A comparison of Ashtekar's and Friedrich's formalisms of spatial infinity

Mariem Magdy Ali Mohamed *[1] and Juan A. Valiente Kroon †[1]

[1] *School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London E1 4NS, United Kingdom.*

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Abstract

Penrose's idea of asymptotic flatness provides a framework for understanding the asymptotic structure of gravitational fields of isolated systems at null infinity. However, the studies of the asymptotic behaviour of fields near spatial infinity are more challenging due to the singular nature of spatial infinity in a regular point compactification for spacetimes with non-vanishing ADM mass. Two different frameworks that address this challenge are Friedrich's cylinder at spatial infinity and Ashtekar's definition of *asymptotically Minkowskian spacetimes at spatial infinity* that give rise to the 3-dimensional *asymptote at spatial infinity* $\mathcal{H}$. Both frameworks address the singularity at spatial infinity although the link between the two approaches had not been investigated in the literature. This article aims to show the relation between Friedrich's cylinder and the asymptote as spatial infinity. To do so, we initially consider this relation for Minkowski spacetime. It can be shown that the solution to the conformal geodesic equations provides a conformal factor that links the cylinder and the asymptote. For general spacetimes satisfying Ashtekar's definition, the conformal factor cannot be determined explicitly. However, proof of the existence of this conformal factor is provided in this article. Additionally, the conditions satisfied by physical fields on the asymptote $\mathcal{H}$ are derived systematically using the conformal constraint equations. Finally, it is shown that a solution to the conformal geodesic equations on the asymptote can be extended to a small neighbourhood of spatial infinity by making use of the stability theorem for ordinary differential equations. This solution can be used to construct a conformal Gaussian system in a neighbourhood of $\mathcal{H}$.

1 Introduction

The theory of isolated systems plays a central role in astrophysical applications of Einstein's theory of General Relativity. A particularly influential approach to this theory is through Penrose's definition of *asymptotic simplicity* —see e.g. [22, 29]:

Definition 1. *A vacuum spacetime* $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ *is said to be asymptotically simple if there exists a smooth, oriented, time-oriented and causal spacetime* $(\tilde{M}, \boldsymbol{g})$ *and a* C^∞ *function* Ξ *on* $\mathcal{M}$ *such that:*

(i) $\mathcal{M}$ *is a manifold with boundary* $\mathscr{I} \equiv \partial\mathcal{M}$*;*

(ii) $\Xi > 0$ *on* $\mathcal{M}\backslash\mathscr{I}$ *and* $\Xi = 0$, $\mathbf{d}\Xi \neq 0$ *on* $\mathscr{I}$*;*

(iii) the manifolds $\tilde{\mathcal{M}}$ *and* $\mathcal{M}$ *are related by an embedding* $\phi : \tilde{\mathcal{M}} \to \mathcal{M}$ *such that* $\phi(\tilde{\mathcal{M}}) = \mathcal{M}\backslash\mathscr{I}$ *and*

$$\phi^*\boldsymbol{g} = \Xi^2\tilde{\boldsymbol{g}}; \tag{1}$$

*E-mail address:m.m.a.mohamed@qmul.ac.uk

†E-mail address:j.a.valiente-kroon@qmul.ac.uk

(iv) each null geodesic of $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ *starts and ends on* $\mathscr{I}$*.*

Identifying the interior of $\mathcal{M}$ with $\tilde{\mathcal{M}}$ one can write equation (1), in a slight abuse of notation, as $\boldsymbol{g} = \Xi^2 \tilde{\boldsymbol{g}}$. Condition *(iv)* is of global nature which strictly speaking makes black hole spacetimes like the Schwarzschild solution not asymptotically simple but rather *weakly asymptotically simple*.

Roughly speaking, a spacetime is said to be asymptotically simple if it admits a conformal extension similar to that of the Minkowski spacetime whereby a null hypersurface (*null infinity*) $\mathscr{I}$ is attached to the spacetime. Generically, $\mathscr{I}$ consists of two disjoints components $\mathscr{I}^+$ (*future null infinity*) and $\mathscr{I}^-$ (*past null infinity*) representing, respectively, the endpoints and startpoints of null geodesics. As stressed by Geroch [18], the central aim of the definition of asymptotically simple spacetimes (and, in fact, any approach to define isolated systems) is to identify Universal structures in a wide class of spacetimes which can, in turn, be used to introduce notions of physical interest —say, conserved quantities, radiation states. The definition tries to strike a balance between ensuring the existence of enough structures to be able to extract useful conclusions and not being too restrictive to avoid that only a handful of exact solutions satisfy it.

A central aspect of the definition of asymptotic flatness is the assumed regularity of the conformal boundary. The classic definition of asymptotic flatness assumed smoothness. However, it is now widely accepted that this is an unnecessarily strong requirement — see e.g. [13]. Although to first sight, the issue of the regularity of the conformal boundary may seem like a technicality, it is in fact, it has great physical content —an important insight of the conformal approach is that regularity questions in the conformal point of view translate on assertions on the decay of fields. It is now known that many of the key structures supplied by the notion of asymptotic simplicity are present under more relaxed regularity assumptions —see e.g. [13, 7].

The definition of asymptotical simplicity only postulates the existence of null infinity, $\mathscr{I}$. However, the conformal compactification of the Minkowski spacetime in, say, the Einstein cylinder (see e.g. [29], Section 6.2) shows that the conformal boundary of this spacetime contains a further point, *spatial infinity* i^0, corresponding to the endpoints of spacelike geodesics. While $\mathscr{I}$ is central in the discussion of radiation properties of isolated bodies in General Relativity, i^0 is key in the discussion of conserved quantities —see e.g. [18, 4]. Also, as evidenced by the stability theorems of the Minkowski spacetime, the properties of the gravitational field in a neighbourhood of spatial infinity is closely related to the regularity of null infinity.

While in the Minkowski spacetime i^0 is a regular point of the conformal structure, it is well known that for spacetimes with non-vanishing (ADM) mass one has a singularity of the conformal structure —see e.g. [29], Chapter 20. This fact makes the analysis of the gravitational field particularly challenging. To address this difficulty it is necessary to introduce a different representation of spatial infinity which, in turn, allows to suitably resolve the structure of the gravitational field in this region. In particular, to avoid having to deal with directional dependent limits at spatial infinity it is natural to blow up the point i^0 to a 2-sphere.

The hyperboloid at spatial infinity. In an attempt to overcome the difficulties posed by the fact that the classic definition of asymptotic simplicity (see Definition 1 above) makes no reference to the the behaviour of the gravitational field at spatial infinity, in [3] Ashtekar & Hansen introduced a new definition of asymptotic simplicity in both null and spacelike directions —see also [2]. This definition (*asymptotically empty and flat at null infinity and spatial infinity, AEFANSI*) combines the conditions on the conformal extension given in the definition of asymptotical flatness with further assumptions on the conformal factor at spatial infinity. Essentially, it is further required that spatial infinity admits a so-called *point compactification* —see, e.g. [29], Section 11.6 for a discussion of this notion. The strategy behind the approach put forward in [3] is to make use of spacetime notions rather than, say, making use of the initial value problem. Their definition of asymptotic flatness allows them to blow up i^0 to the timelike unit 3-hyperboloid —the *hyperboloid at spatial infinity.* The sections of the hyperboloid give the blowing up of i^0 to 2-spheres while time direction along the (timelike) generators of the hyperboloid can be, roughly speaking, associated with all the possible ways in which the asymptotic region of a Cauchy hypersurface can be boosted. The definition of AEFANSI spacetimes is geared towards the discussion of asymptotic symmetries and conserved quantities at spatial infinity.

In a slightly different context, in [6, 5] the hyperboloid at spatial infinity is used, in conjunction with the vacuum Einstein field equations to obtain asymptotic expansions of the gravitational field near spatial infinity in negative powers of a radial coordinate. The main observation is that the field equations give rise to a hierarchy of linear equations for the coefficients of the expansion. It turns out that this hierarchy can be solved provided the initial data satisfies certain conditions. This work can be regarded as the first serious attempt to investigate the consistency between the Einstein field equations and geometric notions of asymptotic flatness.

The notion of the hyperboloid at spatial infinity has been revisited in [4] where the definition of *asymptotically Minkowskian spacetimes at spatial infinity (AMSI)* has been introduced —see Definition 2 in Section 4 of the main body of the present article. As in the analysis of [3], the approach in [4] makes use of spacetime structures and, thus, it is not geared towards the discussion of initial value problems. However, in contrast to the definition of AEFANSI spacetimes in [3], the definition of AMSI spacetimes focuses entirely on spatial infinity and, thus, it makes no assumptions about the properties of null infinity. A consequence of the concept of AMSI spacetimes is the existence of a 3-dimensional manifold $\mathcal{H}$, the *asymptote at spatial infinity*, which generalises the notion of the hyperboloid at spatial infinity.

Friedrich's cylinder at spatial infinity. In [10] Friedrich puts forward an alternative conformal representation of the region of spacetime in the neighbourhood of spatial infinity —*the F-gauge*. The aim of this representation is the formulation of a *regular initial value problem at spatial infinity* for the *conformal Einstein field equations*. This initial value problem is key in the programme to analyse the genericity of asymptotically simple spacetimes and relies heavily on the properties of certain conformal invariants (*conformal geodesics*) and it is such that both the equations and the initial data are regular at the conformal boundary. A central structure in this representation of spatial infinity is the *cylinder at spatial infinity* which, in broad terms, corresponds to the blow-up of the traditional point at spatial infinity to a 2-sphere plus a time dimension —hence, the cylinder at spatial infinity. In contrast to the hyperboloid at spatial infinity, the cylinder at spatial infinity has a finite extension in the time direction. The endpoints of the cylinder, the *critical sets* correspond to the points where spatial infinity meets null infinity. Thus, the F-gauge is ideally suited to the analysis of the connection between the behaviour of the gravitational field at spatial infinity and radiative properties. This idea has been elaborated in [14] to express the Newman-Penrose constants in terms of the initial data for the Einstein field equation and to study the behaviour of the Bondi mass as one approaches spatial infinity —see [23]; also [28]. The key property of the cylinder at spatial infinity which allows connecting properties of the Cauchy initial data in a neighbourhood of spatial infinity with the behaviour near null infinity is that the cylinder at spatial infinity is a total characteristic of the conformal Einstein field equations —that is, the associated evolution equations reduce in its entirety to a system of transport equations on this hypersurface. This property precludes the possibility of prescribing boundary data on the cylinder but allows computing a particular type of asymptotic expansions which make it possible to understand the role of certain pieces of the initial data have on the regularity properties of the gravitational at null infinity —the so-called *peeling behaviour*; see [10, 14, 25, 24, 26, 27, 17].

The F-gauge used in Friedrich's representation of spatial infinity is a gauge prescription based on the properties of conformal geodesics —i.e. a pair consisting of a curve and a covector along the curve satisfying conformally invariant properties. A remarkable property of conformal geodesics is that in Einstein spaces, they give rise to a canonical factor that has a quadratic dependence on the parameter of the curve —see [15, 29]. The resulting expression depends on certain initial data which can be chosen in such a way that the curves reach the conformal boundary and that no caustics are formed. In this way, one can obtain a conformal Gaussian gauge in which coordinates and a frame defined on an initial hypersurface are propagated along the conformal. In effect, this procedure provides a *canonical way* of obtaining conformal extensions of Einstein spaces. In the context of initial value problem for the conformal Einstein equations, one has a gauge in which the location of the conformal boundary is known *a priori*.

Melrose-type compactifications. In the context of this introduction, it is worth mentioning the proof of the stability of the Minkowski spacetime by Hintz & Vasy in which the existence of spacetimes with *polyhomogeneous asymptotic expansions* is established —see [20]— makes use of of a compactification of spacetime into a manifold with boundary and corners. This procedure is inspired by the methods of Melrose's geometric scattering theory programme —see e.g. [21]. This construction has a very strong resemblance to that introduced by Friedrich in [10]. The precise relation between these two seemingly connected representations of spatial infinity will be elaborated elsewhere.

Main results of this article

A cursory glance at Ashtekar's and Friedrich's constructions of spatial infinity suggests that they should be closely related. That this is the case has been part of the longstanding folklore of mathematical relativity. It is the purpose of this article to establish, in a rigorous way, the connection between these two representations of spatial infinity.

In order to carry out the objective outlined in the previous paragraph, we first analyse this relation for the Minkowski spacetime where everything can be explicitly computed. The outcome of this analysis is that Ashtekar's and Friedrich's representations are related to each other through a conformal transformation which preserves the asymptotic behaviour of the metric along spatial infinity but compactifies the time direction. In particular, the hyperboloid at spatial infinity is compactified into the $1+2$ Einstein Universe in the same way as the de Sitter spacetime is compactified into the $1+3$ Einstein Universe. The transformation between the two representations can be expressed in terms of properties of conformal geodesics —although in the case of the Minkowski spacetime, because of its simplicity, this does not play an essential role.

The intuition gained in the analysis of the Minkowski spacetime is, in turn used to establish the relation between Ashtekar's notion of an asymptote at spatial infinity given by the definition of an AMSI spacetime —see Definition 2 in the main text. As the definition of an AMSI spacetime does not imply the existence of null infinity, to carry out our analysis an extra assumption is required —see Assumption 1 in the main text. In essence, we assume that a sufficiently regular null infinity can be attached to the neighbourhood of spacetime around Ashtekar's asymptote. Our main result, whose proof is based on a stability argument for the solutions to the conformal geodesic equations in a neighbourhood of the asymptote is the following:

Theorem. *Given an AMSI spacetime that can be conformally extended to null infinity, there exists a sufficiently small neighbourhood of the asymptote at spatial infinity in which is possible to construct conformal Gaussian coordinates based on curves which extend beyond null infinity.*

Associated to the conformal Gaussian system whose existence is ensured by the above statement, there exists a conformal factor that provides the precise relation between Ashtekar's representation of spatial infinity and F-gauge representation.

Outline of the article

This article is structured as follows. In Section 2 we provide a brief summary of the conformal methods that are required in the analysis of this article. These include: conformal geodesics, conformal Gaussian systems, the conformal Einstein field equations and their constraints. For a full account of the associated literature, the reader is referred to the monograph [29]. Section 3 provides a discussion of the various representations of spatial infinity in the Minkowski spacetime. In particular, it contains the explicit connection between the F-gauge and Ashtekar's representation at spatial infinity representation. Furthermore, it also contains a discussion on how this connection can be expressed in terms of conformal geodesics. Section 4 reviews Ashtekar's definition of asymptotically Minkowskian spacetimes at spatial infinity (AMSI) and the associated notion of an asymptote. It further contains a discussion of the consequences of this definition in the light of the conformal Einstein field equations and the associated constraint equations on timelike hypersurfaces. Subsection 4.4 provides a detailed discussion of the relation between Ashtekar's asymptote and Friedrich's cylinder viewed as intrinsic 3-manifolds. Section 5 contains the main analysis in the article —namely, the construction of a conformal Gaussian gauge system in a neighbourhood of an asymptote. It also provides a more detailed statement of the main theorem of this article —Theorem 1. In addition, the article contains several appendices. Appendix A provides the stability theorem for solutions to a system of ordinary differential equations and their existence times which is used in the analysis of the solutions to the conformal geodesic equations in a neighbourhood of the asymptote. Appendix B provides details of the proof of a technical lemma (Lemma 4) on the regularity of geometric fields at the asymptote. Finally Appendix C provides details on a certain class of conformal geodesics on the Minkowski spacetime.

Notations and conventions

In what follows a, b, $c\dots$ will denote spacetime abstract tensorial indices, while i, j, $k,\dots$ are spatial tensorial indices ranging from 1 to 3. By contrast, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c},\dots$ and $\boldsymbol{i}$, $\boldsymbol{j},\boldsymbol{k},\dots$ will correspond, respectively, to spacetime and spatial coordinate indices.

The signature convention for spacetime metrics is $(+,-,-,-)$. Thus, the induced metrics on spacelike hypersurfaces are negative definite. The analysis of this article involves several conformally related metrics. To differentiate between them we adhere to the following conventions: the *tilde* ($\tilde{}$) is used to denote metrics satisfying the physical Einstein field equations; the *overline* ($\bar{}$) is used to denote metrics in the F-gauge; metrics *without an adornment* (e.g. $\boldsymbol{g}$) are in the conformal gauge given by the Definition 2 of a spacetime asymptotically Minkowskian at spatial infinity (AMSI); metrics in a gauge with compact asymptote are denoted by a *grave accent* ($\grave{}$). Finally, metrics with a *hat* ($\hat{}$) correspond to conformal representations in which spatial infinity corresponds to a point.

