

VAN DER CORPUT LEMMAS FOR MITTAG-LEFFLER FUNCTIONS. II. α -DIRECTIONS

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Dedicated to Johannes Gaultherus van der Corput on the occasion of 100 years of his lemma

ABSTRACT. The paper is devoted to study analogues of the van der Corput lemmas involving Mittag-Leffler functions. The generalisation is that we replace the exponential function with the Mittag-Leffler-type function, to study oscillatory integrals appearing in the analysis of time-fractional partial differential equations. More specifically, we study integral of the form $I_{\alpha,\beta}(\lambda) = \int_{\mathbb{R}} E_{\alpha,\beta}(i^\alpha \lambda \phi(x)) \psi(x) dx$, for the range $0 < \alpha \leq 2$, $\beta > 0$. This extends the variety of estimates obtained in the first part, where integrals with functions $E_{\alpha,\beta}(i\lambda\phi(x))$ have been studied. Several generalisations of the van der Corput lemmas are proved. As an application of the above results, the generalised Riemann-Lebesgue lemma, the Cauchy problem for the time-fractional Klein-Gordon and time-fractional Schrödinger equations are considered.

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1. INTRODUCTION

In this paper we continue the study of oscillatory-type integrals involving Mittag-Leffler functions $E_{\alpha,\beta}$ initiated in [RT20a]. In the case of $\alpha = \beta = 1$, we have $E_{1,1}(z) = e^z$, thus reducing the integral to the classical question of decay of oscillatory integrals.

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Indeed, the estimate obtained by the Dutch mathematician Johannes Gaultherus van der Corput [vdC21] and named in his honour, following Stein [St93], can be stated as follows:

- **van der Corput lemma.** Suppose ϕ is a real-valued and smooth function in $[a, b]$. If ψ is a smooth function and $|\phi^{(k)}(x)| \geq 1$, $k \geq 1$, for all $x \in (a, b)$, then

$$\left| \int_a^b e^{i\lambda\phi(x)} \psi(x) dx \right| \leq C\lambda^{-1/k}, \quad \lambda \rightarrow \infty, \quad (1.1)$$

for $k = 1$ and ϕ' is monotonic, or $k \geq 2$. Here C does not depend on λ .

Various generalisations of the van der Corput lemmas have been investigated over the years [Gr05, SW70, St93, PS92, PS94, Rog05, Par08, Xi17]. Multidimensional analogues of the van der Corput lemmas were studied in [BG11, CCW99, CLTT05, GPT07, PSS01, KR07], while in [Ruz12] the multi-dimensional van der Corput lemma was obtained with constants independent of the phase and amplitude. Some estimates (essentially one-dimensional) in multi-dimensions for Mittag-Leffler functions were obtained in [RT20b].

The main goal of the present paper is to study van der Corput lemmas for the oscillatory integral defined by

$$I_{\alpha,\beta}(\lambda) = \int_{\mathbb{R}} E_{\alpha,\beta}(i^\alpha \lambda \phi(x)) \psi(x) dx, \quad (1.2)$$

where $0 < \alpha < 2$, $\beta > 0$, ϕ is a phase and ψ is an amplitude, and λ is a positive real number that can vary. Here $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function defined as (see e.g. [KST06, GKMR14])

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \quad \beta \in \mathbb{R},$$

with the property that

$$E_{1,1}(z) = e^z. \quad (1.3)$$

Since the function $E_{\alpha,\beta}(i^\alpha z)$, $z \in \mathbb{R}$, has a set of real zeros [GKMR14], the integral (1.2) is oscillating.

Here we can point out already one extension of (1.1) in view of (1.3), namely, an extension (in Theorem 3.5) to the range $0 < \alpha < 2$ in the form

$$\left| \int_a^b E_{\alpha,\alpha}(i^\alpha \lambda \phi(x)) \psi(x) dx \right| \leq C\lambda^{-1/k}, \quad \lambda \rightarrow \infty, \quad (1.4)$$

for $k = 1$ and ϕ' is monotonic, or $k \geq 2$.

This present paper is a continuation of [RT20a], where a variety of van der Corput type lemmas were obtained for the integral defined by

$$\tilde{I}_{\alpha,\beta}(\lambda) = \int_{\mathbb{R}} E_{\alpha,\beta}(i\lambda\phi(x)) \psi(x) dx, \quad (1.5)$$

where $0 < \alpha < 1$, $\beta > 0$.

As we see above, the integral (1.5) is different from the integral (1.2), since in (1.5) there is a purely imaginary number i before the phase function, and in (1.2) the fractional power of the imaginary number, i.e. i^α . Since the asymptotic behaviour of Mittag-Leffler function in these cases is also different, yielding different decay rates, in the integral (1.2) we will assume $0 < \alpha \leq 2, \beta \geq 1$, while the integral (1.5) was studied for $0 < \alpha \leq 1, \beta > 0$. Another difference between the function $E_{\alpha,\beta}(i^\alpha z)$, $z \in \mathbb{R}$, and the function $E_{\alpha,\beta}(iz)$, $z \in \mathbb{R}$, is that the function $E_{\alpha,\beta}(i^\alpha z)$ has a set of real zeros, while the function $E_{\alpha,\beta}(iz)$ has no zeros and is just bounded.

Such integrals as in (1.2) arise in the study of decay estimates of solutions of the time-fractional Schrödinger and the time-fractional wave equations (for example see [DX08, Gr19, Nab04, SZ20]). In Section 4 we will give several immediate applications of the obtained estimates to time-fractional Klein-Gordon and Schrödinger equations.

As in the case of (1.5) studied in [RT20a], we find that the decay rates of (1.2) as $\lambda \rightarrow \infty$ depend not only on the assumptions on the phase but also on the ranges of parameters α and β . We also obtain more results in the case of bounded intervals. For the convenience of the reader, let us briefly summarise the results of this paper, distinguishing between different sets of assumptions:

van der Corput lemmas on \mathbb{R} : consider $I_{\alpha,\beta}$ defined by (1.2).

- Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function and let $\psi \in L^1(\mathbb{R})$. Suppose that $0 < \alpha \leq 2, \beta > 1$, and $m = \text{ess inf}_{x \in \mathbb{R}} |\phi(x)| > 0$, then
 - (i): for $0 < \alpha < 2$ and $\beta \geq \alpha + 1$ we have

$$|I_{\alpha,\beta}(\lambda)| \leq \frac{M_1}{1 + \lambda m} \|\psi\|_{L^1(\mathbb{R})}, \lambda \geq 1,$$

where M_1 does not depend on ϕ, ψ and λ ;

- (ii): for $0 < \alpha < 2$ and $1 < \beta < \alpha + 1$ we have

$$|I_{\alpha,\beta}(\lambda)| \leq \frac{M_2}{(1 + \lambda m)^{\frac{\beta-1}{\alpha}}} \|\psi\|_{L^1(\mathbb{R})}, \lambda \geq 1,$$

where M_2 does not depend on ϕ, ψ and λ ;

- (iii): for $\alpha = 2$ and $\beta > 1$ we have

$$|I_{2,\beta}(\lambda)| \leq \frac{M_3}{(1 + \lambda m)^{\frac{\beta-1}{2}}} \|\psi\|_{L^1(\mathbb{R})}, \lambda \geq 1,$$

where M_3 does not depend on ϕ, ψ and λ .

- Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be an invertible and differentiable function, and let $\psi \in L^1(\mathbb{R})$. Suppose that $0 < \alpha \leq 2, \beta = 1$, and $m = \inf_{x \in \mathbb{R}} |\phi'(x)| > 0$, then

$$|I_{\alpha,1}(\lambda)| \leq \frac{M}{\lambda m} \|\psi\|_{L^1(\mathbb{R})}, \lambda \geq 1,$$

where M does not depend on ϕ, ψ and λ .

van der Corput lemmas on $I = [a, b] \subset \mathbb{R}$: consider

$$I_{\alpha,\beta}(\lambda) = \int_I E_{\alpha,\beta}(i^\alpha \lambda \phi(x)) \psi(x) dx,$$

where $I = [a, b] \subset \mathbb{R}$ with $-\infty < a < b < +\infty$.

- Let $0 < \alpha < 2$, $\beta > 1$, ϕ be a real-valued function such that $\phi \in C^k(I)$, $k \geq 1$, and let $\psi \in C^1(I)$. If $|\phi^{(k)}(x)| \geq 1$ for all $x \in I$, then
 - (i): for $0 < \alpha < 2$ and $\beta \geq \alpha + 1$ we have

$$|I_{\alpha,\beta}(\lambda)| \leq M_k \left[|\psi(b)| + \int_I |\psi'(x)| dx \right] \lambda^{-\frac{1}{k}} \log^{\frac{1}{k}}(1 + \lambda), \quad \lambda \geq 1,$$

where M_k does not depend on λ ;

- (ii): for $0 < \alpha < 2$ and $1 < \beta < \alpha + 1$ we have

$$|I_{\alpha,\beta}(\lambda)| \leq M_k \left[|\psi(b)| + \int_I |\psi'(x)| dx \right] \lambda^{-\frac{1}{k}} (1 + \lambda)^{\frac{\alpha+1-\beta}{\alpha k}}, \quad \lambda \geq 1,$$

where M_k does not depend on λ .

- Let $-\infty < a < b < +\infty$ and $I = [a, b] \subset \mathbb{R}$. Let $0 < \alpha < 2$ and let ϕ be a real-valued function such that $\phi \in C^k(I)$, $k \geq 1$. Let $\psi \in C^1(I)$ and $|\phi^{(k)}(x)| \geq 1$ for all $x \in I$. Then

$$|I_{\alpha,\alpha}(\lambda)| \leq M_k \left[|\psi(b)| + \int_I |\psi'(x)| dx \right] \lambda^{-1/k}, \quad \lambda \geq 1,$$

for $k = 1$ and ϕ' is monotonic, or $k \geq 2$. Here M_k does not depend on λ .

- Let $-\infty < a < b < +\infty$ and $I = [a, b] \subset \mathbb{R}$. Let $0 < \alpha < 2$ and let ϕ be a real-valued function such that $\phi \in C^2(I)$. Let $|\phi'(x)| \geq 1$ for all $x \in I$. Then

$$|I_{\alpha,\alpha}(\lambda)| \leq M \left[|\psi(b)| + \int_I |\psi'(x)| dx \right] \lambda^{-1}, \quad \lambda \geq 1,$$

where M does not depend on λ .

We will often make use of the following estimate.

Proposition 1.1 ([Pod99]). *If $0 < \alpha < 2$, β is an arbitrary real number, μ is such that $\pi\alpha/2 < \mu < \min\{\pi, \pi\alpha\}$, then there is $C_1, C_2 > 0$, such that we have*

$$|E_{\alpha,\beta}(z)| \leq C_1(1 + |z|)^{(1-\beta)/\alpha} \exp(\operatorname{Re}(z^{1/\alpha})) + \frac{C_2}{1 + |z|}, \quad z \in \mathbb{C}, \quad |\arg(z)| \leq \mu. \quad (1.6)$$

We are interested in particular in the behavior of $I_{\alpha,\beta}(\lambda)$ when λ is large, as for small λ the integral is just bounded.

2. VAN DER CORPUT LEMMA ON \mathbb{R}

In this section we consider $I_{\alpha,\beta}$ defined by (1.2), that is,

$$I_{\alpha,\beta}(\lambda) = \int_{\mathbb{R}} E_{\alpha,\beta}(i^\alpha \lambda \phi(x)) \psi(x) dx.$$

As for small λ the integral (1.2) is just bounded, we consider the case $\lambda \geq 1$.

Theorem 2.1. *Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function and let $\psi \in L^1(\mathbb{R})$. Suppose that $0 < \alpha \leq 2$, $\beta > 1$, and $m = \operatorname{ess\,inf}_{x \in \mathbb{R}} |\phi(x)| > 0$, then*

(i): *for $0 < \alpha \leq 2$ and $\beta \geq \alpha + 1$ we have*

$$|I_{\alpha,\beta}(\lambda)| \leq \frac{M_1}{1 + \lambda m} \|\psi\|_{L^1(\mathbb{R})}, \quad \lambda \geq 1, \quad (2.1)$$

where M_1 does not depend on ϕ , ψ and λ ;

(ii): *for $0 < \alpha \leq 2$ and $1 < \beta < \alpha + 1$ we have*

$$|I_{\alpha,\beta}(\lambda)| \leq \frac{M_2}{(1 + \lambda m)^{\frac{\beta-1}{\alpha}}} \|\psi\|_{L^1(\mathbb{R})}, \quad \lambda \geq 1, \quad (2.2)$$

where M_2 does not depend on ϕ , ψ and λ .

Proof. Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function and $\psi \in L^1(\mathbb{R})$. As $|\arg(i^\alpha \lambda \phi(x))| = \frac{\pi\alpha}{2}$ and $\operatorname{Re}(i\lambda^{1/\alpha}(\phi(x))^{1/\alpha}) = 0$, then using estimate (1.6) we have that

$$\begin{aligned} |I_{\alpha,\beta}(\lambda)| &\leq \int_{\mathbb{R}} |E_{\alpha,\beta}(i^\alpha \lambda \phi(x))| |\psi(x)| dx \\ &\leq C_1 \int_{\mathbb{R}} (1 + \lambda |\phi(x)|)^{(1-\beta)/\alpha} |\psi(x)| dx + C_2 \int_{\mathbb{R}} \frac{|\psi(x)|}{1 + \lambda |\phi(x)|} dx. \end{aligned}$$

As ϕ and ψ do not depend on λ , and $m = \operatorname{ess\,inf}_{x \in \mathbb{R}} |\phi(x)| > 0$, then for $\beta \geq \alpha + 1$ we have

$$\begin{aligned} |I_{\alpha,\beta}(\lambda)| &\leq \int_{\mathbb{R}} |E_{\alpha,\beta}(i^\alpha \lambda \phi(x))| |\psi(x)| dx \\ &\leq \max\{C_1, C_2\} \int_{\mathbb{R}} \frac{|\psi(x)|}{1 + \lambda |\phi(x)|} dx \\ &\leq \frac{M_1}{1 + \lambda m} \|\psi\|_{L^1(\mathbb{R})}. \end{aligned}$$

In the case $1 < \beta < \alpha + 1$ we have that

$$\begin{aligned} |I_{\alpha,\beta}(\lambda)| &\leq \int_{\mathbb{R}} |E_{\alpha,\beta}(i^\alpha \lambda \phi(x))| |\psi(x)| dx \\ &\leq \max\{C_1, C_2\} \int_{\mathbb{R}} \frac{|\psi(x)|}{(1 + \lambda |\phi(x)|)^{\frac{\beta-1}{\alpha}}} dx \\ &\leq \frac{M_2}{(1 + \lambda m)^{\frac{\beta-1}{\alpha}}} \|\psi\|_{L^1(\mathbb{R})}. \end{aligned}$$

The cases (i) and (ii) for $0 < \alpha < 2$, $\beta > 1$ are proved.

