + \int_0^\tau e^{(\tau-s)\mathfrak{L}_M} f(\vartheta(s)) ds \right) + \int_0^t e^{(t-s)\mathfrak{L}_M} f(\tilde{\vartheta}(s)) ds \\ &= e^{t\mathfrak{L}_M} \vartheta(\tau) + \int_0^t e^{(t-s)\mathfrak{L}_M} f(\tilde{\vartheta}(s)) ds. \end{aligned}$$

Hence, Lemma 2.4 with  $\vartheta_0$  replaced by  $\vartheta(\tau)$  and  $T$  replaced by  $T-\tau$  for a.e.  $\tau \in (0, T)$  completes the proof.  $\square$

Now we are ready to prove Theorem 2.3.

*Proof of Theorem 2.3.* i) By way of contradiction let us assume that there exists a global solution of (2.6). Then, by Lemma 2.4 when  $M = \mathbb{G}$  (where we have  $\|h_t\|_{L^1(\mathbb{G})} = 1$  by (2.3)) we get

$$|u_0 * t^{\frac{1}{p-1}} h_t| \leq A_p. \quad (2.16)$$

For  $t > 1$  from Theorem 2.1 we get that

$$h_t(x) \geq C_1 t^{-\frac{D}{2}} \exp\left(-C_2 \frac{(\rho(x))^2}{t}\right)$$

for some positive constants  $C_1$  and  $C_2$ . Then, using this for a given measurable function  $v : \mathbb{G} \rightarrow [0, \infty]$ , we have

$$\lim_{t \rightarrow \infty} v * t^{\frac{D}{2}} h_t \geq C \|v\|_{L^1(\mathbb{G})} \quad (2.17)$$

pointwise in  $\mathbb{G}$ , where  $\|v\|_{L^1(\mathbb{G})} := \infty$  if  $v \notin L^1(\mathbb{G})$ . In the case  $p < p_F$ , (2.16) gives  $t^{D/2} \|u_0 * h_t\|_{L^\infty(\mathbb{G})} \rightarrow 0$  as  $t \rightarrow \infty$  which contradicts (2.17) with  $v = u_0$ .

ii) In the case  $p = p_F$ , again by way of contradiction we assume that there exists a global solution of (2.7). We redefine  $u$  on a null set, then assuming that (2.7) actually holds everywhere in  $(0, \infty) \times \mathbb{G}$ , we get

$$u(t + t_0) = u(t_0) * h_t + \int_0^t f(u(s + t_0)) * h_{t-s} ds, \quad (2.18)$$

for all  $t, t_0 > 0$ . Note that Corollary 2.5 when  $M = \mathbb{G}$  (where we have  $\|h_t\|_{L^1(\mathbb{G})} = 1$  by (2.3)) and (2.17) guarantee the existence of positive constant  $C$  such that

$$\|u(\tau)\|_{L^1(\mathbb{G})} \leq C \quad (2.19)$$

for a.e.  $\tau > 0$ . On the other hand, since the Carnot-Carathéodory distance satisfies the triangle inequality, we have  $(\rho(x, z))^2/(4t) \leq ((\rho(x))^2 + (\rho(z))^2)/(2t)$ , and by Theorem 2.1 we obtain

$$u(t, x) \geq (u_0 * h_t)(x) \geq C_1 t^{-\frac{D}{2}} e^{-2C_2 \frac{(\rho(x))^2}{t}} \int_{\mathbb{G}} e^{-2C_2 \frac{(\rho(z))^2}{t}} u_0(z) dz$$

for  $t \geq 1$ , and

$$u(t, x) \geq (u_0 * h_t)(x) \geq C_1 t^{-\frac{d}{2}} e^{-2C_2 \frac{(\rho(x))^2}{t}} \int_{\mathbb{G}} e^{-2C_2 \frac{(\rho(z))^2}{t}} u_0(z) dz$$

for  $0 < t < 1$ , which imply with (2.4) that

$$u(2C_2/C_1, x) \geq C_3 h_1, \quad x \in \mathbb{G}.$$

Using this, and the property  $h_{t+s} = h_t * h_s$  for  $s, t > 0$  and (2.18), we deduce that

$$u(s + 2C_2/C_1) \geq u(2C_2/C_1) * h_s \geq C_3 h_1 * h_s = C_3 h_{s+1}, \quad s > 0. \quad (2.20)$$