An index-free notation will be often used. Given a 1-form $\boldsymbol{\omega}$ and a vector $\boldsymbol{v}$, we denote the action of $\boldsymbol{\omega}$ on $\boldsymbol{v}$ by $\langle\boldsymbol{\omega},\boldsymbol{v}\rangle$. Furthermore, $\boldsymbol{\omega}^\sharp$ and $\boldsymbol{v}^\flat$ denote, respectively, the contravariant version of $\boldsymbol{\omega}$ and the covariant version of $\boldsymbol{v}$ (raising and lowering of indices) with respect to a given Lorentzian metric. This notation can be extended to tensors of higher rank (raising and lowering of all the tensorial indices). The conventions for the curvature tensors will be fixed by the relation

$$(\nabla_a\nabla_b-\nabla_b\nabla_a)v^c = R^c{}_{dab}v^d.$$

Also, one can write the decomposition of the curvature tensor as

$$R^c{}_{dab} = C^c{}_{dab} + 2\left(\delta^c{}_{[a}L_{b]d} - g_{d[a}L_{b]}{}^c\right) \tag{2}$$

where $C^c{}_{dab}$ is the *Weyl* tensor and L_{bd} is the *Schouten* tensor of the metric g_{ab}.

2 Tools of conformal methods

The purpose of this section is to introduce the tools of conformal methods that will be required in this article.

General conventions and notation

In the following assume that $(\tilde{\mathcal{M}},\tilde{\boldsymbol{g}})$ denotes a spacetime satisfying the vacuum Einstein field equation

$$\tilde{R}_{ab}=0. \tag{3}$$

Following standard usage, we call the pair $(\tilde{\mathcal{M}},\tilde{\boldsymbol{g}})$ the *physical spacetime*, while any conformally related spacetime $(\mathcal{M},\boldsymbol{g})$ with

$$\boldsymbol{g}\equiv\Xi^2\tilde{\boldsymbol{g}}$$

will be referred to as the *unphysical spacetime.*

2.1 Conformal geodesics

Given an interval $I\subseteq\mathbb{R}$ and $\tau\in I$, the curve $x(\tau)$ is said to be a *conformal geodesic* if there exists a 1-form $\boldsymbol{\beta}(\tau)$ along $x(\tau)$ such that

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}\dot{\boldsymbol{x}} = -2\langle\boldsymbol{\beta},\dot{\boldsymbol{x}}\rangle\dot{\boldsymbol{x}} + \tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}},\dot{\boldsymbol{x}})\boldsymbol{\beta}^\sharp, \tag{4a}$$

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}\boldsymbol{\beta} = \langle \boldsymbol{\beta}, \dot{\boldsymbol{x}}\rangle \boldsymbol{\beta} - \tfrac{1}{2}\tilde{\boldsymbol{g}}^{\sharp}(\boldsymbol{\beta},\boldsymbol{\beta})\dot{\boldsymbol{x}}^{\flat} + \tilde{\boldsymbol{L}}(\dot{\boldsymbol{x}},\cdot), \tag{4b}$$

are satisfied. For spacetimes $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ satisfying the vacuum Einstein equations 3, one can explicitly determine a canonical conformal factor given initial data on an initial hypersurface —see [29, 9], Proposition 5.1. Specifically,

Proposition 1. *Given an Einstein spacetime* $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$*, a solution* $(x(\tau), \boldsymbol{\beta}(\tau))$ *to the conformal geodesic equations* (4a)-(4b) *and an unphysical spacetime* $\boldsymbol{g} = \Theta^2\tilde{\boldsymbol{g}}$ *defined such that* $\boldsymbol{g}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) = 1$*. Then the conformal factor* Θ *can be written as a quadratic polynomial in terms of* τ*, i.e.*

$$\Theta(\tau) = \Theta_\star + \dot{\Theta}_\star(\tau - \tau_\star) + \frac{1}{2}\ddot{\Theta}_\star(\tau - \tau_\star)^2 \tag{5}$$

with

$$\dot{\Theta}_\star = \langle \boldsymbol{\beta}_\star, \dot{\boldsymbol{x}}_\star\rangle \Theta_\star, \quad \Theta_\star\ddot{\Theta}_\star = \frac{1}{2}\tilde{\boldsymbol{g}}^{\sharp}(\boldsymbol{\beta}_\star, \boldsymbol{\beta}_\star) + \frac{1}{6}\lambda \tag{6}$$

where λ *is the* cosmological constant.

Conformal Gaussian systems

Conformal geodesics can be used to construct the so-called *Conformal Gaussian Systems* in which coordinates and adapted frames are propagated off an initial hypersurface $\mathcal{S}$. One constructs a conformal Gaussian system by initially introducing a $\boldsymbol{g}$-orthonormal Weyl-propagated frame $\{\boldsymbol{e}_{\boldsymbol{a}}\}$ along a congruence of conformal geodesics $(\dot{\boldsymbol{x}}(\tau), \boldsymbol{\beta}(\tau))$ on $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}} = \Xi^{-2}\boldsymbol{g})$ and choosing the time coordinate such that $\boldsymbol{e}_{\boldsymbol{0}} = \partial_\tau$. Then if (x^i) denotes the coordinates of a point p on $\mathcal{S}$, one can propagate the spatial coordinates off $\mathcal{S}$ by requiring them to be constant along the conformal geodesic intersecting $\mathcal{S}$ at p. Then the conformal Gaussian system is given by (τ, x^i).

2.2 Conformal field equations

The vacuum *conformal Einstein field equations* on the unphysical spacetime are given by

$$\nabla_a\nabla_b\Xi = -\Xi L_{ab} + s g_{ab}, \tag{7a}$$
$$\nabla_a s = -L_{ac}\nabla^c\Xi, \tag{7b}$$
$$\nabla_c L_{db} - \nabla_d L_{cb} = \nabla_a\Xi d^a{}_{bcd}, \tag{7c}$$
$$\nabla_a d^a{}_{bcd} = 0, \tag{7d}$$
$$6\Xi s - 3\nabla_c\Xi\nabla^c\Xi = 0 \tag{7e}$$

where $d^a{}_{bcd} = \Xi^{-1}C^a{}_{bcd}$ is the *rescaled Weyl* tensor and s is the *Friedrich's scalar* given by

$$s \equiv \frac{1}{4}\nabla^c\nabla_c\Xi + \frac{1}{24}R\Xi.$$

The conformal Einstein field equations constitute a set of differential conditions for the fields Ξ, s, L_{ab} and $d^a{}_{bcd}$. A solution to the vacuum conformal field equations (7a)-(7e) implies, whenever $\Xi \neq 0$, a solution to the vacuum Einstein field equations (3) on the physical spacetime.

Conformal constraint equations

For latter use, we give here the constraint equations implied by the conformal Einstein field equations (7a)-(7e) on a timelike hypersurface $\mathcal{T}$. These equations can be obtained through a projection formalism and details of the derivation can be found in [29].

In the following let q_{ij} denote the intrinsic metric of the hypersurface $\mathcal{T}$, let D_i be the associated Levi-Civita connection and ω the restriction of the conformal factor Ξ to $\mathcal{T}$. Moreover, let n^a denote the (spacelike) normal to $\mathcal{T}$ with associated extrinsic curvature given by K_{ij}. We use the shorthand

$$\Sigma \equiv n^a \nabla_a \Xi. \tag{8}$$

Finally, the symbol $_\perp$ will indicate contraction with respect to the unit normal n^a. In terms of these quantities, the *conformal Einstein constraint equations* in vacuum are given by:

$$D_i D_j \omega + \Sigma K_{ij} + \omega L_{ij} - s q_{ij} = 0, \tag{9a}$$
$$D_i \Sigma + \omega L_{i\perp} = 0, \tag{9b}$$
$$D_i s - \Sigma L_{i\perp} = 0, \tag{9c}$$
$$D_i L_{jk} - D_j L_{ik} + \Sigma d_{ij\perp k} = 0, \tag{9d}$$
$$D_i L_{j\perp} - K_i^k L_{jk} - D_j L_{i\perp} + K_j^k L_{ik} = 0, \tag{9e}$$
$$D^k d_{k\perp ij} = 0, \tag{9f}$$
$$D^k d_{k\perp j\perp} - K^{ik} d_{i\perp jk} = 0, \tag{9g}$$
$$6\Omega s - 3 D_i \omega D^i \omega - 3\Sigma^2 = 0. \tag{9h}$$

In the above expressions the terms $d_{k\perp j\perp}$ and $d_{k\perp ij}$ correspond, essentially, to the electric and magnetic parts of the rescaled Weyl tensor with respect to the normal n^a —respectively.

3 Representations of spatial infinity in the Minkowski spacetime

In this section, we review several representations of spatial infinity for the Minkowski spacetime. This analysis will provide insight and motivate the analysis in curved spacetimes.

In the following let $(\mathbb{R}^4, \tilde{\boldsymbol{\eta}})$ denote the Minkowski spacetime. Let $(\overline{x}) = (x^\mu)$ denote the standard Cartesian coordinates and write $x^0 = \tilde{t}$, etc. We will also make use of spherical coordinates $(\tilde{t}, \tilde{r}, \theta^{\mathcal{A}})$ where $(\theta^{\mathcal{A}})$ denotes some choice of spherical coordinates over $\mathbb{S}^2$. One has that

$$\begin{aligned}\tilde{\boldsymbol{\eta}} &= \eta_{\mu\nu} \mathbf{d}x^\mu \otimes \mathbf{d}x^\nu, \\ &= \mathbf{d}\tilde{t} \otimes \mathbf{d}\tilde{t} - \mathbf{d}\tilde{x} \otimes \mathbf{d}\tilde{x} - \mathbf{d}\tilde{y} \otimes \mathbf{d}\tilde{y} - \mathbf{d}\tilde{z} \otimes \mathbf{d}\tilde{z}, \\ &= \mathbf{d}\tilde{t} \otimes \mathbf{d}\tilde{t} - \mathbf{d}\tilde{r} \otimes \mathbf{d}\tilde{r} - \tilde{r}^2 \boldsymbol{\sigma},\end{aligned}$$

where $\eta_{\mu\nu} \equiv \mathrm{diag}(1, -1, -1, -1)$, and $\boldsymbol{\sigma}$ is the standard round metric over $\mathbb{S}^2$.

3.1 Spatial infinity as point

In first instance, we consider the standard representation of spatial infinity as a point. Intuitively, the region in the Minkowski spacetime associated to spatial infinity is contained in the domain

$$\tilde{\mathcal{D}} \equiv \{ p \in \mathbb{R}^4 \mid \eta_{\mu\nu} x^\mu(p) x^\nu(p) < 0 \}$$

—the *complement of the light cone through the origin*, see Figure 1. Now, introducing the *inversion coordinates* $\overline{x} = (x^\mu)$ defined by

$$y^\mu = -\frac{x^\mu}{X^2}, \qquad X^2 \equiv \eta_{\mu\nu} x^\mu x^\nu,$$

it follows that

$$\eta_{\mu\nu} \mathbf{d}y^\mu \otimes \mathbf{d}y^\nu = X^{-4} \eta_{\mu\nu} \mathbf{d}x^\mu \otimes \mathbf{d}x^\nu.$$

The latter suggests introducing the conformal factor $\Xi \equiv 1/X^2$ so that

$$\hat{\boldsymbol{\eta}} \equiv \Xi^2 \tilde{\boldsymbol{\eta}} = \Xi^2 \mathbf{d}x^\mu \otimes \mathbf{d}x^\nu.$$

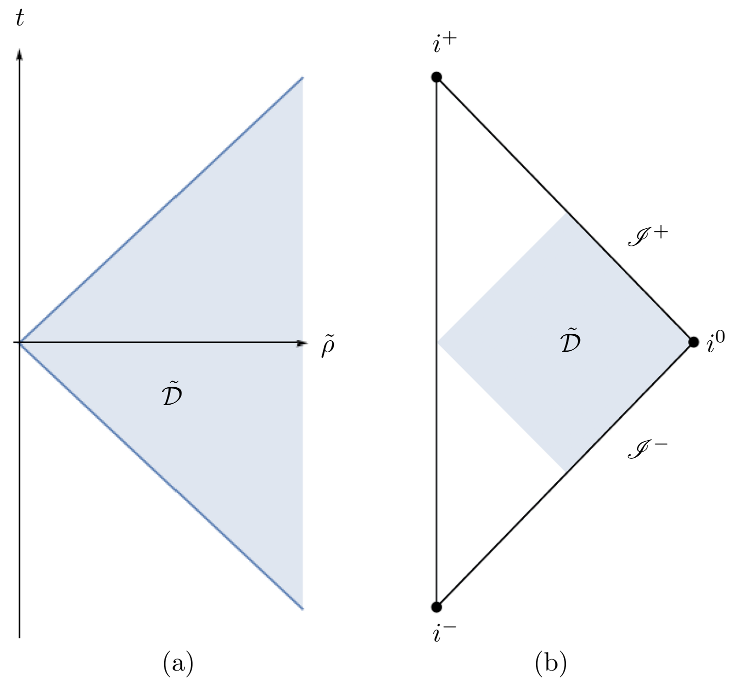

Figure 1: (a) Diagrammatic depiction of the domain $\tilde{\mathcal{D}}$ containing spatial infinity, (b) The domain $\tilde{\mathcal{D}}$ on the conformal diagram of Minkowski spacetime.

This is a conformal representation of the Minkowski spacetime which is flat. The conformal boundary defined by Ξ decomposes into the sets

$$\mathscr{I}^+ \equiv \{p \in \mathbb{R}^4 | y^0(p) > 0,\ \eta_{\mu\nu} y^\mu(p) y^\nu(p) = 0\},$$
$$\mathscr{I}^- \equiv \{p \in \mathbb{R}^4 | y^0(p) < 0,\ \eta_{\mu\nu} y^\mu(p) y^\nu(p) = 0\},$$
$$i^0 \equiv \{p \in \mathbb{R}^4 | (y^\mu(p)) = (0,0,0,0)\}.$$

The sets $\mathscr{I}^+$ ($\mathscr{I}^-$), can be shown to be the endpoints of future (past) null geodesics while spatial geodesics end up in the point i^0 —*spatial infinity*, located in this representation at the origin. Observe that $\mathscr{I}^+$ and $\mathscr{I}^-$ do not contain the whole of null infinity, only the part of the conformal boundary close to spatial infinity —this is a peculiarity of this conformal representation.

Remark 1. In literature, $\mathscr{I}^+$ and $\mathscr{I}^-$ are usually used to denote the whole of future and past null infinity, respectively. In this setting, in a slight abuse of notation, we use $\mathscr{I}^+$ and $\mathscr{I}^-$ to denote the parts of the conformal boundary close to spatial infinity.

It can be verified that while $\mathbf{d}\Xi|_{\mathscr{I}^\pm} \neq 0$, for i^0 it holds that

$$\Xi(i^0) = 0, \qquad \mathbf{d}\Xi(i^0) = 0, \qquad \text{Hess}\,\Xi(i^0) \neq 0. \tag{10}$$

Introducing spherical coordinates $(t, \rho, \theta^{\mathcal{A}})$, with $\rho^2 \equiv (y^1)^2 + (y^2)^2 + (y^3)^2$, in the unphysical spacetime one can write

$$\hat{\boldsymbol{\eta}} = \mathbf{d}t \otimes \mathbf{d}t - \mathbf{d}\rho \otimes \mathbf{d}\rho - \rho^2 \boldsymbol{\sigma}, \qquad \Xi = t^2 - \rho^2. \tag{11}$$

For future use, it is noticed that

$$\tilde{t} = -\frac{t}{t^2 - \rho^2}, \qquad \tilde{\rho} = -\frac{\rho}{t^2 - \rho^2}.$$

3.2 The cylinder at spatial infinity

A different representation of spatial infinity can be obtained from the rescaling

$$\bar{\boldsymbol{\eta}} \equiv \frac{1}{\rho^2}\hat{\boldsymbol{\eta}}. \tag{12}$$

In this representation it is convenient to introduce a new time coordinate τ via the relation

$$t = \rho\tau, \tag{13}$$

so that

$$\bar{\boldsymbol{\eta}} = \mathbf{d}\tau\otimes\mathbf{d}\tau + \frac{\tau}{\rho}\big(\mathbf{d}\tau\otimes\mathbf{d}\rho + \mathbf{d}\rho\otimes\mathbf{d}\tau\big) - \frac{(1-\tau^2)}{\rho^2}\mathbf{d}\rho\otimes\mathbf{d}\rho - \boldsymbol{\sigma}, \tag{14}$$

with contravariant metric given by

$$\bar{\boldsymbol{\eta}}^\sharp = (1-\tau^2)\boldsymbol{\partial}_\tau\otimes\boldsymbol{\partial}_\tau + \rho\tau\big(\boldsymbol{\partial}_\tau\otimes\boldsymbol{\partial}_\rho + \boldsymbol{\partial}_\rho\otimes\boldsymbol{\partial}_\tau\big) - \rho^2\boldsymbol{\partial}_\rho\otimes\boldsymbol{\partial}_\rho - \boldsymbol{\sigma}^\sharp.$$

Introducing the coordinate $\varrho \equiv -\ln\rho$, the metric $\bar{\boldsymbol{\eta}}$ can be rewritten as

$$\bar{\boldsymbol{\eta}} = \mathbf{d}\tau\otimes\mathbf{d}\tau - (1-\tau^2)\mathbf{d}\varrho\otimes\mathbf{d}\varrho - \tau\big(\mathbf{d}\tau\otimes\mathbf{d}\varrho + \mathbf{d}\varrho\otimes\mathbf{d}\tau\big) - \boldsymbol{\sigma}.$$

From this metric and observing equations (11) and (12), one can readily see that spatial infinity i^0, which corresponds to the condition $\rho = 0$ lies at an infinite distance as measured by the metric $\bar{\boldsymbol{\eta}}$.