Now we will prove the case (ii), when $\alpha = 2$, $\beta > 1$. Applying the asymptotic estimate (see [KST06, page 43])

$$E_{2,\beta}(z) = \frac{1}{2}z^{(1-\beta)/2} \left(e^{\sqrt{z}} + e^{-\sqrt{z}-\pi i(1-\beta)\text{sign}(\arg z)} \right) - \sum_{k=1}^N \frac{z^{-k}}{\Gamma(\beta-2k)} + O\left(\frac{1}{z^{N+1}}\right), \quad |z| \rightarrow \infty, \quad |\arg(z)| \leq \pi,$$

we have the following estimate in the case $1 < \beta < 3$

$$\begin{aligned} |I_{2,\beta}(\lambda)| &\leq \int_{\mathbb{R}} |E_{2,\beta}(-\lambda\phi(x))| |\psi(x)| dx \\ &\leq M_2 \lambda^{(1-\beta)/2} \int_{\mathbb{R}} |\phi(x)|^{(1-\beta)/2} |\psi(x)| dx \\ &\leq M_2 m^{(1-\beta)/2} \lambda^{(1-\beta)/2} \int_{\mathbb{R}} |\psi(x)| dx \\ &\leq \frac{M_2 \|\psi\|_{L^1(\mathbb{R})}}{(1+m\lambda)^{(\beta-1)/2}}. \end{aligned}$$

Similarly, when $\beta \geq 3$ we have

$$|I_{2,\beta}(\lambda)| \leq \frac{M_1 \|\psi\|_{L^1(\mathbb{R})}}{1+m\lambda}.$$

The proof is complete. \square

It is easy to see from estimate (1.6) that, for $\beta = 1$, the function $E_{\alpha,1}(\cdot)$ does not decrease, and it will be only bounded function. In this case, the method of proving the Theorem 2.1 is not suitable. Below we give an estimate for the integral (1.2) in the case $\beta = 1$.

Theorem 2.2. *Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be an invertible and differentiable function, and let $\psi \in L^1(\mathbb{R})$. Suppose that $0 < \alpha \leq 2$, $\beta = 1$, and $m = \inf_{x \in \mathbb{R}} |\phi'(x)| > 0$, then*

$$|I_{\alpha,1}(\lambda)| \leq \frac{M}{\lambda m} \|\psi\|_{L^1(\mathbb{R})}, \quad \lambda \geq 1, \quad (2.3)$$

where M does not depend on ϕ , ψ and λ .

Proof. First, we prove the case $0 < \alpha < 2$ and $\beta = 1$. Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be an invertible and differentiable function, and let $\psi \in L^1(\mathbb{R})$. Then replacing $\lambda\phi(x)$ by y we get

$$I_{\alpha,1}(\lambda) = \int_{\mathbb{R}} E_{\alpha,1}(i^\alpha \lambda \phi(x)) \psi(x) dx = \frac{1}{\lambda} \int_{\mathbb{R}} E_{\alpha,1}(i^\alpha y) \psi((\phi^{-1}(y/\lambda))) \frac{dy}{\phi'(\phi^{-1}(y/\lambda))}.$$

Since $x \in \mathbb{R}$, then $\phi^{-1}(y/\lambda) \in \mathbb{R}$. As $|\arg(i^\alpha \lambda \phi(x))| = \frac{\pi\alpha}{2}$ and $\operatorname{Re}(i\lambda^{1/\alpha}(\phi(x))^{1/\alpha}) = 0$, then using (1.6) and $m = \inf_{x \in \mathbb{R}} |\phi'(x)| > 0$ we have

$$\begin{aligned} |I_{\alpha,1}(\lambda)| &\leq \frac{1}{m\lambda} \int_{\mathbb{R}} |E_{\alpha,1}(i^\alpha y)| |\psi((\phi^{-1}(y/\lambda)))| dy \\ &\leq \frac{M}{m\lambda} \int_{\mathbb{R}} |\psi((\phi^{-1}(y/\lambda)))| dy = \frac{M}{m\lambda} \|\psi\|_{L^1(\mathbb{R})}. \end{aligned}$$

If $\alpha = 2$ and $\beta = 1$, we have $E_{2,1}(i^2 \lambda \phi(x)) = \cos \sqrt{\lambda \phi(x)}$. As $|\cos \sqrt{\lambda \phi(x)}| \leq 1$, then repeating above calculations we have

$$|I_{2,1}(\lambda)| \leq \frac{1}{m\lambda} \|\psi\|_{L^1(\mathbb{R})}.$$

The proof is complete. \square

3. VAN DER CORPUT LEMMA ON FINITE INTERVALS

In this section we consider integral (1.2) in the finite interval $I = [a, b] \subset \mathbb{R}$, $-\infty < a < b < +\infty$, i.e.

$$I_{\alpha,\beta}(\lambda) = \int_I E_{\alpha,\beta}(i^\alpha \lambda \phi(x)) \psi(x) dx. \quad (3.1)$$

Since $I_{\alpha,\beta}(\lambda)$ is bounded for small λ , further we can assume that $\lambda \geq 1$.

Theorem 3.1. *Let $0 < \alpha < 2$, $\beta > 1$, ϕ be a real-valued function such that $\phi \in C^k(I)$, $k \geq 1$. If $|\phi^{(k)}(x)| \geq 1$ for all $x \in I$, then*

(i): *for $0 < \alpha < 2$ and $\beta \geq \alpha + 1$ we have*

$$\left| \int_I E_{\alpha,\beta}(i^\alpha \lambda \phi(x)) dx \right| \leq M_k \lambda^{-\frac{1}{k}} \log^{\frac{1}{k}}(1 + \lambda), \quad \lambda \geq 1, \quad (3.2)$$

where M_k does not depend on λ ;

(ii): *for $0 < \alpha < 2$ and $1 < \beta < \alpha + 1$ we have*

$$\left| \int_I E_{\alpha,\beta}(i^\alpha \lambda \phi(x)) dx \right| \leq M_k \lambda^{-\frac{1}{k}} (1 + \lambda)^{\frac{\alpha+1-\beta}{\alpha k}}, \quad \lambda \geq 1, \quad (3.3)$$

where M_k does not depend on λ .

Proof. Proof of (i). First we will prove the case $k = 1$. Since $|\phi'(x)| \geq 1$ for all $x \in I$, then $|\phi|$ is monotonic, and it can have one zero at $c \in [a, b]$. Let $c = a$ or $c = b$ and $\phi(x) \neq 0$ for all $x \in [a, b] \setminus \{c\}$. If $0 < \alpha < 2$ and $\beta \geq \alpha + 1$, then by (1.6) we have

$$\int_I |E_{\alpha,\beta}(i^\alpha \lambda \phi(x))| dx \leq C \int_I \frac{1}{1 + \lambda |\phi(x)|} dx,$$

where C is independent of ϕ and λ .

Let $c = a$. Since $\phi \in C^1(I)$ and $|\phi'(x)| \geq 1$, then

$$\begin{aligned} \int_I |E_{\alpha,\beta}(i^\alpha \lambda \phi(x))| dx &\leq \frac{C}{\lambda} \int_I \frac{1}{\phi'(x)} \frac{\lambda \phi'(x)}{1 + \lambda |\phi(x)|} dx \leq \frac{C}{\lambda} \int_I \frac{d(1 + \lambda |\phi(x)|)}{1 + \lambda |\phi(x)|} \\ &\leq \frac{C}{\lambda} \log(1 + \lambda |\phi(b)|) \leq \frac{M_1}{\lambda} \log(1 + \lambda), \end{aligned} \quad (3.4)$$

where M_1 is independent of λ .