Now, Theorem 2.1,  $(p-1)D/2 = 1$  and (2.3) imply that

$$\begin{aligned} \|h_{s+1}^p\|_{L^1(\mathbb{G})} &\geq C_1^p (s+1)^{-pD/2} \left(\frac{s+1}{p}\right)^{D/2} \left(\frac{s+1}{p}\right)^{-D/2} \int_{\mathbb{G}} e^{-C_2 p \frac{(\rho(x))^2}{s+1}} dx \\ &\geq C_4 (s+1)^{-1} \|h_{\frac{C_1(s+1)}{C_2 p}}\|_{L^1(\mathbb{G})} = C_5 (s+1)^{-1} \end{aligned} \quad (2.21)$$

for all  $s > 1$  and some  $C_4, C_5 > 0$ . As in [QP07, Proposition 48.4], one can note that from (2.3) and Fubini's theorem we have  $e^{t\mathcal{L}}\psi \geq 0$  and

$$\|e^{t\mathcal{L}}\psi\|_{L^1(\mathbb{G})} = \int_{\mathbb{G}} \int_{\mathbb{G}} \psi(\zeta) h_t(\zeta^{-1}\eta) d\zeta d\eta = \int_{\mathbb{G}} \psi(\zeta) \left( \int_{\mathbb{G}} h_t(\zeta^{-1}\eta) d\eta \right) d\zeta = \|\psi\|_{L^1(\mathbb{G})}$$

for any  $\psi \geq 0$ . This calculation, (2.21), (2.18) with  $t_0 = 2C_2/C_1$  and (2.20) imply

$$\begin{aligned} \|u(t + 2C_2/C_1)\|_{L^1(\mathbb{G})} &\geq \int_0^t \|f(u(s + 2C_2/C_1)) * h_{t-s}\|_{L^1(\mathbb{G})} ds \\ &\geq K_2 \int_0^t \|u^p(s + 2C_2/C_1) * h_{t-s}\|_{L^1(\mathbb{G})} ds \\ &\geq K_2 \int_0^t \|(C_3 h_{s+1})^p * h_{t-s}\|_{L^1(\mathbb{G})} ds \\ &= K_2 C_3^p \int_0^t \|h_{s+1}^p\|_{L^1(\mathbb{G})} ds \\ &\geq K_2 C_3^p \int_1^t \|h_{s+1}^p\|_{L^1(\mathbb{G})} ds \\ &\geq K_2 C_3^p C_5 \int_1^t (s+1)^{-1} ds \rightarrow \infty \end{aligned}$$

as  $t \rightarrow \infty$ , which contradicts (2.19). □

Now we prove Theorem 1.7.

*Proof of Theorem 1.7.* To prove Theorem 1.7 we see by Theorem 1.1 that it is enough to show (1.3). Since by the assumption (1.8) we have  $u_0(x) \leq \varepsilon h_\gamma(x)$  for all  $x \in \mathbb{G}$ ,

and noting that  $\|h_t\|_{L^1(\mathbb{G})} = 1$ ,  $\forall t > 0$ , and Theorem 2.1, we obtain

