A conformal approach to the stability of Einstein spaces with spatial sections of negative scalar curvature

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Abstract

It is shown how the extended conformal Einstein field equations and a gauge based on the properties of conformal geodesics can be used to analyse the non-linear stability of Einstein spaces with negative scalar curvature. This class of spacetimes admits a smooth conformal extension with a spacelike conformal boundary. Central to the analysis is the use of conformal Gaussian systems to obtain a hyperbolic reduction of the conformal Einstein field equations for which standard Cauchy stability results for symmetric hyperbolic systems can be employed. The use of conformal methods allows to rephrase the question of global existence of solutions to the Einstein field equations into considerations of finite existence time for the conformal evolution system.

1 Introduction

In the mathematical Relativity literature, for a *Cosmological spacetime* it is usually understood a spacetime with compact spatial sections. Understanding the long-time evolution of generic examples of these spacetimes in, say the *vacuum case*, is one of the open challenges in the area. Although generic initial data is expected to form singularities towards the future, it is nevertheless essential to adress the stability of those solutions which are known to be geodesically complete. The fundamental example of a geodesically complete Cosmological spacetime is given by the *de Sitter spacetime*. Its non-linear stability was analysed in the seminal work by Friedrich [8, 7]. A central aspect of this result is the use of conformal methods to transform the question of the global existence of solutions to a finite existence problem. An alternative approach to the study of the non-linear stability of vacuum Cosmological solutions to the Einstein field equations by means of so-called *CMC foliations* has been used by Andersson & Moncrief [2, 3] to prove the non-linear stability of 4-dimensional Friedmann-Lemaître-Robinson-Walker (FLRW) vacuum solutions. Using similar methods, in [5] Fajman & Kröncke studied the non-linear stability of large classes of Cosmological solutions to the vacuum Einstein field equations with a positive Cosmological constant in arbitrary dimensions. These solutions are characterised by having spatial sections with constant scalar curvature which can be either positive or negative. *The purpose of this article is to show that, in four dimensions, the stability results for spacetimes with spatial sections of constant negative curvature given in [5] can be addressed via a generalisation of the conformal methods developed by Friedrich [8, 9, 10, 12] —see also [19].* The analysis of the case positive constant curvature is essentially contained in the original results in [8] —see also [17]. The use of conformal methods in the stability problem considered in this article provides alternative information and insights into the evolution of Cosmological spacetimes.

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Conformal methods in the analysis of de Sitter-like spacetimes

In what follows, for a *de Sitter-like spacetime* it is understood a vacuum Einstein spacetime with a positive value of the Cosmological constant and compact spatial sections. General results on conformal geometry show that *if* these spacetimes admit a conformal compactification *à la* Penrose then the conformal boundary of the spacetime must be spacelike —see e.g. [19], Theorem 10.1. Following the standard usage, we refer to the conformal extension of a de Sitter-like spacetime as the *unphysical spacetime*. The usefulness of this conformal extension lies in the fact that points representing the infinity of the physical spacetime (e.g. the endpoints if timelike geodesics) are mapped to a finite location in the unphysical spacetime. These points are characterised by the vanishing of the conformal factor.

In the particular case considered in the present article, we consider de Sitter-like spacetimes which can be conformally embedded into a portion of a cylinder whose sections have negative scalar curvature. The conformal embedding is realised by means of a conformal factor Θ which depends quadratically on the affine parameter τ of special curves which are invariants of the conformal structure. These curves are known as *conformal geodesics*, and the affine parameter is used as a time coordinate for the physical metric.

Key in the conformal approach is that the unphysical metric provides a solution to the *conformal Einstein field equations* —i.e. a conformal representation of the vacuum Einstein field equations which provides equations which are regular up and beyond the conformal boundary [6, 19]. These equations allow to avoid the difficulties produced by the fact that the direct application of conformal transformation laws into the Einstein equations leads to equations with singular terms. In the present article we make use of a more general version of these equations, the *extended conformal Einstein field equations* expressed in terms of a *Weyl connection* —i.e. a non-metric torsion free connection which preserves the causal structure. This version of the conformal equations allows the use of *conformal Gaussian coordinate systems* in which coordinates are propagated along conformal geodesics —rather than along standard geodesics as is done in the usual Gaussian systems.

As already mentioned, the appeal of conformal methods in the study of solutions to the Einstein field equations lies in the observation that local results for the unphysical spacetime can, in principle, be translated into global results for the physical spacetime. In original formulation of the conformal Einstein field equations the conformal factor realising the conformal embedding of the physical spacetime into a compact manifold is an unknown of the problem. However, remarkably, the use of conformal Gaussian coordinate systems provide a natural conformal factor which singles out a representative in the conformal class of the spacetime. Accordingly, the location of the conformal boundary is known *a priori*, thus simplifying further the analysis of the evolution equations. The extended conformal Einstein field equations expressed in terms of a conformal Gaussian system can be shown to imply a conformal evolution system which takes the form of a symmetric hyperbolic system —i.e. a class of evolution systems for which there exists a well-developed existence, uniqueness and stability theory [16].

The main result

In the following, let $(\mathcal{S}, \mathring{\boldsymbol{\gamma}})$ denote a compact and complete 3-dimensional Riemannian manifold with negative constant curvature. Then the Lorentzian metric given

$$\mathring{\tilde{\boldsymbol{g}}} = -\mathbf{d}t \otimes \mathbf{d}t + \sinh^2 t \mathring{\boldsymbol{\gamma}} \tag{1}$$

is an Einstein space over $\mathbb{R} \times \mathcal{S}$ which is geodesically complete and for which the Cosmological constant takes the value $\lambda = 3$. Our main result can be formulated, in formally, as follows:

Theorem. *Given smooth initial data* $(\boldsymbol{h}, \boldsymbol{K})$ *for the Einstein field equations on* $\mathcal{S}$ *which is suitably close (as measured by a suitable Sobolev norm) to the data implied by the metric* (1)*, there exists a smooth metric* $\tilde{\boldsymbol{g}}$ *defined over* $[0,\infty) \times \mathcal{S}$ *which is close to* $\mathring{\tilde{\boldsymbol{g}}}$ *(again, in the sense of Sobolev norms) and solves the vacuum Einstein field equations with Cosmological constant* $\lambda = 3$*. The spacetime* $([0,\infty) \times \mathcal{S}, \mathring{\tilde{\boldsymbol{g}}})$ *is future geodesically complete.*

A precise formulation of the result is given in Theorem 1 in Section 8. The construction of the initial data required in the above result has been analysed in [20].

Outline of the article

The present article is structured as follows: Section 2 provides the required background on the extended conformal Einstein field equations required for the analysis in this article —this discussion is not only restricted to the equations but also involves the associated constraint equations and the notion of conformal geodesics which will be used to fix the gauge. Section 3 provides an analysis of the background spacetimes (Einstein spaces with spatial sections of constant negative curvature) in the light of the conformal Einstein field equations. In particular, this section gives a conformal extension of these spacetimes arising from a certain class congruences of conformal geodesics. Section 4 provides an analysis of the conformal evolution system which will be used in the main stability argument and its relation to the actual extended conformal Einstein field equations, including the so-called *propagation of the constraints*. Section 5 contains a brief discussion of the initial data for the conformal evolution equations and how it can be constructed. Section 6 contains the main existence and stability analysis of the conformal evolution equations. Section 7 provides a discussion of the future geodesic completeness of the perturbed spacetimes. Finally, Section 8 contains a precise statement of the main Theorem of this article and some concluding remarks. In addition, the article contains two appendices: Appendix A provides a summary of the main technical tool in this article —Kato's existence and stability result for symmetric hyperbolic systems. Appendix B provides a brief discussion of the geodesic completeness of the background solutions.

Notations and conventions

Throughout we mostly follow the notations and conventions of [19] except from the fact that the sign of the Cosmological constant for de Sitter-like spacetimes is taken to be positive. The signature of Lorentzian metrics is taken to be $(-+++)$. Throughout the Latin letters a, b, c, ... denote spacetime abstract indices indicating the tensorial character of the various objects while the letters i, j, k, ... correspond to spatial abstract indices. The boldface indices $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, ... will be used as spacetime frame indices ranging $\mathbf{0}$, $\mathbf{1}$, $\mathbf{2}$, $\mathbf{3}$ while $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, ... range over $\mathbf{1}$, $\mathbf{2}$, $\mathbf{3}$. The Greek indices μ, ν, λ, ... play the role of spacetime coordinate indices and α, β, γ, ... are spatial coordinate indices. In addition to the index notation described above, when convenient, we also make use of an index-free notation —e.g. a metric tensor can be described, alternatively, by $\boldsymbol{g}$ or g_{ab}. Associated to a given metric $\boldsymbol{g}$ we also make use of the *musical isomorphisms* $\sharp$ and $\flat$ to denote the raising and lowering of indices of tensorial objects away from their natural position.

Our conventions for the curvature are given by the equation

$$\nabla_a \nabla_b v^c - \nabla_b \nabla_a v^c = R^c{}_{dab} v^d.$$

2 The extended conformal Einstein field equations

The main technical tool of this article is given by the *extended conformal Einstein field equations* —see [10, 11]; also [19]. This system of equations constitute a conformal representation of the vacuum Einstein field equations written in terms of *Weyl connections*. A solution to the extended conformal equations implies a solution to the vacuum Einstein field equations away from the conformal boundary. In this section we provide a brief discussion of this system geared towards the applications of this article. A derivation and further discussion of the general properties of these equations can be found in [19], Chapter 8.

Throughout this article let $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ with $\tilde{\mathcal{M}}$ a 4-dimensional manifold and $\tilde{\boldsymbol{g}}$ a Lorentzian metric denote a vacuum spacetime satisfying the Einstein field equations with Cosmological constant

$$\tilde{R}_{ab} = \lambda \tilde{g}_{ab}. \tag{2}$$

Let $\boldsymbol{g}$ denote an unphysical Lorentzian metric conformally related to $\tilde{\boldsymbol{g}}$ via the relation

$$\boldsymbol{g} = \Xi^2 \tilde{\boldsymbol{g}}$$

with Ξ a suitable conformal factor. Let ∇_a and $\tilde{\nabla}_a$ denote, respectively, the Levi-Civita connections of the metrics $\boldsymbol{g}$ and $\tilde{\boldsymbol{g}}$.

2.1 Weyl connections

A Weyl connection is a torsion-free connection $\hat{\nabla}_a$ such that

$$\hat{\nabla}_a g_{bc} = -2 f_a g_{bc}.$$

It follows from the above that the connections ∇_a and $\hat{\nabla}_a$ are related to each other by

$$\hat{\nabla}_a v^b - \nabla_a v^b = S_{ac}{}^{bd} f_d v^c, \qquad S_{ac}{}^{bd} \equiv \delta_a{}^b \delta_c{}^d + \delta_a{}^d \delta_c{}^b - g_{ac} g^{bd}, \tag{3}$$

where f_a is a fixed smooth covector and v^a is an arbitrary vector. Given that

$$\nabla_a v^b - \tilde{\nabla}_a v^b = S_{ac}{}^{bd} (\Xi^{-1} \nabla_a \Xi) v^c,$$

one has that

$$\hat{\nabla}_a v^b - \tilde{\nabla}_a v^b = S_{ac}{}^{bd} \beta_d v^c, \qquad \beta_d \equiv f_d + \Xi^{-1} \nabla_d \Xi.$$

In the following, it will be convenient to define

$$d_a \equiv \Xi f_a + \nabla_a \Xi. \tag{4}$$

In the following $\hat{R}^a{}_{bcd}$ and $\hat{L}_{ab}$ will denote, respectively, the Riemann tensor and Schouten tensor of the Weyl connection $\hat{\nabla}_a$. Observe that for a generic Weyl connection one has that $\hat{L}_{ab} \neq \hat{L}_{ba}$. One has the decomposition

$$\hat{R}^c{}_{dab} = 2 S_{d[a}{}^{ce} \hat{L}_{b]e} + C^c{}_{dab},$$

where $C^c{}_{dab}$ denotes the conformally invariant *Weyl tensor*. The (vanishing) torsion of $\hat{\nabla}_a$ is denoted by $\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$. In the context of the conformal Einstein field equations it is convenient to define the *rescaled Weyl tensor* $d^c{}_{dab}$ via the relation

$$d^c{}_{dab} \equiv \Xi^{-1} C^c{}_{dab}.$$

2.1.1 A frame formalism

Let $\{\boldsymbol{e}_{\boldsymbol{a}}\}$, $\boldsymbol{a} = \boldsymbol{0}, \ldots, \boldsymbol{3}$ denote a $\boldsymbol{g}$-orthogonal frame with associated coframe $\{\boldsymbol{\omega}^{\boldsymbol{a}}\}$. Thus, one has that

$$\boldsymbol{g}(\boldsymbol{e}_{\boldsymbol{a}}, \boldsymbol{e}_{\boldsymbol{b}}) = \eta_{\boldsymbol{ab}}, \qquad \langle \boldsymbol{\omega}^{\boldsymbol{a}}, \boldsymbol{\omega}_{\boldsymbol{b}} \rangle = \delta_{\boldsymbol{b}}{}^{\boldsymbol{a}}.$$

Given a vector v^a, its components with respect to the frame $\{\boldsymbol{e}_{\boldsymbol{a}}\}$ are denoted by $v^{\boldsymbol{a}}$.

Let $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$ and $\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$ denote, respectively, the connection coefficients of $\nabla_{\boldsymbol{a}}$ and $\hat{\nabla}_a$ with respect to the frame $\{\boldsymbol{e}_{\boldsymbol{a}}\}$. It follows then from equation (3) that

$$\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}} + S_{\boldsymbol{ab}}{}^{\boldsymbol{cd}} f_{\boldsymbol{d}}.$$

In particular, one has that

$$f_{\boldsymbol{a}} = \frac{1}{4} \hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{b}}.$$

Denoting by $\partial_{\boldsymbol{a}} \equiv \boldsymbol{e}_{\boldsymbol{a}}{}^{\mu} \partial_\mu$ the directional partial derivative in the direction of $\boldsymbol{e}_{\boldsymbol{a}}$, it follows then that

$$\begin{aligned} \nabla_{\boldsymbol{a}} T^{\boldsymbol{b}}{}_{\boldsymbol{c}} &\equiv e_{\boldsymbol{a}}{}^a \omega^{\boldsymbol{b}}{}_b \omega^{\boldsymbol{c}}{}_c (\nabla_a T^b{}_c), \\ &= \partial_{\boldsymbol{a}} T^{\boldsymbol{b}}{}_{\boldsymbol{c}} + \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{d}} T^{\boldsymbol{d}}{}_{\boldsymbol{c}} - \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}} T^{\boldsymbol{b}}{}_{\boldsymbol{d}}, \end{aligned}$$

with the natural extensions for higher rank tensors and other covariant derivatives.

2.2 The frame version of the extended conformal Einstein field equations

In this article we will make use of a frame version of the extended conformal Einstein field equations. In order to formulate these equations it is convenient to define the following *zero-quantities*:

$$\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}\boldsymbol{e}_{\boldsymbol{c}} \equiv [\boldsymbol{e}_{\boldsymbol{a}}, \boldsymbol{e}_{\boldsymbol{b}}] - (\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} - \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{a}})e_{\boldsymbol{c}}, \tag{5a}$$

$$\hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{dab}} \equiv \hat{P}^{\boldsymbol{c}}{}_{\boldsymbol{dab}} - \hat{\rho}^{\boldsymbol{c}}{}_{\boldsymbol{dab}}, \tag{5b}$$

$$\hat{\Delta}_{\boldsymbol{cdb}} \equiv \nabla_{\boldsymbol{c}}\hat{L}_{\boldsymbol{db}} - \nabla_{\boldsymbol{d}}\hat{L}_{\boldsymbol{cb}} - d_{\boldsymbol{a}}\hat{d}^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}, \tag{5c}$$

$$\Lambda_{\boldsymbol{bcd}} \equiv \nabla_{\boldsymbol{a}} d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}, \tag{5d}$$

where the components of the *geometric curvature* $\hat{P}^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ and the *algebraic curvature* $\hat{\rho}^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ are given, respectively, by

$$\hat{P}^{\boldsymbol{c}}{}_{\boldsymbol{dab}} \equiv \partial_{\boldsymbol{a}}(\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}) - \partial_{\boldsymbol{b}}(\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}) + \hat{\Gamma}_{\boldsymbol{f}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}(\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{a}} - \hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{b}}) + \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}}\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{f}} - \hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}}\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{f}},$$

$$\hat{\rho}^{\boldsymbol{c}}{}_{\boldsymbol{dab}} \equiv \Xi\hat{d}^{\boldsymbol{c}}{}_{\boldsymbol{dab}} + 2S_{\boldsymbol{d}[\boldsymbol{a}}{}^{\boldsymbol{ce}}\hat{L}_{\boldsymbol{b}]\boldsymbol{e}},$$

where $\hat{L}_{\boldsymbol{ab}}$ and $d^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ denote, respectively, the components of the Schouten tensor of $\hat{\nabla}_a$ and the rescaled Weyl tensor with respect to the frame $\{\boldsymbol{e}_{\boldsymbol{a}}\}$. In terms of the zero-quantities (5a)-(5d), the *extended vacuum conformal Einstein field equations* are given by the conditions

$$\hat{\Sigma}_{a}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}\boldsymbol{e}_{\boldsymbol{c}} = 0, \qquad \hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{dab}} = 0, \qquad \hat{\Delta}_{\boldsymbol{cdb}} = 0, \qquad \hat{\Lambda}_{\boldsymbol{bcd}} = 0. \tag{6}$$

In the above equations the fields Ξ and $d_{\boldsymbol{a}}$ —cfr. (4)— are regarded as *conformal gauge fields* which are determined by supplementary conditions. In the present article these gauge conditions will be determined through conformal geodesics —see Subsection 3.1.1 below. In order to account for this it is convenient to define

$$\delta_{\boldsymbol{a}} \equiv d_{\boldsymbol{a}} - \Xi f_{\boldsymbol{a}} - \hat{\nabla}_{\boldsymbol{a}}\Xi, \tag{7a}$$

$$\gamma_{\boldsymbol{ab}} \equiv \hat{L}_{\boldsymbol{ab}} - \hat{\nabla}_{\boldsymbol{a}}(\Xi^{-1}d_{\boldsymbol{b}}) - \frac{1}{2}\Xi^{-1}S_{\boldsymbol{ab}}{}^{\boldsymbol{cd}}d_{\boldsymbol{c}}d_{\boldsymbol{d}} + \frac{1}{6}\lambda\Xi^{-2}\eta_{\boldsymbol{ab}}, \tag{7b}$$

$$\varsigma_{\boldsymbol{ab}} \equiv \hat{L}_{[\boldsymbol{ab}]} - \hat{\nabla}_{[\boldsymbol{a}}f_{\boldsymbol{b}]}. \tag{7c}$$

The conditions

$$\delta_{\boldsymbol{a}} = 0, \qquad \gamma_{\boldsymbol{ab}} = 0, \qquad \varsigma_{\boldsymbol{ab}} = 0, \tag{8}$$

will be called the *supplementary conditions.* They play a role in relating the Einstein field equations to the extended conformal Einstein field equations and also in the propagation of the constraints.

The correspondence between the Einstein field equations and the extended conformal Einstein field equations is given by the following —see Proposition 8.3 in [19]:

Proposition 1. *Let*

$$(\boldsymbol{e}_{\boldsymbol{a}}, \hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}, \hat{L}_{\boldsymbol{ab}}, d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}})$$

denote a solution to the extended conformal Einstein field equations (6) *for some choice of the conformal gauge fields* (Ξ, $d_{\boldsymbol{a}}$) *satisfying the supplementary conditions* (8). *Furthermore, suppose that*

$$\Xi \neq 0, \qquad \det(\eta^{\boldsymbol{ab}}\boldsymbol{e}_{\boldsymbol{a}} \otimes \boldsymbol{e}_{\boldsymbol{b}}) \neq 0$$

on an open subset $\mathcal{U}$. Then the metric

$$\tilde{\boldsymbol{g}} = \Xi^{-2}\eta_{\boldsymbol{ab}}\boldsymbol{\omega}^{\boldsymbol{a}} \otimes \boldsymbol{\omega}^{\boldsymbol{b}}$$

is a solution to the Einstein field equations (2) *on $\mathcal{U}$.*

2.3 The conformal constraint equations

The analysis in this article will make use of the *conformal constraint Einstein equations* —i.e. the intrinsic equations implied by the (standard) vacuum conformal Einstein field equations on a spacelike hypersurface. A derivation of these equations in its frame form can be found in [19], Section 11.4.

Let $\mathcal{S}$ denote a spacelike hypersurface in an unphysical spacetime $(\mathcal{M}, \boldsymbol{g})$. In the following let $\{\boldsymbol{e_a}\}$ denote a $\boldsymbol{g}$-orthonormal frame adapted to $\mathcal{S}$. That is, the vector $\boldsymbol{e_0}$ is chosen to coincide with the unit normal vector to the hypersurface and while the spatial vectors $\{\boldsymbol{e_i}\}$, $\boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2}, \boldsymbol{3}$ are intrinsic to $\mathcal{S}$. In our signature conventions we have that $\boldsymbol{g}(\boldsymbol{e_0}, \boldsymbol{e_0}) = -1$. The extrinsic curvature is described by the components $\chi_{\boldsymbol{ij}}$ of the Weingarten tensor. One has that $\chi_{\boldsymbol{ij}} = \chi_{\boldsymbol{ji}}$ and, moreover

$$\chi_{\boldsymbol{ij}} = -\Gamma_{\boldsymbol{i}}{}^{\boldsymbol{0}}{}_{\boldsymbol{j}}.$$

We denote by Ω the restriction of the spacetime conformal factor Ξ to $\mathcal{S}$ and by Σ the normal component of the gradient of Ξ. The field $l_{\boldsymbol{ij}}$ denote the components of the Schouten tensor of the induced metric h_{ij} on $\mathcal{S}$.

