A COMPARISON PRINCIPLE FOR HIGHER ORDER NONLINEAR HYPOELLIPTIC HEAT OPERATORS ON GRADED LIE GROUPS

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Abstract. In this paper we present a comparison principle for higher order nonlinear hypoelliptic heat operators on graded Lie groups. Moreover, using the comparison principle we obtain blow-up type results and global in t-boundedness of solutions of nonlinear equations for the heat p-sub-Laplacian on stratified Lie groups.

1. Introduction

A connected simply connected Lie group $\mathbb{G}$ is called a graded Lie group if its Lie algebra admits a gradation. The graded Lie groups form the subclass of homogeneous nilpotent Lie groups admitting homogeneous hypoelliptic left-invariant differential operators ([Mil80], [tER97], see also a discussion in [FR16, Section 4.1]). These operators are called Rockland operators from the Rockland conjecture, solved by Helffer and Nourrigat [HN79]. So, we understand by a Rockland operator *any left-invariant homogeneous hypoelliptic differential operator on* $\mathbb{G}$. For example, for the Heisenberg group, the sub-Laplacian and its powers are Rockland operators. If $\mathbb{G}$ is a stratified Lie group with a given basis $X_1, \ldots, X_n$ for the first stratum of its Lie algebra, then the operators

$$\mathcal{R} = (-1)^m \sum_{j=1}^{n} a_j X_j^{2m}, \quad a_j > 0,$$

are positive Rockland operators for any $m \in \mathbb{N}$, yielding the sub-Laplacian for $m = 1$. More generally, for any graded Lie group $\mathbb{G} \sim \mathbb{R}^n$ with dilation weights $\nu_1, \ldots, \nu_n$ and a basis $X_1, \ldots, X_n$ of the corresponding Lie algebra $\mathfrak{g}$ with the property

$$D_r X_j = r^{\nu_j} X_j, \quad j = 1, \ldots, n, \ r > 0,$$

the operator

$$\mathcal{R} = \sum_{j=1}^{n} (-1)^{\frac{\nu_0}{\nu_j}} a_j X_j^{2\frac{\nu_0}{\nu_j}}, \quad a_j > 0, \tag{1.1}$$

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is a Rockland operator of homogeneous degree $2\nu_0$, where ν_0 is any common multiple of $\nu_1, \ldots, \nu_n$. For many other examples and a detailed discussion of Rockland operators we can refer to [FR16, Section 4.1.2].

Thus, the considered setting includes the higher order operators on $\mathbb{R}^n$ as well as higher order hypoelliptic invariant differential operators on the Heisenberg group, on general stratified Lie groups, and on general graded Lie groups.

Let us also recall that the standard Lebesgue measure is the Haar measure for $\mathbb{G}$. Let $\Omega \subset \mathbb{G}$ be a bounded set with smooth boundary. We denote the Sobolev space by $S^{a,p}(\Omega) = S^{a,p}_{\mathcal{R}}(\Omega)$, for $a > 0$ and $p \in (1, \infty) \cup \{\infty_o\}$, defined by the norm

$$\|u\|_{S^{a,p}(\Omega)} := \left(\int_{\Omega} \left(\left|\mathcal{R}^{\frac{a}{\nu}} u(x)\right|^p + |u(x)|^p \right) dx \right)^{\frac{1}{p}}, \tag{1.2}$$

where ν is the homogeneous order of the Rockland operator $\mathcal{R}$. We have allowed ourselves to write $\|\cdot\|_{L^\infty(\mathbb{G})} = \|\cdot\|_{L^{\infty_o}(\mathbb{G})}$ for the supremum norm, in the notation of [FR16, Chapter 4]. Let us also define the functional class $S^{a,p}_0(\Omega)$ to be the completion of $C^\infty_0(\Omega)$ in the norm (1.2). For a general discussion of Sobolev spaces on graded Lie groups we refer to [FR16, Chapter 4] and [FR17].

In this paper we study the higher order nonlinear hypoelliptic heat equation for $u = u(t, x)$,

$$u_t - \sum_{j=1}^{n_2} \mathcal{R}_1^{\frac{a_1}{\nu_1}} \left(\left| \mathcal{R}_1^{\frac{a_1}{\nu_1}} u \right|^{p_j - 2} \mathcal{R}_1^{\frac{a_1}{\nu_1}} u \right) u = \alpha \sum_{i=1}^{n_1} |u|^{q_i - 1} u + \beta \sum_{j=1}^{n_2} |\mathcal{R}_2^{\frac{a_2}{\nu_2}} u|^{r_j} + \gamma \sum_{k=1}^{n_3} |u|^{s_k - 1} u \tag{1.3}$$

for $x \in \Omega$ and $t > 0$, with the initial-boundary conditions

$$u = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{1.4}$$

$$u(0, x) = u_0(x), \quad x \in \Omega, \tag{1.5}$$

where $a_1, a_2 \geq 0$, and $\alpha, \beta, \gamma \in \mathbb{R}$, and $n_1, n_2, n_3 \in \mathbb{N}$, $p_j \geq 2$ and

$$q_i \begin{cases} \geq 1, & \text{if} \quad \alpha > 0, \\ > 0, & \text{if} \quad \alpha < 0, \end{cases} \quad s_k \begin{cases} \geq 1, & \text{if} \quad \gamma > 0, \\ > 0, & \text{if} \quad \gamma < 0, \end{cases} \quad r_j \begin{cases} > 1, & \text{if} \quad \beta > 0, \\ > 0, & \text{if} \quad \beta < 0. \end{cases} \tag{1.6}$$

Here, ν_1 and ν_2 are the homogeneous orders of the Rockland operators $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively. We also assume that the initial data satisfies

$$u_0 \in S^{a,\infty}_0(\Omega), \quad u_0 \geq 0,$$

where $a = \max\{a_1, a_2\}$.