$$\begin{aligned}
\int_0^\infty \|e^{s\mathcal{L}}u_0\|_{L^\infty(\mathbb{G})}^{p-1} ds &= \int_0^\infty \|u_0 * h_s\|_{L^\infty(\mathbb{G})}^{p-1} ds \\
&\leq \varepsilon \int_0^\infty \|h_\gamma * h_s\|_{L^\infty(\mathbb{G})}^{p-1} ds \\
&= \varepsilon \int_0^\infty \|h_{s+\gamma}\|_{L^\infty(\mathbb{G})}^{p-1} ds \\
&= \varepsilon \int_\gamma^\infty \|h_s\|_{L^\infty(\mathbb{G})}^{p-1} ds \\
&< C\varepsilon \left( \int_{\min(\gamma,1)}^1 s^{-\frac{d(p-1)}{2}} ds + \int_1^\infty s^{-\frac{D(p-1)}{2}} ds \right) \\
&< \frac{1}{K_1(p-1)}
\end{aligned}$$

for small  $\varepsilon > 0$  since  $D(p-1)/2 > 1$  and  $\gamma > 0$ .  $\square$

Now let us prove Theorem 1.8.

*Proof of Theorem 1.8.* Actually, the proof of this theorem is similar to the proof of Theorem 1.7, we use Theorem 2.2 instead of Theorem 2.1.

By (1.10),  $\|h_t\|_{L^1(\mathbb{G})} = 1$ ,  $\forall t > 0$ , and Theorem 2.2, one has

$$\begin{aligned}
\int_0^\infty \|e^{s\mathcal{L}}u_0\|_{L^\infty(\mathbb{G})}^{p-1} ds &= \int_0^\infty \|u_0 * h_s\|_{L^\infty(\mathbb{G})}^{p-1} ds \\
&\leq \varepsilon \int_0^\infty \|h_\gamma * h_s\|_{L^\infty(\mathbb{G})}^{p-1} ds \\
&= \varepsilon \int_0^\infty \|h_{s+\gamma}\|_{L^\infty(\mathbb{G})}^{p-1} ds \\
&< C\varepsilon \int_0^\infty (s+\gamma)^{-\frac{n(p-1)}{2}} ds
\end{aligned} \tag{2.22}$$

for small  $\varepsilon > 0$  and for every  $n \geq d$ . So, letting  $n \rightarrow \infty$  we observe that in this case the condition (1.3) holds for  $1 < p < \infty$ .

Thus, Theorem 1.1 concludes the proof.  $\square$

### 3. THE GLOBAL WELL-POSEDNESS ON SUB-RIEMANNIAN MANIFOLDS

In this section we discuss the obtained results on unimodular groups in more general settings, namely, on sub-Riemannian manifolds  $M$ . To have an analogue of Part (i) of Theorem 1.3 on  $M$ , we need to assume that the following estimate for the heat kernel from below holds on  $M$  (see the proof of Part (i) of Theorem 2.3): assume that there exist constants  $C_1, C_2 > 0$  and  $a \geq 0$  such that

$$h_t(x, y) \geq C_1 t^{-\frac{a}{2}} \exp\left(-C_2 \frac{(\rho(x, y))^2}{t}\right), \tag{3.1}$$

for all  $t > 1$  and  $x, y \in M$ . Therefore, we have

**Theorem 3.1.** *Assume that (3.1) holds on  $M$  for some  $a \geq 0$  and that  $\int_M h_t(x, y) d\mu_y \leq 1$  for all  $x \in M$  and  $t > 0$ . Let  $1 < p < 1 + 2/a$ . Let  $f : [0, \infty) \rightarrow [0, \infty)$  be a locally integrable function such that  $f(u) \geq K_2 u^p$  for some positive constant  $K_2 > 0$ . Then the differential inequality*

$$u_t - \mathfrak{L}_M u \geq f(u) \quad (3.2)$$

*does not admit any nontrivial distributional solution  $u \geq 0$  in  $(0, \infty) \times M$ .*

By the proof of Part (ii) of Theorem 2.3, we note that to obtain an analogue of Part (ii) of Theorem 1.3 on  $M$  one needs the following properties:

- (1)  $h_{t+s}(x, y) = \int_M h_t(x, z) h_s(z, y) d\mu_z$  for all  $x, y \in M$  and  $s, t > 0$ ;
- (2)  $\int_M h_t(x, y) d\mu_y = 1$ ,  $\forall x \in M$ ,  $\forall t > 0$ ;
- (3) There exist constants  $\{C_i\}_{i=1}^8 > 0$  and  $a > 0$ ,  $b \geq 0$  such that

$$C_1 t^{-\frac{b}{2}} \exp\left(-C_2 \frac{(\rho(x, y))^2}{t}\right) \leq h_t(x, y) \leq C_3 t^{-\frac{b}{2}} \exp\left(-C_4 \frac{(\rho(x, y))^2}{t}\right), \quad (3.3)$$

for all  $0 < t < 1$  and  $x, y \in M$ , and

$$C_5 t^{-\frac{a}{2}} \exp\left(-C_6 \frac{(\rho(x, y))^2}{t}\right) \leq h_t(x, y) \leq C_7 t^{-\frac{a}{2}} \exp\left(-C_8 \frac{(\rho(x, y))^2}{t}\right), \quad (3.4)$$

for all  $t \geq 1$  and  $x, y \in M$ .

Note that we always have the above property (1) on  $M$  whenever the heat kernel exists.

Therefore, the following theorem can be an analogue of Part (ii) of Theorem 1.3 on  $M$ :

**Theorem 3.2.** *Assume that (2)-(3) hold on  $M$  for some  $a > 0$  and  $b \geq 0$ . Let  $p = 1 + 2/a < \infty$ . Let  $f : [0, \infty) \rightarrow [0, \infty)$  be a locally integrable function such that  $f(u) \geq K_2 u^p$  for some positive constant  $K_2 > 0$ . Then the equation*

$$u_t - \mathfrak{L}_M u = f(u) \quad (3.5)$$

*does not admit any nontrivial distributional solution  $u \geq 0$  in  $(0, \infty) \times M$ .*

**Remark 3.3.** By a distributional solution, we mean in Theorems 3.1 and 3.2 a function  $u \in L_{loc}^p(Q^M)$  which satisfies (3.2) and (3.5) in  $\mathcal{D}'(Q^M)$ , respectively, where  $Q^M := (0, \infty) \times M$ .

As for an analogue of Theorem 1.7 on  $M$ , since we already have Theorem 1.1 on  $M$ , we only need to check (1.3). For this, since we have used the estimate for the heat kernel from above in the proof of Theorem 1.7, then to obtain an analogue of Theorem 1.7 on  $M$  one needs to assume that the following estimates hold on  $M$ : there exist constants  $C_3, C_4, C_7, C_8 > 0$  and  $a > 0$ ,  $b \geq 0$  such that

$$h_t(x, y) \leq C_3 t^{-\frac{b}{2}} \exp\left(-C_4 \frac{(\rho(x, y))^2}{t}\right), \quad (3.6)$$

for all  $0 < t < 1$  and  $x, y \in M$ , and

$$h_t(x, y) \leq C_7 t^{-\frac{a}{2}} \exp\left(-C_8 \frac{(\rho(x, y))^2}{t}\right), \quad (3.7)$$

for all  $t \geq 1$  and  $x, y \in M$ . Therefore, we have the following theorem on  $M$ :

**Theorem 3.4.** *Assume that (3.6) and (3.7) hold on  $M$  for some  $a > 0$  and  $b \geq 0$ . Consider the problem (1.2) with  $1 + 2/a < p < \infty$ . Let  $u_0 \in L^q(M)$  with  $1 \leq q < \infty$  and  $\gamma > 0$ . Let  $f : [0, \infty) \rightarrow [0, \infty)$  be a continuous increasing function such that  $f(u) \leq K_1 u^p$  for some positive constant  $K_1 > 0$ . There exists  $\varepsilon = \varepsilon(\gamma) > 0$  such that, for every  $y \in M$  if*

$$0 \leq u_0(x) \leq \varepsilon h_\gamma(x, y), \quad x \in M, \quad (3.8)$$

*then there exists a non-negative continuous curve  $u : [0, \infty) \rightarrow L^q(M)$  which is a global solution to (1.2) with initial value  $u_0$ . Moreover, we have*

$$0 \leq u(t, x) \leq C h_{t+\gamma}(x, y), \quad x, y \in M, \quad t \in (0, \infty), \quad (3.9)$$