It follows from the previous discussion that one can write

$$\bar{\boldsymbol{\eta}} = \Theta^2\tilde{\boldsymbol{\eta}}, \qquad \Theta \equiv \rho(1-\tau^2). \tag{15}$$

Consistent with the above one can define the set

$$\bar{\mathcal{M}} \equiv \big\{p\in\mathbb{R}^4 \mid -1\leq\tau(p)\leq 1,\ \rho(p)\geq 0\big\},$$

which gives rise to a conformal extension $(\bar{\mathcal{M}}, \bar{\boldsymbol{\eta}})$ of the Minkowski spacetime. In this representation the following sets play a role in our discussion:

$$I \equiv \big\{p\in\bar{\mathcal{M}} \mid |\tau(p)|<1,\ \rho(p)=0\big\}, \qquad I^0 \equiv \big\{p\in\bar{\mathcal{M}} \mid \tau(p)=0,\ \rho(p)=0\big\},$$

and

$$I^+ \equiv \big\{p\in\bar{\mathcal{M}} \mid \tau(p)=1,\ \rho(p)=0\big\}, \qquad I^- \equiv \big\{p\in\bar{\mathcal{M}} \mid \tau(p)=-1,\ \rho(p)=0\big\}.$$

Moreover, future and past null infinity are given by:

$$\mathscr{I}^\pm \equiv \big\{p\in\bar{\mathcal{M}} \mid \tau(p)=\pm 1\big\}.$$

Remark 2. In the following, we call the above conformal representation of the neighbourhood of spatial infinity the *F-gauge horizontal representation.*

Remark 3. Although the metric (14) is singular on I, it induces the Lorentzian 3-metric

$$\bar{\boldsymbol{q}} = \mathbf{d}\tau\otimes\mathbf{d}\tau - \boldsymbol{\sigma}, \tag{16}$$

which is regular. This metric can be regarded as the $1+2$-dimensional version of the Einstein Universe metric. In particular, its Ricci tensor is proportional to the metric —i.e. one has an Einstein space.

3.2.1 Conformal geodesics

A central aspect of the conformal representation of the Minkowski spacetime given by $(\bar{\mathcal{M}}, \bar{\boldsymbol{\eta}})$ is its relation to conformal geodesics. More precisely, one has the following:

Lemma 1. *The pair $(x(s), \bar{\boldsymbol{\beta}}(s))$, $s \in [-1, 1]$ with*

$$x(s) = (s, \rho_\star, \theta^{\mathcal{A}}_\star), \qquad \bar{\boldsymbol{\beta}} = \frac{1}{\rho_\star}\mathbf{d}\rho, \tag{17}$$

for fixed $(\rho_\star, \theta^{\mathcal{A}}_\star) \in \mathcal{S}_\star$ constitutes a non-intersecting congruence of conformal geodesics in $\bar{\mathcal{M}}$.

The details of the calculations providing the proof of the above lemma can be found in Appendix C.

Remark 4. In particular, the pair as given by the relations in (17) is a solution to the $\bar{\boldsymbol{g}}$-conformal geodesic equations.

Remark 5. In the following, in a slight abuse of notation, we identify the affine parameter s of the conformal geodesics with the time coordinate τ. The conformal geodesics given by Lemma 1 have tangent vector given by $\boldsymbol{\partial}_\tau$.

The conformal factor Θ given in equation (15) can be deduced from the solution to the conformal geodesic equation given by Lemma 1 and Proposition 1. More precisely, writing the canonical conformal factor in terms of the parameter s as

$$\Theta(s) = \Theta_\star + \dot{\Theta}_\star s + \frac{1}{2}\ddot{\Theta}_\star s^2$$

with $\dot{\Theta}_\star$ and $\ddot{\Theta}_\star$ given by the relations in (6). From Lemma 1, it readily follows that

$$\dot{\Theta}_\star = 0.$$

Moreover,

$$\Theta_\star = \rho, \qquad \ddot{\Theta}_\star = -2\rho.$$

Thus, to recover the conformal factor in (15) one identifies the parameters s and τ.

Remark 6. While on the one hand one has $\Theta_\star|_{\rho=0} = 0$, on the other hand $\mathbf{d}\Theta|_{\rho=0} \neq 0$. Thus, the choice of $\Theta_\star$ in this conformal representation is not that one of a point compactification. More generally, if Ω is a conformal factor giving rise to a point compactification of an asymptotic end of an asymptotically Euclidean manifold (cf. the conditions in (10)) the prescription of $\Theta_\star$ is of the form $\kappa^{-1}\Omega$ with κ a smooth function of the form $\kappa = \varkappa\rho$ and $\varkappa(i^0) = 1$.

3.2.2 Gauge freedom

The conformal representation of the Minkowski spacetime described in the previous subsections can be generalised to obtain a description in which null infinity does not coincide with hypersurfaces of constant τ. For this, instead of relation (13) one rather considers

$$t = \kappa\tau, \qquad \kappa = \varkappa\rho, \qquad \varkappa = O(\rho^0),$$

with $\varkappa$ a smooth function of the spatial coordinates. This leads to the conformal factor

$$\Theta = \frac{\rho}{\varkappa}\big(1 - \varkappa^2\tau^2\big),$$

with associated metric $\bar{\boldsymbol{g}} = \Theta^2\tilde{\boldsymbol{g}}$

$$\bar{\boldsymbol{g}} = \mathbf{d}\tau \otimes \mathbf{d}\tau + \frac{\tau\kappa'}{\kappa}(\mathbf{d}\tau \otimes \mathbf{d}\rho + \mathbf{d}\rho \otimes \mathbf{d}\tau) - \frac{1}{\kappa^2}(1 - \tau^2\kappa'^2)\mathbf{d}\rho \otimes \mathbf{d}\rho - \frac{\rho^2}{\kappa^2}\boldsymbol{\sigma}, \qquad \kappa' \equiv \frac{\partial\kappa}{\partial\rho}. \tag{18}$$

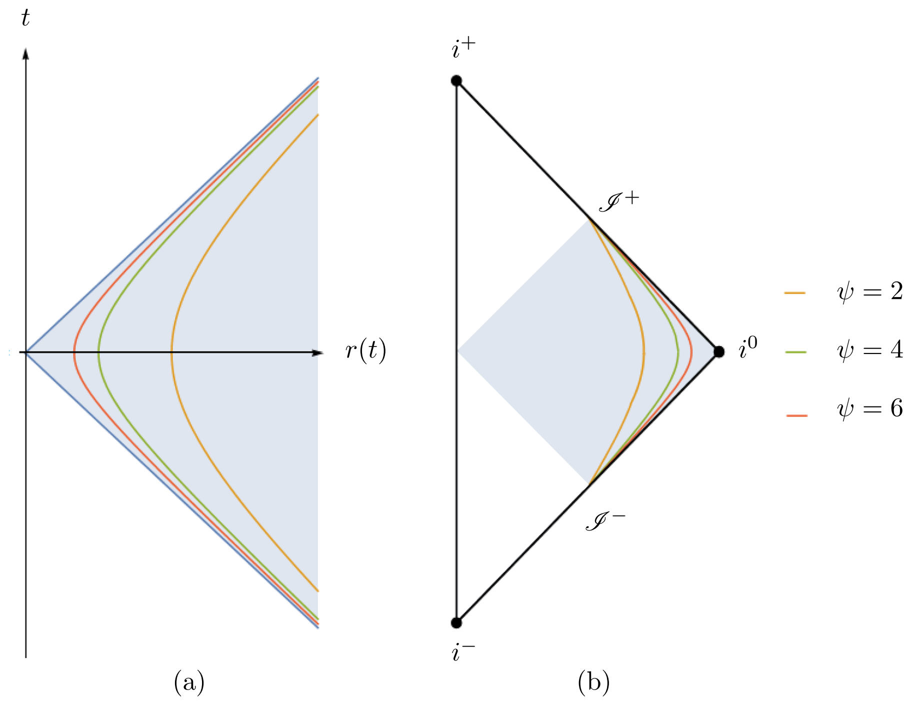

Figure 2: (a) The timelike hyperboloids in the domain $\tilde{\mathcal{D}}$ of the Minkowski spacetime used in the construction of Ashtekar's hyperboloid at spatial infinity. (b) The timelike hyperboloids shown on the conformal diagram of Minkowski spacetime.

In this case, the neighbourhood of spatial infinity is given by

$$\bar{\mathcal{M}} \equiv \{p \in \mathbb{R}^4 \mid -\frac{1}{\varkappa(p)} \leq \tau(p) \leq \frac{1}{\varkappa(p)},\ \rho(p) \geq 0\},$$

while null infinity is described by the sets

$$\mathscr{I}^{\pm} \equiv \left\{p \in \bar{\mathcal{M}} \mid \tau(p) = \pm\frac{1}{\varkappa(p)}\right\}.$$

Remark 7. The general F-gauge representation of Minkowski spacetime is related to the horizontal representation associated with the line element (14) via a conformal transformation which is the identity on I. Moreover, also notice that the parameter of the conformal geodesics in both representations do not coincide and are related to each other via a reparametrisation (Möbius transformation).

Remark 8. A key difference between the horizontal representation line element (14) and the more general F-gauge line element (18) is that the former metric loses rank (i.e. degenerates) all along null infinity ($\mathscr{I}^+$) while the latter does it only at the critical sets ($I^{\pm}$). Thus, horizontal representation provides a slightly more singular representation of the neighbourhood of spatial infinity. This singular behaviour does not play a role in the subsequent discussion of this article. Accordingly, given its relative analytic simplicity, all further discussion of the F-gauge is done in the horizontal representation.

3.3 The hyperboloid at spatial infinity

Following the discussion in [4] a different (albeit related) representation of spatial infinity as an extended set can be obtained by considering *hyperbolic coordinates* $(\psi, \chi, \theta, \varphi)$ via the relations

$$x^0 = \psi \sinh \chi, \tag{19a}$$

$$
\begin{aligned}
x^1 &= \psi \cosh\chi \sin\theta \cos\varphi, &\text{(19b)}\\
x^2 &= \psi \cosh\chi \sin\theta \sin\varphi, &\text{(19c)}\\
x^3 &= \psi \cosh\chi \cos\theta. &\text{(19d)}
\end{aligned}
$$

It readily follows that

$$\tilde{\rho}^2 - \tilde{t}^2 = \psi^2.$$

Accordingly, the hypersurfaces of constant ψ are timelike hyperboloids —see Figure 2. As in Section 3.1 one has that

$$X^2 = \eta_{\mu\nu}x^\mu x^\nu = \tilde{t}^2 - \tilde{\rho}^2 = -\psi^2,$$

so that, as before, spatial infinity is contained in the region

$$\tilde{\mathcal{D}} \equiv \{\overline{x} \in \mathbb{R}^4 \mid X^2 < 0\}.$$

It follows that the hyperbolic coordinates only cover the domain $\tilde{\mathcal{D}}$ —see Figure 1. The Minkowski metric in hyperbolic coordinates takes the form

$$\tilde{\boldsymbol{\eta}} = -\mathbf{d}\psi \otimes \mathbf{d}\psi + \psi^2 \boldsymbol{\ell},$$

where

$$\boldsymbol{\ell} \equiv \mathbf{d}\chi \otimes \mathbf{d}\chi - \cosh^2\chi \boldsymbol{\sigma} \tag{20}$$

is the (Lorentzian) metric of the *unit hyperboloid*. This metric can be obtained from the pullback of $\tilde{\boldsymbol{\eta}}$ to the hypersurface defined by the condition $\tilde{\rho}^2 - \tilde{t}^2 = 1$.

3.3.1 Standard representation of spatial infinity

In order to discuss the behaviour near spatial infinity it is convenient to introduce a new coordinate

$$\zeta \equiv \frac{1}{\psi}.$$

It follows then that

$$\tilde{\boldsymbol{\eta}} = -\frac{1}{\zeta^4}\mathbf{d}\zeta \otimes \mathbf{d}\zeta + \frac{1}{\zeta^2}\boldsymbol{\ell}. \tag{21}$$

Thus, it seems natural to introduce a conformal factor of the form

$$\Omega \equiv \zeta^2$$

so as to obtain

$$
\begin{aligned}
\hat{\boldsymbol{\eta}} &\equiv \Omega^2 \tilde{\boldsymbol{\eta}}\\
&= -\mathbf{d}\zeta \otimes \mathbf{d}\zeta + \zeta^2 \boldsymbol{\ell}.
\end{aligned}
$$

This leads to the *standard* representation of spatial infinity as a point. This is shown by noting that the metric of the 2-surfaces of constant ζ is given by $\zeta^2\boldsymbol{\ell}$. Hence, the set given by $\zeta = 0$ has zero volume and is forced to be a single point by the choice of conformal factor $\Omega = \zeta^2$. Moreover, one can confirm that at $\zeta = 0$, we have $\Omega = 0$, $\mathbf{d}\Omega = 0$, $\operatorname{Hess}\Omega = -2\hat{\boldsymbol{\eta}}$.

3.3.2 The hyperboloid at spatial infinity

In the present case, it is better to define the conformal factor

$$H \equiv \zeta$$

so that

$$
\begin{aligned}
\boldsymbol{\eta} &\equiv H^2 \tilde{\boldsymbol{\eta}},\\
&= -\frac{1}{\zeta^2}\mathbf{d}\zeta \otimes \mathbf{d}\zeta + \boldsymbol{\ell}.
\end{aligned} \tag{22}
$$

This particular rescaling readily connects with Friedrich's framework already discussed in Section 3.4.

Consider in the following the timelike hyperboloids defined by the condition $\zeta = \zeta_\bullet$ where $\zeta_\bullet$ is a constant. The key observation in [4] is that although the conformal metric (22) is singular at $\zeta = 0$, the conformal 3-metric

$$\boldsymbol{q} = H^2 \tilde{\boldsymbol{q}} = \boldsymbol{\ell}$$

is well defined. Observe also, that the contravariant 4-dimensional metric

$$\boldsymbol{\eta}^\sharp = -\zeta^2 \boldsymbol{\partial}_\zeta \otimes \boldsymbol{\partial}_\zeta + \boldsymbol{\ell}^\sharp$$

is well defined —although, it losses rank at the set where $\zeta = 0$. In addition to the above, observe that

$$(\mathbf{d}\zeta)^\sharp = \tilde{\boldsymbol{\eta}}^\sharp(\mathbf{d}\zeta) = -\zeta^4 \boldsymbol{\partial}_\zeta.$$

The later suggests introducing a *rescaled unit normal vector* through the relation

$$\boldsymbol{N} = H^{-4}(\mathbf{d}\zeta)^\sharp = -\boldsymbol{\partial}_\zeta.$$

Remark 9. Starting directly from the (singular) metric $\boldsymbol{\eta}$ one readily finds that the unit normal covector is given by $\zeta^{-1}\mathbf{d}\zeta$. Moreover, one has that

$$\left(\zeta^{-1}\mathbf{d}\zeta\right)^\sharp = -\zeta \boldsymbol{\partial}_\zeta$$

which, despite being well defined at $\zeta = 0$ vanishes. As it will be seen in the following, vectors with this behaviour at infinity play a key role in Friedrich's framework of spatial infinity.

Remark 10. In the following, the timelike hyperboloid $\mathcal{H}$ described by the condition $\zeta = 0$ together with the induced metric $\boldsymbol{q} = \boldsymbol{\ell}$ will be known as *the hyperboloid at spatial infinity of the Minkowski spacetime*. This hyperboloid is an example of the general notion of *asymptote at spatial infinity* introduced in the definition of an asymptotically Minkowskian spacetime at spatial infinity (AMSI) —see Definition 2 in Section 4.

3.4 Relating the Ashtekar and F-gauge construction in the Minkowski spacetime

In this subsection, we obtain the explicit relation between the description of spatial infinity in terms of the hyperboloidal coordinates and that based on the F-gauge. The procedure for relating these two representations will serve as a template for an analogous computation in more general classes of spacetimes.

3.4.1 From the hyperboloid at infinity to the cylinder at infinity

In order to relate the pair $(\mathcal{H}, \boldsymbol{\ell})$ corresponding to Ashtekar's hyperboloid at spatial infinity to the pair $(I, \bar{\boldsymbol{q}})$ with $\bar{\boldsymbol{q}}$ as given by equation (16), it is observed while the former can be thought of as a $1+2$ version of the *de Sitter spacetime*, the latter is a $1+2$ version of the Einstein static Universe —see Remark 3. As both metrics $\boldsymbol{\ell}$ and $\bar{\boldsymbol{q}}$ are Einstein, their Cotton tensor vanishes and thus, they are conformally flat. Accordingly, $\boldsymbol{\ell}$ and $\bar{\boldsymbol{q}}$ are conformally related.