With the above conventions, the conformal constraint equations in the vacuum case are given by —see [19]:

$$D_{\boldsymbol{i}}D_{\boldsymbol{j}}\Omega = \Sigma\chi_{\boldsymbol{ij}} - \Omega L_{\boldsymbol{ij}} + s h_{\boldsymbol{ij}}, \tag{9a}$$
$$D_{\boldsymbol{i}}\Sigma = \chi_{\boldsymbol{i}}{}^{\boldsymbol{k}} D_{\boldsymbol{k}}\Omega - \Omega L_{\boldsymbol{i}}, \tag{9b}$$
$$D_{\boldsymbol{i}} s = L_{\boldsymbol{i}}\Sigma - L_{\boldsymbol{ik}} D^{\boldsymbol{k}}\Omega, \tag{9c}$$
$$D_{\boldsymbol{i}}L_{\boldsymbol{jk}} - D_{\boldsymbol{j}}L_{\boldsymbol{ik}} = \Sigma d_{\boldsymbol{kij}} + D^{\boldsymbol{l}}\Omega d_{\boldsymbol{lkij}}(\chi_{\boldsymbol{ik}}L_{\boldsymbol{j}} - \chi_{\boldsymbol{jk}}L_{\boldsymbol{i}}), \tag{9d}$$
$$D_{\boldsymbol{i}}L_{\boldsymbol{j}} - D_{\boldsymbol{j}}L_{\boldsymbol{i}} = D^{\boldsymbol{l}}\Omega d_{\boldsymbol{lij}} + \chi_{\boldsymbol{i}}{}^{\boldsymbol{k}} L_{\boldsymbol{jk}} - \chi_{\boldsymbol{j}}{}^{\boldsymbol{k}} L_{\boldsymbol{ik}}, \tag{9e}$$
$$D^{\boldsymbol{k}} d_{\boldsymbol{kij}} = -(\chi^{\boldsymbol{k}}{}_{\boldsymbol{i}} d_{\boldsymbol{jk}} - \chi^{\boldsymbol{k}}{}_{\boldsymbol{j}} d_{\boldsymbol{ik}}), \tag{9f}$$
$$D^{\boldsymbol{i}} d_{\boldsymbol{ij}} = \chi^{\boldsymbol{ik}} d_{\boldsymbol{ijk}}, \tag{9g}$$
$$\lambda = 6\Omega s + 3\Sigma^2 - 3D_{\boldsymbol{k}}\Omega D^{\boldsymbol{k}}\Omega, \tag{9h}$$
$$D_{\boldsymbol{j}}\chi_{\boldsymbol{ki}} - D_{\boldsymbol{k}}\chi_{\boldsymbol{ji}} = \Omega d_{\boldsymbol{ijk}} + h_{\boldsymbol{ij}}L_{\boldsymbol{k}} - h_{\boldsymbol{ik}}L_{\boldsymbol{j}}, \tag{9i}$$
$$l_{\boldsymbol{ij}} = \Omega d_{\boldsymbol{ij}} + L_{\boldsymbol{ij}} - \chi(\chi_{\boldsymbol{ij}} - \frac{1}{4}\chi h_{\boldsymbol{ij}}) + \chi_{\boldsymbol{ki}}\chi_{\boldsymbol{j}}{}^{\boldsymbol{k}} - \frac{1}{4}\chi_{\boldsymbol{kl}}\chi^{\boldsymbol{kl}} h_{\boldsymbol{ij}}, \tag{9j}$$

with the understanding that

$$h_{\boldsymbol{ij}} \equiv g_{\boldsymbol{ij}} = \delta_{\boldsymbol{ij}}$$

and where we have defined

$$L_{\boldsymbol{i}} \equiv L_{\boldsymbol{0i}}, \qquad d_{\boldsymbol{ij}} \equiv d_{\boldsymbol{0i0j}}, \qquad d_{\boldsymbol{ijk}} \equiv d_{\boldsymbol{i0jk}}.$$

The fields $d_{\boldsymbol{ij}}$ and $d_{\boldsymbol{ijk}}$ correspond, respectively, to the electric and magnetic parts of the rescaled Weyl tensor. The scalar s denotes the *Friedrich scalar* defined as

$$s \equiv \frac{1}{4}\nabla_a\nabla^a\Xi + \frac{1}{24}R\Xi,$$

with R the Ricci scalar of the metric $\boldsymbol{g}$. Finally, $L_{\boldsymbol{ij}}$ denote the spatial components of the Schouten tensor of $\boldsymbol{g}$.

3 Properties of the background solution

In the following let $(\tilde{\mathcal{M}}, \mathring{\tilde{\boldsymbol{g}}})$ denote the solution to the vacuum Einstein field equations with positive Cosmological constant

$$\tilde{R}_{ab} = \lambda\tilde{g}_{ab}, \qquad \lambda = 3,$$

given by $\tilde{\mathcal{M}} = \mathbb{R} \times \mathcal{S}$ and

$$\mathring{\tilde{\boldsymbol{g}}} = -\mathbf{d}t \otimes \mathbf{d}t + \sinh^2 t\, \mathring{\boldsymbol{\gamma}} \tag{10}$$

where $\mathring{\gamma}$ is a (positive definite) Riemannian metric of constant negative curvature over a compact manifold $\mathcal{S}$ such that

$$r[\mathring{\gamma}] = -6.$$

The spacetime $(\tilde{\mathcal{M}}, \mathring{\tilde{\boldsymbol{g}}})$ is future geodesically complete —see Appendix B.

Remark 1. The value $\lambda = 3$ for the Cosmological constant is conventional and set for convenience. The analysis in this article can be carried out for any other (positive) value.

The Riemann curvature tensor $r^i{}_{jkl}[\mathring{\gamma}]$ of $\mathring{\boldsymbol{\gamma}}$ is given by

$$r_{ijkl}[\mathring{\gamma}] = \mathring{\gamma}_{il}\mathring{\gamma}_{jk} - \mathring{\gamma}_{ik}\mathring{\gamma}_{jk}.$$

From the above expressions it follows that

$$\tilde{R} = 12,$$

so that

$$\tilde{L}_{ab} = \frac{1}{2}\tilde{g}_{ab} \tag{11}$$

In the following, a spacetime of the form given by $(\tilde{\mathcal{M}}, \mathring{\tilde{\boldsymbol{g}}})$ will be known as a *background solution*. In the rest of this section we will perform an analysis of this class of solutions to the Einstein field equations from the point of view of conformal geometry. In particular, we will make use of conformal geodesics to provide a *canonical* conformal extension —see Proposition 2.

3.1 A class of conformal geodesics

In the following we will consider (metric) geodesics $x(s)$ on $(\tilde{\mathcal{M}}, \mathring{\tilde{\boldsymbol{g}}})$ whose tangent vector is proportional to $\boldsymbol{\partial}_t$ —i.e. $\dot{\boldsymbol{x}} = \alpha\boldsymbol{\partial}_t$ for some proportionality function α and where the overdot denotes differentiation with respect to the affine parameter $s \in \mathbb{R}$. The geodesic equation

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}\dot{\boldsymbol{x}} = 0$$

implies that

$$\begin{aligned}\tilde{\nabla}_{\boldsymbol{\partial}_t}(\alpha\boldsymbol{\partial}_t) &= \tilde{\nabla}_t\alpha + \tilde{\nabla}_{\boldsymbol{\partial}_t}\boldsymbol{\partial}_t \\ &= \tilde{\nabla}_t\alpha + \Gamma_t{}^{\mu}{}_t\boldsymbol{\partial}_\mu.\end{aligned}$$

A direct calculation for the metric (10) shows that $\Gamma_t{}^{\mu}{}_t = 0$ so that one concludes that $\partial_t\alpha = 0$ —that is, α is constant along the integral curves of $\boldsymbol{\partial}_t$. Without loss of generality we then set $\alpha = 1$ so that $\boldsymbol{g}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) = 1$. In summary, we have that the curves

$$x(t) = (t, \underline{x}_\star), \qquad \underline{x}_\star \in \mathcal{S},$$

are non-intersecting timelike $\tilde{\boldsymbol{g}}$-geodesics over $\tilde{\mathcal{M}}$. In a slight abuse of notation the coordinate t has been used as parameter of the curve.

3.1.1 Conformal geodesics

The extended conformal Einstein field equations are naturally suited to the use of a gauge based on conformal geodesics.

A *conformal geodesic* on a spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ is a pair $(x(\tau), \tilde{\boldsymbol{\beta}}(\tau))$ consisting of a curve $x(\tau)$ on $\tilde{\mathcal{M}}$, with parameter $\tau \in I \subset \mathbb{R}$, tangent $\dot{\boldsymbol{x}}(\tau)$ and a covector $\tilde{\boldsymbol{\beta}}(\tau)$ along $x(\tau)$ satisfying the equations (in index-free notation)

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}\dot{\boldsymbol{x}} = -2\langle\tilde{\boldsymbol{\beta}}, \dot{\boldsymbol{x}}\rangle\dot{\boldsymbol{x}} + \tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})\tilde{\boldsymbol{\beta}}^\sharp, \tag{12a}$$

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}\tilde{\boldsymbol{\beta}} = \langle\tilde{\boldsymbol{\beta}}, \dot{\boldsymbol{x}}\rangle\tilde{\boldsymbol{\beta}} - \frac{1}{2}\tilde{\boldsymbol{g}}^\sharp(\tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\beta}})\dot{\boldsymbol{x}}^\flat + \tilde{\boldsymbol{L}}(\dot{\boldsymbol{x}}, \cdot), \tag{12b}$$

where $\tilde{\boldsymbol{L}}$ denotes the Schouten tensor of the Levi-Civita connection $\tilde{\nabla}_a$. Associated to a conformal geodesic, it is natural to consider a frame $\{\boldsymbol{e_a}\}$ which is *Weyl propagated* along $x(\tau)$ according to the law

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}\boldsymbol{e_a} = -\langle\tilde{\boldsymbol{\beta}}, \boldsymbol{e_a}\rangle\dot{\boldsymbol{x}} - \langle\tilde{\boldsymbol{\beta}}, \dot{\boldsymbol{x}}\rangle\boldsymbol{e_a} + \tilde{\boldsymbol{g}}(\boldsymbol{e_a}, \dot{\boldsymbol{x}})\tilde{\boldsymbol{\beta}}^\sharp. \tag{13}$$

3.1.2 Reparametrisation as conformal geodesic

In the following, we will make use of the methods in the proof of Lemma 5.2 in [19] to recast the family of geodesics discussed in Subsection 3.1 as conformal geodesics. Accordingly, we consider a reparametrisation of the form

$$\tau \mapsto t(\tau),$$

while we look for a 1-form $\boldsymbol{\beta}$ given by the Ansatz

$$\tilde{\boldsymbol{\beta}} = \alpha(\tau)\boldsymbol{x}'^{\flat} = \alpha(t)\mathbf{d}t,$$

where $'$ denotes derivatives with respect to s. From the chain rule it follows that

$$\dot{\boldsymbol{x}} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{t}\boldsymbol{x}', \qquad \dot{t} \equiv \frac{\mathrm{d}t}{\mathrm{d}\tau}.$$

In particular, one readily has that

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}\dot{\boldsymbol{x}} = \dot{t}^2\tilde{\nabla}_{\boldsymbol{x}'}\boldsymbol{x}' + \ddot{t}\boldsymbol{x}'.$$

Substituting the previous expressions into equations (12a) and (12b), taking into account expression (11) for the components of the Schouten tensor one obtains the system of ordinary differential equations

$$\ddot{t} + \alpha\dot{t}^2 = 0, \tag{14a}$$

$$\dot{\alpha} = \frac{1}{2}\dot{t}(\alpha^2 - 1). \tag{14b}$$

The general solution to the above system can be found to be

$$\alpha(\tau) = c_1\tau + c_2,$$
$$t(\tau) = -2\mathrm{arctanh}(c_1\tau + c_2) + c_3,$$

with c_1, c_2, $c_3 \in \mathbb{R}$ constants. For simplicity one can, e.g. set $c_1 = -1$, $c_2 = c_3 = 0$ to get the simpler expressions

$$\alpha(\tau) = -\tau,$$
$$t(\tau) = 2\mathrm{arctanh}\,\tau.$$

Thus, observing that

$$\sinh\big(2\mathrm{arctanh}\,\tau\big) = \frac{2\tau}{1-\tau^2}, \qquad \frac{\mathrm{d}}{\mathrm{d}\tau}\big(2\mathrm{arctanh}\,\tau\big) = \frac{2}{1-\tau^2},$$

it follows that the pair $(x(\tau), \tilde{\boldsymbol{\beta}}(\tau))$, $\tau \in (-1, 1)$ with

$$x(\tau) = (2\,\mathrm{arctanh}\,\tau, \underline{x}_\star), \qquad \tilde{\boldsymbol{\beta}}(\tau) = -\frac{2\tau}{1-\tau^2}\mathbf{d}\tau,$$

give rise to a congruence of non-intersecting conformal geodesics on the background spacetime $(\tilde{\mathcal{M}}, \mathring{\tilde{\boldsymbol{g}}})$. Using the parameter τ as new coordinate in the metric (10) one concludes that

$$\mathring{\tilde{\boldsymbol{g}}} = \frac{4}{(1-\tau^2)^2}\bigg(-\mathbf{d}\tau\otimes\mathbf{d}\tau + \tau^2\mathring{\boldsymbol{\gamma}}\bigg). \tag{15}$$

Notice that the metric is singular at $\tau \pm 1$.

3.1.3 The canonical factor associated to the congruence of conformal geodesics

The line element (15) readily suggest the conformal factor

$$\Theta \equiv \frac{1}{2}(1-\tau^2).$$

Remark 2. Alternatively, we can make use of the equation

$$\dot{\Theta} = \langle \tilde{\boldsymbol{\beta}}, \dot{\boldsymbol{x}} \rangle \Theta, \qquad \langle \tilde{\boldsymbol{\beta}}, \dot{\boldsymbol{x}} \rangle = \alpha \dot{t} = -\frac{2\tau}{1-\tau^2}$$

implied by the condition $\Theta^2 \tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) = -1$. Integrating one readily finds that

$$\frac{\Theta}{\Theta_\star} = \frac{1-\tau^2}{1-\tau_\star^2}$$

where $\Theta_\star$ is the value of the conformal factor at a fiduciary time $\tau_\star$. Observe, also, that

$$\begin{aligned} \tilde{\boldsymbol{\beta}}(\tau) &= -\frac{2\tau}{1-\tau^2} \mathbf{d}\tau, \\ &= \mathbf{d}\big(\ln \Theta(\tau)\big). \end{aligned} \tag{16}$$

Following expression (15) we introduce a new unphysical metric $\mathring{\boldsymbol{g}}$ via the relation

$$\mathring{\boldsymbol{g}} = \Theta^2 \mathring{\tilde{\boldsymbol{g}}}, \qquad \Theta \equiv \frac{1}{2}(1-\tau^2),$$

so as to ensure that $\Theta \geq 0$ for $|\tau| \leq 1$. It follows then that

$$\mathring{\boldsymbol{g}} = -\mathbf{d}\tau \otimes \mathbf{d}\tau + \tau^2 \mathring{\boldsymbol{\gamma}} \tag{17}$$

is well defined for $\tau \in [\tau_*, \infty)$ with $\tau_\star > 0$. For future use we define the *spatial metric* $\mathring{\boldsymbol{h}}$

$$\mathring{\boldsymbol{h}} \equiv \tau^2 \mathring{\boldsymbol{\gamma}},$$

with associated Levi-Civita connection to be denoted by $\mathring{D}$. Also, denote by $\mathring{\mathfrak{D}}$ the Levi-Civita connection of the metric $\mathring{\boldsymbol{\gamma}}$.

Remark 3. Observe that as the metrics $\mathring{\boldsymbol{\gamma}}$ and $\mathring{\boldsymbol{h}}$ are conformally related via a conformal factor (i.e. τ) independent of the spatial coordinates, it follows then that expressed in terms of local (spatial) coordinates one has that

$$\mathring{D}_\alpha = \mathring{\mathfrak{D}}_\alpha.$$

Remark 4. A computation readily shows that the integral curves of the vector field $\boldsymbol{\partial}_\tau$ are geodesics of the metric $\mathring{\boldsymbol{g}}$ given by equation (17) —that is, one has that

$$\nabla_{\boldsymbol{\partial}_\tau} \boldsymbol{\partial}_\tau = 0.$$

Remark 5. Taking into account the expression (16), the conformal transformation law for conformal geodesics gives that

$$\boldsymbol{\beta} = \tilde{\boldsymbol{\beta}} - \mathbf{d}\big(\ln \Theta(\tau)\big) = 0.$$

To any (non-singular) congruence of conformal geodesics one can associate a Weyl connection $\hat{\nabla}$ via the rule

$$\hat{\nabla} - \tilde{\nabla} = \boldsymbol{S}(\tilde{\boldsymbol{\beta}}).$$

In the present case, $\tilde{\boldsymbol{\beta}}$ is a closed 1-form and, thus, the Weyl connection is, in fact, a Levi-Civita connection which coincides with ∇.

3.2 The background spacetime as a solution to the conformal Einstein field equations

In this subsection we show how to recast the *unphysical spacetime* $(\mathcal{M}, \mathring{\boldsymbol{g}})$ with $\mathcal{M} = [\tau_\star, \infty) \times \mathcal{S}$ as a solution to the conformal Einstein field equations. This construction is conveniently done using an adapted frame formalism.

3.2.1 The frame

Let $\{\mathring{\boldsymbol{c}}_{\boldsymbol{i}}\}$, $\boldsymbol{i} = 1, 2, 3$, denote a $\mathring{\boldsymbol{\gamma}}$-orthonormal frame over $\mathcal{S}$ with associated cobasis $\{\mathring{\boldsymbol{\alpha}}^{\boldsymbol{i}}\}$. Accordingly, one has that

$$\mathring{\boldsymbol{\gamma}}(\mathring{\boldsymbol{c}}_i, \mathring{\boldsymbol{c}}_j) = \delta_{\boldsymbol{ij}}, \qquad \langle \mathring{\boldsymbol{\alpha}}^{\boldsymbol{j}}, \mathring{\boldsymbol{c}}_{\boldsymbol{i}} \rangle = \delta_{\boldsymbol{i}}{}^{\boldsymbol{j}},$$

so that

$$\mathring{\boldsymbol{\gamma}} = \delta_{\boldsymbol{ij}} \mathring{\boldsymbol{\alpha}}^{\boldsymbol{i}} \otimes \mathring{\boldsymbol{\alpha}}^{\boldsymbol{j}}.$$

The above frame is used to introduce a $\mathring{\boldsymbol{g}}$-orthonormal frame $\{\mathring{\boldsymbol{e}}_{\boldsymbol{a}}\}$ with associated cobasis $\{\mathring{\boldsymbol{\omega}}^{\boldsymbol{b}}\}$ so that $\langle \mathring{\boldsymbol{\omega}}^{\boldsymbol{b}}, \mathring{\boldsymbol{e}}_{\boldsymbol{a}} \rangle = \delta_{\boldsymbol{a}}{}^{\boldsymbol{b}}$. We do this by setting

$$\begin{aligned} \mathring{\boldsymbol{e}}_0 &\equiv \boldsymbol{\partial}_\tau, \qquad & \mathring{\boldsymbol{e}}_{\boldsymbol{i}} &\equiv \frac{1}{\tau} \mathring{\boldsymbol{c}}_{\boldsymbol{i}}, \\ \mathring{\boldsymbol{\omega}}^0 &\equiv \mathbf{d}\tau, \qquad & \mathring{\boldsymbol{\omega}}^{\boldsymbol{i}} &= \tau \mathring{\boldsymbol{\alpha}}^{\boldsymbol{i}}, \end{aligned}$$

so that

$$\mathring{\boldsymbol{g}} = \eta_{\boldsymbol{ab}} \mathring{\boldsymbol{\omega}}^{\boldsymbol{a}} \otimes \mathring{\boldsymbol{\omega}}^{\boldsymbol{b}}.$$

Remark 6. It follows that all the coefficients of the frame are smooth (C^∞) over $[\tau_\star, \infty) \times \mathcal{S}$, $\tau_\star > 0$.

3.2.2 The connection coefficients

The connection coefficients $\mathring{\gamma}_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}}$ of the Levi-Civita connection $\mathring{D}$ with respect to the frame $\{\mathring{\boldsymbol{c}}_{\boldsymbol{i}}\}$ are defined through the relations

$$\mathring{D}_{\boldsymbol{i}} \mathring{\boldsymbol{c}}_{\boldsymbol{j}} = \mathring{\gamma}_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}} \mathring{\boldsymbol{c}}_{\boldsymbol{k}}, \qquad \gamma_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}} \equiv \langle \mathring{\boldsymbol{\alpha}}^{\boldsymbol{k}}, \mathring{D}_{\boldsymbol{i}} \mathring{\boldsymbol{c}}_{\boldsymbol{j}} \rangle.$$

Similarly, for the connection coefficients $\mathring{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}}$ of the Levi-Civita connection $\mathring{\nabla}$ with respect to the frame $\{\mathring{\boldsymbol{e}}_{\boldsymbol{a}}\}$ one has that

$$\mathring{\nabla}_{\boldsymbol{a}} \mathring{\boldsymbol{e}}_{\boldsymbol{c}} = \mathring{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} \mathring{\boldsymbol{e}}_{\boldsymbol{c}}, \qquad \mathring{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} \equiv \langle \mathring{\boldsymbol{\omega}}^{\boldsymbol{c}}, \mathring{\nabla}_{\boldsymbol{a}} \mathring{\boldsymbol{e}}_{\boldsymbol{c}} \rangle.$$

We now proceed to compute the various connection coefficients.