*for some  $C = C(\gamma) > 0$ .*

In particular, Theorem 3.4 implies an analogue of Part (iii) of Theorem 1.3 on  $M$ :

**Theorem 3.5.** *Assume that (3.6) and (3.7) hold on  $M$  for some  $a > 0$  and  $b \geq 0$ . Let  $1 + 2/a < p < \infty$ . Let  $f : [0, \infty) \rightarrow [0, \infty)$  be a continuous increasing function such that  $f(u) \leq K_1 u^p$  for some positive constant  $K_1 > 0$ . Then, for any  $1 \leq q < \infty$  the Cauchy problem (1.2) has a global, classical solution for some positive  $u_0 \in L^q(M)$ .*

Now we give some examples. Let us first recall the following result from [Sal10] (see also [Gri91] and [Sal92]) on weighted Riemannian manifolds, that is, complete non-compact Riemannian manifolds equipped with a measure  $\mu(dy) = \sigma(y)v(dy)$ ,  $0 < \sigma \in C^\infty(M)$ , and the associated weighted Laplacian  $\mathfrak{L}_M^\sigma := \sigma^{-1} \operatorname{div}(\sigma \operatorname{grad})$ :

**Theorem 3.6.** [Sal10, Theorem 3.1] *Let  $M$  be a weighted complete Riemannian manifold. Then the following three properties are equivalent:*

- *The parabolic Harnack inequality (PHI).*
- *The two-sided heat kernel bound  $((t, x, y) \in (0, \infty) \times M \times M)$ :*

$$\frac{\tilde{C}_1}{V(x, \sqrt{t})} e^{-\tilde{C}_1 \frac{(\rho(x, y))^2}{t}} \leq h_t(x, y) \leq \frac{\tilde{C}_2}{V(x, \sqrt{t})} e^{-\tilde{C}_2 \frac{(\rho(x, y))^2}{t}}. \quad (3.10)$$

- *The conjunction of*
  - *The volume doubling property*

$$\forall x \in M, r > 0, V(x, 2r) \leq DV(x, r).$$

- *The Poincaré inequality  $(\forall x \in M, r > 0, B = B(x, r))$*

$$\forall f \in \operatorname{Lip}(B), \int_B |f - f_B|^2 d\mu \leq Pr^2 \int_B |\nabla f|^2 d\mu,$$

*where  $f_B$  is the mean of  $f$  over  $B$ .*

**Remark 3.7.** Note that a complete weighted manifold  $M$  satisfies (PHI) if and only if the Riemannian product  $\mathbb{R} \times M$  satisfies the elliptic Harnack inequality (see [HS01]).

We refer to [Sal10, Section 3.2] for more details.

We note by the proof of Theorem 1.1, Lemma 2.4 and Corollary 2.5 that we also have Theorem 1.1, Lemma 2.4 and Corollary 2.5 on weighted Riemannian manifolds satisfying the two-sided heat kernel bound (3.10) (hence also on weighted Riemannian manifolds satisfying (PHI) by virtue of Theorem 3.6) with the weighted Laplacian, since (3.10) also implies that we have the positivity of the heat kernel on such

weighted Riemannian manifolds. Examples of such weighted Riemannian manifolds are complete Riemannian manifolds with non-negative Ricci curvature, convex domains in Euclidean space, complements of any convex domain, connected Lie groups with polynomial volume growth, Riemannian manifolds which cover a compact manifold with deck transformation group  $\Gamma$ , complete Riemannian manifolds  $M$  and  $N$  such that  $M/G = N$ , where  $G$  is a group of isometries of  $M$ , the Euclidean space  $\mathbb{R}^n$ ,  $n \geq 2$ , with weight  $(1 + |x|^2)^{\alpha/2}$  and  $\alpha > -n$ . Hence, they are also examples of weighted Riemannian manifolds satisfying (PHI) because of Theorem 3.6 (see e.g. [Sal10, Section 3.3]).