Definition 1.1. Set $Q_T = (0, T) \times \Omega$, $S_T = (0, T) \times \partial\Omega$, $\partial Q_T = S_T \cup \{\{0\} \times \bar{\Omega}\}$, $p = \max\{p_j\}$ and $m = \max\{p_j, q_i, r_j, s_k\}$. A nonnegative function $u(t, x)$ is called a weak super- (sub-) solution of (1.3)-(1.5) on Q_T if it satisfies

$$u \in C([0, T) \times \overline{\Omega}) \cap L^m \left((0, T); S^{a,m}_0(\Omega) \right), \partial_t u \in L^2 \left((0, T); L^2(\Omega) \right),$$
$$u(0, x) \geq (\leq) u_0, \ u|_{\partial\Omega} \geq (\leq) 0,$$

$$\iint_{Q_T}\left(\partial_t u\phi+\sum_{j=1}^{n_2}|\mathcal{R}_1^{\frac{a_1}{\nu_1}}u|^{p_j-2}\mathcal{R}_1^{\frac{a_1}{\nu_1}}u\cdot\mathcal{R}_1^{\frac{a_1}{\nu_1}}\phi\right)dxdt$$
$$\geq(\leq)\iint_{Q_T}\left(\alpha\sum_{i=1}^{n_1}|u|^{q_i-1}u+\beta\sum_{j=1}^{n_2}|\mathcal{R}_2^{\frac{a_2}{\nu_2}}u|^{r_j}+\gamma\sum_{k=1}^{n_3}|u|^{s_k-1}u\right)\phi dxdt,$$

for all $\phi\in C(\overline{Q_T})\cap L^p\left((0,T);S_0^{a_1,p}(\Omega)\right)$ such that $\phi\geq 0$, $\phi|_{S_T}=0$. Then u is called a weak solution if it is a super-solution and a sub-solution. Here and after, we use $T_{\max}$ to denote the maximal existence time.

Our goal in this paper is to give a simple proof of a comparison principle for the initial boundary value problem for higher order nonlinear hypoelliptic heat operators on graded Lie groups using pure algebraic relations, inspired by the works [Att12] and [ZL13].

The structure of this paper is as follows. Section 2 establishes a comparison principle for the problem (1.3)-(1.5). Then, in Section 3, using the comparison principle, we investigate the blow-up or the boundedness of solution of (1.3)-(1.5) depending on the signs of α, β, γ, and relations between parameters p_j, q_i, r_j, s_k, and on u_0.

2. A comparison principle on graded Lie groups

In this section we state a comparison principle for the problem (1.3)-(1.5).

Theorem 2.1. *Assume that $u,v\in L^\infty_{loc}\left((0,T);S^{a,\infty}(\Omega)\right)$ are sub- and super-solutions of (1.3)-(1.5), respectively. Assume also that at least one of the parameters α, β and γ be positive or $\alpha=\beta=\gamma=0$. Let $r_j\geq\frac{p_j}{2}$ if $\beta>0$. Then we have $u\leq v$ on Q_T.*

Remark 2.2. In the special case $n_1=n_2=n_3=1$, $\beta=0$ and $\alpha\gamma\leq 0$, Theorem 2.1 was obtained in [RS18].

The proof of the comparison principle mostly based on the following algebraic lemma (see e.g. [Att12, Lemma 2.1] or [Lin06, [Section 10]).

Lemma 2.3. *Let $\sigma\geq 2$. For all $\vec{a}$, $\vec{b}\in\mathbb{R}^N$, we have*

$$\left\langle|\vec{a}|^{\sigma-2}\vec{a}-|\vec{b}|^{\sigma-2}\vec{b},\vec{a}-\vec{b}\right\rangle\geq\frac{4}{\sigma^2}\left||\vec{a}|^{\frac{\sigma-2}{2}}\vec{a}-|\vec{b}|^{\frac{\sigma-2}{2}}\vec{b}\right|^2.$$

Proof of Theorem 2.1. First, let us consider the case α,β and $\gamma>0$. Denote $\phi:=\max\{u-v,0\}$, hence $\phi(0,x)=0$ and $\phi(t,x)|_{x\in\partial\Omega}=0$. By the definitions of sub- and super-solutions, using ϕ as the test function, for any $\tau\in(0,T)$, we have

$$\begin{aligned}\frac{1}{2}\int_\Omega\phi^2(\tau,x)dx&=\int_0^\tau\int_\Omega\frac{1}{2}\partial_t\left(\phi^2(t,x)\right)dxdt=\int_0^\tau\int_\Omega\partial_t\phi\phi dxdt\\&\leq\underbrace{-\sum_{j=1}^{n_2}\int_0^\tau\int_{\{\phi(t,\cdot)>0\}}\left(|\mathcal{R}_1^{\frac{a_1}{\nu_1}}u|^{p_j-2}\mathcal{R}_1^{\frac{a_1}{\nu_1}}u-|\mathcal{R}_1^{\frac{a_1}{\nu_1}}v|^{p_j-2}\mathcal{R}_1^{\frac{a_1}{\nu_1}}v\right)\cdot\mathcal{R}_1^{\frac{a_1}{\nu_1}}\phi dxdt}_{I_1}\\&\quad\underbrace{+\beta\sum_{j=1}^{n_2}\int_0^\tau\int_{\{\phi(t,\cdot)>0\}}\left(|\mathcal{R}_2^{\frac{a_2}{\nu_2}}u|^{r_j}-|\mathcal{R}_2^{\frac{a_2}{\nu_2}}v|^{r_j}|\right)\phi dxdt}_{I_2}\end{aligned}\tag{2.1}$$

$$+\underbrace{\alpha\sum_{i=1}^{n_1}\int_0^\tau\int_{\{\phi(t,\cdot)>0\}}\left(|u|^{q_i-1}u-|v|^{q_i-1}v\right)\phi dxdt}_{I_3}$$

$$+\underbrace{\gamma\sum_{k=1}^{n_3}\int_0^\tau\int_{\{\phi(t,\cdot)>0\}}\left(|u|^{s_k-1}u-|v|^{s_k-1}v\right)\phi dxdt}_{I_4}.$$

By Lemma 2.3, for I_1 we have

$$I_1\geq\sum_{j=1}^{n_2}\frac{4}{p_j^2}\int_0^\tau\int_{\{\phi(t,\cdot)>0\}}\left||\mathcal{R}_1^{\frac{a_1}{\nu_1}}u|^{\frac{p_j-2}{2}}\mathcal{R}_1^{\frac{a_1}{\nu_1}}u-|\mathcal{R}_1^{\frac{a_1}{\nu_1}}v|^{\frac{p_j-2}{2}}\mathcal{R}_1^{\frac{a_1}{\nu_1}}v\right|^2dxdt. \tag{2.2}$$

Let us now estimate the term I_2. We put $h(s)=s^{\frac{2r_j}{p_j}}$ for $s\geqslant 0$. Given that $r_j\geqslant\frac{p_j}{2}$, we have $h'(s)=\frac{2r_j}{p_j}s^{\frac{2r_j-p_j}{p_j}}$. Then, by the mean value theorem we have

$$\left||\mathcal{R}_2^{\frac{a_2}{\nu_2}}u|^{r_j}-|\mathcal{R}_2^{\frac{a_2}{\nu_2}}v|^{r_j}\right|^2\leq Ch'(\theta)^2\left||\mathcal{R}_2^{\frac{a_2}{\nu_2}}u|^{p_j/2}-|\mathcal{R}_2^{\frac{a_2}{\nu_2}}v|^{p_j/2}\right|^2,$$

for some $0\leqslant\theta\leqslant\max\left\{|\mathcal{R}_2^{\frac{a_2}{\nu_2}}u|^{p_j/2},|\mathcal{R}_2^{\frac{a_2}{\nu_2}}v|^{p_j/2}\right\}$.