Let $\phi(c) = 0$, $c \in (a, b)$. Then

$$|I_{\alpha,\beta}(\lambda)| \leq \left(\int_a^c + \int_c^b \right) |E_{\alpha,\beta}(i^\alpha \lambda \phi(x))| dx.$$

Further, repeating the above calculations we have (3.2) for $k = 1$.

Let $\lambda \geq 1$ and $k = 2$. We assume that (3.2) is true for $k = 1$ and let $|\phi''(x)| \geq 1$, for all $x \in I$, we prove the estimate (3.2) for $k = 2$. Let $d \in [a, b]$ be a unique point where $|\phi'(d)| \leq |\phi'(x)|$ for all $x \in I$. If $\phi'(d) = 0$, then we obtain $|\phi'(x)| \geq \epsilon$ on I outside $(d - \epsilon, d + \epsilon)$. Further, we will write $\int_a^b E_{\alpha,\beta}(i^\alpha \lambda \phi(x)) dx$ as

$$\int_a^b E_{\alpha,\beta}(i^\alpha \lambda \phi(x)) dx = \left(\int_a^{c-\epsilon} + \int_{c-\epsilon}^{c+\epsilon} + \int_{c+\epsilon}^b \right) E_{\alpha,\beta}(i^\alpha \lambda \phi(x)) dx.$$

As $|\phi'(x)| \geq \epsilon$ on I outside $(d - \epsilon, d + \epsilon)$, then by the case $k = 1$, we have

$$\left| \int_a^{c-\epsilon} E_{\alpha,\beta}(i^\alpha \lambda \phi(x)) dx \right| \leq M_2 \frac{\log(1 + \lambda)}{(\epsilon \lambda)},$$

and

$$\left| \int_{c+\epsilon}^b E_{\alpha,\beta}(i^\alpha \lambda \phi(x)) dx \right| \leq M_2 \frac{\log(1 + \lambda)}{(\epsilon \lambda)}.$$

As

$$\left| \int_{c-\epsilon}^{c+\epsilon} E_{\alpha,\beta}(i^\alpha \lambda \phi(x)) dx \right| \leq 2\epsilon,$$

we have

$$\left| \int_a^b E_{\alpha,\beta}(i^\alpha \lambda \phi(x)) dx \right| \leq 2M_2 \frac{\log(1 + \lambda)}{(\epsilon \lambda)} + 2\epsilon.$$

Taking $\epsilon = \lambda^{-\frac{1}{2}} \log^{\frac{1}{2}}(1 + \lambda)$ we obtain the estimate (3.2) for $k = 2$, which proves the result.

Let us prove the estimate (3.2) by the induction method for $k \geq 2$. We assume that (3.2) is true for $k \geq 2$. And assuming $|\phi^{(k+1)}(x)| \geq 1$, for all $x \in I$, we prove the estimate (3.2) for $k + 1$. Let $d \in [a, b]$ be a unique point where $|\phi^{(k)}(d)| \leq |\phi^{(k)}(x)|$

for all $x \in I$. If $\phi^{(k)}(d) = 0$, then we obtain $|\phi^{(k)}(x)| \geq \epsilon$ on I outside $(d - \epsilon, d + \epsilon)$.

Further, we will write $\int_a^b E_{\alpha,\beta}(i^\alpha \lambda \phi(x)) dx$ as

$$\int_a^b E_{\alpha,\beta}(i^\alpha \lambda \phi(x)) dx = \left(\int_a^{d-\epsilon} + \int_{d-\epsilon}^{d+\epsilon} + \int_{d+\epsilon}^b \right) E_{\alpha,\beta}(i^\alpha \lambda \phi(x)) dx.$$

By inductive hypothesis

$$\left| \int_a^{d-\epsilon} E_{\alpha,\beta}(i^\alpha \lambda \phi(x)) dx \right| \leq M_k(\epsilon \lambda)^{-\frac{1}{k}} \log^{\frac{1}{k}}(1 + \lambda),$$

and

$$\left| \int_{d+\epsilon}^b E_{\alpha,\beta}(i^\alpha \lambda \phi(x)) dx \right| \leq M_k(\epsilon \lambda)^{-\frac{1}{k}} \log^{\frac{1}{k}}(1 + \lambda).$$

As

$$\left| \int_{d-\epsilon}^{d+\epsilon} E_{\alpha,\beta}(i^\alpha \lambda \phi(x)) dx \right| \leq 2\epsilon,$$

we have

$$\left| \int_a^b E_{\alpha,\beta}(i^\alpha \lambda \phi(x)) dx \right| \leq 2M_k(\epsilon \lambda)^{-\frac{1}{k}} \log^{\frac{1}{k}}(1 + \lambda) + 2\epsilon.$$

Taking $\epsilon = \lambda^{-\frac{1}{k+1}} \log^{\frac{1}{k+1}}(1 + \lambda)$ we obtain the estimate (3.2) for $k + 1$, which proves the result.

Proof of (ii). Let $k = 1$. Since $|\phi'(x)| \geq 1$ for all $x \in I$, then $|\phi|$ is monotonic, and it can have one zero at $c \in [a, b]$. Let $c = a$ or $c = b$ and $\phi(x) \neq 0$ for all $x \in [a, b] \setminus \{c\}$. If $0 < \alpha < 2$ and $1 < \beta < \alpha + 1$, then by (1.6) we have

$$\begin{aligned} \left| \int_I E_{\alpha,\beta}(i^\alpha \lambda \phi(x)) dx \right| &\leq \int_I |E_{\alpha,\beta}(i^\alpha \lambda \phi(x))| dx \\ &\leq C \int_I \frac{1}{(1 + \lambda |\phi(x)|)^{\frac{\beta-1}{\alpha}}} dx, \end{aligned}$$

where C is independent of ϕ and λ . Let $c = a$, then by $\phi \in C^1(I)$ and $|\phi'(x)| \geq 1$, we have

$$\begin{aligned} \left| \int_I E_{\alpha,\beta} (i^\alpha \lambda \phi(x)) dx \right| &\leq \frac{C}{\lambda} \int_I \frac{1}{\phi'(x)} \frac{\lambda \phi'(x)}{(1 + \lambda |\phi(x)|)^{\frac{\beta-1}{\alpha}}} dx \\ &\leq \frac{C}{\lambda} \int_I \frac{d(1 + \lambda |\phi(x)|)}{(1 + \lambda |\phi(x)|)^{\frac{\beta-1}{\alpha}}} \\ &\leq \frac{C}{\lambda} \left[(1 + \lambda |\phi(b)|)^{\frac{\alpha+1-\beta}{\alpha}} - 1 \right] \\ &\leq \frac{C}{\lambda} \left[(1 + \lambda |\phi(b)|)^{\frac{\alpha+1-\beta}{\alpha}} \right] \\ &\leq \frac{M_2}{\lambda} (1 + \lambda)^{\frac{\alpha+1-\beta}{\alpha}}, \end{aligned}$$

where M_2 is independent of λ .

Let $\phi(c) = 0$, $c \in (a, b)$. Then

$$|I_{\alpha,\beta}(\lambda)| \leq \left(\int_a^c + \int_c^b \right) |E_{\alpha,\beta} (i^\alpha \lambda \phi(x))| dx.$$

Further, repeating the above calculations we have (3.3) for $k = 1$.