The coefficients $\mathring{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}}$. Recalling the definition the connection coefficients and of the basis fields $\{\mathring{\boldsymbol{e}}_i\}$ and $\{\mathring{\boldsymbol{\omega}}^j\}$ one has that

$$\begin{aligned} \mathring{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}} &= \langle \boldsymbol{\omega}^k, \mathring{\nabla}_{\boldsymbol{i}} \mathring{\boldsymbol{e}}_{\boldsymbol{j}} \rangle = \langle \boldsymbol{\omega}^{\boldsymbol{k}}, \mathring{e}_{\boldsymbol{i}}{}^\alpha \mathring{\nabla}_\alpha \mathring{\boldsymbol{e}}_{\boldsymbol{j}} \rangle \\ &= \frac{1}{\tau} \langle \mathring{\boldsymbol{\alpha}}^{\boldsymbol{k}}, \mathring{c}_{\boldsymbol{i}}{}^\alpha \mathring{\nabla}_\alpha \mathring{\boldsymbol{c}}_{\boldsymbol{j}} \rangle = \frac{1}{\tau} \langle \mathring{\boldsymbol{\alpha}}^{\boldsymbol{k}}, \mathring{c}_{\boldsymbol{i}}{}^\alpha \mathring{D}_\alpha \mathring{\boldsymbol{c}}_{\boldsymbol{j}} \rangle = \frac{1}{\tau} \langle \mathring{\boldsymbol{\alpha}}^{\boldsymbol{k}}, \mathring{D}_{\boldsymbol{i}} \mathring{\boldsymbol{c}}_{\boldsymbol{j}} \rangle \\ &= \frac{1}{\tau} \mathring{\gamma}_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}}. \end{aligned}$$

The coefficients $\mathring{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{a}}{}_{\boldsymbol{0}}$. Recall that $\mathring{\boldsymbol{e}}_{\boldsymbol{0}} = \boldsymbol{\partial}_\tau$ is tangent to geodesics —see Remark 4. Thus,

$$\mathring{\nabla}_{\boldsymbol{0}} \mathring{\boldsymbol{e}}_{\boldsymbol{0}} = \mathring{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{0}} \mathring{\boldsymbol{e}}_{\boldsymbol{c}},$$

from where it follows that

$$\mathring{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{a}}{}_{\boldsymbol{0}} = 0.$$

The coefficients $\mathring{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{0}}$ and $\mathring{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{0}}{}_{\boldsymbol{j}}$. In this case we have that

$$\mathring{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{0}} = \langle \mathring{\boldsymbol{\omega}}^{\boldsymbol{j}}, \mathring{\nabla}_{\boldsymbol{i}} \mathring{\boldsymbol{e}}_{\boldsymbol{0}} \rangle = \mathring{\chi}_{\boldsymbol{i}}{}^{\boldsymbol{j}},$$

where $\chi_{\boldsymbol{i}}{}^{\boldsymbol{j}}$ denote the components of the *Weingarten tensor*. Defining $\mathring{\chi}_{\boldsymbol{ij}} \equiv \eta_{\boldsymbol{jk}} \mathring{\chi}_{\boldsymbol{i}}{}^{\boldsymbol{k}}$, one has that $\mathring{\chi}_{\boldsymbol{ij}} = \mathring{\chi}_{(\boldsymbol{ij})}$ as the congruence defined by $\boldsymbol{\partial}_\tau$ can readily be verified to be hypersurface orthogonal. Thus, in this case $\mathring{\chi}_{\boldsymbol{ij}}$ coincides with the components of the extrinsic curvature of the hypersurfaces of constant τ. To compute $\mathring{\chi}_{\boldsymbol{ij}}$ recall that

$$\chi_{ab} = -\frac{1}{2} \mathcal{L}_{\boldsymbol{\partial}_\tau} h_{ab},$$

where $\mathcal{L}_{\boldsymbol{\partial}_\tau}$ denotes the Lie derivative along the direction of $\boldsymbol{\partial}_t$. As

$$\mathcal{L}_{\boldsymbol{\partial}_\tau} \mathring{\boldsymbol{h}} = \frac{1}{2} \mathcal{L}_{\boldsymbol{\partial}_\tau} \left(\tau^2 \mathring{\boldsymbol{\gamma}} \right) = 2\tau \mathring{\boldsymbol{\gamma}} = \frac{2}{\tau} \mathring{\boldsymbol{h}},$$

one concludes that

$$\mathring{\chi}_{\boldsymbol{ij}} = -\frac{1}{\tau} \delta_{\boldsymbol{ij}}.$$

Exploiting the metricity of the connection $\mathring{\nabla}$ one finds that, moreover,

$$\mathring{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{0}}{}_{\boldsymbol{j}} = -\mathring{\chi}_{\boldsymbol{ij}} = \frac{1}{\tau} \delta_{\boldsymbol{ij}}.$$

The coefficients $\mathring{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{j}}{}_{\boldsymbol{i}}$. In this case one readily finds that

$$\begin{aligned}
\mathring{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{j}}{}_{\boldsymbol{i}} &= \langle \mathring{\boldsymbol{\omega}}^{\boldsymbol{j}}, \mathring{\nabla}_{\boldsymbol{0}} \mathring{\boldsymbol{e}}_{\boldsymbol{i}} \rangle = \langle \mathring{\boldsymbol{\omega}}^{\boldsymbol{j}}, \mathring{\nabla}_{\boldsymbol{0}} \left(\frac{1}{\tau} \mathring{\boldsymbol{c}}_{\boldsymbol{i}} \right) \rangle \\
&= -\frac{1}{\tau} \langle \mathring{\boldsymbol{\alpha}}^{\boldsymbol{j}}, \mathring{\boldsymbol{c}}_{\boldsymbol{i}} \rangle = -\frac{1}{\tau^2} \langle \tau \mathring{\boldsymbol{\alpha}}^{\boldsymbol{j}}, \mathring{\boldsymbol{c}}_{\boldsymbol{i}} \rangle \\
&= -\frac{1}{\tau} \delta_{\boldsymbol{i}}{}^{\boldsymbol{j}}.
\end{aligned}$$

The coefficients $\mathring{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{0}}{}_{\boldsymbol{0}}$. In this case, one readily finds that

$$\mathring{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{0}}{}_{\boldsymbol{i}} = \langle \mathring{\boldsymbol{\omega}}^{\boldsymbol{0}}, \mathring{\nabla}_{\boldsymbol{0}} \mathring{\boldsymbol{e}}_{\boldsymbol{i}} \rangle = \langle \mathring{\boldsymbol{\omega}}^{\boldsymbol{0}}, \mathring{\nabla}_{\boldsymbol{0}} \left(\frac{1}{\tau} \mathring{\boldsymbol{c}}_{\boldsymbol{i}} \right) \rangle = -\frac{1}{\tau^2} \langle \mathbf{d}\tau, \mathring{\boldsymbol{c}}_{\boldsymbol{i}} \rangle = 0.$$

The coefficients $\mathring{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{0}}{}_{\boldsymbol{0}}$. Observing that $[\mathring{\boldsymbol{e}}_i, \mathring{\boldsymbol{e}}_{\boldsymbol{0}}] = 0$ and recalling that in the absence of torsion one has that

$$[\mathring{\boldsymbol{e}}_{\boldsymbol{i}}, \mathring{\boldsymbol{e}}_{\boldsymbol{0}}] = \left(\mathring{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{c}}{}_{\boldsymbol{0}} - \mathring{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{i}} \right) \boldsymbol{e}_{\boldsymbol{c}},$$

it follows from the previous results that

$$\mathring{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{0}}{}_{\boldsymbol{0}} = 0.$$

Remark 7. It follows that all the coefficients of the connection are smooth (C^∞) over $[\tau_\star, \infty) \times \mathcal{S}$.

Remark 8. For latter use it is observed that the extrinsic curvature (Weingarten tensor) can be written in abstract index notation as

$$\mathring{\chi}_{ij} = -\frac{1}{\tau} \mathring{h}_{ij}. \tag{18}$$

3.2.3 Conformal fields

The next step is the computation of the components of the conformal fields appearing in the extended conformal Einstein field equations. To this end, we make use of the conformal Einstein constraints discussed in Section 2.3.

We make use of an adapted frame with $\boldsymbol{e}_0 = \boldsymbol{\partial}_\tau$ and make the identification $\Omega \mapsto \Theta$ in equations (9a)-(9j). Observe that one has that

$$\mathring{D}_i\Omega = 0.$$

The scalars Σ and s. By definition one has that

$$\mathring{\Sigma} \equiv \boldsymbol{n}(\Theta) = -\partial_\tau\Theta = \tau.$$

The minus sign arises from the fact that in our conventions $(\mathbf{d}\tau)^\sharp = -\boldsymbol{\partial}_\tau$. Using the later in the conformal equation (9h) with $\lambda = 3$ one readily concludes

$$\mathring{s} = 1.$$

Components of the Schouten tensor. The constraint (9b) readily yields for $\Theta \geq 0$ that

$$\mathring{L}_i = 0.$$

The spatial components, $\mathring{L}_{ij}$, are computed using the constraint (9j). Observing that in our case $\mathring{D}_i\mathring{D}_j\Theta = 0$ one readily concludes that

$$\mathring{L}_{ij} = 0.$$

Thus, all the components of the Schouten tensor, except for its trace, vanish. This trace is proportional to the Ricci scalar of the metric (17).

Components of the rescaled Weyl tensor. The constraint (9i) offers an easy way of computing the magnetic part of the rescaled Weyl tensor. As $\mathring{D}_j\mathring{\chi}_{ki} = 0$ and we already know that $\mathring{L}_i = 0$, it follows then that $\mathring{d}_{ijk} = 0$ so that, in fact,

$$\mathring{d}^*_{ij} = 0.$$

To compute the electric part of the rescaled Weyl tensor we make use of the constraint equation (9j). This equation requires knowing the value of the Schouten tensor $\mathring{l}_{ij}$ of the metric $\mathring{\boldsymbol{h}}$. From the definition of the 3-dimensional Schouten tensor one readily finds that if $r[\mathring{\boldsymbol{\gamma}}] = -6$, then

$$\boldsymbol{Schouten}[\mathring{\boldsymbol{\gamma}}] = -\frac{1}{2}\mathring{\boldsymbol{\gamma}}.$$

Now, we have that $\mathring{\boldsymbol{h}} = \tau^2\mathring{\boldsymbol{\gamma}}$ so that $\mathring{\boldsymbol{h}}$ and $\mathring{\boldsymbol{\gamma}}$ are conformally related. However, the conformal factor does not depend on the spatial coordinates. It follows then, from the conformal transformation rule of the Schouten tensor that

$$\boldsymbol{Schouten}[\mathring{\boldsymbol{\gamma}}] = \boldsymbol{Schouten}[\mathring{\boldsymbol{h}}].$$

Hence, one has that

$$\mathring{l}_{ij} = -\frac{1}{2}\mathring{\gamma}_{ij} = -\frac{1}{\tau^2}\mathring{h}_{ij}.$$

Now, a calculation using equation (18) reveals that

$$\mathring{l}_{ij} = -\mathring{\chi}\big(\mathring{\chi}_{ij} - \frac{1}{4}\mathring{\chi}\mathring{h}_{ij}\big) + \mathring{\chi}_{ki}\mathring{\chi}_j{}^k - \frac{1}{4}\mathring{\chi}_{kl}\mathring{\chi}^{kl}\mathring{h}_{ij}$$

so that

$$d_{ij} = 0.$$

Remark 9. In summary, on has that the metric (17) is conformally flat.

Ricci scalar. Finally, although it does not appear as an unknown in the extended conformal Einstein equations, it is of interest to compute the Ricci scalar of the metric. To do this we observe that from the definition of the Friedrich scalar one has that

$$\mathring{R}\Theta = 24\big(s - \frac{1}{4}\mathring{\nabla}_c\mathring{\nabla}^c\Theta\big).$$

A computation readily yields $\mathring{\nabla}_c\mathring{\nabla}^c\Theta = -2$ so that one concludes that

$$\mathring{R} = \frac{72}{1-\tau^2}.$$

That is, the Ricci scalar is singular at $\tau = 1$.

Remark 10. Although the Ricci scalar of the background solution solution is singular, this will not pose any difficulty in our subsequent analysis as the Ricci scalar does not appear as an unknown in the extended conformal Einstein field equations.

4 Evolution equations

In this section we discuss the evolution system associated to the extended conformal Einstein equations (6) written in terms of a conformal Gaussian system. This evolution system is central in the discussion of the stability of the background spacetime. In addition, we also discuss the subsidiary evolution system satisfied by the zero-quantities associated to the field equations, (5a)-(5d), and the supplementary zero-quantities (7a)-(7c). The subsidiary system is key in the analysis of the so-called *propagation of the constraints* which allows to establish the relation between a solution to the extended conformal Einstein equations (6) and the Einstein field equations (2).

4.1 The conformal Gaussian gauge

In order to obtain suitable evolution equations for the conformal fields, we make use of a *conformal Gaussian gauge*. More precisely, we assume that we are working on a region $\mathcal{U} \subset \mathcal{M}$ which can be covered by a congruence of non-intersecting conformal geodesics. Choosing

$$\Theta_\star = \frac{1}{2}, \qquad \dot{\Theta}_\star = 0, \qquad \ddot{\Theta}_\star = -\frac{1}{2},$$

for $\tau = \tau_\star$, $\tau_\star \in (0,1)$, the following Proposition gives a conformal factor associated to the curves of the congruence —see e.g. [19], Proposition 5.1, page 133:

Proposition 2. *Let $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ denote an Einstein spacetime. Suppose that $(x(\tau), \boldsymbol{\beta}(\tau))$ is a solution to the conformal geodesic equations* (12a)-(12b) *and that $\{\boldsymbol{e_a}\}$ is a $\boldsymbol{g}$-orthogonal frame propagated along the curve according to* (13). *If*

$$\boldsymbol{g} = \Theta^2\tilde{\boldsymbol{g}}, \qquad \textit{such that}, \quad \boldsymbol{g}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) = -1,$$

then the conformal factor Θ satisfies

$$\Theta(\tau) = \Theta_\star + \dot{\Theta}_\star(\tau - \tau_\star) + \frac{1}{2}\ddot{\Theta}_\star(\tau - \tau_\star)^2,$$

where the coefficients $\Theta_\star$, $\dot{\Theta}_\star$, $\ddot{\Theta}_\star$ are constant along the conformal geodesic and are subject to the constraints

$$\dot{\Theta}_\star = \langle \boldsymbol{\beta}_\star, \dot{x}_\star \rangle \Theta_\star, \qquad \Theta_\star\ddot{\Theta}_\star = \frac{1}{2}\tilde{\boldsymbol{g}}^\sharp(\boldsymbol{\beta}_\star, \boldsymbol{\beta}_\star) + \frac{1}{6}\lambda.$$

Furthermore, along each conformal geodesic one has

$$\Theta\boldsymbol{\beta_0} = \dot{\Theta}_\star, \qquad \Theta\beta_{\boldsymbol{i}} = \Theta_\star\boldsymbol{\beta_{i\star}}.$$

Remark 11. Thus, if one has a congruence of non-intersecting conformal geodesics in a region $\mathcal{U}$ of spacetime, then the above proposition provides a *canonical way* of obtaining a conformal extension. This strategy naturally leads to a so-called *conformal Gaussian gauge.*

The Proposition 2 gives the conformal factor

$$\Theta(\tau) = \frac{1}{2}\big(1 - (\tau - \tau_\star)^2\big) \tag{19}$$

along the curves of the congruences. The choice of initial data for the conformal factor is associated to a congruence that leaves orthogonally a fiduciary initial hypersurface $\mathcal{S}_\star$ with $\tau = \tau_\star$ —notice, however, that the congruence of conformal geodesics is, in general, not hypersurface orthogonal.

Remark 12. Since the conformal factor Θ given by equation (19) does not depend on the initial data for the evolution equations it can be regarded as universal —i.e. valid not only for the background solution but also for perturbations thereof. Similarly, a consequence of Proposition 2, it follows that the components $d_{\boldsymbol{a}}$ of the the covector $\boldsymbol{d}$ are, in the same sense, universal.

Along the congruence of conformal geodesics one considers a $\boldsymbol{g}$-orthogonal frame $\{\boldsymbol{e_0}\}$ which is Weyl-propagated and such that $\boldsymbol{\tau} = \boldsymbol{e_0}$. The Weyl connection $\hat{\nabla}_a$ associated to the congruence then satisfies

$$\hat{\nabla}_{\boldsymbol{\tau}}\boldsymbol{e_a} = 0, \qquad \hat{\boldsymbol{L}}(\tau, \cdot) = 0,$$

which is equivalent to

$$\hat{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} = 0, \qquad f_{\boldsymbol{0}} = 0, \qquad \hat{L}_{\boldsymbol{0a}} = 0,$$

—see e.g. [19], Section 13.4, page 366. By choosing the parameter, τ, of the conformal geodesics as time coordinate one gets the additional gauge condition

$$\boldsymbol{e_0} = \boldsymbol{\partial}_\tau, \qquad e_{\boldsymbol{0}}{}^\mu = \delta_0{}^\mu.$$

On $\mathcal{S}_\star$ we choose some local coordinates $\underline{x} = (x^\alpha)$. Assuming that each curve of the congruence intersects $\mathcal{S}_\star$ only once, one can extend the coordinates off the initial hypersurface by requiring them to be constant along the conformal geodesic which intersects $\mathcal{S}_\star$ at the point with coordinates $\underline{x}$. The coordinates $(\tau, \underline{x})$ thus obtained are known as *conformal Gaussian coordinates.*

4.2 The main evolution system

One of the main advantages of writing the conformal field equations in terms of zero-quantities and using a frame formalism is that the various evolution equations can be readily identified as certain components of the zero-quantities.

The required evolution equations for the frame components, connection coefficients and components of the Schouten tensor are obtained from the conditions

$$\hat{\Sigma}_{\boldsymbol{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}\boldsymbol{e_c} = 0, \qquad \hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{d0b}} = 0, \qquad \hat{\Delta}_{\boldsymbol{0bc}} = 0. \tag{20}$$

In particular, the evolution equation for components of the covector f_a defining the Weyl connection is given by

$$\hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{c0b}} = 0.$$

In the following we analyse each of these equations in more detail.

4.2.1 Evolution equations for the components of the frame

Now, starting from equation (5a)

$$\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}\boldsymbol{e_c} \equiv [\boldsymbol{e_a}, \boldsymbol{e_b}] - (\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} - \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{a}})\boldsymbol{e_c}$$

and writing $\boldsymbol{e_a} = e_{\boldsymbol{a}}{}^\mu \boldsymbol{\partial}_\mu$, it follows that the condition $\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}\boldsymbol{e_c} = 0$ implies

$$(\partial_{\boldsymbol{a}} e_{\boldsymbol{b}}{}^\nu - \partial_{\boldsymbol{b}} e_{\boldsymbol{a}}{}^\nu) = (\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} - \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{a}}) e_{\boldsymbol{c}}{}^\nu, \qquad \partial_{\boldsymbol{a}} \equiv e_{\boldsymbol{a}}{}^\mu \partial_\mu.$$

Setting $\boldsymbol{a} = \boldsymbol{0}$ it follows that the evolution equation for the components of the frame takes the form

$$\partial_{\boldsymbol{0}} e_{\boldsymbol{b}}{}^\nu = -\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{0}} e_{\boldsymbol{c}}{}^\nu. \tag{21}$$

4.2.2 Evolution equations for the components of the connection

In order to obtain the evolution equation for the components of the frame not determined by the gauge conditions one considers the condition $\hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{d0b}} = 0$.