A direct computation yields that

$$\left||\mathcal{R}_2^{\frac{a_2}{\nu_2}}u|^{p_j/2}-|\mathcal{R}_2^{\frac{a_2}{\nu_2}}v|^{p_j/2}\right|^2\leqslant\left||\mathcal{R}_2^{\frac{a_2}{\nu_2}}u|^{(p_j-2)/2}\mathcal{R}_2^{\frac{a_2}{\nu_2}}u-|\mathcal{R}_2^{\frac{a_2}{\nu_2}}v|^{(p_j-2)/2}\mathcal{R}_2^{\frac{a_2}{\nu_2}}v\right|^2.$$

Taking into account $u,v\in L^\infty\left((0,T);S^{a,\infty}(\Omega)\right)$, it follows that

$$\left||\mathcal{R}_2^{\frac{a_2}{\nu_2}}u|^{r_j}-|\mathcal{R}_2^{\frac{a_2}{\nu_2}}v|^{r_j}\right|^2\leqslant C\left||\mathcal{R}_2^{\frac{a_2}{\nu_2}}u|^{(p_j-2)/2}\mathcal{R}_2^{\frac{a_2}{\nu_2}}u-|\mathcal{R}_2^{\frac{a_2}{\nu_2}}v|^{(p_j-2)/2}\mathcal{R}_2^{\frac{a_2}{\nu_2}}v\right|^2, \tag{2.3}$$

where C is a positive constant depending on r_j,p_j and $\max\{|\mathcal{R}_2^{\frac{a_2}{\nu_2}}u|^{p_j/2},|\mathcal{R}_2^{\frac{a_2}{\nu_2}}v|^{p_j/2}\}$.

On the other hand, by Young's inequality we have

$$I_2\leqslant\sum_{j=1}^{n_2}\epsilon_j\int_0^\tau\int_{\{\phi(t,\cdot)>0\}}\left||\mathcal{R}_2^{\frac{a_2}{\nu_2}}u|^{r_j}-|\mathcal{R}_2^{\frac{a_2}{\nu_2}}v|^{r_j}\right|^2dxdt$$
$$+\sum_{j=1}^{n_2}C(\epsilon_j)\int_0^\tau\int_{\{\phi(t,\cdot)>0\}}\phi^2dxdt. \tag{2.4}$$

A combination of (2.3) and (2.4) leads to

$$I_2\leqslant C\sum_{j=1}^{n_2}\epsilon_j\int_0^\tau\int_{\{\phi(t,\cdot)>0\}}\left||\mathcal{R}_2^{\frac{a_2}{\nu_2}}u|^{(p_j-2)/2}\mathcal{R}_2^{\frac{a_2}{\nu_2}}u-|\mathcal{R}_2^{\frac{a_2}{\nu_2}}v|^{(p_j-2)/2}\mathcal{R}_2^{\frac{a_2}{\nu_2}}v\right|^2dxdt$$
$$+\sum_{j=1}^{n_2}C(\epsilon_j)\int_0^\tau\int_{\{\phi(t,\cdot)>0\}}\phi^2dxdt. \tag{2.5}$$

For I_3, by the mean value theorem we obtain

$$I_3 \leq \sum_{i=1}^{n_1} q_i \|u\|_{L^\infty}^{q_i-1} \int_0^\tau \int_{\{\phi(t,\cdot)>0\}} \phi^2 dx dt. \tag{2.6}$$

Similarly, for I_4 we have

$$I_4 \leq \sum_{k=1}^{n_3} s_k \|u\|_{L^\infty}^{s_k-1} \int_0^\tau \int_{\{\phi(t,\cdot)>0\}} \phi^2 dx dt. \tag{2.7}$$

Choosing $0 < \epsilon_j < 4/(\beta C p_j^2)$ and combining the estimates (2.1), (2.2), (2.5), (2.6) and (2.7), we obtain for any $\tau \in (0, T)$ that

$$\begin{aligned}&\int_\Omega \phi^2(\tau, x) dx \\ &\leq C\left(\alpha, \beta, \epsilon, q_i, s_k, r_j, p_j, \|u\|_{L^\infty}, \max\{|\mathcal{R}_2^{\frac{a_2}{\nu_2}} u|^{p_j/2}, |\mathcal{R}_2^{\frac{a_2}{\nu_2}} v|^{p_j/2}\}\right) \int_0^\tau \int_\Omega \phi^2 dx dt. \end{aligned}\tag{2.8}$$

Then by Gronwall's lemma we conclude that $\phi \equiv 0$ almost everywhere.

The case $\alpha = \beta = \gamma = 0$ is trivial.

Now, we discuss the case, when not all, but at least one of the parameters α, β, γ is positive. Note that I_3 is positive, since for $q_i > 0$ we have

$$\begin{cases} |u|^{q_i-1}u - |v|^{q_i-1}v = u^{q_i} - v^{q_i} > 0, & \text{for} \quad u > v > 0 \\ |u|^{q_i-1}u - |v|^{q_i-1}v = u^{q_i} + |v|^{q_i} > 0, & \text{for} \quad u > 0 > v \\ |u|^{q_i-1}u - |v|^{q_i-1}v = -|u|^{q_i} + |v|^{q_i} > 0, & \text{for} \quad 0 > u > v. \end{cases}$$

Similarly, one can verify that I_4 is positive for $s_k > 0$. Therefore, in the case when $\alpha < 0$ or $\beta < 0$ (or $\gamma < 0$) by dropping I_3 or I_2 (or I_4), respectively, we can always get (2.8). □

3. Some applications to nonlinear equations for the heat p-sub-Laplacian

In this section, we give some applications of Theorem 2.1 to nonlinear equations for the heat p-sub-Laplacian on stratified Lie groups. These groups are an important class of graded Lie groups, investigated thoroughly by Folland [Fol75]. There are many different, equivalent ways to define a stratified Lie group (see, for example, [BLU07, FS82] or [FR16, RS19] for the Lie group and Lie algebra points of view, respectively). A Lie group $\mathbb{G} = \left(\mathbb{R}^N, \circ\right)$ is called a stratified Lie group if it satisfies the following two conditions:

- for every $\lambda > 0$ the dilation $\delta_\lambda : \mathbb{R}^N \to \mathbb{R}^N$ defined by
$$\delta_\lambda(x) \equiv \delta_\lambda\left(x', \ldots, x^{(r)}\right) := \left(\lambda x', \ldots, \lambda^r x^{(r)}\right)$$
is an automorphism of the group $\mathbb{G}$, where $x' \equiv x^{(1)} \in \mathbb{R}^{N_1}$ and $x^{(k)} \in \mathbb{R}^{N_k}$ for $k = 2, \ldots, r$ with $N_1 + \cdots + N_r = N$ and $\mathbb{R}^N = \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_r}$.
- let $X_1, \ldots, X_{N_1}$ be the left invariant vector fields on $\mathbb{G}$ such that $X_j(0) = \frac{\partial}{\partial x_j}|_0$ for $j = 1, \ldots, N_1$. Then, for every $x \in \mathbb{R}^N$ the Hörmander condition
$$\operatorname{rank}\left(\operatorname{Lie}\{X_1, \ldots, X_{N_1}\}\right) = N$$

holds, that is, $X_1, \ldots, X_{N_1}$ with their iterated commutators span the whole Lie algebra of the group $\mathbb{G}$.