Let $\lambda \geq 1$, $0 < \alpha < 2$ and $1 < \beta < \alpha + 1$, $k = 2$. We assume that (3.3) holds for $k = 1$. Assuming $|\phi''(x)| \geq 1$, for all $x \in I$, we prove the estimate (3.3) for $k = 2$. Let $d \in [a, b]$ be a unique point where $|\phi'(d)| \leq |\phi'(x)|$ for all $x \in I$. If $\phi'(d) = 0$, then we obtain $|\phi'(x)| \geq \epsilon$ on I outside $(d - \epsilon, d + \epsilon)$. Further, we will write $\int_a^b E_{\alpha,\beta} (i^\alpha \lambda \phi(x)) dx$ as

$$\int_a^b E_{\alpha,\beta} (i^\alpha \lambda \phi(x)) dx = \left(\int_a^{d-\epsilon} + \int_{d-\epsilon}^{d+\epsilon} + \int_{d+\epsilon}^b \right) E_{\alpha,\beta} (i^\alpha \lambda \phi(x)) dx.$$

As $|\phi'(x)| \geq \epsilon$ on I outside $(d - \epsilon, d + \epsilon)$, by the case $k = 1$, we have

$$\left| \int_a^{d-\epsilon} E_{\alpha,\beta} (i^\alpha \lambda \phi(x)) dx \right| \leq M_1 \frac{(1 + \lambda)^{\frac{\alpha+1-\beta}{\alpha}}}{(\epsilon \lambda)},$$

and

$$\left| \int_{d+\epsilon}^b E_{\alpha,\beta} (i^\alpha \lambda \phi(x)) dx \right| \leq M_1 \frac{(1 + \lambda)^{\frac{\alpha+1-\beta}{\alpha}}}{(\epsilon \lambda)}.$$

As

$$\left| \int_{d-\epsilon}^{d+\epsilon} E_{\alpha,\beta} (i^\alpha \lambda \phi(x)) dx \right| \leq 2\epsilon,$$

we have

$$\left| \int_a^b E_{\alpha,\beta}(i^\alpha \lambda \phi(x)) dx \right| \leq 2M_1 \frac{(1+\lambda)^{\frac{\alpha+1-\beta}{\alpha}}}{(\epsilon\lambda)} + 2\epsilon.$$

Taking $\epsilon = \lambda^{-\frac{1}{2}}(1+\lambda)^{\frac{1+\alpha-\beta}{2\alpha}}$ we obtain the estimate (3.3) for $k = 2$, which proves the result. The cases when $c = a$ or $c = b$ can be proved similarly.

Let us prove the estimate (3.3) by induction method on $k \geq 2$. We assume that (3.3) holds for $k \geq 2$. Assuming $|\phi^{(k+1)}(x)| \geq 1$, for all $x \in I$, we prove the estimate (3.3) for $k+1$. Let $d \in [a, b]$ be a unique point where $|\phi^{(k)}(d)| \leq |\phi^{(k)}(x)|$ for all $x \in I$. If $\phi^{(k)}(d) = 0$, then we obtain $|\phi^{(k)}(x)| \geq \epsilon$ on I outside $(d - \epsilon, d + \epsilon)$. Further, we will write $\int_a^b E_{\alpha,\beta}(i^\alpha \lambda \phi(x)) dx$ as

$$\int_a^b E_{\alpha,\beta}(i^\alpha \lambda \phi(x)) dx = \left(\int_a^{d-\epsilon} + \int_{d-\epsilon}^{d+\epsilon} + \int_{d+\epsilon}^b \right) E_{\alpha,\beta}(i^\alpha \lambda \phi(x)) dx.$$

By inductive hypothesis

$$\left| \int_a^{d-\epsilon} E_{\alpha,\beta}(i^\alpha \lambda \phi(x)) dx \right| \leq M_k (\epsilon\lambda)^{\frac{1}{k}} (1+\lambda)^{\frac{\alpha+1-\beta}{\alpha k}},$$

and

$$\left| \int_{d+\epsilon}^b E_{\alpha,\beta}(i^\alpha \lambda \phi(x)) dx \right| \leq M_k (\epsilon\lambda)^{\frac{1}{k}} (1+\lambda)^{\frac{\alpha+1-\beta}{\alpha k}}.$$

As

$$\left| \int_{d-\epsilon}^{d+\epsilon} E_{\alpha,\beta}(i^\alpha \lambda \phi(x)) dx \right| \leq 2\epsilon,$$

we have

$$\left| \int_a^b E_{\alpha,\beta}(i^\alpha \lambda \phi(x)) dx \right| \leq 2M_k (\epsilon\lambda)^{\frac{1}{k}} (1+\lambda)^{\frac{\alpha+1-\beta}{\alpha k}} + 2\epsilon.$$

Taking $\epsilon = \lambda^{\frac{1}{k+1}}(1+\lambda)^{\frac{\alpha+1-\beta}{\alpha(k+1)}}$ we obtain the estimate (3.3) for $k+1$, which proves the result. The cases when $c = a$ or $c = b$ can be proved similarly. \square

Theorem 3.2. *Let $-\infty < a < b < +\infty$ and $I = [a, b] \subset \mathbb{R}$. Let $0 < \alpha < 2$, $\beta > 1$, ϕ is a real-valued function such that $\phi \in C^k(I)$, $k \geq 1$ and let $\psi \in C^1(I)$. If $|\phi^{(k)}(x)| \geq 1$ for all $x \in I$, then*

(i): *for $0 < \alpha < 2$ and $\beta \geq \alpha + 1$ we have*

$$|I_{\alpha,\beta}(\lambda)| \leq M_k \left[|\psi(b)| + \int_I |\psi'(x)| dx \right] \lambda^{-\frac{1}{k}} \log^{\frac{1}{k}}(1+\lambda), \quad \lambda \geq 1, \quad (3.5)$$

where M_k does not depend on λ ;

(ii): for $0 < \alpha < 2$ and $1 < \beta < \alpha + 1$ we have

$$|I_{\alpha,\beta}(\lambda)| \leq M_k \left[|\psi(b)| + \int_I |\psi'(x)| dx \right] \lambda^{-\frac{1}{k}} (1 + \lambda)^{\frac{\alpha+1-\beta}{\alpha k}}, \quad \lambda \geq 1, \quad (3.6)$$

where M_k does not depend on λ .

Proof. We write (3.1) as

$$\int_I E'(x) \psi(x) dx,$$

where

$$E(x) = \int_a^x E_{\alpha,\beta}(i^\alpha \lambda \phi(s)) ds.$$

Let $0 < \alpha < 2$ and $\beta \geq \alpha + 1$. Integrating by parts and applying the estimate of part (i) of Theorem 3.1 we obtain

$$|I_{\alpha,\beta}(\lambda)| \leq M_k \frac{\log(1 + \lambda)}{(1 + \lambda)^{1/k}} \left[|\psi(b)| + \int_I |\psi'(x)| dx \right].$$

The case (ii) can be proved similarly by applying results of part (ii) of Theorem 3.1. \square

Below we are interested in a particular case of the integral (3.1), when $0 < \alpha < 2$, $\beta = \alpha$, that is, the integral $I_{\alpha,\alpha}(\lambda)$. For smoother ϕ and ψ we get a better estimate than (3.3) and (3.6).