Since now we have Lemma 2.4 and Corollary 2.5 on weighted Riemannian manifolds satisfying the two-sided heat kernel bound (3.10) (hence also on weighted Riemannian manifolds satisfying (PHI) by virtue of Theorem 3.6), then taking into account the above discussions for Theorems 3.1, 3.2, 3.4 and 3.5, we obtain these Theorems 3.1, 3.2, 3.4, 3.5 on weighted Riemannian manifolds satisfying the two-sided heat kernel bound (3.10) with the volume growth, such that ultimately an estimate for the heat kernel has to have a form as in (3.1) for Theorem 3.1, (3.3)-(3.4) for Theorem 3.2, and (3.6)-(3.7) for Theorems 3.4-3.5. Here, we want to note that the volume growth does not have to be polynomial, see for example Theorem 1.8, if the volume growth is exponential but we have an estimate of the type (2.5).

## REFERENCES

- [BBN12] D. Barilari, U. Boscain, R. W. Neel. Small-time heat kernel asymptotics at the sub-Riemannian cut locus. *J. Differential Geom.*, 92(3):373–416, 2012.
- [BBN16] D. Barilari, U. Boscain, R. W. Neel. Heat kernel asymptotics on sub-Riemannian manifolds with symmetries and applications to the bi-Heisenberg group. *Ann. Fac. Sci. Toulouse Math*, 28(4):707–732, 2019.
- [BP85] P. Baras and M. Pierre. Critère d’existence de solutions positives pour des équations semi-linéaires non monotones. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 2, 2(3):185–212, 1985.
- [CRT01] T. Coulhon, E. Russ and V. Tardivel-Nachef. Sobolev algebras on Lie groups and Riemannian manifolds. *Amer. J. Math.*, 123(2):283–342, 2001.
- [CM96] M. Christ and D. Müller. On  $L^p$  spectral multipliers for a solvable Lie group. *Geometric and Functional Analysis*, 6(5):860–876, 1996.
- [Fuj66] H. Fujita. On the blowing up of solutions of the Cauchy problem for  $u_t = \Delta u + u^{1+\alpha}$ . *J. Fac. Sci. Univ. Tokyo Sec. IA Math.*, 13:109–124, 1966.
- [GP19] V. Georgiev and A. Palmieri. Upper bound estimates for local in time solutions to the semilinear heat equation on stratified lie groups in the sub-Fujita case. *AIP Conference Proceedings*, 2159, 020003, 2019. DOI:10.1063/1.5127465
- [Gri91] A. Grigor’yan. The heat equation on noncompact Riemannian manifolds. *(Russian) Mat. Sb.*, 182:55–87, 1991; translation in *Math. USSR-Sb.* 72:47–77, 1992.
- [Gui73] Y. Guivarc’h. Croissance polynomiale et périodes des fonctions harmoniques. *Bull. Soc. Math. France*, 101:333–379, 1973.
- [GP21] V. Georgiev and A. Palmieri. Lifespan estimates for local in time solutions to the semilinear heat equation on the Heisenberg group. *Annali di Matematica*, 200:999–1032, 2021.
- [HM04] M. Hesaaraki and A. Moameni. Blow-up of positive solutions for a family of nonlinear parabolic equations in general domain in  $\mathbb{R}^N$ . *Michigan Math. J.*, 52:375–389, 2004.
- [HS01] W. Hebisch and L. Saloff-Coste. On the relation between elliptic and parabolic Harnack inequalities. *Ann. Inst. Fourier (Grenoble)*, 51:1437–1481, 2001.