Let us also recall that the left invariant vector field X_j has an explicit form given by (see, e.g. [FR16, Section 3.1.5])

$$X_j = \frac{\partial}{\partial x'_j} + \sum_{l=2}^{r} \sum_{m=1}^{N_l} a_{j,m}^{(l)} \left(x', \ldots, x^{(l-1)}\right) \frac{\partial}{\partial x_m^{(l)}}. \tag{3.1}$$

Throughout this section, we will also use the following notations

$$\nabla_H := (X_1, \ldots, X_{N_1})$$

for the horizontal gradient,

$$\mathcal{L}_p f := \nabla_H \left(|\nabla_H f|^{p-2} \nabla_H f\right), \quad 2 \leq p < \infty, \tag{3.2}$$

for the p-sub-Laplacian, and

$$|x'| = \sqrt{x_1'^2 + \cdots + x_{N_1}'^2}$$

for the Euclidean norm on $\mathbb{R}^{N_1}$.

Using (3.1) one can observe that (see, e.g. [RS17])

$$|\nabla_H |x'|^b| = b|x'|^{b-1}, \tag{3.3}$$

and

$$\nabla_H \left(\frac{x'}{|x'|^b}\right) = \frac{N_1 - b}{|x'|^b} \tag{3.4}$$

for all $b \in \mathbb{R}, x' \in \mathbb{R}^{N_1}$ and $|x'| \neq 0$.

Let us first consider the following initial boundary value problem for the p-sub-Laplacian, $2 \leq p < \infty$,

$$\begin{cases} u_t - \mathcal{L}_p u = \alpha |u|^{q-1} u + \beta |\nabla_H u|^r, & x \in \Omega, \quad t > 0, \\ u(t,x) = 0, \quad x \in \partial\Omega, \quad t > 0, \\ u(0,x) = u_0(x), \quad x \in \Omega, \end{cases} \tag{3.5}$$

where $u_0(x) \geqslant 0$, $u_0(x) \not\equiv 0$, $u_0 \in S_0^{1,\infty}(\Omega)$, and the parameters α, β, q and r will be determined later. By Definition 1.1 let us recall that $T_{\max}$ is the maximal existence time of a weak solution of (3.5).

Theorem 3.1. *Let $\Omega \subset \mathbb{G}$ be a bounded open set in a stratified Lie group with N_1 being the dimension of the first stratum. Assume that α, β, p, q and r in (3.5) satisfy one of the following conditions:*

(i) $\alpha < 0$, $\beta > 0$, and $p \geq 2$, $r > 1$ with $p \leq r + 1$ and $p/2 \leq r < q$;
(ii) $\alpha > 0$, $\beta < 0$, and $p \geq 2$, $q \geq 1$ with $p < r + 1$ and $q \leq r$.

Then a weak solution of (3.5) is globally in t-bounded, that is, there exists a constant M depending only on p, q, r, α, β, N_1, Ω and u_0 such that for every $T > 0$ we have $0 \leq u \leq M$ on $(0,T)$.

Proof of Theorem 3.1. Part (i). For convenience, we assume that $\beta = -\alpha = 1$. Set $R' := \max_{x=(x',x'')\in\Omega} |x'|$. Then, since Ω is bounded, we get $R' < \infty$. For any $x = (x', x'') \in \Omega$, let $x_0 = (x_0', x_0'') \in \mathbb{G}\backslash\Omega$ and $\varepsilon \in (0, 1)$ be such that $\varepsilon \leqslant |x_0' - x'| < R' + 1$. We also introduce the following notations

$$V_1(t, x) := K_1 e^{\sigma_1 R}, \quad R = |x' - x_0'|, \quad x = (x', x'') \in \Omega,$$

and

$$\mathcal{M}_p w := w_t - \mathcal{L}_p w + w^q - |\nabla_H w|^r.$$

Let us now find suitable positive K_1 and σ_1 such that $V_1(t, x)$ is a super-solution of (3.5). By using the identities (3.3) and (3.4), we observe that

$$|\nabla_H V_1|^r = K_1^r e^{\sigma_1 R r} \sigma_1^r$$

and

$$\begin{aligned}
\mathcal{L}_p V_1 &= \nabla_H \left(\left| \nabla_H K_1 e^{\sigma_1 R} \right|^{p-2} \nabla_H K_1 e^{\sigma_1 R} \right) \\
&= \nabla_H \left(K_1^{p-2} \sigma_1^{p-2} e^{\sigma_1 |x'-x_0'|(p-2)} K_1 \sigma_1 e^{\sigma_1 |x'-x_0'|} \frac{x' - x_0'}{|x' - x_0'|} \right) \\
&= \nabla_H \left(K_1^{p-1} \sigma_1^{p-1} e^{\sigma_1 |x'-x_0'|(p-1)} \frac{x' - x_0'}{|x' - x_0'|} \right) \\
&= K_1^{p-1} \sigma_1^{p-1} \sigma_1 (p-1) e^{\sigma_1 |x'-x_0'|(p-1)} + K_1^{p-1} \sigma_1^{p-1} e^{\sigma_1 |x'-x_0'|(p-1)} \frac{N_1 - 1}{|x' - x_0'|} \\
&= (p-1) \sigma_1^p K_1^{p-1} e^{(p-1)\sigma_1 R} + \frac{N_1 - 1}{R} \sigma_1^{p-1} K_1^{p-1} e^{\sigma_1 R (p-1)}.
\end{aligned}$$

Thus, we have

$$\mathcal{M}_p V_1 = -(p-1) \sigma_1^p K_1^{p-1} e^{(p-1)\sigma_1 R} - \frac{N_1 - 1}{R} \sigma_1^{p-1} K_1^{p-1} e^{(p-1)\sigma_1 R} + K_1^q e^{q\sigma_1 R} - K_1^r e^{r\sigma_1 R} \sigma_1^r.$$