Theorem 3.3. *Let $-\infty < a < b < +\infty$ and $I = [a, b] \subset \mathbb{R}$. Let $0 < \alpha < 2$ and let ϕ be a real-valued function such that $\phi \in C^k(I)$, $k \geq 1$. Let $|\phi^{(k)}(x)| \geq 1$ for all $x \in I$, then*

$$\left| \int_I E_{\alpha,\alpha}(i^\alpha \lambda \phi(x)) dx \right| \leq M_k \lambda^{-\frac{1}{k}}, \quad \lambda \geq 1, \quad (3.7)$$

for $k = 1$ and ϕ' is monotonic, or $k \geq 2$. Here M_k does not depend on λ .

We note that the classical van der Corput lemma (1.1) is covered by (3.7) with $\alpha = 1$.

Proof. First we will prove the case $k = 1$. Let $0 < \alpha < 2$, $\lambda \geq 1$ and let ϕ has one zero $c \in [a, b]$. Let us consider the integral

$$\int_I E_{\alpha,\alpha}(i^\alpha \lambda \phi(x)) dx.$$

Then integrating by parts gives

$$\begin{aligned}
\int_I E_{\alpha,\alpha}(i^\alpha \lambda \phi(x)) dx &= \frac{\alpha}{i^\alpha \lambda} \int_I \frac{d}{d\phi(x)} (E_{\alpha,1}(i^\alpha \lambda \phi(x))) dx \\
&= \frac{\alpha}{i^\alpha \lambda} \int_I \frac{1}{\phi'(x)} \frac{d}{dx} (E_{\alpha,1}(i^\alpha \lambda \phi(x))) dx \\
&= \frac{\alpha}{i^\alpha \lambda} E_{\alpha,1}(i^\alpha \lambda \phi(b)) \frac{1}{\phi'(b)} - \frac{\alpha}{i^\alpha \lambda} E_{\alpha,1}(i^\alpha \lambda \phi(a)) \frac{1}{\phi'(a)} \\
&\quad - \frac{\alpha}{i^\alpha \lambda} \int_I E_{\alpha,1}(i^\alpha \lambda \phi(x)) \frac{d}{dx} \left(\frac{1}{\phi'(x)} \right) dx,
\end{aligned}$$

thanks to property

$$\frac{d}{dz} E_{\alpha,1}(z) = \frac{1}{\alpha} E_{\alpha,\alpha}(z)$$

and $|\phi'(x)| \geq 1$, $x \in I$. Then we have

$$\begin{aligned}
\left| \int_I E_{\alpha,\alpha}(i^\alpha \lambda \phi(x)) dx \right| &\leq \frac{\alpha}{\lambda} \int_I |E_{\alpha,1}(i^\alpha \lambda \phi(x))| \left| \frac{d}{dx} \left(\frac{1}{\phi'(x)} \right) \right| dx \\
&\quad + \frac{\alpha}{\lambda} |E_{\alpha,1}(i^\alpha \lambda \phi(b))| \frac{1}{|\phi'(b)|} \\
&\quad + \frac{\alpha}{\lambda} |E_{\alpha,1}(i^\alpha \lambda \phi(a))| \frac{1}{|\phi'(a)|}.
\end{aligned} \tag{3.8}$$

As ϕ' is monotonic and $\phi'(x) \geq 1$ for all $x \in [a, b]$, then $\frac{1}{\phi'}$ is also monotonic, and $\frac{d}{dx} \frac{1}{\phi'(x)}$ has a fixed sign. Hence estimate (1.6) and $\phi(c) = 0$, $c \in [a, b]$ implies

$$\begin{aligned}
\left| \int_I E_{\alpha,\alpha}(i^\alpha \lambda \phi(x)) dx \right| &\leq \frac{C\alpha}{\lambda} \int_I (1 + \lambda |\phi(x)|)^{\frac{1-\alpha}{\alpha}} \left| \frac{d}{dx} \left(\frac{1}{\phi'(x)} \right) \right| dx \\
&\quad + \frac{C\alpha}{\lambda} (1 + \lambda |\phi(b)|)^{\frac{1-\alpha}{\alpha}} + \frac{C\alpha}{\lambda} (1 + \lambda |\phi(a)|)^{\frac{1-\alpha}{\alpha}} \\
&\stackrel{|\phi(x)| \geq 0}{\leq} \frac{C\alpha}{\lambda} \int_I \left| \frac{d}{dx} \left(\frac{1}{\phi'(x)} \right) \right| dx + \frac{2C\alpha}{\lambda} \\
&\leq \frac{C\alpha}{\lambda} \left| \int_I \frac{d}{dx} \left(\frac{1}{\phi'(x)} \right) dx \right| + \frac{2C\alpha}{\lambda} \\
&\leq \frac{C\alpha}{\lambda} \left[2 + \frac{1}{|\phi'(b)|} + \frac{1}{|\phi'(b)|} \right] \leq \frac{M_1}{\lambda},
\end{aligned}$$

thanks to fixed sign of $\frac{d}{dx} \frac{1}{\phi'(x)}$. Here M_1 does not depend on λ .

We prove (3.7) for $k = 2$. Let for $d \in [a, b]$ satisfies $|\phi'(d)| \leq |\phi'(x)|$ for all $x \in [a, b]$. Then $|\phi'(x)| \geq \epsilon$ on $[a, b] \setminus (d - \epsilon, d + \epsilon)$. Hence

$$\int_I E_{\alpha, \alpha} (i^\alpha \lambda \phi(x)) dx = \left(\int_a^{d-\epsilon} + \int_{d-\epsilon}^{d+\epsilon} + \int_{d+\epsilon}^b \right) E_{\alpha, \alpha} (i^\alpha \lambda \phi(x)) dx.$$

Then $|\phi'(x)| \geq \epsilon$ on $[a, b] \setminus (d - \epsilon, d + \epsilon)$, implies

$$\left| \int_a^{d-\epsilon} E_{\alpha, \alpha} (i^\alpha \lambda \phi(x)) dx \right| \leq M_1(\epsilon \lambda)^{-1},$$

and

$$\left| \int_{d+\epsilon}^b E_{\alpha, \alpha} (i^\alpha \lambda \phi(x)) dx \right| \leq M_1(\epsilon \lambda)^{-1}.$$

As

$$\left| \int_{d-\epsilon}^{d+\epsilon} E_{\alpha, \alpha} (i^\alpha \lambda \phi(x)) dx \right| \leq 2\epsilon,$$

we have

$$\left| \int_a^b E_{\alpha, \alpha} (i^\alpha \lambda \phi(x)) dx \right| \leq 2M_1(\epsilon \lambda)^{-1} + 2\epsilon.$$

Taking $\epsilon = \lambda^{-\frac{1}{2}}$ we obtain the estimate (3.7) for $k = 2$.

We prove the case $k \geq 2$ by induction method. Let (3.7) is true for k , and suppose $|\phi^{(k+1)}(x)| \geq 1$, for all $x \in [a, b]$, we prove (3.7) for $k + 1$.