- [JKS16] M. Jleli, M. Kirane and B. Samet. A Fujita-type theorem for a multitime evolutionary  $p$ -Laplace inequality in the Heisenberg group. *Electron. J. Differential Equations*, 2016(303):1–8, 2016.
- [LP76] H. A. Levine and L. E. Payne. Nonexistence of global weak solutions for classes of nonlinear wave and parabolic equations. *J. Math. Anal. Appl.*, 55:329–334, 1976.
- [LZZ18] Y. Li, Z. Zhang and L. Zhu. Classification of certain qualitative properties of solutions for the quasilinear parabolic equations. *Sci. China Math.*, 61:855–868, 2018.
- [MP01] E. Mitidieri and S. I. Pohozaev. A priori estimates and blow-up of solutions of nonlinear partial differential equations and inequalities. *Proc. Steklov Inst. Math.*, 234:1–362, 2001.
- [Pal21a] A. Palmieri. On the blow-up of solutions to semilinear damped wave equations with power nonlinearity in compact Lie groups. *J. Diff. Equations*, 281: 85–104, 2021. DOI:10.1016/j.jde.2021.02.002
- [Pal21b] A. Palmieri. Semilinear wave equation on compact Lie groups. *J. Pseudo-Differ. Oper. Appl.*, 12: 43, 2021. DOI:10.1007/s11868-021-00414-x
- [Pas98] A. Pascucci. Semilinear equations on nilpotent Lie groups: global existence and blow-up of solutions. *LE MATEMATICHE*, Vol.LIII - Fasc.II: 345–357, 1998.
- [Pun12] F. Punzo. Blow-up of solutions to semilinear parabolic equations on Riemannian manifolds with negative sectional curvature. *J. Math. Anal. Appl.*, 387(2):815–827, 2012.
- [Qui91] P. Quittner. Blow-up for semilinear parabolic equations with a gradient term. *Math. Methods Appl. Sci.*, 14:413–417, 1991.
- [QP07] P. Quittner and P. Souplet. *Superlinear Parabolic Problems. Blow-up, Global Existence and Steady States*. Birkhuser Adv. Texts. Basl. Lehrbcher, Birkhuser Verlag, Basel, 2007.
- [RS18] M. Ruzhansky and D. Suragan. A comparison principle for nonlinear heat Rockland operators on graded groups. *Bull. London Math. Soc.*, 50:753–758, 2018.
- [RY20] M. Ruzhansky and N. Yessirkegenov. A comparison principle for higher order nonlinear hypoelliptic heat operators on graded Lie groups. to appear in *Nonlinear Analysis*, arXiv:2005.11503 , 2020. <https://doi.org/10.1016/j.na.2021.112621>
- [Sal92] L. Saloff-Coste. A note on Poincaré, Sobolev, and Harnack inequalities. *Int. Math. Res. Not. IMRN*, 2:27–38, 1992.
- [Sal10] L. Saloff-Coste. The heat kernel and its estimates. *Adv. Stud. Pure Math.*, 57:405–436, 2010.
- [Sch93] L. B. Schweitzer. Dense  $m$ -convex Fréchet subalgebras of operator algebra crossed products by Lie groups. *Internat. J. Math.*, 4(4):601–673, 1993.
- [Str86] R. S. Strichartz. Sub-Riemannian geometry. *J. Differential Geom.*, 24(2):221–263, 1986.
- [SW97] Ph. Souplet and F. B. Weissler. Self-similar subsolutions and blowup for nonlinear parabolic equations. *J. Math. Anal. Appl.*, 212:60–74, 1997.
- [VCS92] N. Th. Varopoulos, T. Coulhon and L. Saloff-Coste. *Analysis and geometry on groups*. Cambridge Tracts in Mathematics, 100. Cambridge University Press, Cambridge, 1992.
- [Wei80] F. B. Weissler. Local existence and nonexistence for semilinear parabolic equations in  $L^p$ . *Indiana Univ. Math. J.*, 29:79–102, 1980.
- [Wei81] F. B. Weissler. Existence and non-existence of global solutions for a semilinear heat equation. *Israel J. Math.*, 38:29–40, 1981.
- [Zha99] Q. S. Zhang. Blow-up results for nonlinear parabolic equations on manifolds. *Duke Math. J.*, 97(3):515–539, 1999.

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