Now, we need to find σ_1 and K_1 such that $\mathcal{M}_p V_1 \geqslant 0$, that is,

$$(p-1) \sigma_1^p K_1^{p-1} e^{(p-1)\sigma_1 R} + \frac{N_1 - 1}{R} \sigma_1^{p-1} K_1^{p-1} e^{(p-1)\sigma_1 R} + K_1^r e^{r\sigma_1 R} \sigma_1^r \leqslant K_1^q e^{q\sigma_1 R}.$$

Multiplying both sides of the inequality by $K_1^{-p+1} e^{-(p-1)\sigma_1 R}$, we derive that

$$(p-1) \sigma_1^p + \frac{N_1 - 1}{R} \sigma_1^{p-1} + K_1^{r-p+1} e^{(r-p+1)\sigma_1 R} \sigma_1^r \leqslant K_1^{q+1-p} e^{(q+1-p)\sigma_1 R}. \tag{3.6}$$

Taking into account $\varepsilon \leqslant R < R' + 1$, we see that in order to prove (3.6) it is sufficient to show

$$(p-1) \sigma_1^p + \frac{N_1 - 1}{\varepsilon} \sigma_1^{p-1} + K_1^{r-p+1} e^{(r-p+1)\sigma_1 (R'+1)} \sigma_1^r \leqslant K_1^{q+1-p}.$$

Thus, to have $\mathcal{M}_p V_1 \geqslant 0$ we can choose

$$\begin{gathered}
\sigma_1 = \tfrac{1}{(r-p+1)(R'+1)}, \\
K_1 = \max \left\{ (2e\sigma_1^r)^{1/(q-r)}, \left(2 \left((p-1)\sigma_1^p + \tfrac{N_1 - 1}{\varepsilon} \sigma_1^{p-1} \right) \right)^{1/(q+1-p)} \right\}
\end{gathered}$$

when $r+1>p$, and

$$\sigma_1=1, \quad K_1=\max\left\{2^{1/(q-r)}, \left(2\left(p-1+\frac{N_1-1}{\varepsilon}\right)\right)^{1/(q+1-p)}\right\}$$

when $r+1=p$. We also need that $K_1 \geqslant \|u_0\|_{L^\infty(\Omega)}$ such that $V_1(0,x)=K_1e^{\sigma_1 R} \geqslant u_0(x)$. Obviously, we also have $V_1(t,x)\geq 0=u(t,x)$ on $\partial\Omega$. Therefore, $V_1(t,x)$ is a super-solution of (3.5). Then, Theorem 2.1 concludes that

$$0\leq u(t,x)\leqslant K_1e^{\sigma_1(R'+1)}<\infty, \quad R'=\max_{x=(x',x'')\in\Omega}|x'|. \tag{3.7}$$

Note that the right-hand side of (3.7) is independent of t, hence $u(t,x)$ is globally in t-bounded.

Part (ii). In this case, we may assume that $\alpha=-\beta=1$. We recall from Part (i) that $R'=\max_{x=(x',x'')\in\Omega}|x'|<\infty$ and $\varepsilon\leqslant|x_0'-x'|<R'+1$ for any $x=(x',x'')\in\Omega$, where $x_0=(x_0',x_0'')\in\mathbb{G}\backslash\Omega$ and $\varepsilon\in(0,1)$.

First, let us consider the case $r>q$. Here, we will use the following notations

$$V_2(t,x):=\frac{K_2}{\sigma_2}R^{\sigma_2}, \quad \sigma_2=\frac{p}{p-1}, \quad R=|x'-x_0'|, \quad x\in\Omega,$$

and

$$\mathcal{N}_p w:=w_t-\mathcal{L}_p w-w^q+|\nabla_H w|^r.$$

Now, we need to find a suitable positive K_2 such that $V_2(t,x)$ is a super-solution of (3.5). By using the identities (3.3) and (3.4), we observe that

$$\begin{aligned}
\mathcal{L}_p V_2 &= \nabla_H\left(\left|\nabla_H\left(\frac{K_2}{\sigma_2}R^{\sigma_2}\right)\right|^{p-2}\nabla_H\left(\frac{K_2}{\sigma_2}R^{\sigma_2}\right)\right)\\
&=\left(\frac{K_2}{\sigma_2}\right)^{p-1}\nabla_H\left(\sigma_2^{p-2}R^{(\sigma_2-1)(p-2)}\sigma_2R^{\sigma_2-1}\frac{x'-x_0'}{|x'-x_0'|}\right)\\
&=K_2^{p-1}\nabla_H\left(R^{(\sigma_2-1)(p-1)}\frac{x'-x_0'}{|x'-x_0'|}\right)\\
&=N_1K_2^{p-1}.
\end{aligned}$$

Then we have

$$\mathcal{N}_pV_2=-N_1K_2^{p-1}+K_2^rR^{\frac{r}{p-1}}-\left(\frac{K_2}{\sigma_2}\right)^qR^{\frac{qp}{p-1}}.$$

From this, we have

$$\mathcal{N}_pV_2\geq 0 \Longleftrightarrow K_2^rR^{\frac{r}{p-1}}\geq N_1K_2^{p-1}+\left(\frac{K_2}{\sigma_2}\right)^qR^{\frac{qp}{p-1}}.$$

Thus, it is sufficient to choose K_2 such that

$$K_2^rR^{\frac{r}{p-1}}\geq 2N_1K_2^{p-1}, \tag{3.8}$$

$$K_2^rR^{\frac{r}{p-1}}\geq 2\left(\frac{K_2}{\sigma_2}\right)^qR^{\frac{qp}{p-1}}. \tag{3.9}$$

Note that the inequality (3.8) is satisfied if we take

$$K_2 \geq \left(\frac{2N_1}{\varepsilon^{\frac{r}{p-1}}} \right)^{\frac{1}{r-p+1}},$$

provided that $r > p-1$. We divide inequality (3.9) by $K_2^q R^{\frac{r}{p-1}}$ to derive

$$K_2^{r-q} \geq \frac{2}{\sigma_2^q} R^{\frac{qp-r}{p-1}}.$$

For $qp \geq r$, we can set

$$K_2 \geq \left(\frac{2}{\sigma_2^q} \right)^{\frac{1}{r-q}} (R'+1)^{\frac{qp-r}{(p-1)(r-q)}},$$

while for $qp < r$, we can set

$$K_2 \geq \left(\frac{2}{\sigma_2^q} \right)^{\frac{1}{r-q}} \varepsilon^{\frac{qp-r}{(p-1)(r-q)}}.$$

