

Fractional Schrödinger equations with singular potentials of higher order. II: Hypoelliptic case

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Abstract

In this paper we consider the space-fractional Schrödinger equation with a singular potential for a wide class of fractional hypoelliptic operators. Such analysis can be conveniently realised in the setting of graded Lie groups. The paper is a continuation and extension of the first part [2] where the classical Schrödinger equation on \mathbb{R}^n with singular potentials was considered.

Keywords: Schrödinger equation, generalised solution, singular potential, regularisation, Rockland operator, distributional coefficient, very weak solution, graded Lie group.

1 Introduction

This paper is devoted to the fractional Schrödinger equation for positive (left) Rockland operator \mathcal{R} (left-invariant hypoelliptic partial differential operator which is homogeneous of positive degree ν) on a general graded Lie group \mathbb{G} , with a possibly singular potential; that is for $T > 0$, and for $s > 0$ we consider the Cauchy problem

$$iu_t(t, x) + \mathcal{R}^s u(t, x) + p(x)u(t, x) = 0, \quad (t, x) \in [0, T] \times \mathbb{G}, \quad (1)$$

$$u(0, x) = u_0(x), u_t(0, x) = u_1(x), x \in \mathbb{G},$$

where p is a non-negative distributional function.

The main idea of this paper is to relax the regularity assumptions on the potential p in (1). The coefficient p is allowed to have δ -function type singularities. But the question of having a suitable notion of solutions to (1) is still open. The situation is reaching an impasse by the well-known impossibility problem on the multiplication of distributions stated by Schwartz in [28]. To deal with it, in this paper we use the concept of very weak solutions introduced in [16] to work with the wave equations with irregular coefficients. Later, the developed tools were applied to other equations with singular coefficients [20] and [23]. In all these papers authors work with the time-dependent equations and in the recent works [1], [2], [3] and [15] one started a development of the very weak solutions for partial differential equations with (strongly singular) space-depending coefficients. The wave-type equations on graded Lie groups with time dependent coefficients were analysed in [24, 25].

In this paper, we will show in details that the notion of very weak solutions is applicable to the Cauchy problem (1) for the fractional Schrödinger equation for the Rockland operator \mathcal{R} on the graded Lie group \mathbb{G} with a strongly singular potential-coefficient depending on the spacial variable. Indeed, the present work is an improvement and extension of the results obtained in the first part [2] addressed to the fractional Schrödinger equation. It should be mentioned that the setting of [2] was the equation (1) for $\mathbb{G} = \mathbb{R}^d$ and $\mathcal{R} = (-\Delta)^s$ being the positive fractional Laplacian on the Euclidean space. Consequently, the results of [2] can be considered as a special case of the results obtained here.

At the same time, we also give here some corrections and clarifications to statements of the first part [2], see Remark 2 and Remark 3. Also, the arguments around the Sobolev embedding techniques starting from Proposition 3 are new here, giving a new result also for the setting in [2].

Let us briefly recall the necessary notions in the setting of graded groups. For a more detailed exposition we refer to Folland and Stein [Chapter 1 in [14]], or to the more recent one by Fischer and the second author [Chapter 3 in [13]].

A connected simply connected Lie group \mathbb{G} is called a graded Lie group if its Lie algebra \mathfrak{g} has a vector space decomposition of the following form

$$\mathfrak{g} = \bigoplus_{i=1}^{\infty} \mathfrak{g}_i,$$

where all, but finitely many \mathfrak{g}_i 's, are equal to $\{0\}$, and we have

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}^1, \quad \forall i, j \in \mathbb{N}.$$

For a graded Lie group $\mathbb{G} \sim \mathbb{R}^n$ with Lie algebra \mathfrak{g} , we fix a basis $\{X_1, \dots, X_n\}$ of \mathfrak{g} adapted to the gradation of the above form. Then, the exponential map $\exp_{\mathbb{G}} : \mathfrak{g} \rightarrow \mathbb{G}$ defined as

$$x := \exp_{\mathbb{G}}(x_1 X_1 + \dots + x_n X_n),$$

is a global diffeomorphism from \mathfrak{g} onto \mathbb{G} .

Let A be a diagonalisable linear operator on \mathfrak{g} with positive eigenvalues. Then, a family $\{D_r\}_{r>0}$ of dilations of \mathfrak{g} is a collection of linear mappings of the form

$$D_r = \text{Exp}(A \ln r) = \sum_{k=0}^{\infty} \frac{1}{k!} (\ln(r)A)^k,$$

where Exp denotes the exponential of matrices. Moreover, the exponential mapping $\exp_{\mathbb{G}}$ on \mathbb{G} transports the dilations $\{D_r\}_{r>0}$ on the group side; i.e., we have

$$D_r(x) = rx = (r^{\nu_1} x_1, \dots, r^{\nu_n} x_n), \quad x \in \mathbb{G}, \quad (2)$$

where ν_1, \dots, ν_n are the weights of the dilations. Additionally, each D_r , $r > 0$, is a morphism of \mathfrak{g} , and consequently, also of \mathbb{G} .

Finally, let us note that a connected simply connected Lie group \mathbb{G} that can be equipped with such a family of automorphisms is called a homogeneous Lie group; for such groups, the quantity

$$Q := \text{Tr}A = \nu_1 + \dots + \nu_n,$$

is called the homogeneous dimension of \mathbb{G} .

Recall that graded Lie groups are naturally also homogeneous Lie groups. Let us illustrate the above ideas with the following examples in the settings of two well-studied homogeneous Lie groups; see [5, 6] for the case of the Engel group as in Example 2.

Example 1. *Heisenberg group* \mathbb{H}_n , $n \in \mathbb{N}$: Let \mathfrak{h}_n be the Lie algebra of the Heisenberg group $\mathbb{H}_n \sim \mathbb{R}^{2n+1}$ with elements

$$\mathfrak{h}_n = \{X_1, \dots, X_n, Y_1, \dots, Y_n, T\},$$

that satisfy the (non-zero) commutator relations

$$[X_i, Y_i] = T, \quad i = 1, \dots, n.$$

Therefore, \mathfrak{h}_n admits the following gradation

$$\mathfrak{h}_n = V_1 \oplus V_2 = \text{span}\{X_1, \dots, X_n, Y_1, \dots, Y_n\} \oplus \mathbb{R}T.$$

¹For the vector spaces V, W we denote by $[V, W]$ the vector space $\{[v, w] : v \in V, w \in W\}$, where $[v, w] := vw - wv$ is the Lie bracket.

The dilations on the elements of \mathfrak{h}_n are given by

$$D_r(X_i) = rX_i, D_r(Y_i) = rY_i, i = 1, \dots, n, D_r(T) = r^2T,$$

and subsequently \mathbb{H}_n is homogeneous when equipped with the dilations

$$rh = (rx, ry, r^2t), \quad h = (x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R},$$

with homogeneous dimension $Q_{\mathbb{H}_n} = n + n + 2 = 2n + 2$.

Example 2. *Engel group \mathcal{B}_4 :* Let \mathfrak{l}_4 be the Lie algebra of the Engel group $\mathcal{B}_4 \sim \mathbb{R}^4$ with elements

$$\mathfrak{l}_4 = \{X_1, X_2, X_3, X_4\},$$

that satisfy the (non-zero) commutator relations

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4.$$

The Lie algebra \mathfrak{l}_4 admits the gradation

$$\mathfrak{l}_4 = V_1 \oplus V_2 \oplus V_3 = \text{span}\{X_1, X_2\} \oplus \mathbb{R}X_3 \oplus \mathbb{R}X_4,$$

and the natural dilations on \mathfrak{l}_4 are given by

$$D_r(X_1) = rX_1, D_r(X_2) = rX_2, D_r(X_3) = r^2X_3, D_r(X_4) = r^3X_4,$$

which, transported to the group side, yield

$$rx = (rx_1, rx_2, r^2x_3, r^3x_4), x = (x_1, x_2, x_3, x_4) \in \mathcal{B}_4.$$

Its homogeneous dimension is $Q_{\mathcal{B}_4} = 1 + 1 + 2 + 3 = 7$.