Now, since

$$\hat{P}^{\boldsymbol{c}}{}_{\boldsymbol{d0b}} = \partial_{\boldsymbol{0}}(\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}) - \partial_{\boldsymbol{b}}(\hat{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}) + (\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}}\hat{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{f}} - \hat{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}}\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{f}}) + \hat{\Gamma}_{\boldsymbol{f}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}(\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{0}} - \hat{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{f}}{}_{\boldsymbol{b}}),$$

then using the gauge condition $\hat{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}} = 0$ one has that

$$\hat{P}^{\boldsymbol{c}}_{\boldsymbol{d0b}} = \boldsymbol{e_0}(\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}) + \hat{\Gamma}_{\boldsymbol{f}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{0}}.$$

In addition, observing that

$$S_{\boldsymbol{d}[\boldsymbol{0}}{}^{\boldsymbol{ce}}\hat{L}_{\boldsymbol{b}]\boldsymbol{e}} = \delta_{\boldsymbol{d}}{}^{\boldsymbol{c}}\hat{L}_{\boldsymbol{b0}} + \delta_{\boldsymbol{0}}{}^{c}\hat{L}_{\boldsymbol{bd}} - g_{\boldsymbol{d0}}g^{\boldsymbol{ce}}\hat{L}_{\boldsymbol{be}} - \delta_{\boldsymbol{d}}{}^{\boldsymbol{c}}\hat{L}_{\boldsymbol{0b}} - \delta_{\boldsymbol{b}}{}^{\boldsymbol{c}}\hat{L}_{\boldsymbol{b0}} + g_{\boldsymbol{db}}g^{\boldsymbol{ce}}\hat{L}_{\boldsymbol{0e}},$$

together with the gauge condition $\hat{L}_{\boldsymbol{0a}} = 0$, it follows that

$$\hat{\rho}^{\boldsymbol{c}}{}_{\boldsymbol{d0b}} = \Theta\hat{d}^{\boldsymbol{c}}{}_{\boldsymbol{d0b}} + 2\delta_{\boldsymbol{d}}{}^{\boldsymbol{c}}\hat{L}_{\boldsymbol{b0}} + 2\delta_{\boldsymbol{0}}{}^{\boldsymbol{c}}\hat{L}_{\boldsymbol{bd}} - 2\eta_{\boldsymbol{d0}}\eta^{\boldsymbol{ce}}\hat{L}_{\boldsymbol{be}},$$

where it has been used that $g_{\boldsymbol{d0}}g^{\boldsymbol{ce}} = \eta_{\boldsymbol{d0}}\eta^{\boldsymbol{ce}}$. It follows that the evolution equation for the coefficients of the connection not determined by the gauge is given by

$$\partial_{\boldsymbol{0}}(\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}) + \hat{\Gamma}_{\boldsymbol{f}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{0}} = 2\eta_{\boldsymbol{d0}}\eta^{\boldsymbol{ce}}\hat{L}_{\boldsymbol{be}} - 2\delta_{\boldsymbol{d}}{}^{\boldsymbol{c}}\hat{L}_{\boldsymbol{b0}} - 2\delta_{\boldsymbol{0}}{}^{\boldsymbol{c}}\hat{L}_{\boldsymbol{bd}} - \Theta\hat{d}^{\boldsymbol{c}}{}_{\boldsymbol{d0b}}.$$

The above expression can be written in terms of the Levi-Civita connection coefficients $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$ and the 1-form $f_{\boldsymbol{a}}$ through the relation

$$\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} = \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} + S_{\boldsymbol{ab}}{}^{\boldsymbol{cd}}f_{\boldsymbol{d}}.$$

In particular, since

$$f_{\boldsymbol{a}} = \frac{1}{4}\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{b}},$$

it follows from the gauge condition $f_{\boldsymbol{0}} = 0$ and $\hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{c0b}} = 0$ that

$$\partial_{\boldsymbol{0}}f_{\boldsymbol{i}} + f_{\boldsymbol{j}}\hat{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{0}} = \hat{L}_{\boldsymbol{i0}}.$$

4.2.3 Evolution equations for the components of the Schouten tensor

The evolution equations for the components of the Schouten tensor not determined by the gauge are obtained from the condition $\hat{\Delta}_{\boldsymbol{0db}} = 0$. It follows then that

$$\nabla_{\boldsymbol{0}}\hat{L}_{\boldsymbol{db}} - \nabla_{\boldsymbol{d}}\hat{L}_{\boldsymbol{0b}} - d_{\boldsymbol{a}}d^{\boldsymbol{a}}{}_{\boldsymbol{b0d}} = 0.$$

However, in the conformal Gaussian gauge one has that $\hat{L}_{\boldsymbol{0b}} = 0$ so that the evolution equation for the components of the Schouten tensor can be simplified to

$$\partial_{\boldsymbol{0}}\hat{L}_{\boldsymbol{db}} = \hat{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}\hat{L}_{\boldsymbol{cb}} + \hat{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}\hat{L}_{\boldsymbol{dc}} + d_{\boldsymbol{a}}d^{\boldsymbol{a}}{}_{\boldsymbol{b0d}} = 0,$$

as $\hat{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}} = 0$.

4.2.4 Evolution equations for the components of the rescaled Weyl tensor

The evolution equations for the components of the Weyl tensor are extracted from the decomposition of the zero-quantity $\Lambda_{\boldsymbol{bcd}}$. As this zero-quantity contains a contracted derivative, the decomposition is more involved than for the other zero-quantities. As in the case of the conformal constraint equations, this analysis is best done using the decomposition of the rescaled Weyl tensor in its electric and magnetic parts with respect to the tangent to the congruence of conformal geodesics on which our gauge is based.

In the following, let $h_a{}^b$ denote the projector to the hyperplanes orthogonal to the tangent vector field τ^a to the congruence of conformal geodesics. One has that

$$h_a{}^b = \delta_a{}^b - \tau_a\tau^b,$$

so that

$$\begin{aligned}\Lambda_{\boldsymbol{bcd}} &= \nabla^{\boldsymbol{a}}(\delta_{\boldsymbol{a}}{}^{\boldsymbol{f}} d_{\boldsymbol{fbcd}}) = \delta_{\boldsymbol{a}}{}^{\boldsymbol{f}}\nabla^{\boldsymbol{a}} d_{\boldsymbol{fbcd}} \\ &= \tau^{\boldsymbol{f}}\tau_{\boldsymbol{a}}\nabla^{\boldsymbol{a}} d_{\boldsymbol{fbcd}} + h_{\boldsymbol{a}}{}^{\boldsymbol{f}}\nabla^{\boldsymbol{a}} d_{\boldsymbol{fbcd}} \\ &= \mathcal{D} d_{\boldsymbol{fbcd}} + \mathcal{D}^{\boldsymbol{f}} d_{\boldsymbol{fbcd}}\tau^{\boldsymbol{f}},\end{aligned}$$

where $\mathcal{D}_a \equiv h_a{}^b\nabla_b$ and $\mathcal{D} \equiv \tau^a\nabla_a$ denote, respectively, the Sen and Fermi covariant derivatives associated to the congruence. Now, observing that the acceleration and Weingarten tensor of the congruence are given, respectively by

$$\begin{aligned}a_a &\equiv \tau^b\nabla_b\tau_a = \mathcal{D}\tau_a, \\ \chi_{ab} &\equiv h_a{}^c\nabla_c\tau_b = \mathcal{D}_a\tau_b,\end{aligned}$$

it follows that

$$\begin{aligned}\Lambda_{\boldsymbol{bcd}}\tau^{\boldsymbol{c}} = \Lambda_{\boldsymbol{b0d}} &= \tau^{\boldsymbol{c}}\mathcal{D}(\tau^{\boldsymbol{f}} d_{\boldsymbol{fbcd}}) + \tau^{\boldsymbol{c}}\mathcal{D}^{\boldsymbol{f}} d_{\boldsymbol{fbcd}} - a^{\boldsymbol{f}}\tau^{\boldsymbol{c}} d_{\boldsymbol{fbcd}} \\ &= \mathcal{D}(\tau^{\boldsymbol{f}}\tau^{\boldsymbol{c}} d_{\boldsymbol{fbcd}}) + \mathcal{D}^{\boldsymbol{f}} d_{\boldsymbol{fb0d}} - a^{\boldsymbol{f}} d_{\boldsymbol{fb0d}} - a^{\boldsymbol{c}} d_{\boldsymbol{0bcd}} - \chi^{\boldsymbol{fc}} d_{\boldsymbol{fbcd}},\end{aligned}$$

so that

$$\Lambda_{\boldsymbol{b0d}} = \mathcal{D} d_{\boldsymbol{0b0d}} + \mathcal{D}^{\boldsymbol{f}} d_{\boldsymbol{fb0d}} - a^{\boldsymbol{f}} d_{\boldsymbol{fb0d}} - a^{\boldsymbol{c}} d_{\boldsymbol{0bcd}} - \chi^{\boldsymbol{fc}} d_{\boldsymbol{fbcd}}.$$

To further simplify we make use of the decomposition

$$d_{\boldsymbol{abcd}} = 2(l_{\boldsymbol{b}[\boldsymbol{c}} d_{\boldsymbol{d}]\boldsymbol{a}} - l_{\boldsymbol{a}[\boldsymbol{c}} d_{\boldsymbol{d}]\boldsymbol{b}}) - 2(\tau_{[\boldsymbol{c}} d^*{}_{\boldsymbol{d}]\boldsymbol{e}}\epsilon^{\boldsymbol{e}}{}_{\boldsymbol{ab}} + \tau_{[\boldsymbol{a}} d^*{}_{\boldsymbol{b}]\boldsymbol{e}}\epsilon^{\boldsymbol{e}}{}_{\boldsymbol{cd}}),$$

of the rescaled Weyl tensor in terms of its electric part $d_{\boldsymbol{ab}}$ and magnetic part $d^*_{\boldsymbol{ab}}$ with respect to the vector field t^a where $l_{ab} = h_{ab} - \tau_a\tau_b$ to obtain

$$\begin{aligned}\Lambda_{\boldsymbol{b0d}} =& \mathcal{D} d_{\boldsymbol{bd}} + \mathcal{D}^{\boldsymbol{f}} d_{\boldsymbol{fbd}} - a^{\boldsymbol{f}} d_{\boldsymbol{fbd}} - a^{\boldsymbol{c}} d_{\boldsymbol{bcd}} - 2\chi^{\boldsymbol{fc}}(l_{\boldsymbol{b}[\boldsymbol{c}} d_{\boldsymbol{d}]\boldsymbol{f}} - l_{\boldsymbol{f}[\boldsymbol{c}} d_{\boldsymbol{d}]\boldsymbol{b}}) \\ &+ 2\chi^{\boldsymbol{fc}}(\tau_{[\boldsymbol{c}} d^*{}_{\boldsymbol{d}]\boldsymbol{e}}\epsilon^{\boldsymbol{e}}{}_{\boldsymbol{fb}} + \tau_{[\boldsymbol{f}} d^*{}_{\boldsymbol{b}]\boldsymbol{e}}\epsilon^{\boldsymbol{e}}{}_{\boldsymbol{cd}}).\end{aligned}$$

To finally extract the required evolution equation we consider $\Lambda_{(\boldsymbol{b}|\boldsymbol{0}|\boldsymbol{d})}$. Observing that all the involved tensors are spatial one obtains, after some simplification, that

$$\Lambda_{(\boldsymbol{b}|\boldsymbol{0}|\boldsymbol{d})} = \partial_{\boldsymbol{0}} d_{\boldsymbol{ij}} + \epsilon^{\boldsymbol{kl}}{}_{(\boldsymbol{i}} D_{|\boldsymbol{l}|} d^*{}_{\boldsymbol{j})\boldsymbol{k}} - 2a_{\boldsymbol{l}}\epsilon^{\boldsymbol{kl}}{}_{(\boldsymbol{i}} d^*{}_{\boldsymbol{j})\boldsymbol{k}} + \chi d_{\boldsymbol{ij}} - 2\chi^{\boldsymbol{k}}{}_{(\boldsymbol{i}} d_{\boldsymbol{j})\boldsymbol{k}} = 0. \tag{22}$$

To complete the system of evolution equations for the components of the Weyl tensor one carries out a completely analogous calculation with the zero-quantity

$$\Lambda^*_{\boldsymbol{bcd}} \equiv D^{\boldsymbol{a}} d^*{}_{\boldsymbol{abcd}}$$

and the decomposition

$$d^*_{\boldsymbol{abcd}} = 2(l_{\boldsymbol{b}[\boldsymbol{c}} d^*{}_{\boldsymbol{d}]\boldsymbol{a}} - l_{\boldsymbol{f}[\boldsymbol{c}} d^*{}_{\boldsymbol{d}]\boldsymbol{b}}) + 2(\tau_{[\boldsymbol{c}} d_{\boldsymbol{d}]\boldsymbol{e}}\epsilon^{\boldsymbol{e}}{}_{\boldsymbol{ab}} + \tau_{[\boldsymbol{a}} d_{\boldsymbol{b}]\boldsymbol{e}}\epsilon^{\boldsymbol{e}}{}_{\boldsymbol{cd}}),$$

where the Hodge dual of the rescaled Weyl tensor is defined as

$$d^*_{abcd} \equiv \frac{1}{2}\epsilon^{ef}_{ab} d_{cdef}.$$

More precisely, the decomposition

$$\Lambda^*_{\boldsymbol{bcd}} = \tau^{\boldsymbol{a}}\mathcal{D} d^*_{\boldsymbol{abcd}} + \mathcal{D}^{\boldsymbol{a}} d^*{}_{\boldsymbol{abcd}},$$

leads, after a lengthy computation, to the evolution equation

$$\Lambda^*{}_{(\boldsymbol{i}|\boldsymbol{0}|\boldsymbol{j})} = \partial_{\boldsymbol{0}} d^*_{\boldsymbol{ij}} - \epsilon^{\boldsymbol{k}}{}_{\boldsymbol{l}(\boldsymbol{i}} D^{\boldsymbol{l}} d_{\boldsymbol{j})\boldsymbol{k}} - 2a^{\boldsymbol{l}}\epsilon_{\boldsymbol{l}(\boldsymbol{i}}{}^{\boldsymbol{k}} d_{\boldsymbol{j})\boldsymbol{k}} + \chi d^*_{\boldsymbol{ij}} - 2\chi^{\boldsymbol{k}}{}_{(\boldsymbol{i}} d^*_{\boldsymbol{j})\boldsymbol{k}} = 0, \tag{23}$$

in which all the fields are spatial.

Remark 13. The zero-quantities $\Lambda_{\boldsymbol{bcd}}$ and $\Lambda^*_{\boldsymbol{bcd}}$ are no independent. In fact, $\Lambda_{\boldsymbol{bcd}} = 0$ if and only if $\Lambda^*_{\boldsymbol{bcd}} = 0$.

Remark 14. Equations (22) and (23) imply a symmetric hyperbolic evolution system for the (ten) independent components of the fields $E_{\boldsymbol{ab}}$ and $B_{\boldsymbol{ab}}$ —see e.g. [1] for explicit expressions of the associated matrices.

4.3 The subsidiary evolution system

The analysis of the relation between the solutions to the evolution equations and actual solutions to the full conformal Einstein field equations, the so-called *propagation of the constraints*, requires the construction of a system of *subsidiary evolution equations for the zero-quantities* associated to the conformal equations, (5a)-(5d), and the gauge conditions (7a)-(7c). for the standard argument of the propagation of the constraints to follow through, the subsidiary system is required to be homogeneous in the zero-quantities. If this is the case, then it follows from the uniqueness of solutions to symmetric hyperbolic systems that if the zero-quantities vanish initially, then they will vanish for all later times as the vanishing (zero) solution is always a solution of a homogeneous evolution equation.

4.3.1 General remarks

The basic assumption in the construction of the subsidiary evolution system is that the evolution equations associated to the extended conformal field equations are satisfied. Hence, we assume that

$$\hat{\Sigma}_{\boldsymbol{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = 0, \qquad \hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{d0b}} = 0, \qquad \hat{\Delta}_{\boldsymbol{0bc}} = 0,$$

together with

$$\Lambda_{(\boldsymbol{i}|\boldsymbol{0}|\boldsymbol{j})} = 0, \qquad \Lambda^*{}_{(\boldsymbol{i}|\boldsymbol{0}|\boldsymbol{j})} = 0.$$

These evolution equations have been constructed using the gauge conditions

$$f_{\boldsymbol{0}} = 0, \qquad \hat{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} = 0, \qquad \hat{L}_{\boldsymbol{0b}} = 0.$$

These gauge conditions will also be used in the construction of the subsidiary evolution system. Accordingly, the construction requires the evolution equations for the additional zero-quantities δ_a, γ_{ab} and ς_{ab} which are associated to the gauge. In our gauge $d_0 = 0$ so that

$$\delta_{\boldsymbol{0}} = 0.$$

Since $\hat{L}_{\boldsymbol{0b}} = 0$, by virtue of the definition of $S_{ab}{}^{cd}$ and the evolution equation for the covector β_a, namely,

$$\hat{\nabla}_{\boldsymbol{0}}\beta_{\boldsymbol{a}} + \beta_{\boldsymbol{0}}\beta_{\boldsymbol{a}} - \frac{1}{2}\eta_{\boldsymbol{0a}}(\beta_{\boldsymbol{e}}\beta^{\boldsymbol{e}} - 2\lambda\Theta^{-2}) = 0,$$

it follows that

$$\gamma_{\boldsymbol{0b}} = \hat{L}_{\boldsymbol{0b}} - \hat{\nabla}_{\boldsymbol{0}}\beta_{\boldsymbol{b}} - \frac{1}{2}S_{\boldsymbol{0b}}{}^{\boldsymbol{ef}}\beta_{\boldsymbol{e}}\beta_{\boldsymbol{f}} + \lambda\Theta^{-2}\eta_{\boldsymbol{0b}} = 0.$$

As a result of the Θ^{-2} in the last term of this equation, it can only be used away from the conformal boundary —this is, however, not a problem in our analysis as the propagation of the constraints only need to be considered in the regions where $\Theta \neq 0$. Moreover, by virtue of the gauge conditions and the evolution equation for the covector f_a, we have

$$\varsigma_{\boldsymbol{0b}} = -\hat{L}_{\boldsymbol{b0}} - \hat{\nabla}_{\boldsymbol{0}}f_{\boldsymbol{b}} + \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}f_{\boldsymbol{e}} = 0.$$

4.3.2 The subsidiary equation for the torsion

To obtain the subsidiary equation for the no-torsion condition we consider the totally antisymmetric covariant derivative $\hat{\nabla}_{[\boldsymbol{a}}\hat{\Sigma}_{\boldsymbol{b}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}]}$ and observe that

$$3\hat{\nabla}_{[\boldsymbol{0}}\hat{\Sigma}_{\boldsymbol{b}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}]} = \hat{\nabla}_{\boldsymbol{0}}\hat{\Sigma}_{\boldsymbol{b}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}} - \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\hat{\Sigma}_{\boldsymbol{c}}{}^{\boldsymbol{d}}{}_{\boldsymbol{e}} - \hat{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\hat{\Sigma}_{\boldsymbol{e}}{}^{\boldsymbol{d}}{}_{\boldsymbol{b}}. \tag{24}$$

On the other hand, from the first Bianchi identity

$$\hat{R}^{\boldsymbol{d}}{}_{[\boldsymbol{cab}]} + \hat{\nabla}_{[\boldsymbol{a}}\hat{\Sigma}_{\boldsymbol{b}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}} + \hat{\Sigma}_{[\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{b}}\hat{\Sigma}_{\boldsymbol{c}]}{}^{\boldsymbol{d}}{}_{\boldsymbol{e}} = 0,$$

and the definition of $\hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{cab}}$ one obtains

$$\hat{\nabla}_{[\boldsymbol{a}}\hat{\Sigma}_{\boldsymbol{b}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}]} = -\hat{\Xi}^{\boldsymbol{d}}{}_{[\boldsymbol{cab}]} - \hat{\Sigma}_{[\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{b}}\hat{\Sigma}_{\boldsymbol{c}]}{}^{\boldsymbol{d}}{}_{\boldsymbol{e}}, \tag{25}$$

where it has been used that, by construction, $\hat{\rho}^{\boldsymbol{d}}{}_{[\boldsymbol{cab}]} = 0$. The desired evolution equation is obtained combining equations (24) and (25) to yield

$$\hat{\nabla}_{\boldsymbol{0}}\hat{\Sigma}_{\boldsymbol{b}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}} = -\frac{1}{3}\hat{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\hat{\Sigma}_{\boldsymbol{e}}{}^{\boldsymbol{d}}{}_{\boldsymbol{b}} - \frac{1}{3}\hat{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\hat{\Sigma}_{\boldsymbol{e}}{}^{\boldsymbol{d}}{}_{\boldsymbol{b}} - \hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{0bc}}. \tag{26}$$

This evolution equation is homogeneous in the various zero-quantities.