In order to find the conformal factor relating $\boldsymbol{\ell}$ and $\bar{\boldsymbol{q}}$ we follow the same procedure used to show that the (4-dimensional) de Sitter spacetime can be conformally embedded in the (4-dimensional) Einstein static Universe —see e.g. Section 6.3 in [29]. More precisely, starting from the metric $\boldsymbol{\ell}$ of the unit hyperboloid, introduce the coordinate transformation given by the relation

$$\mathbf{d}\chi = \cosh \chi \mathbf{d}\tau.$$

It follows readily that

$$\boldsymbol{\ell} = \cosh^2 \chi \big(\mathbf{d}\tau \otimes \mathbf{d}\tau - \boldsymbol{\sigma}\big)$$

from where one can indeed see that $\boldsymbol{\ell}$ and $\bar{\boldsymbol{q}}$ are conformally related. In particular, it can be shown that $\cosh \chi = \sec \tau$.

Remark 11. It follows from the previous discussion that Friedrich's cylinder at spatial infinity is indeed a time compactified version of Ashtekar's hyperboloid. In particular, the critical sets $I^{\pm}$ correspond to the limits $\chi \to \pm\infty$.

3.4.2 Relating the neighbourhoods of spatial infinity

Now, to relate the two constructions away from spatial infinity we recall that

$$\bar{\boldsymbol{\eta}} = \Theta^2 \tilde{\boldsymbol{\eta}}, \qquad \boldsymbol{\eta} = H^2 \tilde{\boldsymbol{\eta}}. \tag{23}$$

Now, as $\zeta = 1/\psi$, a direct computation then gives that

$$\rho = \frac{1}{\psi}\cosh\chi, \qquad \tau = \tanh\chi, \tag{24}$$

so that

$$\zeta = \rho\,\mathrm{sech}\,\chi.$$

Moreover, from the rescalings in (23) it follows that

$$\boldsymbol{\eta} = \varpi^2 \bar{\boldsymbol{\eta}}, \qquad \varpi \equiv H\Theta^{-1}.$$

In terms of coordinates, one has

$$\varpi = \frac{\zeta}{\rho(1-\tau^2)} = \cosh\chi.$$

Observing that $\chi = \operatorname{arctanh}\tau$ it follows that

$$\varpi = \frac{1}{\sqrt{1-\tau^2}}.$$

Notice that $\varpi|_{\mathcal{S}_\star} = 1$. Moreover, $\varpi \to \infty$ as $\tau \to \pm 1$. Thus, ϖ gives rise to a conformal representation of Minkowski spacetime that does not include null infinity.

Remark 12. Using equation (24) and setting $\psi = \psi_\bullet$ where $\psi_\bullet$ is a constant. One has that $(\tanh\chi, \psi_\bullet^{-1}\cosh\chi)$ describes a curve in the (τ,ρ)-plane —see the Figure 3. Observe, in particular, that

$$(\tanh\chi, \psi_\bullet^{-1}\cosh\chi) \longrightarrow (\pm 1, \infty) \qquad \text{as} \qquad \chi \to \pm\infty.$$

Accordingly, the hyperboloids of constant ψ never reach the conformal boundary —i.e. $\mathscr{I}^{\pm}$; see Figure 3. *Hence, this representation of spatial infinity cannot be used to study the effects of gravitational radiation and the relation between asymptotic charges at null infinity and conserved quantities at spatial infinity.*

3.4.3 Conformal geodesics and construction of a conformal Gaussian system

The unphysical metric $\bar{\boldsymbol{\eta}}$ is conformally related to the physical metric $\tilde{\boldsymbol{\eta}}$ via the conformal factor $\Theta = \rho(1-\tau^2)$. Thus, a solution to the conformal geodesic equations on $\bar{\boldsymbol{\eta}}$ implies a solution to the conformal geodesic equation on the physical metric $\tilde{\boldsymbol{\eta}}$. Recalling that the pair given by $x(\tau) = (\tau, \rho_\star, \theta^{\mathcal{A}}_\star)$ and $\bar{\boldsymbol{\beta}} = \mathbf{d}\rho/\rho$ satisfies the conformal geodesic equations for the metric $\bar{\boldsymbol{\eta}}$, it follows, using the coordinate transformation to hyperbolic coordinates given by (19a)-(19d), that the tangent vector to the curve $x(\tau)$ is given in the (ζ, χ, θ^A) coordinates by

$$\dot{\boldsymbol{x}} = -\frac{\rho_\star \tau}{\sqrt{1-\tau^2}}\boldsymbol{\partial}_\zeta + \frac{1}{1-\tau^2}\boldsymbol{\partial}_\chi.$$

Then, $(\dot{\boldsymbol{x}}, \boldsymbol{\beta})$ provides a solution to the conformal geodesic equation, where $\boldsymbol{\beta}$ is given by

$$\boldsymbol{\beta} = \bar{\boldsymbol{\beta}} - \varpi^{-1}\mathbf{d}\varpi$$

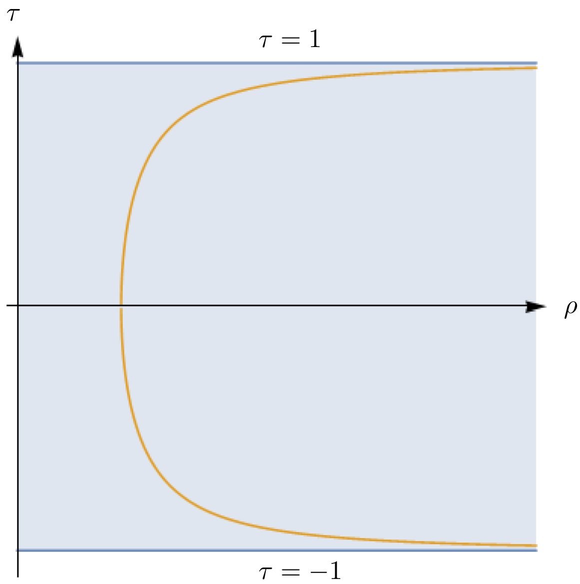

Figure 3: Example of one of the timelike hyperboloids used in Ashtekar's representation of spatial infinity for the Minkowski as seen from the point of view of the F-gauge. Observe that the hyperboloid asymptotes the sets described by the conditions $\tau = \pm 1$. Notice, however, that in this description the hyperboloid at spatial infinity is compact and corresponds to the portion of the vertical axis between the values $\tau = -1$ and $\tau = 1$.

$$= \frac{1}{\zeta}\mathbf{d}\zeta = \mathbf{d}\ln H,$$

where it has been used that

$$\bar{\boldsymbol{\beta}} = \frac{1}{\zeta}\mathbf{d}\zeta + \tanh\chi\mathbf{d}\chi, \qquad \mathbf{d}\ln\varpi = \tanh\chi\mathbf{d}\chi.$$

Setting up a conformal Gaussian system. To conclude this discussion, we show how to compute a conformal Gaussian system on top of Ashtekar's conformal representation of spatial infinity in the Minkowski spacetime. As we have already shown that we have a non-intersecting congruence of conformal geodesics, one can invoke Proposition 1 in Section 2.1 so that one can associate a conformal factor

$$\Lambda(\tau) = \Lambda_\star + \dot{\Lambda}_\star s + \frac{1}{2}\ddot{\Lambda}_\star s^2,$$

along each of the curves of the congruence and where s is the natural parameter of the curves. The coefficients in the above expression satisfy the relations

$$\dot{\Lambda}_\star = \langle\tilde{\boldsymbol{\beta}}_\star, \dot{\boldsymbol{x}}_\star\rangle\Lambda_\star, \qquad \Lambda_\star\ddot{\Lambda}_\star = \frac{1}{2}\tilde{\boldsymbol{\eta}}^\sharp(\tilde{\boldsymbol{\beta}}_\star, \tilde{\boldsymbol{\beta}}_\star).$$

Consistent with the conformal metric (22), for $\Lambda_\star$ one prescribes $\Lambda_\star = \zeta$. For the covector $\tilde{\boldsymbol{\beta}}$ one would like to prescribe a behaviour of the form $\tilde{\boldsymbol{\beta}}_\star = 2\mathbf{d}\tilde{r}/\tilde{r}$. Note that the above form of $\tilde{\boldsymbol{\beta}}$ is consistent with the conformal geodesic solution obtained on the F-gauge construction with $\bar{\boldsymbol{\beta}} \propto 1/\rho$. Also, this type of behaviour near spatial infinity is obtained from the conformal factor $\omega = \zeta^2 = 1/\psi^2 \sim 1/r^2$, which renders the *standard* point compactification of spatial infinity. Consistent with this discussion set

$$\tilde{\boldsymbol{\beta}} = \mathbf{d}\ln\omega = \frac{2}{\zeta}\mathbf{d}\zeta.$$

it follows from the above discussion that $\dot{\Lambda}_\star = 0$ as $\langle\mathbf{d}\zeta, \boldsymbol{\partial}_\tau\rangle = 0$. Also, notice that

$$\tilde{\boldsymbol{\eta}}^\sharp(\tilde{\boldsymbol{\beta}}_\star, \tilde{\boldsymbol{\beta}}_\star) = \tilde{\boldsymbol{h}}^\sharp(\tilde{\boldsymbol{\beta}}_\star, \tilde{\boldsymbol{\beta}}_\star).$$

A quick computation gives the metric on $\chi = 0$ hypersurface

$$\tilde{\boldsymbol{h}} = -\frac{1}{\zeta^4}\mathbf{d}\zeta \otimes \mathbf{d}\zeta - \frac{1}{\zeta^2}\boldsymbol{\sigma}, \qquad \tilde{\boldsymbol{h}}^\sharp = -\zeta^4 \boldsymbol{\partial}_\zeta \otimes \boldsymbol{\partial}_\zeta + \zeta^2 \boldsymbol{\sigma}^\sharp,$$

so that $\tilde{\boldsymbol{h}}^\sharp(\tilde{\boldsymbol{\beta}}_\star, \tilde{\boldsymbol{\beta}}_\star) = -4\zeta^2$. From the latter it follows then that $\ddot{\Lambda}_\star = -2\zeta$. Accordingly, the conformal factor associated to the congruence of conformal geodesics has the form

$$\Lambda = \zeta(1 - s^2).$$

Finally, defining $\breve{\boldsymbol{\eta}} \equiv \breve{\Lambda}^2 \tilde{\boldsymbol{\eta}}$ so that

$$\breve{\boldsymbol{\beta}} = \tilde{\boldsymbol{\beta}} - \mathbf{d} \ln \breve{\Lambda},$$

one concludes that $\breve{\boldsymbol{\beta}}_\star = \mathbf{d}\zeta/\zeta$. This shows the consistency of the prescription of initial data for the covector $\tilde{\boldsymbol{\beta}}$ given above.

4 Asymptotically Minkowskian spacetimes at spatial infinity

In [4] the notion of spacetimes which are *asymptotically Minkowskian at spatial infinity* has been introduced. We want to analyse this definition in the light of Friedrich's framework. Hence, we briefly review the relevant definitions and properties.

4.1 Definitions

Following the discussion in [4], in the following we will consider the following definition:

Definition 2. *A vacuum spacetime* $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ *is said to* ***possess an asymptote at spatial infinity*** *if there exists a manifold with boundary* $\mathcal{H}$*, a smooth function* Ω *defined on* $\mathcal{M}$ *and a diffeomorphism from* $\tilde{\mathcal{M}}$ *to* $\mathcal{M} \setminus \mathcal{H}$ *(which is used to identify* $\tilde{\mathcal{M}}$ *with its image in* $\mathcal{M}$*; in particular* Ω *is non-vanishing in* $\tilde{\mathcal{M}}$*) such that:*

(i) $\Omega = 0$ *and* $\mathbf{d}\Omega \neq 0$ *on* $\mathcal{H}$*;*

(ii) the fields

$$\boldsymbol{q} \equiv \Omega^2 \big(\tilde{\boldsymbol{g}} + \Omega^{-4} \mathbf{d}\Omega \otimes \mathbf{d}\Omega\big) \tag{25}$$

and

$$\boldsymbol{N} \equiv \Omega^{-4} \tilde{\boldsymbol{g}}^\sharp(\mathbf{d}\Omega, \cdot)$$

admit smooth limits to $\mathcal{H}$*. In particular, the pullback of* $\boldsymbol{q}$ *(to be denoted again by* $\boldsymbol{q}$*) to* $\mathcal{H}$ *is also well defined and has signature* $(+--)$*.*

In addition, if $\mathcal{H}$ *has the topology of* $\mathbb{R} \times \mathbb{S}^2$ *then* $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ *is said to be* ***asymptotically flat at spatial infinity****. Moroever, if* $\mathcal{H}$ *is geodesically complete with respect to* $\boldsymbol{q}$ *we say that the spacetime is* ***asymptotically Minkowskian at spatial infinity (AMSI).***

Notation. In the following, for convenience, we will make use of the symbol $\simeq$ to denote equality on $\mathcal{H}$. With this notation the conditions in point *(i)* of the Definition 2 are written as

$$\Omega \simeq 0, \qquad \mathbf{d}\Omega \not\simeq 0.$$

Remark 13. The above definition involves some conformal gauge freedom in the sense that if $(\mathcal{M}, \Omega)$ satisfy Definition 2 and α is a smooth function which is a non-zero constant on $\mathcal{H}$ and non-vanishing in $\tilde{\mathcal{M}}$, then $(\mathcal{M}, \Omega' = \alpha\Omega)$ also satisfy the definition.

4.2 Properties and consequences of the definition of an AMSI spacetime

In the following, we explore some direct consequences of Definition 2 which will be used repeatedly in the rest of this article. Accordingly, throughout we assume that one has a vacuum AMSI spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$.

For conceptual clarity, let ζ denote a coordinate such that in a neighbourhood of $\mathcal{H}$ one has

$$\Omega = \zeta.$$

It follows then that the metric

$$\boldsymbol{q} \equiv \zeta^2\big(\tilde{\boldsymbol{g}} + \zeta^{-4}\mathbf{d}\zeta \otimes \mathbf{d}\zeta\big)$$

has a smooth limit as $\zeta \to 0$. From this assumption it follows that

$$\boldsymbol{q} = \mathring{\boldsymbol{q}} + \breve{\boldsymbol{q}} \tag{26}$$

with $\mathring{\boldsymbol{q}}$ independent of ζ and, moreover, $\mathring{\boldsymbol{q}}(\boldsymbol{\partial}_\zeta, \cdot) = 0$. The tensor $\breve{\boldsymbol{q}}$ has smooth components such that

$$\breve{\boldsymbol{q}} = o(\zeta).$$

Remark 14. Recall that the notation $f(\zeta) = o(\zeta^\alpha)$ means that $f(\zeta)/\zeta^\alpha \to 0$ as $\zeta \to 0$.

The statement (26) can be regarded as a *zeroth-order* Taylor expansion of the metric $\boldsymbol{q}$ with respect to ζ. From (25) and (26), it follows that the physical metric $\tilde{\boldsymbol{g}}$ has, close to $\mathcal{H}$, the form

$$\tilde{\boldsymbol{g}} = -\frac{1}{\zeta^4}\mathbf{d}\zeta \otimes \mathbf{d}\zeta + \frac{1}{\zeta^2}\big(\mathring{\boldsymbol{q}} + \breve{\boldsymbol{q}}\big).$$

Following the analogy of the Minkowski spacetime, define the conformal metric

$$\boldsymbol{g} \equiv \Omega^2 \tilde{\boldsymbol{g}},$$

so that one has

$$\boldsymbol{g} = -\frac{1}{\zeta^2}\mathbf{d}\zeta \otimes \mathbf{d}\zeta + \big(\mathring{\boldsymbol{q}} + \breve{\boldsymbol{q}}\big) \tag{27}$$

—cf. the line element (22) for the conformal Minkowski spacetime in hyperboloidal coordinates.

Remark 15. As in the case of the Minkowski spacetime, the conformal metric (27) is singular at $\zeta = 0$. Dealing with this singular behaviour will be the main challenge in the subsequent analysis of this section.

Remark 16. As it will be seen, the metric (27) can be further specialised by choosing coordinates on $\mathcal{H}$ so that

$$\mathring{\boldsymbol{q}} = \boldsymbol{\ell} = \mathbf{d}\chi \otimes \mathbf{d}\chi - \cosh^2\chi\boldsymbol{\sigma},$$

the metric of the unit timelike hyperboloid.

4.3 The conformal constraint equations on $\mathcal{H}$

In this section, we discuss the implications Definition 2 has on the conformal Einstein constraint equations (9a)-(9h) when evaluated on the asymptote $\mathcal{H}$. In this way, we recover systematically the conditions satisfied by the gravitational field on the hyperboloid as discussed in [4].