Let π be a representation of the group \mathbb{G} on the separable Hilbert space \mathcal{H}_π . We say that a vector $v \in \mathcal{H}_\pi$ is a smooth vector and we write $v \in \mathcal{H}_\pi^\infty$, if the function

$$\mathbb{G} \ni x \mapsto \pi(x)v \in \mathcal{H}_\pi,$$

is of class C^∞ . If $X \in \mathfrak{g}$, then for $v \in \mathcal{H}_\pi^\infty$, the limit

$$d\pi(X)v := \lim_{t \rightarrow 0} \frac{1}{t} (\pi(\exp_{\mathbb{G}}(tX))v - v),$$

exists, and the mapping $d\pi : \mathfrak{g} \rightarrow \text{End}(\mathcal{H}_\pi^\infty)$ is the infinitesimal representation of \mathfrak{g} on \mathcal{H}_π^∞ associated to π . With an abuse of notation we shall write π for the infinitesimal representation of \mathfrak{g} . Setting $\pi(X)^\alpha = \pi(X^\alpha)$, $\alpha \in \mathbb{N}^n$, we can extend the infinitesimal representation of \mathfrak{g} to elements of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$; i.e., we can write

$$d\pi(T) := \pi(T), \quad T \in \mathfrak{U}(\mathfrak{g}),$$

where T has been identified with a left-invariant operator on \mathbb{G} , and the set of infinitesimal representations $\{\pi(T) : \pi \in \widehat{\mathbb{G}}\}$ is a fields of operators that turns out to be the symbol associated to T .

Recall that an immediate consequence of the so-called *Poincaré-Birkhoff-Witt* Theorem is that $\mathfrak{U}(\mathfrak{g})$ can be identified with the space of left-invariant differential operators on \mathbb{G} , and moreover, any left-invariant vector field can be written in a unique way as the sum

$$\sum_{\alpha \in \mathbb{N}^n} c_\alpha X^\alpha, \quad (3)$$

where all but finite $c_\alpha \in \mathbb{C}$ are zero, and where for $X_j \in \mathfrak{g}$ we have defined $X^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n}$, for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$.

If $\pi \in \widehat{\mathbb{G}}$; that is, if π is an element of the unitary dual of \mathbb{G} , then we say that the left-invariant differential operator \mathcal{R} on \mathbb{G} , which is homogeneous of a positive degree, is a Rockland operator, if it satisfies the following Rockland condition:

(\mathcal{R}) for every non-trivial representation $\pi \in \widehat{\mathbb{G}}$ the operator $\pi(\mathcal{R})$ is injective on \mathcal{H}_π^∞ , i.e.,

$$\forall v \in \mathcal{H}_\pi^\infty, \pi(\mathcal{R})v = 0 \implies v = 0.$$

For a more detailed discussion of the above condition as appeared in the work of Rockland [22] we refer to [Sections 1.7 and 4.1 in [13]]. Equivalent to (\mathcal{R}) conditions have appeared in the works of Beals [4] and Helffer and Nourrigat [17], with the latter characterising Rockland operators as being the left-invariant hypoelliptic differential operators on \mathbb{G} . Spectral properties of the infinitesimal representations of Rockland operators have been considered in [27].

For $\pi \in \widehat{\mathbb{G}}$, and for \mathcal{R} being a positive Rockland operator of homogeneous degree ν , from (3) we obtain the following representation of symbol associated to \mathcal{R} ,

$$\pi(\mathcal{R}) = \sum_{[\alpha]=\nu} c_\alpha \pi(X)^\alpha,$$

where

$$[\alpha] = \nu_1 \alpha_1 + \cdots + \nu_n \alpha_n$$

is the homogeneous length of the multi-index α , and

$$\pi(X)^\alpha = \pi(X^\alpha) = \pi(X_1^{\alpha_1} \cdots X_n^{\alpha_n}),$$

where X_j is of homogeneous degree ν_j .

Recall that \mathcal{R} and $\pi(\mathcal{R})$ are densely defined on $\mathcal{D}(\mathbb{G}) \subset L^2(\mathbb{G})$ and on $\mathcal{H}_\pi^\infty \subset \mathcal{H}_\pi$, respectively (see, e.g. [Proposition 4.1.15 in [13]]). Additionally, let us mention that, for the groups we consider here, in the case where $\mathcal{H}_\pi = L^2(\mathbb{R}^m)$ we have $\mathcal{H}_\pi^\infty = \mathcal{S}(\mathbb{R}^m)$, see [Corollary 4.1.2 in [9]]. From now on, let us denote by \mathcal{R} , and by $\pi(\mathcal{R})$ the self-adjoint extensions of the above on the spaces $L^2(\mathbb{G})$, and \mathcal{H}_π , respectively.

By the spectral theorem for unbounded operators (see, e.g. Theorem VIII.6 in [21]) we can write

$$\mathcal{R} = \int_{\mathbb{R}} \lambda dE(\lambda), \quad \text{and} \quad \pi(\mathcal{R}) = \int_{\mathbb{R}} \lambda dE_{\pi}(\lambda),$$

where E and E_{π} stand for the spectral measures associated to \mathcal{R} and to $\pi(\mathcal{R})$.

For our purposes, we have required the positivity of the Rockland operator \mathcal{R} that should be regarded in the operator sense. In particular, the Rockland operator \mathcal{R} is positive on $L^2(\mathbb{G})$, if it is formally self-adjoint; that is we have $\mathcal{R} = \mathcal{R}^*$ in the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$, and \mathcal{R} satisfies the condition

$$\int_{\mathbb{G}} \mathcal{R}f(x) \overline{f(x)} dx \geq 0, \quad \forall f \in \mathcal{D}(\mathbb{G}).$$

For a positive Rockland operator \mathcal{R} , the infinitesimal representations $\pi(\mathcal{R})$ are also positive because of the relations between the spectral measures.

A standard example of a Rockland operator on a stratified Lie group \mathbb{G} is the so-called sub-Laplacian on \mathbb{G} that is of homogeneous degree $\nu = 2$, and is defined as follows:

If \mathbb{G} is a stratified Lie group with a given basis Z_1, \dots, Z_k for the first stratum of its Lie algebra, then the left-invariant differential operator on \mathbb{G} given by

$$Z_1^2 + \dots + Z_k^2,$$

is called the sub-Laplacian on \mathbb{G} .

The infinitesimal representations of such operators on the particular cases of the Heisenberg and Engel groups, as introduced in Examples 1 and 2, are given in the following examples.

Example 3. Heisenberg group \mathbb{H}_n : Using the Schrödinger representations (see e.g. [26]) of \mathbb{H}_n , the infinitesimal representation, parametrised by $\lambda \in \mathbb{R} \setminus \{0\}$, of the sub-Laplacian $\mathcal{L}_{\mathbb{H}}$ on \mathbb{H}_n , is the operator on $\mathcal{H}_{\pi_{\lambda}}^{\infty} = \mathcal{S}(\mathbb{R}^n)$ given by

$$\mathcal{A} := \pi_{\lambda}(\mathcal{L}_{\mathbb{H}}) = |\lambda| \sum_{j=1}^n \left(\partial_{u_j}^2 - u_j^2 \right).$$

Now, \mathcal{A} is the harmonic oscillator, and if we keep the same notation for its self-adjoint extension on $\mathcal{H}_{\pi_{\lambda}} = L^2(\mathbb{R}^n)$, then the spectrum of $-\mathcal{A}$ is explicitly known as

$$\{2|\ell| + n, \ell \in \mathbb{N}^n\},$$

where $|\ell| = \ell_1 + \dots + \ell_n$, see, e.g. [Section 6.4 in [13]].

Example 4. Engel group \mathcal{B}_4 : Using the representations of \mathcal{B}_4 proved by Dixmier [p.333 in [10]] we see that the infinitesimal representation, parametrised

by $\lambda \in \mathbb{R} \setminus \{0\}, \mu \in \mathbb{R}$, of the sub-Laplacian $\mathcal{L}_{\mathcal{B}_4}$ on \mathcal{B}_4 , is the operator on $\mathcal{H}_{\pi_{\lambda,\mu}} = \mathcal{S}(\mathbb{R})$ given by

$$\mathcal{A} := \pi_{\lambda,\mu}(\mathcal{L}_{\mathcal{B}_4}) = \frac{d^2}{du^2} - \frac{1}{4} \left(\lambda u^2 - \frac{\mu}{\lambda} \right)^2.$$

The operator \mathcal{A} here is an anharmonic oscillator that admits a self-adjoint extension on $\mathcal{H}_{\pi_{\lambda}} = L^2(\mathbb{R}^n)$ and has discrete spectrum, see e.g. [7].

More generally, for a positive Rockland operator \mathcal{R} , Hulanicki, Jenkins and Ludwig [18] proved that the spectrum of $\pi(\mathcal{R})$, with $\pi \in \widehat{\mathbb{G}} \setminus \{1\}$, is discrete and lies in $(0, \infty)$, which allows us to choose an orthonormal basis for \mathcal{H}_{π} such that the self-adjoint operator $\pi(\mathcal{R})$ admits an infinite matrix representation of the form

$$\pi(\mathcal{R}) = \text{diag}(\pi_1^2, \pi_2^2, \dots) \quad (4)$$

where $\pi \in \widehat{\mathbb{G}} \setminus \{1\}$ and $\pi_j > 0$.