4.3.3 The subsidiary equation for the Ricci identity

To obtain a subsidiary equation for the Ricci identity, we consider the totally symmetrised covariant derivative $\hat{\nabla}_{[\boldsymbol{a}}\hat{\Xi}^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]}$ and observe that

$$3\hat{\nabla}_{[\boldsymbol{0}}\hat{\Xi}^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]} = \hat{\nabla}_{\boldsymbol{0}}\hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{ebc}} - \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{0}}\hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{ecf}} - \hat{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{f}}{}_{\boldsymbol{0}}\hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{efb}}. \tag{27}$$

Using the second Bianchi identity

$$\hat{\nabla}_{[\boldsymbol{a}}\hat{R}^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]} + \hat{\Sigma}_{[\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{b}}\hat{R}^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{c}]\boldsymbol{f}} = 0$$

and the definition of $\hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{ebc}}$ it follows that

$$\hat{\nabla}_{[\boldsymbol{a}}\hat{\Xi}^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]} = -\hat{\Sigma}_{[\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{b}}\hat{R}^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{c}]\boldsymbol{f}} - \hat{\nabla}_{[\boldsymbol{a}}\hat{\rho}^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]}. \tag{28}$$

The first term on the right-hand side is already of the required form. The second one needs to be analysed in more detail. For this, it is recalled that

$$\hat{\rho}^{\boldsymbol{d}}{}_{\boldsymbol{ebc}} \equiv C^{\boldsymbol{d}}{}_{\boldsymbol{ebc}} + 2S_{\boldsymbol{e}[\boldsymbol{b}}{}^{\boldsymbol{df}}\hat{L}_{\boldsymbol{c}]\boldsymbol{f}}.$$

Thus,

$$\hat{\nabla}_{[\boldsymbol{a}}\hat{\rho}^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]} = \hat{\nabla}_{[\boldsymbol{a}}C^{\boldsymbol{d}}_{|\boldsymbol{e}|\boldsymbol{bc}]} + 2S_{\boldsymbol{e}[\boldsymbol{b}}{}^{\boldsymbol{df}}\hat{\nabla}_{\boldsymbol{a}}\hat{L}_{\boldsymbol{c}]\boldsymbol{f}}.$$

To further expand this expression we consider the combination $\epsilon_{\boldsymbol{f}}{}^{\boldsymbol{abc}}\hat{\nabla}_{\boldsymbol{a}}\hat{\rho}^{\boldsymbol{d}}{}_{\boldsymbol{ebc}}$. A direct computation shows that

$$\hat{\nabla}_{[\boldsymbol{a}}C^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]} = \nabla_{[\boldsymbol{a}}C^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]} + \delta_{[\boldsymbol{a}}{}^{\boldsymbol{d}}f_{|\boldsymbol{f}}C^{\boldsymbol{f}}{}_{\boldsymbol{e}|\boldsymbol{bc}]} + \eta_{\boldsymbol{e}[\boldsymbol{a}}f^{\boldsymbol{f}}C^{\boldsymbol{d}}{}_{|\boldsymbol{f}|\boldsymbol{bc}]}.$$

Moreover, one has

$$\epsilon_{\boldsymbol{f}}{}^{\boldsymbol{abc}}\nabla_{\boldsymbol{a}}C^{\boldsymbol{d}}{}_{\boldsymbol{ebc}} = -\epsilon_{\boldsymbol{e}}{}^{\boldsymbol{dgh}}\nabla_{\boldsymbol{a}}C^{\boldsymbol{a}}{}_{\boldsymbol{fgh}}.$$

Thus, using that $C^{\boldsymbol{c}}{}_{\boldsymbol{dab}} = \Theta d^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ and the definition of the zero quantity $\Lambda_{\boldsymbol{abc}}$ it follows that

$$\epsilon_{\boldsymbol{f}}{}^{\boldsymbol{abc}}\hat{\nabla}_{\boldsymbol{a}}C^{\boldsymbol{d}}{}_{\boldsymbol{ebc}} = \Theta\epsilon_{\boldsymbol{e}}{}^{\boldsymbol{dgh}}\Lambda_{\boldsymbol{fgh}} + 2\nabla^{\boldsymbol{g}}\Theta d^{*\boldsymbol{d}}{}_{\boldsymbol{efg}} + 2\Theta f^{\boldsymbol{g}}d^{*}{}_{\boldsymbol{gef}}{}^{\boldsymbol{d}} + 2\Theta f^{\boldsymbol{g}}d^{*\boldsymbol{d}}{}_{\boldsymbol{gfe}}.$$

A similar computation using the definition of $\hat{\Delta}_{\boldsymbol{abc}}$ yields

$$2\epsilon_{\boldsymbol{f}}{}^{\boldsymbol{abc}}S_{\boldsymbol{eb}}{}^{\boldsymbol{dg}}\hat{\Delta}_{\boldsymbol{acg}} = 2\Theta\beta_{\boldsymbol{g}}d^{*\boldsymbol{g}}{}_{\boldsymbol{ef}}{}^{\boldsymbol{d}} - 2\Theta\beta_{\boldsymbol{g}}d^{*\boldsymbol{gd}}{}_{\boldsymbol{fe}}.$$

Thus, using the symmetries of $d^{*}{}_{\boldsymbol{cdab}}$ and the definition of $\delta_{\boldsymbol{a}}$ one concludes that

$$\epsilon_{\boldsymbol{f}}{}^{\boldsymbol{abc}}\hat{\nabla}_{\boldsymbol{a}}\hat{\rho}^{\boldsymbol{d}}{}_{\boldsymbol{ebc}} = \Theta\epsilon_{\boldsymbol{e}}{}^{\boldsymbol{dgh}}\Lambda_{\boldsymbol{fgh}} - 2\Theta\delta^{\boldsymbol{g}}d^{*\boldsymbol{d}}{}_{\boldsymbol{efg}} + \epsilon_{\boldsymbol{f}}{}^{\boldsymbol{abc}}S_{\boldsymbol{eb}}{}^{\boldsymbol{dg}}\hat{\Delta}_{\boldsymbol{acg}}.$$

Alternatively, using the properties of the generalised Hodge duals we can write

$$\hat{\nabla}_{[\boldsymbol{a}}\hat{\rho}^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]} = \frac{1}{6}\Theta\epsilon^{\boldsymbol{f}}{}_{\boldsymbol{abc}}\epsilon_{\boldsymbol{e}}{}^{\boldsymbol{dgh}}\Lambda_{\boldsymbol{fgh}} - \frac{1}{3}\Theta\epsilon^{\boldsymbol{f}}{}_{\boldsymbol{abc}}\delta^{\boldsymbol{g}}d^{*\boldsymbol{d}}{}_{\boldsymbol{efg}} - S_{\boldsymbol{e}[\boldsymbol{b}}{}^{\boldsymbol{dg}}\hat{\Delta}_{\boldsymbol{ac}]\boldsymbol{g}}.$$

Combining the expressions, we obtain the following evolution equation

$$\begin{aligned}\hat{\nabla}_{\boldsymbol{0}}\hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{ebc}} =& \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{0}}\hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{ecf}} + \hat{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{f}}{}_{\boldsymbol{0}}\hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{efb}} - \hat{\Sigma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{c}}\hat{R}^{\boldsymbol{d}}{}_{\boldsymbol{e0f}} - \frac{1}{2}\Theta\epsilon^{\boldsymbol{f}}{}_{\boldsymbol{0bc}}\epsilon_{\boldsymbol{e}}{}^{\boldsymbol{dgh}}\Lambda_{\boldsymbol{fgh}} \\ &+ \epsilon^{\boldsymbol{f}}{}_{\boldsymbol{0bc}}\delta^{\boldsymbol{g}}d^{*\boldsymbol{d}}{}_{\boldsymbol{efg}} + 3S_{\boldsymbol{e0}}{}^{\boldsymbol{dg}}\hat{\Delta}_{\boldsymbol{cbg}},\end{aligned} \tag{29}$$

which is homogeneous in the zero-quantities.

4.3.4 Subsidiary equation for the Cotton equation

Now, to compute the subsidiary equation for the Cotton equation we consider $\hat{\nabla}_{[\boldsymbol{a}}\hat{\Delta}_{\boldsymbol{bc}]\boldsymbol{d}}$. On the one hand, a direct computation yields

$$3\hat{\nabla}_{[\boldsymbol{0}}\hat{\Delta}_{\boldsymbol{bc}]\boldsymbol{d}} = \hat{\nabla}_{\boldsymbol{0}}\hat{\Delta}_{\boldsymbol{bcd}} - \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\hat{\Delta}_{\boldsymbol{ced}} - \hat{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\hat{\Delta}_{\boldsymbol{ebd}}.$$

On the other hand, using the definition of $\hat{\Xi}^{\boldsymbol{e}}{}_{\boldsymbol{cab}}$ and the symmetries of $\hat{\rho}^{\boldsymbol{e}}{}_{\boldsymbol{cab}}$ one obtains

$$\hat{\nabla}_{[\boldsymbol{a}}\hat{\Delta}_{\boldsymbol{bc}]\boldsymbol{d}} = -\hat{\Xi}^{\boldsymbol{e}}{}_{[\boldsymbol{cab}]}\hat{L}_{\boldsymbol{ed}} - \hat{\Xi}^{\boldsymbol{e}}{}_{\boldsymbol{d}[\boldsymbol{ab}}\hat{L}_{\boldsymbol{c}]\boldsymbol{e}} - \hat{\rho}^{\boldsymbol{e}}{}_{\boldsymbol{d}[\boldsymbol{ab}}\hat{L}_{\boldsymbol{c}]\boldsymbol{e}} + \hat{\Sigma}_{[\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{b}}\hat{\nabla}_{|\boldsymbol{e}|}\hat{L}_{\boldsymbol{c}]\boldsymbol{d}} - \hat{\nabla}_{[\boldsymbol{a}}d_{|\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{d}|\boldsymbol{bc}]} - d_{\boldsymbol{e}}\hat{\nabla}_{[\boldsymbol{a}}d^{\boldsymbol{e}}{}_{|\boldsymbol{d}|\boldsymbol{bc}]}.$$

Using the definition of δ_a and γ_{ab} one finds that

$$\hat{\nabla}_{[\boldsymbol{a}}d_{|\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{d}|\boldsymbol{bc}]} = -\Theta\delta_{[\boldsymbol{a}}\beta_{|\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{d}|\boldsymbol{bc}]} - \Theta\gamma_{[\boldsymbol{a}|\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{d}|\boldsymbol{bc}]} - \Theta f_{[\boldsymbol{a}}\beta_{|\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{d}|\boldsymbol{bc}]} + \Theta\hat{L}_{[\boldsymbol{a}|\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{d}|\boldsymbol{bc}]}.$$

Finally, a calculation shows that $\epsilon_{\boldsymbol{f}}{}^{\boldsymbol{abc}}\nabla_{\boldsymbol{a}}d^{\boldsymbol{e}}{}_{\boldsymbol{dbc}} = \epsilon_{\boldsymbol{d}}{}^{\boldsymbol{egh}}\nabla_{\boldsymbol{a}}d^{\boldsymbol{e}}{}_{\boldsymbol{fgh}}$, so that using

$$\hat{\nabla}_{[\boldsymbol{a}}C^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]} = \nabla_{[\boldsymbol{a}}C^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]} + \delta_{[\boldsymbol{a}}{}^{\boldsymbol{d}}f_{|\boldsymbol{f}}C^{\boldsymbol{f}}{}_{\boldsymbol{e}|\boldsymbol{bc}]} + \eta_{\boldsymbol{e}[\boldsymbol{a}}f^{\boldsymbol{f}}C^{\boldsymbol{d}}{}_{|\boldsymbol{f}|\boldsymbol{bc}]},$$

and the properties of the generalised duals we find that

$$\hat{\nabla}_{[\boldsymbol{a}}d^{\boldsymbol{e}}{}_{|\boldsymbol{d}|\boldsymbol{bc}]} = \frac{1}{6}\epsilon_{\boldsymbol{abc}}{}^{\boldsymbol{f}}\epsilon_{\boldsymbol{d}}{}^{\boldsymbol{egh}}\Lambda_{\boldsymbol{fgh}} + \delta_{[\boldsymbol{a}}{}^{\boldsymbol{e}}f_{|\boldsymbol{f}}d^{\boldsymbol{f}}{}_{\boldsymbol{d}|\boldsymbol{bc}]} + \eta_{\boldsymbol{d}[\boldsymbol{a}}f^{\boldsymbol{f}}d^{\boldsymbol{e}}{}_{|\boldsymbol{f}|\boldsymbol{bc}]}.$$

Combining the above expressions and using the properties of the decomposition of $\hat{\rho}^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ we obtain the expression

$$\hat{\nabla}_{[\boldsymbol{a}}\hat{\Delta}_{\boldsymbol{bc}]\boldsymbol{d}} = -\hat{\Xi}^{\boldsymbol{e}}{}_{[\boldsymbol{cab}]}\hat{L}_{\boldsymbol{ed}} - \hat{\Xi}^{\boldsymbol{e}}{}_{\boldsymbol{d}[\boldsymbol{ab}}\hat{L}_{\boldsymbol{c}]\boldsymbol{e}} + \hat{\Sigma}_{[\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{b}}\hat{\nabla}_{|\boldsymbol{e}|}\hat{L}_{\boldsymbol{c}]\boldsymbol{d}} + \Theta\delta_{[\boldsymbol{a}}\beta_{|\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{d}|\boldsymbol{bc}]} + \Theta\gamma_{[\boldsymbol{a}|\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{d}|\boldsymbol{bc}]} - \frac{1}{6}\epsilon_{\boldsymbol{abc}}{}^{\boldsymbol{f}}\epsilon_{\boldsymbol{d}}{}^{\boldsymbol{egh}}\Lambda_{\boldsymbol{fgh}}\beta_{\boldsymbol{e}}$$

and, eventually, the evolution equation

$$\begin{aligned}\hat{\nabla}_{\boldsymbol{0}}\hat{\Delta}_{\boldsymbol{bcd}} =& \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\hat{\Delta}_{\boldsymbol{ced}} + \hat{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\hat{\Delta}_{\boldsymbol{ebd}} - \hat{\Xi}^{\boldsymbol{e}}{}_{\boldsymbol{0bc}}\hat{L}_{\boldsymbol{ed}} + \delta_{\boldsymbol{b}}d_{\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{dc0}} + \delta_{\boldsymbol{c}}d_{\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{d0b}} \\ &+ \Theta\gamma_{\boldsymbol{be}}d^{\boldsymbol{e}}{}_{\boldsymbol{dc0}} + \Theta\gamma_{\boldsymbol{ce}}d^{\boldsymbol{e}}{}_{\boldsymbol{d0b}} - \frac{1}{2}\epsilon_{\boldsymbol{0bc}}{}^{\boldsymbol{f}}\epsilon_{\boldsymbol{d}}{}^{\boldsymbol{egh}}\Lambda_{\boldsymbol{fgh}}\beta_{\boldsymbol{e}},\end{aligned}$$

which is homogeneous in zero-quantities as required.

4.3.5 Subsidiary equations for the Bianchi identity

Finally, we are left to show the propagation of the physical Bianchi identity. In view of the contracted derivative appearing in this equation, the construction of suitable subsidiary equations is more involved.

Since $h_a{}^b = \delta_a{}^b + \tau_a\tau^b$, it follows then that

$$\Lambda_{abc} = \delta_a{}^d\Lambda_{dbc} = (h_a{}^d - \tau_a\tau^d)\Lambda_{dbc} = h_a{}^d\Lambda_{dbc} - \tau_a\tau^d\Lambda_{dbc}. \qquad (30)$$

Now, let

$$\Omega_{abc} \equiv h_a{}^d\Lambda_{dbc}, \qquad \Omega_{bc} \equiv \tau^d\Lambda_{dbc}.$$

By construction, the tensor Ω_{bc} is antisymmetric, hence it admits a decomposition in *electric* and *magnetic parts*. That is, one can write

$$\Omega_{bc} = \Omega_{[bc]} = \Omega^*_e\epsilon^e{}_{bc} - 2\Omega_{[b}\tau_{c]},$$

where

$$\Omega_a \equiv \Omega_{cb}\tau^b h_a{}^c, \qquad \Omega^*_a \equiv \Omega^*_{cb}\tau^b h_a{}^c.$$

Furthermore, one also has that

$$\Omega_{dbc} = \Omega_{d[bc]} = H^*_{de}\epsilon^e{}_{dc} - 2H_{d[b}\tau_{c]},$$

where

$$H_{da} \equiv \Omega_{dcb}\tau^b h_a{}^c, \qquad H^*{}_{da} \equiv \Omega^*_{dcb}\tau^b h_a{}^c.$$

Substituting the above expressions for Ω_{bc} and Ω_{dbc} into equation (30) it follows then that

$$\Lambda_{abc} = h_a{}^d(H^*_{de}\epsilon^e{}_{dc} - 2H_{d[b}n_{c]}) - n_a(\Omega^*_e\epsilon^e{}_{bc} - 2\Omega_{[b}n_{c]}). \tag{31}$$

Crucially, it can be verified that if the evolution equations (22) and (23) for the electric and magnetic part sof the rescaled Weyl tensor are satisfied then

$$H_{da} = 0, \qquad H^*{}_{da} = 0.$$

If the above holds, then equation (31) reduces to

$$\Lambda_{abc} = n_a(2\Omega_{[b}n_{c]} - \Omega^*{}_e\epsilon^e{}_{bc}) = n_a\Omega_{bc}.$$

Remark 15. The tensors Ω_a and Ω^*_a encode, respectively, the *Gauss constraints* for the electric and magnetic parts of the Weyl tensor —that is, the equations

$$\mathcal{D}^a d_{ab} = 0, \qquad \mathcal{D}^a d^*_{ab} = 0.$$

To conclude the computation, it remains to compute $\nabla^a\Lambda_{abc}$. A direct calculation gives

$$\nabla^a\Lambda_{abc} = \nabla^a\tau_a\Omega_{bc} + \tau_a\nabla^a\Omega_{bc} = \nabla^a\tau_a\Omega_{bc} + \partial_\tau\Omega_{bc}. \tag{32}$$

An alternative computation of $\nabla^a\Lambda_{abc}$ using the commutator of the covariant derivative ∇ gives

$$2\nabla^b\Lambda_{bcd} = 2\nabla^{[b}\nabla^{a]}d_{abcd} = 2R^e{}_{[c}{}^{ba}d_{d]eab} - 2R^e{}_a{}^{ba}d_{ebcd} + \Sigma_b{}^e{}_a\nabla_e d^{ab}{}_{cd}.$$

Observing that $\hat{\Sigma}_a{}^c{}_b = \Sigma_a{}^c{}_b$ as $\hat{\nabla} - \nabla = S(f)$, it follows that the equation

$$\hat{R}^a{}_{bcd} - R^a{}_{bcd} = 2(\delta^a{}_{[c}\hat{\nabla}_{d]}\hat{f}_b + \hat{\nabla}_{[c}\hat{f}^a\hat{g}_{d]b} - \delta^a{}_b\hat{\nabla}_{[c}\hat{f}_{d]} - \delta^a{}_{[c}\hat{f}_{d]}\hat{f}_b + \hat{g}_{b[c}\hat{f}_{d]}\hat{f}^a + \delta^a{}_{[c}\hat{g}_{d]b}\hat{f}_e\hat{f}^e)$$

together with the definitions of the zero quantities $\hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ and $\varsigma_{\boldsymbol{ab}}$ and the symmetries of $d_{\boldsymbol{abcd}}$ so that after projecting the equations with respect to the frame one obtains

$$\nabla^{\boldsymbol{b}}\Lambda_{\boldsymbol{bcd}} = \hat{\Xi}^{\boldsymbol{e}}{}_{[\boldsymbol{c}}{}^{\boldsymbol{ba}}d_{\boldsymbol{d}]\boldsymbol{eab}} - \hat{\Xi}^{\boldsymbol{e}}{}_{\boldsymbol{a}}{}^{\boldsymbol{ba}}d_{\boldsymbol{ebcd}} + \frac{1}{2}\hat{\Sigma}_{\boldsymbol{b}}{}^{\boldsymbol{e}}{}_{\boldsymbol{a}}\nabla_{\boldsymbol{e}}d^{\boldsymbol{ab}}{}_{\boldsymbol{cd}} + \varsigma^{\boldsymbol{ab}}d_{\boldsymbol{abcd}}, \tag{33}$$

which is homogeneous in zero-quantities. Hence, combining equations (32) and (33), we obtain the following equation for the components of Ω_{ab}:

$$\partial_{\boldsymbol{0}}\Omega_{\boldsymbol{bc}} = \hat{\Xi}^{\boldsymbol{e}}{}_{[\boldsymbol{b}}{}^{\boldsymbol{af}}d_{\boldsymbol{c}]\boldsymbol{efa}} - \hat{\Xi}^{\boldsymbol{e}}{}_{\boldsymbol{f}}{}^{\boldsymbol{af}}d_{\boldsymbol{eabc}} + \frac{1}{2}\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{f}}\nabla_{\boldsymbol{e}}d^{\boldsymbol{fa}}{}_{\boldsymbol{bc}} + \varsigma^{\boldsymbol{fa}}d_{\boldsymbol{fabc}} - \chi\Omega_{\boldsymbol{bc}}.$$

4.3.6 Subsidiary equations for the gauge conditions

To conclude our discussion of the subsidiary equations, we are left with the task of providing evolution equations for the zero-quantities associated to the gauge. In order to do so we expand $\hat{\nabla}_{[\boldsymbol{0}}\delta_{\boldsymbol{b}]}$, $\hat{\nabla}_{[\boldsymbol{0}}\gamma_{\boldsymbol{b}]\boldsymbol{c}}$ and $\hat{\nabla}_{[\boldsymbol{0}}\varsigma_{\boldsymbol{bc}]}$ to get

$$\begin{aligned}
2\hat{\nabla}_{[\boldsymbol{0}}\delta_{\boldsymbol{b}]} &= \hat{\nabla}_{\boldsymbol{0}}\delta_{\boldsymbol{b}} + \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{e}}{}_{\boldsymbol{e}}\delta_{\boldsymbol{e}},\\
2\hat{\nabla}_{[\boldsymbol{0}}\gamma_{\boldsymbol{b}]\boldsymbol{c}} &= \hat{\nabla}_{\boldsymbol{0}}\gamma_{\boldsymbol{bc}} + \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\gamma_{\boldsymbol{ec}},\\
3\hat{\nabla}_{[\boldsymbol{0}}\varsigma_{\boldsymbol{bc}]} &= \hat{\nabla}_{\boldsymbol{0}}\varsigma_{\boldsymbol{bc}} - \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\varsigma_{\boldsymbol{ce}} - \hat{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\varsigma_{\boldsymbol{eb}}.
\end{aligned}$$