In order to evaluate the conformal Einstein constraints on $\mathcal{H}$, we first consider the equations (9a)-(9h) on timelike hypersurfaces for which the conformal factor Ω is constant. On these hypersurfaces one has, in adapted coordinates, that $D_i\zeta = 0$ and $D_iD_j\zeta = 0$. Following the discussion in [4] define

$$F \equiv N^a \nabla_a \Omega.$$

Note that the unit normal is related to Ashtekar's normal by $n^a = \nu N^a$ with $\nu \equiv \Omega \digamma^{-\frac{1}{2}} = O(\Omega)$ i.e. as $\zeta \to 0$, we have $|\nu| \leq M|\Omega|$, where M is a positive constant. From (8), we have $\Sigma = \nu \digamma$. Taking the limit of equations (9a)-(9h) as $\zeta \to 0$, one obtains the following equations on $\mathcal{H}$:

$$D_i \digamma \simeq 0, \tag{28a}$$
$$D_a s \simeq 0, \tag{28b}$$
$$D_i L_{jk} - D_j L_{ik} \simeq 0, \tag{28c}$$
$$D_i L_{j\perp} - D_j L_{i\perp} - K^l_i L_{jl} + K^l_j L_{il} - n_j K^l_i L_{l\perp} + n_i K^l_j L_{l\perp} \simeq 0, \tag{28d}$$
$$D^k d_{k\perp ij} \simeq 0, \tag{28e}$$
$$D^k d_{k\perp j\perp} \simeq 0. \tag{28f}$$
$$s \simeq 0, \tag{28g}$$

As in the previous section, the symbol $\simeq$ is used to indicate an equality which holds at $\mathcal{H}$ and these equations to be understood as the limit as $\zeta \to 0$. For example, equation (28a) is to be understood as

$$\lim_{\zeta \to 0} D_i \digamma = 0.$$

The conformal field equations imply an additional constraint $D_i D_j \Omega \simeq 0$ which is satisfied identically on $\mathcal{H}$. In addition to the above relations, one can also consider the Gauss-Codazzi and Codazzi-Mainardi equations on $\mathcal{H}$. Assuming vacuum, these give the relations

$$q_i{}^a q_j{}^b L_{ab} \simeq l_{ij} + \tfrac{1}{4} q_{ij} \left(K_{kl} K^{kl} - K^2 \right) + K K_{ij} - K^k_i K_{jk}, \tag{29a}$$
$$D_i K_{jk} - D_j K_{ik} \simeq 0, \tag{29b}$$

where l_{ij} denotes the *Schouten tensor* of the metric q_{ij} and one uses q^{ij} to raise and lower indices on $\mathcal{H}$.

A direct consequence of the above relations is that $\digamma$ is constant on $\mathcal{H}$. Moreover, from the above relations one obtains the following:

Lemma 2. *For a spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ satisfying Definition 2 it follows that on the asymptote $\mathcal{H}$ one has the relation*

$$r_{ij} \simeq 2\digamma q_{ij}.$$

In other words, $(\mathcal{H}, \boldsymbol{q})$ is an Einstein space.

Proof. Starting from the transformation rule for the Schouten tensor

$$L_{ab} - \tilde{L}_{ab} = -\frac{1}{\Xi} \nabla_a \nabla_b \Xi + \frac{1}{2\Xi^2} \nabla_c \Xi \nabla^c \Xi g_{ab},$$

contracting with $q_i{}^a q_j{}^b$ and using $K_{ij} = -q_i{}^c q_j{}^d \nabla_c n_d$ and $\nabla_c \Xi \nabla^c \Xi = \Xi^2 \digamma$, the relation between L_{ab} and $\tilde{L}_{ab}$ can be written as

$$q_i{}^a q_j{}^b L_{ab} = q_i{}^a q_j{}^b \tilde{L}_{ab} + \digamma^{\frac{1}{2}} K_{ij} + \frac{1}{2} \digamma q_{ij}.$$

The first term on the right hand side vanishes for a spacetime satisfying Definition 2. Substituting into the Gauss-codazzi equation (29a) and using $K_{ij} \simeq -\digamma^{\frac{1}{2}} q_{ij}$, we get

$$l_{ij} = \digamma q_{ij}.$$

Then, defining the 3-dimensional Schouten tensor as $l_{ij} \equiv r_{ij} - \frac{1}{4} r q_{ij}$, we can confirm that $r_{ij} = 4\digamma q_{ij}$. Making use of the conformal gauge freedom, $\digamma$ can be redefined so that $r_{ij} = 2\digamma q_{ij}$. □

In addition, one has the following *peeling-type* behaviour for the components of the Weyl tensor:

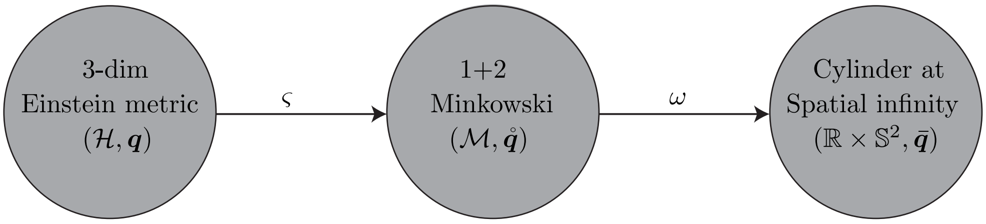

Figure 4: Schematic summary of the argument showing that the an asymptote $\mathcal{H}$ satisfying Definition 2 and the $1+2$ Einstein static Universe are conformally related.

Lemma 3. *For a spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ satisfying Definition 2, the electric E_{ab} and magnetic B_{ab} parts of the Weyl satisfy*

$$E_{ab} \simeq 0$$
$$B_{ab} \simeq 0.$$

Proof. Starting with the definition of the electric and magnetic part of the Weyl tensor

$$E_{ab} = C_{acbd}n^c n^d \quad B_{ab} =^* C_{acbd}n^c n^d,$$

where ${}^*C_{acbd}$ is the left hodge dual of C_{acbd}. Using the decomposition of the curvature tensor (2) and after a lengthy calculation, one can write

$$E_{ab} = \mathcal{L}_n K_{ab} + F^{\frac{1}{2}} D_a D_b F^{-\frac{1}{2}} - K^c_b K_{ac} + q^c_a q^d_b L_{cd} - q_{ab} n^c n^d L_{cd}$$
$$B_{ab} = -\epsilon_a^{cd}\left(D_{[d}K_{c]b} + \frac{1}{2}\left(g_{db}L_{ec}n^e - g_{bc}L_{ed}n^e\right)\right)$$

where $\mathcal{L}_n$ denotes the Lie derivative in the direction of the unit normal n. Then making use of the transformation law of the Schouten tensor and the fact that $\tilde{L}_{ab} = 0$ and taking the limit as $\zeta \to 0$ and again using $K_{ij} \simeq -F^{\frac{1}{2}} q_{ij}$, it can be shown that $\lim_{\zeta\to 0} E_{ab} = 0$ and $\lim_{\zeta\to 0} B_{ab} = 0$. □

4.4 Relating a general asymptote to Friedrich's cylinder

The purpose of this section is to show that, as in the case of the Minkowski spacetime, the 3-dimensional Lorentzian manifold $(\mathcal{H}, \boldsymbol{q})$ is conformally related to the 3-dimensional Einstein Universe (Friedrich's cylinder) $(\mathbb{R}\times\mathbb{S}^2, \bar{\boldsymbol{q}} \equiv \mathbf{d}\tau \otimes \mathbf{d}\tau - \boldsymbol{\sigma})$.

In the following we assume one is given an asymptote $\mathcal{H}$ as given by Definition 2. From Lemma 2, it follows then that

$$r_{ij} = 2F q_{ij}$$

with F constant on $\mathcal{H}$. In order to find the conformal transformation between $\boldsymbol{q}$ and $\bar{\boldsymbol{q}}$ we proceed in two steps: (i) first we show that q_{ij} is conformally related to the $1+2$-dimensional Minkowski spacetime; and (ii) use the fact that the $1+2$-dimensional Minkowski spacetime is conformally related to the 3-dimensional Einstein cylinder. The composition of these two conformal rescalings gives the relation between $\boldsymbol{q}$ and $\bar{\boldsymbol{q}}$ —see Figure 4.

4.4.1 From the Einstein metric to $1+2$-Minkowski spacetime

We begin by observing that if $\boldsymbol{q}$ is conformally related to the $1+2$-Minkowski metric $\mathring{\boldsymbol{q}}$ via a rescaling of the form $\boldsymbol{q} = \varsigma^2\mathring{\boldsymbol{q}}$, then the Schouten tensors of $\boldsymbol{q}$ and $\mathring{\boldsymbol{q}}$ are related via

$$l_{ij} - \mathring{l}_{ij} = -\frac{1}{\varsigma}D_i D_j \varsigma + \frac{1}{2\varsigma^2}D_k\varsigma D^k\varsigma q_{ij}. \tag{30}$$

Now, defining

$$\alpha_i \equiv \varsigma^{-1} D_i \varsigma,$$

then we can rewrite equation (30) as

$$l_{ij} - \mathring{l}_{ij} = -D_i\alpha_j - \alpha_i\alpha_j + \frac{1}{2}\alpha_k\alpha^k q_{ij}.$$

Given that $\mathring{l}_{ij} = 0$, one finds that

$$D_i\alpha_j = -l_{ij} - \alpha_i\alpha_j + \frac{1}{2}\alpha_k\alpha^k q_{ij}. \tag{31}$$

Multiplying by q^{il} and expanding the covariant derivative in local coordinates $\underline{x} = (x^\alpha)$, one ends up with the expression

$$\partial_\alpha \alpha^\beta = -l_\alpha{}^\beta - \alpha_\alpha\alpha^\beta + \frac{1}{2}\alpha_\gamma\alpha^\gamma\delta_\alpha^\beta - \Gamma_\alpha{}^\beta{}_\gamma\alpha^\gamma. \tag{32}$$

This is an overdetermined partial differential equation for the components of the covector $\boldsymbol{\alpha}$. To ensure the existence of a solution to this equation, we make use of *Frobenius theorem* —see e.g. [8], Appendix C. Specifically, for a system of partial differential equations of N dependent variables given in terms of n independent variables

$$\frac{\partial\alpha^{\mathcal{A}}}{\partial x^i} = \psi_i^{\mathcal{A}}(\underline{x}, \boldsymbol{\alpha}), \qquad i = 1, \ldots, n, \qquad \mathcal{A} = 1, ..., N,$$

the *necessary and sufficient condition* to find a unique solution $\alpha^{\mathcal{A}} = \alpha^{\mathcal{A}}(\underline{x})$ is given by

$$\frac{\partial\psi_\alpha^{\mathcal{A}}}{\partial x^\beta} - \frac{\partial\psi_\beta^{\mathcal{A}}}{\partial x^\alpha} + \sum_{\mathcal{B}}\left(\frac{\partial\psi_\alpha^{\mathcal{A}}}{\partial\alpha^{\mathcal{B}}}\psi_\beta^{\mathcal{B}} - \frac{\partial\psi_\beta^{\mathcal{A}}}{\partial\alpha^{\mathcal{B}}}\psi_\alpha^{\mathcal{B}}\right) = 0. \tag{33}$$

In the case of equation (32) one has that $n = 3$ and $N = 9$. Thus, making the identification $\psi_\alpha^{\mathcal{A}} \mapsto \partial_\alpha\alpha^{\mathcal{A}}$ and after a lengthy calculation, we find that equation (33) can be rewritten as

$$r^l{}_{kji}\alpha_l - 2\alpha_{[i}D_{j]}\alpha_k + 2\alpha_l q_{k[i}D_{j]}\alpha^l = 0 \tag{34}$$

with $r^l{}_{kji}$ denoting the components of the Riemann tensor of $\boldsymbol{q}$. Now, in 3 dimensions the Riemann tensor is completely determined by the Schouten tensor —more precisely

$$r^k{}_{lji} = 2\left(\delta^k{}_{[j}l_{i]l} - q_{l[j}l_{i]}{}^k\right).$$

Using this expression for the Riemann tensor, together with the condition $l_{ij} = \digamma q_{ij}$, valid for an Einstein space, in equation (34) we find that the integrability condition is automatically satisfied. Accordingly, there exists a solution α_α to equation (32). From equation (31), one can show that $D_{[i}\alpha_{j]} = 0$, so α_j is a closed covector. Hence, it is locally exact. Thus, one can guarantee the existence of a conformal factor locally relating the 3-dimensional Lorentzian metric $\boldsymbol{q}$ and the $1+2$-Minkowski metric $\mathring{\boldsymbol{q}}$.

4.4.2 From the $1+2$-Minkowski spacetime to Friedrich's cylinder

The next step is to find the conformal factor relating the $1+2$-Minkowski to the cylinder at spatial infinity. Starting with the $1+2$-Minkowski metric in the form

$$\mathring{\boldsymbol{q}} = \mathbf{d}t \otimes \mathbf{d}t - \mathbf{d}r \otimes \mathbf{d}r - r^2\mathbf{d}\theta \otimes \mathbf{d}\theta,$$

we can write this metric in terms of the double null coordinates $u \equiv t - r$ and $v \equiv t + r$ as

$$\mathring{\boldsymbol{q}} = \frac{1}{2}\left(\mathbf{d}v \otimes \mathbf{d}u + \mathbf{d}u \otimes \mathbf{d}v\right) - \frac{1}{4}(v-u)^2\mathbf{d}\theta \otimes \mathbf{d}\theta.$$

If we introduce the two coordinate transformations

$$u = \tan U, \qquad v = \tan V$$

and

$$R = V - U, \qquad T = V + U,$$

the $1+2$-Minkowski metric transforms to

$$\begin{aligned}\mathring{\boldsymbol{q}} &= \omega^{-2}\left(\mathbf{d}T\otimes\mathbf{d}T - \mathbf{d}R\otimes\mathbf{d}R - \sin^2 R\mathbf{d}\theta\otimes\mathbf{d}\theta\right)\\ &= \omega^{-2}\left(\mathbf{d}T\otimes\mathbf{d}T - \boldsymbol{\sigma}\right)\\ &= \omega^{-2}\bar{\boldsymbol{q}}\end{aligned}$$

where $\bar{\boldsymbol{q}}$ is the metric of Friedrich's cylinder and Ξ is given by

$$\omega \equiv 2\cos U\cos V \equiv \cos T + \cos R.$$

4.4.3 Summary

The discussion of the previous subsections can be summarised as follows:

Proposition 2. *The metric $\boldsymbol{q}$ of an asymptote $\mathcal{H}$ satisfying Definition 2 is conformally related to the standard metric of Friedrich's cylinder $\mathbb{R}\times\mathbb{S}^2$.*

In the following, we make this relation more precise by recasting the neighbourhood of the asymptote $\mathcal{H}$ in terms of the F-gauge in which the cylinder at spatial infinity is described.

5 Conformal Gaussian gauge systems in a neighbourhood of an asymptote

In this section, we provide the main analysis of the article: the construction of conformal Gaussian systems in a (spacetime) neighbourhood of an asymptote $\mathcal{H}$ which satisfies Definition 2. More precisely, we show that Definition 2 provides enough regularity in the conformal geometric fields to run a stability argument to show that the neighbourhood of the asymptote $\mathcal{H}$ can be covered by a non-intersecting congruence of conformal geodesics extending up to null infinity (and beyond). This congruence is used, in turn, to build on top a representation of spatial infinity *à la* Friedrich.

The construction proceeds in various steps and mimics, to some extent, the analysis of the conformal extension of static and stationary spacetimes given in [12, 1].

5.1 A compactified version of the asymptote $\mathcal{H}$

The construction of a conformal Gaussian system is based on the conformal representation of the asymptotic region of the spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ given by the metric (27). Moreover, using arguments similar to those used in Section 4.4 it can be assumed, without loss of generality that the metric $\mathring{\boldsymbol{q}}$ is, in fact, the metric of the unit hyperboloid $\boldsymbol{\ell}$. Accordingly, in the following we consider a metric of the form

$$\boldsymbol{g} = -\frac{1}{\zeta^2}\mathbf{d}\zeta\otimes\mathbf{d}\zeta + \left(\boldsymbol{\ell} + \breve{\boldsymbol{q}}\right), \qquad \breve{\boldsymbol{q}} = o(\zeta). \tag{35}$$

In order to construct a conformal Gaussian system in a neighbourhood of the asymptote $\mathcal{H}$ it is convenient to consider a representation in which the time dimension has compact extension. There are several ways of doing this, however, for the present purposes probably the most convenient approach is to mimic the discussion of the relation between the F-gauge and hyperboloid representations of spatial infinity for the Minkowski spacetime given in Section 3.4.2.