We will now briefly recall the group Fourier transform: If we identify the irreducible unitary representations with their equivalence classes, then for $f \in L^1(\mathbb{G})$ and for $\pi \in \widehat{\mathbb{G}}$, the group Fourier transform of f at π is the map

$$\mathcal{F}_{\mathbb{G}} f : \pi \mapsto \mathcal{F}_{\mathbb{G}} f(\pi),$$

that is a linear endomorphism on \mathcal{H}_{π} , defined by

$$\mathcal{F}_{\mathbb{G}} f(\pi) \equiv \widehat{f}(\pi) \equiv \pi(f) := \int_{\mathbb{G}} f(x) \pi(x)^* dx,$$

where the integration on \mathbb{G} is taken with respect to the binvariant Haar measure dx on \mathbb{G} . By the above we can also write

$$\mathcal{F}_{\mathbb{G}}(\mathcal{R}f)(\pi) = \pi(\mathcal{R})\widehat{f}(\pi),$$

and, using the basis in the representation of \mathcal{H}_{π} given in (4), the latter can be rewritten as

$$\left\{ \pi_k^2 \cdot \widehat{f}(\pi)_{k,l} \right\}_{k,l \in \mathbb{N}}.$$

For graded Lie groups, or more generally for connected simply connected nilpotent Lie groups, the orbit method or, more particularly, the geometry of co-adjoint orbits [9, 11, 19], identifies the unitary dual $\widehat{\mathbb{G}}$ with a subset of a Euclidean space which is equipped with a concrete measure μ , called the Plancherel measure, that allows for the Fourier inversion formula. Furthermore, the operator $\pi(f)$ is in the Hilbert-Schmidt class, and its Hilbert-Schmidt norm depends only on the class of π ; the map

$$\widehat{\mathbb{G}} \ni \pi \mapsto \|\pi(f)\|_{\text{HS}}^2$$

is integrable against μ and we have the following isometry, known as the Plancherel formula

$$\int_{\mathbb{G}} |f(x)|^2 dx = \int_{\widehat{\mathbb{G}}} \|\pi(f)\|_{\text{HS}}^2 d\mu(\pi). \quad (5)$$

For a detailed discussion on this topic we refer to [Section 1.8, Appendix B.2 in [13]].

Finally, since the action of a Rockland operator \mathcal{R} is involved in our analysis, let us make a brief overview of some related properties.

Definition 1. [Homogeneous Sobolev spaces] For $s > 0$, $p > 1$, and \mathcal{R} a positive homogeneous Rockland operator of degree ν , we define the \mathcal{R} -Sobolev spaces as the space of tempered distributions $\mathcal{S}'(\mathbb{G})$ obtained by the completion of $\mathcal{S}(\mathbb{G}) \cap \text{Dom}(\mathcal{R}_p^{\frac{s}{\nu}})$ for the norm

$$\|f\|_{\dot{L}_s^p(\mathbb{G})} := \|\mathcal{R}_p^{\frac{s}{\nu}} f\|_{L^p(\mathbb{G})}, \quad f \in \mathcal{S}(\mathbb{G}) \cap \text{Dom}(\mathcal{R}_p^{\frac{s}{\nu}}),$$

where \mathcal{R}_p is the maximal restriction of \mathcal{R} to $L^p(\mathbb{G})$.²

Let us mention that, the above \mathcal{R} -Sobolev spaces do not depend on the specific choice of \mathcal{R} , in the sense that, different choices of the latter produce equivalent norms, see [Proposition 4.4.20 in [13]].

In the scale of these Sobolev spaces, we recall the next proposition as in [Proposition 4.4.13 in [13]].

Proposition 1. [Sobolev embeddings] For $1 < \tilde{q}_0 < q_0 < \infty$ and for $a, b \in \mathbb{R}$ such that

$$b - a = Q \left(\frac{1}{\tilde{q}_0} - \frac{1}{q_0} \right),$$

we have the continuous inclusions

$$\dot{L}_b^{\tilde{q}_0}(\mathbb{G}) \subset \dot{L}_a^{q_0}(\mathbb{G}),$$

that is, for every $f \in \dot{L}_b^{\tilde{q}_0}(\mathbb{G})$, we have $f \in \dot{L}_a^{q_0}(\mathbb{G})$, and there exists some positive constant $C = C(\tilde{q}_0, q_0, a, b)$ (independent of f) such that

$$\|f\|_{\dot{L}_a^{q_0}(\mathbb{G})} \leq C \|f\|_{\dot{L}_b^{\tilde{q}_0}(\mathbb{G})}. \quad (6)$$

In the sequel we will make use of the following notation:

- When we write $a \leq b$, we will mean that there exists some constant $c > 0$ (independent of any involved parameter) such that $a \leq cb$;
- if $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is some multi-index, then we denote by

$$[\alpha] = \sum_{i=1}^n v_i \alpha_i,$$

²When $p = 2$, we will write $\mathcal{R}_2 = \mathcal{R}$ for the self-adjoint extension of \mathcal{R} on $L^2(\mathbb{G})$.

its *homogeneous length*, where the v_i 's stand for the dilations' weights as in (2), and by

$$|\alpha| = \sum_{i=1}^n \alpha_i,$$

the length of it;

- for suitable $f \in \mathcal{S}'(\mathbb{G})$ we have introduced the following norm

$$\|f\|_{H^s(\mathbb{G})} := \|f\|_{L^2_s(\mathbb{G})} + \|f\|_{L^2(\mathbb{G})};$$

- when regularisations of functions/distributions on \mathbb{G} are considered, they must be regarded as arising via convolution with Friedrichs-mollifiers; that is, ψ is a Friedrichs-mollifier, if it is a compactly supported smooth function with $\int_{\mathbb{G}} \psi dx = 1$. Then the regularising net is defined as

$$\psi_\epsilon(x) = \epsilon^{-Q} \psi(D_{\epsilon^{-1}}(x)), \quad \epsilon \in (0, 1], \quad (7)$$

where Q is the homogeneous dimension of \mathbb{G} .

2 Estimates for the classical solution

Here and thereafter, we consider a fixed power $s > 0$ of a fixed, positive Rockland operator \mathcal{R} that is assumed to be of homogeneous degree ν . Moreover, the coefficient p in (1) will be regarded to be non-negative on \mathbb{G} .

The next two propositions prove the existence and uniqueness of the classical solution to the Cauchy problem (1), in the cases where the potential p is in the space $L^\infty(\mathbb{G})$ or $L^{\frac{2Q}{\nu s}}(\mathbb{G})$, where, in the second case, the condition $Q > \nu s$ must be satisfied.

Proposition 2. Let $p \in L^\infty(\mathbb{G})$ (or $p \in L^{\frac{2Q}{\nu s}} \cap L^{\frac{Q}{\nu s}}$ if also $Q > \nu s$), where $p \geq 0$, and suppose that $u_0 \in H^{\frac{s\nu}{2}}(\mathbb{G})$. Then, there exists a unique solution $u \in C([0, T]; H^{\frac{s\nu}{2}}(\mathbb{G}))$ to the Cauchy problem (1), that satisfies the estimate

$$\|u(t, \cdot)\|_{H^{\frac{s\nu}{2}}(\mathbb{G})} \leq (1 + \|p\|_{L^\infty(\mathbb{G})}) \|u_0\|_{H^{\frac{s\nu}{2}}(\mathbb{G})}, \quad (8)$$

uniformly in $t \in [0, T]$.