We then compute $\hat{\nabla}_{[a}\delta_{b]}$, $\hat{\nabla}_{[a}\gamma_{b]c}$ and $\hat{\nabla}_{[a}\varsigma_{bc]}$ explicitly making use of the definitions of the zero-quantities and re-expressing the result in terms of zero-quantities so as to obtain

$$\begin{aligned}
2\hat{\nabla}_{[\boldsymbol{a}}\delta_{\boldsymbol{b}]} &= -\gamma_{[\boldsymbol{ab}]} + \varsigma_{\boldsymbol{ab}} - \frac{1}{2}\Theta^{-1}\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{b}}\hat{\nabla}_{\boldsymbol{e}}\Theta,\\
2\hat{\nabla}_{[\boldsymbol{a}}\gamma_{\boldsymbol{b}]\boldsymbol{c}} &= \hat{\Delta}_{\boldsymbol{abc}} + \beta_{\boldsymbol{e}}\hat{\Xi}^{\boldsymbol{e}}{}_{\boldsymbol{cab}} - \hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{b}}\hat{\nabla}_{\boldsymbol{e}}\beta_{\boldsymbol{c}} + 2\beta_{\boldsymbol{c}}\gamma_{[\boldsymbol{ab}]} - 2\beta_{[\boldsymbol{a}}\gamma_{\boldsymbol{b}]\boldsymbol{c}}\\
&\quad +\eta_{\boldsymbol{c}[\boldsymbol{a}}\beta^{\boldsymbol{e}}\gamma_{\boldsymbol{b}]\boldsymbol{e}} + 2\lambda\Theta^{-2}\delta_{[\boldsymbol{a}}\eta_{\boldsymbol{b}]\boldsymbol{c}} + \beta_{[\boldsymbol{a}}\eta_{\boldsymbol{b}]\boldsymbol{c}}\beta_{\boldsymbol{e}}\beta^{\boldsymbol{e}} - 2\lambda\Theta^{-2}\eta_{\boldsymbol{c}[\boldsymbol{a}}\beta_{\boldsymbol{b}]},\\
\hat{\nabla}_{[\boldsymbol{a}}\varsigma_{\boldsymbol{bc}]} &= \frac{1}{2}\hat{\Delta}_{[\boldsymbol{abc}]} + \frac{1}{2}\hat{\Xi}^{\boldsymbol{e}}{}_{[\boldsymbol{cab}]}f_{\boldsymbol{e}} - \frac{1}{2}\hat{\Sigma}_{[\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{b}}\hat{\nabla}_{|\boldsymbol{e}|}f_{\boldsymbol{c}]}.
\end{aligned}$$

From the above expressions it follows that the evolution equations for δ_a, γ_{ab} and ς_{ab} are given by

$$\hat{\nabla}_{\mathbf{0}}\delta_{\boldsymbol{i}} = \gamma_{\boldsymbol{i}\mathbf{0}} - \hat{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{e}}{}_{\mathbf{0}}\delta_{\boldsymbol{e}}, \tag{34a}$$

$$\hat{\nabla}_{\mathbf{0}}\gamma_{\boldsymbol{ic}} = -\gamma_{\boldsymbol{jc}}\hat{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\mathbf{0}} - \beta_{\mathbf{0}}\gamma_{\boldsymbol{ic}} - \beta_{\boldsymbol{c}}\gamma_{\boldsymbol{i}\mathbf{0}} + \eta_{\mathbf{0}\boldsymbol{c}}(\beta^{\boldsymbol{e}}\gamma_{\boldsymbol{ie}} - 2\lambda\Theta^{-2}\delta_{\boldsymbol{i}}), \tag{34b}$$

$$\hat{\nabla}_{\mathbf{0}}\varsigma_{\boldsymbol{jk}} = \hat{\Gamma}_{\boldsymbol{j}}{}^{\boldsymbol{e}}{}_{\mathbf{0}}\varsigma_{\boldsymbol{ke}} + \Gamma_{\boldsymbol{k}}{}^{\boldsymbol{e}}{}_{\mathbf{0}}\varsigma_{\boldsymbol{ej}} + \frac{1}{2}\hat{\Delta}_{\boldsymbol{jk}\mathbf{0}} + \frac{1}{2}\hat{\Xi}^{\boldsymbol{e}}{}_{\mathbf{0}\boldsymbol{jk}}f_{\boldsymbol{e}} + \frac{1}{2}\hat{\Sigma}_{\boldsymbol{j}}{}^{\boldsymbol{e}}{}_{\boldsymbol{k}}\hat{\Gamma}_{\boldsymbol{e}}{}^{\boldsymbol{f}}{}_{\mathbf{0}}f_{\boldsymbol{f}}, \tag{34c}$$

where, in particular, the evolution equation for the covector β_a,

$$\hat{\nabla}_{\mathbf{0}}\beta_{\boldsymbol{a}} + \beta_{\mathbf{0}}\beta_{\boldsymbol{a}} - \frac{1}{2}\eta_{\mathbf{0}\boldsymbol{a}}(\beta_{\boldsymbol{e}}\beta^{\boldsymbol{e}} - 2\lambda\Theta^{-2}) = 0,$$

has been used in the derivation of equation (34b). Again, as required, the equations (34a)-(34c) are homogeneous in various zero-quantities.

Remark 16. Observe that equation (34b) contains the potentially singular term $\lambda\Theta^{-2}\delta_{\boldsymbol{i}}$. As such, this equation can only be used away from the conformal boundary where $\Theta \neq 0$. This is a consequence of the use of a conformal Gaussian gauge hinged on a standard Cauchy hypersurface. This singular behaviour is of no consequence in our analysis as one is only interested on solutions to the subsidiary equations away from the conformal boundary.

4.4 Summary: structural properties of the evolution and subsidiary equations

As conclusion of the long computations in this section, we now provide a summary of the conformal evolution equations, the associated subsidiary system and the structural properties of these systems which will be required in the reminder of our analysis.

The computations discussed in the previous subsections show that in a conformal Gaussian gauge the various fields associated to the extended vacuum conformal Einstein field equations satisfy the evolution equations

$$\partial_\tau e_{\boldsymbol{b}}{}^{\nu} = -\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\mathbf{0}}e_{\boldsymbol{c}}{}^{\nu}, \tag{35a}$$

$$\partial_\tau \hat{L}_{\boldsymbol{db}} = \hat{\Gamma}_{\mathbf{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}\hat{L}_{\boldsymbol{cb}} + \hat{\Gamma}_{\mathbf{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}\hat{L}_{\boldsymbol{dc}} + d_{\boldsymbol{a}}\hat{d}^{\boldsymbol{a}}{}_{\boldsymbol{b}\mathbf{0}\boldsymbol{d}}, \tag{35b}$$

$$\partial_\tau f_{\boldsymbol{i}} = -f_{\boldsymbol{j}}\hat{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\mathbf{0}} + \hat{L}_{\boldsymbol{i}\mathbf{0}}, \tag{35c}$$

$$\partial_\tau(\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}) = -\hat{\Gamma}_{\boldsymbol{f}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\mathbf{0}} - \Xi\hat{d}^{\boldsymbol{c}}{}_{\boldsymbol{d}\mathbf{0}\boldsymbol{b}} - 2\delta_{\boldsymbol{d}}{}^{\boldsymbol{c}}\hat{L}_{\boldsymbol{b}\mathbf{0}} - 2\delta_{\mathbf{0}}{}^{\boldsymbol{c}}\hat{L}_{\boldsymbol{bd}} + 2g_{\boldsymbol{d}\mathbf{0}}g^{\boldsymbol{ce}}\hat{L}_{\boldsymbol{be}}, \tag{35d}$$

$$\partial_\tau d_{\boldsymbol{bd}} + \epsilon^{\boldsymbol{ef}}{}_{(\boldsymbol{d}}D_{\boldsymbol{f}}d^{*}_{\boldsymbol{b})\boldsymbol{e}} = 2a_{\boldsymbol{f}}\epsilon^{\boldsymbol{ef}}_{(\boldsymbol{d}}d^{*}_{\boldsymbol{b})\boldsymbol{e}} - \chi d_{\boldsymbol{bd}} + 2\chi^{\boldsymbol{f}}{}_{(\boldsymbol{b}}d_{\boldsymbol{d})\boldsymbol{f}}, \tag{35e}$$

$$\partial_\tau d^{*}_{\boldsymbol{bd}} - \epsilon^{\boldsymbol{e}}{}_{\boldsymbol{f}(\boldsymbol{d}}D^{\boldsymbol{f}}d_{\boldsymbol{b})\boldsymbol{e}} = 2a^{\boldsymbol{f}}\epsilon_{\boldsymbol{f}(\boldsymbol{d}}{}^{\boldsymbol{e}}d_{\boldsymbol{b})\boldsymbol{e}} - \chi d^{*}_{\boldsymbol{bd}} + 2\chi^{\boldsymbol{f}}_{(\boldsymbol{b}}d^{*}_{\boldsymbol{d})\boldsymbol{f}}. \tag{35f}$$

Letting $\boldsymbol{e}$, $\boldsymbol{\Gamma}$, $\hat{\boldsymbol{L}}$ and $\boldsymbol{\phi}$ denote, respectively, the independent components of the coefficients of the frame, the connection coefficients, the Schouten tensor of the Weyl connection and the rescaled Weyl tensor and setting, for convenience, $\hat{\mathbf{u}} \equiv (\hat{\boldsymbol{\upsilon}}, \boldsymbol{\phi})$, $\hat{\boldsymbol{\upsilon}} \equiv (\boldsymbol{e}, \hat{\boldsymbol{\Gamma}}, \hat{\boldsymbol{L}})$, one has that equations (35a)-(35f) can be written, schematically, in the form

$$\partial_\tau\hat{\boldsymbol{\upsilon}} = \mathbf{K}\hat{\boldsymbol{\upsilon}} + \mathbf{Q}(\hat{\boldsymbol{\upsilon}}, \hat{\boldsymbol{\upsilon}}) + \mathbf{L}(\bar{x})\boldsymbol{\phi}, \tag{36a}$$

$$(\mathbf{I} + \mathbf{A}^0(\boldsymbol{e}))\partial_\tau\boldsymbol{\phi} + \mathbf{A}^\alpha(\boldsymbol{e})\partial_\alpha\boldsymbol{\phi} = \mathbf{B}(\hat{\boldsymbol{\Gamma}})\boldsymbol{\phi}, \tag{36b}$$

where $\mathbf{K}$ and $\mathbf{Q}$ denote, respectively, a matrix and a quadratic form, both with constant coefficients while $\mathbf{L}$ is a matrix with coefficients depending smoothly on the coordinates. Moreover, $\mathbf{A}^\mu(\boldsymbol{e})$ denote, for $\mu = 0, \ldots, 3$ Hermitian matrix-valued functions depending smoothly on $\boldsymbol{e}$. In particular $\mathbf{I} + \mathbf{A}^0(\boldsymbol{e})$ is positive definite for suitable small $\boldsymbol{e}$. Finally, $\mathbf{B}(\hat{\boldsymbol{\Gamma}})$ denotes a smooth matrix-value function of the component of the connection.

Remark 17. Altogether, the conformal evolution system described by equations (36a)-(36b) constitutes a quasilinear symmetric hyperbolic system for which a well-posedness theory is available —see [16], also [19] for an abridged version. This theory will be used in the remaining sections of this article to establish the stability of the solution to the Einstein field equations given by the metric (10).

Remark 18. A remarkable structural property of the conformal evolution system (36a)-(36b) is that the equations in (36a) are, in fact, mere transport equations along conformal geodesics. The true hyperbolic content of the system is contained in the *Bianchi subsystem* (36b). This property does not play any particular role in our analysis, but it may prove key in, for example, the analysis of formation of singularities.

Regarding the subsidiary evolution system, the key conclusion from the system

$$\hat{\nabla}_{\boldsymbol{0}}\hat{\Sigma}_{\boldsymbol{b}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}} = -\frac{1}{3}\hat{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\hat{\Sigma}_{\boldsymbol{e}}{}^{\boldsymbol{d}}{}_{\boldsymbol{b}} - \frac{1}{3}\hat{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\hat{\Sigma}_{\boldsymbol{e}}{}^{\boldsymbol{d}}{}_{\boldsymbol{b}} - \hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{0bc}}, \tag{37a}$$

$$\hat{\nabla}_{\boldsymbol{0}}\hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{ebc}} = \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{0}}\hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{ecf}} + \hat{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{f}}{}_{\boldsymbol{0}}\hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{efb}} - \hat{\Sigma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{c}}\hat{R}^{\boldsymbol{d}}{}_{\boldsymbol{e0f}} - \frac{1}{2}\Theta\epsilon^{\boldsymbol{f}}{}_{\boldsymbol{0bc}}\epsilon_{\boldsymbol{e}}{}^{\boldsymbol{dgh}}\Lambda_{\boldsymbol{fgh}} \tag{37b}$$

$$+\epsilon^{\boldsymbol{f}}{}_{\boldsymbol{0bc}}\delta^{\boldsymbol{g}}d^{*\boldsymbol{d}}{}_{\boldsymbol{efg}} + 3S_{\boldsymbol{e0}}{}^{\boldsymbol{dg}}\hat{\Delta}_{\boldsymbol{cbg}}, \tag{37c}$$

$$\hat{\nabla}_{\boldsymbol{0}}\hat{\Delta}_{\boldsymbol{bcd}} = \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\hat{\Delta}_{\boldsymbol{ced}} + \hat{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\hat{\Delta}_{\boldsymbol{ebd}} - \hat{\Xi}^{\boldsymbol{e}}{}_{\boldsymbol{0bc}}\hat{L}_{\boldsymbol{ed}} + \delta_{\boldsymbol{b}}d_{\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{dc0}} + \delta_{\boldsymbol{c}}d_{\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{d0b}} \tag{37d}$$

$$+\Theta\gamma_{\boldsymbol{be}}d^{\boldsymbol{e}}{}_{\boldsymbol{dc0}} + \Theta\gamma_{\boldsymbol{ce}}d^{\boldsymbol{e}}{}_{\boldsymbol{d0b}} - \frac{1}{2}\epsilon_{\boldsymbol{0bc}}{}^{\boldsymbol{f}}\epsilon_{\boldsymbol{d}}{}^{\boldsymbol{egh}}\Lambda_{\boldsymbol{fgh}}\beta_{\boldsymbol{e}}, \tag{37e}$$

$$\hat{\nabla}_{\boldsymbol{0}}\hat{\Omega}_{\boldsymbol{bc}} = \hat{\Xi}^{\boldsymbol{e}}{}_{[\boldsymbol{b}}{}^{\boldsymbol{af}}d_{\boldsymbol{c}]\boldsymbol{efa}} - \hat{\Xi}^{\boldsymbol{e}}{}_{\boldsymbol{f}}{}^{\boldsymbol{af}}d_{\boldsymbol{eabc}} + \frac{1}{2}\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{f}}\nabla_{\boldsymbol{e}}d^{\boldsymbol{fa}}{}_{\boldsymbol{bc}} + \varsigma^{\boldsymbol{fa}}d_{\boldsymbol{fabc}} - \chi\Omega_{\boldsymbol{bc}}, \tag{37f}$$

$$\hat{\nabla}_{\boldsymbol{0}}\delta_{\boldsymbol{i}} = \gamma_{\boldsymbol{i0}} - \hat{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\delta_{\boldsymbol{e}}; \tag{37g}$$

$$\hat{\nabla}_{\boldsymbol{0}}\gamma_{\boldsymbol{ic}} = -\gamma_{\boldsymbol{jc}}\hat{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{0}} - \beta_{\boldsymbol{0}}\gamma_{\boldsymbol{ic}} - \beta_{\boldsymbol{c}}\gamma_{\boldsymbol{i0}} + \eta_{\boldsymbol{0c}}(\beta^{\boldsymbol{e}}\gamma_{\boldsymbol{ie}} - 2\lambda\Theta^{-2}\delta_{\boldsymbol{i}}), \tag{37h}$$

$$\hat{\nabla}_{\boldsymbol{0}}\varsigma_{\boldsymbol{jk}} = \hat{\Gamma}_{\boldsymbol{j}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\varsigma_{\boldsymbol{ke}} + \Gamma_{\boldsymbol{k}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\varsigma_{\boldsymbol{ej}} + \frac{1}{2}\hat{\Delta}_{\boldsymbol{jk0}} + \frac{1}{2}\hat{\Xi}^{\boldsymbol{e}}{}_{\boldsymbol{0jk}}f_{\boldsymbol{e}} + \frac{1}{2}\hat{\Sigma}_{\boldsymbol{j}}{}^{\boldsymbol{e}}{}_{\boldsymbol{k}}\hat{\Gamma}_{\boldsymbol{e}}{}^{\boldsymbol{f}}{}_{\boldsymbol{0}}f_{\boldsymbol{f}}, \tag{37i}$$

is that the zero-quantities $\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$, $\hat{\Xi}^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}$, $\hat{\Delta}_{\boldsymbol{abc}}$, $\hat{\Lambda}_{\boldsymbol{abc}}$, $\delta_{\boldsymbol{ab}}$, $\gamma_{\boldsymbol{ab}}$ and $\varsigma_{\boldsymbol{ab}}$ satisfy, if the conformal evolution equations (35a)-(35e) hold, a symmetric hyperbolic system which is homogeneous in the zero-quantities —accordingly, the particular situation in which all the zero-quantities vanish identically gives rise to the subsidiary evolution system. The subsidiary system is regular away from the conformal boundary —i.e. the sets for which the conformal factor vanishes.

5 Initial data for the evolution equations

Given a solution $(S, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$ to the Einstein constraint equations (i.e the Hamiltonian and the momentum constraints), there exists an algebraic procedure to compute initial data for the conformal evolution equations —see e.g. [19], Lemma 11.1, page 265. Now, a suitable perturbative existence theorem which covers perturbations of the initial data implied by the metric (10) on the hypersurfaces of constant t has been provided in [20] —see Theorem 1. From this result one can deduce the following assertion:

Proposition 3. *Let* $(\mathcal{S}, \mathring{\boldsymbol{h}}, \mathring{\boldsymbol{K}})$ *with* $\mathcal{S}$ *compact,* $\mathring{\tilde{\boldsymbol{h}}}$ *a smooth Riemannian metric of constant negative curvature and* $\mathring{\boldsymbol{K}} = \varkappa\mathring{\boldsymbol{h}}$ *with* $\varkappa$ *a constant, denote a initial data set for the vacuum Einstein field equations with positive Cosmological constant. Then for each pair of sufficiently small (in the sense of suitable Sobolev norms) tensors* T_{ij} *and* $\bar{T}_{ij}$ *over* $\mathcal{S}$*, transverse-tracefree with respect to* $\mathring{\boldsymbol{h}}$*, and each sufficiently small scalar field* ϕ *over* $\mathcal{S}$*, there exists a solution of the Einstein constraint equations* $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K})$ *with positive Cosmological constant which is suitably close to* $(\mathcal{S}, \mathring{\boldsymbol{h}}, \mathring{\boldsymbol{K}})$ *and such that* $tr_{\mathring{\boldsymbol{h}}}(\boldsymbol{K} - \mathring{\boldsymbol{K}}) = \phi$ *and for which the electric and magnetic parts of the Weyl tensor (restricted to* $\mathcal{S}$*) of the resulting spacetime development take the form*

$$d_{ij} = \mathring{L}(\boldsymbol{X})_{ij} + T_{ij} - \frac{1}{3}tr_{\boldsymbol{h}}(\mathring{L}(\boldsymbol{X}) + \boldsymbol{T})h_{ij},$$

$$d^*_{ij} = \mathring{L}(\bar{\boldsymbol{X}})_{ij} + \bar{T}_{ij} - \frac{1}{3}tr_{\boldsymbol{h}}(\mathring{L}(\bar{\boldsymbol{X}}) + \bar{\boldsymbol{T}})h_{ij},$$

for some covectors $\boldsymbol{X}$*,* $\bar{\boldsymbol{X}}$ *over* $\mathcal{S}$ *and where* $\mathring{L}$ *denotes the conformal Killing operator with respect to* $\mathring{\boldsymbol{h}}$*.*

Remark 19. Thus, choosing the free data T_{ij}, $\bar{T}_{ij}$ and ϕ suitably small one can ensure that the *perturbed data* $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K})$ is as close to $(\mathcal{S}, \mathring{\boldsymbol{h}}, \mathring{\boldsymbol{K}})$. Accordingly, the associated initial data for the conformal evolution equations will be close to initial data for the background solution.

Remark 20. Theorem 1 in [20] applies to the broader class of *conformally rigid hyperbolic* compact manifolds —that is, Einstein manifolds with negative Ricci scalar which do not admit a non-trivial Codazzi tensor; see the discussion in Section 3.4.3 of this reference. The precise statement of the result also excludes values of $\varkappa$ which are related in a specific manner to the eigenvalues of the Laplacian of $\mathring{\boldsymbol{h}}$ —however, we do not require this level of detail in the subsequent discussion.

6 Analysis of the existence and stability of solutions

In this section we develop the theory of the existence, uniqueness and stability of solutions to the Einstein field equations which can be regarded as perturbations of the background solution. The argument proceeds in several steps: first, the Cauchy stability of solutions to symmetric hyperbolic systems is used to conclude the existence of solutions to the conformal evolution system (35a)-(35f); in a second step the uniqueness of solutions to the subsidiary system (37a)-(37i) to argue the propagation of constraints; finally general theory of the conformal Einstein field equations is invoked to establish the connection between solutions to the conformal equations and actual solutions to the Einstein field equations.