Let for $d \in [a, b]$ satisfies $|\phi^{(k)}(d)| \leq |\phi^{(k)}(x)|$ for all $x \in [a, b]$. Then $|\phi^{(k)}(x)| \geq \epsilon$ on $[a, b] \setminus (c - \epsilon, c + \epsilon)$. Therefore

$$\int_I E_{\alpha, \alpha} (i^\alpha \lambda \phi(x)) dx = \left(\int_a^{d-\epsilon} + \int_{d-\epsilon}^{d+\epsilon} + \int_{d+\epsilon}^b \right) E_{\alpha, \alpha} (i^\alpha \lambda \phi(x)) dx.$$

By inductive hypothesis

$$\left| \int_a^{d-\epsilon} E_{\alpha, \alpha} (i^\alpha \lambda \phi(x)) dx \right| \leq M_k(\epsilon \lambda)^{-\frac{1}{k}},$$

and

$$\left| \int_{d+\epsilon}^b E_{\alpha, \alpha} (i^\alpha \lambda \phi(x)) dx \right| \leq M_k(\epsilon \lambda)^{-\frac{1}{k}}.$$

As

$$\left| \int_{d-\epsilon}^{d+\epsilon} E_{\alpha, \alpha} (i^\alpha \lambda \phi(x)) dx \right| \leq 2\epsilon,$$

we have

$$\left| \int_a^b E_{\alpha,\alpha}(i^\alpha \lambda \phi(x)) dx \right| \leq 2M_k(\epsilon \lambda)^{-\frac{1}{k}} + 2\epsilon.$$

Taking $\epsilon = \lambda^{-\frac{1}{k+1}}$ we obtain the estimate (3.7) for $k+1$. \square

Below we show that if ϕ' is not monotonic, then to obtain estimate (3.7) when $k=1$, it is necessary to increase the smoothness of function ϕ .

Theorem 3.4. *Let $-\infty < a < b < +\infty$ and $I = [a, b] \subset \mathbb{R}$. Let $0 < \alpha < 2$ and let ϕ be a real-valued function such that $\phi \in C^2(I)$. Let $|\phi'(x)| \geq 1$ for all $x \in I$, then*

$$\left| \int_I E_{\alpha,\alpha}(i^\alpha \lambda \phi(x)) dx \right| \leq M \lambda^{-1}, \quad \lambda \geq 1,$$

where M does not depend on λ .

Proof. Suppose that $\phi \in C^2(I)$ and $|\phi'(x)| \geq 1$ for all $x \in I$, then from (3.8) we have

$$\begin{aligned} \left| \int_I E_{\alpha,\alpha}(i^\alpha \lambda \phi(x)) dx \right| &\leq \frac{\alpha}{\lambda} \int_I |E_{\alpha,1}(i^\alpha \lambda \phi(x))| \left| \frac{d}{dx} \left(\frac{1}{\phi'(x)} \right) \right| dx \\ &\quad + \frac{\alpha}{\lambda} |E_{\alpha,1}(i^\alpha \lambda \phi(b))| \\ &\quad + \frac{\alpha}{\lambda} |E_{\alpha,1}(i^\alpha \lambda \phi(a))|. \end{aligned}$$

Since $\phi \in C^2(I)$ and $|\phi'(x)| \geq 1$ for all $x \in I$, then the function $\frac{d}{dx} \left(\frac{1}{\phi'(x)} \right)$ will be continuous and bounded, and therefore by (1.6) we have

$$\begin{aligned} \left| \int_I E_{\alpha,\alpha}(i^\alpha \lambda \phi(x)) dx \right| &\leq \frac{C\alpha}{\lambda} \left[\int_I \left| \frac{d}{dx} \left(\frac{1}{\phi'(x)} \right) \right| dx + 2 \right] \\ &\leq \frac{C\alpha}{\lambda} (M_1(b-a) + 2) \leq \frac{M}{\lambda}, \end{aligned}$$

where $M_1 = \left\| \frac{d}{dx} \left(\frac{1}{\phi'(x)} \right) \right\|_{L^\infty(I)}$, $C \geq |E_{\alpha,1}(z)|$ and M is a constant independent of λ . \square

Theorem 3.5. *Let $0 < \alpha < 2$ and let ϕ be a real-valued function such that $\phi \in C^k(I)$, $k \geq 1$. Let $\psi \in C^1(I)$ and $|\phi^{(k)}(x)| \geq 1$ for all $x \in I$, then*

$$|I_{\alpha,\alpha}(\lambda)| \leq M_k \left[|\psi(b)| + \int_I |\psi'(x)| dx \right] \lambda^{-1/k}, \quad \lambda \geq 1,$$

for $k=1$ and ϕ' is monotonic, or $k \geq 2$. Here M_k does not depend on λ .

Theorem 3.5 can be proved similarly as Theorem 3.1.

The case of $\alpha=1$ corresponds to the classical van der Corput lemma (1.1).

Also, for $k=1$, Theorem 3.5 holds if we replace the condition that ϕ' is monotonic by $\phi \in C^2(I)$.

4. APPLICATIONS

In this section we give some applications of van der Corput lemmas involving Mittag-Leffler function.

4.1. Applications to the fractional evolution equations.

4.1.1. *Decay estimates for the time-fractional Klein-Gordon equation.* Consider the time-fractional Klein-Gordon equation

$$D_{0+,t}^\alpha u(t, x) + i^\alpha u_{xx}(t, x) - i^\alpha \mu u(t, x) = 0, \quad t > 0, \quad x \in \mathbb{R}, \quad (4.1)$$

with initial data

$$I_{0+,t}^{2-\alpha} u(0, x) = 0, \quad x \in \mathbb{R}, \quad (4.2)$$

$$\partial_t I_{0+,t}^{2-\alpha} u(0, x) = \psi(x), \quad x \in \mathbb{R}, \quad (4.3)$$

where $\mu > 0$, and,

$$I_{0+,t}^\alpha u(t, x) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s, x) ds$$

and

$$D_{0+,t}^\alpha u(t, x) = \frac{1}{\Gamma(2-\alpha)} \partial_t^2 \int_0^t (t-s)^{1-\alpha} u(s, x) ds$$

is the Riemann-Liouville fractional integral and derivative of order $1 < \alpha \leq 2$.

If $\alpha = 2$, then from (4.1) we obtain the classical Klein-Gordon equation.

Applying the Fourier transform \mathcal{F} to problem (4.1)-(4.3) with respect to space variable x yields

$$D_{0+,t}^\alpha \hat{u}(t, \xi) - i^\alpha (\xi^2 + \mu) \hat{u}(t, \xi) = 0, \quad t > 0, \quad \xi \in \mathbb{R}, \quad (4.4)$$

$$I_{0+,t}^{2-\alpha} \hat{u}(0, \xi) = 0, \quad \xi \in \mathbb{R}, \quad (4.5)$$

$$\partial_t I_{0+,t}^{2-\alpha} \hat{u}_t(0, \xi) = \hat{\psi}(\xi), \quad \xi \in \mathbb{R}, \quad (4.6)$$

due to $\mathcal{F}\{u_{xx}(t, x)\} = -\xi^2 \hat{u}(t, \xi)$. The general solution of equation (4.4) can be represented as

$$\hat{u}(t, \xi) = C_1(\xi) t^{\alpha-1} E_{\alpha,\alpha}(i^\alpha (\xi^2 + \mu) t^\alpha) + C_2(\xi) t^{\alpha-2} E_{\alpha,\alpha-1}(i^\alpha (\xi^2 + \mu) t^\alpha),$$

where $C_1(\xi)$ and $C_2(\xi)$ are unknown coefficients. Then by initial conditions (4.5)-(4.6) we have

$$\hat{u}(t, \xi) = \hat{\psi}(\xi) t^{\alpha-1} E_{\alpha,\alpha}(i^\alpha (\xi^2 + \mu) t^\alpha).$$

By applying the inverse Fourier transform \mathcal{F}^{-1} we have

$$u(t, x) = \int_{\mathbb{R}} e^{ix\xi} t^{\alpha-1} E_{\alpha,\alpha}(i^\alpha (\xi^2 + \mu) t^\alpha) \hat{\psi}(\xi) d\xi, \quad (4.7)$$

where $\hat{\psi}(\xi) = \frac{1}{\pi} \int_{\mathbb{R}} e^{-iy\xi} \psi(y) dy$.