We also need that $K_2 \geq \frac{\sigma_2 \|u_0\|_{L^\infty}}{\varepsilon^{\sigma_2}}$ to have $V_2(0,x) \geq u_0$. Thus, taking K_2 as follows

$$K_2 \geq \max \left\{ \frac{\sigma_2 \|u_0\|_{L^\infty}}{\varepsilon^{\sigma_2}}, \left(\frac{2N_1}{\varepsilon^{\frac{r}{p-1}}} \right)^{\frac{1}{r-p+1}}, \left(\frac{2}{\sigma_2^q} \right)^{\frac{1}{r-q}} (R'+1)^{\frac{qp-r}{(p-1)(r-q)}}, \left(\frac{2}{\sigma_2^q} \right)^{\frac{1}{r-q}} \varepsilon^{\frac{qp-r}{(p-1)(r-q)}} \right\},$$

we obtain $\mathcal{N}_p V_2 \geq 0$ and $V_2(0,x) \geq u_0$. It is clear that $V_2(t,x) \geq 0 = u(t,x)$ on $\partial\Omega$. Therefore, $V_2(t,x)$ is a super-solution of (3.5). Then, Theorem 2.1 concludes that

$$0 \leq u(t,x) \leq \frac{K_2 (R'+1)^{\frac{p}{p-1}}}{\sigma_2} < \infty.$$

In the case when $r = q$, we can take

$$\sigma_3 \geq \max \left\{ 1, 2^{1/r}(R'+1) \right\},$$

and

$$K_3 \geq \max \left\{ \varepsilon^{-\sigma_3} \|u_0\|_{L^\infty}, \left(\frac{2((p-1)(\sigma_3-1)+N_1-1)}{\varepsilon^{(r-p+1)(\sigma_3-1)+1}} \right)^{\frac{1}{r-p+1}} \right\},$$

such that the function $V_3(t,x) = K_3 R^{\sigma_3}$ is a super-solution of (3.5). By the same procedure, one can obtain the uniform boundedness of $u(t,x)$. □

Theorem 3.2. *Let $\alpha > 0$, $\beta < 0$, $p \geq 2$ and $r > 0$. If $q > \max\{p-1, r\}$, then the solution of the problem (3.5) blows up in finite time for some large $u_0 > 0$.*

Proof of Theorem 3.2. For convenience, let us assume that $\alpha = -\beta = 1$. Set

$$v(t,|x'|) := \frac{1}{(1-\delta t)^{k_1}} F\left(\frac{|x'|}{(1-\delta t)^{k_2}} \right), \quad t_0 \leq t < \frac{1}{\delta}, \tag{3.10}$$

where

$$F(y) := 1 + \frac{A}{\sigma} - \frac{y^\sigma}{\sigma A^{\sigma-1}}, \quad y \geq 0, \quad \sigma = \frac{p}{p-1},$$

and

$$k_1=\frac{1}{q-1},\quad 0<k_2<\min\left\{\frac{q-p+1}{p(q-1)},\frac{q-r}{r(q-1)}\right\},\quad A>\frac{k_1}{k_2},\quad \delta<\frac{1}{k_1\left(1+\frac{A}{\sigma}\right)}. \tag{3.11}$$

Then, it can be noted that $v(t,|x'|)$ is positive and smooth when $t\in[t_0,\frac{1}{\delta})$ and $|x'|<R_1(1-\delta t)^{k_2}$, where $R_1:=(A^{\sigma-1}(A+\sigma))^{1/\sigma}$.

We want to show that $v(t,|x'|)$ is a sub-solution of (3.5). For $y=\frac{|x'|}{(1-\delta t)^{k_2}}$, by a direct calculation we have

$$\begin{aligned}\mathcal{N}_p v =&v_t-\mathcal{L}_p v-v^q+|\nabla_H v|^r\\ =&\frac{\delta\left(k_1F+k_2yF'\right)}{(1-\delta t)^{k_1+1}}-\frac{\left(|F'|^{p-2}F'\right)'+\frac{N_1-1}{y}|F'|^{p-2}F'}{(1-\delta t)^{(p-2)(k_1+k_2)+(k_1+2k_2)}}\\ &-\frac{F^q}{(1-\delta t)^{qk_1}}+\frac{|F'|^r}{(1-\delta t)^{r(k_1+k_2)}}.\end{aligned}$$

Note that $k_1q=k_1+1>(p-2)(k_1+k_2)+k_1+2k_2$ and $k_1+1>(k_1+k_2)r$ by (3.11). Observe that

$$\left(|F'|^{p-2}F'\right)'+\frac{N_1-1}{y}|F'|^{p-2}F'=-\frac{N_1}{A},\quad 0<y<R_1.$$

Let us now show that $\mathcal{N}_p v\le 0$ for all $t\in[t_0,\frac{1}{\delta})$ and $0\le y\le R_1$. In the case $0\le y\le A$, from the representation of $F(y)$ we note that

$$1\le F(y)\le 1+\frac{A}{\sigma}\quad\text{and}\quad -1\le F'(y)\le 0.$$

Then, we can take $t_0=t_0(p,q,r,\delta,N_1,A)$ close to $\frac{1}{\delta}$ such that

$$\begin{aligned}\mathcal{N}_p v\le\frac{1}{(1-\delta t)^{k_1+1}}\Bigg(\delta k_1\left(1+\frac{A}{\sigma}\right)-1+\frac{N_1}{A}(1-\delta t_0)^{1-2k_2-(p-2)(k_1+k_2)}\\ +(1-\delta t_0)^{k_1+1-r(k_1+k_2)}\Bigg)\le 0.\end{aligned} \tag{3.12}$$

In the case $A\le y\le R_1$, we have

$$0\le F(y)\le 1\quad\text{and}\quad -\left(\frac{R_1}{A}\right)^{\sigma-1}\le F'(y)\le -1.$$

Similarly as above, one verifies that

$$\begin{aligned}\mathcal{N}_p v\le\frac{1}{(1-\delta t)^{k_1+1}}\Bigg(\delta(k_1-k_2A)+\frac{N_1}{A}(1-\delta t_0)^{1-2k_2-(p-2)(k_1+k_2)}\\ +\left(\frac{R_1}{A}\right)^{r(\sigma-1)}(1-\delta t_0)^{k_1+1-r(k_1+k_2)}\Bigg)\le 0.\end{aligned} \tag{3.13}$$

From (3.12) and (3.13), we conclude that $\mathcal{N}_p v\le 0$ for all $t\in[t_0,\frac{1}{\delta})$ and $|x'|<R_1(1-\delta t)^{k_2}$.