In the following introduce new coordinates (τ, ρ, θ^A) in the line element (35) via the relations

$$\grave{\rho} = \zeta\cosh\chi, \qquad \grave{\tau} = \tanh\chi.$$

This coordinate transformation is formally identical to the one used in Section 3.4.2 but the geometric interpretation does not follow through as the vector field $\boldsymbol{\partial}_\tau$ is no longer tangent to a congruence of conformal geodesics. Observe that $\grave{\tau}=0$ if and only if $\chi=0$. A computation then shows that

$$\boldsymbol{g}=\varpi^2\bar{\boldsymbol{\eta}}+\breve{\boldsymbol{q}}, \qquad \varpi\equiv\frac{1}{\sqrt{1-\grave{\tau}^2}}.$$

This suggests introducing a new conformal metric $\grave{\boldsymbol{g}}$ via

$$\grave{\boldsymbol{g}}\equiv\varpi^{-2}\grave{\boldsymbol{g}},$$

so that

$$\grave{\boldsymbol{g}}=\bar{\boldsymbol{\eta}}+\varpi^{-2}\breve{\boldsymbol{q}}, \tag{36}$$

where $\bar{\boldsymbol{\eta}}$ is the expression for the Minkowski metric in the F-gauge, equation (14), with the replacements $\tau\mapsto\grave{\tau}$ and $\rho\mapsto\grave{\rho}$. Observe that

$$\begin{aligned}\varpi^{-2}\breve{\boldsymbol{q}}&=(1-\grave{\tau}^2)\,\breve{\boldsymbol{q}},\\ &=\operatorname{sech}^2\chi\,\breve{\boldsymbol{q}}.\end{aligned}$$

In view of the above expression we make further assumption independently of Definition 2:

Assumption 1. (i) The field

$$(1-\grave{\tau}^2)\,\breve{\boldsymbol{q}} \qquad (=\operatorname{sech}^2\chi\,\breve{\boldsymbol{q}})$$

has a suitably regular limit as $\grave{\tau}\to\pm1$ (i.e. $\chi\to\pm\infty$).

(ii) Moreover, it is required that

$$(1-\grave{\tau}^2)\breve{\boldsymbol{q}}(\mathbf{d}\grave{\tau},\mathbf{d}\grave{\tau})\to0, \qquad \text{as} \qquad \grave{\tau}\to\pm1.$$

Remark 17. The above assumption is, in fact, a statement about the regularity of the conformal metric in the sets where spatial infinity meets null infinity. A programme to analyse this issue has been started in [10] —see also [29], Chapter 20 for further discussions on the subject.

Remark 18. Condition (ii) above ensures that $\grave{\boldsymbol{g}}(\mathbf{d}\grave{\tau},\mathbf{d}\grave{\tau})=0$ for $\tau=\pm1$, so that the conformal boundary is null as it is to be expected. *Suitably regular in the present context means that the limit is assumed to be sufficiently regular for the subsequent argument to hold.* The analysis of how stringent these conditions are or how this can be encoded goes beyond the scope of this article.

5.2 Regularisation of the conformal fields

The metric $\grave{\boldsymbol{g}}$ shares with $\boldsymbol{g}$ the property of being singular at the asymptote $\mathcal{H}$. In order to deal with this difficulty we follow the spirit of the analysis of [12], Section 6, and consider a frame $\{\boldsymbol{c_a}\}$, $\boldsymbol{a}=\mathbf{0},\mathbf{1},+,-$, with

$$\boldsymbol{c_0}\equiv\boldsymbol{\partial}_{\grave{\tau}}, \qquad \boldsymbol{c_1}\equiv\grave{\rho}\boldsymbol{\partial}_{\grave{\rho}}, \qquad \boldsymbol{c_A}\equiv\boldsymbol{\partial}_A=(\boldsymbol{\partial}_+,\boldsymbol{\partial}_-).$$

where $(\boldsymbol{\partial}_+,\boldsymbol{\partial}_-)$ is a complex null frame on $\mathbb{S}^2$. The associated coframe $\{\boldsymbol{\alpha^a}\}$ is given by

$$\boldsymbol{\alpha^0}\equiv\mathbf{d}\grave{\tau}, \qquad \boldsymbol{\alpha^1}\equiv\frac{1}{\grave{\rho}}\mathbf{d}\grave{\rho}, \qquad \boldsymbol{\alpha^A}=\mathbf{d}\theta^A=(\boldsymbol{\omega}^+,\boldsymbol{\omega}^-).$$

In this frame, the standard round metric of $\mathbb{S}^2$ is given by

$$\boldsymbol{\sigma}=2\left(\boldsymbol{\omega}^+\otimes\boldsymbol{\omega}^-+\boldsymbol{\omega}^-\otimes\boldsymbol{\omega}^+\right)$$

Clearly, $\langle\boldsymbol{\alpha^a},\boldsymbol{c_b}\rangle=\delta_{\boldsymbol{b}}{}^{\boldsymbol{a}}$. However, notice that $\{\boldsymbol{c_a}\}$ is not $\grave{\boldsymbol{g}}$-orthogonal. In the following we ignore the complications arising from the fact that there is no globally defined basis over $\mathbb{S}^2$ as there are standard methods to deal with them —see e.g. [11, 1, 16]. The components $\grave{g}_{\boldsymbol{ab}} \equiv \grave{\boldsymbol{g}}(\boldsymbol{c_a}, \boldsymbol{c_b})$ of $\grave{\boldsymbol{g}}$ with respect to the frame $\{\boldsymbol{c_a}\}$ are given by

$$\grave{g}_{\boldsymbol{ab}} = \bar{\eta}_{\boldsymbol{ab}} + (1-\grave{\tau}^2)\breve{q}_{\boldsymbol{ab}},$$

where

$$(\bar{\eta}_{\boldsymbol{ab}}) = \begin{pmatrix} 1 & \grave{\tau} & 0 & 0 \\ \grave{\tau} & -(1-\grave{\tau}^2) & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & 0 \end{pmatrix}, \qquad \breve{q}_{\boldsymbol{ab}} = o(\grave{\rho}).$$

Similarly, defining $\grave{g}^{\boldsymbol{ab}} \equiv \grave{\boldsymbol{g}}^\sharp(\boldsymbol{\alpha^a}, \boldsymbol{\alpha^b})$ one finds that

$$\grave{g}^{\boldsymbol{ab}} = \bar{\eta}^{\boldsymbol{ab}} + Q^{\boldsymbol{ab}},$$

where

$$(\bar{\eta}^{\boldsymbol{ab}}) = \begin{pmatrix} (1-\grave{\tau}^2) & \grave{\tau} & 0 & 0 \\ \grave{\tau} & -1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix}, \qquad Q^{\boldsymbol{ab}} = o(\grave{\rho}^3).$$

In the following let $\grave{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} \equiv \langle \boldsymbol{\omega^c}, \grave{\nabla}_{\boldsymbol{a}}\boldsymbol{e_c}\rangle$ denote the connection coefficients of the Levi-Civita connection of the metric $\grave{\boldsymbol{g}}$ with respect to the frame $\{\boldsymbol{c_a}\}$, and $\grave{L}_{\boldsymbol{ab}}$ the components of the associated Schouten tensor. A detailed computation leads to the following:

Lemma 4. *The fields $\grave{g}_{\boldsymbol{ab}}$, $\grave{g}^{\boldsymbol{ab}}$, $\grave{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$ and $\grave{L}_{\boldsymbol{ab}}$ extend smoothly to $\mathcal{H}$.*

The details of the proof of the above result can be found in Appendix B.

5.3 Analysis of the $\grave{\boldsymbol{g}}$-conformal geodesic equations

In the following, we consider the conformal geodesic equations for the metric $\grave{\boldsymbol{g}}$ in terms of the basis $\{\boldsymbol{c_a}\}$ and study their solution in a neighbourhood of the asymptote $\mathcal{H}$ with the aim of establishing the existence of a conformal Gaussian system.

5.3.1 The equations

The conformal geodesic for the metric $\grave{\boldsymbol{g}}$ is given by a spacetime curve $\big(x^\mu(\tau)\big) = \big(\grave{\tau}(\tau), \grave{\rho}(\tau), \theta^A(\tau)\big)$ with tangent vector $\boldsymbol{v}(\tau)$ and a 1-form $\grave{\boldsymbol{\beta}}(\tau)$ along the curve such that:

$$\dot{\boldsymbol{x}} = \boldsymbol{v}, \tag{37a}$$

$$\grave{\nabla}_{\boldsymbol{v}}\boldsymbol{v} = -2\langle\grave{\boldsymbol{\beta}}, \boldsymbol{v}\rangle\boldsymbol{v} + \grave{\boldsymbol{g}}(\boldsymbol{v},\boldsymbol{v})\grave{\boldsymbol{\beta}}^\sharp, \tag{37b}$$

$$\grave{\nabla}_{\boldsymbol{v}}\grave{\boldsymbol{\beta}} = \langle\grave{\boldsymbol{\beta}}, \boldsymbol{v}\rangle\grave{\boldsymbol{\beta}} - \frac{1}{2}\grave{\boldsymbol{g}}(\grave{\boldsymbol{\beta}},\grave{\boldsymbol{\beta}})\boldsymbol{v}^\flat + \grave{L}(\boldsymbol{v},\cdot). \tag{37c}$$

Equation (37a) is just the definition of the tangent vector. In order to analyse these equations consider the expansions

$$\begin{aligned} &\boldsymbol{v} = v^{\boldsymbol{a}}\boldsymbol{c_a}, \\ &\grave{\boldsymbol{\beta}} = \grave{\beta}_{\boldsymbol{a}}\boldsymbol{\alpha^a}, \\ &\grave{\boldsymbol{g}} = \grave{g}_{\boldsymbol{ab}}\boldsymbol{\alpha^a}\otimes\boldsymbol{\alpha^b}, \qquad \grave{\boldsymbol{g}}^\sharp = \grave{g}^{\boldsymbol{ab}}\boldsymbol{c_a}\otimes\boldsymbol{c_b}, \\ &\grave{L} = \grave{L}_{\boldsymbol{ab}}\boldsymbol{\alpha^a}\otimes\boldsymbol{\alpha^b}. \end{aligned}$$

Using the above expansions equation (37a) gives the components

$$\frac{\mathrm{d}\grave{\tau}}{\mathrm{d}\tau} = v^{\boldsymbol{0}}, \qquad \frac{\mathrm{d}\grave{\rho}}{\mathrm{d}\tau} = \grave{\rho}v^{\boldsymbol{1}}, \qquad \frac{\mathrm{d}\theta^A}{\mathrm{d}\tau} = v^A. \tag{38}$$

From equations (37b) and (37c) one gets

$$\frac{\mathrm{d}v^{\boldsymbol{a}}}{\mathrm{d}\tau} + \grave{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{a}}{}_{\boldsymbol{c}}v^{\boldsymbol{b}}v^{\boldsymbol{c}} = -2\grave{\beta}_{\boldsymbol{d}}v^{\boldsymbol{d}}v^{\boldsymbol{a}} + \grave{g}_{\boldsymbol{bc}}v^{\boldsymbol{b}}v^{\boldsymbol{c}}\grave{g}^{\boldsymbol{ad}}\grave{\beta}_{\boldsymbol{d}}, \tag{39a}$$

$$\frac{\mathrm{d}\grave{\beta}_{\boldsymbol{a}}}{\mathrm{d}\tau} - \grave{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{a}}v^{\boldsymbol{b}}\grave{\beta}_{\boldsymbol{c}} = \grave{\beta}_{\boldsymbol{c}}v^{\boldsymbol{c}}\grave{\beta}_{\boldsymbol{a}} + \frac{1}{2}\grave{g}^{\boldsymbol{cd}}\grave{\beta}_{\boldsymbol{c}}\grave{\beta}_{\boldsymbol{d}}\grave{g}_{\boldsymbol{ab}}\boldsymbol{v^b} + \grave{L}_{\boldsymbol{ba}}v^{\boldsymbol{b}}. \tag{39b}$$

5.3.2 The initial data

Following the discussion of the Minkowski spacetime, in the sequel the initial data of the curve $\big(x^\mu(\tau)\big)$ is chosen so that $\dot{\boldsymbol{x}}$ is $\grave{\boldsymbol{g}}$-normalised and orthogonal to the hypersurface

$$\mathcal{S}_\star \equiv \{\grave{\tau} = 0\}.$$

The normal covector to this hypersurface is given by $\mathbf{d}\grave{\tau}$ from where it follows that the unit normal vector is given by

$$\begin{aligned}\grave{\boldsymbol{n}} &\equiv \frac{1}{\sqrt{\grave{\boldsymbol{g}}(\mathbf{d}\grave{\tau},\mathbf{d}\grave{\tau})}}\grave{\boldsymbol{g}}^\sharp(\mathbf{d}\grave{\tau},\cdot),\\ &= \frac{1}{\sqrt{(1-\grave{\tau}^2)(1-\breve{\boldsymbol{q}}^\sharp(\mathbf{d}\grave{\tau},\mathbf{d}\grave{\tau}))}}\Big((1-\grave{\tau}^2)\boldsymbol{\partial}_{\grave{\tau}} + \grave{\rho}\grave{\tau}\boldsymbol{\partial}_{\grave{\rho}} - (1-\grave{\tau}^2)\breve{\boldsymbol{q}}^\sharp(\mathbf{d}\grave{\tau},\cdot)\Big).\end{aligned}$$

Thus, on $\mathcal{S}_\star$ one has that

$$\grave{\boldsymbol{n}}_\star = \frac{1}{\sqrt{1-\breve{\boldsymbol{q}}^\sharp(\mathbf{d}\grave{\tau},\mathbf{d}\grave{\tau})}}\Big(\boldsymbol{\partial}_{\grave{\tau}} - \breve{\boldsymbol{q}}^\sharp_\star(\mathbf{d}\grave{\tau},\cdot)\Big),$$

from where one can readily compute

$$v_\star^{\boldsymbol{0}} \equiv \langle \boldsymbol{\alpha}^{\boldsymbol{0}}, \grave{\boldsymbol{n}}_\star\rangle, \qquad v_\star^{\boldsymbol{1}} \equiv \langle \boldsymbol{\alpha}^{\boldsymbol{1}}, \grave{\boldsymbol{n}}_\star\rangle, \qquad v_\star^{\boldsymbol{A}} \equiv \langle \boldsymbol{\alpha}^{\boldsymbol{A}}, \grave{\boldsymbol{n}}_\star\rangle. \tag{40}$$

The prescription of initial data for the 1-form $\tilde{\boldsymbol{\beta}}$ in the $\tilde{\boldsymbol{g}}$-conformal geodesic equations is that

$$\tilde{\boldsymbol{\beta}}_\star = \Omega_\star^{-1}\mathbf{d}\Omega_\star,$$

where $\Omega_\star$ is a 3-dimensional conformal factor satisfying the *point compactification* conditions

$$\Omega_\star(i) = 0, \qquad \mathbf{d}\Omega_\star(i) = 0, \qquad \operatorname{Hess}\Omega_\star(i) \neq 0.$$

The restriction of the conformal factor $\Omega = \zeta$ to the hypersurface $\mathcal{S}_\star$ does not satisfy these conditions. However, the choice $\Omega_\star = \zeta^2$ does. In this case one has that

$$\tilde{\boldsymbol{\beta}}_\star = \frac{2}{\zeta}\mathbf{d}\zeta. \tag{41}$$

Given that $\grave{\boldsymbol{g}} = \zeta^4\tilde{\boldsymbol{g}}$, one has that

$$\begin{aligned}\grave{\boldsymbol{\beta}}_\star &= \tilde{\boldsymbol{\beta}} - \frac{1}{\zeta}\mathbf{d}\zeta,\\ &= \frac{1}{\zeta}\mathbf{d}\zeta.\end{aligned}$$

Remark 19. Observe that $\grave{\boldsymbol{\beta}}_\star$ is singular at $\zeta = 0$. However, defining $\grave{\beta}_{\boldsymbol{i}\star} \equiv \langle\grave{\boldsymbol{\beta}}_\star, \boldsymbol{c}_{\boldsymbol{i}}\rangle$ with $\boldsymbol{i} = \boldsymbol{1}, \boldsymbol{A}$, one readily finds that

$$\grave{\beta}_{\boldsymbol{i}\star} = \delta_{\boldsymbol{i}}{}^{\boldsymbol{1}}, \tag{42}$$

so that, in fact, $\grave{\boldsymbol{\beta}}_\star = \boldsymbol{\alpha}^{\boldsymbol{1}}$. Thus, the components of $\grave{\boldsymbol{\beta}}_\star$ as measured by the frame $\{\boldsymbol{c}_{\boldsymbol{i}}\}$ are regular.