Suppose that $\psi \in L^1(\mathbb{R})$ and $\hat{\psi} \in L^1(\mathbb{R})$. As

$$\inf_{\xi \in \mathbb{R}} (\xi^2 + \mu) = \mu > 0,$$

then using Theorem 2.1 (ii) we obtain the dispersive estimate

$$\begin{aligned} \|u(t, \cdot)\|_{L^\infty(\mathbb{R})} &\leq Ct^{\alpha-1}(1+t^\alpha)^{\frac{1-\alpha}{\alpha}} \|\hat{\psi}\|_{L^1(\mathbb{R})} \\ &\leq Ct^{\alpha-1}(1+t)^{1-\alpha} \|\hat{\psi}\|_{L^1(\mathbb{R})}, \quad t > 0. \end{aligned}$$

4.1.2. *Decay estimates for the time-fractional Schrödinger equation.* Consider the time-fractional Schrödinger equation

$$\mathcal{D}_{0+,t}^\alpha u(t, x) + i^\alpha u_{xx}(t, x) - i^\alpha \mu u(t, x) = t^{-\gamma} \psi(x), \quad t > 0, \quad x \in \mathbb{R}, \quad (4.8)$$

with Cauchy data

$$u(0, x) = 0, \quad x \in \mathbb{R}, \quad (4.9)$$

where $\mu > 0$, $0 \leq \gamma < \alpha$, and,

$$\mathcal{D}_{0+,t}^\alpha u(t, x) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{1-\alpha} \partial_s u(s, x) ds$$

is the Caputo fractional derivative of order $0 < \alpha \leq 1$.

When $\alpha = 1$, we have the classical Schrödinger equation.

Applying the Fourier transform \mathcal{F} to problem (4.8)-(4.9) with respect to space variable x yields

$$D_{0+,t}^\alpha \hat{u}(t, \xi) - i^\alpha (\xi^2 + \mu) \hat{u}(t, \xi) = t^{-\gamma} \hat{\psi}(\xi), \quad t > 0, \quad \xi \in \mathbb{R}, \quad (4.10)$$

$$\hat{u}(0, \xi) = 0, \quad \xi \in \mathbb{R}. \quad (4.11)$$

The solution of problem (4.10)-(4.11) can be represented as

$$\hat{u}(t, \xi) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} (i^\alpha (\xi^2 + \mu) (t-s)^\alpha) s^{-\gamma} \hat{\psi}(\xi) ds.$$

Calculating the above integral we obtain

$$\hat{u}(t, \xi) = \Gamma(1-\gamma) \hat{\psi}(\xi) t^{\alpha-\gamma} E_{\alpha,\alpha-\gamma+1} (i^\alpha (\xi^2 + \mu) t^\alpha).$$

By applying the inverse Fourier transform \mathcal{F}^{-1} we have

$$u(t, x) = \Gamma(1-\gamma) \int_{\mathbb{R}} e^{ix\xi} t^{\alpha-\gamma} E_{\alpha,\alpha-\gamma+1} (i^\alpha (\xi^2 + \mu) t^\alpha) \hat{\psi}(\xi) d\xi, \quad (4.12)$$

where $\hat{\psi}(\xi) = \frac{1}{\pi} \int_{\mathbb{R}} e^{-iy\xi} \psi(y) dy$.

Suppose that $\psi \in L^1(\mathbb{R})$ and $\hat{\psi} \in L^1(\mathbb{R})$. As

$$\inf_{\xi \in \mathbb{R}} (\xi^2 + \mu) = \mu > 0$$

and $\alpha - \gamma + 1 < \alpha + 1$, then using Theorem 2.1 (ii) we obtain the estimate

$$\begin{aligned} \|u(t, \cdot)\|_{L^\infty(\mathbb{R})} &\leq Ct^{\alpha-\gamma}(1+t^\alpha)^{\frac{\gamma-\alpha}{\alpha}} \|\hat{\psi}\|_{L^1(\mathbb{R})} \\ &\leq Ct^{\alpha-\gamma}(1+t)^{\gamma-\alpha} \|\hat{\psi}\|_{L^1(\mathbb{R})}, \quad t > 0. \end{aligned}$$

4.2. Generalised Riemann-Lebesgue lemma. The Riemann-Lebesgue lemma is the classical result of harmonic and asymptotic analysis. The simplest form of the Riemann-Lebesgue lemma states that for a function $f \in L^1(\mathbb{R})$ we have

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}} e^{ikx} f(x) dx = 0. \quad (4.13)$$

If $f \in C^1([a, b])$, then

$$\int_a^b e^{ikx} f(x) dx = \mathcal{O}\left(\frac{1}{k}\right), \text{ at } k \rightarrow \infty. \quad (4.14)$$

As a generalisation of Fourier integral we consider the following integral

$$\int_D E_{\alpha, \beta}(i^\alpha kx) f(x) dx, \quad (4.15)$$

where $0 < \alpha \leq 2$, $\beta \geq 1$, and $D \subseteq \mathbb{R}$.

Suppose that $D = \mathbb{R}$ and $f \in L^1(\mathbb{R})$, then Theorem 2.2 yields

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}} E_{\alpha, 1}(i^\alpha kx) f(x) dx = 0.$$

For $\alpha = 1$ this reduces to (4.13).

If $f \in C^1([a, b])$, then from the van der Corput lemmas we obtain

- for $0 < \alpha < 2$ and $\beta \geq \alpha + 1$, by Theorem 3.2 (i) we have

$$\int_a^b E_{\alpha, \beta}(i^\alpha kx) f(x) dx = \mathcal{O}\left(\frac{\log(1+k)}{k}\right);$$

- for $0 < \alpha < 2$ and $1 < \beta < \alpha + 1$, by Theorem 3.2 (ii) we have

$$\int_a^b E_{\alpha, \beta}(i^\alpha kx) f(x) dx = \mathcal{O}\left(k^{-1}(1+k)^{\frac{\alpha+1-\beta}{\alpha}}\right);$$

- for $0 < \alpha < 2$ and $\beta = \alpha$, by Theorem 3.5 we have

$$\int_a^b E_{\alpha, \alpha}(i^\alpha kx) f(x) dx = \mathcal{O}(k^{-1}).$$

CONCLUSION

The main goal of the paper was to study van der Corput lemmas for the integral defined by

$$I_{\alpha, \beta}(\lambda) = \int_a^b E_{\alpha, \beta}(i^\alpha \lambda \phi(x)) \psi(x) dx,$$

with $0 < \alpha \leq 2$, $\beta \geq 1$.

For $I_{\alpha,\beta}(\lambda)$, van der Corput type lemmas were obtained, for the following cases of parameters α and β :

- $0 < \alpha < 2$ and $\beta \geq \alpha + 1$;
- $0 < \alpha < 2$ and $1 < \beta < \alpha + 1$;
- $\alpha = 2$ and $\beta \geq 1$;
- $0 < \alpha < 2$ and $\beta = \alpha$.

As an immediate application of the obtained results, time-estimates of the solutions of time-fractional Klein-Gordon and time-fractional Schrödinger equations and generalisations of the Riemann-Lebesgue lemma were considered.

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