Proof. Multiplying the equation (1) by u_t and integrating over \mathbb{G} , we get

$$\Re(\langle iu_t(t, \cdot), u_t(t, \cdot) \rangle_{L^2(\mathbb{G})} + \langle \mathcal{R}^s u(t, \cdot), u_t(t, \cdot) \rangle_{L^2(\mathbb{G})} + \langle p(\cdot)u(t, \cdot), u_t(t, \cdot) \rangle_{L^2(\mathbb{G})}) = 0, \quad (9)$$

for all $t \in [0, T]$. It is easy to see that

$$\Re(\langle \mathcal{R}^s u(t, \cdot), u_t(t, \cdot) \rangle_{L^2(\mathbb{G})}) = \frac{1}{2} \partial_t \|\mathcal{R}^{\frac{s}{2}} u(t, \cdot)\|_{L^2(\mathbb{G})}^2,$$

and

$$\Re(\langle p(\cdot)u(t, \cdot), u_t(t, \cdot) \rangle_{L^2(\mathbb{G})}) = \frac{1}{2} \partial_t \|\sqrt{p}(\cdot)u(t, \cdot)\|_{L^2(\mathbb{G})}^2,$$

so that, denoting by

$$E(t) := \|\mathcal{R}^{\frac{s}{2}}u(t, \cdot)\|_{L^2(\mathbb{G})}^2 + \|\sqrt{p}(\cdot)u(t, \cdot)\|_{L^2(\mathbb{G})}^2,$$

the real part of the functional estimate of (9), the equation (9) implies that $\partial_t E(t) = 0$, and consequently also that

$$E(t) = E(0), \quad \forall t \in [0, T]. \quad (10)$$

Therefore, taking into consideration the estimate

$$\|\sqrt{p}u_0\|_{L^2(\mathbb{G})}^2 \leq \|p\|_{L^\infty(\mathbb{G})} \|u_0\|_{L^2(\mathbb{G})}^2,$$

the equation (10) implies that for all $t \in [0, T]$ we have

$$\|\sqrt{p}u(t, \cdot)\|_{L^2(\mathbb{G})}^2 \leq \|\mathcal{R}^{\frac{s}{2}}u_0\|_{L^2(\mathbb{G})}^2 + \|p\|_{L^\infty} \|u_0\|_{L^2(\mathbb{G})}^2, \quad (11)$$

and

$$\|\mathcal{R}^{\frac{s}{2}}u(t, \cdot)\|_{L^2(\mathbb{G})}^2 \leq \|\mathcal{R}^{\frac{s}{2}}u_0\|_{L^2(\mathbb{G})}^2 + \|p\|_{L^\infty} \|u_0\|_{L^2(\mathbb{G})}^2. \quad (12)$$

Now since

$$\|\mathcal{R}^{\frac{s}{2}}u_0\|_{L^2(\mathbb{G})}^2, \|u_0\|_{L^2(\mathbb{G})}^2 \leq \|u_0\|_{H^{\frac{s\nu}{2}}(\mathbb{G})}^2,$$

we can estimate (11) and (12) further by

$$\|\sqrt{p}u(t, \cdot)\|_{L^2(\mathbb{G})} \leq \left(1 + \|p\|_{L^\infty(\mathbb{G})}^{\frac{1}{2}}\right) \|u_0\|_{H^{\frac{s\nu}{2}}(\mathbb{G})}, \quad (13)$$

and

$$\|\mathcal{R}^{\frac{s}{2}}u(t, \cdot)\|_{L^2(\mathbb{G})} \leq \left(1 + \|p\|_{L^\infty(\mathbb{G})}^{\frac{1}{2}}\right) \|u_0\|_{H^{\frac{s\nu}{2}}(\mathbb{G})}, \quad (14)$$

respectively.

Now, to prove (8), it remains to show the desired estimate for the norm $\|u(t, \cdot)\|_{L^2(\mathbb{G})}$. To this end, we first apply the group Fourier transform to (1) with respect to $x \in \mathbb{G}$ and for all $\pi \in \widehat{\mathbb{G}}$, and we get

$$i\widehat{u}_t(t, \pi) + \pi(\mathcal{R})^s \widehat{u}(t, \pi) = \widehat{f}(t, \pi); \quad \widehat{u}(0, \pi)_{k,l} = \widehat{u}_0(\pi)_{k,l}, \quad (15)$$

where $\widehat{f}(t, \pi)$ denotes the group Fourier transform of the function $f(t, x) := -p(x)u(t, x)$. Taking into account the matrix representation of $\pi(\mathcal{R})$, we rewrite the matrix equation (15) componentwise as the infinite system of equations of the form

$$i\widehat{u}_t(t, \pi)_{k,l} + \pi_k^{2s} \cdot \widehat{u}(t, \pi)_{k,l} = \widehat{f}(t, \pi)_{k,l}, \quad (16)$$

for all $\pi \in \widehat{\mathbb{G}}$ and for any $k, l \in \mathbb{N}$, where now $\widehat{f}(t, \pi)_{k,l}$ can be regarded as the source term of the second order differential equation as in (16).

Now, let us decouple the matrix equation in (16) by fixing $\pi \in \widehat{\mathbb{G}}$, and treat each of the equations represented in (16) individually. If we denote by

$$v(t) := \widehat{u}(t, \pi)_{k,l}, \beta^{2s} := \pi_k^{2s}, f(t) := \widehat{f}(t, \pi)_{k,l} \quad \text{and} \quad v_0 := \widehat{u}_0(\pi)_{k,l},$$

then (16) becomes

$$iv'(t) + \beta^{2s} \cdot v(t) = f(t); \quad v(0) = v_0, \quad (17)$$

with $\beta > 0$. By solving first the homogeneous version of (17), and then by applying Duhamel's principle (see e.g. [12]), we get the following representation of the solution of (17)

$$v(t) = v_0 \exp(-i\beta^{2s}t) + \int_0^t \exp(-i\beta^{2s}(t-s))f(s) ds.$$

Therefore, if we substitute back our initial conditions in t , then we get the estimate

$$|\widehat{u}(t, \pi)_{k,l}|^2 \leq |\widehat{u}_0(\pi)_{k,l}|^2 + \int_0^T |\widehat{f}(t, \pi)_{k,l}|^2 dt, \quad (18)$$

which holds uniformly in $\pi \in \widehat{\mathbb{G}}$ and for each $k, l \in \mathbb{N}$, where we have used that $L^2([0, T]) \subset L^1([0, T])$. Now, recall that since for any Hilbert-Schmidt operator A one has

$$\|A\|_{\text{HS}}^2 = \sum_{k,l} |\langle A\varphi_k, \varphi_l \rangle|^2,$$

where $\{\varphi_1, \varphi_2, \dots\}$ is some orthonormal basis, summing over $k, l \in \mathbb{N}$ the inequalities (18) we get

$$\|\widehat{u}(t, \pi)\|_{\text{HS}}^2 \leq \|\widehat{u}_0(\pi)\|_{\text{HS}}^2 + \sum_{k,l} \int_0^T |\widehat{f}(t, \pi)_{k,l}|^2 dt.$$

Next we integrate the last inequality with respect to the Plancherel measure μ on $\widehat{\mathbb{G}}$, so that using the Plancherel identity (5), we obtain

$$\|u(t, \cdot)\|_{L^2(\mathbb{G})}^2 \leq \|u_0\|_{L^2(\mathbb{G})}^2 + \int_{\mathbb{G}} \sum_{k,l} \int_0^T |\widehat{f}(t, \pi)_{k,l}|^2 dt d\mu(\pi), \quad (19)$$

and if we use Lebesgue's dominated convergence theorem, Fubini's theorem and the Plancherel formula we have

$$\int_{\mathbb{G}} \sum_{k,l} \int_0^T |\widehat{f}(t, \pi)_{k,l}|^2 dt d\mu = \int_0^T \int_{\mathbb{G}} \sum_{k,l} |\widehat{f}(t, \pi)_{k,l}|^2 d\mu dt = \int_0^T \|f(t, \cdot)\|_{L^2(\mathbb{G})}^2 dt. \quad (20)$$

Now, since $f(t, x) = -p(x)u(t, x)$, using the estimate (13) we get

$$\|f(t, \cdot)\|_{L^2(\mathbb{G})} \leq \|p\|_{L^\infty(\mathbb{G})}^{\frac{1}{2}} \|\sqrt{p}u(t, \cdot)\|_{L^2(\mathbb{G})} \leq (1 + \|p\|_{L^\infty(\mathbb{G})}) \|u_0\|_{H^{\frac{s\nu}{2}}(\mathbb{G})}, \quad (21)$$

so that by (19) we arrive at

$$\|u(t, \cdot)\|_{L^2(\mathbb{G})} \leq (1 + \|p\|_{L^\infty(\mathbb{G})}) \|u_0\|_{H^{\frac{s\nu}{2}}(\mathbb{G})}. \quad (22)$$

Finally, combining the inequalities (14) and (22) we get

$$\|u(t, \cdot)\|_{L^2(\mathbb{G})} \leq (1 + \|p\|_{L^\infty(\mathbb{G})}) \|u_0\|_{H^{\frac{s\nu}{2}}(\mathbb{G})}, \quad (23)$$

uniformly in $t \in [0, T]$, and this shows the estimate (8) while the uniqueness of u also follows. This completes the proof of Proposition 2.