6.1 A symmetric hyperbolic evolution system

In the following we look for solutions to the system (36a)-(36b) of the form

$$\hat{\mathbf{u}} = \mathring{\mathbf{u}} + \breve{\mathbf{u}}$$

where $\mathring{\mathbf{u}}$ is the solution to the conformal evolution equations (35a)-(35f) implied by a background solution, while $\breve{u}$ denotes a (small) perturbation. Accordingly, making use of the schematic notation of equations (36a)-(36b) one can set

$$\hat{\boldsymbol{\upsilon}} = \mathring{\boldsymbol{\upsilon}} + \breve{\boldsymbol{\upsilon}}, \qquad \hat{\boldsymbol{\phi}} = \breve{\boldsymbol{\phi}}, \tag{38a}$$

$$\hat{\boldsymbol{e}} = \mathring{\boldsymbol{e}} + \breve{\boldsymbol{e}}, \qquad \hat{\boldsymbol{\Gamma}} = \mathring{\boldsymbol{\Gamma}} + \breve{\boldsymbol{\Gamma}}. \tag{38b}$$

Now, we have found that on the initial surface $\mathcal{S}_\star$ described by the condition $\tau = \tau_\star$ one can write $\mathring{\mathbf{u}}_\star = (\mathring{\boldsymbol{\upsilon}}_\star, \mathring{\boldsymbol{\phi}}_\star) = (\mathring{\boldsymbol{\upsilon}}_\star, 0)$. As the conformal factor Θ and the covector $\boldsymbol{d}$ are universal, it follows that

$$\partial_\tau \mathring{\boldsymbol{\upsilon}} = \mathbf{K}\mathring{\boldsymbol{\upsilon}} + \mathbf{Q}(\mathring{\boldsymbol{\upsilon}}, \mathring{\boldsymbol{\upsilon}}).$$

Substituting (38a) and (38b) into equations (36a) and (36b) yields evolution equations for $\breve{\mathbf{u}} = (\breve{\boldsymbol{\upsilon}}, \breve{\boldsymbol{\upsilon}})$ which, schematically, take the form

$$\partial_\tau \breve{\boldsymbol{\upsilon}} = \mathbf{K}\breve{\boldsymbol{\upsilon}} + \mathbf{Q}(\mathring{\boldsymbol{\Gamma}} + \breve{\boldsymbol{\Gamma}})\breve{\boldsymbol{\upsilon}} + \mathbf{Q}(\breve{\boldsymbol{\Gamma}})\mathring{\boldsymbol{\upsilon}} + \mathbf{L}(\bar{x})\breve{\boldsymbol{\phi}}, \tag{39a}$$

$$(\mathbf{I} + \mathbf{A}^0(\mathring{\boldsymbol{e}} + \breve{\boldsymbol{e}}))\partial_\tau \breve{\boldsymbol{\phi}} + \mathbf{A}^\alpha(\mathring{\boldsymbol{e}} + \breve{\boldsymbol{e}})\partial_\alpha \breve{\boldsymbol{\phi}} = \mathbf{B}(\mathring{\boldsymbol{\Gamma}} + \breve{\boldsymbol{\Gamma}})\breve{\boldsymbol{\phi}}. \tag{39b}$$

Now, in the following it is convenient to define

$$\bar{\mathbf{A}}^0(\tau, \underline{x}, \breve{\mathbf{u}}) \equiv \begin{pmatrix} \mathrm{I} & 0 \\ 0 & \mathrm{I} + \mathbf{A}^0(\mathring{\boldsymbol{e}} + \breve{\boldsymbol{e}}) \end{pmatrix}, \qquad \bar{\mathbf{A}}^\alpha(\tau, \underline{x}, \breve{\mathbf{u}}) \equiv \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{A}^\alpha(\mathring{\boldsymbol{e}} + \breve{\boldsymbol{e}}) \end{pmatrix}$$

and

$$\bar{\mathbf{B}}(\tau, \underline{x}, \breve{\mathbf{u}}) \equiv \breve{\mathbf{u}}\bar{\mathbf{Q}}\breve{\mathbf{u}} + \bar{\mathbf{L}}(\bar{x})\breve{\mathbf{u}} + \bar{\mathbf{K}}\breve{\mathbf{u}},$$

where

$$\breve{\mathbf{u}}\bar{\mathbf{Q}}\breve{\mathbf{u}} \equiv \begin{pmatrix} \breve{\boldsymbol{\upsilon}}\mathbf{Q}\breve{\boldsymbol{\upsilon}} & 0 \\ 0 & \mathbf{B}(\breve{\boldsymbol{\Gamma}})\breve{\boldsymbol{\phi}} \end{pmatrix}, \qquad \bar{\mathbf{L}}(\bar{x})\breve{\mathbf{u}} \equiv \begin{pmatrix} \mathring{\boldsymbol{\upsilon}}\mathbf{Q}\breve{\boldsymbol{\upsilon}} + \mathbf{Q}(\breve{\boldsymbol{\Gamma}})\mathring{\boldsymbol{\upsilon}} & \mathbf{L}(\bar{x})\breve{\boldsymbol{\phi}} \\ 0 & 0 \end{pmatrix}, \qquad \bar{\mathbf{K}}\breve{\mathbf{u}} \equiv \begin{pmatrix} \mathbf{K}\breve{\boldsymbol{\upsilon}} & 0 \\ 0 & \mathbf{B}(\mathring{\Gamma})\breve{\boldsymbol{\phi}} \end{pmatrix},$$

denote, respectively, quadratic, linear and constant terms in the unknowns. In terms of the latter it is possible to rewrite the system (39a) and (39b) in the form

$$\bar{\mathbf{A}}^0(\tau, \underline{x}, \breve{\mathbf{u}})\partial_\tau \breve{\mathbf{u}} + \bar{\mathbf{A}}^\alpha(\tau, \underline{x}, \breve{\mathbf{u}})\partial_\alpha \breve{\mathbf{u}} = \bar{\mathbf{B}}(\tau, \underline{x}, \breve{\mathbf{u}}). \tag{40}$$

From the discussion in the previous sections, it follows that the system described by (40) is a symmetric hyperbolic system for which the theory of [16] can be applied. The natural domain of the solutions to this system is of the form

$$\mathcal{M} = [\tau_\star, \tau_\bullet) \times \mathcal{S}, \qquad \tau_\star \in (0,1), \qquad \tau_\bullet \geq 1.$$

6.2 The existence, uniqueness and Cauchy stability of the solution

The existence of de Sitter-like solutions to the conformal evolution system (40) is given by the following proposition:

Proposition 4 (***existence and uniqueness of the solutions to the perturbed de Sitter–like evolution equations***). *Given* $\mathbf{u}_\star = \mathring{\mathbf{u}}_\star + \breve{\mathbf{u}}_\star$ *and* $m \geq 4$*, one has that:*

(i) There exists $\varepsilon > 0$ *such that if*

$$||\breve{\mathbf{u}}_\star||_{\mathcal{S},m} < \varepsilon, \tag{41}$$

then there exists a unique solution $\breve{\mathbf{u}} \in C^{m-2}\big([\tau_\star, \tfrac{3}{2}) \times \mathcal{S}, \mathbb{R}^N\big)$ *to the Cauchy problem for the conformal evolution equations* (40) *with initial data* $\mathbf{u}(0, \underline{x}) = \breve{\mathbf{u}}_\star$*,* $\tau_\star > 0$ *and with* N *denoting the dimension of the vector* $\mathbf{u}$*.*

(ii) Given a sequence of initial data $\breve{\mathbf{u}}^{(n)}_\star$ *such that*

$$||\breve{\mathbf{u}}^{(n)}_\star||_{\mathcal{S},m} < \varepsilon, \qquad \textit{and} \qquad ||\breve{\mathbf{u}}^{(n)}{}_\star||_{\mathcal{S},m} \xrightarrow{n\to\infty} 0,$$

then for the corresponding solutions $\breve{\mathbf{u}}^{(n)} \in C^{m-2}\big([\tau_\star, \tfrac{3}{2}) \times \mathcal{S}, \mathbb{R}^N\big)$*, one has* $||\breve{\mathbf{u}}^{(n)}||_{\mathcal{S},m} \to 0$ *uniformly in* $\tau \in [\tau_\star, \tfrac{3}{2})$ *as* $n \to \infty$*.*

Remark 21. In the above proposition $||\breve{\mathbf{u}}_\star||_{\mathcal{S},m}$ denotes the standard L^2-Sobolev norm over $\mathcal{S}$ of order $m \geq 4$ of the independent components of the vector $\breve{\mathbf{u}}_\star$.

Proof. The proof is an application of the existence and stability results for symmetric hyperbolic systems with compact spatial sections —see e.g. [19], Section 12.3 which, in turn, follows from Kato's theory for symmetric hyperbolic systems over $\mathbb{R}^n$ [16]. More precisely, since the 3-dimensional manifold $\mathcal{S}$ is compact, there exists a finite cover consisting of open sets $\mathcal{R}_1, \ldots, \mathcal{R}_M \subset \mathcal{S}$ such that $\cup_{i=1}{}^M \mathcal{R}_i = \mathcal{S}$. On each of the open sets $\mathcal{R}_i$ it is possible to introduce coordinates $\underline{x}_i \equiv (x^\alpha{}_i)$ which allow one to identify $\mathcal{R}_i$ with open subsets $\mathcal{B}_i \subset \mathbb{R}^3$. As $\mathcal{S}$ is assumed to be a smooth manifold, the coordinate patches can be chosen so that the change of coordinates on intersecting sets is smooth. The initial data $\breve{\mathbf{u}}_\star : \mathcal{S} \to \mathbb{R}^N$ is a smooth function on $\mathcal{S}$ and can be restricted to a particular open set $\mathcal{R}_i$. The restriction $\breve{\mathbf{u}}_{i\star}$, in local coordinates x_i, can be regarded as a function $\breve{\mathbf{u}}_{i\star} : \mathcal{B}_i \to \mathbb{R}^N$. Now, assuming that $\mathcal{R} \subset \mathbb{R}^3$ is bounded with smooth boundary $\partial\mathcal{R}$, it is possible to extend $\breve{\mathbf{u}}_{i\star}$ to a function $\mathcal{E}\breve{\mathbf{u}}_{i\star} : \mathbb{R}^3 \to \mathbb{R}^N$ —see e.g. Proposition 12.2 in [19]. Using these extensions it is possible to define the Sobolev norm

$$||\breve{\mathbf{u}}_\star||_{\mathcal{S},m} \equiv \sum_{i=1}^{M} ||\breve{\mathbf{u}}_{i\star}||_{\mathbb{R}^3,m}.$$

Now, for each of the $\mathcal{E}\breve{\mathbf{u}}_{i\star}$ one can formulate an initial value problem of the form

$$\bar{\mathbf{A}}^0(\tau, \underline{x}, \breve{\mathbf{u}})\partial_\tau \breve{\mathbf{u}} + \bar{\mathbf{A}}^\alpha(\tau, \underline{x}, \breve{\mathbf{u}})\partial_\alpha \breve{\mathbf{u}} = \mathcal{B}(\tau, \underline{x}, \breve{\mathbf{u}}),$$
$$\breve{\mathbf{u}}(0, \underline{x}) = \mathcal{E}\breve{\mathbf{u}}_{i\star}(\underline{x}) \in H^m(\mathcal{S}, \mathbb{R}^N) \quad \text{for } m \geq 4.$$

For this initial value problem it is observed that:

(a) The matrices $\bar{\mathbf{A}}^\mu(\tau, \underline{x}, \mathcal{E}\breve{\mathbf{u}}_{i\star})$ are positive definite and depend linearly on the solution $\breve{\mathbf{u}}_i$ with coefficients which are constant.

(b) The dependence of $\mathcal{B}$ on $\breve{\mathbf{u}}_i$ is at most quadratic: there are linear and quadratic terms for the connection coefficients; linear terms for the components of the Schouten tensor. The explicit dependence on $(\tau, \underline{x})$ comes from the conformal factor and the covector d_a —this dependence is smooth.

(c) The connection coefficients and the components of the Schouten tensor of the background solution are smooth functions (C^∞) of $(\tau, \underline{x})$.

(d) The dependence of the frame coefficients of the background solution is smooth (C^∞) on τ for $\tau \in [\tau_\star, \frac{3}{2}]$ with $\tau_\star \neq 0$.

It follows from the above observations that the system considered in the present article satisfies the conditions of Kato's theorems —see Appendix A. This theory implies existence, uniqueness and stability —i.e. points (i) and (ii) in the theorem. Notice, however, that strictly speaking, this theorem only applies to settings in which the spatial sections are diffeomorphic to $\mathbb{R}^3$. To address this one makes use of the following strategy: standard results on causality theory imply that

$$D^+(\mathcal{R}_i) \cap I^+(\mathcal{S} \setminus \mathcal{R}_i) = \emptyset,$$

where $D^+(\mathcal{R}_i)$ denotes the causal future of $\mathcal{R}_i$ —see e.g. [19], Theorem 14.1. Accordingly, the value of $\breve{\mathbf{u}}$ on $\mathcal{D}_i \equiv D^+(\mathcal{R}_i)$ is determined only by the data on $\mathcal{R}_i$. Then the solution on $\mathcal{D}_i$ is independent of the particular extension $\mathcal{E}\breve{\mathbf{u}}_{i\star}$ being used. Hence, one can speak of a solution $\breve{\mathbf{u}}_i$ on a domain $\mathcal{D}_i \subset [\tau_\star, \tau_i] \times \mathcal{R}_i$. Since the manifold is smooth and as a consequence of uniqueness, it follows that given two solutions $\breve{\mathbf{u}}_i$ and $\breve{\mathbf{u}}_j$ defined, respectively, on intersecting domains $\mathcal{D}_i$ and $\mathcal{D}_j$ they must coincide on $\mathcal{D}_i \bigcap \mathcal{D}_j$. Proceeding in the same manner over the whole finite cover of $\mathcal{S}$ and since the compactness of $\mathcal{S}$ ensures the existence of a minimum non-zero existence time for the whole of the domains $\mathcal{D}_i$, then there is a unique solution $\breve{\mathbf{u}}$ on $[\tau_\star, \frac{3}{2}] \times \mathcal{S}$ with $\frac{3}{2} = \min_{i=1,\ldots,M}\{\tau_i\}$ which is constructed by patching together the localised solutions $\breve{\mathbf{u}}_1, \ldots, \breve{\mathbf{u}}_M$ defined, respectively on the domains $\mathcal{D}_i, \ldots, \mathcal{D}_M$. The existence interval $[\tau_\star, \frac{3}{2})$ follows from the fact that the background solution $\mathring{\mathbf{u}}$ has this existence interval.

□

Remark 22. The existence and Cauchy stability of the solution to the initial value problem for the original conformal evolution problem

$$\mathbf{A}^0(\tau, \underline{x}, \hat{\mathbf{u}})\partial_\tau \hat{\mathbf{u}} + \mathbf{A}^\alpha(\tau, \underline{x}, \hat{\mathbf{u}})\partial_\alpha \hat{\mathbf{u}} = \mathbf{B}(\tau, \underline{x}, \hat{\mathbf{u}}),$$
$$\hat{\mathbf{u}}|_\star = \mathring{\mathbf{u}}_\star + \breve{\mathbf{u}}_\star \in H^m(\mathcal{S}, \mathbb{R}^N) \quad \text{for} \quad m \geq 4$$

follows from the fact that $\hat{\mathbf{u}}$ satisfies the same properties as $\breve{\mathbf{u}}$ in Proposition 4 and then it exists in the same solution manifold and with the same regularity properties, existence and uniqueness.

6.3 Propagation of the constraints

In this section we discuss the so-called *propagation of the constraints*. This argument is essential to establish the connection between solutions to the conformal evolution systems and actual solutions to the Einstein field equations. More precisely, one has the following:

Proposition 5 (***propagation of the constraints***)**.** *Let $\hat{\mathbf{u}}_\star = \mathring{\mathbf{u}}_\star + \breve{\mathbf{u}}_\star$ denote initial data for the conformal evolution equations on a 3-manifold $\mathcal{S}_\star \approx \mathcal{S}$ such that*

$$\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}|_{\mathcal{S}_\star} = 0, \quad \hat{\Xi}^c{}_{\boldsymbol{dab}}|_{\mathcal{S}_\star} = 0, \quad \hat{\Delta}_{\boldsymbol{abc}}|_{\mathcal{S}_\star} = 0, \quad \hat{\Lambda}_{\boldsymbol{abc}}|_{\mathcal{S}_\star} = 0,$$

and

$$\delta_{\boldsymbol{a}}|_{\mathcal{S}_\star} = 0, \quad \gamma_{\boldsymbol{ab}}|_{\mathcal{S}_\star} = 0, \quad \varsigma_{\boldsymbol{ab}}|_{\mathcal{S}_\star} = 0,$$

then the solution $\breve{\mathbf{u}}$ to the conformal evolution equations given by Proposition 4 implies a C^{m-2} solution $\hat{\mathbf{u}} = \mathring{\mathbf{u}} + \breve{\mathbf{u}}$ to the extended conformal field equations on $\big[\tau_\star, 1\big) \times \mathcal{S}$.

Proof. The proof follows from the properties of the subsidiary evolution system. First, it is observed that by assumption

$$\hat{\Sigma}_{\boldsymbol{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = 0, \qquad \hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{d0b}} = 0, \qquad \hat{\Delta}_{\boldsymbol{0bc}} = 0,$$

hold —cfr. the equations in (20). Moreover, the associated evolution equations are expressed in terms of a conformal Gaussian gauge system and the independent components of the rescaled Weyl tensor satisfy either the evolution system (22) and (23). Now, following the discussion of Section 4.3, the independent components of the zero-quantities

$$\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}, \quad \hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{dab}}, \quad \hat{\Delta}_{\boldsymbol{abc}}, \quad \hat{\Lambda}_{\boldsymbol{abc}}, \quad \delta_{\boldsymbol{a}}, \quad \gamma_{\boldsymbol{ab}}, \quad \varsigma_{\boldsymbol{ab}},$$

which are not determined by either the evolution equations or gauge conditions satisfy a symmetric hyperbolic system which is homogeneous in the zero-quantities. More precisely, defining $\hat{\mathbf{X}} \equiv (\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}, \hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{dab}}, \hat{\Delta}_{\boldsymbol{abc}}, \hat{\Lambda}_{\boldsymbol{abc}}, \delta_{\boldsymbol{a}}, \gamma_{\boldsymbol{ab}}, \varsigma_{\boldsymbol{ab}})$, these equations can be recast as a symmetric hyperbolic system of the form

$$\partial_\tau \hat{\mathbf{X}} = \mathbf{H}(\hat{\mathbf{X}}),$$

where $\mathbf{H}$ is a homogeneous function of its arguments —i.e. $\mathbf{H}(\mathbf{0}) = \mathbf{0}$. It follows then that a solution to the initial value problem

$$\partial_\tau \hat{\mathbf{X}} = \mathbf{H}(\hat{\mathbf{X}}),$$
$$\hat{\mathbf{X}}_\star = 0.$$

is given (trivially) by $\hat{\mathbf{X}} = 0$. Moreover, following Kato's theorem it follows this is the unique solution. Thus, the zero-quantities must vanish on $\big[\tau_\star, 1\big) \times \mathcal{S}$. That is, the solution $\check{\mathbf{u}}$ to the conformal evolution equations implies a solution to the extended conformal Einstein field equations over the latter domain. □

From the above statement, making use of the relation between the extended conformal Einstein field equations and the actual Einstein field equations —see Proposition 8.3 in 208 — in [19] it follows the following:

Corollary 1. *The metric*

$$\tilde{\boldsymbol{g}} = \Theta^{-2} \boldsymbol{g}$$

obtained from the solution to the conformal evolution equations given in Proposition 4 implies a solution to the vacuum Einstein field equations with $\lambda = 3$.

7 Future geodesic completeness

In this section we discuss the future geodesic completeness of the spacetimes obtained in the previous section. Our analysis distinguishes two cases: null geodesics and timelike geodesics.

7.1 Null geodesics

As a consequence of the compactness of the unphysical manifold

$$\mathcal{M} = \Big\{ (\tau, \underline{x}) \in \mathbb{R} \times \mathcal{S} \mid \tau_\bullet \leq \tau \leq 1 \Big\},$$

null geodesics in the unphysical manifold starting at the initial hypersurface $\mathcal{S}_\star$, reach the conformal boundary in a finite amount of affine parameter. Furthermore, null geodesics with respect to the unphysical metric $\boldsymbol{g}$ coincide, up to a reparametrisation, with null geodesics respect to the physical metric $\tilde{\boldsymbol{g}}$ on $\tilde{\mathcal{M}}$. More precisely, let γ be a null geodesic in $(\mathcal{M}, \boldsymbol{g})$ with affine parameter v such that $v = 0$ on $\partial\tilde{\mathcal{M}}$. The equations for γ are

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}v^2} + \Gamma^\mu{}_{\nu\lambda} \frac{\mathrm{d}x\nu}{\mathrm{d}v} \frac{\mathrm{d}x^\lambda}{\mathrm{d}v} = 0.$$

Let $\tilde{\gamma}$ denote the corresponding geodesics in $\tilde{\mathcal{M}}$. Using a different parameter $\tilde{v} = \tilde{v}(v)$ and the relation between the Christoffel symbols $\Gamma^{\mu}{}_{\nu\lambda}$ and $\tilde{\Gamma}^{\mu}{}_{\nu\lambda}$ it follows that

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tilde{v}^2} + \tilde{\Gamma}^{\mu}{}_{\nu\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tilde{v}}\frac{\mathrm{d}x^\lambda}{d\tilde{v}} = -\frac{1}{\tilde{v}'}\left(\frac{\tilde{v}''}{\tilde{v}'} + 2\frac{\Theta'}{\Theta}\right)\frac{\mathrm{d}x^\mu}{\mathrm{d}\tilde{v}}.$$

By requiring that $\tilde{v}$ to be an affine parameter the right hand side must vanish. This implies $\tilde{v}' = \mathrm{const}/\Theta^2$, and absorbing the constant into $\tilde{v}$ we obtain

$$\frac{\mathrm{d}\tilde{v}}{\mathrm{d}v} = \frac{1}{\Theta^2}.$$

Furthermore, at $\mathscr{I}^+$, $\Theta = 0$ and $\mathbf{d}\Theta \neq 0$, and we may choose v so that near $\partial\tilde{M}$, $v \sim -\Theta$. Thus $\tilde{v} \sim -1/v$ becomes unbounded —i.e. the physical affine parameter for the physical geodesic must blow up as $\Theta \to 0$. Thus, $\tilde{\gamma}$ never reaches $\partial\tilde{\mathcal{M}}$ and the null geodesic must be complete —see also the discussion in [18], Chapter 3.