From the above equations (40) and (42), it follows that the complete initial data for the equations (38) and (39a)-(39b) takes the form

$$(x_\star^\mu) = \big(0, \grave{\rho}_\star, \theta_\star^A\big), \tag{43a}$$

$$v_\star^{\boldsymbol{0}} = \sqrt{1-\breve{\boldsymbol{q}}^\sharp_\star(\mathbf{d}\grave{\tau},\mathbf{d}\grave{\tau})}, \tag{43b}$$

$$v_\star^{\boldsymbol{1}} = -\frac{\breve{\boldsymbol{q}}^\sharp_\star(\mathbf{d}\grave{\tau},\mathbf{d}\grave{\rho})}{\grave{\rho}\sqrt{1-\breve{\boldsymbol{q}}^\sharp_\star(\mathbf{d}\grave{\tau},\mathbf{d}\grave{\tau})}}, \tag{43c}$$

$$v_\star^A = -\frac{\breve{\boldsymbol{q}}_\star^\sharp(\mathbf{d}\grave{\tau}, \mathbf{d}\grave{\rho})}{\grave{\rho}\sqrt{1-\breve{\boldsymbol{q}}_\star^\sharp(\mathbf{d}\grave{\tau}, \mathbf{d}\theta^A)}}, \tag{43d}$$

$$\grave{\beta}_{\mathbf{0}} = 0, \qquad \grave{\beta}_{\mathbf{1}} = 1, \qquad \grave{\beta}_A = 0. \tag{43e}$$

As $\breve{\boldsymbol{q}}_\star = o(\grave{\rho})$, it follows then that

$$v_\star^{\mathbf{0}} \to 1, \qquad v_\star^{\mathbf{1}},\ v_\star^A \to 0, \qquad \text{as} \quad \grave{\rho} \to 0.$$

Accordingly, the data described by conditions (43a)-(43e) are regular at $\mathcal{H}$.

5.3.3 The solution to the conformal geodesic equations in a neighbourhood of $\mathcal{H}$

In this section, we make use of the information gathered in the previous sections to show the existence, in a neighbourhood of the asymptote $\mathcal{H}$, of a congruence of non-intersecting conformal geodesics extending smoothly beyond the null infinity. This congruence gives rise, in a natural way, to a conformal Gaussian gauge system. The argument in this section makes use of the theory of perturbations of ordinary differential equations.

The first step in the analysis is the observation that the system (38) and (39a)-(39b) with initial data given by (43a)-(43e) can be solved exactly on $\mathcal{H}$.

Lemma 5. *The unique solution to the conformal geodesic equations* (38) *and* (39a)-(39b) *with initial data given by* (43a)-(43e) *for* $\grave{\rho} = 0$ *is given by*

$$\begin{aligned}
&\big(x^\mu(\tau)\big) = (\tau, 0, \theta^A),\\
&v^{\mathbf{0}}(\tau) = 1, \qquad v^{\mathbf{1}}(\tau) = 0, \qquad v^A(\tau) = 0,\\
&\grave{\beta}_{\mathbf{0}}(\tau) = 0, \qquad \grave{\beta}_{\mathbf{1}}(\tau) = 1, \qquad \grave{\beta}_A(\tau) = 0.
\end{aligned}$$

This solution is smooth for all $\tau \in \mathbb{R}$. *In particular, it extends smoothly beyond the interval* $[-1, 1]$.

Proof. The proof of the lemma follows from the observation that on $\mathcal{H}$ (i.e. $\grave{\rho}$) the metric $\grave{\boldsymbol{g}}$ coincides with the metric $\bar{\boldsymbol{\eta}}$. As discussed in Section 3.2, Lemma 1, the solution to the conformal geodesic equations with the given data is given by

$$\big(x^\mu(\tau)\big) = (\tau, \rho_\star, \theta_\star^A), \qquad \dot{\boldsymbol{x}} = \boldsymbol{\partial}_\tau, \qquad \bar{\boldsymbol{\beta}} = \frac{\mathbf{d}\rho}{\rho_\star}.$$

Contracting with the frame $\{\boldsymbol{c_a}\}$ and coframe $\{\boldsymbol{\alpha^a}\}$ as necessary one obtains the result. □

From the above result, making use of the regularity of the fields appearing in the conformal geodesic equations (38) and (39a)-(39b) and the initial conditions (43a)-(43e) in a neighbourhood of $\mathcal{H}$ one obtains the following:

Lemma 6. *There exists* $\grave{\rho}_\bullet > 0$ *such that if* $\grave{\rho}_\star \in [0, \grave{\rho}_\bullet)$ *then the system* (38) *and* (39a)-(39b) *with initial conditions* (43a)-(43e) *has a unique smooth solution with existence interval extending beyond* $[-1, 1]$ *—e.g.* $\tau \in [-\frac{3}{2}, \frac{3}{2}]$.

Proof. The result follows from the stability theory of ordinary differential equations. In particular, the regularity of the components $\grave{g}_{\boldsymbol{ab}}$, $\grave{g}^{\boldsymbol{ab}}$ and $\grave{L}_{\boldsymbol{ab}}$ in a neighbourhood of the asymptote $\mathcal{H}$ and up to and beyond the sets given by the conditions $\tau = \pm 1$ allows making use of Theorem 2 in Appendix A. □

Remark 20. By varying the starting point $p \in \mathcal{S}_\star$ subject to the condition $\grave{\rho}_\star(p) \in [0, \grave{\rho}_\bullet)$ ensures that a neighbourhood of $\mathcal{H}$ can be covered by conformal geodesics. In the next subsection it will be shown that, possibly by reducing $\grave{\rho}_\bullet$ this congruence is non-intersecting.

Remark 21. The curves of the congruence extend beyond $\mathscr{I}^\pm$ as their existence interval is $[-\frac{3}{2}, \frac{3}{2}]$.

5.4 The construction of the conformal Gaussian system

In this section, we discuss how the congruence of conformal geodesics obtained in Lemma 5 implies the existence of a conformal Gaussian system in a neighbourhood of the asymptote $\mathcal{H}$.

5.4.1 General considerations

Given a point $p \in \mathcal{S}_\star$ with coordinates $\underline{x}(p) = (\grave{\rho}(p), \grave{\theta}^A(p))$ such that $\grave{\rho}(p) < \grave{\rho}_\bullet$, these coordinates are propagated off $\mathcal{S}_\star$ by requiring them to be constant along the *unique conformal geodesic* with data as given in Lemma 6 passing through p. In order to differentiate between the propagated coordinates and those on $\mathcal{S}_\star$ we denote the former by (ρ, θ^A). Points along the conformal geodesic passing through p are labelled using the parameter τ of the curve. In this way, by varying the point p and as long as the congruence of conformal geodesics is non-intersecting, one obtains *conformal Gaussian coordinates* $\bar{x} = (\tau, \rho, \theta^A)$.

Remark 22. It should be stressed that given a point q in the neighbourhood of $\mathcal{H}$ covered by the curves of Lemma 6 and described, respectively, by coordinates $\big(\grave{\tau}(q), \grave{\rho}(q), \grave{\theta}^A(q)\big)$ and $\big(\tau(q), \rho(q), \theta^A(q)\big)$, one has, in general, that $\grave{\rho}(q) \neq \rho(q)$, $\grave{\theta}^A(q) \neq \theta^A(q)$.

5.4.2 Analysis of the Jacobian

In order to ensure that the collection (τ, ρ, θ^A) gives rise to a well defined coordinate system in a neighbourhood of $\mathcal{H}$ we need to consider the Jacobian determinant of the change of coordinates

$$(\tau, \rho, \theta^A) \mapsto (\grave{\tau}, \grave{\rho}, \grave{\theta}^A).$$

In order to ease the presentation in the following, we restrict the discussion to the transformation between non-angular coordinates $(\grave{\tau}, \grave{\rho}) \mapsto (\tau, \rho)$. The full analysis follows in a similar manner at the expense of lengthier computations. Writing

$$\grave{\tau} = \grave{\tau}(\tau, \rho), \qquad \grave{\rho} = \grave{\rho}(\tau, \rho),$$

the associated Jacobian determinant is given by

$$\frac{\partial(\grave{\tau}, \grave{\rho})}{\partial(\tau, \rho)} = \begin{vmatrix} \dfrac{\partial \grave{\tau}}{\partial \tau} & \dfrac{\partial \grave{\tau}}{\partial \rho} \\ \dfrac{\partial \grave{\rho}}{\partial \tau} & \dfrac{\partial \grave{\rho}}{\partial \rho} \end{vmatrix}$$

Now, from the solution to the conformal geodesic equations (4a)-(4b) on $\mathcal{H}$ as given by Lemma 5 it follows that

$$\left.\frac{\partial \grave{\tau}}{\partial \tau}\right|_{\mathcal{H}} = 1, \qquad \left.\frac{\partial \grave{\tau}}{\partial \rho}\right|_{\mathcal{H}} = 0,$$

so that, in fact, one has

$$\left.\frac{\partial(\grave{\tau}, \grave{\rho})}{\partial(\tau, \rho)}\right|_{\mathcal{H}} = \left.\frac{\partial \grave{\rho}}{\partial \rho}\right|_{\mathcal{H}}.$$

Differentiating the second equation in (38) one obtains

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial \grave{\rho}}{\partial \rho}\right) = v^{\mathbf{1}} \frac{\partial \grave{\rho}}{\partial \rho} + \grave{\rho} \frac{\partial v^{\mathbf{1}}}{\partial \rho},$$

from where it follows

$$\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial \grave{\rho}}{\partial \rho}\right)\right|_{\mathcal{H}} = 0.$$

In other words, the partial derivative $\partial \grave{\rho}/\partial \rho$ is constant on $\mathcal{H}$. To evaluate the constant it is observed that by construction $\grave{\rho} = \rho$ on $\mathcal{S}_\star$. Accordingly, one concludes that

$$\left.\frac{\partial \grave{\rho}}{\partial \rho}\right|_{\mathcal{H} \cap \mathcal{S}_\star} = 1.$$

It then follows from the above argument that

$$\left.\frac{\partial(\grave{\tau},\grave{\rho})}{\partial(\tau,\rho)}\right|_{\mathcal{H}} = 1.$$

By continuity, it follows that the Jacobian determinant is non-zero in a neighbourhood of $\mathcal{H}$. In order to ensure that this condition holds it may be necessary to reduce the constant $\grave{\rho}_\bullet$ in Lemma 6.

Remark 23. It follows from the fact that the Jacobian of the transformation between the coordinates $(\grave{\tau},\grave{\rho},\grave{\theta}^A)$ and (τ,ρ,θ^A) is non-zero that the congruence of conformal geodesics in a neighbourhood of $\mathcal{H}$ is non-intersecting as the intersection of curves would indicate a singularity in the coordinate system given by (τ,ρ,θ^A).

5.4.3 The conformal factor associated to the congruence

The existence of a congruence of conformal geodesics in a neighbourhood of the asymptote $\mathcal{H}$ ensures that the family of *conformal factors* given by Proposition 1 is well defined. This, in turn, completes the construction of a conformal Gaussian system in a neighbourhood of $\mathcal{H}$.

Consistent with the discussion of the initial data for the congruence of conformal geodesics in Subsection 5.3.2 one has that

$$\Theta_\star = \Omega_\star = \rho.$$

Moreover, from (41) it then follows that

$$\dot{\Theta}_\star = 0.$$

Finally, from the relations in (6) of Proposition 1 one has that

$$\ddot{\Theta}_\star = \frac{2}{\rho}\tilde{\boldsymbol{h}}^\sharp(\mathbf{d}\rho,\mathbf{d}\rho).$$

Let $\bar{\boldsymbol{h}}$ denote the pull-back to $\mathcal{S}_\star$ of the the metric $\grave{\boldsymbol{g}}$ as given in equation (36). One then has that

$$\bar{\boldsymbol{h}} = -\frac{1}{\rho^2}\mathbf{d}\rho\otimes\mathbf{d}\rho - \boldsymbol{\sigma} + \breve{\boldsymbol{k}}, \qquad \bar{\boldsymbol{h}} = -\rho^2\boldsymbol{\partial}_\rho\otimes\boldsymbol{\partial}_\rho - \boldsymbol{\sigma}^\sharp + \breve{\boldsymbol{k}}^\sharp,$$

where $\breve{\boldsymbol{k}}$ is the pull-back to $\mathcal{S}_\star$ of $\breve{\boldsymbol{q}}$. It follows then that

$$\ddot{\Theta}_\star = -2\rho + \frac{2}{\rho}\breve{\boldsymbol{k}}^\sharp(\mathbf{d}\rho,\mathbf{d}\rho).$$

Observe that, in particular, one has

$$\frac{2}{\rho}\breve{\boldsymbol{k}}^\sharp(\mathbf{d}\rho,\mathbf{d}\rho)\to 0, \qquad \text{as} \qquad \rho\to 0.$$

Accordingly, in the neighbourhood of the asymptote $\mathcal{H}$ where the congruence of conformal geodesics is well defined one has the conformal factor

$$\Theta = \rho\left(1-\left(1-\frac{\breve{\boldsymbol{k}}^\sharp(\mathbf{d}\rho,\mathbf{d}\rho)}{\rho^2}\right)\tau^2\right).$$

In particular, it is noticed that:

$$\pm\frac{\rho}{\sqrt{\rho^2-\breve{\boldsymbol{k}}^\sharp(\mathbf{d}\rho,\mathbf{d}\rho)}}\to\pm 1, \qquad \text{as} \qquad \rho\to 0.$$

Thus, one defines, in analogy to the case of the Minkowski spacetime in the F-gauge (see Section 3.2) the sets

$$\bar{\mathcal{M}}_{\rho_\bullet} \equiv \left\{ p \in \mathbb{R}^4 \;\middle|\; -\frac{\rho}{\sqrt{\rho^2 - \breve{\boldsymbol{k}}^\sharp(\mathbf{d}\rho,\mathbf{d}\rho)}} \leq \tau(p) \leq \frac{\rho}{\sqrt{\rho^2 - \breve{\boldsymbol{k}}^\sharp(\mathbf{d}\rho,\mathbf{d}\rho)}},\; 0 \leq \rho(p) < \rho_\bullet \right\},$$

and

$$I \equiv \big\{ p \in \bar{\mathcal{M}} \mid \; |\tau(p)| < 1,\; \rho(p) = 0 \big\}$$

$$I^+ \equiv \big\{ p \in \bar{\mathcal{M}} \mid \tau(p) = 1,\; \rho(p) = 0 \big\}, \qquad I^- \equiv \big\{ p \in \bar{\mathcal{M}} \mid \tau(p) = -1,\; \rho(p) = 0 \big\}.$$

Moreover, future and past null infinity are given by:

$$\mathscr{I}^\pm \equiv \left\{ p \in \bar{\mathcal{M}} \mid \tau(p) = \pm \frac{\rho}{\sqrt{\rho^2 - \breve{\boldsymbol{k}}^\sharp(\mathbf{d}\rho,\mathbf{d}\rho)}},\; 0 \leq \rho(p) < \rho_\bullet \right\}.$$

Remark 24. The set I (the cylinder at spatial infinity) coincides with the asymptote $\mathcal{H}$.

Remark 25. Observe that in contrast to the representation of the Minkowski in Section 3.2, the representation obtained in this section *is not horizontal* in the sense that the location of null infinity is not given by the condition $\tau = \pm 1$. A horizontal representation can be obtained by considering a more general initial conformal factor of the form $\Theta_\star = \varkappa\rho$, $\varkappa|_{\rho=0} = 1$, with $\varkappa$ suitably chosen.

5.4.4 Main statement

We summarise the discussion of the previous subsections in the following theorem which is a more detailed version of the statement presented in the introductory section:

Theorem 1. *Let $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ denote a spacetime satisfying Definition 2 of an asymptotically Minkowskian spacetime at spatial infinity with asymptote $\mathcal{H}$ which, in addition, satisfies Assumption 1. Then there exists a neighbourhood $\mathcal{M}_{\rho_\bullet}$ of $\mathcal{H}$ which can be covered by a conformal Gaussian coordinate system $(x^\mu) \equiv (\tau, \rho, \theta^A)$. The domain $\mathcal{M}_{\rho_\bullet}$ includes portions of future and past null infinity which meet with the asymptote $\mathcal{H}$. In this gauge the asymptote $\mathcal{H}$ coincides with the $1+2$-dimensional Einstein static cylinder (Friedrich's cylinder).*

A The basic stability theorem for ordinary differential equations depending on a parameter

In the following let

$$\mathbf{X}' = \mathbf{F}(t, \mathbf{X}, \lambda), \qquad \mathbf{X}(0) = \mathbf{X}_\star, \tag{44}$$

denote an initial value problem for a N-dimensional vector-value ordinary differential equation depending on a parameter λ. One has the following result —see [19], Theorem 2.1 in page 94 and Corollary 4.1 in page 101.

Theorem 2. *Let $\mathbf{F}(t, \mathbf{X}, \lambda)$ be continuous on an open set $\mathcal{E} \subset \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$ consisting of points $(t, \mathbf{X}, \lambda)$ with the property that for every $(t_\star, \mathbf{X}_\star, \lambda) \in \mathcal{E}$, the initial value problem (44) with λ fixed has a unique solution $\mathbf{X}(t) = \mathbf{X}(t, t_\star, \mathbf{X}_\star, \lambda)$. Let $\omega_- < t < \omega_+$ be the maximal interval of existence of $\mathbf{X}(t, t_\star, \mathbf{X}_\star, \lambda)$. Then $\omega_+ = \omega_+(t_\star, \mathbf{X}_\star, \lambda)$ (respectively $\omega_- = \omega_-(t_\star, \mathbf{X}_\star, \lambda)$) is a lower (respectively upper) semicontinuous function[1] of $(t_\star, \mathbf{X}_\star, \lambda) \in \mathcal{E}$ and $\mathbf{X}(t, t_\star, \mathbf{X}_\star, \lambda)$ is continuous on the set*

$$\{\omega_- < t < \omega_+,\; (t_\star, \mathbf{X}_\star, \lambda) \in \mathcal{E}\}.$$

Moreover, if $\mathbf{F}(t, \mathbf{X}, \lambda)$ is of class C^m, $m \geq 1$ on $\mathcal{E}$, then $\mathbf{X}(t, t_\star, \mathbf{X}_\star, \lambda)$ is of class C^m on its domain of existence.