Proposition 3. Assume that $Q > \nu s$, and let $p \in L^{\frac{2Q}{\nu s}}(\mathbb{G}) \cap L^{\frac{Q}{\nu s}}(\mathbb{G})$, $p \geq 0$. If we suppose that $u_0 \in H^{\frac{s\nu}{2}}(\mathbb{G})$, then there exists a unique solution $u \in C([0, T]; H^{\frac{s\nu}{2}}(\mathbb{G}))$ to the Cauchy problem (1) satisfying the estimate

$$\|u(t, \cdot)\|_{H^{\frac{s\nu}{2}}(\mathbb{G})} \leq \|u_0\|_{H^{\frac{s\nu}{2}}(\mathbb{G})} \left\{ \left(1 + \|p\|_{L^{\frac{2Q}{\nu s}}(\mathbb{G})}\right) \left(1 + \|p\|_{L^{\frac{Q}{\nu s}}(\mathbb{G})}\right)^{\frac{1}{2}} \right\}, \quad (24)$$

uniformly in $t \in [0, T]$.

Proof. Proceeding as in the proof of Proposition 2, we have

$$E(t) = E(0), \quad \forall t \in [0, T], \quad (25)$$

where the energy estimate E is given by

$$E(t) = \|\mathcal{R}^{\frac{s}{2}} u(t, \cdot)\|_{L^2(\mathbb{G})}^2 + \|\sqrt{p}(\cdot)u(t, \cdot)\|_{L^2(\mathbb{G})}^2.$$

Now, applying Hölder's inequality, we get

$$\|\sqrt{p}u_0\|_{L^2(\mathbb{G})}^2 \leq \|p\|_{L^{q'}(\mathbb{G})} \|u_0\|_{L^{2q}(\mathbb{G})}^2, \quad (26)$$

where $1 < q, q' < \infty$, and (q, q') conjugate exponents, to be chosen later. Observe that if we apply (6) for $u_0 \in H^{\frac{s\nu}{2}}(\mathbb{G})$, $b = \frac{s\nu}{2}$, $a = 0$, and $q_0 = \frac{2Q}{Q-\nu s}$, then $\tilde{q}_0 = 2$, and we have

$$\|u_0\|_{L^{q_0}(\mathbb{G})} \leq \|\mathcal{R}^{\frac{s}{2}} u_0\|_{L^2(\mathbb{G})} < \infty. \quad (27)$$

Choosing $2q = q_0$ in (26) so that $q = \frac{Q}{Q-\nu s}$, we get $q' = \frac{Q}{\nu s}$, so that

$$\|\sqrt{p}u_0\|_{L^2(\mathbb{G})}^2 \leq \|p\|_{L^{\frac{Q}{\nu s}}(\mathbb{G})} \|\mathcal{R}^{\frac{s}{2}} u_0\|_{L^2(\mathbb{G})}^2 < \infty, \quad (28)$$

and by (25) we can estimate

$$\begin{aligned} \|\sqrt{p}(\cdot)u(t, \cdot)\|_{L^2(\mathbb{G})}^2 &\leq \|u_0\|_{H^{\frac{s\nu}{2}}(\mathbb{G})}^2 + \|\sqrt{p}u_0\|_{L^2(\mathbb{G})}^2 \\ &\leq \|u_0\|_{H^{\frac{s\nu}{2}}(\mathbb{G})}^2 + \|p\|_{L^{\frac{Q}{\nu s}}(\mathbb{G})} \|u_0\|_{H^{\frac{s\nu}{2}}(\mathbb{G})}^2 \\ &\leq \left(1 + \|p\|_{L^{\frac{Q}{\nu s}}(\mathbb{G})}\right) \|u_0\|_{H^{\frac{s\nu}{2}}(\mathbb{G})}^2, \end{aligned} \quad (29)$$

uniformly in $t \in [0, T]$. Additionally, (25), using the estimate (29), implies

$$\|\mathcal{R}^{\frac{s}{2}}u(t, \cdot)\|_{L^2(\mathbb{G})}^2 \leq \left(1 + \|p\|_{L^{\frac{Q}{\nu s}}(\mathbb{G})}\right) \|u_0\|_{H^{\frac{s\nu}{2}}(\mathbb{G})}^2. \quad (30)$$

To show our claim (24), it suffices to show the desired estimate for the solution norm $\|u(t, \cdot)\|_{L^2(\mathbb{G})}$. To this end, observe that by the Sobolev embeddings (6) and Hölder's inequality, using (28) with p instead of \sqrt{p} , and $\|p^2\|_{L^{\frac{Q}{\nu s}}(\mathbb{G})} = \|p\|_{L^{\frac{2Q}{\nu s}}(\mathbb{G})}^2$, one obtains

$$\|pu(t, \cdot)\|_{L^2(\mathbb{G})}^2 \leq \|p\|_{L^{\frac{2Q}{\nu s}}(\mathbb{G})}^2 \|\mathcal{R}^{\frac{s}{2}}u(t, \cdot)\|_{L^2(\mathbb{G})}^2,$$

where the last combined with (30) yields

$$\|pu(t, \cdot)\|_{L^2(\mathbb{G})}^2 \leq \|p\|_{L^{\frac{2Q}{\nu s}}(\mathbb{G})}^2 \left(1 + \|p\|_{L^{\frac{Q}{\nu s}}(\mathbb{G})}\right) \|u_0\|_{H^{\frac{s\nu}{2}}(\mathbb{G})}^2. \quad (31)$$

Finally, using arguments similar to those we developed in Proposition 2, together with the estimate (31) we get

$$\begin{aligned} \|u(t, \cdot)\|_{L^2(\mathbb{G})}^2 &\leq \|u_0\|_{L^2(\mathbb{G})}^2 + \|p(\cdot)u(t, \cdot)\|_{L^2(\mathbb{G})}^2 \\ &\leq \|u_0\|_{H^{\frac{s\nu}{2}}(\mathbb{G})}^2 \left\{ \left(1 + \|p\|_{L^{\frac{2Q}{\nu s}}(\mathbb{G})}\right) \left(1 + \|p\|_{L^{\frac{Q}{\nu s}}(\mathbb{G})}\right) \right\}, \end{aligned}$$

uniformly in $t \in [0, T]$. The uniqueness of u is immediate by the estimate (24), and this finishes the proof of Proposition 3.

3 Existence and uniqueness of the very weak solution

Proving the existence and the uniqueness of the very weak solution to the Cauchy problem (1) requires to assume that the potential p and the initial data u_0 in (1) satisfy some moderateness properties. Regarding the potential p , we have in mind cases where p is strongly singular; like for instance when $p = \delta$ or $p = \delta^2$. In the first case the moderate properties of p follow by Proposition 4, while, in the second case, we understand δ^2 as an approximating family or in the Colombeau sense.

Definition 2. [Moderateness]

1. Let X be a normed space of functions on \mathbb{G} . A net of functions $(f_\epsilon)_\epsilon \in X$ is said to be *X-moderate* if there exists $N \in \mathbb{N}$ such that

$$\|f_\epsilon\|_X \leq \epsilon^{-N},$$

uniformly in $\epsilon \in (0, 1]$.

2. A net of functions $(u_\epsilon)_\epsilon$ in $C([0, T]; H^{\frac{s\nu}{2}}(\mathbb{G}))$ is said to be $C([0, T]; H^{\frac{s\nu}{2}}(\mathbb{G}))$ -*moderate* if there exists $N \in \mathbb{N}$ such that

$$\sup_{t \in [0, T]} \|u(t, \cdot)\|_{H^{\frac{s\nu}{2}}(\mathbb{G})} \leq \epsilon^{-N},$$

uniformly in $\epsilon \in (0, 1]$.

Definition 3. [Negligibility] Let Y be a normed space of functions on \mathbb{G} . Let $(f_\epsilon)_\epsilon, (\tilde{f}_\epsilon)_\epsilon$ be two nets. Then, the net $(f_\epsilon - \tilde{f}_\epsilon)_\epsilon$ is called Y -*negligible*, if the following condition is satisfied

$$\|f_\epsilon - \tilde{f}_\epsilon\|_Y \leq \epsilon^k, \quad (32)$$

for all $k \in \mathbb{N}$, $\epsilon \in (0, 1]$. In the case where $f = f(t, x)$ is a function also depending on $t \in [0, T]$, then the *negligibility condition* (32) can be regarded as

$$\|f_\epsilon(t, \cdot) - \tilde{f}_\epsilon(t, \cdot)\|_Y \leq \epsilon^k, \quad \forall k \in \mathbb{N},$$

uniformly in $t \in [0, T]$. The constant in the inequality (32) may depend on k but not on ϵ .

Definitions 4 and 5 introduce the notion of the unique very weak solution to the Cauchy problem (1). Our definitions resembles the ones in [16], but here we measure moderateness and negligibility in terms of $L^p(\mathbb{G})$ or $H^{\frac{s\nu}{2}}(\mathbb{G})$ -norms rather than in terms of Gevrey-seminorms.