Next, we estimate u_0. By the group translation, without loss of generality we may assume that Ω contains the unit element of the group $\mathbb{G}$. Then, we can take suitable

t_0 such that $R_1(1-\delta t_0)^{k_2} < \max_{x=(x',x'')\in\Omega} |x'|$ and $u_0 \geq v(t_0,\cdot)$ in $\Omega \cap \{x=(x',x'') : |x'| < R_1(1-\delta t_0)^{k_2}\}$ for some large $u_0 > 0$. Then, taking into account $v \leq 0$ when $|x'| \geq R_1(1-\delta t)^{k_2}$, we obtain that $u_0 \geq v(t_0,\cdot)$ in $\overline{\Omega}$. Obviously, we also have $v \leq 0$ when $(t,x) \in \left(t_0, \frac{1}{\delta}\right) \times \partial\Omega$. Thus, the comparison principle (Theorem 2.1) implies that

$$u(t,x) \geq v(t+t_0,x), \quad t \in \left[t_0, \frac{1}{\delta}\right), \quad |x'| < R_1(1-\delta t)^{k_2}.$$

On the other hand, by the definition of v we have $\lim_{t\to 1/\delta} v(t,0) \to \infty$. Consequently, u must blow up at a finite time $T \leq \frac{1}{\delta} - t_0 < \infty$. □

Theorem 3.3. *Assume that $\alpha < 0$, $\beta > 0$, $p \geq 2$, $r > 1$ and $q > 0$ in (3.5) satisfy one of the following conditions:*

- $r > \max\{p,q\}$*;*
- $r = q > p$*, and* $\beta \gg |\alpha|$*.*

There exists $M > 0$ such that if $\int_\Omega u_0^{\frac{2r-p}{r-p}} dx > M$, then $T_{max} < \infty$.

Proof of Theorem 3.3. Assume for a contradiction that $T_{\max} = \infty$. By C_1 and C_2 we denote positive constants which may vary from line to line. Set $\kappa = r/(r-p)$ and $y(t) = \frac{1}{\kappa+1}\int_\Omega u^{\kappa+1} dx$. Then, using $\kappa - 1 = \frac{p}{r-p} = \frac{p\kappa}{r}$, we have

$$\begin{aligned} y'(t) &= \beta \int_\Omega u^\kappa |\nabla_H u|^r dx - \kappa \int_\Omega u^{\kappa-1}|\nabla_H u|^p dx - |\alpha| \int_\Omega u^{q+\kappa} dx \\ &= \beta \int_\Omega u^\kappa |\nabla_H u|^r dx - \kappa \int_\Omega \left(u^\kappa |\nabla_H u|^r\right)^{p/r} dx - |\alpha| \int_\Omega u^{q+\kappa} dx. \end{aligned}$$

For $r > q$, using Hölder's and Young's inequalities we get

$$\begin{aligned} \int_\Omega \left(u^\kappa |\nabla_H u|^r\right)^{p/r} dx &\leq \left(\int_\Omega u^\kappa |\nabla_H u|^r dx\right)^{p/r} |\Omega|^{(r-p)/r} \\ &\leq \epsilon \frac{p}{r} \int_\Omega u^\kappa |\nabla_H u|^r dx + C(\epsilon)\frac{r-p}{r}|\Omega| \end{aligned}$$

and

$$\begin{aligned} \int_\Omega u^{q+\kappa} dx &= \int_\Omega \left(u^{r+\kappa}\right)^{\frac{q+\kappa}{r+\kappa}} dx \leq \left(\int_\Omega u^{r+\kappa} dx\right)^{\frac{q+\kappa}{r+\kappa}} |\Omega|^{\frac{r-q}{r+\kappa}} \\ &\leq \varepsilon \frac{q+\kappa}{r+\kappa} \int_\Omega u^{r+\kappa} dx + C(\varepsilon)\frac{r-q}{r+\kappa}|\Omega|. \end{aligned}$$

Then, by Poincaré's (see, e.g. [RS17, Formula 1.10]) and reverse Hölder's inequalities we obtain

$$\begin{aligned} y'(t) &\geq \frac{\beta r-\epsilon p}{r}\int_{\Omega}u^{\kappa}|\nabla_H u|^r dx-|\alpha|\varepsilon\frac{q+\kappa}{r+\kappa}\int_{\Omega}u^{r+\kappa}dx-C \\ &= \frac{\beta r-\epsilon p}{r}\left(\frac{r}{r+\kappa}\right)^r\int_{\Omega}|\nabla_H u^{\frac{r+\kappa}{r}}|^r dx-|\alpha|\varepsilon\frac{q+\kappa}{r+\kappa}\int_{\Omega}u^{r+\kappa}dx-C \\ &\geq \left(\frac{\beta r-\epsilon p}{r}\left(\frac{r}{r+\kappa}\right)^r C'-|\alpha|\varepsilon\frac{q+\kappa}{r+\kappa}\right)\int_{\Omega}u^{r+\kappa}dx-C \\ &= C_1\int_{\Omega}u^{r+\kappa}dx-C \\ &\geq C_1\left(\int_{\Omega}u^{\kappa+1}dx\right)^{\frac{r+\kappa}{\kappa+1}}|\Omega|^{\frac{1-r}{\kappa+1}}-C_2 \\ &\geq C_1\left(\int_{\Omega}u^{\kappa+1}dx\right)^{\frac{r+\kappa}{\kappa+1}}-C_2. \end{aligned}$$

Thus, we have obtained

$$y'(t)\geq C_1 y^{\frac{r+\kappa}{\kappa+1}}(t)-C_2,$$

where $C_1=C_1(p,r,q,\alpha,\beta,\epsilon,\varepsilon,\Omega,N_1)$ and $C_2=C_2(p,r,q,\alpha,\beta,\epsilon,\varepsilon,\Omega,N_1)>0$ with suitable ϵ and ε, and N_1 is the dimension of the first stratum of the group $\mathbb{G}$. Set

$$M>\left(\frac{2C_2}{C_1}\right)^{\frac{\kappa+1}{r+\kappa}},$$

then if $y(0)>M$, we have

$$y'(t)\geq\frac{C_1 y^{\frac{r+\kappa}{\kappa+1}}(t)}{2}. \tag{3.14}$$

A contradiction then follows by integrating (3.14), hence $T_{\max}<\infty$.

In the case $r=q$, the proof above is still valid for $\beta\gg|\alpha|$. □

As another application of the comparison principle, we now investigate the following initial boundary value problem for the p-sub-Laplacian, $2\leq p<\infty$,

$$\begin{cases} u_t-\mathcal{L}_p u=\alpha\sum_{i=1}^{n_1}|u|^{q_i-1}u+\gamma\sum_{i=1}^{n_1}|u|^{s_i-1}u, & x\in\Omega, \quad t>0, \\ u(t,x)=0, \quad x\in\partial\Omega, \quad t>0, \\ u(0,x)=u_0(x), \quad x\in\Omega, \end{cases} \tag{3.15}$$

where $u_0(x)\geqslant 0$, $u_0(x)\not\equiv 0$, $u_0\in S_0^{1,\infty}(\Omega)$, and the parameters α, γ, q_i and s_i will be determined later.