7.2 Timelike geodesics

The argument used for null geodesics cannot readily be applied to the discussion of timelike geodesics as these are not conformally invariant. Instead, we make use of timelike conformal geodesics.

Every timelike metric geodesic on the physical spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ can be recast, after a reparametrisation, as a conformal geodesics $(x, \tilde{\boldsymbol{\beta}})$ —see e.g. [19], Lemma 5.2 in page 131; also [14]. Under the rescaling $\boldsymbol{g} = \Theta^2\tilde{\boldsymbol{g}}$, the conformal geodesic $(x, \tilde{\boldsymbol{\beta}})$ transforms into a geodesic (x, β) in the unphysical spacetime $(\mathcal{M}, \boldsymbol{g})$. Now, it is known that any $\boldsymbol{g}$-conformal geodesic that leaves $\mathscr{I}^+$ orthogonally into the past, is up to a reparametrisation, a timelike future complete geodesic for the physical metric $\tilde{g}$ —see e.g. [14, 13]. Moreover, a conformal geodesic through a point of $\mathscr{I}^+$ which is not orthogonal to the conformal boundary cannot represent a geodesic in the physical spacetime.

Now, from the $\tilde{\boldsymbol{g}}$-future geodesic completeness of the background solution (see Appendix B) it follows that every conformal geodesic in the background spacetime starting orthogonal to the initial hypersurface $\mathcal{S}_\star$ must reach the conformal boundary $\mathscr{I}^+$. Hence, every timelike $\tilde{\boldsymbol{g}}$-geodesic is, up to a reparametrisation, a timelike conformal curve reaching $\mathscr{I}^+$ orthogonally. Moreover, let us consider a pair $(x(\tau), \tilde{\boldsymbol{\beta}}(\tau))$ with parameter $\tau \in \mathbb{R}$. Furthermore, let us suppose that this geodesic starts at $\tau = \tau_\star$, i.e. the initial hypersurface $\mathcal{S}$, and it reaches the conformal boundary $\mathscr{I}^+$ at $\tau = 1$. Now, consider a small perturbation of the quantities $(x, \tilde{\boldsymbol{\beta}})$ so that

$$\hat{x} = x + \breve{x},$$
$$\hat{\boldsymbol{\beta}} = \tilde{\boldsymbol{\beta}} + \breve{\boldsymbol{\beta}},$$

where $\breve{x}$ and $\breve{\boldsymbol{\beta}}$ are small perturbations. In this case, the perturbed conformal geodesic equations read

$$\tilde{\nabla}_{\boldsymbol{x}'}(\boldsymbol{x}' + \breve{\boldsymbol{x}}') = -2\langle(\tilde{\boldsymbol{\beta}} + \breve{\boldsymbol{\beta}}), (\boldsymbol{x}' + \breve{\boldsymbol{x}}')\rangle(\boldsymbol{x}' + \breve{\boldsymbol{x}}') + \tilde{\boldsymbol{g}}((\boldsymbol{x}' + \breve{\boldsymbol{x}}'), (\boldsymbol{x}' + \breve{\boldsymbol{x}}'))(\tilde{\boldsymbol{\beta}} + \breve{\boldsymbol{\beta}})^\sharp,$$
$$\tilde{\nabla}_{\boldsymbol{x}'}(\tilde{\boldsymbol{\beta}} + \breve{\boldsymbol{\beta}}) = \langle(\tilde{\boldsymbol{\beta}} + \breve{\boldsymbol{\beta}}), (\boldsymbol{x}' + \breve{\boldsymbol{x}}')\rangle(\tilde{\boldsymbol{\beta}} + \breve{\boldsymbol{\beta}}) - \frac{1}{2}\boldsymbol{g}^\sharp((\tilde{\boldsymbol{\beta}} + \breve{\boldsymbol{\beta}}), (\tilde{\boldsymbol{\beta}} + \breve{\boldsymbol{\beta}}))(\boldsymbol{x} + \breve{\boldsymbol{x}})'^\flat + \tilde{\boldsymbol{L}}((\boldsymbol{x}' + \breve{\boldsymbol{x}}'), \cdot),$$

where the metric, covariant derivative and Schouten tensor are those obtained from the solution to the Einstein field equations given in Corollary 1. These equations can be read as a system of ordinary differential equations for the fields $\breve{x}$ and $\breve{\boldsymbol{\beta}}$. Because of the smoothness of the perturbed spacetime it follows that one can make use of the stability theory for ordinary differential equations —see e.g. [15], Theorem 2.1 in page 94 and Corollary 4.1 in page 101. In particular, these conformal geodesics will have the same existence interval as those in the background spacetime. Accordingly, it follows that $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ is future $\tilde{\boldsymbol{g}}$-geodesically complete.

Remark 23. An alternative way of concluding the future geodesic completeness of the solutions to the Einstein field equations provided by Corollary 1 is to make use of the theory in [4] —see also Appendix B. By choosing the $\varepsilon > 0$ in condition (41) of Proposition 4 sufficiently small, it can be shown that the physical metric $\tilde{\boldsymbol{g}}$ satisfies the bounds required to shown geodesic completeness.

8 The main result

We summarise the discussion of the preceding sections with a more detailed formulation of the main result of this article:

Theorem 1. *Let $\hat{\mathbf{u}}_\star = \mathbf{u}_\star + \breve{\mathbf{u}}_\star$ denote smooth initial data for the conformal evolution equations satisfying the conformal constraint equations on a hypersurface $\mathcal{S}_\star$. Then, there exists $\varepsilon > 0$ such that if*

$$||\breve{\mathbf{u}}_\star||_{\mathcal{S}_\star,m} < \varepsilon, \qquad m \geq 4$$

then there exists a unique C^{m-2} solution $\tilde{\boldsymbol{g}}$ to the vacuum Einstein field equation with positive Cosmological constant over $[\tau_\star, \infty) \times \mathcal{S}_\star$ for $\tau_\star > 0$ which is future geodesically complete and whose restriction to $\mathcal{S}_\star$ implies the initial data $\hat{\mathbf{u}}_\star$. Moreover, the solution $\hat{\mathbf{u}}$ remains suitably close (in the Sobolev norm $\| \cdot \|_{\mathcal{S},m}$) to the background solution $\mathring{u}$.

Remark 24. It follows from Proposition 3 that there exists an open set of initial data for the Einstein field equations satisfying the hypothesis of the above theorem.

A On Kato's existence and stability result for symmetric hyperbolic systems

In this appendix we make some remarks concerning the hypothesis in Kato's existence, uniqueness and stability result for symmetric hyperbolic equations in [16]. The results in this reference and, in particular the main Theorem II, are very general and presented in an abstract manner. This abstract presentation hinders the direct applicability of the theory. The purpose of this Appendix is to provide a guide to the use of this theorem and to verify that the main evolution system in this article satisfies the hypothesis of the result.

Kato's theory is concerned with symmetric hyperbolic systems in which the unknown $\mathbf{u}$ is regarded as a $\mathcal{P}$-valued function over $\mathbb{R}^m$ where $\mathcal{P}$ is a Hilbert space. The Hilbert space can be real or complex and, in fact, infinite dimensional. In the present article we are interested in the case where $\mathcal{P}$ is finite dimensional —say, of dimension N. In this case the symmetric hyperbolic system becomes a *standard* partial differential equation. For concreteness we set here $\mathcal{P} = \mathbb{R}^N$ and $m = 3$. The following discussion of Kato's theorem will be made with this particular choice. in mind.

Kato's theorem is concerned with (N-dimensional) symmetric hyperbolic quasilinear systems of the form

$$\mathbf{A}^0(t, \underline{x}, \mathbf{u})\partial_t \mathbf{u} + \mathbf{A}^\alpha(t, \underline{x}, \mathbf{u})\partial_\alpha \mathbf{u} = \mathbf{F}(t, \underline{x}, \mathbf{u}). \tag{42}$$

for $0 \leq t \leq T$, $\underline{x} \in \mathbb{R}^3$, $\alpha = 1, 2, 3$, and initial conditions

$$\mathbf{u}(0, x) = \mathbf{u}_\star(\underline{x}). \tag{43}$$

In Kato's theory it is convenient to regard the coefficients $\mathbf{A}^0(t, \underline{x}, \mathbf{u})$ and $\mathbf{A}^\alpha(t, \underline{x}, \mathbf{u})$ as non-linear operators depending on t sending $\mathbb{R}^N$-valued functions (i.e. the vector $\mathbf{u}$) over $\mathbb{R}^3$ into $(N \times N)$-matrix valued functions on $\mathbb{R}^3$ —in Kato's terminology these are the elements of $\mathcal{B}(\mathcal{P})$, the space of bounded linear operators over $\mathcal{P}$. Similarly, $\mathbf{F}(t, \underline{x}, \mathbf{u})$ is regarded as an non-linear operator depending on t sending $\mathbb{R}^N$-valued functions on $\mathbb{R}^3$ into $\mathbb{R}^N$-valued functions on $\mathbb{R}^3$.

Consider now $H^s(\mathbb{R}^3, \mathbb{R}^N)$, the space of ($\mathbb{R}^N$)-vector valued functions over $\mathbb{R}^3$ such that their entries have finite Sobolev norm of order s. Let $\mathcal{D}$ be a bounded open subset of $H^s(\mathbb{R}^3, \mathbb{R}^N)$. Writing

$$\mathbf{A}^\mu(t, \underline{x}, \mathbf{u}) = \big(a^\mu_{ij}(t, \underline{x}, \mathbf{u})\big), \qquad \mathbf{F}(t, \underline{x}, \mathbf{u}) = \big(f_i(t, \underline{x}, \mathbf{u})\big), \qquad i, j = 1, \ldots N, \qquad \mu = 0, \ldots, 3,$$

one has that for fixed t and $\mathbf{u} \in \mathcal{D}$

$$a^\mu_{ij}(t, \underline{x}, \mathbf{u}) : \mathbb{R}^m \to \mathbb{R},$$
$$f_i(t, \underline{x}, \mathbf{u}) : \mathbb{R}^m \to \mathbb{R}.$$

Key in Kato's analysis are the *uniformly local Sobolev spaces* H^s_{ul}. Let $C^\infty_0(\mathbb{R}^3,\mathbb{R})$ denote the sets of smooth functions of compact support from $\mathbb{R}^3$ to $\mathbb{R}$. Given any non-zero $\phi \in C^\infty_0(\mathbb{R}^3,\mathbb{R})$ not identically zero, then $\mathbf{u} \in H^s_{ul}$ if and only if

$$\sup_{\underline{x}\in\mathbb{R}^3} \parallel \phi_x \mathbf{u} \parallel_s < \infty, \qquad \phi_x(\underline{y}) \equiv \phi(\underline{y}-\underline{x}).$$

Remark 25. In other words, the vector-valued function $\mathbf{u}$ is in H^s_{ul} if its Sobolev norm of order s over any compact set over $\mathbb{R}^3$ is finite and remains finite as one considers larger and larger compact sets on $\mathbb{R}^3$.

Remark 26. The spaces H^s_{ul} satisfy nice embedding properties analogous to those of H^s —see Lemma 2.7 in [16].

In the following it will be assumed that, for fixed t and $\mathbf{u} \in \mathcal{D}$, the coefficients $a^\mu_{ij}(t,\underline{x},\mathbf{u}(\underline{x}))$ are functions from $\mathcal{D}$ to $H^s_{ul}(\mathbb{R}^3,\mathbb{R})$. For $f_i(t,\underline{x},\mathbf{u}(\underline{x}))$ one has the more relaxed condition of being a function from $\mathcal{D}$ to $H^s(\mathbb{R}^3,\mathbb{R})$. In Kato's more abstract terminology this is equivalent to requiring that $\mathbf{A}^\mu$ is a function from $\mathcal{D}$ to $H^s_{ul}(\mathbb{R}^3,\mathcal{B}(\mathcal{P}))$ and $\mathbf{F}$ from $\mathcal{D}$ to $H^s(\mathbb{R}^3,\mathcal{P})$.

One has the following reformulation of Theorem II in [16]:

Theorem 2. *Let s be a positive integer such that $s > 3/2 + 1 = 5/2$. Let $\mathbf{A}^\mu(t,\underline{x},\mathbf{v}(\underline{x}))$, $\mathbf{F}(t,\underline{x},\mathbf{v}(\underline{x}))$ and $\mathbf{v} \in \mathcal{D}$ as above with $0 \le t \le T$. Assume that the following conditions hold:*

(i) The components $a^\mu_{ij}(t,\underline{x},\mathbf{v}(\underline{x}))$ (respectively, $f_i(t,\underline{x},\mathbf{v}(\underline{x}))$) are bounded in the H^s_{ul}-norm (respectively H^s-norm) for $\mathbf{v} \in \mathcal{D}$, uniformly in t.

(ii) For each t, the map $\mathbf{v}(\underline{x}) \mapsto \mathbf{A}^\alpha(t,\underline{x},\mathbf{v}(\underline{x}))$ is uniformly Lipschitz continuous on $\mathcal{D}$ from the H^0-norm to the H^0_{ul}-norm, uniformly in t. Similarly, the map $\mathbf{v}(\underline{x}) \mapsto \mathbf{F}(t,\underline{x},\mathbf{v}(\underline{x}))$ is Lipschitz continuous from the H^0-norm to the H^0-norm, again uniformly in t.

(iii) The map $\mathbf{v}(\underline{x}) \mapsto \mathbf{A}^0(t,\underline{x},\mathbf{v}(\underline{x}))$ is Lipschitz continuous on $\mathcal{D}$ from the H^{s-1}-norm to the H^{s-1}_{ul}-norm, uniformly in t.

(iv) The maps $t \mapsto \mathbf{A}^\alpha(t,\underline{x},\mathbf{v}(\underline{x}))$ are continuous in the H^0_{ul}-norm for each $\mathbf{v} \in \mathcal{D}$. Similarly, the map $t \mapsto \mathbf{F}(t,\underline{x},\mathbf{v}(\underline{x}))$ is continuous in the H^0-norm for each $\mathbf{v} \in \mathcal{D}$.

(v) The map $t \mapsto \mathbf{A}^0(t,\underline{x},\mathbf{v}(\underline{x}))$ is Lipschitz-continuous on $[0,T]$ in the H^{s-1}_{ul}-norm, uniformly for $\mathbf{v} \in \mathcal{D}$.

(vi) For each $\mathbf{v} \in \mathcal{D}$ the matrix-valued functions $\mathbf{A}^\mu(t,\underline{x},\mathbf{v}(\underline{x}))$ are symmetric for each $(t,\underline{x}) \in [0,T] \times \mathbb{R}^m$.

(vii) The matrix $\mathbf{A}^0(t,\underline{x},\mathbf{v}(\underline{x}))$ is positive definite with eigenvalues larger that, say, 1 for each $(t,\underline{x})$ and each $\mathbf{v} \in \mathcal{D}$.

(viii) $\mathbf{u}_\star \in \mathcal{D}$.

Then there is a unique solution $\mathbf{u}$ to (42)-(43) defined on $[0,T']$ where $0 < T' \le T$ such that

$$\mathbf{u} \in C[0,T';\mathcal{D}] \cup C^1[0,T';H^{s-1}(\mathbb{R}^3,\mathbb{R}^N)],$$

where T' can be chosen common to all initial conditions $\mathbf{u}_\star$ in a suitably small condition of a given point in $\mathcal{D}$.

In practice, the conditions of the above theorem are hard to verify. Kato provides sufficient conditions ensuring that conditions in the above theorem are satisfied (Theorem IV in [16]:

Theorem 3. *Suppose that $s > 3/2+1 = 5/2$. Let Ω be the subset of $\mathbb{R}^3 \times \mathbb{R}^N$ consisting of pairs $(\underline{x},\underline{v})$ such that*

$$|v - v_\star(x)| < \omega, \qquad \underline{x} \in \mathbb{R}^3$$

where $\omega > 0$ and $v_\star \in H^s(\mathbb{R}^3, \mathbb{R}^N) \subset C^1(\mathbb{R}^3, \mathbb{R}^N)$ are fixed. Let, as before,

$$\mathbf{A}^\mu : [0,T] \times \Omega \longrightarrow \mathcal{B}(\mathbb{R}^N),$$
$$\mathbf{F} : [0,T] \times \Omega \longrightarrow \mathbb{R}^N,$$

where $\mathcal{B}(\mathbb{R}^N)$ denotes the set of $(N \times N)$-matrix valued functions over $\mathbb{R}^3$ with the properties

(a) $\mathbf{A}^\alpha \in C[0,T; C_b^s(\Omega, \mathcal{B}(\mathbb{R}^N))]$,

(b) $\mathbf{A}^0 \in \mathrm{Lip}[0,T; C_b^{s-1}(\Omega, \mathcal{B}(\mathbb{R}^N))]$,

(c) $\mathbf{F} \in C[0,T; C_b^{s+1}(\Omega, \mathbb{R}^N)]$,

(d) $\mathbf{F}_\star \in L^\infty[0,T; H^s(\mathbb{R}^3, \mathbb{R}^N)] \cap C[0,T; H^0(\mathbb{R}^3, \mathbb{R}^N)]$,

where $\mathbf{F}_\star(t, \underline{x}) \equiv \mathbf{F}(t, \underline{x}, v_\star(\underline{x}))$. Then conditions (i)-(v) *in Theorem 2 are satisfied by $\mathbf{A}^\mu$, $\mathbf{F}$ provided that $\mathcal{D}$ is chosen as a ball in $H^s(\mathbb{R}^3, \mathbb{R}^N)$ with $v_\star$ as center and a sufficiently small radius $R_\star$. In addition,* (ix) *is satisfied if* (a) *is assumed to hold with s replaced by $s+1$.*

Remark 27. The sets $C_b^r(\Omega, \mathcal{B}(\mathbb{R}^N))$ and $C_b^r(\Omega, \mathbb{R}^N)$ denote the spaces of functions having derivatives up to the r-th order which are continuous and bounded in the supremum norm.

Remark 28. If the $\mathbf{A}^\mu$ are polynomials in $\underline{p}$ it actually suffices that the coefficients only be in $C[0,T; H^s_{ul}]$ and also in $C^1[0,T; H^{s-1}_{ul}]$ for $\mathbf{A}^0$.

B Geodesic completeness of the background solution

The geodesic completeness of the metric (10) can be shown using the theory developed in [4] —in particular, Corollary 3.3 in this reference applies to the present situation.

More precisely, the theory in [4] applies to spacetimes $(\mathcal{M}, \boldsymbol{g})$ such that $\mathcal{M} = [t_\bullet, \infty) \times \mathcal{S}$ where $t_\bullet > 0$ and $\mathcal{S}$ is a smooth 3-dimensional manifold. The metric $\boldsymbol{g}$ has the $3+1$ split

$$\boldsymbol{g} = -\alpha^2 \boldsymbol{\omega}^0 \otimes \boldsymbol{\omega}^0 + h_{\boldsymbol{ij}} \boldsymbol{\omega}^{\boldsymbol{i}} \otimes \boldsymbol{\omega}^{\boldsymbol{j}},$$

with

$$\boldsymbol{\omega}^0 = \mathbf{d}t, \qquad \boldsymbol{\omega}^{\boldsymbol{i}} = \mathbf{d}x^{\boldsymbol{i}} + \beta^{\boldsymbol{i}} \mathbf{d}t.$$

There exist numbers $0 < \alpha_-$, α_+ such that

$$0 < \alpha_- \leq \alpha \leq \alpha_+.$$

The metric $\boldsymbol{h} \equiv h_{\boldsymbol{ij}} \mathbf{d}x^{\boldsymbol{i}} \otimes \mathbf{d}x^{\boldsymbol{j}}$ is a *geodesically complete Riemannian metric* on $\mathcal{S}_t \equiv \{t\} \times \mathcal{S}$ such that there exists a constant $C_1 > 0$ such that

$$C_1 h_{\boldsymbol{ij}}(t_\bullet) v^{\boldsymbol{i}} v^{\boldsymbol{j}} \leq h_{\boldsymbol{ij}}(t) v^{\boldsymbol{i}} v^{\boldsymbol{j}}$$

for all vectors on $T\mathcal{S}$ and $t \in [t_\bullet, \infty)$. Furthermore, there exists another constant C_2 such that

$$\beta_i \beta^i \leq C_2, \qquad t \in [t_\bullet, \infty).$$

In the following let K_{ij} denote the extrinsic curvature of the hypersurfaces $\mathcal{S}_t$, $K_{\{ij\}}$ is tracefree part and K its trace.

With the above conditions, the metric $\boldsymbol{g}$ is future geodesically complete if the following two conditions hold:

(i) $D_i \alpha D^i \alpha$ is bounded by a function of t which is integrable on $[t_\bullet, \infty)$;

(ii) $K < 0$ and $K_{ij} K^{ij}$ is integrable on $[t_\bullet, \infty)$.

The metric (10) can be readily seen to satisfy the above conditions. In particular, as $\alpha = 1$, the norm of the spatial gradient of the lapse vanishes and, accordingly, it is integrable —this verifies condition (i) above. Moreover, the extrinsic curvature of the hypersurfaces of constant t is given by

$$K_{ij} = -\sinh t \cosh t \mathring{\gamma}_{ij},$$

so that it is pure trace. Moreover, one has that

$$K = -3 \coth t < 0, \qquad t \in [t_\bullet, \infty).$$

As $K_{\{ij\}} = 0$ in this case one has that (ii) is also satisfied. It follows then that the background metric $\mathring{\tilde{\boldsymbol{g}}}$ is future geodesically complete.

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