Definition 4.[Very weak solution] If there exists a $L^\infty(\mathbb{G})$ -moderate, or (provided that $Q > \nu s$) a $L^{\frac{2Q}{\nu s}}(\mathbb{G}) \cap L^{\frac{Q}{\nu s}}(\mathbb{G})$ -moderate *approximating net* $(p_\epsilon)_\epsilon$, $p_\epsilon \geq 0$ to p , and a $H^{\frac{s\nu}{2}}(\mathbb{G})$ -moderate regularising net $(u_{0,\epsilon})_\epsilon$ to u_0 , then the net $(u_\epsilon)_\epsilon \in C([0, T]; H^{\frac{s\nu}{2}}(\mathbb{G}))$ which solves the ϵ -parametrised problem

$$i\partial_t u_\epsilon(t, x) + \mathcal{R}^s u_\epsilon(t, x) + p_\epsilon(x) u_\epsilon(t, x) = 0, \quad (t, x) \in [0, T] \times \mathbb{G}, \quad (33)$$

$$u_\epsilon(0, x) = u_{0,\epsilon}(x), \quad x \in \mathbb{G},$$

for all $\epsilon \in (0, 1]$, is said to be a *very weak solution to the Cauchy problem (1)* if it is $H^{\frac{s\nu}{2}}(\mathbb{G})$ -moderate.

Remark 1. Let us mention that in Definition 4 the approximating net p_ϵ includes the case where p_ϵ is a *regularisation* of p in the case where $p \in \mathcal{D}'(\mathbb{G})$ is a distribution, i.e., for a Friedrichs mollifier $\psi \geq 0$ we define $p_\epsilon = p * \psi_\epsilon$. In singular cases, like for instance when $p = \delta^2$, we can think of p_ϵ as $p_\epsilon = \psi_\epsilon^2$; see also Remark 2 for additional clarifications.

Next we formulate the very weak existence result in compatibility with the two possible moderateness assumptions on the approximating nets $(p_\epsilon)_\epsilon$

as stated in Definition 4. Before doing that, let us mention that, regarding the moderateness assumption of the regularisations (or approximations), the global structure of \mathcal{E}' -distributions, implies that, for any regularisation of them taken via convolutions with a mollifier as in (7), the assumption on the L^p -moderateness, for $p \in [1, \infty]$, is natural. Formally we have the following proposition as in Proposition 4.8 in [8].

Proposition 4. Let $v \in \mathcal{E}'(\mathbb{G})$, and let $v_\epsilon = v * \psi_\epsilon$ be obtained as the convolution of v with a mollifier ψ_ϵ as in (7). Then the regularising net $(v_\epsilon)_\epsilon$ is $L^p(\mathbb{G})$ -moderate for any $p \in [1, \infty]$.

As an immediate consequence of Proposition 4 is that, for the existence of the very weak solution to the Cauchy problem (1), we do not require that the initial data u_0 is necessarily an element of the space $H^{\frac{s\nu}{2}}(\mathbb{G})$ as Proposition 2 and Proposition 3 on the existence of the classical solution of (1) indicate. Indeed, we also allow that $u_0 \in \mathcal{E}'(\mathbb{G})$ is compactly supported distribution.

Theorem 1. Let $u_0 \in H^{\frac{s\nu}{2}}(\mathbb{G}) \cup \mathcal{E}'(\mathbb{G})$. Then the Cauchy problem (1) has a very weak solution.

Proof. Let u_0 be as in the hypothesis. If $(p_\epsilon)_\epsilon$ is $L^\infty(\mathbb{G})$ -moderate (or $L^{\frac{2Q}{\nu s}}(\mathbb{G}) \cap L^{\frac{Q}{\nu s}}(\mathbb{G})$ -moderate) and $(u_{0,\epsilon})_\epsilon$ is $H^{\frac{s\nu}{2}}(\mathbb{G})$ -moderate, then, since also $p_\epsilon \geq 0$, by using (8) (or (24), respectively) we get

$$\|u_\epsilon(t, \cdot)\|_{H^{\frac{s\nu}{2}}(\mathbb{G})} \leq \epsilon^{-N}, \quad N \in \mathbb{N},$$

for all $t \in [0, T]$ and for any $\epsilon \in (0, 1]$. This means that the family of solutions $(u_\epsilon)_\epsilon$ is $H^{\frac{s\nu}{2}}(\mathbb{G})$ -moderate, and completes the proof of Theorem 1.

Roughly speaking, proving well-posedness in the very weak sense amount to proving that a very weak solution exists and it is unique modulo negligible nets. For the Cauchy problem (1) that we consider here, this notion can be formalised as follows.

Definition 5. Let X and Y be normed spaces of functions on \mathbb{G} . We say that the Cauchy problem (1) has an (X, Y) -unique very weak solution, if for all X -moderate nets $p_\epsilon \geq 0, \tilde{p}_\epsilon \geq 0$, such that $(p_\epsilon - \tilde{p}_\epsilon)_\epsilon$ is Y -negligible, and for all $H^{\frac{s\nu}{2}}(\mathbb{G})$ -moderate regularisations $u_{0,\epsilon}, \tilde{u}_{0,\epsilon}$ such that $(u_{0,\epsilon} - \tilde{u}_{0,\epsilon})_\epsilon$ is $H^{\frac{s\nu}{2}}(\mathbb{G})$ -negligible, it follows that

$$\|u_\epsilon(t, \cdot) - \tilde{u}_\epsilon(t, \cdot)\|_{L^2(\mathbb{G})} \leq C_N \epsilon^N, \quad \forall N \in \mathbb{N},$$

uniformly in $t \in [0, T]$, and for all $\epsilon \in (0, 1]$, where $(u_\epsilon)_\epsilon$ and $(\tilde{u}_\epsilon)_\epsilon$ are the families of solutions corresponding to the ϵ -parametrised problems

$$i\partial_t u_\epsilon(t, x) + \mathcal{R}^s u_\epsilon(t, x) + p_\epsilon(x) u_\epsilon(t, x) = 0, \quad (t, x) \in [0, T] \times \mathbb{G}, \quad (34)$$

$$u_\epsilon(0, x) = u_{0,\epsilon}(x), \quad x \in \mathbb{G},$$

and

$$i\partial_t \tilde{u}_\epsilon(t, x) + \mathcal{R}^s \tilde{u}_\epsilon(t, x) + \tilde{p}_\epsilon(x) \tilde{u}_\epsilon(t, x) = 0, \quad (t, x) \in [0, T] \times \mathbb{G}, \quad (35)$$

$$\tilde{u}_\epsilon(0, x) = \tilde{u}_{0,\epsilon}(x), \quad x \in \mathbb{G},$$

respectively.

Remark 2. Definition 5 is a rigorous version of Definition 2.2 in the previous paper [2] regarding the uniqueness of the very weak solution to the Cauchy problem (1) in the Euclidean setting. In particular, in [2] we assume that p_ϵ and \tilde{p}_ϵ are regularisations of $p \in \mathcal{D}'(\mathbb{G})$, and so they approximate p in some suitable sense. Instead, in Definition 5 we do not require the nets $p_\epsilon, \tilde{p}_\epsilon$ to approximate p ; for instance, if p_ϵ is some regularisation of p and \tilde{p}_ϵ is given as

$$\tilde{p}_\epsilon = p_\epsilon + e^{-1/\epsilon}, \quad (36)$$

then the net $(p_\epsilon - \tilde{p}_\epsilon)_\epsilon$ is L^∞ -negligible, and so satisfies the assumption described in Definition 5. Moreover, the absence of the approximation requirement, allows to consider singular cases of p ; cf. Remark 1 where we take $p = \delta^2$ and $p_\epsilon = \psi_\epsilon^2$. Thus, under the choice of \tilde{p}_ϵ as in (36), the implied net $(p_\epsilon - \tilde{p}_\epsilon)$ is suitable for our purposes.

To summarise the above, the meaning of the conditions regarding the nets p_ϵ and \tilde{p}_ϵ in Definition 5 should be interpreted as a requirement for the stability of the very weak solution under negligible changes on the potential p ; see also Theorem 2 and Theorem 3 where no approximating assumption has been regarded.

The following theorems show the uniqueness of the very weak solution to the Cauchy problem (1) under different assumptions on the nets $(p_\epsilon)_\epsilon$. In order to do this, we need the following technical lemma, that shall also be used to prove the consistency of the very weak solution with the classical one.

Lemma 1. Let $u_0 \in L^2(\mathbb{G})$ and assume that p is non-negative. Then, for the unique solution u to the Cauchy problem (1) we have the energy conservation

$$\|u(t, \cdot)\|_{L^2(\mathbb{G})} = \text{constant}, \quad (37)$$

for all $t \in [0, T]$.