Theorem 3.4. *Let $\Omega\subset\mathbb{G}$ be a bounded open set in a stratified Lie group with N_1 being the dimension of the first stratum. Let $\widetilde{s}=\min\{s_i\}$ and $\widetilde{q}=\min\{q_i\}$. Assume that α, γ, q_i and s_i in (3.15) satisfy one of the following conditions:*

(i) $\alpha>0$, $\gamma<0$, and $q_i\geq 1$ with $2\leq p<\widetilde{s}+1$ and $s_i<q_i$;

(ii) $\alpha<0$, $\gamma>0$, and $s_i\geq 1$ with $2\leq p<\widetilde{q}+1$ and $s_i>q_i$.

Then a weak solution of (3.15) is globally in t-bounded, that is, there exists a constant M depending only on p, q_i, s_i, α, γ, N_1, Ω and u_0 such that for every $T>0$ we have $0\leq u\leq M$ on $(0,T)$.

Remark 3.5. We refer to [RS18, Section 3] for a similar investigation when $\alpha = -\gamma = 1$, $n_1 = 1$.

Proof of Theorem 3.4. We only prove Part (i), since Part (ii) is actually the same, but only α and q_i are swapped by γ and s_i, respectively. For convenience, we assume that $\alpha = -\gamma = 1$. We recall that $R' = \max\limits_{x=(x',x'')\in\Omega} |x'| < \infty$ and $\varepsilon \leqslant |x_0' - x'| < R' + 1$ for any $x = (x', x'') \in \Omega$, where $x_0 = (x_0', x_0'') \in \mathbb{G}\backslash\Omega$ and $\varepsilon \in (0, 1)$. We also employ the following notations

$$V_4(t,x) := \frac{K_4}{\sigma_4} R^{\sigma_4}, \quad \sigma_4 = \frac{p}{p-1}, \quad R = |x' - x_0'|, \quad x = (x', x'') \in \Omega,$$

and

$$\mathcal{K}_p w := w_t - \mathcal{L}_p w - \sum_{i=1}^{n_1} w^{q_i} + \sum_{i=1}^{n_1} w^{s_i}.$$

Now, we look for a suitable positive K_4 such that $V_4(t,x)$ is a super-solution of (3.15). Then we have

$$\mathcal{K}_p V_4 = -N_1 K_4^{p-1} - \sum_{i=1}^{n_1} \left(\frac{K_4}{\sigma_4}\right)^{q_i} R^{\frac{q_i p}{p-1}} + \sum_{i=1}^{n_1} \left(\frac{K_4}{\sigma_4}\right)^{s_i} R^{\frac{s_i p}{p-1}}.$$

From this, we note that

$$\mathcal{K}_p V_4 \geq 0 \Longleftrightarrow \sum_{i=1}^{n_1} \left(\frac{K_4}{\sigma_4}\right)^{s_i} R^{\frac{s_i p}{p-1}} \geq N_1 K_4^{p-1} + \sum_{i=1}^{n_1} \left(\frac{K_4}{\sigma_4}\right)^{q_i} R^{\frac{q_i p}{p-1}}.$$

So, it is sufficient to choose K_4 such that

$$\sum_{i=1}^{n_1} \left(\frac{K_4}{\sigma_4}\right)^{s_i} R^{\frac{s_i p}{p-1}} \geq 2N_1 K_4^{p-1}, \tag{3.16}$$

$$\left(\frac{K_4}{\sigma_4}\right)^{s_i} R^{\frac{s_i p}{p-1}} \geq 2 \left(\frac{K_4}{\sigma_4}\right)^{q_i} R^{\frac{q_i p}{p-1}}. \tag{3.17}$$

The inequality (3.16) is satisfied if we take

$$K_4 \geq (2N_1)^{\frac{1}{\tilde{s}-p+1}} \left(\sum_{i=1}^{n_1} \frac{\varepsilon^{\frac{s_i p}{p-1}}}{\sigma_4^{s_i}} \right)^{-\frac{1}{\tilde{s}-p+1}},$$

provided that $\widetilde{s} = \min\{s_i\} > p-1$. Dividing the inequality (3.17) by $K_4^{q_i} \frac{R^{\frac{s_i p}{p-1}}}{\sigma_4^{s_i}}$ we get

$$K_4^{s_i - q_i} \geq 2\sigma_4^{s_i - q_i} R^{\frac{p(q_i - s_i)}{p-1}},$$

that is,

$$K_4 \geq 2\sigma_4 \varepsilon^{-\frac{p}{p-1}}.$$

We also need that $K_4 \geq \frac{\sigma_4 \|u_0\|_{L^\infty}}{\varepsilon^{\sigma_4}}$ to ensure $V_4(0,x) \geq u_0$. Thus, choosing K_4 as follows

$$K_4 \geq \max\left\{ \frac{\sigma_4 \|u_0\|_{L^\infty}}{\varepsilon^{\sigma_4}}, (2N_1)^{\frac{1}{\tilde{s}-p+1}} \left(\sum_{i=1}^{n_1} \frac{\varepsilon^{\frac{s_i p}{p-1}}}{\sigma_4^{s_i}} \right)^{-\frac{1}{\tilde{s}-p+1}}, 2\sigma_4 \varepsilon^{-\frac{p}{p-1}} \right\},$$

we obtain $\mathcal{K}_p V_4 \geq 0$ and $V_4(0,x) \geq u_0$. Clearly, we also have $V_4(t,x) \geq 0 = u(t,x)$ on $\partial\Omega$. Therefore, we can conclude that $V_4(t,x)$ is a super-solution of (3.15). Then, the comparison principle yields that

$$0 \leq u(t,x) \leq \frac{K_4(R'+1)^{\frac{p}{p-1}}}{\sigma_4} < \infty, \quad R' = \max_{x=(x',x'')\in\Omega} |x'|\,. \tag{3.18}$$

Since the right-hand side of (3.18) is independent of t, we can conclude that $u(t,x)$ is globally in t-bounded. $\square$

By the same procedure as in the proof of Theorem 3.2, one can obtain the following result for the problem (3.15) when $n_1 = 1$:

Theorem 3.6. *Let $\alpha > 0$, $\gamma < 0$, $p \geq 2$ and $s > 0$. If $q > \max\{s, p-1\}$, then the solution of the problem (3.15) blows up in finite time for some large $u_0 > 0$.*

Proof of Theorem 3.6. As in the proof of Theorem 3.2, one can show that the same function v from (3.10) is a sub-solution of the problem (3.15). Then, the comparison principle (Theorem 2.1) concludes the proof. $\square$

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