[1]The lower semicontinuity of ω_+ at $(t_\bullet, \mathbf{X}_\bullet, \lambda_\bullet)$ means that if $t_\sharp < \omega_+(t_\bullet, \mathbf{X}_\bullet, \lambda_\bullet)$ then $\omega_+(t, \mathbf{X}, \lambda) \geq t_\sharp$ for all $(t, \mathbf{X}, \lambda)$ near $(t_\bullet, \mathbf{X}_\bullet, \lambda_\bullet)$. Upper semicontinuity is similarly defined.

So, it follows from the above theorem that if $\mathbf{F}$ satisfies the conditions of the theorem above, and the problem

$$\mathbf{X}' = \mathbf{F}(t, \mathbf{X}, 0), \qquad \mathbf{X}(0) = \mathbf{X}_\star,$$

has a solution with existence interval $t \in (\omega_-, \omega_+)$, then any other solution to (44) with λ sufficiently close to 0 will have, at least the same existence interval. In the above statement the functions ω_+ and ω_- can take the values ∞ and $-\infty$.

B Details of the proof of Lemma 4

In this section, we show that the fields $\grave{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$ and $\grave{L}_{\boldsymbol{ab}}$ extend smoothly to $\mathcal{H}$.

B.0.1 Connection coefficients

The connection coefficients of the Levi-Civita connection $\grave{\nabla}$ of the metric $\grave{\boldsymbol{g}}$ with respect to the frame $\{\boldsymbol{c}_{\boldsymbol{a}}\}$ are defined by the relation

$$\grave{\nabla}_{\boldsymbol{a}} \boldsymbol{c}_{\boldsymbol{b}} = \grave{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} \boldsymbol{c}_{\boldsymbol{c}}.$$

It can be readily verified that $[\boldsymbol{c}_{\boldsymbol{a}}, \boldsymbol{c}_{\boldsymbol{b}}] = 0$. Thus, it follows from the Kulkarni formula that

$$\grave{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} = \frac{1}{2}\grave{g}^{\boldsymbol{bd}}\big(\boldsymbol{c}_{\boldsymbol{c}}(\grave{g}_{\boldsymbol{ad}}) + \boldsymbol{c}_{\boldsymbol{a}}(\grave{g}_{\boldsymbol{dc}}) - \boldsymbol{c}_{\boldsymbol{b}}(\grave{g}_{\boldsymbol{ac}})\big). \tag{45}$$

In the following it will be shown that the connection coefficients $\grave{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$ are regular at $\grave{\rho} = 0$. The analysis of the connection coefficients requires further specification of the coordinates. In the following we will make use of Gaussian coordinates of the asymptote $\mathcal{H}$ —that is, the coordinates $(\grave{\tau}, \theta^A)$ on $\mathcal{H}$ are propagated on the bulk of the spacetime using the relation:

$$\grave{\nabla}_{\boldsymbol{\partial}_{\grave{\rho}}} \boldsymbol{\partial}_{\boldsymbol{i}} = 0, \qquad \boldsymbol{i} = \mathbf{0}, \mathbf{2}, \mathbf{3}. \tag{46}$$

Using this condition one can readily verify that

$$\grave{\Gamma}_{\mathbf{1}}{}^{\boldsymbol{a}}{}_{\boldsymbol{i}} = 0.$$

The other components of the connection coefficients not determined by the gauge condition (46) can be directly computed using Kulkarni's formula (45). One finds that

$$\begin{aligned}
&\grave{\Gamma}_{\mathbf{0}}{}^{\mathbf{0}}{}_{\mathbf{0}} = O(\grave{\rho})\\
&\grave{\Gamma}_{\mathbf{0}}{}^{\mathbf{1}}{}_{\mathbf{0}} = -1 + O(\grave{\rho}^2)\\
&\grave{\Gamma}_{\mathbf{0}}{}^{\boldsymbol{A}}{}_{\mathbf{0}} = o(\grave{\rho}^2)\\
&\grave{\Gamma}_{\mathbf{1}}{}^{\mathbf{0}}{}_{\mathbf{1}} = -\grave{\tau}(1 - \grave{\tau}^2) + O(\grave{\rho})\\
&\grave{\Gamma}_{\mathbf{1}}{}^{\mathbf{1}}{}_{\mathbf{1}} = -\grave{\tau}^2 + O(\grave{\rho})\\
&\grave{\Gamma}_{\mathbf{1}}{}^{\boldsymbol{A}}{}_{\mathbf{1}} = O(\grave{\rho}^2)\\
&\grave{\Gamma}_{\mathbf{0}}{}^{\mathbf{0}}{}_{\boldsymbol{A}} = O(\grave{\rho}^3)\\
&\grave{\Gamma}_{\mathbf{0}}{}^{\mathbf{1}}{}_{\boldsymbol{A}} = O(\grave{\rho}^2)\\
&\grave{\Gamma}_{\boldsymbol{A}}{}^{\mathbf{1}}{}_{\boldsymbol{B}} = O(\grave{\rho})\\
&\grave{\Gamma}_{\mathbf{0}}{}^{\boldsymbol{A}}{}_{\boldsymbol{B}} = O(\grave{\rho})\\
&\grave{\Gamma}_{\boldsymbol{A}}{}^{\mathbf{0}}{}_{\boldsymbol{B}} = O(\grave{\rho})\\
&\grave{\Gamma}_{\boldsymbol{A}}{}^{\boldsymbol{A}}{}_{\boldsymbol{B}} = o(\grave{\rho}^3)\\
&\grave{\Gamma}_{\boldsymbol{B}}{}^{\boldsymbol{A}}{}_{\boldsymbol{B}} = o(\grave{\rho}^4)\\
&\grave{\Gamma}_{\boldsymbol{A}}{}^{\boldsymbol{A}}{}_{\boldsymbol{A}} = o(\grave{\rho}^4)
\end{aligned}$$

Thus, the connection coefficients $\grave{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$ are regular with respect to the coordinates $(\grave{\tau}, \grave{\rho})$ in a neighbourhood of $\mathcal{H}$. In particular, the components coincide, on $\mathcal{H}$, with those of the Schouten tensor of the Minkowski spacetime in the F-representation.

B.0.2 The components of the Schouten tensor

The Schouten tensors $\tilde{L}_{ab}$ and $\grave{L}_{ab}$ are related by the formula

$$\grave{L}_{ab} - \tilde{L}_{ab} = -\frac{1}{\Theta}\grave{\nabla}_a\grave{\nabla}_b\Theta + \frac{1}{2\Theta^2}\grave{\nabla}_c\Theta\grave{\nabla}^c\Theta\grave{g}_{ab}.$$

Where Θ is the conformal factor relating $\tilde{\boldsymbol{g}}$ and $\grave{\boldsymbol{g}}$. In view of the Einstein field equations we have that $\tilde{L}_{ab} = 0$ so that

$$\grave{L}_{ab} = -\frac{1}{\Theta}\grave{\nabla}_a\grave{\nabla}_b\Theta + \frac{1}{2\Theta^2}\grave{\nabla}_c\Theta\grave{\nabla}^c\Theta\grave{g}_{ab}. \tag{47}$$

Remark 26. The previous expression is singular at $\grave{\rho} = 0$. We need to compute the components $\grave{L}_{\boldsymbol{ab}}$ with respect to the coefficients $\{\boldsymbol{c_a}\}$.

A direct computation shows that

$$\grave{\nabla}_{\boldsymbol{a}}\Theta = \boldsymbol{c_a}\Theta = \begin{cases} -2\grave{\tau}\grave{\rho} & \text{if } \boldsymbol{a} = \boldsymbol{0} \\ \grave{\rho}(1-\grave{\tau}^2) & \text{if } \boldsymbol{a} = \boldsymbol{1} \\ 0 & \text{if } \boldsymbol{a} \neq \boldsymbol{0}, \boldsymbol{1} \end{cases}$$

$$\grave{\nabla}_{\boldsymbol{a}}\grave{\nabla}_{\boldsymbol{b}}\Theta = \boldsymbol{c_a}(\boldsymbol{c_b}(\Theta)) - \grave{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}\grave{\nabla}_{\boldsymbol{c}}\Theta$$

where $\boldsymbol{c_a}(\boldsymbol{c_b}(\Theta))$ is given by

$$\boldsymbol{c_a}(\boldsymbol{c_b}(\Theta)) = \begin{cases} -2\grave{\tau}\grave{\rho} & \text{if } \boldsymbol{a} \neq \boldsymbol{b} \text{ and } \boldsymbol{a} = \boldsymbol{0}, \boldsymbol{1} \\ -2\grave{\rho} & \text{if } \boldsymbol{a} = \boldsymbol{b} = \boldsymbol{0} \\ \grave{\rho}(1-\grave{\tau}^2) & \text{if } \boldsymbol{a} = \boldsymbol{b} = \boldsymbol{1} \\ 0 & \text{if } \boldsymbol{a},\, \boldsymbol{b} \neq \boldsymbol{0}, \boldsymbol{1} \end{cases}$$

Thus, the first term in the right-hand side of (47) is regular. Moreover,

$$\grave{\nabla}_{\boldsymbol{c}}\grave{\rho}\grave{\nabla}^{\boldsymbol{c}}\grave{\rho} = -\grave{\rho}^2(1-\grave{\tau}^2)^2$$

so that the second term in (47) is regular hence, the components $\grave{L}_{\boldsymbol{ab}}$ are regular with respect to the coordinates $(\grave{\tau}, \grave{\rho})$ in a neighbourhood of $\mathcal{H}$.

C Conformal geodesics in the Minkowski spacetime

In this section we show that the vector $\boldsymbol{\partial}_\tau$ in the F-gauge representation of the Minkowski spacetime of Section 3.2 is a conformal geodesic —see Lemma 1. Thus, it should satisfy the equations

$$\bar{\nabla}_{\boldsymbol{\partial}_\tau}\boldsymbol{\partial}_\tau = -2\langle\bar{\boldsymbol{\beta}}, \boldsymbol{\partial}_\tau\rangle\boldsymbol{\partial}_\tau + \bar{\boldsymbol{\eta}}(\boldsymbol{\partial}_\tau, \boldsymbol{\partial}_\tau)\bar{\boldsymbol{\beta}}^\sharp, \tag{48a}$$

$$\bar{\nabla}_{\boldsymbol{\partial}_\tau}\bar{\boldsymbol{\beta}} = \langle\bar{\boldsymbol{\beta}}, \boldsymbol{\partial}_\tau\rangle\bar{\boldsymbol{\beta}} - \frac{1}{2}\bar{\boldsymbol{\eta}}^\sharp(\bar{\boldsymbol{\beta}}, \bar{\boldsymbol{\beta}})(\boldsymbol{\partial}_\tau)^\flat + \bar{\mathbf{L}}(\boldsymbol{\partial}_\tau, \cdot). \tag{48b}$$

In particular, one has that

$$\bar{\boldsymbol{\eta}}(\boldsymbol{\partial}_\tau, \boldsymbol{\partial}_\tau) = 1.$$

In the following, we make use of the expansion

$$\bar{\boldsymbol{\beta}} = \bar{\beta}_0\mathbf{d}\tau + \bar{\beta}_1\mathbf{d}\rho.$$

A computation gives that

$$\bar{\mathbf{L}} = \frac{1}{2}\mathbf{d}\tau\otimes\mathbf{d}\tau + \frac{\tau}{2\rho}(\mathbf{d}\tau\otimes\mathbf{d}\rho + \mathbf{d}\rho\otimes\mathbf{d}\tau) + \frac{\tau^2-1}{2\rho^2}\mathbf{d}\rho\otimes\mathbf{d}\rho + \frac{1}{2}\boldsymbol{\sigma}.$$

So, in particular,

$$\bar{\mathbf{L}}(\boldsymbol{\partial}_\tau, \cdot) = \frac{1}{2}\mathbf{d}\tau + \frac{\tau}{2\rho}\mathbf{d}\rho.$$

One also has that

$$\bar{\nabla}_{\boldsymbol{\partial}_\tau}\mathbf{d}\tau = -\bar{\Gamma}_0{}^0{}_0\mathbf{d}\tau - \bar{\Gamma}_0{}^1{}_0\mathbf{d}\rho,$$
$$\bar{\nabla}_{\boldsymbol{\partial}_\tau}\mathbf{d}\rho = -\bar{\Gamma}_1{}^0{}_1\mathbf{d}\tau - \bar{\Gamma}_1{}^1{}_1\mathbf{d}\rho,$$

so that

$$\bar{\nabla}_{\boldsymbol{\partial}_\tau}\bar{\boldsymbol{\beta}} = (\partial_\tau\bar{\beta}_0 - \bar{\beta}_0\bar{\Gamma}_0{}^0{}_0 - \bar{\beta}_1\bar{\Gamma}_1{}^0{}_1)\mathbf{d}\tau + (\partial_\tau\bar{\beta}_1 - \bar{\beta}_0\bar{\Gamma}_0{}^1{}_0 - \bar{\beta}_1\bar{\Gamma}_1{}^1{}_1)\mathbf{d}\rho,$$

where

$$\bar{\Gamma}_0{}^0{}_0 = \tau, \qquad \bar{\Gamma}_0{}^1{}_0 = -\rho,$$
$$\bar{\Gamma}_1{}^0{}_1 = \frac{\tau}{\rho^2}(\tau^2 - 1), \qquad \bar{\Gamma}_1{}^1{}_1 = \frac{\tau^2 - 1}{\rho}.$$

Substituting all of the above in equation (48a) one concludes that

$$\bar{\beta}_0 = 0, \qquad \bar{\beta}_1 = \frac{1}{\rho},$$

so that the vector $\boldsymbol{\partial}_\tau$ is tangent to timelike conformal geodesics with

$$\bar{\boldsymbol{\beta}} = \frac{1}{\rho}\mathbf{d}\rho.$$

Equation (48b) can be shown to be satisfied identically.

Finally, let us find the connection to physical spacetime. Recalling that

$$\bar{\boldsymbol{\eta}} = \Theta^2\tilde{\boldsymbol{\eta}}, \qquad \Theta = \rho(1 - \tau^2),$$

one has that

$$\tilde{\boldsymbol{\beta}} = \bar{\boldsymbol{\beta}} + \mathbf{d}\ln\Theta,$$

so that

$$\tilde{\boldsymbol{\beta}} = \frac{2}{\rho}\mathbf{d}\rho - \frac{2\tau}{1 - \tau^2}\mathbf{d}\tau.$$

To transform to the physical coordinates it is observed that

$$\tau = \frac{t}{r}, \qquad \rho = \frac{r}{r^2 - t^2},$$

so that

$$\mathbf{d}\tau = \frac{1}{r}\mathbf{d}t - \frac{t}{r^2}\mathbf{d}r, \qquad \mathbf{d}r = -\frac{r^2 + t^2}{(r^2 - t^2)^2}\mathbf{d}r + \frac{2rt}{(r^2 - t^2)^2}\mathbf{d}t.$$

From the above, it follows that

$$\tilde{\boldsymbol{\beta}} = \frac{2}{r^2 - t^2}\big(t\mathbf{d}t - r\mathbf{d}r\big).$$

In particular, one has that

$$\tilde{\boldsymbol{\beta}}\big|_{t=0} = -\frac{2}{r}\mathbf{d}r = \frac{1}{\Omega}\mathbf{d}\Omega, \qquad \Omega \equiv \frac{1}{r^2}.$$

The conformal Ω above realises the *point compactification* of infinity in the Euclidean space. A further computation shows that

$$\boldsymbol{\partial}_\tau = \frac{\rho(1 + \tau^2)}{\Theta^2}\boldsymbol{\partial}_t + \frac{2\tau\rho}{\Theta^2}\boldsymbol{\partial}_r.$$

In particular, one has that

$$\tilde{\boldsymbol{\eta}}(\boldsymbol{\partial}_\tau, \boldsymbol{\partial}_\tau) = \frac{1}{\Theta^2}\big(\rho^2(1+\tau^2)^2 + 4\tau^2\rho^2\big) \to \infty, \qquad \text{as} \quad \tau \to \pm 1.$$

The associated integral curves are

$$y^\mu(\tau) = \left(\frac{\tau}{\rho_\star(1-\tau^2)}, \frac{1}{\rho_\star(1-\tau^2)}, \theta_\star, \varphi_\star \right).$$

As

$$r(\tau)^2 - t(\tau)^2 = \frac{1}{\rho_\star(1-\tau^2)}$$

one has that these curves do not generate (standard) hyperboloids.

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