Proof. If we multiply equation (1) by $-i$, then we obtain

$$u_t(t, x) - i\mathcal{R}^s u(t, x) - ip(x)u(t, x) = 0.$$

If we multiply the above with u , integrate over \mathbb{G} , and consider the real part of the above we get

$$\Re(\langle u_t(t, \cdot), u(t, \cdot) \rangle_{L^2(\mathbb{G})} - i\langle \mathcal{R}^s u(t, \cdot), u(t, \cdot) \rangle_{L^2(\mathbb{G})} - i\langle p(\cdot)u(t, \cdot), u(t, \cdot) \rangle_{L^2(\mathbb{G})}) = 0,$$

or equivalently

$$\Re(\langle u_t(t, \cdot), u(t, \cdot) \rangle_{L^2(\mathbb{G})}) = \frac{1}{2} \partial_t \|u(t, \cdot)\|_{L^2(\mathbb{G})}^2 = 0.$$

The latter means that we have energy conservation, i.e., the norm $\|u(t, \cdot)\|_{L^2(\mathbb{G})}$ remains constants over time, and in particular we have

$$\|u(t, \cdot)\|_{L^2(\mathbb{G})} = \|u_0\|_{L^2(\mathbb{G})}, \quad \forall t \in [0, T],$$

implying (37).

Theorem 2. Suppose that $u_0 \in H^{\frac{s}{2}}(\mathbb{G}) \cup \mathcal{E}'(\mathbb{G})$. Then the very weak solution to the Cauchy problem (1) is $(L^\infty(\mathbb{G}), L^\infty(\mathbb{G}))$ -unique.

Proof. Let $(u_\epsilon)_\epsilon$ and $(\tilde{u}_\epsilon)_\epsilon$ be the families of solutions corresponding to the Cauchy problems (34) and (35), respectively. If we denote by $U_\epsilon(t, \cdot) := u_\epsilon(t, \cdot) - \tilde{u}_\epsilon(t, \cdot)$, then U_ϵ satisfies

$$i\partial_t U_\epsilon(t, x) + \mathcal{R}^s U_\epsilon(t, x) + p_\epsilon(x) U_\epsilon(t, x) = f_\epsilon(t, x), \quad (t, x) \in [0, T] \times \mathbb{G}, \quad (38)$$

$$U_\epsilon(0, x) = (u_{0, \epsilon} - \tilde{u}_{0, \epsilon})(x), \quad x \in \mathbb{G},$$

where $f_\epsilon(t, x) := (\tilde{p}_\epsilon(x) - p_\epsilon(x))\tilde{u}_\epsilon(t, x)$.

The solution of the Cauchy problem (38) can be expressed in terms of the solution to the corresponding homogeneous Cauchy problem using Duhamel's principle. Indeed, if $W_\epsilon(t, x)$, and $V_\epsilon(t, x; \sigma)$, where σ is some fixed parameter in $[0, T]$, are the solutions to the homogeneous Cauchy problems

$$i\partial_t V_\epsilon(t, x; \sigma) + \mathcal{R}^s V_\epsilon(t, x; \sigma) + p_\epsilon V_\epsilon(t, x; \sigma) = 0, \quad \text{in } (\sigma, T] \times \mathbb{G},$$

$$V_\epsilon(t, x; \sigma) = f_\epsilon(\sigma, x) \quad \text{on } \{t = \sigma\} \times \mathbb{G},$$

and

$$i\partial_t W_\epsilon(t, x) + \mathcal{R}^s W_\epsilon(t, x) + p_\epsilon W_\epsilon(t, x) = 0, \quad \text{in } [0, T] \times \mathbb{G},$$

$$W_\epsilon(t, x) = (u_{0, \epsilon} - \tilde{u}_{0, \epsilon})(x) \quad \text{on } \{t = 0\} \times \mathbb{G},$$

respectively, then U_ϵ is given by

$$U_\epsilon(t, x) = W_\epsilon(t, x) + \int_0^t V_\epsilon(t - \sigma, x; \sigma) d\sigma. \quad (39)$$

Taking the L^2 -norm in (39) and using the energy conservation (37) to estimate V_ϵ we get

$$\begin{aligned} \|U_\epsilon(t, \cdot)\|_{L^2(\mathbb{G})} &\leq \|W_\epsilon(t, \cdot)\|_{L^2(\mathbb{G})} + \int_0^t \|V_\epsilon(t - \sigma, \cdot; \sigma)\|_{L^2(\mathbb{G})} d\sigma & (40) \\ &\leq \|u_{0, \epsilon} - \tilde{u}_{0, \epsilon}\|_{L^2(\mathbb{G})} + \int_0^t \|f_\epsilon(\sigma, \cdot)\|_{L^2(\mathbb{G})} d\sigma \\ &\leq \|u_{0, \epsilon} - \tilde{u}_{0, \epsilon}\|_{L^2(\mathbb{G})} + \|\tilde{p}_\epsilon - p_\epsilon\|_{L^\infty(\mathbb{G})} \int_0^t \|\tilde{u}_\epsilon(\sigma, \cdot)\|_{L^2(\mathbb{G})} d\sigma, \end{aligned}$$

for all $t \in [0, T]$, where for the first inequality (40) we have applied Minkowski's integral inequality, i.e., that

$$\left\| \int_0^t V_\epsilon(t - \sigma, \cdot; \sigma) d\sigma \right\|_{L^2(\mathbb{G})} \leq \int_0^t \|V_\epsilon(t - \sigma, \cdot; \sigma)\|_{L^2(\mathbb{G})} d\sigma.$$

Now, using the fact that $(u_{0,\epsilon} - \tilde{u}_{0,\epsilon})_\epsilon$ is $H^{\frac{sv}{2}}(\mathbb{G})$ -negligible, while also that the net $(\tilde{u}_\epsilon)_\epsilon$, as being a very weak solution to the Cauchy problem (34), is $H^{\frac{sv}{2}}(\mathbb{G})$ -moderate and that $(p_\epsilon - \tilde{p}_\epsilon)_\epsilon$ is L^∞ -negligible, we get that

$$\|U_\epsilon(t, \cdot)\|_{L^2(\mathbb{G})} \leq \epsilon^N + \epsilon^{\tilde{N}} \int_0^T \epsilon^{-N_1} d\sigma,$$

for some $N_1 \in \mathbb{N}$, and for all $N, \tilde{N} \in \mathbb{N}$, $\epsilon \in (0, 1]$. That is, we have

$$\|U_\epsilon(t, \cdot)\|_{L^2(\mathbb{G})} \leq \epsilon^k,$$

for all $k \in \mathbb{N}$, and the last shows that the net $(u_\epsilon)_\epsilon$ is the unique very weak solution to the Cauchy problem (1).

Alternative to Theorem 2 conditions on the nets $(p_\epsilon)_\epsilon, (\tilde{p}_\epsilon)_\epsilon$ that guarantee the very weak well-posedness of (1) are given in the following theorem.

Theorem 3. Let $Q > \nu s$, and suppose that $u_0 \in H^{\frac{sv}{2}}(\mathbb{G})$. Then, the very weak solution to the Cauchy problem (1) is $(L^\infty(\mathbb{G}), L^{\frac{2Q}{\nu s}}(\mathbb{G}))$ -unique. Moreover, the very weak solution to the Cauchy problem (1) is also $(L^{\frac{2Q}{\nu s}}(\mathbb{G}) \cap L^{\frac{Q}{\nu s}}(\mathbb{G}), L^{\frac{2Q}{\nu s}}(\mathbb{G}))$ -unique and $(L^{\frac{2Q}{\nu s}}(\mathbb{G}) \cap L^{\frac{Q}{\nu s}}(\mathbb{G}), L^\infty(\mathbb{G}))$ -unique.

Proof. We will only prove the $(L^\infty(\mathbb{G}), L^{\frac{2Q}{\nu s}}(\mathbb{G}))$ -uniqueness as the other two uniqueness statements are similar. Using arguments similar to those developed in Theorem 2, we arrive at

$$\begin{aligned} \|U_\epsilon(t, \cdot)\|_{L^2(\mathbb{G})} &\leq \|u_{0,\epsilon} - \tilde{u}_{0,\epsilon}\|_{L^2(\mathbb{G})} + \int_0^T \|f_\epsilon(\sigma, \cdot)\|_{L^2(\mathbb{G})} d\sigma \\ &= \|u_{0,\epsilon} - \tilde{u}_{0,\epsilon}\|_{L^2(\mathbb{G})} + \int_0^T \|(\tilde{p}_\epsilon - p_\epsilon)(\cdot)\tilde{u}_\epsilon(\sigma, \cdot)\|_{L^2(\mathbb{G})} d\sigma. \end{aligned}$$

for all $t \in [0, T]$. Additionally, by applying Hölder's inequality, together with the Sobolev embeddings (6), we have

$$\|(\tilde{p}_\epsilon - p_\epsilon)(\cdot)\tilde{u}_\epsilon(t, \cdot)\|_{L^2(\mathbb{G})} \leq \|\tilde{p}_\epsilon - p_\epsilon\|_{L^{\frac{2Q}{\nu s}}(\mathbb{G})} \|\mathcal{R}^{\frac{s}{2}}\tilde{u}_\epsilon(t, \cdot)\|_{L^2(\mathbb{G})},$$

where since $(\tilde{u}_\epsilon)_\epsilon$, as being the very weak solution corresponding to the $L^\infty(\mathbb{G})$ -moderate net $(\tilde{p}_\epsilon)_\epsilon$, is $H^{\frac{sv}{2}}(\mathbb{G})$ -moderate, we have

$$\|\mathcal{R}^{\frac{s}{2}}\tilde{u}_\epsilon(t, \cdot)\|_{L^2(\mathbb{G})} \leq \epsilon^{-N_1},$$

for some $N_1 \in \mathbb{N}$.

Summarising the above, and since

$$\|u_{0,\epsilon} - \tilde{u}_{0,\epsilon}\|_{L^2(\mathbb{G})}, \|\tilde{p}_\epsilon - p_\epsilon\|_{L^{\frac{2Q}{\nu s}}(\mathbb{G})} \leq \epsilon^N, \quad \forall N \in \mathbb{N},$$

we obtain

$$\|U_\epsilon(t, \cdot)\|_{L^2(\mathbb{G})} \leq \epsilon^k, \quad \forall k \in \mathbb{N},$$

uniformly in t , and this finishes the proof of Theorem 3.

4 Consistency of the very weak solution with the classical one

The next theorems stress the conditions, on the potential p and on the initial data u_0 , under which, the classical solution to the Cauchy problem (1), as given in Proposition 2 or Proposition 3, can be recaptured by its very weak solution. To avoid any possible misunderstanding, let us clarify by a ‘regularisation’ of p we mean the net arising via the convolution of p with non-negative Friedrichs mollifiers as in (7).

Theorem 4. Let $Q > \nu s$. Consider the Cauchy problem (1), and let $u_0 \in H^{\frac{s\nu}{2}}(\mathbb{G})$. Assume also that $p \in L^{\frac{2Q}{\nu s}}(\mathbb{G}) \cap L^{\frac{Q}{\nu s}}(\mathbb{G})$, $p \geq 0$, and that $(p_\epsilon)_\epsilon$, is a regularisation of the potential p . Then the regularised net $(u_\epsilon)_\epsilon$ converges, as $\epsilon \rightarrow 0$, in $L^2(\mathbb{G})$ to the classical solution u given by Proposition 3.

Proof. Let u be the classical solution of (1) given by Proposition 3, and let (u_ϵ) be the very weak solution of the regularised analogue of it as in (34). If we denote by $W_\epsilon(t, x) := u(t, x) - u_\epsilon(t, x)$, then W_ϵ solves the auxiliary Cauchy problem

$$\begin{aligned} i\partial_t W_\epsilon(t, x) + \mathcal{R}^s W_\epsilon(t, x) + p_\epsilon(x)W_\epsilon(t, x) &= \eta_\epsilon(t, x), \\ W_\epsilon(0, x) &= (u_0 - u_{0,\epsilon})(x), \end{aligned} \quad (41)$$

where $\eta_\epsilon(t, x) := (p_\epsilon(x) - p(x))u(t, x)$. Using Duhamel’s principle and arguments similar to Theorem 2 we get the estimates

$$\begin{aligned} \|W_\epsilon(t, \cdot)\|_{L^2(\mathbb{G})} &\leq \|u_0 - u_{0,\epsilon}\|_{L^2(\mathbb{G})} + \int_0^T \|\eta_\epsilon(\sigma, \cdot)\|_{L^2(\mathbb{G})} d\sigma \\ &= \|u_0 - u_{0,\epsilon}\|_{L^2(\mathbb{G})} + \int_0^T \|(p_\epsilon - p)(\cdot)u(\sigma, \cdot)\|_{L^2(\mathbb{G})} d\sigma \\ &\leq \|u_0 - u_{0,\epsilon}\|_{L^2(\mathbb{G})} + \int_0^T \|p_\epsilon - p\|_{L^{\frac{2Q}{\nu s}}(\mathbb{G})} \|\mathcal{R}^{\frac{s}{2}} u(\sigma, \cdot)\|_{L^2(\mathbb{G})} d\sigma, \end{aligned} \quad (42)$$

where to get the last inequality we apply Hölder’s inequality and the Sobolev embeddings (6). Now, since by Proposition 3 we have $u \in H^{\frac{s\nu}{2}}(\mathbb{G})$, while also

$p \in L^{\frac{2Q}{\nu s}}(\mathbb{G})$, $u_0 \in H^{\frac{sv}{2}}(\mathbb{G})$, we get that

$$\|u_0 - u_{0,\epsilon}\|_{L^2(\mathbb{G})}, \|p_\epsilon - p\|_{L^{\frac{2Q}{\nu s}}(\mathbb{G})} \|\mathcal{R}^{\frac{s}{2}}u(\sigma, \cdot)\|_{L^2(\mathbb{G})} \rightarrow 0,$$

as $\epsilon \rightarrow 0$, so that by (42) and Lebesgue's dominated convergence theorem we get

$$\|W_\epsilon(t, \cdot)\|_{L^2(\mathbb{G})} \rightarrow 0, \quad (43)$$

uniformly in $t \in [\sigma, T]$, where $\sigma \in [0, T]$, i.e., the very weak solution converges to the classical one in L^2 , and this finishes the proof of Theorem 4.

In the following theorem we denote by $C_0(\mathbb{G})$ the space of continuous functions on \mathbb{G} vanishing at infinity, that is, such that for every $\epsilon > 0$ there exists a compact set K outside of which we have $|f| < \epsilon$. Note that $C_0(\mathbb{G})$ is a Banach space if endowed with the norm $\|\cdot\|_{L^\infty(\mathbb{G})}$.

Theorem 5. Consider the Cauchy problem (1), and let $u_0 \in H^{\frac{sv}{2}}(\mathbb{G})$. Assume also that $p \in C_0(\mathbb{G})$, $p \geq 0$, and that $(p_\epsilon)_\epsilon$, $p_\epsilon \geq 0$, is a regularisation of the coefficient p . Then the regularised net $(u_\epsilon)_\epsilon$ converges, as $\epsilon \rightarrow 0$, in $L^2(\mathbb{G})$ to the classical solution u given by Proposition 2.

Before giving the proof of Theorem 5, let us make the following observation: If $p \in C_0(\mathbb{G})$, then $\|p_\epsilon\|_{L^\infty(\mathbb{G})} \leq C < \infty$, uniformly in $\epsilon \in (0, 1]$.

Proof of Theorem 5. First observe that for p , $(p_\epsilon)_\epsilon$ as in the hypothesis, we have $p_\epsilon \in L^\infty(\mathbb{G})$ for each $\epsilon \in (0, 1]$. Hence, if we denote by W_ϵ the solution to the problem (41), then, reasoning as we did in Theorems 2 and 4, we obtain

$$\|W_\epsilon(t, \cdot)\|_{L^2(\mathbb{G})} \leq \|u_0 - u_{0,\epsilon}\|_{L^2(\mathbb{G})} + \int_0^T \|(p_\epsilon - p)(\cdot)u(\sigma, \cdot)\|_{L^2(\mathbb{G})} d\sigma,$$

uniformly in $t \in [0, T]$. Now, since

$$\|(p_\epsilon - p)(\cdot)u(\sigma, \cdot)\|_{L^2(\mathbb{G})} \leq \|p_\epsilon - p\|_{L^\infty(\mathbb{G})} \|u(\sigma, \cdot)\|_{L^2(\mathbb{G})},$$

while by Lemmas 3.1.58 and 3.1.59 in [13] we have

$$\|p_\epsilon - p\|_{L^\infty(\mathbb{G})} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0,$$

summarising the above we get

$$\|W_\epsilon(t, \cdot)\|_{L^2(\mathbb{G})} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0,$$

and this completes the proof of Theorem 5.

Remark 3. In the consistency result [2, Theorem 2.3] the assumption on the potential p is regarded as $p \in L^\infty(\mathbb{R}^d)$. However, this assumption is not a sufficient one; we should instead ask for p to be in the subspace $C_0(\mathbb{R}^d)$ of $L^\infty(\mathbb{R}^d)$, as follows by Theorem 5 in the particular case where $\mathbb{G} = \mathbb{R}^d$ and $\mathcal{R} = -\Delta$.

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